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Absolute profinite rigidity and hyperbolic geometry: Supplemental Magma code

By M. R. Bridson, D. B. McReynolds, A. W. Reid, and R. Spitler

1. Introduction

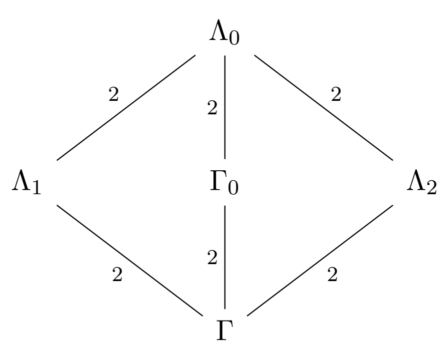
This document is a supplement to our paper [1]. It contains the Magma codes used to verify several group theoretic facts needed in the proofs of the main results of [1]. Before giving the Magma codes and their output, we recall some notation from [1].

In what follows, Λ_0 is the group of isometries of hyperbolic 3-space generated by reflections in the faces of the tetrahedron $T_0 = T[3,2,2;6,2,3]$. The index 2 subgroup of Λ_0 formed by the orientation-preserving isometries is $\Gamma_0 \cong \mathrm{PGL}(2,\mathbb{Z}[\omega])$.

The group Λ_1 is generated by reflections in the face of the tetrahedron $T_1 = T[3,2,2;3,3,3]$; this is an index 2 subgroup of Λ_0. The index 2 subgroup of Λ_1 formed by the orientation-preserving isometries is $\Gamma = \mathrm{PSL}(2,\mathbb{Z}[\omega])$.

The group Λ_0 has a third subgroup of index 2 that will be denoted by Λ_2. One can check that Λ_2 contains Γ as a subgroup of index 2:

(1)



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The groups Λ_0, Γ_0, Λ_1, Γ, and Λ_2 can be presented as follows:

$$\begin{aligned}
\Lambda_0 &= \big\langle x,y,z,w \ \mid \ x^2=y^2=z^3=w^2=(xy)^3=(xz)^2=(xw)^2 \\
&\qquad\qquad\qquad = (yz)^3=(yw)^2=(zw)^6=1\big\rangle \\
\Gamma_0 &= \big\langle x,y,z \ \mid \ x^3=y^2=z^2=(yz)^6=(zx^{-1})^2=(yx^{-1})^3=1\big\rangle \\
\Lambda_1 &= \big\langle x,y,z,w \ \mid \ x^2=y^2=z^2=w^2=(xy)^2=(xz)^2=(xw)^3 \\
&\qquad\qquad\qquad = (yz)^3=(yw)^3=(zw)^3=1\big\rangle \\
\Gamma &= \big\langle x,y,z \ \mid \ x^3=y^2=z^2=(yz)^3=(zx^{-1})^3=(yx^{-1})^3=1\big\rangle \\
\Lambda_2 &= \big\langle x,y,z \ \mid \ y^2=x^2=z^3=zyz^{-1}y=(zx)^3=(xy)^6=1\big\rangle .
\end{aligned}$$

Finally, Γ_W is the fundamental group of the Weeks manifold; it has the following presentation

$$\Gamma_W = \big\langle a,b \ \mid \ ababa^{-1}b^2a^{-1}b = abab^{-1}a^2b^{-1}ab = 1\big\rangle .$$

We remind the reader that Magma uses the notation $[m,n,0]$ to declare that the abelianization of a group is $\mathbb{Z}/m\mathbb{Z}\times\mathbb{Z}/n\mathbb{Z}\times\mathbb{Z}$.

2. **The proof of [1, Lemma 7.2]**

What follows is the Magma routine to complete the proof of [1, Lemma 7.2]. Here, `g` is the group Γ, presented as in (1). The routine computes subgroups of index n less than or equal to 12, prints the number of such and computes their abelianizations.

```
> g<a,b,c>:=Group<a,b,c|a^3,b^2,c^2,(b*c)^3,(c*a^-1)^3,(b*a^-1)^3>;
> print AbelianQuotientInvariants(g);
[ 3 ]
> l:=LowIndexSubgroups(g,<3,3>);
> print #l;
1
> print AbelianQuotientInvariants(l[1]);
[ 2, 2 ]
> l:=LowIndexSubgroups(g,<4,4>);
> print #l;
1
> print AbelianQuotientInvariants(l[1]);
[ 3, 3 ]
> l:=LowIndexSubgroups(g,<5,5>);
> print #l;
1
> print AbelianQuotientInvariants(l[1]);
[ 3, 3 ]
> l:=LowIndexSubgroups(g,<6,6>);
> print #l;
```

```
2
> print AbelianQuotientInvariants(l[1]);
[ 2, 0 ]
> print AbelianQuotientInvariants(l[2]);
[ 6 ]
> l:=LowIndexSubgroups(g,<7,7>);
> print #l;
4
> print AbelianQuotientInvariants(l[1]);
[ 6 ]
> print AbelianQuotientInvariants(l[2]);
[ 6 ]
> print AbelianQuotientInvariants(l[3]);
[ 6 ]
> print AbelianQuotientInvariants(l[4]);
[ 6 ]
> l:=LowIndexSubgroups(g,<8,8>);
> print #l;
2
> print AbelianQuotientInvariants(l[1]);
[ 3, 3 ]
> print AbelianQuotientInvariants(l[2]);
[ 3, 3 ]
> l:=LowIndexSubgroups(g,<9,9>);
> print #l;
1
> print AbelianQuotientInvariants(l[1]);
[ 2, 2 ]
> l:=LowIndexSubgroups(g,<10,10>);
> print #l;
1
> print AbelianQuotientInvariants(l[1]);
[ 6, 0 ]
> l:=LowIndexSubgroups(g,<11,11>);
> print #l;
0
> l:=LowIndexSubgroups(g,<12,12>);
> print #l;
7
> print AbelianQuotientInvariants(l[1]);
[ 0 ]
> print AbelianQuotientInvariants(l[2]);
```

```
[ 5, 0 ]
> print AbelianQuotientInvariants(l[3]);
[ 3, 9 ]
> print AbelianQuotientInvariants(l[4]);
[ 3, 9 ]
> print AbelianQuotientInvariants(l[5]);
[ 3, 3, 3 ]
> print AbelianQuotientInvariants(l[6]);
[ 2, 0 ]
> print AbelianQuotientInvariants(l[7]);
[ 3, 3, 3 ]
```

3. The proof of [1, Proposition 7.3]

What follows is the Magma routine to complete the proof of [1, Proposition 7.3]. This runs through all of the index 5 subgroups in the groups under consideration and calculates their abelianizations. We again use the presentation of Γ from (1). We consider those subgroups listed in the previous Magma routine that have first Betti number $b_1 = 1$, except that we ignore the index 12 subgroup of Γ with abelianization $\mathbb{Z}$, because this is the fundamental group of the figure-eight knot complement, which can be eliminated from consideration because its unique 5–fold cyclic cover is a once-punctured torus bundle over the circle with $b_1 = 1$. The last group whose index 5 subgroups are analysed below is the fundamental group of the sister of the figure-eight knot complement.

```
g<a,b,c>:=Group<a,b,c|a^3,b^2,c^2,(b*c)^3,(c*a^-1)^3,(b*a^-1)^3>;
> l:=LowIndexSubgroups(g,<6,6>);
> print #l;
2
> print AbelianQuotientInvariants(l[1]);
[ 2, 0 ]
> h:=Rewrite(g,l[1]);
> k:=LowIndexSubgroups(h,<5,5>);
> print #k;
4
> print AbelianQuotientInvariants(k[1]);
[ 2, 0, 0 ]
> print AbelianQuotientInvariants(k[2]);
[ 2, 0 ]
> print AbelianQuotientInvariants(k[3]);
[ 2, 0, 0, 0 ]
> print AbelianQuotientInvariants(k[4]);
[ 2, 0, 0 ]
> l:=LowIndexSubgroups(g,<10,10>);
> print #l;
```

```
1
> print AbelianQuotientInvariants(l[1]);
[ 6, 0 ]
> h:=Rewrite(g,l[1]);
> k:=LowIndexSubgroups(h,<5,5>);
> print #k;
6
> print AbelianQuotientInvariants(k[1]);
[ 3, 6, 0 ]
> print AbelianQuotientInvariants(k[2]);
[ 3, 6, 0 ]
> print AbelianQuotientInvariants(k[3]);
[ 3, 6, 0 ]
> print AbelianQuotientInvariants(k[4]);
[ 6, 0 ]
> print AbelianQuotientInvariants(k[5]);
[ 2, 6, 0, 0 ]
> print AbelianQuotientInvariants(k[6]);
[ 3, 6, 0, 0 ]
> l:=LowIndexSubgroups(g,<12,12>);
> print #l;
7
> print AbelianQuotientInvariants(l[6]);
[ 2, 0 ]
> h:=Rewrite(g,l[6]);
> k:=LowIndexSubgroups(h,<5,5>);
> print #k;
4
> print AbelianQuotientInvariants(k[1]);
[ 2, 0, 0, 0 ]
> print AbelianQuotientInvariants(k[2]);
[ 2, 0, 0, 0 ]
> print AbelianQuotientInvariants(k[3]);
[ 2, 0, 0, 0 ]
> print AbelianQuotientInvariants(k[4]);
[ 2, 0 ]
> print AbelianQuotientInvariants(l[2]);
[ 5, 0 ]
> h:=Rewrite(g,l[2]);
> k:=LowIndexSubgroups(h,<5,5>);
> print #k;
8
```

```
> print AbelianQuotientInvariants(k[1]);
[ 0, 0, 0, 0, 0 ]
> print AbelianQuotientInvariants(k[2]);
[ 5, 5, 0 ]
> print AbelianQuotientInvariants(k[3]);
[ 5, 5, 0 ]
> print AbelianQuotientInvariants(k[4]);
[ 5, 5, 0 ]
> print AbelianQuotientInvariants(k[5]);
[ 5, 25, 0 ]
> print AbelianQuotientInvariants(k[6]);
[ 0, 0, 0 ]
> print AbelianQuotientInvariants(k[7]);
[ 0, 0, 0 ]
> print AbelianQuotientInvariants(k[8]);
[ 5, 5, 0 ]
```

4. The proof of [1, Theorem 8.2]

The following routine is used in the proof of [1, Theorem 8.2], where we promote the profinite rigidity of Γ to profinite rigidity for each of its index 2 extensions, Γ_0, Λ_1, Λ_2 and $\Gamma \times \mathbb{Z}/2\mathbb{Z}$. The group g below is Λ_0, presented as in (1), while z is the group Λ_2, presented by applying "Rewrite" to the group l[2], and bia is the group $\Gamma \times \mathbb{Z}/2\mathbb{Z}$.

```
> g<r1,r2,r3,r4>:=Group<r1,r2,r3,r4|r1^2,r2^2,r3^2,r4^2,(r1*r2)^3,(r1*r3)^2,(r1*r4)^2,
> (r2*r3)^3,(r2*r4)^2,(r3*r4)^6>;
> print AbelianQuotientInvariants(g);
[ 2, 2 ]
> l:=LowIndexSubgroups(g,<2,2>);
> print AbelianQuotientInvariants(l[1]);
[ 2 ]
> print AbelianQuotientInvariants(l[2]);
[ 6 ]
> print AbelianQuotientInvariants(l[3]);
[ 2 ]
> print l[1];
Finitely presented group on 3 generators, Index in group g is 2,
Generators as words in group g
    $.1 = r2 * r1, $.2 = r3 * r1, $.3 = r4 * r1
> print l[2];
Finitely presented group on 3 generators, Index in group g is 2,
Generators as words in group g
    $.1 = r2 * r1, $.2 = r3 * r1, $.3 = r4
> print l[3];
```

```
Finitely presented group on 4 generators, Index in group g is 2,
Generators as words in group g
    $.1 = r1, $.2 = r2, $.3 = r3, $.4 = r4 * r3 * r4
> z:=Rewrite(g,l[2]); print z;
Finitely presented group z on 3 generators, Generators as words in group g
    z.1 = r2 * r1, z.2 = r3 * r1, z.3 = r4
Relations
    z.3^2 = Id(z), z.2^2 = Id(z), z.1^3 = Id(z), z.1 * z.3 * z.1^-1 * z.3 = Id(z)
    (z.1 * z.2)^3 = Id(z), (z.2 * z.3)^6 = Id(z)
> bia<a,b,c,t>:=Group<a,b,c,t|(t,a),(t,b),(t,c),t^2,a^3,b^2,c^2,(b*c)^3,
(c*a^-1)^3,(b*a^-1)^3>;
> q:=LowIndexSubgroups(z,<7,7>);
> print #q;
0
> qq:=LowIndexSubgroups(bia,<7,7>);
> print #qq;
4
```

The proof of [1, Theorem 8.2] also relies on the following routine which distinguishes $\widehat{\Gamma}_0$ from $\widehat{\Lambda}_1$ by counting conjugacy classes of index 8 subgroups.

```
> g<x,y,z,w>:=Group<x,y,z,w|x^2,y^2,z^2,w^2,(x*y)^2,(x*z)^2,(x*w)^3,
(y*z)^3,(y*w)^3,(z*w)^3>; l:=LowIndexSubgroups(g,<8,8>);
print #l;
1
> h<a,b,c>:=Group<a,b,c|a^3,b^2,c^2,(b*c)^6,(c*a^-1)^2,(b*a^-1)^3>;
> l:=LowIndexSubgroups(h,<8,8>);
print #l;
3
```

5. **The proof of** [1, Theorem 8.3]

[1, Theorem 8.3] relies on the following Magma routine which shows that `g` $= \Lambda_0$ has fewer index 8 subgroups, up to conjugacy, than `bia` $= \Gamma_0 \times \mathbb{Z}/2\mathbb{Z}$.

```
> g<r1,r2,r3,r4>:=Group<r1,r2,r3,r4|r1^2,r2^2,r3^2,r4^2,(r1*r2)^3,(r1*r3)^2,
(r1*r4)^2, (r2*r3)^3,(r2*r4)^2,(r3*r4)^6>; bia<a,b,c,t>:=Group<a,b,c,t|(t,a),
(t,b),(t,c),t^2,a^3,b^2,c^2,(b*c)^6,(c*a^-1)^2, (b*a^-1)^3>;
> l:=LowIndexSubgroups(g,<8,8>);
k:=LowIndexSubgroups(bia,<8,8>);
print #l;
3
> print #k;
5
```

6. Calculations for Γ_W

This section contains the Magma calculations used in [1, §9] in proving that the fundamental group of the Weeks manifold is profinitely rigid; see in particular the proof of [1, Lemma 9.3]. The group **g** in this routine is Γ_W, given by the presentation from (1).

```
> g<a,b>:=Group<a,b| a*b*a*b*a^-1*b^2*a^-1*b,a*b*a*b^-1*a^2*b^-1*a*b>;
> print AbelianQuotientInvariants(g);
[ 5, 5 ]
> l:=LowIndexSubgroups(g,<24,24>);
> print #l;
11
> print AbelianQuotientInvariants(l[1]);
[ 5, 55, 0 ]
> f,i:=CosetAction(g,l[1]);
> print Order(i);
6072
> IsSimple(i);
true
> print AbelianQuotientInvariants(l[2]);
[ 2, 2, 2, 10, 110 ]
> f,i:=CosetAction(g,l[2]);
> print Order(i);
6072
> IsSimple(i);
true
> print AbelianQuotientInvariants(l[3]);
[ 5, 30, 0 ]
> f,i:=CosetAction(g,l[3]);
> print Order(i);
2204496
> print AbelianQuotientInvariants(l[4]);
[ 90, 90 ]
> f,i:=CosetAction(g,l[4]);
> print Order(i);
2204496
> print AbelianQuotientInvariants(l[5]);
[ 5, 30, 0 ]
> f,i:=CosetAction(g,l[5]);
> print Order(i);
2204496
> print AbelianQuotientInvariants(l[6]);
[ 2, 2, 2, 70, 70 ]
```

```
> f,i:=CosetAction(g,l[6]);
> print Order(i);
168
> print AbelianQuotientInvariants(l[7]);
[ 90, 90 ]
> f,i:=CosetAction(g,l[7]);
> print Order(i);
2204496
> print AbelianQuotientInvariants(l[8]);
[ 5, 5, 10 ]
> f,i:=CosetAction(g,l[8]);
> print Order(i);
1320
> print AbelianQuotientInvariants(l[9]);
[ 5, 30 ]
> f,i:=CosetAction(g,l[9]);
> print Order(i);
310224200866619719680000
> print AbelianQuotientInvariants(l[10]);
> f,i:=CosetAction(g,l[10]);
> print Order(i);
310224200866619719680000
> print AbelianQuotientInvariants(l[11]);
[ 5, 30 ]
> f,i:=CosetAction(g,l[11]);
> print Order(i);
310224200866619719680000
```

The following enumeration of index 8 subgroups of Γ_W is also required in the proof of [1, Lemma 9.3]. Again g denotes Γ_W.

```
> g<a,b>:=Group<a,b| a*b*a*b*a^-1*b^2*a^-1*b,a*b*a*b^-1*a^2*b^-1*a*b>;
> print AbelianQuotientInvariants(g);
[ 5, 5 ]
> l:=LowIndexSubgroups(g,<8,8>); print #l;
1
> print AbelianQuotientInvariants(l[1]);
[ 5, 30 ]
```

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References

[1] M. R. Bridson, D. B. McReynolds, A. W. Reid, R. Spitler, *Absolute profinite rigidity and hyperbolic geometry*, *Annals of Math.* (2) **192** no. 3 (2020), 679–719. https://doi.org/10.4007/annals.2020.192.3.1.

Mathematical Institute, University of Oxford, Oxford, UK
E-mail: bridson@maths.ox.ac.uk

Department of Mathematics, Purdue University West Lafayette, IN, USA
E-mail: dmcreyno@purdue.edu

Department of Mathematics, Rice University, Houston, TX, USA
E-mail: alan.reid@rice.edu

Department of Mathematics, McMaster University Hamilton,
Ontario, Canada
E-mail: spitlerr@mcmaster.ca