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Shtukas and the Taylor expansion of L-functions (II)

By Zhiwei Yun and Wei Zhang

Abstract

For arithmetic applications, we extend and refine our previously published results to allow ramifications in a minimal way. Starting with a possibly ramified quadratic extension F'/F of function fields over a finite field in odd characteristic, and a finite set of places Σ of F that are unramified in F', we define a collection of Heegner–Drinfeld cycles on the moduli stack of PGL_2-Shtukas with r-modifications and Iwahori level structures at places of Σ. For a cuspidal automorphic representation π of $\mathrm{PGL}_2(\mathbb{A}_F)$ with square-free level Σ, and $r \in \mathbb{Z}_{\geq 0}$ whose parity matches the root number of $\pi_{F'}$, we prove a series of identities between

(1) the product of the central derivatives of the normalized L-functions

$$\mathscr{L}^{(a)}\left(\pi, \frac{1}{2}\right)\mathscr{L}^{(r-a)}\left(\pi\otimes\eta, \frac{1}{2}\right),$$

where η is the quadratic idèle class character attached to F'/F, and $0 \leq a \leq r$;

(2) the self intersection number of a linear combination of Heegner–Drinfeld cycles.

In particular, we can now obtain global L-functions with odd vanishing orders. These identities are function-field analogues of the formulae of Waldspurger and Gross–Zagier for higher derivatives of L-functions.

Contents

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1. Introduction

1.1. *Main results.* Let X be a smooth projective and geometrically connected curve over a finite field $k = \mathbb{F}_q$ of characteristic $p \neq 2$. Let $F = k(X)$ be the function field of X and $\mathbb{A}_F$ be the ring of adèles of F. Let $G = \mathrm{PGL}_2$. Let π be a cuspidal automorphic representation of $G(\mathbb{A}_F)$. Let X' be another smooth projective and geometrically connected curve over k together with a double cover $\nu : X' \to X$.

In [10], under the assumption that both π and ν are everywhere unramified, we proved an analogue of the formulae of Waldspurger [9] and Gross–Zagier [4] for higher order central derivatives of the base change L-function $L(\pi_{F'}, s)$. Our formula reads

$$\text{(1.1)} \qquad \frac{|\omega_X|}{2(\log q)^r L(\pi, \mathrm{Ad}, 1)} \mathscr{L}^{(r)}(\pi_{F'}, \frac{1}{2}) = \left([\mathrm{Sht}_T^\mu]_\pi, [\mathrm{Sht}_T^\mu]_\pi\right)_{\mathrm{Sht}_G'^r}.$$

Here $r \geq 0$ is an *even* integer. This formula relates the r-th central derivative of a certain normalization[1] $\mathscr{L}(\pi_{F'}, s)$ of the L-function of the base change $\pi_{F'}$ to the self-intersection number of a certain algebraic cycle $[\mathrm{Sht}_T^\mu]_\pi$ on the moduli stack of G-Shtukas $\mathrm{Sht}_G'^r$ with r modifications. The cycles $[\mathrm{Sht}_T^\mu]_\pi$ are analogous to the Heegner points on modular curves.

In this paper, we generalize the formula (1.1) to the case where the double cover ν is allowed to be ramified and the automorphic representation π is allowed to have square-free level. Moreover, we refine the formula (1.1) to give a geometric expression of derivatives of the form $\mathscr{L}^{(a)}\left(\pi, \frac{1}{2}\right)\mathscr{L}^{(b)}(\pi \otimes \eta, \frac{1}{2})$. Below we set up some notation for the statement of our main results.

1.1.1. *Ramifications of the automorphic representation.* Let Σ be a finite set of closed points of X. Let π be a cuspidal automorphic representation of $G(\mathbb{A})$ that is unramified away from Σ and, locally at each $x \in \Sigma$, isomorphic to an unramified twist of the Steinberg representation.

Let R be the ramification locus of the double cover ν, and let $\rho = \deg R$. Then the genus g' of X' and the genus g of X are related by $g' - 1 = 2(g-1) + \rho/2$. Let $\eta = \eta_{F'/F} : F^\times \backslash \mathbb{A}_F^\times \to \{\pm 1\}$ be the idèle class character corresponding to the extension F'/F.

[1] In [10], the definition of $\mathscr{L}(\pi_{F'}, s)$ included the denominator $L(\pi, \mathrm{Ad}, 1)$; in the current paper, we separate $L(\pi, \mathrm{Ad}, 1)$ from $\mathscr{L}(\pi_{F'}, s)$.

We assume

The sets R and Σ are disjoint.

The normalized L-functions

$$\mathscr{L}\left(\pi, s+\frac{1}{2}\right) = q^{(2g-2+N/2)s} L\left(\pi, s+\frac{1}{2}\right),$$

$$\mathscr{L}\left(\pi\otimes\eta, s+\frac{1}{2}\right) = q^{(2g-2+\rho+N/2)s} L\left(\pi\otimes\eta, s+\frac{1}{2}\right)$$

are either even or odd functions in s depending on the root numbers of π and $\pi\otimes\eta$. We define a normalized L-function in two variables

$$\mathscr{L}_{F'/F}(\pi, s_1, s_2) := \mathscr{L}\left(\pi, s_1+s_2+\frac{1}{2}\right)\mathscr{L}\left((\pi\otimes\eta, s_1-s_2+\frac{1}{2}\right)$$

so that its specialization to $s_1 = s, s_2 = 0$ gives the normalized base change L-function $\mathscr{L}(\pi_{F'}, s+\frac{1}{2})$. Then $\mathscr{L}_{F'/F}(\pi, s_1, s_2)$ satisfies a function equation

$$\mathscr{L}_{F'/F}(\pi, s_1, s_2) = (-1)^{r(\pi_{F'})}\mathscr{L}_{F'/F}(\pi, -s_1, -s_2),$$

where $(-1)^{r(\pi_{F'})}$ is the root number for the base change $\pi_{F'}$, and

$$r(\pi_{F'}) = \#\left\{x\in\Sigma \,\middle|\, x \text{ is inert in } X'\right\}.$$

For $r_+, r_- \in \mathbb{Z}_{\geq 0}$, we define

$$\mathscr{L}^{(r_+,r_-)}_{F'/F}(\pi) := \left(\frac{\partial}{\partial s_1}\right)^{r_+}\left(\frac{\partial}{\partial s_2}\right)^{r_-}\mathscr{L}_{F'/F}(\pi, s_1, s_2)\Big|_{s_1=s_2=0}.$$

From the functional equation of $\mathscr{L}_{F'/F}(\pi, s_1, s_2)$, we see that $\mathscr{L}^{(r_+,r_-)}_{F'/F}(\pi) = 0$ unless

$$r_+ + r_- \equiv r(\pi_{F'}) \mod 2.$$

1.1.2. *The moduli of Shtukas with Iwahori level structure.* On the geometric side, we will consider the moduli stack of G-Shtukas with Iwahori level structures. The points with Iwahori level structure come in two kinds: those resembling the finite primes dividing the level N for a modular curve $X_0(N)$ and those resembling the Archimedean place. In fact, starting with a finite subset $\Sigma\subset|X|$ together with a disjoint union decomposition $\Sigma = \Sigma_f \sqcup \Sigma_\infty$ and a non-negative integer r such that $r \equiv \#\Sigma_\infty \mod 2$, we will define in Sections 3.2.1 and 3.2.8 a moduli stack $\mathrm{Sht}^r_G(\Sigma;\Sigma_\infty)$ equipped with a map

$$\Pi^r_G\colon \mathrm{Sht}^r_G(\Sigma;\Sigma_\infty) \longrightarrow X^r\times\mathfrak{S}_\infty,$$

where $\mathfrak{S}_\infty = \prod_{x\in\Sigma_\infty}\mathrm{Spec}\, k(x)$. Then $\mathrm{Sht}^r_G(\Sigma;\Sigma_\infty)$ is a smooth $2r$-dimensional Deligne–Mumford (DM for short) stack locally of finite type over k (see Proposition 3.9). We will also consider the base change

$$\mathrm{Sht}'^r_G(\Sigma;\Sigma_\infty) := \mathrm{Sht}^r_G(\Sigma;\Sigma_\infty)\times_{(X^r\times\mathfrak{S}_\infty)}(X'^r\times\mathfrak{S}'_\infty),$$

where $\mathfrak{S}'_\infty = \prod_{x'\in\Sigma'_\infty} \operatorname{Spec} k(x')$ and $\Sigma'_\infty = \nu^{-1}(\Sigma_\infty)$. If we base change $\mathrm{Sht}'^r_G(\Sigma;\Sigma_\infty)$ to $\overline{k}$, it decomposes as

$$\mathrm{Sht}'^r_G(\Sigma;\Sigma_\infty)\otimes\overline{k} = \coprod_\xi \mathrm{Sht}'^r_G(\Sigma;\xi),$$

where $\xi = (\xi_{x'})_{x'\in\Sigma'_\infty}$ runs over the choices of a $\overline{k}$-point $\xi_{x'}$ over each $x'\in\Sigma'_\infty$. We fix such a ξ.

There is an action of the spherical Hecke algebra $\mathscr{H}^\Sigma_G = \otimes_{x\in|X|-\Sigma}\mathscr{H}_x$ on the cohomology groups $\mathrm{H}^*_c(\mathrm{Sht}'^r_G(\Sigma;\xi),\mathbb{Q}_\ell)$, which is infinite-dimensional in the middle degree. We have an Eisenstein ideal $\mathcal{I}_{\mathrm{Eis}}\subset\mathscr{H}^\Sigma_G$ defined in the same way as in [10, §4.1]. We prove a spectral decomposition similar to the unramified case.

THEOREM 1.1. *There is a canonical decomposition of $\mathscr{H}^\Sigma_G$-modules*

$$\mathrm{H}^{2r}_c(\mathrm{Sht}'^r_G(\Sigma;\xi),\overline{\mathbb{Q}}_\ell) = \Big(\bigoplus_{\mathfrak{m}} V'(\xi)_{\mathfrak{m}}\Big)\oplus V'(\xi)_{\mathrm{Eis}}, \tag{1.2}$$

where

- $\mathfrak{m}$ *runs over a finite set of maximal ideals of $\mathscr{H}^\Sigma_G$ that do not contain the Eisenstein ideal, and $V'(\xi)_{\mathfrak{m}}$ is the generalized eigenspace of the $\mathscr{H}^\Sigma_G$-action on $\mathrm{H}^{2r}_c(\mathrm{Sht}'^r_G(\Sigma;\xi),\overline{\mathbb{Q}}_\ell)$ corresponding to $\mathfrak{m}$. Moreover, $V'(\xi)_{\mathfrak{m}}$ is finite-dimensional over $\overline{\mathbb{Q}}_\ell$.*
- *$V'(\xi)_{\mathrm{Eis}}$ is a finitely generated $\mathscr{H}^\Sigma_G$-module on which the action of $\mathscr{H}^\Sigma_G$ factors through $\mathscr{H}^\Sigma_G/\mathcal{I}^m_{\mathrm{Eis}}$ for some $m>0$.*

Using the cup product, we have a perfect pairing

$$(\cdot,\cdot)_{\mathrm{Sht}'^r_G(\Sigma;\xi)} : V'(\xi)_{\mathfrak{m}}\times V'(\xi)_{\mathfrak{m}}\longrightarrow\overline{\mathbb{Q}}_\ell. \tag{1.3}$$

1.1.3. *The Heegner–Drinfeld cycle.* We make the following assumptions, which are analogous to the Heegner hypothesis:

$$\textit{all places in } \Sigma_f \textit{ are split in } X'; \tag{1.4}$$

$$\textit{all places in } \Sigma_\infty \textit{ are inert in } X'. \tag{1.5}$$

By considering rank one Shtukas on X', we obtain a moduli stack $\mathrm{Sht}^{\underline{\mu}}_T(\mu_\infty\cdot\Sigma'_\infty)$ that depends on the data $\underline{\mu}\in\{\pm1\}^r$ and $\mu_\infty\in\{\pm1\}^{\Sigma_\infty}$. The stack $\mathrm{Sht}^{\underline{\mu}}_T(\mu_\infty\cdot\Sigma'_\infty)$ is a finite étale cover of $X'^r\times\mathfrak{S}'_\infty$.

To map $\mathrm{Sht}^{\underline{\mu}}_T(\mu_\infty\cdot\Sigma'_\infty)$ to $\mathrm{Sht}'^r_G(\Sigma;\Sigma_\infty)$ we need an extra choice μ_f, which is a section to the two-to-one map $\Sigma'_f:=\nu^{-1}(\Sigma_f)\to\Sigma_f$. Altogether we have chosen an element

$$\mu = (\underline{\mu},\mu_f,\mu_\infty)\in\mathfrak{T}_{r,\Sigma} := \{\pm1\}^r\times\mathrm{Sect}(\Sigma'_f/\Sigma_f)\times\{\pm1\}^{\Sigma_\infty}. \tag{1.6}$$

From this choice we have a map (cf. Section 4.2.2)

$$\theta'^\mu : \mathrm{Sht}^{\underline{\mu}}_T(\mu_\infty\cdot\Sigma'_\infty)\longrightarrow\mathrm{Sht}'^r_G(\Sigma;\Sigma_\infty).$$

Base-changing to $\overline{k}$ and taking the ξ-component, we get a map

$$\theta'^{\mu}_{\xi} : \mathrm{Sht}^{\mu}_{T}(\mu_\infty \cdot \xi) \longrightarrow \mathrm{Sht}'^{r}_{G}(\Sigma;\xi).$$

We define the *Heegner–Drinfeld cycle* to be the algebraic cycle with proper support

$$\mathcal{Z}^{\mu}(\xi) := \theta'^{\mu}_{\xi,*}[\mathrm{Sht}^{\mu}_{T}(\mu_\infty \cdot \xi)] \in \mathrm{Ch}_{c,r}(\mathrm{Sht}'^{r}_{G}(\Sigma;\xi))_{\mathbb{Q}}.$$

Its cycle class in cohomology is denoted by

$$Z^{\mu}(\xi) := \mathrm{cl}(\mathcal{Z}^{\mu}(\xi)) \in \mathrm{H}^{2r}_{c}(\mathrm{Sht}'^{r}_{G}(\Sigma;\xi), \mathbb{Q}_\ell).$$

1.1.4. *Main result.* Our main theorem is the following.

THEOREM 1.2 (Main result, first formulation). *Let π be a cuspidal automorphic representation of $G(\mathbb{A}_F)$ ramified at a finite set of places Σ. Assume*

- *for each $x \in \Sigma$, π_x is isomorphic to an unramified twist of the Steinberg representation;*
- *the ramification locus R of the double cover $\nu : X' \to X$ is disjoint from Σ.*

We decompose Σ as $\Sigma_f \sqcup \Sigma_\infty$ in a unique way so that the conditions (1.4) and (1.5) hold. Let r be a non-negative integer such that

$$r \equiv \#\Sigma_\infty \mod 2.$$

Let $\mu, \mu' \in \mathfrak{T}_{r,\Sigma}$. Let

$$r_+ = \{1 \le i \le r \mid \mu_i = \mu'_i\}, \quad r_- = \{1 \le i \le r \mid \mu_i \ne \mu'_i\}.$$

Then

$$\frac{|\omega_X| q^{\rho/2-N} \varepsilon_-(\pi\otimes\eta)}{2(-\log q)^r L(\pi,\mathrm{Ad},1)} \mathscr{L}^{(r_+,r_-)}_{F'/F}(\pi) = \left(Z^{\mu}_{\pi}(\xi), Z^{\mu'}_{\pi}(\xi)\right)_{\mathrm{Sht}'^{r}_{G}(\Sigma;\xi)}. \tag{1.7}$$

Here,

- $|\omega_X| = q^{-(2g-2)}$.
- $\varepsilon_-(\pi\otimes\eta) \in \{\pm 1\}$ *is the product of the Atkin–Lehner eigenvalues of $\pi\otimes\eta$ at $x \in \Sigma_-(\mu,\mu')$, where $\Sigma_-(\mu,\mu') \subset \Sigma$ is defined in (4.6).*
- *The automorphic representation π gives a character λ_π of $\mathscr{H}^{\Sigma}_{G}$ that does not factor through the Eisenstein ideal; we denote by $V'(\xi)_\pi$ the direct summand in (1.2) corresponding to the maximal ideal $\mathfrak{m}_\pi = \ker(\lambda_\pi)$ and let $Z^{\mu}_{\pi}(\xi)$ be the projection of $Z^{\mu}(\xi)$ to $V'(\xi)_\pi$.*
- *The pairing $(\cdot,\cdot)_{\mathrm{Sht}'^{r}_{G}(\Sigma;\xi)}$ on the right side of (1.7) is (1.3).*

The Galois involution for the double cover X'/X induces an action of $(\mathbb{Z}/2\mathbb{Z})^r$ on X'^r, hence on $\mathrm{Sht}'^{r}_{G}(\Sigma;\xi)$ by acting only on the X'^r-factor. Let $\sigma_i \in (\mathbb{Z}/2\mathbb{Z})^r$ be the element with only the i-th coordinate non-trivial. For $0 \le r_1 \le r$, we define an idempotent in the group algebra $\mathbb{Q}[(\mathbb{Z}/2\mathbb{Z})^r]$ by

$$\varepsilon_{r_1} = \prod_{i=1}^{r_1} \frac{1+\sigma_i}{2} \prod_{j=r_1+1}^{r} \frac{1-\sigma_i}{2}.$$

THEOREM 1.3 (Main result, second formulation). *Keep the same assumptions as Theorem 1.2. Let $0 \le r_1 \le r$ be an integer, and let $\mu \in \mathfrak{T}_{r,\Sigma}$. Then*

$$\frac{|\omega_X| q^{\rho/2-N}}{2(-\log q)^r L(\pi, \mathrm{Ad}, 1)} \mathscr{L}^{(r_1)}\left(\pi, \frac{1}{2}\right) \mathscr{L}^{(r-r_1)}\left(\pi \otimes \eta, \frac{1}{2}\right) = \left(\varepsilon_{r_1} Z^{\mu}_{\pi}(\xi), \varepsilon_{r_1} Z^{\mu}_{\pi}(\xi)\right)_{\mathrm{Sht}'^r_G(\Sigma;\xi)}.$$

In the special case $r_1 = r$, we may further reformulate the theorem as follows.

COROLLARY 1.4. *Keep the same assumptions as Theorem 1.2. Let $Y^{\mu}_{\pi}(\xi) \in \mathrm{H}^{2r}_c(\mathrm{Sht}^r_G(\Sigma;\xi), \overline{\mathbb{Q}}_\ell)$ be the class of the push-forward of $Z^{\mu}_{\pi}(\xi)$ to $\mathrm{Sht}^r_G(\Sigma;\xi) = \mathrm{Sht}^r_G(\Sigma;\Sigma_\infty) \times_{\mathfrak{S}_\infty} \xi$. Then $Y^{\mu}_{\pi}(\xi)$ depends only on (r, μ_f, μ_∞), and*

$$\frac{2^{r-1}|\omega_X| q^{\rho/2-N}}{(-\log q)^r L(\pi, \mathrm{Ad}, 1)} \mathscr{L}^{(r)}\left(\pi, \frac{1}{2}\right) \mathscr{L}\left(\pi \otimes \eta, \frac{1}{2}\right) = \left(Y^{\mu}_{\pi}(\xi), Y^{\mu}_{\pi}(\xi)\right)_{\mathrm{Sht}^r_G(\Sigma;\xi)}.$$

Remark 1.5. Consider the case where Σ_∞ consists of a single place ∞, $r = 1$, and $\mu = \mu'$. In this case the moduli stack $\mathrm{Sht}^1_G(\Sigma;\Sigma_\infty)$ over X is closely related to the moduli space of elliptic modules originally defined by Drinfeld [2] (see the discussion in Section 3.2.3), the latter being a perfect analogue of a semistable integral model for modular curves $X_0(N)$. In this special case, Theorem 1.2 reads

$$(1.8) \qquad -\frac{|\omega_X| q^{\rho/2-N}}{2\log q \cdot L(\pi, \mathrm{Ad}, 1)} \mathscr{L}'(\pi_{F'}, \frac{1}{2}) = (Z^{\mu}_{\pi}(\xi), Z^{\mu}_{\pi}(\xi))_{\mathrm{Sht}'^1_G(\Sigma;\xi)}.$$

This is a direct analogue of the Gross-Zagier formula for modular curves [4]. We understand that D. Ulmer has an unpublished proof of a formula similar to (1.8). The method of our proof is quite different from that in [4] in that we do not need to explicitly compute either side of the formula.

1.2. *What is new.* We highlight both the new results and new techniques in this paper compared to the unramified case treated in [10].

1.2.1. First we compare our results with our previous ones in [10]. Theorems 1.2 and 1.3 have much wider applicability than the ones in [10]. In particular, for a non-isotrivial elliptic curve E over F with semistable reductions, its L-function $L(E, s)$ is the automorphic L-function $L(\pi, s + 1/2)$ for some π satisfying the conditions of our theorems. Therefore, our results in this paper give a geometric interpretation of Taylor coefficients of L-functions of semistable elliptic curves over function fields. For potential applications to the arithmetic of elliptic curves, see the discussion in Section 1.3.

In addition, in this paper we study the intersection of different Heegner–Drinfeld cycles by varying the discrete datum μ. As a result we get products of derivatives of $\mathscr{L}(\pi, s)$ and $\mathscr{L}(\pi \otimes \eta, s)$, as opposed to just the derivatives of their product $\mathscr{L}(\pi_{F'}, s)$. So Theorems 1.2 and 1.3 are new even in the unramified case.

1.2.2. Next we comment on the proof. To prove Theorem 1.2, we follow the general strategy of relative trace formulae comparison as in [10]. In this paper, we have tried to avoid repeating similar arguments from [10] and only write new arguments in detail. Here are some highlights of the new techniques compared to the unramified case.

The key identity between relative traces takes the form

$$\left(\frac{\partial}{\partial s_1}\right)^{r_+}\left(\frac{\partial}{\partial s_2}\right)^{r_-}\left(q^{N_+ s_1 + N_- s_2}\mathbb{J}(f', s_1, s_2)\right)\Big|_{s_1=s_2=0} = \left(Z^{\mu}(\xi), f * Z^{\mu'}(\xi)\right)_{\mathrm{Sht}'^{r}_{G}(\Sigma;\xi)},$$

where $f \in \mathscr{H}_G^{\Sigma\cup R}$ and $f' \in C_c(G(\mathbb{A}))$ is a "matching function," and where $N_\pm = \deg \Sigma_\pm(\mu, \mu')$ (see (4.5) and (4.6)). In the unramified case, we simply took $f' = f$. At places $x \in \Sigma$, the corresponding factors of f' are not surprising: they are essentially characteristic functions of the Iwahori. However, it is not obvious what to put at places $x \in R$ (where R is the ramification locus of F'/F). This is one of the main difficulties of this work.

In Section 2.4.1 we give a somewhat surprising formula for the test function $h_x^\square$ to be put at $x \in R$ in f'. The discovery of the function $h_x^\square$ was guided by the geometric interpretation of orbital integrals. We wanted a moduli space $\mathcal{N}_d$ that looked like the counterpart of $\mathcal{M}_d$ (see Definition 5.1) for a split quadratic extension $F \times F$ but somehow remembered the ramification locus R. Once we realized the correct candidate for $\mathcal{N}_d$ (see Definition 6.1), the formula for $h_x^\square$ fell out quite naturally as counting points on $\mathcal{N}_d$. From the spectral calculation, we get another characterization of $h_x^\square$ (see Section 2.4.2), which justifies its canonicity from a different perspective. The idea should be applicable to other situations of relative trace formulae where one needs explicit *ramified* test functions. We hope to return to this topic in the future.

The presence of Iwahori structures makes the geometry of the horocycles in $\mathrm{Sht}'^{r}_{G}(\Sigma; \Sigma_\infty)$ much more complicated than in the unramified case, which explains the length of Section 3.4. The study of the horocycles is needed in order to establish a cohomological spectral decomposition. Also, the proof of the key finiteness results leading to the cohomological spectral decomposition in Section 3.5 uses a new strategy: we introduce "almost isomorphisms" between ind-perverse sheaves (i.e., we work with a quotient category of ind-perverse sheaves). Compared to our approach in [10], this strategy is more robust in showing qualitative results for spaces of infinite type and should work for the cohomological spectral decomposition for higher rank groups.

1.3. *Potential arithmetic applications.*

1.3.1. *Determinant of the Frobenius eigenspace.* Let π be a cuspidal automorphic representation of $G(\mathbb{A})$ as in Theorem 1.2. By the global Langlands correspondence proved by Drinfeld [3], there is a rank two irreducible $\overline{\mathbb{Q}}_\ell$-local system ρ_π attached to π over an open subset of X. Our convention is that $\det(\rho_\pi) \cong \overline{\mathbb{Q}}_\ell(-1)$; in particular, ρ_π is pure of weight 1. Let $j_{!*}\rho_\pi$ be the middle extension of ρ_π to the complete curve X. The base change $\pi_{F'}$ corresponds to the local system $\nu^*\rho_\pi$ on an open subset of X', and we denote by $j'_{!*}\nu^*\rho_\pi$ its middle extension to X'. Let

$$W'_\pi := \mathrm{H}^1(X' \otimes \overline{k},\, j'_{!*}\nu^*\rho_\pi).$$

This is a $\overline{\mathbb{Q}}_\ell$-vector space with the geometric Frobenius automorphism Fr of weight 2. The L-function $L(\pi_{F'}, s)$ is related to $\nu^*\rho_\pi$ by

$$L(\pi_{F'}, s - \frac{1}{2}) = \det\left(1 - q^{-s}\,\mathrm{Fr} \,\middle|\, W'_\pi\right).$$

Let $\Pi^r_G\colon \mathrm{Sht}^r_G(\Sigma) \to X^r \times \mathfrak{S}_\infty$ be the projection map. It is expected that under the $\mathscr{H}^\Sigma_G$-action, the λ_π-isotypical component of the complex $\mathbf{R}\Pi^r_{G,!}\overline{\mathbb{Q}}_\ell$ on $X^r \times \mathfrak{S}_\infty$ takes the form

$$(1.9)\quad (\mathbf{R}\Pi^r_{G,!}\overline{\mathbb{Q}}_\ell)_\pi = \pi^K \otimes \Big(\underbrace{j_{!*}\rho_\pi[-1] \boxtimes \cdots \boxtimes j_{!*}\rho_\pi[-1]}_{r \text{ times}}\Big) \boxtimes \Big(\boxtimes_{x\in\Sigma_\infty} \rho^{I_x}_{\pi,x}\Big),$$

where $K = \prod_{x\notin\Sigma} G(\mathcal{O}_x)\times\prod_{x\in\Sigma}\mathrm{Iw}_x$, and $\rho_{\pi,x}$ is the restriction of ρ_π to $\operatorname{Spec} F_x$ and $I_x < \mathrm{Gal}(F^{\mathrm{sep}}_x/F_x)$ is the inertial group at x. Pulling back to $X'^r \times \mathfrak{S}'_\infty$, (1.9) implies that the generalized eigenspace $V'(\xi)_\pi := V'(\xi)_{\ker(\lambda_\pi)}$ in (1.2) should take the form

$$V'(\xi)_\pi \cong \pi^K \otimes W'^{\otimes r}_\pi \otimes \ell_{\pi,\xi},$$

where $\ell_{\pi,\xi}$ is the geometric stalk of $\boxtimes_{x\in\Sigma_\infty}\rho^{I_x}_{\pi,x}$ at ξ. Note that both π^K and $\ell_{\pi,\xi}$ are one-dimensional since π is an unramified twist of the Steinberg representation at $x \in \Sigma$.

Then the cohomology class of the Heegner–Drinfeld cycle gives rise to an element in $Z^\mu_\pi(\xi) \in \pi^K \otimes W'^{\otimes r}_\pi \otimes \ell_{\pi,\xi}$. It can be shown that $Z^\mu_\pi(\xi)$ is an eigenvector for the operator $\mathrm{id}\otimes\mathrm{Fr}^{\otimes r}\otimes\mathrm{id}$, with eigenvalue q^r. Our main result (Theorem 1.2) together with the super-positivity proved in [10, Th. B.2] shows that $Z^\mu_\pi(\xi)$ does not vanish when $r \geq \mathrm{ord}_{s=1/2} L(\pi_{F'}, s)$, provided that $L(\pi_{F'}, s)$ is not a constant (i.e., $2(4g-4+N+\rho) > 0$).

Partly motivated by the standard conjecture about Frobenius semisimplicity, we propose

CONJECTURE 1.6. *Let $r = \mathrm{ord}_{s=1/2} L(\pi_{F'}, s)$ (i.e., r is the dimension of the* generalized *eigenspace of* Fr *on W'_π with eigenvalue q) and $\mu \in \mathfrak{T}_{r,\Sigma}$. Then the class $Z^\mu_\pi(\xi)$ belongs to $\pi^K \otimes \wedge^r\left(W'^{\mathrm{Fr}=q}_\pi\right) \otimes \ell_{\pi,\xi}$.*

In particular, for the eigenvalue q, the generalized eigenspace of the Fr*-action on W'_π coincides with the eigenspace, and $Z^\mu_\pi(\xi)$ gives a basis of the line $\pi^K \otimes \wedge^r \left(W'^{\mathrm{Fr}=q}_\pi\right) \otimes \ell_{\pi,\xi}$.*

In a forthcoming work, the authors plan to prove the following (assuming that (1.9) holds):

(i) If $r_0 \geq 0$ is the smallest integer r such that $Z^\mu_\pi \neq 0$ for some $\mu \in \{\pm 1\}^r$, then $\dim W'^{\mathrm{Fr}=q}_\pi = r_0$ and the class $Z^\mu_\pi(\xi)$ gives a basis of the line $\pi^K \otimes \wedge^{r_0} \left(W'^{\mathrm{Fr}=q}_\pi\right) \otimes \ell_{\pi,\xi}$.

(ii) $\mathrm{ord}_{s=1/2}\, L(\pi_{F'}, s) = 1$ *if and only if* $\dim W'^{\mathrm{Fr}=q}_\pi = 1$. Moreover, if $\mathrm{ord}_{s=1/2}\, L(\pi_{F'}, s) = 3$, then $\dim W'^{\mathrm{Fr}=q}_\pi = 3$.

1.3.2. *Elliptic curves.* Let E be a non-isotrivial semistable elliptic curve over F. Attached to E is a cuspidal automorphic representation π of $G(\mathbb{A}_F)$ such that $\rho_\pi \cong V_\ell(E)^* \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell$ as representations of $\mathrm{Gal}(F^{\mathrm{sep}}/F)$. In particular, $L(E, s) = L(\pi, s - \frac{1}{2})$, and $L(E_{F'}, s) = L(\pi_{F'}, s - \frac{1}{2})$. Moreover, after choosing a semistable model $\mathcal{E}'$ over X', we may identify W'_π with a subquotient of $\mathrm{H}^2(\mathcal{E}' \otimes \overline{k}, \overline{\mathbb{Q}}_\ell)$, and we think of it as the ℓ-adic Selmer group of E. The function-field analogue of the conjecture of Birch and Swinnerton-Dyer, as formulated by Artin and Tate [7], predicts that the q-eigenspace of Fr on W'_π is the same as the generalized eigenspace and is spanned by classes of sections of $\mathcal{E}'$. The expected result (ii) above would imply that if $\mathrm{ord}_{s=1}\, L(E_{F'}, s) = 3$, then the q-eigenspace of Fr on W'_π is the same as the generalized eigenspace.

While it is difficult to construct algebraic cycles on $\mathcal{E}'$ spanning $W'^{\mathrm{Fr}=q}_\pi$, it may be easier to construct a basis of the line $\wedge^r(W'^{\mathrm{Fr}=q}_\pi)$. Conjecture 1.6 proposes a candidate generator for $\wedge^r(W'^{\mathrm{Fr}=q}_\pi)$, namely, the cycle $Z^\mu_\pi(\xi)$. It is not clear though how to relate the ambient space of $Z^\mu_\pi(\xi)$, namely $\mathrm{Sht}'^r_G(\Sigma;\xi)$, to powers of $\mathcal{E}'$.

1.4. *Notation.*

1.4.1. *Function field notation.* Throughout this paper, we fix a finite field $k = \mathbb{F}_q$ of characteristic $p \neq 2$. We fix a smooth, projective and geometrically connected curve X over k. Let $F = k(X)$ be the function field of X. Let $|X|$ denote the set of closed points of X.

For $x \in |X|$, let $\mathcal{O}_x$ (resp. F_x) denote the completed local ring of X at x (resp. the fraction field of $\mathcal{O}_x$). Let $\mathfrak{m}_x \subset \mathcal{O}_x$ be the maximal ideal. We typically denote a uniformizer of $\mathcal{O}_x$ by ϖ_x. Let $\mathbb{A}_F$ denote the ring of adèles of F, and let $\mathbb{O} = \prod_{x \in |X|} \mathcal{O}_x$. Let $k(x)$ denote the residue field of $\mathcal{O}_x$, and let

$$d_x = [k(x) : k], \quad q_x = q^{d_x} = \#k(x).$$

Let $v_x : F_x^\times \to \mathbb{Z}$ be the valuation normalized by $v_x(\varpi_x) = 1$.

We will also consider a double covering $\nu : X' \to X$ where X' is also a smooth, projective and geometrically connected curve X over k. The function field of X' is denoted by F'. Other notation for X extend to their counterparts for X'.

1.4.2. *Group-theoretic notation.* Except for in Sections 3.1 and 3.2, the letter G always denotes the algebraic group PGL_2 over k. Let $A \subset G$ be the diagonal torus. For $x \in |X|$, the standard Iwahori subgroup Iw_x of $G(F_x)$ is the image of the following subgroup of $\mathrm{GL}_2(\mathcal{O}_x)$:

$$\widetilde{\mathrm{Iw}}_x = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\mathcal{O}_x) \,\middle|\, c \in \mathfrak{m}_x \right\}.$$

For an algebraic group H over F, we denote

$$[H] := H(F)\backslash H(\mathbb{A}).$$

1.4.3. *Algebro-geometric notation.* Most of the algebraic stacks that appear in this paper are over the finite field k (with exceptions of affine $\mathbb{Q}_\ell$-schemes appearing in Theorem 3.41), and the product $S \times S'$ (without subscript) of such stacks S and S' is always understood to be the fiber product of S and S' over $\operatorname{Spec} k$.

For any stack S over k, $\mathrm{Fr}_S : S \to S$ denotes the k-linear Frobenius that raises functions to the q-th power.

For an S-point $x : S \to X$, we denote by $\Gamma_x \subset X \times S$ the graph of x, which is a Cartier divisor of $X \times S$.

We fix a prime ℓ different from p, and we fix an algebraic closure $\overline{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$. The étale cohomology groups in this paper are with $\mathbb{Q}_\ell$ or $\overline{\mathbb{Q}}_\ell$ coefficients.

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2. **The analytic side: relative trace formula**

We extend the results in [10, §§2, 4] on Jacquet's relative trace formula (RTF) [5] to our current setting. Since most arguments in [10] extend without any difficulty, we will not repeat them, but simply indicate the necessary changes.

A new phenomenon is that we need to choose a new test function at the places where F'/F is ramified. This is done in Section 2.4 and is the most non-obvious point of the analytic part of this paper.

By convention, the automorphic representations we consider in this section are on $\mathbb{C}$-vector spaces.

2.1. *Jacquet's* RTF. For $f \in C_c^\infty(G(\mathbb{A}))$, we consider the automorphic kernel function

$$\mathbb{K}_f(g_1, g_2) = \sum_{\gamma \in G(F)} f(g_1^{-1}\gamma g_2), \quad g_1, g_2 \in G(\mathbb{A}). \tag{2.1}$$

Let $A \subset G$ be the diagonal torus. We define a distribution given by a regularized integral, for $(s_1, s_2) \in \mathbb{C}^2$,

$$\mathbb{J}(f, s_1, s_2) = \int_{[A]\times[A]}^{\mathrm{reg}} \mathbb{K}_f(h_1, h_2)|h_1|^{s_1+s_2}|h_2|^{s_1-s_2}\eta(h_2)\, dh_1\, dh_2. \tag{2.2}$$

Here the measure on $[A] = A(F)\backslash A(\mathbb{A})$ is induced from the Haar measure on $A(\mathbb{A})$ such that $\mathrm{vol}(A(\mathbb{O})) = 1$.

The regularization is the same as in [10, §§2.2–2.5], i.e., as the limit of the integral over a certain sequence of increasing bounded subsets that cover $[A] \times [A]$. Moreover, we define a two-variable orbital integral

$$\mathbb{J}(\gamma, f, s_1, s_2) = \int_{A(\mathbb{A})\times A(\mathbb{A})} f(h_1^{-1}\gamma h_2)|h_1 h_2|^{s_1}|h_1/h_2|^{s_2}\eta(h_2)\, dh_1\, dh_2.$$

Recall the function $\mathrm{inv} : G(F) \to \mathbb{P}^1(F) - \{1\}$ defined in [10, (2.1)]. When $u = \mathrm{inv}(\gamma) \in \mathbb{P}^1(F) \setminus \{0, 1, \infty\}$, the integral $\mathbb{J}(\gamma, f, s_1, s_2)$ is absolutely convergent. When $u = \mathrm{inv}(\gamma) \in \{0, \infty\}$, the integral defining $\mathbb{J}(\gamma, f, s_1, s_2)$ requires regularization as in [10, §2.5], and the proof in loc. cit. goes through in our two-variable setting.

Now $\mathbb{J}(f, s_1, s_2)$ and $\mathbb{J}(\gamma, f, s_1, s_2)$ are in $\mathbb{C}[q^{\pm s_1}, q^{\pm s_2}]$; i.e., each of them is a *finite* sum of the form

$$\sum_{(n_1,n_2)\in\mathbb{Z}^2} a_{n_1,n_2}\, q^{n_1 s_1 + n_2 s_2}, \quad a_{n_1,n_2} \in \mathbb{C}.$$

We have an expansion of $\mathbb{J}(f, s_1, s_2)$ into a sum of orbital integrals

$$\mathbb{J}(f, s_1, s_2) = \sum_{\gamma \in A(F)\backslash G(F)/A(F)} \mathbb{J}(\gamma, f, s_1, s_2). \tag{2.3}$$

We also define

$$\mathbb{J}(u, f, s_1, s_2) = \sum_{\gamma \in A(F)\backslash G(F)/A(F),\, \mathrm{inv}(\gamma)=u} \mathbb{J}(\gamma, f, s_1, s_2), \quad u \in \mathbb{P}^1(F) - \{1\}. \tag{2.4}$$

2.2. *The Eisenstein ideal.* For $x \in |X|$, let

$$\mathscr{H}_x = C_c(G(\mathcal{O}_x)\backslash G(F_x)/G(\mathcal{O}_x))$$

be the spherical Hecke algebra of $G(F_x)$. For a finite set S of closed points of X , define $\mathscr{H}_G^S = \otimes_{x\in|X|-S}\mathscr{H}_x$. In [10, §4.1] we defined the Eisenstein ideal

$\mathcal{I}_{\mathrm{Eis}} \subset \mathscr{H}_G$ for the full spherical Hecke algebra $\mathscr{H}_G = \otimes_{x\in|X|}\mathscr{H}_x$ as the kernel of the composition of ring homomorphisms

$$a_{\mathrm{Eis}}\colon \mathscr{H}_G \xrightarrow{\mathrm{Sat}} \mathbb{Q}[\mathrm{Div}(X)] \twoheadrightarrow \mathbb{Q}[\mathrm{Pic}_X(k)].$$

Here the first map Sat is the tensor product of Satake transforms $\mathscr{H}_x \to \mathbb{Q}[t_x, t_x^{-1}]$. We restrict the homomorphism to the subalgebra $\mathscr{H}_G^S$

$$a_{\mathrm{Eis}}^S\colon \mathscr{H}_G^S \xrightarrow{\mathrm{Sat}} \mathbb{Q}[\mathrm{Div}(X-S)] \longrightarrow \mathbb{Q}[\mathrm{Pic}_X(k)]$$

and define

$$\mathcal{I}_{\mathrm{Eis}}^S\colon = \mathrm{Ker}\left(a_{\mathrm{Eis}}^S : \mathscr{H}_G^S \longrightarrow \mathbb{Q}[\mathrm{Pic}_X(k)]\right).$$

Recall from [10, 4.1.2] that the image of a_{Eis}, hence that of a_{Eis}^S, lies in $\mathbb{Q}[\mathrm{Pic}_X(k)]^{\iota_{\mathrm{Pic}}}$ for an involution ι_{Pic} on $\mathbb{Q}[\mathrm{Pic}_X(k)]$. We have the following analogue of [10, Lemma 4.2] with the same proof.

LEMMA 2.1. *The map* $a_{\mathrm{Eis}}^S : \mathscr{H}_G^S \to \mathbb{Q}[\mathrm{Pic}_X(k)]^{\iota_{\mathrm{Pic}}}$ *is surjective.*

We have a generalization of [10, Th. 4.3].

THEOREM 2.2. *Let* $f^S \in \mathcal{I}_{\mathrm{Eis}}^S$, *and let* $f_S \in C_c^\infty(G(\mathbb{A}_S))$ *be left invariant under the Iwahori* $\mathrm{Iw}_S = \prod_{x\in S}\mathrm{Iw}_x$. *Then for* $f = f_S \otimes f^S \in C_c^\infty(G(\mathbb{A}))$, *we have*

$$\mathbb{K}_f = \mathbb{K}_{f,\mathrm{cusp}} + \mathbb{K}_{f,\mathrm{sp}}.$$

Here $\mathbb{K}_{f,\mathrm{cusp}}$ (*resp.* $\mathbb{K}_{f,\mathrm{sp}}$) *is the projection of* $\mathbb{K}_f$ *to the cuspidal spectrum* (*resp. residual spectrum*, *i.e.*, *one-dimensional representations*); *see* [10, §4.2].

Proof. We indicate how to the modify the proof of [10, Th. 4.3]. Let $K^S = \prod_{x\notin S} G(\mathcal{O}_x)$, and let $K = K_S \cdot K^S$ be a compact open subgroup of $G(\mathbb{A})$ such that $K_S \subset \mathrm{Iw}_S$ and that f is bi-K-invariant. The analogue of equation [10, (4.9)] now reads

(2.5)
$$\mathbb{K}_{f,\mathrm{Eis},\chi}(x,y) = \frac{\log q}{2\pi i}\sum_{\alpha,\beta}\int_0^{\frac{2\pi i}{\log q}} (\rho_{\chi,u}(f)\phi_\alpha, \phi_\beta)\, E(x,\phi_\alpha,u,\chi)\overline{E(y,\phi_\beta,u,\chi)}\, du,$$

where $\{\phi_\alpha\}$ is an orthonormal basis of V_χ^K. Since f is left invariant under the Iwahori $\mathrm{Iw}_S \times K^S$, $(\rho_{\chi,u}(f)\phi_\alpha,\phi_\beta) = 0$ unless the $\mathrm{Iw}_S \times K^S$-average of ϕ_β is non-zero; i.e., $(\rho_{\chi,u}(f)\phi_\alpha,\phi_\beta) = 0$ unless $V_\chi^{\mathrm{Iw}_S\times K^S} \neq 0$, which happens if and only if χ is everywhere unramified. When χ is everywhere unramified, we have

$$(\rho_{\chi,u}(f)\phi_\alpha,\phi_\beta) = \chi_{u+1/2}(a_{\mathrm{Eis}}^S(f^S))(\rho_{\chi,u}(f_S\otimes 1_{K^S})\phi_\alpha,\phi_\beta).$$

In particular, if f^S lies in the Eisenstein ideal, then $a_{\mathrm{Eis}}^S(f^S) = 0$, and hence the integrand in (2.5) vanishes. This completes the proof. □

2.3. *The spherical character*: *global and local.*

2.3.1. *Global spherical characters and period integral.* We first recall from [10, §4.3] the global spherical character. Let π be a cuspidal automorphic representation of $G(\mathbb{A})$, endowed with the natural Hermitian form given by the Petersson inner product: $\langle \phi, \phi' \rangle$ for $\phi, \phi' \in \pi$.

For a character $\chi : F^\times \backslash \mathbb{A}^\times \to \mathbb{C}^\times$, the (A, χ)-period integral for $\phi \in \pi$ is defined as

$$\mathscr{P}_\chi(\phi, s) := \int_{[A]} \phi(h)\chi(h)\left|h\right|^s dh. \tag{2.6}$$

We simply write $\mathscr{P}(\phi, s)$ if $\chi = \mathbf{1}$ is trivial. The global spherical character (relative to $(A \times A, 1 \times \eta)$) associated to π is a distribution on $G(\mathbb{A})$ defined by

$$\mathbb{J}_\pi(f, s_1, s_2) = \sum_{\{\phi\}} \frac{\mathscr{P}(\pi(f)\phi, s_1 + s_2)\mathscr{P}_\eta(\overline{\phi}, s_1 - s_2)}{\langle \phi, \phi \rangle}, \quad f \in C_c^\infty(G(\mathbb{A})), \tag{2.7}$$

where the sum runs over an *orthogonal* basis $\{\phi\}$ of π. This expression is independent of the choice of the measure on $G(\mathbb{A})$ as long as we use the same measure to define the operator $\pi(f)$ and the Petersson inner product. The function $\mathbb{J}_\pi(f, s_1, s_2)$ defines an element in $\mathbb{C}[q^{\pm s_1}, q^{\pm s_2}]$.

Using Theorem 2.2, the same argument of [10, Lemma 4.4] proves the following lemma.

LEMMA 2.3. *Let f be the same as in Theorem 2.2. Then*

$$\mathbb{J}(f, s_1, s_2) = \sum_{\pi} \mathbb{J}_\pi(f, s_1, s_2),$$

where the sum runs over all cuspidal *automorphic representations π of $G(\mathbb{A})$ and the summand $\mathbb{J}_\pi(f, s)$ is zero for all but finitely many π.*

2.3.2. *Local spherical characters.* We now recall the factorization of the global spherical character (2.7) into a product of local spherical characters. For unexplained notation and convention, we refer to the proof of [10, Prop. 4.5].

Let $\psi : F\backslash\mathbb{A} \to \mathbb{C}^\times$ be a non-trivial character, and let ψ_x be its restriction to F_x. For the discussion of the local spherical characters, we will use Tamagawa measures on various groups, which differ from our earlier convention. Strictly speaking, as in loc. cit., the measure on $A(\mathbb{A}) = \mathbb{A}^\times$ is not the Tamagawa measure, but rather an unnormalized (decomposable) one $\prod_{x \in |X|} d^\times t_x$ where $d^\times t_x = \zeta_x(1)\frac{dt_x}{|t_x|}$ for the self-dual measure dt_x (with respect to ψ_x). In particular, we have $\mathrm{vol}(\mathcal{O}_x^\times) = 1$ when ψ_x is unramified (i.e., the conductor of ψ_x is $\mathcal{O}_x$). A similar remark applies to the measure $G(\mathbb{A})$; cf. [10, p.804].

We consider the Whittaker model of π_x with respect to the character ψ_x, denoted by $\mathcal{W}_{\psi_x}(\pi_x)$. For $\phi = \otimes_{x\in|X|}\phi_x \in \pi = \otimes'_{x\in|X|}\pi_x$, the ψ-Whittaker coefficient W_ϕ decomposes as a product $\otimes_{x\in|X|}W_x$, where $W_x \in \mathcal{W}_{\psi_x}(\pi_x)$. We define a normalized linear functional

$$\lambda_x^\natural(W_x,\eta_x,s) := \frac{1}{L(\pi_x\otimes\eta_x, s+1/2)}\int_{F_x^\times} W_x\left(\begin{bmatrix} a & \\ & 1\end{bmatrix}\right)\eta_x(a)|a|^s\,d^\times a.$$

We define a local (invariant) inner product $\theta_x^\natural$ on the Whittaker model $\mathcal{W}_{\psi_x}(\pi_x)$

$$\theta_x^\natural(W_x,W'_x) := \frac{1}{L(\pi_x\times\widetilde{\pi}_x,1)}\int_{F_x^\times} W_x\left(\begin{bmatrix} a & \\ & 1\end{bmatrix}\right)\overline{W'_x}\left(\begin{bmatrix} a & \\ & 1\end{bmatrix}\right)d^\times a.$$

Now we define the local spherical character as

$$\begin{aligned}(2.8)\qquad &\mathbb{J}_{\pi_x}(f_x,s_1,s_2)\\ &:= \sum_{\{W_i\}} \frac{\lambda_x^\natural(\pi_x(f_x)W_i,\mathbf{1}_x,s_1+s_2)\lambda_x^\natural(\overline{W_i},\eta_x,s_1-s_2)}{\theta_x^\natural(W_i,W_i)},\end{aligned}$$

where the sum runs over an *orthogonal* basis $\{W_i\}$ of $\mathcal{W}_{\psi_x}(\pi_x)$. By the product decomposition of the period integrals (2.6) and the Petersson inner product (cf. the proof of [10, Prop. 4.5]), the global spherical character decomposes into a product of local ones (cf. [10, (4.16)]):

(2.9)

$$\begin{aligned}&\mathbb{J}_\pi(f,s_1,s_2)\\ &\quad = |\omega_X|^{-1}\,\frac{L(\pi,s_1+s_2+\frac12)L(\pi\otimes\eta,s_1-s_2+\frac12)}{2\,L(\pi,\mathrm{Ad},1)}\prod_{x\in|X|}\mathbb{J}_{\pi_x}(f_x,s_1,s_2).\end{aligned}$$

We note that the factor $|\omega_X|^{-1}$ is due to the fact that in our earlier definition (2.2) of $\mathbb{J}(f,s_1,s_2)$, the measure on $A(\mathbb{A})$ gives $\mathrm{vol}(A(\mathbb{O}))=1$, while the (unnormalized) Tamagawa measure gives $\mathrm{vol}(A(\mathbb{O}))=|\omega_X|^{1/2}$.

2.4. *Local test functions.* Our test function $f\in C_c^\infty(G(\mathbb{A}))$ will be a pure tensor $f=\otimes_{x\in|X|}f_x$, where $f_x\in\mathscr{H}_x$ is in the spherical Hecke algebra for $x\notin\Sigma\cup R$. Below we define the local components f_x for $x\in R$ (in Sections 2.4.1–2.4.2) and for $x\in\Sigma$ (in Section 2.4.3).

For any place $x\in|X|$, let $p_x:\mathrm{GL}_2(F_x)\to G(F_x)$ be the projection. The fibers of p_x are torsors under $F_x^\times$ and are equipped with $F_x^\times$-invariant measures such that any $\mathcal{O}_x^\times$-orbit has volume 1. Let $p_{x,*}:C_c^\infty(\mathrm{GL}_2(F_x))\to C_c^\infty(G(F_x))$ be the map defined by integration along the fibers of p_x with the above-defined measure.

2.4.1. *The function $h_x^\square$.* For $a\in\mathcal{O}_x$, we denote by $\overline{a}$ its image in $k(x)$. For any $n\in\mathbb{Z}$, let $\mathrm{Mat}_2(\mathcal{O}_x)_{v_x(\det)=n}$ be the set of 2-by-2 matrices M with entries in $\mathcal{O}_x$ such that $v_x(\det(M))=n$.

At $x\in R$, the character $\eta_x|_{\mathcal{O}_x^\times}$ factors through the unique non-trivial character $\overline{\eta}_x: k(x)^\times\to\{\pm1\}$. We also denote by $\overline{\eta}_x: k(x)\to\{0,\pm1\}$ its extension by zero to the whole $k(x)$.

When $x \in R$, let $\widetilde{h}_x^{\square} \in C_c^{\infty}(\mathrm{GL}_2(F_x))$ be the function supported on $\mathrm{Mat}_2(\mathcal{O}_x)_{v_x(\det)=1}$ given by

$$\widetilde{h}_x^{\square}((a_{ij})) = \begin{cases} \frac{1}{2}\prod_{i,j\in\{1,2\}}(1+\overline{\eta}_x(\overline{a}_{ij})) & \text{if } a_{ij} \in \mathcal{O}_x^{\times}\ \forall i,j \in \{1,2\}, \\ \prod_{i,j\in\{1,2\}}(1+\overline{\eta}_x(\overline{a}_{ij})) & \text{otherwise.} \end{cases} \tag{2.10}$$

Define

$$h_x^{\square} = p_{x,*}\widetilde{h}_x^{\square} \in C_c^{\infty}(G(F_x)).$$

We give an interpretation of the formula (2.10) as counting the number of certain "square-roots" of (a_{ij}). Let Ξ_x be the set of pairs of matrices $\left(\left[\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{smallmatrix}\right], \left[\begin{smallmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22}\end{smallmatrix}\right]\right) \in \mathrm{Mat}_2(\mathcal{O}_x) \times \mathrm{Mat}_2(k(x))$ such that

(1) for $1 \le i,j \le 2$, $\alpha_{ij}^2 = \overline{a}_{ij}$, the image of a_{ij} in $k(x)$;
(2) $\det(\alpha_{ij}) = 0$;
(3) $v_x(\det(a_{ij})) = 1$.

LEMMA 2.4. *Let $\mu_x : \Xi_x \to \mathrm{Mat}_2(\mathcal{O}_x)$ be the projection to the first factor (a_{ij}). We have*

$$\widetilde{h}_x^{\square} = \mu_{x,*}\mathbf{1}_{\Xi_x}. \tag{2.11}$$

Proof. Let $(a_{ij}) \in \mathrm{Mat}_2(\mathcal{O}_x)_{v_x(\det)=1}$ be such that all a_{ij} are squares. Then its preimage in $\Xi_{D,x}$ consists of $(\alpha_{ij}) \in \mathrm{Mat}_2(k(x))$ where α_{ij} is a square root of $\overline{a}_{ij}$, such that $\det(\alpha_{ij}) = 0$. If all a_{ij} are units, among the $\prod_{i,j}(1+\overline{\eta}_x(\overline{a}_{ij})) = 2^4 = 16$ choices of (α_{ij}), only half of them satisfy $\det(\alpha_{ij}) = 0$. Hence the preimage of such (a_{ij}) in Ξ_x consists of eight elements. If at least one of a_{ij} is non-unit, then the condition $v_x(\det(a_{ij})) = 1$ implies $\det(\alpha_{ij}) = 0$. Therefore, the preimage of such $(a_{ij}) \in \mathrm{Mat}_2(\mathcal{O}_x)_{v_x(\det)=1}$ in Ξ_x has cardinality given by $\prod_{i,j}(1+\overline{\eta}_x(\overline{a}_{ij}))$, as desired by (2.10). □

2.4.2. *The function $f_x^{\square}$.* We introduce another test function, closely related to $h_x^{\square}$, which will be useful in the calculation of its action on representations.

For $x \in R$, let $\widetilde{f}_x^{\square}$ be the function supported on $\mathrm{Mat}_2(\mathcal{O}_x)_{v_x(\det)=1}$ given by the formula

$$\widetilde{f}_x^{\square}((a_{ij})) = \begin{cases} \overline{\eta}_x(\overline{a}_{11}\overline{a}_{12}) & \text{if } a_{11}, a_{12} \in \mathcal{O}_x^{\times}, \\ \overline{\eta}_x(\overline{a}_{21}\overline{a}_{22}) & \text{if } a_{21}, a_{22} \in \mathcal{O}_x^{\times}, \\ 0 & \text{otherwise.} \end{cases}$$

Note that the first two cases above are not mutually exclusive, but when all $a_{ij} \in \mathcal{O}_x^{\times}$, we have $\overline{\eta}_x(\overline{a}_{11}\overline{a}_{12}) = \overline{\eta}_x(\overline{a}_{21}\overline{a}_{22})$ because the rank of $(\overline{a}_{ij}) \in \mathrm{Mat}_2(k(x))$ is one.

We then define

$$f_x^{\square} = p_{x*}\widetilde{f}_x^{\square} \in C_c^{\infty}(G(F_x)).$$

LEMMA 2.5. *The function $\widetilde{f}_x^{\Box}$ is characterized up to a scalar by the following three properties*:

(1) *its support is contained in* $\mathrm{Mat}_2(\mathcal{O}_x)_{v_x(\det)=1}$;
(2) *it is left invariant under* $\mathrm{GL}_2(\mathcal{O}_x)$;
(3) *under the action of the diagonal torus $\widetilde{A}(\mathcal{O}_x)$ by right multiplication, it is an eigenfunction with eigencharacter* $\mathrm{diag}(a,d)\mapsto \overline{\eta}_x(a/d)$.

Furthermore, we have

$$\widetilde{f}_x^{\Box}=\sum_{u\in k(x)^\times}\overline{\eta}_x(u)\cdot \mathbf{1}_{\mathrm{GL}_2(\mathcal{O}_x)\left[\begin{smallmatrix}1 & u\\ & \varpi_x\end{smallmatrix}\right]}. \tag{2.12}$$

Proof. Let $\mathcal{F}$ be the space $\mathbb{C}$-valued functions satisfying the above conditions. The coset space $\mathrm{GL}_2(\mathcal{O}_x)\backslash\mathrm{Mat}_2(\mathcal{O}_x)_{v_x(\det)=1}$ has representatives given by

$$\begin{bmatrix}\varpi_x & 0\\ 0 & 1\end{bmatrix},\qquad \begin{bmatrix}1 & u\\ 0 & \varpi_x\end{bmatrix},\quad u\in k(x).$$

We have a bijection $\mathrm{GL}_2(\mathcal{O}_x)\backslash\mathrm{Mat}_2(\mathcal{O}_x)_{v_x(\det)=1}\cong\mathbb{P}^1(k(x))=k(x)\cup\{\infty\}$ by sending $\left[\begin{smallmatrix}\varpi_x & 0\\ 0 & 1\end{smallmatrix}\right]$ to ∞ and $\left[\begin{smallmatrix}1 & u\\ 0 & \varpi_x\end{smallmatrix}\right]$ to u. The right multiplication of $\widetilde{A}(\mathcal{O}_x)$ on $\mathrm{GL}_2(\mathcal{O}_x)\backslash\mathrm{Mat}_2(\mathcal{O}_x)_{v_x(\det)=1}$ factors through $\widetilde{A}(\mathcal{O}_x)\to\widetilde{A}(k(x))$, and $\mathrm{diag}(a,d)$ acts as $u\mapsto(\overline{d}/\overline{a})\cdot u$ ($u\in\mathbb{P}^1(k(x))$). Therefore, $\mathcal{F}$ is isomorphic to the $\overline{\eta}_x$-eigenspace of $\widetilde{A}(k(x))$ on $C(\mathbb{P}^1(k(x)))$ under right translation. The latter space is one-dimensional and is spanned by $f_\eta: u\mapsto\overline{\eta}_x(u)$ for $u\in k(x)^\times$ and zero for $u=0$ or ∞. Hence $\dim_{\mathbb{C}}\mathcal{F}=1$.

The right-hand side of the expression (2.12) is the function in $\mathcal{F}$ corresponding to f_η, therefore it is a constant multiple of $\widetilde{f}_x^{\Box}$. But both sides take value 1 at $\left[\begin{smallmatrix}1 & 1\\ 0 & \varpi_x\end{smallmatrix}\right]$, so they must be equal. This proves the lemma. $\Box$

We compare the test functions $h_x^{\Box}$ and $f_x^{\Box}$.

LEMMA 2.6. *The difference $h_x^{\Box}-f_x^{\Box}$ is a sum of two functions*, *one is invariant under the right translation by $A(\mathcal{O}_x)$*, *and the other is η-eigen under the left translation by $A(\mathcal{O}_x)$*.

Proof. The function $\widetilde{h}_x^{\Box}$ can be written as

$$\widetilde{h}_x^{\Box}=\Phi_0-\frac{1}{2}\Phi_1,$$

where both Φ_0 and Φ_1 are supported on $\mathrm{Mat}_2(\mathcal{O}_x)_{v_x(\det)=1}$:

$$\Phi_0((a_{ij}))=\prod_{i,j\in\{1,2\}}(1+\overline{\eta}_x(\overline{a}_{ij}))$$

and

$$\Phi_1((a_{ij})) = \begin{cases} \prod_{i,j\in\{1,2\}}(1+\overline{\eta}_x(\overline{a}_{ij})) & \text{if } a_{ij}\in\mathcal{O}_x^\times\ \forall i,j\in\{1,2\},\\ 0 & \text{otherwise.}\end{cases}$$

For any subset $S\subset\{(1,1),(1,2),(2,1),(2,2)\}$, define the following functions supported on $\mathrm{Mat}_2(\mathcal{O}_x)_{v_x(\det)=1}$:

$$\widetilde{\delta}_{0,S}((a_{ij})) := \prod_{(i,j)\in S}\overline{\eta}_x(\overline{a}_{ij}),$$

$$\widetilde{\delta}_{1,S}((a_{ij})) := \begin{cases} \prod_{(i,j)\in S}\overline{\eta}_x(\overline{a}_{ij}) & \text{if } a_{ij}\in\mathcal{O}_x^\times\ \forall i,j\in\{1,2\},\\ 0 & \text{otherwise.}\end{cases}$$

Then

$$\Phi_0=\sum_S\widetilde{\delta}_{0,S},\quad \Phi_1=\sum_S\widetilde{\delta}_{1,S},$$

and hence

$$\widetilde{h}_x^\square=\sum_S\widetilde{\delta}_{0,S}-\frac{1}{2}\sum_S\widetilde{\delta}_{1,S}. \tag{2.13}$$

On the other hand, let $S_{1*}=\{(1,1),(1,2)\}$ (entries in the first row) and $S_{2*}=\{(2,1),(2,2)\}$ (entries in the second row). From the definition of $\widetilde{f}_x^\square$, we have

$$\widetilde{f}_x^\square=\widetilde{\delta}_{0,S_{1*}}+\widetilde{\delta}_{0,S_{2*}}-\frac{1}{2}\left(\widetilde{\delta}_{1,S_{1*}}+\widetilde{\delta}_{1,S_{2*}}\right). \tag{2.14}$$

In fact, the only non-obvious part of the equality is when all four entries are units, in which cases all four functions $\widetilde{\delta}_{0,S_{1*}}$, $\widetilde{\delta}_{0,S_{2*}}$, $\widetilde{\delta}_{1,S_{1*}}$ and $\widetilde{\delta}_{1,S_{2*}}$ take the same value. Comparing (2.13) and (2.14), we see that $\widetilde{h}_x^\square-\widetilde{f}_x^\square$ is a linear combination of $\widetilde{\delta}_{0,S}$ and $\widetilde{\delta}_{1,S}$ for S in one of the three cases

(1) $|S|$ is odd;
(2) S is either a column, or contains every entry;
(3) S is one of the two diagonals.

Therefore, $h_x^\square-f_x^\square$ is a linear combination of $\delta_{0,S}=p_{x*}\widetilde{\delta}_{0,S}$ and $\delta_{1,S}=p_{x*}\widetilde{\delta}_{1,S}$ for S in one of the above three cases.

In case (1), $\widetilde{\delta}_{0,S}$ and $\widetilde{\delta}_{1,S}$ are eigenfunctions under the translation by scalar matrices in $\mathcal{O}_x^\times$ with non-trivial eigenvalue η_x, and therefore $\delta_{0,S}=\delta_{1,S}=0$.

In case (2), $\widetilde{\delta}_{0,S}$ and $\widetilde{\delta}_{1,S}$ are right invariant under $\widetilde{A}(\mathcal{O}_x)$. Therefore, $\delta_{0,S}$ and $\delta_{1,S}$ are right invariant under $A(\mathcal{O}_x)$.

In case (3), $\widetilde{\delta}_{0,S}$ and $\widetilde{\delta}_{1,S}$ are eigen under the left translation by $\widetilde{A}(\mathcal{O}_x)$ with respect to the character $\mathrm{diag}(a,d)\mapsto\eta_x(a/d)$, and hence $\delta_{0,S}$ and $\delta_{1,S}$ are η_x-eigen under the left translation by $A(\mathcal{O}_x)$.

Combining these calculations, we have proved the lemma. □

2.4.3. We fix a decomposition

$$\Sigma = \Sigma_+ \sqcup \Sigma_-. \tag{2.15}$$

Let $N_\pm = \deg \Sigma_\pm$. Later such a decomposition will come from a pair $\mu, \mu' \in \mathfrak{T}_{r,\Sigma}$ (see (4.5), (4.6)).

For each $x \in \Sigma$, we define a subset $\mathbf{J}_x \subset G(\mathcal{O}_x)$ by

$$\mathbf{J}_x = \begin{cases} \left\{ g \in G(\mathcal{O}_x) | g \equiv \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \mod \mathfrak{m}_x \right\} = \mathrm{Iw}_x & \text{if } x \in \Sigma_+, \\ \left\{ g \in G(\mathcal{O}_x) | g \equiv \begin{bmatrix} * & * \\ * & 0 \end{bmatrix} \mod \mathfrak{m}_x \right\} = \mathrm{Iw}_x \cdot w & \text{if } x \in \Sigma_-. \end{cases} \tag{2.16}$$

Here $w = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}$ is the Weyl element. The local component f_x of our test function f at $x \in \Sigma$ will be the characteristic function of $\mathbf{J}_x$.

2.5. *Calculations of local spherical characters.* In this subsection we compute the local distributions $\mathbb{J}_{\pi_x}(f_x, s_1, s_2)$ for certain pairs (π_x, f_x). We always assume that the additive character ψ_x is unramified. It follows that our measure $d^\times t_x = \zeta_x(1)\frac{dt_x}{|t_x|}$ on $A(F_x) = F_x^\times$ gives $\mathrm{vol}(\mathcal{O}_x^\times) = 1$.

2.5.1. *The case $x \in R$ and π_x unramified.* We consider the test function introduced in Section 2.4.2:

$$\widetilde{f}_x = \widetilde{f}_x^\square, \quad f_x = f_x^\square.$$

We need an equivalent expression of the local spherical character (2.8):

$$\mathbb{J}_{\pi_x}(f_x, s_1, s_2) = \sum_{\{W_i\}} \frac{\lambda_x^\natural(W_i, \mathbf{1}, s_1 + s_2)\lambda_x^\natural\left(\overline{\pi_x(f_x^\vee)W_i}, \eta_x, s_1 - s_2\right)}{\theta_x^\natural(W_i, W_i)}, \tag{2.17}$$

where

$$f_x^\vee(g) := \overline{f_x(g^{-1})}.$$

A similar definition applies to the test function $\widetilde{f}_x$ on $\mathrm{GL}_2(F_x)$. By (2.12), we have

$$\widetilde{f}_x^\vee = \sum_{u \in k(x)^\times} \overline{\eta}_x(u) \cdot \mathbf{1}_{\begin{bmatrix} 1 & \frac{u}{\varpi_x} \\ & \end{bmatrix}^{-1} \mathrm{GL}_2(\mathcal{O}_x)}.$$

LEMMA 2.7. *Let π_x be unramified and $K_x = G(\mathcal{O}_x)$. Let $W_0 \in \mathcal{W}_{\psi_x}(\pi_x)^{K_x}$ be the unique element such that $W_0(1_2) = 1$. Then*

$$\pi(f_x^\vee)W_0\left(\begin{bmatrix} a & \\ & 1 \end{bmatrix}\right) = \begin{cases} \mathrm{vol}(K_x)\eta_x(-a) \cdot q_x^{1/2}\epsilon(\eta_x, 1/2, \psi_x & \textit{if } v_x(a) = -1, \\ 0 & \textit{otherwise.} \end{cases}$$

Here the local ϵ-factor for the quadratic character η_x is given by

$$\epsilon(\eta_x, 1/2, \psi_x) = q_x^{-1/2} \sum_{u\in k(x)^\times} \eta_x(a'u)\psi_x(a'u),$$

where $a' \in F_x^\times$ is any element with $v_x(a') = -1$.

Proof. Let $\begin{bmatrix} \alpha & \\ & \beta \end{bmatrix} \in \mathrm{SL}_2(\mathbb{C})$ (i.e., $\alpha\beta = 1$) be the Satake parameter of π. By Casselman–Shalika formula, we have

$$W_0\left(\begin{bmatrix} \varpi_x^n & \\ & 1 \end{bmatrix}\right) = \begin{cases} q_x^{-n/2}\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}, & n \geq 0, \\ 0, & n < 0. \end{cases}$$

On the other hand, we have

$$\begin{aligned}\pi_x\left(\begin{bmatrix} 1 & u \\ & \varpi_x \end{bmatrix}^{-1}\right) W_0\left(\begin{bmatrix} a & \\ & 1 \end{bmatrix}\right) &= W_0\left(\begin{bmatrix} a & \\ & 1 \end{bmatrix}\begin{bmatrix} 1 & u \\ & \varpi_x \end{bmatrix}^{-1}\right) \\ &= W_0\left(\begin{bmatrix} 1 & -au \\ & 1 \end{bmatrix}\begin{bmatrix} a & \\ & \varpi_x^{-1} \end{bmatrix}\right) = \psi_x(-ua)W_0\left(\begin{bmatrix} a\varpi_x & \\ & 1 \end{bmatrix}\right).\end{aligned}$$

It follows that

$$\begin{aligned}&\pi_x(f_x^\vee)W_0\left(\begin{bmatrix} a & \\ & 1 \end{bmatrix}\right) \\ &\qquad = \mathrm{vol}(K_x)\left(\sum_{u\in k(x)^\times} \overline{\eta}_x(u)\psi_x(-ua)\right) W_0\left(\begin{bmatrix} a\varpi_x & \\ & 1 \end{bmatrix}\right).\end{aligned}$$

By the support of W_0, the second factor in the right-hand side vanishes if $v_x(a) \leq -2$. Since ψ_x is unramified, the first factor in the right-hand side vanishes if $v_x(a) \geq 0$. When $v_x(a) = -1$, we have

$$\begin{aligned}\pi_x(f_x^\vee)W_0\left(\begin{bmatrix} a & \\ & 1 \end{bmatrix}\right) &= \mathrm{vol}(K_x)\left(\sum_{u\in k(x)^\times} \overline{\eta}_x(u)\psi_x(-au)\right) \\ &= \mathrm{vol}(K_x)\eta_x(-a)\left(\sum_{u\in k(x)^\times} \eta_x(-au)\psi_x(-au)\right) \\ &= \mathrm{vol}(K_x)\eta_x(-a) \cdot q_x^{1/2}\epsilon(\eta_x, 1/2, \psi_x).\end{aligned}$$

This completes the proof. □

Proposition 2.8. *Let π_x be unramified, and let F_x'/F_x be ramified. Then*

$$\begin{aligned}\mathbb{J}_{\pi_x}(h_x^\square, s_1, s_2) &= \mathbb{J}_{\pi_x}(f_x^\square, s_1, s_2) \\ &= \mathrm{vol}(G(\mathcal{O}_x))\zeta_x(2) \cdot \eta_x(-1)\epsilon(\eta_x, 1/2, \psi_x) \cdot q_x^{s_1-s_2+1/2}.\end{aligned}$$

Proof. We use the formula (2.17) for the local spherical character evaluated at $f_x = f_x^\square$. Now we note that $f_x^\vee$ is right invariant under $K_x = G(\mathcal{O}_x)$. Therefore, we may simplify the sum into one term involving only the spherical vector $W_0 \in \mathcal{W}_{\psi_x}(\pi_x)^{K_x}$ (normalized so that $W_0(1_2) = 1$):

$$\mathbb{J}_{\pi_x}(f_x, s_1, s_2) = \frac{\lambda_x^\natural(W_0, \mathbf{1}, s_1 + s_2)\lambda_x^\natural\left(\overline{\pi(f^\vee)W_0}, \eta_x, s_1 - s_2\right)}{\theta_x^\natural(W_0, W_0)}. \tag{2.18}$$

Since π_x is unramified, we have

$$\lambda_x^\natural(W_0, \mathbf{1}, s) = 1. \tag{2.19}$$

The quadratic character η_x is ramified and hence

$$L(\pi_x \otimes \eta_x, s) = 1.$$

Using this and Lemma 2.7, we get

$$\lambda_x^\natural\left(\overline{\pi_x(f_x^\vee)W_0}, \eta_x, s\right) = \mathrm{vol}(K_x)\eta_x(-1)q_x^{1/2}\varepsilon(\eta_x, 1/2, \psi_x) \cdot q_x^s. \tag{2.20}$$

Again since π_x is unramified (and ψ_x unramified), we have

$$\theta_x^\natural(W_0, W_0) = 1 - q_x^{-2} = \zeta_x(2)^{-1}. \tag{2.21}$$

Plugging (2.19), (2.20) and (2.21) into (2.18), we get the desired formula for $\mathbb{J}_{\pi_x}(f_x^\square, s_1, s_2)$.

To show $\mathbb{J}_{\pi_x}(h_x^\square, s_1, s_2) = \mathbb{J}_{\pi_x}(f_x^\square, s_1, s_2)$, by Lemma 2.6, it suffices to show that $\mathbb{J}_{\pi_x}(f, s_1, s_2) = 0$ when f is either

(1) invariant under right translation by $A(\mathcal{O}_x)$, or
(2) η_x-eigen under left translation by $A(\mathcal{O}_x)$.

In the first case, $f^\vee$ is invariant under the left translation by $A(\mathcal{O}_x)$. The desired vanishing follows from the formula (2.17), and the fact that the linear functional $\lambda_x^\natural(-, \eta_x, s)$ of π_x is η_x-eigen under $A(\mathcal{O}_x)$. In the second case, the desired vanishing follows from the formula (2.8), and the fact that the linear functional $\lambda_x^\natural(-, \mathbf{1}, s)$ of π_x is invariant under $A(\mathcal{O}_x)$. □

2.5.2. *The case $x \in \Sigma$ and π_x a twisted Steinberg.* Let St be the Steinberg representation of $G(F_x)$.

PROPOSITION 2.9. *Let $\pi_x = \mathrm{St}_\chi = \mathrm{St} \otimes \chi$ be an unramified twist of Steinberg representation, where χ is an unramified quadratic character of $F_x^\times$. Then we have*

$$\mathbb{J}_{\pi_x}(1_{\mathrm{Iw}_x}, s_1, s_2) = \mathrm{vol}(G(\mathcal{O}_x))\zeta_x(2) \cdot q_x^{-1}, \tag{2.22}$$

$$\mathbb{J}_{\pi_x}(1_{\mathrm{Iw}_x \cdot w}, s_1, s_2) = \mathrm{vol}(G(\mathcal{O}_x))\zeta_x(2) \cdot \epsilon(\pi_x \otimes \eta_x, 1/2, \psi_x)q_x^{s_1 - s_2 - 1}. \tag{2.23}$$

Proof. We first prove (2.22). By (2.8), the local spherical character evaluated at $f = 1_{\mathrm{Iw}_x}$ simplifies into one term

$$\mathbb{J}_{\pi_x}(1_{\mathrm{Iw}_x}, s_1, s_2) = \mathrm{vol}(\mathrm{Iw}_x)\,\frac{\lambda_x^{\natural}(W_0, \mathbf{1}, s_1 + s_2)\lambda_x^{\natural}(W_0, \eta_x, s_1 - s_2)}{\theta_x^{\natural}(W_0, W_0)}, \tag{2.24}$$

where W_0 is any non-zero element in the line $\mathcal{W}_{\psi_x}(\mathrm{St}_\chi)^{\mathrm{Iw}_x}$. We normalized W_0 so that $W_0(1_2) = 1$. Then explicitly we have

$$W_0\left(\begin{bmatrix} a & \\ & 1 \end{bmatrix}\right) = \begin{cases} \chi(a)|a|, & v_x(a) \geq 0, \\ 0, & v_x(a) < 0. \end{cases}$$

For any unramified character $\chi' : F_x^\times \to \mathbb{C}^\times$, we have

$$\lambda_x^{\natural}(W_0, \chi', s) = 1. \tag{2.25}$$

We compute the inner product $\theta_x^{\natural}(W_0, W_0)$. First we note

$$\int_{F_x^\times} W_0\left(\begin{bmatrix} a & \\ & 1 \end{bmatrix}\right) \overline{W}_0\left(\begin{bmatrix} a & \\ & 1 \end{bmatrix}\right) d^\times a = \sum_{i=0}^{\infty} q_x^{-2i} = (1 - q_x^{-2})^{-1}.$$

For $\pi_x = \mathrm{St}_\chi$, the local L-factor is

$$L(\pi_x \times \widetilde{\pi}_x, s) = (1 - q_x^{-1-s})^{-1}(1 - q_x^{-s})^{-1}.$$

It follows that the normalized inner product is

$$\theta_x^{\natural}(W_0, W_0) = 1 - q_x^{-1}.$$

Finally we note

$$\mathrm{vol}(\mathrm{Iw}_x) = (1 + q_x)^{-1}\,\mathrm{vol}(G(\mathcal{O}_x)).$$

Hence

$$\mathrm{vol}(\mathrm{Iw}_x)\theta_x^{\natural}(W_0, W_0)^{-1} = \mathrm{vol}(G(\mathcal{O}_x))\zeta_x(2)q_x^{-1}. \tag{2.26}$$

Plugging (2.25), (2.26) into (2.24), we get (2.22).

Now we prove (2.23). By definition, we have

$$\begin{aligned}
&\mathbb{J}_{\pi_x}(1_{\mathrm{Iw}_x \cdot w}, s_1, s_2) \\
&\quad = \sum_{\{W_i\}} \frac{\lambda_x^{\natural}(\pi_x(1_{\mathrm{Iw}_x \cdot w})W_i, \mathbf{1}, s_1 + s_2)\lambda_x^{\natural}\left(\overline{W}_i, \eta, s_1 - s_2\right)}{\theta_x^{\natural}(W_i, W_i)} \\
&\quad = \sum_{\{W_i\}} \frac{\lambda_x^{\natural}(\pi_x(1_{\mathrm{Iw}_x})\pi_x(w)W_i, \mathbf{1}, s_1 + s_2)\lambda_x^{\natural}\left(\overline{\pi_x(w)\pi_x(w)W_i}, \eta_x, s_1 - s_2\right)}{\theta_x^{\natural}(\pi_x(w)W_i, \pi_x(w)W_i)}.
\end{aligned}$$

Note that $\{\pi(w)W_i\}$ is another orthogonal basis for $\mathcal{W}_{\psi_x}(\mathrm{St}_\chi)$; therefore, we may rename it by $\{W_i\}$ and rewrite the above as

$$\mathbb{J}_{\pi_x}(1_{\mathrm{Iw}_x \cdot w}, s_1, s_2) = \sum_{\{W_i\}} \frac{\lambda_x^{\natural}(\pi_x(1_{\mathrm{Iw}_x})W_i, \mathbf{1}, s_1 + s_2)\lambda_x^{\natural}\left(\overline{\pi_x(w)W_i}, \eta_x, s_1 - s_2\right)}{\theta_x^{\natural}(W_i, W_i)},$$

which again simplifies into one single term corresponding to the unique $W_0 \in \mathcal{W}_{\psi_x}(\mathrm{St}_\chi)^{\mathrm{Iw}_x}$ with $W_0(1_2) = 1$:

(2.27)

$$\mathbb{J}_{\pi_x}(1_{\mathrm{Iw}_x\cdot w}, s_1, s_2) = \mathrm{vol}(\mathrm{Iw}_x)\,\frac{\lambda^\natural_x(W_0, \mathbf{1}, s_1+s_2)\lambda^\natural_x\left(\overline{\pi_x(w)W_0}, \eta_x, s_1-s_2\right)}{\theta^\natural_x(W_0, W_0)}.$$

We have an explicit formula

$$(\pi_x(w)W_0)\left(\begin{bmatrix} a & \\ & 1\end{bmatrix}\right) = W_0\left(\begin{bmatrix} & a \\ -1 & \end{bmatrix}\right) = \begin{cases} -q_x^{-1}\chi(a)|a|, & v_x(a) \geq -1, \\ 0, & v_x(a) \leq -2.\end{cases}$$

Using this we can calculate

$$\lambda^\natural_x(\overline{\pi_x(w)W_0}, \eta_x, s) = -(\chi\eta_x)(\varpi_x)q_x^s. \tag{2.28}$$

Plugging (2.26), (2.25) and (2.28) into (2.27), we get

$$\mathbb{J}_{\pi_x}(1_{\mathrm{Iw}_x\cdot w}, s_1, s_2) = -\mathrm{vol}(G(\mathcal{O}_x))\zeta_x(2)(\chi\eta_x)(\varpi_x)q_x^{s_1-s_2-1}. \tag{2.29}$$

Finally recall the ε-factor for the twisted Steinberg $\pi_x \otimes \eta_x = \mathrm{St}\otimes\chi\eta_x$, and recall that the unramified ψ_x is the Atkin–Lehner eigenvalue

$$\epsilon(\pi_x\otimes\eta_x, 1/2, \psi_x) = \varepsilon(\mathrm{St}\otimes\chi\eta_x, 1/2, \psi_x) = -(\chi\eta_x)(\varpi_x).$$

Using this we can rewrite (2.29) in the form of (2.23). □

2.6. *The global spherical character for our test functions.*

2.6.1. *Assumptions on* π. Let $\pi = \otimes'_{x\in|X|}\pi_x$ be a cuspidal automorphic representation of $G(\mathbb{A})$ that is ramified exactly at the set Σ. Assume that π_x is isomorphic to an unramified twist of the Steinberg representation at each $x \in \Sigma$.

Recall that $R \subset |X|$ is the ramification locus of the double cover $\nu : X' \to X$. Assume $\Sigma \cap R = \varnothing$. Let $\Sigma = \Sigma_f \sqcup \Sigma_\infty$ be the decomposition determined by the conditions (1.4) and (1.5).

The degrees of the L-functions $L(\pi, s)$ and $L(\pi\otimes\eta, s)$ as a polynomials of q^{-s} are

$$\deg L(\pi, s) = 4g-4+N, \quad \deg L(\pi\otimes\eta, s) = 4g-4+2\rho+N.$$

We set

$$\begin{aligned}\mathscr{L}_{F'/F}(\pi, s_1, s_2) &:= q^{(2g-2+N/2)(s_1+s_2)+(2g-2+\rho+N/2)(s_1-s_2)} \\ &\qquad\times\frac{L\left(\pi, s_1+s_2+\frac12\right)L\left((\pi\otimes\eta, s_1-s_2+\frac12\right)}{L(\pi, \mathrm{Ad}, 1)} \\ &= |\omega_X|^{-2s_1}q^{\rho(s_1-s_2)}q^{Ns_1}\frac{L\left(\pi, s_1+s_2+\frac12\right)L\left((\pi\otimes\eta, s_1-s_2+\frac12\right)}{L(\pi, \mathrm{Ad}, 1)}.\end{aligned}$$

Then we have

$$\mathscr{L}_{F'/F}(\pi, s_1, s_2) = (-1)^{\#\Sigma_\infty}\mathscr{L}_{F'/F}(\pi, -s_1, -s_2).$$

Indeed, the sign that appears above is the root number of the base change L-function $L(\pi_{F'}, s)$, which is the parity of the number of places in F' at which the base change of π_x is the Steinberg representation. If $x \in \Sigma_f$, then x is split in F', and its contribution to the root number is always $+1$; if $x \in \Sigma_\infty$, then x is inert in F', the base change of π_x is always the Steinberg representation, and hence it contributes -1 to the root number.

Recall that in (2.15) we have a decomposition $\Sigma = \Sigma_+ \sqcup \Sigma_-$ (right now arbitrary). We set

$$\epsilon_-(\pi \otimes \eta) \colon = \prod_{x \in \Sigma_-} \epsilon(\pi_x \otimes \eta_x, 1/2).$$

Note that this is the Atkin–Lehner eigenvalue at the set of places Σ_-.

For each $f \in \mathscr{H}_G^{\Sigma \cup R}$, we define

$$f^{\Sigma_\pm} = f \otimes \left(\bigotimes_{x \in R} h_x^{\square}\right) \otimes \left(\bigotimes_{x \in \Sigma} \mathbf{1}_{\mathbf{J}_x}\right) \in C_c^\infty(G(\mathbb{A})). \tag{2.30}$$

Proposition 2.10. *Let π be a cuspidal automorphic representation of $G(\mathbb{A})$ satisfying the conditions in Section 2.6.1. Let $\lambda_\pi : \mathscr{H}_G^{\Sigma \cup R} \to \mathbb{C}$ be the homomorphism associated to π. Then for $f \in \mathscr{H}_G^{\Sigma \cup R}$, we have*

$$q^{N_+ s_1 + N_- s_2}\mathbb{J}_\pi(f^{\Sigma_\pm}, s_1, s_2) = \frac{1}{2}\lambda_\pi(f) \cdot \epsilon_-(\pi \otimes \eta) \cdot |\omega_X| q^{\rho/2 - N}\mathscr{L}_{F'/F}(\pi, s_1, s_2).$$

Proof. We choose a non-trivial $\psi : F\backslash\mathbb{A} \to \mathbb{C}^\times$. Such a character ψ comes from a non-zero rational differential form c on X, so that the conductor of ψ_x is $\mathfrak{m}_x^{v_x(c)}$, where $v_x(c)$ is the order of c at x. We choose such a c so that c has no zeros or poles at $\Sigma \cup R$, so that ψ_x is unramified at $x \in \Sigma \cup R$.

When $x \notin \Sigma \cup R$, f_x is in the spherical Hecke algebra $\mathscr{H}_x$, and therefore

$$\mathbb{J}_{\pi_x}(f_x, s_1, s_2) = \lambda_{\pi_x}(f_x)\,\mathrm{vol}(G(\mathcal{O}_x))\frac{\lambda_x^\natural(W_0, \mathbf{1}, s_1 + s_2)\lambda_x^\natural(\overline{W}_0, \eta_x, s_1 - s_2)}{\theta_x^\natural(W_0, W_0)}$$

for $W_0 \in \mathcal{W}_{\psi_x}(\pi_x)^{G(\mathcal{O}_x)}$ normalized by $W_0(1_2) = 1$. By the same proof as [10, Lemma 4.6], we obtain

$$\frac{\lambda_x^\natural(W_0, \mathbf{1}, s_1 + s_2)\lambda_x^\natural(\overline{W}_0, \eta_x, s_1 - s_2)}{\theta_x^\natural(W_0, W_0)} = \eta_x(c)|c|_x^{-2s_1 + 1/2}\zeta_x(2).$$

Therefore

$$\mathbb{J}_{\pi_x}(f_x, s_1, s_2) = \mathrm{vol}(G(\mathcal{O}_x))\zeta_x(2) \cdot \eta_x(c)|c|_x^{-2s_1+1/2}\lambda_{\pi_x}(f_x). \tag{2.31}$$

Now we use the calculation of local spherical characters at $x \in \Sigma \cup R$ given in Propositions 2.8 and 2.9 together with (2.31), and we plug them into (2.9) to obtain

$$\begin{aligned}\mathbb{J}_\pi(f^{\Sigma_\pm}, s_1, s_2) &= |\omega_X|^{-1} C_{\mathrm{vol}} C_0 C_{\Sigma_+} C_{\Sigma_-} C_R \\ &\quad \times \frac{L\left(\pi, s_1+s_2+\frac{1}{2}\right) L\left((\pi\otimes\eta, s_1-s_2+\frac{1}{2}\right)}{2\, L(\pi, \mathrm{Ad}, 1)},\end{aligned} \tag{2.32}$$

where

$$C_{\mathrm{vol}} = \prod_{x\in|X|} \mathrm{vol}(G(\mathcal{O}_x))\zeta_x(2) = \mathrm{vol}(G(\mathbb{O}))\zeta_F(2) = |\omega_X|^{3/2},$$

$$\begin{aligned}C_0 &= \lambda_\pi(f) \prod_{x\notin R\cup\Sigma} \eta_x(c)|c|_x^{1/2-2s_1} \\ &= \lambda_\pi(f)|\omega_X|^{1/2-2s_1} \prod_{x\notin R\cup\Sigma} \eta_x(c),\end{aligned} \tag{2.33}$$

$$C_{\Sigma_+} = \prod_{x\in\Sigma_+} q_x^{-1} = q^{-N_+},$$

$$C_{\Sigma_-} = \prod_{x\in\Sigma_-} \varepsilon(\pi_x\otimes\eta_x, 1/2, \psi_x) q_x^{s_1-s_2-1} = \varepsilon_-(\pi\otimes\eta) q^{N_-(s_1-s_2)-N_-},$$

$$\begin{aligned}C_R &= \prod_{x\in R} \eta_x(-1)\epsilon(\eta_x, 1/2, \psi_x) q_x^{s_1-s_2+1/2} \\ &= q^{\rho(s_1-s_2)+\rho/2} \prod_{x\in R} \epsilon(\eta_x, 1/2, \psi_x).\end{aligned} \tag{2.34}$$

Here, in (2.33) we used that c is a differential form with no zeros or poles along $\Sigma \cup R$; in (2.34) we have used $\prod_{x\in R} \eta_x(-1) = \eta(-1) = 1$ since $\eta_x(-1)$ is trivial for $x \notin R$. Taking the product and using (2.32), we get

$$\begin{aligned}&\mathbb{J}_\pi(f^{\Sigma_\pm}, s_1, s_2) \\ &= \frac{1}{2}\lambda_\pi(f)|\omega_X|\varepsilon_-(\pi\otimes\eta)\cdot C_\eta \cdot |\omega_X|^{-2s_1} q^{\rho(s_1-s_2)+\rho/2} q^{-N} q^{N_-(s_1-s_2)} \\ &\quad \times \frac{L\left(\pi, s_1+s_2+\frac{1}{2}\right) L\left((\pi\otimes\eta, s_1-s_2+\frac{1}{2}\right)}{L(\pi, \mathrm{Ad}, 1)},\end{aligned} \tag{2.35}$$

where

$$C_\eta = \prod_{x\in R} \epsilon(\eta_x, 1/2, \psi_x) \prod_{x\notin R\cup\Sigma} \eta_x(c).$$

We claim that $C_\eta = 1$. In fact, for $x \notin R$ we have

$$\epsilon(\eta_x, 1/2, \psi_x) = \eta_x(c).$$

It follows that

$$C_\eta = \epsilon(\eta, 1/2, \psi).$$

Recall that $\epsilon(\eta, s) = \epsilon(\eta, s, \psi) = \prod_{x\in|X|} \epsilon(\eta_x, s, \psi_x)$ is the ϵ-factor in the functional equation $L(\eta, s) = \epsilon(\eta, s)L(\eta, 1-s)$. It follows from the expression $L(\eta, s) = \frac{\zeta_{F'}(s)}{\zeta_F(s)}$ that $\epsilon(\eta, 1/2) = 1$. This proves $C_\eta = 1$. Comparing the other terms in (2.35) and in the definition of $\mathscr{L}_{F'/F}(\pi, s_1, s_2)$, we get

$$\mathbb{J}_\pi(f^{\Sigma_\pm}, s_1, s_2) = \frac{1}{2}\lambda_\pi(f)\varepsilon_-(\pi\otimes\eta)|\omega_X|q^{\rho/2-N}q^{N_-(s_1-s_2)-Ns_1}\mathscr{L}_{F'/F}(\pi, s_1, s_2).$$

Multiplying both sides by $q^{N_+s_1+N_-s_2}$, the proposition follows. □

3. **Shtukas with Iwahori level structures**

In this section we will define various moduli stacks of Shtukas with Iwahori level structure and "supersingular legs" at ∞. We study the geometric properties of such moduli stacks and establish the spectral decomposition of their cohomology under the action of the Hecke algebra.

3.1. *Bundles with Iwahori level structures.* Let n be a positive integer. Let $G = \mathrm{PGL}_n$. Let $\Sigma \subset |X|$ be finite set of closed points of X.

Definition 3.1. Let $\mathrm{Bun}_n(\Sigma)$ be the moduli stack whose S-points is the groupoid of

$$\mathcal{E}^\dagger = \left(\mathcal{E}, \{\mathcal{E}\left(-\frac{j}{n}x\right)\}_{1\le j\le n-1, x\in\Sigma}\right),$$

where

- $\mathcal{E}$ is a rank n vector bundle over $X \times S$;
- for each $x \in \Sigma$, $\{\mathcal{E}(-\frac{j}{n}x)\}_{1\le j\le n-1}$, form a chain of coherent subsheaves of $\mathcal{E}$ such that

$$\mathcal{E} \supset \mathcal{E}\left(-\frac{1}{n}x\right) \supset \mathcal{E}\left(-\frac{2}{n}x\right) \supset \cdots \supset \mathcal{E}\left(-\frac{n-1}{n}x\right) \supset \mathcal{E}(-x) = \mathcal{E}\otimes_{\mathcal{O}_X}\mathcal{O}_X(-x)$$

  and that the quotient $\mathcal{E}(-\frac{j-1}{n}x)/\mathcal{E}(-\frac{j}{n}x)$ is scheme theoretically supported at $\{x\}\times S = \mathrm{Spec}\, k(x)\times S$ and is locally free of rank one on $\{x\}\times S$.

The Picard stack Pic_X acts on $\mathrm{Bun}_n(\Sigma)$ by tensoring on both $\mathcal{E}$ and the $\mathcal{E}(-\frac{j}{n}x)$'s. We define

$$\mathrm{Bun}_G(\Sigma) := \mathrm{Bun}_n(\Sigma)/\mathrm{Pic}_X.$$

3.1.1. *Fractional twists.* Let $\mathcal{E}^\dagger = (\mathcal{E}; \{\mathcal{E}(-\frac{j}{n}x)\}_{x\in\Sigma}) \in \mathrm{Bun}_n(\Sigma)(S)$. For any rational divisor

$$D = \sum_{x\in|X|} c_x \cdot x$$

on X satisfying

$$c_x \in \frac{1}{n}\mathbb{Z} \text{ for } x\in\Sigma, \quad c_x\in\mathbb{Z} \text{ otherwise,} \tag{3.1}$$

we may define a vector bundle $\mathcal{E}(D)$ in the following way. There is a unique way to write $D = D_0 - D_1$ where $D_0 \in \mathrm{Div}(X)$ and $D_1 = \sum_{x\in\Sigma} \frac{i_x}{n} x$ for integers $0 \leq i_x \leq n-1$. We define $\mathcal{E}(-D_1) \subset \mathcal{E}$ to be the kernel of the sum of projections

$$\mathcal{E} \longrightarrow \bigoplus_{x\in\Sigma} \mathcal{E}/\mathcal{E}\left(-\frac{i_x}{n}x\right).$$

Then we define $\mathcal{E}(D) = \mathcal{E}(-D_1)\otimes_X \mathcal{O}_X(D_0)$. It is easy to check that $\mathcal{E}(D+D') = (\mathcal{E}(D))(D')$ whenever both D and D' satisfy (3.1).

3.1.2. *Variant of fractional twists.* Now suppose Σ is decomposed into a disjoint union of two subsets

$$\Sigma = \Sigma_\infty \coprod \Sigma_f. \tag{3.2}$$

Let

$$\mathfrak{S}_\infty = \prod_{x\in\Sigma_\infty} \mathrm{Spec}\, k(x) \quad \text{(product over } k).$$

We now consider the base change

$$\mathrm{Bun}_n(\Sigma) \times \mathfrak{S}_\infty.$$

An S-point of $\mathfrak{S}_\infty$ is a collection $\{x^{(1)}\}_{x\in\Sigma_\infty}$ where $x^{(1)} : S \to \mathrm{Spec}\, k(x) \hookrightarrow X$ for each $x \in \Sigma_\infty$. It will be convenient to introduce $x^{(i)}$ for all integers i inductively such that

$$x^{(i)} = x^{(i-1)} \circ \mathrm{Fr}_S : S \xrightarrow{\mathrm{Fr}_S} S \xrightarrow{x^{(i-1)}} \mathrm{Spec}\, k(x) \hookrightarrow X, \quad i \in \mathbb{Z}. \tag{3.3}$$

Clearly we have $x^{(i)} = x^{(i+d_x)}$, where $d_x = [k(x) : k]$.

For each $x \in \Sigma_\infty$, we have a canonical point

$$\mathbf{x}^{(1)} : \mathfrak{S}_\infty \longrightarrow \mathrm{Spec}\, k(x) \longrightarrow X$$

given by projection to the x-factor. We define $\mathbf{x}^{(i)}$ as in (3.3) with S replaced by $\mathfrak{S}_\infty$. Then the graph $\Gamma_{\mathbf{x}^{(i)}}$ of $\mathbf{x}^{(i)}$ is a divisor in $X \times \mathfrak{S}_\infty$. We abuse the notation to abbreviate $\Gamma_{\mathbf{x}^{(i)}}$ by $\mathbf{x}^{(i)}$. Then we have a decomposition

$$\{x\} \times \mathfrak{S}_\infty = \mathrm{Spec}\, k(x) \times \mathfrak{S}_\infty = \coprod_{i=1}^{d_x} \mathbf{x}^{(i)}.$$

Now let $\{x^{(1)}\}_{x\in\Sigma_\infty}$ be an S-point of $\mathfrak{S}_\infty$. Then the graphs of $x^{(i)}$ ($x \in \Sigma_\infty$, $1 \leq i \leq d_x$) are divisors in $X \times S$ pulled back from the divisors $\mathbf{x}^{(i)}$ on $X \times \mathfrak{S}_\infty$. For $\mathcal{E}^\dagger \in \mathrm{Bun}_n(\Sigma)(S)$, the quotient $\mathcal{E}/\mathcal{E}(-\frac{i}{n}x)$ then splits as a direct sum $\oplus_{j=1}^{d_x} \mathcal{Q}_i^{(j)}$ where $\mathcal{Q}_i^{(j)}$ is supported on $\Gamma_{x^{(j)}}$ (with rank i). We define $\mathcal{E}(-\frac{i}{n}x^{(j)})$ to be the kernel

$$\mathcal{E}\left(-\frac{i}{n}x^{(j)}\right) := \ker\left(\mathcal{E} \longrightarrow \mathcal{E}/\mathcal{E}\left(-\frac{i}{n}x\right) \twoheadrightarrow \mathcal{Q}_i^{(j)}\right).$$

In other words, $\{\mathcal{E}(-\frac{i}{n}x^{(j)})\}_{1\leq i\leq n-1}$ give an Iwahori level structure of $\mathcal{E}$ at $x^{(j)}$. With these definitions, for $\mathcal{E}^\dagger \in \mathrm{Bun}_n(\Sigma)(S)$, the construction in Section 3.1.1 then allows us to make sense of $\mathcal{E}(D)$, where D is a divisor on $X \times \mathfrak{S}_\infty$ of the form

$$D = \sum_{x\in\Sigma_\infty, 1\leq j\leq d_x} c_x^{(j)}\mathbf{x}^{(j)} + \sum_{x\in|X|-\Sigma_\infty} c_x(\{x\}\times\mathfrak{S}_\infty), \tag{3.4}$$

where

$$\begin{aligned} c_x^{(j)} &\in \frac{1}{n}\mathbb{Z} && \text{for } x\in\Sigma_\infty, 1\leq j\leq d_x,\\ c_x &\in \frac{1}{n}\mathbb{Z} && \text{for } x\in\Sigma_f,\\ c_x &\in \mathbb{Z} && \text{otherwise.}\end{aligned}$$

More precisely, we can uniquely write $D = D_0 - D_1$, where $D_0 \in \mathrm{Div}(X\times\mathfrak{S}_\infty)$ has $\mathbb{Z}$-coefficients and D_1 is of the form

$$D_1 = \sum_{x\in\Sigma_\infty, 1\leq j\leq d_x} \frac{i_x^{(j)}}{n}\mathbf{x}^{(j)} + \sum_{x\in\Sigma_f}\frac{i_x}{n}(\{x\}\times\mathfrak{S}_\infty),$$

and the coefficients $\frac{i_x^{(j)}}{n}$ (for $x\in\Sigma_\infty$) and $\frac{i_x}{n}$ (for $x\in\Sigma_f$) are in $\{\frac{1}{n},\frac{2}{n},\dots,\frac{n-1}{n}\}$. We define $\mathcal{E}(-D_1)$ to be the kernel of the sum of the projections

$$\mathcal{E}\longrightarrow\left(\bigoplus_{x\in\Sigma_\infty,1\leq j\leq d_x}\mathcal{E}/\mathcal{E}\left(-\frac{i_x^{(j)}}{n}x^{(j)}\right)\right)\bigoplus\left(\bigoplus_{x\in\Sigma_f}\mathcal{E}/\mathcal{E}\left(-\frac{i_x}{n}x\right)\right).$$

Finally let $\mathcal{E}(D) := \mathcal{E}(-D_1)\otimes_{\mathcal{O}_{X\times\mathfrak{S}_\infty}}\mathcal{O}_{X\times\mathfrak{S}_\infty}(D_0)$.

Definition 3.2. Let D be a $\mathbb{Q}$-divisor of $X\times\mathfrak{S}_\infty$ satisfying the conditions as in (3.4). The *Atkin–Lehner automorphisms* for $\mathrm{Bun}_n(\Sigma)$ and $\mathrm{Bun}_G(\Sigma)$ are maps

$$\begin{aligned}\widetilde{\mathrm{AL}}(D) &: \mathrm{Bun}_n(\Sigma)\times\mathfrak{S}_\infty\longrightarrow\mathrm{Bun}_n(\Sigma),\\ \mathrm{AL}(D) &: \mathrm{Bun}_G(\Sigma)\times\mathfrak{S}_\infty\longrightarrow\mathrm{Bun}_G(\Sigma)\end{aligned}$$

sending $\mathcal{E}^\dagger = (\mathcal{E};\{\mathcal{E}(-\frac{j}{n}x)\}_{x\in\Sigma};\{x^{(1)}\}_{x\in\Sigma_\infty})$ to

$$\mathcal{E}^\dagger(D) = \left(\mathcal{E}(D);\{\mathcal{E}(D-\frac{j}{n}(\{x\}\times\mathfrak{S}_\infty))\}_{x\in\Sigma}\right),$$

which makes sense by the discussion in Section 3.1.2.

The maps $\widetilde{\mathrm{AL}}(D)$ and $\mathrm{AL}(D)$ are analogous to the Atkin–Lehner automorphisms on the modular curves, hence their name. From the definition we see that $\mathrm{AL}(D)$ depends only on $D_\infty \mod \mathbb{Z}$.

3.1.3. Let $r \geq 0$ be an integer, and let $\underline{\mu} = (\mu_1, \dots, \mu_r) \in \{\pm 1\}^r$. We define the Hecke stack with Iwahori level structures.

Definition 3.3. Let $\mathrm{Hk}_n^{\mu}(\Sigma)$ be the stack whose S-points is the groupoid of the following data:

- a sequence of S-points $\mathcal{E}_i^{\dagger} = (\mathcal{E}_i; \{\mathcal{E}_i(-\frac{j}{n}x)\}_{x\in\Sigma}) \in \mathrm{Bun}_n(\Sigma)(S)$ for $i = 0, 1, \dots, r$;
- morphisms $x_i : S \to X$ for $i = 1, \dots, r$, with graphs $\Gamma_{x_i} \subset X \times S$;
- isomorphisms of vector bundles

$$f_i : \mathcal{E}_{i-1}|_{X\times S-\Gamma_{x_i}} \xrightarrow{\sim} \mathcal{E}_i|_{X\times S-\Gamma_{x_i}}, \quad i = 1, 2, \dots, r. \tag{3.5}$$

These data are required to satisfy the following conditions:

(1) If $\mu_i = 1$, then f_i extends to an injective map $\mathcal{E}_{i-1} \to \mathcal{E}_i$ whose cokernel is an invertible sheaf on the graph Γ_{x_i}. Moreover, f_i sends $\mathcal{E}_{i-1}(-\frac{j}{n}x)$ to $\mathcal{E}_i(-\frac{j}{n}x)$ for all $x \in \Sigma$ and $1 \leq j \leq n-1$.

(2) If $\mu_i = -1$, then f_i^{-1} extends to an injective map $\mathcal{E}_i \to \mathcal{E}_{i-1}$ whose cokernel is an invertible sheaf on the graph Γ_{x_i}. Moreover, f_i^{-1} sends $\mathcal{E}_i(-\frac{j}{n}x)$ to $\mathcal{E}_{i-1}(-\frac{j}{n}x)$ for all $x \in \Sigma$ and $1 \leq j \leq n-1$.

We have a morphism $\pi_{\mathrm{Hk}}^{\mu} : \mathrm{Hk}_n^{\mu}(\Sigma) \to X^r$ recording the points $x_1, \dots, x_r$ in the above definition. For $0 \leq i \leq r$, let

$$\widetilde{p}_i : \mathrm{Hk}_n^{\mu}(\Sigma) \longrightarrow \mathrm{Bun}_n(\Sigma)$$

be the morphism recording the i-th point $\mathcal{E}_i^{\dagger} \in \mathrm{Bun}_n(\Sigma)$.

There is an action of Pic_X on $\mathrm{Hk}_n^{\mu}(\Sigma)$ by tensoring. We form the quotient

$$\mathrm{Hk}_G^{\mu}(\Sigma) = \mathrm{Hk}_n^{\mu}(\Sigma)/\,\mathrm{Pic}_X$$

with maps recording $\mathcal{E}_i^{\dagger}$

$$p_i : \mathrm{Hk}_G^{\mu}(\Sigma) \longrightarrow \mathrm{Bun}_G(\Sigma), \quad i = 0, \dots, r.$$

PROPOSITION 3.4.

(1) *For $0 \leq i \leq r$, the morphism $\widetilde{p}_i : \mathrm{Hk}_n^{\mu}(\Sigma) \to \mathrm{Bun}_n(\Sigma)$ is smooth of relative dimension rn.*

(2) *For $0 \leq i \leq r$, the morphism $(\widetilde{p}_i, \pi_{\mathrm{Hk}}^{\mu}) : \mathrm{Hk}_n^{\mu}(\Sigma) \to \mathrm{Bun}_n(\Sigma) \times X^r$ is smooth of relative dimension $r(n-1)$ when restricted to $\mathrm{Bun}_n(\Sigma) \times (X - \Sigma)^r$.*

(3) *For $0 \leq i \leq r$, the morphism $(\widetilde{p}_i, \pi_{\mathrm{Hk}}^{\mu}) : \mathrm{Hk}_n^{\mu}(\Sigma) \to \mathrm{Bun}_n(\Sigma) \times X^r$ is flat of relative dimension $r(n-1)$.*

(4) *The statements of (1)–(3) hold when $\mathrm{Hk}_n^{\mu}(\Sigma)$ is replaced with $\mathrm{Hk}_G^{\mu}(\Sigma)$ and $\mathrm{Bun}_n(\Sigma)$ is replaced with $\mathrm{Bun}_G(\Sigma)$.*

Proof. We first make some reductions. Once (1)–(3) are proved, (4) follows by dividing out by Pic_X. By the iterative nature of $\mathrm{Hk}_n^{\mu}(\Sigma)$, it is enough to treat the case $r = 1$. We consider the case $\underline{\mu} = 1$ and $i = 1$; the other cases can be treated similarly. We also base change the situation to $\overline{k}$ without changing notation (i.e., X now means $X_{\overline{k}}$, Σ means $\Sigma(\overline{k})$, and the products are over $\overline{k}$, etc). Moreover, if $x \in \Sigma$ and $\Sigma^x = \Sigma - \{x\}$, we observe that over $X - \Sigma^x$ there is an isomorphism $\mathrm{Hk}^1_n(\Sigma)|_{X-\Sigma^x} \cong (\mathrm{Hk}^{\underline{\mu}}_n(\{x\})|_{X-\Sigma^x}) \times_{\mathrm{Bun}_n(\{x\})} \mathrm{Bun}_n(\Sigma)$ such that the projection p_1 is the projection to the second factor. Therefore, to show the statements over $X - \Sigma^x$, it suffices to show the same statements for $\Sigma = \{x\}$. Since the $X - \Sigma^x$ cover X as x runs over Σ, we reduce to the case where Σ is a singleton $\{x\}$. In other words, we are concerned with the map

$$(\widetilde{p}_1, \pi^1_{\mathrm{Hk}}) : \mathrm{Hk}^1_n(\{x\}) \longrightarrow \mathrm{Bun}_n(\{x\}) \times X.$$

(2) Since $\mathrm{Hk}^1_n(\{x\})|_{X-\{x\}} \cong (\mathrm{Hk}^1_n|_{X-\{x\}}) \times_{\mathrm{Bun}_n} \mathrm{Bun}_n(\{x\})$, we have a Cartesian diagram

$$\begin{array}{ccc} \mathrm{Hk}^1_n(\{x\})|_{X-\{x\}} & \xrightarrow{\widetilde{p}_1|_{X-\{x\}}} & \mathrm{Bun}_n(\{x\}) \times (X - \{x\}) \\ \downarrow & & \downarrow \\ \mathrm{Hk}^1_n & \xrightarrow{\widetilde{p}_1} & \mathrm{Bun}_n \times X. \end{array}$$

Since the bottom horizontal map $\mathrm{Hk}^1_n \to \mathrm{Bun}_n \times X$ is the projectivization of the universal bundle over $\mathrm{Bun}_n \times X$, it is smooth of relative dimension $n-1$. Therefore, the same is true for the top horizontal map.

(1) and (3). Let $S = \operatorname{Spec} R$, where R is a local $\overline{k}$-algebra. Let $\mathcal{E}^\dagger \in \mathrm{Bun}_n(\{x\})(S)$. For an S-scheme S', the S'-points of the fiber $\widetilde{p}_1^{-1}(\mathcal{E}^\dagger)$ are $\mathcal{F}^\dagger \in \mathrm{Bun}_n(\{x\})(S')$ such that for each $0 \le i \le n-1$, $\mathcal{F}(-\frac{i}{n}x) \subset \mathcal{E}(-\frac{i}{n}x)$ with quotients an invertible sheaf supported on the graph of some map $y : S' \to X$. Such $\mathcal{F}(-\frac{i}{n}x)$ are classified by the projectivization $\mathbb{P}(\mathcal{E}(-\frac{i}{n}x))$ over $X \times S$. The fiber $\widetilde{p}_1^{-1}(\mathcal{E}^\dagger)$ is then a closed subscheme of

$$\mathbb{P}(\mathcal{E}) \times_{X\times S} \mathbb{P}\left(\mathcal{E}\left(-\frac{1}{n}x\right)\right) \times_{X\times S} \cdots \times \mathbb{P}\left(\mathcal{E}\left(-\frac{n-1}{n}x\right)\right).$$

We will write down defining equations of this closed subscheme. Let $U_x \subset X$ be an open affine neighborhood of x, and let $t \in \mathcal{O}(U_x)$ be a coordinate at x. Shrinking U_x we may assume t only vanishes at x. Since we know (2) already, to show (1) and (3), it is enough to show the corresponding statements over U_x.

After étale localizing S, we may assume that $\mathcal{E}^\dagger$ is trivialized on $U_x \times S$. Thus we fix a trivialization $\iota : \mathcal{E}|_{U_x\times S} \xrightarrow{\sim} \mathcal{O}^n_{U_x\times S}$ so that

$$\iota\left(\mathcal{E}\left(-\frac{i}{n}x\right)\Big|_{U_x\times S}\right) = t\mathcal{O}_{U_x\times S} \oplus \cdots \oplus t\mathcal{O}_{U_x\times S} \oplus \mathcal{O}_{U_x\times S} \oplus \cdots \oplus \mathcal{O}_{U_x\times S}, \tag{3.6}$$

where the first i summands are $t\mathcal{O}_{U_x\times S}$ and the last $n-i$ are $\mathcal{O}_{U_x\times S}$. Using the decomposition (3.6), we may canonically identify $\mathbb{P}(\mathcal{E}(-\frac{i}{n}x))|_{U_x\times S} \cong \mathbb{P}^{n-1} \times U_x \times S$. Let $S' = \operatorname{Spec} R'$, where R' is a local R-algebra. Then an

R' point in $\widetilde{p}^{-1}(\mathcal{E}^\dagger)|_{U_x\times S}$ may be expressed using homogeneous coordinates $a^{(i)} = [a^{(i)}_0, \dots, a^{(i)}_{n-1}] \in \mathbb{P}^{n-1}(R')$ for $i = 0, \dots, n-1$ (which gives $\mathcal{F}(-\frac{i}{n}x)$) and a point $y \in U_x(R)$. The superscripts and subscripts of $a^{(i)}_j$ are understood as elements in $\mathbb{Z}/n\mathbb{Z}$, so $a^{(0)}_j = a^{(n)}_j$ etc.

The condition $\mathcal{F}(-\frac{i}{n}x) \subset \mathcal{F}(-\frac{i-1}{n}x)$ means that the following diagram can be completed into a commutative diagram by a choice of $\lambda \in R'$:

$$\begin{array}{ccccc}
\mathcal{E}(-\frac{i}{n}x) & \xrightarrow{\mathrm{ev}_y} & R'^n & \xrightarrow{a^{(i)}} & R' \\
\big\downarrow & & \big\downarrow{\scriptstyle \tau_{i-1}:=\mathrm{diag}(1,\dots,t(y),\dots,1)} & & \big\downarrow{\scriptstyle \lambda} \\
\mathcal{E}(-\frac{i-1}{n}x) & \xrightarrow{\mathrm{ev}_y} & R'^n & \xrightarrow{a^{(i-1)}} & R',
\end{array}$$

where the middle vertical map τ_{i-1} is the diagonal matrix with $t(y) \in R'$ on the (i,i)-entry and 1 elsewhere on the diagonal (so $\tau_{i-1}(a^{(i-1)})$ multiplies $a^{(i-1)}_{i-1}$ by $t(y)$ and leaves the other coordinates unchanged). This gives the closed condition

$$\tau_{i-1}(a^{(i-1)}) \text{ is in the line spanned by } a^{(i)}. \tag{3.7}$$

We study the special fiber of $\widetilde{p}_i$ over $(\mathcal{E}^\dagger, x)$. Fix a $\overline{k}$-point of $\mathcal{F}^\dagger \in \widetilde{p}_1^{-1}(\mathcal{E}^\dagger)$ over $y = x$ with coordinates $\mathbf{a}^{(i)} = [\mathbf{a}^{(i)}_0, \dots, \mathbf{a}^{(i)}_{n-1}], i \in \mathbb{Z}/n\mathbb{Z}$. Let $[e_i] \in \mathbb{P}^{n-1}$ be the coordinate line where only the i-th coordinate can be non-zero. Define

$$I = \{i \in \mathbb{Z}/n\mathbb{Z} | \mathbf{a}^{(i)} = [e_i]\}.$$

It is easy to see from condition (3.7) that $I \neq \varnothing$. The points in I cut the cyclically ordered set $\mathbb{Z}/n\mathbb{Z}$ into intervals. (Think about the n-th roots of unity on the unit circle.) For neighboring $i_1, i_2 \in I$, we have an interval $(i_1, i_2]$ (excluding i_1 and containing i_2 and not containing any other elements in I). When I is a singleton i_1, we understand $(i_1, i_1]$ to be the whole $\mathbb{Z}/n\mathbb{Z}$. These intervals give a partition of $\mathbb{Z}/n\mathbb{Z}$. By (3.7), the homogeneous coordinates $[\mathbf{a}^{(i)}_0, \dots, \mathbf{a}^{(i)}_{n-1}]$ for $\mathcal{F}(-\frac{i}{n}x)$ satisfy

$$\text{if } i \text{ is in the interval } (i_1, i_2], \text{ then } \mathbf{a}^{(i)}_j = 0 \text{ unless } j \in [i, i_2].$$

Moreover, by the definition of I, $\mathbf{a}^{(i)}_i$ is non-zero when $i \in I$. The relation (3.7) implies that whenever $i \in (i_1, i_2]$, where $i_1, i_2 \in I$ are neighbors, $\mathbf{a}^{(i)}_{i_2}$ is non-zero.

Now we give equations defining $\widetilde{p}_1^{-1}(\mathcal{E}^\dagger)$ near the point $\mathcal{F}^\dagger$. Let $a^{(i)} = [a^{(i)}_0, \dots, a^{(i)}_{n-1}], 0 \le i \le n-1$ be the coordinates of such an R'-valued point that specializes to $\mathcal{F}^\dagger$. For an interval $(i_1, i_2]$ and $i \in (i_1, i_2]$, since $\mathbf{a}^{(i)}_{i_2} \neq 0$, $a^{(i)}_{i_2}$ is interval in R', therefore we may assume $a^{(i)}_{i_2} = 1$ for $i \in (i_1, i_2]$. We now use the following affine coordinates: for any interval $(i_1, i_2]$ formed by neighboring elements $i_1, i_2 \in I$, we consider

$$a_{i_1+1}^{(i_1+1)}, \dots, a_{i_2-1}^{(i_1+1)}, \text{ and } a_{i_2}^{(i_1)}. \tag{3.8}$$

There are n such variables. Condition (3.7) implies that

$$\prod_{i_1 \in I} a_{i_2}^{(i_1)} = t(y), \tag{3.9}$$

where i_1 runs over I and i_2 is its immediate successor. It turns out that the other coordinates can be uniquely determined by the ones in (3.8) using condition (3.7) and that (3.9) is the only relation implied by (3.7). From this description we conclude that étale locally near $\mathcal{F}^\dagger$, $\widetilde{p}_1^{-1}(\mathcal{E}^\dagger)|_{U_x}$ is isomorphic to $\mathbb{A}_S^n$ with the map $\widetilde{p}_1^{-1}(\mathcal{E}^\dagger)|_{U_x} \xrightarrow{\pi^1_{\mathrm{Hk}}} U_x \times S \xrightarrow{t} \mathbb{A}_S^1$ corresponding to $\mathbb{A}_S^n \to \mathbb{A}_S^1$ given by the product of a subset of coordinates. Therefore, (1) and (3) follow. □

3.2. *Shtukas with Iwahori level structures.*

3.2.1. *Moduli of rank n Shtukas with Iwahori level structures.* Let $\underline{\mu} \in \{\pm 1\}^r$. Fix a $\mathbb{Q}$-divisor D_∞ on $X \times \mathfrak{S}_\infty$ supported at $\Sigma_\infty \times \mathfrak{S}_\infty$ of the form

$$D_\infty = \sum_{x \in \Sigma_\infty, 1 \le i \le d_x} c_x^{(i)} \mathbf{x}^{(i)}, \quad c_x^{(i)} \in \frac{1}{n}\mathbb{Z}. \tag{3.10}$$

We assume that $\underline{\mu}$ satisfies the following condition:

$$\sum_{i=1}^r \mu_i = \sum_{x \in \Sigma_\infty, 1 \le i \le d_x} n c_x^{(i)} = n \deg D_\infty. \tag{3.11}$$

Definition 3.5. We define the stack $\mathrm{Sht}_n^{\underline{\mu}}(\Sigma; D_\infty)$ by the Cartesian diagram

$$\begin{array}{ccc} \mathrm{Sht}_n^{\underline{\mu}}(\Sigma; D_\infty) & \longrightarrow & \mathrm{Hk}_n^{\underline{\mu}}(\Sigma) \times \mathfrak{S}_\infty \\ \downarrow & & \downarrow {\scriptstyle (\widetilde{p}_0, \widetilde{\mathrm{AL}}(-D_\infty) \circ (\widetilde{p}_r \times \mathrm{id}_{\mathfrak{S}_\infty}))} \\ \mathrm{Bun}_n(\Sigma) & \xrightarrow{(\mathrm{id}, \mathrm{Fr})} & \mathrm{Bun}_n(\Sigma) \times \mathrm{Bun}_n(\Sigma). \end{array} \tag{3.12}$$

Concretely, for a k-scheme S, an S-point of $\mathrm{Sht}_n^{\underline{\mu}}(\Sigma; D_\infty)$ consists of the following data:

- for each $0 \le i \le r$, a point $\mathcal{E}_i^\dagger = (\mathcal{E}_i; \{\mathcal{E}_i(-\frac{j}{n}x)\}_{x \in \Sigma}) \in \mathrm{Bun}_n(\Sigma)(S)$;
- for each $x \in \Sigma_\infty$, a morphism $x^{(1)} : S \to \mathrm{Spec}\, k(x)$;
- for each $1 \le i \le r$, a morphism $x_i : S \to X$;
- maps $f_1, \dots, f_r$ as in the definition of $\mathrm{Hk}_n^{\underline{\mu}}(\Sigma)$;
- an isomorphism $\iota : \mathcal{E}_r \cong ({}^\tau\mathcal{E}_0)(D_\infty)$ (first pullback by Frobenius, then fractional twist by D_∞) respecting the Iwahori level structures at all $x \in \Sigma$.

By definition, we have a morphism recording x_i and $\{x^{(1)}\}_{x\in\Sigma_\infty}$ in the definition above:

$$\Pi^{\mu}_{n,D_\infty} : \mathrm{Sht}^{\mu}_{n}(\Sigma; D_\infty) \longrightarrow X^r \times \mathfrak{S}_\infty. \tag{3.13}$$

LEMMA 3.6. *Let D_∞ be a $\mathbb{Q}$-divisor of the form (3.10). Then up to canonical isomorphisms*, $\mathrm{Sht}^{\mu}_{n}(\Sigma; D_\infty)$ *depends only on the sum $\sum_{1\le i\le d_x} c^{(i)}_x$ for each $x \in \Sigma_\infty$.*

Proof. Let $D'_\infty = \sum_{x\in\Sigma_\infty}(\sum_{1\le i\le d_x} c^{(i)}_x)\mathbf{x}^{(1)}$. It suffices to give a canonical isomorphism $\alpha : \mathrm{Sht}^{\mu}_{n}(\Sigma; D_\infty) \xrightarrow{\sim} \mathrm{Sht}^{\mu}_{n}(\Sigma; D'_\infty)$. Let $(\mathcal{E}^\dagger_i; x_i; \{x^{(1)}\}; \iota)$ be an S-point of $\mathrm{Sht}^{\mu}_{n}(\Sigma; D_\infty)$. For $0 \le i \le r$, let

$$\mathcal{F}^\dagger_i = \mathcal{E}^\dagger_i\left(-\sum_{2\le j\le j'\le d_x} c^{(j')}_x \mathbf{x}^{(j)}\right).$$

One checks that ι induces an isomorphism $\iota' : \mathcal{F}^\dagger_r \cong {}^\tau\mathcal{F}^\dagger_0(D'_\infty)$. Define

$$\alpha(\mathcal{E}^\dagger_i; x_i; \{x^{(1)}\}; \iota) = (\mathcal{F}^\dagger_i; x_i; \{x^{(1)}\}; \iota'),$$

which is easily seen to be an isomorphism. □

3.2.2. *The case $r = 0$.* When $r = 0$, $\mathrm{Sht}^{\varnothing}_{n}(\Sigma; \Sigma_\infty)$ is zero dimensional. We describe the groupoid of $\overline{k}$-points of $\mathrm{Sht}^{\varnothing}_{n}(\Sigma; \Sigma_\infty)$. For any $\xi : \mathfrak{S}_\infty \to \overline{k}$ (which amounts to choosing a $\overline{k}$-point $x^{(1)}$ over each $x \in \Sigma_\infty$), let $\mathrm{Sht}^{\varnothing}_{n}(\Sigma; \xi)$ be the fiber of $\mathrm{Sht}^{\varnothing}_{n}(\Sigma; \Sigma_\infty)$ over ξ.

Let B be the central simple algebra over F of dimension n^2, which is split at points away from Σ_∞ and has Hasse invariant $\mathrm{inv}_x(B) = \sum_{1\le i\le d_x} c^{(i)}_x$ for $x \in \Sigma_\infty$. Since $\sum_{x\in\Sigma_\infty}\sum_{1\le i\le d_x} c^{(i)}_x = 0$ by (3.11), such a central simple algebra B exists. Let $B^\times$ denote the algebraic group over F of the multiplicative group of units in B. For $x \in \Sigma$, let $K_x \subset B^\times(F_x)$ be a minimal parahoric subgroup (so for $x \in \Sigma_f$, K_x is an Iwahori subgroup of $B^\times(F_x) \cong \mathrm{GL}_n(F_x)$). For $x \in |X - \Sigma|$, let K_x be a maximal parahoric of $B^\times(F_x) \cong \mathrm{GL}_n(F_x)$ such that almost all of them come from an integral model of B over X. Then we have an isomorphism of groupoids

$$\mathrm{Sht}^{\varnothing}_{n}(\Sigma; \xi)(\overline{k}) \cong B^\times(F)\backslash B^\times(\mathbb{A}_F)/\prod_{x\in|X|} K_x.$$

3.2.3. *The case $r = 1$ and Drinfeld modules.* We consider the special case where $r = 1$, $\mu = -1$, Σ_∞ consists of a single point ∞, and $D_\infty = -\frac{1}{n}\Gamma_{\infty^{(1)}}$. In this case the stack $\mathrm{Sht}^{\mu}_{n}(\Sigma; D_\infty)$ is closely related to the moduli stack $\mathrm{DrMod}_n(\Sigma_f)$ of Drinfeld $A = \Gamma(X - \{\infty\}, \mathcal{O}_X)$-modules with Iwahori level structure at Σ_f. In fact, in [1, Th. 3.1.4] it is shown that $\mathrm{DrMod}_n(\Sigma_f)$ can be identified with the open and closed substack of $\mathrm{Sht}^{\mu}_{n}(\Sigma; D_\infty)|_{X-\{\infty\}}$ consisting of those $(\mathcal{E}_i^\dagger;\dots)$ where $\mathcal{E}_0$ has degree $n(g-1)+1$. This implies an isomorphism over $X-\{\infty\}$:

$$\mathrm{DrMod}_n(\Sigma_f)/\operatorname{Pic}^0_X(k)\cong \mathrm{Sht}^{\mu}_G(\Sigma;D_\infty)|_{X-\{\infty\}}.$$

3.2.4. *Relation with the usual Shtukas.* We explain how $\mathrm{Sht}^{\mu}_n(\Sigma;D_\infty)$ is related to the Shtukas in the sense of [8]. Let $\Sigma_\infty=\{y_1,\dots,y_s\}$ and $d_i=[k(y_i):k]$. Let $r'=r+\sum_{i=1}^s d_i$. For each $c\in\frac{1}{n}\mathbb{Z}$, we have a unique coweight $\underline{\mu}(c)=(a_1,\dots,a_n)\in\mathbb{Z}^n$ of GL_n such that $\sum_i a_i=nc$ and $a_n\le a_{n-1}\le\dots\le a_1\le a_n+1$. (In other words, $\mu(c)$ is a minuscule coweight.) Let D_∞ take the form (3.10). Let

$$\underline{\mu}'=\big(\mu_1,\dots,\mu_r,\mu(c^{(1)}_{y_1}),\dots,\mu(c^{(d_1)}_{y_1}),\mu(c^{(1)}_{y_2}),\\ \dots,\mu(c^{(d_2)}_{y_2}),\dots,\mu(c^{(1)}_{y_s}),\dots,\mu(c^{(d_s)}_{y_s})\big).$$

This is an r'-tuple of minuscule dominant coweights of GL_n. We consider the stack $\mathrm{Sht}^{\underline{\mu}'}_n(\Sigma)$ of rank n Shtukas with modification types given by $\underline{\mu}'$ and Iwahori level structure at Σ: it is given by the Cartesian diagram

$$\begin{array}{ccc} \mathrm{Sht}^{\underline{\mu}'}_n(\Sigma) & \longrightarrow & \mathrm{Hk}^{\underline{\mu}'}_n(\Sigma)\\ \downarrow & & \downarrow{\scriptstyle(\widetilde{p}_0,\widetilde{p}_{r'})}\\ \mathrm{Bun}_n(\Sigma) & \xrightarrow{(\mathrm{id},\mathrm{Fr})} & \mathrm{Bun}_n(\Sigma)\times\mathrm{Bun}_n(\Sigma), \end{array}$$

where $\mathrm{Hk}^{\underline{\mu}'}_n(\Sigma)$ is defined similarly as $\mathrm{Hk}^{\mu}_n(\Sigma)$. There is a natural map $\pi^{\underline{\mu}'}_n:\mathrm{Sht}^{\underline{\mu}'}_n(\Sigma)\to X^{r'}$. We have a map

$$e_{\Sigma_\infty}:X^r\times\mathfrak{S}_\infty\longmapsto X^{r'}$$

given by sending

$$(x_1,\dots,x_r,y_1^{(1)},\dots,y_s^{(1)})$$

to

$$(x_1,\dots,x_r,y_1^{(1)},\dots,y_1^{(d_1)},y_2^{(1)},\dots,y_s^{(d_s)}).$$

Lemma 3.7. *There is a canonical closed embedding* $\widetilde{e}:\mathrm{Sht}^{\mu}_n(\Sigma;D_\infty)\hookrightarrow\mathrm{Sht}^{\underline{\mu}'}_n(\Sigma)$ *making the following diagram commutative*:

$$\begin{array}{ccc} \mathrm{Sht}^{\mu}_n(\Sigma;D_\infty) & \xrightarrow{\widetilde{e}} & \mathrm{Sht}^{\underline{\mu}'}_n(\Sigma)\\ \downarrow{\scriptstyle\Pi^{\mu}_{n,D_\infty}} & & \downarrow{\scriptstyle\pi^{\underline{\mu}'}_n}\\ X^r\times\mathfrak{S}_\infty & \xrightarrow{e_{\Sigma_\infty}} & X^{r'}. \end{array}$$

Proof. The map $\widetilde{e}$ is defined by sending $(\mathcal{E}_i^\dagger, f_i, \iota) \in \mathrm{Sht}_n^\mu(\Sigma; D_\infty)$ over $(x_1, \dots, x_r, y_1^{(1)}, \dots, y_s^{(1)}) \in X^r \times \mathfrak{S}_\infty$ to the following point $(\mathcal{F}_i^\dagger, f_i', \iota')$ over $e_{\Sigma_\infty}(x_1, \dots, x_r, y_1^{(1)}, \dots, y_s^{(1)})$. We define

$$\mathcal{F}_i^\dagger = \begin{cases} \mathcal{E}_i^\dagger, \text{ if } 0 \le i \le r; \\ ({}^\tau\mathcal{E}_0^\dagger)(D_\infty - \sum_{h=1}^{j_1} c_{y_h}^{(h)} y_1^{(h)}), \\ \quad \text{if } i = r + j_1, 1 \le j_1 \le d_1; \\ ({}^\tau\mathcal{E}_0^\dagger)(D_\infty - \sum_{h=1}^{d_1} c_{y_1}^{(h)} y_1^{(h)} - \sum_{h=1}^{j_2} c_{y_2}^{(h)} y_2^{(h)}), \\ \quad \text{if } i = r + d_1 + j_2, 1 \le j_2 \le d_2; \\ \dots \\ ({}^\tau\mathcal{E}_0^\dagger)(\sum_{h=j_s+1}^{d_s} c_{y_s}^{(h)} y_s^{(h)}), \\ \quad \text{if } i = r + d_1 + \dots + d_{s-1} + j_s, 1 \le j_s \le d_s. \end{cases}$$

The map f_r' is $\mathcal{E}_r^\dagger \xrightarrow{\iota} ({}^\tau\mathcal{E}_0^\dagger)(D_\infty) \dashrightarrow ({}^\tau\mathcal{E}_0^\dagger)(D_\infty - c_{y_1}^{(1)} y_1)$, and the other maps f_i', ι' are the obvious ones. The above equation for $\mathcal{F}_i^\dagger$ gives a closed condition on $\mathrm{Sht}_n^{\mu'}(\Sigma)$ without changing automorphisms, realizing $\mathrm{Sht}_n^\mu(\Sigma; D_\infty)$ as a closed substack of $\mathrm{Sht}_n^{\mu'}(\Sigma)$. ☐

3.2.5. $\mathrm{Sht}_G^\mu(\Sigma; D_\infty)$ *and its geometric properties.* The groupoid $\mathrm{Pic}_X(k)$ acts on $\mathrm{Sht}_n^\mu(\Sigma; D_\infty)$ by tensoring. We define the quotient (see [10, 5.2.1] for the explanation why the quotient makes sense as a stack)

$$\mathrm{Sht}_G^\mu(\Sigma; D_\infty) := \mathrm{Sht}_n^\mu(\Sigma; D_\infty)/\,\mathrm{Pic}_X(k).$$

We have a Cartesian diagram

$$\begin{array}{ccc} \mathrm{Sht}_G^\mu(\Sigma; D_\infty) & \longrightarrow & \mathrm{Hk}_G^\mu(\Sigma) \times \mathfrak{S}_\infty \\ \Big\downarrow \omega_0 & & \Big\downarrow (p_0, \mathrm{AL}(-D_\infty)\circ(p_r \times \mathrm{id}_{\mathfrak{S}_\infty})) \\ \mathrm{Bun}_G(\Sigma) & \xrightarrow{(\mathrm{id},\mathrm{Fr})} & \mathrm{Bun}_G(\Sigma) \times \mathrm{Bun}_G(\Sigma). \end{array} \tag{3.14}$$

The map Π_{n,D_∞}^μ in (3.13) induces a map

$$\Pi_{G,D_\infty}^\mu = (\pi_G^\mu, \pi_{G,\infty}) : \mathrm{Sht}_G^\mu(\Sigma; D_\infty) \longrightarrow X^r \times \mathfrak{S}_\infty. \tag{3.15}$$

Since the action $\mathrm{AL}(D_\infty)$ on $\mathrm{Bun}_G(\Sigma)$ depends only on $D_\infty \mod \mathbb{Z}$, combined with Lemma 3.6 we conclude that

LEMMA 3.8. *The moduli stack* $\mathrm{Sht}_G^\mu(\Sigma; D_\infty)$ *depends only on the image of* D_∞ *in* $\mathrm{Div}(\Sigma_\infty) \otimes_{\mathbb{Z}} (\frac{1}{n}\mathbb{Z}/\mathbb{Z})$.

PROPOSITION 3.9.

(1) *The stack* $\mathrm{Sht}_G^\mu(\Sigma; D_\infty)$ *is a smooth* DM *stack of dimension* rn.

(2) *The morphism* Π^{μ}_{G,D_∞} *is separated and is smooth of relative dimension* $r(n-1)$ *over* $(X-\Sigma)^r \times \mathfrak{S}_\infty$.

Proof. To show the smoothness statements in (1) and (2), we adapt the argument of [6, Prop. 2.11] and apply [6, Lemme 2.13] to the diagram (3.14). Without giving all the details, the same argument of [6, Prop. 2.11] shows that after an étale base change, the fibration $p_r : \mathrm{Hk}^{\mu}_{G}(\Sigma) \to \mathrm{Bun}_G(\Sigma)$ can be trivialized. Therefore, the same is true for $q_r := \mathrm{AL}(-D_\infty) \circ (p_r \times \mathrm{id}_{\mathfrak{S}_\infty}) : \mathrm{Hk}^{\mu}_{G}(\Sigma) \times \mathfrak{S}_\infty \to \mathrm{Bun}_G(\Sigma)$ because $\mathrm{AL}(-D_\infty)$ is étale. Then [6, Lemme 2.13] applied to the diagram (3.14) implies that $\mathrm{Sht}^{\mu}_{G}(\Sigma; D_\infty)$ is étale locally isomorphic to a fiber of q_r. More precisely, for a fixed choice of $\mathcal{E}^\dagger \in \mathrm{Bun}_G(\Sigma)(k)$ (for example the trivial bundle with any Iwahori level structure at Σ), there exists an étale covering $\{U_i\}$ of $\mathrm{Sht}^{\mu}_{G}(\Sigma; D_\infty)$ together with étale maps $U_i \to q_r^{-1}(\mathcal{E}^\dagger)$ over $X^r \times \mathfrak{S}_\infty$.

Since p_r is smooth of relative dimension rn by Proposition 3.4(1), so is q_r and hence $q_r^{-1}(\mathcal{E}^\dagger)$ is smooth over k of dimension rn. This implies that $\mathrm{Sht}^{\mu}_{G}(\Sigma; D_\infty)$ is smooth of dimension rn.

By Proposition 3.4(2), $p_r^{-1}(\mathcal{E}^\dagger)|_{(X-\Sigma)^r}$ is smooth of relative of dimension $r(n-1)$ over $(X-\Sigma)^r$. Therefore, the same is true for $q_r^{-1}(\mathcal{E}^\dagger)|_{(X-\Sigma)^r\times\mathfrak{S}_\infty}$. By the discussion in the first paragraph, this implies that $\mathrm{Sht}^{\mu}_{G}(\Sigma; D_\infty)|_{(X-\Sigma)^r\times\mathfrak{S}_\infty}$ is smooth over $(X-\Sigma)^r \times \mathfrak{S}_\infty$ of relative dimension $r(n-1)$.

We now show that $\mathrm{Sht}^{\mu}_{G}(\Sigma; D_\infty)$ is DM. By Lemma 3.7, $\mathrm{Sht}^{\mu}_{G}(\Sigma; D_\infty)$ is a closed substack of $\mathrm{Sht}^{\mu'}_{G}(\Sigma) := \mathrm{Sht}^{\mu'}_{n}(\Sigma)/\mathrm{Pic}_X(k)$. The map $\mathrm{Sht}^{\mu'}_{G}(\Sigma) \to \mathrm{Sht}^{\mu'}_{G}$ (forgetting the level structure) is clearly representable. By [8, Prop. 2.16(a)], $\mathrm{Sht}^{\mu'}_{G}$ is DM, hence so are $\mathrm{Sht}^{\mu'}_{G}(\Sigma)$ and its closed substack $\mathrm{Sht}^{\mu}_{G}(\Sigma; D_\infty)$.

Finally we show Π^{μ}_{G,D_∞} is separated. The map $\mathrm{Sht}^{\mu'}_{G} \to X^{r'}$ is separated, as can be seen from the same argument following [10, Th. 5.4]. This implies that $\pi^{\mu'}_{n} : \mathrm{Sht}^{\mu'}_{G}(\Sigma) \to X^{r'} \times \mathfrak{S}_\infty$ is also separated as $\mathrm{Sht}^{\mu'}_{G}(\Sigma) \to \mathrm{Sht}^{\mu'}_{G}$ is proper. Since $\mathrm{Sht}^{\mu}_{G}(\Sigma; D_\infty)$ is a closed substack of $\mathrm{Sht}^{\mu'}_{G}(\Sigma)$, Π^{μ}_{G,D_∞} is also separated. □

3.2.6. *The base-change situation.* Now let X' be another smooth, projective curve over k with a map $\nu : X' \to X$ satisfying

$$\text{The map } \nu \text{ is unramified over } \Sigma. \tag{3.16}$$

Let

$$\mathfrak{S}'_\infty = \prod_{x' \in \nu^{-1}(\Sigma_\infty)} \mathrm{Spec}\, k(x').$$

Then we have a natural map induced by ν

$$\nu'^r : X'^r \times \mathfrak{S}'_\infty \longrightarrow X^r \times \mathfrak{S}_\infty. \tag{3.17}$$

Define the base change of $\mathrm{Sht}^{\mu}_{G}(\Sigma; D_\infty)$:

$$\mathrm{Sht}'^{\mu}_{G}(\Sigma; D_\infty) := \mathrm{Sht}^{\mu}_{G}(\Sigma; D_\infty) \times_{(X^r\times\mathfrak{S}_\infty)} (X'^r \times \mathfrak{S}'_\infty).$$

Proposition 3.10. *Under the assumption* (3.16), *the stack* $\mathrm{Sht}'^{\underline{\mu}}_G(\Sigma; D_\infty)$ *is a smooth* DM *stack of dimension* rn.

Proof. Only the smoothness of $\mathrm{Sht}'^{\underline{\mu}}_G(\Sigma; D_\infty)$ requires an argument. Let $\mathrm{Hk}'^{\underline{\mu}}_G(\Sigma) = \mathrm{Hk}^{\underline{\mu}}_G(\Sigma) \times_{X^r} X'^r$. As in the proof of Proposition 3.9(1), we reduce to showing that $p'_r : \mathrm{Hk}'^{\underline{\mu}}_G(\Sigma) \to \mathrm{Bun}_G(\Sigma)$ is smooth of relative dimension rn. As in the proof of Proposition 3.4, it suffices to treat the case where $r = 1$ and $\underline{\mu} = 1$.

Let R' be the ramification locus of ν. Then $\mathrm{Hk}'^{\underline{\mu}}_G(\Sigma)|_{X'-R'} \to \mathrm{Hk}^{\underline{\mu}}_G(\Sigma)$ is étale. Hence by Proposition 3.4(1), $p'_1 : \mathrm{Hk}'^{\underline{\mu}}_G(\Sigma) \to \mathrm{Bun}_G(\Sigma)$ is smooth of relative dimension n when restricted to $\mathrm{Hk}'^{\underline{\mu}}_G(\Sigma)|_{X'-R'}$. On the other hand, let $\Sigma' = \nu^{-1}(\Sigma)$. By Proposition 3.4(2),

$$(p_1, \pi^{\underline{\mu}}_{\mathrm{Hk}}) : \mathrm{Hk}^{\underline{\mu}}_G(\Sigma)|_{X-\Sigma} \longrightarrow \mathrm{Bun}_G(\Sigma) \times (X - \Sigma)$$

is smooth of relative dimension $n-1$ By base change along $\nu|_{X'-\Sigma'} : X' - \Sigma' \to X - \Sigma$, we see that $\mathrm{Hk}^{\underline{\mu}}_G(\Sigma)|_{X'-\Sigma'} \to \mathrm{Bun}_G(\Sigma) \times (X' - \Sigma')$ is smooth of relative dimension $n-1$, hence p'_1 is smooth of relative dimension n when restricted to $\mathrm{Hk}'^{\underline{\mu}}_G(\Sigma)|_{X'-\Sigma'}$. By assumption (3.16), $R' \cap \Sigma' = \varnothing$, hence $X' - \Sigma'$ and $X' - R'$ cover X', and we conclude that p'_1 is smooth of relative dimension n, which finishes the proof. □

3.2.7. *Atkin–Lehner for* $\mathrm{Sht}^{\underline{\mu}}_G(\Sigma; D_\infty)$. For $x \in \Sigma$, fractional twisting by $\frac{1}{n}x$ gives an automorphism of $\mathrm{Bun}_n(\Sigma)$ and $\mathrm{Hk}^{\underline{\mu}}_n(\Sigma)$. By the diagram (3.12), we have an induced automorphism on $\mathrm{Sht}^{\underline{\mu}}_n(\Sigma; D_\infty)$,

$$\widetilde{\mathrm{AL}}_{\mathrm{Sht},x} : \mathrm{Sht}^{\underline{\mu}}_n(\Sigma; D_\infty) \longrightarrow \mathrm{Sht}^{\underline{\mu}}_n(\Sigma; D_\infty),$$

sending $(\mathcal{E}^\dagger_i, x_i, \dots)$ to $(\mathcal{E}^\dagger_i(-\frac{1}{n}x), x_i, \dots)$. This also induces an automorphism on $\mathrm{Sht}^{\underline{\mu}}_G(\Sigma; D_\infty)$:

$$\mathrm{AL}_{\mathrm{Sht},x} : \mathrm{Sht}^{\underline{\mu}}_G(\Sigma; D_\infty) \longrightarrow \mathrm{Sht}^{\underline{\mu}}_G(\Sigma; D_\infty).$$

3.2.8. *The case* $n = 2$ *and a specific choice of* D_∞. We specialize to the case $n = 2$ and hence $G = \mathrm{PGL}_2$. Let $\mathscr{D}_\infty$ be the group of $\mathbb{Z}$-valued divisors on $X \times \mathfrak{S}_\infty$ supported on $\Sigma_\infty \times \mathfrak{S}_\infty$, which is the union of the graphs of $\mathbf{x}^{(i)}$ for $x \in \Sigma_\infty$ and $1 \le i \le d_x$. Let $\frac{1}{2}\mathscr{D}_\infty = \frac{1}{2}\mathbb{Z} \otimes_{\mathbb{Z}} \mathscr{D}_\infty$. Then $\mathrm{Sht}^{\underline{\mu}}_G(\Sigma; D_\infty)$ is defined for $D_\infty \in \frac{1}{2}\mathscr{D}_\infty$ satisfying (3.11) for $n = 2$. As in [10, Lemma 5.5], one can show that $\mathrm{Hk}^{\underline{\mu}}_G(\Sigma)$ is canonically independent of $\underline{\mu}$. In this case we denote $\mathrm{Hk}^{\underline{\mu}}_G(\Sigma)$ by $\mathrm{Hk}^r_G(\Sigma)$. This implies

Lemma 3.11. *For fixed* r *and* $D_\infty \in \frac{1}{2}\mathscr{D}_\infty$, *and for any two* $\underline{\mu}, \underline{\mu}' \in \{\pm 1\}^r$ *satisfying the same condition* (3.11), *there is a canonical isomorphism of stacks* $\mathrm{Sht}^{\underline{\mu}}_G(\Sigma; D_\infty) \cong \mathrm{Sht}^{\underline{\mu}'}_G(\Sigma; D_\infty)$ *over* X^r.

Lemma 3.8 implies that $\mathrm{Sht}_G^{\underline{\mu}}(\Sigma; D_\infty)$ depends only on the image of D_∞ in $\mathrm{Div}(\Sigma_\infty) \otimes \frac{1}{2}\mathbb{Z}/\mathbb{Z}$. We consider the following specific choice of D_∞:

$$D_\infty^{(1)} = \sum_{x \in \Sigma_\infty} \frac{1}{2} \mathbf{x}^{(1)}.$$

Definition 3.12. Assume r satisfies the parity condition

$$r \equiv \#\Sigma_\infty \mod 2. \tag{3.18}$$

Let $\underline{\mu} = (\mu_1, \dots, \mu_r) \in \{\pm 1\}^r$. For any $D_\infty \in \frac{1}{2}\mathscr{D}_\infty$ such that

$$D_\infty \equiv D_\infty^{(1)} \mod \mathscr{D}_\infty, \quad \text{and } \sum_{i=1}^r \mu_i = 2 \deg D_\infty, \tag{3.19}$$

we define

$$\mathrm{Sht}_G^r(\Sigma; \Sigma_\infty) := \mathrm{Sht}_G^{\underline{\mu}}(\Sigma; D_\infty).$$

By Lemmas 3.11 and 3.8, this is independent of the choice of $\underline{\mu}$ and D_∞ satisfying condition (3.19), justifying the notation.

We remark that the parity condition (3.18) guarantees that for any $\underline{\mu} \in \{\pm 1\}^r$, the $D_\infty \in \frac{1}{2}\mathscr{D}_\infty$ satisfying (3.19) exists.

We denote $\mathrm{AL}(-D_\infty^{(1)})$ simply by

$$\mathrm{AL}_{G,\infty} := \mathrm{AL}(-D_\infty^{(1)}) : \mathrm{Bun}_G(\Sigma) \times \mathfrak{S}_\infty \longrightarrow \mathrm{Bun}_G(\Sigma). \tag{3.20}$$

Then the diagram (3.14) becomes

$$\begin{array}{ccc} \mathrm{Sht}_G^r(\Sigma; \Sigma_\infty) & \longrightarrow & \mathrm{Hk}_G^r(\Sigma) \times \mathfrak{S}_\infty \\ \Big\downarrow \omega_0 & & \Big\downarrow (p_0, \mathrm{AL}_{G,\infty} \circ (p_r \times \mathrm{id}_{\mathfrak{S}_\infty})) \\ \mathrm{Bun}_G(\Sigma) & \xrightarrow{(\mathrm{id}, \mathrm{Fr})} & \mathrm{Bun}_G(\Sigma) \times \mathrm{Bun}_G(\Sigma). \end{array} \tag{3.21}$$

For D_∞ satisfying (3.19), we denote the morphism $\Pi_{G,D_\infty}^{\underline{\mu}}$ in (3.15) by

$$\Pi_G^r = (\pi_G^r, \pi_{G,\infty}) : \mathrm{Sht}_G^r(\Sigma; \Sigma_\infty) \longrightarrow X^r \times \mathfrak{S}_\infty.$$

3.3. *Hecke symmetry.* For the rest of the paper, we will use G to denote PGL_2. We will focus on the the stack $\mathrm{Sht}_G^r(\Sigma; \Sigma_\infty)$ for r and Σ_∞ satisfying the parity condition (3.18).

3.3.1. *Hecke correspondence.* For $x \in |X - \Sigma|$, let $\mathscr{H}_x$ be the spherical Hecke algebra

$$\mathscr{H}_x = C_c(G(\mathcal{O}_x) \backslash G(F_x) / G(\mathcal{O}_x), \mathbb{Q}).$$

Let $\mathscr{H}_G^\Sigma = \otimes_{x \in |X - \Sigma|} \mathscr{H}_x$. Then $\mathscr{H}_G^\Sigma$ has a $\mathbb{Q}$-basis $\{h_D\}$ indexed by effective divisors $D \in \mathrm{Div}^+(X - \Sigma)$, where h_D is defined in [10, §3.1].

Let D be an effective divisor on $X-\Sigma$. For $\underline{\mu}\in\{\pm1\}^r$ and $D_\infty=\sum_{x\in\Sigma_\infty}c_x\mathbf{x}^{(1)}$ as in Definition 3.12, we define a stack $\mathrm{Sht}_2^{\underline{\mu}}(\Sigma;D_\infty;h_D)$ whose S-points classify the following data:

- two objects $(\mathcal{E}_i^\dagger,f_i,\iota,\dots)$ and $(\mathcal{E}_i'^\dagger,f_i',\iota',\dots)$ of $\mathrm{Sht}_2^{\underline{\mu}}(\Sigma;D_\infty)(S)$ that map to the same S-point of $(x_1,\dots,x_r,\{x^{(1)}\})\in(X^r\times\mathfrak{S}_\infty)(S)$;
- for each $i=0,1,\dots,r$, an embedding of coherent sheaves $\varphi_i:\mathcal{E}_i\to\mathcal{E}_i'$ compatible with the Iwahori level structures, such that $\det(\varphi_i):\det(\mathcal{E}_i)\to\det(\mathcal{E}_i')$ has divisor $D\times S\subset X\times S$, and such that the following diagram is commutative:

$$\begin{array}{ccccccccc}\mathcal{E}_0 & \overset{f_1}{\dashrightarrow} & \mathcal{E}_1 & \overset{f_2}{\dashrightarrow} & \cdots & \overset{f_r}{\dashrightarrow} & \mathcal{E}_r & \xrightarrow{\iota} & ({}^\tau\mathcal{E}_0)(D_\infty)\\ \downarrow{\scriptstyle\varphi_0} & & \downarrow{\scriptstyle\varphi_1} & & & & \downarrow{\scriptstyle\varphi_r} & & \downarrow{\scriptstyle{}^\tau\varphi_0}\\ \mathcal{E}_0' & \overset{f_1'}{\dashrightarrow} & \mathcal{E}_1' & \overset{f_2'}{\dashrightarrow} & \cdots & \overset{f_r'}{\dashrightarrow} & \mathcal{E}_r' & \xrightarrow{\iota'} & ({}^\tau\mathcal{E}_0')(D_\infty).\end{array}\tag{3.22}$$

Let $\mathrm{Sht}_G^r(\Sigma;\Sigma_\infty;h_D)=\mathrm{Sht}_2^{\underline{\mu}}(\Sigma;D_\infty;h_D)/\mathrm{Pic}_X(k)$, which is independent of the choice of $(\underline{\mu},D_\infty)$ as it is for $\mathrm{Sht}_G^r(\Sigma;\Sigma_\infty)$. Then $\mathrm{Sht}_G^r(\Sigma;\Sigma_\infty;h_D)$ can be viewed as a self-correspondence of $\mathrm{Sht}_G^r(\Sigma;\Sigma_\infty)$ over $X^r\times\mathfrak{S}_\infty$,

$$\mathrm{Sht}_G^r(\Sigma;\Sigma_\infty)\xleftarrow{\overleftarrow{p}}\mathrm{Sht}_G^r(\Sigma;\Sigma_\infty;h_D)\xrightarrow{\overrightarrow{p}}\mathrm{Sht}_G^r(\Sigma;\Sigma_\infty),\tag{3.23}$$

where the maps $\overleftarrow{p}$ and $\overrightarrow{p}$ record the first and the second row of (3.22).

LEMMA 3.13. *Let D be an effective divisor on $X-\Sigma$.*

(1) *The two maps $\overleftarrow{p},\overrightarrow{p}:\mathrm{Sht}_G^r(\Sigma;\Sigma_\infty;h_D)\to\mathrm{Sht}_G^r(\Sigma;\Sigma_\infty)$ are representable and proper.*
(2) *The restrictions of $\overleftarrow{p}$ and $\overrightarrow{p}$ over $(X-D)^r$ are finite étale.*
(3) *The fibers of $\Pi_G^r(h_D):\mathrm{Sht}_G^r(\Sigma;\Sigma_\infty;h_D)\to X^r\times\mathfrak{S}_\infty$ all have dimension r.*

Proof. (1) For a rank two vector bundle $\mathcal{E}$ over $X\times S$, let $\mathrm{Quot}^D_{X\times S/S}(\mathcal{E})$ be the S-scheme classifying quotients $\mathcal{E}\twoheadrightarrow\mathcal{Q}$, flat over S and with divisor D. (Namely, for every geometric point $s\in S$, $\mathcal{Q}|_s$ is a torsion sheaf on $X\times s$ with length n_x at $x\times s$ for any $x\in|X|$, where n_x is the coefficient of x in D.) Then $\mathrm{Quot}^D_{X\times S/S}(\mathcal{E})$ is a closed subscheme of the Quot-scheme of $\mathcal{E}$, hence projective over S. The fiber of $\overrightarrow{p}$ over any point $(\mathcal{E}_i'^\dagger,x_i,f_i',\iota')\in\mathrm{Sht}_G^r(\Sigma;\Sigma_\infty)(S)$ is a closed subscheme of $\mathrm{Quot}^D_{X\times S/S}(\mathcal{E}_1')\times_S\mathrm{Quot}^D_{X\times S/S}(\mathcal{E}_2')\times\cdots\times_S\mathrm{Quot}^D_{X\times S/S}(\mathcal{E}_r')$, hence projective over S. This shows that $\overrightarrow{p}$ are representable and proper. The argument for $\overleftarrow{p}$ is similar.

(2) When $(\mathcal{E}_i^\dagger,x_i,f_i,\iota)\in\mathrm{Sht}_G^r(\Sigma;\Sigma_\infty)(S)$ and x_i are disjoint from D (which is assumed to be disjoint from Σ), the restriction $\mathcal{E}|_{D\times S}$ carries a Frobenius structure $\iota|_{D\times S}:\mathcal{E}|_{D\times S}\xrightarrow{\sim}{}^\tau\mathcal{E}|_{D\times S}$ and hence descends to a G_D-torsor $\mathcal{E}_D$ over S, with $G_D=\mathrm{Res}_k^{\mathcal{O}_D}G$ the Weil restriction. Recording this G_D-torsor

defines a map

$$\omega_D : \mathrm{Sht}^r_G(\Sigma;\Sigma_\infty)|_{(X-D)^r} \longrightarrow \mathbb{B}G_D.$$

Let $\widetilde{L}_D$ be the moduli stack whose S-points are triples $(\mathcal{F}_D, \mathcal{F}'_D, \varphi_D)$, where $\mathcal{F}_D, \mathcal{F}'_D$ are lisse sheaves over S that are locally free $\mathcal{O}_D$-modules of rank two, and $\varphi_D : \mathcal{F}_D \to \mathcal{F}'_D$ is an $\mathcal{O}_D$-linear map whose cokernel at each geometric point of S has divisor D. (That is, if $D = \sum_x n_x x$, then the cokernel as an $\mathcal{O}_D$-module has length n_x when localized at x.) Let $L_D = \widetilde{L}_D/\mathbb{B}\mathcal{O}_D^\times$ where $\mathbb{B}\mathcal{O}_D^\times$ acts by simultaneously tensoring. The stack L_D itself is the quotient of a finite discrete scheme over k by a finite group, hence is finite étale over k, and it has two maps to $\mathbb{B}G_D$ recording $\mathcal{F}_D$ and $\mathcal{F}'_D$,

$$\mathbb{B}G_D \xleftarrow{\overleftarrow{\ell}} L_D \xrightarrow{\overrightarrow{\ell}} \mathbb{B}G_D,$$

which are also finite étale.

There is a natural map

$$\widetilde{\omega}_D : \mathrm{Sht}^r_G(\Sigma;\Sigma_\infty;h_D)|_{(X-D)^r} \longrightarrow L_D.$$

In fact, each point $(\mathcal{E}_i^\dagger, x_i, \dots, \mathcal{E}_i'^\dagger, \dots, \varphi_i) \in \mathrm{Sht}^r_G(\Sigma;\Sigma_\infty;h_D)(S)$ gives a pair of G_D-torsors $\mathcal{E}_D$ and $\mathcal{E}'_D$ over S. If we lift $\mathcal{E}_i$ and $\mathcal{E}'_i$ to rank two vector bundles on $X \times S$, then $\mathcal{E}_D$ and $\mathcal{E}'_D$ have associated $\mathcal{O}_D^{\oplus 2}$-torsors $\mathcal{F}_D$ and $\mathcal{F}'_D$ over S, well defined up to simultaneous twisting by $\mathcal{O}_D^\times$-torsors on S. The φ_i then induces an $\mathcal{O}_D$-linear map $\varphi_D : \mathcal{E}_D \to \mathcal{E}'_D$ whose cokernel has divisor D.

When the points x_i are disjoint from D, knowing the top row (or the bottom row) of (3.22) and any of the vertical arrows recovers the whole diagram. Any vertical arrow $\varphi_i : \mathcal{E}_i \to \mathcal{E}'_i$ is in turn determined by $\mathcal{E}_i$ (or $\mathcal{E}'_i$) together with its image in L_D. Therefore, the whole diagram is uniquely determined by the top row (or the bottom row) and its image in L_D. Moreover, since D is disjoint from Σ, the level structures of the top row determines that of the bottom row, and vice versa. This shows the two squares below are Cartesian:

$$\begin{array}{ccccc}
\mathrm{Sht}^r_G(\Sigma;\Sigma_\infty)|_{(X-D)^r} & \xleftarrow{\overleftarrow{p}} & \mathrm{Sht}^r_G(\Sigma;\Sigma_\infty;h_D)|_{(X-D)^r} & \xrightarrow{\overrightarrow{p}} & \mathrm{Sht}^r_G(\Sigma;\Sigma_\infty)|_{(X-D)^r} \\
\downarrow{\scriptstyle \omega_D} & & \downarrow{\scriptstyle \widetilde{\omega}_D} & & \downarrow{\scriptstyle \omega_D} \\
\mathbb{B}G_D & \xleftarrow{\overleftarrow{\ell}} & L_D & \xrightarrow{\overrightarrow{\ell}} & \mathbb{B}G_D.
\end{array}$$

This implies that both $\overleftarrow{p}$ and $\overrightarrow{p}$ are finite étale, because the maps $\overleftarrow{\ell}$ and $\overrightarrow{\ell}$ are.

(3) The argument is similar to that of [10, Lemma 5.9], so we only give a sketch.

Fix a geometric point $\underline{x} = (x_1, \dots, x_r) \in X^r$. We will show that the fiber $\mathrm{Sht}^r_G(\Sigma;\Sigma_\infty;h_D)_{\underline{x}}$ has dimension r. We introduce the moduli stack $H_D(\Sigma)$

classifying $(\mathcal{E}^\dagger, \mathcal{E}'^\dagger, \varphi)$ up to the action of Pic_X, where $\mathcal{E}^\dagger, \mathcal{E}'^\dagger \in \mathrm{Bun}_2(\Sigma)$ and $\varphi : \mathcal{E} \to \mathcal{E}'$ is an injective map with divisor D. Let $\mathrm{Hk}^r_{H,D}(\Sigma)$ classify diagrams

$$\begin{array}{ccccccc} \mathcal{E}_0 & \overset{f_1}{\dashrightarrow} & \mathcal{E}_1 & \overset{f_1}{\dashrightarrow} & \cdots & \overset{f_r}{\dashrightarrow} & \mathcal{E}_r \\ \downarrow{\scriptstyle \varphi_0} & & \downarrow{\scriptstyle \varphi_1} & & & & \downarrow{\scriptstyle \varphi_r} \\ \mathcal{E}'_0 & \overset{f'_1}{\dashrightarrow} & \mathcal{E}'_1 & \overset{f'_2}{\dashrightarrow} & \cdots & \overset{f'_r}{\dashrightarrow} & \mathcal{E}'_r \end{array} \tag{3.24}$$

satisfying the same conditions as the diagram (3.22) without the last column, modulo simultaneous tensoring by Pic_X. We have a Cartesian diagram

$$\begin{array}{ccc} \mathrm{Sht}^r_G(\Sigma; \Sigma_\infty; h_D)_{\underline{x}} & \longrightarrow & \mathrm{Hk}^r_{H,D}(\Sigma)_{\underline{x}} \\ \downarrow & & \downarrow{\scriptstyle (p_0,p_r)} \\ H_D(\Sigma) \times \mathfrak{S}_\infty & \xrightarrow{(\mathrm{id}, \mathrm{AL}_{H,\infty} \circ \mathrm{Fr})} & H_D(\Sigma) \times H_D(\Sigma). \end{array}$$

Here $\mathrm{AL}_{H,\infty} : H_D(\Sigma) \times \mathfrak{S}_\infty \to H_D(\Sigma)$ is given by applying $\mathrm{AL}_{G,\infty}$ to both $\mathcal{E}^\dagger$ and $\mathcal{E}'^\dagger$. The stacks $H_D(\Sigma)$ and $\mathrm{Hk}^r_{H,D}(\Sigma)$ will turn out to be fibers of the stacks $H_d(\Sigma)$ and $\mathrm{Hk}^r_{H,d}(\Sigma)$ over $D \in X_d$, to be introduced in Section 5.2.1.

We introduce the analog $H^\natural_D(\Sigma)$ of the $H_{D,D}$ introduced in [10, 6.4.4], which is an open substack of $H_D(\Sigma)$ where φ does not land in $\mathcal{E}'(-x)$ for any $x \in D$. We claim that the map $H^\natural_D(\Sigma) \to \mathrm{Bun}_G(\Sigma)$ sending $(\mathcal{E}^\dagger, \mathcal{E}'^\dagger, \varphi)$ to $\mathcal{E}'^\dagger$ is smooth. Indeed, its fiber over $\mathcal{E}'^\dagger \in \mathrm{Bun}_G(\Sigma)(S)$ is $\mathrm{Res}^{D\times S}_S(\mathbb{P}_{D\times S}(\mathcal{E}'_{D\times S}))$, the restriction of scalars of the projectivization of the rank two bundle $\mathcal{E}'_{D\times S}$ over $D \times S$. (The Σ-level structure on $\mathcal{E}^\dagger$ is automatically inherited from $\mathcal{E}'^\dagger$, since D is disjoint from Σ.) In particular, $H^\natural_D(\Sigma)$ is smooth over k.

Similarly we introduce the open substack $\mathrm{Hk}^{r,\natural}_{H,D}(\Sigma)_{\underline{x}} \subset \mathrm{Hk}^r_{H,D}(\Sigma)_{\underline{x}}$ by requiring each column of (3.24) to be in $H^\natural_D(\Sigma)$. We define the open substack $\mathrm{Sht}^{r,\natural}_G(\Sigma; \Sigma_\infty; h_D)_{\underline{x}} \subset \mathrm{Sht}^r_G(\Sigma; \Sigma_\infty; h_D)_{\underline{x}}$ to fit into a Cartesian diagram

$$\begin{array}{ccc} \mathrm{Sht}^{r,\natural}_G(\Sigma; \Sigma_\infty; h_D)_{\underline{x}} & \longrightarrow & \mathrm{Hk}^{r,\natural}_{H,D}(\Sigma)_{\underline{x}} \\ \downarrow & & \downarrow{\scriptstyle (p_0,p_r)} \\ H^\natural_D(\Sigma) \times \mathfrak{S}_\infty & \xrightarrow{(\mathrm{id}, \mathrm{AL}_{H,\infty} \circ \mathrm{Fr})} & H^\natural_D(\Sigma) \times H^\natural_D(\Sigma). \end{array}$$

As in [10, 6.4.4], it suffices to show that $\dim \mathrm{Sht}^{r,\natural}_G(\Sigma; \Sigma_\infty; h_D)_{\underline{x}} = r$. As in the case without level structures, $p_r : \mathrm{Hk}^{r,\natural}_{H,D}(\Sigma)_{\underline{x}} \to H^\natural_D(\Sigma)$ is an étale locally trivial fibration. Using a slight variant of [6, Lemme 2.13], $\mathrm{Sht}^{r,\natural}_G(\Sigma; \Sigma_\infty; h_D)_{\underline{x}}$ is étale locally isomorphic to a fiber of p_r. It remains to show that the geometric fibers of p_r have dimension r. The iterative nature of $\mathrm{Hk}^{r,\natural}_{H,D}(\Sigma)_{\underline{x}}$ allows us to reduce to the case $r = 1$.

First consider the case $x_1 \notin D$. Then the diagram (3.24) is determined by its top row and the last column, which means that the fibers of p_1 are the same as the fibers of $p_1 : \mathrm{Hk}^1_G(\Sigma)_{x_1} \to \mathrm{Bun}_G(\Sigma)$, which are 1-dimensional by Proposition 3.4(3).

Next consider the case $x_1 \in D$. Since Σ is disjoint from D, the Iwahori level structures along Σ of $\mathcal{E}_1$ and $\mathcal{E}'_1$ uniquely determine the Iwahori level structures along Σ of all bundles in the diagram (3.24). Thus the fibers of p_1 are the same as the fibers of $p_1 : \mathrm{Hk}^{1,\natural}_{H,D,\underline{x}} \to H^\natural_D$ (the version without level structure); this latter map was denoted $\mathrm{Hk}^1_{D,D,\underline{x}} \to H_{D,D}$ in [10, 6.4.4], and in the last paragraph of [10, 6.4.4] it was shown that its fibers are 1-dimensional. We are done. □

3.3.2. *Hecke symmetry on the Chow group.* Using the dimension calculation in Lemma 3.13, the same argument as in [10, Prop 5.10] proves the following result.

Proposition 3.14. *The assignment*

$$h_D \longmapsto (\overleftarrow{p} \times \overrightarrow{p})_*[\mathrm{Sht}^r_G(\Sigma; \Sigma_\infty; h_D)]$$

extends linearly to a ring homomorphism

$$\mathscr{H}^\Sigma_G \longrightarrow {}_c\mathrm{Ch}_{2r}(\mathrm{Sht}^r_G(\Sigma;\Sigma_\infty) \times \mathrm{Sht}^r_G(\Sigma;\Sigma_\infty))_{\mathbb{Q}}.$$

In particular, we get an action of $\mathscr{H}^\Sigma_G$ on the Chow group of proper cycles $\mathrm{Ch}_{c,*}(\mathrm{Sht}^r_G(\Sigma;\Sigma_\infty))_{\mathbb{Q}}$.

3.3.3. *Hecke symmetry on cohomology.* We shall define an action of $\mathscr{H}^\Sigma_G$ on $\mathrm{H}^*_c(\mathrm{Sht}^r_G(\Sigma;\Sigma_\infty) \otimes \overline{k}, \mathbb{Q}_\ell)$ following the strategy in [10, 7.1.4]. For this we need a presentation of $\mathrm{Sht}^r_G(\Sigma;\Sigma_\infty)$ as an increasing union of open substacks of finite type. Here we are satisfied with a minimal version of such a presentation, and we postpone a more refined version to Section 3.4. For $N \geq 0$, we define ${}^{\leq N}\mathrm{Sht}$ to be the open substack of $\mathrm{Sht}^r_G(\Sigma;\Sigma_\infty)$ consisting of those $(\mathcal{E}^\dagger_i; \dots)$ where $\mathrm{inst}(\mathcal{E}_0) \leq N$. Since the forgetful map $\mathrm{Sht}^r_G(\Sigma;\Sigma_\infty) \to \mathrm{Bun}_G$ recording $\mathcal{E}_0$ is of finite type, ${}^{\leq N}\mathrm{Sht}$ is of finite type over k. As N increases, $\mathrm{Sht}^r_G(\Sigma;\Sigma_\infty)$ is the union of the increasing sequence of open substacks ${}^{\leq N}\mathrm{Sht}$.

With the finite-type open substacks ${}^{\leq N}\mathrm{Sht}$, we can copy the construction of [10, 7.1.4] by first defining the action of h_D as a map $\mathbf{R}\pi_{\leq N,!}\mathbb{Q}_\ell \to \mathbf{R}\pi_{\leq N',!}\mathbb{Q}_\ell$ (where $\pi_{\leq N} : {}^{\leq N}\mathrm{Sht} \to X^r \times \mathfrak{S}_\infty$) for $N' - N \geq \deg D$, and then pass to cohomology and pass to inductive limits. Using the dimension calculation in Lemma 3.13(3), the same argument as in [10, Prop. 7.1] shows

Proposition 3.15. *The assignment $h_D \mapsto C(h_D)$, extended linearly, defines an action of $\mathscr{H}^\Sigma_G \otimes \mathbb{Q}_\ell$ on $\mathrm{H}^i_c(\mathrm{Sht}^r_G(\Sigma;\Sigma_\infty) \otimes \overline{k}, \mathbb{Q}_\ell)$ for each $i \in \mathbb{Z}$.*

The following two results are analogues of [10, Lemmas 5.12, 7.2, and 7.3], with the same proofs.

LEMMA 3.16. *Let $f \in \mathscr{H}_G^{\Sigma}$. Then the action of f on the Chow group* $\mathrm{Ch}_{c,*}(\mathrm{Sht}_G^r(\Sigma;\Sigma_\infty))_{\mathbb{Q}}$ *(resp. on the cohomology* $\mathrm{H}_c^{2r}(\mathrm{Sht}_G^r(\Sigma;\Sigma_\infty)\otimes\overline{k},\mathbb{Q}_\ell)(r))$ *is self-adjoint with respect to the intersection pairing (resp. cup product pairing).*

LEMMA 3.17. *The cycle class map*

$$\mathrm{cl}:\mathrm{Ch}_{c,i}(\mathrm{Sht}_G^r(\Sigma;\Sigma_\infty))_{\mathbb{Q}}\longrightarrow \mathrm{H}_c^{4r-2i}(\mathrm{Sht}_G^r(\Sigma;\Sigma_\infty)\otimes\overline{k},\mathbb{Q}_\ell)(2r-i)$$

is equivariant under the $\mathscr{H}_G^{\Sigma}$-actions for all i.

3.3.4. *The base-change situation.* Consider another curve X' as in Section 3.2.6. Let

$$\mathrm{Sht}_G^{\prime r}(\Sigma;\Sigma_\infty)=\mathrm{Sht}_G^r(\Sigma;\Sigma_\infty)\times_{(X^r\times\mathfrak{S}_\infty)}(X^{\prime r}\times\mathfrak{S}'_\infty).$$

We may define the Hecke correspondence $\mathrm{Sht}_G^{\prime r}(\Sigma;\Sigma_\infty;h_D)$ for $\mathrm{Sht}_G^{\prime r}(\Sigma;\Sigma_\infty)$ as the base change of $\mathrm{Sht}_G^r(\Sigma;\Sigma_\infty)$ from $X^r\times\mathfrak{S}_\infty$ to $X^{\prime r}\times\mathfrak{S}'_\infty$. The smoothness of $\mathrm{Sht}_G^{\prime r}(\Sigma;\Sigma_\infty)$ proved in Proposition 3.10 allows us to apply the formalism of correspondences acting on Chow groups; see [10, A.1.6]. The same argument as in [10, Prop. 5.10] gives an analogue of Proposition 3.14: there is an action of $\mathscr{H}_G^{\Sigma}$ on the Chow group of proper cycles $\mathrm{Ch}_{c,*}(\mathrm{Sht}_G^{\prime r}(\Sigma;\Sigma_\infty))_{\mathbb{Q}}$, where h_D acts via the fundamental class of $\mathrm{Sht}_G^{\prime r}(\Sigma;\Sigma_\infty;h_D)$.

Similarly, with the smoothness of $\mathrm{Sht}_G^{\prime r}(\Sigma;\Sigma_\infty)$ proved in Proposition 3.10, analogues of Proposition 3.15 and Lemmas 3.16 and 3.17 make sense and continue to hold true for $\mathrm{Sht}_G^{\prime r}(\Sigma;\Sigma_\infty)$ in place of $\mathrm{Sht}_G^r(\Sigma;\Sigma_\infty)$.

Remark 3.18. Besides the action of $\mathscr{H}_G^{\Sigma}$, the Atkin–Lehner involutions $\mathrm{AL}_{\mathrm{Sht},x}$ for $x\in\Sigma$ (see Section 3.2.7) also act on $\mathrm{Sht}_G^r(\Sigma;\Sigma_\infty)$ and $\mathrm{Sht}_G^{\prime r}(\Sigma;\Sigma_\infty)$. Therefore, they induce involutions on the Chow groups and cohomology groups of $\mathrm{Sht}_G^r(\Sigma;\Sigma_\infty)$ and $\mathrm{Sht}_G^{\prime r}(\Sigma;\Sigma_\infty)$, which we still denote by $\mathrm{AL}_{\mathrm{Sht},x}$. These involutions commute with the action of $\mathscr{H}_G^{\Sigma}$.

3.4. *Horocycles.* This subsection studies the geometry of $\mathrm{Sht}_G^r(\Sigma;\Sigma_\infty)$ "near infinity." It serves as technical preparation for the proof of the spectral decomposition in the next subsection.

To alleviate notation in this subsection we introduce the notation

$$\mathrm{Sht}:=\mathrm{Sht}_G^r(\Sigma;\Sigma_\infty)\otimes\overline{k}.$$

3.4.1. *Index of instability.* Let us first introduce the notion of instability for points in $\mathrm{Bun}_2(\Sigma)$. For a rank two bundle $\mathcal{E}$ on X, $\mathrm{inst}(\mathcal{E})\in\mathbb{Z}$ is defined as in [10, §7.1.1]: it is the maximum of $2\deg\mathcal{L}-\deg\mathcal{E}$ when $\mathcal{L}$ runs over line subbundles of $\mathcal{E}$. For a geometric point $\mathcal{E}^\dagger=(\mathcal{E},\{\mathcal{E}(-\tfrac{1}{2}x)\}_{x\in\Sigma})\in\mathrm{Bun}_2(\Sigma)(K)$, we have a bundle $\mathcal{E}(\frac{1}{2}D)$ for any divisor $D \subset X_K$ supported in $\Sigma(K)$. We call $\mathcal{E}^\dagger$ *purely unstable* if $\mathrm{inst}(\mathcal{E}(\frac{1}{2}D)) > 0$ for all $D \leq \Sigma(K)$. Note that the condition $\mathrm{inst}(\mathcal{E}(\frac{1}{2}D)) > 0$ depends only on the class of D modulo 2; i.e., we may think of D as an element in $\mathbb{Z}/2\mathbb{Z}[\Sigma(K)]$, the free $\mathbb{Z}/2\mathbb{Z}$-module with basis given by $\Sigma(K)$. Define

$$\mathrm{inst}(\mathcal{E}^\dagger) := \min\left\{\mathrm{inst}(\mathcal{E}(\frac{1}{2}D)); D \in \mathbb{Z}/2\mathbb{Z}[\Sigma(K)]\right\}.$$

Both the notion of pure instability and the number $\mathrm{inst}(\mathcal{E}^\dagger)$ depends only on the image of $\mathcal{E}^\dagger$ in $\mathrm{Bun}_G(\Sigma)$.

Suppose $\mathcal{F} \in \mathrm{Bun}_2(K)$ is unstable, with maximal line bundle $\mathcal{L}$ and quotient $\mathcal{M} := \mathcal{F}/\mathcal{L}$. For any effective divisor D', we denote by $\mathcal{F}_{\lrcorner D'}$ the resulting rank two bundle by pushing out the exact sequence $0 \to \mathcal{L} \to \mathcal{F} \to \mathcal{M} \to 0$ along $\mathcal{L} \hookrightarrow \mathcal{L}(D')$. Similarly let ${}^{\ulcorner}{}_{D'}\mathcal{F}$ be the pullback of the same exact sequence along $\mathcal{M}(-D') \hookrightarrow \mathcal{M}$. Note that we have a canonical isomorphism $\mathcal{F}_{\lrcorner D'} \cong ({}^{\ulcorner}{}_{D'}\mathcal{F})(D')$, which means that $\mathcal{F}_{\lrcorner D'}$ and ${}^{\ulcorner}{}_{D'}\mathcal{F}$ have the same image in Bun_G.

LEMMA 3.19. *Let K be an algebraically closed field containing k, and let $\mathcal{E}^\dagger \in \mathrm{Bun}_G(\Sigma)(K)$ be purely unstable.*

(1) *There is a unique $D \in \mathbb{Z}/2\mathbb{Z}[\Sigma(K)]$ such that $\mathrm{inst}(\mathcal{E}^\dagger) = \mathrm{inst}(\mathcal{E}(\frac{1}{2}D))$. (Note that $\mathcal{E}(\frac{1}{2}D)$ is a well-defined point of $\mathrm{Bun}_G(\Sigma)$ when $D \in \mathbb{Z}/2\mathbb{Z}[\Sigma(K)]$.)*
(2) *The point $\mathcal{E}^\dagger$ is uniquely determined by $\mathcal{E}(\frac{1}{2}D)$ (D as in (1)) in the following way: for any effective divisor D' supported on $\Sigma(K)$, $\mathcal{E}(\frac{1}{2}D + \frac{1}{2}D') = \mathcal{E}(\frac{1}{2}D)_{\lrcorner D'}$. Moreover, we have*

$$\mathrm{inst}(\mathcal{E}(\frac{1}{2}D + \frac{1}{2}D')) = \mathrm{inst}(\mathcal{E}^\dagger) + |D'|, \tag{3.25}$$

where $|D'| = \#\{x \in \Sigma(K) | x \text{ has non-zero coefficient in } D'\}$.

Proof. We prove all statements simultaneously. Let $D \in \mathbb{Z}/2\mathbb{Z}[\Sigma(K)]$ be some divisor such that $\mathrm{inst}(\mathcal{E}^\dagger) = \mathrm{inst}(\mathcal{E}(\frac{1}{2}D))$. (We do not assume D is unique for now.) Write $\mathcal{F} = \mathcal{E}(\frac{1}{2}D)$. For any $x \in \Sigma(K)$, we have $\mathrm{inst}(\mathcal{F}(\frac{1}{2}x)) = \mathrm{inst}(\mathcal{F}) \pm 1$. Since $\mathcal{F}$ achieves the minimal index of instability, we must have $\mathrm{inst}(\mathcal{F}(\frac{1}{2}x)) = \mathrm{inst}(\mathcal{F}) + 1$. This means that $\mathcal{F}(\frac{1}{2}x) = \mathcal{F}_{\lrcorner x}$. For any effective D' supported on $\Sigma(K)$ and *multiplicity-free*, $\mathcal{F}(\frac{1}{2}D')$ is the union of $\mathcal{F}(\frac{1}{2}x)$ for $x \in D'$, and we get $\mathcal{F}(\frac{1}{2}D') = \mathcal{F}_{\lrcorner D'}$. This implies that

$$\mathrm{inst}(\mathcal{F}(\frac{1}{2}D')) = \mathrm{inst}(\mathcal{F}) + \deg D' = \mathrm{inst}(\mathcal{F}) + |D' \mod 2|. \tag{3.26}$$

Since the set of points $\{\mathcal{F}(\frac{1}{2}D')\}_{D' \leq \Sigma(K)}$, as points of $\mathrm{Bun}_G(\Sigma)$, is exactly $\{\mathcal{E}(\frac{1}{2}D')\}_{D' \leq \Sigma(K)}$, we see that $\mathrm{inst}(\mathcal{E}(\frac{1}{2}D'))$ achieves its minimum exactly when $D' = D$ and nowhere else. The equality (3.25) follows from (3.26). □

By the above lemma, for a purely unstable $\mathcal{E}^\dagger \in \mathrm{Bun}_G(\Sigma)(K)$, we may define an invariant

$$\kappa(\mathcal{E}^\dagger) = (D, \mathrm{inst}(\mathcal{E}^\dagger)) \in \mathbb{Z}/2\mathbb{Z}[\Sigma(K)] \times \mathbb{Z}_{>0},$$

where $D \in \mathbb{Z}/2\mathbb{Z}[\Sigma(K)]$ is the unique element such that $\mathrm{inst}(\mathcal{E}^\dagger) = \mathrm{inst}(\mathcal{E}(\frac{1}{2}D))$.

3.4.2. *Strata in* $\mathrm{Bun}_G(\Sigma)$. For $N > 0$, we also denote by ${}^N\mathrm{Bun}_G$ the locally closed substack of Bun_G whose geometric points are exactly those $\mathcal{E}$ with $\mathrm{inst}(\mathcal{E}) = N$.

For any field K containing $\overline{k}$, we have a canonical bijection $\Sigma(\overline{k}) \xrightarrow{\sim} \Sigma(K)$. For $\kappa \in \mathbb{Z}/2\mathbb{Z}[\Sigma(\overline{k})] \times \mathbb{Z}_{>0}$, there is a locally closed substack ${}^\kappa\mathrm{Bun}_G(\Sigma) \subset \mathrm{Bun}_G(\Sigma) \otimes \overline{k}$ whose geometric points are exactly those geometric points $\mathcal{E}^\dagger$ with $\kappa(\mathcal{E}^\dagger) = \kappa$ (under the identification $\Sigma(\overline{k}) \xrightarrow{\sim} \Sigma(K)$).

We define a partial order on $\mathbb{Z}/2\mathbb{Z}[\Sigma(\overline{k})] \times \mathbb{Z}$ by saying that $\kappa = (D, N) \le \kappa' = (D', N')$ if and only if

$$N' - N \ge |D - D'|.$$

For $\kappa = (D, N) \in \mathbb{Z}/2\mathbb{Z}[\Sigma(\overline{k})] \times \mathbb{Z}_{>0}$, let ${}^{\le\kappa}\mathrm{Bun}_G(\Sigma) \subset \mathrm{Bun}_G(\Sigma) \otimes \overline{k}$ be the open substack consisting of $\mathcal{E}^\dagger$ such that for any $D' \in \mathbb{Z}/2\mathbb{Z}[\Sigma(\overline{k})]$, $\mathrm{inst}(\mathcal{E}(\frac{1}{2}D')) \le N + |D' - D|$. We see that ${}^\kappa\mathrm{Bun}_G(\Sigma) \subset {}^{\le\kappa'}\mathrm{Bun}_G(\Sigma)$ if and only if $\kappa \le \kappa'$. Moreover, ${}^\kappa\mathrm{Bun}_G(\Sigma)$ is closed in ${}^{\le\kappa}\mathrm{Bun}_G(\Sigma)$, with open complement denoted by ${}^{<\kappa}\mathrm{Bun}_G(\Sigma)$.

COROLLARY 3.20 (of Lemma 3.19). *For* $\kappa = (D, N) \in \mathbb{Z}/2\mathbb{Z}[\Sigma(\overline{k})] \times \mathbb{Z}_{>0}$, *the map* $\mathcal{E}^\dagger \mapsto \mathcal{E}(\frac{1}{2}D)$ *gives an isomorphism of* $\overline{k}$*-stacks*

$${}^\kappa\mathrm{Bun}_G(\Sigma) \xrightarrow{\sim} {}^N\mathrm{Bun}_G \otimes \overline{k}.$$

3.4.3. *Elementary modifications.* In this section we study how the invariant κ changes under an elementary modification of bundles. Recall the stack $\mathrm{Hk}^1_G(\Sigma)$ classifying $(\mathcal{E}^\dagger, \mathcal{F}^\dagger, y, \varphi)$ modulo tensoring with line bundles, where $\mathcal{E}^\dagger, \mathcal{F}^\dagger \in \mathrm{Bun}_2(\Sigma)$ and $\varphi : \mathcal{E} \hookrightarrow \mathcal{F}$ is an injective map compatible with Iwahori structures whose cokernel is an invertible sheaf on the graph of $y : S \to X$. Recording y gives a map $\pi^1_{\mathrm{Hk}} : \mathrm{Hk}^1_G(\Sigma) \to X$.

For two elements $\kappa = (D, N), \kappa' = (D', N') \in \mathbb{Z}/2\mathbb{Z}[\Sigma(\overline{k})] \times \mathbb{Z}_{>0}$, we define

$$|\kappa - \kappa'| := |D - D'| + |N - N'| \in \mathbb{Z}_{\ge 0}$$

with $|D - D'|$ defined in Lemma 3.19(3).

LEMMA 3.21. *Suppose* $(\mathcal{E}^\dagger, \mathcal{F}^\dagger, y, \varphi) \in \mathrm{Hk}^1_G(\Sigma)(K)$ (*where* K *is an algebraically closed field,* $\mathcal{E}^\dagger, \mathcal{F}^\dagger$ *are lifted to* $\mathrm{Bun}_2(\Sigma)(K)$, $\varphi : \mathcal{E} \hookrightarrow \mathcal{F}$ *and* y *is the support of* $\mathrm{coker}(\varphi)$), *and suppose* $\mathcal{E}^\dagger$ *and* $\mathcal{F}^\dagger$ *are both purely unstable. Write* $\kappa(\mathcal{E}^\dagger) = (D, N)$, $\kappa(\mathcal{F}^\dagger) = (D', N')$.

(1) $|\kappa(\mathcal{E}^\dagger) - \kappa(\mathcal{F}^\dagger)| = 1.$

(2) *If $N = N'$, then D and D' differ at a unique point $x \in \Sigma(K)$, and we have $y = x$. The points $\mathcal{E}^\dagger$ and $\mathcal{F}^\dagger$ are uniquely determined by the triple $(\mathcal{E}(\frac{1}{2}D), \mathcal{F}(\frac{1}{2}D'), \alpha)$, where α is an isomorphism of G-bundles*

$$\alpha : \mathcal{E}(\frac{1}{2}D)_{\lrcorner x} \cong \mathcal{F}(\frac{1}{2}D')_{\lrcorner x}.$$

(3) *If $N = N' - 1$, then $D = D'$, and $\mathcal{E}^\dagger$ and $\mathcal{F}^\dagger$ are determined by the single bundle $\mathcal{E}(\frac{1}{2}D)$ in the following way:*
  - *$\mathcal{E}^\dagger$ is determined by $\mathcal{E}(\frac{1}{2}D)$ as in Lemma 3.19(2);*
  - *$\mathcal{F}(\frac{1}{2}D) = \mathcal{E}(\frac{1}{2}D)_{\lrcorner y}$, and $\mathcal{F}^\dagger$ is determined by $\mathcal{F}(\frac{1}{2}D)$ again by Lemma 3.19(2).*

(4) *If $N = N' + 1$, then $D = D'$, and $\mathcal{E}^\dagger$ and $\mathcal{F}^\dagger$ are determined by the single bundle $\mathcal{F}(\frac{1}{2}D)$ in the following way:*
  - *$\mathcal{F}^\dagger$ is determined by $\mathcal{F}(\frac{1}{2}D)$ as in Lemma 3.19(2);*
  - *$\mathcal{E}(\frac{1}{2}D) = {}^{\ulcorner}{}_{y}(\mathcal{F}(\frac{1}{2}D))$, and $\mathcal{E}^\dagger$ is determined by $\mathcal{E}(\frac{1}{2}D)$ again by Lemma 3.19(2).*

Proof. For any $D'' \in \mathbb{Z}/2\mathbb{Z}[\Sigma(\overline{k})]$, we have $\mathrm{inst}(\mathcal{E}(\frac{1}{2}D'')) = \mathrm{inst}(\mathcal{F}(\frac{1}{2}D''))$ ± 1, and therefore $N - N' \in \{0, 1, -1\}$.

When $N - N' = -1$, $\mathcal{E}(\frac{1}{2}D)$ achieves the minimal index of instability among all the bundles $\{\mathcal{E}(\frac{1}{2}D''), \mathcal{F}(\frac{1}{2}D'')\}_{D'' \in \mathbb{Z}/2\mathbb{Z}[\Sigma(K)]}$. Since $\mathrm{inst}(\mathcal{F}(\frac{1}{2}D)) = \mathrm{inst}(\mathcal{E}(\frac{1}{2}D)) \pm 1$, we must have $\mathrm{inst}(\mathcal{F}(\frac{1}{2}D)) = N + 1$, therefore $\mathrm{inst}(\mathcal{F}(\frac{1}{2}D))$ $= N'$ and $D' = D$. The same argument as Lemma 3.19(2) shows that $\mathcal{F}(\frac{1}{2}D)$ is determined by $\mathcal{E}(\frac{1}{2}D)$. This proves (3).

The analysis of the case $N - N' = 1$ is similar, which takes care of (4).

Finally consider the case $N = N'$. Since $\mathrm{inst}(\mathcal{F}(\frac{1}{2}D)) = \mathrm{inst}(\mathcal{E}(\frac{1}{2}D)) \pm 1$ and $\mathrm{inst}(\mathcal{F}(\frac{1}{2}D)) \geq N' = N = \mathrm{inst}(\mathcal{E}(\frac{1}{2}D))$, we must have $\mathrm{inst}(\mathcal{F}(\frac{1}{2}D)) = N + 1$. On the other hand, we have $\mathrm{inst}(\mathcal{F}(\frac{1}{2}D')) = N' = N$ by definition. By Lemma 3.19(3), we have $|D - D'| = (N+1) - N = 1$; that is, D' and D differ by one point $x \in \Sigma(K)$. We show that y must be equal to x. Suppose not. Consider the bundle $\mathcal{G} = \mathcal{F}(\frac{1}{2}D)$ (represented by a rank two bundle on X_K) with subsheaves

$$\mathcal{G}\left(-\frac{1}{2}y\right) := \mathcal{E}\left(\frac{1}{2}D\right) \text{ and } \mathcal{G}\left(-\frac{1}{2}x\right) := \mathcal{F}\left(\frac{1}{2}D - \frac{1}{2}x\right).$$

Then $\mathcal{G}^\dagger := (\mathcal{G}, \mathcal{G}(-\frac{1}{2}y), \mathcal{G}(-\frac{1}{2}x))$ defines a point in $\mathrm{Bun}_2(\{x, y\})(K)$. Note that $\mathrm{inst}(\mathcal{G}(-\frac{1}{2}y)) = N$ by definition and

$$\mathrm{inst}\left(\mathcal{G}\left(-\frac{1}{2}x\right)\right) = \mathrm{inst}\left(\mathcal{F}\left(\frac{1}{2}D - \frac{1}{2}x\right)\right) = \mathrm{inst}\left(\mathcal{F}\left(\frac{1}{2}D'\right)\right) = N;$$

also $\mathrm{inst}(\mathcal{G}) = N + 1$ and $\mathrm{inst}(\mathcal{G}(-\frac{1}{2}x - \frac{1}{2}y)) = \mathrm{inst}(\mathcal{E}(\frac{1}{2}D - \frac{1}{2}x)) = N + 1$. It follows that $\mathcal{G}^\dagger$ is purely unstable. This contradicts Lemma 3.19(1) because both $\mathcal{G}(-\frac{1}{2}x)$ and $\mathcal{G}(-\frac{1}{2}y)$ achieve the minimal index of instability. This contradiction proves $y = x$. The isomorphism α comes from the fact that $\mathcal{G}(-\frac{1}{2}y)_{\lrcorner x} = \mathcal{G} = \mathcal{G}(-\frac{1}{2}x)_{\lrcorner x}$. The triple $(\mathcal{E}(\frac{1}{2}D), \mathcal{F}(\frac{1}{2}D'), \alpha)$ first determines $\mathcal{E}^\dagger$ and $\mathcal{F}^\dagger$ by Lemma 3.19(2). Now we represent D and D' by multiplicity-free effective divisors on $\Sigma(K)$. When $D' = D + x$, the map α then determines the injective map $\psi : \mathcal{E}(-\frac{1}{2}D) \hookrightarrow \mathcal{F}(-\frac{1}{2}D')_{\lrcorner x}$, which then gives

$$\varphi : \mathcal{E} = \mathcal{E}\left(-\frac{1}{2}D\right)_{\lrcorner D} \xrightarrow{\psi} \mathcal{F}\left(-\frac{1}{2}D'\right)_{\lrcorner x+D} = \mathcal{F}\left(-\frac{1}{2}D'\right)_{\lrcorner D'} = \mathcal{F}.$$

When $D' = D - x$, the map α gives the injective map $\psi : \mathcal{E}(-\frac{1}{2}D)_{\lrcorner x} \hookrightarrow \mathcal{F}(-\frac{1}{2}D')$, which then gives

$$\varphi : \mathcal{E} = \mathcal{E}\left(-\frac{1}{2}D\right)_{\lrcorner D} = \left(\mathcal{E}\left(-\frac{1}{2}D\right)_{\lrcorner x}\right)_{\lrcorner D'} \xrightarrow{\psi} \mathcal{F}\left(-\frac{1}{2}D'\right)_{\lrcorner D'} = \mathcal{F}.$$

Part (2) is proved.

All three cases above satisfy $|\kappa(\mathcal{E}^\dagger) - \kappa(\mathcal{F}^\dagger)| = 1$, which verifies (1). □

For $\kappa = (D, N)$ and $\kappa' = (D', N')$ in $\mathbb{Z}/2\mathbb{Z}[\Sigma(\overline{k})] \times \mathbb{Z}_{>0}$, let ${}^{\kappa,\kappa'}\mathrm{Hk}^1_G(\Sigma)$ be the locally closed substack of $\mathrm{Hk}^1_G(\Sigma) \otimes \overline{k}$ whose geometric points are exactly those $(\mathcal{E}^\dagger, \mathcal{F}^\dagger, y, \varphi)$ such that $\kappa(\mathcal{E}^\dagger) = \kappa$ and $\kappa(\mathcal{F}^\dagger) = \kappa'$.

Corollary 3.22 (of Lemma 3.21).

(1) *The stack* ${}^{\kappa,\kappa'}\mathrm{Hk}^1_G(\Sigma)$ *is empty unless* $|\kappa - \kappa'| = 1$.

(2) *When* $N = N'$ *and* D *and* D' *differ only at* $x \in \Sigma(\overline{k})$, *the map* π^1_{Hk} *maps* ${}^{\kappa,\kappa'}\mathrm{Hk}^1_G(\Sigma)$ *to a single point* x, *and there is an isomorphism*

$$^{\kappa,\kappa'}\mathrm{Hk}^1_G(\Sigma) \xrightarrow{\sim} ({}^{N}\mathrm{Bun}_G \times_{{}^{N+1}\mathrm{Bun}_G} {}^{N}\mathrm{Bun}_G) \otimes \overline{k},$$

with both maps ${}^N\mathrm{Bun}_G \to {}^{N+1}\mathrm{Bun}_G$ *given by* $(-)_{\lrcorner x}$. *The above isomorphism is given by*

$$(\mathcal{E}^\dagger, \mathcal{F}^\dagger, x, \varphi) \longmapsto \left(\mathcal{E}\left(\frac{1}{2}D\right), \mathcal{F}\left(\frac{1}{2}D'\right), \alpha\right)$$

as in Lemma 3.21(2).

(3) *When* $N = N' - 1$ *and* $D = D'$, *we have an isomorphism*

$$^{\kappa,\kappa'}\mathrm{Hk}^1_G(\Sigma) \xrightarrow{\sim} ({}^{N}\mathrm{Bun}_G \times X) \otimes \overline{k}$$

given by $(\mathcal{E}^\dagger, \mathcal{F}^\dagger, y, \varphi) \mapsto (\mathcal{E}(\frac{1}{2}D), y)$.

(4) *When* $N = N' + 1$ *and* $D = D'$, *we have an isomorphism*

$$^{\kappa,\kappa'}\mathrm{Hk}^1_G(\Sigma) \xrightarrow{\sim} ({}^{N'}\mathrm{Bun}_G \times X) \otimes \overline{k}$$

given by $(\mathcal{E}^\dagger, \mathcal{F}^\dagger, y, \varphi) \mapsto (\mathcal{F}(\frac{1}{2}D'), y)$.

Definition 3.23. Let $\underline{\kappa} = (\kappa_0, \kappa_1, \dots, \kappa_r)$ be a sequence of elements in $\mathbb{Z}/2\mathbb{Z}[\Sigma(\overline{k})] \times \mathbb{Z}_{>0}$.

(1) The *horocycle of type* $\underline{\kappa}$ of Sht is the locally closed substack ${}^{\underline{\kappa}}\mathrm{Sht} \subset \mathrm{Sht}$ whose geometric points are exactly those $(\mathcal{E}_i^\dagger; \dots) \in \mathrm{Sht}$ such that each $\mathcal{E}_i^\dagger$ is purely unstable with $\kappa(\mathcal{E}_i^\dagger) = \kappa_i$, for $i = 0, 1, \dots, r$.
(2) The *truncation up to* $\underline{\kappa}$ of Sht is the open substack of Sht consisting of $(\mathcal{E}_i^\dagger; \dots)$ such that $\mathcal{E}_i^\dagger \in {}^{\le \kappa_i}\mathrm{Bun}_G(\Sigma)$ for all $0 \le i \le r$.

Then ${}^{\underline{\kappa}}\mathrm{Sht}$ is closed in ${}^{\le\underline{\kappa}}\mathrm{Sht}$, and we denote its open complement by ${}^{<\underline{\kappa}}\mathrm{Sht}$.

3.4.4. *The index set for horocycles.* Above we defined horocycles for any r-tuple of elements $\underline{\kappa}$ in $\mathbb{Z}/2\mathbb{Z}[\Sigma(\overline{k})] \times \mathbb{Z}_{>0}$. However, for many such $\underline{\kappa}$, ${}^{\underline{\kappa}}\mathrm{Sht}$ turns out to be empty.

LEMMA 3.24. *Let* $\underline{\kappa} = (\kappa_0, \kappa_1, \dots, \kappa_r)$ *be a sequence of elements in*

$$\mathbb{Z}/2\mathbb{Z}[\Sigma(\overline{k})] \times \mathbb{Z}_{>0}.$$

If ${}^{\underline{\kappa}}\mathrm{Sht}$ *is non-empty, then*

(1) *for each* $i = 1, \dots, r$, $|\kappa_{i-1} - \kappa_i| = 1$;
(2) *if we write* $\kappa_i = (D_i, N_i)$, *then* $N_0 = N_r$, *and* $\mathrm{Fr}(D_0)$ (*applying the arithmetic Frobenius to each point appearing* D_0) *and* D_r *differ at exactly one* $\overline{k}$*-point above each place of* Σ_∞ *and nowhere else.*

Proof. Suppose $(\mathcal{E}_i^\dagger, \dots) \in {}^{\underline{\kappa}}\mathrm{Sht}$ is a geometric point over $\{x^{(1)}\}_{x\in\Sigma_\infty} \in \mathfrak{S}_\infty$. Then $|\kappa_{i-1} - \kappa_i| = 1$ by Corollary 3.22(1). The isomorphism $\mathcal{E}_r \cong ({}^\tau\mathcal{E}_0)(\frac{1}{2}\sum_{x\in\Sigma_\infty} x^{(1)})$ implies $N_0 = N_r$ and $\mathrm{Fr}(D_0) + \sum_{x\in\Sigma_\infty} x^{(1)} \equiv D_r \mod 2$, which implies the second condition. □

Definition 3.25. Let $\mathfrak{K}_r$ be the set of $\underline{\kappa} = (\kappa_0, \kappa_1, \dots, \kappa_r)$, where each $\kappa_i \in \mathbb{Z}/2\mathbb{Z}[\Sigma(\overline{k})] \times \mathbb{Z}$, satisfying the two conditions in Lemma 3.24. (For technical reasons we do not impose $\kappa_i > 0$ in the definition of $\mathfrak{K}_r$.)

From the definition and Lemma 3.24 we see that

$$\mathrm{Sht} = \bigcup_{\underline{\kappa}\in\mathfrak{K}_r} {}^{\le\underline{\kappa}}\mathrm{Sht}.$$

The partial order on $\mathbb{Z}/2\mathbb{Z}[\Sigma(\overline{k})] \times \mathbb{Z}_{>0}$ extends to one on $\mathfrak{K}_r$: we say that $(\kappa_0, \dots, \kappa_r) \le (\kappa'_0, \dots, \kappa'_r)$ if and only if $\kappa_i \le \kappa'_i$ for all $0 \le i \le r$. Then it is easy to check that, for $\underline{\kappa}, \underline{\kappa}' \in \mathfrak{K}_r$, ${}^{\underline{\kappa}}\mathrm{Sht} \subset {}^{\le\underline{\kappa}'}\mathrm{Sht}$ if and only if $\underline{\kappa} \le \underline{\kappa}'$.

For $\underline{\kappa} \in \mathfrak{K}_r$ and $N \in \mathbb{Z}$, we write $\underline{\kappa} > N$ if $N_i(\underline{\kappa}) > N$ for all $0 \le i \le r$. (Here $N_i(\underline{\kappa})$ denotes the $\mathbb{Z}_{>0}$-part of the i-th component of $\underline{\kappa}$.)

3.4.5. $I(\underline{\kappa})$ *and* $X(\underline{\kappa})$. For $\underline{\kappa} = (\kappa_0, \dots, \kappa_r) \in \mathfrak{K}_r$ with $\underline{\kappa}_i = (D_i, N_i)$, we define the subset $I(\underline{\kappa}) \subset \{1, 2, \dots, r\}$ as

$$I(\underline{\kappa}) = \{1 \le i \le r | N_{i-1} \ne N_i\}.$$

For $i \in \{1, 2, \dots, r\} - I(\underline{\kappa})$, there is a unique point $x \in \Sigma(\overline{k})$ such that D_{i-1} and D_i differ at x. We denote this point x by $x_i(\underline{\kappa})$. Also, by the second condition on $\underline{\kappa}$ above, the difference between D_r and $\mathrm{Fr}(D_0)$ consists of a $\overline{k}$-point $x^{(1)}(\underline{\kappa})$ over each $x \in \Sigma_\infty$.

For $i \in I(\underline{\kappa})$, we have $N_i = N_{i-1} \pm 1$. Since $N_r = N_0$, we see that $\#I(\underline{\kappa})$ is even.

We define $X(\underline{\kappa}) \subset (X^r \times \mathfrak{S}_\infty) \otimes \overline{k}$ to be the coordinate subspace

$$X(\underline{\kappa}) = \{(x_1, \dots, x_r, \{x^{(1)}\}_{x \in \Sigma_\infty}) | x_i = x_i(\underline{\kappa}) \text{ for all } i \notin I(\underline{\kappa}); \\ x^{(1)} = x^{(1)}(\underline{\kappa}) \text{ for all } x \in \Sigma_\infty\}.$$

The projection to the $I(\underline{\kappa})$-coordinates gives an isomorphism

$$X(\underline{\kappa}) \xrightarrow{\sim} X^{I(\underline{\kappa})} \otimes \overline{k}.$$

Viewing $\mathbb{Z}/2\mathbb{Z}[\Sigma]$ as a subgroup of $\mathbb{Z}/2\mathbb{Z}[\Sigma(\overline{k})]$ by $\Sigma \ni x \mapsto \sum_{\Sigma(\overline{k}) \ni \overline{x} \mapsto x} \overline{x}$, there is an action of $\mathbb{Z}/2\mathbb{Z}[\Sigma]$ on $\mathbb{Z}/2\mathbb{Z}[\Sigma(\overline{k})]$ by translation. This induces a diagonal action of $\mathbb{Z}/2\mathbb{Z}[\Sigma]$ on $\mathfrak{K}_r$ by acting only on the divisor parts of each κ_i. For $\underline{\kappa}, \underline{\kappa}' \in \mathfrak{K}_r$, we say $\underline{\kappa} \sim \underline{\kappa}'$ if the divisor parts of $\underline{\kappa}$ and $\underline{\kappa}'$ are in the same $\mathbb{Z}/2\mathbb{Z}[\Sigma]$-orbit (no other condition on the $\mathbb{Z}$-factors). This defines an equivalence relation on $\mathfrak{K}_r$. Let $[\mathfrak{K}_r]$ be the quotient

$$[\mathfrak{K}_r] := \mathfrak{K}_r / \sim .$$

The following lemma is a direct calculation.

LEMMA 3.26. *The map*

$$X(\cdot) : \mathfrak{K}_r \longrightarrow \{\textit{subschemes of } (X^r \times \mathfrak{S}_\infty) \otimes \overline{k}\} \\ \underline{\kappa} \longmapsto X(\underline{\kappa})$$

factors through $[\mathfrak{K}_r]$ *and induces an injective map*

$$X(\cdot) : [\mathfrak{K}_r] \hookrightarrow \{\textit{subschemes of } (X^r \times \mathfrak{S}_\infty) \otimes \overline{k}\}.$$

By the above lemma, for $\sigma \in [\mathfrak{K}_r]$, we may write

$$X(\sigma), \quad I(\sigma)$$

for $X(\underline{\kappa})$ and $I(\underline{\kappa})$, where $\underline{\kappa}$ is any element in the orbit σ.

COROLLARY 3.27 (of Lemma 3.21 and Corollary 3.22). *For $\underline{\kappa} \in \mathfrak{K}_r$ and $\underline{\kappa} > 0$, the restriction of the map $\Pi^r_G : \mathrm{Sht} \to X^r \times \mathfrak{S}_\infty$ to ${}^{\underline{\kappa}}\mathrm{Sht}$ has image in $X(\underline{\kappa})$. We denote the resulting map by*

$$\pi_{\underline{\kappa}} : {}^{\underline{\kappa}}\mathrm{Sht} \longrightarrow X(\underline{\kappa}).$$

3.4.6. *Geometry of horocycles.* For any $N > 0$, we have a map

$$\Delta : {}^{N}\mathrm{Bun}_G \longrightarrow \mathrm{Pic}_X^N$$

sending $\mathcal{E}$ to the line bundle $\Delta(\mathcal{E}) = \mathcal{L} \otimes \mathcal{M}^{-1}$ of degree N on X, where $\mathcal{L} \subset \mathcal{E}$ is the maximal line subbundle and $\mathcal{M} = \mathcal{E}/\mathcal{L}$.

Now if $\underline{\kappa} \in \mathfrak{K}_r$ and $\underline{\kappa} > 0$, for $(\mathcal{E}_i^{\dagger}; \dots) \in {}^{\underline{\kappa}}\mathrm{Sht}$, we have a sequence of line bundles $\Delta_i := \Delta(\mathcal{E}_i(\frac{1}{2}D_i))$ by the above construction applied to $\mathcal{E}_i(\frac{1}{2}D_i) \in {}^{N_i}\mathrm{Bun}_G$ (recall $\kappa_i = (D_i, N_i)$, so $\mathcal{E}_i(\frac{1}{2}D_i)$ has the smallest index of instability among all fractional twists of $\mathcal{E}_i$). By Lemma 3.21, these line bundles are related by canonical isomorphisms:

$$\Delta_i \cong \begin{cases} \Delta_{i-1} & \text{if } N_i = N_{i-1}, \\ \Delta_{i-1}(x_i) & \text{if } N_i = N_{i-1} + 1, \\ \Delta_{i-1}(-x_i) & \text{if } N_i = N_{i-1} - 1. \end{cases}$$

Finally $\Delta_r \cong {}^{\tau}\Delta_0$. Thus $\Delta = (\Delta_0, \dots, \Delta_r)$ together with the above isomorphisms give a point in $\mathrm{Sht}_1^{N(\underline{\kappa})}$, the moduli of rank one Shtukas $(\mathcal{L}_0, \mathcal{L}_1, \dots, \mathcal{L}_r)$ over X with $\deg(\mathcal{L}_i) = N_i$. (When $N_{i-1} = N_i$, we have an isomorphism $\mathcal{L}_{i-1} \xrightarrow{\sim} \mathcal{L}_i$.) This gives a morphism

$$q_{\underline{\kappa}} : {}^{\underline{\kappa}}\mathrm{Sht} \longrightarrow \mathrm{Sht}_1^{N(\underline{\kappa})} \otimes \overline{k}$$

through which the canonical map $\Pi_G^r : {}^{\underline{\kappa}}\mathrm{Sht}_G^r(\Sigma; \Sigma_\infty) \to X(\underline{\kappa}) \cong X^{I(\underline{\kappa})} \otimes \overline{k}$ factors.

Lemma 3.28. *Suppose $\underline{\kappa} \in \mathfrak{K}_r$ and $\underline{\kappa} > \max\{2g-2, 0\}$. Then the map $q_{\underline{\kappa}}$ is smooth of relative dimension $r - \#I(\underline{\kappa})/2$. The geometric fibers of $q_{\underline{\kappa}}$ are isomorphic to $[\mathbb{G}_a^{r-\#I(\underline{\kappa})/2}/Z]$ for some finite étale group scheme Z acting on $\mathbb{G}_a^{r-\#I(\underline{\kappa})/2}$ via a homomorphism $Z \to \mathbb{G}_a^{r-\#I(\underline{\kappa})/2}$.*

Proof. The argument is similar to [10, Lemma 7.5], so we only sketch the difference with the situation without level structures. We define ${}^{\underline{\kappa}}\mathrm{Hk}_G^r(\Sigma) \subset \mathrm{Hk}_G^r(\Sigma) \otimes \overline{k}$ to be the locally closed substack where $\kappa(\mathcal{E}_i^{\dagger}) = \kappa_i$ for $0 \le i \le r$. Then ${}^{\underline{\kappa}}\mathrm{Hk}_G^r(\Sigma)$ is the iterated fiber product of ${}^{\kappa_{i-1}, \kappa_i}\mathrm{Hk}_G^1(\Sigma)$. By definition, we have a Cartesian diagram

$$\begin{array}{ccc} {}^{\underline{\kappa}}\mathrm{Sht}_G^r(\Sigma; \Sigma_\infty) & \longrightarrow & {}^{\underline{\kappa}}\mathrm{Hk}_G^r(\Sigma) \\ \downarrow{\scriptstyle p_0} & & \downarrow{\scriptstyle (p_0, \mathrm{AL}_{G,\infty} \circ p_r)} \\ {}^{\kappa_0}\mathrm{Bun}_G(\Sigma) & \xrightarrow{(\mathrm{id}, \mathrm{Fr}_{/\overline{k}})} & {}^{\kappa_0}\mathrm{Bun}_G(\Sigma) \times {}^{\mathrm{Fr}(\kappa_0)}\mathrm{Bun}_G(\Sigma), \end{array} \tag{3.27}$$

where the map $\mathrm{Fr}_{/\overline{k}} : {}^{\kappa_0}\mathrm{Bun}_G(\Sigma) \to {}^{\mathrm{Fr}(\kappa_0)}\mathrm{Bun}_G(\Sigma)$ is the restriction of the $\overline{k}$-linear Frobenius $\mathrm{Fr} \times \mathrm{id}_{\overline{k}} : \mathrm{Bun}_G(\Sigma) \otimes \overline{k} \to \mathrm{Bun}_G(\Sigma) \otimes \overline{k}$ to the stratum

${}^{\kappa_0}\mathrm{Bun}_G(\Sigma)$. Using Corollary 3.20, we may replace the bottom row by

$$(\mathrm{id}, \mathrm{Fr}\times \mathrm{id}_{\overline{k}}) : {}^{N_0}\mathrm{Bun}_G \otimes \overline{k} \longrightarrow ({}^{N_0}\mathrm{Bun}_G \otimes \overline{k}) \times_{\overline{k}} ({}^{N_0}\mathrm{Bun}_G \otimes \overline{k}).$$

The diagram (3.27) now reads

$$\begin{array}{ccc} {}^{\underline{\kappa}}\mathrm{Sht}^r_G(\Sigma;\Sigma_\infty) & \longrightarrow & {}^{\underline{\kappa}}\mathrm{Hk}^r_G(\Sigma) \\ \downarrow{\scriptstyle h_0} & & \downarrow{\scriptstyle (h_0,h_r)} \\ {}^{N_0}\mathrm{Bun}_G \otimes \overline{k} & \xrightarrow{(\mathrm{id},\mathrm{Fr}\times\mathrm{id}_{\overline{k}})} & ({}^{N_0}\mathrm{Bun}_G \otimes \overline{k}) \times_{\overline{k}} ({}^{N_0}\mathrm{Bun}_G \otimes \overline{k}), \end{array} \tag{3.28}$$

where $h_i : {}^{\underline{\kappa}}\mathrm{Hk}^r_G(\Sigma) \to {}^{N_i}\mathrm{Bun}_G \otimes \overline{k}$ is the composition of p_i with the isomorphism ${}^{\kappa_i}\mathrm{Bun}_G(\Sigma) \xrightarrow{\sim} {}^{N_i}\mathrm{Bun}_G \otimes \overline{k}$ in Corollary 3.20.

Let S be a $\overline{k}$-algebra. Fix an S-point $\underline{y} = (y_1, \dots, y_r) \in X(\underline{\kappa})$, and denote by ${}^{\underline{\kappa}}\mathrm{Hk}^r_G(\Sigma)_{\underline{y}}$ the fiber over $\underline{y}$. Let ${}^N\mathrm{Bun}_{G,S}$ be the base change of ${}^N\mathrm{Bun}_G$ from $\mathrm{Spec}\, k$ to S.

For $1 \le i \le r$, let

$$M_i = \min\{N_{i-1}, N_i\} + 1.$$

Then using the description of ${}^{\kappa_{i-1},\kappa_i}\mathrm{Hk}^1_G(\Sigma)$ in Corollary 3.22, we get an isomorphism

$$\begin{aligned} {}^{\underline{\kappa}}\mathrm{Hk}^r_G(\Sigma)_{\underline{y}} \cong {}^{N_0}\mathrm{Bun}_{G,S} \times_{{}^{M_1}\mathrm{Bun}_{G,S}} {}^{N_1}\mathrm{Bun}_{G,S} \times_{{}^{M_2}\mathrm{Bun}_{G,S}} {}^{N_2}\mathrm{Bun}_{G,S} \\ \times \cdots \times_{{}^{M_r}\mathrm{Bun}_{G,S}} {}^{N_r}\mathrm{Bun}_{G,S}, \end{aligned} \tag{3.29}$$

where the maps ${}^{N_{i-1}}\mathrm{Bun}_{G,S} \to {}^{M_i}\mathrm{Bun}_{G,S}$ and ${}^{N_i}\mathrm{Bun}_{G,S} \to {}^{M_i}\mathrm{Bun}_{G,S}$ are either the identity map or the pushout $\lrcorner_{y_i}$.

There is a map $\Delta_{\mathrm{Hk},\underline{y}} : {}^{\underline{\kappa}}\mathrm{Hk}^r_G(\Sigma)_{\underline{y}} \to \mathrm{Pic}^{N_0}_{X,S} \times \mathrm{Pic}^{N_r}_{X,S}$, which is induced by the map $\Delta : {}^{N_i}\mathrm{Bun}_G \to \mathrm{Pic}^{N_i}_X$ on each factor in (3.29). Now we fix an S-point $\underline{\Delta} = (\Delta_0, \Delta_1, \dots, \Delta_r) \in \mathrm{Sht}^{N(\underline{\kappa})}_1(S)$ over $\underline{y}$, namely, $\deg \Delta_i = N_i$ and $\Delta_i = \Delta_{i-1}((N_i - N_{i-1})y_i)$ for $1 \le i \le r$. Let $E_i \subset {}^{N_i}\mathrm{Bun}_{G,S}$ be the preimage of $\Delta_i \in \mathrm{Pic}^{N_i}_X(S)$ under Δ (so E_i is an S-stack). Since $N_i > \max\{2g-2, 0\}$, we have that $E_i \cong \mathbb{B}H_i$ is the classifying space of the vector bundle $H_i = p_{S*}\Delta_i$ over S (where $p_S : X \times S \to S$). Similarly, we let $C_i \subset {}^{M_i}\mathrm{Bun}_{G,S}$ be the preimage of the following line bundle under Δ:

$$\Delta'_i := \begin{cases} \Delta_i(y_i) & \text{if } N_i = N_{i+1}, \\ \Delta_i & \text{if } N_i = N_{i-1}+1, \\ \Delta_{i-1} & \text{if } N_i = N_{i-1}-1. \end{cases}$$

We have $C_i \cong \mathbb{B}J_i$ for the vector bundle $J_i = p_{S*}\Delta'_i$ over S. The canonical embeddings $\Delta_{i-1}, \Delta_i \hookrightarrow \Delta'_i$ induce embeddings $H_{i-1} \hookrightarrow J_i$ and $H_i \hookrightarrow J_i$, hence maps $E_{i-1} \to C_i$ and $E_i \to C_i$ for $1 \le i \le r$. By (3.29), the preimage of $\underline{\Delta}$ under $\Delta_{\mathrm{Hk},\underline{y}}$ is

$$E_0 \times_{C_1} E_1 \times_{C_2} \cdots \times_{C_r} E_r,$$

which is isomorphic to the stack over S,

$$H_0 \backslash J_1 \overset{H_1}{\times} J_2 \overset{H_2}{\times} \cdots \overset{H_{r-1}}{\times} J_r / H_r,$$

which is the quotient of $J_1 \times \cdots \times J_r$ (product over S) by the action of H_0 on J_1, the diagonal action of H_1 on J_1 and $J_2, \ldots$, the diagonal action of H_i on J_i and $J_{i+1}, \ldots$, and the action of H_r on J_r.

Using the Cartesian diagram (3.28), we get

$$q_{\underline{\kappa}}^{-1}(\underline{\Delta}) \cong (J_1 \overset{H_1}{\times} J_2 \overset{H_2}{\times} \cdots \overset{H_{r-1}}{\times} J_r)/H_0,$$

where the action of H_0 is by translation on J_1 and on J_r, via composing with the relative Frobenius $\mathrm{Fr}_{H_0/S} : H_0 \to H_r$ and the H_r-translation on J_r. This presentation shows that $q_{\underline{\kappa}}^{-1}(\underline{\Delta})$ is smooth over S. Hence $q_{\underline{\kappa}}$ is smooth.

To calculate the relative dimension of $q_{\underline{\kappa}}$, we take $S = \operatorname{Spec} K$ to be a geometric point, and

$$\dim q_{\underline{\kappa}}^{-1}(\underline{\Delta}) = \sum_{i=1}^{r} \dim J_i - \sum_{i=0}^{r-1} \dim H_i.$$

Since

$$\begin{aligned} \dim J_i - \dim H_{i-1} &= \dim \mathrm{H}^0(X_K, \Delta_i') - \dim \mathrm{H}^0(X_K, \Delta_{i-1}) \\ &= \begin{cases} 1 & \text{if } N_i = N_{i-1} \text{ or } N_i = N_{i-1} - 1, \\ 0 & \text{if } N_i = N_{i-1} + 1, \end{cases} \end{aligned}$$

we see that

$$\dim q_{\underline{\kappa}}^{-1}(\underline{\Delta}) = r - \#\{1 \le i \le r | N_i = N_{i-1} - 1\} = r - \#I(\underline{\kappa})/2.$$

This proves the dimension part of the statement. The rest of the argument is the same as the last part of the proof of [10, Lemma 7.5], using the fact that the translation of H_0 on J_1 induces a free action on the vector space $J_1 \overset{H_1}{\times} J_2 \overset{H_2}{\times} \cdots \overset{H_{r-1}}{\times} J_r$. □

COROLLARY 3.29 (of Lemma 3.28). *Suppose $\underline{\kappa} \in \mathfrak{K}_r$ and $\underline{\kappa} > \max\{2g-2, 0\}$. Let $\pi_1^{N(\underline{\kappa})} : \mathrm{Sht}_1^{N(\underline{\kappa})} \otimes \overline{k} \to X(\underline{\kappa})$ be the projection. Then we have a canonical isomorphism*

$$\mathbf{R}\pi_{\underline{\kappa},!}\mathbb{Q}_\ell \cong \mathbf{R}\pi_{1,!}^{N(\underline{\kappa})}\mathbb{Q}_\ell[-2r + \#I(\underline{\kappa})](-r + \#I(\underline{\kappa})/2).$$

In particular, $\mathbf{R}\pi_{\underline{\kappa},!}\mathbb{Q}_\ell$ is a local system shifted in degree $2r - \#I(\underline{\kappa})$, and

$$P_{\underline{\kappa}} := \mathbf{R}\pi_{\underline{\kappa},!}\mathbb{Q}_\ell[2r](r) \in D^b(X(\underline{\kappa}), \mathbb{Q}_\ell) \tag{3.30}$$

is a perverse sheaf on $X(\underline{\kappa})$ with full support and pure of weight 0.

3.4.7. *When is* $\mathrm{Sht}^r_G(\Sigma;\Sigma_\infty)$ *of finite type*? Let

$$\mathfrak{K}^\sharp_r = \{\underline{\kappa}\in\mathfrak{K}_r|\underline{\kappa}>\max\{2g-2,0\}\},$$
$$^\sharp\mathrm{Sht}_{\overline{k}} = \cup_{\underline{\kappa}\in\mathfrak{K}^\sharp_r}{}^{\underline{\kappa}}\mathrm{Sht}.$$

Then $^\sharp\mathrm{Sht}$ consists of $(\mathcal{E}^\dagger_i;\dots)$ where all $\mathrm{inst}(\mathcal{E}^\dagger_i)>\max\{2g-2,0\}$, therefore it is a closed substack of Sht. Let $^\flat\mathrm{Sht}=\mathrm{Sht}-{}^\sharp\mathrm{Sht}$ be its open complement.

LEMMA 3.30. *The substack* $^\flat\mathrm{Sht}$ *is of finite type over* $\overline{k}$.

Proof. Let $(\mathcal{E}^\dagger_i;\dots)$ be a geometric point of $^\flat\mathrm{Sht}$. Then for some i_0, $\mathrm{inst}(\mathcal{E}^\dagger_{i_0})\le\max\{2g-2,0\}$, hence $\mathrm{inst}(\mathcal{E}_{i_0})\le\max\{2g-2,0\}+\deg\Sigma$. Since $\mathcal{E}_0$ is related to $\mathcal{E}_{i_0}$ by at most r steps of elementary modifications, we have $\mathrm{inst}(\mathcal{E}_0)\le r+\max\{2g-2,0\}+\deg\Sigma=:c$ for any i. Then $^\flat\mathrm{Sht}$ is contained in the preimage of $^{\le c}\mathrm{Bun}_G$ under the map $\underline{p}_0:\mathrm{Sht}\to\mathrm{Bun}_G$ (recording only $\mathcal{E}_0$). Since $\underline{p}_0$ is of finite type and $^{\le c}\mathrm{Bun}_G$ is of finite type over k, so is $^\flat\mathrm{Sht}$. □

COROLLARY 3.31 (of Lemmas 3.30 and 3.28). *The stack* $\mathrm{Sht}^r_G(\Sigma;\Sigma_\infty)$ *is of finite type over* k *if and only if* $r<\#\Sigma_\infty$.

Proof. If $r<\#\Sigma_\infty$, then the set $\mathfrak{K}_r$ is empty. In fact, if $\underline{\kappa}=(\kappa_0,\dots,\kappa_r)\in\mathfrak{K}_r$, then the first condition defining $\mathfrak{K}_r$ implies $|D_r-D_0|\le r$ (D_i is the divisor part of κ_i), while the second condition implies that for each $x\in\Sigma_\infty$, D_0 and D_r must differ at a geometric point above x, hence $|D_r-D_0|\ge\#\Sigma_\infty$. Therefore, $\mathrm{Sht}={}^\flat\mathrm{Sht}$, which is of finite type over $\overline{k}$ by Lemma 3.30. This implies that Sht is of finite type over k.

Conversely, if $r\ge\#\Sigma_\infty$, then the set $\mathfrak{K}^\sharp_r$ is infinite as can be seen in the following way. Write $\Sigma_\infty=\{x_1,\dots,x_m\}$, and fix $x^{(1)}_i\in X(\overline{k})$ above each x_i. Let $D_0=0$, $D_i=x^{(1)}_1+\dots+x^{(1)}_i$ for $1\le i\le m$, and $D_m=D_{m+1}=\dots=D_r$. Then take $N_0=\dots=N_m$ and $N_j=N_{j-1}\pm1$ for $m<j\le r$ such that $N_r=N_m$ and $N_i>\max\{2g-2,0\}$ for all $0\le i\le r$. (There are infinitely many such sequences (N_i).) Let $\kappa_i=(D_i,N_i)$, then $\underline{\kappa}=(\underline{\kappa}_1,\dots,\underline{\kappa}_r)\in\mathfrak{K}^\sharp_r$. For each $\underline{\kappa}\in\mathfrak{K}^\sharp_r$, $^{\underline{\kappa}}\mathrm{Sht}$ is non-empty by Lemma 3.28. Therefore, Sht is not of finite type over $\overline{k}$ in this case. □

3.5. *Cohomological spectral decomposition.* In this subsection, we continue to use the abbreviations Sht, $^{\underline{\kappa}}\mathrm{Sht}$ as in Section 3.4. Let

$$V=\mathrm{H}^{2r}_c(\mathrm{Sht},\mathbb{Q}_\ell)(r).$$

Since Sht is the union of open substacks $^{\le\underline{\kappa}}\mathrm{Sht}$ for $\underline{\kappa}\in\mathfrak{K}_r$, we have by definition

$$V=\varinjlim_{\underline{\kappa}\in\mathfrak{K}_r,\underline{\kappa}>0}\mathrm{H}^{2r}_c({}^{\le\underline{\kappa}}\mathrm{Sht},\mathbb{Q}_\ell)(r).$$

For $\underline{\kappa} \in \mathfrak{K}_r, \underline{\kappa} > 0$, let $\pi_{\leq \underline{\kappa}} : {}^{\leq \underline{\kappa}}\mathrm{Sht} \to (X^r \times \mathfrak{S}_\infty) \otimes \overline{k}$ be the restriction of Π^r_G. Let

$$K_{\leq \underline{\kappa}} = \mathbf{R}\pi_{\leq \underline{\kappa},!}\mathbb{Q}_\ell[2r](r) \in D^b((X^r \times \mathfrak{S}_\infty) \otimes \overline{k}, \mathbb{Q}_\ell).$$

For $0 < \underline{\kappa} \leq \underline{\kappa}' \in \mathfrak{K}_r$, the open inclusion ${}^{\leq \underline{\kappa}}\mathrm{Sht} \hookrightarrow {}^{\leq \underline{\kappa}'}\mathrm{Sht}$ induces a map

$$\iota_{\underline{\kappa},\underline{\kappa}'} : K_{\leq \underline{\kappa}} \longrightarrow K_{\leq \underline{\kappa}'}.$$

3.5.1. *Ind-perverse sheaves.* The perverse sheaves $\{{}^p\mathrm{H}^i K_{\leq \underline{\kappa}}\}_{\underline{\kappa} \in \mathfrak{K}_r}$ form an inductive system indexed by the directed set $\mathfrak{K}_r$. Consider the inductive limit

$${}^p\mathrm{H}^i K := \varinjlim_{\underline{\kappa}} {}^p\mathrm{H}^i K_{\leq \underline{\kappa}} \in \mathrm{indPerv}((X^r \times \mathfrak{S}_\infty) \otimes \overline{k}, \mathbb{Q}_\ell).$$

Here the right side is the category of ind-objects in the abelian category $\mathrm{Perv}((X^r \times \mathfrak{S}_\infty) \otimes \overline{k}, \mathbb{Q}_\ell)$ of perverse constructible sheaves on $(X^r \times \mathfrak{S}_\infty) \otimes \overline{k}$, which is again an abelian category. Note that the notation ${}^p\mathrm{H}^i K$ comes as a whole, as we are not defining K as the inductive limit of $K_{\leq \underline{\kappa}}$, but only defining the ind-perverse sheaves ${}^p\mathrm{H}^i K$.

Definition 3.32. Let $\varphi : P \to P'$ be a morphism in $\mathrm{indPerv}((X^r \times \mathfrak{S}_\infty) \otimes \overline{k}, \mathbb{Q}_\ell)$.

(1) We say φ is an *mc-isomorphism* (mc for modulo constructibles) if the kernel and cokernel of φ are in the essential image of the natural embedding $\mathrm{Perv}((X^r \times \mathfrak{S}_\infty) \otimes \overline{k}, \mathbb{Q}_\ell) \hookrightarrow \mathrm{indPerv}((X^r \times \mathfrak{S}_\infty) \otimes \overline{k}, \mathbb{Q}_\ell)$.
(2) We say φ is *mc-zero* if its image is in the essential image of the natural embedding $\mathrm{Perv}((X^r \times \mathfrak{S}_\infty) \otimes \overline{k}, \mathbb{Q}_\ell) \hookrightarrow \mathrm{indPerv}((X^r \times \mathfrak{S}_\infty) \otimes \overline{k}, \mathbb{Q}_\ell)$.

Likewise we have the notion of an mc-commutative square of ind-perverse sheaves; i.e., the appropriate difference of the compositions is mc-zero. Concatenation of mc-commutative squares is still mc-commutative.

LEMMA 3.33. *Let $0 < \underline{\kappa} \leq \underline{\kappa}' \in \mathfrak{K}_r$. Then the map $\iota_{\underline{\kappa},\underline{\kappa}'}$ on the perverse cohomology sheaves*

$${}^p\mathrm{H}^i\iota_{\underline{\kappa},\underline{\kappa}'} : {}^p\mathrm{H}^i K_{\leq \underline{\kappa}} \longrightarrow {}^p\mathrm{H}^i K_{\leq \underline{\kappa}'}$$

is injective for $i = 0$, surjective for $i = 1$ and an isomorphism for $i \neq 0, 1$.

In particular, ${}^p\mathrm{H}^i K$ is eventually stable when $i \neq 0$. (That is, the natural map ${}^p\mathrm{H}^i K_{\leq \underline{\kappa}} \to {}^p\mathrm{H}^i K$ is an isomorphism for sufficiently large $\underline{\kappa}$.)

Proof. Let ${}^{(\underline{\kappa},\underline{\kappa}']}\mathrm{Sht} = {}^{\leq \underline{\kappa}'}\mathrm{Sht} - {}^{\leq \underline{\kappa}}\mathrm{Sht}$, which is a union of horocycles ${}^{\underline{\kappa}''}\mathrm{Sht}$ for $\underline{\kappa}'' \leq \underline{\kappa}'$ but $\underline{\kappa}'' \not\leq \underline{\kappa}$. The horocycles form a stratification of ${}^{\leq \underline{\kappa}'}\mathrm{Sht} - {}^{\leq \underline{\kappa}}\mathrm{Sht}$. Let $\pi_{(\underline{\kappa},\underline{\kappa}']} : {}^{(\underline{\kappa},\underline{\kappa}']}\mathrm{Sht} \to (X^r \times \mathfrak{S}_\infty) \otimes \overline{k}$ be the projection. Then $K_{(\underline{\kappa},\underline{\kappa}']} := \mathbf{R}\pi_{(\underline{\kappa},\underline{\kappa}'],!}\mathbb{Q}_\ell[2r](r)$ is the cone of $\iota_{\underline{\kappa},\underline{\kappa}'}$, and it is a successive extension of $P_{\underline{\kappa}''}$ (see (3.30)), viewed as a complex on $(X^r \times \mathfrak{S}_\infty) \otimes \overline{k}$. By Corollary 3.29, $P_{\underline{\kappa}''}$ is a perverse sheaf; therefore, so is $K_{(\underline{\kappa},\underline{\kappa}']}$. The long exact sequence for the perverse cohomology sheaves attached to the triangle $K_{\leq\underline{\kappa}} \to K_{\leq\underline{\kappa}'} \to K_{(\underline{\kappa},\underline{\kappa}']} \to K_{\leq\underline{\kappa}}[1]$ then gives the desired statements. □

3.5.2. *Hecke symmetry on ind-perverse sheaves.* A variant of the construction in Section 3.3.3 gives an $\mathscr{H}_G^\Sigma$-action on ${}^p\mathrm{H}^iK$ for any $i \in \mathbb{Z}$. Namely, for each effective divisor D on $X - \Sigma$, the fundamental cycle of the Hecke correspondence $\mathrm{Sht}_G^r(\Sigma; \Sigma_\infty; h_D)$ (as a cohomological correspondence between constant sheaves on truncated $\mathrm{Sht}_G^r(\Sigma; \Sigma_\infty)$) induces a map $K_{\leq\underline{\kappa}} \to K_{\leq\underline{\kappa}'}$ for $\kappa' - \kappa \geq d$. Passing to perverse cohomology sheaves and passing to inductive limits, we get a map in $\mathrm{indPerv}((X^r \times \mathfrak{S}_\infty) \otimes \overline{k}, \mathbb{Q}_\ell)$:

$${}^p\mathrm{H}^i(h_D) : {}^p\mathrm{H}^iK \longrightarrow {}^p\mathrm{H}^iK.$$

The same argument as [10, Prop. 7.1], using the dimension calculation in Lemma 3.13(3), shows that the assignment $h_D \mapsto {}^p\mathrm{H}^i(h_D)$, extended linearly, gives an action of $\mathscr{H}_G^\Sigma$ on ${}^p\mathrm{H}^iK$.

3.5.3. *The constant term map.* Recall the closed substack ${}^\sharp\mathrm{Sht}$ of Sht and its open complement ${}^\flat\mathrm{Sht}$ from Section 3.4.7. Let $\pi_\flat : {}^\flat\mathrm{Sht} \to (X^r \times \mathfrak{S}_\infty) \otimes \overline{k}$ and $K_\flat = \mathbf{R}\pi_{\flat,!}\mathbb{Q}_\ell[2r](r) \in D^b((X^r \times \mathfrak{S}_\infty) \otimes \overline{k}, \mathbb{Q}_\ell)$.

We have a stratification of ${}^\sharp\mathrm{Sht}$ by locally closed substacks ${}^{\underline{\kappa}}\mathrm{Sht}$. Therefore, we may similarly define ${}^p\mathrm{H}^iK_\sharp$ as the inductive limit of the perverse sheaves ${}^p\mathrm{H}^iK_{\sharp,\leq\underline{\kappa}}$ as $\underline{\kappa}$ runs over $\mathfrak{K}_r$, where $K_{\sharp,\leq\underline{\kappa}}$ is the direct image complex of ${}^\sharp\mathrm{Sht} \cap {}^{\leq\underline{\kappa}}\mathrm{Sht} \to (X^r \times \mathfrak{S}_\infty) \otimes \overline{k}$.

Lemma 3.34.

(1) *The restriction map associated to the closed inclusion* ${}^\sharp\mathrm{Sht} \hookrightarrow \mathrm{Sht}$ *induces an mc-isomorphism of ind-perverse sheaves*

$${}^p\mathrm{H}^0K \longrightarrow {}^p\mathrm{H}^0K_\sharp.$$

(2) *We have* ${}^p\mathrm{H}^iK_\sharp = 0$ *for all* $i \neq 0$. *Moreover, there is a* canonical *isomorphism of perverse sheaves on* $(X^r \times \mathfrak{S}_\infty) \otimes \overline{k}$:

$${}^p\mathrm{H}^0K_\sharp \cong \oplus_{\underline{\kappa}\in\mathfrak{K}_r^\sharp} P_{\underline{\kappa}}.$$

Proof. (1) By definition, we have ${}^\flat\mathrm{Sht} \subset {}^{\leq\underline{\kappa}}\mathrm{Sht}$ for $\underline{\kappa}$ large enough, with the complement $\cup_{\underline{\kappa}'\in\mathfrak{K}_r^\sharp,\underline{\kappa}'\leq\underline{\kappa}} {}^{\underline{\kappa}'}\mathrm{Sht}$. This gives a distinguished triangle $K_\flat \to K_{\leq\underline{\kappa}} \to K_{\sharp,\leq\underline{\kappa}} \to$. The long exact sequence of perverse cohomology sheaves gives

$${}^p\mathrm{H}^0K_\flat \longrightarrow {}^p\mathrm{H}^0K_{\leq\underline{\kappa}} \longrightarrow {}^p\mathrm{H}^0K_{\sharp,\leq\underline{\kappa}} \longrightarrow {}^p\mathrm{H}^1K_\flat.$$

Taking inductive limit we get an exact sequence

$${}^p\mathrm{H}^0K_\flat \longrightarrow {}^p\mathrm{H}^0K \longrightarrow {}^p\mathrm{H}^0K_\sharp \longrightarrow {}^p\mathrm{H}^1K_\flat.$$

By Lemma 3.30, ${}^\flat\mathrm{Sht}$ is a DM stack of finite type over $\overline{k}$, hence $K_\flat$ is constructible, and the middle map above is an mc-isomorphism.

To show (2), it suffices to give a canonical isomorphism (again $\underline{\kappa}$ is large enough so that ${}^{\flat}\mathrm{Sht} \subset {}^{\leq\underline{\kappa}}\mathrm{Sht}$)

$$K_{\sharp,\leq\underline{\kappa}} \cong \oplus_{\underline{\kappa}'\in\mathfrak{K}_r^{\sharp},\underline{\kappa}'\leq\underline{\kappa}} P_{\underline{\kappa}'},$$

compatible with the transition maps when $\underline{\kappa}$ grows. Since $K_{\sharp,\leq\underline{\kappa}}$ is a successive extension of $P_{\underline{\kappa}'}$ for $\underline{\kappa}' \in \mathfrak{K}_r^{\sharp}$ and $\underline{\kappa}' \leq \underline{\kappa}$, we have a canonical decomposition according support

$$K_{\sharp,\leq\underline{\kappa}} \cong \oplus_{\sigma\in[\mathfrak{K}_r]}(K_{\sharp,\leq\underline{\kappa}})_{\sigma},$$

where we recall from Lemma 3.26 that the support of $P_{\underline{\kappa}}$ is determined by the image of $\underline{\kappa}$ in $[\mathfrak{K}_r]$, and different classes in $[\mathfrak{K}_r]$ give different supports. Each $(K_{\sharp,\leq\underline{\kappa}})_{\sigma}$ is then a successive extension of those $P_{\underline{\kappa}'}$ where $\underline{\kappa}' \in \mathfrak{K}_r^{\sharp} \cap \sigma$ and $\underline{\kappa}' \leq \underline{\kappa}$. Hence $(K_{\sharp,\leq\underline{\kappa}})_{\sigma}$ is a local system on $X(\sigma)$ shifted in degree $-\dim X(\sigma) = -\#I(\sigma)$. Let η_{σ} be a geometric generic point of $X(\sigma)$. It suffices to give a canonical decomposition of the stalks at η_{σ}:

$$(K_{\sharp,\leq\underline{\kappa}})_{\sigma}|_{\eta_{\sigma}} \cong \oplus_{\underline{\kappa}'\in\mathfrak{K}_r^{\sharp}\cap\sigma,\underline{\kappa}'\leq\underline{\kappa}} P_{\underline{\kappa}'}|_{\eta_{\sigma}}. \tag{3.31}$$

Now

$$K_{\sharp,\leq\underline{\kappa}}|_{\eta_{\sigma}} \cong \mathrm{H}_c^{2r-\#I(\sigma)}({}^{\sharp}\mathrm{Sht}_{\eta_{\sigma}} \cap {}^{\leq\underline{\kappa}}\mathrm{Sht}_{\eta_{\sigma}}, \mathbb{Q}_{\ell})(r)$$

and

$${}^{\sharp}\mathrm{Sht}_{\eta_{\sigma}} \cap {}^{\leq\underline{\kappa}}\mathrm{Sht}_{\eta_{\sigma}} = \cup_{\underline{\kappa}'\leq\underline{\kappa}} {}^{\underline{\kappa}'}\mathrm{Sht}_{\eta_{\sigma}}.$$

If ${}^{\underline{\kappa}'}\mathrm{Sht}_{\eta_{\sigma}} \neq \varnothing$, we must have $X(\underline{\kappa}') \supset X(\sigma)$, hence $\dim {}^{\underline{\kappa}'}\mathrm{Sht}_{\eta_{\sigma}} = r - \#I(\underline{\kappa}')/2 \leq r - \#I(\sigma)/2$ with equality if and only if $\underline{\kappa}' \in \sigma$. Hence $\dim {}^{\sharp}\mathrm{Sht}_{\eta_{\sigma}} \cap {}^{\leq\underline{\kappa}}\mathrm{Sht}_{\eta_{\sigma}} \leq r - \#I(\sigma)/2$, with top-dimensional components given by ${}^{\underline{\kappa}'}\mathrm{Sht}_{\eta_{\sigma}}$ for those $\underline{\kappa}' \in \mathfrak{K}_r^{\sharp} \cap \sigma$ and $\underline{\kappa}' \leq \underline{\kappa}$. This implies a canonical isomorphism

$$\mathrm{H}_c^{2r-\#I(\sigma)}({}^{\sharp}\mathrm{Sht}_{\eta_{\sigma}} \cap {}^{\leq\underline{\kappa}}\mathrm{Sht}_{\eta_{\sigma}}, \mathbb{Q}_{\ell})(r) \cong \oplus_{\underline{\kappa}'\in\mathfrak{K}_r^{\sharp}\cap\sigma,\underline{\kappa}'\leq\underline{\kappa}} \mathrm{H}_c^{2r-\#I(\sigma)}({}^{\underline{\kappa}'}\mathrm{Sht}_{\eta_{\sigma}}, \mathbb{Q}_{\ell})(r),$$

which is exactly (3.31). □

Combining the two maps in the above lemma, we get a canonical map of ind-perverse sheaves that is an mc-isomorphism

$$\gamma : {}^{p}\mathrm{H}^0 K \longrightarrow \oplus_{\underline{\kappa}\in\mathfrak{K}_r^{\sharp}} P_{\underline{\kappa}}. \tag{3.32}$$

This can be called the *cohomological constant term operator*.

Remark 3.35. Compared to the treatment in [10, §7.3.1], we do not need the generic fibers of the horocycles to be closed in Sht. In fact the horocycle ${}^{\underline{\kappa}}\mathrm{Sht}$ is not necessarily closed when restricted to the generic point of $X(\underline{\kappa})$; for example, this fails when $X(\underline{\kappa})$ is a point.

3.5.4. *Constant term intertwines with Satake.* Recall from Corollary 3.29 that whenever $\underline{\kappa} \in \mathfrak{K}_r^{\sharp}$, we have an isomorphism

$$P_{\underline{\kappa}} \cong \mathbf{R}\pi_{1,!}^{N(\underline{\kappa})}\mathbb{Q}_\ell[-\#I(\underline{\kappa})](-\#I(\underline{\kappa})/2).$$

The map $\pi_1^{N(\underline{\kappa})} : \mathrm{Sht}_1^{N(\underline{\kappa})} \otimes \overline{k} \to X(\underline{\kappa})$ is a $\mathrm{Pic}_X^0(k)$-torsor.

Now for any $\underline{\kappa} \in \mathfrak{K}_r$ (without assuming $\underline{\kappa} > 0$), the stack $\mathrm{Sht}_1^{N(\underline{\kappa})}$ is always defined, and $\pi_1^{N(\underline{\kappa})} : \mathrm{Sht}_1^{N(\underline{\kappa})} \otimes \overline{k} \to X(\underline{\kappa})$ is a $\mathrm{Pic}_X(k)$-torsor. Moreover, the union

$$\coprod_{\underline{\kappa}' \in \underline{\kappa}+\mathbb{Z}} \mathrm{Sht}_1^{N(\underline{\kappa}')} \otimes \overline{k} \longrightarrow X(\underline{\kappa})$$

is a $\mathrm{Pic}_X(k)$-torsor, extending the $\mathrm{Pic}_X^0(k)$-torsor structure on each component of the left-hand side. Here we write $\underline{\kappa} + \mathbb{Z}$ for $\mathbb{Z}$-orbit of $\underline{\kappa}$ in $\mathfrak{K}_r$, and $\mathbb{Z}$ acts by translating the degree parts of $\underline{\kappa} \in \mathfrak{K}_r$ simultaneously. (Note that $X(\underline{\kappa})$ is unchanged under the $\mathbb{Z}$-action.) The $\mathrm{Pic}_X(k)$-action then gives an action on the ind-perverse sheaf

$$\oplus_{\underline{\kappa}' \in \underline{\kappa}+\mathbb{Z}} \mathbf{R}\pi_{1,!}^{N(\underline{\kappa})}\mathbb{Q}_\ell[-\#I(\underline{\kappa})](-\#I(\underline{\kappa})/2).$$

Summing over all $\mathbb{Z}$-orbits of $\mathfrak{K}_r$ we get a canonical $\mathrm{Pic}_X(k)$-action on

$$\oplus_{\underline{\kappa} \in \mathfrak{K}_r} \mathbf{R}\pi_{1,!}^{N(\underline{\kappa})}\mathbb{Q}_\ell[-\#I(\underline{\kappa})](-\#I(\underline{\kappa})/2).$$

For any $u \in \mathrm{Pic}_X(k)$, restricting the source to $\oplus_{\underline{\kappa} \in \mathfrak{K}_r^{\sharp}} P_{\underline{\kappa}}$ and projecting the target to $\oplus_{\underline{\kappa} \in \mathfrak{K}_r^{\sharp}} P_{\underline{\kappa}}$, the u-action gives a map

$$\alpha(u) : \oplus_{\underline{\kappa} \in \mathfrak{K}_r^{\sharp}} P_{\underline{\kappa}} \longrightarrow \oplus_{\underline{\kappa} \in \mathfrak{K}_r^{\sharp}} P_{\underline{\kappa}}.$$

However, this no longer gives an action of $\mathrm{Pic}_X(k)$. Instead, it is an mc-action: for $u, v \in \mathrm{Pic}_X(k)$, the endomorphism $a(uv) - a(u)a(v)$ of $\oplus_{\underline{\kappa} \in \mathfrak{K}_r^{\sharp}} P_{\underline{\kappa}}$ is zero on $P_{\underline{\kappa}}$ for $\underline{\kappa}$ large enough, hence a mc-zero map. This mc-action extends to an mc-action of $\mathbb{Q}_\ell[\mathrm{Pic}_X(k)]$ on $\oplus_{\underline{\kappa} \in \mathfrak{K}_r^{\sharp}} P_{\underline{\kappa}}$, which we also denote by α.

Recall the ring homomorphism

$$a_{\mathrm{Eis}} : \mathscr{H}_G^{\Sigma} \xrightarrow{\mathrm{Sat}} \mathscr{H}_A^{\Sigma} = \mathbb{Q}[\mathrm{Div}(X - \Sigma)] \longrightarrow \mathbb{Q}[\mathrm{Pic}_X(k)].$$

LEMMA 3.36. *For any $f \in \mathscr{H}_G^{\Sigma}$, we have an mc-commutative diagram*

$$\begin{array}{ccc} {}^p\mathrm{H}^0 K & \xrightarrow{{}^p\mathrm{H}^0(f)} & {}^p\mathrm{H}^0 K \\ \downarrow \gamma & & \downarrow \gamma \\ \oplus_{\underline{\kappa} \in \mathfrak{K}_r^{\sharp}} P_{\underline{\kappa}} & \xrightarrow{\alpha(a_{\mathrm{Eis}}(f))} & \oplus_{\underline{\kappa} \in \mathfrak{K}_r^{\sharp}} P_{\underline{\kappa}}. \end{array}$$

In particular, if $f \in \mathcal{I}_{\mathrm{Eis}}$, then the action ${}^p\mathrm{H}^0(f) : {}^p\mathrm{H}^0 K \to {}^p\mathrm{H}^0 K$ is mc-zero.

Proof. Since $\{h_y\}_{y\in|X-\Sigma|}$ generate $\mathscr{H}_G^\Sigma$ as an algebra, it suffices to check the lemma for $f = h_y$. (We are also using the fact that $u \mapsto \alpha(u)$ is an mc-action of $\mathbb{Q}_\ell[\mathrm{Pic}_X(k)]$ on $\oplus_{\underline{\kappa}\in\mathfrak{K}_r^\sharp} P_{\underline{\kappa}}$.) Let $d_y = [k(y) : k]$. We will show that $\gamma \circ {}^p\mathrm{H}^0(f)$ and $\alpha(a_{\mathrm{Eis}}(f)) \circ \gamma : {}^p\mathrm{H}^0 K \to \oplus_{\underline{\kappa}\in\mathfrak{K}_r^\sharp} P_{\underline{\kappa}}$ agree on the factors $P_{\underline{\kappa}}$ whenever $\underline{\kappa} > \max\{2g-2, 0\} + d_y$. Since we are checking whether two maps ${}^p\mathrm{H}^0 K \to P_{\underline{\kappa}}$ agree, and since $P_{\underline{\kappa}}$ is a perverse sheaf all of whose simple constituents have full support on $X(\underline{\kappa})$, it suffices to check at a geometric generic point η of $X(\underline{\kappa})$.

Since $a_{\mathrm{Eis}}(h_y) = \mathbf{1}_{\mathcal{O}(y)} + q^{d_y}\mathbf{1}_{\mathcal{O}(-y)}$, we see that $a_{\mathrm{Eis}}(h_y)P_{\underline{\kappa}'}$ has a $P_{\underline{\kappa}}$-component only when $\underline{\kappa}' > \max\{2g-2, 0\}$ and $\underline{\kappa}' \in \underline{\kappa} + \mathbb{Z}$. In particular, $\underline{\kappa}' \in \mathfrak{K}_r^\sharp$. Thus we only need to check that the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{H}_c^{2r-\#I(\underline{\kappa})}(\mathrm{Sht}_\eta) & \xrightarrow{h_y} & \mathrm{H}_c^{2r-\#I(\underline{\kappa})}(\mathrm{Sht}_\eta) \\ \downarrow{\scriptstyle \gamma_\eta} & & \downarrow{\scriptstyle \gamma_{\eta,\underline{\kappa}}} \\ \oplus_{\underline{\kappa}'\in\mathfrak{K}_r^\sharp, \underline{\kappa}'\in\underline{\kappa}+\mathbb{Z}} \mathrm{H}_c^0(\mathrm{Sht}_{1,\eta}^{N(\underline{\kappa}')}) & \xrightarrow{(a_{\mathrm{Eis}}(h_y))_{\underline{\kappa}}} & \mathrm{H}_c^0(\mathrm{Sht}_{1,\eta}^{N(\underline{\kappa})}). \end{array} \tag{3.33}$$

Here the $\underline{\kappa}'$ component of γ_η is the composition (where the first one is induced by the closed embedding of the closure of ${}^{\underline{\kappa}'}\mathrm{Sht}_\eta$)

$$\gamma_{\eta,\kappa'} : \mathrm{H}_c^{2r-\#I(\underline{\kappa})}(\mathrm{Sht}_\eta) \longrightarrow \mathrm{H}_c^{2r-\#I(\underline{\kappa})}(\overline{{}^{\underline{\kappa}'}\mathrm{Sht}_\eta}) \cong \mathrm{H}_c^{2r-\#I(\underline{\kappa})}({}^{\underline{\kappa}'}\mathrm{Sht}_\eta) \cong \mathrm{H}_c^0(\mathrm{Sht}_{1,\eta}^{N(\underline{\kappa}')}).$$

The proof of (3.33) is similar to that of [10, Lemma 7.8]. The key point is that if we restrict the Hecke correspondence $\mathrm{Sht}(h_y)_\eta$,

$$\mathrm{Sht}_\eta \xleftarrow{\overleftarrow{p}_\eta} \mathrm{Sht}(h_y)_\eta \xrightarrow{\overrightarrow{p}_\eta} \mathrm{Sht}_\eta,$$

over the horocycle ${}^{\underline{\kappa}}\mathrm{Sht}_\eta$ via $\overleftarrow{p}_\eta$, then it decomposes into two pieces, one mapping isomorphically to ${}^{\underline{\kappa}-d_y}\mathrm{Sht}_\eta$ via $\overrightarrow{p}_\eta$, and the other one is a finite étale cover of ${}^{\underline{\kappa}+d_y}\mathrm{Sht}_\eta$ of degree q^{d_y} via $\overrightarrow{p}_\eta$. We omit details. □

3.5.5. *Key finiteness results.* For $i \in \mathbb{Z}$, let

$$V_{\leq i} := \varinjlim_{\underline{\kappa}} \mathrm{H}^0((X^r \times \mathfrak{S}_\infty) \otimes \overline{k}, {}^p\tau_{\leq i} K_{\leq \underline{\kappa}}).$$

Then we have natural maps

$$\cdots \longrightarrow V_{\leq -1} \longrightarrow V_{\leq 0} \longrightarrow V_{\leq 1} \longrightarrow \cdots \longrightarrow V,$$

which are not necessarily injective. Since the action of f comes from a cohomological correspondence, the same cohomological correspondence also acts on each $V_{\le i}$ making the above maps equivariant under the action of $\mathscr{H}_G^\Sigma$. We also have an $\mathscr{H}_G^\Sigma$-module map

$$V_{\le i} \longrightarrow \mathrm{H}^{-i}((X^r \times \mathfrak{S}_\infty) \otimes \overline{k}, {}^p\mathrm{H}^i K).$$

Lemma 3.37.

(1) *The kernel and the cokernel of* $V_{\le 0} \to V$ *are finite-dimensional.*
(2) *The kernel and the cokernel of* $V_{\le 0} \to \mathrm{H}^0((X^r \times \mathfrak{S}_\infty) \otimes \overline{k}, {}^p\mathrm{H}^0 K)$ *are finite-dimensional.*

Proof. (1) Since ${}^p\mathrm{H}^i K = 0$ for i large, $V_{\le i} \stackrel{\sim}{\to} V$ for i sufficiently large. Similarly, $V_i = 0$ for i sufficiently small. Therefore, it suffices to show that $V_{\le i}/V_{\le i-1}$ (namely, modulo the image of $V_{\le i-1}$) is finite-dimensional for $i \neq 0$.

The triangle ${}^p\tau_{\le i-1}K_{\le \underline{\kappa}} \to {}^p\tau_{\le i}K_{\le \underline{\kappa}} \to {}^p\mathrm{H}^i K_{\le \underline{\kappa}}[-i] \to 0$ induces an injective map

$$\mathrm{H}^0({}^p\tau_{\le i}K_{\le \underline{\kappa}})/\mathrm{H}^0({}^p\tau_{\le i-1}K_{\le \underline{\kappa}}) \hookrightarrow \mathrm{H}^{-i}((X^r \times \mathfrak{S}_\infty) \otimes \overline{k}, {}^p\mathrm{H}^i K_{\le \underline{\kappa}}).$$

Taking inductive limit over $\underline{\kappa}$, we have an injection

$$\begin{aligned} V_{\le i}/V_{\le i-1} &\hookrightarrow \varinjlim_{\underline{\kappa}} \mathrm{H}^{-i}((X^r \times \mathfrak{S}_\infty) \otimes \overline{k}, {}^p\mathrm{H}^i K_{\le \underline{\kappa}}) \\ &= \mathrm{H}^{-i}((X^r \times \mathfrak{S}_\infty) \otimes \overline{k}, {}^p\mathrm{H}^i K). \end{aligned} \tag{3.34}$$

(We use that $\varinjlim_{\underline{\kappa}}$ commutes with taking cokernel.) By Lemma 3.33, the right side stabilizes as $\{{}^p\mathrm{H}^i K_{\le \underline{\kappa}}\}$ stabilizes for $i \neq 0$ and hence is finite-dimensional. Therefore, for $i \neq 0$, $V_{\le i}/V_{\le i-1}$ is finite-dimensional. In particular, $V_{\le -1}$ is finite-dimensional.

(2) The injection (3.34) is still valid for $i = 0$, and it can be extended to an exact sequence

$$\begin{aligned} 0 \longrightarrow V_{\le 0}/V_{\le -1} \longrightarrow & \mathrm{H}^0((X^r \times \mathfrak{S}_\infty) \otimes \overline{k}, {}^p\mathrm{H}^0 K) \\ & \longrightarrow \varinjlim_{\underline{\kappa}} \mathrm{H}^1((X^r \times \mathfrak{S}_\infty) \otimes \overline{k}, {}^p\tau_{\le -1}K_{\underline{\kappa}}). \end{aligned}$$

By Lemma 3.33, ${}^p\tau_{\le -1}K_{\underline{\kappa}}$ is eventually stable (in fact a constant inductive system), hence the last term above is finite-dimensional. Since $V_{\le -1}$ is also finite-dimensional, $V_{\le 0} \to \mathrm{H}^0((X^r \times \mathfrak{S}_\infty) \otimes \overline{k}, {}^p\mathrm{H}^0 K)$ has finite-dimensional kernel and the cokernel. □

Corollary 3.38 (of Lemmas 3.36 and 3.37). *If* $f \in \mathcal{I}_{\mathrm{Eis}}$, *then the image of the Hecke action* $f : V \to V$ (*defined in Proposition* 3.15) *is finite-dimensional.*

Proof. By Lemma 3.37(1), it suffices to show that the f-action on $V_{\le 0}$ has finite rank. By Lemma 3.37(2), it suffices to show that ${}^p\mathrm{H}^0(f) : {}^p\mathrm{H}^0 K \to {}^p\mathrm{H}^0 K$ induces a finite-rank map after applying $\mathrm{H}^0((X^r \times \mathfrak{S}_\infty) \otimes \overline{k}, -)$. However, by

Lemma 3.36, ${}^p\mathrm{H}^0(f)$ is mc-zero since $a_{\mathrm{Eis}}(f) = 0$, and the conclusion follows. □

Proposition 3.39. *For any place $y \in |X| - \Sigma$, V is a finitely generated $\mathscr{H}_y \otimes \mathbb{Q}_\ell$-module.*

Proof. By Lemma 3.37, it suffices to show that $\mathrm{H}^0((X \times \mathfrak{S}_\infty) \otimes \overline{k}, {}^p\mathrm{H}^0 K)$ is a finitely generated $\mathscr{H}_y \otimes \mathbb{Q}_\ell$-module.

The ind-perverse sheaf ${}^p\mathrm{H}^0 K$ has an increasing filtration given by ${}^p\mathrm{H}^0 K_{\leq \underline{\kappa}}$ (by Lemma 3.33) with associated graded $P_{\underline{\kappa}}$. Let $F_{\leq N}({}^p\mathrm{H}^0 K) \subset {}^p\mathrm{H}^0 K$ be the sum of ${}^p\mathrm{H}^0 K_{\leq \underline{\kappa}}$ for $\underline{\kappa} \in \mathfrak{K}^\sharp_r$ and $\underline{\kappa} \leq N d_y$. Then $\{F_{\leq N}({}^p\mathrm{H}^0 K)\}$ gives an increasing filtration on ${}^p\mathrm{H}^0 K$. The map γ in (3.32) induces

$$\mathrm{Gr}^F_N(\gamma) : \mathrm{Gr}^F_N({}^p\mathrm{H}^0 K) \longrightarrow \oplus_{\underline{\kappa} \in \mathfrak{K}^\sharp_r, \underline{\kappa} \leq N d_y, \underline{\kappa} \nleq (N-1) d_y} P_{\underline{\kappa}},$$

which is an isomorphism for large N, by Lemma 3.34.

Now h_y sends $F_{\leq N}({}^p\mathrm{H}^0 K)$ to $F_{\leq N+1}({}^p\mathrm{H}^0 K)$. By Lemma 3.36, for N large enough, the induced map

$$\mathrm{Gr}^F_N(h_y) : \mathrm{Gr}^F_N({}^p\mathrm{H}^0 K) \longrightarrow \mathrm{Gr}^F_{N+1}({}^p\mathrm{H}^0 K)$$

is the same as the action of $\mathbf{1}_{\mathcal{O}(y)} \in \mathrm{Pic}_X(k)$

$$\mathbf{1}_{\mathcal{O}(y)} : \oplus_{\underline{\kappa} \in \mathfrak{K}^\sharp_r, \underline{\kappa} \leq N d_y, \underline{\kappa} \nleq (N-1) d_y} P_{\underline{\kappa}} \longrightarrow \oplus_{\underline{\kappa} \in \mathfrak{K}^\sharp_r, \underline{\kappa} \leq (N+1) d_y, \underline{\kappa} \nleq N d_y} P_{\underline{\kappa}}. \tag{3.35}$$

Since $\mathbf{1}_{\mathcal{O}(y)}$ maps $P_{\underline{\kappa}}$ isomorphically to $P_{\underline{\kappa}+d_y}$, (3.35) is an isomorphism. Therefore, $\mathrm{Gr}^F_N(h_y)$ is an isomorphism for large N.

Next we apply $\mathrm{H}^0((X^r \times \mathfrak{S}_\infty) \otimes \overline{k}, -)$ to $F_{\leq N}({}^p\mathrm{H}^0 K)$ and ${}^p\mathrm{H}^0 K$, which we abbreviate as $\mathrm{H}^0(F_{\leq N}({}^p\mathrm{H}^0 K))$ and $\mathrm{H}^0({}^p\mathrm{H}^0 K)$. Note that each $F_{\leq N}({}^p\mathrm{H}^0 K)$ has a Weil structure, $\mathrm{H}^0(F_{\leq N}{}^p\mathrm{H}^0 K)$ is a Frobenius module and we can talk about its weight. We have an exact sequence

$$\begin{aligned} &\mathrm{H}^0(\mathrm{Gr}^F_N({}^p\mathrm{H}^0 K)) \longrightarrow \mathrm{H}^1(F_{\leq N-1}({}^p\mathrm{H}^0 K)) \\ &\qquad \longrightarrow \mathrm{H}^1(F_{\leq N}({}^p\mathrm{H}^0 K)) \longrightarrow \mathrm{H}^1(\mathrm{Gr}^F_N({}^p\mathrm{H}^0 K)). \end{aligned} \tag{3.36}$$

Since $\mathrm{Gr}^F_N({}^p\mathrm{H}^0 K)$ is a sum of $P_{\underline{\kappa}}$, it is pure of weight 0 by Corollary 3.29. Therefore, $\mathrm{H}^0(\mathrm{Gr}^F_N({}^p\mathrm{H}^0 K))$ is pure of weight 0 and $\mathrm{H}^1(\mathrm{Gr}^F_N({}^p\mathrm{H}^0 K))$ is pure of weight 1. For weight reasons, (3.36) gives a long exact sequence

$$\mathrm{H}^0(F_{\leq N-1}({}^p\mathrm{H}^0 K)) \longrightarrow \mathrm{H}^0(F_{\leq N}({}^p\mathrm{H}^0 K)) \longrightarrow \mathrm{H}^0(\mathrm{Gr}^F_N({}^p\mathrm{H}^0 K)) \tag{3.37}$$

$$\longrightarrow W_{\leq 0}\mathrm{H}^1(F_{\leq N-1}({}^p\mathrm{H}^0 K)) \longrightarrow W_{\leq 0}\mathrm{H}^1(F_{\leq N}({}^p\mathrm{H}^0 K)) \longrightarrow 0, \tag{3.38}$$

where $W_{\leq 0}(-)$ means the sub Frobenius-module of weight ≤ 0. The surjectivity of (3.38) implies $W_{\leq 0}\mathrm{H}^1(F_{\leq N}({}^p\mathrm{H}^0 K))$ is eventually stable for N large, and hence the last arrow in (3.37) is surjective for N large. As $\mathrm{Gr}^F_N(h_y)$ is an isomorphism for large N, it induces an isomorphism $\mathrm{H}^0(\mathrm{Gr}^F_N({}^p\mathrm{H}^0 K)) \xrightarrow{\sim}$

$\mathrm{H}^0(\mathrm{Gr}^F_{N+1}({}^p\mathrm{H}^0K))$ for large N. This implies that for large N, the image of $\mathrm{H}^0(F_{\le N}({}^p\mathrm{H}^0K))$ in $\mathrm{H}^0({}^p\mathrm{H}^0K)$ generates it as an $\mathscr{H}_y\otimes\mathbb{Q}_\ell$-module. □

Let $\overline{\mathscr{H}}^\Sigma_\ell$ be the image of the ring homomorphism

$$\mathscr{H}^\Sigma_G\otimes\mathbb{Q}_\ell\longrightarrow \mathrm{End}_{\mathbb{Q}_\ell}(V)\times\mathbb{Q}_\ell[\mathrm{Pic}_X(k)]^{\iota_{\mathrm{Pic}}}$$

given by the product of the action map on V and a^Σ_{Eis}.

Corollary 3.40 (of Proposition 3.39).

(1) *$\overline{\mathscr{H}}^\Sigma_\ell$ is a finitely generated $\mathbb{Q}_\ell$-algebra of Krull dimension one.*
(2) *V is finitely generated as a $\overline{\mathscr{H}}^\Sigma_\ell$-module.*

Proof. Part (2) is an obvious consequence of Proposition 3.39. The proof of part (1) is the same as [10, Lemma 7.13(2)]. □

Theorem 3.41 (Cohomological spectral decomposition).

(1) *There is a decomposition of the reduced scheme of $\mathrm{Spec}\,\overline{\mathscr{H}}^\Sigma_\ell$ into a disjoint union*

$$\mathrm{Spec}(\overline{\mathscr{H}}^\Sigma_\ell)^{\mathrm{red}}=Z_{\mathrm{Eis},\mathbb{Q}_\ell}\coprod Z^\Sigma_{0,\ell},$$

where $Z_{\mathrm{Eis},\mathbb{Q}_\ell}=\mathrm{Spec}\,\mathbb{Q}_\ell[\mathrm{Pic}_X(k)]^{\iota_{\mathrm{Pic}}}$ and $Z^\Sigma_{0,\ell}$ consists of a finite set of closed points.
(2) *There is a unique decomposition*

$$V=V_0\oplus V_{\mathrm{Eis}}$$

into $\mathscr{H}^\Sigma_G\otimes\mathbb{Q}_\ell$-submodules, such that $\mathrm{Supp}(V_{\mathrm{Eis}})\subset Z_{\mathrm{Eis},\mathbb{Q}_\ell}$, and $\mathrm{Supp}(V_0)=Z^\Sigma_{0,\ell}$.
(3) *The subspace V_0 is finite dimensional over $\mathbb{Q}_\ell$.*

Proof. (1) By Lemma 2.1, a^Σ_{Eis} induces a closed embedding $Z_{\mathrm{Eis},\mathbb{Q}_\ell}\hookrightarrow\mathrm{Spec}\,\overline{\mathscr{H}}^\Sigma_\ell$. We are going to show that the complement of $Z_{\mathrm{Eis},\mathbb{Q}_\ell}$ in $\mathrm{Spec}\,\overline{\mathscr{H}}^\Sigma_\ell$ is a finite set of closed points.

Let $\overline{\mathcal{I}}_{\mathrm{Eis}}$ be the image of $\mathcal{I}_{\mathrm{Eis}}$ in $\overline{\mathscr{H}}^\Sigma_\ell$. Then by Corollary 3.40, $\overline{\mathscr{H}}^\Sigma_\ell$ is noetherian and hence $\overline{\mathcal{I}}_{\mathrm{Eis}}$ is finitely generated, say by $f_1,\dots,f_N$. By Corollary 3.38, each $f_i\cdot V$ is finite-dimensional, therefore so is $\overline{\mathcal{I}}_{\mathrm{Eis}}\cdot V=f_1\cdot V+\cdots+f_N\cdot V$. Now let $Z'_0\subset\mathrm{Spec}(\mathscr{H}^\Sigma_\ell)^{\mathrm{red}}$ be the support of the finite-dimensional $\overline{\mathscr{H}}^\Sigma_\ell$-module $\overline{\mathcal{I}}_{\mathrm{Eis}}\cdot V$. Hence Z'_0 is a finite set of closed points. The same argument as that of [10, Th. 7.14] shows that $\mathrm{Spec}(\mathscr{H}^\Sigma_\ell)^{\mathrm{red}}$ is the union of $Z_{\mathrm{Eis},\mathbb{Q}_\ell}$ and Z'_0. Finally we let $Z^\Sigma_{0,\ell}$ be the complement of $Z_{\mathrm{Eis},\mathbb{Q}_\ell}$ in $\mathrm{Spec}(\overline{\mathscr{H}}^\Sigma_\ell)^{\mathrm{red}}$.

The argument for (2) and (3) is the same as that of [10, Th. 7.14]. □

3.5.6. *The base-change situation.* Consider the situation as in Section 3.2.6. We argue that the analogue of Theorem 3.41 holds for $\mathrm{Sht}'^r_G(\Sigma;\Sigma_\infty)$ in place of $\mathrm{Sht}^r_G(\Sigma;\Sigma_\infty)$. Let

$$V' = \mathrm{H}^{2r}_c(\mathrm{Sht}'^r_G(\Sigma;\Sigma_\infty)\otimes\overline{k},\mathbb{Q}_\ell)(r).$$

Then V' is also a $\mathscr{H}^\Sigma_G$-module; see the discussion in Section 3.3.4. The results in this subsection for the $\mathscr{H}^\Sigma_G$-module V have obvious analogues for V' because most of these results are consequences of finiteness results on ${}^p\mathrm{H}^iK$ and similar results formally hold for its pullback to $X'^r\times\mathfrak{S}'_\infty$. There is one place in the proof of Proposition 3.39 where we used purity argument for the cohomology $\mathrm{H}^*((X^r\times\mathfrak{S}_\infty)\otimes\overline{k},P_{\underline{\kappa}})$, which continues to hold for $\mathrm{H}^*((X'^r\times\mathfrak{S}'_\infty)\otimes\overline{k},\nu'^{r,*}P_{\underline{\kappa}})$. Therefore, all results in this subsection hold for V' in place of V. In particular, Theorem 1.1 holds.

4. **The Heegner–Drinfeld cycles**

In this section we define Heegner–Drinfeld cycles in the ramified case. All the notation appearing on the geometric side of our main Theorem 1.2 will be explained in this section.

4.1. *T-Shtukas.*

4.1.1. *The double cover.* Let X' be another smooth, projective and geometrically connected curve over k, and let $\nu: X'\to X$ be a finite morphism of degree 2. Let $R'\subset X'$ be the (reduced) ramification locus of ν, and let $R\subset X$ be its image under ν. Then ν induces an isomorphism $R'\stackrel{\sim}{\to}R$. Let $\sigma: X'\to X'$ be the non-trivial involution over X.

We always assume that conditions (1.4) and (1.5) hold. In particular, they imply that

$$R\cap\Sigma=\varnothing.$$

Let

$$\Sigma'_\infty=\nu^{-1}(\Sigma_\infty)\subset|X'|.$$

Then $\nu:\Sigma'_\infty\to\Sigma_\infty$ is a bijection. For $x\in\Sigma_\infty$, we denote its preimage in Σ'_∞ by x'. Set

$$\mathfrak{S}'_\infty=\prod_{x'\in\Sigma'_\infty}\mathrm{Spec}\,k(x').$$

An S-point of $\mathfrak{S}'_\infty$ is $\{x'^{(1)}\}_{x'\in\Sigma'_\infty}$, where $x'^{(1)}: S\to\mathrm{Spec}\,k(x')\hookrightarrow X'$. We introduce the notation $x'^{(i)}$ for all $i\in\mathbb{Z}$ as before.

4.1.2. *Hecke stack for T-bundles.* Let

$$\mathrm{Bun}_T=\mathrm{Pic}_{X'}/\,\mathrm{Pic}_X\,.$$

As a special case of [10, Def. 5.1], for $\underline{\mu} \in \{\pm 1\}^r$, we have the Hecke stack $\mathrm{Hk}^{\underline{\mu}}_{1,X'}$ classifying a chain of $r+1$ line bundles on X',

$$\mathcal{L}_0 \xrightarrow{f'_1} \mathcal{L}_1 \xrightarrow{f'_2} \cdots \xrightarrow{f'_r} \mathcal{L}_r,$$

with modification type of f'_i given by μ_i. Then $\mathrm{Hk}^{\underline{\mu}}_{1,X'} \cong \mathrm{Pic}_{X'} \times X'^r$, where the projection to $\mathrm{Pic}_{X'}$ records $\mathcal{L}_0$, and the projection to X'^r records the locus of modification of $f_i : \mathcal{L}_{i-1} \dashrightarrow \mathcal{L}_i$. We define

$$\mathrm{Hk}^{\underline{\mu}}_{T} := \mathrm{Hk}^{\underline{\mu}}_{1,X'} / \mathrm{Pic}_X$$

together with maps recording $\mathcal{L}_i$

$$p^{\underline{\mu}}_{T,i} : \mathrm{Hk}^{\underline{\mu}}_{T} \longrightarrow \mathrm{Bun}_T, \quad i = 0, \dots, r.$$

4.1.3. *T-Shtukas.* For $x' \in \Sigma'_\infty$ and $i \in \mathbb{Z}$, we have a map

$$\mathbf{x}'^{(i)} : \mathfrak{S}'_\infty \longrightarrow \mathrm{Spec}\, k(x') \xrightarrow{\mathrm{Fr}^{i-1}} \mathrm{Spec}\, k(x') \hookrightarrow X', \quad 1 \le i \le d_{x'} = 2d_x,$$

where the first map is the projection to the x'-factor and the last one is the natural embedding. Again we denote the graph of $\mathbf{x}'^{(i)}$ (as a divisor on $X' \times \mathfrak{S}'_\infty$) by the same notation $\mathbf{x}'^{(i)}$.

Let $\mathscr{D}'_\infty$ be the group of $\mathbb{Z}$-valued divisors on $X' \times \mathfrak{S}'_\infty$ supported on $\Sigma'_\infty \times \mathfrak{S}'_\infty$, which is the union of the graphs of $\mathbf{x}'^{(i)}$ for $x' \in \Sigma'_\infty, 1 \le i \le d_{x'}$. For any $D'_\infty \in \mathscr{D}'_\infty$ as above, we have morphisms

$$\begin{aligned}
\widetilde{\mathrm{AL}}(D'_\infty) &: \mathrm{Pic}_{X'} \times \mathfrak{S}'_\infty \longrightarrow \mathrm{Pic}_{X'},\\
\mathrm{AL}(D'_\infty) &: \mathrm{Bun}_T \times \mathfrak{S}'_\infty \longrightarrow \mathrm{Bun}_T,\\
&(\mathcal{L}, \{x'^{(1)}\}_{x' \in \Sigma'_\infty}) \longmapsto \mathcal{L}\Big(\sum_{x' \in \Sigma'_\infty, 1 \le i \le d_{x'}} c^{(i)}_{x'} \Gamma_{x'^{(i)}}\Big).
\end{aligned}$$

Suppose $\underline{\mu} \in \{\pm 1\}^r$ and $D'_\infty \in \mathscr{D}'_\infty$ satisfy

$$\sum_{i=1}^r \mu_i = \deg D'_\infty = \sum_{x' \in \Sigma'_\infty, 1 \le i \le d_{x'}} c^{(i)}_{x'}. \tag{4.1}$$

We then apply the definition of $\mathrm{Sht}^{\underline{\mu}}_n(\Sigma; D_\infty)$ to the case $n = 1$, the curve being X', and Σ and Σ_∞ are both replaced by Σ'_∞. Denote the resulting moduli stack by $\mathrm{Sht}^{\underline{\mu}}_{1,X'}(D'_\infty)$.

The groupoid $\mathrm{Pic}_X(k)$ acts on $\mathrm{Sht}^{\underline{\mu}}_{1,X'}(D'_\infty)$ by tensoring all the line bundles in the data with the pullback of $\mathcal{K} \in \mathrm{Pic}_X(k)$ to X'. We define

$$\mathrm{Sht}^{\underline{\mu}}_T(D'_\infty) = \mathrm{Sht}^{\underline{\mu}}_{1,X'}(D'_\infty) / \mathrm{Pic}_X(k).$$

We have a morphism

$$\Pi^{\underline{\mu}}_{T, D'_\infty} : \mathrm{Sht}^{\underline{\mu}}_T(D'_\infty) \longrightarrow X'^r \times \mathfrak{S}'_\infty.$$

From the definition we have a Cartesian diagram

$$\begin{array}{ccc} \mathrm{Sht}^{\underline{\mu}}_T(D'_\infty) & \longrightarrow & \mathrm{Hk}^{\underline{\mu}}_T \times \mathfrak{S}'_\infty \\ \Big\downarrow{\scriptstyle \omega_{T,0}} & & \Big\downarrow{\scriptstyle (p^{\underline{\mu}}_{T,0}, \mathrm{AL}(-D'_\infty)\circ(p^{\underline{\mu}}_{T,r}\times \mathrm{id}_{\mathfrak{S}'_\infty}))} \\ \mathrm{Bun}_T & \xrightarrow{(\mathrm{id},\mathrm{Fr})} & \mathrm{Bun}_T \times \mathrm{Bun}_T. \end{array} \tag{4.2}$$

From the diagram we get the following statement.

LEMMA 4.1. *The moduli stack* $\mathrm{Sht}^{\underline{\mu}}_T(D'_\infty)$ *depends only on the image of* D'_∞ *in* $\mathscr{D}'_\infty/\nu^*\mathscr{D}_\infty$.

The following alternative description of $\mathrm{Sht}^{\underline{\mu}}_T(D'_\infty)$ follows easily from the definitions.

LEMMA 4.2. *We have a Cartesian diagram*

$$\begin{array}{ccc} \mathrm{Sht}^{\underline{\mu}}_T(D'_\infty) & \xrightarrow{\omega_{T,0}} & \mathrm{Bun}_T \\ \Big\downarrow{\scriptstyle \Pi^{\underline{\mu}}_{T,D'_\infty}} & & \Big\downarrow{\scriptstyle \lambda} \\ X'^r \times \mathfrak{S}'_\infty & \xrightarrow{\alpha^{\underline{\mu}}_{D'_\infty}} & \mathrm{Bun}_T, \end{array}$$

where $\lambda : \mathcal{L} \mapsto \mathcal{L}^{-1} \otimes {}^\tau\mathcal{L}$ *is the Lang map for* Bun_T *and* $\alpha^{\underline{\mu}}_{D'_\infty}$ *sends*

$$(x'_1, \dots, x'_r; \{x'^{(1)}\}_{x' \in \Sigma'_\infty})$$

to the image of the line bundle

$$\mathcal{O}_{X'}\left(\sum_{i=1}^r \mu_i \Gamma_{x'_i} - \sum_{x' \in \Sigma'_\infty, 1 \le i \le d_{x'}} c^{(i)}_{x'} \Gamma_{x'^{(i)}}\right)$$

in Bun_T.

COROLLARY 4.3 (of Lemma 4.2). *The morphism* $\Pi^{\underline{\mu}}_{T,D'_\infty}$ *is a torsor under the (finite discrete) groupoid* $\mathrm{Bun}_T(k)$. *In particular,* $\mathrm{Sht}^{\underline{\mu}}_T(D'_\infty)$ *is a smooth and proper DM stack over* k *of dimension* r.

4.1.4. *Specific choice of* D'_∞. For each $\mu_\infty = (\mu_x)_{x\in\Sigma_\infty} \in \{\pm1\}^{\Sigma_\infty}$, define the following element in $\mathscr{D}'_\infty$

$$\mu_\infty \cdot \Sigma'_\infty := \sum_{x \in \Sigma_\infty} \mu_x \mathbf{x}'^{(1)} \in \mathscr{D}'_\infty.$$

Definition 4.4. Fix r satisfying the parity condition (3.18). Let $\underline{\mu} \in \{\pm1\}^r$, $\mu_\infty \in \{\pm1\}^{\Sigma_\infty}$. For any $D'_\infty \in \mathscr{D}'_\infty$ satisfying $D'_\infty \equiv \mu_\infty \cdot \Sigma'_\infty \mod \nu^*\mathscr{D}_\infty$ and (4.1), define

$$\mathrm{Sht}^{\underline{\mu}}_T(\mu_\infty \cdot \Sigma'_\infty) := \mathrm{Sht}^{\underline{\mu}}_T(D'_\infty).$$

The notation is justified because the right side above depends only on μ_∞ by Lemma 4.1. We denote the projection $\Pi^{\underline{\mu}}_{T,D'_\infty}$ for such D'_∞ by

$$\Pi^{\underline{\mu}}_{T,\mu_\infty} : \mathrm{Sht}^{\underline{\mu}}_T(\mu_\infty \cdot \Sigma'_\infty) \longrightarrow X'^r \times \mathfrak{S}'_\infty.$$

Remark 4.5. Whenever r satisfies the parity condition (3.18), for any $(\underline{\mu}, \mu_\infty) \in \{\pm 1\}^r \times \{\pm 1\}^{\Sigma_\infty}$, the divisor $D'_\infty \in \mathscr{D}'_\infty$ satisfying the conditions in Definition 4.4 always exists. Therefore, $\mathrm{Sht}^{\underline{\mu}}_T(\mu_\infty \cdot \Sigma'_\infty)$ is always defined (and non-empty).

The following lemma is a direct consequence of the diagram (4.2).

LEMMA 4.6. *The following diagram is Cartesian*:

$$\begin{array}{ccc} \mathrm{Sht}^{\underline{\mu}}_T(\mu_\infty \cdot \Sigma'_\infty) & \longrightarrow & \mathrm{Hk}^{\underline{\mu}}_T \times \mathfrak{S}'_\infty \\ \downarrow & & \downarrow{\scriptstyle (p^{\underline{\mu}}_{T,0} \times \mathrm{id}_{\mathfrak{S}'_\infty}, \mathrm{AL}^{\sharp}_{T,\mu_\infty} \circ (p^{\underline{\mu}}_{T,r} \times \mathrm{id}_{\mathfrak{S}'_\infty}))} \\ \mathrm{Bun}_T \times \mathfrak{S}'_\infty & \xrightarrow{(\mathrm{id},\mathrm{Fr})} & (\mathrm{Bun}_T \times \mathfrak{S}'_\infty) \times (\mathrm{Bun}_T \times \mathfrak{S}'_\infty), \end{array} \tag{4.3}$$

where $\mathrm{AL}^{\sharp}_{T,\mu_\infty}$ *is the map*

$$\mathrm{AL}^{\sharp}_{T,\mu_\infty} = (\mathrm{AL}(-\mu_\infty \cdot \Sigma'_\infty), \mathrm{Fr}_{\mathfrak{S}'_\infty}) : \mathrm{Bun}_T \times \mathfrak{S}'_\infty \longrightarrow \mathrm{Bun}_T \times \mathfrak{S}'_\infty. \tag{4.4}$$

4.1.5. *Relation to T-Shtukas in* [10]. For $(\underline{\mu}, \mu_\infty) \in \{\pm 1\}^r \times \{\pm 1\}^{\Sigma_\infty}$, let $\widetilde{\mu} = (\underline{\mu}, -\mu_\infty)$. Then $\mathrm{Sht}^{\widetilde{\mu}}_T$ is defined as in [10, §5.4] (the loc. cit. also applies to a ramified cover X'/X), with a map $\pi^{\widetilde{\mu}}_T : \mathrm{Sht}^{\widetilde{\mu}}_T \to X'^r \times X'^{\Sigma_\infty}$. Let $\mathfrak{S}'_\infty \hookrightarrow X'^{\Sigma_\infty}$ be the product of the natural embeddings $\mathrm{Spec}\, k(x') \hookrightarrow X'$ for each $x \in \Sigma_\infty$. From the definitions, we see that $\mathrm{Sht}^{\underline{\mu}}_T(\mu_\infty \cdot \Sigma'_\infty)$ fits into a Cartesian diagram

$$\begin{array}{ccc} \mathrm{Sht}^{\underline{\mu}}_T(\mu_\infty \cdot \Sigma'_\infty) & \hookrightarrow & \mathrm{Sht}^{\widetilde{\mu}}_T \\ \downarrow{\scriptstyle \Pi^{\underline{\mu}}_{T,\mu_\infty}} & & \downarrow{\scriptstyle \pi^{\widetilde{\mu}}_T} \\ X'^r \times \mathfrak{S}'_\infty & \hookrightarrow & X'^r \times X'^{\Sigma_\infty}. \end{array}$$

4.2. *The Heegner–Drinfeld cycles*. In this subsection we will define a map from $\mathrm{Sht}^{\underline{\mu}}_T(\mu_\infty \cdot \Sigma'_\infty)$ to $\mathrm{Sht}^r_G(\Sigma; \Sigma_\infty)$ depending on an auxiliary choice.

Recall that condition (1.4) is assumed. Let $\Sigma'_f = \nu^{-1}(\Sigma_f)$. Let $\mathrm{Sect}(\Sigma'_f/\Sigma_f)$ be the set of sections of the two-to-one map $\Sigma'_f \to \Sigma_f$. Then $\mathrm{Sect}(\Sigma'_f/\Sigma_f)$ is a torsor under $\{\pm 1\}^{\Sigma_f}$. The auxiliary choice we need is an element $\mu_f \in \mathrm{Sect}(\Sigma'_f/\Sigma_f)$.

4.2.1. *The map* $\theta^{\mu_\Sigma}_{\mathrm{Bun}}$. Let $\mu_\Sigma = (\mu_f, \mu_\infty) \in \mathrm{Sect}(\Sigma'_f/\Sigma_f) \times \{\pm 1\}^{\Sigma_\infty}$. We define a map

$$\widetilde{\theta}^{\mu_\Sigma}_{\mathrm{Bun}} : \mathrm{Pic}_{X'} \times \mathfrak{S}'_\infty \longrightarrow \mathrm{Bun}_2(\Sigma).$$

To an S-point $(\mathcal{L}, \{x'^{(1)}\}_{x' \in \Sigma'_\infty})$ of $\mathrm{Pic}_{X'} \times \mathfrak{S}'_\infty$, we assign the following S-point of $\mathrm{Bun}_2(\Sigma)$:

$$\mathcal{E}^\dagger = (\mathcal{E}, \{\mathcal{E}(-\frac{1}{2}x)\}_{x \in \Sigma}),$$

where

- $\mathcal{E} = \nu_{S,*}\mathcal{L}$, where $\nu_S = \nu \times \mathrm{id}_S : X' \times S \to X \times S$;
- for $x \in \Sigma_f$, denote the value of μ_f at x by $\mu_x \in \nu^{-1}(x)$ — then $\mathcal{E}(-\frac{1}{2}x) = \nu_{S,*}(\mathcal{L}(-\mu_x))$;
- for $x \in \Sigma_\infty$,

$$\mathcal{E}(-\frac{1}{2}x) = \begin{cases} \nu_{S,*}(\mathcal{L}(-\Gamma_{x'^{(1)}} - \Gamma_{x'^{(2)}} - \cdots - \Gamma_{x'^{(d_x)}})), & \mu_x = 1, \\ \nu_{S,*}(\mathcal{L}(-\Gamma_{x'^{(d_x+1)}} - \Gamma_{x'^{(d_x+2)}} - \cdots - \Gamma_{x'^{(2d_x)}})), & \mu_x = -1. \end{cases}$$

Note here that for $x \in \Sigma_\infty$, the divisors $\Gamma_{x'^{(1)}} + \Gamma_{x'^{(2)}} + \cdots + \Gamma_{x'^{(d_x)}}$ and $\Gamma_{x'^{(d_x+1)}} + \Gamma_{x'^{(d_x+2)}} + \cdots + \Gamma_{x'^{(2d_x)}}$ in the above formulae are "half" of the divisor $\{x'\} \times S \subset X' \times S$.

Dividing by Pic_X we get a morphism

$$\theta^{\mu_\Sigma}_{\mathrm{Bun}} : \mathrm{Bun}_T \times \mathfrak{S}'_\infty \longrightarrow \mathrm{Bun}_G(\Sigma).$$

The next lemma is a direct calculation.

LEMMA 4.7. *Let $\mu_\Sigma = (\mu_f, \mu_\infty)$. The following diagram is commutative:*

$$\begin{array}{ccc} \mathrm{Bun}_T \times \mathfrak{S}'_\infty & \xrightarrow{\mathrm{AL}^\sharp_{T,\mu_\infty}} & \mathrm{Bun}_T \times \mathfrak{S}'_\infty \\ \downarrow{\scriptstyle (\theta^{\mu_\Sigma}_{\mathrm{Bun}}, \nu_\infty)} & & \downarrow{\scriptstyle \theta^{\mu_\Sigma}_{\mathrm{Bun}}} \\ \mathrm{Bun}_G(\Sigma) \times \mathfrak{S}_\infty & \xrightarrow{\mathrm{AL}_{G,\infty}} & \mathrm{Bun}_G(\Sigma), \end{array}$$

where $\nu_\infty : \mathfrak{S}'_\infty \to \mathfrak{S}_\infty$ is the map induced from ν.

4.2.2. *Heegner–Drinfeld cycle.* We define

$$\mathfrak{T}_{r,\Sigma} := \{\pm 1\}^r \times \mathrm{Sect}(\Sigma'_f / \Sigma_f) \times \{\pm 1\}^{\Sigma_\infty}.$$

For $\mu = (\underline{\mu}, \mu_f, \mu_\infty) \in \mathfrak{T}_{r,\Sigma}$, we have a map

$$\theta^\mu_{\mathrm{Hk}} : \mathrm{Hk}^\mu_T \times \mathfrak{S}'_\infty \longrightarrow \mathrm{Hk}^r_G(\Sigma)$$

by applying $\theta^{\mu_\Sigma}_{\mathrm{Bun}}$ (where $\mu_\Sigma = (\mu_f, \mu_\infty)$) to each member of the chain $\{\mathcal{L}_i\}_{0 \le i \le r}$ classified by Hk^μ_T. By construction we have $p_i \circ \theta^\mu_{\mathrm{Hk}} = \theta^{\mu_\Sigma}_{\mathrm{Bun}} \circ (p^\mu_{T,i} \times \mathrm{id}_{\mathfrak{S}'_\infty}) : \mathrm{Hk}^\mu_T \times \mathfrak{S}'_\infty \to \mathrm{Bun}_G(\Sigma)$ for $1 \le i \le r$.

Now compare the Cartesian diagrams (4.3) and (3.21). Each corner of the diagram (4.3) except the upper left corner maps to the corresponding corner of (3.21) by θ_{Bun} and θ^μ_{Hk}; Lemma 4.7 says that the corresponding maps in the two diagrams are intertwined. Therefore, we get a morphism between the upper left corners since both diagrams are Cartesian:

$$\theta^{\mu} : \mathrm{Sht}_T^{\underline{\mu}}(\mu_\infty \cdot \Sigma'_\infty) \longrightarrow \mathrm{Sht}_G^r(\Sigma; \Sigma_\infty).$$

We have a commutative diagram

$$\begin{array}{ccc} \mathrm{Sht}_T^{\underline{\mu}}(\mu_\infty \cdot \Sigma'_\infty) & \xrightarrow{\theta^{\mu}} & \mathrm{Sht}_G^r(\Sigma; \Sigma_\infty) \\ \downarrow{\scriptstyle \Pi_{T,\mu_\infty}^{\underline{\mu}}} & & \downarrow{\scriptstyle \Pi_G^r} \\ X'^r \times \mathfrak{S}'_\infty & \xrightarrow{(\nu^r, \nu_\infty)} & X^r \times \mathfrak{S}_\infty, \end{array}$$

which induces a morphism

$$\theta'^{\mu} : \mathrm{Sht}_T^{\underline{\mu}}(\mu_\infty \cdot \Sigma'_\infty) \longrightarrow \mathrm{Sht}'^r_G(\Sigma; \Sigma_\infty) := \mathrm{Sht}_G^r(\Sigma; \Sigma_\infty) \times_{X^r \times \mathfrak{S}_\infty} (X'^r \times \mathfrak{S}'_\infty).$$

Since $\mathrm{Sht}_T^{\underline{\mu}}(\mu_\infty \cdot \Sigma'_\infty)$ is proper over k of dimension r by Corollary 4.3, its image in $\mathrm{Sht}'^r_G(\Sigma; \Sigma_\infty)$ defines an element in the Chow group of proper cycles.

Definition 4.8. The *Heegner–Drinfeld cycle of type* $\mu = (\underline{\mu}, \mu_f, \mu_\infty) \in \mathfrak{T}_{r,\Sigma}$ is the class

$$\mathcal{Z}^{\mu} := \theta'^{\mu}_*[\mathrm{Sht}_T^{\underline{\mu}}(\mu_\infty \cdot \Sigma'_\infty)] \in \mathrm{Ch}_{c,r}(\mathrm{Sht}'^r_G(\Sigma; \Sigma_\infty))_{\mathbb{Q}}.$$

Definition 4.9. Let $\mu, \mu' \in \mathfrak{T}_{r,\Sigma}$. Define a linear functional $\mathbb{I}^{\mu,\mu'}$ on $\mathscr{H}_G^{\Sigma}$ by

$$\mathbb{I}^{\mu,\mu'}(f) = \left(\prod_{x' \in \Sigma'_\infty} d_{x'} \right)^{-1} \langle \mathcal{Z}^{\mu}, f * \mathcal{Z}^{\mu'} \rangle_{\mathrm{Sht}'^r_G(\Sigma; \Sigma_\infty)} \in \mathbb{Q}. \quad f \in \mathscr{H}_G^{\Sigma}.$$

Here we are using the $\mathscr{H}_G^{\Sigma}$-action on $\mathrm{Ch}_{c,r}(\mathrm{Sht}'^r_G(\Sigma; \Sigma_\infty))_{\mathbb{Q}}$ defined in Section 3.3.4.

4.3. *Symmetry among Heegner–Drinfeld cycles.* Let $\mu = (\underline{\mu}, \mu_f, \mu_\infty) \in \mathfrak{T}_{r,\Sigma}$. We study how $\mathcal{Z}^{\mu}$ changes when we vary μ.

4.3.1. *Changing* $\underline{\mu}$. As in [10, §5.4.6], for two choices $\underline{\mu}, \underline{\mu}' \in \{\pm 1\}^r$, there is a canonical isomorphism $\iota_{\underline{\mu},\underline{\mu}'} : \mathrm{Sht}_T^{\underline{\mu}}(\mu_\infty \cdot \Sigma'_\infty) \cong \mathrm{Sht}_T^{\underline{\mu}'}(\mu_\infty \cdot \Sigma'_\infty)$ preserving the T-bundle $\mathcal{L}_i$ and the projection to $\mathfrak{S}'_\infty$. However, $\iota_{\underline{\mu},\underline{\mu}'}$ does not preserve the projections $\Pi_{T,\mu_\infty}^{\underline{\mu}}$ and $\Pi_{T,\mu_\infty}^{\underline{\mu}'}$. Instead, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Sht}_T^{\underline{\mu}}(\mu_\infty \cdot \Sigma'_\infty) & \xrightarrow{\iota(\underline{\mu},\underline{\mu}')} & \mathrm{Sht}_T^{\underline{\mu}'}(\mu_\infty \cdot \Sigma'_\infty) \\ \downarrow{\scriptstyle \Pi_{T,\mu_\infty}^{\underline{\mu}}} & & \downarrow{\scriptstyle \Pi_{T,\mu_\infty}^{\underline{\mu}'}} \\ X'^r \times \mathfrak{S}'_\infty & \xrightarrow{\sigma(\underline{\mu},\underline{\mu}') \times \mathrm{id}} & X'^r \times \mathfrak{S}'_\infty, \end{array}$$

where the involution $\sigma(\underline{\mu},\underline{\mu}') : X'^r \to X'^r$ sends a point $(x'_1,\dots,x'_r)$ to the point $(x''_1,\dots,x''_r)$ where, for $1 \le i \le r$,

$$x''_i = \begin{cases} x'_i & \text{if } \mu_i = \mu'_i, \\ \sigma(x'_i) & \text{if } \mu_i \neq \mu'_i. \end{cases}$$

Letting $\mu' = (\underline{\mu}', \mu_f, \mu_\infty)$, it is easy to check that $\iota(\underline{\mu},\underline{\mu}')$ intertwines the map θ^μ and $\theta^{\mu'}$.

4.3.2. *Changing* μ_f. Let $\mu'_f = \{\mu'_x\}_{x\in\Sigma_f} \in \mathrm{Sect}(\Sigma'_f/\Sigma_f)$ be another element. Consider the following divisor on X':

$$D(\mu_f,\mu'_f) = \sum_{x\in\Sigma_f, \mu_x\neq\mu'_x} \mu_x.$$

We have an automorphism

$$\iota(\mu_f,\mu'_f) : \mathrm{Sht}^{\underline{\mu}}_T(\mu_\infty\cdot\Sigma'_\infty) \longrightarrow \mathrm{Sht}^{\underline{\mu}}_T(\mu_\infty\cdot\Sigma'_\infty)$$

sending $(\mathcal{L}_i; x_i; \{x'^{(1)}\})$ to $(\mathcal{L}_i(-D(\mu_f,\mu'_f)); x_i; \{x'^{(1)}\})$. Letting

$$\mu' = (\underline{\mu}, \mu'_f, \mu_\infty),$$

direct calculation shows that the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{Sht}^{\underline{\mu}}_T(\mu_\infty\cdot\Sigma'_\infty) & \xrightarrow{\iota(\mu_f,\mu'_f)} & \mathrm{Sht}^{\underline{\mu}}_T(\mu_\infty\cdot\Sigma'_\infty) \\ \downarrow{\scriptstyle\theta^\mu} & & \downarrow{\scriptstyle\theta^{\mu'}} \\ \mathrm{Sht}^r_G(\Sigma;\Sigma_\infty) & \xrightarrow{\mathrm{AL}_{\mathrm{Sht}}(\mu_f,\mu'_f)} & \mathrm{Sht}^r_G(\Sigma;\Sigma_\infty), \end{array}$$

where $\mathrm{AL}_{\mathrm{Sht}}(\mu_f,\mu'_f)$ is the composition of $\mathrm{AL}_{\mathrm{Sht},x}$ (see Section 3.2.7) for $x\in\Sigma_f$ such that $\mu_x \neq \mu'_x$.

4.3.3. *Changing* μ_∞. Let $\mu'_\infty \in \{\pm 1\}^{\Sigma_\infty}$ be another element. Consider the following divisor on $X' \times \mathfrak{S}'_\infty$:

$$D(\mu_\infty,\mu'_\infty) = \sum_{\mu_x=1,\mu'_x=-1} (\mathbf{x}'^{(1)}+\cdots+\mathbf{x}'^{(d_x)}) + \sum_{\mu_x=-1,\mu'_x=1} (\mathbf{x}'^{(d_x+1)}+\cdots+\mathbf{x}'^{(2d_x)}),$$

where both sums are over $x\in\Sigma_\infty$. Define an isomorphism

$$\iota(\mu_\infty,\mu'_\infty) : \mathrm{Sht}^{\underline{\mu}}_T(\mu_\infty\cdot\Sigma'_\infty) \longrightarrow \mathrm{Sht}^{\underline{\mu}}_T(\mu'_\infty\cdot\Sigma'_\infty)$$

sending $(\mathcal{L}_i; x_i; \{x'^{(1)}\})$ to $(\mathcal{L}_i(-D(\mu_\infty,\mu'_\infty)); x_i; \{x'^{(1)}\})$. Letting

$$\mu' = (\underline{\mu}, \mu_f, \mu'_\infty),$$

direct calculation shows that the following diagram is commutative:

$$\begin{array}{ccc}\mathrm{Sht}_T^{\underline{\mu}}(\mu_\infty\cdot\Sigma'_\infty) & \xrightarrow{\iota(\mu_\infty,\mu'_\infty)} & \mathrm{Sht}_T^{\underline{\mu}}(\mu'_\infty\cdot\Sigma'_\infty)\\ \downarrow{\scriptstyle\theta^\mu} & & \downarrow{\scriptstyle\theta^{\mu'}}\\ \mathrm{Sht}_G^r(\Sigma;\Sigma_\infty) & \xrightarrow{\mathrm{AL}_{\mathrm{Sht}}(\mu_\infty,\mu'_\infty)} & \mathrm{Sht}_G^r(\Sigma;\Sigma_\infty),\end{array}$$

where $\mathrm{AL}_{\mathrm{Sht}}(\mu_\infty,\mu'_\infty)$ is the composition of $\mathrm{AL}_{\mathrm{Sht},x}$ for $x\in\Sigma_\infty$ such that $\mu_x\neq\mu'_x$.

4.3.4. *The action of* $\mathfrak{A}_{r,\Sigma}$. We observe that $\mathfrak{T}_{r,\Sigma}$ is a torsor under the group $\mathfrak{A}_{r,\Sigma}:=(\mathbb{Z}/2\mathbb{Z})^{\{1,2,\dots,r\}\sqcup\Sigma}$. We denote the action of $a\in\mathfrak{A}_{r,\Sigma}$ on $\mathfrak{T}_{r,\Sigma}$ by $a\cdot(-)$.

We also have an action of $\mathfrak{A}_{r,\Sigma}$ on $\mathrm{Sht}'^r_G(\Sigma;\Sigma_\infty)$ defined as follows. The factor of $\mathbb{Z}/2\mathbb{Z}$ indexed by $1\le i\le r$ acts on the i-th factor of X' by Galois involution over X. For $x\in\Sigma$, the non-trivial element in the factor of $\mathbb{Z}/2\mathbb{Z}$ indexed by x acts by the involution $\mathrm{AL}_{\mathrm{Sht},x}$ defined in Section 3.2.7 on the $\mathrm{Sht}^r_G(\Sigma;\Sigma_\infty)$-factor and identity on $X'^r\times\mathfrak{S}'_\infty$. We denote this action by

$$\mathfrak{A}_{r,\Sigma}\ni a\longmapsto \mathrm{AL}_{\mathrm{Sht}',a}.$$

The following lemma summarizes the calculations in Sections 4.3.1, 4.3.2 and 4.3.3.

LEMMA 4.10. *For any* $\mu\in\mathfrak{T}_{r,\Sigma}$ *and* $a\in\mathfrak{A}_{r,\Sigma}$, *the following diagram is commutative*:

$$\begin{array}{ccc}\mathrm{Sht}_T^{\underline{\mu}}(\mu_\infty\cdot\Sigma'_\infty) & \xrightarrow{\iota(\mu,a\cdot\mu)} & \mathrm{Sht}_T^{\underline{a\cdot\mu}}((a\cdot\mu_\infty)\cdot\Sigma'_\infty)\\ \downarrow{\scriptstyle\theta'^\mu} & & \downarrow{\scriptstyle\theta'^{a\cdot\mu}}\\ \mathrm{Sht}'^r_G(\Sigma;\Sigma_\infty) & \xrightarrow{\mathrm{AL}_{\mathrm{Sht}',a}} & \mathrm{Sht}'^r_G(\Sigma;\Sigma_\infty).\end{array}$$

Here the upper horizontal arrow is the composition of $\iota(\underline{\mu},\underline{\mu}'),\iota(\mu_f,\mu'_f)$ *and* $\iota(\mu_\infty,\mu'_\infty)$ *defined in Sections* 4.3.1, 4.3.2 *and* 4.3.3. *In particular, we have*

$$\mathcal{Z}^\mu=\mathrm{AL}^*_{\mathrm{Sht}',a}(\mathcal{Z}^{a\cdot\mu}),\quad\forall\mu\in\mathfrak{T}_{r,\Sigma},a\in\mathfrak{A}_{r,\Sigma}.$$

Let $\mu=(\underline{\mu},\mu_f,\mu_\infty),\mu'=(\underline{\mu}',\mu'_f,\mu'_\infty)\in\mathfrak{T}_{r,\Sigma}$. Let

$$\Delta(\mu,\mu'):=\{1\le i\le r|\mu_i\neq\mu'_i\},$$

$$\Sigma_-(\mu,\mu'):=\{x\in\Sigma|\mu_x\neq\mu'_x\}\subset\Sigma,\tag{4.5}$$

$$\Sigma_+(\mu,\mu'):=\{x\in\Sigma|\mu_x=\mu'_x\}=\Sigma-\Sigma_-(\mu,\mu').\tag{4.6}$$

COROLLARY 4.11 (of Lemma 4.10). *Let* $\mu,\mu'\in\mathfrak{T}_{r,\Sigma}$. *Then* $\mathbb{I}^{\mu,\mu'}$ *depends only on the sets* $\Delta(\mu,\mu')$ *and* $\Sigma_-(\mu,\mu')$.

Proof. Let $a(\mu,\mu')\in\mathfrak{A}_{r,\Sigma}$ be the unique element such that $a(\mu,\mu')\cdot\mu=\mu'$. Then $\Delta(\mu,\mu')$ and $\Sigma_-(\mu,\mu')$ determines $a(\mu,\mu')$ and vice versa. Therefore, we only need to show that $\mathbb{I}^{\mu,\mu'}$ depends only on $a(\mu,\mu')$.

Suppose μ,μ' and $\widehat{\mu},\widehat{\mu}'$ satisfy $a(\mu,\mu')=a(\widehat{\mu},\widehat{\mu}')$. We will show that $\mathbb{I}^{\mu,\mu'}=\mathbb{I}^{\widehat{\mu},\widehat{\mu}'}$. Since $\mathfrak{T}_{r,\Sigma}$ is a torsor under $\mathfrak{A}_{r,\Sigma}$, there is a unique $b\in\mathfrak{A}_{r,\Sigma}$ such that $\widehat{\mu}=b\cdot\mu$, $\widehat{\mu}'=b\cdot\mu'$. Since $\mathrm{AL}_{\mathrm{Sht}',b}$ commutes with the action of any $f\in\mathscr{H}_G^{\Sigma}$, we have

$$\begin{aligned}\langle\mathcal{Z}^{\widehat{\mu}},f*\mathcal{Z}^{\widehat{\mu}'}\rangle&=\langle\mathrm{AL}^*_{\mathrm{Sht}',b}(\mathcal{Z}^{\widehat{\mu}}),\mathrm{AL}^*_{\mathrm{Sht}',b}(f*\mathcal{Z}^{\widehat{\mu}'})\rangle\\&=\langle\mathrm{AL}^*_{\mathrm{Sht}',b}(\mathcal{Z}^{\widehat{\mu}}),f*\mathrm{AL}^*_{\mathrm{Sht}',b}(\mathcal{Z}^{\widehat{\mu}'})\rangle.\end{aligned}$$

By Lemma 4.10, we have

$$\mathrm{AL}^*_{\mathrm{Sht}',b}(\mathcal{Z}^{\widehat{\mu}})=\mathcal{Z}^{\mu},\quad \mathrm{AL}^*_{\mathrm{Sht}',b}(\mathcal{Z}^{\widehat{\mu}'})=\mathcal{Z}^{\mu'}.$$

Therefore, we get

$$\langle\mathcal{Z}^{\widehat{\mu}},f*\mathcal{Z}^{\widehat{\mu}'}\rangle=\langle\mathcal{Z}^{\mu},f*\mathcal{Z}^{\mu'}\rangle;$$

i.e., $\mathbb{I}^{\mu,\mu'}(f)=\mathbb{I}^{\widehat{\mu},\widehat{\mu}'}(f)$ for all $f\in\mathscr{H}_G^{\Sigma}$. □

We will see later (in Theorem 5.6) that in fact $\mathbb{I}^{\mu,\mu'}$ only depends on $\Sigma_-(\mu,\mu')$ and the *cardinality* of $\Delta(\mu,\mu')$.

4.3.5. *Heegner–Drinfeld cycles over* $\overline{k}$. Fix a $\overline{k}$-point $\xi\in\mathfrak{S}'_\infty(\overline{k})$. Concretely this means a collection of field embeddings

$$\xi=(\xi_{x'})_{x'\in\Sigma'_\infty},\quad \xi_{x'}:k(x')\hookrightarrow\overline{k}.$$

Then ξ also determines a $\overline{k}$-point of $\mathfrak{S}_\infty$ by the projection $\mathfrak{S}'_\infty\to\mathfrak{S}_\infty$, which we still denote by ξ. We denote

$$\begin{aligned}\mathrm{Sht}^r_G(\Sigma;\xi)&:=\mathrm{Sht}^r_G(\Sigma;\Sigma_\infty)\times_{\mathfrak{S}_\infty}\xi,\\\mathrm{Sht}'^r_G(\Sigma;\xi)&:=\mathrm{Sht}'^r_G(\Sigma;\Sigma_\infty)\times_{\mathfrak{S}'_\infty}\xi\cong\mathrm{Sht}^r_G(\Sigma;\xi)\times_{X^r}X'^r,\\\mathrm{Sht}^{\underline{\mu}}_T(\mu_\infty\cdot\xi)&:=\mathrm{Sht}^{\underline{\mu}}_T(\mu_\infty\cdot\Sigma'_\infty)\times_{\mathfrak{S}'_\infty}\xi.\end{aligned}$$

Then we have maps

$$\begin{array}{ccc} & \mathrm{Sht}^{\underline{\mu}}_T(\mu_\infty\cdot\xi) & \\ {\scriptstyle\theta'^{\mu}_{\xi}}\swarrow & & \searrow{\scriptstyle\theta^{\mu}_{\xi}} \\ \mathrm{Sht}'^r_G(\Sigma;\xi) & \longrightarrow & \mathrm{Sht}^r_G(\Sigma;\xi).\end{array}$$

Definition 4.12. The *Heegner–Drinfeld cycle of type* $\mu=(\underline{\mu},\mu_f,\mu_\infty)\in\mathfrak{T}_{r,\Sigma}$ *over* ξ is the class

$$\mathcal{Z}^{\mu}(\xi):=\theta'^{\mu}_{\xi,*}[\mathrm{Sht}^{\underline{\mu}}_T(\mu_\infty\cdot\xi)]\in\mathrm{Ch}_{c,r}(\mathrm{Sht}'^r_G(\Sigma;\xi))_{\mathbb{Q}}.$$

By definition, the pullback of $\mathcal{Z}^\mu$ to $\mathrm{Sht}'^r_G(\Sigma;\Sigma_\infty)\otimes\overline{k}$ is the disjoint union of $\mathcal{Z}^\mu(\xi)$ for various $\xi\in\mathfrak{S}'_\infty(\overline{k})$.

COROLLARY 4.13 (of Lemma 4.10). *For $\mu=(\underline{\mu},\mu_f,\mu_\infty)\in\mathfrak{T}_{r,\Sigma}$ and $a\in\mathfrak{A}_{r,\Sigma}$, we have*

$$\mathcal{Z}^\mu(\xi)=\mathrm{AL}^*_{\mathrm{Sht}',a}(\mathcal{Z}^{a\cdot\mu}(\xi)).$$

LEMMA 4.14. *For any $\xi\in\mathfrak{S}'_\infty(\overline{k})$, any $\mu,\mu'\in\mathfrak{T}_{r,\Sigma}$ and any $f\in\mathscr{H}^\Sigma_G$, we have an identity*

$$\mathbb{I}^{\mu,\mu'}(f)=\langle\mathcal{Z}^\mu(\xi),f*\mathcal{Z}^{\mu'}(\xi)\rangle_{\mathrm{Sht}'^r_G(\Sigma;\xi)}. \tag{4.7}$$

In particular, by Corollary 4.11, the right side depends only on the sets $\Delta(\mu,\mu')$ and $\Sigma_-(\mu,\mu')$.

Proof. Since $\mathrm{Sht}'^r_G(\Sigma;\Sigma_\infty)\otimes\overline{k}$ is the disjoint union of $\mathrm{Sht}'^r_G(\Sigma;\xi)$ for a total of $\prod_{x'\in\Sigma'_\infty}d_{x'}$ different choices of ξ, it suffices to show that the right side of (4.7) is independent of the choice of ξ. To compare a general ξ' to ξ, we may reduce to the case where $\xi'\in\mathfrak{S}'_\infty(\overline{k})$ is obtained by changing $\xi_{x'}$ to $\mathrm{Fr}(\xi_{x'})$ for a unique $x'\in\Sigma'_\infty$, and keeping the other coordinates.

Consider the isomorphism

$$\jmath_{x'}:\mathrm{Sht}^{\underline{\mu}}_T(\mu_\infty\cdot\Sigma'_\infty)\xrightarrow{\sim}\mathrm{Sht}^{\underline{\mu}}_T(\mu_\infty\cdot\Sigma'_\infty)$$

sending

$$(\mathcal{L}_i;x'_i;x'^{(1)},\{y'^{(1)}\}_{y'\in\Sigma'_\infty,y'\neq x'})$$

to

$$(\mathcal{L}_i(-\mu_x x'^{(1)});x'_i;x'^{(2)},\{y'^{(1)}\}_{y'\in\Sigma'_\infty,y'\neq x'}).$$

Direct calculation shows that the following diagram is commutative:

$$\begin{array}{ccc}\mathrm{Sht}^{\underline{\mu}}_T(\mu_\infty\cdot\Sigma'_\infty) & \xrightarrow{\jmath_{x'}} & \mathrm{Sht}^{\underline{\mu}}_T(\mu_\infty\cdot\Sigma'_\infty)\\ \downarrow{\scriptstyle\theta'^\mu} & & \downarrow{\scriptstyle\theta'^\mu}\\ \mathrm{Sht}'^r_G(\Sigma;\Sigma_\infty) & \xrightarrow{\mathrm{AL}^{(1)}_{x'}} & \mathrm{Sht}'^r_G(\Sigma;\Sigma_\infty),\end{array} \tag{4.8}$$

where $\mathrm{AL}^{(1)}_{x'}$ sends

$$(\mathcal{E}^\dagger_i;x'_i;x'^{(1)},\{y'^{(1)}\}_{y'\in\Sigma'_\infty,y'\neq x'})$$

to

$$(\mathcal{E}^\dagger_i(-\frac{1}{2}x^{(1)});x'_i;x'^{(2)},\{y'^{(1)}\}_{y'\in\Sigma'_\infty,y'\neq x'}).$$

(Here $x^{(1)}$ is the image of $x'^{(1)}$.) The diagram (4.8) implies that

$$(\mathrm{AL}^{(1)}_{x'})^*\mathcal{Z}^\mu(\xi')=\mathcal{Z}^\mu(\xi).$$

Therefore, using that $\mathrm{AL}_{x'}^{(1)}$ commutes with the $\mathscr{H}_G^{\Sigma}$-action, we have

$$\begin{aligned}\langle \mathcal{Z}^{\mu}(\xi), f * \mathcal{Z}^{\mu'}(\xi)\rangle_{\mathrm{Sht}'^r_G(\Sigma;\xi)} &= \langle (\mathrm{AL}_{x'}^{(1)})^*(\mathcal{Z}^{\mu}(\xi')), f * (\mathrm{AL}_{x'}^{(1)})^* \mathcal{Z}^{\mu'}(\xi')\rangle_{\mathrm{Sht}'^r_G(\Sigma;\xi)} \\ &= \langle (\mathrm{AL}_{x'}^{(1)})^*(\mathcal{Z}^{\mu}(\xi')), (\mathrm{AL}_{x'}^{(1)})^*(f * \mathcal{Z}^{\mu'}(\xi'))\rangle_{\mathrm{Sht}'^r_G(\Sigma;\xi)} \\ &= \langle \mathcal{Z}^{\mu}(\xi'), f * \mathcal{Z}^{\mu'}(\xi')\rangle_{\mathrm{Sht}'^r_G(\Sigma;\xi')}. \end{aligned}$$

□

5. The moduli stack $\mathcal{M}_d$ and intersection numbers

The goal of this subsection is to give a Lefschetz-type formula for the intersection number $\mathbb{I}_r^{\mu,\mu'}(h_D)$; see Theorem 5.6. This is parallel to [10, §6] in the unramified case.

Recall that Σ' and R' are the preimages of Σ and R under ν. We introduce the notation

$$U = X - \Sigma - R,$$
$$U' = X' - \Sigma' - R'.$$

Our construction below will rely on variants of the Picard stack with an extra choice of a square root along the divisor R, which naturally appears in the geometric class field theory of X with ramification along R. We refer to Appendix A for the definitions and properties of such variants of the Picard stack.

5.1. *Definition of $\mathcal{M}_d$ and statement of the formula.* Let d be an integer. We shall define an analog of the moduli stacks $\mathcal{M}_d$ and $\mathcal{A}_d$ in [10, §6.1] for the possibly ramified double cover $\nu : X' \to X$.

5.1.1. *The stack $\mathcal{M}_d$.* For any divisor D of X disjoint from R, $\mathcal{O}_X(D)$ has a canonical lift $\mathcal{O}_X(D)^{\natural} = (\mathcal{O}_X(D), \mathcal{O}_R, 1) \in \mathrm{Pic}_X^{\sqrt{R}}(k)$ and a canonical lift $\dot{\mathcal{O}}_X(D) = (\mathcal{O}_X(D), \mathcal{O}_R, 1, 1) \in \mathrm{Pic}_X^{\sqrt{R};\sqrt{R}}(k)$.

Suppose we are given a decomposition

$$\Sigma = \Sigma_+ \sqcup \Sigma_-.$$

Let

$$\rho = \deg R = \deg R'; \quad N = \deg \Sigma; \quad N_{\pm} = \deg \Sigma_{\pm}.$$

Definition 5.1. Let $\mathcal{M}_d = \mathcal{M}_d(\Sigma_{\pm})$ be the moduli stack whose S-points consist of tuples $(\mathcal{I}, \mathcal{J}, \alpha, \beta, \jmath)$, where

- $\mathcal{I}$ is a line bundle on $X' \times S$ with fiber-wise degree $d + \rho - N_-$, and α is a section of $\mathcal{I}$.
- $\mathcal{J}$ is a line bundle on $X' \times S$ with fiber-wise degree $d + \rho - N_+$, and β is a section of $\mathcal{J}$.

- $\jmath$ is an isomorphism $\mathrm{Nm}^{\sqrt{R}}_{X'/X}(\mathcal{I})\otimes\mathcal{O}_X(\Sigma_-)^\natural\xrightarrow{\sim}\mathrm{Nm}^{\sqrt{R}}_{X'/X}(\mathcal{J})\otimes\mathcal{O}_X(\Sigma_+)^\natural$, as S-points of $\mathrm{Pic}^{\sqrt{R},d+\rho}_X$. Concretely, $\jmath$ is a collection of isomorphisms

$$\begin{aligned}\jmath_{\mathrm{Nm}}:\mathrm{Nm}_{X'/X}(\mathcal{I})\otimes\mathcal{O}_X(\Sigma_-)&\xrightarrow{\sim}\mathrm{Nm}_{X'/X}(\mathcal{J})\otimes\mathcal{O}_X(\Sigma_+),\\ \jmath_x:\mathcal{I}|_{x'\times S}&\xrightarrow{\sim}\mathcal{J}|_{x'\times S}\quad\forall x\in R\end{aligned}\tag{5.1}$$

  such that the following diagram is commutative for all $x\in R$:

$$\begin{array}{ccc}\mathcal{I}^{\otimes 2}|_{x'\times S} & \xrightarrow[\sim]{\jmath_x^{\otimes 2}} & \mathcal{J}^{\otimes 2}|_{x'\times S}\\ \wr\downarrow & & \wr\downarrow\\ \mathrm{Nm}_{X'/X}(\mathcal{I})|_{x\times S} & \xrightarrow[\sim]{\jmath_{\mathrm{Nm}}|_{x\times S}} & \mathrm{Nm}_{X'/X}(\mathcal{J})|_{x\times S}.\end{array}\tag{5.2}$$

  Here the vertical maps are the tautological isomorphisms.

These data are required to satisfy the following conditions:

(1) $\alpha|_{\nu^{-1}(\Sigma_+)\times S}$ is nowhere vanishing.
(2) $\beta|_{\nu^{-1}(\Sigma_-)\times S}$ is nowhere vanishing.
(3) For each $x\in R$, we have

$$\jmath_x(\alpha|_{x'\times S})=\beta|_{x'\times S}.$$

  Moreover, $\mathrm{Nm}(\alpha)-\mathrm{Nm}(\beta)$ vanishes only to the first order along $R\times S$.
(4) This condition is non-void only when $\Sigma=\varnothing$ and $R=\varnothing$; for each geometric point $s\in S$, the restriction $(\mathrm{Nm}(\alpha)-\mathrm{Nm}(\beta))|_{X\times s}$ is not identically zero.

From the definition we have an open embedding

$$\iota_d:\mathcal{M}_d\hookrightarrow\widehat{X}'_{d+\rho-N_-}\times_{\mathrm{Pic}_X^{\sqrt{R};\sqrt{R},d+\rho}}\widehat{X}'_{d+\rho-N_+},\tag{5.3}$$

where the fiber product is taken over

$$\nu_\alpha:\widehat{X}'_{d+\rho-N_-}\xrightarrow{\widehat{\nu}^{\sqrt{R}}}\widehat{X}^{\sqrt{R}}_{d+\rho-N_-}\xrightarrow{\widehat{\mathrm{AJ}}^{\sqrt{R};\sqrt{R}}_{d+\rho-N_-}}\mathrm{Pic}_X^{\sqrt{R};\sqrt{R},d+\rho-N_-}\xrightarrow{\otimes\dot{\mathcal{O}}_X(\Sigma_-)}\mathrm{Pic}_X^{\sqrt{R};\sqrt{R},d+\rho}$$

and

$$\nu_\beta:\widehat{X}'_{d+\rho-N_+}\xrightarrow{\widehat{\nu}^{\sqrt{R}}}\widehat{X}^{\sqrt{R}}_{d+\rho-N_+}\xrightarrow{\widehat{\mathrm{AJ}}^{\sqrt{R};\sqrt{R}}_{d+\rho-N_+}}\mathrm{Pic}_X^{\sqrt{R};\sqrt{R},d+\rho-N_+}\xrightarrow{\otimes\dot{\mathcal{O}}_X(\Sigma_+)}\mathrm{Pic}_X^{\sqrt{R};\sqrt{R},d+\rho}.$$

Here the Abel-Jacobi maps $\widehat{\mathrm{AJ}}^{\sqrt{R};\sqrt{R}}_{d+\rho-N_\pm}$ are defined in Section A.1.5.

Remark 5.2. When $\Sigma=\varnothing$ and $R=\varnothing$, there is a slight difference between the current definition of $\mathcal{M}_d$ and the one in [10]. In [10], we only require that

$\alpha|_{X'\times s}$ and $\beta|_{X'\times s}$ are not both zero for any geometric point $s \in S$; here we impose a stronger open condition that $\mathrm{Nm}(\alpha) - \mathrm{Nm}(\beta)$ is non-zero on $X \times s$ for any geometric point $s \in S$. Therefore, the current version of $\mathcal{M}_d$ is the one denoted by $\mathcal{M}_d^{\heartsuit}$ in [10]. A similar remark applies to the space $\mathcal{A}_d$ to be defined below.

5.1.2. *The base* $\mathcal{A}_d$.

Definition 5.3. Let $\mathcal{A}_d = \mathcal{A}_d(\Sigma_\pm)$ be the moduli stack whose S-points consist of tuples

$$(\Delta, \Theta_R, \iota, a, b, \vartheta_R),$$

where

- $(\Delta, \Theta_R, \iota) \in \mathrm{Pic}_X^{\sqrt{R}, d+\rho}(S)$ — namely, Δ is a line bundle on $X \times S$ of fiberwise degree $d + \rho$, Θ_R a line bundle over $R \times S$ and ι an isomorphism $\Theta_R^{\otimes 2} \cong \Delta|_{R\times S}$;
- a and b are sections of Δ;
- ϑ_R is a section of Θ_R.

These data are required to satisfy the following conditions:

(1) $a|_{\Sigma_-\times S} = 0$, and $a|_{\Sigma_+\times S}$ is nowhere vanishing.
(2) $b|_{\Sigma_+\times S} = 0$, and $b|_{\Sigma_-\times S}$ is nowhere vanishing.
(3) $a|_{R\times S} = \iota(\vartheta_R^{\otimes 2}) = b|_{R\times S}$. Moreover, $a - b$ vanishes only to the first order along $R \times S$.
(4) This condition is only non-void when $\Sigma = \varnothing$ and $R = \varnothing$; for every geometric point s of S, $(a - b)|_{X\times s} \neq 0$.

The assignment $(\Delta, \Theta_R, \iota, a, b, \vartheta_R) \mapsto (\Delta(-\Sigma_-), \Theta_R, \iota, a, \vartheta_R)$ gives a map

$$\mathcal{A}_d \longrightarrow \widehat{X}_{d+\rho-N_-}^{\sqrt{R}}.$$

Similarly, the assignment $(\Delta, \Theta_R, \iota, a, b, \vartheta_R) \mapsto (\Delta(-\Sigma_+), \Theta_R, \iota, b, \vartheta_R)$ gives a map

$$\mathcal{A}_d \longrightarrow \widehat{X}_{d+\rho-N_+}^{\sqrt{R}}.$$

Combining these maps, we get an open embedding

$$\omega_d : \mathcal{A}_d \hookrightarrow \widehat{X}_{d+\rho-N_-}^{\sqrt{R}} \times_{\mathrm{Pic}_X^{\sqrt{R};\sqrt{R},d+\rho}} \widehat{X}_{d+\rho-N_+}^{\sqrt{R}}, \tag{5.4}$$

where the the fiber product is formed using the Abel-Jacobi maps

$$\nu_a : \widehat{X}_{d+\rho-N_-}^{\sqrt{R}} \xrightarrow{\widehat{\mathrm{AJ}}_{d+\rho-N_-}^{\sqrt{R};\sqrt{R}}} \mathrm{Pic}_X^{\sqrt{R};\sqrt{R},d+\rho-N_-} \xrightarrow{\otimes \dot{\mathcal{O}}_X(\Sigma_-)} \mathrm{Pic}_X^{\sqrt{R};\sqrt{R},d+\rho},$$

$$\nu_b : \widehat{X}_{d+\rho-N_+}^{\sqrt{R}} \xrightarrow{\widehat{\mathrm{AJ}}_{d+\rho-N_+}^{\sqrt{R};\sqrt{R}}} \mathrm{Pic}_X^{\sqrt{R};\sqrt{R},d+\rho-N_+} \xrightarrow{\otimes \dot{\mathcal{O}}_X(\Sigma_+)} \mathrm{Pic}_X^{\sqrt{R};\sqrt{R},d+\rho}.$$

5.1.3. *The base* $\mathcal{A}_d^\flat$. Later we will need to use another base space $\mathcal{A}_d^\flat$.

Definition 5.4. Let $\mathcal{A}_d^\flat = \mathcal{A}_d^\flat(\Sigma_\pm)$ be the moduli stack whose S-points consist of tuples (Δ, a, b) where

- Δ is a line bundle on $X \times S$ of fiber-wise degree $d + \rho$,
- a and b are sections of Δ,

such that the same conditions (1)–(4) hold as in Definition 5.3.

Similar to the case of $\mathcal{A}_d$, we have an open embedding

$$\omega_d^\flat : \mathcal{A}_d^\flat \hookrightarrow \widehat{X}_{d+\rho-N_-} \times_{\mathrm{Pic}_X^{d+\rho}} \widehat{X}_{d+\rho-N_+}. \tag{5.5}$$

By [10, §3.2.3], $\mathcal{A}_d^\flat$ is a scheme over k. Later it will be technically more convenient to apply the Lefschetz trace formula to the base scheme $\mathcal{A}_d^\flat$ instead of the stack $\mathcal{A}_d$.

There is a forgetful map

$$\Omega : \mathcal{A}_d \longrightarrow \mathcal{A}_d^\flat$$

that corresponds to the forgetful maps $\widehat{X}^{\sqrt{R}}_{d+\rho-N_\pm} \to \widehat{X}_{d+\rho-N_\pm}$ under the embeddings (5.4) and (5.5).

We have a morphism

$$\delta : \mathcal{A}_d^\flat \longrightarrow U_d$$

sending (Δ, a, b) to the divisor of $a - b$ as a non-zero section of $\Delta(-R)$, the latter having degree d. The conditions (1), (2) and (3) in Definition 5.3 imply that the divisor of $a - b$ does not meet Σ or R.

For D to be an effective divisor on U of degree d, let

$$\mathcal{A}_D^\flat = \delta^{-1}(D) \subset \mathcal{A}_d^\flat. \tag{5.6}$$

5.1.4. *Geometric properties of* $\mathcal{M}_d$. We have a morphism

$$f_d : \mathcal{M}_d \longrightarrow \mathcal{A}_d$$

defined by applying $\widehat{\nu}^{\sqrt{R}}$ to both $\widehat{X}'_{d+\rho-N_-}$ and $\widehat{X}'_{d+\rho-N_+}$. In other words, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{M}_d & \hookrightarrow & \widehat{X}'_{d+\rho-N_-} \times_{\nu_\alpha, \mathrm{Pic}_X^{\sqrt{R};\sqrt{R},d+\rho}, \nu_\beta} \widehat{X}'_{d+\rho-N_+} \\ \downarrow f_d & & \downarrow \widehat{\nu}^{\sqrt{R}}_{d+\rho-N_-} \times \widehat{\nu}^{\sqrt{R}}_{d+\rho-N_+} \\ \mathcal{A}_d & \hookrightarrow & \widehat{X}^{\sqrt{R}}_{d+\rho-N_-} \times_{\nu_a, \mathrm{Pic}_X^{\sqrt{R};\sqrt{R},d+\rho}, \nu_b} \widehat{X}^{\sqrt{R}}_{d+\rho-N_+}. \end{array} \tag{5.7}$$

We denote by $f_d^\flat$ the composition

$$f_d^\flat : \mathcal{M}_d \xrightarrow{f_d} \mathcal{A}_d \xrightarrow{\Omega} \mathcal{A}_d^\flat.$$

The following is a generalization of [10, Prop 6.1] to the ramified situation.

Proposition 5.5.

(1) *When* $d \geq 2g' - 1 + N = 4g - 3 + \rho + N$, *the stack* $\mathcal{M}_d$ *is a smooth DM stack pure of dimension* $m = 2d + \rho - N - g + 1$.
(2) *The diagram* (5.7) *is Cartesian.*
(3) *The morphisms* f_d *and* $f_d^\flat$ *are proper.*
(4) *When* $d \geq 3g - 2 + N$, *the morphism* f_d *is small; it is generically finite, and for any* $n > 0$, $\{a \in \mathcal{A}_d | \dim f_d^{-1}(a) \geq n\}$ *has codimension* $\geq 2n + 1$ *in* $\mathcal{A}_d$.
(5) *The stack* $\mathcal{M}_d$ *admits a finite flat presentation in the sense of* [10, Def. A.1].

Proof. (1) To show that $\mathcal{M}_d$ is smooth DM, it suffices to show that both of following stacks are smooth DM:

$$\widehat{X}'_{d+\rho-N_-} \times_{\nu_\alpha, \mathrm{Pic}_X^{\sqrt{R};\sqrt{R},d+\rho}, \nu_\beta} X'_{d+\rho-N_+}, \tag{5.8}$$

$$X'_{d+\rho-N_-} \times_{\nu_\alpha, \mathrm{Pic}_X^{\sqrt{R};\sqrt{R},d+\rho}, \nu_\beta} \widehat{X}'_{d+\rho-N_+}. \tag{5.9}$$

Let $Q_{X'}^{R'}$ be the moduli stack of pairs $(\mathcal{L}', \vartheta_{R'})$, where $\mathcal{L}' \in \mathrm{Pic}_{X'}$ and $\vartheta_{R'}$ is a section of $\mathcal{L}'|_{R'}$. Then $Q_{X'}^{R'} \cong \mathrm{Pic}_{X'} \times_{\mathrm{Pic}_X^{\sqrt{R}}} \mathrm{Pic}_X^{\sqrt{R};\sqrt{R}}$. In particular, the norm map $Q_{X'}^{R'} \to \mathrm{Pic}_X^{\sqrt{R};\sqrt{R}}$ is smooth and relative DM.

For any geometric point s and line bundle $\mathcal{L}$ on $X' \times s$ of degree $n \geq 2g' + \rho - 1$, the restriction map $\mathrm{H}^0(X' \times s, \mathcal{L}) \to \mathrm{H}^0(R' \times s, \mathcal{L}|_{R' \times s})$ is surjective with kernel dimension $n - g' + 1 - \rho$. This implies $\widehat{X}'_n \to Q_{X'}^{R'}$ is a vector bundle of rank $n - g' + 1 - \rho$, whenever $n \geq 2g' - 1 + \rho$, in which case $\widehat{X}'_n$ itself is also smooth.

If $d \geq 2g' - 1 + N \geq 2g' - 1 + N_+$, then $d + \rho - N_+ \geq 2g' - 1 + \rho$, the map $\nu_\beta : \widehat{X}'_{d+\rho-N_+} \to Q_{X'}^{R'} \to \mathrm{Pic}_X^{\sqrt{R},\sqrt{R}}$ is then smooth and relative DM by the above discussion, and therefore the fiber product (5.9) is smooth over its first factor $X'_{d+\rho-N_-}$. Since $X'_{d+\rho-N_-}$ is a scheme smooth over k, the fiber product (5.9) is smooth DM over k. The argument for (5.8) is the same.

For the dimension, we have

$$\begin{aligned}\dim \mathcal{M}_d &= \dim \widehat{X}'_{d+\rho-N_-} + \dim \widehat{X}'_{d+\rho-N_+} - \dim \mathrm{Pic}_X^{\sqrt{R},\sqrt{R}} \\ &= (d + \rho - N_-) + (d + \rho - N_+) - (g - 1 + \rho) \\ &= 2d + \rho - N - g + 1.\end{aligned}$$

Condition (2) follows directly by comparing the four conditions in Definition 5.1 and in Definition 5.3.

(3) Since Ω is proper, it suffices to show that f_d is proper. By (2), it suffices to show that $\widehat{\nu}_n^{\sqrt{R}} : \widehat{X}'_n \to \widehat{X}_n^{\sqrt{R}}$ is proper for any $n \geq 0$. We consider

the factorization of the usual norm map

$$\widehat{\nu}_n : \widehat{X}'_n \xrightarrow{\widehat{\nu}_n^{\sqrt{R}}} \widehat{X}_n^{\sqrt{R}} \xrightarrow{\widehat{\omega}_n^{\sqrt{R}}} \widehat{X}_n.$$

The same argument of [10, Prop. 6.1(4)] shows that $\widehat{\nu}_n$ is proper. On the other hand, $\widehat{\omega}_n^{\sqrt{R}}$ is separated because it is obtained by base change from the separated map $[2] : [\mathrm{Res}_k^R \mathbb{A}^1 / \mathrm{Res}_k^R \mathbb{G}_m] \to [\mathrm{Res}_k^R \mathbb{A}^1 / \mathrm{Res}_k^R \mathbb{G}_m]$; see the diagram (A.1). Therefore, $\widehat{\nu}_n^{\sqrt{R}}$ is proper.

(4) Over $\mathcal{A}_d^{\diamondsuit} := (X^{\sqrt{R}}_{d+\rho-N_-} \times_{\mathrm{Pic}_X^{\sqrt{R};\sqrt{R},d+\rho}} X^{\sqrt{R}}_{d+\rho-N_+}) \cap \mathcal{A}_d$, f_d is finite. The complement $\mathcal{A}_d - \mathcal{A}_d^{\diamondsuit}$ is the disjoint union of $\mathcal{A}_d^{a=0}$ and $\mathcal{A}_d^{b=0}$ corresponding to the locus $a = 0$ or $b = 0$. Note that $\mathcal{A}_d^{a=0} = \varnothing$ unless $\Sigma_+ = \varnothing$ and $\mathcal{A}_d^{b=0} = \varnothing$ unless $\Sigma_- = \varnothing$.

We first analyze the fibers over $\mathcal{A}_d^{b=0}$ when $\Sigma_- = \varnothing$. The coarse moduli space of $\mathcal{A}_d^{b=0}$ is U_d (by taking $\mathrm{div}(a) - R$; note that $\Sigma = \Sigma_+$). Hence $\dim \mathcal{A}_d^{b=0} = d$, and $\mathrm{codim}_{\mathcal{A}_d}(\mathcal{A}_d^{b=0}) = d - g + 1 + \rho - N$. The restriction of f_d to $\mathcal{A}_d^{b=0}$ is, up to passing to coarse moduli spaces, given by the norm map with respect to the double cover $U' \to U$,

$$U'_d \times_{\mathrm{Pic}_X^{\sqrt{R},d+\rho}} \mathrm{Pic}_{X'}^{d+\rho-N_+} \longrightarrow U_d.$$

From this we see that the fiber dimension of f_d over $\mathcal{A}_d^{b=0}$ is the same as that of the norm map $\mathrm{Pic}_{X'} \to \mathrm{Pic}_X^{\sqrt{R}}$, which is $g' - g$.

Similar argument shows that when $\Sigma_+ = \varnothing$, $\mathrm{codim}_{\mathcal{A}_d}(\mathcal{A}_d^{a=0}) = d - g + 1 + \rho - N$ and the fiber dimension of f_d over $\mathcal{A}_d^{a=0}$ is still $g' - g$. In either case, since $d \geq 3g - 2 + N$, we have

$$d - g + 1 + \rho - N \geq 2g - 1 + \rho = 2(g' - g) + 1,$$

which checks the smallness of f_d.

(5) We need to show that there is a finite flat map $Y \to \mathcal{M}_d$ from an algebraic space Y of finite type over k. As in [10, proof of Prop. 6.1(1)], by introducing a rigidification at some closed point $y \in U'$, we may define a schematic map

$$\mathcal{M}_d \longrightarrow J_{X'}^{d+\rho} \times \mathrm{Prym}_{X'/X},$$

where $J_{X'}^{d+\rho}$ is the Picard scheme of X' of degree $d + \rho$, and $\mathrm{Prym}_{X'/X} := \ker(\mathrm{Nm}^{\sqrt{R}}_{X'/X} : \mathrm{Pic}^0_{X'} \to \mathrm{Pic}_X^{\sqrt{R},0})$. Since $J_{X'}^{d+\rho}$ is a scheme and $\mathrm{Prym}_{X'/X}$ is a global finite quotient of an abelian variety, $J_{X'}^{d+\rho} \times \mathrm{Prym}_{X'/X}$ admits a finite flat presentation; therefore, the same is true for $\mathcal{M}_d$. □

5.1.5. *The incidence correspondences.* To state the formula for $\mathbb{I}^{\mu,\mu'}(h_D)$, we need to introduce two self-correspondences of $\mathcal{M}_d$. We define $\mathcal{H}_+$ to be the substack of $\mathcal{M}_d \times X'$ consisting of those $(\mathcal{I}, \mathcal{J}, \alpha, \beta, \jmath, x')$ such that β vanishes on $\Gamma_{x'}$. We have the natural projection

$$\overleftarrow{\gamma}_+ : \mathcal{H}_+ \longrightarrow \mathcal{M}_d$$

recording $(\mathcal{I}, \mathcal{J}, \alpha, \beta, \jmath)$. We also have another projection

$$\overrightarrow{\gamma}_+ : \mathcal{H}_+ \longrightarrow \mathcal{M}_d$$

sending $(\mathcal{I}, \mathcal{J}, \alpha, \beta, \jmath, x')$ to $(\mathcal{I}, \mathcal{J}(\Gamma_{\sigma x'} - \Gamma_{x'}), \alpha, \beta, \jmath)$. This makes sense since twisting by $\mathcal{O}_{X'}(\Gamma_{\sigma x'} - \Gamma_{x'})$ does not affect the image under $\mathrm{Nm}^{\sqrt{R}}_{X'/X}$ and the fact that β can be viewed as a section of $\mathcal{J}(\Gamma_{\sigma x'} - \Gamma_{x'})$ since it vanishes along $\Gamma_{x'}$. Via $(\overleftarrow{\gamma}_+, \overrightarrow{\gamma}_+)$, we view $\mathcal{H}_+$ as a self-correspondence of $\mathcal{M}_d$. We have a commutative diagram

(5.10)
$$\begin{array}{ccccc} & & \mathcal{H}_+ & & \\ & \overset{\overleftarrow{\gamma}_+}{\swarrow} & & \overset{\overrightarrow{\gamma}_+}{\searrow} & \\ \mathcal{M}_d & & & & \mathcal{M}_d \\ & \underset{f_d}{\searrow} & & \underset{f_d}{\swarrow} & \\ & & \mathcal{A}_d. & & \end{array}$$

Similarly, we define $\mathcal{H}_-$ to be the substack of $\mathcal{M}_d \times X'$ consisting of those $(\mathcal{I}, \mathcal{J}, \alpha, \beta, \jmath, x')$ such that α vanishes on $\Gamma_{x'}$. We view $\mathcal{H}_-$ as a self-correspondence of $\mathcal{M}_d$ over $\mathcal{A}_d$,

(5.11)
$$\begin{array}{ccccc} & & \mathcal{H}_- & & \\ & \overset{\overleftarrow{\gamma}_-}{\swarrow} & & \overset{\overrightarrow{\gamma}_-}{\searrow} & \\ \mathcal{M}_d & & & & \mathcal{M}_d \\ & \underset{f_d}{\searrow} & & \underset{f_d}{\swarrow} & \\ & & \mathcal{A}_d, & & \end{array}$$

where

$$\overleftarrow{\gamma}_-(\mathcal{I}, \mathcal{J}, \alpha, \beta, \jmath, x') = (\mathcal{I}, \mathcal{J}, \alpha, \beta, \jmath)$$

and

$$\overrightarrow{\gamma}_-(\mathcal{I}, \mathcal{J}, \alpha, \beta, \jmath, x') = (\mathcal{I}(\Gamma_{\sigma x'} - \Gamma_{x'}), \mathcal{J}, \alpha, \beta, \jmath).$$

Let $\mathcal{A}^{\diamondsuit}_d = (X^{\sqrt{R}}_{d+\rho-N_-} \times_{\mathrm{Pic}^{\sqrt{R};\sqrt{R}}_X} X^{\sqrt{R}}_{d+\rho-N_+}) \cap \mathcal{A}_d$ be the locus where $a, b \neq 0$.[2] Let $\mathcal{M}^{\diamondsuit}_d \subset \mathcal{M}_d$ be the preimage of $\mathcal{A}^{\diamondsuit}_d$. Let $\mathcal{H}^{\diamondsuit}_+$ and $\mathcal{H}^{\diamondsuit}_-$ be the restriction of $\mathcal{H}_+$ and $\mathcal{H}_-$ to $\mathcal{A}^{\diamondsuit}_d$.

[2] The definition of $\mathcal{A}^{\diamondsuit}_d$ is different from the one in [10].

Consider the incidence correspondence

$$\begin{array}{ccc} & I'_{d+\rho-N_+} & \\ \overleftarrow{i} \swarrow & & \searrow \overrightarrow{i} \\ X'_{d+\rho-N_+} & & X'_{d+\rho-N_+}. \end{array} \tag{5.12}$$

Here $I'_{d+\rho-N_+} = \{(D,x') \in X'_{d+\rho-N_+} \times X' | x' \in D\}$, $\overleftarrow{i}(D,x') = D$ and $\overrightarrow{i}(D,x') = D + \sigma(x') - x'$.

By definition, over $\mathcal{M}_d^{\diamondsuit}$, $\mathcal{H}_+^{\diamondsuit}$ is obtained from the incidence correspondence $I'_{d+\rho-N_+}$ by applying $X'_{d+\rho-N_-} \times_{\mathrm{Pic}_X^{\sqrt{R};\sqrt{R}}} (-)$ and then restricting to $\mathcal{A}_d^{\diamondsuit}$. Similarly, $\mathcal{H}_-^{\diamondsuit}$ is obtained from the incidence correspondence $I'_{d+\rho-N_-}$ by applying $(-) \times_{\mathrm{Pic}_X^{\sqrt{R};\sqrt{R}}} X'_{d+\rho-N_+}$ and then restricting to $\mathcal{A}_d^{\diamondsuit}$ (cf. [10, Lemma 6.3]).

From this description, we see that $\dim \mathcal{H}_\pm^{\diamondsuit} = \dim \mathcal{M}_d^{\diamondsuit} = 2d+\rho-N-g+1$. Let $\overline{\mathcal{H}}_\pm^{\diamondsuit}$ be the closure of $\mathcal{H}_\pm^{\diamondsuit}$, and let $[\overline{\mathcal{H}}_\pm^{\diamondsuit}]$ denote its cycle class as an element in $\mathrm{H}^{\mathrm{BM}}_{2(2d+\rho-N-g+1)}(\mathcal{H}_\pm)$. Then $[\overline{\mathcal{H}}_\pm^{\diamondsuit}]$ is a cohomological correspondence between the constant sheaf on $\mathcal{M}_d$ and itself, which then induces an endomorphism of $\mathbf{R}f_{d,!}\mathbb{Q}_\ell$:

$$f_{d,!}[\overline{\mathcal{H}}_\pm^{\diamondsuit}] : \mathbf{R}f_{d,!}\mathbb{Q}_\ell \longrightarrow \mathbf{R}f_{d,!}\mathbb{Q}_\ell.$$

Taking direct image under $\Omega : \mathcal{A}_d \to \mathcal{A}_d^\flat$, we get an endomorphism

$$f^\flat_{d,!}[\overline{\mathcal{H}}_\pm^{\diamondsuit}] : \mathbf{R}f^\flat_{d,!}\mathbb{Q}_\ell \longrightarrow \mathbf{R}f^\flat_{d,!}\mathbb{Q}_\ell.$$

For $a \in \mathcal{A}_d^\flat(k)$, let $(f^\flat_{d,!}[\overline{\mathcal{H}}_\pm^{\diamondsuit}])_a$ be the action of $f^\flat_{d,!}[\overline{\mathcal{H}}_\pm^{\diamondsuit}]$ on the geometric stalk $(\mathbf{R}f_{d,!}\mathbb{Q}_\ell)_a$.

5.1.6. *The formula.* For the rest of the section, we fix a pair

$$\mu = (\underline{\mu}, \mu_f, \mu_\infty), \mu' = (\underline{\mu}', \mu'_f, \mu'_\infty) \in \mathfrak{T}_{r,\Sigma}.$$

We let

$$\Sigma_+ := \Sigma_+(\mu,\mu'), \quad \Sigma_- := \Sigma_-(\mu,\mu')$$

be defined as in (4.5) and (4.6). Thus $\mathcal{M}_d = \mathcal{M}_d(\Sigma_\pm)$ is defined. We also let

$$r_+ = \{1 \le i \le r | \mu_i = \mu'_i\}; \quad r_- = \{1 \le i \le r | \mu_i \neq \mu'_i\}. \tag{5.13}$$

The following is the main theorem of this section, parallel to [10, Th. 6.5].

THEOREM 5.6. *Suppose D is an effective divisor on U of degree $d \ge \max\{2g'-1+N, 2g\}$. Under the above notation, we have*

$$\mathbb{I}^{\mu,\mu'}(h_D) = \sum_{a \in \mathcal{A}_D^\flat(k)} \mathrm{Tr}\left((f^\flat_{d,!}[\overline{\mathcal{H}}_+^{\diamondsuit}])_a^{r_+} \circ (f^\flat_{d,!}[\overline{\mathcal{H}}_-^{\diamondsuit}])_a^{r_-} \circ \mathrm{Fr}_a, (\mathbf{R}f^\flat_{d,!}\mathbb{Q}_\ell)_a \right), \tag{5.14}$$

where Fr_a is the geometric Frobenius at a.

5.1.7. *Outline of the proof.* The rest of the section is devoted to the proof of Theorem 5.6. The proof consists of three steps

I. Introduce a moduli stack $\mathcal{M}_d(\mu_\Sigma, \mu'_\Sigma)$ and a Hecke correspondence $\mathrm{Hk}^{\mu,\mu'}_{\mathcal{M},d}$ for $\mathcal{M}_d(\mu_\Sigma, \mu'_\Sigma)$.

   This step is done in Section 5.2. We also introduce certain auxiliary spaces that form the "master diagram" (5.18). Later we will apply the octahedron lemma [10, Th. A.10] to this diagram.

II. Relate $\mathcal{M}_d(\mu_\Sigma, \mu'_\Sigma)$ and $\mathcal{M}_d$; relate $\mathrm{Hk}^{\mu,\mu'}_{\mathcal{M},d}$ and a composition of $\mathcal{H}_\pm$.

   This is done in Section 5.3. This step is significantly more complicated than the unramified case treated in [10]. It amounts to showing that $\mathcal{M}_d$ is a descent of $\mathcal{M}_d(\mu_\Sigma, \mu'_\Sigma)$ from $\mathfrak{S}'_\infty$ to $\operatorname{Spec} k$.

III. Show that $\mathbb{I}^{\mu,\mu'}(h_D)$ can be expressed as the intersection number of a cycle class supported on $\mathrm{Hk}^{\mu,\mu'}_{\mathcal{M},d}$ and the graph of Frobenius of $\mathcal{M}_d(\mu_\Sigma, \mu'_\Sigma)$, and rewrite this intersection number into a trace as in the right-hand side of (5.14).

   This step is done in Section 5.4. The argument is quite similar to the proof of [10, Th. 6.6], together with a standard application of a version of the Lefschetz trace formula reviewed in [10, Prop A.12].

5.2. *Auxiliary moduli stacks.*

5.2.1. *The stack $H_d(\Sigma)$.*

Definition 5.7.

(1) Let $\widetilde{H}_d(\Sigma)$ be the moduli stack whose S-points consist of triples $(\mathcal{E}^\dagger, \mathcal{E}'^\dagger, \varphi)$, where
   - $\mathcal{E}^\dagger = (\mathcal{E}; \{\mathcal{E}(-\frac{1}{2}x)\})$ and $\mathcal{E}'^\dagger = (\mathcal{E}'; \{\mathcal{E}'(-\frac{1}{2}x)\})$ are S-points of $\mathrm{Bun}_2(\Sigma)$ such that $\deg(\mathcal{E}'|_{X\times s}) - \deg(\mathcal{E}|_{X\times s}) = d$ for all geometric points $s \in S$;
   - $\varphi : \mathcal{E} \to \mathcal{E}'$ is a map of coherent sheaves that is injective when restricted to $X \times s$ for all geometric points $s \in S$ and maps $\mathcal{E}(-\frac{1}{2}x)$ to $\mathcal{E}'(-\frac{1}{2}x)$ for all $x \in \Sigma$;
   - the restriction $\varphi|_{(\Sigma\sqcup R)\times S}$ is an isomorphism.

(2) We define
$$H_d(\Sigma) = \widetilde{H}_d(\Sigma)/\operatorname{Pic}_X,$$
where Pic_X acts by tensoring on $\mathcal{E}^\dagger$ and $\mathcal{E}'^\dagger$ simultaneously.

We have a map
$$\overleftrightarrow{p_H} = (\overleftarrow{p_H}, \overrightarrow{p_H}) : H_d(\Sigma) \longrightarrow \mathrm{Bun}_G(\Sigma)^2$$

recording $\mathcal{E}^\dagger$ and $\mathcal{E}'^\dagger$. We also have a map

$$s : H_d(\Sigma) \longrightarrow U_d \tag{5.15}$$

recording the vanishing divisor of $\det(\varphi)$ as a section of $\det(\mathcal{E})^{-1} \otimes \det(\mathcal{E}')$.

We also have an Atkin–Lehner operator

$$\mathrm{AL}_{H,\infty} : H_d(\Sigma) \times \mathfrak{S}_\infty \longrightarrow H_d(\Sigma) \tag{5.16}$$

defined by applying $\mathrm{AL}_{G,\infty}$ (see (3.20)) to both $\mathcal{E}$ and $\mathcal{E}'$, and keeping φ.

5.2.2. *The Hecke correspondence for $H_d(\Sigma)$.*

Definition 5.8. Let $\underline{\mu} \in \{\pm 1\}^r$.

(1) Let $\widetilde{\mathrm{Hk}}^{\underline{\mu}}_{H,d}(\Sigma)$ [3] be the moduli stack of $(\{\mathcal{E}_i^\dagger\}_{0\le i\le r}, \{\mathcal{E}_i'^\dagger\}_{0\le i\le r}, \{x_i\}_{1\le i\le r})$ together with a diagram

$$\begin{array}{ccccccc} \mathcal{E}_0 & \xrightarrow{f_1} & \mathcal{E}_1 & \xrightarrow{f_1} & \cdots & \xrightarrow{f_r} & \mathcal{E}_r \\ \downarrow{\scriptstyle\varphi_0} & & \downarrow{\scriptstyle\varphi_1} & & & & \downarrow{\scriptstyle\varphi_r} \\ \mathcal{E}'_0 & \xrightarrow{f'_1} & \mathcal{E}'_1 & \xrightarrow{f'_2} & \cdots & \xrightarrow{f'_r} & \mathcal{E}'_r, \end{array} \tag{5.17}$$

where
- each $\mathcal{E}_i$ and $\mathcal{E}'_i$ are underlying rank two vector bundles of points $\mathcal{E}_i^\dagger, \mathcal{E}_i'^\dagger$ of $\mathrm{Bun}_2(\Sigma)$;
- the upper and lower rows form objects in $\mathrm{Hk}^{\underline{\mu}}_2(\Sigma)$ with modifications at $\{x_i\}_{1\le i\le r} \in X^r$;
- the vertical maps φ_i are such that $(\mathcal{E}_i^\dagger, \mathcal{E}_i'^\dagger, \varphi_i) \in \widetilde{H}_d(\Sigma)$.

(2) Let

$$\mathrm{Hk}^r_{H,d}(\Sigma) := \widetilde{\mathrm{Hk}}^{\underline{\mu}}_{H,d}(\Sigma)/\operatorname{Pic}_X,$$

where Pic_X acts on $\widetilde{\mathrm{Hk}}^{\underline{\mu}}_{H,d}(\Sigma)$ by simultaneously tensoring on all $\mathcal{E}_i^\dagger$ and $\mathcal{E}_i'^\dagger$.

The notation for $\mathrm{Hk}^r_{H,d}(\Sigma)$ is justified because one can check, as in the case of $\mathrm{Hk}^{\underline{\mu}}_G(\Sigma)$, that $\widetilde{\mathrm{Hk}}^{\underline{\mu}}_{H,d}(\Sigma)/\operatorname{Pic}_X$ is canonically independent of $\underline{\mu}$.

We have projections

$$p_{H,i} : \mathrm{Hk}^r_{H,d}(\Sigma) \longrightarrow H_d(\Sigma), \quad i = 0, \dots, r$$

recording the i-th column of the diagram (5.17). We also have projections recording the upper and lower rows of the diagram (5.17):

$$\overleftrightarrow{q} = (\overleftarrow{q}, \overrightarrow{q}) : \mathrm{Hk}^r_{H,d}(\Sigma) \longrightarrow \mathrm{Hk}^r_G(\Sigma)^2.$$

[3] In [10], the analogue of $\widetilde{\mathrm{Hk}}^{\mu}_{H,d}(\Sigma)$ was denoted by $\widetilde{\mathrm{Hk}}^{\mu}_{G,d}$.

Let

$$\begin{aligned}\mathrm{Hk}'^{r}_{H,d}(\Sigma) &:= \mathrm{Hk}^{r}_{H,d}(\Sigma) \times_{X^r} X'^{r},\\ \mathrm{Hk}'^{r}_{G}(\Sigma) &:= \mathrm{Hk}^{r}_{G}(\Sigma) \times_{X^r} X'^{r}.\end{aligned}$$

The maps $p_{H,i}$ and $\overleftrightarrow{q}$ induce maps

$$\begin{aligned}p'_{H,i} &: \mathrm{Hk}'^{r}_{H,d}(\Sigma) \longrightarrow \mathrm{Hk}^{r}_{H,d}(\Sigma) \xrightarrow{p_{H,i}} H_d(\Sigma), \quad i = 0, \dots, r,\\ \overleftrightarrow{q}' = (\overleftarrow{q}', \overrightarrow{q}') &: \mathrm{Hk}'^{r}_{H,d}(\Sigma) \longrightarrow \mathrm{Hk}'^{r}_{G}(\Sigma)^2.\end{aligned}$$

5.2.3. *The master diagram.* Recall $\mu = (\underline{\mu}, \mu_\Sigma), \mu' = (\underline{\mu}', \mu'_\Sigma) \in \mathfrak{T}_{r,\Sigma}$. We consider the following diagram in which each square is commutative:

(5.18)

$$\begin{array}{ccccc}
(\mathrm{Hk}^{\underline{\mu}}_T \times \mathrm{Hk}^{\underline{\mu}'}_T)_{\mathfrak{S}'_\infty} & \xrightarrow{\theta^{\mu,\mu'}_{\mathrm{Hk}} \times \mathrm{id}_{\mathfrak{S}'_\infty}} & \mathrm{Hk}'^{r}_G(\Sigma)^2_{\mathfrak{S}'_\infty} & \xleftarrow{\overleftrightarrow{q}' \times \mathrm{id}_{\mathfrak{S}'_\infty}} & \mathrm{Hk}'^{r}_{H,d}(\Sigma)_{\mathfrak{S}'_\infty} \\
\downarrow{\scriptstyle (p^{\underline{\mu}}_{T,0} \times p^{\underline{\mu}'}_{T,0} \times \mathrm{id}_{\mathfrak{S}'_\infty}, \alpha_T)} & & \downarrow{\scriptstyle (p'^{2}_{G,0}, \alpha_G)} & & {\scriptstyle (p'_{H,0}, \alpha_H)}\downarrow \\
(\mathrm{Bun}^2_T)_{\mathfrak{S}'_\infty} \times (\mathrm{Bun}^2_T)_{\mathfrak{S}'_\infty} & \xrightarrow[\times \theta^{\mu,\mu'}_{\mathrm{Bun}}]{\theta^{\mu,\mu'}_{\mathrm{Bun}}} & \mathrm{Bun}_G(\Sigma)^2 \times \mathrm{Bun}_G(\Sigma)^2 & \xleftarrow[\times \overleftrightarrow{p_H}]{\overleftrightarrow{p_H}} & H_d(\Sigma) \times H_d(\Sigma) \\
{\scriptstyle (\mathrm{id},\mathrm{Fr})}\uparrow & & {\scriptstyle (\mathrm{id},\mathrm{Fr})}\uparrow & & {\scriptstyle (\mathrm{id},\mathrm{Fr})}\uparrow \\
(\mathrm{Bun}^2_T)_{\mathfrak{S}'_\infty} & \xrightarrow{\theta^{\mu,\mu'}_{\mathrm{Bun}}} & \mathrm{Bun}_G(\Sigma)^2 & \xleftarrow{\overleftrightarrow{p_H}} & H_d(\Sigma).
\end{array}$$

Here we use subscript $\mathfrak{S}'_\infty$ to denote the product with $\mathfrak{S}'_\infty$ over k. The map $\theta^{\mu,\mu'}_{\mathrm{Bun}} : \mathrm{Bun}^2_T \times \mathfrak{S}'_\infty \to \mathrm{Bun}_G(\Sigma)^2$ is given by $\theta^{\mu_\Sigma}_{\mathrm{Bun}} \times \theta^{\mu'_\Sigma}_{\mathrm{Bun}}$, using a common copy of $\mathfrak{S}'_\infty$; here $\theta^{\mu,\mu'}_{\mathrm{Hk}} : \mathrm{Hk}^{\underline{\mu}}_T \times \mathrm{Hk}^{\underline{\mu}'}_T \times \mathfrak{S}'_\infty \to \mathrm{Hk}'^{r}_G(\Sigma)^2$ is similarly defined using $\theta^{\mu}_{\mathrm{Hk}}$ and $\theta^{\mu'}_{\mathrm{Hk}}$.

Let us explain the three maps α_T, α_G and α_H that appear as the second components of the vertical maps connecting the first and the second rows.

- The map α_T is the composition

$$\mathrm{Hk}^{\underline{\mu}}_T \times \mathrm{Hk}^{\underline{\mu}'}_T \times \mathfrak{S}'_\infty \xrightarrow{p^{\underline{\mu}}_{T,r} \times p^{\underline{\mu}'}_{T,r} \times \mathrm{id}_{\mathfrak{S}'_\infty}} \mathrm{Bun}^2_T \times \mathfrak{S}'_\infty \xrightarrow{\mathrm{AL}_{T,\mu_\infty,\mu'_\infty}} \mathrm{Bun}^2_T \times \mathfrak{S}'_\infty,$$

  where $\mathrm{AL}_{T,\mu_\infty,\mu'_\infty}$ is defined as

(5.19)

$$\begin{aligned}&\mathrm{AL}_{T,\mu_\infty,\mu'_\infty}(\mathcal{L}_1, \mathcal{L}_2, \{x'^{(1)}\})\\ &\qquad = \left(\mathcal{L}_1\left(-\sum_{x \in \Sigma_\infty} \mu_x x'^{(1)}\right), \mathcal{L}_2\left(-\sum_{x \in \Sigma_\infty} \mu'_x x'^{(1)}\right), \{x'^{(2)}\}\right).\end{aligned}$$

  Hence on the $\mathfrak{S}'_\infty$-factor, α_T is the Frobenius morphism.
- The map α_G is the composition

$$\mathrm{Hk}'^{r}_G(\Sigma)^2 \times \mathfrak{S}'_\infty \xrightarrow{p'^{2}_{G,r} \times \nu_\infty} \mathrm{Bun}_G(\Sigma)^2 \times \mathfrak{S}_\infty \xrightarrow{\mathrm{AL}^{(2)}_{G,\infty}} \mathrm{Bun}_G(\Sigma)^2,$$

where $\mathrm{AL}^{(2)}_{G,\infty}$ is $\mathrm{AL}_{G,\infty}$ on both copies of $\mathrm{Bun}_G(\Sigma)$ using a common copy of $\mathfrak{S}_\infty$.

- The map α_H is the composition

$$\mathrm{Hk}'^{r}_{H,d}(\Sigma)\times\mathfrak{S}'_\infty \xrightarrow{p'_{H,r}\times\nu_\infty} H_d(\Sigma)\times\mathfrak{S}_\infty \xrightarrow{\mathrm{AL}_{H,\infty}} H_d(\Sigma).$$

5.2.4. We define $\mathrm{Sht}'^{r}_{H,d}(\Sigma;\Sigma_\infty)$ to be the fiber product of the third column of (5.18); i.e., the following diagram is Cartesian:

$$\begin{array}{ccc}\mathrm{Sht}'^{r}_{H,d}(\Sigma;\Sigma_\infty) & \longrightarrow & \mathrm{Hk}'^{r}_{H,d}(\Sigma)\times\mathfrak{S}'_\infty\\ \downarrow & & \downarrow{\scriptstyle (p'_{H,0},\alpha_H)}\\ H_d(\Sigma) & \xrightarrow{(\mathrm{id},\mathrm{Fr})} & H_d(\Sigma)\times H_d(\Sigma).\end{array}\tag{5.20}$$

Then the fiber product of the three columns are

(5.21)

$$\mathrm{Sht}^{\mu}_{T}(\mu_\infty\cdot\Sigma'_\infty)\times_{\mathfrak{S}'_\infty}\mathrm{Sht}^{\mu'}_{T}(\mu'_\infty\cdot\Sigma'_\infty)\xrightarrow{\theta'^{\mu}\times\theta'^{\mu'}}\mathrm{Sht}'^{r}_{G}(\Sigma;\Sigma_\infty)\times_{\mathfrak{S}'_\infty}\mathrm{Sht}'^{r}_{G}(\Sigma;\Sigma_\infty)$$
$$\longleftarrow \mathrm{Sht}'^{r}_{H,d}(\Sigma;\Sigma_\infty).$$

Recall the map $s: H_d(\Sigma)\to U_d$ from (5.15). The Hecke correspondence $\mathrm{Hk}'^{r}_{H,d}(\Sigma)$ preserves the map s while the Frobenius map on $H_d(\Sigma)$ covers the Frobenius map of U_d. Therefore, from the definition of $\mathrm{Sht}'^{r}_{H,d}(\Sigma;\Sigma_\infty)$, we get canonical decomposition of it indexed by k-points of U_d, i.e., effective divisors of degree d on U. As in [10, Lemma 6.12], one shows that the piece indexed by $D\in U_d(k)$ is exactly the Hecke correspondence $\mathrm{Sht}'^{r}_{G}(\Sigma;\Sigma_\infty;h_D)$ for $\mathrm{Sht}'^{r}_{G}(\Sigma;\Sigma_\infty)$. In other words, we have a decomposition

$$\mathrm{Sht}'^{r}_{H,d}(\Sigma;\Sigma_\infty)=\coprod_{D\in U_d(k)}\mathrm{Sht}'^{r}_{G}(\Sigma;\Sigma_\infty;h_D).\tag{5.22}$$

5.2.5. *The stack $\mathcal{M}_d(\mu_\Sigma,\mu'_\Sigma)$ and its Hecke correspondence.* Now we consider the fiber product of the three rows of the master diagram (5.18).

Definition 5.9. Let $\mathcal{M}_d(\mu_\Sigma,\mu'_\Sigma)$ be the fiber product of the bottom row of (5.18); i.e., we have the following Cartesian diagram:

$$\begin{array}{ccc}\mathcal{M}_d(\mu_\Sigma,\mu'_\Sigma) & \longrightarrow & H_d(\Sigma)\\ \downarrow & & \downarrow{\scriptstyle \overleftrightarrow{p}_H}\\ \mathrm{Bun}^2_T\times\mathfrak{S}'_\infty & \xrightarrow{\theta^{\mu,\mu'}_{\mathrm{Bun}}} & \mathrm{Bun}_G(\Sigma)^2.\end{array}\tag{5.23}$$

Our notation suggests that $\mathcal{M}_d(\mu_\Sigma,\mu'_\Sigma)$ depends only on μ_Σ and μ'_Σ. This is indeed the case, because $\theta^{\mu,\mu'}_{\mathrm{Bun}}$ depends only on μ_Σ and μ'_Σ.

From the definition of $\mathcal{M}_d(\mu_\Sigma,\mu'_\Sigma)$, the Atkin–Lehner automorphisms $\mathrm{AL}_{G,\infty}$ (see (3.20)), $\mathrm{AL}_{H,\infty}$ (see (5.16)) and $\mathrm{AL}_{T,\mu_\infty,\mu'_\infty}$ (see (5.19)) together with Lemma 4.7 induce an Atkin–Lehner automorphism for $\mathcal{M}_d(\mu_\Sigma, \mu'_\Sigma)$:

$$\mathrm{AL}_{\mathcal{M},\infty} : \mathcal{M}_d(\mu_\Sigma, \mu'_\Sigma) \longrightarrow \mathcal{M}_d(\mu_\Sigma, \mu'_\Sigma).$$

Definition 5.10. Let $\mathrm{Hk}^{\mu,\mu'}_{\mathcal{M},d}$ be the fiber product of the top row of (5.18). Equivalently, we have the following Cartesian diagram:

$$\begin{CD} \mathrm{Hk}^{\mu,\mu'}_{\mathcal{M},d} @>>> \mathrm{Hk}'^{r}_{H,d}(\Sigma) \\ @VVV @VV{\overleftrightarrow{q}}V \\ \mathrm{Hk}^{\mu}_{T} \times \mathrm{Hk}^{\mu'}_{T} \times \mathfrak{S}'_\infty @>{\theta^{\mu,\mu'}_{\mathrm{Hk}}}>> \mathrm{Hk}'^{r}_{G}(\Sigma)^2. \end{CD} \tag{5.24}$$

Comparing the diagrams (5.23) and (5.24), we get projections

$$p_{\mathcal{M},i} : \mathrm{Hk}^{\mu,\mu'}_{\mathcal{M},d} \longrightarrow \mathcal{M}_d(\mu_\Sigma, \mu'_\Sigma), \quad i = 0, \dots, r$$

as the fiber product of $p^{\mu}_{T,i} \times p^{\mu'}_{T,i} \times \mathrm{id}_{\mathfrak{S}'_\infty}$ and $p'_{H,i}$ over $p'^{2}_{G,i}$. We also let

$$\alpha_{\mathcal{M}} = \mathrm{AL}_{\mathcal{M},\infty} \circ p_{\mathcal{M},r} : \mathrm{Hk}^{\mu,\mu'}_{\mathcal{M},d} \longrightarrow \mathcal{M}_d(\mu_\Sigma, \mu'_\Sigma).$$

The fiber products of the three rows of (5.18) now read

$$\begin{CD} \mathrm{Hk}^{\mu,\mu'}_{\mathcal{M},d} \\ @VV{(p_{\mathcal{M},0},\alpha_{\mathcal{M}})}V \\ \mathcal{M}_d(\mu_\Sigma, \mu'_\Sigma) \times \mathcal{M}_d(\mu_\Sigma, \mu'_\Sigma) \\ @A{(\mathrm{id},\mathrm{Fr})}AA \\ \mathcal{M}_d(\mu_\Sigma, \mu'_\Sigma) \end{CD} \tag{5.25}$$

5.2.6. *The stack* $\mathrm{Sht}^{\mu,\mu'}_{\mathcal{M},d}$.

Definition 5.11. Let $\mathrm{Sht}^{\mu,\mu'}_{\mathcal{M},d}$ be the fiber product of the maps in (5.25); i.e., we have a Cartesian diagram

$$\begin{CD} \mathrm{Sht}^{\mu,\mu'}_{\mathcal{M},d} @>>> \mathrm{Hk}^{\mu,\mu'}_{\mathcal{M},d} \\ @VVV @VV{(p_{\mathcal{M},0},\alpha_{\mathcal{M}})}V \\ \mathcal{M}_d(\mu_\Sigma, \mu'_\Sigma) @>{(\mathrm{id},\mathrm{Fr})}>> \mathcal{M}_d(\mu_\Sigma, \mu'_\Sigma) \times \mathcal{M}_d(\mu_\Sigma, \mu'_\Sigma). \end{CD} \tag{5.26}$$

By the diagram (5.18), $\mathrm{Sht}^{\mu,\mu'}_{\mathcal{M},d}$ is also the fiber product of the maps in (5.21); i.e., the following diagram is also Cartesian:

(5.27)

$$\begin{array}{ccc} \mathrm{Sht}^{\mu,\mu'}_{\mathcal{M},d} & \longrightarrow & \mathrm{Sht}'^{r}_{H,d}(\Sigma;\Sigma_\infty) \\ \big\downarrow & & \big\downarrow \\ \mathrm{Sht}^{\mu}_{T}(\mu_\infty\cdot\Sigma'_\infty)\times_{\mathfrak{S}'_\infty}\mathrm{Sht}^{\mu'}_{T}(\mu'_\infty\cdot\Sigma'_\infty) & \xrightarrow{\theta'^{\mu}\times\theta'^{\mu'}} & \mathrm{Sht}'^{r}_{G}(\Sigma;\Sigma_\infty)\times_{\mathfrak{S}'_\infty}\mathrm{Sht}'^{r}_{G}(\Sigma;\Sigma_\infty). \end{array}$$

According to the decomposition (5.22), we get a corresponding decomposition of $\mathrm{Sht}^{\mu,\mu'}_{\mathcal{M},d}$,

$$\mathrm{Sht}^{\mu,\mu'}_{\mathcal{M},d}=\coprod_{D\in U_d(k)}\mathrm{Sht}^{\mu,\mu'}_{\mathcal{M},D}, \tag{5.28}$$

where $\mathrm{Sht}^{\mu,\mu'}_{\mathcal{M},D}$ is the preimage of $\mathrm{Sht}'^{r}_{G}(\Sigma;\Sigma_\infty;h_D)\subset\mathrm{Sht}'^{r}_{H,d}(\Sigma;\Sigma_\infty)$ under the upper horizontal map in (5.27). We have a Cartesian diagram

(5.29)

$$\begin{array}{ccc} \mathrm{Sht}^{\mu,\mu'}_{\mathcal{M},D} & \longrightarrow & \mathrm{Sht}'^{r}_{G}(\Sigma;\Sigma_\infty;h_D) \\ \big\downarrow & & \big\downarrow{\scriptstyle(\overleftarrow{p}',\overrightarrow{p}')} \\ \mathrm{Sht}^{\mu}_{T}(\mu_\infty\cdot\Sigma'_\infty)\times_{\mathfrak{S}'_\infty}\mathrm{Sht}^{\mu'}_{T}(\mu'_\infty\cdot\Sigma'_\infty) & \xrightarrow{\theta'^{\mu}\times\theta'^{\mu'}} & \mathrm{Sht}'^{r}_{G}(\Sigma;\Sigma_\infty)\times_{\mathfrak{S}'_\infty}\mathrm{Sht}'^{r}_{G}(\Sigma;\Sigma_\infty). \end{array}$$

Here the maps $\overleftarrow{p}',\overrightarrow{p}':\mathrm{Sht}'^{r}_{G}(\Sigma;\Sigma_\infty;h_D)\to\mathrm{Sht}'^{r}_{G}(\Sigma;\Sigma_\infty)$ are the base changes of the maps $\overleftarrow{p}$ and $\overrightarrow{p}$ in (3.23).

5.3. *Relation between* $\mathcal{M}_d$ *and* $\mathcal{M}_d(\mu_\Sigma,\mu'_\Sigma)$. In this subsection, we relate $\mathcal{M}_d(\mu_\Sigma,\mu'_\Sigma)$ to the moduli stack $\mathcal{M}_d$ that was defined earlier. For this, we first give an alternative description of $\mathcal{M}_d(\mu_\Sigma,\mu'_\Sigma)$ in the style of the definition of $\mathcal{M}_d$ in [10, §6.1.1].

5.3.1. *Some preparation.* Let S be any scheme, and let $\mathcal{L}$ and $\mathcal{L}'$ be two line bundles over $X'\times S$. We denote by $H_{R'}(\mathcal{L},\mathcal{L}')$ be the set of pairs (α,β), where

$$\alpha:\mathcal{L}\longrightarrow\mathcal{L}'(R'):=\mathcal{L}'\otimes_{\mathcal{O}_{X'}}\mathcal{O}_{X'}(R'), \tag{5.30}$$

$$\beta:\sigma^*\mathcal{L}\longrightarrow\mathcal{L}'(R') \tag{5.31}$$

such that their restrictions to $R'\times S$ satisfy

$$\alpha|_{R'\times S}=\beta|_{R'\times S}. \tag{5.32}$$

Note that $\mathcal{L}$ and $\sigma^*\mathcal{L}$ are the same when restricted to $R'\times S$, hence the above equality makes sense.

Recall $\nu_S=\nu\times\mathrm{id}_S:X'\times S\to X\times S$.

LEMMA 5.12. *There is a canonical bijection*

$$\mathrm{Hom}_{X\times S}(\nu_{S,*}\mathcal{L},\nu_{S,*}\mathcal{L}')\xrightarrow{\sim} H_{R'}(\mathcal{L},\mathcal{L}')$$

such that, if $\varphi:\nu_{S,*}\mathcal{L}\to\nu_{S,*}\mathcal{L}'$ *corresponds to* (α,β) *under this bijection, we have*

$$\det(\varphi)=\mathrm{Nm}(\alpha)-\mathrm{Nm}(\beta) \tag{5.33}$$

as sections of $\det(\nu_{S,*}\mathcal{L})^{-1}\otimes\det(\nu_{S,*}\mathcal{L}')\cong\mathrm{Nm}_{X'/X}(\mathcal{L})^{-1}\otimes\mathrm{Nm}_{X'/X}(\mathcal{L}')$.

Proof. By adjunction a map $\varphi:\nu_{S,*}\mathcal{L}\to\nu_{S,*}\mathcal{L}'$ is equivalent to a map $\nu_S^*\nu_{S,*}\mathcal{L}\to\mathcal{L}'$. Note that $\nu_S^*\nu_{S,*}\mathcal{L}\cong\mathcal{O}_{X'}\otimes_{\mathcal{O}_X}\mathcal{L}\cong(\mathcal{O}_{X'}\otimes_{\mathcal{O}_X}\mathcal{O}_{X'})\otimes_{\mathcal{O}_{X'}}\mathcal{L}$, whose $\mathcal{O}_{X'}$-module structure is given by the first factor of $\mathcal{O}_{X'}$.

We have an injective map $\jmath:\mathcal{O}_{X'}\otimes_{\mathcal{O}_X}\mathcal{O}_{X'}\to\mathcal{O}_{X'}\oplus\mathcal{O}_{X'}$ sending $a\otimes b\mapsto ab+a\sigma(b)$. By a local calculation at points in R' we see that the image of $\jmath$ is $\mathcal{O}_{X'}\oplus_{R'}\mathcal{O}_{X'}:=\ker(\mathcal{O}_{X'}\oplus\mathcal{O}_{X'}\xrightarrow{(i^*,-i^*)}\mathcal{O}_{R'})$ (the difference of two restriction maps $i^*:\mathcal{O}_{X'}\to\mathcal{O}_{R'}$). Therefore, $\nu_S^*\nu_{S,*}\mathcal{L}\cong(\mathcal{O}_{X'}\oplus_{R'}\mathcal{O}_{X'})\otimes_{\mathcal{O}_{X'}}\mathcal{L}=\mathcal{L}\oplus_{R'}\sigma^*\mathcal{L}=\ker(\mathcal{L}\oplus\sigma^*\mathcal{L}\xrightarrow{(i^*,-i^*)}\mathcal{L}_{R'\times S})$. Hence the map φ is equivalent to a map

$$\psi:\mathcal{L}\oplus_{R'}\sigma^*\mathcal{L}\longrightarrow\mathcal{L}'.$$

Since $\mathcal{L}(-R')\oplus\sigma^*\mathcal{L}(-R')\subset\mathcal{L}\oplus_{R'}\sigma^*\mathcal{L}$, the map ψ restricts to a map

$$\mathcal{L}(-R')\oplus\sigma^*\mathcal{L}(-R')\longrightarrow\mathcal{L}'$$

or

$$\mathcal{L}\oplus\sigma^*\mathcal{L}\longrightarrow\mathcal{L}'(R').$$

We then define the two components of above map to be α and $-\beta$. Condition (5.32) is equivalent to that the map $\alpha\oplus(-\beta):\mathcal{L}\oplus\sigma^*\mathcal{L}\to\mathcal{L}'(R')$, when restricted to $\mathcal{L}\oplus_{R'}\sigma^*\mathcal{L}$, lands in $\mathcal{L}'$.

If φ corresponds to (α,β), we may pullback φ to X' so it becomes the map $\mathcal{L}\oplus_{R'}\sigma^*\mathcal{L}\to\mathcal{L}'\oplus_{R'}\sigma^*\mathcal{L}'$ given by the matrix

$$\begin{bmatrix}\alpha & -\beta\\ -\sigma^*\beta & \sigma^*\alpha\end{bmatrix}.$$

Therefore, $\det(\varphi)=\mathrm{Nm}(\alpha)-\mathrm{Nm}(\beta)$. □

5.3.2. *Alternative description of* $\mathcal{M}_d(\mu_\Sigma,\mu'_\Sigma)$. We define $\widetilde{\mathcal{M}}_d(\mu_\Sigma,\mu'_\Sigma)$ by the Cartesian diagram

$$\begin{array}{ccc}\widetilde{\mathcal{M}}_d(\mu_\Sigma,\mu'_\Sigma) & \longrightarrow & \widetilde{H}_d(\Sigma)\\ \downarrow & & \downarrow{\scriptstyle \overleftrightarrow{p}_H}\\ \mathrm{Pic}_{X'}\times\mathrm{Pic}_{X'}\times\mathfrak{S}'_\infty & \xrightarrow{\widetilde{\theta}^{\mu,\mu'}_{\mathrm{Bun}}} & \mathrm{Bun}_2(\Sigma)\times\mathrm{Bun}_2(\Sigma).\end{array}$$

Here $\widetilde{\theta}^{\mu,\mu'}_{\mathrm{Bun}}$ is given by $\widetilde{\theta}^{\mu_\Sigma}_{\mathrm{Bun}} \times \widetilde{\theta}^{\mu'_\Sigma}_{\mathrm{Bun}}$, using a common copy of $\mathfrak{S}'_\infty$, and $\overleftrightarrow{p}_H$ sends $(\mathcal{E}^\dagger, \mathcal{E}'^\dagger, \varphi) \in \widetilde{H}_d(\Sigma)(S)$ to $(\mathcal{E}^\dagger, \mathcal{E}'^\dagger) \in (\mathrm{Bun}_2(\Sigma)(S))^2$. Comparing with Definition 5.9, we have

$$\mathcal{M}_d(\mu_\Sigma, \mu'_\Sigma) \cong \widetilde{\mathcal{M}}_d(\mu_\Sigma, \mu'_\Sigma)/\operatorname{Pic}_X .$$

For $x' \in \Sigma'_\infty$ and $x'^{(1)} : S \to \operatorname{Spec} k(x') \xrightarrow{x'} X'$, recall that we inductively defined $x'^{(j)}$ using $x'^{(j)} = x'^{(j-1)} \circ \mathrm{Fr}_S$ for $j \geq 2$. We have a morphism

$$\mathfrak{D}_+ : \mathfrak{S}'_\infty \longrightarrow X'_{N_+}$$

that sends $\{x'^{(1)}\}_{x' \in \Sigma'_\infty} \in \mathfrak{S}'_\infty(S)$ to the following divisor of $X' \times S$ of degree N_+:

$$\mathfrak{D}_+(\{x'^{(1)}\}) := \sum_{x \in \Sigma_f \cap \Sigma_+} \mu'_x \times S + \sum_{x \in \Sigma_\infty \cap \Sigma_+} \begin{cases} (\Gamma_{x'^{(1)}} + \Gamma_{x'^{(2)}} + \cdots + \Gamma_{x'^{(d_x)}}) & \text{if } \mu'_x = 1, \\ (\Gamma_{x'^{(d_x+1)}} + \Gamma_{x'^{(d_x+2)}} + \cdots + \Gamma_{x'^{(2d_x)}}) & \text{if } \mu'_x = -1. \end{cases}$$

Similarly, we define

$$\mathfrak{D}_- : \mathfrak{S}'_\infty \longrightarrow X'_{N_-}$$

by sending $\{x'^{(1)}\}_{x' \in \Sigma'_\infty} \in \mathfrak{S}'_\infty(S)$ to the following divisor of $X' \times S$ of degree N_-:

$$\mathfrak{D}_-(\{x'^{(1)}\}) := \sum_{x \in \Sigma_f \cap \Sigma_-} \mu'_x \times S + \sum_{x \in \Sigma_\infty \cap \Sigma_-} \begin{cases} (\Gamma_{x'^{(1)}} + \Gamma_{x'^{(2)}} + \cdots + \Gamma_{x'^{(d_x)}}) & \text{if } \mu'_x = 1, \\ (\Gamma_{x'^{(d_x+1)}} + \Gamma_{x'^{(d_x+2)}} + \cdots + \Gamma_{x'^{(2d_x)}}) & \text{if } \mu'_x = -1. \end{cases}$$

Now we can state the alternative description of $\mathcal{M}_d(\mu_\Sigma, \mu'_\Sigma)$.

LEMMA 5.13. *For a scheme S, $\widetilde{\mathcal{M}}_d(\mu_\Sigma, \mu'_\Sigma)(S)$ is canonically equivalent to the groupoid of tuples $(\mathcal{L}, \mathcal{L}', \alpha, \beta, \{x'^{(1)}\}_{x' \in \Sigma'_\infty})$, where*

- *$\mathcal{L}$ and $\mathcal{L}'$ are line bundles on $X' \times S$ such that $\deg(\mathcal{L}'|_{X' \times s}) - \deg(\mathcal{L}|_{X' \times s}) = d$ for all geometric points $s \in S$;*
- *$\alpha : \mathcal{L} \to \mathcal{L}'(R')$, $\beta : \sigma^*\mathcal{L} \to \mathcal{L}'(R')$.*

These data are required to satisfy the following conditions:

(1) *$\alpha|_{\mathfrak{D}_-(\{x'^{(1)}\})} = 0$, and $\alpha|_{\nu^{-1}(\Sigma_+) \times S}$ is an isomorphism.*
(2) *$\beta|_{\mathfrak{D}_+(\{x'^{(1)}\})} = 0$, and $\beta|_{\nu^{-1}(\Sigma_-) \times S}$ is an isomorphism.*
(3) *$\alpha|_{R' \times S} = \beta|_{R' \times S}$. Moreover, $\mathrm{Nm}(\alpha) - \mathrm{Nm}(\beta)$, viewed as a section of $\mathrm{Nm}_{X'/X}(\mathcal{L})^{-1} \otimes \mathrm{Nm}_{X'/X}(\mathcal{L}')$, is nowhere vanishing along $R \times S$.*
(4) *This is non-void only when $\Sigma = \varnothing$ and $R = \varnothing$: for every geometric point s of S, $\mathrm{Nm}(\alpha) - \mathrm{Nm}(\beta)$ is not identically zero on $X \times s$.*

Proof. By definition, S-points of $\widetilde{\mathcal{M}}_d(\mu_\Sigma, \mu'_\Sigma)$ consist of tuples

$$(\mathcal{L}, \mathcal{L}', \varphi, \{x'^{(1)}\}_{x' \in \Sigma'_\infty}),$$

where

- $\mathcal{L}$ and $\mathcal{L}'$ are line bundles on $X' \times S$ such that $\deg(\mathcal{L}'|_{X\times s}) - \deg(\mathcal{L}|_{X\times s}) = d$ for all geometric points $s \in S$;
- $\varphi : \nu_{S,*}\mathcal{L} \to \nu_{S,*}\mathcal{L}'$ is an injective map when restricted to $X \times s$ for every geometric point $s \in S$ — moreover, φ is an isomorphism along $(\Sigma \sqcup R) \times S$;
- for each $x' \in \Sigma'_\infty$, $x'^{(1)}$ is a map $S \to \operatorname{Spec} k(x') \xrightarrow{x'} X'$.

These data are required to satisfy the following condition. We have two S-points of $\mathrm{Bun}_2(\Sigma)$:

$$\mathcal{E}^\dagger = \widetilde{\theta}^{\mu_\Sigma}_{\mathrm{Bun}}(\mathcal{L}, \{x'^{(1)}\}_{x' \in \Sigma'_\infty}),$$
$$\mathcal{E}'^\dagger = \widetilde{\theta}^{\mu'_\Sigma}_{\mathrm{Bun}}(\mathcal{L}', \{x'^{(1)}\}_{x' \in \Sigma'_\infty}).$$

Then $\varphi : \mathcal{E} = \nu_{S,*}\mathcal{L} \to \mathcal{E}' = \nu_{S,*}\mathcal{L}'$ should respect the level structures of $\mathcal{E}^\dagger$ and $\mathcal{E}'^\dagger$.

By Lemma 5.12, the map $\varphi : \nu_{S,*}\mathcal{L} \to \nu_{S,*}\mathcal{L}'$ becomes a pair $\alpha : \mathcal{L} \to \mathcal{L}'(R')$ and $\beta : \sigma^*\mathcal{L} \to \mathcal{L}'(R')$ satisfying $\alpha|_{R'\times S} = \beta|_{R'\times S}$. Since $\varphi|_{R\times S}$ is an isomorphism, formula (5.33) implies that $\mathrm{Nm}(\alpha) - \mathrm{Nm}(\beta)$ is nowhere vanishing along $R \times S$, hence condition (3) in the statement of the lemma is verified. Condition (4) also follows from (5.33) and the condition on φ above.

Since φ respects the Iwahori level structures of $\nu_{S,*}\mathcal{L}$ and $\nu_{S,*}\mathcal{L}'$, it sends $\nu_{S,*}(\mathcal{L}(-\mu_x))$ to $\nu_{S,*}(\mathcal{L}'(-\mu'_x))$ for all $x \in \Sigma_f$. (Recall μ_x is the value of μ_f at x.) A local calculation shows that α should vanish along $\mu'_x \times S$ for those $x \in \Sigma_f$ such that $\mu_x \neq \mu'_x$, and β should vanish along $\mu'_x \times S$ for those $x \in \Sigma_f$ such that $\mu_x = \mu'_x$. A similar local calculation at $x \in \Sigma_\infty$ implies the vanishing of α and β along the corresponding parts of $\mathfrak{D}_-$ and $\mathfrak{D}_+$. For example, if $\mu_x = \mu'_x = 1$, then φ should send $\nu_{S,*}(\mathcal{L}(-\Gamma_{x'^{(1)}} - \cdots - \Gamma_{x'^{(d_x)}}))$ to $\nu_{S,*}(\mathcal{L}'(-\Gamma_{x'^{(1)}} - \cdots - \Gamma_{x'^{(d_x)}}))$, which implies that β vanishes along $\Gamma_{x'^{(1)}} + \Gamma_{x'^{(2)}} + \cdots + \Gamma_{x'^{(d_x)}}$. These verify the vanishing parts of conditions (1) and (2).

Finally, since $\varphi|_{\Sigma\times S}$ is an isomorphism, $\det(\varphi) = \mathrm{Nm}(\alpha) - \mathrm{Nm}(\beta)$ is nowhere vanishing on $\Sigma \times S$. Since $\mathrm{Nm}(\alpha)|_{\Sigma_-\times S} = 0$ and $\mathrm{Nm}(\beta)|_{\Sigma_+\times S} = 0$ by the vanishing parts of (1) and (2), $\mathrm{Nm}(\alpha)|_{\Sigma_+\times S}$ and $\mathrm{Nm}(\beta)|_{\Sigma_-\times S}$ are nowhere vanishing. These verify the non-vanishing parts of conditions (1) and (2). We have verified all the desired conditions for $(\mathcal{L}, \mathcal{L}', \alpha, \beta, \{x'^{(1)}\}_{x'\in\Sigma'_\infty})$. ☐

Using the description of $\mathcal{M}_d(\mu_\Sigma, \mu'_\Sigma)$ given in Lemma 5.13, we can describe its Atkin–Lehner automorphism $\mathrm{AL}_{\mathcal{M},\infty}$ as follows.

LEMMA 5.14. *Let $(\mathcal{L}, \mathcal{L}', \alpha, \beta, \{x'^{(1)}\}_{x'\in\Sigma'_\infty})$ be an S-point of $\widetilde{\mathcal{M}}_d(\mu_\Sigma, \mu'_\Sigma)$ as described in Lemma 5.13. We use the same notation to denote its image in*

$\mathcal{M}_d(\mu_\Sigma,\mu'_\Sigma)$. *Then*

$$\mathrm{AL}_{\mathcal{M},\infty}(\mathcal{L},\mathcal{L}',\alpha,\beta,\{x'^{(1)}\}_{x'\in\Sigma'_\infty}) = \Bigg(\mathcal{L}\Bigg(-\sum_{x\in\Sigma_\infty}\mu_x\Gamma_{x'^{(1)}}\Bigg),$$
$$\mathcal{L}'\Bigg(-\sum_{x\in\Sigma_\infty\cap\Sigma_+}\mu_x\Gamma_{x'^{(1)}} - \sum_{x\in\Sigma_\infty\cap\Sigma_-}\mu_x\Gamma_{x'^{(d_x+1)}}\Bigg),\alpha',\beta',\{x'^{(2)}\}_{x'\in\Sigma'_\infty}\Bigg).$$

Here, α' is induced from α using the fact that $\alpha|_{\mathfrak{D}_-}=0$ and β' is induced from β using the fact that $\beta|_{\mathfrak{D}_+}=0$.

The proof is by tracking the definitions and we omit it.

The next result clarifies the relation between $\mathcal{M}_d$ and $\mathcal{M}_d(\mu_\Sigma,\mu'_\Sigma)$.

PROPOSITION 5.15. *There is a canonical isomorphism over $\mathfrak{S}'_\infty$,*

$$\Xi_{\mathcal{M}}:\mathcal{M}_d\times\mathfrak{S}'_\infty \xrightarrow{\sim} \mathcal{M}_d(\mu_\Sigma,\mu'_\Sigma), \tag{5.34}$$

such that

(1) *the automorphism* $\mathrm{id}\times\mathrm{Fr}_{\mathfrak{S}'_\infty}$ *on the left corresponds to the automorphism* $\mathrm{AL}_{\mathcal{M},\infty}$ *on the right*;

(2) *the following diagram is commutative*:

$$\begin{array}{ccc}
\mathcal{M}_d\times\mathfrak{S}'_\infty & \xrightarrow{\mathrm{Fr}\times\mathrm{id}} & \mathcal{M}_d\times\mathfrak{S}'_\infty \\
\wr\downarrow\Xi_{\mathcal{M}} & & \wr\downarrow\Xi_{\mathcal{M}} \\
\mathcal{M}_d(\mu_\Sigma,\mu'_\Sigma) & \xrightarrow{\mathrm{AL}^{-1}_{\mathcal{M},\infty}\circ\mathrm{Fr}} & \mathcal{M}_d(\mu_\Sigma,\mu'_\Sigma).
\end{array}$$

Proof. We first define a map

$$\imath_d:\mathcal{M}_d(\mu_\Sigma,\mu'_\Sigma)\longrightarrow\mathcal{M}_d\times\mathfrak{S}'_\infty\subset(\widehat{X}'_{d+\rho-N_-}\times_{\mathrm{Pic}_X^{\sqrt{R};\sqrt{R},d+\rho}}\widehat{X}'_{d+\rho-N_+})\times\mathfrak{S}'_\infty.$$

Using the description of points of $\widetilde{\mathcal{M}}_d(\mu_\Sigma,\mu'_\Sigma)$ in Lemma 5.13, we have a morphism

$$\imath_\alpha:\mathcal{M}_d(\mu_\Sigma,\mu'_\Sigma)\longrightarrow\widehat{X}'_{d+\rho-N_-}$$

sending $(\mathcal{L},\mathcal{L}',\alpha,\beta,\{x'^{(1)}\}_{x'\in\Sigma'_\infty})$ to the line bundle $\mathcal{L}^{-1}\otimes\mathcal{L}'(R'-\mathfrak{D}_-(\{x'^{(1)}\}))$ and its section given by α. Similarly we have a morphism

$$\imath_\beta:\mathcal{M}_d(\mu_\Sigma,\mu'_\Sigma)\longrightarrow\widehat{X}'_{d+\rho-N_+}$$

sending $(\mathcal{L},\mathcal{L}',\alpha,\beta,\{x'^{(1)}\}_{x'\in\Sigma'_\infty})$ to the line bundle

$$\sigma^*\mathcal{L}^{-1}\otimes\mathcal{L}'(R'-\mathfrak{D}_+(\{x'^{(1)}\}))$$

and its section given by β. We have a canonical isomorphism $\nu_\alpha\circ\imath_\alpha\cong\nu_\beta\circ\imath_\beta$ using $\alpha|_{R'}=\beta|_{R'}$. The map $\imath_d$ is given by $(\imath_\alpha,\imath_\beta)$ and the natural projection to $\mathfrak{S}'_\infty$. It is easy to see that the image of $\imath_d$ lies in the open substack $\mathcal{M}_d\times\mathfrak{S}'_\infty$.

Next we construct the desired map $\Xi_{\mathcal{M}}$ as in (5.34). Start with a point $(\mathcal{I},\mathcal{J},\alpha,\beta,\jmath)\in\mathcal{M}_d(S)$, and let $\{x'^{(1)}\}_{x'\in\mathfrak{S}'_\infty}\in\mathfrak{S}'_\infty(S)$. Let $D_\pm=\mathfrak{D}_\pm(\{x'^{(1)}\})$ (a divisor of degree $N_\pm$ on $X'\times S$ with image $\Sigma_\pm\times S$ in $X\times S$), and let $\mathcal{I}'=\mathcal{I}(D_-)$ and $\mathcal{J}':=\mathcal{J}(D_+)$. The isomorphism $\jmath$ then gives an $\mathrm{Nm}^{\sqrt{R}}_{X'/X}(\mathcal{I}')\cong\mathrm{Nm}^{\sqrt{R}}_{X'/X}(\mathcal{J}')\in\mathrm{Pic}^{\sqrt{R},d+\rho}_X(S)$, or a trivialization of $\mathrm{Nm}^{\sqrt{R}}_{X'/X}(\mathcal{I}'^{\otimes-1}\otimes\mathcal{J}')$ as an S-point of $\mathrm{Pic}^{\sqrt{R},d+\rho}_X$. The exact sequence (A.11) then implies, upon localizing S in the étale topology, that there exists a line bundle $\mathcal{L}\in\mathrm{Pic}_{X'}(S)$ together with an isomorphism $\tau:\mathcal{L}^{-1}\otimes\sigma^*\mathcal{L}\cong\mathcal{I}'\otimes\mathcal{J}'^{-1}$, and such a pair $(\mathcal{L},\tau)$ is unique up to tensoring with $\mathrm{Pic}_X(S)$ (upon further localizing S). Let $\mathcal{L}'=\mathcal{L}\otimes\mathcal{I}'(-R')$. Then α can be viewed as a section of $\mathcal{L}^{-1}\otimes\mathcal{L}'(R')$, or a map $\mathcal{L}\to\mathcal{L}'(R')$ that vanishes along D_-. Since $\mathcal{J}'\cong\mathcal{I}'\otimes\mathcal{L}\otimes\sigma^*\mathcal{L}^{-1}\cong\sigma^*\mathcal{L}^{-1}\otimes\mathcal{L}'(R')$, β can be viewed as a section of $\sigma^*\mathcal{L}^{-1}\otimes\mathcal{L}'(R')$, or a map $\sigma^*\mathcal{L}\to\mathcal{L}'(R')$ that vanishes along D_+. Moreover, the equality $\alpha|_{R'\times S}=\beta|_{R'\times S}$ is built into the definition of $\mathcal{M}_d$. This way we get an S-point $(\mathcal{L},\mathcal{L}',\alpha,\beta,\{x'^{(1)}\})$ of $\mathcal{M}_d(\mu_\Sigma,\mu'_\Sigma)$ using the description of $\widetilde{\mathcal{M}}_d(\mu_\Sigma,\mu'_\Sigma)$ given in Lemma 5.13.

It is easy to see that $\Xi_{\mathcal{M}}$ is inverse to $\imath_d$. Therefore, $\Xi_{\mathcal{M}}$ is an isomorphism. This finishes the construction of the isomorphism $\Xi_{\mathcal{M}}$.

Now property (1) follows from Lemma 5.14 by a direct calculation.

To check property (2), observe that the total Frobenius morphisms $\mathrm{Fr}\times\mathrm{Fr}$ on $\mathcal{M}_d\times\mathfrak{S}'_\infty$ and Fr on $\mathcal{M}_d(\mu_\Sigma,\mu'_\Sigma)$ correspond to each other under $\Xi_{\mathcal{M}}$. On the other hand, by (1), $\mathrm{id}\times\mathrm{Fr}$ on $\mathcal{M}_d\times\mathfrak{S}'_\infty$ corresponds to $\mathrm{AL}_{\mathcal{M},\infty}$ on $\mathcal{M}_d(\mu_\Sigma,\mu'_\Sigma)$. Therefore, $\mathrm{Fr}\times\mathrm{id}=(\mathrm{id}\times\mathrm{Fr}^{-1})\circ(\mathrm{Fr}\times\mathrm{Fr})$ on $\mathcal{M}_d\times\mathfrak{S}'_\infty$ corresponds to $\mathrm{AL}^{-1}_{\mathcal{M},\infty}\circ\mathrm{Fr}$ on $\mathcal{M}_d(\mu_\Sigma,\mu'_\Sigma)$. □

5.3.3. *Comparison of Hecke correspondences for $\mathcal{M}_d(\mu_\Sigma,\mu'_\Sigma)$ and for $\mathcal{M}_d$.* We have already defined two self-correspondences $\mathcal{H}_+$ and $\mathcal{H}_-$ of $\mathcal{M}_d$ in Section 5.1.5. For $\underline{\lambda}=(\lambda_1,\dots,\lambda_r)\in\{\pm1\}^r$, let

$$\mathcal{H}_{\lambda_i}=\begin{cases}\mathcal{H}_+, & \lambda_i=1,\\ \mathcal{H}_-, & \lambda_i=-1.\end{cases}$$

Let $\overleftarrow{\gamma}_i,\overrightarrow{\gamma}_i:\mathcal{H}_{\lambda_i}\to\mathcal{M}_d$ be the two projections. Then define $\mathcal{H}_{\underline{\lambda}}$ to be the composition of $\mathcal{H}_{\lambda_i}$ as follows:

$$\mathcal{H}_{\underline{\lambda}}:=\mathcal{H}_{\lambda_1}\times_{\overrightarrow{\gamma}_1,\mathcal{M}_d,\overleftarrow{\gamma}_2}\mathcal{H}_{\lambda_2}\times_{\overrightarrow{\gamma}_2,\mathcal{M}_d,\overleftarrow{\gamma}_3}\cdots\times_{\overrightarrow{\gamma}_{r-1},\mathcal{M}_d,\overleftarrow{\gamma}_r}\mathcal{H}_{\lambda_r}.$$

We apply this construction to $\underline{\lambda}=\underline{\mu\mu'}=(\mu_1\mu'_1,\dots,\mu_r\mu'_r)$. Then we have $(r+1)$ projections

$$\gamma_i:\mathcal{H}_{\underline{\mu\mu'}}\longrightarrow\mathcal{M}_d,\quad i=0,1,\dots,r.$$

Proposition 5.16. *There is a canonical isomorphism over $\mathfrak{S}'_\infty$,*

$$\Xi_{\mathcal{H}}:\mathcal{H}_{\underline{\mu\mu'}}\times\mathfrak{S}'_\infty\xrightarrow{\sim}\mathrm{Hk}^{\mu,\mu'}_{\mathcal{M},d},\tag{5.35}$$

such that the following diagram is commutative for $i = 0, 1, \dots, r$*:*

$$\begin{array}{ccc} \mathcal{H}_{\underline{\mu\mu'}} \times \mathfrak{S}'_\infty & \xrightarrow[\sim]{\Xi_{\mathcal{H}}} & \mathrm{Hk}^{\mu,\mu'}_{\mathcal{M},d} \\ \Big\downarrow{\scriptstyle \gamma_i \times \mathrm{id}_{\mathfrak{S}'_\infty}} & & \Big\downarrow{\scriptstyle p_{\mathcal{M},i}} \\ \mathcal{M}_d \times \mathfrak{S}'_\infty & \xrightarrow[\sim]{\Xi_{\mathcal{M}}} & \mathcal{M}_d(\mu_\Sigma, \mu'_\Sigma). \end{array}$$

Proof. By the iterative nature of $\mathrm{Hk}^{\mu,\mu'}_{\mathcal{M},d}$, it suffices to prove the case $r = 1$. (At this point we may drop the assumption $r \equiv \#\Sigma_\infty \mod 2$ because everything makes sense without this condition, before passing to Shtukas.) We distinguish two cases.

Case 1: $\mu_1 = \mu'_1$. We treat only the case $\mu_1 = \mu'_1 = 1$; the other case is similar. In this case, $\mathrm{Hk}^{\mu,\mu'}_{\mathcal{M},d}(S)$ classifies the following data up to the action of Pic_X:

- A map $x'_1 : S \to X'$ with graph $\Gamma_{x'_1}$.
- For each $x' \in \Sigma'_\infty$, an S-point $x'^{(1)} : S \to \operatorname{Spec} k(x') \xrightarrow{x'} X'$.
- Line bundles $\mathcal{L}_0$ and $\mathcal{L}'_0$ on $X' \times S$ such that $\deg(\mathcal{L}'_0|_{X\times s}) - \deg(\mathcal{L}_0|_{X\times s}) = d$ for all geometric points $s \in S$. Let
$$\mathcal{L}_1 = \mathcal{L}_0(\Gamma_{x'_1}), \quad \mathcal{L}'_1 = \mathcal{L}'_0(\Gamma_{x'_1}).$$
- A map $\varphi_1 : \nu_{S,*}\mathcal{L}_1 \to \nu_{S,*}\mathcal{L}'_1$ that restricts to a map $\varphi_0 : \nu_{S,*}\mathcal{L}_0 \to \nu_{S,*}\mathcal{L}'_0$. Moreover, for $i = 0$ and 1, we require the tuple $(\mathcal{L}_i, \mathcal{L}'_i, \varphi_i, \{x'^{(1)}\})$ to give a point of $\mathcal{M}_d(\mu_\Sigma, \mu'_\Sigma)$. In other words,
  - φ_i preserves the level structures of $\nu_{S,*}\mathcal{L}_i$ and $\nu_{S,*}\mathcal{L}'_i$ given in Section 4.2.1;
  - φ_i is injective when restricted to $X \times s$ for every geometric point $s \in S$; and
  - $\varphi_i|_{(\Sigma\cup R)\times S}$ is an isomorphism.

Using Lemma 5.13, we may replace the data φ_i above by a pair of maps (α_i, β_i), where $\alpha_i : \mathcal{L}_i \to \mathcal{L}'_i(R')$, $\beta_i : \sigma^*\mathcal{L}_i \to \mathcal{L}'_i(R')$ satisfying certain conditions. Let $D_\pm = \mathfrak{D}_\pm(\{x'^{(1)}\})$. Then $\alpha_i|_{D_-} = 0$ and $\beta_i|_{D+} = 0$. Denote by

$$\alpha_i^\natural : \mathcal{L}_i \longrightarrow \mathcal{L}'_i(R' - D_-),$$
$$\beta_i^\natural : \sigma^*\mathcal{L}_i \longrightarrow \mathcal{L}'_i(R' - D_+)$$

the maps induced by α_i and β_i.

The relation between φ_0 and φ_1 implies that the following two diagrams are commutative:

$$\begin{array}{ccccc} \mathcal{L}_0 & \hookrightarrow & \mathcal{L}_1 & = & \mathcal{L}_0(\Gamma_{x'_1}) \\ \Big\downarrow{\scriptstyle \alpha_0^\natural} & & \Big\downarrow{\scriptstyle \alpha_1^\natural} & & \\ \mathcal{L}'_0(R' - D_-) & \hookrightarrow & \mathcal{L}'_1(R' - D_-) & = & \mathcal{L}'_0(R' - D_- + \Gamma_{x'_1}), \end{array} \tag{5.36}$$

$$\begin{array}{ccccc} \sigma^*\mathcal{L}_0 & \hookrightarrow & \sigma^*\mathcal{L}_1 & = & (\sigma^*\mathcal{L}_0)(\Gamma_{\sigma(x_1')}) \\ \downarrow \beta_0^\natural & & \downarrow \beta_1^\natural & & \\ \mathcal{L}_0'(R'-D_+) & \hookrightarrow & \mathcal{L}_1'(R'-D_+) & = & \mathcal{L}_0'(R'-D_++\Gamma_{x_1'}). \end{array} \tag{5.37}$$

The diagram (5.36) simply says that $\alpha_1^\natural$ is determined by $\alpha_0^\natural$ (no condition on $\alpha_0^\natural$, hence no condition on α_0). The diagram (5.37) imposes a non-trivial condition on $\beta_0^\natural$, as claimed below.

Claim. *$\beta_0^\natural$ vanishes along $\Gamma_{\sigma(x_1')}$.*

Proof of Claim. The argument for this claim is more complicated than the argument in [10, Lemma 6.3] because of the ramification of ν. To prove the claim, it suffices to argue for the similar statement for the restriction of $\beta_0^\natural$ to $(X'-R')\times S$ and to the formal completions $\operatorname{Spec}\mathcal{O}_{x'}\widehat{\times}S$ for each $x'\in R'$.

Computing the divisors of the maps in the first square of (5.37), we get

$$\operatorname{div}(\beta_0^\natural)+\Gamma_{x_1'}=\operatorname{div}(\beta_1^\natural)+\Gamma_{\sigma(x_1')}. \tag{5.38}$$

Restricting both sides to $(X'-R')\times S$, and observing that $\Gamma_{x_1'}$ and $\Gamma_{\sigma(x_1')}$ are disjoint when restricted to $(X'-R')\times S$, we see that $\Gamma_{\sigma(x_1')}\cap((X'-R')\times S)$ is contained in $\operatorname{div}(\beta_0^\natural)\cap((X'-R')\times S)$.

Now we consider the restriction of the diagram (5.37) to the formal completion $\operatorname{Spec}\mathcal{O}_{x'}\widehat{\times}S$ at any $x'\in R'$. Since $D_\pm$ is disjoint from R', after restricting to $\operatorname{Spec}\mathcal{O}_{x'}\widehat{\times}S$ we may identify β_i and $\beta_i^\natural$. We may assume S is affine, and by extending k we may assume $k(x')=k$. Choose a uniformizer ϖ at x' such that $\sigma(\varpi)=-\varpi$, then $\operatorname{Spec}\mathcal{O}_{x'}\widehat{\times}S=\operatorname{Spec}\mathcal{O}_S[[\varpi]]$. After trivializing $\mathcal{L}_i,\mathcal{L}_i'(R')$ near $x'\times S$, we may assume $f_1=f_1'=\varpi-a$ for some $a\in\mathcal{O}_S$, $\alpha_0=\alpha_1\in\mathcal{O}_S[[\varpi]]$. The diagram (5.37) implies the equation in $\mathcal{O}_S[[\varpi]]$,

$$f_1'\cdot\beta_0=\sigma^*f_1\cdot\beta_1,$$

where $\beta_0,\beta_1\in\mathcal{O}_S[[\varpi]]$. This equation is the same as

$$(\varpi-a)\beta_0(\varpi)=(-\varpi-a)\beta_1(\varpi). \tag{5.39}$$

Recall that we also have the condition $\beta_i|_{R'\times S}=\alpha_i|_{R'\times S}$ for $i=0,1$, which implies that $\beta_0(0)=\alpha_0(0)=\alpha_1(0)=\beta_1(0)$, or $\beta_1(\varpi)=\varpi\gamma(\varpi)+\beta_0(\varpi)$ for some $\gamma\in\mathcal{O}_S[[\varpi]]$. Combining this with (5.39) we get

$$2\varpi\beta_0(\varpi)=(-\varpi-a)\varpi\gamma(\varpi).$$

Since ϖ is not a zero divisor, we conclude that $\beta_0(\varpi)=-(\varpi+a)\gamma(\varpi)/2$, hence $\varpi+a$ divides $\beta_0(\varpi)$. This implies that $\Gamma_{\sigma(x_1')}\cap(\operatorname{Spec}\mathcal{O}_{x'}\widehat{\times}S)$ is contained in $\operatorname{div}(\beta_0)\cap(\operatorname{Spec}\mathcal{O}_{x'}\widehat{\times}S)=\operatorname{div}(\beta_0^\natural)\cap(\operatorname{Spec}\mathcal{O}_{x'}\widehat{\times}S)$. The proof of the claim is complete. □

On the other hand, the condition that $\beta_0^\natural$ vanishes along $\Gamma_{\sigma(x'_1)}$ is sufficient for the existence of β_1 making (5.37) commutative. Therefore, in this case, $\mathrm{Hk}^{\mu,\mu'}_{\mathcal{M}}$ is the incidence correspondence for the divisor of $\beta^\natural$ in $\mathcal{M}_d(\mu_\Sigma, \mu'_\Sigma)$ under the description of Lemma 5.13. This gives the isomorphism $\Xi_{\mathcal{H}} : \mathcal{H}_{\underline{\mu\mu'}} \times \mathfrak{S}'_\infty \cong \mathrm{Hk}^{\mu,\mu'}_{\mathcal{M}}$.

Case 2. $\mu_1 \neq \mu'_1$. Let us consider only the case $\mu_1 = 1, \mu'_1 = -1$. We only indicate the modifications from the previous case. In this case, $\mathcal{L}_1 = \mathcal{L}_0(\Gamma_{x'_1})$ but $\mathcal{L}'_1 = \mathcal{L}'_0(-\Gamma_{x'_1})$. We may change $\mathcal{L}'_1$ to $\mathcal{L}'_0(\Gamma_{\sigma(x'_1)})$ (which has the same image as $\mathcal{L}'_0(-\Gamma_{x'_1})$ in Bun_T) so that $\deg \mathcal{L}'_1 - \deg \mathcal{L}_1 = d$ still holds. The diagrams (5.36) and (5.37) now become

$$\begin{array}{ccccc} \mathcal{L}_0 & \hookrightarrow & \mathcal{L}_1 & = & \mathcal{L}_0(\Gamma_{x'_1}) \\ \downarrow{\scriptstyle \alpha_0^\natural} & & \downarrow{\scriptstyle \alpha_1^\natural} & & \\ \mathcal{L}'_0(R' - D_-) & \hookrightarrow & \mathcal{L}'_1(R' - D_-) & = & \mathcal{L}'_0(R' - D_- + \Gamma_{\sigma(x'_1)}), \end{array} \tag{5.40}$$

$$\begin{array}{ccccc} \sigma^*\mathcal{L}_0 & \hookrightarrow & \sigma^*\mathcal{L}_1 & = & (\sigma^*\mathcal{L}_0)(\Gamma_{\sigma(x'_1)}) \\ \downarrow{\scriptstyle \beta_0^\natural} & & \downarrow{\scriptstyle \beta_1^\natural} & & \\ \mathcal{L}'_0(R' - D_+) & \hookrightarrow & \mathcal{L}'_1(R' - D_+) & = & \mathcal{L}'_0(R' - D_+ + \Gamma_{\sigma(x'_1)}). \end{array} \tag{5.41}$$

Now (5.41) imposes no condition on β_0, but (5.40) gives

$$\mathrm{div}(\alpha_0^\natural) + \Gamma_{\sigma(x'_1)} = \mathrm{div}(\alpha_1^\natural) + \Gamma_{x'_1}.$$

An analog of the claim in Case 1 says that $\alpha_0^\natural$ must vanish along $\Gamma_{x'_1}$. Therefore, in this case, $\mathrm{Hk}^{\mu,\mu'}_{\mathcal{M}}$ is the incidence correspondence for the divisor of $\alpha^\natural$ in $\mathcal{M}_d(\mu_\Sigma, \mu'_\Sigma)$ under the description of Lemma 5.13. This gives the isomorphism $\Xi_{\mathcal{H}}$. □

5.4. *Proof of Theorem* 5.6.

5.4.1. *Geometric facts.* We first collect some geometric facts about the stacks involved in the constructions in Section 5.2.

PROPOSITION 5.17.

(1) *The stack* $\mathrm{Bun}_G(\Sigma)$ *is smooth of pure dimension* $3(g-1)+N$.
(2) *The stack* $\mathrm{Hk}^r_G(\Sigma)$ *is smooth of pure dimension* $3(g-1)+N+2r$.
(3) *The stack* Bun_T *is smooth*, DM *and proper over* k *of pure dimension* $g'-g = g-1+\frac{1}{2}\rho$.
(4) *The stack* Hk^μ_T *is smooth*, DM *and proper over* k *of pure dimension* $g-1+\frac{1}{2}\rho+r$.

(5) *The morphisms* $\overleftarrow{p_H}, \overrightarrow{p_H} : H_d(\Sigma) \to \mathrm{Bun}_G(\Sigma)$ *are representable and smooth of pure relative dimension* $2d$. *In particular,* $H_d(\Sigma)$ *is a smooth algebraic stack over* k *of pure dimension* $2d + 3(g-1) + N$.
(6) *The stack* $\mathrm{Hk}^r_{H,d}(\Sigma)$ *has dimension* $2d + 2r + 3(g-1) + N$.
(7) *For* $d \geq 2g' - 1 + N$, $\mathcal{M}_d(\mu_\Sigma, \mu'_\Sigma)$ *is a smooth and separated* DM *stack pure of dimension* $m = 2d + \rho - N - g + 1$.
(8) *Let* D *be an effective divisor on* U. *The stack* $\mathrm{Sht}^{\mu,\mu'}_{\mathcal{M},D}$ *is proper over* k.

Proof. (1), (3) and (4) are standard facts. (2) follows from Proposition 3.4(4).

(5) Recall the stack H_d defined in [10, §6.3.2], with two maps $\overleftarrow{p}, \overrightarrow{p}$ to Bun_G. We have an open embedding $H_d(\Sigma) \hookrightarrow \mathrm{Bun}_G(\Sigma) \times_{\mathrm{Bun}_G, \overleftarrow{p}} H_d$ because once the Σ-level structure of $\mathcal{E}$ is chosen, it induces a unique Σ-level structure on $\mathcal{E}'$ via φ (which is assumed to be an isomorphism near Σ). Since $\overleftarrow{p} : H_d \to \mathrm{Bun}_G$ is smooth of relative dimension $2d$ by [10, Lemma 6.8(1)], so is its base change $\overleftarrow{p_H}$. A similar argument works for $\overrightarrow{p_H}$.

(6) As in [10, §6.3.4], we have a map $\mathrm{Hk}^r_{H,d}(\Sigma) \to \mathrm{Bun}_G(\Sigma) \times U_d \times X^r$. (The first factor records $\mathcal{E}_0^\dagger$, the second records the divisor of $\det(\varphi_0)$ and the third records x_i.) The same argument as [10, Lemma 6.10] shows that all geometric fibers of this map have dimension $d + r$. (Note that the horizontal maps are allowed to vanish at points in Σ, but this does not complicate the argument because the vertical maps do not vanish at Σ.) Therefore, $\dim \mathrm{Hk}^r_{H,d}(\Sigma) = d + r + d + r + \dim \mathrm{Bun}_G(\Sigma) = 2d + 2r + 3(g-1) + N$.

(7) By Proposition 5.15, $\mathcal{M}_d(\mu_\Sigma, \mu'_\Sigma) \cong \mathcal{M}_d \times \mathfrak{S}'_\infty$. Therefore, the required geometric properties of $\mathcal{M}_d(\mu_\Sigma, \mu'_\Sigma)$ follow from those of $\mathcal{M}_d$ proved in Proposition 5.5(1).

(8) Consider the Cartesian diagram (5.29). Since $\mathrm{Sht}'^r_G(\Sigma; \Sigma_\infty)$ is separated over $\mathfrak{S}'_\infty$ by Proposition 3.9 and $\overleftarrow{p}' : \mathrm{Sht}'^r_G(\Sigma; \Sigma_\infty; h_D) \to \mathrm{Sht}'^r_G(\Sigma; \Sigma_\infty)$ is proper by Lemma 3.13(1), the map

$$(\overleftarrow{p}', \overrightarrow{p}') : \mathrm{Sht}'^r_G(\Sigma; \Sigma_\infty; h_D) \longrightarrow \mathrm{Sht}'^r_G(\Sigma; \Sigma_\infty) \times_{\mathfrak{S}'_\infty} \mathrm{Sht}'^r_G(\Sigma; \Sigma_\infty)$$

is proper. This implies that

$$\mathrm{Sht}^{\mu,\mu'}_{\mathcal{M},D} \longrightarrow \mathrm{Sht}^{\underline{\mu}}_T(\mu_\infty \cdot \Sigma'_\infty) \times_{\mathfrak{S}'_\infty} \mathrm{Sht}^{\underline{\mu}'}_T(\mu'_\infty \cdot \Sigma'_\infty)$$

is proper. Since $\mathrm{Sht}^{\underline{\mu}}_T(\mu_\infty \cdot \Sigma'_\infty)$ and $\mathrm{Sht}^{\underline{\mu}'}_T(\mu'_\infty \cdot \Sigma'_\infty)$ are proper over k by Corollary 4.3, so is $\mathrm{Sht}^{\mu,\mu'}_{\mathcal{M},D}$. □

PROPOSITION 5.18. *Suppose* D *is an effective divisor on* U *of degree* $d \geq \max\{2g' - 1 + N, 2g\}$. *Then the diagram* (5.18) *satisfies all the conditions for applying the Octahedron Lemma* [10, Th. A.10].

Proof. We refer to [10, Th. A.10] for the statement of the conditions.

Condition (1): We need to show the smoothness of all members in the diagram (5.18) except for $\mathrm{Hk}'^{r}_{H,d}(\Sigma)$. This is done in Proposition 5.17.

Condition (2): We need to check that

$$\mathcal{M}_d(\mu_\Sigma,\mu'_\Sigma), \mathcal{M}_d(\mu_\Sigma,\mu'_\Sigma)^2, \mathrm{Sht}^{\underline{\mu}}_T(\mu_\infty\cdot\Sigma'_\infty)\times_{\mathfrak{S}'_\infty}\mathrm{Sht}^{\underline{\mu}'}_T(\mu'_\infty\cdot\Sigma'_\infty)$$

and

$$\mathrm{Sht}'^{r}_G(\Sigma;\Sigma_\infty)\times_{\mathfrak{S}'_\infty}\mathrm{Sht}'^{r}_G(\Sigma;\Sigma_\infty)$$

are smooth of the expected dimensions. These facts follow from Proposition 5.17(7), Corollary 4.3 and Proposition 3.9.

Condition (3): We need to show that the diagrams (5.24) and (5.20) satisfy either the conditions in [10, §A.2.8], or the conditions in [10, §A.2.10].

We first show that (5.24) satisfies the conditions in [10, §A.2.8]. We claim that $\mathrm{Hk}^{\mu,\mu'}_{\mathcal{M},d}$ is a DM stack that admits a finite flat presentation. By Proposition 5.15, $\mathcal{M}_d(\mu_\Sigma,\mu'_\Sigma)\cong\mathcal{M}_d\times\mathfrak{S}'_\infty$. By Proposition 5.5(5), $\mathcal{M}_d$ is DM and admits a finite flat presentation; therefore, the same is true for $\mathcal{M}_d(\mu_\Sigma,\mu'_\Sigma)$. Since the map $p_{\mathcal{M},0}:\mathrm{Hk}^{\mu,\mu'}_{\mathcal{M},d}\to\mathcal{M}_d(\mu_\Sigma,\mu'_\Sigma)$ is schematic, the same is true for $\mathcal{M}_d(\mu_\Sigma,\mu'_\Sigma)$. It remains to check that $\theta^{\mu,\mu'}_{\mathrm{Hk}}$ can be factored into a regular local immersion and a smooth relative DM map. It suffices to show the same thing for $\theta^{\mu}_{\mathrm{Hk}}:\mathrm{Hk}^{\underline{\mu}}_T\times\mathfrak{S}'_\infty\to\mathrm{Hk}'^{r}_G(\Sigma)$ (and the same result applies to μ' as well). The argument is similar to that in [10, Lemma 6.11(1)], and we only give a sketch here. We may enlarge the set Σ to $\widetilde{\Sigma}\subset|X-R|$ such that $\deg\widetilde{\Sigma}>\rho/2$. By enlarging the base field k, we may assume that all points in $\nu^{-1}(\widetilde{\Sigma})$ are defined over k. Choose a section of $\nu^{-1}(\widetilde{\Sigma})\to\widetilde{\Sigma}$ extending the existing section μ_f, and call this section $\widetilde{\Sigma}'$. Using $\widetilde{\Sigma}'$ we have a map $\widetilde{\theta}^{\mu}_{\mathrm{Hk}}:\mathrm{Hk}^{\underline{\mu}}_T\to\mathrm{Hk}'^{r}_G(\widetilde{\Sigma})$. Since the projection $\mathrm{Hk}'^{r}_G(\widetilde{\Sigma})\to\mathrm{Hk}'^{r}_G(\Sigma)$ is smooth and schematic, it suffices to show that $\widetilde{\theta}^{\mu}_{\mathrm{Hk}}:\mathrm{Hk}^{\underline{\mu}}_T=\mathrm{Bun}_T\times X'^r\to\mathrm{Hk}'^{r}_G(\widetilde{\Sigma})$ is a regular local embedding. To check this, we calculate the tangent map of $\widetilde{\theta}^{\mu}_{\mathrm{Hk}}$ at a geometric point $b=(\mathcal{L},x'_1,\dots,x'_r)\in\mathrm{Bun}_T(K)\times X'^r(K)$. Or rather we calculate the relative tangent map with respect to the projections to X'^r. We base change to K without changing notation. The relative tangent complex of $\mathrm{Hk}^{\underline{\mu}}_T$ at b is $\mathrm{H}^*(X,\mathcal{O}_{X'}/\mathcal{O}_X)[1]$. The relative tangent complex of $\mathrm{Hk}'^{r}_G(\widetilde{\Sigma})$ at $\widetilde{\theta}^{\mu}_{\mathrm{Hk}}(b)$ is $\mathrm{H}^*(X,\mathrm{Ad}^{\underline{x}',\widetilde{\Sigma}}(\nu_*\mathcal{L}))[1]$, where $\mathrm{Ad}^{\underline{x}',\widetilde{\Sigma}}(\nu_*\mathcal{L})=\underline{\mathrm{End}}^{\underline{x}',\widetilde{\Sigma}}(\nu_*\mathcal{L})/\mathcal{O}_X$, and $\mathrm{End}^{\underline{x}',\widetilde{\Sigma}}(\nu_*\mathcal{L})$ is the endomorphism sheaf of the chain $\nu_*\mathcal{L}\to\nu_*(\mathcal{L}(x'_1))\to\cdots$ preserving the level structures at $\widetilde{\Sigma}$. The tangent map of $\widetilde{\theta}^{\mu}_{\mathrm{Hk}}$ is induced by a natural embedding $e:\nu_*\mathcal{O}_{X'}/\mathcal{O}_X\hookrightarrow\mathrm{Ad}^{\underline{x}',\widetilde{\Sigma}'}(\nu_*\mathcal{L})$. A calculation similar to Lemma 5.12 gives

$$\mathrm{End}^{\underline{x}',\widetilde{\Sigma}}(\nu_*\mathcal{L})\subset\nu_*(\mathcal{O}_{X'}(R'))\oplus_{R'}\nu_*(\sigma^*\mathcal{L}^{-1}\otimes\mathcal{L}(R'-\widetilde{\Sigma}'')),$$

where $\widetilde{\Sigma}'' = \sigma(\widetilde{\Sigma}')$. Therefore, we have

$$\mathrm{Ad}^{\underline{x}',\widetilde{\Sigma}}(\nu_*\mathcal{L}) \subset (\nu_*(\mathcal{O}_{X'}(R'))/\mathcal{O}_X) \oplus_{R'} \nu_*(\sigma^*\mathcal{L}^{-1} \otimes \mathcal{L}(R' - \widetilde{\Sigma}'')),$$

under which e corresponds to the embedding of $\nu_*\mathcal{O}_{X'}/\mathcal{O}_X$ into the first factor. One checks that the projection $\mathrm{coker}(e) \to \nu_*(\sigma^*\mathcal{L}^{-1} \otimes \mathcal{L}(R' - \widetilde{\Sigma}''))$ is injective. The latter having degree $\rho/2 - \deg \widetilde{\Sigma} < 0$, we have $\mathrm{H}^0(X, \mathrm{coker}(e)) = 0$, which implies that the tangent map of $\widetilde{\theta}^{\mu}_{\mathrm{Hk}}$ is injective.

Next we show that (5.20) satisfies the conditions in [10, §A.2.10]. The argument is similar to that of [10, Lemma 6.14(1)], using the smoothness of $H_d(\Sigma)$ proved in Proposition 5.17(5).

Condition (4): We need to show that (5.26) and (5.27) both satisfy the conditions in [10, §A.2.8]. Again the argument is completely similar to the corresponding argument in the proof of [10, Th. 6.6]. We omit details here. □

5.4.2. *The cycle* ζ. Using the dimension calculations in Proposition 5.17(6), (4) and (2), we have

$$\dim \mathrm{Hk}'^{r}_{H,d}(\Sigma) + \dim(\mathrm{Hk}^{\underline{\mu}}_{T} \times \mathrm{Hk}^{\underline{\mu}'}_{T} \times \mathfrak{S}'_{\infty}) - 2\dim \mathrm{Hk}'^{r}_{G}(\Sigma) = m = 2d + \rho - N - g + 1.$$

Therefore, the Cartesian diagram (5.24) defines a cycle

$$\zeta = (\theta^{\mu,\mu'}_{\mathrm{Hk}})^{!}[\mathrm{Hk}'^{r}_{H,d}(\Sigma)] \in \mathrm{Ch}_m(\mathrm{Hk}^{\mu,\mu'}_{\mathcal{M},d}). \tag{5.42}$$

LEMMA 5.19. *Assume* $d \geq \max\{2g' - 1 + N, 2g + N\}$. *Let*

$$\zeta^{\sharp} \in \mathrm{Ch}_*(\mathcal{H}_{\underline{\mu\mu'}} \times \mathfrak{S}'_{\infty})$$

be the pullback of ζ *under the isomorphism* $\Xi_{\mathcal{H}}$. *Then when restricted over* $\mathcal{A}^{\diamondsuit}_d$, $\zeta^{\sharp}$ *coincides with the fundamental class of* $\mathcal{H}_{\underline{\mu\mu'}} \times \mathfrak{S}'_{\infty}$.

Proof. We have a map $\mathrm{Hk}^{r}_{H,d}(\Sigma) \to U_d \times X^r$ similar to the one defined in [10, §6.3.4]. Let $(U_d \times X^r)^{\circ}$ be the open subset consisting of $(D, x_1, \dots, x_r)$ such that each x_i is disjoint from the support of D. Let $\mathrm{Hk}'^{r,\circ}_{H,d}(\Sigma)$ be the preimage of $(U_d \times X^r)^{\circ}$. Similarly, let $\mathrm{Hk}^{\mu,\mu',\circ}_{\mathcal{M},d}$ be the preimage of $(U_d \times X^r)^{\circ}$ in $\mathrm{Hk}^{\mu,\mu'}_{\mathcal{M},d}$, which corresponds under $\Xi_{\mathcal{H}}$ to an open subset of the form $\mathcal{H}^{\circ}_{\underline{\mu\mu'}} \times \mathfrak{S}'_{\infty}$.

We have a map $\mathrm{Hk}^{r}_{H,d}(\Sigma) \to \mathrm{Hk}^{r}_{G}(\Sigma) \times_{\mathrm{Bun}_G(\Sigma)} H_d(\Sigma)$ by considering the top row and left column of the diagram (5.17). When restricted to $(U_d \times X^r)^{\circ}$, this map is an isomorphism. Therefore, $\mathrm{Hk}^{r,\circ}_{H,d}(\Sigma)$, and hence $\mathrm{Hk}'^{r,\circ}_{H,d}(\Sigma)$ is smooth of dimension $3(g-1) + N + 2r + 2d$. Restricting the diagram (5.24) to $(U_d \times X^r)^{\circ}$, $\mathrm{Hk}^{\mu,\mu',\circ}_{\mathcal{M},d}$ is the intersection of smooth stacks with the expected dimension $\dim \mathrm{Hk}'^{r,\circ}_{H,d}(\Sigma) + \dim(\mathrm{Hk}^{\underline{\mu}}_{T} \times \mathrm{Hk}^{\underline{\mu}'}_{T} \times \mathfrak{S}'_{\infty}) - \dim \mathrm{Hk}'^{r}_{G}(\Sigma) = m$. Therefore, ζ is the fundamental class when restricted to $\mathrm{Hk}^{\mu,\mu',\circ}_{\mathcal{M},d} = \mathcal{H}^{\circ}_{\underline{\mu\mu'}} \times \mathfrak{S}'_{\infty}$.

It remains to show that $\dim(\mathcal{H}^{\diamondsuit}_{\underline{\mu\mu'}} - \mathcal{H}^{\circ}_{\underline{\mu\mu'}}) < \dim \mathcal{H}^{\diamondsuit}_{\underline{\mu\mu'}}$. The maps $\mathcal{H}^{\diamondsuit}_{\underline{\mu\mu'}} \to \mathcal{M}^{\diamondsuit}_d \to \mathcal{A}^{\diamondsuit}_d$ are finite surjective. On the other hand, as in [10, §6.4.3], the image of $\mathcal{H}^{\diamondsuit}_{\underline{\mu\mu'}} - \mathcal{H}^{\circ}_{\underline{\mu\mu'}}$ in $\mathcal{A}^{\diamondsuit}_d$ lies in the closed substack $\mathcal{C}_d$ consisting of those $(\Delta, \Theta_R, \iota, a, b, \vartheta_R)$ where $\mathrm{div}(a)$ and $\mathrm{div}(b)$ (both are divisors of degree $d+\rho$ on X) have one point in common that lies in U. Therefore, it suffices to show that $\dim \mathcal{C}_d < \dim \mathcal{A}_d = m$. Now $\mathcal{C}_d$ is contained in the image of a map $U \times (X^{\sqrt{R}}_{d+\rho-N_--1} \times_{\mathrm{Pic}_X^{\sqrt{R};\sqrt{R}}} X^{\sqrt{R}}_{d+\rho-N_+-1}) \to X^{\sqrt{R}}_{d+\rho-N_-} \times_{\mathrm{Pic}_X^{\sqrt{R};\sqrt{R}}} X^{\sqrt{R}}_{d+\rho-N_+}$. Using $d \geq 2g+N$ we may calculate the dimension of $X^{\sqrt{R}}_{d+\rho-N_--1} \times_{\mathrm{Pic}_X^{\sqrt{R};\sqrt{R}}} X^{\sqrt{R}}_{d+\rho-N_+-1}$ by Riemann-Roch, from which we conclude again that $\dim \mathcal{C}_d \leq m-1$. This completes the proof. □

5.4.3. Consider the cycle

$$(\mathrm{id}, \mathrm{Fr}_{\mathcal{M}_d(\mu_\Sigma, \mu'_\Sigma)})^! \zeta \in \mathrm{Ch}_0(\mathrm{Sht}^{\mu,\mu'}_{\mathcal{M},d}).$$

This is well defined because $\mathcal{M}_d(\mu_\Sigma, \mu'_\Sigma)$ is smooth DM by Proposition 5.17(7), and hence $(\mathrm{id}, \mathrm{Fr})$ is a regular local immersion. Let

$$((\mathrm{id}, \mathrm{Fr}_{\mathcal{M}_d(\mu_\Sigma, \mu'_\Sigma)})^! \zeta)_D \in \mathrm{Ch}_0(\mathrm{Sht}^{\mu,\mu'}_{\mathcal{M},D})$$

be its D-component. Since $\mathrm{Sht}^{\mu,\mu'}_{\mathcal{M},D}$ is proper by Proposition 5.17(8), it makes sense to take degrees of 0-cycles on it. Hence we define

$$\langle \zeta, \Gamma(\mathrm{Fr}_{\mathcal{M}_d(\mu_\Sigma, \mu'_\Sigma)}) \rangle_D := \deg((\mathrm{id}, \mathrm{Fr}_{\mathcal{M}_d(\mu_\Sigma, \mu'_\Sigma)})^! \zeta)_D \in \mathbb{Q}.$$

THEOREM 5.20. *Suppose D is an effective divisor on U of degree $d \geq \max\{2g'-1+N, 2g\}$. We have*

$$\left(\prod_{x' \in \Sigma'_\infty} d_{x'} \right) \cdot \mathbb{I}^{\mu,\mu'}(h_D) = \langle \zeta, \Gamma(\mathrm{Fr}_{\mathcal{M}_d(\mu_\Sigma, \mu'_\Sigma)}) \rangle_D. \tag{5.43}$$

Proof. From the definition of Heegner–Drinfeld cycles, it is easy to see using the diagram (5.29) that

$$\left(\prod_{x' \in \Sigma'_\infty} d_{x'} \right) \cdot \mathbb{I}^{\mu,\mu'}(h_D) = \deg\left((\theta'^{\mu} \times \theta'^{\mu'})^! [\mathrm{Sht}'^{r}_G(\Sigma; \Sigma_\infty; h_D)] \right). \tag{5.44}$$

On the other hand, applying the Octahedron Lemma [10, Th. A.10] to (5.18), we get that

$$\begin{aligned}
&(\theta'^{\mu} \times \theta'^{\mu'})^! (\mathrm{id}, \mathrm{Fr}_{H_d(\Sigma)})^! [\mathrm{Hk}'^{r}_{H,d}(\Sigma) \times \mathfrak{S}'_\infty] \\
&= (\mathrm{id}, \mathrm{Fr}_{\mathcal{M}_d(\mu_\Sigma, \mu'_\Sigma)})^! (\theta^{\mu,\mu'}_{\mathrm{Hk}} \times \mathrm{id}_{\mathfrak{S}'_\infty})^! [\mathrm{Hk}'^{r}_{H,d}(\Sigma) \times \mathfrak{S}'_\infty] \\
&= (\mathrm{id}, \mathrm{Fr}_{\mathcal{M}_d(\mu_\Sigma, \mu'_\Sigma)})^! \zeta \in \mathrm{Ch}_0(\mathrm{Sht}^{\mu,\mu'}_{\mathcal{M},d}).
\end{aligned} \tag{5.45}$$

If we can show that

$$(\mathrm{id}, \mathrm{Fr}_{H_d(\Sigma)})^! [\mathrm{Hk}'^{r}_{H,d}(\Sigma) \times \mathfrak{S}'_\infty] = [\mathrm{Sht}'^{r}_{H,d}(\Sigma; \Sigma_\infty)], \tag{5.46}$$

then extracting the D-components of (5.45) and (5.46) identifies

$$[(\theta'^{\mu} \times \theta'^{\mu'})^![\mathrm{Sht}'^r_G(\Sigma; \Sigma_\infty; h_D)]$$

with the cycle $((\mathrm{id}, \mathrm{Fr}_{\mathcal{M}_d(\mu_\Sigma, \mu'_\Sigma)})^! \zeta)_D$. Taking degrees then identifies the right side of (5.44) with the right side of (5.43), and we are done. Therefore, it remains to show (5.46). The argument is similar to [10, Lemma 6.14(2)]. Let $\mathrm{Sht}'^{r,\circ}_{H,d}(\Sigma; \Sigma_\infty) \subset \mathrm{Sht}'^r_{H,d}(\Sigma; \Sigma_\infty)$ be the preimage of $(U_d \times X^r)^\circ$. By (5.22), $\mathrm{Sht}'^{r,\circ}_{H,d}(\Sigma; \Sigma_\infty)$ is the disjoint union over $D \in U_d(k)$ of

$$(\mathrm{Sht}^r_G(\Sigma; \Sigma_\infty; h_D))|_{(X-D)^r} \times_{X^r} X'^r.$$

By Lemma 3.13(2), $\mathrm{Sht}^r_G(\Sigma; \Sigma_\infty; h_D)|_{(X-D)^r}$ is smooth of dimension $2r$, which is the expected dimension from the diagram (5.20). Therefore, the restriction of $(\mathrm{id}, \mathrm{Fr}_{H_d(\Sigma)})^![\mathrm{Hk}'^r_{H,d}(\Sigma) \times \mathfrak{S}'_\infty]$ to $\mathrm{Sht}'^{r,\circ}_{H,d}(\Sigma; \Sigma_\infty)$ is the fundamental class. By Lemma 3.13(3), $\mathrm{Sht}^r_G(\Sigma; \Sigma_\infty; h_D)$ has the same dimension as its restriction over $(X - D)^r$, hence $\dim \mathrm{Sht}'^{r,\circ}_{H,d}(\Sigma; \Sigma_\infty) = \mathrm{Sht}'^r_{H,d}(\Sigma; \Sigma_\infty)$, therefore (5.46) holds as cycles on the whole of $\mathrm{Sht}'^r_{H,d}(\Sigma; \Sigma_\infty)$. This finishes the proof. □

5.4.4. *Proof of Theorem* 5.6. Now we can deduce Theorem 5.6 from Theorem 5.20.

Consider the diagram (5.26). Moving the Atkin–Lehner automorphism of $\mathcal{M}_d(\mu_\Sigma, \mu'_\Sigma)$ from the vertical arrow to the horizontal arrow, we get another Cartesian diagram:

(5.47)
$$\begin{array}{ccc} \mathrm{Sht}^{\mu,\mu'}_{\mathcal{M},d} & \longrightarrow & \mathrm{Hk}^{\mu,\mu'}_{\mathcal{M}} \\ \downarrow & & \downarrow{\scriptstyle (p_{\mathcal{M},0}, p_{\mathcal{M},r})} \\ \mathcal{M}_d(\mu_\Sigma, \mu'_\Sigma) & \xrightarrow{(\mathrm{id}, \mathrm{AL}^{-1}_{\mathcal{M},\infty} \circ \mathrm{Fr})} & \mathcal{M}_d(\mu_\Sigma, \mu'_\Sigma) \times \mathcal{M}_d(\mu_\Sigma, \mu'_\Sigma). \end{array}$$

From this we get

(5.48)
$$(\mathrm{id}, \mathrm{Fr}_{\mathcal{M}_d(\mu_\Sigma, \mu'_\Sigma)})^! \zeta = (\mathrm{id}, \mathrm{AL}^{-1}_{\mathcal{M},\infty} \circ \mathrm{Fr})^! \zeta \in \mathrm{Ch}_0(\mathrm{Sht}^{\mu,\mu'}_{\mathcal{M},d}).$$

Define $\mathcal{S}_{\underline{\mu\mu'}}$ by the Cartesian diagram

(5.49)
$$\begin{array}{ccc} \mathcal{S}_{\underline{\mu\mu'}} & \longrightarrow & \mathcal{H}_{\underline{\mu\mu'}} \\ \downarrow & & \downarrow{\scriptstyle (p_{\mathcal{H},0}, p_{\mathcal{H},r})} \\ \mathcal{M}_d & \xrightarrow{(\mathrm{id}, \mathrm{Fr})} & \mathcal{M}_d \times \mathcal{M}_d. \end{array}$$

Using the isomorphisms $\Xi_{\mathcal{M}}$ and $\Xi_{\mathcal{H}}$ established in Propositions 5.15 and 5.16, (5.47) is isomorphic to the Cartesian diagram

$$\begin{array}{ccc} \mathcal{S}_{\underline{\mu\mu'}} \times \mathfrak{S}'_\infty & \longrightarrow & \mathcal{H}_{\underline{\mu\mu'}} \times \mathfrak{S}'_\infty \\ \downarrow & & \downarrow {\scriptstyle (p_{\mathcal{H},0}\times \mathrm{id}_{\mathfrak{S}'_\infty}, p_{\mathcal{H},r}\times \mathrm{id}_{\mathfrak{S}'_\infty})} \\ \mathcal{M}_d \times \mathfrak{S}'_\infty & \xrightarrow{(\mathrm{id}, \mathrm{Fr}_{\mathcal{M}_d}\times \mathrm{id}_{\mathfrak{S}'_\infty})} & (\mathcal{M}_d \times \mathfrak{S}'_\infty) \times (\mathcal{M}_d \times \mathfrak{S}'_\infty). \end{array} \tag{5.50}$$

Here we are using Proposition 5.15(2) to identify $\mathrm{AL}^{-1}_{\mathcal{M},\infty} \circ \mathrm{Fr}$ on $\mathcal{M}_d(\mu_\Sigma, \mu'_\Sigma)$ with $\mathrm{Fr}_{\mathcal{M}_d} \times \mathrm{id}_{\mathfrak{S}'_\infty}$ on $\mathcal{M}_d \times \mathfrak{S}'_\infty$. In particular, we get an isomorphism

$$\Xi_{\mathcal{S}} : \mathcal{S}_{\underline{\mu\mu'}} \times \mathfrak{S}'_\infty \xrightarrow{\sim} \mathrm{Sht}^{\mu,\mu'}_{\mathcal{M},d}.$$

Recall that $\zeta^\sharp \in \mathrm{Ch}_m(\mathcal{H}_{\underline{\mu\mu'}} \times \mathfrak{S}'_\infty)$ is the transport of ζ under the isomorphism $\Xi_{\mathcal{H}}$. Then we have

$$(\mathrm{id}, \mathrm{AL}^{-1}_{\mathcal{M},\infty} \circ \mathrm{Fr})^! \zeta = (\mathrm{id}, \mathrm{Fr}_{\mathcal{M}_d} \times \mathrm{id}_{\mathfrak{S}'_\infty})^! \zeta^\sharp \in \mathrm{Ch}_0(\mathcal{S}_{\underline{\mu\mu'}} \times \mathfrak{S}'_\infty). \tag{5.51}$$

By Lemma 5.19, $\zeta^\sharp$ is the fundamental cycle of $\mathcal{H}_{\underline{\mu\mu'}} \times \mathfrak{S}'_\infty$ when restricted to $\mathcal{A}^\diamondsuit_d$. By Proposition 5.5(4), the complement of $\mathcal{M}^\diamondsuit_d \times_{\mathcal{A}^\diamondsuit_d} \mathcal{M}^\diamondsuit_d$ in $\mathcal{M}_d \times_{\mathcal{A}_d} \mathcal{M}_d$ has dimension strictly smaller than $\dim \mathcal{M}_d$. (The condition $d \geq 2g' - 1 + N = 4g - 3 + \rho + N$ implies $d \geq 3g - 2 + N$.) Therefore, we may replace $\zeta^\sharp$ with the fundamental cycle of the closure of $\mathcal{H}_{\underline{\mu\mu'}}|_{\mathcal{A}^\diamondsuit_d} \times \mathfrak{S}'_\infty$, and the intersection number on the right-hand side of (5.51) does not change. We denote the latter by $\overline{\mathcal{H}}^\diamondsuit_{\underline{\mu\mu'}} \times \mathfrak{S}'_\infty$. Combining (5.48) and (5.51), we get

$$\begin{aligned} &(\mathrm{id}, \mathrm{Fr}_{\mathcal{M}_d} \times \mathrm{id}_{\mathfrak{S}'_\infty})^! \zeta^\sharp \\ &= (\mathrm{id}, \mathrm{Fr}_{\mathcal{M}_d} \times \mathrm{id}_{\mathfrak{S}'_\infty})^! [\overline{\mathcal{H}}^\diamondsuit_{\underline{\mu\mu'}} \times \mathfrak{S}'_\infty] \\ &= ((\mathrm{id}, \mathrm{Fr}_{\mathcal{M}_d})^! [\overline{\mathcal{H}}^\diamondsuit_{\underline{\mu\mu'}}]) \times [\mathfrak{S}'_\infty] \in \mathrm{Ch}_0(\mathcal{S}_{\underline{\mu\mu'}} \times \mathfrak{S}'_\infty). \end{aligned}$$

Taking the degree of the D-component, we get

$$\langle \zeta, \Gamma(\mathrm{Fr}_{\mathcal{M}_d(\mu_\Sigma, \mu'_\Sigma)}) \rangle_D = \deg(\mathfrak{S}'_\infty) \cdot \langle [\overline{\mathcal{H}}^\diamondsuit_{\underline{\mu\mu'}}], \Gamma(\mathrm{Fr}_{\mathcal{M}_d}) \rangle_D.$$

Using Theorem 5.20, we get

$$\begin{aligned} \mathbb{I}^{\mu,\mu'}(h_D) &= \left(\prod_{x' \in \Sigma'_\infty} d_{x'} \right)^{-1} \langle \zeta, \Gamma(\mathrm{Fr}_{\mathcal{M}_d(\mu_\Sigma, \mu'_\Sigma)}) \rangle_D \\ &= \left(\prod_{x' \in \Sigma'_\infty} d_{x'} \right)^{-1} \deg(\mathfrak{S}'_\infty) \cdot \langle [\overline{\mathcal{H}}^\diamondsuit_{\underline{\mu\mu'}}], \Gamma(\mathrm{Fr}_{\mathcal{M}_d}) \rangle_D \\ &= \langle [\overline{\mathcal{H}}^\diamondsuit_{\underline{\mu\mu'}}], \Gamma(\mathrm{Fr}_{\mathcal{M}_d}) \rangle_D. \end{aligned}$$

It remains to calculate $\langle[\overline{\mathcal{H}}^{\diamondsuit}_{\underline{\mu\mu'}}],\Gamma(\mathrm{Fr}_{\mathcal{M}_d})\rangle_D$. Note that $\mathcal{H}_{\underline{\mu\mu'}}$ is a self-correspondence of $\mathcal{M}_d$ over $\mathcal{A}_d$. By the discussion in [10, §A.4.5], the map $\mathcal{S}_{\underline{\mu\mu'}}\to\mathcal{M}_d\xrightarrow{f^\flat_d}\mathcal{A}^\flat_d$ lands in the rational points $\mathcal{A}^\flat_d(k)$, hence we have a decomposition

$$\mathcal{S}_{\underline{\mu\mu'}}=\coprod_{a\in\mathcal{A}^\flat_d(k)}\mathcal{S}_{\underline{\mu\mu'}}(a).$$

Under the isomorphism $\Xi_{\mathcal{M}}$, this gives a refinement of the decomposition (5.28), namely,

$$\mathrm{Sht}^{\mu,\mu'}_{\mathcal{M},D}\xleftarrow[\sim]{\Xi_{\mathcal{S}}}\textstyle\coprod_{a\in\mathcal{A}^\flat_D(k)}\mathcal{S}_{\underline{\mu\mu'}}(a)\times\mathfrak{S}'_\infty.$$

The fundamental cycle $[\overline{\mathcal{H}}^{\diamondsuit}_{\underline{\mu\mu'}}]$ gives a cohomological correspondence between the constant sheaf on $\mathcal{M}_d$ and itself. It induces an endomorphism of the complex $\mathbf{R}f_{d,!}\mathbb{Q}_\ell$:

$$f_{d,!}[\overline{\mathcal{H}}^{\diamondsuit}_{\underline{\mu\mu'}}]:\mathbf{R}f_{d,!}\mathbb{Q}_\ell\longrightarrow\mathbf{R}f_{d,!}\mathbb{Q}_\ell.$$

Taking direct image under Ω, we also get an endomorphism of $\mathbf{R}f^\flat_{d,!}\mathbb{Q}_\ell$

$$f^\flat_{d,!}[\overline{\mathcal{H}}^{\diamondsuit}_{\underline{\mu\mu'}}]:\mathbf{R}f^\flat_{d,!}\mathbb{Q}_\ell\longrightarrow\mathbf{R}f^\flat_{d,!}\mathbb{Q}_\ell.$$

Applying the Lefschetz trace formula [10, Prop. A.12] to the diagram (5.49) (which is stated for S being a scheme, so we apply it to the map $f^\flat_d$ rather than f_d), we get that

$$\langle[\overline{\mathcal{H}}^{\diamondsuit}_{\underline{\mu\mu'}}],\Gamma(\mathrm{Fr}_{\mathcal{M}_d})\rangle_D=\sum_{a\in\mathcal{A}^\flat_D(k)}\mathrm{Tr}(f^\flat_{d,!}[\overline{\mathcal{H}}^{\diamondsuit}_{\underline{\mu\mu'}}]\circ\mathrm{Fr}_a,(\mathbf{R}f^\flat_{d,!}\mathbb{Q}_\ell)_a).\tag{5.52}$$

Since $\mathcal{H}_{\underline{\mu\mu'}}$ is the composition of r_+ times $\mathcal{H}_+$ and r_- times $\mathcal{H}_-$, the cohomological correspondence $[\overline{\mathcal{H}}^{\diamondsuit}_{\underline{\mu\mu'}}]$ is equal to the composition of r_+ times $[\overline{\mathcal{H}}^{\diamondsuit}_+]$ and r_- times $[\overline{\mathcal{H}}^{\diamondsuit}_-]$ over $\mathcal{A}^{\diamondsuit}_d$. By Proposition 5.5(4), the complement of $\mathcal{M}^{\diamondsuit}_d\times_{\mathcal{A}^{\diamondsuit}_d}\mathcal{M}^{\diamondsuit}_d$ in $\mathcal{M}_d\times_{\mathcal{A}_d}\mathcal{M}_d$ has dimension strictly smaller than $\dim\mathcal{M}_d$; therefore, $[\overline{\mathcal{H}}^{\diamondsuit}_{\underline{\mu\mu'}}]$ and the composition of r_+ times $[\overline{\mathcal{H}}^{\diamondsuit}_+]$ and r_- times $[\overline{\mathcal{H}}^{\diamondsuit}_-]$ induce the same endomorphism on $f_{d,!}\mathbb{Q}_\ell$. This implies

$$f_{d,!}[\overline{\mathcal{H}}^{\diamondsuit}_{\underline{\mu\mu'}}]=(f_{d,!}[\overline{\mathcal{H}}^{\diamondsuit}_+])^{r_+}\circ(f_{d,!}[\overline{\mathcal{H}}^{\diamondsuit}_-])^{r_-}\in\mathrm{End}(\mathbf{R}f_{d,!}\mathbb{Q}_\ell).$$

Taking direct image under Ω, we get

$$f^\flat_{d,!}[\overline{\mathcal{H}}^{\diamondsuit}_{\underline{\mu\mu'}}]=(f^\flat_{d,!}[\overline{\mathcal{H}}^{\diamondsuit}_+])^{r_+}\circ(f^\flat_{d,!}[\overline{\mathcal{H}}^{\diamondsuit}_-])^{r_-}\in\mathrm{End}(\mathbf{R}f^\flat_{d,!}\mathbb{Q}_\ell).$$

This combined with (5.52) gives (5.14). The proof of Theorem 5.6 is now complete.

6. The moduli stack $\mathcal{N}_d$ and orbital integrals

In this section we introduce another moduli stack $\mathcal{N}_d$, similar to $\mathcal{M}_d$. The point-counting on $\mathcal{N}_d$ is closely related to orbital integrals appearing in Jacquet's RTF we set up in Section 2 for our specific test functions.

6.1. *Definition of $\mathcal{N}_d$.*

6.1.1. Our moduli space $\mathcal{N}_d$ depends on the ramification set R with degree ρ, a fixed finite set Σ and a decomposition

$$\Sigma = \Sigma_+ \sqcup \Sigma_-, \quad N_\pm = \deg \Sigma_\pm.$$

In our application, such a decomposition comes from a pair $\mu, \mu' \in \mathfrak{T}_{r,\Sigma}$, for which we take $\Sigma_\pm = \Sigma_\pm(\mu, \mu')$ as in (4.5) and (4.6). We are also assuming that $\Sigma \cap R = \varnothing$.

Let $d \geq 0$ be an integer. Let Q_d be the set of quadruples

$$\underline{d} = (d_{11}, d_{12}, d_{21}, d_{22}) \in \mathbb{Z}^4_{\geq 0}$$

satisfying $d_{11} + d_{22} = d_{12} + d_{21} = d + \rho$.

Definition 6.1. Let $\underline{d} \in Q_d$. Let $\widetilde{\mathcal{N}}_{\underline{d}} = \widetilde{\mathcal{N}}_{\underline{d}}(\Sigma_\pm)$ be the stack whose S-points consist of

$$(\mathcal{L}_1^\natural, \mathcal{L}_2^\natural, \mathcal{L}_1'^\natural, \mathcal{L}_2'^\natural, \varphi, \psi_R),$$

where

- For $i = 1, 2$, $\mathcal{L}_i^\natural = (\mathcal{L}_i, \mathcal{K}_{i,R}, \iota_i)$ and $\mathcal{L}_i'^\natural = (\mathcal{L}_i', \mathcal{K}_{i,R}', \iota_i') \in \mathrm{Pic}_X^{\sqrt{R}}(S)$, such that for any geometric point $s \in S$, $\deg(\mathcal{L}_i'|_{X \times s}) - \deg(\mathcal{L}_j|_{X \times s}) = d_{ij}$ for $i, j \in \{1, 2\}$.
- φ is an $\mathcal{O}_{X \times S}$-linear map $\mathcal{L}_1 \oplus \mathcal{L}_2 \to \mathcal{L}_1' \oplus \mathcal{L}_2'$. We write it as a matrix

  $$\varphi = \begin{bmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{bmatrix},$$

  where $\varphi_{ij} : \mathcal{L}_j \to \mathcal{L}_i'$.
- ψ_R is an $\mathcal{O}_{R \times S}$-linear map $\mathcal{K}_{1,R} \oplus \mathcal{K}_{2,R} \to \mathcal{K}_{1,R}' \oplus \mathcal{K}_{2,R}'$. Again we write ψ_R as a matrix

  $$\psi_R = \begin{bmatrix} \psi_{11,R} & \psi_{12,R} \\ \psi_{21,R} & \psi_{22,R} \end{bmatrix}$$

  with $\psi_{ij,R} : \mathcal{K}_{j,R} \to \mathcal{K}_{i,R}'$.

These data are required to satisfy the following conditions:

(0) The following diagram is commutative for $1 \le i, j \le 2$:

$$\begin{array}{ccc} \mathcal{K}_{j,R}^{\otimes 2} & \xrightarrow{\psi_{ij,R}^{\otimes 2}} & \mathcal{K}_{i,R}'^{\otimes 2} \\ \downarrow{\iota_j} & & \downarrow{\iota_i'} \\ \mathcal{L}_j|_{R\times S} & \xrightarrow{\varphi_{ij}|_{R\times S}} & \mathcal{L}_i'|_{R\times S}. \end{array} \tag{6.1}$$

(1) $\varphi_{22}|_{\Sigma_-\times S} = 0$; $\varphi_{11}|_{\Sigma_+\times S}$ and $\varphi_{22}|_{\Sigma_+\times S}$ are nowhere vanishing.
(2) $\varphi_{21}|_{\Sigma_+\times S} = 0$; $\varphi_{12}|_{\Sigma_-\times S}$ and $\varphi_{21}|_{\Sigma_-\times S}$ are nowhere vanishing.
(3) $\det(\psi_R) = 0$. Moreover, $\det(\varphi)$ vanishes only to the first order along $R\times S$. (By (6.1) and $\det(\psi_R) = 0$, $\det(\varphi)$ does vanish along $R \times S$.)
(4) This condition is only non-void when $\Sigma = \varnothing$ and $R = \varnothing$: $\det(\varphi)$ is not identically zero on $X \times s$ for any geometric point s of S.
(5) For each geometric point $s \in S$, the following conditions hold. If $d_{11} < d_{22} - N_-$, then $\varphi_{11}|_{X\times s} \neq 0$; if $d_{11} \ge d_{22} - N_-$, then $\varphi_{22}|_{X\times s} \neq 0$. If $d_{12} < d_{21} - N_+$, then $\varphi_{12}|_{X\times s} \neq 0$; if $d_{12} \ge d_{21} - N_+$, then $\varphi_{21}|_{X\times s} \neq 0$.

There is an action of $\mathrm{Pic}_X^{\sqrt{R}}$ on $\widetilde{\mathcal{N}}_{\underline{d}}$ by twisting each $\mathcal{L}_i^\natural$ and $\mathcal{L}_i'^\natural$ simultaneously ($i = 1, 2$). Let $\mathcal{N}_{\underline{d}}$ be the quotient

$$\mathcal{N}_{\underline{d}} := \widetilde{\mathcal{N}}_{\underline{d}} / \mathrm{Pic}_X^{\sqrt{R}}.$$

Let $\mathcal{N}_d$ be the disjoint union

$$\mathcal{N}_d = \coprod_{\underline{d}\in Q_d} \mathcal{N}_{\underline{d}}.$$

6.1.2. Next we give an alternative description of $\mathcal{N}_d$ in the style of [10, §3], which makes its similarity with $\mathcal{M}_d$ more transparent.

Let $(\mathcal{L}_1^\natural, \mathcal{L}_2^\natural, \mathcal{L}_1'^\natural, \mathcal{L}_2'^\natural, \varphi, \psi_R) \in \mathcal{N}_{\underline{d}}(S)$. For $i, j \in \{1, 2\}$, define

$$\mathcal{L}_{ij}^\natural = \mathcal{L}_j^{\natural,\otimes -1} \otimes \mathcal{L}_i'^\natural = (\mathcal{L}_j^{\otimes -1} \otimes \mathcal{L}_i', \mathcal{K}_{j,R}^{\otimes -1} \otimes \mathcal{K}_{i,R}', \iota_j^{-1} \otimes \iota_i').$$

We have $\mathcal{L}_{ij}^\natural \in \mathrm{Pic}_X^{\sqrt{R}}(S)$. By the diagram (6.1), $(\mathcal{L}_{ij}^\natural, \varphi_{ij}, \psi_{ij,R})$ defines a point in $\widehat{X}_{d_{ij}}^{\sqrt{R}}(S)$.

For $(i, j) = (1, 1)$ or $(1, 2)$, we thus have a morphism $\jmath_{ij} : \mathcal{N}_{\underline{d}} \to \widehat{X}_{d_{ij}}^{\sqrt{R}}$ sending the data $(\mathcal{L}_1^\natural, \mathcal{L}_2^\natural, \mathcal{L}_1'^\natural, \mathcal{L}_2'^\natural, \varphi, \psi_R) \in \mathcal{N}_{\underline{d}}(S)$ to $(\mathcal{L}_{ij}^\natural, \varphi_{ij}, \psi_{ij,R}) \in \widehat{X}_{d_{ij}}^{\sqrt{R}}(S)$.

The condition $\varphi_{21}|_{\Sigma_+\times S} = 0$ allows us to view φ_{21} as a section of $\mathcal{L}_{21}(-\Sigma_+)$, which has degree $d_{21} - N_+$ and extends to a point $\mathcal{L}_{21}^\natural(-\Sigma_+) \in \mathrm{Pic}_X^{\sqrt{R}}(S)$ using the original $\mathcal{K}_{21,R} = \mathcal{K}_1^{\otimes -1} \otimes \mathcal{K}_2'$ and $\iota_1^{-1} \otimes \iota_2'$ (because $\Sigma_+ \cap R = \varnothing$). We then define a morphism $\jmath_{21} : \mathcal{N}_{\underline{d}} \to \widehat{X}_{d_{21}-N_+}^{\sqrt{R}}$ sending $(\mathcal{L}_1^\natural, \mathcal{L}_2^\natural, \mathcal{L}_1'^\natural, \mathcal{L}_2'^\natural, \varphi, \psi_R)$ to $(\mathcal{L}_{21}^\natural(-\Sigma_+), \varphi_{21}, \psi_{21,R})$. Similarly we can define $\jmath_{22} : \mathcal{N}_{\underline{d}} \to \widehat{X}_{d_{22}-N_-}^{\sqrt{R}}$. We have constructed a morphism

$$\jmath_{\underline{d}} = (\jmath_{ij})_{i,j\in\{1,2\}} : \mathcal{N}_{\underline{d}} \longrightarrow \widehat{X}^{\sqrt{R}}_{d_{11}} \times \widehat{X}^{\sqrt{R}}_{d_{22}-N_-} \times \widehat{X}^{\sqrt{R}}_{d_{12}} \times \widehat{X}^{\sqrt{R}}_{d_{21}-N_+}.$$

In the above construction, we have canonical isomorphisms $\mathcal{L}_{11}\otimes\mathcal{L}_{22}\cong\mathcal{L}_{12}\otimes\mathcal{L}_{21}$ and $\mathcal{K}_{11,R}\otimes\mathcal{K}_{22,R}\cong\mathcal{K}_{12,R}\otimes\mathcal{K}_{21,R}$, which give a canonical isomorphism

$$\mathcal{L}^{\natural}_{11}\otimes\mathcal{L}^{\natural}_{22}\cong\mathcal{L}^{\natural}_{12}\otimes\mathcal{L}^{\natural}_{21}\in \mathrm{Pic}^{\sqrt{R},d+\rho}_X(S). \tag{6.2}$$

Moreover, the condition that $\det(\psi_R)=0$ implies that $\psi_{11,R}\psi_{22,R}=\psi_{12,R}\psi_{21,R}$. Therefore, the isomorphism (6.2) extends to an isomorphism

$$(\mathcal{L}^{\natural}_{11}\otimes\mathcal{L}^{\natural}_{22}, \psi_{11,R}\psi_{22,R})\cong(\mathcal{L}^{\natural}_{12}\otimes\mathcal{L}^{\natural}_{21}, \psi_{12,R}\psi_{21,R})\in \mathrm{Pic}^{\sqrt{R};\sqrt{R},d+\rho}_X(S).$$

Therefore, $\jmath_{\underline{d}}$ lifts to a morphism

$$\jmath_{\underline{d}} : \mathcal{N}_{\underline{d}} \longrightarrow (\widehat{X}^{\sqrt{R}}_{d_{11}} \times \widehat{X}^{\sqrt{R}}_{d_{22}-N_-}) \times_{\mathrm{Pic}^{\sqrt{R};\sqrt{R},d+\rho}_X} (\widehat{X}^{\sqrt{R}}_{d_{12}} \times \widehat{X}^{\sqrt{R}}_{d_{21}-N_+}). \tag{6.3}$$

Here the fiber product is formed using the following maps:

$$\widehat{X}^{\sqrt{R}}_{d_{11}} \times \widehat{X}^{\sqrt{R}}_{d_{22}-N_-} \xrightarrow{(\widehat{\mathrm{AJ}}^{\sqrt{R},\sqrt{R}}_{d_{11}},\widehat{\mathrm{AJ}}^{\sqrt{R},\sqrt{R}}_{d_{22}-N_-})} \mathrm{Pic}^{\sqrt{R};\sqrt{R},d_{11}}_X \times \mathrm{Pic}^{\sqrt{R};\sqrt{R},d_{22}-N_-}_X$$
$$\xrightarrow{(\mathrm{id},\otimes\dot{\mathcal{O}}_X(\Sigma_-))} \mathrm{Pic}^{\sqrt{R};\sqrt{R},d_{11}}_X \times \mathrm{Pic}^{\sqrt{R};\sqrt{R},d_{22}}_X \xrightarrow{\mathrm{mult}} \mathrm{Pic}^{\sqrt{R};\sqrt{R},d+\rho}_X$$

(where mult is the multiplication map for $\mathrm{Pic}^{\sqrt{R};\sqrt{R}}_X$) and

$$\widehat{X}^{\sqrt{R}}_{d_{12}} \times \widehat{X}^{\sqrt{R}}_{d_{21}-N_+} \xrightarrow{(\widehat{\mathrm{AJ}}^{\sqrt{R};\sqrt{R}}_{d_{12}},\widehat{\mathrm{AJ}}^{\sqrt{R};\sqrt{R}}_{d_{21}-N_+})} \mathrm{Pic}^{\sqrt{R};\sqrt{R},d_{12}}_X \times \mathrm{Pic}^{\sqrt{R};\sqrt{R},d_{21}-N_+}_X$$
$$\xrightarrow{(\mathrm{id},\otimes\dot{\mathcal{O}}_X(\Sigma_+))} \mathrm{Pic}^{\sqrt{R};\sqrt{R},d_{12}}_X \times \mathrm{Pic}^{\sqrt{R};\sqrt{R},d_{21}}_X \xrightarrow{\mathrm{mult}} \mathrm{Pic}^{\sqrt{R};\sqrt{R},d+\rho}_X.$$

6.1.3. We have a morphism to the base (cf. Section 5.1.2)

$$g_{\underline{d}} : \mathcal{N}_{\underline{d}} \longrightarrow \mathcal{A}_d = \mathcal{A}_d(\Sigma_\pm)$$

sending $(\mathcal{L}^{\natural}_1, \mathcal{L}^{\natural}_2, \mathcal{L}'^{\natural}_1, \mathcal{L}'^{\natural}_2, \varphi, \psi_R)$ to $(\Delta, \Theta_R, \iota, a, b, \vartheta_R)$, where

$$\Delta = \mathcal{L}^{\otimes -1}_1 \otimes \mathcal{L}^{\otimes -1}_2 \otimes \mathcal{L}'_1 \otimes \mathcal{L}'_2,$$
$$\Theta_R = \mathcal{K}^{\otimes -1}_{1,R} \otimes \mathcal{K}^{\otimes -1}_{2,R} \otimes \mathcal{K}'_{1,R} \otimes \mathcal{K}'_{2,R}, \iota_R$$

is the obvious product of $\iota_1\iota_2$ and $\iota'_1\iota'_2$, $a = \varphi_{11}\varphi_{22}$, $b = \varphi_{12}\varphi_{21}$, $\vartheta_R = \psi_{11,R}\psi_{22,R} = \psi_{12,R}\psi_{21,R}$. We also have the composition

$$g^{\flat}_{\underline{d}} = \Omega\circ g_{\underline{d}} \colon \mathcal{N}_{\underline{d}} \xrightarrow{g_{\underline{d}}} \mathcal{A}_d \xrightarrow{\Omega} \mathcal{A}^{\flat}_d.$$

PROPOSITION 6.2. *Let $\underline{d}\in\Sigma_d$. Then*

(1) *the morphism $\jmath_{\underline{d}}$ in (6.3) is an open embedding, and $\mathcal{N}_{\underline{d}}$ is geometrically connected;*

(2) *if* $d \geq 4g-3+\rho+N$, *then* $\mathcal{N}_{\underline{d}}$ *is a smooth* DM *stack of dimension* $2d+\rho-g-N+1=m$;

(3) *the following diagram is commutative*:

$$\begin{array}{ccc} \mathcal{N}_{\underline{d}} & \xrightarrow{\jmath_{\underline{d}}} & (\widehat{X}^{\sqrt{R}}_{d_{11}} \times \widehat{X}^{\sqrt{R}}_{d_{22}-N_-}) \times_{\mathrm{Pic}_X^{\sqrt{R};\sqrt{R},d+\rho}} (\widehat{X}^{\sqrt{R}}_{d_{12}} \times \widehat{X}^{\sqrt{R}}_{d_{21}-N_+}) \\ \downarrow g_{\underline{d}} & & \downarrow \widehat{\mathrm{add}}^{\sqrt{R}} \times \widehat{\mathrm{add}}^{\sqrt{R}} \\ \mathcal{A}_d & \xrightarrow{\omega_d} & \widehat{X}^{\sqrt{R}}_{d+\rho-N_-} \times_{\mathrm{Pic}_X^{\sqrt{R};\sqrt{R},d+\rho}} \widehat{X}^{\sqrt{R}}_{d+\rho-N_+}; \end{array} \tag{6.4}$$

(4) *the morphisms* $g_{\underline{d}}$ *and* $g^\flat_{\underline{d}}$ *are proper.*

Proof. The proofs of (1) and (3) are similar to their counterparts in [10, Prop 3.1].

(2) We first show that $\mathcal{N}_{\underline{d}}$ is a DM stack. By conditions (4) and (5) of Definition 6.1, at most one of φ_{ij} can be identically zero, so $\mathcal{N}_{\underline{d}}$ is covered by four open substacks U_{ij}, $i,j \in \{1,2\}$, in which only φ_{ij} is allowed to be zero. (In fact, two of these will be empty by condition (5).) We will show that U_{11} is a DM stack, and the argument for other U_{ij} is similar. Since U_{11} is open in

$$V_{11} = \left(\widehat{X}^{\sqrt{R}}_{d_{11}} \times X^{\sqrt{R}}_{d_{22}-N_-}\right) \times_{\mathrm{Pic}_X^{\sqrt{R};\sqrt{R}}} \left(X^{\sqrt{R}}_{d_{12}} \times X^{\sqrt{R}}_{d_{21}-N_+}\right),$$

it suffices to show V_{11} is DM. The projection $V_{11} \to X^{\sqrt{R}}_{d_{22}-N_-} \times X^{\sqrt{R}}_{d_{12}} \times X^{\sqrt{R}}_{d_{21}-N_+}$ is schematic. By Lemma A.4(2), $X^{\sqrt{R}}_n$ is DM for any n, therefore V_{11}, hence U_{11} is also DM.

We now prove the smoothness of $\mathcal{N}_{\underline{d}}$ in the case $d_{11} < d_{22}-N_-$ and $d_{12} < d_{21}-N_+$; the other cases are similar. In this case the image of $\jmath_{\underline{d}}$ lies in the open substack

$$\left(X^{\sqrt{R}}_{d_{11}} \times \widehat{X}^{\sqrt{R}}_{d_{22}-N_-}\right) \times_{\mathrm{Pic}_X^{\sqrt{R};\sqrt{R}}} \left(X^{\sqrt{R}}_{d_{12}} \times \widehat{X}^{\sqrt{R}}_{d_{21}-N_+}\right).$$

Since $d_{12}+(d_{21}-N) = d+\rho-N \geq 2(2g-1+\rho)-1$ by assumption on d, and $d_{12} < d_{21}-N_+$, we have $d_{21}-N_+ \geq 2g-1+\rho$. Similarly, we have $d_{22}-N_- \geq 2g-1+\rho$. Therefore, the Abel-Jacobi maps $\widehat{X}^{\sqrt{R}}_{d_{22}-N_-} \to \mathrm{Pic}_X^{\sqrt{R};\sqrt{R},d_{22}-N_-}$ and $\widehat{X}^{\sqrt{R}}_{d_{21}-N_+} \to \mathrm{Pic}_X^{\sqrt{R};\sqrt{R},d_{21}-N_+}$ are affine space bundles by Riemann-Roch, hence smooth. It therefore suffices to show the smoothness of

$$\mathcal{Q} := \left(X^{\sqrt{R}}_{d_{11}} \times \mathrm{Pic}_X^{\sqrt{R};\sqrt{R},d_{22}-N_-}\right) \times_{\mathrm{Pic}_X^{\sqrt{R};\sqrt{R}}} \left(X^{\sqrt{R}}_{d_{12}} \times \mathrm{Pic}_X^{\sqrt{R};\sqrt{R},d_{21}-N_+}\right). \tag{6.5}$$

We have the evaluation maps (by recording the square root line along R and its section)

$$\mathrm{ev}^{\sqrt{R}}_{d_{ij}} : X^{\sqrt{R}}_{d_{ij}} \longrightarrow [\mathrm{Res}^R_k \mathbb{A}^1/\mathrm{Res}^R_k \mathbb{G}_m],$$
$$\mathrm{ev}^{\sqrt{R}}_{\mathrm{Pic}} : \mathrm{Pic}^{\sqrt{R};\sqrt{R}}_X \longrightarrow [\mathrm{Res}^R_k \mathbb{A}^1/\mathrm{Res}^R_k \mathbb{G}_m],$$

which are both smooth, by Lemma A.4. To simplify notation, we write

$$[\mathrm{Res}^R_k \mathbb{A}^1/\mathrm{Res}^R_k \mathbb{G}_m] = [\mathbb{A}^1/\mathbb{G}_m]_R.$$

Then the fiber product of these maps give a smooth map

$$\mathrm{ev}^{\sqrt{R}}_{\mathcal{Q}} : \mathcal{Q} \longrightarrow ([\mathbb{A}^1/\mathbb{G}_m]_R \times [\mathbb{A}^1/\mathbb{G}_m]_R) \times_{[\mathbb{A}^1/\mathbb{G}_m]_R} ([\mathbb{A}^1/\mathbb{G}_m]_R \times [\mathbb{A}^1/\mathbb{G}_m]_R).$$

Let $C_R := \mathrm{Res}^R_k \mathbb{A}^2 \times_{\mathrm{Res}^R_k \mathbb{A}^1} \mathrm{Res}^R_k \mathbb{A}^2$ with the two maps $\mathrm{Res}^R_k \mathbb{A}^2 \to \mathrm{Res}^R_k \mathbb{A}^1$ both given by $(u, v) \mapsto uv$. Then the target of $\mathrm{ev}^{\sqrt{R}}_{\mathcal{Q}}$ can be written as $[C_R/\mathrm{Res}^R_k \mathbb{G}^3_m]$, where the torus $\mathbb{G}^3_m$ is the subtorus of $\mathbb{G}^4_m$ consisting of (u, v, s, t) such that $uv = st$. After base change to $\bar{k}$, we have $C_{R,\bar{k}} \cong \prod_{x\in R(\bar{k})} C_x$, where $C_x \subset \mathbb{A}^4_{\bar{k}}$ is the cone defined by $uv - st = 0$. Note that $C^\circ_x = C_x - \{(0, 0, 0, 0)\}$ is smooth over $\bar{k}$. The product $\prod_{x\in R(\bar{k})} C^\circ_x$ defines a smooth open subset $C^\circ_R \subset C_R$. We claim that the image of $\mathrm{ev}^{\sqrt{R}}_{\mathcal{Q}}$ lies in $[C^\circ_R/\mathrm{Res}^R_k \mathbb{G}^3_m]$. For otherwise, there would be a point $(\mathcal{L}_i, \dots, \varphi, \psi_R) \in \mathcal{N}_{\underline{d}}(\bar{k})$ and some $x \in R(\bar{k})$ such that $\psi_{ij,R}$ (hence φ_{ij}) vanishes at x for all $i, j \in \{1, 2\}$, implying that $\det(\varphi)$ vanished twice at x and contradicting condition (3). Therefore, the image of $\mathrm{ev}^{\sqrt{R}}_{\mathcal{Q}}$ lies in the smooth locus of $[C_R/\mathrm{Res}^R_k \mathbb{G}^3_m]$, showing that $\mathcal{Q}$ is itself smooth over k. This implies that $\mathcal{N}_{\underline{d}}$ is smooth over k. The dimension calculation is similar to Proposition 5.5(1) for $\dim \mathcal{M}_d$, and we omit it here.

(4) Since Ω is proper, it suffices to show that $g_{\underline{d}}$ is proper. As in the proof of [10, Prop. 3.1(3)], it suffices to show that the restriction of $\widehat{\mathrm{add}}^{\sqrt{R}}_{d_1,d_2}$

$$X^{\sqrt{R}}_{d_1} \times \widehat{X}^{\sqrt{R}}_{d_2} \longrightarrow \widehat{X}^{\sqrt{R}}_{d_1+d_2} \tag{6.6}$$

is proper for any $d_1, d_2 \geq 0$. Since $\widehat{X}^{\sqrt{R}}_n \to \widehat{X}_n$ is finite (hence proper), the properness of (6.6) follows from the properness of $\widehat{\mathrm{add}}_{d_1,d_2} : X_{d_1} \times \widehat{X}_{d_2} \to \widehat{X}_{d_1+d_2}$, which was shown in the proof of [10, Prop. 3.1(3)]. $\square$

6.2. *Relation with orbital integrals.*

6.2.1. *The rank one local system.* Recall the double cover $\nu : X' \to X$ from Section 4.1.1. Let $\sigma : X' \to X'$ be the non-trivial involution over X. The direct image sheaf $\nu_*\mathbb{Q}_\ell$ has a decomposition $\nu_*\mathbb{Q}_\ell = \mathbb{Q}_\ell \oplus L_{X'/X}$ into σ eigenspaces of eigenvalue 1 and -1. Then $L_{X'/X}|_{X-R}$ is a local system of rank one with geometric monodromy of order 2 around each $\bar{k}$-point of the ramification locus R.

Starting with $L = L_{X'/X}$, in Section A.2.2 we construct a rank one local system L^{Pic} on $\mathrm{Pic}_X^{\sqrt{R}}$ whose corresponding trace function is the quadratic idèle class character $\eta = \eta_{F'/F}$ (Proposition A.12). Via pullback along $\widehat{\mathrm{AJ}}_d^{\sqrt{R}} : \widehat{X}_d^{\sqrt{R}} \to \mathrm{Pic}_X^{\sqrt{R},d}$, it gives a rank one local system $\widehat{L}_d$ on $\widehat{X}_d^{\sqrt{R}}$ for each $d \in \mathbb{Z}$ extending the local system L_d on $X_d^{\sqrt{R}}$ defined in Lemma A.7.

For $\underline{d} \in Q_d$, we define a local system $L_{\underline{d}}$ on $\mathcal{N}_{\underline{d}}$ by

$$L_{\underline{d}} = \jmath_{\underline{d}}^*(\widehat{L}_{d_{11}} \boxtimes \mathbb{Q}_\ell \boxtimes \widehat{L}_{d_{12}} \boxtimes \mathbb{Q}_\ell).$$

6.2.2. Recall that, for each $f \in \mathscr{H}_G^{\Sigma\cup R}$, by (2.30) we have defined

$$f^{\Sigma_\pm} = f \cdot \left(\bigotimes_{x\in R} h_x^{\square}\right) \otimes \left(\bigotimes_{x\in\Sigma} \mathbf{1}_{\mathbf{J}_x}\right) \in C_c^\infty(G(\mathbb{A})).$$

Let D be an effective divisor on $U = X - \Sigma - R$ of degree d. In [10, §3.1] we have defined a spherical Hecke function $h_D \in \mathscr{H}_G^{\Sigma\cup R}$. Therefore, the element $h_D^{\Sigma_\pm} \in C_c^\infty(G(\mathbb{A}))$ is defined.

For $u \in \mathbb{P}^1(F) - \{1\}$ and $h \in C_c^\infty(G(\mathbb{A}))$, let

$$\mathbb{J}(u, h, s_1, s_2) = \sum_{\gamma\in A(F)\backslash G(F)/A(F),\mathrm{inv}(\gamma)=u} \mathbb{J}(\gamma, h, s_1, s_2). \tag{6.7}$$

Note that when $u \notin \{0, 1, \infty\}$, the right-hand side of (6.7) has only one term; when $u = 0$ or ∞, the right-hand side of (6.7) has three terms (cf. [10, 3.3.2]).

Recall the space $\mathcal{A}_D^\flat$ defined in (5.6). Then we have a map

$$\mathrm{inv}_D : \mathcal{A}_D^\flat(k) \longrightarrow \mathbb{P}^1(F) - \{1\}$$

sending (Δ, a, b) to the rational function $b/a \in \mathbb{P}^1(F)$. As in [10, 3.3.2], the map inv_D is injective.

THEOREM 6.3. *Let D be an effective divisor on $U = X-\Sigma-R$ of degree d. Let $u \in \mathbb{P}^1(F) - \{1\}$.*

(1) *If u is not in the image of $\mathrm{inv}_D : \mathcal{A}_D^\flat(k) \hookrightarrow \mathbb{P}^1(F) - \{1\}$, then*

$$\mathbb{J}(u, h_D^{\Sigma_\pm}, s_1, s_2) = 0.$$

(2) *If $u \notin \{0, 1, \infty\}$ and $u = \mathrm{inv}_D(a)$ for $a \in \mathcal{A}_D^\flat(k)$ (which is then unique), then*

$$\mathbb{J}(u, h_D^{\Sigma_\pm}, s_1, s_2) = \sum_{\underline{d}\in Q_d} q^{(2d_{12}-d-\rho)s_1+(2d_{11}-d-\rho)s_2}\,\mathrm{Tr}(\mathrm{Fr}_a, (\mathbf{R}g^\flat_{\underline{d},!}L_{\underline{d}})_{\overline{a}}). \tag{6.8}$$

(3) *Assume $d \geq 4g-3+\rho+N$. If $u = 0$ or ∞, and $u = \mathrm{inv}_D(a)$ for $a \in \mathcal{A}_D^\flat(k)$ (which is then unique), then (6.8) still holds.*

The proof of this theorem will occupy the rest of this subsection. From now on, we fix an effective divisor D on U of degree d.

6.2.3. *The set* $\mathfrak{X}_{D,\widetilde{\gamma}}$. Recall from Section A.1.6 the definition of $\mathbb{O}^{\times}_{\sqrt{R}}$, which maps to $\mathbb{O}^{\times}$ and hence acts on $\mathbb{A}^{\times}$ by translation. Define a groupoid

$$\mathrm{Div}^{\sqrt{R}}(X) = \mathbb{A}^{\times}/\mathbb{O}^{\times}_{\sqrt{R}}.$$

There are natural maps

$$\mathrm{AJ}^{\sqrt{R}}(k) : \mathrm{Div}^{\sqrt{R}}(X) \longrightarrow F^{\times}\backslash\mathbb{A}^{\times}/\mathbb{O}^{\times}_{\sqrt{R}} = \mathrm{Pic}^{\sqrt{R}}_X(k),$$
$$\omega : \mathrm{Div}^{\sqrt{R}}(X) \longrightarrow \mathbb{A}^{\times}/\mathbb{O}^{\times} = \mathrm{Div}(X).$$

We denote an element in $\mathrm{Div}^{\sqrt{R}}(X)$ by $E^{\natural}$ and denote its image in $\mathrm{Div}(X)$ by E. We denote the multiplication in $\mathrm{Div}^{\sqrt{R}}(X)$ by $+$. For $E^{\natural} \in \mathrm{Div}^{\sqrt{R}}(X)$, the line bundle $\mathcal{O}_X(-E)$, when restricted to R, carries a canonical square root, which we denote by $\mathcal{O}_X(-E^{\natural})_{\sqrt{R}}$ (an invertible $\mathcal{O}_R$-module). The character $\eta = \eta_{F'/F}$ on $\mathrm{Pic}^{\sqrt{R}}_X(k)$ can also be viewed as a character on $\mathrm{Div}^{\sqrt{R}}(X)$ by pullback.

Let $\widetilde{\gamma} \in \mathrm{GL}_2(F)$. Let $\widetilde{\mathfrak{X}}_{D,\widetilde{\gamma}}$ be the groupoid of $(E_1^{\natural}, E_2^{\natural}, E_1'^{\natural}, E_2'^{\natural}, \psi_R)$, where

- $E_i^{\natural}, E_i'^{\natural} \in \mathrm{Div}^{\sqrt{R}}(X)$ for $i = 1, 2$.
- $\psi_R : \mathcal{O}_X(-E_1^{\natural})_{\sqrt{R}} \oplus \mathcal{O}_X(-E_2^{\natural})_{\sqrt{R}} \to \mathcal{O}_X(-E_1'^{\natural})_{\sqrt{R}} \oplus \mathcal{O}_X(-E_2'^{\natural})_{\sqrt{R}}$ is an $\mathcal{O}_R$-linear map. Write ψ_R as a matrix $\begin{bmatrix} \psi_{11,R} & \psi_{12,R} \\ \psi_{21,R} & \psi_{22,R} \end{bmatrix}$.

These data are required to satisfy the following conditions:

(0) The rational map $\widetilde{\gamma} : \mathcal{O}_X^2 \dashrightarrow \mathcal{O}_X^2$ given by the matrix $\widetilde{\gamma}$ induces an everywhere defined map

$$\varphi : \mathcal{O}_X(-E_1) \oplus \mathcal{O}_X(-E_2) \longrightarrow \mathcal{O}_X(-E_1') \oplus \mathcal{O}_X(-E_2').$$

We write φ as a matrix $\begin{bmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{bmatrix}$. Moreover, $\psi_{ij,R}^2 = \varphi_{ij}|_R$ for $1 \le i, j \le 2$.

(1) φ_{22} vanishes along Σ_-.

(2) φ_{21} vanishes along Σ_+.

(3) $\det(\varphi)$ has divisor $D + R$.

Define the groupoid

$$\mathfrak{X}_{D,\widetilde{\gamma}} = \widetilde{\mathfrak{X}}_{D,\widetilde{\gamma}}/\mathrm{Div}^{\sqrt{R}}(X)$$

with the action of $\mathrm{Div}^{\sqrt{R}}(X)$ given by simultaneous translation on $E_i^{\natural}$ and $E_i'^{\natural}$. We may identify $\mathfrak{X}_{D,\widetilde{\gamma}}$ with the sub groupoid of $\widetilde{\mathfrak{X}}_{D,\widetilde{\gamma}}$ where $E_2'^{\natural}$ is equal to the identity element in $\mathrm{Div}^{\sqrt{R}}(X)$.

LEMMA 6.4. *We have*

$$\mathbb{J}(\gamma, h_D^{\Sigma_\pm}, s_1, s_2) = \sum_{\Lambda=(E_1^\natural,\dots,E_2'^\natural,\psi_R)\in\mathfrak{X}_{D,\widetilde{\gamma}}} \times \frac{1}{\#\operatorname{Aut}(\Lambda)} q^{-\deg(E_1-E_2+E_1'-E_2')s_1} q^{-\deg(-E_1+E_2+E_1'-E_2')s_2}\eta(E_1^\natural - E_2^\natural). \tag{6.9}$$

Proof. Let $\widetilde{A}\subset \mathrm{GL}_2$ be the diagonal torus, and let $Z\subset \mathrm{GL}_2$ be the center. Let

$$\widetilde{h}_D^{\Sigma_\pm} = \widetilde{h}_D \cdot \left(\bigotimes_{x\in R}\widetilde{h}_x^{\square}\right)\otimes\left(\bigotimes_{x\in\Sigma}\mathbf{1}_{\widetilde{\mathbf{J}}_x}\right).$$

Here $\widetilde{h}_D\in\mathscr{H}_{\mathrm{GL}_2}$ is as defined in [10, proof of Prop 3.2], and $\widetilde{\mathbf{J}}_x\subset\mathrm{GL}_2(\mathcal{O}_x)$ is defined by the same formulae as $\mathbf{J}_x$ (see (2.16)), with G replaced by GL_2. Then we have $h_D^{\Sigma_\pm} = p_*\widetilde{h}_D^{\Sigma_\pm}$, where $p_*: C_c^\infty(\mathrm{GL}_2)\to C_c^\infty(G(\mathbb{A}))$ is the tensor product of $p_{x,*}$. This allows us to convert the integral $\mathbb{J}(\gamma, h_D^{\Sigma_\pm}, s_1, s_2)$ into an integral on GL_2, i.e.,

$$\begin{aligned}&\mathbb{J}(\gamma, h_D^{\Sigma_\pm}, s_1, s_2)\\ &= \int_{\Delta(Z(\mathbb{A}))\backslash(\widetilde{A}(\mathbb{A})\times\widetilde{A}(\mathbb{A}))}\widetilde{h}_D^{\Sigma_\pm}(t'^{-1}\widetilde{\gamma}t)|\alpha(t)\alpha(t')|^{s_1}|\alpha(t')/\alpha(t)|^{s_2}\eta(\alpha(t))dtdt'.\end{aligned}$$

Here $\alpha:\widetilde{A}\to\mathbb{G}_m$ is the positive root $\left[\begin{smallmatrix} t_1 & 0\\ 0 & t_2\end{smallmatrix}\right]\mapsto t_1/t_2$, and the measure on $\mathbb{A}^\times$ is such that $\mathrm{vol}(\mathbb{O}^\times)=1$. We may identify $\Delta(Z)\backslash\widetilde{A}\times\widetilde{A}$ with $\mathbb{G}_m^3$ such that $(\left[\begin{smallmatrix} t_1 & 0\\ 0 & t_2\end{smallmatrix}\right],\left[\begin{smallmatrix} t_1' & 0\\ 0 & 1\end{smallmatrix}\right])$ corresponds to $(t_1,t_2,t_1')\in\mathbb{G}_m^3$, and we rewrite the above integral as

$$\begin{aligned}&\mathbb{J}(\gamma, h_D^{\Sigma_\pm}, s_1, s_2)\\ &= \int_{(\mathbb{A}^\times)^3}\widetilde{h}_D^{\Sigma_\pm}(\left[\begin{smallmatrix} t_1'^{-1} & 0\\ 0 & 1\end{smallmatrix}\right]\widetilde{\gamma}\left[\begin{smallmatrix} t_1 & 0\\ 0 & t_2\end{smallmatrix}\right])|t_1t_2^{-1}t_1'|^{s_1}|t_2t_1^{-1}t_1'|^{s_2}\eta(t_1t_2^{-1})dt_1dt_2dt_1'.\end{aligned}\tag{6.10}$$

For $x\in|X|$, define a set $\Xi_{D,x}$ as follows:

- for $x\in R$, let $\Xi_{D,x}=\Xi_x$ defined in Section 2.4.1;
- for $x\in\Sigma$, $\Xi_{D,x}=\widetilde{\mathbf{J}}_x$;
- for $x\in|X|-R-\Sigma$, $\Xi_{D,x}=\mathrm{Mat}_2(\mathcal{O}_x)_{v_x(\det)=n_x}$, where n_x is the coefficient of x in D.

Let $\Xi_D=\prod_{x\in|X|}\Xi_{D,x}$. Then there is a projection map

$$\mu:\Xi_D\longrightarrow\mathrm{Mat}_2(\mathbb{O})_{\mathrm{div}(\det)=D+R}.$$

We have

$$\widetilde{h}_D^{\Sigma_\pm}=\mu_*\mathbf{1}_{\Xi_D}. \tag{6.11}$$

In fact, this can be checked place by place. The assertion is trivial when $x \notin R$, and it follows from Lemma 2.4 when $x \in R$.

By (6.11), we may rewrite (6.10) as

(6.12)
$$\begin{aligned}&\mathbb{J}(\gamma, h_D^{\Sigma_\pm}, s_1, s_2)\\ &= \int_{(\mathbb{A}^\times)^3} \#\mu^{-1}\left(\begin{bmatrix} t_1'^{-1} & 0\\ 0 & 1\end{bmatrix}\widetilde{\gamma}\begin{bmatrix} t_1 & 0\\ 0 & t_2\end{bmatrix}\right)|t_1t_2^{-1}t_1'|^{s_1}|t_2t_1^{-1}t_1'|^{s_2}\eta(t_1t_2^{-1})dt_1dt_2dt_1'.\end{aligned}$$

Here $\mu^{-1}(g) = \varnothing$ if $g \notin \mathrm{Mat}_2(\mathbb{O})_{\mathrm{div}(\det)=D+R}$.

Note that the integrand in (6.12) is invariant under translating each of the variables by $\mathbb{O}_{\sqrt{R}}^\times$, therefore we may turn $\mathbb{J}(\gamma, h_D^{\Sigma_\pm}, s_1, s_2)$ into an integration over $\mathrm{Div}^{\sqrt{R}}(X)^3$. To do this, we first write the integrand as a function on $\mathrm{Div}^{\sqrt{R}}(X)^3$. Denote the images of t_1, t_2, t_1' and $t_2' = 1$ in $\mathrm{Div}^{\sqrt{R}}(X)$ by $E_1^\natural, E_2^\natural, E_1'^\natural$ and $E_2'^\natural = 0$. One checks that the set $\mu^{-1}(\begin{bmatrix} t_1'^{-1} & 0\\ 0 & 1\end{bmatrix}\widetilde{\gamma}\begin{bmatrix} t_1 & 0\\ 0 & t_2\end{bmatrix})$ is in natural bijection with the fiber of $\widetilde{\lambda} : \widetilde{\mathfrak{X}}_{D,\widetilde{\gamma}} \to \mathrm{Div}^{\sqrt{R}}(X)^4$ over $(E_1^\natural, E_2^\natural, E_1'^\natural, E_2'^\natural)$, or equivalently the fiber of $\lambda : \mathfrak{X}_{D,\gamma} \to \mathrm{Div}^{\sqrt{R}}(X)^3$. Moreover, we have

$$|t_i| = q^{-\deg E_i}, \quad |t_i'| = q^{-\deg E_i'}, \quad i, j \in \{1, 2\}.$$

Hence the integrand in (6.12) descends to the following function on $\mathrm{Div}^{\sqrt{R}}(X)^3$ (with $E_2^\natural = 0$):

$$\lambda_! \mathbf{1}_{\mathfrak{X}_{D,\gamma}} q^{-\deg(E_1-E_2+E_1'-E_2')s_1} q^{-\deg(-E_1+E_2+E_1'-E_2')s_2} \eta(E_1^\natural - E_2^\natural). \tag{6.13}$$

To finish the argument we need some general remarks about integrating a function over a groupoid:

(i) If $\mathcal{G}$ is a groupoid with finite automorphisms, and if f is a function on $\mathcal{G}$ with finite support, then define
$$\int_{\mathcal{G}} f := \sum_{g \in \mathcal{G}} \frac{1}{\#\,\mathrm{Aut}(g)} f(g).$$

(ii) The integration above is compatible with push-forward of functions. If $\varphi : \mathcal{G} \to \mathcal{G}'$ is a map of groupoids with finite automorphisms, and if f is a function with finite support on $\mathcal{G}'$, then
$$\int_{\mathcal{G}} f = \int_{\mathcal{G}'} \varphi_! f,$$
where $(\varphi_! f)(g') = \int_{\varphi^{-1}(g')} f|_{\varphi^{-1}(g')}$, where $\varphi^{-1}(g')$ is the fiber groupoid of φ over g'.

(iii) Suppose we have a topological group H with Haar measure dh and a homomorphism $\varphi : H_1 \to H$ from a compact topological group H_1 such that the image of φ is open and φ has finite kernel. Then the groupoid $\mathcal{G} = H/H_1$ has discrete topology with finite automorphism groups equal to $\ker(\varphi)$. For a function f on H invariant under right translation by $\varphi(H_1)$, we have

$$\int_H f(h)dh = \mathrm{vol}(\varphi(H_1), dh)\#\ker(\varphi)\cdot\int_{H/H_1}\overline{f}, \tag{6.14}$$

where $\overline{f}$ is the pullback of the descent of f from $H/\varphi(H_1)$ to H/H_1.

Apply (iii) above to $H = (\mathbb{A}^\times)^3$ and $H_1 = (\mathbb{O}^\times_{\sqrt{R}})^3$ with the natural map $H_1 \to (\mathbb{O}^\times)^3 \hookrightarrow H$. Note that the kernel and the cokernel of the map $\mathbb{O}^\times_{\sqrt{R}} \to \mathbb{O}^\times$ have the same finite cardinality $2^{\#R}$. Since $\mathrm{vol}(\mathbb{O}^\times) = 1$ under the Haar measure on $\mathbb{A}^\times$, the constant factor on the right side of (6.14) is 1 in this case. Therefore, by (6.14), (6.12) can be written as the integration over $\mathrm{Div}^{\sqrt{R}}(X)^3$ of the function (6.13). Applying (ii) above to $\lambda : \mathfrak{X}_{D,\gamma} \to \mathrm{Div}^{\sqrt{R}}(X)^3$, we further turn the integration over $\mathrm{Div}^{\sqrt{R}}(X)^3$ into an integration over $\mathfrak{X}_{D,\gamma}$,

$$\begin{aligned}&\mathbb{J}(\gamma, h_D^{\Sigma_\pm}, s_1, s_2)\\ &= \int_{\mathfrak{X}_{D,\gamma}} q^{-\deg(E_1-E_2+E_1'-E_2')s_1} q^{-\deg(-E_1+E_2+E_1'-E_2')s_2}\eta(E_1^\natural - E_2^\natural),\end{aligned} \tag{6.15}$$

where $(E_1^\natural, E_2^\natural, E_1'^\natural, E_2'^\natural)$ is the image of a variable point of $\mathfrak{X}_{D,\gamma}$ in

$$\mathrm{Div}^{\sqrt{R}}(X)^4/\mathrm{Div}^{\sqrt{R}}(X).$$

Now the formula (6.9) follows from (6.15) by the definition in (i). □

6.2.4. *Proof of Theorem 6.3 for $u \notin \{0, 1, \infty\}$.* For $u \notin \{0, 1, \infty\}$, let $\widetilde{\gamma}(u) = \begin{bmatrix} 1 & u \\ 1 & 1\end{bmatrix}$, which represents the unique $\widetilde{A}(F)$ double coset in $\mathrm{GL}_2(F)$ with invariant u. We define a map

$$\lambda : \mathfrak{X}_{D,\widetilde{\gamma}(u)} \longrightarrow \mathcal{N}_d(k),$$
$$(E_1^\natural, E_2^\natural, E_1'^\natural, E_2'^\natural, \psi_R) \longmapsto (\mathcal{L}_1^\natural, \mathcal{L}_2^\natural, \mathcal{L}_1'^\natural, \mathcal{L}_2'^\natural, \varphi, \psi_R),$$

where $\mathcal{L}_i^\natural$ (resp. $\mathcal{L}_i'^\natural$) is the image of $-E_i^\natural$ (resp. $-E_i'^\natural$) under

$$\mathrm{AJ}^{\sqrt{R}}(k) : \mathrm{Div}^{\sqrt{R}}(X) \longrightarrow \mathrm{Pic}_X^{\sqrt{R}}(k);$$

the definition of φ is contained in the definition of $\widetilde{\mathfrak{X}}_{D,\widetilde{\gamma}}$. If Λ is in the image of λ, then $a := g^\flat_{\underline{d}}(\Lambda) \in \mathcal{A}^\flat_D(k)$ and $\mathrm{inv}_D(a) = u$. In particular, if u is not in the image of inv_D, then $\mathfrak{X}_{D,\widetilde{\gamma}(u)} = \varnothing$ hence $J(u, h_D^{\Sigma_\pm}, s_1, s_2) = 0$ by Lemma 6.4.

Now we assume $u = \mathrm{inv}_D(a)$ for some (unique) $a \in \mathcal{A}^\flat_D(k)$. Let $\mathcal{N}_{\underline{d},a} = g^{\flat,-1}_{\underline{d}}(a)$ and $\mathcal{N}_{d,a} = \coprod_{\underline{d}\in Q_d}\mathcal{N}_{\underline{d},a}$. Then we can write

$$\lambda : \mathfrak{X}_{D,\widetilde{\gamma}(u)} \longrightarrow \mathcal{N}_{d,a}(k).$$

Let us define an inverse to λ. Let $(\mathcal{L}_1^\natural, \dots, \mathcal{L}_2'^\natural, \varphi, \psi_R) \in \mathcal{N}_{d,a}(k)$. Since the $(\mathcal{L}_1^\natural, \dots, \mathcal{L}_2'^\natural)$ are up to simultaneous tensoring with $\mathrm{Pic}_X^{\sqrt{R}}(k)$, we may fix $\mathcal{L}_2'^\natural$

to be $\dot{\mathcal{O}}_X$, the identity object in $\mathrm{Pic}_X^{\sqrt{R}}(k)$. Since $\mathrm{inv}_D(a) = u \neq 0, \infty$, the maps φ_{ij} are all non-zero. Then $\varphi_{21} : \mathcal{L}_1 \to \mathcal{O}_X = \mathcal{L}'_2$ allows us to write $\mathcal{L}_1 = \mathcal{O}_X(-E_1)$ for an effective divisor E_1. The lifting $\mathcal{L}_1^\natural$ of $\mathcal{L}_1$ gives a canonical lifting $E_1^\natural \in \mathrm{Div}^{\sqrt{R}}(X)$ of E_1, so that $\mathrm{AJ}^{\sqrt{R}}(k)(-E_1^\natural) \cong \mathcal{L}_1^\natural$ canonically. Similarly, using φ_{22} we get $E_2^\natural \in \mathrm{Div}^{\sqrt{R}}(X)$ whose inverse represents $\mathcal{L}_2^\natural$. Using φ_{11} and $E_1^\natural$, we further get $E_1'^\natural \in \mathrm{Div}^{\sqrt{R}}(X)$ whose inverse represents $\mathcal{L}_1'^\natural$. Then $(E_1^\natural, E_2^\natural, E_1'^\natural, 0, \psi_R)$ (0 denotes the identity in $\mathrm{Div}^{\sqrt{R}}(X)$) gives an element in $\mathfrak{X}_{D,\widetilde{\gamma}(u)}$. It is easy to check that this assignment is inverse to λ, hence λ is an isomorphism of groupoids.

Under λ, we have

$$-\deg(E_1 - E_2 + E_1' - E_2') = d_{12} - d_{21} = 2d_{12} - d - \rho, \tag{6.16}$$

$$-\deg(-E_1 + E_2 + E_1' - E_2') = d_{11} - d_{22} = 2d_{11} - d - \rho, \tag{6.17}$$

$$\eta(E_1^\natural - E_2^\natural) = \eta(\mathcal{L}_{11}^\natural)\eta(\mathcal{L}_{12}^\natural) = \eta(\mathcal{L}_{21}^\natural)\eta(\mathcal{L}_{22}^\natural), \tag{6.18}$$

where $\mathcal{L}_{ij}^\natural = \mathcal{L}_j^{\natural,\otimes -1} \otimes \mathcal{L}_i'^\natural$ and $\deg \mathcal{L}_{ij} = d_{ij}$. Therefore, we may rewrite (6.9) as

$$\begin{aligned}&\mathbb{J}(\gamma(u), h_D^{\Sigma_\pm}, s_1, s_2)\\ &= \sum_{\Lambda = (\mathcal{L}_1^\natural, \dots, \mathcal{L}_2'^\natural, \varphi, \psi_R) \in \mathcal{N}_{d,a}(k)} \frac{1}{\# \mathrm{Aut}(\Lambda)} q^{(2d_{12}-d-\rho)s_1 + (2d_{11}-d-\rho)s_2} \eta(\mathcal{L}_{11}^\natural)\eta(\mathcal{L}_{12}^\natural).\end{aligned}$$

By Proposition A.12, the trace function given by L^{Pic} is the character η on $\mathrm{Pic}_X^{\sqrt{R}}(k)$. The formula (6.8) then follows from the Lefschetz trace formula for Frobenius:

$$\sum_{\Lambda = (\mathcal{L}_1^\natural, \dots, \mathcal{L}_2'^\natural, \varphi, \psi_R) \in \mathcal{N}_{\underline{d},a}(k)} \frac{1}{\# \mathrm{Aut}(\Lambda)} \eta(\mathcal{L}_{11}^\natural)\eta(\mathcal{L}_{12}^\natural) = \mathrm{Tr}(\mathrm{Fr}_a, (\mathbf{R}g_{\underline{d},!}^\flat L_{\underline{d}})_{\overline{a}}).$$

6.2.5. *Proof of Theorem 6.3 for* $u = 0$. There are three $A(F)$ double cosets with invariant 0:

$$1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad n_+ = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad n_- = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

We first consider the case when $\Sigma_- = \varnothing$. Then $a_0 = (\mathcal{O}_X(D+R), 1, 0) \in \mathcal{A}_D^\flat(k)$ is the unique point satisfying $\mathrm{inv}_D(a_0) = 0 = u$. Let $\widehat{Q}_d \subset \mathbb{Z}^4$ be the set defined similarly as Q_d except we drop the condition that $d_{ij} \geq 0$. For any $\underline{d} \in \widehat{Q}_d$, we define $\widehat{\mathcal{N}}_{\underline{d}}$ in the same way as $\mathcal{N}_{\underline{d}}$ except that we drop condition (5) in Definition 6.1, but requiring at most one of φ_{ij} is zero. We still have a map $\widehat{g}_d^\flat : \widehat{\mathcal{N}}_{\underline{d}} \to \mathcal{A}_d \to \mathcal{A}_d^\flat$, and we denote the fiber over a_0 by $\widehat{\mathcal{N}}_{\underline{d},a_0}$. Let $\widehat{\mathcal{N}}_{d,a_0} = \coprod_{\underline{d} \in \widehat{Q}_d} \widehat{\mathcal{N}}_{\underline{d},a_0}$. We have a decomposition $\widehat{\mathcal{N}}_{d,a_0} = \widehat{\mathcal{N}}_{d,a_0}^+ \sqcup \widehat{\mathcal{N}}_{d,a_0}^-$, where

$\widehat{\mathcal{N}}^{+}_{d,a_0}$ consists of those $(\mathcal{L}_1^\natural, \dots, \mathcal{L}_2'^\natural, \varphi, \psi_R)$ such that $\varphi_{21} = 0, \varphi_{12} \neq 0$; $\widehat{\mathcal{N}}^{-}_{d,a_0}$ consists of those $(\mathcal{L}_1^\natural, \dots, \mathcal{L}_2'^\natural, \varphi, \psi_R)$ such that $\varphi_{12} = 0, \varphi_{21} \neq 0$.

The same argument as in Section 6.2.4 gives canonical isomorphisms of groupoids $\lambda_\pm : \mathfrak{X}_{D,n_\pm} \xrightarrow{\sim} \widehat{\mathcal{N}}^{\pm}_{d,a_0}(k)$. Using the isomorphism $\lambda_\pm$, (6.16), (6.17) and (6.18), Lemma 6.4 implies

$$
\begin{aligned}
&\mathbb{J}(n_+, h_D^{\Sigma_\pm}, s_1, s_2)\\
&= \sum_{\Lambda=(\mathcal{L}_1^\natural,\dots,\mathcal{L}_2'^\natural,\varphi,\psi_R)\in\widehat{\mathcal{N}}^{+}_{d,a_0}(k)} \frac{1}{\#\operatorname{Aut}(\Lambda)} q^{(2d_{12}-d-\rho)s_1+(2d_{11}-d-\rho)s_2}\eta(\mathcal{L}_{11}^\natural)\eta(\mathcal{L}_{12}^\natural)\\
&= \sum_{\underline{d}\in\widehat{Q}_d} q^{(2d_{12}-d-\rho)s_1+(2d_{11}-d-\rho)s_2}\\
&\quad\times \sum_{\Lambda=(\mathcal{L}_1^\natural,\dots,\mathcal{L}_2'^\natural,\varphi,\psi_R)\in\widehat{\mathcal{N}}^{+}_{\underline{d},a_0}(k)} \frac{1}{\#\operatorname{Aut}(\Lambda)}\eta(\mathcal{L}_{11}^\natural)\eta(\mathcal{L}_{12}^\natural).
\end{aligned}
\tag{6.19}
$$

Similarly,

$$
\begin{aligned}
&\mathbb{J}(n_-, h_D^{\natural}, s_1, s_2)\\
&= \sum_{\underline{d}\in\widehat{Q}_d} q^{(2d_{12}-d-\rho)s_1+(2d_{11}-d-\rho)s_2}\\
&\quad\times \sum_{\Lambda=(\mathcal{L}_1^\natural,\dots,\mathcal{L}_2'^\natural,\varphi,\psi_R)\in\widehat{\mathcal{N}}^{-}_{\underline{d},a_0}(k)} \frac{1}{\#\operatorname{Aut}(\Lambda)}\eta(\mathcal{L}_{21}^\natural)\eta(\mathcal{L}_{22}^\natural).
\end{aligned}
\tag{6.20}
$$

On the other hand, by the Lefschetz trace formula for Frobenius, we have

$$
\begin{aligned}
&\sum_{\underline{d}\in Q_d} q^{(2d_{12}-d-\rho)s_1+(2d_{11}-d-\rho)s_2}\operatorname{Tr}(\mathrm{Fr}_{a_0}, (\mathbf{R}g^\flat_{\underline{d},!}L_{\underline{d}})_{a_0})\\
&= \sum_{\underline{d}\in Q_d} q^{(2d_{12}-d-\rho)s_1+(2d_{11}-d-\rho)s_2} \sum_{\Lambda=(\mathcal{L}_1^\natural,\cdots)\in\mathcal{N}_{\underline{d},a_0}(k)} \frac{1}{\#\operatorname{Aut}(\Lambda)}\eta(\mathcal{L}_{11}^\natural)\eta(\mathcal{L}_{12}^\natural)\\
&= \sum_{\underline{d}\in Q_d} q^{(2d_{12}-d-\rho)s_1+(2d_{11}-d-\rho)s_2}\\
&\quad\times\left(\sum_{\Lambda\in\mathcal{N}^{+}_{\underline{d},a_0}(k)} \frac{1}{\#\operatorname{Aut}(\Lambda)}\eta(\mathcal{L}_{11}^\natural)\eta(\mathcal{L}_{12}^\natural) + \sum_{\Lambda\in\mathcal{N}^{-}_{\underline{d},a_0}(k)} \frac{1}{\#\operatorname{Aut}(\Lambda)}\eta(\mathcal{L}_{21}^\natural)\eta(\mathcal{L}_{22}^\natural)\right).
\end{aligned}
$$

Here $\mathcal{N}^{\pm}_{\underline{d},a_0}$ is defined as $\widehat{\mathcal{N}}^{\pm}_{\underline{d},a_0} \cap \mathcal{N}_{\underline{d},a_0}$. By condition (5) in Definition 6.1, we have $\mathcal{N}^{-}_{\underline{d},a_0} = \varnothing$ if $d_{12} < d_{21} - N$ and $\mathcal{N}^{+}_{\underline{d},a_0} = \varnothing$ if $d_{12} \geq d_{21} - N$. Therefore, the above formula equals

$$\begin{aligned}(6.21)\quad &\sum_{\underline{d}\in Q_d, d_{12}<d_{21}-N} q^{(2d_{12}-d-\rho)s_1+(2d_{11}-d-\rho)s_2} \sum_{\Lambda\in\mathcal{N}^+_{\underline{d},a_0}(k)} \frac{1}{\#\operatorname{Aut}(\Lambda)}\eta(\mathcal{L}^\natural_{11})\eta(\mathcal{L}^\natural_{12})\\ &+\sum_{\underline{d}\in Q_d, d_{12}\geq d_{21}-N} q^{(2d_{12}-d-\rho)s_1+(2d_{11}-d-\rho)s_2} \sum_{\Lambda\in\mathcal{N}^-_{\underline{d},a_0}(k)} \frac{1}{\#\operatorname{Aut}(\Lambda)}\eta(\mathcal{L}^\natural_{21})\eta(\mathcal{L}^\natural_{22}).\end{aligned}$$

Comparing the sum of the right-hand sides of (6.19) and (6.20) with the expression(6.21), the only difference is the range of $\underline{d}$ in the summation; however, many $\underline{d}$'s do not contribute as the following lemma shows.

Lemma 6.5. *Let* $\underline{d}\in\widehat{Q}_d$.

(1) *If* $d_{12}\geq 2g-1+\rho$, *then*

$$\sum_{\Lambda=(\mathcal{L}^\natural_1,\dots,\mathcal{L}'^\natural_2,\varphi,\psi_R)\in\widehat{\mathcal{N}}^+_{\underline{d},a_0}(k)} \frac{1}{\#\operatorname{Aut}(\Lambda)}\eta(\mathcal{L}^\natural_{11})\eta(\mathcal{L}^\natural_{12})=0.$$

(2) *If* $d_{21}-N_+\geq 2g-1+\rho$, *then*

$$\sum_{\Lambda=(\mathcal{L}^\natural_1,\dots,\mathcal{L}'^\natural_2,\varphi,\psi_R)\in\widehat{\mathcal{N}}^-_{\underline{d},a_0}(k)} \frac{1}{\#\operatorname{Aut}(\Lambda)}\eta(\mathcal{L}^\natural_{21})\eta(\mathcal{L}^\natural_{22})=0.$$

(3) *We have*

$$\mathbb{J}\left(\begin{bmatrix}1&0\\0&1\end{bmatrix}, h_D^{\Sigma_\pm}, s_1, s_2\right)=0.$$

Proof. (1) Let $(X^{\sqrt{R}}_{d_{11}}\times X^{\sqrt{R}}_{d_{22}})_{D+R}$ be the fiber over $D+R$ of the map

$$X^{\sqrt{R}}_{d_{11}}\times X^{\sqrt{R}}_{d_{22}} \xrightarrow{\operatorname{add}^{\sqrt{R}}} X^{\sqrt{R}}_{d+\rho} \xrightarrow{\omega^{\sqrt{R}}_{d+\rho}} X_{d+\rho}.$$

We have an isomorphism

$$(6.22)\qquad \widehat{\mathcal{N}}^+_{\underline{d},a_0} \xrightarrow{\sim} (X^{\sqrt{R}}_{d_{11}}\times X^{\sqrt{R}}_{d_{22}-N_-})_{D+R}\times X^{\sqrt{R}}_{d_{12}}$$

by recording $(\mathcal{L}^\natural_{ij}, \varphi_{ij}, \psi_{ij,R})$ for $(i,j) = (1,1),(2,2)$ and $(1,2)$. (Then $\mathcal{L}^\natural_{21}$ is determined uniquely and $\varphi_{21} = 0$.) Using this isomorphism we can write

$$\begin{aligned}
&\sum_{\Lambda=(\mathcal{L}^\natural_1,\dots,\mathcal{L}'^\natural_2,\varphi,\psi_R)\in\widehat{\mathcal{N}}^+_{\underline{d},a_0}(k)} \frac{1}{\#\operatorname{Aut}(\Lambda)}\eta(\mathcal{L}^\natural_{11})\eta(\mathcal{L}^\natural_{12}) \\
(6.23)\qquad &= \sum_{\Lambda'=(\mathcal{L}^\natural_{11},\cdots)\in(X^{\sqrt{R}}_{d_{11}}\times X^{\sqrt{R}}_{d_{22}-N_-})_{D+R}(k)} \frac{1}{\#\operatorname{Aut}(\Lambda')}\eta(\mathcal{L}^\natural_{11}) \\
&\quad\times \sum_{\Lambda''=(\mathcal{L}^\natural_{12},\cdots)\in X^{\sqrt{R}}_{d_{12}}(k)} \frac{1}{\#\operatorname{Aut}(\Lambda'')}\eta(\mathcal{L}^\natural_{12}).
\end{aligned}$$

Since $d_{12} \geq 2g-1+\rho$, the fibers of the map $\operatorname{AJ}^{\sqrt{R}}_{d_{12}}(k) : X^{\sqrt{R}}_{d_{12}}(k) \to \operatorname{Pic}^{\sqrt{R},d_{12}}_X(k)$ have the same cardinality. Since the character η is non-trivial on $\operatorname{Pic}^{\sqrt{R},d_{12}}_X(k)$, the last sum in (6.23) vanishes.

The proof of (2) is similar to (1), using the isomorphism

$$\widehat{\mathcal{N}}^-_{\underline{d},a_0} \xrightarrow{\sim} (X^{\sqrt{R}}_{d_{11}} \times X^{\sqrt{R}}_{d_{22}-N_-})_{D+R} \times X^{\sqrt{R}}_{d_{21}-N_+}$$

instead of (6.22).

(3) The restriction of the character $(t,t') \mapsto |tt'|^{s_1}|t'/t|^{s_2}\eta(t)$ on the stabilizer of 1 under $A(\mathbb{A}) \times A(\mathbb{A})$ (the diagonal $A(\mathbb{A})$) is non-trivial, therefore the integral vanishes. □

By Lemma 6.5(3), we have

$$(6.24)\qquad \mathbb{J}(0, h^{\Sigma_\pm}_D, s_1, s_2) = \mathbb{J}(n_+, h^{\Sigma_\pm}_D, s_1, s_2) + \mathbb{J}(n_-, h^{\Sigma_\pm}_D, s_1, s_2),$$

which is calculated in (6.19) and (6.20). Using Lemma 6.5(1), we may restrict the summation in the right-hand side of (6.19) to those $\underline{d} \in \widehat{Q}_d$ such that $0 \leq d_{12} \leq 2g-2+\rho$ ($d_{12} \geq 0$ for otherwise $\widehat{\mathcal{N}}^+_{\underline{d},a_0} = \varnothing$). Since $d \geq 4g-3+N+\rho$, we have $d_{12} + (d_{21} - N_+) \geq 2(2g-2+\rho)+1$. Therefore, we may alternatively restrict the summation in the right-hand side of (6.19) to those $\underline{d} \in Q_d$ such that $d_{12} < d_{21} - N_+$. Therefore, the right-hand side of (6.19) matches the first term in the right-hand side of (6.21). Similarly, the right-hand side of (6.20) matches the second term in the right-hand side of (6.21). We thus get (6.8) by combining (6.24), (6.19), (6.20) and (6.21).

Finally, we consider the case $\Sigma_- \neq \varnothing$. Then u is not in the image of inv_D. In this case, $\mathfrak{X}_{D,n_\pm} = \varnothing$, hence $\mathbb{J}(n_\pm, h^{\Sigma_\pm}_D, s_1, s_2) = 0$ by Lemma 6.4. Together with Lemma 6.5(3), we get $\mathbb{J}(0, h^{\Sigma_\pm}_D, s_1, s_2) = 0$.

6.2.6. *Proof of Theorem 6.3 for $u = \infty$.* There are three $A(F)$ double cosets with invariant ∞:

$$w_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad n_+ w_0 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad n_- w_0 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

The argument is the same as in the case $u = 0$, which we do not repeat.

7. Proof of the main theorem

7.1. *Comparison of sheaves.*

7.1.1. *The perverse sheaf K_d.* Let $d \geq 0$ be an integer, and consider the direct image complex $\nu_{d,!}^{\sqrt{R}}\mathbb{Q}_\ell$ under $\nu_d^{\sqrt{R}} : X'_d \to X_d^{\sqrt{R}}$ defined in (A.10). Let $X_d^\circ \subset X_d$ be the open locus of multiplicity-free divisors, and let $X_d^{\sqrt{R},\circ}$ (resp. $X_d'^\circ$) be its preimage in $X_d^{\sqrt{R}}$ (resp. X'_d). Restricting $\nu_d^{\sqrt{R}}$ to $X_d^{\sqrt{R},\circ}$ we get a finite étale Galois cover $X_d'^\circ \to X_d^{\sqrt{R},\circ}$ with Galois group $\Gamma_d = (\mathbb{Z}/2\mathbb{Z})^d \rtimes S_d$. (Note that $\nu_d^{\sqrt{R}}$ is still étale when the multiplicity-free divisor meets R, as $X' \to X_1^{\sqrt{R}}$ is étale.) As in [10, §8.1.1], for $0 \leq i \leq d$, we consider the following representation $\rho_{d,i} = \mathrm{Ind}_{\Gamma_d(i)}^{\Gamma_d}(\widetilde{\chi}_i)$ of Γ_d, where $\Gamma_d(i) = (\mathbb{Z}/2\mathbb{Z})^d \rtimes (S_i \times S_{d-i})$, χ_i is the character on $(\mathbb{Z}/2\mathbb{Z})^d$ that is non-trivial on the first i factors and trivial on the rest, and $\widetilde{\chi}_i$ is the extension of χ_i to $\Gamma_d(i)$ that is trivial on $S_i \times S_{d-i}$. As we noted towards the end of the proof of [10, Prop 8.2], there is a canonical isomorphism of Γ_d-representations:

$$\mathrm{Ind}_{S_d}^{\Gamma_d}(\mathbf{1}) \cong \bigoplus_{i=0}^{d} \rho_{d,i}. \tag{7.1}$$

Then ρ_i gives rise to a local system $L(\rho_{d,i})$ on $X_d^{\sqrt{R},\circ}$ (which is smooth over k). Let $j_d : X_d^{\sqrt{R},\circ} \hookrightarrow \widehat{X}_d^{\sqrt{R}}$ be the inclusion. Let

$$K_{d,i} = j_{d,!*}(L(\rho_{d,i})[d])[-d]$$

be the middle extension perverse sheaf on $\widehat{X}_d^{\sqrt{R}}$.

We first study the direct image complex of $f_d : \mathcal{M}_d \to \mathcal{A}_d$. By Proposition 5.5, for $d \geq 2g' - 1 + N$, $\dim \mathcal{M}_d = m = \mathcal{A}_d$.

PROPOSITION 7.1. *Let $d \geq 2g' - 1 + N$.*

(1) *The complex $\mathbf{R}f_{d,!}\mathbb{Q}_\ell[m]$ is a perverse sheaf on $\mathcal{A}_d$, and it is the middle extension of its restriction to any non-empty open subset of $\mathcal{A}_d$.*
(2) *We have a canonical isomorphism*

$$\mathbf{R}f_{d,!}\mathbb{Q}_\ell \cong \bigoplus_{i=0}^{d+\rho-N_-} \bigoplus_{j=0}^{d+\rho-N_+} (K_{d+\rho-N_-,i} \boxtimes K_{d+\rho-N_+,j})|_{\mathcal{A}_d}. \tag{7.2}$$

Here we are identifying $\mathcal{A}_d$ *with an open substack of* $\widehat{X}^{\sqrt{R}}_{d+\rho-N_-}\times_{\mathrm{Pic}_X^{\sqrt{R};\sqrt{R},d+\rho}}\widehat{X}^{\sqrt{R}}_{d+\rho-N_+}$ *using* (5.4).

Proof. (1) We observe that the base $\mathcal{A}_d$ is irreducible (because both maps ν_a and ν_b are vector bundles when $d\geq 2g-1+N$). By Proposition 5.5(1), $\mathcal{M}_d$ is smooth and equidimensional. By Proposition 5.5(3)(4), f_d is proper and small. Therefore, $\mathbf{R}f_{d!}\mathbb{Q}_\ell[m]$ is a middle extension perverse sheaf from any non-empty open subset of $\mathcal{A}_d$.

(2) In fact this part holds under a weaker condition $d\geq 3g-2+N$. By Proposition 5.5(2) and the Künneth formula, we have

$$\mathbf{R}f_{d!}\mathbb{Q}_\ell\cong(\mathbf{R}\widehat{\nu}^{\sqrt{R}}_{d+\rho-N_-,!}\mathbb{Q}_\ell\boxtimes\mathbf{R}\widehat{\nu}^{\sqrt{R}}_{d+\rho-N_+,!}\mathbb{Q}_\ell)|_{\mathcal{A}_d}.$$

Therefore, it suffices to show that for $d'\geq 2g'-g=3g-2+\rho$ (note that $d+\rho-N_\pm\geq 3g-2+\rho$),

$$\mathbf{R}\widehat{\nu}^{\sqrt{R}}_{d'!}\mathbb{Q}_\ell\cong\bigoplus_{i=0}^{d'}K_{d',i}.$$

We claim that $\widehat{\nu}^{\sqrt{R}}_{d'}:\widehat{X}'_{d'}\to\widehat{X}^{\sqrt{R}}_{d'}$ is small when $d'\geq 2g'-g$. In fact, the only positive dimensional fibers are over the zero section $\mathrm{Pic}_X^{\sqrt{R},d'}\hookrightarrow\widehat{X}^{\sqrt{R}}_{d'}$, which has codimension $d'-g+1$ (provided that $d'\geq g-1$). The restriction of $\widehat{\nu}^{\sqrt{R}}_{d'}$ over $\mathrm{Pic}_X^{\sqrt{R},d'}$ is the norm map $\mathrm{Pic}^{d'}_{X'}\to\mathrm{Pic}_X^{\sqrt{R},d'}$, whose fibers have dimension $g'-g$. Since $d'\geq 2g'-g$, we have $d'-g+1\geq 2(g'-g)+1$, which implies that $\widehat{\nu}^{\sqrt{R}}_{d'}$ is small.

Since the source of $\widehat{\nu}^{\sqrt{R}}_{d'}$ is smooth and geometrically connected of dimension d', and since $\widehat{\nu}^{\sqrt{R}}_{d'}$ is proper, $\mathbf{R}\widehat{\nu}^{\sqrt{R}}_{d'!}\mathbb{Q}_\ell[d]$ is a middle extension perverse sheaf from its restriction to $X^{\sqrt{R},\circ}_{d'}$. The rest of the argument is the same as [10, Prop. 8.2], using (7.1). □

Recall from Section 5.1.5 that we have endomorphisms $f_{d,!}[\overline{\mathcal{H}}^{\diamondsuit}_+]$ and $f_{d,!}[\overline{\mathcal{H}}^{\diamondsuit}_-]$ of $\mathbf{R}f_{d,!}\mathbb{Q}_\ell$.

PROPOSITION 7.2. *Suppose* $d\geq 2g'-1+N$. *Then the action of* $f_{d,!}[\overline{\mathcal{H}}^{\diamondsuit}_+]$ (*resp.* $f_{d,!}[\overline{\mathcal{H}}^{\diamondsuit}_-]$) *preserves each direct summand in the decomposition* (7.2) *and acts on the summand* $(K_{d+\rho-N_-,i}\boxtimes K_{d+\rho-N_+,j})|_{\mathcal{A}_d}$ *by the scalar* $d+\rho-N_+-2j$ (*resp.* $d+\rho-N_--2i$).

Proof. By Proposition 7.1(1), any endomorphism of the middle extension perverse sheaf $\mathbf{R}f_{d!}\mathbb{Q}_\ell$ (up to a shift) is determined by its restriction to any non-empty open subset of $\mathcal{A}_d$. Therefore, it suffices to prove the same statements over $\mathcal{A}^{\diamondsuit}_d$, over which $\mathcal{H}^{\diamondsuit}_+$ (resp. $\mathcal{H}^{\diamondsuit}_-$) is the pullback of the incidence correspondence $I'_{d+\rho-N_+}$ (resp. $I'_{d+\rho-N_-}$); see Section 5.1.5. The rest of the argument is the same as [10, Prop. 8.3]. □

Now we turn to the direct image complex of $g_{\underline{d}}:\mathcal{N}_{\underline{d}}\to\mathcal{A}_d$. By Proposition 6.2, when $d\geq 2g'-1+N$ and $\mathcal{N}_{\underline{d}}\neq\varnothing$, $\dim\mathcal{N}_{\underline{d}}=\dim\mathcal{A}_d=m$.

PROPOSITION 7.3. *Let $d\geq 2g'-1+N$ and $\underline{d}\in Q_d$.*

(1) *The complex $\mathbf{R}g_{\underline{d},!}L_{\underline{d}}[m]$ is a perverse sheaf on $\mathcal{A}_d$, and it is the middle extension of its restriction to any non-empty open subset of $\mathcal{A}_d$.*
(2) *We have a canonical isomorphism*

$$\mathbf{R}g_{\underline{d},!}L_{\underline{d}}\cong(K_{d+\rho-N_-,d_{11}}\boxtimes K_{d+\rho-N_+,d_{12}})|_{\mathcal{A}_d}. \tag{7.3}$$

Proof. (1) As in the proof of [10, Prop. 8.5], $g_{\underline{d}}$ is not small; however, by Proposition 6.2(2) and (4), we know that $\mathbf{R}g_{\underline{d},!}L_{\underline{d}}[m]$ is Verdier self-dual. Since $g_{\underline{d}}$ is finite over $\mathcal{A}_d^{\diamondsuit}$, $\mathbf{R}g_{\underline{d},!}L_{\underline{d}}[m]$ is a middle extension perverse sheaf on $\mathcal{A}_d^{\diamondsuit}$. To prove $\mathbf{R}g_{\underline{d},!}L_{\underline{d}}[m]$ is a middle extension perverse sheaf on the whole $\mathcal{A}_d$, we only need to show that the restriction $\mathbf{R}g_{\underline{d},!}L_{\underline{d}}[m]|_{\partial\mathcal{A}_d}$ lies in strictly negative perverse degrees, where $\partial\mathcal{A}_d=\mathcal{A}_d-\mathcal{A}_d^{\diamondsuit}$.

We have $\mathcal{A}_d=\mathcal{A}_d^{a=0}\sqcup\mathcal{A}_d^{b=0}$. (See notation in the proof of Proposition 5.5(4).) Below we will show that $\mathbf{R}g_{\underline{d},!}L_{\underline{d}}[m]|_{\mathcal{A}_d^{b=0}}$ lies in negative perverse degrees, and the argument for $\mathcal{A}_d^{a=0}$ is similar.

When $d_{12}<d_{21}-N_+$, we have a Cartesian diagram

$$\begin{array}{ccc} g_{\underline{d}}^{-1}(\mathcal{A}_d^{b=0}) & \longrightarrow & (X_{d_{11}}^{\sqrt{R}}\times X_{d_{22}-N_-}^{\sqrt{R}})\times_{\mathrm{Pic}_X^{\sqrt{R};\sqrt{R},d+\rho}}(X_{d_{12}}^{\sqrt{R}}\times\mathrm{Pic}_X^{\sqrt{R},d_{21}-N_+}) \\ \downarrow{\scriptstyle g_{\underline{d}}} & & \downarrow{\scriptstyle \mathrm{add}^{\sqrt{R}}_{d_{11},d_{22}-N_-}\times h} \\ \mathcal{A}_d^{b=0} & \longrightarrow & X_{d+\rho-N_-}^{\sqrt{R}}\times_{\mathrm{Pic}_X^{\sqrt{R};\sqrt{R},d+\rho}}\mathrm{Pic}_X^{\sqrt{R},d+\rho-N_+}, \end{array}$$

where the map h is the composition

$$X_{d_{12}}^{\sqrt{R}}\times\mathrm{Pic}_X^{\sqrt{R},d_{21}-N_+}\xrightarrow{\mathrm{AJ}_{d_{12}}^{\sqrt{R}}\times\mathrm{id}}\mathrm{Pic}_X^{\sqrt{R},d_{12}}\times\mathrm{Pic}_X^{\sqrt{R},d_{21}-N_+}\xrightarrow{\mathrm{mult}}\mathrm{Pic}_X^{\sqrt{R},d+\rho-N_+}.$$

We have

$$\mathbf{R}g_{\underline{d},!}L_{\underline{d}}|_{\mathcal{A}_d^{b=0}}\cong\Big(\mathbf{R}\mathrm{add}^{\sqrt{R}}_{d_{11},d_{22}-N_-,!}(L_{d_{11}}\boxtimes\mathbb{Q}_\ell)\boxtimes\mathbf{R}h_!(L_{d_{12}}\boxtimes\mathbb{Q}_\ell)\Big)|_{\mathcal{A}_d^{b=0}}.$$

The first factor $\mathbf{R}\mathrm{add}^{\sqrt{R}}_{d_{11},d_{22}-N_-,!}(L_{d_{11}}\boxtimes\mathbb{Q}_\ell)$ is concentrated in degree 0 since $\mathrm{add}^{\sqrt{R}}_{d_{11},d_{22}-N_-}$ is finite. The second factor is the constant sheaf on $\mathrm{Pic}_X^{\sqrt{R},d+\rho-N_+}$ with geometric stalk isomorphic to $\mathrm{H}^*(X_{d_{12}}^{\sqrt{R}}\otimes\overline{k},L_{d_{12}})$. By Lemma A.6,

$$\mathrm{H}^*(X_{d_{12}}^{\sqrt{R}}\otimes\overline{k},L_{d_{12}})$$

always lies in degrees

$$\leq\dim\mathrm{H}^1(X_1^{\sqrt{R}}\otimes\overline{k},L)=\dim\mathrm{H}^1_c((X-R)\otimes\overline{k},L)=2g-2+\rho.$$

Therefore, $\mathbf{R}g_{\underline{d},!}L_{\underline{d}}|_{\mathcal{A}_d^{b=0}}$ lies in degrees $\leq 2g-2+\rho$. Since $\operatorname{codim}_{\mathcal{A}_d}(\mathcal{A}_d^{b=0}) = d+\rho-N_+-g+1$ (see the proof of Proposition 5.5(4)), which is $\geq (2g-2+\rho)+1$ (for this we only need the weaker condition $d \geq 3g-2+N_+$), we conclude that $\mathbf{R}g_{\underline{d},!}L_{\underline{d}}[m]|_{\mathcal{A}_d^{b=0}}$ lies in cohomological degrees strictly less than $-\dim \mathcal{A}_d^{b=0}$, hence in strictly negative perverse degrees.

When $d_{12} \geq d_{21} - N_+$, the argument is similar. The role of the map h is now played by

$$h' : \operatorname{Pic}_X^{\sqrt{R},d_{12}} \times X_{d_{21}-N_+}^{\sqrt{R}} \xrightarrow{\mathrm{id}\times \mathrm{AJ}^{\sqrt{R}}_{d_{21}-N_+}} \operatorname{Pic}_X^{\sqrt{R},d_{12}} \times \operatorname{Pic}_X^{\sqrt{R},d_{21}-N_+} \xrightarrow{\mathrm{mult}} \operatorname{Pic}_X^{\sqrt{R},d+\rho-N_+}.$$

Using the isomorphism

$$\gamma = (h', \mathrm{pr}_2) : \operatorname{Pic}_X^{\sqrt{R},d_{12}} \times X_{d_{21}-N_+}^{\sqrt{R}} \xrightarrow{\sim} \operatorname{Pic}_X^{\sqrt{R},d+\rho-N_+} \times X_{d_{21}-N_+}^{\sqrt{R}},$$

the map $h'\gamma^{-1}$ becomes the projection to the first factor of

$$\operatorname{Pic}_X^{\sqrt{R},d+\rho-N_+} \times X_{d_{21}-N_+}^{\sqrt{R}}.$$

By Proposition A.11, $\mathrm{mult}^* L^{\mathrm{Pic}}_{d+\rho-N_+} \cong L^{\mathrm{Pic}}_{d_{12}} \boxtimes L^{\mathrm{Pic}}_{d_{21}-N_+}$. Therefore, we have $(\gamma^{-1})^*(L_{d_{12}} \boxtimes \mathbb{Q}_\ell) \cong L^{\mathrm{Pic}}_{d+\rho-N_+} \boxtimes L^{-1}_{d_{21}-N_+} \cong L^{\mathrm{Pic}}_{d+\rho-N_+} \boxtimes L_{d_{21}-N_+}$, and hence

$$h'_!(L_{d_{12}} \boxtimes \mathbb{Q}_\ell) \cong L^{\mathrm{Pic}}_{d+\rho-N_+} \otimes \mathrm{H}^*(X^{\sqrt{R}}_{d_{21}-N_+} \otimes \overline{k}, L_{d_{21}-N_+}).$$

Then we use Lemma A.6 again to conclude that $\mathbf{R}g_{\underline{d},!}L_{\underline{d}}[m]|_{\mathcal{A}_d^{b=0}}$ lies in strictly negative perverse degrees.

(2) By (1), we only need to check (7.3) over the open subset $\mathcal{A}_d^{\diamondsuit}$. By Proposition 6.2(3), the diagram (6.4) is Cartesian over $\mathcal{A}_d^{\diamondsuit}$, and we have

$$\mathbf{R}g_{\underline{d},!}L_{\underline{d}}|_{\mathcal{A}_d^{\diamondsuit}} \cong \left(\mathrm{add}^{\sqrt{R}}_{d_{11},d_{22}-N_-,!}(L_{d_{11}} \boxtimes \mathbb{Q}_\ell) \boxtimes \mathrm{add}^{\sqrt{R}}_{d_{12},d_{21}-N_+,!}(L_{d_{12}} \boxtimes \mathbb{Q}_\ell)\right)|_{\mathcal{A}_d^{\diamondsuit}}.$$

Here $\mathrm{add}^{\sqrt{R}}_{i,j}$ is the addition map (A.2). Therefore, it suffices to show that for any $i, j \geq 0$, there is a canonical isomorphism over $X^{\sqrt{R}}_{i+j}$,

$$\mathrm{add}^{\sqrt{R}}_{i,j,!}(L_i \boxtimes \mathbb{Q}_\ell) \cong K_{i+j,i}|_{X^{\sqrt{R}}_{i+j}}. \tag{7.4}$$

Now both sides are middle extension perverse sheaves (because $\mathrm{add}^{\sqrt{R}}_{i,j}$ is finite surjective with smooth irreducible source). The isomorphism (7.4) then follows from the same isomorphism between the restrictions of both sides to $(X-R)^\circ_{i+j}$; the latter was proved in [10, Prop. 8.5]. □

7.2. *Comparison of traces.* For $\mu,\mu'\in\mathfrak{T}_{r,\Sigma}$, recall the definition of $r_\pm$ from (5.13). For $f\in\mathscr{H}_G^\Sigma$, with $f^{\Sigma_\pm}$ defined in (2.30), let

$$\mathbb{J}^{\mu,\mu'}(f)=\left(\frac{\partial}{\partial s_1}\right)^{r_+}\left(\frac{\partial}{\partial s_2}\right)^{r_-}\left(q^{N_+s_1+N_-s_2}\mathbb{J}(f^{\Sigma_\pm},s_1,s_2)\right)\Big|_{s_1=s_2=0}.$$

THEOREM 7.4. *Suppose D is an effective divisor on U of degree $d\geq\max\{2g'-1+N,2g\}$. Then*

$$(-\log q)^{-r}\mathbb{J}^{\mu,\mu'}(h_D)=\mathbb{I}^{\mu,\mu'}(h_D). \tag{7.5}$$

Proof. By Theorem 6.3, we have

$$\begin{aligned}q^{N_+s_1+N_-s_2}\mathbb{J}(h_D^{\Sigma_\pm},s_1,s_2)=&\sum_{\underline{d}\in Q_d}q^{(2d_{12}-d-\rho+N_+)s_1+(2d_{11}-d-\rho+N_-)s_2}\\&\times\sum_{a\in\mathcal{A}_D^\flat(k)}\mathrm{Tr}(\mathrm{Fr}_a,(\mathbf{R}g^\flat_{\underline{d},!}L_{\underline{d}})_a).\end{aligned}$$

Using $\mathbf{R}g^\flat_{\underline{d},!}L_{\underline{d}}=\mathbf{R}\Omega_!\mathbf{R}g_{\underline{d},!}L_{\underline{d}}$, we have

$$\sum_{a\in\mathcal{A}_D^\flat(k)}\mathrm{Tr}(\mathrm{Fr}_a,(\mathbf{R}g^\flat_{\underline{d},!}L_{\underline{d}})_a)=\sum_{\widetilde{a}\in\mathcal{A}_D(k)}\frac{1}{\#\operatorname{Aut}(\widetilde{a})}\mathrm{Tr}(\mathrm{Fr}_{\widetilde{a}},(\mathbf{R}g_{\underline{d},!}L_{\underline{d}})_{\widetilde{a}}).$$

Here $\mathcal{A}_D\subset\mathcal{A}$ is the preimage of $\mathcal{A}_D^\flat$. Using Proposition 7.3, we can rewrite the above as

$$\sum_{\widetilde{a}\in\mathcal{A}_D(k)}\frac{1}{\#\operatorname{Aut}(\widetilde{a})}\mathrm{Tr}(\mathrm{Fr}_{\widetilde{a}},(K_{d+\rho-N_-,d_{11}}\boxtimes K_{d+\rho-N_+,d_{12}})_{\widetilde{a}}).$$

Therefore, we get

$$\begin{aligned}&q^{N_+s_1+N_-s_2}\mathbb{J}(h_D^{\Sigma_\pm},s_1,s_2)\\=&\sum_{i=0}^{d+\rho-N_-}\sum_{j=0}^{d+\rho-N_+}q^{(2j-d-\rho+N_+)s_1+(2i-d-\rho+N_-)s_2}\\&\times\sum_{\widetilde{a}\in\mathcal{A}_D(k)}\frac{1}{\#\operatorname{Aut}(\widetilde{a})}\mathrm{Tr}(\mathrm{Fr}_{\widetilde{a}},(K_{d+\rho-N_-,i}\boxtimes K_{d+\rho-N_+,j})_{\widetilde{a}}).\end{aligned}$$

Taking derivatives, we get

$$\begin{aligned}(\log q)^{-r}\mathbb{J}^{\mu,\mu'}(h_D)=&\sum_{i=0}^{d+\rho-N_-}\sum_{j=0}^{d+\rho-N_+}(2j-d-\rho+N_+)^{r_+}(2i-d-\rho+N_-)^{r_-}\\&\times\sum_{\widetilde{a}\in\mathcal{A}_D(k)}\frac{1}{\#\operatorname{Aut}(\widetilde{a})}\mathrm{Tr}(\mathrm{Fr}_{\widetilde{a}},(K_{d+\rho-N_-,i}\boxtimes K_{d+\rho-N_+,j})_{\widetilde{a}}).\end{aligned} \tag{7.6}$$

On the other hand, by Theorem 5.6, we have

$$\begin{aligned}&\mathbb{I}^{\mu,\mu'}(h_D)\\&=\sum_{a\in\mathcal{A}^{\flat}_D(k)}\mathrm{Tr}\left((f^{\flat}_{d,!}[\overline{\mathcal{H}}^{\diamondsuit}_{+}])^{r_+}_a\circ(f^{\flat}_{d,!}[\overline{\mathcal{H}}^{\diamondsuit}_{-}])^{r_-}_a\circ\mathrm{Fr}_a,(\mathbf{R}f^{\flat}_{d,!}\mathbb{Q}_\ell)_a\right)\\&=\sum_{\widetilde{a}\in\mathcal{A}_D(k)}\frac{1}{\#\,\mathrm{Aut}(\widetilde{a})}\mathrm{Tr}\left((f_{d,!}[\overline{\mathcal{H}}^{\diamondsuit}_{+}])^{r_+}_{\widetilde{a}}\circ(f_{d,!}[\overline{\mathcal{H}}^{\diamondsuit}_{-}])^{r_-}_{\widetilde{a}}\circ\mathrm{Fr}_{\widetilde{a}},(\mathbf{R}f_{d,!}\mathbb{Q}_\ell)_{\widetilde{a}}\right).\end{aligned}$$

By Propositions 7.1 and 7.2, for $\widetilde{a}\in\mathcal{A}_d(k)$, we have

$$\begin{aligned}&\mathrm{Tr}\left((f_{d,!}[\overline{\mathcal{H}}^{\diamondsuit}_{+}])^{r_+}_{\widetilde{a}}\circ(f_{d,!}[\overline{\mathcal{H}}^{\diamondsuit}_{-}])^{r_-}_{\widetilde{a}}\circ\mathrm{Fr}_{\widetilde{a}},(\mathbf{R}f_{d,!}\mathbb{Q}_\ell)_{\widetilde{a}}\right)\\&\quad=\sum_{i=0}^{d+\rho-N_-}\sum_{j=0}^{d+\rho-N_+}(d+\rho-N_+-2j)^{r_+}(d+\rho-N_--2i)^{r_-}\\&\quad\quad\times\mathrm{Tr}\left(\mathrm{Fr}_{\widetilde{a}},(K_{d+\rho-N_-,i}\boxtimes K_{d+\rho-N_+,j})_{\widetilde{a}}\right).\end{aligned}$$

Therefore

$$\begin{aligned}\mathbb{I}^{\mu,\mu'}(h_D)=&\sum_{i=0}^{d+\rho-N_-}\sum_{j=0}^{d+\rho-N_+}(d+\rho-N_+-2j)^{r_+}(d+\rho-N_--2i)^{r_-}\\&\times\sum_{\widetilde{a}\in\mathcal{A}_D(k)}\frac{1}{\#\,\mathrm{Aut}(\widetilde{a})}\mathrm{Tr}\left(\mathrm{Fr}_{\widetilde{a}},(K_{d+\rho-N_-,i}\boxtimes K_{d+\rho-N_+,j})_{\widetilde{a}}\right).\end{aligned}\tag{7.7}$$

Comparing (7.6) and (7.7), we get (7.5). The extra sign $(-1)^r$ in (7.5) comes from the fact that

$$\begin{aligned}&(d+\rho-N_+-2j)^{r_+}(d+\rho-N_--2i)^{r_-}\\&\qquad=(-1)^r(2j-d-\rho+N_+)^{r_+}(2i-d-\rho+N_-)^{r_-}.\end{aligned}$$ □

7.2.1. Fix $\xi\in\mathfrak{S}'_\infty(\overline{k})$. Let $V'(\xi)=\mathrm{H}^{2r}_c(\mathrm{Sht}'^{r}_G(\Sigma;\xi)\otimes\overline{k},\mathbb{Q}_\ell)(r)$. By the discussion in Section 3.5.6, the finiteness results proved in Section 3.5.5 for the cohomology of $\mathrm{Sht}^r_G(\Sigma;\Sigma_\infty)$ as a $\mathscr{H}^{\Sigma}_G$-module are also valid for V', hence for its summand $V'(\xi)$.

Let

$$K=\prod_{x\notin\Sigma}G(\mathcal{O}_x)\times\prod_{x\in\Sigma}\mathrm{Iw}_x.$$

Denote by $\mathcal{A}(K)$ the space of compactly supported, $\mathbb{Q}$-valued functions on the double coset $G(F)\backslash G(\mathbb{A})/K$. The moduli stack $\mathrm{Sht}^0_G(\Sigma)$ is exactly the discrete groupoid $G(F)\backslash G(\mathbb{A})/K$. Therefore, $\mathcal{A}(K)\otimes\mathbb{Q}_\ell$ is identified with

$$\mathrm{H}^0_c(\mathrm{Sht}^0_G(\Sigma)\otimes\overline{k},\mathbb{Q}_\ell).$$

Corollary 3.40 implies that the image of the action map $\mathscr{H}^{\Sigma}_G\to\mathrm{End}(\mathcal{A}(K))$ is a finitely generated $\mathbb{Q}$-algebra with Krull dimension one. Theorem 3.41 allows us to write

$$\mathcal{A}(K)\otimes\overline{\mathbb{Q}}_\ell=\mathcal{A}(K)_{\mathrm{Eis}}\otimes\overline{\mathbb{Q}}_\ell\oplus(\oplus_{\pi\in\Pi_\Sigma(\overline{\mathbb{Q}}_\ell)}\mathcal{A}(K)_\pi).$$

Here $\Pi_\Sigma(\overline{\mathbb{Q}}_\ell)$ is the set of cuspidal automorphic representations (with $\overline{\mathbb{Q}}_\ell$-coefficients) of $G(\mathbb{A})$ with level K. Each π determines a character $\lambda_\pi:\mathscr{H}_G^\Sigma\to\overline{\mathbb{Q}}_\ell$. By strong multiplicity one for G, the character λ_π determined π. Therefore, we may identify $\Pi_\Sigma(\overline{\mathbb{Q}}_\ell)$ as a subset of $\mathrm{Spec}\,\mathscr{H}_G^\Sigma\otimes\overline{\mathbb{Q}}_\ell$.

Let

$$\widetilde{\mathscr{H}_\ell^\Sigma}=\mathrm{Im}(\mathscr{H}_G^\Sigma\otimes\mathbb{Q}_\ell\longrightarrow\mathrm{End}_{\mathbb{Q}_\ell}(V'(\xi))\times\mathrm{End}_{\mathbb{Q}_\ell}(\mathcal{A}(K)\otimes\mathbb{Q}_\ell)\times\mathbb{Q}_\ell[\mathrm{Pic}_X(k)]^{\iota_{\mathrm{Pic}}}).$$

Then by Corollary 3.40, $\widetilde{\mathscr{H}_\ell^\Sigma}$ is again a finitely generated $\mathbb{Q}_\ell$-algebra with Krull dimension one.

THEOREM 7.5. *Let $\mu,\mu'\in\{\pm1\}^r$. Then for all $f\in\mathscr{H}_G^\Sigma$, we have the identity*

$$(-\log q)^{-r}\mathbb{J}^{\mu,\mu'}(f)=\mathbb{I}^{\mu,\mu'}(f).$$

The proof is the same as that of [10, Th. 9.2], using the finiteness property of $\widetilde{\mathscr{H}_\ell^\Sigma}$ and [10, Lemma 9.1].

7.3. *Conclusion of the proofs.*

7.3.1. *Proof of Theorem 1.2*. Both $\mathbb{I}^{\mu,\mu'}(h)$ and $\mathbb{J}^{\mu,\mu'}(h)$ depend only on the image of h in $\widetilde{\mathscr{H}_\ell^\Sigma}$.

Let $\mathcal{Y}=\mathrm{Spec}\,\widetilde{\mathscr{H}_\ell^\Sigma}$. By Theorem 3.41, we have a decomposition

$$\mathcal{Y}^{\mathrm{red}}=Z_{\mathrm{Eis},\mathbb{Q}_\ell}\coprod\mathcal{Y}_0,$$

where $\mathcal{Y}_0$ is a finite set of closed points. Under this decomposition, we have a corresponding decomposition of $\widetilde{\mathscr{H}_\ell^\Sigma}$,

$$\widetilde{\mathscr{H}_\ell^\Sigma}=\widetilde{\mathscr{H}_{\ell,\mathrm{Eis}}^\Sigma}\times\widetilde{\mathscr{H}_{\ell,0}^\Sigma},\tag{7.8}$$

such that $\mathrm{Spec}\,\widetilde{\mathscr{H}_{\ell,\mathrm{Eis}}^{\Sigma,\mathrm{red}}}=Z_{\mathrm{Eis},\mathbb{Q}_\ell}$ and $\mathrm{Spec}\,\widetilde{\mathscr{H}_{\ell,0}^{\Sigma,\mathrm{red}}}=\mathcal{Y}_0$. We have a decomposition

$$V'(\xi)\otimes\overline{\mathbb{Q}}_\ell=V'(\xi)_{\mathrm{Eis}}\otimes\overline{\mathbb{Q}}_\ell\oplus(\oplus_{\mathfrak{m}\in\mathcal{Y}_0(\overline{\mathbb{Q}}_\ell)}V'(\xi)_{\mathfrak{m}}),$$

where $\mathrm{Supp}(V'(\xi)_{\mathrm{Eis}})\subset Z_{\mathrm{Eis},\mathbb{Q}_\ell}$ and $V'(\xi)_{\mathfrak{m}}$ is the generalized eigenspace of $V'(\xi)\otimes\overline{\mathbb{Q}}_\ell$ under the character $\mathfrak{m}$ of $\widetilde{\mathscr{H}_\ell^\Sigma}$. Under this decomposition, let $Z^\mu_{\mathfrak{m}}(\xi)$ be the projection of $Z^\mu(\xi)\in V'(\xi)$ (the cycle class of $\theta'^{\mu}_*[\mathrm{Sht}^\mu_T(\mu_\infty\cdot\xi)]$) to the direct summand $V'(\xi)_{\mathfrak{m}}$.

Let $h\in\widetilde{\mathscr{H}_{\ell,0}^\Sigma}$, viewed as $(0,h)\in\widetilde{\mathscr{H}_\ell^\Sigma}$ under the decomposition (7.8). Since the $\mathscr{H}_G^\Sigma$-action on $V'(\xi)$ is self-adjoint with respect to the cup product pairing, we have

$$\mathbb{I}^{\mu,\mu'}(h)=\sum_{\mathfrak{m}\in\mathcal{Y}_0(\overline{\mathbb{Q}}_\ell)}(Z^\mu_{\mathfrak{m}}(\xi),h*Z^{\mu'}_{\mathfrak{m}}(\xi)).\tag{7.9}$$

On the other hand, we have

(7.10)
$$\begin{aligned}&\mathbb{J}^{\mu,\mu'}(h)\\&=\sum_{\pi\in\Pi_\Sigma(\overline{\mathbb{Q}}_\ell)}\lambda_\pi(h)\left(\frac{\partial}{\partial s_1}\right)^{r_+}\left(\frac{\partial}{\partial s_2}\right)^{r_-}\left(q^{N_+s_1+N_-s_2}\mathbb{J}_\pi(h^{\Sigma_\pm},s_1,s_2)\right)\Big|_{s_1=s_2=0}.\end{aligned}$$

By the discussion in Section 7.2.1, $\Pi_\Sigma(\overline{\mathbb{Q}}_\ell)$ can be viewed as a subset of $\mathcal{Y}_0(\overline{\mathbb{Q}}_\ell)$. Now let π be as in the statement of Theorem 1.2. Let $h=e_\pi$ be the idempotent in $\widetilde{\mathscr{H}}^\Sigma_{\ell,0}\otimes\overline{\mathbb{Q}}_\ell$ corresponding to $\pi\in\Pi_\Sigma(\overline{\mathbb{Q}}_\ell)\subset\mathcal{Y}_0(\overline{\mathbb{Q}}_\ell)$. In (7.9) and (7.10) we plug in $h=e_\pi$, and we get

$$\begin{aligned}&\mathbb{I}^{\mu,\mu'}(e_\pi)=(Z^\mu_\pi(\xi),Z^\mu_\pi(\xi)),\\&\mathbb{J}^{\mu,\mu'}(e_\pi)=\left(\frac{\partial}{\partial s_1}\right)^{r_+}\left(\frac{\partial}{\partial s_2}\right)^{r_-}\left(q^{N_+s_1+N_-s_2}\mathbb{J}_\pi(h^{\Sigma_\pm},s_1,s_2)\right)\Big|_{s_1=s_2=0}.\end{aligned}$$

Applying Theorem 7.5 to e_π,

$$\begin{aligned}(-\log q)^{-r}\left(\frac{\partial}{\partial s_1}\right)^{r_+}\left(\frac{\partial}{\partial s_2}\right)^{r_-}\left(q^{N_+s_1+N_-s_2}\mathbb{J}_\pi(h^{\Sigma_\pm},s_1,s_2)\right)\Big|_{s_1=s_2=0}\\=(Z^\mu_\pi(\xi),Z^\mu_\pi(\xi)).\end{aligned}$$

By Proposition 2.10, the left side above is the left side of (1.7). The proof of Theorem 1.2 is complete.

7.3.2. *Proof of Theorem 1.3*. Make a change of variables $t_1=s_1+s_2$, $t_2=s_1-s_2$. We have

$$\begin{aligned}\left(\frac{\partial}{\partial t_1}\right)^{r_1}\left(\frac{\partial}{\partial t_2}\right)^{r-r_1}&=\frac{1}{2^r}\left(\frac{\partial}{\partial s_1}+\frac{\partial}{\partial s_2}\right)^{r_1}\left(\frac{\partial}{\partial s_1}-\frac{\partial}{\partial s_2}\right)^{r-r_1}\\&=\frac{1}{2^r}\sum_{I\subset\{1,2,\dots,r\}}(-1)^{\#(I\cap\{r_1+1,\dots,r\})}\left(\frac{\partial}{\partial s_1}\right)^{r-\#I}\left(\frac{\partial}{\partial s_2}\right)^{\#I}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\mathscr{L}^{(r_1)}\left(\pi,\frac{1}{2}\right)\mathscr{L}^{(r-r_1)}\left(\pi\otimes\eta,\frac{1}{2}\right)\\&=\left(\frac{\partial}{\partial t_1}\right)^{r_1}\left(\frac{\partial}{\partial t_2}\right)^{r-r_1}(\mathscr{L}(\pi,t_1+\frac{1}{2})\mathscr{L}(\pi\otimes\eta,t_2+\frac{1}{2}))\Big|_{t_1=t_2=0}\\&=\frac{1}{2^r}\sum_{I\subset\{1,2,\dots,r\}}(-1)^{\#(I\cap\{r_1+1,\dots,r\})}\\&\qquad\times\left(\frac{\partial}{\partial s_1}\right)^{r-\#I}\left(\frac{\partial}{\partial s_2}\right)^{\#I}\mathscr{L}_{F'/F}(\pi,s_1,s_2)\Big|_{s_1=s_2=0}.\end{aligned}$$

For $I\subset\{1,2,\dots,r\}$, let $\sigma_I\in\{\pm1\}^r$ be the element that is -1 on the i-th coordinate if $i\in I$ and 1 elsewhere. We may view σ_I as an element in $\mathfrak{A}_{r,\Sigma}$.

Let $\mu \in \mathfrak{T}_{r,\Sigma}$. By Theorem 1.2,

$$\left(\frac{\partial}{\partial s_1}\right)^{r-\#I}\left(\frac{\partial}{\partial s_2}\right)^{\#I}\mathscr{L}_{F'/F}(\pi, s_1, s_2)\Big|_{s_1=s_2=0} = (Z^{\mu}_{\pi}(\xi), Z^{\sigma_I\cdot\mu}_{\pi}(\xi)) = (Z^{\mu}_{\pi}(\xi), \sigma_I\cdot Z^{\mu}_{\pi}(\xi)),$$

where the second equality follows from Lemma 4.10. Therefore

$$\begin{aligned}&\mathscr{L}^{(r_1)}\left(\pi,\frac{1}{2}\right)\mathscr{L}^{(r-r_1)}\left(\pi\otimes\eta,\frac{1}{2}\right)\\ &=\frac{1}{2^r}\sum_{I\subset\{1,2,\dots,r\}}(-1)^{\#(I\cap\{r_1+1,\dots,r\})}(Z^{\mu}_{\pi}(\xi),\quad \sigma_I\cdot Z^{\mu}_{\pi}(\xi))\\ &=\left(Z^{\mu}_{\pi}(\xi),\quad \frac{1}{2^r}\sum_{I\subset\{1,2,\dots,r\}}(-1)^{\#(I\cap\{r_1+1,\dots,r\})}\sigma_I\cdot Z^{\mu}_{\pi}(\xi)\right)\\ &=\left(Z^{\mu}_{\pi}(\xi),\quad \prod_{i=1}^{r_1}\frac{1+\sigma_i}{2}\prod_{j=r_1}^{r}\frac{1-\sigma_j}{2}\cdot Z^{\mu}_{\pi}(\xi)\right)=(Z^{\mu}_{\pi}(\xi),\varepsilon_{r_1}\cdot Z^{\mu}_{\pi}(\xi)).\end{aligned}$$

Since ε_{r_1} is an idempotent in $\mathbb{Q}[(\mathbb{Z}/2\mathbb{Z})^r]$ that is self-adjoint with respect to the intersection pairing on $\mathrm{Sht}'^{r}_{G}(\Sigma;\xi)$, we have

$$(Z^{\mu}_{\pi}(\xi),\varepsilon_{r_1}\cdot Z^{\mu}_{\pi}(\xi)) = (\varepsilon_{r_1}\cdot Z^{\mu}_{\pi}(\xi),\varepsilon_{r_1}\cdot Z^{\mu}_{\pi}(\xi)).$$

The theorem is proved.

Appendix A. Picard stack with ramifications

In this appendix we record some constructions in the geometric class field theory with ramifications of order two, which will be used in the descriptions of the moduli spaces in Sections 5 and 6.

A.1. *The Picard stack and Abel-Jacobi map with ramifications.* Let $R \subset X$ be a reduced finite subscheme.

Definition A.1. Let $\mathrm{Pic}_X^{\sqrt{R}}$ be the functor on k-schemes whose S-valued points is the groupoid of triples $\mathcal{L}^{\natural} = (\mathcal{L}, \mathcal{K}_R, \iota)$, where

- $\mathcal{L}$ is a line bundle over $X \times S$;
- $\mathcal{K}_R$ is a line bundle over $R \times S$;
- $\iota : \mathcal{K}_R^{\otimes 2} \xrightarrow{\sim} \mathcal{L}|_{R\times S}$ is an isomorphism of line bundles over $R \times S$.

We have a decomposition $\mathrm{Pic}_X^{\sqrt{R}} = \sqcup_{d\in\mathbb{Z}} \mathrm{Pic}_X^{\sqrt{R},d}$, where $\mathrm{Pic}_X^{\sqrt{R},d}$ is the subfunctor defined by imposing that $\deg(\mathcal{L}_s) = d$ for each geometric point $s \in S$.

A.1.1. We present $\mathrm{Pic}_X^{\sqrt{R}}$ as a quotient stack. Let $\mathrm{Pic}_{X,R}$ be the moduli stack classifying $(\mathcal{L}, \gamma)$, where $\mathcal{L}$ is a line bundle over X and γ is a trivialization of $\mathcal{L}_R$. The Weil restriction $\mathrm{Res}_k^R \mathbb{G}_m$ acts on $\mathrm{Pic}_{X,R}$ by changing the trivialization γ, whose quotient is naturally isomorphic to Pic_X. From the definition of $\mathrm{Pic}_X^{\sqrt{R}}$ we see there is a natural isomorphism of stacks

$$\mathrm{Pic}_X^{\sqrt{R}} \cong [\mathrm{Pic}_{X,R} /_{[2]} \mathrm{Res}_k^R \mathbb{G}_m].$$

Here the quotient is obtained by making $\mathrm{Res}_k^R \mathbb{G}_m$ act on $\mathrm{Pic}_{X,R}$ via the square of the usual action, and the notation $/_{[2]}$ is to emphasize the square action. When $R = \varnothing$, $\mathrm{Res}_k^R \mathbb{G}_m = \mathrm{Spec}\, k$ by convention, and the above discussion is still valid.

The forgetful map $(\mathcal{L}, \mathcal{K}_R, \iota) \mapsto \mathcal{L}$ gives a morphism of stacks

$$\mathrm{Pic}_X^{\sqrt{R}} \longrightarrow \mathrm{Pic}_X,$$

which is a $\mathrm{Res}_k^R \mu_2$-gerbe.

A.1.2. *Variant of* $\mathrm{Pic}_X^{\sqrt{R}}$. We will also need the following variant of $\mathrm{Pic}_X^{\sqrt{R}}$. Let $\mathrm{Pic}_X^{\sqrt{R};\sqrt{R}}$ be the stack whose S-points consist of $(\mathcal{L}, \mathcal{K}_R, \iota, \alpha_R)$, where $(\mathcal{L}, \mathcal{K}_R, \iota) \in \mathrm{Pic}_X^{\sqrt{R}}(S)$ and α_R is a section of $\mathcal{K}_R$. Then we have

$$\mathrm{Pic}_X^{\sqrt{R};\sqrt{R}} \cong \mathrm{Pic}_{X,R} \overset{[2],\mathrm{Res}_k^R \mathbb{G}_m}{\times} \mathrm{Res}_k^R \mathbb{A}^1.$$

Here the action of $\mathrm{Res}_k^R \mathbb{G}_m$ on $\mathrm{Pic}_{X,R}$ is the square action and its action on $\mathrm{Res}_k^R \mathbb{A}^1$ is by dilation.

Definition A.2. For each integer $d \geq 0$, let $\widehat{X}_d^{\sqrt{R}}$ be the k-stack whose S-points is the groupoid of tuples $(\mathcal{L}^\natural, a, \alpha_R)$, where

- $\mathcal{L}^\natural = (\mathcal{L}, \mathcal{K}_R, \iota) \in \mathrm{Pic}_X^{\sqrt{R},d}(S)$ — in particular, ι is an isomorphism $\mathcal{K}_R^{\otimes 2} \xrightarrow{\sim} \mathcal{L}_R$;
- a is a global section of $\mathcal{L}$;
- α_R is a section of $\mathcal{K}_R$ such that $\iota(\alpha_R^{\otimes 2}) = a_R$, where a_R is the restriction of a_R to $R \times S$.

We let $X_d^{\sqrt{R}} \subset \widehat{X}_d^{\sqrt{R}}$ be the open substack defined by requiring that a is nonzero along the geometric fiber $X \times \{s\}$ for all geometric points $s \in S$.

A.1.3. Forgetting the square roots $(\mathcal{K}_R, \iota, \alpha_R)$ we get a morphism to the stack $\widehat{X}_d$ defined in [10, §3.2.1]:

$$\widehat{\omega}_d^{\sqrt{R}} : \widehat{X}_d^{\sqrt{R}} \longrightarrow \widehat{X}_d.$$

Over a geometric point $(\mathcal{L}, a \in \Gamma(X_K, \mathcal{L})) \in \widehat{X}_d(K)$, the fiber of $\widehat{\omega}_d^{\sqrt{R}}$ is a product $\prod_{x \in R(K)} \mathcal{P}_x$, where $\mathcal{P}_x \cong \mathrm{Spec}\, K$ if $a(x) \neq 0$, and $\mathcal{P}_x \cong [\mathrm{Spec}\, K/\mu_{2,K}]$

if $a(x)=0$. In particular, the restriction of $\widehat{\omega}_d^{\sqrt{R}}$ to $X_d^{\sqrt{R}}$,

$$\omega_d^{\sqrt{R}} : X_d^{\sqrt{R}} \longrightarrow X_d,$$

realizes X_d as the coarse moduli scheme of $X_d^{\sqrt{R}}$. When $d=1$, $X_1^{\sqrt{R}}$ is the DM curve with coarse moduli space X and automorphic group μ_2 along R.

Definition A.3. For an open subset $U \subset X$, we define $U_d^{\sqrt{R}}$ to be the subset of $X_d^{\sqrt{R}}$ that is the preimage of U_d under the map $\omega_d^{\sqrt{R}}$.

We have another description of $\widehat{X}_d^{\sqrt{R}}$ as follows. Evaluating a section of a line bundle along R gives a morphism

$$\mathrm{ev}_d^R : \widehat{X}_d \longrightarrow [\mathrm{Res}_k^R \mathbb{A}^1 / \mathrm{Res}_k^R \mathbb{G}_m].$$

From the construction of $\widehat{X}_d^{\sqrt{R}}$ we get a Cartesian diagram

$$\begin{array}{ccc} \widehat{X}_d^{\sqrt{R}} & \xrightarrow{\mathrm{ev}_d^{\sqrt{R}}} & [\mathrm{Res}_k^R \mathbb{A}^1 / \mathrm{Res}_k^R \mathbb{G}_m] \\ \Big\downarrow \widehat{\omega}_d^{\sqrt{R}} & & \Big\downarrow [2] \\ \widehat{X}_d & \xrightarrow{\mathrm{ev}_d^R} & [\mathrm{Res}_k^R \mathbb{A}^1 / \mathrm{Res}_k^R \mathbb{G}_m]. \end{array} \tag{A.1}$$

Here the vertical map [2] is the square map on both $\mathrm{Res}_k^R \mathbb{A}^1$ and $\mathrm{Res}_k^R \mathbb{G}_m$.

Lemma A.4.

(1) *The map* ev_d^R *is smooth when restricted to* X_d.

(2) $X_d^{\sqrt{R}}$ *is a smooth* DM *stack over* k.

Proof. (1) We may argue by base changing to $\overline{k}$. We have

$$[\mathrm{Res}_k^R \mathbb{A}^1 / \mathrm{Res}_k^R \mathbb{G}_m]_{\overline{k}} \cong \prod_{x \in R(\overline{k})} [\mathbb{A}^1/\mathbb{G}_m],$$

and the map $\mathrm{ev}_{d,\overline{k}}^R : X_{d,\overline{k}} \to \prod_{x\in R(\overline{k})} [\mathbb{A}^1/\mathbb{G}_m]$ is the product of the evaluation maps ev_x for $x \in R(\overline{k})$. The following general statement follows from an easy calculation of tangent spaces.

Claim. *Let Z be a smooth and irreducible $\overline{k}$-scheme and $f_i : Z \to [\mathbb{A}^1/\mathbb{G}_m]$ be a collection of morphisms, $1 \le i \le n$. Assume the image of each f_i does not lie entirely in $[\{0\}/\mathbb{G}_m]$, so the scheme-theoretic preimage of $[\{0\}/\mathbb{G}_m]$ under f_i is a divisor $D_i \subset Z$. Let $f : Z \to \prod_{i=1}^n [\mathbb{A}^1/\mathbb{G}_m] \cong [\mathbb{A}^n/\mathbb{G}_m^n]$ be the fiber product of the f_i's. Then f is a smooth morphism if and only if the divisors $D_1, \dots, D_n$ are smooth and intersect transversely.*

We apply this claim to $Z = X_{d,\overline{k}}$ and the maps ev_x for $x \in R(\overline{k})$. The divisor D_x in this case is the locus in $X_{d,\overline{k}}$ classifying those degree d divisors

D of X containing x. For a subset $I \subset R(\overline{k})$, the intersection $D_I = \cap_{x\in I} D_x$ is the locus classifying those degree d divisors D of X containing all points in I. This is non-empty only if $\#I \le d$. When this is the case, we have an isomorphism $X_{d-\#I} \cong D_I$ given by $D \mapsto D + \sum_{x\in I} x$. (The fact that this is an isomorphism can be checked by an étale local calculation, reducing to the case X is $\mathbb{A}^1$.) In particular, $D_I \subset X_{d,\overline{k}}$ is smooth of codimension $\#I$. This shows that the divisors $\{D_x\}_{x\in R(\overline{k})}$ intersect transversely. By the claim above, the map $\mathrm{ev}^R_{d,\overline{k}}$ is smooth when restricted to $X_{d,\overline{k}}$.

(2) Since $\mathrm{ev}^R_d|_{X_d}$ is smooth by part (1), so is $\mathrm{ev}^{\sqrt{R}}_d|_{X^{\sqrt{R}}_d}$ by the Cartesian diagram (A.1). Therefore, $X^{\sqrt{R}}_d$ is a smooth algebraic stack over k. Since the square map $[\mathrm{Res}^R_k \mathbb{A}^1/\mathrm{Res}^R_k \mathbb{G}_m] \to [\mathrm{Res}^R_k \mathbb{A}^1/\mathrm{Res}^R_k \mathbb{G}_m]$ is relative DM and X_d is a scheme, we see that $X^{\sqrt{R}}_d$ is a DM stack again from (A.1). □

A.1.4. *The addition map.* Suppose $d_1, d_2 \in \mathbb{Z}_{\ge 0}$. Then we have a map

$$\widehat{\mathrm{add}}^{\sqrt{R}}_{d_1,d_2} : \widehat{X}^{\sqrt{R}}_{d_1} \times \widehat{X}^{\sqrt{R}}_{d_2} \longrightarrow \widehat{X}^{\sqrt{R}}_{d_1+d_2}$$

sending $(\mathcal{L}^\natural_1, a_1, \alpha_{R,1}, \mathcal{L}^\natural_2, a_2, \alpha_{R,2})$ to $(\mathcal{L}^\natural_1 \otimes \mathcal{L}^\natural_2, a_1 \otimes a_2, \alpha_{R,1} \otimes \alpha_{R,2})$. It restricts to a map

$$\mathrm{add}^{\sqrt{R}}_{d_1,d_2} : X^{\sqrt{R}}_{d_1} \times X^{\sqrt{R}}_{d_2} \longrightarrow X^{\sqrt{R}}_{d_1+d_2}. \tag{A.2}$$

In particular, applying this construction iteratively, we get a map (for $d \ge 0$)

$$p^{\sqrt{R}}_d : (X^{\sqrt{R}}_1)^d \longrightarrow X^{\sqrt{R}}_d, \tag{A.3}$$

which is S_d-invariant with respect to the permutation action on the source.

A.1.5. *The Abel-Jacobi map.* Forgetting the sections a we get a morphism

$$\widehat{\mathrm{AJ}}^{\sqrt{R};\sqrt{R}}_d : \widehat{X}^{\sqrt{R}}_d \longrightarrow \mathrm{Pic}^{\sqrt{R};\sqrt{R},d}_X.$$

We also get a map

$$\widehat{\mathrm{AJ}}^{\sqrt{R}}_d : \widehat{X}^{\sqrt{R}}_d \longrightarrow \mathrm{Pic}^{\sqrt{R},d}_X$$

by further forgetting α_R. Let $\mathrm{AJ}^{\sqrt{R};\sqrt{R}}_d$ and $\mathrm{AJ}^{\sqrt{R}}_d$ be the restrictions of $\widehat{\mathrm{AJ}}^{\sqrt{R};\sqrt{R}}_d$ and $\widehat{\mathrm{AJ}}^{\sqrt{R}}_d$ to $X^{\sqrt{R}}_d$. When $R = \varnothing$, $\mathrm{AJ}^{\sqrt{R}}_d$ reduces to the usual Abel-Jacobi map.

A.1.6. *Presentation of* $\mathrm{Pic}^{\sqrt{R}}_X(k)$. For $x \in R$, let

$$\mathcal{O}_{\sqrt{x}} = \mathcal{O}_x \times_{k(x)} k(x), \quad \mathcal{O}^\times_{\sqrt{x}} = \mathcal{O}^\times_x \times_{k(x)^\times} k(x)^\times,$$

where the second projections $k(x) \to k(x)$ and $k(x)^\times \to k(x)^\times$ are the square maps. Let $\mathbb{O}^\times_{\sqrt{R}} = \prod_{x\in R} \mathcal{O}^\times_{\sqrt{x}} \times \prod_{x\in|X-R|} \mathcal{O}^\times_x$. We have a homomorphism $\mathbb{O}^\times_{\sqrt{R}} \to \mathbb{O}^\times = \prod_{x\in|X|} \mathcal{O}^\times_x \to \mathbb{A}^\times_F$.

LEMMA A.5. *There is a canonical isomorphism of Picard groupoids*

$$F^\times\backslash\mathbb{A}^\times_F/\mathbb{O}^\times_{\sqrt{R}} \xrightarrow{\sim} \mathrm{Pic}^{\sqrt{R}}_X(k) \tag{A.4}$$

sending ϖ_x^{-1} *(where* ϖ_x *is a uniformizer at* $x \in |X-R|$*) to the point* $\mathcal{O}_X(x)^\natural = (\mathcal{O}_X(x), \mathcal{O}_R, 1) \in \mathrm{Pic}^{\sqrt{R}}_X(k)$.

Proof. Consider the groupoid $\widehat{\mathrm{Pic}}^{\sqrt{R}}_X(k)$ classifying

$$(\mathcal{L}, \tau_\eta, \{\tau_x\}_{x\in|X|}, \mathcal{K}_R, \iota, t_R = \{t_x\}_{x\in R}),$$

where $(\mathcal{L}, \mathcal{K}_R, \iota) \in \mathrm{Pic}^{\sqrt{R}}_X(k)$, $\tau_\eta : \mathcal{L}|_{\mathrm{Spec}\, F} \cong F$ is a trivialization of $\mathcal{L}$ at the generic point, and $\tau_x : \mathcal{L}|_{\mathrm{Spec}\,\mathcal{O}_x} \cong \mathcal{O}_x$ is a trivialization of $\mathcal{L}$ in the formal neighborhood of x, $t_x : \mathcal{K}_x \xrightarrow{\sim} k(x)$ is a trivialization of $\mathcal{K}_x$ for every $x \in R$, such that the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{K}_x^{\otimes 2} & \xrightarrow{\iota_x} & \mathcal{L}_x \\ \downarrow{\scriptstyle t_x^{\otimes 2}} & & \downarrow{\scriptstyle \tau_x|_x} \\ k(x)^{\otimes 2} & = & k(x). \end{array}$$

Similarly, we define $\widehat{\mathrm{Pic}}_X(k)$ to classify part of the data $(\mathcal{L}, \tau_\eta, \{\tau_x\}_{x\in|X|})$ as above. The forgetful map $\widehat{\mathrm{Pic}}^{\sqrt{R}}_X(k) \to \widehat{\mathrm{Pic}}_X(k)$ is an equivalence; the choices of the extra data $(\mathcal{K}_R, \iota, \tau_R)$ are unique up to a unique isomorphism.

We have an isomorphism $\widehat{\mathrm{Pic}}_X(k) \xrightarrow{\sim} \mathbb{A}^\times_F$ sending $(\mathcal{L}, \tau_\eta, \{\tau_x\}_{x\in|X|})$ to $(\tau_x \circ \tau_\eta^{-1})_{x\in|X|} \in \mathbb{A}^\times$. Therefore, we get a canonical isomorphism

$$\alpha : \mathbb{A}^\times_F \xrightarrow{\sim} \widehat{\mathrm{Pic}}^{\sqrt{R}}_X(k).$$

It is easy to see that for $x \in |X - R|$, $\alpha(\varpi_x^{-1})$ has image $\mathcal{O}_X(x)^\natural$ in $\mathrm{Pic}^{\sqrt{R}}_X(k)$.

There is an action of $F^\times$ on $\widehat{\mathrm{Pic}}^{\sqrt{R}}_X(k)$ by changing τ_η. For $x \in |X - R|$, there is an action of $\mathcal{O}^\times_x$ on $\widehat{\mathrm{Pic}}^{\sqrt{R}}_X(k)$ by changing τ_x. For $x \in R$, there is an action of $\mathcal{O}^\times_{\sqrt{x}} = \mathcal{O}^\times_x \times_{k(x)^\times} k(x)^\times$ on $\widehat{\mathrm{Pic}}^{\sqrt{R}}_X(k)$ by changing τ_x and t_x compatibly. Therefore, we get an action of $F^\times \times \mathbb{O}^\times_{\sqrt{R}}$ on $\widehat{\mathrm{Pic}}^{\sqrt{R}}_X(k)$. The isomorphism α is equivariant with respect to these actions. The forgetful map $\widehat{\mathrm{Pic}}^{\sqrt{R}}_X(k) \to \mathrm{Pic}^{\sqrt{R}}_X(k)$ is a torsor for the action of $F^\times \times \mathbb{O}^\times_{\sqrt{R}}$. Therefore, α induces the equivalence (A.4). $\square$

A.2. *Geometric class field theory.* In this subsection, we fix L to be a rank one $\overline{\mathbb{Q}}_\ell$-local system on $X_1^{\sqrt{R}}$. Since $X_1^{\sqrt{R}}$ is a smooth DM curve with coarse moduli space X and automorphic group μ_2 along R, such a local system is the same datum as a rank one $\overline{\mathbb{Q}}_\ell$-local system on $X-R$ with monodromy of order at most 2 at the $x \in R$.

Starting from L, we will give a canonical construction of local systems L_d on $X_d^{\sqrt{R}}$ for $d \geq 0$ and show that it descends to $\mathrm{Pic}_X^{\sqrt{R},d}$. In the case $R=\varnothing$, such a construction goes back to Deligne.

A.2.1. *The local system L_d.* Consider the S_d-invariant map $p_d^{\sqrt{R}}$ in (A.3). The complex $p_{d,!}^{\sqrt{R}} L^{\boxtimes d}$ is a middle extension perverse sheaf on $X_d^{\sqrt{R}}$ (i.e., it is the middle extension of a local system from a dense open subset of $X_d^{\sqrt{R}}$) because $p_d^{\sqrt{R}}$ is a finite map from a smooth and geometrically connected DM stack. Therefore, the S_d-invariant part

$$L_d := (p_{d,!}^{\sqrt{R}} L^{\boxtimes d})^{S_d}$$

is also a middle extension perverse sheaf on $X_d^{\sqrt{R}}$.

Lemma A.6. *Suppose the local system L is geometrically non-trivial. Then*

$$\mathrm{H}^i(X_d^{\sqrt{R}} \otimes \overline{k}, L_d) = \begin{cases} \wedge^d \left(\mathrm{H}^1(X_1^{\sqrt{R}} \otimes \overline{k}, L)\right), & i = d, \\ 0, & i \neq d. \end{cases}$$

Proof. By construction, the graded vector space $\mathrm{H}^*(X_d^{\sqrt{R}} \otimes \overline{k}, L_d)$ is the S_d-invariants of the graded vector space $\mathrm{H}^*(X_1^{\sqrt{R}}, L)^{\otimes d}$. (Here S_d acts by permuting the factors with the Koszul sign convention.) Since L is geometrically non-trivial, $\mathrm{H}^*(X_1^{\sqrt{R}}, L)$ is concentrated in degree 1. Hence $\mathrm{H}^*(X_d^{\sqrt{R}} \otimes \overline{k}, L_d)$ is concentrated in degree d and is equal to $\wedge^d \left(\mathrm{H}^1(X_1^{\sqrt{R}} \otimes \overline{k}, L)\right)$ in that degree. ☐

Lemma A.7. *The perverse sheaf L_d is a local system of rank one on $X_d^{\sqrt{R}}$.*

Proof. Since L_d is a middle extension perverse sheaf on $X_d^{\sqrt{R}}$, to show it is a local system of rank one, it suffices to check the stalks of L_d at any geometric point of $X_d^{\sqrt{R}}$ is one-dimensional. Consider a geometric point $(\mathcal{L}^\natural, a, \alpha_R) \in X_d^{\sqrt{R}}$ with $\mathrm{div}(a) = D$. By factorizing the situation according to the points in D, we reduce to show that for $x \in R(\overline{k})$, L_d has one-dimensional stalk at the geometric point $dx \in X_d^{\sqrt{R}}(\overline{k})$. The point dx has automorphism μ_2, and the restriction of $p_d^{\sqrt{R}}$ to the preimage of this orbifold point is

$$p_{dx} : [\mathrm{pt}/\mu_2]^d \longrightarrow [\mathrm{pt}/\mu_2]$$

induced by the multiplication map $m:\mu_2^d\to\mu_2$. The restriction of L to $x=[\mathrm{pt}/\mu_2]\in X_1^{\sqrt{R}}$ is given by either the trivial or the sign representation of μ_2 on $\overline{\mathbb{Q}}_\ell$. Therefore, $p_{dx,!}L_x^{\boxtimes d}$ is the $K_d=\ker(m:\mu_2^d\to\mu_2)$-coinvariants on $L_x^{\boxtimes d}$, which is $L_x^{\boxtimes d}$ itself since K_d always acts trivially on it. Therefore, the stalk of L_d at dx is one-dimensional. □

Lemma A.8. *For $d_1,d_2\ge 0$, there is a canonical isomorphism of local systems on $X_{d_1}^{\sqrt{R}}\times X_{d_2}^{\sqrt{R}}$,*

$$\alpha_{d_1,d_2}:\mathrm{add}_{d_1,d_2}^{\sqrt{R},*}L_{d_1+d_2}\cong L_{d_1}\boxtimes L_{d_2},$$

which is commutative and associative in the obvious sense.

Proof. Let $d=d_1+d_2$. Since both $\mathrm{add}_{d_1,d_2}^{\sqrt{R},*}L_d$ and $L_{d_1}\boxtimes L_{d_2}$ are local systems, it suffices to give such an isomorphism over a dense open substack of $X_{d_1}^{\sqrt{R}}\times X_{d_2}^{\sqrt{R}}$. Let $U=X-R$. Let $U_d^\circ\subset X_d^{\sqrt{R}}$ be the open subscheme consisting of multiplicity-free divisors on U. Let $(U_{d_1}\times U_{d_2})^\circ\subset X_{d_1}^{\sqrt{R}}\times X_{d_2}^{\sqrt{R}}$ be the preimage of U_d° under $\mathrm{add}_{d_1,d_2}^{\sqrt{R}}$.

The monodromy representation of the local system $L|_U$ is given by a homomorphism

$$\chi:\pi_1(U)\longrightarrow\{\pm1\}.$$

For any $n\in\mathbb{Z}_{\ge0}$, there is a canonical homomorphism

$$\varphi_n:\pi_1(U_n^\circ)\longrightarrow\pi_1(U)^n\rtimes S_n$$

given by the branched S_n-cover $U^n\to U_n$.

The monodromy representation of the local system $L_{d_1}\boxtimes L_{d_2}|_{(U_{d_1}\times U_{d_2})^\circ}$ is given by

$$\begin{aligned}\pi_1((U_{d_1}\times U_{d_2})^\circ)&\xrightarrow{(p_{1*},p_{2*})}\pi_1(U_{d_1}^\circ)\times\pi_1(U_{d_2}^\circ)\\&\xrightarrow{\varphi_{d_1}\times\varphi_{d_2}}(\pi_1(U)^{d_1}\rtimes S_{d_1})\times(\pi_1(U)^{d_2}\rtimes S_{d_2})\\&=\pi_1(U)^d\rtimes(S_{d_1}\times S_{d_2})\xrightarrow{(\chi,\dots,\chi)\times1}\{\pm1\}.\end{aligned}\tag{A.5}$$

The last map is χ on all the $\pi_1(U)$-factors and trivial on $S_{d_1}\times S_{d_2}$.

On the other hand, the local system $\mathrm{add}_{d_1,d_2}^{*}L_d|_{U_d^\circ}$ is given by the character

$$\pi_1((U_{d_1}\times U_{d_2})^\circ)\xrightarrow{\mathrm{add}_*}\pi_1(U_d^\circ)\xrightarrow{\varphi_d}\pi_1(U)^d\rtimes S_d\xrightarrow{(\chi,\dots,\chi)\times1}\{\pm1\}.\tag{A.6}$$

Observe that (A.5) and (A.6) are the same homomorphisms. This gives the desired isomorphism α_{d_1,d_2}. We leave the verification of the commutativity and associativity properties of α_{d_1,d_2} as an exercise. □

Lemma A.9. *For $d\ge\rho+\max\{2g-1,1\}$, the local system L_d on $X_d^{\sqrt{R}}$ descends to $\mathrm{Pic}_X^{\sqrt{R},d}$ via the map $\mathrm{AJ}_d^{\sqrt{R}}$.*

Proof. The case $R = \varnothing$ is well known; we treat only the case $R \neq \varnothing$.

When $d \geq 2g-1+\rho$, by Riemann-Roch, $\mathrm{AJ}_d^{\sqrt{R}}$ is a locally trivial fibration, and therefore it suffices to show that the restriction of L_d to geometric fibers of $\mathrm{AJ}_d^{\sqrt{R}}$ are trivial.

Fix a geometric point $\mathcal{L}^\natural = (\mathcal{L}, \mathcal{K}_R, \iota) \in \mathrm{Pic}_X^{\sqrt{R},d}(K)$ for some algebraically closed field K. We base change the situation from k to K without changing notation. The fiber of $\mathrm{AJ}_d^{\sqrt{R}}$ over $\mathcal{L}^\natural$ is

$$M = \mathrm{H}^0(X, \mathcal{L})^\circ \times_{\mathrm{H}^0(R,\mathcal{L}_R)} \mathrm{H}^0(R, \mathcal{K}_R),$$

where $\mathrm{H}^0(X,\mathcal{L})^\circ = \mathrm{H}^0(X,\mathcal{L}) - \{0\}$, and the map $\mathrm{H}^0(R,\mathcal{K}_R) \to \mathrm{H}^0(R,\mathcal{L}_R)$ is the square map via ι. The torus $\mathbb{G}_m$ acts on M by weight 2 on $\mathrm{H}^0(X,\mathcal{L})$ and weight 1 on $\mathrm{H}^0(R,\mathcal{K}_R)$. Then the map $M \to X_d^{\sqrt{R}}$ factors through the quotient $[M/\mathbb{G}_m]$. The triviality of $L_d|_{[M/\mathbb{G}_m]}$ follows from the claim below.

Claim. *$[M/\mathbb{G}_m]$ is simply-connected.*

It remains to prove the claim. Choosing a basis for $\mathrm{H}^0(R,\mathcal{L}_R)$ and extending it to $\mathrm{H}^0(X,\mathcal{L})$, we may identify M with a punctured affine space $\mathbb{A}^n - \{0\}$, and the action of $\mathbb{G}_m$ has weights 2 (on the first $n-\rho$ coordinates) and 1 (on the last ρ coordinates). Since $n = d-g+1 \geq \rho+1$, the weight 2 appears at least once.

Suppose $Y \to [M/\mathbb{G}_m]$ is a finite étale map with Y connected. Consider the map $\pi : \mathbb{P}^{n-1} \to [M/\mathbb{G}_m]$ given by $[x_1,\dots,x_{n-\rho},y_1,\dots,y_\rho] \mapsto [x_1^2,\dots,x_{n-\rho}^2,y_1,\dots,y_\rho]$. Then π is a branched Galois cover with Galois group $\mu_2^{n-\rho}$. Since $\mathbb{P}^{n-1}$ is simply-connected, π lifts to $\widetilde{\pi} : \mathbb{P}^{n-1} \to Y$. Therefore, the function field $K(Y) \subset K(\mathbb{P}^{n-1})$ corresponds to a subgroup $\Gamma \subset \mu_2^{n-\rho}$ so that Y is the normalization of $[M/\mathbb{G}_m]$ in $\mathrm{Spec}\, K(Y)$. We consider the open subset M°, where the last coordinate $y_\rho \neq 0$; then $M^\circ/\mathbb{G}_m \cong \mathbb{A}^{n-1}$. Let Y° be the preimage of $M^\circ/\mathbb{G}_m$ in Y, and let $(\mathbb{P}^{n-1})^\circ \cong \mathbb{A}^{n-1}$ be the preimage in $\mathbb{P}^{n-1}$. Then Y° is the GIT quotient of $(\mathbb{P}^{n-1})^\circ$ by Γ. If $\Gamma \neq \mu_2^{n-\rho}$, then there is a non-empty subset $I \subset \{1,\dots,n-\rho\}$ such that Γ is contained in the kernel of $e_I^* : \mu_2^{n-\rho} \to \mu_2$ given by $e_I^*(\varepsilon_i) = \varepsilon_i$ if $i \in I$ and 1 is $i \notin I$. In this case, $x_I = \prod_{i\in I} x_i$ is fixed by Γ, hence $x_I \in \mathcal{O}(Y^\circ)$. However, $x_I \notin \mathcal{O}(M^\circ/\mathbb{G}_m)$ (only $x_I^2 \in \mathcal{O}(M^\circ/\mathbb{G}_m)$). This implies that $Y^\circ \to M^\circ/\mathbb{G}_m$ is ramified along the divisor $x_I = 0$ in Y°, a contradiction. Therefore, $\Gamma = \mu_2^{n-\rho}$ and $Y = [M/\mathbb{G}_m]$. □

A.2.2. *Construction of L_d^{Pic} for all $d \in \mathbb{Z}$.* Let L_d^{Pic} be the descent of L_d to $\mathrm{Pic}_X^{\sqrt{R},d}$ when $d \geq \rho + \max\{2g-1, 1\}$. Next we extend the local systems $\{L_d^{\mathrm{Pic}}\}$ to all components of $\mathrm{Pic}_X^{\sqrt{R}}$.

Fix any integer d. For any divisor $D = \sum_{x\in|X-R|} n_x \cdot x \in \mathrm{Div}(X-R)$ of degree d', we have a canonical line $L_D = \otimes L_x^{\otimes n_x}$. Tensoring with $\mathcal{O}_X(D)^\natural$

(the canonical lift of $\mathcal{O}_X(D)$ to $\mathrm{Pic}_X^{\sqrt{R}}$) defines an isomorphism $t_D : \mathrm{Pic}_X^{\sqrt{R},d} \to \mathrm{Pic}_X^{\sqrt{R},d+d'}$. If $d'+d \geq \max\{2g-1,1\}+\rho$, then $L^{\mathrm{Pic}}_{d+d'}$ is already defined, and we define L^{Pic}_d to be the local system $t_D^* L^{\mathrm{Pic}}_{d+d'} \otimes L_D^{\otimes -1}$ on $\mathrm{Pic}_X^{\sqrt{R},d}$. We claim that L^{Pic}_d thus defined is canonically independent of the choice of D, as long as the degree d' of D satisfies $d' \geq \max\{2g-1,1\}+\rho-d$. To show this, it suffices to show that for any $n, n' \geq \max\{2g-1,1\}+\rho$ (so that L^{Pic}_n and $L^{\mathrm{Pic}}_{n'}$ are both defined as the descent of L_n and $L_{n'}$) and any $D \in \mathrm{Div}^{n'-n}(X-R)$, there is a canonical isomorphism $t_D^* L^{\mathrm{Pic}}_{n'} \cong L^{\mathrm{Pic}}_n \otimes L_D$ as local systems on $\mathrm{Pic}_X^{\sqrt{R},n}$. It is easy to reduce to the case D effective. Since $\mathrm{AJ}_n^{\sqrt{R}}$ has connected geometric fibers, it is enough to give such an isomorphism after pulling back to $X_n^{\sqrt{R}}$; i.e., we need to give a canonical isomorphism of local systems on $X_n^{\sqrt{R}}$,

$$T_D^* L_{n'} \cong L_n \otimes L_D, \tag{A.7}$$

where $T_D : X_n^{\sqrt{R}} \to X_{n'}^{\sqrt{R}}$ is the addition by D. Such an isomorphism is given by Lemma A.8 by taking restricting $\alpha_{n,n'-n}$ to $X_n^{\sqrt{R}} \times \{D\}$.

We have therefore defined a canonical local system L^{Pic}_d on $\mathrm{Pic}_d^{\sqrt{R}}$ for each $d \in \mathbb{Z}$. Let L^{Pic} be the local system on $\mathrm{Pic}_X^{\sqrt{R}}$ whose restriction to $\mathrm{Pic}_d^{\sqrt{R}}$ is L^{Pic}_d.

LEMMA A.10. *For $d \geq 0$, we have a canonical isomorphism of local systems on $X_d^{\sqrt{R}}$*

$$\mathrm{AJ}_d^{\sqrt{R},*} L^{\mathrm{Pic}}_d \cong L_d.$$

Proof. Let D be a divisor on $X-R$ of degree $d' \geq \max\{2g-1,1\}+\rho-d$. By construction we have $L^{\mathrm{Pic}}_d = t_D^* L^{\mathrm{Pic}}_{d+d'} \otimes L_D^{\otimes -1}$. Pulling back both sides to $X_d^{\sqrt{R}}$, and noting $\mathrm{AJ}_{d+d'}^{\sqrt{R}} \circ T_D = t_D \circ \mathrm{AJ}_d^{\sqrt{R}}$, we get

$$\begin{aligned}\mathrm{AJ}_d^{\sqrt{R}*} L^{\mathrm{Pic}}_d = \mathrm{AJ}_d^{\sqrt{R}*} t_D^* L^{\mathrm{Pic}}_{d+d'} \otimes L_D^{\otimes -1} &= T_D^* \mathrm{AJ}_{d+d'}^{\sqrt{R}*} L^{\mathrm{Pic}}_{d+d'} \otimes L_D^{\otimes -1} \\ &= T_D^* L_{d+d'} \otimes L_D^{\otimes -1},\end{aligned}$$

which is canonically isomorphic to L_d by (A.7). □

PROPOSITION A.11. *The local system L^{Pic} is a character sheaf on $\mathrm{Pic}_X^{\sqrt{R}}$. More precisely, this means the following*:

(1) *There is a canonical trivialization $\iota : L^{\mathrm{Pic}}|_e \cong \overline{\mathbb{Q}}_\ell$, where e is the origin of $\mathrm{Pic}_X^{\sqrt{R}}$.*

(2) *There is a canonical isomorphism of local systems on $\mathrm{Pic}_X^{\sqrt{R}} \times \mathrm{Pic}_X^{\sqrt{R}}$,*

$$\mu : \mathrm{mult}^* L^{\mathrm{Pic}} \cong L^{\mathrm{Pic}} \boxtimes L^{\mathrm{Pic}},$$

where $\mathrm{mult} : \mathrm{Pic}_X^{\sqrt{R}} \times \mathrm{Pic}_X^{\sqrt{R}} \to \mathrm{Pic}_X^{\sqrt{R}}$ *is the multiplication map.*

(3) *The isomorphism μ is commutative and associative in the obvious sense, and its restrictions to $\{e\} \times \mathrm{Pic}_X^{\sqrt{R}}$ and $\mathrm{Pic}_X^{\sqrt{R}} \times \{e\}$ are the identity maps on L^{Pic} (after using ι to trivialize $L^{\mathrm{Pic}}|_e$).*

Proof. By construction, $L^{\mathrm{Pic}}|_e \cong L_d^{\mathrm{Pic}}|_{\mathcal{O}(D)^\natural} \otimes L_D^{\otimes -1} \cong L_d|_D \otimes L_D^{\otimes -1}$ for any effective divisor $D \in \mathrm{Div}(X-R)$ of large degree d. (We are viewing D as a k-point of $(X-R)_d \subset X_d^{\sqrt{R}}$, so $L_d|_D$ means the stalk of L_d at this k-point D.) If we write $D = \sum_{x \in |X-R|} n_x \cdot x$, then by construction we have a canonical isomorphism $L_d|_D \cong \otimes_{x\in|X-R|} L_x^{\otimes n_x} = L_D$, which gives a trivialization $\iota_D : L^{\mathrm{Pic}}|_e \cong \overline{\mathbb{Q}}_\ell$. We leave it as an exercise to check that ι_D is independent of the choice of D.

Now we construct the isomorphism μ, i.e., a system of isomorphisms

$$\mu_{d_1,d_2} : \mathrm{mult}^*_{d_1,d_2} L^{\mathrm{Pic}}_{d_1+d_2} \cong L^{\mathrm{Pic}}_{d_1} \boxtimes L^{\mathrm{Pic}}_{d_2}$$

for all $d_1, d_2 \in \mathbb{Z}$. When $d_1, d_2 \geq \rho + \max\{2g-1, 1\}$, $L^{\mathrm{Pic}}_{d_i}$ and $L^{\mathrm{Pic}}_{d_1+d_2}$ come by descent from L_{d_i} and $L_{d_1+d_2}$. Since $\mathrm{AJ}^{\sqrt{R}}_{d_1+d_2}$ has connected geometric fibers, it suffices to give μ_{d_1,d_2} after pulling back both sides to $X^{\sqrt{R}}_{d_1+d_2}$, in which case the desired isomorphism is given by α_{d_1,d_2} constructed in Lemma A.8.

For general d_1, d_2, let $D_1, D_2 \in \mathrm{Div}(X-R)$ with degrees $\deg D_i = n_i$ such that $n_i + d_i \geq \rho + \max\{2g-1, 1\}$ for $i = 1, 2$. Then by construction,

$$L^{\mathrm{Pic}}_{d_1} \boxtimes L^{\mathrm{Pic}}_{d_2} \cong (t^*_{D_1} L^{\mathrm{Pic}}_{d_1+n_1} \boxtimes t^*_{D_2} L^{\mathrm{Pic}}_{d_2+n_2}) \otimes (L^{\otimes -1}_{D_1} \otimes L^{\otimes -1}_{D_2}). \tag{A.8}$$

On the other hand, $L^{\mathrm{Pic}}_{d_1+d_2} \cong t^*_{D_1+D_2} L_{d_1+d_2+n_1+n_2} \otimes L^{\otimes -1}_{D_1+D_2}$, hence

$$\begin{aligned} \mathrm{mult}^*_{d_1,d_2} L^{\mathrm{Pic}}_{d_1+d_2} &\cong \mathrm{mult}^*_{d_1,d_2} t^*_{D_1+D_2} L_{d_1+d_2+n_1+n_2} \otimes L^{\otimes -1}_{D_1+D_2} \\ &\cong ((t_{D_1} \times t_{D_2})^* \mathrm{mult}^*_{d_1+n_1,d_2+n_2} L_{d_1+d_2+n_1+n_2}) \\ &\quad \otimes (L^{\otimes -1}_{D_1} \otimes L^{\otimes -1}_{D_2}). \end{aligned} \tag{A.9}$$

Comparing the right-hand sides of (A.8) and (A.9), the desired isomorphism μ_{d_1,d_2} is induced from the already-constructed $\mu_{d_1+n_1,d_2+n_2}$. Again we leave it as an exercise to check that μ_{d_1,d_2} is independent of the choices of D_1, D_2, and it satisfies commutativity, associativity, and compatibility with ι. □

A.3. *Ramified double cover.* Let $\nu : X' \to X$ be a double cover with ramification locus $R \subset X$, where X' is also a smooth projective and geometrically connected curve over k. Let $\sigma : X' \to X'$ be the non-trivial involution over X. Let $R' \subset X'$ be the reduced preimage of R, then ν induces an isomorphism $R' \xrightarrow{\sim} R$.

A.3.1. *The norm map on Picard.* Let $i_R : R \hookrightarrow X$ be the inclusion. We consider the étale sheaf $\mathbb{G}_{m,R}$ on R as an étale sheaf on X via $i_{R,*}$. There is a restriction map $\mathbb{G}_{m,X} \to \mathbb{G}_{m,R}$. Consider the following étale sheaf on X:

$$\mathbb{G}_{m,X}^{\sqrt{R}} = \mathbb{G}_{m,X} \times_{\mathbb{G}_{m,R},[2]} \mathbb{G}_{m,R},$$

where the map $\mathbb{G}_{m,R} \to \mathbb{G}_{m,R}$ is the square map. By construction, $\mathrm{Pic}_X^{\sqrt{R}}$ is the moduli stack of $\mathbb{G}_{m,X}^{\sqrt{R}}$-torsors over X.

We have the sheaf homomorphism induced by the norm map $\mathrm{Nm} : \nu_*\mathbb{G}_{m,X'} \to \mathbb{G}_{m,X}$ and the restriction map $r_{R'} : \nu_*\mathbb{G}_{m,X'} \to \nu_*\mathbb{G}_{m,R'} = \mathbb{G}_{m,R}$. Computing with local coordinates at R, we see that the composition $\nu_*\mathbb{G}_{m,X'} \xrightarrow{\mathrm{Nm}} \mathbb{G}_{m,X} \xrightarrow{r_R} \mathbb{G}_{m,R}$ (the latter r_R is given by restriction) is the square of the restriction map $r_{R'}$. Therefore, $(\mathrm{Nm}, r_{R'})$ induces a sheaf homomorphism

$$\underline{\mathrm{Nm}}_{X'/X}^{\sqrt{R}} : \nu_*\mathbb{G}_{m,X'} \longrightarrow \mathbb{G}_{m,X}^{\sqrt{R}},$$

which is easily seen to be surjective by local calculation at R. The map $\underline{\mathrm{Nm}}_{X'/X}^{\sqrt{R}}$ on sheaves induces a morphism of Picard stacks

$$\mathrm{Nm}_{X'/X}^{\sqrt{R}} : \mathrm{Pic}_{X'} \longrightarrow \mathrm{Pic}_X^{\sqrt{R}},$$

which lifts the usual norm map $\mathrm{Nm}_{X'/X} : \mathrm{Pic}_{X'} \to \mathrm{Pic}_X$.

A.3.2. *The norm map on symmetric powers.* There is also a natural lifting of the norm map $\widehat{\nu}_d : \widehat{X}'_d \to \widehat{X}_d$:

$$\widehat{\nu}_d^{\sqrt{R}} : \widehat{X}'_d \longrightarrow \widehat{X}_d^{\sqrt{R}}. \tag{A.10}$$

In fact, for $(\mathcal{L}', a') \in \widehat{X}'_d(S)$, where $\mathcal{L}'$ is a line bundle over $X' \times S$ and a' a global section of $\mathcal{L}'$, $\mathcal{L} = \mathrm{Nm}_{X'/X}(\mathcal{L}')$ is a line bundle over $X \times S$, and $a = \mathrm{Nm}(a')$ is a section of $\mathcal{L}$. We have a canonical isomorphism $\iota : (\mathcal{L}'|_{R' \times S})^{\otimes 2} \cong (\mathcal{L}' \otimes \sigma^*\mathcal{L}')|_{R' \times S} \cong \mathcal{L}|_{R \times S}$. Under ι, $a'|_{R' \times S}$ gives a square root of the restriction $a|_{R \times S}$. We then send $(\mathcal{L}', a')$ to $(\mathcal{L}, \mathcal{L}'|_{R' \times S}, \iota, a, a'|_{R' \times S}) \in \widehat{X}_d^{\sqrt{R}}(S)$.

By construction, we have a commutative diagram

$$\begin{array}{ccc}
\widehat{X}'_d & \xrightarrow{\widehat{\nu}_d^{\sqrt{R}}} & \widehat{X}_d^{\sqrt{R}} \\
\Big\downarrow \widehat{\mathrm{AJ}}'_d & & \Big\downarrow \widehat{\mathrm{AJ}}_d^{\sqrt{R}} \\
\mathrm{Pic}_{X'} & \xrightarrow{\mathrm{Nm}_{X'/X}^{\sqrt{R}}} & \mathrm{Pic}_X^{\sqrt{R}},
\end{array}$$

where $\widehat{\mathrm{AJ}}'_d$ is the Abel-Jacobi map for X'.

A.3.3. *Descent of line bundles.* A local calculation shows that the image of $1 - \sigma : \nu_*\mathbb{G}_{m,X'} \to \nu_*\mathbb{G}_{m,X'}$ is equal to the kernel of $\underline{\mathrm{Nm}}_{X'/X}^{\sqrt{R}}$. Therefore, we have an exact sequence of étale sheaves on X:

$$1 \longrightarrow \mathbb{G}_{m,X} \longrightarrow \nu_*\mathbb{G}_{m,X'} \xrightarrow{1-\sigma} \nu_*\mathbb{G}_{m,X'} \xrightarrow{\underline{\mathrm{Nm}}_{X'/X}^{\sqrt{R}}} \mathbb{G}_{m,X}^{\sqrt{R}} \longrightarrow 1.$$

Taking the corresponding Picard stacks we get an exact sequence of Picard stacks:

$$1 \longrightarrow \mathrm{Pic}_X \xrightarrow{\nu^*} \mathrm{Pic}_{X'} \xrightarrow{1-\sigma} \mathrm{Pic}_{X'} \xrightarrow{\mathrm{Nm}^{\sqrt{R}}_{X'/X}} \mathrm{Pic}^{\sqrt{R}}_X \longrightarrow 1. \tag{A.11}$$

A.3.4. *The local system L.* The direct image sheaf $\nu_*\mathbb{Q}_\ell$ has a decomposition $\nu_*\mathbb{Q}_\ell = \overline{\mathbb{Q}}_\ell \oplus L_{X'/X}$ into σ-eigensheaves with eigenvalues 1 and -1. Then $L_{X'/X}|_{X-R}$ is a $\mathbb{Q}_\ell$-local system of rank one with monodromy in $\{\pm 1\}$ ramified exactly along R. Let L be the local system on $X_1^{\sqrt{R}}$ corresponding to $(L_{X'/X}\otimes\overline{\mathbb{Q}}_\ell)|_{X-R}$. Associated to L is a local system L^{Pic} on $\mathrm{Pic}^{\sqrt{R}}_X$ constructed in Section A.2.

Let $F' = k(X')$, a quadratic extension of F unramified away from R. By class field theory, F'/F gives rise to an idèle class character

$$\eta_{F'/F} : F^\times\backslash\mathbb{A}_F^\times/\mathbb{O}^\times_{\sqrt{R}} \longrightarrow \{\pm 1\}.$$

For the notation $\mathbb{O}^\times_{\sqrt{R}}$, see Section A.1.6.

Proposition A.12. *Under the sheaf-to-function correspondence, the function on $\mathrm{Pic}^{\sqrt{R}}_X(k)$ given by L^{Pic} is the idèle class character $\eta_{F'/F}$ under the isomorphism* (A.4).

Proof. Let $f_L : \mathrm{Pic}^{\sqrt{R}}_X(k) \to \overline{\mathbb{Q}}_\ell^\times$ be the function attached to L^{Pic}. By Proposition A.11, f_L is a group homomorphism. We know that $\eta_{F'/F}$ is characterized by the property that for a uniformizer ϖ_x at $x \in |X-R|$,

$$\eta_{F'/F}(\varpi_x^{-1}) = \begin{cases} 1 & \text{if } x \text{ is split in } F', \\ -1 & \text{if } x \text{ is inert in } F'. \end{cases}$$

Now x is split (resp. inert) in F' if and only if $\mathrm{Tr}(\mathrm{Fr}_x, L_x) = 1$ (resp. $\mathrm{Tr}(\mathrm{Fr}_x, L_x) = -1$). Therefore

$$\eta_{F'/F}(\varpi_x^{-1}) = \mathrm{Tr}(\mathrm{Fr}_x, L_x).$$

We only need to check that f_L enjoys the same property as $\eta_{F'/F}$. Since ϖ_x^{-1} corresponds to $\mathcal{O}(x)^\natural \in \mathrm{Pic}^{\sqrt{R},d_x}_X(k)$ under (A.4), we need to show

$$\mathrm{Tr}(\mathrm{Fr}_{\mathcal{O}(x)^\natural}, L^{\mathrm{Pic}}|_{\mathcal{O}(x)^\natural}) = \mathrm{Tr}(\mathrm{Fr}_x, L_x) \quad \forall x \in |X-R|.$$

Let $d = d_x$. By Lemma A.10, L^{Pic}_d pulls back to L_d on $X^{\sqrt{R}}_d$; viewing x as a divisor of degree d on $X-R$ (and denoted $[x]$), it maps to $\mathcal{O}(x)^\natural$ via $\mathrm{AJ}^{\sqrt{R}}_d$, hence the left side above is equal to $\mathrm{Tr}(\mathrm{Fr}_k, L_d|_{[x]})$. Therefore, it suffices to show

$$\mathrm{Tr}(\mathrm{Fr}_k, L_d|_{[x]}) = \mathrm{Tr}(\mathrm{Fr}_x, L_x). \tag{A.12}$$

By the construction of L_d, there is an isomorphism $L_d|_{[x]} \cong L_x^{\otimes d}$ such that the Fr_k-action on $L_d|_{[x]}$ corresponds to the automorphism $\ell_1\otimes\ell_2\otimes\cdots\otimes\ell_d \mapsto$

$\ell_2 \otimes \cdots \otimes \ell_d \otimes \mathrm{Fr}_x(\ell_1)$ on $L_x^{\otimes d}$. This shows (A.12) and finishes the proof of the proposition. □

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Massachusetts Institute of Technology, Cambridge, MA
E-mail: zyun@mit.edu

Massachusetts Institute of Technology, Cambridge, MA
E-mail: weizhang@mit.edu