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Global existence of weak solutions for compressible Navier–Stokes equations: Thermodynamically unstable pressure and anisotropic viscous stress tensor

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Abstract

We prove global existence of appropriate weak solutions for the compressible Navier–Stokes equations for a more general stress tensor than those previously covered by P.-L.Lions and E. Feireisl's theory. More precisely we focus on more *general pressure laws* that are *not thermodynamically stable*; we are also able to handle some *anisotropy in the viscous stress tensor*. To give answers to these two longstanding problems, we revisit the classical compactness theory on the density by obtaining precise quantitative regularity estimates: This requires a more precise analysis of the structure of the equations combined to a novel approach to the compactness of the continuity equation. These two cases open the theory to important physical applications, for instance to describe solar events (virial pressure law), geophysical flows (eddy viscosity) or biological situations (anisotropy).

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1. **Introduction**

The question of global in time existence of solutions to fluid dynamics' models goes back to the pioneering work by J. Leray [45] (1934), where he introduced the concept of weak (turbulent) solutions to the Navier–Stokes systems describing the motion of an incompressible fluid; this work has become the basis of the underlying mathematical theory up to the present day. The theory for viscous compressible fluids in a barotropic regime has, in comparison, been developed more recently in the monograph by P.-L. Lions [48] (1993-1998), later extended by E. Feireisl and collaborators [34] (2001) and has been since then a very active field of study.

When changes in temperature are not taken into account, the barotropic Navier–Stokes system reads

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}\sigma = \rho f, \end{cases} \tag{1.1}$$

where ρ, u denote respectively the density and the velocity field and f is the given external force. The stress tensor σ of a general fluid obeys Stokes' law $\sigma = \mathcal{S} - P\,\mathrm{Id}$, where P is a scalar function termed pressure (depending on the density in the compressible barotropic setting or being an unknown in the incompressible setting) and $\mathcal{S}$ denotes the viscous stress tensor that characterizes the measure of resistance of the fluid to flow.

Our approach should also apply to the Navier–Stokes–Fourier system, as we will explain in a future work; this system is considered more physically relevant. But our main purpose here is to explain how the new regularity method that we introduce can be applied to a wide range of Navier–Stokes-like models and not to focus on a particular system. For this reason, we discuss the main features of our new theory on the simpler (1.1).

In comparison with Leray's work on incompressible flows, which is nowadays relatively "simple" at least from the point of view of modern functional analysis (and in the linear viscous stress tensor case), the mathematical theory of weak solutions to compressible fluids is quite involved, bearing many common aspects with the theory of non-linear conservation laws.

Our focus is on the global existence of weak solutions. For this reason we will not refer to the important question of existence of strong solutions or the corresponding uniqueness issues.

Several important problems about global existence of weak solutions for compressible flows remain open. In this article we consider the following questions:

- general pressure laws, in particular without any monotonicity assumption;
- anisotropy in the viscous stress tensor, which is especially important in geophysics.

In the current Lions–Feireisl theory, the pressure law P is often assumed to be of the form $P(\rho) = a\rho^\gamma$ but this can be generalized, a typical example being

$$\begin{gathered} P \in \mathcal{C}^1([0,+\infty)), \quad P(0) = 0 \text{ with} \\ a\rho^{\gamma-1} - b \le P'(\rho) \le \frac{1}{a}\rho^{\gamma-1} + b \text{ with } \gamma > d/2 \end{gathered} \tag{1.2}$$

for some constants $a > 0, b \geq 0$; see B. Ducomet, E. Feireisl, H. Petzeltova, I. Straskraba [27] or E. Feireisl [31] for slightly more general assumptions. However it is always required that $P(\rho)$ *be increasing* after a certain fixed critical value of ρ.

This monotonicity of P is connected to several well-known difficulties:

- The monotonicity of the pressure law is required for the stability of the thermodynamical equilibrium. Changes in monotonicity in the pressure are typically connected to intricate phase transition problems.
- At the level of compressible Euler, i.e., when $\mathcal{S} = 0$, non-monotone pressure laws may lead to a loss of hyperbolicity in the system, possibly leading to corrected systems (in particular, as by Korteweg).

In spite of these issues, we are able to show that compressible Navier–Stokes systems like (1.1) are *globally well posed without monotonicity assumptions on the pressure law*; instead, only rough growth estimates are required. This allows us to *consider for the first time several famous physical laws such as modified virial expansions.*

As for the pressure law, the theory initiated in the multi-dimensional setting by P.-L. Lions and E. Feireisl requires that the stress tensor has the very specific form

$$\sigma = 2\mu D(u) + \lambda \mathrm{div} u \, \mathrm{Id} - P(\rho) \, \mathrm{Id}$$

with $D(u) = (\nabla u + \nabla u^T)/2$, μ and λ such that $\lambda + 2\mu/d \geq 0$. The coefficients λ and μ do not need to be constant but require some explicit regularity; see, for instance, [33] for temperature dependent coefficients.

Unfortunately several physical situations involve some anisotropy in the stress tensor; geophysical flows, for instance, use and require such constitutive laws; see, for instance, [56] and [14] with eddy viscosity in turbulent flows.

In this article we present the first results able to handle *more general viscous stress tensor* of the form

$$\sigma = A(t)\nabla u + \lambda \mathrm{div} u \, \mathrm{Id} - P(\rho) \, \mathrm{Id}$$

with a $d \times d$ symmetric matrix A with regular enough coefficients. The matrix A can incorporate anisotropic phenomena in the fluid. Note that our result also applies to the case

$$\sigma = A(t)D(u) + \lambda \mathrm{div} u \, \mathrm{Id} - P(\rho) \, \mathrm{Id},$$

where $D(u) = (\nabla u + \nabla u^T)/2$ still.

Our new results therefore significantly expand the reach of the current theory for compressible Navier–Stokes and make it more robust with respect to the large variety of laws of state and stress tensors that are used. This is achieved through a complete revisiting of the classical compactness theory by obtaining *quantitative regularity estimates.* The idea is inspired by estimates obtained for non-linear continuity equations in [6], though with a different method than the one introduced here. Those estimates correspond to critical spaces, also developed and used, for instance, in works by J. Bourgain, H. Brézis and P. Mironescu and by A.C. Ponce; see [11] and [53].

Because of the weak regularity of the velocity field, the corresponding norm of the critical space cannot be propagated. Instead the norm has to be modified by weights based on an auxiliary function that solves a kind of dual equation adapted to the compressible Navier–Stokes system under consideration. After proving appropriate properties of the weights, we can prove compactness on the density.

The article is organized as follows:

- Section 2 presents the classical theory by P.-L. Lions and E. Feireisl, with the basic energy estimates. It explains why the classical proof of compactness does not seem able to handle the more general equations of state that concern us here. We also summarize the basic physical discussions on pressure laws and stress tensors choices that motivate our study. This section can be skipped by readers who are already familiar with the state of the art.
- In Section 3, we present the equations and the corresponding main results concerning global existence of weak solutions for non-monotone pressure law and then for anisotropic viscous stress tensor. Those are given in the barotropic setting.
- Section 4 is devoted to an introduction to our new method. We give our quantitative compactness criterion, and we show the basic ideas in the simple context of linear uncoupled transport equations and a very rough sketch of proof in the compressible Navier–Stokes setting.
- Section 5 states the stability results that constitute the main contribution of the paper.
- Section 6 states technical lemmas that are needed in the main proof and are based on classical harmonic analysis tools: maximal and square functions properties, and translation of operators. It also includes a basic presentation of the theory of renormalized solutions that we rely on in our calculations.
- Sections 7 and 8 constitute the heart of the proof. Section 7 is devoted to the renormalized equation with definitions and properties of the weights. Section 8 is devoted to the proof of the stability results of Section 5, both concerning more general pressure laws and concerning the anisotropic stress tensor.
- Section 9 concerns the construction of the approximate solutions. It uses the stability results of Section 5 to conclude the proof of the existence theorems of Section 3.
- Section 10 is a list of some of the notations that we use.
- Section 11 is an appendix recalling basic facts on Besov spaces that are used in the article.

2. **Classical theory by E. Feireisl and P.-L. Lions, open problems and physical considerations**

For the moment we consider compressible fluid dynamics in a general domain Ω that can be the whole space $\mathbb{R}^d$, a periodic box $\mathbb{T}^d$ or a bounded smooth domain with adequate boundary conditions. We do not specify the boundary conditions and instead leave those various choices open as they may depend on the problem, and we want to insist in this section on the common difficulties and approaches. We will later present our precise estimates in the periodic setting for simplicity.

2.1. *A priori estimates.* We collect the main physical a priori estimates for very general barotropic systems on $\mathbb{R}_+ \times \Omega$,

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \mathcal{D}\, u + \nabla P(\rho) = \rho f, \end{cases} \tag{2.1}$$

where $\mathcal{D}$ is only assumed to be a negative differential operator in divergence form on u such that

$$\int_\Omega u \cdot \mathcal{D}\, u \, dx \sim -\int_\Omega |\nabla u|^2 \, dx, \tag{2.2}$$

and for any ϕ and u,

$$\int_\Omega \phi \cdot \mathcal{D}\, u \, dx \leq C \, \|\nabla \phi\|_{L^2} \, \|\nabla u\|_{L^2}. \tag{2.3}$$

The following estimates form the basis of the classical theory of existence of weak solutions, and we will use them in our own proof. We only give the formal derivation of the estimates at the time being.

First of all, the total energy of the fluid is dissipated. This energy is the sum of the kinetic energy and the potential energy (due to the compressibility), namely,

$$E(\rho, u) = \int_\Omega \left(\rho \frac{|u|^2}{2} + \rho e(\rho) \right) dx,$$

where

$$e(\rho) = \int_{\rho_{\rm ref}}^{\rho} P(s)/s^2 ds$$

with $\rho_{\rm ref}$ a constant reference density. Observe that formally from (2.1),

$$\partial_t \big(\rho \frac{|u|^2}{2} \big) + \operatorname{div} \left(\rho\, u \frac{|u|^2}{2} \right) - u \cdot \mathcal{D} u + u \cdot \nabla P(\rho) = \rho\, f \cdot u,$$

and thus

$$\frac{d}{dt} \int_\Omega \rho \frac{|u|^2}{2} - \int_\Omega u \cdot \mathcal{D}\, u - \int_\Omega P(\rho) \operatorname{div}\, u = \int_\Omega \rho f \cdot u.$$

On the other hand, by the definition of e, the continuity equation on ρ implies

$$\partial_t(\rho e(\rho)) + \operatorname{div}(\rho e(\rho)\, u) + P(\rho) \operatorname{div} u = 0.$$

Integrating and combining with the previous equality leads to the energy equality

$$\frac{d}{dt} E(\rho, u) - \int_\Omega u \cdot \mathcal{D}\, u = \int_\Omega \rho f \cdot u. \tag{2.4}$$

Let us quantify further the estimates that follow from (2.4). Assume that $P(\rho)$ behaves roughly like ρ^γ in the following weak sense:

$$C^{-1}\,\rho^\gamma - C \le P(\rho) \le C\,\rho^\gamma + C; \tag{2.5}$$

then $\rho e(\rho)$ also behaves like ρ^γ. Note that (2.5) does not imply any monotonicity on P that could keep oscillating. One could also work with an even more general assumption than (2.5): Different exponents γ on the left-hand side and the right-hand side, for instance. But for simplicity, we use (2.5).

Assuming that f is bounded (or with enough integrability), one now deduces from (2.4) the following uniform bounds:

$$\begin{gathered} \sup_t \int_\Omega \rho\, |u|^2\, dx \le C + E(\rho^0, u^0), \\ \sup_t \int_\Omega \rho^\gamma\, dx \le C, \\ \int_0^T \int_\Omega |\nabla u|^2\, dx \le C. \end{gathered} \tag{2.6}$$

We can now improve on the integrability of ρ, as it was first observed by P.-L. Lions. Choose any smooth, positive $\chi(t)$ with compact support, and test the momentum equation by $\chi\, g = \chi\, \mathcal{B}\, \rho^a$, where $\mathcal{B}$ is a linear operator (in x) such that

$$\begin{gathered} \operatorname{div}\, g = (\rho^a - \overline{\rho^a}), \quad \|\nabla g\|_{L^p} \le C_p\, \|\rho^a - \overline{\rho^a}\|_{L^p}, \\ \|\mathcal{B}\, \phi\|_{L^p} \le C_p\, \|\phi\|_{L^p}, \ \forall\ 1 < p < \infty, \end{gathered}$$

where we denote by $\overline{\rho^a}$ the average of ρ^a over Ω. Finding g is straightforward in the whole space but more delicate in bounded domain as the right boundary conditions must also be imposed. This is where E. Feireisl et al. introduce the BOGOVSKI operator. We obtain that

$$\begin{aligned} \int \chi(t) \int_\Omega \rho^a\, P(\rho)\, dx\, dt \le \int \chi(t) \int_\Omega g\, (\partial_t(\rho\, u) + \operatorname{div}(\rho\, u \otimes u) - \mathcal{D}\, u - \rho\, f)\, dx\, dt \\ + \int \chi(t) \int_\Omega \overline{\rho^a} P(\rho). \end{aligned}$$

By (2.5), the left-hand side dominates

$$\int \chi(t) \int_\Omega \rho^{a+\gamma}\, dx\, dt.$$

It is possible to bound the terms in the right-hand side. For instance, by (2.3)

$$-\int \chi(t)\, g\, \mathcal{D}\, u\, dx\, dt \le C\, \|\nabla u\|_{L^2([0,\, T],\, L^2(\Omega))}\, \|\chi\, \nabla g\|_{L^2([0,\, T],\, L^2(\Omega))}$$
$$\le C\, \|\nabla u\|_{L^2([0,\, T],\, L^2(\Omega))}\, \|\chi\, (\rho^a - \overline{\rho^a})\|_{L^2([0,\, T],\, L^2(\Omega))},$$

by the choice of g. Given the bound (2.6) on ∇u, this term does not pose any problem if $2\,a < a + \gamma$. Next

$$\int \chi\, g\, \partial_t(\rho\, u)\, dx\, dt = -\int (g\, \chi'(t) + \chi(t)\, \mathcal{B}\, (\partial_t(\rho^a - \overline{\rho^a})))\, \rho\, u\, dx\, dt. \tag{2.7}$$

The first term in the right-hand side is easy to bound; as for the second one, the continuity equation implies

$$\begin{aligned} \int \chi(t)\, \mathcal{B}(\partial_t(\rho^a - \overline{\rho^a}))\, \rho\, u\, dx\, dt = &-\int \chi\, \left[\mathcal{B}\, (\mathrm{div}\, (u\, \rho^a))\right]\, \rho\, u \\ &- \int \chi\, \Big[(a-1)\, \mathcal{B}\, \Big(\rho^a\, \mathrm{div}\ u - \overline{\mathrm{div}(u\rho^a) + (a-1)\rho^a \mathrm{div}\, u)}\Big)\Big]\, \rho\, u. \end{aligned} \tag{2.8}$$

Using the properties of $\mathcal{B}$ and the energy estimates (2.6), it is possible to control those terms as well as the last one in (2.7), provided $a \le 2\gamma/d - 1$ and $\gamma > d/2$, which leads to

$$\int_0^T \int_\Omega \rho^{\gamma+a}\, dx\, dt \le C(T, E(\rho^0, u^0)). \tag{2.9}$$

2.2. *Heuristic presentation of the method by E. Feireisl and P.-L. Lions.* Let us explain, briefly and only heuristically, the main steps to prove global existence of weak solutions in the barotropic case with constant viscosities and power γ pressure law. Our purpose is to highlight why a specific form of the pressure or of the stress tensor is needed in the classical approaches. For such a general presentation of the theory, we also refer to the book by A. Novotny and I. Straskraba [50], the monograph Etats de la Recherche edited by D. Bresch [49], or the book by P. Plotnikov and J. Sokolowski [52].

Let us first consider the simplest model with constant viscosity coefficients μ and λ, before discussing the limitations of the classical approach to other settings. In that case, the compressible Navier–Stokes equation reads, on $\mathbb{R}_+ \times \Omega$,

$$\begin{cases} \partial_t\rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) - \mu\Delta u - (\lambda+\mu)\nabla \mathrm{div} u + \nabla P(\rho) = \rho f, \end{cases} \tag{2.10}$$

with $P(\rho) = a\rho^\gamma$. For simplicity, we work in a smooth, Lipschitz regular, bounded domain Ω with homogeneous Dirichlet boundary conditions on the velocity

$$u|_{\partial\Omega} = 0. \tag{2.11}$$

A key concept for the existence of weak solutions is the notion of renormalized solution to the continuity equation, as per the theory for linear transport equations by R. J. DiPerna and P.-L. Lions, which we briefly recall in Section 6. Assuming ρ and u are smooth and satisfy the continuity equation, for all $b \in \mathcal{C}([0,+\infty))$, one may multiply the equation by $b'(\rho)$ to find that (ρ, u) also solve

$$\partial_t b(\rho) + \mathrm{div}(b(\rho)u) + (b'(\rho)\rho - b(\rho))\mathrm{div} u = 0. \tag{2.12}$$

This leads to the following definition:

Definition 2.1. For any $T \in (0,+\infty)$, f, ρ_0, m_0 satisfying some technical assumptions (defined later on in theorems), we say that a couple (ρ, u) is a weak renormalized solution with bounded energy if it has the following properties:

- $\rho \in L^\infty(0,T; L^\gamma(\Omega)) \cap \mathcal{C}^0([0,T], L^\gamma_{\text{weak}}(\Omega))$, $\rho \geq 0$ a.e. in $(0,T)\times\Omega$, $\rho|_{t=0} = \rho_0$ a.e. in Ω;
- $u \in L^2(0,T; H^1_0(\Omega))$, $\rho|u|^2 \in L^\infty(0,T; L^1(\Omega))$, ρu is continuous in time with value in the weak topology of $L^{2\gamma/(\gamma+1)}_{\text{weak}}(\Omega)$, $(\rho u)|_{t=0} = m_0$ a.e. Ω;
- $(\rho,\, u)$ extended by zero out of Ω solves $(2.10)_1$ in $\mathcal{D}'((0,T)\times\mathbb{R}^d)$;
- (ρ, u) solves the momentum equation $(2.10)_2$ in $\mathcal{D}'((0,T)\times\Omega)$;
- for any smooth b with appropriate monotonicity properties, $b(\rho)$ solves the renormalized equation (2.12);
- for almost all $\tau \in (0,T)$, (ρ, u) satisfies the energy inequality

$$E(\rho,u)(\tau) + \int_0^\tau \int_\Omega (\mu|\nabla u|^2 + (\lambda+\mu)\,|\mathrm{div} u|^2) \leq E_0 + \int_0^\tau \int_\Omega \rho f \cdot u.$$

In this inequality,

$$E(\rho,u)(\tau) = \int_\Omega (\rho|u|^2/2 + \rho e(\rho))(\tau),$$

with $e(\rho) = \int_{\rho_{\text{ref}}}^\rho P(s)/s^2 ds$ (ρ_{ref} being any constant reference density), denotes the total energy at time τ and $E_0 = \int_\Omega |m_0|^2/2\rho_0 + \rho_0 e(\rho_0)$ denotes the initial total energy.

Assuming $P(\rho) = a\rho^\gamma$ (in that case $e(\rho)$ may equal to $a\rho^{\gamma-1}/(\gamma-1)$), the theory developed by P.-L. Lions to prove the global existence of renormalized weak solution with bounded energy asks for some limitation on the adiabatic constant γ, namely, $\gamma > 3d/(d+2)$. E. Feireisl et al. have generalized this approach in order to cover the range $\gamma > 3/2$ in dimension 3 and more generally $\gamma > d/2$, where d is the space dimension.

We present the initial proof due to P.-L. Lions and indicate quickly at the end how it was improved by E. Feireisl et al. The method relies on the construction of a sequence of approximate solution, derivation of a priori estimates and passage to the limit, which requires delicate compactness estimates.

For the time being, we skip the construction of such an approximate sequence; see, for instance, the book by A. Novotny and I. Straskraba for details.

The approximate sequence, denoted by $(\rho_k,\ u_k)$, should satisfy the energy inequality leading to a first uniform *a priori bound*, using that $\mu > 0$ and $\lambda + 2\,\mu/d > 0$,

$$\sup_t \int_\Omega (\rho_k\,|u_k|^2/2 + a\rho_k^\gamma/(\gamma-1))\,dx + \mu\int_0^t\int_\Omega |\nabla u_k|^2\,dx\,dt \le C,$$

for some constant independent of n.

For $\gamma > d/2$, we also have the final a priori estimate (2.9) explained in the previous subsection, namely,

$$\int_0^\infty\int_\Omega \rho_k^{\gamma+a} \le C(R,T) \text{ for } a \le \frac{2}{d}\gamma - 1.$$

When needed for clarification, we denote by $\overline{U}$ the weak limit of a general sequence U_k (up to a subsequence). Using the energy estimate and the extra integrability property proved on the density, and by extracting subsequences, one obtains the following convergence:

$$\begin{aligned}
\rho_k &\rightharpoonup \rho &&\text{in } \mathcal{C}^0([0,T]; L^\gamma_{\text{weak}}(\Omega)),\\
\rho_k^\gamma &\rightharpoonup \overline{\rho^\gamma} &&\text{in } L^{(\gamma+a)/\gamma}((0,T)\times\Omega),\\
\rho_k u_k &\rightharpoonup \rho u &&\text{in } \mathcal{C}^0([0,T]; L^{2\gamma/(\gamma+1)}(\Omega)),\\
\rho_k u_k^i u_k^j &\rightharpoonup \rho u^i\,u^j &&\text{in } \mathcal{D}'((0,T)\times\Omega) \text{ for } i,j = 1,2,3.
\end{aligned}$$

The convergence of the non-linear terms $\rho_k\,u_k$ and $\rho_k\,u_k\otimes u_k$ uses the compactness in time of ρ_k deduced from the uniform estimate on $\partial_t\rho_k$ given by the continuity equation and the compactness in time of $\sqrt{\rho_k}u_k$ deduced from the uniform estimate on $\partial_t(\rho_k u_k)$ given by the momentum equation. This is combined with the L^2 estimate on ∇u_k. Consequently, $\rho, u, \overline{\rho^\gamma}$ solve the momentum equation

$$\partial_t(\rho u) + \text{div}(\rho u\otimes u) - \mu\Delta u - (\lambda+\mu)\nabla\text{div}u + a\nabla\overline{\rho^\gamma} = \rho f.$$

The extensions by zero to $(0,T)\times\mathbb{R}^d/\Omega$ of (ρ,u) (again denoted (ρ,u)) satisfy the mass equation in $\mathbb{R}_+\times\mathbb{R}^d$,

$$\partial_t\rho + \text{div}(\rho u) = 0. \tag{2.13}$$

The difficulty consists in proving that (ρ,u) is a renormalized weak solution with bounded energy and the main point is showing that $\overline{\rho^\gamma} = \rho^\gamma$ a.e. in $(0,T)\times\Omega$.

This requires compactness on the density sequence that cannot follow from the previous a priori estimates only. Instead P.-L. Lions uses a weak compactness of the sequence $\{F_k\}_{k\in\mathbb{N}^*} = \{a\rho_k^\gamma - (\lambda+2\mu)\text{div}u_k\}_{k\in\mathbb{N}^*}$, which is usually called the effective viscous flux. This property was previously identified in one space dimension by D. Hoff and D. Serre. More precisely, we have the following property for all functions $b \in \mathcal{C}^1([0,+\infty))$ satisfying some increasing properties at infinity:

$$\begin{aligned}\lim_{k\to+\infty} \int_0^T \int_\Omega (a\rho_k^\gamma - (2\mu+\lambda)\mathrm{div} u_k) b(\rho_k)\varphi dx dt \\ = \int_0^T \int_\Omega (a\overline{\rho^\gamma} - (2\mu+\lambda)\mathrm{div} u)\overline{b(\rho)}\varphi dx dt,\end{aligned} \tag{2.14}$$

where the over-line quantities design the weak limit of the corresponding quantities and $\varphi \in \mathcal{D}((0,T)\times\Omega)$. Note that such a property is reminiscent of compensated compactness as the weak limit of a product is shown to be the product of the weak limits. In particular, the previous property implies that

$$\overline{\rho \mathrm{div} u} - \rho \mathrm{div} u = \frac{\overline{P(\rho)\rho} - \overline{P(\rho)}\,\rho}{2\mu+\lambda}. \tag{2.15}$$

Note that taking the divergence of the momentum equation, we get the relation

$$\Delta[(2\mu+\lambda)\mathrm{div} u_k - P(\rho_k)] = \mathrm{div}[\partial_t(\rho_k u_k) + \mathrm{div}(\rho_k u_k\otimes u_k)] - \mathrm{div}(\rho_k f)$$

written as

$$(2\mu+\lambda)\mathrm{div} u_k - P(\rho_k) = F_k + R_k, \tag{2.16}$$

where

$$F_k = \Delta^{-1}\mathrm{div}[\partial_t(\rho_k u_k) + \mathrm{div}(\rho_k u_k\otimes u_k)], \qquad R_k = \Delta^{-1}\mathrm{div}(\rho_k f). \tag{2.17}$$

We call F_k the effective viscous flux that has the same compactness property in space as $(2\mu+\lambda)\,\mathrm{div} u_k - P(\rho_k)$. Note that here the form of the stress tensor (isotropy and linearity) has been strongly used to get this expression. From this identity, P.-L. Lions proves the property (2.14) based on harmonic analysis due to R. Coifman and Y. Meyer (regularity properties of commutators) and takes the observations by D. Serre made in the one-dimensional case into account. The proof by E. Feireisl is based on div-curl lemma introduced by F. Murat and L. Tartar.

To simplify the remaining calculations, we assume $\gamma \geq 3d/(d+2)$, and in that case, due to the extra integrability on the density, we get that $\rho_k \in L^2((0,T)\times\Omega)$. This lets us choose $b(s) = s\log s$ in the renormalized formulation for ρ_k and ρ and take the difference of the two equations. Then we pass to the limit $k\to+\infty$ and use the identity of weak compactness on the effective viscous flux to replace terms with divergence of velocity by terms with density using (2.15), leading to

$$\partial_t(\overline{\rho\log\rho} - \rho\log\rho) + \mathrm{div}((\overline{\rho\log\rho} - \rho\log\rho)u) = \frac{1}{2\mu+\lambda}(\overline{P(\rho)}\rho - \overline{P(\rho)\rho}).$$

Observe that the monotonicity of the pressure $P(\rho) = a\rho^\gamma$ implies that

$$\overline{P(\rho)}\rho - \overline{P(\rho)\rho} \leq 0.$$

This is the one point where the monotonicity assumption is used. It allows us to show that the defect measure for the sequence of density satisfies

$$\mathrm{dft}[\rho_k - \rho](t) = \int_\Omega \overline{\rho\log\rho}(t) - \rho\log\rho(t)\,dx \leq \mathrm{dft}[\rho_k - \rho](t=0).$$

On the other hand, the strict-convexity of the function $s \mapsto s\log s$, $s \geq 0$ implies that $\mathrm{dft}[\rho_k - \rho] \geq 0$. If initially this quantity vanishes, it then vanishes at every later time.

Finally the commutation of the weak convergence with a strictly convex function yields *the strong convergence* of the density ρ_k in L^1_{loc}. Combined with the uniform bound of ρ_k in $L^{\gamma+a}((0,T)\times\Omega)$, we get the strong convergence of the pressure term ρ_k^γ.

This concludes the proof in the case $\gamma \geq 3d/(d+2)$. The proof of E. Feireisl works even if the density is not a priori square integrable. For that, E. Feireisl observes that it is possible to control the amplitude of the possible oscillations on the density in a norm L^p with $p > 2$ allowing the use of an effective viscous flux property with some truncature. Namely, he introduced the following oscillation measure:

$$\mathrm{osc}_p[\rho_k - \rho] = \sup_{n\geq 1}\,[\limsup_{k\to+\infty}\|T_n(\rho_k) - T_n(\rho)\|_{L^p((0,T)\times\Omega)}],$$

where T_n are cut-off functions defined as

$$T_n(z) = nT\Big(\frac{z}{n}\Big), \qquad n \geq 1$$

with $T \in \mathcal{C}^2(R)$,

$$T(z) = z \text{ for } z \leq 1, \qquad T(z) = 2 \text{ for } z \geq 3, \qquad T \text{ concave on } R.$$

The existence result can then obtained up to $\gamma > d/2$; see again the review by A. Novotny and I. Straskraba [50].

To the author's knowledge there exist few extensions of the previous study to more general pressure laws or the more general stress tensor. Concerning a generalization of the pressure law, as explained in the introduction there exist the works by B. Ducomet, E. Feireisl, H. Petzeltova, I. Straskraba [27] and E. Feireisl [31] where the hypothesis imposed on the pressure P imply that

$$P(z) = r_3(z) - r_4(z),$$

where r_3 is non-decreasing in $[0,+\infty)$ with $r_4 \in \mathcal{C}^2([0,+\infty))$ satisfying $r_4 \geq 0$ and $r_4(z) \equiv 0$ when $z \geq Z$ for a certain $Z \geq 0$. The form is used to show that it is possible respectively to continue to control the amplitude of the oscillations $\mathrm{osc}_p[\rho_k - \rho]$ and then to show that the defect measure vanishes if initially it vanishes. The two papers [31] and [27] we refer to allow us to consider, for instance, two important cases: the Van der Waals equation of state and some cold nuclear equations of state with finite number of monomial (see the subsection on the physical discussion).

2.3. *The limitations of the Lions–Feireisl theory.* The previous heuristical part makes explicit the difficulty in extending the global existence result for the more general non-monotone pressure law or for the non-isotropic stress tensor. First of all the key point in the previous approach was

$$\overline{P(\rho)}\,\rho - \overline{P(\rho)\,\rho} \leq 0.$$

This property is intimately connected to the monotonicity of $P(\rho)$ or of $P(\rho)$ for $\rho \geq \rho_c$ with truncation operators as in [31] or [27]. Non-monotone pressure terms cannot satisfy such an inequality and are therefore completely outside the current theory.

The difficulty with an anisotropic stress tensor is that we are losing the other key relation in the previous proof, namely, (2.15). For a non-isotropic stress tensor with an additional vertical component and power pressure law, for instance, we get instead the following relation:

$$\overline{\rho \mathrm{div} u} - \rho \mathrm{div} u \leq a \frac{\overline{\rho}\,\overline{A_\mu \rho^\gamma} - \overline{\rho A_\mu \rho^\gamma}}{\mu_x + \lambda}$$

with some non-local anisotropic operator $A_\mu = (\Delta - (\mu_z - \mu_x)\partial_z^2)^{-1}\partial_z^2$ where Δ is the total Laplacian in terms of (x, z) with variables $x = (x_1, \cdots, x_{d-1})$, $z = x_d$.

Unfortunately, we are again losing the structure and, in particular, the sign of the right-hand side as observed, in particular, in [14]. Furthermore even small anisotropic perturbations of an isotropic stress tensor cannot be controlled in terms of the defect measure introduced by E. Feireisl and collaborators: Note the non-local behavior in the right-hand side due to the term A_μ. For this reason, the anisotropic case seems to fall completely out the theory developed by P.-L. Lions and E. Feireisl.

Those two open questions are the main objective of this monograph.

2.4. *Physical discussions on pressure laws and stress tensors .* The derivation of the compressible Navier–Stokes system from first principles is delicate and goes well beyond the scope of this manuscript. In several respects the system is only an approximation, and this should be kept in mind in any discussion of the precise form of the equations, which should allow for some uncertainty.

2.4.1. *Equations of state.* In general it is a non-straightforward question to decide what kind of pressure law should be used depending on the many possible applications: mixtures of fluids, solids, and even the interior of stars. Among possible equation of state, one can find several well-known laws such as Dalton's law of partial pressures (1801), ideal gas law (Clapeyron 1834), the

Van der Waals equation of state (Van De Waals 1873), the virial equation of state (H. Hamerlingh Onnes 1901).

In general the pressure law $P(\rho, \vartheta)$ can depend on both the density ρ and the temperature ϑ. While we focus on barotropic systems in this article, we include the temperature already in the present discussion to emphasize its relevance and importance.

Let us give some important examples of equations of state.

- State equations are barotropic if $P(\rho)$ depends only on the density. As explained in the book by E. Feireisl [32] (see pages 8–10 and 13–15), the simplest example of a barotropic flow is an isothermal flow where the temperature is assumed to be constant. If both conduction of heat and its generation by dissipation of mechanical energy can be neglected, then the temperature is uniquely determined as a function of the density (if initially the entropy is constant) yielding a barotropic state equation for the pressure $P(\rho) = a\rho^\gamma$ with $a > 0$ and $\gamma = (R + c_v)/c_v > 1$. Another barotropic flow was discussed in [27].
- The classical Van der Waals equation reads
$$(P + a\,\rho^2)\,(b - \rho) = c\,\rho\,\vartheta,$$
where a, b, c are constants. The pressure law is non-monotone if the temperature ϑ is below a critical value, $\vartheta < \vartheta_c$, but it satisfies (1.2). In compressible fluid dynamics, the Van der Waals equation of state is sometimes simplified by neglecting specific volume changes and becomes
$$(P + a)\,(b - \rho) = c\,\rho\,\vartheta,$$
with similar properties.
- Using finite-temperature Hartree-Fock theory, it is possible to obtain a temperature dependent equation of state of the following form:
$$P(\rho, \vartheta) = a_3(1 + \sigma)\rho^{2+\sigma} - a_0\rho^2 + k\vartheta \sum_{n\geq 1} B_n \rho^n, \tag{2.18}$$
where k is the Boltzmann's constant, and where the last expansion (a simplified virial series) converges rapidly because of the rapid decrease of the B_n.
- Equations of state can include other physical mechanism. A good example is found in the article [27], where radiation comes into play: a photon assembly is superimposed to the nuclear matter background. If this radiation is in quasi-local thermodynamical equilibrium with the (nuclear) fluid, the resulting mixture nucleons+photons can be described by a one-fluid heat-conducting Navier–Stokes system, provided one adds to the equation of state a Stefan–Boltzmann contribution of black-body type
$$P_R(\vartheta) = a\vartheta^4 \text{ with } a > 0,$$

and provided one adds a corresponding contribution to the energy equation. The corresponding models are more complex and do not satisfy (1.2) in general.

- In the context of the previous example, a further simplification can be introduced leading to the so-called Eddington's standard model. This approximation assumes that the ratio between the total pressure $P(\rho,\vartheta) = P_G(\rho,\vartheta) + P_R(\vartheta)$ and the radiative pressure $P_R(\vartheta)$ is a pure constant

$$\frac{P_R(\vartheta)}{P_G(\rho,\vartheta)+P_R(\vartheta)} = 1-\beta,$$

where $0<\beta<1$ and P_G is given, for instance, by (2.18). Although crude, this model is in good agreement with more sophisticated models — in particular, for the sun.

One case where this model leads to a pressure law satisfying (1.2) is when one keeps only the low order term into the virial expansion. Suppose that $\sigma = 1$, and let us plug the expression of the two pressure laws in this relation,

$$2a_3\rho^3 - a_0\rho^2 + kB_1\vartheta\rho = \frac{\beta}{1-\beta}\frac{a}{3}\vartheta^4.$$

By solving this algebraic equation to leading order (high temperature), one gets

$$\vartheta \approx \Big(\frac{6a_3(1-\beta)}{a\beta}\Big)\rho^{3/4},$$

leading to the pressure law

$$P(\rho,\vartheta) = \frac{2a_3}{\beta}\rho^3 - a_0\rho^2 + kB_1\Big(\frac{6a_3(1-\beta)}{a\beta}\Big)\rho^{7/4},$$

which satisfies (1.2) because of the constant coefficients.

However in this approximation, only the higher order terms were kept. Considering the non-constant coefficient or keeping the whole virial sum in the pressure law was out of the scope of [27] and leads to precisely the type of non-monotone pressure laws that we consider in the present work.

- The virial equation of state for heat conducting Navier–Stokes equations can be derived from statistical physics and reads

$$P(\rho,\vartheta) = \rho\,\vartheta\Big(\sum_{n\ge 0} B_n(\vartheta)\rho^n\Big)$$

with $B_0 = \text{cst}$, and the coefficients $B_n(\vartheta)$ have to be specified for $n \ge 1$.

We will the treat truncated virial with appropriate assumptions or pressure laws of the type $P(\rho,\vartheta) = P_e(\rho) + P_{th}(\rho,\vartheta)$ for the Navier–Stokes–Fourier system in a future work.

- Pressure laws can also incorporate many other type of phenomena. Compressible fluids may include or model biological agents that have their own type of interactions. In addition, as explained later, our techniques also apply to other types of "momentum" equations. The range of possible pressure laws is then even wider.

Based on these examples, the possibilities of pressure laws are many. Most are not monotone and several do not satisfy (1.2), proving the need for a theory able to handle all sort of behaviors.

2.4.2. *Stress tensors.* One finds a similar variety of stress tensors as for pressure laws. We recall that we denote $D(u) = (\nabla u + (\nabla u)^T)/2$.

- The isotropic stress tensor with constant coefficients reads as
$$\mathcal{D} = \mu\,\Delta u + (\lambda + \mu)\,\nabla\,\mathrm{div}\,u,$$
which is the classical example that can be handled by the Lions–Feireisl theory; see, for instance, [48], [50] and [52] with $\gamma > d/2$. See also the recent interesting work by P.I. Plotnikov and W. Weigant (see [51]) in the two-dimensional in space case with $\gamma = 1$.
- Isotropic stress tensors with non-constant coefficients better represent the physics of the fluid however. Those coefficients can be temperature ϑ dependent
$$\mathcal{D} = 2\,\mathrm{div}\,\left(\mu(\vartheta)\,D(u)\right) + \nabla\left(\lambda(\vartheta)\mathrm{div}\,u\right).$$
Provided adequate non-degeneracy conditions are made on μ and λ, this case can still be efficiently treated by the Lions–Feireisl theory under some assumptions on the pressure law; see, for instance, [32] or [33].
- The coefficients of the isotropic stress tensors may also depend on the density
$$\mathcal{D} = 2\,\mathrm{div}\,\left(\mu(\rho,\vartheta)\,D(u)\right) + \nabla\left(\lambda(\rho,\vartheta)\,\mathrm{div}\,u\right).$$
This is a very difficult problem in general. The almost only successful insight in this case can be found in [12], [14], [15], [58], [46] with no dependency with respect to the temperature; see the recent review paper [13]. Those articles require a very special form of $\mu(\rho)$ and $\lambda(\rho)$, and without such precise assumptions, almost nothing is known. Note also the very nice paper concerning global existence of strong solutions in two-dimension by A. Kazhikhov and V. A. Vaigant where μ is constant but $\lambda = \rho^\beta$ with $\beta \geq 3$; see [57].
- Geophysical flow cannot in general be assumed to be isotropic, but instead some directions have different behaviors; this can be due to gravity in large scale fluids for instance. A nice example is found in the handbook written by R. Temam and M. Ziane, where the eddy-viscous term $\mathcal{D}$ is given by
$$\mathcal{D} = \mu_h \Delta_x u + \mu_z \partial_z^2 u + (\lambda + \mu)\,\nabla\mathrm{div}\,u,$$
with $\mu_h \neq \mu_z$. While such an anisotropy only requires minor modifications for the incompressible Navier–Stokes system, it is not compatible with the Lions–Feireisl approach (see, for instance, [14]) and requires $\mu, \mu_h, \mu_z > 0$ and $\lambda + 2\min(\mu_h, \mu_z, \mu)/d > 0$.

3. New results for the compressible Navier-Stokes system

From now on we will work on the torus $\mathbb{T}^d$. This is only for simplicity in order to avoid discussing boundary conditions or the behavior at infinity. The proofs would easily extend to other cases as mentioned at the end of the paper.

3.1. *Statements of the results*: *Theorems* 3.1 *and* 3.2. In this section we present our main existence results. As usual for global existence of weak solutions to non-linear PDEs, one has to prove stability estimates for sequences of approximate solutions and construct such approximate sequences. The main contribution in this paper and the major part of the proofs concern the stability procedure and more precisely the compactness of the density.

(I) *Isotropic compressible Navier–Stokes equations with general pressure.* Let us consider the isotropic compressible Navier–Stokes equations in $(0,T) \times \mathbb{T}^d$:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \mu \Delta u - (\lambda + \mu)\nabla \operatorname{div} u + \nabla P(\rho) = \rho f, \end{cases} \tag{3.1}$$

with $2\,\mu/d + \lambda > 0$, a pressure law P that is continuous on $[0, +\infty)$, P locally Lipschitz on $(0, +\infty)$ with $P(0) = 0$ such that there exists $C > 0$ with

$$C^{-1}\rho^\gamma - C \le P(\rho) \le C\rho^\gamma + C \tag{3.2}$$

and for all $s \ge 0$,

$$|P'(s)| \le \bar{P} s^{\tilde{\gamma}-1} \tag{3.3}$$

with two constants $\gamma > d/2, \tilde{\gamma} > 1$. System (3.1) is complemented with the initial conditions

$$\rho|_{t=0} = \rho_0, \qquad (\rho u)|_{t=0} = \rho_0\, u_0. \tag{3.4}$$

One then has global existence.

THEOREM 3.1. *Assume that the initial data u_0 and $\rho_0 \ge 0$ with $\int_{\mathbb{T}^d} \rho_0 = M_0 > 0$ satisfies the bound*

$$E_0 = \int_{\mathbb{T}^d} \Big(\frac{|(\rho u)_0|^2}{2\rho_0} + \rho_0 e(\rho_0) \Big)\, dx < +\infty,$$

where $e(s) = \int_1^s p(\tau)/\tau^2 d\tau$. Let the pressure law P satisfy (3.2) *and* (3.3) *with*

$$\gamma > \Big(\max(2, \tilde{\gamma}) + 1 \Big) \frac{d}{d+2}, \tag{3.5}$$

and let f be bounded in $L^1(0,T; L^{2\gamma/(\gamma-1)}(\mathbb{T}^d))$. Then there exists a global weak solution of the compressible Navier–Stokes system (3.1) *with the initial condition* (3.4) *in the sense of Definition* 2.1.

Remark. Let us note that the solution satisfies the explicit regularity estimate

$$\sup_{t\in[0,T]} \int_{\mathbb{T}^{2d}} \mathbb{I}_{\rho_k(x,t)\geq\eta}\, \mathbb{I}_{\rho_k(y,t)\geq\eta}\, K_h(x-y)\, \chi(\delta\rho_k)(t) \leq \frac{C\, \|K_h\|_{L^1}}{\eta^{1/2}\, |\log h|^{\theta/2}}$$

for some $\theta > 0$, where K_h is defined as in Proposition 4.1 and $\delta\rho_k$ and χ are defined as in Section 8; see (8.1).

(II) *Non-isotropic compressible Navier–Stokes equations.* We consider an example of non-isotropic compressible Navier–Stokes equations in $(0,T)\times\mathbb{T}^d$:

(3.6)
$$\begin{cases} \partial_t\rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u\otimes u) - \mathrm{div}\,\left(A(t)\,\nabla u\right) - (\mu+\lambda)\nabla\mathrm{div} u + \nabla P(\rho) = \rho f, \end{cases}$$

with $A(t)$ a given smooth and symmetric matrix, satisfying

$$A(t) = \mu\,\mathrm{Id} + \delta A(t), \quad \mu > 0, \quad \frac{2}{d}\,\mu + \lambda - \|\delta A(t)\|_{L^\infty} > 0, \tag{3.7}$$

where δA will be a perturbation around $\mu\,\mathrm{Id}$. We again take P locally Lipschitz on $[0,+\infty)$ with $P(0)=0$ but require it to be monotone after a certain point

$$C^{-1}\,\rho^{\gamma-1} - C \leq P'(\rho) \leq C\,\rho^{\gamma-1} + C, \tag{3.8}$$

with $\gamma > d/2$. System (3.6) is supplemented with the initial conditions

$$\rho|_{t=0} = \rho_0, \qquad (\rho u)|_{t=0} = \rho_0\, u_0. \tag{3.9}$$

The second main result that we obtain is

THEOREM 3.2. *Assume that the initial data u_0 and $\rho_0 \geq 0$ with $\int_{\mathbb{T}^d}\rho_0 = M_0 > 0$ satisfies the bound*

$$E_0 = \int_{\mathbb{T}^d} \left(\frac{|(\rho u)_0|^2}{2\rho_0} + \rho_0 e(\rho_0) \right) dx < +\infty,$$

where $e(s) = \int_1^s p(\tau)/\tau^2\, d\tau$. Let the pressure P satisfy (3.8) *with*

$$\gamma > \frac{d}{2}\left[\left(1+\frac{1}{d}\right) + \sqrt{1+\frac{1}{d^2}} \right],$$

and let f be bounded in $L^1(0,T;L^{2\gamma/(\gamma-1)}(\mathbb{T}^d))$. There exists a universal constant $C_\star > 0$ such that if

$$\|\delta A\|_\infty \leq C_\star\,(2\mu+\lambda),$$

then there exists a global weak solution of the compressible Navier–Stokes equation (3.6) *with the boundary condition* (3.9) *in the sense of Definition* 2.1 *but replacing the isotropic energy inequality by the following anisotropic energy*:

$$E(\rho,u)(\tau) + \int_0^\tau \int_\Omega (\nabla u^T\, A(t)\,\nabla u + (\mu+\lambda)\,|\mathrm{div} u|^2) \leq E_0.$$

Remark. Let us note that the constraint on γ corresponds to the constraint on p: $p > \gamma + \gamma/(\gamma - 1)$, where p is the extra integrability property on ρ.

3.2. *Important comments/comparison with previous results.* The choice was made to focus on explaining the new method instead of trying to write results as general as possible but at the cost of further burdening the proofs. For this reason, Theorems 3.1 and 3.2 are only two examples of what can be done.

As we mentioned in the introduction, our new method should also apply to the Navier–Stokes–Fourier system (with an additional equation for temperature). The Navier–Stokes–Fourier system is physically more relevant than the barotropic case and moreover, as seen from the discussion in Section 2.4, it exhibits even more examples of non-monotone pressure laws.

(I) *Possible extensions.* Applications may be done to various other important models — in particular, in the Bio-Sciences where the range of possible pressure laws (or what plays their role such as chemical attraction/repulsion) is wide. But there are many other possible extensions; for instance, (3.2) could be replaced with a more general

$$C^{-1}\,\rho^{\gamma_1} - C \le P(\rho) \le C\,\rho^{\gamma_2} + C,$$

with different exponents $\gamma_1 \neq \gamma_2$. While the proofs would essentially remain the same, the assumption (3.5) would then have to be replaced and would involve γ_1 and γ_2. Similarly, it is possible to consider the spatially dependent stress tensor $A(t,x)$ in Theorem 3.2. This introduces additional terms in the proof, but those can easily be handled as long as A is smooth by classical methods for pseudo differential operators.

(II) *Comparison with previous results.*

(III-1) *Non-monotone pressure laws.* Theorem 3.1 is the first result to allow for completely non-monotone pressure laws. Among many important previous contributions, we refer to [27], [31], [17], [48] and [32], [33], [50], [18] for the Navier–Stokes–Fourier system. These references are our main point of comparison, and they all require $\partial_\rho P > 0$ after a certain point and, in fact, typically a condition like (3.8). The removal of the key assumption of monotonicity has important consequences:

- From the physical and modeling point of view, it opens the possibility of working with a wider range of equations of state as discussed in Section 2.4, and it makes the current theory on viscous, compressible fluids more robust to perturbation of the model.
- Changes of monotonicity in P can create and develop oscillations in the density ρ (because some “regions” of large density become locally attractive). It was a major question whether such oscillations remain under control at

least over bounded time intervals. This shows that the stability for bounded times is very different from uniform in time stability as $t \to +\infty$. Only the latter requires assumptions of a thermodynamical nature such as the monotonicity of P.

- Obviously well posedness for non-monotone P could not be obtained as is done here for the compressible Euler system. As can be seen from the proofs, the viscous stress tensor in the compressible Navier–Stokes system has precisely the critical scaling to control the oscillations created by the non-monotonicity. This implies, for instance, that in phase transition phenomena, the transition occurs smoothly precisely at the scale of the viscosity.
- Our results could have further consequences, for instance, to show convergence of numerical schemes (or for other approximate systems). Typical numerical schemes for compressible Navier–Stokes raise issues of oscillations in the density that are reminiscent of the ones faced in this article. The question of convergence of numerical schemes to compressible Navier–Stokes is an important and delicate subject in its own, going well beyond the scope of this short comment. We refer, for instance, to the works by R. Eymard, T. Gallouët, R. Herbin, J.-C. Latché and T. K. Karper; see, for instance, [36], [29] for the simpler Stokes case, [28], [35], [37] for Navier–Stokes, and more recently to the work [19], [42].

Concerning the requirement on the growth of the pressure at ∞, that is, on the coefficient γ in (3.5), we have the following remarks:

- In the typical case where $\tilde\gamma = \gamma$, (3.5) leads to the same constraint as in P.-L. Lions [48] for a similar reason: the need to have $\rho \in L^2$ to make sense of $\rho \operatorname{div} u$. It is worse than the $\gamma > d/2$ required, for instance, in [32]. In $3d$, we hence need $\gamma > 9/5$ versus only $\gamma > 3/2$ in [32].
- It may be possible to improve on (3.5) while still using the method introduced here but propagating compactness on appropriate truncation $T_K(\rho)$ of ρ; for instance, by writing an equivalent of Lemma 7.1 on $T_K(\rho(t,x)) - T_K(\rho(t,y))$ as in the multi-dimensional setting by E. Feireisl. This possibility was left to future works. Note that the requirement on $\gamma > d/2$ comes from the need to gain integrability as per (2.9) along the strategy presented in Section 2.1. Our new method still relies on this estimate and therefore has no hope, on its own, to improve on the condition $\gamma > d/2$.
- In the context of general pressure laws, and even more so for possible later applications to the Navier–Stokes–Fourier system, assumption (3.5) is not a strong limitation. Virial-type pressure laws, where $P(\rho)$ is a polynomial expansion, automatically satisfy it, for instance, as do many other examples discussed in Section 2.4.

(II-2) *Anisotropic stress tensor.* Theorem 3.2 is so far the only result of global existence of weak solutions that is able to handle anisotropy in the stress tensor. It applies, for instance, to the eddy-viscous tensor mentioned above for geophysical flows

$$\mathcal{D} = \mu_h \Delta_x u + \mu_z \partial_z^2 u + (\lambda + \mu)\, \nabla \mathrm{div}\, u,$$

where $\mu_h \neq \mu_z$ and corresponding to

$$A_{ij} = \mu_h\, \delta_{ij} \quad \text{for } i,\, j = 1,\, 2, \qquad A_{33} = \mu_z, \qquad A_{ij} = 0 \quad \text{otherwise.} \tag{3.10}$$

This satisfies the assumptions of Theorem 3.2 provided $|\mu_h - \mu_x|$ is not too large, which is usually the case in the context of geophysical flows.

We also wish to emphasize here that it is also possible to have a fully symmetric anisotropy, namely, $\mathrm{div}\,(A\, D\, u)$ with $D(u) = \nabla u + \nabla u^T$ in the momentum equation. This is the equivalent of the anisotropic case in linear elasticity, and it is also an important case for compressible fluids. Note that it leads to a different form of the stress tensor. With the above choice of A, equation (3.10), one would instead obtain

$$\mathrm{div}\,(A\, D\, u) = \mu_h \Delta_x u + \mu_z \partial_z^2 u + \mu_z\, \nabla \partial_z u_z + (\lambda + \mu)\, \nabla \mathrm{div}\, u.$$

Accordingly we choose to state Theorem 3.2 with the non-symmetric anisotropy $\mathrm{div}\,(A\, \nabla u)$ as it corresponds to the eddy-viscous term by R. Temam and M. Ziane mentioned above. But the extension to the symmetric anisotropy is possible although it introduces some minor complications. For instance, one cannot simply obtain $\mathrm{div}\, u$ by solving a scalar elliptic system, but one has to solve a vector valued one instead; we refer the interested readers to the remark just after (5.10) and at the end of the proof of Theorem 3.2 in Section 9.

Ideally one would like to obtain an equivalent of Theorem 3.2 assuming only uniform elliptic bounds on $A(t)$ and a much lower bound on γ. Theorem 3.2 is a first attempt in that direction, which can hopefully later be improved.

However the reach of Theorem 3.2 should not be minimized because non-isotropy in the stress tensor appears to be a level of difficulty above even non-monotone pressure laws. Losing the pointwise relation between $\mathrm{div}\, u$ and $P(\rho)$ is *a major hurdle*, as it can also be seen from the proofs later in the article. Instead one has to work with

$$\mathrm{div}\, u = P(\rho) + L\, P(\rho) + \text{ effective viscous flux},$$

with L a non-local operator of order 0. The difficulty is to appropriately control this non-local term so that its contribution can eventually be bounded by the dissipation due to the local pressure term.

Notation. For simplicity, in the rest of the article, C will denote a numerical constant whose value may change from line to line. It may depend on some uniform estimates on the sequences of functions considered (as per bounds (5.5) or (5.4) for instance) but it will never depend on the sequence under consideration (denoted with index k) or the scaling parameters h or h_0.

4. **Sketch of the new compactness method**

The standard compactness criteria used in the compressible Navier–Stokes framework is the Aubin–Lions–Simon lemma to get compactness on the terms ρu and $\rho u \otimes u$. A more complex trick is used to get the strong convergence of the density. More precisely it combines extra integrability estimates on the density and the effective viscous flux property (a kind of weak compactness) and then a convexity-monotonicity tool to conclude.

Here we present a tool that will be the cornerstone in our study to prove compactness on the density and that will be appropriate to cover the more general equation of the state or stress tensor form.

In order to give the main idea of the method, we present it first in this section for the well-known case of linear transport equations, i.e., assuming that u is given. We then give a rough sketch of the main ideas we will use in the rest of the article. This presents the steps we will follow for proofs in the more general setting.

4.1. *The compactness criterion.* We start with a well-known result providing compactness of a sequence:

PROPOSITION 4.1. *Let ρ_k be a sequence uniformly bounded in $L^p((0,T) \times \mathbb{T}^d)$ for some $1 \le p < \infty$. Assume that $\mathcal{K}_h$ is a sequence of positive, bounded functions such that*

(i) *for all $\eta > 0$, $\sup_h \int_{\mathbb{T}^d} \mathcal{K}_h(x)\, \mathbb{1}_{\{x\,:\,|x|\ge\eta\}} dx < \infty$;*
(ii) *$\|\mathcal{K}_h\|_{L^1(\mathbb{T}^d)} \longrightarrow +\infty$ as $h \to +\infty$.*

If $\partial_t \rho_k \in L^q([0,\ T] \times W^{-1,q}(\mathbb{T}^d))$ (with $q \ge 1$) uniformly in k and

$$\limsup_k \Big[\frac{1}{\|\mathcal{K}_h\|_{L^1}} \int_0^T \int_{\mathbb{T}^{2d}} \mathcal{K}_h(x-y)\, |\rho_k(t,x) - \rho_k(t,y)|^p\, dx\, dy\, dt\Big] \longrightarrow 0$$

as $h \to 0$, then ρ_k is compact in $L^p([0,\ T] \times \mathbb{T}^d)$. Conversely, if ρ_k is compact in $L^p([0,\ T] \times \mathbb{T}^d)$, then the above quantity converges to 0 with h.

For the reader's convenience, we just quickly recall that the compactness in space is connected to the classical approximation by convolution. Denote by $\bar{\mathcal{K}}_h$ the normalized kernel

$$\bar{\mathcal{K}}_h = \frac{\mathcal{K}_h}{\|\mathcal{K}_h\|_{L^1}}.$$

Write

$$\|\rho_k - \bar{\mathcal{K}}_h \star_x \rho_k\|_{L^p}^p \leq \frac{1}{\|\mathcal{K}_h\|_{L^1}^p} \int_{\mathbb{T}^d} \Big(\int_{\mathbb{T}^d} \mathcal{K}_h(x-y) |\rho_k(t,x) - \rho_k(t,y)| dx \Big)^p dy$$
$$\leq \frac{1}{\|\mathcal{K}_h\|_{L^1}} \int_{\mathbb{T}^{2d}} \mathcal{K}_h(x-y) |\rho_k(t,x) - \rho_k(t,y)|^p dx\, dy,$$

which converges to zero as $h \to 0$ uniformly in k by assumption. On the other hand, for a fixed h, the sequence $\overline{K}_h \star_x u_k$ in k is compact in x. This completes the compactness in space. Concerning the compactness in time, we just have to couple everything and use the uniform bound on $\partial_t \rho_k$ as per the usual Aubin-Lions-Simon lemma.

In all the paper the following important choice of Kernel K_h and its associated $\mathcal{K}_{h_0}$ functions are used.

Definition 4.2. We define the positive, bounded and symmetric function K_h such that

$$K_h(x) = \frac{1}{(h+|x|)^a} \quad \text{for } |x| \leq 1/2,$$

with some $a > d$ and K_h positive, independent of h for $|x| \geq 2/3$, K_h positive constant outside $B(0,3/4)$ and periodized so as to belong in $C^\infty(\mathbb{T}^d \backslash B(0,3/4))$.

For convenience, we denote

$$\overline{K}_h(x) = \frac{K_h(x)}{\|K_h\|_{L^1(\mathbb{T}^d)}}.$$

This kernel K_h is enough for linear transport equations to prove compactness. For compressible Navier–Stokes, for $0 < h_0 < 1$, the following important quantity will play a crucial role:

$$\mathcal{K}_{h_0}(x) = \int_{h_0}^1 \overline{K}_h(x) \, \frac{dh}{h}.$$

Important remarks. The weights defined in Definition 4.2 satisfy the properties

$$K_h(x) = K_h(-x), \qquad |x||\nabla K_h(x)| \leq c\, K_h(x)$$

for some constant $c > 0$ and

$$\|\mathcal{K}_{h_0}\|_{L^1(\mathbb{T}^d)} \sim |\log h_0|.$$

These properties will be strongly used throughout the paper.

4.2. *Compactness for linear transport equation.* Consider a sequence of solutions ρ_k, on the torus $\mathbb{T}^d$ (so as to avoid any discussion of boundary conditions or behavior at infinity) to

$$\partial_t \rho_k + \operatorname{div}(\rho_k\, u_k) = 0, \qquad \rho_k|_{t=0} = \rho_k^0, \tag{4.1}$$

where u_k (a given velocity field) is assumed to satisfy, for some $1 < p \leq \infty$,

$$\sup_k \|u_k\|_{L^p_t W^{1,p}_x} < \infty. \tag{4.2}$$

Defining

$$\varepsilon_k(h) = \frac{1}{\|K_h\|_{L^1}} \int_0^T \int_{\mathbb{T}^{2d}} K_h(x-y)\,|\mathrm{div}_x u_k(t,x) - \mathrm{div}_y u_k(t,y)|^p\,dx\,dy, \tag{4.3}$$

we assume $\mathrm{div}\, u_k$ compact in x, i.e.,

$$\limsup_k \varepsilon_k(h) \longrightarrow 0 \text{ as } h \to 0. \tag{4.4}$$

Moreover, defining

$$\tilde\varepsilon_k(h) = \frac{1}{\|K_h\|_{L^1}} \int_{\mathbb{T}^{2d}} K_h(x-y)\,|\rho_k^0(x) - \rho_k^0(y)|\,dx\,dy, \tag{4.5}$$

we assume compactness of the initial data, namely,

$$\limsup_k \tilde\varepsilon_k(h) \to 0 \text{ as } h \to 0. \tag{4.6}$$

The condition on the divergence is replaced by bounds on ρ_k,

$$0 < \frac{1}{C} \leq \inf_{\mathbb{T}^d} \rho_k \leq \sup_{\mathbb{T}^d} \rho_k \leq C < +\infty, \quad \forall\, t \in [0,\ T], \tag{4.7}$$

where C does not dependent on k. One then has the well-known

Proposition 4.3. *Assume ρ_k solves* (4.1) *with the bounds* (4.2), *and* (4.7). *Assume, moreover, that the initial data ρ_k^0 is compact — namely,* (4.6) *— and that the divergence of the velocity u_k is compact in space — namely,* (4.4). *Then ρ_k is compact and, more precisely,*

$$\sup_{t\in[0,T]} \int_{\mathbb{T}^{2d}} K_h(x-y)\,|\rho_k(t,x) - \rho_k(t,y)|\,dx\,dy \leq C\,\frac{\|K_h\|_{L^1}}{|\log(h + \varepsilon_k(h) + \tilde\varepsilon_k(h))|}, \tag{4.8}$$

where $\varepsilon_k(h)$, $\tilde\varepsilon_k(h)$ are given respectively by (4.3) *and* (4.5).

This type of results for *non-Lipschitz* velocity fields u_k was first obtained by R. J. Di Perna and P.-L. Lions in [26] with the introduction of renormalized solutions for $u_k \in W^{1,1}$ and appropriate bounds on div u_k. This was extended to $u_k \in BV$, first by F. Bouchut in [8] in the kinetic context (see also M. Hauray in [39]) and then by L. Ambrosio in [3] in the most general case. We also refer to C. Le Bris and P.-L. Lions in [44], [48] and to the nice lecture notes written by C. De Lellis in [23]. In general, $u_k \in BV$ is the optimal regularity as shown by N. Depauw in [25]. This can only be improved with specific additional structure, such as provided by low dimension; see [2], [10], [20], [21], [38], Hamiltonian properties [16], [40], or as a singular integral [9].

Of more specific interest for us are the results that do not require bounds on $\operatorname{div} u_k$ (which are not available for compressible Navier–Stokes) but replace them by bounds on ρ_k, such as (4.7). The compactness in Proposition 4.3 was first obtained in [4].

Explicit regularity estimates of ρ_k were first derived by G. Crippa and C. De Lellis in [22]. (See also [41] for the $W^{1,1}$ case.) These are based on explicit control on the characteristics. While it is quite convenient to work on the characteristics in many settings, this is not the case here — in particular, due to the coupling between div u_k and $p(\rho_k)$.

In many respects the proof of Proposition 4.3 is an equivalent approach to the method of G. Crippa and C. De Lellis in [22] at the PDE level, instead of the ODE level. Its interest will be manifest later in the article when dealing with the full Navier–Stokes system. The idea of controlling the compactness of solutions to transport equations through estimates such as provided by Proposition 4.1 was first introduced in [6] but relied on a very different method. Note that in the linear case with a given vector field u_k sequence, the compactness of the $\operatorname{div} u_k$ is strictly required to obtain the compactness of ρ_k; see, for instance, [24].

Proof. One does not try to directly propagate

$$\int_{\mathbb{T}^{2d}} K_h(x-y)\,|\rho_k(t,x)-\rho_k(t,y)|\,dx\,dy.$$

Instead one introduces the weight w_k solution to the auxiliary equations

$$\partial_t w_k + u_k\cdot\nabla w_k = -\lambda\, M\,|\nabla u_k|\,w_k, \qquad w_k|_{t=0}=1, \tag{4.9}$$

where $M\,f$ denotes the maximal function of f (recalled in Section 6) and λ is a constant to be chosen large enough.

First step: *Propagation of a weighted regularity.* Here and in the following, we use the convenient notation $(\rho_k^x,u_k^x)=(\rho_k(t,x),u_k(t,x))$, $(\rho_k^y,u_k^y)=(\rho_k(t,y),u_k(t,y))$ and $(\rho_k^x)_0=\rho_k^x|_{t=0}$, $(\rho_k^y)_0=\rho_k^y|_{t=0}$. We prove that we propagate in time the following quantity:

$$R_k(t)=\int_{\mathbb{T}^{2d}} K_h(x-y)\,|\rho_k^x-\rho_k^y|\,w_k^x\,w_k^y\,dx\,dy,$$

where as before we denote $w_k^x=w_k(t,x)$ and $w_k^y=w_k(t,y)$.

The starting point is essentially a doubling of variables argument, going to Kruzkov's seminal work [43]. Using (4.1), we note that densities ρ_k^x and ρ_k^y respectively satisfy

$$\begin{aligned}&\partial_t\rho_k^x+\operatorname{div}_x(\rho_k^x u_k^x)=0, \qquad \rho_k^x|_{t=0}=(\rho_k^x)_0,\\ &\partial_t\rho_k^y+\operatorname{div}_y(\rho_k^y u_k^y)=0, \qquad \rho_k^y|_{t=0}=(\rho_0^y)_0,\end{aligned} \tag{4.10}$$

and from (4.9) weights w_k^x and w_k^y respectively satisfy

$$\partial_t w_k^x + u_k^x \cdot \nabla_x w_k^x = -\lambda\, M\, |\nabla u_k|^x\, w_k^x, \qquad w_k^x|_{t=0} = 1 \tag{4.11}$$

and

$$\partial_t w_k^y + u_k^y \cdot \nabla_y w_k^y = -\lambda\, M\, |\nabla u_k|^y\, w_k^y, \qquad w_k^y|_{t=0} = 1. \tag{4.12}$$

Using (4.10), we obtain that

$$\begin{aligned}\partial_t|\rho_k^x - \rho_k^y| &+ \operatorname{div}_x (u_k^x\, |\rho_k^x - \rho_k^y|) + \operatorname{div}_y (u_k^y\, |\rho_k^x - \rho_k^y|)\\ &= \frac{1}{2}(\operatorname{div}_x u_k^x + \operatorname{div}_y u_k^y)\, |\rho_k^x - \rho_k^y|\\ &\quad - \frac{1}{2}(\operatorname{div}_x u_k^x - \operatorname{div}_y u_k^y)\, (\rho_k^x + \rho_k^y)\, s_k,\end{aligned}$$

where $s_k = \operatorname{sign}(\rho_k^x - \rho_k^y)$. We refer to Section 7.1 for the details of this calculation, which is rigorously justified for a fixed k through the theory of renormalized solutions in [26] as recalled in Section 6. From this equation on $|\rho_k^x - \rho_k^y|$, we deduce

$$\begin{aligned}\frac{d}{dt}R_k(t) &= \int_{\mathbb{T}^{2d}} \nabla K_h(x-y) \cdot (u_k^x - u_k^y)\, |\rho_k^x - \rho_k^y|\, w_k^x\, w_k^y\, dx\, dy\\ &- \frac{1}{2}\int_{\mathbb{T}^{2d}} K_h(x-y)\, (\operatorname{div}_x u_k^x - \operatorname{div}_y u_k^y)\, (\rho_k^x + \rho_k^y)\, s_k\, w_k^x\, w_k^y\, dx\, dy\\ &+ \int_{\mathbb{T}^{2d}} K_h(x-y)\, |\rho_k^x - \rho_k^y| \left(\partial_t w_k^x + u_k^x \cdot \nabla_x w_k^x + \frac{1}{2}\operatorname{div}_x u_k^x\, w_k^x\right) w_k^y\, dx\, dy\\ &+ \text{ symmetric of the last term.}\end{aligned}$$

Note that the weights w_k^x and w_k^y satisfy (4.11) and (4.12) and are uniformly bounded; thus $\partial_t w_k^x + u_k^x \cdot \nabla_x w_k^x + \operatorname{div}_x u_k^x\, w_k^x/2$ belongs to L^p uniformly in k for some $1 < p < +\infty$, even though we may not be able to make sense of each term individually. Observe that by (4.9), $w_k \leq 1$ and therefore by (4.7) and the definition of ε_k in (4.4), the second term in the right-hand side is easily bounded:

$$\begin{aligned}\int_0^s \int_{\mathbb{T}^{2d}} K_h(x-y)\, (\operatorname{div} u_k^x - \operatorname{div} u_k^y)\, (\rho_k^x + \rho_k^y)\, s_k\, w_k^x\, w_k^y\, dx\, dy&\\ \leq C\, \|K_h\|_{L^1(\mathbb{T}^d)}\, \varepsilon_k(h).&\end{aligned}$$

For the first term, one uses the well-known inequality (see [54], [55] or Section 6)

$$|u_k^x - u_k^y| \leq C_d\, |x-y|\, (M\, |\nabla u_k|^x + M\, |\nabla u_k|^y),$$

combined with the remark that from the choice of K_h,

$$|\nabla K_h(x-y)|\, |x-y| \leq C\, K_h(x-y).$$

Therefore,

$$\int_0^s \int_{\mathbb{T}^{2d}} \nabla K_h(x-y)\cdot(u_k^x-u_k^y)\,|\rho_k^x-\rho_k^y|\,w_k^x\,w_k^y\,dx\,dy\,dt$$
$$\leq C\int_0^s \int_{\mathbb{T}^{2d}} K_h(x-y)\,(M\,|\nabla u_k|^x+M\,|\nabla u_k|^y)\,|\rho_k^x-\rho_k^y|\,w_k^x\,w_k^y\,dx\,dy\,dt,$$

combining everything and integrating the equation on $R(t)$ from 0 to s:

$$R_k|_{t=s}-R_k|_{t=0}\leq C\,\|K_h\|_{L^1}\,\varepsilon_k(h)$$
$$+\int_0^s\int_{\mathbb{T}^{2d}} K_h(x-y)\left(\partial_t w_k^x+u_k^x\cdot\nabla_x w_k^x+C\,(\operatorname{div}_x u_k^x+M\,|\nabla u_k|^x)\,w_k^x\right)$$
$$|\rho_k^x-\rho_k^y|\,w_k^y\,dx\,dy\,dt+\text{ symmetric of the last term.}$$

Since $\operatorname{div} u_k\leq d, |\nabla u_k|\leq d\,M|\nabla u_k|$, by taking the constant λ large enough in (4.9),

$$\partial_t w_k^x+u_k^x\cdot\nabla_x w_k^x+C\,(\operatorname{div}_x u_k^x+M\,|\nabla u_k|^x)\,w_k^x\leq 0,$$

and hence

$$\tag{4.13} \begin{aligned} R_k|_{t=s}&\leq\int_{\mathbb{T}^{2d}} K_h(x-y)\,|\rho_k^0(x)-\rho_k^0(y)|\,dx\,dy+C\|K_h\|_{L^1(\mathbb{T}^d)}\,\varepsilon_k(h)\\ &\leq C\,\|K_h\|_{L^1(\mathbb{T}^d)}\left(\varepsilon_k(h)+\tilde\varepsilon_k(h)\right). \end{aligned}$$

Second step: *property of the weight.* We need to control the measure of the set where the weight w is small. Obviously if w were to vanish everywhere, then the control of $R(t)$ would be trivial but of very little interest. From equation (4.9) one formally obtains

$$\partial_t(\rho_k^x\,|\log w_k^x|)+\operatorname{div}_x(\rho_k^x\,u_k|\log w_k^x|)=\lambda\,\rho_k^x\,M\,|\nabla u_k|^x.$$

And thus, integrating in space on $\mathbb{T}^d$,

$$\tag{4.14} \begin{aligned} \frac{d}{dt}\int_{\mathbb{T}^d}|\log w_k^x|\,\rho_k^x\,dx&=\lambda\int_{\mathbb{T}^d}\rho_k^x\,M\,|\nabla u_k|^x\,dx\\ &\leq\lambda\,\|\rho_k^x\|_{L^{p^*}(\mathbb{T}^d)}\,\|M\,|\nabla u_k|^x\|_{L^p(\mathbb{T}^d)}. \end{aligned}$$

Of course at this point, the calculations are only formal. In particular, since w_k may vanish, one has to be especially careful when trying to work with $\log w_k$. Instead one may rigorously derive (4.14) directly from the theory of renormalized solutions; we refer more precisely to Lemma 6.11 later in the article.

Equation (4.14) gives

$$\sup_{t\in[0,T]}\int_{\mathbb{T}^d}|\log w_k^x|\,\rho_k^x\,dx\leq C$$

uniformly with respect to k by (4.2) (which implies that the maximal function is bounded on $L^p((0,T)\times\mathbb{T}^d)$ for $p>1$) and from (4.7). This estimate may be written in the (t,y) variable, namely,

$$\sup_{t\in[0,T]}\int_{\mathbb{T}^d} |\log w_k^y|\,\rho_k^y\,dx \le C$$

uniformly with respect to k.

Third step: *Conclusion of the proof.* Assume t is fixed, again using (4.7), and let η be a small enough parameter. Then

$$|\{x,\ w_k^x\le\eta\}| \le \frac{C}{|\log\eta|}\int_{\mathbb{T}^d}|\log w_k^x|\,\rho_k^x\,dx \le \frac{C}{|\log\eta|}.$$

Note that the same holds in the (t,y) variables. Let us now write

$$\begin{aligned}&\int_{\mathbb{T}^{2d}} K_h(x-y)\,|\rho_k^x-\rho_k^y|\,dx\,dy\\ &\qquad=\int_{w_k^x>\eta,\ w_k^y>\eta} K_h\,|\rho_k^x-\rho_k^y|\,dx\,dy+\int_{w_k^x\le\eta\ \ or\ \ w_k^y\le\eta} K_h^x\,|\rho_k^x-\rho_k^y|\,dx\,dy,\end{aligned}$$

and so

$$\begin{aligned}&\sup_{t\in[0,T]}\int_{\mathbb{T}^{2d}} K_h(x-y)\,|\rho_k^x-\rho_k^y|\,dx\,dy\\ &\qquad\le\frac{1}{\eta^2}\sup_{t\in[0,T]}\int_{\mathbb{T}^{2d}} K_h\,|\rho_k^x-\rho_k^y|\,w_k^x\,w_k^y\,dx\,dy+\frac{C}{|\log\eta|}\,\|K_h\|_{L^1}\\ &\qquad\le C\,\|K_h\|_{L^1}\left(\frac{\varepsilon_k(h)+\tilde\varepsilon_k(h)}{\eta^2}+\frac{1}{|\log\eta|}\right),\end{aligned}$$

which by minimizing in η finishes the proof of (4.8). The compactness is a consequence of the compactness criterion, taking $\limsup_k$ and checking the convergence to zero when h goes to zero using (4.4) and (4.6). □

4.3. *A rough sketch of the extension to compressible Navier–Stokes.* We will only consider the case of general pressure laws and assume that the stress tensor is isotropic. When considering the compressible Navier–Stokes system, the divergence $\operatorname{div} u_k$ is not given anymore but has to be calculated from ρ_k and the total time derivative of the velocity itself through the relation (2.16), where R_k includes the force applied on the fluid and F_k is the effective viscous flux encoding the total time derivative of the velocity itself (see (2.17)). For the moment, we will assume to have the compactness property in space for R_k and $F_k\equiv 0$ even if for the compressible Navier–Stokes system it is not the case. More precisely, we consider that

$$\operatorname{div} u_k = P(\rho_k)+R_k. \tag{4.15}$$

The aim of this subsection is to provide a rough idea of how to extend the previous method when the velocity field u_k is not given but linked to density

ρ_k and to prove compactness far from the vacuum. To keep things as simple as possible here, we temporarily assume that

(4.16)
$$\sup_k \|R_k\|_{L^\infty((0,T)\times\mathbb{T}^d)} < \infty, \quad \limsup_k \varepsilon_k(h) \longrightarrow 0 \text{ as } h \to 0,$$
$$\text{where } \varepsilon_k(h) = \frac{1}{\|K_h\|_{L^1}} \int_0^T \int_{\mathbb{T}^{2d}} K_h(x-y)\,|R_k(t,x) - R_k(t,y)|^p\,dx\,dy\,dt.$$

Denoting

$$\tilde{\varepsilon}_k(h) = \frac{1}{\|K_h\|_{L^1}} \int_{\mathbb{T}^{2d}} K_h(x-y)\,|\rho_k^0(x) - \rho_k^0(y)|\,dx\,dy, \tag{4.17}$$

we assume compactness of the initial data, namely,

$$\limsup_k \tilde{\varepsilon}_k(h) \to 0 \text{ as } h \to 0. \tag{4.18}$$

We do not assume monotonicity on the pressure P but simply the control

$$|P'(\rho)| \le C\,\rho^{\gamma-1}. \tag{4.19}$$

A modification of the previous proof then yields

PROPOSITION 4.4. *Assume ρ_k solves* (4.1) *and the bounds*

$$\sup_k \|\rho_k\|_{L^\infty(0,T;L^1(\mathbb{T}^d))} < \infty, \qquad \sup_k \|\rho_k\|_{L^p((0,T)\times\mathbb{T}^d)} < \infty \quad \text{with } p \ge \gamma + 1.$$

Assume that $\sup_k \|u_k\|_{L^2(0,T;H^1(\mathbb{T}^d))} < \infty$ *and that* (4.15) *holds with the bounds* (4.16) *on R_k and* (4.19) *on P. Assume moreover that the initial data ρ_k^0 is compact, namely,* (4.18). *Then ρ_k is compact away from the vacuum and, more precisely,*

$$\sup_{t\in(0,T)} \int_{\rho_k(t,x)\ge\eta,\ \rho_k(t,y)\ge\eta} K_h(x-y)\,|\rho_k(t,x) - \rho_k(t,y)|\,dx\,dy$$
$$\le C_\eta \frac{\|K_h\|_{L^1}}{|\log(h + (\varepsilon_k(h)) + \tilde{\varepsilon}_k(h))|},$$

where $\varepsilon_k(h)$ and $\tilde{\varepsilon}_k(h)$ are respectively given by (4.16) *and* (4.17).

Unfortunately Proposition 4.4 is only a rough and unsatisfactory attempt for the following reasons:

- The main problem with Proposition 4.4 is that it does not imply compactness on the sequence ρ_k because it only controls oscillations of ρ_k for large enough values but we do not have any lower bounds on ρ_k. In fact not only can ρ_k vanish, but for weak solutions, a vacuum could even form; that is, there may be a set of non-vanishing measures where $\rho_k = 0$. This comes from the fact that the proof only gives an estimate on
$$\int_{\mathbb{T}^{2d}} K_h(x-y)\,|\rho_k^x - \rho_k^y|\,w_k^x\,w_k^y\,dx\,dy,$$

but since there is no lower bound on ρ_k^x and ρ_k^y anymore, estimates like (4.14) only control the set where $w_k^x\, w_k^y$ is small and both ρ_k^x and ρ_k^y are small. Unfortunately $|\rho_k^x - \rho_k^y|$ could be large while only one of ρ_k^x and ρ_k^y is small (and hence $w_k^x\, w_k^y$ is small as well).

- The solution is to work with $w_k^x + w_k^y$ instead of $w_k^x\, w_k^y$. Now the sum $w_k^x + w_k^y$ can only be small if $|\rho_k^x - \rho_k^y|$ is small as well, meaning that a bound on

$$\int_{\mathbb{T}^{2d}} K_h(x-y)\, |\rho_k^x - \rho_k^y|\, (w_k^x + w_k^y)\, dx\, dy,$$

together with estimates like (4.14), would control the compactness on ρ_k. Unfortunately this leads to various additional difficulties because some terms are now not localized at the right point. For instance, one has problems estimating the commutator term in $\nabla K_h \cdot (u_k^x - u_k^y)$ or one cannot directly control terms like $\operatorname{div}_x u_k^x\, w_k^y$ by the penalization that would now be of the form $M\, |\nabla u_k|^x\, w_k^x$. Some of these problems are solved by using more elaborate harmonic analysis tools, while others require a more precise analysis of the structure of the equations. Those difficulties are even magnified for anisotropic stress tensors that add even trickier non-local terms.

- The integrability assumption on ρ_k, $p > \gamma + 1$ is not very realistic and too demanding. If $p = \gamma(1 + 2/d) - 1$ as for the compressible Navier–Stokes equations with power law $P(\rho) = a\rho^\gamma$, then this requires $\gamma > d$. Improving it creates important difficulty in the interaction with the penalization. It forces us to modify the penalization and prevents us from getting an inequality like (4.14) and, in fact, only modified inequalities can be obtained, of the type

$$\sup_k \int_{\mathbb{T}^d} |\log w_k^x|^\theta\, \rho_k^x\, dx < \infty.$$

- The bounds (4.16) that we have assumed for simplicity on R_k cannot be deduced from the equations. The effective viscous flux F_k is not zero, is not bounded in L^∞, and is not a priori compact. (It will only be so at the very end as a consequence of ρ_k being compact.) Instead we will have to establish regularity bounds on the effective viscous flux when integrated against specific test functions, but in a manner more precise than the existing Lions–Feireisl theory; see Lemma 8.3 later.
- This is of course only a stability result; in order to get existence one has to work with an appropriate approximate system. This will be the subject of Section 9.

Proof. One now works with a different equation for the weight

$$\partial_t w_k + u_k \cdot \nabla w_k = -\lambda\, \left(M\, |\nabla u_k| + \rho_k^\gamma\right), \quad w_k|_{t=0} = 1, \tag{4.20}$$

where $M\, f$ is again the maximal function of f.

First step: *Propagation of some weighted regularity.* Recall, here and in the following, that we use the notation $G_k^x = G_k(t,x)$ and $G_k^y = G_k(t,y)$ for all $G_k \in D'((0,T)\times\mathbb{T}^d)$. The beginning of the first step essentially remains the same as in the proof of Proposition 4.1: one propagates

$$R_k(t) = \int_{\mathbb{T}^{2d}} K_h(x-y)\,|\rho_k^x - \rho_k^y|\,w_k^x\,w_k^y\,dx\,dy.$$

The initial calculations are nearly identical. The only difference is that we do not have (4.4) any more, so we simply keep the term with $\operatorname{div}_x u_k^x - \operatorname{div}_y u_k^y$ for the time being. We thus obtain

$$\begin{aligned}\frac{d}{dt}R_k(t) \le &-\frac12\int_{\mathbb{T}^{2d}} K_h(x-y)\,(\operatorname{div}_x u_k^x - \operatorname{div}_y u_k^y)(\rho_k^x+\rho_k^y)\,s_k\,w_k^x\,w_k^y\,dx\,dy\\ &-\lambda\int_{\mathbb{T}^{2d}}((\rho_k^x)^\gamma + (\rho_k^y)^\gamma)\,|\rho_k^x-\rho_k^y|\,w_k^x\,w_k^y\,dx\,dy\end{aligned} \tag{4.21}$$

by taking the additional term in equation (4.20) into account. This is of course where the coupling between u_k and ρ_k comes into play, here only through the simplified equation (4.15). Thus

$$\begin{aligned}&\int_{\mathbb{T}^{2d}} K_h(x-y)\,(\operatorname{div}_x u_k^x - \operatorname{div}_y u_k^y)\,(\rho_k^x+\rho_k^y)\,s_k\,w_k^x\,w_k^y\,dx\,dy\\ &\quad=\int_{\mathbb{T}^{2d}} K_h(x-y)\,(P(\rho_k^x)-P(\rho_k^y))\,(\rho_k^x+\rho_k^y)\,s_k\,w_k^x\,w_k^y\,dx\,dy\\ &\qquad+\int_{\mathbb{T}^{2d}} K_h(x-y)\,(R_k^x-R_k^y)\,(\rho_k^x+\rho_k^y)\,s_k\,w_k^x\,w_k^y\,dx\,dy.\end{aligned} \tag{4.22}$$

By the uniform L^p bound on ρ_k and the estimate (4.16), one has

$$\begin{aligned}&\int_0^s\int_{\mathbb{T}^{2d}} K_h(x-y)\,(R_k^x-R_k^y)\,(\rho_k^x+\rho_k^y)\,s_k\,w^x\,w^y\,dx\,dy\,dt\\ &\qquad\le C\,\|K_h\|_{L^1}\,(\varepsilon_k(h))^{1-1/p}.\end{aligned} \tag{4.23}$$

Now using (4.19), it is possible to bound

$$\begin{aligned}|P(\rho_k^x)-P(\rho_k^y)| &\le |\rho_k^x-\rho_k^y|\int_0^1 |P'(s\,\rho_k^x+(1-s)\,\rho_k^y)|\,ds\\ &\le C\,((\rho_k^x)^{\gamma-1}+(\rho_k^y)^{\gamma-1})\,|\rho_k^x-\rho_k^y|,\end{aligned}$$

leading to

$$\begin{aligned}&\int_{\mathbb{T}^{2d}} K_h(x-y)\,(P(\rho_k^x)-P(\rho_k^y))\,(\rho_k^x+\rho_k^y)\,s_k\,w_k^x\,w_k^y\,dx\,dy\\ &\le C\int_{\mathbb{T}^{2d}} K_h(x-y)\,((\rho_k^x)^{\gamma-1}+(\rho_k^y)^{\gamma-1})\,(\rho_k^x+\rho_k^y)|\rho_k^x-\rho_k^y|\,w_k^x\,w_k^y\,dx\,dy\\ &\le C\int_{\mathbb{T}^{2d}} K_h(x-y)\,((\rho_k^x)^{\gamma}+(\rho_k^y)^{\gamma})\,|\rho_k^x-\rho_k^y|\,w_k^x\,w_k^y\,dx\,dy.\end{aligned}$$

Now using this estimate, equality (4.22), the compactness (4.23), and by taking λ large enough, one finds from (4.21) that

$$R_k|_{t=s} \leq R_k|_{t=0} + C\, \|K_h\|_{L^1(\mathbb{T}^d)}\, (\varepsilon_k(h))^{1-1/p}.$$

Second step and third steps: *Property of the weight and conclusion.* The starting point is again the same and gives

$$\int_{\mathbb{T}^d} |\log w_k^x|\, \rho_k^x\, dx \leq C, \qquad \int_{\mathbb{T}^d} |\log w_k^y|\, \rho_k^y\, dy \leq C$$

with C independent of k but where we now need $\rho_k \in L^p$ with $p \geq \gamma + 1$ because of the additional term in equation (4.20). By splitting the integration,

$$\begin{aligned}\int_{\rho_k^x\geq\eta,\ \rho_k^y\geq\eta} & K_h(x-y)\, |\rho_k^x - \rho_k^y|\, dx\, dy \\ &= \int_{\rho_k^x\geq\eta,\ \rho_k^y\geq\eta,\ w_k^x\geq\eta',\ w_k^y\geq\eta'} K_h(x-y)\, |\rho_k^x - \rho_k^y|\, dx\, dy \\ &\quad + \|K_h\|_{L^1} \int_{\rho_k^x\geq\eta,\ w_k^x\leq\eta'} (1+\rho_k^x)\, dx.\end{aligned}$$

On the one hand,

$$\begin{aligned}\int_{\rho_k^x\geq\eta,\ w_k^x\leq\eta'} (1+\rho_k^x)\, dx &\leq \left(\frac{1}{\eta}+1\right) \int_{w_k^x\leq\eta'} \rho_k^x\, dx \\ &\leq \left(\frac{1}{\eta}+1\right) \frac{1}{|\log\eta'|} \int_{\mathbb{T}^d} |\log w_k^x|\, \rho_k^x\, dx \leq \left(\frac{1}{\eta}+1\right) \frac{C}{|\log\eta'|}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\int_{\rho_k^x\geq\eta,\ \rho_k^y\geq\eta,\ w_k^x\geq\eta',\ w_k^y\geq\eta'} & K_h(x-y)\, |\rho_k^x - \rho_k^y|\, dx\, dy \\ &\leq \frac{1}{(\eta')^2} \int_{\mathbb{T}^{2d}} K_h(x-y)\, |\rho_k^x - \rho_k^y|\, w_k^x\, w_k^y\, dx\, dy.\end{aligned}$$

Therefore,

$$\begin{aligned}&\int_{\rho_k^x\geq\eta,\ \rho_k^y\geq\eta} K_h(x-y)\, |\rho_k^x - \rho_k^y|\, dx\, dy \\ &\leq \|K_h\|_{L^1} \left(\frac{1}{\eta}+1\right) \frac{C}{|\log\eta'|} + \frac{C}{(\eta')^2}\, \|K_h\|_{L^1}\, \left(\tilde{\varepsilon}_k(h) + (\varepsilon_k(h))^{1-1/p}\right),\end{aligned} \tag{4.24}$$

which concludes the proof by optimizing in η' as we get the estimate

$$\begin{aligned}\int_{\rho_k(x)\geq\eta,\ \rho_k(y)\geq\eta} & K_h(x-y)\, |\rho_k(t,x) - \rho_k(t,y)|\, dx\, dy \\ &\leq C_\eta\, \frac{\|K_h\|_{L^1}}{|\log(h + (\varepsilon_k(h)) + \tilde{\varepsilon}_k(h))|},\end{aligned}$$

where $\varepsilon_k(h)$ and $\tilde{\varepsilon}_k(h)$ are respectively given by (4.16) and (4.17). Now taking the lim sup in k, the resulting quantity converges to zero as $h \to 0$. Therefore we get compactness far from the vacuum applying the compactness lemma. □

5. Stability results

5.1. *The equations.* Here and in the following, we will often use the notation $G_k^{\cdot_1} = G_k(t, \cdot_1)$ and $H_k^{\cdot_1,\cdot_2} = H_k(t, \cdot_1, \cdot_2)$ for any space variable $\cdot_1$, $\cdot_2$ belonging to $\mathbb{T}^d$. For instance, $\rho_k^x = \rho_k(t,x)$, $P_k^{x,\rho_k^x} = P_k(t, x, \rho_k(t,x))$.

5.1.1. *General pressure law.* We consider a sequence (ρ_k, u_k) of global weak solutions (with the uniform bounds given below, which allows us to make sense of the equations in the sense of distributions). Here ρ_k solves the continuity equation

$$\partial_t \rho_k + \operatorname{div}(\rho_k\, u_k) = \alpha_k \Delta \rho_k \qquad \text{in } (0,T) \times \mathbb{T}^d, \tag{5.1}$$

and we ask that ρ_k be a renormalized solution to (5.1). (We recall this notion in Section 6.2.) Here u_k solves

$$\mu_k^x \operatorname{div}_x u_k^x - P_k^{x,\rho_k^x} = F_k^x + R_k^x \qquad \text{in } (0,T) \times \mathbb{T}^d, \tag{5.2}$$

where

$$F_k^x = \Delta_x^{-1} \operatorname{div}_x(\partial_t(\rho_k^x\, u_k^x) + \operatorname{div}_x(\rho_k^x\, u_k^x \otimes u_k^x))$$

and R_k^x represents terms that will be assumed to be compact as an external force with the initial conditions

$$\rho_k|_{t=0} = \rho_0, \qquad \rho_k u_k|_{t=0} = \rho_0 u_0 \qquad \text{in } \mathbb{T}^d$$

and periodic boundary conditions in space.

Remark. Note that the term F_k^x represents the effective viscous flux part coming from the total time derivative. Compactness of such a quantity is a priori not known and will be treated in a section. Readers who are interested by the method without too much complexity may skip this term and the corresponding parts. This will provide compactness with the vacuum state compared to the previous section, which focused on compactness far from the vacuum.

Important remark. Note that here we allow a possible explicit dependence on t and x in P_k, namely, $P_k^{x,\rho_k^x} = P_k(\rho_k(t,x), t, x)$. This does not really affect our stability results, and it can be critical for treating non-homogeneous settings or cases such as the Navier–Stokes–Fourier system. This general form is hence a good illustration of the flexibility of our compactness method.

In this part, we will assume the viscosity of the fluid to depend on time and space for the first time. To handle with more general viscosity of this kind, we assume it to be bounded from below and above, namely, that there exists a constant $\bar\mu$ independent on k such that

$$0 < \frac{1}{\bar\mu} \le \mu_k^x \le \bar\mu < +\infty. \tag{5.3}$$

We consider the following control on the density (for $p>1$):

$$\sup_k \left[\|\rho_k\|_{L^\infty([0,\,T],\;L^\gamma(\mathbb{T}^d))} + \|\rho_k\|_{L^p([0,\,T]\times\mathbb{T}^d)}\right] < \infty. \tag{5.4}$$

We consider the following control for u_k:

$$\sup_k \Big[\|\rho_k\,|u_k|^2\|_{L^\infty([0,\,T],\;L^1(\mathbb{T}^d))} + \|\nabla u_k\|_{L^2(0,T;L^2(\mathbb{T}^d))}\Big] < +\infty. \tag{5.5}$$

We also need some control on the time derivative of $\rho_k u_k$ through

$$\exists \bar p > 1, \quad \sup_k\ \|\partial_t(\rho_k u_k)\|_{L^2_t W_x^{-1,\bar p}} < \infty \tag{5.6}$$

and on the time derivative of ρ_k, namely,

$$\exists q > 1, \quad \sup_k\ \|\partial_t \rho_k\|_{L^q_t W_x^{-1,q}} < \infty. \tag{5.7}$$

Remark. Note that usually (see, for instance, [32], [48]), (5.6) and (5.7) are consequences of the momentum equation and the mass equation using the uniform estimates given by the energy estimates and the extra integrability on the density.

Concerning the equation of state, we will consider that for every x, s: $P_k^{x,s}$ continuous in s on $[0,+\infty)$ and positive, $P_k^{x,s}$ locally Lipschitz in s on $(0,+\infty)$ with $P_k^{x,0} = \Theta_k^x$ and with one of the two following cases:

(i) *Pressure laws with a quasi-monotone property*: There exists $\bar P$, ρ_0 independent of t,x such that if $s \ge \rho_0$, then $P_k^{x,s}$ is a function $\tilde P_k^x$ plus a function independent of t,x and

$$\begin{aligned}
&\partial_s\left(\left[P_k^{x,s} - \tilde P_k^x\right]/s\right) \ge 0 \text{ for all } s \ge \rho_0, \qquad \lim_{s\to+\infty} P_k^{x,s} = +\infty,\\
&|P_k^{x,r} - P_k^{y,s}| \le \bar P\,|r-s| + Q_k^{x,y} \text{ for all } r,\ s \le \rho_0,\ x,\ y \in \mathbb{T}^d,\\
&\limsup_{h\to 0}\ \sup_k \int_0^T \int_{\mathbb{T}^{2d}} \frac{K_h(x-y)}{\|K_h\|_{L^1}} \left(|\tilde P_k^x - \tilde P_k^y| + Q_k^{x,y}\right) dx\,dy\,dt = 0.
\end{aligned} \tag{5.8}$$

(ii) *Non-monotone pressure laws* (*with very general Lipschitz pressure laws*): There exist $\bar P > 0$, $\tilde\gamma > 0$ and $\tilde P_k^x$ in $L^2((0,\,T)\times\mathbb{T}^d)$, $Q_k^{x,y}$ in $L^1((0,\,T)\times\mathbb{T}^{2d})$

and Θ_k^x in $L^1((0,\ T)\times\mathbb{T}^d)$ such that for all t,x, y,

$$
\begin{aligned}
&|P_k^{x,\rho_k^x} - P_k^{y,\rho_k^y}| \le Q_k^{x,y} + \left[\bar P\ \left((\rho_k^x)^{\tilde\gamma-1} + (\rho_k^y)^{\tilde\gamma-1}\right) + \tilde P_k^x + \tilde P_k^y\right]\ |\rho_k^x - \rho_k^y|,\\
&P_k^{x,\rho_k^x} \le \bar P\ (\rho_k^x)^{\tilde\gamma} + \Theta_k^x \text{ with } \Theta_k^x \ge 0,\\
&\sup_k \Big(\|\tilde P_k\|_{L^2((0,\ T)\times\mathbb{T}^d)} + \|\Theta_k\|_{L^1([0,\ T]\times\mathbb{T}^d)}\Big) < \infty,\\
&\limsup_{h\to 0}\sup_k \int_0^T\int_{\mathbb{T}^{2d}} \frac{K_h(x-y)}{\|K_h\|_{L^1}}\Big(|\tilde P_k^x - \tilde P_k^y|\\
&\qquad\qquad + |\Theta_k^x - \Theta_k^y| + Q_k^{x,y}\Big)\,dx\,dy\,dt = 0.
\end{aligned}
\tag{5.9}
$$

Two important remarks. (1) The general hypothesis on pressure laws should prove quite useful in many extensions of our results, such as the heat-conducting compressible Navier–Stokes case. The pressure law can then include, for instance, a radiative part (namely, a part depending only on the temperature as in [33]) and a pressure law in density with coefficients depending on temperature. (See the comments in the section under consideration.)

But this general form can also cover various inhomogeneous settings where the pressure law may have some explicit spatial dependence.

(2) In the basic case where P_k does not depend explicitly on t or x (namely, $\tilde P_k \equiv 0$, $Q_k \equiv 0$ and $\Theta_k \equiv 0$), then (5.9) reduces to the very simple condition

$$|P_k(r) - P_k(s)| \le \bar P\, r^{\tilde\gamma-1}|r-s|.$$

Note that this assumption is satisfied if P is locally Lipschitz on $(0,+\infty)$ with

$$|P'(s)| \le \bar P s^{\tilde\gamma-1},$$

namely, with the hypothesis mentioned in Theorem 3.1.

Remark. Note that (i) with the lower bound $P(\rho) \ge C^{-1}\rho^\gamma - C$ provides the same assumptions as in the article [31] by E. Feireisl. Point (i) will be used to construct approximate solutions in the non-monotone case.

5.1.2. *A non-isotropic stress tensor.* In that case, assume (ρ_k, u_k) to be a sequence of global weak solutions solving (5.1) with $\alpha_k = 0$,

$$\partial_t\rho_k + \operatorname{div}(\rho_k\, u_k) = 0 \qquad \text{in } (0,T)\times\mathbb{T}^d.$$

and

$$\operatorname{div} u_k = \nu_k\, P_k(\rho_k) + \nu_k\, a_\mu A_\mu\, P_k(\rho_k) + F_{k,a}, \tag{5.10}$$

where

$$A_\mu = (\Delta - a_\mu\, E_k)^{-1}\, E_k$$

with E_k a given integral or differential operator discussed later on and where the anisotropic effective viscous flux is given by

$$F_{k,a} = \nu_k\,(\Delta - a_\mu\,E_k)^{-1}\,\mathrm{div}\,(\partial_t(\rho_k\,u_k) + \mathrm{div}\,(\rho_k\,u_k \otimes u_k)).$$

The equations are supplemented with the initial conditions

$$\rho_k|_{t=0} = \rho_0, \qquad \rho_k u_k|_{t=0} = \rho_0 u_0 \qquad \text{in } \mathbb{T}^d$$

and are periodic in space boundary conditions.

Remark. If one considers a symmetric anisotropy, $\mathrm{div}\,(A\,D\,u)$ in Theorem 3.2, then instead of (5.10), we have the more complicated formula

$$\begin{aligned}\mathrm{div}\,u_k =& \nu_k\,P_k(\rho_k) + \nu_k\,a_\mu A_\mu\,P_k(\rho_k)\\ &+ \nu_k\,\mathrm{div}\,(\Delta\,I - a_\mu\,E_k)^{-1}\,(\partial_t(\rho_k\,u_k) + \mathrm{div}\,(\rho_k\,u_k \otimes u_k)),\end{aligned} \tag{5.11}$$

where $A_\mu = (\Delta\,I - a_\mu\,E_k)^{-1} \cdot \tilde{E}_k$. But now E_k and $\tilde{E}_k$ may be different and are vector-valued operators so that, in particular, $(\Delta\,I - a_\mu\,E_k)^{-1}$ means inverting a vector valued elliptic system. Except for the formulation there would however be no actual difference in the rest of the proof.

Coming back to (5.10), we assume ellipticity on ν_k:

$$0 < \underline{\nu} \le \nu_k \le \overline{\nu} < \infty. \tag{5.12}$$

We assume that E_k is a given operator (differential or integral) such that

- $(\Delta - a_\mu\,E_k)^{-1}\,\Delta$ is bounded on every L^p space;
- $A_\mu = (\Delta - a_\mu\,E_k)^{-1}\,E_k$ is bounded of norm less than 1 on every L^p space and can be represented by a convolution with a singular integral still denoted by A_μ:

$$A_\mu\,f = A_\mu \star_x f, \quad |A_\mu(x)| \le \frac{C}{|x|^d}, \quad \int A_\mu(x)\,dx = 0.$$

Note here that to make more apparent the smallness of the non-isotropic part, we explicitly scale it with a_μ. We consider again the control (5.4) on the density but for $p > \gamma^2/(\gamma - 1)$, and the bound (5.5) for u_k. We also need the same controls: (5.6) on the time derivative of $\rho_k u_k$ and (5.7) on the time of the ρ_k.

The main idea here is to investigate the compactness for an anisotropic viscous stress obtained as the perturbation of the usual isotropic viscous stress tensor, namely, $-\mathrm{div}\,(A\,\nabla u) + (\lambda + \mu)\,\nabla \mathrm{div} u$ assuming $A = \mu\,Id + \delta A$ and $a_\mu = \|\delta A\| \le \varepsilon$ for some small enough ε.

5.2. *The main stability results*: *Theorems* 5.1, 5.2 *and* 5.3. First note that as in Lions' scenario, the main point in the proof of both results is the coupling of the specific renormalized form of the continuity equation for the new quantity measuring density oscillations with the specific compensated compactness properties of the commutator involving the effective viscous flux coming from the total time derivative. Readers who are interested in models without total time derivative in the equation that gives the velocity field in terms of the pressure may skip the parts called effective viscous flux.

5.2.1. *General pressure laws.* The main step in that case is to prove the two compactness results

THEOREM 5.1. *Assume that ρ_k solves* (5.1), *u_k solves* (5.2) *with the bounds* (5.3), (5.5), (5.6), (5.7), *and that μ_k and R_k are compact in $L^1((0,T)\times\mathbb{T}^d)$. Moreover,*

(i) *if $\alpha_k > 0$ (with $\alpha_k \to 0$ when $k \to +\infty$), we assume the estimate* (5.4) *on ρ_k with $\gamma > 3/2$ and $p > 2$ and quasi-monotonicity on P_k through* (5.8);
(ii) *if $\alpha_k = 0$, then it is enough to assume* (5.4) *with $\gamma > 3/2$ and $p > \max(2,\tilde\gamma)$ and only* (5.9) *on P_k.*

Then the sequence ρ_k is compact in $L^1((0,T)\times\mathbb{T}^d)$.

We also provide a complementary result that is a more precise rate of compactness away from the vacuum; namely,

THEOREM 5.2. *Assume again that ρ_k solves* (5.1) *with $\alpha_k = 0$, u_k solves* (5.2) *with the bounds* (5.3), (5.5), (5.6), (5.7) *and that μ_k and R_k are compact in $L^1((0,T)\times\mathbb{T}^d)$. Assume that* (5.4) *holds with $\gamma > d/2$ and $p > \max(2,\tilde\gamma)$ and that P_k satisfies* (5.9). *Then there exists $\theta > 0$ and a continuous function ε with $\varepsilon(0) = 0$, depending only on μ_k and R_k such that*

$$\limsup_k \sup_{s\in[0,T]} \Big[\int_{\mathbb{T}^{2d}} \mathbb{I}_{\rho_k(x,t)\geq\eta}\, \mathbb{I}_{\rho_k(y,t)\geq\eta}\, K_h(x-y)\,\chi(\delta\rho_k)\,dx\,dy\Big]\Big|_{t=s} \leq \frac{C\,\|K_h\|_{L^1}}{\eta^{1/2}\,|\log(\varepsilon(h)+h^\theta)|^{\theta/2}}.$$

For instance, if $\tilde P_k$, Θ_k, μ_k and R_k are uniformly in $W^{s,1}$ for $s > 0$, then for some constant $C > 0$,

$$\limsup_k \sup_{s\in[0,T]} \Big[\int_{\mathbb{T}^{2d}} \mathbb{I}_{\rho_k(x,t)\geq\eta}\, \mathbb{I}_{\rho_k(y,t)\geq\eta}\, K_h(x-y)\,\chi(\delta\rho_k)\,dx\,dy\Big]\Big|_{t=s} \leq C\,\frac{\|K_h\|_{L^1}}{\eta^{1/2}\,|\log h|^{\theta/2}}.$$

Since those results depend on the regularity of μ_k and R_k, we denote $\varepsilon_0(h)$ a continuous function with $\varepsilon_0(0) = 0$ such that

$$\begin{aligned}\int_0^T \int_{\mathbb{T}^{2d}} K_h(x-y) \Big(|R_k^x - R_k^y| + |\mu_k^x - \mu_k^y| \\ + |\tilde{P}_k^x - \tilde{P}_k^y| + |\Theta_k^x - \Theta_k^y| + |Q_k^{x,y}|\Big)\, dx\, dy\, dt \leq \varepsilon_0(h)\, \|K_h\|_{L^1}.\end{aligned} \tag{5.13}$$

5.2.2. *Non-isotropic stress tensor.* In that case our result reads

THEOREM 5.3. *Assume that ρ_k solves* (5.1) *and that u_k solves* (5.10) *with the bounds* (5.5), (5.6), (5.7) *and* (5.12) *together with all the assumptions on E_k below* (5.10). *Assume as well that P_k satisfies* (5.8) *and that* (5.4) *with $\gamma > d/2$ and $p > \gamma^2/(\gamma-1)$. There exists a universal constant $C_* > 0$ such that if*

$$a_\mu \leq C_*,$$

then ρ_k is compact in $L^1((0,T) \times \mathbb{T}^d)$.

Remarks. Theorems 5.1, 5.2, and 5.3 are really the main contributions of this article. For instance, deducing Theorems 3.1 and 3.2 follows usual and straightforward approximation procedures.

As such the main improvements with respect to the existing theory can be seen in the fact that point (ii) in Theorem 5.1 does not require monotonicity on P_k and in the fact that Theorem 5.3 does not require isotropy on the stress tensor.

Our starting approximate system involves diffusion, $\alpha_k \neq 0$, in the continuity equation (5.1). As can be seen from point (i) of Theorem 5.1, our compactness result in that case requires an isotropic stress tensor and a pressure P_k that is monotone after a certain point by (5.8). This limitation is the reason why we also have to consider approximations P_k and E_k of the pressure and the stress tensor. While it may superficially appear that we did not improve the existing theory in that case with diffusion, we want to point out the following:

- We could not have used P.-L. Lions' approach because this requires strict monotonicity: $P_k' > 0$ everywhere. Instead, any non-monotone pressure P satisfying (5.9) can be approximated by P_k satisfying (5.8) simply by considering $P_k = P + \varepsilon_k\, \rho^{\bar{\gamma}}$ as long as $\bar{\gamma} > \tilde{\gamma}$ and thus without changing the requirements on γ.
- E. Feireisl et al. can handle "quasi-monotone" pressure laws satisfying (5.8) together with diffusion, but they require higher integrability on ρ_k for this: $p \geq 4$ in (5.4). This in turn leads to a more complex approximation procedure.

6. Technical lemmas and renormalized solutions

6.1. *Useful technical lemmas.* We recall the well-known inequality, which we used in Section 4.2 and will use several times in the following (see, for instance, [54]): For any $|x-y| \le 1$,

$$|\Phi(x) - \Phi(y)| \le C\,|x-y|\,(M|\nabla\Phi|(x) + M|\nabla\Phi|(y)), \tag{6.1}$$

where M is the localized maximal operator

$$M\,f(x) = \sup_{r\le 1} \frac{1}{|B(0,r)|} \int_{B(0,r)} f(x+z)\,dz. \tag{6.2}$$

As will be seen later, there is a technical difficulty in the proof, which would lead us to try (and fail) to control $M|\nabla u_k|(y)$ by $M|\nabla u_k|(x)$. Instead we have to be more precise than (6.1) in order to avoid this. To deal with such problems, we use more sophisticated tools. First,

LEMMA 6.1. *There exists $C > 0$ such that for any $u \in W^{1,1}(\mathbb{T}^d)$, one has*

$$|u(x) - u(y)| \le C\,|x-y|\,(D_{|x-y|}u(x) + D_{|x-y|}u(y)),$$

where we denote

$$D_h u(x) = \frac{1}{h}\int_{|z|\le h} \frac{|\nabla u(x+z)|}{|z|^{d-1}}\,dz.$$

Proof. A full proof of such a well-known result can, for instance, be found in [41] in a more general setting, namely, $u \in BV$. The idea is simply to consider trajectories $\gamma(t)$ from x to y that stay within the ball of diameter $|x-y|$ to control

$$|u(x) - u(y)| \le \int_0^1 \gamma'(t)\cdot\nabla u(\gamma(t))\,dt,$$

and then to average over all such trajectories with length of order $|x-y|$. Similar calculations are also present, for instance, in [30]. □

Note that this result implies the estimate (6.1) as

LEMMA 6.2. *There exists $C > 0$ for any $u \in W^{1,p}(\mathbb{T}^d)$ with $p \ge 1$:*

$$D_h\,u(x) \le C\,M|\nabla u|(x).$$

Proof. Do a dyadic decomposition, and define i_0 such that $2^{-i_0-1} < h \leq 2^{-i_0}$:

$$\begin{aligned} D_h\, u(x) &\leq \frac{1}{h} \sum_{i\geq i_0} \int_{2^{-i-1}<|z|\leq 2^{-i}} \frac{|\nabla u(x+z)|}{|z|^{d-1}}\, dz \\ &\leq \sum_{i\geq i_0} \frac{2^{(i+1)\,(d-1)}}{h} \int_{2^{-i-1}<|z|\leq 2^{-i}} |\nabla u(x+z)|\, dz \\ &\leq 2^{d-1} \sum_{i\geq i_0} |B(0,1)|\, \frac{2^{-i}}{h} M\, |\nabla u|(x) \leq C\, M\, |\nabla u|(x). \end{aligned}$$
□

The key improvement in using D_h is that small translations of the operator D_h are actually easy to control

LEMMA 6.3. *For any $1 < p < \infty$, there exists $C > 0$ such that for any $u \in H^1(\mathbb{T}^d)$,*

$$\int_{h_0}^1 \int_{\mathbb{T}^d} \overline{K}_h(z)\, \|D_{|z|}\, u(.) - D_{|z|}\, u(.+z)\|_{L^p}\, dz\, \frac{dh}{h} \leq C\, \|u\|_{B^1_{p,1}}, \tag{6.3}$$

where the definition and basic properties of the Besov space $B^1_{p,1}$ are recalled in Section 11. As a consequence,

$$\int_{h_0}^1 \int_{\mathbb{T}^d} \overline{K}_h(z)\, \|D_{|z|}\, u(.) - D_{|z|}\, u(.+z)\|_{L^2}\, dz\, \frac{dh}{h} \leq C\, |\log h_0|^{1/2}\, \|u\|_{H^1}. \tag{6.4}$$

It is also possible to disconnect the shift from the radius in $D_r u$ and obtain, for instance,

$$\begin{aligned} \int_{h_0}^1 \int_{\mathbb{T}^{2d}} \overline{K}_h(z)\, \overline{K}_h(w) \|D_{|z|}\, u(.) - D_{|z|}\, u(.+w)\|_{L^2}\, dz\, dw\, \frac{dh}{h} \\ \leq C\, |\log h_0|^{1/2}\, \|u\|_{H^1}. \end{aligned} \tag{6.5}$$

We can in fact write a more general version of Lemma 6.3 for any kernel:

LEMMA 6.4. *For any $1 < p < \infty$ and any family $N_r \in W^{s,1}(\mathbb{T}^d)$ for some $s > 0$ such that*

$$\begin{aligned} \sup_{|\omega|\leq 1} \sup_r r^{-s} \int_{\mathbb{T}^d} |z|^s\, |N_r(z) - N_r(z + r\,\omega)|\, dz < \infty, \\ \sup_r \left(\|N_r\|_{L^1} + r^s\, \|N_r\|_{W^{s,1}}\right) < \infty, \end{aligned}$$

there exists $C > 0$ such that for any $u \in L^p(\mathbb{T}^d)$,

$$\int_{h_0}^1 \int_{\mathbb{T}^d} \overline{K}_h(z)\, \|N_h \star u(.) - N_h \star u(.+z)\|_{L^p}\, dz\, \frac{dh}{h} \leq C\, |\log h_0|^{1/2}\, \|u\|_{L^p}. \tag{6.6}$$

We will mostly use the specific version in Lemma 6.3 but will need the more general Lemma 6.4 to handle the anisotropic case in Lemma 8.6. Both lemmas are in fact a corollary of a classical result:

LEMMA 6.5. *For any* $1 < p < \infty$, *any family* L_r *of kernels satisfying for some* $s > 0$,

(6.7)
$$\int L_r = 0,\ \sup_r \left(\|L_r\|_{L^1} + r^s\, \|L_r\|_{W^{s,1}}\right) \leq C_L,\ \sup_r r^{-s} \int |z|^s\, |L_r(z)|\, dz \leq C_L,$$

then there exists $C > 0$ *depending only on* C_L *above such that for any* $u \in L^p(\mathbb{T}^d)$,

$$\int_{h_0}^1 \|L_r \star u\|_{L^p} \frac{dr}{r} \leq C\, \|u\|_{B^0_{p,1}}. \tag{6.8}$$

As a consequence, for $p \leq 2$,

$$\int_{h_0}^1 \|L_r \star u\|_{L^p} \frac{dr}{r} \leq C\, |\log h_0|^{1/2}\, \|u\|_{L^p}. \tag{6.9}$$

Note that by a simple change of variables in r one has, for instance, for any fixed power l,

$$\int_{h_0}^1 \|L_{r^l} \star u\|_{L^p} \frac{dr}{r} \leq C_l\, |\log h_0|^{1/2}\, \|u\|_{L^p}.$$

Remark. The bounds (6.4) and (6.9) could also be obtained by straightforward application of the so-called square function; see the book written by E. M. Stein [54]. We instead use Besov spaces as this yields the interesting and optimal inequalities (6.3)-(6.8) as an intermediary step.

Proof of Lemmas 6.3 *and* 6.4 *assuming Lemma* 6.5. First of all, observe that $D_h\, u = N_h \star u$ with

$$N_h = \frac{1}{h\, |z|^{d-1}}\, \mathbb{I}_{|z| \leq h},$$

which satisfies all the assumptions of Lemma 6.4. Therefore the proofs of Lemmas 6.3 and 6.4 are identical, just by replacing D_h by $N_h\star$. Hence we only give the proof of Lemma 6.3.

Calculate

$$\int_{h_0}^1 \overline{K}_h(z)\, \frac{dh}{h} \leq \int_{h_0}^1 \frac{C\, h^{\nu-d}}{(h + |z|)^\nu}\, \frac{dh}{h} \leq \frac{C}{(|z| + h_0)^d}.$$

Note also for future use that the same calculation provides

$$\int_{h_0}^1 \overline{K}_h(z)\, \frac{dh}{h} \geq \frac{1}{C\, (|z| + h_0)^d}. \tag{6.10}$$

We hence observe that we are essentially working here with the kernel

$$\frac{1}{(|z|+h_0)^d},$$

which has the critical exponent (equal to the dimension d). Indeed in this section, we could replace $\int_{h_0}^1 \bar K_h \frac{dh}{h}$ by $\frac{1}{(|z|+h_0)^d}$ and obtain similar results. Some formulations though are more natural with $\bar K_h$, such as (6.5) in Lemma 6.3.

Therefore, using spherical coordinates,

$$\int_{h_0}^1 \int_{\mathbb{T}^d} \overline{K}_h(z)\, \|D_{|z|}\, u(.) - D_{|z|}\, u(.+z)\|_{L^p}\, dz\, dh \\ \leq C \int_{S^{d-1}} \int_{h_0}^1 \|D_r\, u(.) - D_r\, u(.+r\,\omega)\|_{L^p} \frac{dr}{r+h_0}\, d\omega.$$

Denote

$$L_\omega(x) = \frac{\mathbb{I}_{|x|\leq 1}}{|x|^{d-1}} - \frac{\mathbb{I}_{|x-\omega|\leq 1}}{|x-\omega|^{d-1}}, \quad L_{\omega,r}(x) = r^{-d}\, L_\omega(x/r),$$

and remark that $L_\omega \in W^{s,1}$ for some $s>0$ with a norm uniform in ω and with support in $B(0,2)$. Moreover,

$$D_r u(x) - D_r u(x+r\omega) = \int |\nabla u|(x-r\,z)\, L_\omega(z)\, dz = L_{\omega,r} \star |\nabla u|.$$

We hence apply Lemma 6.5 since the family $L_{\omega,r}$ satisfies the required hypothesis, and we get

$$\int_{h_0}^1 \|L_{\omega,r} \star \nabla\, u\|_{L^p} \frac{dr}{r} \leq C\, \|u\|_{B^1_{1,p}},$$

with a constant C independent of ω, and so

$$\int_{h_0}^1 \int_{\mathbb{T}^d} \overline{K}_h(z)\, \|D_{|z|}\, u(.) - D_{|z|}\, u(.+z)\|_{L^p}\, dz\, dh \\ \leq C \int_{S^{d-1}} \int_{h_0}^1 \|L_{\omega,r} \star \nabla u\|_{L^p} \frac{dr}{r}\, d\omega \leq C \int_{S^{d-1}} \|u\|_{B^1_{1,p}}\, d\omega,$$

yielding (6.3). The bound (6.4) is deduced in the same manner. The proof of the bound (6.5) follows the same steps; the only difference is that the average over the sphere is replaced by a smoother integration against the weight $1/(1+|w|)^a$. □

Proof of Lemma 6.5. First remark that L_r is not smooth enough to be used as the basic kernels Ψ_k in the classical Littlewood–Paley decomposition (see Section 11) as, in particular, the Fourier transform of L_r is not necessarily compactly supported. We use instead the Littlewood–Paley decomposition of u. Denote

$$U_k = \Psi_k \star u.$$

The kernel L_r has 0 average, and so

$$L_r \star U_k = \int_{\mathbb{T}^d} L_r(x-y)\,(U_k(y)-U_k(x))\,dy.$$

Therefore,

$$\begin{aligned}\|L_r \star U_k\|_{L^p} &\le \int_{\mathbb{T}^d} |L_r(z)|\,\|U_k(.)-U_k(.+z)\|_{L^p}\,dz\\ &\le \int_{\mathbb{T}^d} |L_r(z)|\,|z|^s\,\|U_k\|_{W^{s,p}}\,dz,\end{aligned}$$

yielding by the assumption on L_r, for $k < |\log_2 r|$,

$$\|L_r \star U_k\|_{L^p} \le C\,r^s\,2^{k\,s}\,\|U_k\|_{L^p}, \tag{6.11}$$

by Proposition 11.2. Note that C only depends on $\int |z|^s\,|L_r(z)|\,dz$.

Similarly, we now use that $L_r \in W^{s,1}$ and deduce for $k \ge |\log_2 r|$ by Proposition 11.2,

$$\|L_r \star U_k\|_{L^p} \le \|L_r\|_{W^{s,1}}\,\|U_k\|_{W^{-s,p}} \le C\,r^{-s}\,2^{-k\,s}\,\|U_k\|_{L^p}, \tag{6.12}$$

where C only depends on $\sup_r r^s\,\|L_r\|_{W^{s,1}}$. From the decomposition of f,

$$\begin{aligned}\int_{h_0}^1 \|L_r \star u\|_{L^p}\,\frac{dr}{r} &= \sum_{k=0}^\infty \int_{h_0}^1 \|L_r \star U_k\|_{L^p}\,\frac{dr}{r}\\ &\le C\,\sum_{k=0}^\infty \|U_k\|_{L^p}\left(\mathbb{I}_{k\le|\log_2 h_0|}\int_{h_0}^{2^{-k}} r^s\,2^{k\,s}\,\frac{dr}{r} + \int_{\max(h_0,2^{-k})}^1 r^{-s}\,2^{-k\,s}\,\frac{dr}{r}\right),\end{aligned}$$

by using (6.11) and (6.12). This shows that

$$\int_{h_0}^1 \|L_r \star u\|_{L^p}\,\frac{dr}{r} \le C \sum_{k\le|\log_2 h_0|} \|U_k\|_{L^p} + C \sum_{k>|\log_2 h_0|} \frac{2^{-k\,s}}{h_0^s}\,\|U_k\|_{L^p}. \tag{6.13}$$

Now simply bound

$$\begin{aligned}\sum_{k\le|\log_2 h_0|} \|U_k\|_{L^p} + \sum_{k>|\log_2 h_0|} \frac{2^{-ks}}{h_0^s}\,\|U_k\|_{L^p} &\le C\,\sum_{k=0}^\infty 2^k\,\|U_k\|_{L^p}\\ &= C\,\|u\|_{B^0_{p,1}},\end{aligned}$$

which gives (6.8).

Next remark that

$$\sum_{k>|\log_2 h_0|} \frac{2^{-ks}}{h_0^s}\,\|U_k\|_{L^p} \le C\,\sup_k \|U_k\|_{L^p} \le C\,\|u\|_{B^0_{p,\infty}}.$$

Therefore (6.13) combined with Lemma 11.3 yields

$$\int_{h_0}^1 \|L_r \star u\|_{L^p}\,\frac{dr}{r} \le C\,\sqrt{|\log_2 h_0|}\,\|u\|_{L^p} + C\,\|u\|_{B^0_{p,\infty}},$$

which gives (6.9) by Proposition 11.2. □

Finally we emphasize that

Lemma 6.6. *The kernel*

$$\mathcal{K}_{h_0}(z) = \int_{h_0}^1 \overline{K}_h(z)\,\frac{dh}{h}$$

also satisfies (i) *and* (ii) *of Proposition* 4.1.

Proof. This is a straightforward consequence of using (6.10). □

6.2. *A brief presentation of renormalized solutions.* Many steps in our proofs manipulate solutions to the transport equation, either under the conservative form

$$\partial_t \rho + \operatorname{div}(\rho\, u) = 0, \tag{6.14}$$

or under the advective form

$$\partial_t w + u\cdot\nabla w = F. \tag{6.15}$$

We will also consider the particular form of (6.15),

$$\partial_t w + u\cdot\nabla w = f\,w, \tag{6.16}$$

which can directly be obtained from (6.15) by taking $F = f\,w$.

However since u is not Lipschitz, we do not have strong solutions to these equations, and one should in principle be careful with using them. Those manipulations can be justified using the theory of renormalized solutions as introduced in [26]. Instead of having to justify every time, we briefly explain in this subsection how one may proceed. The reader more familiar with the theory of renormalized solutions may safely skip most of the presentation below.

Assume for the purpose of this subsection that u is a given vector field in $L^2_t H^1_x$. The basic idea behind the renormalized solution is the commutator estimate.

Lemma 6.7. *Assume that* $\rho \in L^2_{t,x}$ *and* $w \in L^2_{t,x}$. *Consider any convolution kernel* $L \in C^1$, *compactly supported in some* $B(0,r)$ *with* $\int_{\mathbb{T}^d} L\,dx = 1$. *Then*

$$\|\operatorname{div}(L_\varepsilon \star_x (\rho\, u) - u\, L_\varepsilon \star u)\|_{L^1_{t,x}} \longrightarrow 0 \quad \text{as } \varepsilon \to 0,$$
$$\|(L_\varepsilon \star_x (u\cdot\nabla_x w) - u\cdot\nabla_x L_\varepsilon \star w\|_{L^1_{t,x}} \longrightarrow 0 \quad \text{as } \varepsilon \to 0.$$

The proof of Lemma 6.7 is straightforward and can be found in [26]. Note however that the techniques we introduce here could also be used, a variant of Proposition 4.3, to make the estimates even more explicit. From Lemma 6.7, one may simply prove

LEMMA 6.8. *Assume that $\rho \in L^2_{t,x}$ is a solution in the sense of distribution to* (6.14). *Assume $w \in L^2_{t,x}$, with $F \in L^1$, a solution in the sense of distribution to* (6.15). *Then for any $\chi \in W^{1,\infty}(\mathbb{R})$, one has in the sense of distribution that*

$$\partial_t \chi(\rho) + \operatorname{div}(\chi(\rho)\, u) = (\chi(\rho) - \rho \chi'(\rho)) \operatorname{div} u,$$
$$\partial_t \chi(w) + u \cdot \nabla \chi(w) = F\, \chi'(w).$$

Finally, if in addition $\rho \in L^{p_1}$, $w \in L^{p_2}$, $u \in L^{p_3}$ with $1/p_1 + 1/p_2 + 1/p_3 \le 1$ and $F \in L^q_{t,x}$ with $1/p_1 + 1/q \le 1$, then in the sense of distribution for any $\chi \in W^{1,\infty}(\mathbb{R})$,

$$\partial_t(\rho\, \chi(w)) + \operatorname{div}(\rho\, \chi(w)\, u) = F\, \chi'(w)\, \rho.$$

Of course Lemma 6.8 applies to (6.16) in the exact same manner just replacing F by $f\, w$, provided that $f \in L^p$ and $w \in L^{p^*}$ with $1/p^* + 1/p = 1$ (so that $F \in L^1$) and $f\, w \in L^q_{t,x}$.

Lemma 6.8 can be used to justify most of our manipulations later on. Remark that all terms in the equation make sense in $\mathcal{D}'$: For instance, $u \cdot \nabla w = \operatorname{div}(u\, w) - w \operatorname{div} u$, which is well defined since u, $\operatorname{div} u$ and w belong to L^2. The proof of Lemma 6.8 is essentially found in [26] and consists simply in writing approximate equations on $L_\varepsilon \star \rho$, $L_\varepsilon \star w$, performing the required manipulation on those quantities, and then simply passing to the limit in ε.

As a straightforward consequence, we can easily obtain uniqueness for (6.14). Consider two solutions $\rho_1,\ \rho_2 \in L^2_{t,x}$ to (6.14) with same initial data. Apply the previous lemma to $\rho = \rho_1 - \rho_2$ and $\chi(\rho) = |\rho|$, and simply integrate the equation over $\mathbb{T}^d$ to find

$$\frac{d}{dt} \int_{\mathbb{T}^d} |\rho_1(t,x) - \rho_2(t,x)|\, dx = 0.$$

Thus,

LEMMA 6.9. *For a given $\rho^0 \in L^2_x$, there exists at most one solution $\rho \in L^2_{t,x}$ to* (6.14).

The uniqueness for the dual problem (6.15) or (6.16) is, however, more delicate and, in particular, the previous strategy cannot work unless $\operatorname{div} u \in L^\infty_x$. The estimates are now slightly different from (6.15) or (6.16), and we present them for (6.16) as we use this form more later on.

If one considers two solutions w_1 and w_2 to (6.16) and a solution ρ to (6.14), one has

$$\frac{d}{dt} \int_{\mathbb{T}^d} \rho(t,x)\, |w_1(t,x) - w_2(t,x)|\, dx = \int_{\mathbb{T}^d} f\, \rho(t,x)\, |w_1(t,x) - w_2(t,x)|\, dx,$$

leading to

LEMMA 6.10. *Assume that*

- *$\rho \in L^2_{t,x}$ solves* (6.14);

- *w_1 and w_2 are two solutions in $L^2_{t,x}$ to* (6.16) *with $w_1(t=0)=w_2(t=0)$ for a given $f\in L^p_{t,x}$ and $w_i\in L^{p^*}$, $i=1,\ 2$, with $1/p^*+1/p=1$;*
- *$\rho\in L^{p_1}$, $w\in L^{p_2}$, $u\in L^{p_3}$ with $1/p_1+1/p_2+1/p_3\le 1$ and $f\,w\in L^q_{t,x}$ with $1/p_1+1/q\le 1$;*
- *finally either $f\in L^\infty$ or $f\le 0$ and $\rho\ge 0$.*

Then $w_1=w_2$ ρ almost everywhere.

Of course if $\rho>0$ everywhere, then Lemma 6.10 provides the uniqueness of the solution to (6.15). But in general ρ could vanish on a set of non-zero measure. (This is the difficult vacuum problem for compressible Navier–Stokes.) In that case in general one cannot expect uniqueness for (6.15).

We will use the same strategy of integrating against a solution ρ to the conservative equation (6.14) to obtain some bounds on $\log w$ for w a solution to (6.16).

Lemma 6.11. *Assume that*

- *$\rho\ge 0$ in $L^2_{t,x}$ solves* (6.14);
- *w_1 is a solution in $L^2_{t,x}$ to* (6.16) *with $0\le w_1\le 1$, $w_1(t=0)=w^0$ for a given $f\in L^p_{t,x}$ and $w\in L^{p^*}$, $i=1,\ 2$, with $1/p^*+1/p=1$;*
- *$\rho\in L^{p_1}$, $w\in L^{p_2}$, $u\in L^{p_3}$ with $1/p_1+1/p_2+1/p_3\le 1$ and $f\,w\in L^q_{t,x}$ with $1/p_1+1/q\le 1$.*

Then one has for any $0\le\theta\le 1$,

$$\int_{\mathbb{T}^d}(1+|\log w(t,x)|)^\theta\,\rho(t,x)\,dx\le\int_{\mathbb{T}^d}|\log w^0|^\theta\,\rho(t,x)\,dx$$
$$+\theta\int_0^t\int_{\mathbb{T}^d}|f(s,x)|\,(1+|\log w(s,x)|)^{\theta-1}\,\rho(s,x)\,dx\,ds.$$

The lemma is proved simply by applying Lemma 6.8 (the last point) to a sequence $\chi_\varepsilon(w)=(1+|\log(\varepsilon+w)|)^\theta$, as for a fixed $\varepsilon>0$, χ_ε is Lipschitz. One then integrates in t and x and finally passes to the limit $\varepsilon\to 0$ by the monotone convergence theorem.

Note that the log transform allows one to derive (6.15) from (6.16) but requires in addition $\log w\in L^2$ while Lemma 6.11 does not require any a priori estimates on $\log w$.

Let us finish this subsection by briefly mentioning the existence question. This does not use renormalized solutions per se, although as we saw using the solutions once they are obtained requires the theory.

For uniqueness, the conservative form was well behaved and the advective form delicate. Hence for existence, things are reversed. Unless $\operatorname{div} u\in L^\infty$, it is not possible to have a general existence result for (6.14). In general, a solution to (6.14) with only $\operatorname{div} u\in L^2$ may concentrate, forming Dirac masses, for instance.

But it is quite simple to obtain a general existence result for (6.15)

LEMMA 6.12. *Assume that $w^0 \in L^\infty(\mathbb{T}^d)$ and either that $f \in L^\infty(\mathbb{T}^d)$ or that $f \leq 0,\ f \in L^1_{t,x}$ and $w^0 \geq 0$. Then there exists $w \in L^\infty([0,\ T] \times \mathbb{T}^d)$ for any $T > 0$ solution to* (6.16) *in the sense of distributions.*

Proof. Consider a sequence $u_n \in C^\infty$ such that u_n converges to u in $L^2_t H^1_x$. Define the solution w_n to

$$\partial_t w_n + u_n \cdot \nabla_x w_n = f\, w_n, \quad w_n(t=0) = w^0.$$

This solution w_n is easy to construct by using the characteristics flow based on u_n. Now if $f \in L^\infty$, then

$$\|w_n(t,.)\|_{L^\infty_x} \leq \|w^0\|_{L^\infty_x}\, e^{\,t\,\|f\|_{L^\infty_{t,x}}}.$$

In the other case, if $w^0 \geq 0$, then $w_n \geq 0$. Furthermore, if $f \leq 0$, then

$$\|w_n(t,.)\|_{L^\infty_x} \leq \|w^0\|_{L^\infty_x}.$$

So in both cases w_n is uniformly bounded in $L^\infty([0,\ T] \times \mathbb{T}^d)$ for any $T > 0$. Extracting a subsequence, still denoted by w_n for simplicity, w_n converges to w in the weak-* topology of $L^\infty([0,\ T] \times \mathbb{T}^d)$.

It only remains to pass to the limit in $u_n \cdot \nabla_x w_n = \operatorname{div}(u_n\, w_n) - w_n \operatorname{div} u_n$, which follows from the strong convergence in L^2 of u_n and $\operatorname{div} u_n$. Similarly one may pass to the limit in $f\, w_n$. □

7. Renormalized equation and weights

We explain here the various renormalizations of the transport equation satisfied by ρ_k. We then define the weights we will consider and give their properties.

7.1. *Renormalized equation.* We explain in this subsection how to obtain the equation satisfied by various quantities that we will need and of the form $Z_k^{x,y} \chi(\rho_k^x - \rho_k^y)$ where $Z_k^{x,y}$ is chosen as $Z_k^{x,y} = K_h(x-y) W_{k,h}^{x,y}$ with $W_{k,h}^{x,y} = W_{k,h}(t,x,y)$. The weights $W_{k,h}^{x,y}$ are assumed to satisfy

$$W_{k,h}^{x,y} \in L^\infty((0,T) \times \mathbb{T}^{2d}),$$
$$\partial_t W_{k,h}^{x,y} + u_k^x \cdot \nabla_x W_{k,h}^{x,y} + u_k^x \cdot \nabla_x W_{k,h}^{x,y} \in L^1((0,T) \times \mathbb{T}^{2d})$$

and

$$\alpha_k W_{k,h}^{x,y} \in L^1(0,T; H^2(\mathbb{T}^{2d})).$$

LEMMA 7.1. *Assume ρ_k solves* (5.1) *with* (5.4) *for $p > 2$ and that u_k satisfies* (5.5). *Then for any convex function $\chi \in W^{2,\infty}$,*

$$\begin{aligned}
&\Big[\int_{\mathbb{T}^{2d}} K_h(x-y) W_{k,h}^{x,y}\, \chi(\delta\rho_k)\, dx\, dy\Big]\Big|_{t=s} - \Big[\int_{\mathbb{T}^{2d}} K_h(x-y) W_{k,h}^{x,y}\, \chi(\delta\rho_k)\, dx\, dy\Big]\Big|_{t=0} \\
&\quad + \int_0^s \int_{\mathbb{T}^{2d}} \big(\chi'(\delta\rho_k)\delta\rho_k - \chi(\delta\rho_k)\big)\big(\operatorname{div}_x u_k^x + \operatorname{div}_y u_k^y\big) K_h(x-y) W_{k,h}^{x,y}\, dx\, dy\, dt \\
&\quad - \int_0^s \int_{\mathbb{T}^{2d}} \chi(\delta\rho_k)\big[u_k^x \cdot \nabla_x K_h(x-y) + u_k^y \cdot \nabla_y K_h(x-y) \\
&\qquad\qquad + \alpha_k(\Delta_x + \Delta_y) K_h(x-y)\big] W_{k,h}^{x,y}\, dx\, dy\, dt \\
&\quad - 2\alpha_k \int_0^s \int_{\mathbb{T}^{2d}} \nabla K_h(x-y)\, \chi(\delta\rho_k)\, \Big[\nabla_x W_{k,h}^{x,y} - \nabla_y W_{k,h}^{x,y}\Big]\, dx\, dy\, dt \\
&\quad - 2\alpha_k \int_0^s \int_{\mathbb{T}^{2d}} K_h(x-y)\, \chi(\delta\rho_k)\, \Big[\Delta_x W_{k,h}^{x,y} + \Delta_y W_{k,h}^{x,y}\Big]\, dx\, dy\, dt \\
&\leq \int_0^s \int_{\mathbb{T}^{2d}} \chi(\delta\rho_k)[\partial_t W_{k,h}^{x,y} + u_k^x \cdot \nabla_x W_{k,h}^{x,y} + u_k^y \cdot \nabla_y W_{k,h}^{x,y} \\
&\qquad\qquad - \alpha_k(\Delta_x + \Delta_y) W_{k,h}^{x,y}] K_h(x-y)\, dx\, dy\, dt \\
&\quad - \frac{1}{2} \int_0^s \int_{\mathbb{T}^{2d}} \chi'(\delta\rho_k) K_h(x-y) W_{k,h}^{x,y} \big(\operatorname{div}_x u_k^x - \operatorname{div}_y u_k^y\big) \bar\rho_k\, dx\, dy\, dt \\
&\quad + \frac{1}{2} \int_0^s \int_{\mathbb{T}^{2d}} \chi'(\delta\rho_k) K_h(x-y) W_{k,h}^{x,y} (\operatorname{div}_x u_k^x + \operatorname{div}_y u_k^y) \delta\rho_k\, dx\, dy\, dt.
\end{aligned}$$

Proof. The result essentially relies on a doubling of variable argument and straightforward algebraic calculations (up to Di Perna–Lions techniques). Since ρ_k solves (5.1), one has that

$$\partial_t \rho_k^x + \operatorname{div}_x(\rho_k^x u_k^x) = \alpha_k \Delta_x \rho_k^x, \qquad \partial_t \rho_k^y + \operatorname{div}_y(\rho_k^y u_k^y) = \alpha_k \Delta_y \rho_k^y.$$

Recalling $\delta\rho_k = \rho_k^x - \rho_k^y$, and using that $\rho_k \in L^p_{t,x}$ with $p > 2$ and hence $\rho_k \operatorname{div} u_k$ is well defined, one can check that

$$\partial_t \delta\rho_k + \operatorname{div}_x(u_k^x\, \delta\rho_k) + \operatorname{div}_y(u_k^y\, \delta\rho_k) = \alpha_k(\Delta_x + \Delta_y)\, \delta\rho_k - \rho_k^y \operatorname{div}_x u_k^x + \rho_k^x \operatorname{div}_y u_k^y.$$

Then, recalling the notation $\bar\rho_k = \rho_k^x + \rho_k^y$, we observe that

$$\begin{aligned}
-\rho_k^y \operatorname{div}_x u_k^x + \rho_k^x \operatorname{div}_y u_k^y &= \frac{1}{2}\Big(\operatorname{div}_x u_k^x\, \rho_k^x - \operatorname{div}_x u_k^x\, \rho_k^y + \operatorname{div}_y u_k^y\, \rho_k^x - \operatorname{div}_y u_k^y \rho_k^y \\
&\qquad - \operatorname{div}_x u_k^x\, \rho_k^x - \operatorname{div}_x u_k^x\, \rho_k^y + \operatorname{div}_y u_k^y\, \rho_k^x + \operatorname{div}_y u_k^y\, \rho_k^y\Big) \\
&= \frac{1}{2}(\operatorname{div}_x u_k^x + \operatorname{div}_y u_k^y)\, \delta\rho_k - \frac{1}{2}(\operatorname{div} u_k^x - \operatorname{div} u_k^y)\, \bar\rho_k.
\end{aligned}$$

Consequently, we can write

$$\begin{aligned}
\partial_t \delta\rho_k + \operatorname{div}_x(u_k^x\, \delta\rho_k) + \operatorname{div}_y(u_k^y\, \delta\rho_k) &= \alpha_k\, (\Delta_x + \Delta_y)\, \delta\rho_k \\
&\quad + \frac{1}{2}(\operatorname{div}_x u_k^x + \operatorname{div}_y u_k^y)\, \delta\rho_k - \frac{1}{2}(\operatorname{div} u_k^x - \operatorname{div} u_k^y)\, \bar\rho_k.
\end{aligned} \tag{7.1}$$

We now turn to the renormalized equation, i.e., the equation satisfied by $\chi(\delta_k)$ for a non-linear function $s \mapsto \chi(s)$. Formally the equation can be obtained by multiplying (7.1) by $\chi'(\delta\rho_k)$. If $\alpha_k = 0$ and ρ_k is not smooth, then the formal calculation can be justified following Di Perna–Lions techniques using regularizing by convolution and the estimate (5.5), i.e., $u_k \in L^2_t H^1_x$. Then

$$\begin{aligned}
&\partial_t \chi(\delta\rho_k) + \operatorname{div}_x(u_k^x \chi(\delta\rho_k)) + \operatorname{div}_y(u_k^y \chi(\delta\rho_k)) \\
&\qquad + \Big(\chi'(\delta\rho_k)\delta\rho_k - \chi(\delta\rho_k)\Big)\Big(\operatorname{div}_x u_k^x + \operatorname{div}_y u_k^y\Big) \\
&\quad = \alpha_k(\Delta_x + \Delta_y)\chi(\delta\rho_k) - \frac{1}{2}\chi'(\delta\rho_k)\Big(\operatorname{div}_x u_k^x - \operatorname{div}_y u_k^y\Big)\bar\rho_k \\
&\qquad + \frac{1}{2}\chi'(\delta\rho_k)(\operatorname{div}_x u_k^x + \operatorname{div}_y u_k^y)\delta\rho_k - \alpha_k \chi''(\rho_k)\Big(|\nabla_x \delta\rho_k|^2 + |\nabla_y \delta\rho_k|^2\Big).
\end{aligned}$$

For any V_k^x, V_k^y and smooth enough $Z_{k,h}^{x,y}$, in the sense of distributions one has

$$\begin{aligned}
Z_{k,h}^{x,y}\Delta_x V_k^x &= \operatorname{div}_x(Z_{k,h}^{x,y}\nabla_x V_k^x) - \operatorname{div}_x(V_k^x \nabla_x Z_k^{x,y}) + V_k^x \Delta_x Z_{k,h}^{x,y}, \\
Z_{k,h}^{x,y}\Delta_y V_k^y &= \operatorname{div}_y(Z_{k,h}^{x,y}\nabla_y V_k^y) - \operatorname{div}_y(V_k^y \nabla_y Z_{k,h}^{x,y}) + V_k^y \Delta_y Z_{k,h}^{x,y}.
\end{aligned}$$

Consequently, we get the following equation for $Z_{k,h}^{x,y}\chi(\delta\rho_k)$:

$$\begin{aligned}
&\partial_t[Z_{k,h}^{x,y}\chi(\delta\rho_k)] + \operatorname{div}_x(u_k^x \chi(\delta\rho_k) Z_{k,h}^{x,y}) + \operatorname{div}_y(u_k^y \chi(\delta\rho_k) Z_{k,h}^{x,y}) \\
&\quad + \Big(\chi'(\delta\rho_k)\delta\rho_k - \chi(\delta\rho_k)\Big)\Big(\operatorname{div}_x u_k^x + \operatorname{div}_y u_k^y\Big) Z_{k,h}^{x,y} \\
&\quad - \chi(\delta\rho_k)[\partial_t Z_{k,h}^{x,y} + u_k^x \cdot \nabla_x Z_{k,h}^{x,y} + u_k^y \cdot \nabla_y Z_{k,h}^{x,y} - \alpha_k(\Delta_x + \Delta_y) Z_{k,h}^{x,y}] = \text{r.h.s},
\end{aligned}$$

with

$$\begin{aligned}
\text{r.h.s.} = &-\frac{1}{2}\chi'(\delta\rho_k) Z_{k,h}^{x,y}\Big(\operatorname{div}_x u_k^x - \operatorname{div}_y u_k^y\Big)\bar\rho_k \\
&+ \frac{1}{2}\chi'(\delta\rho_k) Z_{k,h}^{x,y}(\operatorname{div}_x u_k^x + \operatorname{div}_y u_k^y)\delta\rho_k \\
&- \alpha_k \chi''(\rho_k) Z_{k,h}^{x,y}\Big(|\nabla_x \delta\rho_k|^2 + |\nabla_y \delta\rho_k|^2\Big) + 2\alpha_k \chi(\delta\rho_k)(\Delta_x + \Delta_y) Z_{k,h}^{x,y} \\
&+ \alpha_k\Big[\operatorname{div}_x(Z_{k,h}^{x,y}\nabla_x \chi(\delta\rho_k)) - \operatorname{div}_x(\chi(\delta\rho_k)\nabla_x Z_{k,h}^{x,y}) \\
&\qquad\qquad + \operatorname{div}_y(Z_{k,h}^{x,y}\nabla_y \chi(\delta\rho_k)) - \operatorname{div}_y(\chi(\delta\rho_k)\nabla_y Z_{k,h}^{x,y})\Big].
\end{aligned}$$

Integrating in time and twice in space (double variable (x,y)) and performing the required integration by parts, we get the desired equality writing $Z_{k,h}^{x,y} = K_h(x-y) W_{k,h}^{x,y}$. □

7.2. *The weights*: *Choice and properties*. In this subsection, we choose the PDEs satisfied by the weights. We state and then prove some of their properties.

7.2.1. *Basic considerations*. We define weights w_k, periodic in space, that satisfy

$$\partial_t w_k + u_k \cdot \nabla w_k = -D_k\, w_k + \alpha_k \Delta w_k, \qquad w_k|_{t=0} = w^0 \tag{7.2}$$

for some appropriate penalization D_k depending on the case under consideration: $D_{0,k}, D_{1,k}, D_{a,k}$. The choice of D_k will be based on the need to control "bad" terms when looking at the propagation of the weighted quantity. The choice will also have to ensure that the weights are not too small, too often.

7.2.2. *Isotropic viscosity, general pressure laws. The case with* $\alpha_k > 0$ *and monotone pressure.* The simplest choice for the penalization D to define w_0 is

$$D_{0,k} = \lambda\, M\, |\nabla u_k|, \tag{7.3}$$

with λ a fixed constant (chosen later on) and M the localized maximal operator as defined by (6.2). In that case we choose accordingly

$$w_{0,k}|_{t=0} = w_0^0 \equiv 1.$$

The case $\alpha_k = 0$ *and non-monotone pressure.* In the absence of diffusion in (5.1) ($\alpha_k = 0$) and when the pressure term P_k is non-monotone, for instance, one needs to add a term $\rho_k^{\tilde\gamma}$ in the penalization. This would lead to very strong assumptions, in particular, on the exponent p in (5.4) (and hence γ) as explained after Proposition 4.4. It is possible to obtain better results using that $\rho \in L^p$ for some $p > 2$, by taking the more refined

$$\frac{D_{1,k}}{\lambda} = \rho_k\, |\mathrm{div}\, u_k| + |\mathrm{div}\, u_k| + M\, |\nabla u_k| + \rho_k^{\tilde\gamma} + \tilde P_k\, \rho_k + R_k \tag{7.4}$$

for the general compactness result. For simplicity we take

$$w_{1,k}|_{t=0} = w_1^0 \equiv \exp\left(-\lambda \sup \rho_k^0\right).$$

The reason for the first term in $D_{1,k}$ compared to $D_{0,k}$ is to ensure that $w_{1,k} \le e^{-\lambda \rho_k}$, which helps compensates the penalization in $\rho_k^{\tilde\gamma}$ to get the property on $\rho_k |\log w_1|^\theta$ for some $\theta > 0$. The three last terms are needed to respectively counterbalance: additional divergence terms in the propagation quantity compared to w_0, the same $M\,|\nabla u_k|$ as for w_0, and the $\rho_k^{\tilde\gamma}$ for terms coming from the pressure.

7.2.3. *Anisotropic stress tensor.* The choice for the penalization, denoted D_a in this case and leading to the weight w_a, is now

$$\frac{D_{a,k}}{\lambda} = M|\nabla u_k| + \overline{K_h} \star (|\mathrm{div} u_k| + |A_\mu \rho_k^\gamma|). \tag{7.5}$$

Note that the second term in the penalization is used to control the non-local part of the pressure terms. As initial condition, we choose accordingly

$$w_{a,k}|_{t=0} = w_a^0 \equiv 1.$$

7.2.4. *The forms of the weights.* Recall that we use the convenient notation $w_k^x = w_k(t,x)$ and $w_k^y = w_k(t,y)$ when we compare expressions at points (t,x) and (t,y). Two types of weights $W^{x,y}$ are used:

$$W^{x,y} = w^x + w^y, \quad \text{or} \quad W^{x,y} = w^x\, w^y.$$

The first one will provide compactness and will be used with (7.3) or (7.4). The second, used with (7.4), gives better explicit regularity estimates but far from the vacuum and is considered for the sake of completeness. Therefore one defines

$$\begin{aligned} W_{0,k}^{x,y} &= w_{0,k}^x + w_{0,k}^y, \quad W_{1,k}^{x,y} = w_{1,k}^x + w_{1,k}^y,\\ W_{2,k}^{x,y} &= w_{1,k}^x w_{1,k}^y, \quad W_{a,k}^{x,y} = w_{a,k}^x + w_{a,k}^y. \end{aligned} \tag{7.6}$$

As for the penalization, we use the notation $W_k^{x,y}$ when the particular choice is not relevant and $W_{i,k}^{x,y}$, $i = 0$, 1, 2 or a otherwise. For all choices, one has

$$\partial_t W_{i,k}^{x,y} + u_k^x \cdot \nabla_x W_{i,k}^{x,y} + u_k^y \cdot \nabla_y W_{i,k}^{x,y} = -Q_{i,k}^{x,y} + \alpha_k \Delta_{x,y} W_{i,k}^{x,y}. \tag{7.7}$$

The term Q_k depends on the choices of penalizations and weights with the four possibilities

$$\begin{aligned} Q_{0,k}^{x,y} &= D_{0,k}^x\, w_{0,k}^x + D_{0,k}^y\, w_{0,k}^y, \qquad & Q_{1,k}^{x,y} &= D_{1,k}^x\, w_{1,k}^x + D_{1,k}^y\, w_{1,k}^y,\\ Q_{2,k}^{x,y} &= (D_{1,k}^x + D_{1,k}^y)\, w_{1,k}^x\, w_{1,k}^y, \qquad & Q_{a,k}^{x,y} &= D_{a,k}^x\, w_{a,k}^x + D_{a,k}^y\, w_{a,k}^y. \end{aligned}$$

7.2.5. *The weight properties.* We summarize the main estimates on the weights previously defined.

Proposition 7.2. *Assume that ρ_k solves* (5.1) *with the bounds* (5.5) *on u_k and* (5.4) *with $p > \max(2,\,\tilde\gamma)$. Assume that $\tilde P_k$, R_k are given by* (5.9) *and, in particular, $\tilde P_k$ is uniformly bounded in $L^2_{t,x}$ and R_k in $L^1_{t,x}$. Then there exist weights w_0, w_1, and w_a that satisfy equation* (7.2) *with initial data respectively,*

$$w_{0,k}|_{t=0} = 1, \quad w_{1,k}|_{t=0} = \exp(-\lambda \sup \rho_k^0), \quad w_{a,k}|_{t=0} = 1,$$

and $D_{0,k}, D_{1,k}, D_{a,k}$ respectively given by (7.3), (7.4) *and* (7.5) *such that*

(i) *For any t,x,*

$$0 \le w_{0,k}(t,x) \le 1, \quad 0 \le w_{a,k}(t,x) \le 1, \quad 0 \le w_{1,k}(t,x) \le e^{-\lambda\, \rho_k(t,x)}. \tag{7.8}$$

(ii) *One has*

$$\sup_{t\in[0,T]} \int_{\mathbb{T}^d} \rho_k(t,x)\, |\log w_{0,k}(t,x)|\, dx \le C\,(1+\lambda).$$

If $\alpha_k = 0$ and $p > \max(2,\,\tilde\gamma)$, then similarly there exists $\theta > 0$ such that

$$\sup_{t\in[0,T]} \int_{\mathbb{T}^d} \rho_k(t,x)\, |\log w_{1,k}(t,x)|^\theta\, dx \le C_\lambda,$$

while finally if $p \geq \gamma + 1$, *then*

$$\sup_{t\in[0,T]} \int_{\mathbb{T}^d} \rho_k(t,x)\,|\log w_{a,k}(t,x)|\,dx \leq C(1+\lambda). \tag{7.9}$$

(iii) *For any* η, *we have the two estimates*

$$\sup_{t\in[0,T]} \int_{\mathbb{T}^d} \rho_k(t,x)\,\mathbb{I}_{(\overline{K}_h \star w_{0,k})(t,x) \leq \eta}\,dx\big] \leq C\,\frac{1+\lambda}{|\log \eta|}$$

and, if $p \geq \gamma + 1$,

$$\sup_{t\in[0,T]} \int_{\mathbb{T}^d} \rho_k(t,x)\,\mathbb{I}_{(\overline{K}_h \star w_{a,k})(t,x) \leq \eta}\,dx \leq C\,\frac{1+\lambda}{|\log \eta|}.$$

(iv) *Denoting* $w_{a,k,h} = \overline{K}_h \star w_{a,k}$, *if* $p > \gamma$, *for some* $0 < \theta < 1$ *we have*

$$\int_{h_0}^1 \int_0^t \Big\| \overline{K}_h \star \big(\overline{K}_h \star (|\operatorname{div} u_k| + |A_\mu \rho_k^\gamma|) w_{a,k}\big) - \big(\overline{K}_h \star (|\operatorname{div} u_k| + |A_\mu \rho_k^\gamma|)\big)\, w_{a,k,h} \Big\|_{L^q} dt\,\frac{dh}{h} \leq C\,|\log h_0|^\theta,$$

with $q = \min(2, p/\gamma)$.

Remark 7.3. Part (i) tells us that $w_{i,k}$ is small at the right points (in particular, when ρ_k is large). On the other hand, we want w_i to be small only on a set of small mass; otherwise, one obviously does not control much. This is the role of part (ii). We use part (iv) to regularize weights in the anisotropic case. Part (iii) is also used to get a control under the form given in 4.1 from the estimates with weights.

Remark 7.4. Even when $\alpha_k > 0$, it would be possible to define D_1 in order to have a bound like $w_{1,k}(t,x) \leq e^{-\lambda \rho_k(t,x)^{q-1}}$. For instance, take

$$\frac{D_1}{\lambda} = \rho_k^{q-1}\,(q-1)\,|\operatorname{div} u_k| + \frac{\alpha_{k,\lambda}}{\lambda}\,|\nabla \log w_{1,k}|^2 + |\operatorname{div} u_k| + M\,|\nabla u_k| + L\,\rho_k^\gamma + \tilde{P}_k\,\rho_k + R_k,$$

with

$$\alpha_{k,\lambda} = \alpha_k \left(1 - \frac{\bar{\alpha}_\lambda}{1 + \lambda\,\rho_k^{q-1}}\right), \qquad \bar{\alpha}_\lambda = \frac{q-2}{(q-1)}.$$

However one needs $q \leq p/2$ and $q > 2$, which already forces $p > 4$. Moreover the main difficulty when $\alpha_k > 0$ comes from the proof of Lemma 8.1 which forces us to work with $\overline{K}_h \star w_{0,k}$ and not $w_{0,k}$. Because of that, any pointwise inequality between $w_k(x)$ and $\rho_k(x)$ is mostly useless.

Proof. Point (i). This point focuses on the construction of the weights satisfying the bounds (7.8).

Construction of $w_{0,k}$. The construction of the weights w_0 is classical since it satisfies a parabolic equation; moreover, since $D_{0,k}$ is positive, then one has $0 \le w_{0,k} \le 1$.

Construction of $w_{a,k}$. Choosing $w_a|_{t=0} = 1$ and $f = -D_a \le 0$ and noticing that $D_a \in L^2$ and we hence easily construct (see Lemmas 6.12, 6.8 and 6.9) $w_{a,k}$ such that $0 \le w_{a,k} \le 1$ solution of

$$\partial_t w_{a,k} + u_k \cdot \nabla w_{a,k} + D_{a,k}\, w_{a,k} = 0, \qquad w_a|_{t=0} = 1.$$

Note that $D_a \ge |\mathrm{div} u_k|$ because $M\,|\nabla u_k| \ge |\nabla u_k| \ge |\mathrm{div} u_k|$ and $w_{a,k}$ also solves

$$\partial_t w_{a,k} + \mathrm{div}(u_k w_{a,k}) + (D_{a,k} - \mathrm{div} u_k)\, w_{a,k} = 0,$$

so that by the maximum principle, we also have $w_{a,k} \le \rho_k$ where $\rho_k > 0$. This means that we can actually uniquely define $w_{a,k}$ by imposing $w_{a,k} = 0$ if $\rho_k = 0$.

Construction of $w_{1,k}$. Choosing $w_{1,k}|_{t=0} = \exp(-\lambda \sup \rho_k^0)$ and $f = -D_1 \le 0$ and noticing that $D_{1,k} \in L^1$ and $D_{1,k} \ge |\mathrm{div} u_k|$, we again construct $w_{1,k}$ just as $w_{a,k}$ such that $0 \le w_{1,k} \le 1$ with $w_{1,k}(t,x) = 0$, where $\rho_k(t,x) = 0$ and $w_{1,k}$ is a solution to

$$\partial_t w_{1,k} + \mathrm{div}(u_k w_{1,k}) + (D_{1,k} - \mathrm{div} u_k)\, w_{1,k} = 0, \qquad w_{1,k}|_{t=0} = \exp(-\lambda \sup \rho_k^0).$$

Note that using the renormalization technique on the mass equation,

$$\partial_t[\exp(-\lambda\rho_k] + \mathrm{div}\Big(u_k[\exp(-\lambda\rho_k)]\Big) + [-\lambda\rho_k \mathrm{div} u_k - \mathrm{div} u_k]\exp(-\lambda\rho_k) = 0.$$

Subtracting the two equations we get the following on $g_k = w_{1,k} - \exp(-\lambda\rho_k)$:

$$\partial_t g_k + \mathrm{div}(u_k g_k) + (D_{1,k} - \mathrm{div} u_k) g_k = -(D_{1,k} + \lambda\rho_k \mathrm{div} u_k)\exp(-\lambda\rho_k)$$

Recall now that $D_{1,k} \ge |\mathrm{div} u_k|$ and $D_{1,k} \ge -\lambda\rho_k \mathrm{div} u_k$; thus using the maximum principle, we get

$$w_{1,k} \le e^{-\lambda\rho_k},$$

recalling that we have $w_{1,k} = 0$ where $\rho_k = 0$.

Point (ii). Let $i = 0, 1$ or a. By point (i), $w_{i,k} \le 1$, hence $|\log w_{i,k}| = -\log w_{i,k}$, and from (7.2), denoting $|\log w_{i,k}| = A_{i,k}$,

$$\begin{aligned}\partial_t(A_{i,k}) + u_k \cdot \nabla_x(A_{i,k}) - D_{i,k} &= -\frac{\alpha_k}{w_{i,k}}\Delta\, w_{i,k} \\ &= \alpha_k \Delta A_{i,k} - \alpha_k\,|\nabla A_{i,k}|^2.\end{aligned} \tag{7.10}$$

For $A_{0,k}$, we directly apply

$$\begin{aligned}\partial_t(\rho_k\, A_{0,k}) + \mathrm{div}(\rho_k\, A_{0,k} u_k) = D_{0,k}\,\rho_k + \alpha_k\Delta(\rho_k\, A_{0,k}) - 2\alpha_k\,\nabla\rho_k \cdot \nabla A_{0,k} \\ - \alpha_k \rho_k |\nabla A_{0,k}|^2\end{aligned}$$

and integrate to find

$$\Big(\int_{\mathbb{T}^d}\rho_k\,A_{0,k}\,dx\Big)|_{t=s} = C+\int_0^s\int_{\mathbb{T}^d}D_{0,k}(t,x)\,\rho_k\,dx\,dt - \alpha_k\int_0^s\int_{\mathbb{T}^d}\rho_k|\nabla A_{0,k}|^2\,dx\,dt - 2\alpha_k\int_0^s\int_{\mathbb{T}^d}\nabla\rho_k\cdot\nabla A_{0,k}\,dx\,dt.$$

Simply bound

$$\int_0^s\int_{\mathbb{T}^d}\nabla\rho_k\;\cdot\nabla A_{0,k}\,dx\,dt\le\frac{1}{4}\int_{\mathbb{T}^d}\rho_k|\nabla A_{0,k}|^2+\int_{\mathbb{T}^d}\frac{|\nabla\rho_k|^2}{\rho_k}.$$

On the other hand, using renormalization techniques,

$$\frac{d}{dt}\int_{\mathbb{T}^d}\rho_k\,\log\rho_k\,dx=-\int_{\mathbb{T}^d}\rho_k\,\mathrm{div}\,u_k\,dx-\alpha_k\int_{\mathbb{T}^d}\frac{|\nabla\rho_k|^2}{\rho_k}\,dx.$$

However as $p\ge 2$, then $\rho_k\,\mathrm{div}\,u_k$ is bounded uniformly in $L^1_{t,x}$ and $\rho_k\,\log\rho_k$ in $L^\infty_t(L^1_x)$. This implies, from the previous equality, that

$$\alpha_k\int_0^s\int_{\mathbb{T}^d}\frac{|\nabla\rho_k|^2}{\rho_k}\,dx\,dt\le C,$$

and consequently

$$-2\alpha_k\int_0^s\int_{\mathbb{T}^d}\nabla\rho_k\;\cdot\nabla A_{i,k}\,dx\,dt\le\frac{\alpha_k}{2}\int_0^s\int_{\mathbb{T}^d}\rho_k|\nabla A_{i,k}|^2\,dx\,dt+C.$$

Using this in the equality on $\displaystyle\int_{\mathbb{T}^d}\rho_k\,A_{0,k}$ given previously, we get

$$\Big(\int_{\mathbb{T}^d}\rho_k\,A_{0,k}\,dx\Big)(s)\le C+\int_0^s\int D_{0,k}(t,x)\,\rho_k\,dx\,dt-\frac{\alpha_k}{2}\int_0^s\int_{\mathbb{T}^d}\rho_k|\nabla A_{0,k}|^2\,dx\,dt.$$

In that case, we know that $\|D_{0,k}\|_{L^2}\le C\,\lambda$, and since $p\ge 2$, we get

$$\Big(\int_{\mathbb{T}^d}\rho_k\,|\log w_{0,k}|\,dx\Big)(s)+\alpha_k\int_0^s\int_{\mathbb{T}^d}\rho_k\,|\nabla A_{0,k}|^2\,dx\,ds\le C(1+\lambda). \tag{7.11}$$

Concerning $w_{a,k}$, to get

$$\Big(\int_{\mathbb{T}^d}\rho_k\,|\log w_{a,k}|\,dx\Big)(s)\le C(1+\lambda),$$

the estimate is similar assuming $p \geq \gamma+1$, and even simpler as $\alpha_k = 0$. Indeed $M|\nabla u_k|$ and $\overline{K}_h \star |\mathrm{div}\, u_k|$ are bounded in L^2 by (5.5). Finally,

$$\begin{aligned}\int_0^T\int_{\mathbb{T}^d} \rho_k\, \overline{K}_h \star (|A_\mu\, \rho_k^\gamma|)\, dx\, dt &\leq \left(\int_0^T\int_{\mathbb{T}^d} \rho_k^{\gamma+1}\, dx\, dt\right)^{1/(\gamma+1)}\\ &\quad \cdot \left(\int_0^T\int_{\mathbb{T}^d} |A_\mu\, \rho_k^\gamma|^{(\gamma+1)/\gamma}\, dx\, dt\right)^{\gamma/(\gamma+1)}\\ &\leq C\int_0^T\int_{\mathbb{T}^d} \rho_k^{\gamma+1}\, dx\, dt,\end{aligned}$$

since A_μ is continuous on any L^q space for $1 < q < +\infty$. The right-hand side is bounded assuming $p > \gamma+1$.

For $w_{1,k}$, as before the estimate is a bit different. We now assume that $\alpha_k = 0$, define $\tilde{A}_{1,k} = (1 + A_{1,k})^\theta$ and from (7.10), obtain

$$\partial_t \tilde{A}_{1,k} + u_k \cdot \nabla \tilde{A}_{1,k} = \theta\, \frac{D_{1,k}}{(1 + A_{1,k})^{1-\theta}}.$$

Integrating and recalling $A_{1,k} \geq \lambda\, \rho_k$, $M\,|\nabla u_k|$ and $\tilde{P}_k$ by (5.9) are uniformly bounded in L^2 and R_k in L^1:

$$\begin{aligned}\left(\int_{\mathbb{T}^d} \rho_k\, \tilde{A}_{1,k}\, dx\right)(s) &\leq C + C\int_0^s\int_{\mathbb{T}^d} \frac{1+\rho_k^2}{1+\rho_k^{(1-\theta)}}\, |\mathrm{div}\, u_k|\, dx\, dt\\ &\quad + C\int_0^s\int_{\mathbb{T}^d} \frac{\rho_k}{1+\rho_k^{(1-\theta)}}\, (M\,|\nabla u_k| + \tilde{P}_k\, \rho_k + \Theta_k)\, dx\, dt\\ &\quad + C\int_0^s\int_{\mathbb{T}^d} \frac{\rho_k^{\tilde{\gamma}+1}}{1+\rho_k^{(1-\theta)}}\, dx\, dt \leq C\end{aligned}$$

for some $\theta > 0$ depending on $p - \max(2,\ \tilde{\gamma})$. This gives the desired control regarding $\rho|\log w_1|^\theta$ for an exponent θ small enough.

Point (iii). Estimate (7.11) will not be enough in the proof, and we will need to control the mass of ρ_k where $\overline{K}_h \star w_{0,k}$ is small. Denote

$$\Omega_{h,\eta} = \{x \in \mathbb{T}^d,\ \overline{K}_h \star w_{0,k}(t,x) \leq \eta\}, \quad \tilde{\Omega}_{h,\eta} = \{x \in \Omega_{h,\eta},\ w_{0,k}(t,x) \geq \sqrt{\eta}\}.$$

The time t is fixed during this argument, and for simplicity we omit it. One cannot easily estimate $|\Omega_{h,\eta}|$ directly, but it is straightforward to bound $|\tilde{\Omega}_{h,\eta}|$. Assume $x \in \Omega_{h,\eta}$ i.e., $\overline{K}_h \star w_{0,k}(x) \leq \eta$. From the expression of K_h, if $|\delta| \leq h$

$$\overline{K}_h(z+\delta) \leq \frac{\|K_h\|_{L^1}^{-1}}{(h + |z+\delta|)^a} \leq C\, \overline{K}_h(z),$$

then we deduce that for any $y \in B(x,h)$,

$$\overline{K}_h \star w_{0,k}^y \leq C\, \eta.$$

Now cover $\tilde\Omega_{\eta,h}$ by $\bigcup_i C_i^h$ with C_i^h disjoint hyper-cubes of diameter h/C. For any i, denote $\tilde\Omega^i_{\eta,h}=\tilde\Omega_{\eta,h}\cap C_i^h$. If $\tilde\Omega^i_{\eta,h}\neq\emptyset$, then $\overline{K}_h\star w_{0,k}(x)\le C\,\eta$ on the whole C_i^h. In that case,

$$\begin{aligned}C\eta\,h^d &\ge \int_{C_i^h}\overline{K}_h\star w_{0,k}\,dx\ge\int_{C_i^h}\int_{\tilde\Omega^i_{\eta,h}}\overline{K}_h(x-y)\,w_{0,k}(y)\,dy\,dx\\ &\ge\frac{\sqrt{\eta}}{C}\,|\tilde\Omega^i_{\eta,h}|.\end{aligned}$$

We conclude that $|\tilde\Omega^i_{\eta,h}|\le C\,\sqrt{\eta}\,h^d$. Summing over the cubes, we deduce that one has

$$|\tilde\Omega_{\eta,h}|\le C\,\sqrt{\eta}.$$

Finally,

$$\int_{\Omega_{\eta,h}}\rho_k\,dx\le\int_{\tilde\Omega_{\eta,h}}\rho_k\,dx+\frac{2}{|\log\eta|}\int_{\mathbb{T}^d}\rho_k\,|\log w_{0,k}|\,dx\le C\,\eta^{1/2-1/2\gamma}+\frac{C}{|\log\eta|},$$

since $\rho_k\in L^\infty_t(L^\gamma_x)$ for some $\gamma>1$. This is the desired bound.

The same bound may be obtained on the quantity $\rho_k\mathbb{I}_{\overline{K}_h\star w_{a,k}\le\eta}$ in a similar way when $p>\gamma+1$ because of bound (7.9) on $\rho_k|\log w_{a,k}|$.

Point (iv). To simplify, denote $f=|\operatorname{div}u_k|+|A_\mu\,\rho_k^\gamma|$. Then by the definition of $w_{a,k,h}$,

$$\begin{aligned}&\int_{h_0}^1\int_0^s\Big\|\overline{K}_h\star\big(\overline{K}_h\star f\,w_{a,k}\big)-\big(\overline{K}_h\star f\big)\,\omega_{a,k,h}\Big\|_{L^q(\mathbb{T}^{2d})}\,dt\,\frac{dh}{h}\\ &\quad\le\int_{h_0}^1\int_0^s\int_{\mathbb{T}^d}\overline{K}_h(z)\Big\|\Big((\overline{K}_h\star f)^{\cdot+z}-(\overline{K}_h\star f)^{\cdot}\Big)\,w_{a,k}^{\cdot+z}\Big\|_{L^q(\mathbb{T}^d)}\,dt\,dz\,\frac{dh}{h}\\ &\quad\le\int_0^s\int_{h_0}^1\int_{\mathbb{T}^d}\overline{K}_h(z)\Big\|\Big((\overline{K}_h\star f)^{\cdot+z}-(\overline{K}_h\star f)^{\cdot}\Big)\Big\|_{L^q(\mathbb{T}^d)}\,dz\,\frac{dh}{h}\,dt\\ &\quad\le C\,|\log h_0|^{1/2}\int_0^s\|f(t,.)\|_{L^q(\mathbb{T}^d)}\,dt\le C\,|\log h_0|^{1/2}\end{aligned}$$

by a direct application of Lemma 6.4 with $N_h=\bar K_h$ provided f is uniformly bounded in $L^1_tL^q_x$, which is guaranteed by $q\le\min(2,\ p/\gamma)$. □

8. Proof of Theorems 5.1, 5.2 and 5.3

We start with the propagation of regularity on the transport equation in terms of the regularity of $\operatorname{div}u_k$; more precisely, $\operatorname{div}_x u_k^x-\operatorname{div}_y u_k^y$. We prove in the second subsection some estimates on the effective pressure. This allows us to write a lemma in the third subsection controlling $\operatorname{div}_x u_k^x-\operatorname{div}_y u_k^y$ and then to close the loop in the fourth subsection, thus concluding the proof. In Sections 8.5 and 8.6 we consider the anisotropic viscous case.

8.1. *The propagation of regularity on the transport equation.* This subsection uses only equation (5.1) on ρ_k without yet specifying the coupling between $\operatorname{div} u_k$ and ρ_k (for instance through (5.2)).

Recall that we denote

$$\delta\rho_k = \rho_k^x - \rho_k^y, \qquad \bar\rho_k = \rho_k^x + \rho_k^y.$$

Choose any C^2 convex function χ such that

$$\left|\chi(\xi) - \frac{1}{2}\chi'(\xi)\,\xi\right| \le \frac{1}{2}\chi'(\xi)\,\xi, \quad \chi'(\xi)\,\xi \le C\,\chi(\xi) \le C\,|\xi|. \tag{8.1}$$

It is, for instance, possible to take $\chi(\xi) = \xi^2$ for $|\xi| \le 1/2$ and $\chi(\xi) = |\xi|$ for $|\xi| \ge 1$.

Similarly for the anisotropic viscous term, for some $\ell > 0$, choose any convex $\chi_a \in C^1$,

$$\begin{aligned} &\left|\chi_a(\xi) - \frac{1}{2}\chi_a'(\xi)\,\xi\right| \le \frac{1-\ell}{2}\chi_a'(\xi)\,\xi,\\ &\chi_a'(\xi)\,\xi \le C\,\chi_a(\xi) \le C\,|\xi|^{1+\ell},\\ &(\xi^\gamma + \tilde\xi^\gamma)\big(-\chi_a'(\xi-\tilde\xi)(\xi-\tilde\xi) + 2\chi_a(\xi-\tilde\xi)\big)\\ &\qquad\qquad \ge -(\xi^\gamma - \tilde\xi^\gamma)\,\frac{\ell-1}{\ell}\chi_a'(\xi-\tilde\xi)(\xi+\tilde\xi). \end{aligned} \tag{8.2}$$

Note that it is possible to simply choose $\chi_a = |\xi|^{1+\ell}$. But to unify the notations and the calculations with the other terms involving χ, we use the abstract χ_a.

The properties on these non-linear functions χ and χ_a will be strongly used to characterize the effect of the pressure law in the contribution of $\operatorname{div}_x u_k(x) - \operatorname{div}_y u_k(y)$. They will play the role of renormalized functions on the difference $|\rho_k^x - \rho_k^y|$.

The form of χ, χ_a and the choice of ℓ will have to be determined very precisely so that the corresponding bad terms will be exactly counterbalanced by the λ terms coming from the penalization: We refer, for instance, in the anisotropic case, to the λ terms appearing in Lemma 8.2.

We write two distinct lemmas concerning respectively the non-monotone pressure law case and the anisotropic tensor case; even though the continuity equation is the same in both cases, we do not use the same weights as we will need different properties for them later in the proof.

Lemma 8.1. *Assume that ρ_k solves* (5.1) *with estimates* (5.4) *and* (5.5) *on u_k.*

(i) *With diffusion, $\alpha_k > 0$, if $p > 2$, there exists $\varepsilon_{h_0}(k) \to 0$ as $k \to \infty$ for a fixed h_0:*

$$\Big[\int_{h_0}^1 \int_{\mathbb{T}^{4d}} \overline{K}_h(x-z)\,\overline{K}_h(y-w)\,W_{0,k}^{z,w}\,K_h(x-y)\,\chi(\delta\rho_k)\,dx\,dy\,dz\,dw\,\frac{dh}{h}\Big]\Big|_{t=s}$$
$$\le C\,(\varepsilon_{h_0}(k) + |\log h_0|^{1/2})$$
$$-\frac{1}{2}\int_{h_0}^1\int_0^s\int_{\mathbb{T}^{4d}} K_h(x-y)\,(\mathrm{div}\,_x u_k^x - \mathrm{div}\,_y u_k^y)\,\chi'(\delta\rho_k)\,\bar\rho_k$$
$$\cdot\, W_{0,k}^{z,w}\,\overline{K}_h(x-z)\,\overline{K}_h(y-w)\,dx\,dy\,dz\,dw\,dt\,\frac{dh}{h}$$
$$-\frac{1}{2}\int_{h_0}^1\int_0^s\int_{\mathbb{T}^{4d}} K_h(x-y)\,(\mathrm{div}\,_x u_k^x + \mathrm{div}\,_y u_k^y)$$
$$\cdot\,(\chi'(\delta\rho_k)\,\delta\rho_k - 2\,\chi(\delta\rho_k))\;W_{0,k}^{z,w}\,\overline{K}_h(x-z)\,\overline{K}_h(y-w)\,dx\,dy\,dz\,dw\,dt\,\frac{dh}{h}$$
$$-\frac{\lambda}{2}\int_{h_0}^1\int_0^s\int_{\mathbb{T}^{3d}} K_h(x-y)\chi(\delta\rho_k)\,\overline{K}_h(x-z)\,M|\nabla u_k|^z\,w_{0,k}^z\,dx\,dy\,dz\,dt\,\frac{dh}{h},$$

where we recall that $W_{0,k}^{z,w} = w_{0,k}^z + w_{0,k}^w$.

(ii) *Without diffusion, $\alpha_k = 0$, if $p \ge 2$, then*

$$\Big[\int_{h_0}^1 \int_{\mathbb{T}^{2d}} \overline{K}_h(x-y)\,\chi(\delta\rho_k)\,(w_{1,k}^x + w_{1,k}^y)\,dx\,dy\,\frac{dh}{h}\Big]\Big|_{t=s}$$
$$\le C\,|\log h_0|^{1/2}\int_0^s \|u_k(t,\cdot)\|_{H^1}\,dt$$
$$-2\,\lambda\int_{h_0}^1\int_0^s\int_{\mathbb{T}^{2d}} \overline{K}_h(x-y)\,((\rho_k^x)^{\tilde\gamma} + \tilde P_k^x\,\rho_k^x + \Theta_k^x)\,w_{1,k}^x\,\chi(\delta\rho_k)\,dx\,dy\,dt\,\frac{dh}{h}$$
$$-2\int_{h_0}^1\int_0^s\int_{\mathbb{T}^{2d}} \overline{K}_h(x-y)\,(\mathrm{div}\,u_k^x - \mathrm{div}\,u_k^y)\,\Big(\frac{1}{2}\chi'(\delta\,\rho_k)\,\bar\rho_k$$
$$+\,\chi(\delta\rho_k) - \frac{1}{2}\chi'(\delta\,\rho_k)\,\delta\rho_k\Big)\,w_{1,k}^x\,dx\,dy\,dt\,\frac{dh}{h}.$$

For the derivation of explicit regularity estimates, we also have the version with the product weight, namely,

$$\Big[\int_{\mathbb{T}^{2d}} K_h(x-y)\chi(\delta\rho_k)\,w_{1,k}^x\,w_{1,k}^y\,dx\,dy\Big]\Big|_{t=s}$$
$$\le C - \int_0^s\int_{\mathbb{T}^{2d}} K_h(x-y)(\mathrm{div}\,_x u_k^x - \mathrm{div}\,_y u_k^y)\,\chi'(\delta\,\rho_k)\,\bar\rho_k\,w_{1,k}^x\,w_{1,k}^y\,dx\,dy\,dt$$
$$-\lambda\int_0^s\int_{\mathbb{T}^{2d}} K_h(x-y)\Big((\rho_k^x)^{\tilde\gamma} + \tilde P_k^x\,\rho_k^x + \Theta_k^x + (\rho_k^y)^{\tilde\gamma} + \tilde P_k^y\,\rho_k^y + \Theta_k^y\Big)\,\chi(\delta\rho_k)$$
$$\cdot\, w_{1,k}^x\,w_{1,k}^y\,dx\,dy\,dt.$$

For convenience, we write separately the result that we will use in the anisotropic case:

LEMMA 8.2. *Assume that ρ_k solves* (5.1) *with estimates* (5.5)–(5.4). *Without diffusion, $\alpha_k = 0$, assume* (8.2) *on χ_a with $p > \gamma + \ell + 1$, and denote $w_{a,h} = \overline{K}_h \star w_a$. There exists θ with $0 < \theta < 1$ such that*

$$\Big[\int_{h_0}^1 \int_{\mathbb{T}^{2d}} \frac{\overline{K}_h(x-y)}{h} (w_{a,k,h}^x + w_{a,k,h}^y)\, \chi_a(\delta\rho_k)(t)\, dx\, dy\, dh\Big]\Big|_{t=s}$$
$$\leq \Big[\int_{h_0}^1 \int_{\mathbb{T}^{2d}} \frac{\overline{K}_h(x-y)}{h} (w_{a,k,h}^x + w_{a,k,h}^y)\, \chi_a(\delta\rho_k)\, dx\, dy\, dh\Big]\Big|_{t=0}$$
$$+ C\,|\log h_0|^\theta + I + II - \mathbb{T}_a,$$

with the dissipation term

$$\mathbb{T}_a = \lambda \int_{h_0}^1 \int_0^s \int_{\mathbb{T}^d} w_{a,k,h}^x\, \chi_a(\delta\rho_k)\, \overline{K}_h \star (|\mathrm{div}\, u_k|^x + |A_\mu(\rho)^\gamma|^x)\, \bar{K}_h\, dx\, dt\, \frac{dh}{h},$$

while

$$I = -\frac{1}{2} \int_{h_0}^1 \int_0^s \int_{\mathbb{T}^{2d}} \frac{\overline{K}_h(x-y)}{h} (\mathrm{div}_x u_k^x - \mathrm{div}_y u_k^y)$$
$$\cdot\, \chi_a'(\delta\rho_k)\, \bar{\rho}_k (w_{a,k,h}^x + w_{a,k,h}^y)\, dx\, dy\, dt\, dh$$

and

$$II = -\frac{1}{2} \int_{h_0}^1 \int_0^s \int_{\mathbb{T}^{2d}} \frac{\overline{K}_h(x-y)}{h} (\mathrm{div}_x u_k^x + \mathrm{div}_y u_k^y)$$
$$\cdot\, (\chi_a'(\delta\rho_k)\delta\rho_k - 2\chi_a(\delta\rho_k))\, (w_{a,k,h}^x + w_{a,k,h}^y)\, dx\, dy\, dt\, dh.$$

Remark. We emphasize that the λ terms in relations (i) and (ii) of Lemma 8.1 come from the penalization in the definition of the weights w_0 and w_1. They will help to counterbalance terms coming from the contribution by $\mathrm{div}_x u_k^x - \mathrm{div}_y u_k^y$. Similarly, the non-local term $\mathbb{T}_a$ follows from the definition of the weights w_a.

Proof. Case (i). Denote

$$W_{0,h}^{x,y} = \int_{\mathbb{T}^{2d}} \overline{K}_h(x-z)\, \overline{K}_h(y-w)\, W_{0,k}^{z,w}\, dz\, dw,$$

and let us use χ in the renormalized equation from Lemma 7.1. We get

$$\Big(\int_{\mathbb{T}^d} W_{0,h}^{x,y}\, K_h(x-y)\, \chi(\delta\rho_k)\, dx\, dy\Big)\Big|_{t=s} \leq A + B + D$$
$$- \frac{1}{2} \int_0^s \int_{\mathbb{T}^{2d}} K_h(x-y)\, (\mathrm{div}_x u_k^x - \mathrm{div}_y u_k^y)\, \chi'(\delta\rho_k)\, \bar{\rho}_k\, W_{0,h}^{x,y}\, dx\, dy\, dt$$
$$- \frac{1}{2} \int_0^s \int_{\mathbb{T}^{2d}} K_h(x-y)\, (\mathrm{div}_x u_k^x + \mathrm{div}_y u_k^y)\, (\chi'(\delta\rho_k)\, \delta\rho_k$$
$$- 2\,\chi(\delta\rho_k))\, W_{0,h}^{x,y}\, dx\, dy\, dt,$$

with, by the symmetry of K_h, $\overline{K}_h$ and $W_{0,h}^{x,y}$ and, in particular, since $\nabla_x W_{0,h}^{x,y} = \nabla_y W_{0,h}^{x,y}$,

$$A = \int_0^s \int_{\mathbb{T}^{2d}} (u_k^x - u_k^y) \cdot \nabla K_h(x-y)\ \chi(\delta\,\rho_k)\, W_{0,h}^{x,y}\, dx\, dy\, dt,$$

$$B = 2\int_0^s \int_{\mathbb{T}^{2d}} K_h(x-y)\big(\partial_t W_{0,h}^{x,y} + u_k \cdot \nabla_x W_{0,h}^{x,y} - \alpha_k \Delta_x W_{0,h}^{x,y}\big) \cdot \chi(\delta\,\rho_k)\, dx\, dy\, dt,$$

$$D = 2\,\alpha_k \int_0^s \int_{\mathbb{T}^{2d}} \chi(\delta\,\rho_k)\, [\Delta_x\, K_h(x-y)\, W_{0,k,h}^{x,y} + 2K_h(x-y)\Delta_x W_{0,k,h}^{x,y}] dx\, dy\, dt.$$

Then using (8.1), simply bound

$$D \le 8\,\alpha_k\ h^{-2}\, \|K_h\|_{L^1(\mathbb{T}^d}\, \|\rho_k\|_{L^1((0,T)\times\mathbb{T}^d)} \le C\,\alpha_k\ h^{-2}\, \|K_h\|_{L^1},$$

leading us to choose

$$\varepsilon_{h_0}(k) = \alpha_k \int_{h_0}^1 h^{-2}\, \frac{dh}{h}. \tag{8.3}$$

As for B, using equation (7.2),

$$B = B_1 - 2\int_0^s \int_{\mathbb{T}^{2d}} K_h(x-y)\, \chi(\delta\rho_k)\, \tilde{K}_h \star_{x,y}\ R_0\, dx\, dy\, dt,$$

with

$$B_1 = 2\int_0^s \int_{\mathbb{T}^{4d}} K_h(x-y)\, \chi(\delta\rho_k)\, (u_k^x - u_k^z) \cdot \nabla_x \overline{K}_h(x-z)\overline{K}_h(y-w) \cdot W_0^{z,w}\, dx\, dy\, dz\, dw\, dt.$$

We recall that $R_0^{x,y} = D_0^x\, w_0^x + D_0^y\, w_0^y$ with $D_0 = \lambda\, \overline{K}_h \star (M\,|\nabla u_k|)$, and we thus only have to bound B_1. By Lemma 6.1, we have

$$\begin{aligned} B_1 &\le C\int_0^s \int_{\mathbb{T}^{4d}} K_h(x-y)\, \chi(\delta\rho_k)\, (D_{|x-y|} u_k^x + D_{|x-y|} u_k^z)|x-z| \\ &\quad \cdot |\nabla \overline{K}_h(x-z)| \overline{K}_h(y-w)\, W_0^{z,w} dx\, dy\, dz\, dw\, dt \\ &\le C\int_0^t \int_{\mathbb{T}^{4d}} K_h(x-y)\, \chi(\delta\rho_k)\, (D_{|x-y|} u_k^x + D_{|x-y|} u_k^z) \\ &\quad \cdot \overline{K}_h(x-z)\overline{K}_h(y-w)\, W_0^{z,w} dx\, dy\, dz\, dw\, dt \end{aligned}$$

as $|x|\, |\nabla K_h| \le C\, K_h$. Next, recalling that $W_0^{z,w} = w_0^z + w_0^w$, by symmetry,

$$\begin{aligned} B_1 &\le C\int_0^s \int_{\mathbb{T}^{4d}} K_h(x-y)\, \chi(\delta\rho_k)\, D_{|x-y|} u_k^z \\ &\quad \cdot \overline{K}_h(x-z)\overline{K}_h(y-w)\, w_0^z dx\, dy\, dz\, dw\, dt \\ &\quad + C\int_0^s \int_{\mathbb{T}^{4d}} K_h(x-y)\, \chi(\delta\rho_k)\, (D_{|x-y|} u_k^x + D_{|x-y|} u_k^w - 2D_{|x-y|} u_k^z) \\ &\quad \cdot \overline{K}_h(x-z)\overline{K}_h(y-w)\ w_0^z dx\, dy\, dz\, dw\, dt\,. \end{aligned}$$

Since $D_{|x-y|}u_k(z) \leq C\, M|\nabla u_k|(z)$, for λ large enough, the first term may be bounded by

$$-\frac{\lambda}{2}\int_0^s \int_{\mathbb{T}^{2d}} K_h(x-y)\,\chi(\delta\rho_k)\,\overline{K}_h \star_x (M|\nabla u_k|\,w_0)\,dx\,dy\,dt.$$

Use the uniform bound on $\|\rho_k\|_{L^p}$ with $p > 2$ to find

$$\int_0^s \int_{\mathbb{T}^{4d}} K_h(x-y)\,\chi(\delta\rho_k)\,(D_{|x-y|}u_k^x + D_{|x-y|}u_k^w - 2D_{|x-y|}u_k^z) \cdot \tilde{K}_h\, w_{0,k}^z dx\,dy\,dz\,dw\,dt$$
$$\leq C\int_0^s \int_{\mathbb{T}^{3d}} \left\|D_{|r|}u_k^{\cdot} + D_{|r|}u_k^{\cdot+r+u} - 2D_{|r|}u_k^{\cdot+v}\right\|_{L^2_\cdot} K_h(r)\, \overline{K}_h(u)\,\overline{K}_h(v)\,dr\,du\,dv\,dt,$$

where we used that $w = x + (y-x) + (w-y)$. We now use Lemma 6.3 and, more precisely, the inequality (6.5) to obtain

$$\int_{h_0}^1\int_0^s \int_{\mathbb{T}^{3d}} \overline{K}_h(x-y)\,\chi(\delta\rho_k)\,(D_{|x-y|}u_k^x + D_{|x-y|}u_k^w - 2D_{|x-y|}u_k^z) \cdot \tilde{K}_h\, w_{0,k}^z \frac{dh}{h} dx\,dy\,dz\,dw\,dt$$
$$\leq C\,|\log h_0|^{1/2}\int_0^s \|u_k(t,.)\|_{H^1}\,dt \leq C\,|\log h_0|^{1/2}.$$

Therefore, we have that

$$\int_{h_0}^1 B\,\frac{\|K_h\|_{L^1}^{-1}}{h}\,dh \leq C\,\varepsilon_{h_0}(k) + C\,|\log h_0|^{1/2}$$
$$-\frac{3\,\lambda}{4}\int_{h_0}^1\int_0^s\int_{\mathbb{T}} \overline{K}_h(x-y)\,\chi(\delta\rho_k)\,M|\nabla u_k|(z)\,\overline{K}_h(x-z)\,w_0(z)\,dz\,dx\,dy\,dt\,\frac{dh}{h}.$$

The computations are similar for A, and we only give the main steps. Again using Lemma 6.1, we have

$$\int_0^s\int_{\mathbb{T}^{2d}} \nabla K_h(x-y)\cdot(u_k^x - u_k^y)\,\chi(\delta\rho_k)\,W_{0,h}^{x,y} dx\,dy\,dt$$
$$\leq C\int_0^s\int_{\mathbb{T}^{2d}} K_h(x-y)\,(D_{|x-y|}\,u_k^x + D_{|x-y|}\,u_k^y)\,\chi(\delta\,\rho_k)\,W_{0,h}^{x,y} dx\,dy\,dt.$$

By decomposing $W_{0,h}$, just as for B_1, we can write

$$\int_0^s\int_{\mathbb{T}^{2d}} \nabla K_h(x-y)\cdot(u_k^x - u_k^y)\,\chi(\delta\rho_k)\,W_{0,h}^{x,y} dx\,dy\,dt$$
$$\leq C\int_0^s\int_{\mathbb{T}^{3d}} K_h(x-y)\,M\,|\nabla u_k|^z\,\chi(\delta\,\rho_k)\,\overline{K}_h(x-z)\,w_{0,k}^z\,dx\,dy\,dz\,dt$$
$$+ C\int_0^s\int_{\mathbb{T}^{4sd}} K_h(x-y)\,(D_{|x-y|}\,u_k^x + D_{|x-y|}\,u_k^y + D_{|x-y|}\,u_k^w - 3\,D_{|x-y|}\,u_k^z) \cdot \chi(\delta\,\rho_k)\,\overline{K}_h(x-z)\,\overline{K}_h(y-w)\,w_{0,k}^z\,dx\,dy\,dz\,dw\,dt.$$

The first term in the right-hand side can again be bounded by

$$-\frac{\lambda}{2}\int_0^s \int_{\mathbb{T}^{2d}} K_h(x-y)\,\chi(\delta\rho_k)\,\overline{K}_h\star_x(M|\nabla u_k|\,w_{0,k})\,dx\,dy\,dt.$$

The second term in the right-hand side is now integrated in h and controlled as before thanks to the bound (6.5) in Lemma 6.3 and the uniform L^p bound on ρ_k and H^1 on u_k. This leads to

$$\int_{h_0}^1 A\,\|K_h\|_{L^1}^{-1}\frac{dh}{h}\le C\,|\log h_0|^{1/2}$$
$$+\frac{\lambda}{4}\int_{h_0}^1\int_0^s\int_{\mathbb{T}^{3d}}\overline{K}_h(x-y)\,M|\nabla u_k|^z\,\chi(\delta\,\rho_k)\,\overline{K}_h(x-z)\,w_{0,k}^z\,dx\,dy\,dz\,dt\,\frac{dh}{h}.$$

Now summing all the contributions we get

$$\int_{h_0}^1(A+B+D)\,\|K_h\|_{L^1}^{-1}\frac{dh}{h}\le C\,\varepsilon_{h_0}(k)+C\,|\log h_0|^{1/2}$$
$$-\frac{\lambda}{2}\int_{h_0}^1\int_0^s\int_{\mathbb{T}^{2d}}K_h(x-y)\chi(\delta\rho_k)\overline{K}_h\star_x(M|\nabla u_k|\,w_{0,k})\,dx\,dy\,dt\frac{dh}{h}.$$

Note that indeed $\varepsilon_{h_0}(k)\to 0$ as $k\to\infty$ for a fixed h_0. This concludes the proof in that first case.

Case (ii). In this part, we assume $\alpha_k=0$. We may not assume that ρ_k is smooth anymore. However by [26], since ρ_k and ∇u_k belong to the space $L^2((0,T)\times\mathbb{T}^d)$, one may use the renormalized relation with $\varphi=\chi$ and choose $W_{h,k}^{x,y}=W_{i,k}^{x,y}$. We then can use the identity given in Lemma 7.1. Denoting

$$\tilde\chi(\xi)=\frac{1}{\chi(\xi)}\left(\chi(\xi)-\frac{1}{2}\chi'(\xi)\,\xi\right),$$

for $i=1$, 2, we get

$$\left(\int_{\mathbb{T}^{2d}}K_h(x-y)\,\chi(\delta\rho_k)\,W_{i,k}^{x,y}\,dx\,dy\right)(s)\le A_i+B_i+D_i,$$

where by the symmetry in x and y,

$$A_1=\int_0^s\int_{\mathbb{T}^{2d}}(u_k^x-u_k^y)\cdot\nabla K_h(x-y)\,\chi(\delta\,\rho_k)\,W_{1,k}^{x,y}\,dx\,dy\,dt$$
$$-\lambda\int_0^s\int_{\mathbb{T}^{2d}}K_h(x-y)\,(M\,|\nabla u_k|^x\,w_{1,k}^x+M\,|\nabla u_k|^y\,w_{1,k}^y)\,\chi(\delta\,\rho_k)\,dx\,dy\,dt,$$

while

$$A_2=\int_0^s\int_{\mathbb{T}^{2d}}(u_k^x-u_k^y)\cdot\nabla K_h(x-y)\,\chi(\delta\,\rho_k)\,W_{2,k}^{x,y}\,dx\,dy\,dt$$
$$-\lambda\int_0^s\int_{\mathbb{T}^{2d}}K_h(x-y)\,(M\,|\nabla u_k|^x+M\,|\nabla u_k|^y)\,w_{1,k}^x\,w_{1,k}^y)\,\chi(\delta\,\rho_k)\,dx\,dy\,dt.$$

Furthermore,

$$B_1 = 2\int_0^s \int_{\mathbb{T}^{2d}} K_h(x-y)\Big(\partial_t w_{1,k}^x + u_k^x\cdot\nabla_x w_{1,k}^x + 2\mathrm{div}_x u_k^x\,\tilde\chi(\delta\rho_k)\,w_{1,k}^x + \lambda\,M|\nabla u_k|^x\,w_{1,k}^x\Big)\,\chi(\delta\,\rho_k)\,dx\,dy\,dt,$$

while

$$B_2 = 2\int_0^s \int_{\mathbb{T}^{2d}} K_h(x-y)\,\Big(\partial_t w_{1,k}^x + u_k^x\cdot\nabla_x w_{1,k}^x + \mathrm{div}_x u_k^x\,\tilde\chi(\delta\rho_k)\,w_{1,k}^x + \lambda\,M|\nabla u_k||^x\,w_{1,k}^x\Big)\,w_{1,k}^y\,\chi(\delta\,\rho_k)\,dx\,dy\,dt.$$

Finally,

$$D_1 = -2\int_0^s \int_{\mathbb{T}^{2d}} K_h(x-y)\,(\mathrm{div}_x u_k^x - \mathrm{div}_y u_k^y)\,\Big(\frac{1}{2}\chi'(\delta\,\rho_k)\,\bar\rho_k + \chi(\delta\rho_k) - \frac{1}{2}\chi'(\delta\,\rho_k)\,\delta\rho_k\Big)\,w_{1,k}^x\,dx\,dy\,dt$$

and

$$D_2 = -\int_0^s \int_{\mathbb{T}^{2d}} K_h(x-y)\,(\mathrm{div}_x u_k^x - \mathrm{div}_y u_k^y)\,\chi'(\delta\,\rho_k)\,\bar\rho_k w_{1,k}^x\,w_{1,k}^y\,dx\,dy\,dt.$$

Note that we have split a null contribution into several non-null parts in terms of the maximal function, namely, the ones with $M|\nabla u_k|^2$. Notice also the additional terms in D_1 that come from cross products such as $\mathrm{div}\,u_k(y)\ \tilde\chi\,\chi\,w_1(x)$, which would pose problems in B_1.

The contributions D_1 and D_2 are already under the right form. Using equation (7.2) with (7.4), one may directly bound

$$B_1 \le -2\,\lambda\int_0^s \int_{\mathbb{T}^{2d}} K_h(x-y)\,((\rho_k^x)^{\tilde\gamma} + \tilde P_k^x\,\rho_k^x + R_k^x)\,w_{1,k}^x\,\chi(\delta\,\rho_k)\,dx\,dy\,dt$$

and

$$B_2 \le -2\,\lambda\int_0^s \int_{\mathbb{T}^{2d}} K_h(x-y)\,((\rho_k^x)^{\tilde\gamma} + \tilde P_k^x\,\rho_k^x + R_k^x)\,w_{1,k}^x\,w_{1,k}^y\,\chi(\delta\,\rho_k)\,dx\,dy\,dt,$$

giving the desired result by symmetry of the expression in x and y. The term A_2 is straightforward to handle as well. Use (6.1) to get

$$A_2 \le \int_0^s \int_{\mathbb{T}^{2d}} |\nabla K_h(x-y)|\,|x-y|\,(M\,|\nabla u_k|^x + M\,|\nabla u_k|^y)\,\chi(\delta\,\rho_k)\,W_{2,k}^{x,y}\,dx\,dy\,dt - \lambda\int_0^s \int_{\mathbb{T}^{2d}} K_h(x-y)\,(M\,|\nabla u_k|^x + M\,|\nabla u_k|^y)\,w_{1,k}^x\,w_{1,k}^y\,\chi(\delta\,\rho_k)\,dx\,dy\,dt.$$

Since $|x|\,|\nabla K_h| \le C K_h$, by taking λ large enough, one obtains

$$A_2 \le 0.$$

The term A_1 is more complex because it has no symmetry. By Lemma 6.1,

$$\begin{aligned}A_1 \leq C \int_0^s \int_{\mathbb{T}^{2d}} &|\nabla K_h(x-y)|\,|x-y|\,((D_{|x-y|}\,u_k)^x + (D_{|x-y|}\,u_k)^y)\\ &\cdot \chi(\delta\,\rho_k)\,w_{1,k}^x\,dx\,dy\,dt\\ &- \lambda \int_0^s \int_{\mathbb{T}^{2d}} K_h(x-y)\,M\,|\nabla u_k|^x\,w_{1,k}^x\,\chi(\delta\,\rho_k)\,dx\,dy\,dt\\ &+ \text{similar terms in } w_{1,k}^y.\end{aligned}$$

The *key problem* here is the $D_h\,u_k(y)\,w_{1,k}^x$ term, which one has to control by the term $M\,|\nabla u_k|(x)\,w_{1,k}^x$. This is where integration over h and the use of Lemma 6.3 is needed. (The other term in $w_{1;k}^y$ is dealt with in a symmetric manner.) For that we will add and subtract an appropriate quantity to see the quantity $(D_{|x-y|}u_k)^x - (D_{|x-y|}u_k)^y$.

By the definition of K_h,

$$|z|\,|\nabla K_h(z)| \leq C\,K_h(z)$$

and by (8.1),

$$\chi(\delta\rho_k) \leq C\,(\rho_k^x + \rho_k^y)$$

with $\rho_k \in L^2$ uniformly and w_1 uniformly bounded. Hence using Cauchy-Schwartz and denoting $z = x - y$,

$$\begin{aligned}\int_{h_0}^1 \frac{A_1}{\|K_h\|_{L^1}}\frac{dh}{h} &\leq 2C \int_{h_0}^1 \int_0^s \int_{\mathbb{T}^d} \overline{K}_h(z)\,\|(D_{|z|}\,u_k)^{\cdot} - (D_{|z|}u_k)^{\cdot+z}\|_{L^2(\mathbb{T}^d)}\,dz\,dt\,\frac{dh}{h}\\ &\quad + 4\,C \int_0^s \int_{\mathbb{T}^{2d}} \mathcal{K}_{h_0}(x-y)\,(D_{|x-y|}\,u_k)^x\,\chi(\delta\,\rho_k)\,W_{1,k}^{x,y}\,dx\,dy\,dt\\ &\quad - 2\lambda \int_0^s \int_{\mathbb{T}^{2d}} \mathcal{K}_{h_0}(x-y)\,(M\,|\nabla u_k|)^x\,w_{1,k}^x\,\chi(\delta\,\rho_k)\,dx\,dy\,dt\\ &\leq 2C \int_{h_0}^1 \int_{\mathbb{T}^d} \overline{K}_h(z)\,\|(D_{|z|}\,u_k)^{\cdot} - (D_{|z|}u_k)^{\cdot+z}\|_{L^2}\,dz\,\frac{dh}{h}\end{aligned}$$

by taking λ large enough since Lemma 6.2 bounds $(D_{|x-y|}\,u_k)^x$ by $(M\,|\nabla u_k|)^x$.

Finally, using Lemma 6.3,

$$\int_{h_0}^1 \frac{A_1}{\|K_h\|_{L^1}}\frac{dh}{h} \leq C\,|\log h_0|^{1/2} \int_0^s \|u_k(t,\cdot)\|_{H^1}\,dt.$$

Summing up $A_i + B_i + D_i$ and integrating against $\dfrac{dh}{\|K_h\|_{L^1}\,h}$ for $i = 1$ concludes the proof. □

Proof of Lemma 8.2. In this part, we again assume $\alpha_k = 0$ and still use [26] to obtain the renormalized relation of Lemma 7.1 with $\varphi = \chi_a$ and $W_{k,h}^{x,y} = \overline{K}_h \star W_{a,k} = w_{a,k,h} + w_{a,k,h}$. With this exception the proof follows the lines of point (i) in Lemma 8.1, so we only sketch it here.

From Lemma 7.1, we get

$$\Big[\int_{h_0}^1\int_{\mathbb{T}^{2d}}\frac{\overline{K}_h(x-y)}{h}(w^x_{a,k,h}+w^y_{a,k,h})\,\chi_a(\delta\rho_k)\,dx\,dy\,dh\Big]\Big|_{t=s}$$
$$\le\Big(\int_{h_0}^1\int_{\mathbb{T}^{2d}}\frac{\overline{K}_h(x-y)}{h}(w^x_{a,k,h}+w^y_{a,k,h})\,\chi_a(\delta\rho_k)\,dx\,dy\,dh\Big)\Big|_{t=0}$$
$$+A+B+D+I+II+\mathbb{T}_a,$$

with the terms

$$A=\int_0^s\int_{\mathbb{T}^{2d}}(u^x_k-u^y_k)\cdot\nabla\overline{K}_h(x-y)\chi_a(\delta\rho_k)(w^x_{a,k,h}+w^y_{a,k,h})\,dx\,dy\,dt,$$

while

$$B=2\Big[\int_0^s\int_{\mathbb{T}^{3d}}K_h(x-y)\chi_a(\delta\rho_k)(u^x_k-u^z_k)\cdot\nabla_x\overline{K}_h(x-z)w^z_{a,k}\,dx\,dy\,dz\,dt$$
$$-\lambda\int_0^s\int_{\mathbb{T}^{2d}}K_h(x-y)\chi_a(\delta\rho_k)(\overline{K}_h\star(M|\nabla u_k|w_{a,k}))^x\,dx\,dy\,dt\Big]$$

and

$$D=-\lambda\int_0^s\int_{h_0}^1\int_{\mathbb{T}^{2d}}\chi_a(\delta\rho_k)\,[\overline{K}_h\star((|\operatorname{div}u_k|+|A_\mu\rho^\gamma|)w_{a,k}\,)]^x$$
$$\cdot\,\overline{K}_h(x-y)\,dx\,dy\,\frac{dh}{h}\,dt$$
$$+\lambda\int_0^s\int_{h_0}^1\int_{\mathbb{T}^{2d}}w_{a,k,h}(x)\,\chi_a(\delta\rho_k)\,(\overline{K}_h\star(|\operatorname{div}u_k|+|A_\mu\rho^\gamma|))^x$$
$$\cdot\,\overline{K_h}(x-y)dx\,dy\,\frac{dh}{h}\,dt.$$

The dissipation term is under the right form

$$\mathbb{T}_a=-\lambda\int_0^s\int_{h_0}^1\int_{\mathbb{T}^{2d}}w^x_{a,k,h}\,\chi_a(\delta\rho_k)\,[\overline{K}_h\star(|\operatorname{div}u_k|+|A_\mu\rho^\gamma|)]^x$$
$$\cdot\,\overline{K_h}(x-y)\,dx\,dy\,\frac{dh}{h}\,dt,$$

and by symmetry, so are

$$I=-\frac12\int_{h_0}^1\int_0^s\int_{\mathbb{T}^{2d}}\frac{\overline{K}_h(x-y)}{h}\,(\operatorname{div}u^x_k-\operatorname{div}u^y_k)\chi'_a(\delta\rho_k)\,\bar\rho_k\,w^x_{a,k,h}\,dx\,dy\,dt\,dh$$

and

$$II=-\frac12\int_{h_0}^1\int_0^s\int_{\mathbb{T}^{2d}}\frac{\overline{K}_h(x-y)}{h}\,(\operatorname{div}u^x_k+\operatorname{div}u^y_k)(\chi'_a(\delta\rho_k)\delta\rho_k-2\chi_a(\delta\rho_k))$$
$$\cdot w^x_{a,k,h}\,dx\,dy\,dt\,dh.$$

The terms A and B are treated exactly as in case (i) of Lemma 8.1; they only require the higher integrability $p>\gamma+1+\ell$.

The only additional term is hence D, which is required in order to write the dissipation term $\mathbb{T}_a$ in the right form. D is bounded directly by point (iv) in Lemma 7.2. Thus

$$A + B + D \le C\,|\log h_0|^\theta$$

for some $0 < \theta < 1$, which concludes the proof. □

8.2. *The control on the effective viscous flux.* Before coupling the previous estimate with the equation on $\operatorname{div} u_k$, we start with a lemma that will be used in every situation as it controls the regularity properties of

$$F_k = \Delta^{-1} \operatorname{div}\ \left(\partial_t(\rho_k\, u_k) + \operatorname{div}(\rho_k\, u_k \otimes u_k)\right),$$

per

Lemma 8.3. *Assume that ρ_k solves* (5.1), *that* (5.5)–(5.6) *hold, and* (5.4) *with $\gamma > d/2$. Assume, moreover, that $\Phi \in L^\infty([0,\ T] \times \mathbb{T}^{2d})$ and that*

$$\begin{aligned} C_\phi := &\left\| \int_{\mathbb{T}^d} \overline{K}_h(x-y)\, \Phi(t,x,y)\, dy \right\|_{W^{1,1}(0,T;\, W_x^{-1,1}(\mathbb{T}^d))} \\ &+ \left\| \int_{\mathbb{T}^d} \overline{K}_h(x-y)\, \Phi(t,x,y)\, dx \right\|_{W^{1,1}(0,T;\, W_y^{-1,1}(\mathbb{T}^d))} < \infty. \end{aligned}$$

Then there exists $\theta > 0$ such that

$$\begin{aligned} &\int_0^s \int_{\mathbb{T}^{2d}} K_h(x-y)\, \Phi(t,x,y)\, (F_k(t,y) - F_k(t,x))\, dx\, dy\, dt \\ &\qquad \le C\, \|K_h\|_{L^1}\, (h^\theta + \varepsilon_h(k))\, \left(\|\Phi\|_{L^\infty((0,T)\times\mathbb{T}^{2d})} + C_\Phi\right), \end{aligned}$$

with $\varepsilon_h(k) \to 0$ as $k \to +\infty$ for a fixed h (and in fact $\varepsilon_h(k) = 0$ if $\alpha_k = 0$).

The proof below makes heavy use of some special notation, which we recall here for convenience. We will often denote exponents $q+0$ or $r-0$. This means that the estimate holds for some exponent $q' > q$ or some exponent $r' < r$. The exact values of q' or r' are irrelevant for our proof. We are indeed using many interpolations between Sobolev spaces that are not exact. (To have the precise exponent one would have to use Besov spaces instead; see, for instance, [47].) We also use θ as a generic but positive exponent whose value may change from line to line. For example, we may say that a function f belongs to $L^{1+0}_{t,x}$, meaning that $f \in L^q_{t,x}$ for some $q > 1$. And for any $r > 1$, we freely use interpolation arguments of the type

$$\|f\|_{L^{1+0}_{t,x}} \le \|f\|^{1-\theta}_{L^1_{t,x}}\, \|f\|^\theta_{L^r_{t,x}}.$$

Proof. The proof is divided in four steps. The first one concerns a control on $\rho_k|u_k - u_{k,\eta}|$, where $u_{k,\eta}$ is a regularization of u_k defined later-on. The second step concerns the proof of an estimate for $\Phi_x = \left(\overline{K}_h \star \Phi\right)(t,x)$ and $\Phi_y = \left(\overline{K}_h \star \Phi\right)(t,y)$ in $L^2_t L^{\bar p'}_x$ with $\bar p' = \bar p/(\bar p - 1)$. The third step concerns a control with respect to h when Φ_x, Φ_y in $L^2_t L^{\bar p'}_x \cap W^{1,+\infty}_t W^{-1,\infty-0}$ with $\bar p' = \bar p/(\bar p - 1)$. The last term is the end of the proof obtained by interpolation.

(i) *A control on $\rho_k |u_k - u_{k,\eta}|$ where $u_{k,\eta}$ is a regularization in space and time defined later-on.* Choose a kernel $\mathcal{L}_\eta \in C^\infty_c(\mathbb{R}_+ \times \mathbb{R}_+)$ with $\mathcal{L}_\eta \geq 0$, with $\int_{\mathbb{R}^+} \mathcal{L}_\eta(t,s)\,ds = 1$ such that $\mathcal{L}_\eta(t,s) = 0$ if $|t-s| \geq \eta$ for smoothing in time. We still denote, with a slight abuse of notation,

$$\mathcal{L}_\eta \star_t u_k(t) = \int_{\mathbb{R}_+} \mathcal{L}_\eta(t,s)\, u_k(s)\, ds.$$

Now u_k is uniformly bounded in $L^2_t H^1_x \subset L^2_t L^q_x$ with $1/q = 1/2 - 1/d$ (and in $L^2_t L^q_x$ for any $q < \infty$ if $d = 2$). Hence $u_k^2 \in L^1_t L^{q/2}$, and since $\gamma > d/2$, all expressions of the type $\rho_k\, u_k$ or $\rho_k\, |u_k|^2$ are well defined and even belong to $L^{1+0}_{t,x}$, respectively $L^1_t\, L^{1+0}_x$. The same applies if we replace u_k by any convolution $L_\eta \star_{t,x} u_k$ uniformly in the parameter η.

For this reason one has

$$\begin{aligned}
&\int \rho_k(t,x)\, \frac{(u_k(t,x) - u_k(s,x))^2}{1 + |u_k(t,x)| + |u_k(s,x)|}\, \mathcal{L}_\eta(t,s)\, dt\, ds\, dx \\
&\leq \int \rho_k(t,x)\, (u_k(t,x) - u_k(s,x))\, \mathcal{L}_\eta(t,s)\, \overline{\mathcal{L}_{\eta'}} \star_x \frac{u_k(t,.) - u_k(s,.)}{1 + |u_k(t,.)| + |u_k(s,.)|} \\
&\quad + \int \rho_k(t,x)\, (u_k(t,x) - u_k(s,x)) \mathcal{L}_\eta(t,s) \left| \overline{\mathcal{L}_{\eta'}} \star_x g(t,s,x) - g(t,s,x) \right| dt\, ds\, dx,
\end{aligned}$$

with $g(t,s,x) = \frac{u_k(t,x) - u_k(s,x)}{1 + |u_k(t,x)| + |u_k(s,x)|}$. Since $\gamma > d/2$,

$$\begin{aligned}
&\int \rho_k(t,x)\, (u_k(t,x) - u_k(s,x))\, \mathcal{L}_\eta(t,s)\, \left| \overline{\mathcal{L}_{\eta'}} \star_x g(t,s,x) - g(t,s,x) \right| dt\, ds\, dx \\
&\qquad \leq 2\, \|\rho_k\, u_k\|_{L^{1+0}_{t,x}} \left(\int \mathcal{L}_\eta(t,s)\, \left| \overline{\mathcal{L}_{\eta'}} \star_x g(t,s,x) - g(t,s,x) \right|^q dt\, ds\, dx \right)^{1/q}
\end{aligned}$$

for some $q < \infty$. Since $\|g\|_{L^\infty} \leq 1$, we have by interpolation

$$\begin{aligned}
&\left(\int \mathcal{L}_\eta(t,s)\, \left| \overline{\mathcal{L}_{\eta'}} \star_x g(t,s,x) - g(t,s,x) \right|^q dt\, ds\, dx \right)^{1/q} \\
&\qquad \leq \|\overline{\mathcal{L}_{\eta'}} \star_x u_k - u_k\|^{2/q}_{L^2_{t,x}} \leq C\, (\eta')^{2/q}\, \|u_k\|^{2/q}_{L^2_t H^1_x}.
\end{aligned}$$

Hence for some $\theta > 0$,

$$\begin{aligned}
&\int \rho_k(t,x)\, (u_k(t,x) - u_k(s,x))\, \mathcal{L}_\eta(t,s) \\
&\qquad \cdot \left| \overline{\mathcal{L}_{\eta'}} \star_x g(t,s,x) - g(t,s,x) \right| dt\, ds\, dx \leq C\, (\eta')^\theta.
\end{aligned}$$

Note that

$$\|\partial_t \rho_k\|_{L^1_t W^{-1,1}_x} \leq C,$$

and by interpolation, as $\gamma > d/2$ and thus $\gamma > 2d/(d+2)$, there exists $\theta > 0$ such that

$$\|\rho_k\|_{H^\theta_t H^{-1}_x} \leq C.$$

Thus

$$\int \rho_k(t,x)\,\frac{(u_k(t,x)-u_k(s,x))^2}{1+|u_k(t,x)|+|u_k(s,x)|}\,\mathcal{L}_\eta(t,s)\,dt\,ds\,dx$$
$$\leq \int (\rho_k(t)\,u_k(t,x)-\rho_k(s)\,u_k(s,x))\,\mathcal{L}_\eta(t,s)\,\overline{\mathcal{L}_{\eta'}}\star\frac{u_k(t,.)-u_k(s,.)}{1+|u_k(t,.)|+|u_k(s,.)|}$$
$$+ C\,(\eta')^\theta + C\,\frac{\eta^\theta}{\eta'^d}\,\|u_k\|_{L^2_t H^1_x}\,.$$

Using (5.6), one deduces that

$$\int \rho_k(t,x)\,\frac{(u_k(t,x)-u_k(s,x))^2}{1+|u_k(t,x)|+|u_k(s,x)|}\,\mathcal{L}_\eta(t,s)\,dt\,ds\,dx \leq C\eta'^\theta + C\,\frac{\eta^\theta}{\eta'^d} + C\,\frac{\eta}{\eta'^d}.$$

Optimizing in η' (taking $\eta' = \eta^{\theta'}$ for the best exponent θ'), one has that for some other exponent $\theta > 0$,

$$\int \rho_k(t,x)\,\frac{(u_k(t,x)-u_k(s,x))^2}{1+|u_k(t,x)|+|u_k(s,x)|}\,\mathcal{L}_\eta(t,s)\,dt\,ds\,dx \leq C\,\eta^\theta.$$

We can now remove the denominator in $1+|u_k(t,x)|+|u_k(s,x)|$ simply by noticing that

$$\frac{(u_k(t,x)-u_k(s,x))^2}{1+|u_k(t,x)|+|u_k(s,x)|} \leq |u_k(t,x)-u_k(s,x)|.$$

Hence

$$\int \rho_k(t,x)\,|u_k(t,x)-u_k(s,x)|\,\mathcal{L}_\eta(t,s)\,dt\,ds\,dx \leq C\eta^\theta,$$

and this directly implies, in particular, that

$$\int \rho_k(t,x)\,|u_k(t,x)-\mathcal{L}_\eta\star_t u_k(t,x)|\,dt\,dx \leq C\eta^\theta. \tag{8.4}$$

For some $\nu > 0$, define

$$u_{k,\eta} = \overline{\mathcal{L}_{\eta^\nu}}\star_x\ \mathcal{L}_\eta\star_t u_k.$$

We recall that since $\gamma > d/2$, one has for any $f \in L^2_t H^1_x$,

$$\int \rho_k(t,x)\,f(t,x)\,dt\,dx \leq \|\rho_k\|_{L^\infty_t L^\gamma_x}\,\|f\|^{1-0}_{L^2_t H^1_x}\,\|f\|_{L^2_{t,x}}.$$

On the other hand,

$$\|u_{k,\eta} - \mathcal{L}_\eta\star_t u_k\|_{L^2_{t,x}} \leq \eta^\nu\,\|u_k\|_{L^2_t H^1_x}.$$

Therefore,

$$\int \rho_k(t,x)\,|u_{k,\eta}(t,x)-\mathcal{L}_\eta\star_t u_k(t,x)|\,dt\,dx \leq C\,\eta^\theta,$$

again for some exponent $\theta > 0$.

Combining this with (8.4), for some $\theta > 0$, we have that

$$\int \rho_k(t,x)\,|u_k(t,x) - u_{k,\eta}(t,x)|\,dx\,dt \leq C\,\eta^\theta.$$

Note again that since $\gamma > d/2$, one actually has that

$$\|\rho_k\, f\|_{L^q_{t,x}} \leq C\,\|\rho_k\|_{L^\infty_t L^\gamma_x}\,\|f\|_{L^2_t H^1_x}$$

for some $q > 1$, and hence by interpolation,

$$\|\rho_k\, f\|_{L^{1+0}_{t,x}} \leq C\,\|\rho_k\, f\|^\theta_{L^1_{t,x}}\,\|\rho_k\|^\theta_{L^\infty_t L^\gamma_x}\,\|f\|^{1-\theta}_{L^2_t H^1_x}$$

still for some (possibly small) positive $\theta > 0$.

Applying this to $f = u_k(t,x) - u_{k,\eta}(t,x)$, we finally deduce that there exists $\theta > 0$ such that

$$\|\rho_k\,(u_k - u_{k,\eta})\|_{L^{1+0}_t L^{1+0}_x} \leq C\,\eta^\theta. \tag{8.5}$$

(ii) *The case where Φ_x, Φ_y is only in $L^2_t L^{\bar p'}_x$ with $\bar p' = \bar p/(\bar p - 1)$.* We recall that $\bar p$ is the exponent in (5.6). Denote

$$I\Phi = \int_0^s \int_{\mathbb{T}^{2d}} K_h(x-y)\,\Phi(t,x,y)\,(F_k(t,y) - F_k(t,x))\,dx\,dy\,dt,$$

which can be seen as a linear form on Φ. Recall as well that

$$\Phi_x = \int_{\mathbb{T}^d} \overline{K}_h(x-y)\,\Phi(t,x,y)\,dy, \quad \Phi_y = \int_{\mathbb{T}^d} \overline{K}_h(x-y)\,\Phi(t,x,y)\,dx.$$

By (5.6), F_k is uniformly bounded in $L^2_t L^{\bar p}_x$. Therefore,

$$|I\,\Phi| \leq C\,\|K_h\|_{L^1}\left(\|\Phi_x\|_{L^2_t L^{\bar p'}_x} + \|\Phi_y\|_{L^2_t L^{\bar p'}_x}\right), \quad \frac{1}{\bar p'} = 1 - \frac{1}{\bar p} > 0. \tag{8.6}$$

(iii) *The case Φ_x, Φ_y in $L^2_t L^{\bar p'}_x \cap W^{1,+\infty}_t W^{-1,\infty-0}$ with $\bar p' = \bar p/(\bar p - 1)$.* Denote

$$\tilde C_\Phi = \|\Phi_x\|_{L^2_t L^{\bar p'}_x} + \|\Phi_y\|_{L^2_t L^{\bar p'}_x} + \|\Phi_x\|_{W^{1,\infty}_t W^{-1,\infty-0}} + \|\Phi_y\|_{W^{1,\infty}_t W^{-1,\infty-0}}$$

and

$$R_1 = \Delta^{-1}\,\mathrm{div}\,\rho_k\,(u_k - u_{k,\eta}).$$

Observe that by (8.5) and integration by part in time,

$$\int_0^s \int_{\mathbb{T}^d} \Phi_x\,\partial_t R_1\,dx\,dt \leq \tilde C_\Phi\,\eta^\theta\,\|K_h\|_{L^1}.$$

The same procedure can be performed with $\mathrm{div}\,(\rho_k\,u_k \otimes u_k)$. Denoting

$$F_{k,\eta} = \Delta^{-1}\,\mathrm{div}\,\left(\partial_t(\rho_k\,u_{k,\eta}) + \mathrm{div}\,(\rho_k\,u_k \otimes u_{k,\eta})\right),$$

one then has

$$\begin{aligned} I\,\Phi \leq{} & \tilde C_\Phi\,\|K_h\|_{L^1}\,\eta^\theta \\ & + \int_0^s \int_{\mathbb{T}^{2d}} K_h(x-y)\,\Phi(t,x,y)\,(F_{k,\eta}(t,y) - F_{k,\eta}(t,x))\,dx\,dy\,dt. \end{aligned}$$

However using (5.1),

$$\partial_t(\rho_k\, u_{k,\eta}) + \mathrm{div}\,(\rho_k\, u_k \otimes u_{k,\eta}) = \rho_k(\partial_t u_{k,\eta} + u_k \cdot \nabla u_{k,\eta}) + \alpha_k\, u_{k,\eta}\, \Delta\rho_k.$$

For some exponent κ,

$$\left\| \Delta^{-1}\mathrm{div}\ \left(\rho_k(\partial_t u_{k,\eta} + u_k\cdot\nabla u_{k,\eta})\right)\right\|_{L^1_t W^{1,1}_x} \leq C\,\eta^{-\kappa}$$

and

$$\alpha_k \left\| \Delta^{-1}\mathrm{div}\ \left(\alpha_k\, u_{k,\eta}\, \Delta\rho_k\right)\right\|_{L^2_t\, L^2_x} \leq C\,\eta^{-\kappa}\,\sqrt{\alpha_k}.$$

Therefore,

$$\int_0^s \int_{\mathbb{T}^{2d}} K_h(x-y)\,\Phi(t,x,y)\,(F_{k,\eta}(t,y) - F_{k,\eta}(t,x))\,dx\,dy\,dt$$
$$\leq \tilde C_\Phi\,\eta^{-\kappa}\,\|K_h\|_{L^1}\,(h+\sqrt{\alpha_k})^{1-0}.$$

Finally

$$I\,\Phi \leq \tilde C_\Phi\,\|K_h\|_{L^1}\,\left(\eta^\theta + \eta^{-\kappa}\,(h+\sqrt{\alpha_k})\right),$$

and by optimizing in η, there exists $\theta>0$ such that

$$\text{(8.7)}\quad I\,\Phi \leq C\,\|K_h\|_{L^1}\,(h^\theta + \varepsilon_h(k))\,\left(\|\Phi_x\|_{W^{1,\infty}_t W^{-1,\infty-0}} + \|\Phi_y\|_{W^{1,\infty}_t W^{-1,\infty-0}}\right),$$

with $\varepsilon_h(k)\to 0$ as $k\to+\infty$ for a fixed h.

(iv) *Interpolation between the two inequalities* (8.6) *and* (8.7). For any $s\in(0,1)$, there exists $\theta>0$ such that

$$I\,\Phi \leq C\,\|K_h\|_{L^1}\,(h^\theta + +\varepsilon_h(k))\,\Big(\|\Phi_x\|_{L^2_t L^{\bar p'}_x} + \|\Phi_y\|_{L^2_t L^{\bar p'}_x}$$
$$+ \|\Phi_x\|_{W^{s,q+0}_t W^{-s,r+0}_x} + \|\Phi_y\|_{W^{s,q+0}_t W^{-s,r+0}_x}\Big),$$

with

$$\frac{1}{q} = \frac{1-s}{2},\qquad \frac{1}{r} = \frac{1-s}{\bar p'}$$

and $\varepsilon_h(k)\to 0$ as $k\to+\infty$ for a fixed h. On the other hand if, for example, Φ_x belongs to $L^\infty_{t,x}$ and to $W^{1,1}_t W^{-1,1}_x$, then by interpolation Φ_x is in $W^{s,1/s-0}_t\, W^{-s,1/s-0}_x$. Hence

$$C_\phi = \|\Phi_x\|_{W^{1,1}_t W^{-1,1}_x} + \|\Phi_y\|_{W^{1,1}_t W^{-1,1}_y}$$

controls the $W^{s,q+0}_t W^{-s,r+0}_x$ norm provided

$$s < 1/q = \frac{1-s}{2},\qquad s < 1/r = \frac{1-s}{\bar p'}.$$

One can readily check that this is always possible by taking s small enough (but strictly positive) as $\bar p>1$ and hence $\bar p'<\infty$. This concludes the proof. □

8.3. *The coupling with the pressure law.* We are able to handle all type of weights at the same time here. For convenience, we denote

$$\chi_1^{x,y} = \frac{1}{2}\chi'(\delta\rho_k)\,\bar\rho_k + \chi(\rho_k) - \frac{1}{2}\chi'(\delta\rho_k)\,\delta\rho_k.$$

(1) In the case without diffusion, one has

LEMMA 8.4. *Assume that ρ_k solves* (5.1) *with $\alpha_k = 0$ and that* (5.6), (5.5), *and* (5.4) *with $\gamma > d/2$ and $p > 2$ hold. Assume, moreover, that u solves* (5.2) *with μ_k compact in L^1 and satisfying* (5.3), *R_k compact in L^1, P_k satisfying* (5.9).

(i) *Then there exists a continuous function $\varepsilon(.)$ with $\varepsilon(0) = 0$, depending only on our uniform bounds and the smoothness of μ_k and R_k such that*

$$-\int_0^s \int_{\mathbb{T}^{2d}} K_h(x-y)\,(\operatorname{div}_x u_k^x - \operatorname{div}_y u_k^y)\,\chi_1^{x,y}\,w_{1,k}^x\,dx\,dy\,dt \le C\,\|K_h\|_{L^1}\,\varepsilon(h)$$
$$+C\int_0^s \int_{\mathbb{T}^{2d}} K_h(x-y)\,\left(1 + (\rho_k^x)^{\tilde\gamma} + \tilde P_k^x\,\rho_k^x + \Theta_k^x\right)\,\chi(\delta\rho_k)\,w_{1,k}^x\,dx\,dy\,dt.$$

(ii) *There exist $\theta > 0$ and a continuous function ε with $\varepsilon(0) = 0$, still depending only on p and the smoothness of μ_k and R_k, such that*

$$-\int_0^s \int_{\mathbb{T}^{2d}} K_h(x-y)\,(\operatorname{div}_x u_k^x - \operatorname{div}_y u_k^y)\,\chi'(\delta\rho_k)\,\bar\rho_k\,w_{1,k}^x\,w_{1,k}^y\,dx\,dy\,dt$$
$$\le C\,\|K_h\|_{L^1}\,\left(\varepsilon(h) + h^\theta\right)$$
$$+\,C\int_0^s \int_{\mathbb{T}^{2d}} K_h(x-y)\left(1 + (\rho_k^x)^{\tilde\gamma} + \tilde P_k^x\,\rho_k^x + \Theta_k^x + (\rho_k^y)^{\tilde\gamma} + \tilde P_k^y\,\rho_k^y + \Theta_k^y\right)$$
$$\cdot\,\chi(\delta\rho_k)\,w_{1,k}^x\,w_{1,k}^y\,dx\,dy\,dt.$$

For instance, if μ_k and R_k belong to $W^{s,1}$ for some $s > 0$, then one may take $\varepsilon(h) = h^\theta$ for some $\theta > 0$.

(2) In the case with diffusion, more terms have to be considered, but one can prove a very similar type of result with

LEMMA 8.5. *Assume that ρ_k solves* (5.1) *and that* (5.6), (5.5), (5.4) *with $\gamma > d/2$ and $p > 2$ hold. Assume, moreover, that u solves* (5.2) *with μ_k compact in L^1 and satisfying* (5.3), *R_k compact in L^1, P_k satisfying* (5.8). *Then there exists a continuous function $\varepsilon(.)$ with $\varepsilon(0) = 0$ and depending only* *on the smoothness of μ_k and R_k such that*

$$
\begin{aligned}
&-\frac{1}{2}\int_0^s\int_{\mathbb{T}^{4d}} K_h(x-y)\,(\mathrm{div}_x u_k^x - \mathrm{div}_y u_k^y)\,\chi'(\delta\rho_k)\\
&\quad\cdot \bar\rho_k\, W_{0,k}^{x,y}\,\overline{K}_h(x-z)\,\overline{K}_h(y-w)\,dx\,dy\,dz\,dw\,dt\\
&-\frac{1}{2}\int_0^t\int_{\mathbb{T}^{4d}} K_h(x-y)(\mathrm{div}_x u_k^x + \mathrm{div}_y u_k^y)(\chi'(\delta\rho_k)\delta\rho_k - 2\chi(\delta\rho_k))\\
&\quad\cdot W_{0,k}^{z,w}\overline{K}_h(x-z)\overline{K}_h(y-w)\,dx\,dy\,dz\,dw\,dt\\
&-\frac{\lambda}{2}\int_0^s\int_{\mathbb{T}^{3d}} K_h(x-y)\chi(\delta\rho_k)\,\overline{K}_h(x-z)\,M|\nabla u_k|^z\,w_{0,k}^z\,dx\,dz\,dt\\
&\quad\le C\,\|K_h\|_{L^1}\,(\varepsilon(h)+\varepsilon_h(k)) + C\int_0^s\int_{\mathbb{T}^{4d}} K_h(x-y)\,\chi(\delta\rho_k)\,W_{0,k}^{z,w}\\
&\qquad\cdot\overline{K}_h(x-z)\,\overline{K}_h(y-w)\,dx\,dy\,dz\,dw\,dt,
\end{aligned}
$$

with $\varepsilon_h(k)\to 0$ as $k\to\infty$ for a fixed h.

Proof of Lemmas 8.4 *and* 8.5. The computations are very similar for (i) and (ii) in Lemma 8.4 and for Lemma 8.5. For simplicity, in order to treat the proofs together as much as possible, we denote

$$
\begin{aligned}
&G_{1,k}^{x,y} = \chi_1^{x,y}\,w_{1,k}^x, \quad G_{2,k}^{x,y} = \chi'(\delta\rho_k)\,\bar\rho_k\,w_{1,k}^x\,w_{1,k}^y,\\
&G_{0,k}^{x,y} = \frac{1}{2}\chi'(\delta\rho_k)\,\bar\rho_k\int_{\mathbb{T}^{2d}} W_{0,k}^{z,w}\,\overline{K}_h(x-z)\,\overline{K}_h(y-w)\,dz\,dw.
\end{aligned}
$$

The first step is to truncate: Denote $I_k^L = \phi(\rho_k^x/L)\,\phi(\rho_k^y/L)$ for some smooth and compactly supported ϕ,

$$
\begin{aligned}
&-\int_0^s\int_{\mathbb{T}^{2d}} K_h(x-y)\,(\mathrm{div}_x u_k^x - \mathrm{div}_y u_k^y)\,G_{i,k}^{x,y}\,dx\,dy\,dt\\
&\quad\le C\,\|K_h\|_{L^1}\,L^{-\theta_0} - \int_0^s\int_{\mathbb{T}^{2d}} K_h(x-y)\,(\mathrm{div}_x u_k^x - \mathrm{div}_y u_k^y)\,G_{i,k}^{x,y}\,I_k^L\,dx\,dy\,dt.
\end{aligned}
$$

Here due to the property of χ, for $i=0,1,2$, $G_{i,k}^{x,y}\le C\,(\rho_k^x+\rho_k^y)$ (even $G_2\le 2$) and consequently, as $\mathrm{div}\,u_k\in L^2$ uniformly, only $p>2$ is required with $\theta_0=(p-2)/2>0$. Introduce an approximation $\mu_{k,\eta}$ of μ_k, satisfying (5.3) and such that

$$
\begin{aligned}
&\|\mu_{k,\eta}\|_{W^{2,\infty}_{t,x}} \le C\,\eta^{-2}, \quad \|\mu_{k,\eta}-\mu_k\|_{L^1}\le\varepsilon_0(\eta),\\
&\int_0^T\int_{\mathbb{T}^{2d}} K_h(x-y)\,|\mu_{k,\eta}^x - \mu_{k,\eta}^y|\,dx\,dy\,dt \le \|K_h\|_{L^1}\,\varepsilon_0(h),
\end{aligned}
\tag{8.8}
$$

from (5.13). Use (5.2) to decompose

$$
-\int_{\mathbb{T}^{2d}} K_h(x-y)\,(\mathrm{div}_x u_k^x - \mathrm{div}_y u_k^y)\,G_{i,k}^{x,y}\,I_k^L\,dx\,dy = 2\,A_i + 2\,B_i + 2\,E_i,
$$

with

$$A_i = -\int_{\mathbb{T}^{2d}} K_h(x-y)\,(P_k(\rho_k^x)) - P_k(\rho_k^y))\,G_{i,k}^{x,y}\,\frac{I_k^L}{\mu_{k,\eta}^x}\,dx\,dy$$

and

$$B_i = \int_{\mathbb{T}^{2d}} K_h(x-y)\,\tilde R_k^{x,y}\,G_{i,k}^{x,y}\,\frac{I_k^L}{\mu_{k,\eta}^x}\,dx\,dy,$$

where

$$\tilde R_k^{x,y} = R_k^x - R_k^y + \mu_k^y\,\mu_{k,\eta}^x\,\mathrm{div}_{\,y} u_k^y\,\left(\frac{1}{\mu_{k,\eta}^x} - \frac{1}{\mu_k^y}\right) - \mu_k^x\,\mu_{k,\eta}^x\,\mathrm{div}_{\,x} u_k^x\,\left(\frac{1}{\mu_{k,\eta}^x} - \frac{1}{\mu_k^x}\right).$$

Finally,

$$E_i = \int K_h(x-y)\,(F_k^y - F_k^x)\,G_{i,k}^{x,y}\,\frac{I_k^L}{\mu_{k,\eta}^x}\,dx\,dy,$$

with F_k the viscous effective flow, namely,

$$F_k = \Delta^{-1}\,\mathrm{div}\,(\partial_t(\rho_k\,u_k) + \mathrm{div}\,(\rho_k\,u_k\otimes u_k)).$$

(I) For B_i by the compactness of R_k, μ_k, estimates (8.8) and (5.13), and by (5.3),

$$\begin{aligned}B_i &\leq C\,L\int_0^t\int_{\mathbb{T}^{2d}} K_h(x-y)\,|\tilde R_k^{x,y}|\,dx\,dy\,dt\\ &\leq C\,L\,(\varepsilon_0(h)+\varepsilon_0(\eta))\,\|K_h\|_{L^1(\mathbb{T}^d)}.\end{aligned}$$

Note that again $|G_{i,k}^{x,y}| \leq C\,(\rho_k(x)+\rho_k(y))$ for $i=0,\,1,\,2$.

(II) For E_i, we use Lemma 8.3 by simply defining

$$\Phi_i(t,x,y) = G_{i,k}^{x,y}\,I_k^L\,\frac{1}{\mu_{k,\eta}^x}.$$

By (5.3),

$$\|\Phi_i\|_{L^\infty((0,T)\times\mathbb{T}^{2d})} \leq C\,L.$$

As for the time derivative of Φ, for $i=1,\,2$, G_i is a combination of functions of $\rho_k(t,x)$, $\rho_k(t,y)$ and w_i, which all satisfy the same transport equation (with different right-hand sides). By (5.1),

$$\begin{aligned}&partial_t G_{i,k}^{x,y} + \mathrm{div}_{\,x}\left(u_k^x\,G_{i,k}^{x,y}\right) + \mathrm{div}_{\,y}\,(u_k^y\,G_i^{x,y})\\ &\qquad = f_{1,i,k}^{x,y}\,\mathrm{div}_{\,x} u_k^x + f_{2,i,k}^{x,y}\,\mathrm{div}_{\,y} u_k^y + f_{3,i,k}^{x,y}\,D_{i,k}^x\, + f_{4,i}\,D_{i,k}^y,\end{aligned}$$

where the $D_{i,k}$ are the penalizations introduced in Section 7.2 and the $f_{n,i,k}^{x,y}$ are again combinations of functions of ρ_k^x, ρ_k^y, $w_{i,k}^x$ and $w_{i,k}^y$. Finally by the smoothness of $\mu_{k,\eta}$,

$$\begin{aligned}&\partial_t\Phi_{i,k}^{x,y} + \mathrm{div}_{\,x}\left(u_k^x\,\Phi_{i,k}^{x,y}\right) + \mathrm{div}_{\,y}\left(u_k^y\,\Phi_{i,k}^{x,y}\right)\\ &\qquad = \tilde f_{1,i}\,\mathrm{div}_{\,x} u_k^x + \tilde f_{2,i}\,\mathrm{div}_{\,y} u_k^y + \tilde f_{3,i}\,D_{i,k}^x\, + \tilde f_{4,i}\,D_{i,k}^y + \Phi_{i,k}^{x,y}\,g_\eta,\end{aligned}$$

where every $\tilde f_{n,i}$ contains a factor I_k^L or a derivative of I_k^L and thus, for instance,

$$\|\tilde f_{n,i,k}\|_{L^\infty} \le C\,L \qquad \forall n,\ i.$$

It is then easy to check that the constant C_{Φ_i} as defined in Lemma 8.3 is bounded by $C\,L\,\eta^{-1}$.

The case $i = 0$ is slightly more complicated as W_0 is integrated against $\overline{K}_h$ so the equation on Φ_0 involves non-local terms and we have to take into account extra terms as mentioned in the statement of Lemma 8.4. By (7.2), denoting $w_{0,k,h} = \overline{K}_h \star w_{0,k}$,

$$\partial_t w^x_{0,k,h} + u^x_k \cdot \nabla_x w^x_{0,k,h} - \alpha_k \Delta_x w^x_{0,k,h} = -\overline{K}_h \star (D_0\, w_{0,k}) + R^x_h - \overline{K}_h \star (\operatorname{div} u_k\, w_{0,k}),$$

with

$$R^x_h = \int_{\mathbb{T}^d} \nabla \overline{K}_h(x-z)\cdot (u^x_k - u^z_k)\, w^z_{0,k}\,dz.$$

Remark that R_h is uniformly bounded in $L^2_{t,x}$ by usual commutator estimates. Finally as $\mu_{k,\eta}$ is smooth in time, one has

$$\begin{aligned}
&\partial_t \Phi^{x,y}_{0,k} + \operatorname{div}_x\left(u^x_k\,\Phi^{x,y}_{0,k}\right) + \operatorname{div}_y\left(u^y_k\,\Phi^{x,y}_{0;k}\right) - \alpha_k\,(\Delta_x + \Delta_y)\,\Phi^{x,y}_{0,k}\\
&= f_{1,0}\operatorname{div}_x u_k(t,x) + f_{2,0}\operatorname{div}_y u_k(t,y) + \alpha_k\,\left(f_{3,0}\,|(\nabla_x\rho_k)^x|^2 + f_{4,0}\,|(\nabla_x\rho_k)^y|^2\right)\\
&\quad - \frac{\Phi_\rho}{\mu_{k,\eta}}\,\left(\overline{K}_h \star (D_0\,w_{0,k}) + R_h - \overline{K}_h\star(\operatorname{div} u_k\,w_{0,k})\right)\\
&\quad - 2\,\frac{\alpha_k}{\mu_{k,\eta}}\,\nabla_x\Phi_\rho\cdot\nabla_x w_{0,k,h} + \Phi_0\,g_\eta,
\end{aligned}$$

where $\Phi_\rho = \delta\rho_k\,\bar\rho_k\,I_k^L$, g_η is a function involving first and second derivatives of $\mu_{k,\eta}$ in t and x and ∇u_k. The $f_{j,0}$ are combinations of functions of $\rho_k(t,x)$ and $\rho_k(t,y)$, multiplied by $w_{0,h}$, and involving $\phi(\rho_k(x)/L)$, $\phi'(\rho_k(x)/L)$, or $\phi''(\rho_k(x)/L)$ and the corresponding term with $\rho_k(y)$. By the L^∞ bounds on Φ_ρ, w_0, each $f_{j,0}$ and by (5.5), one obtains

$$\left\|\partial_t \int_{\mathbb{T}^d} \overline{K}_h(x-y)\,\Phi^{x,y}_{0,k}\,dy\right\|_{L^1_t\,W^{-1,1}_x} \le C\,L\,\eta^{-1}.$$

Therefore $C_\phi \le C\,L\,\eta^{-1}$. Thus for all three cases, Lemma 8.3 yields

$$E_i \le C\,L\,\eta^{-1}\,\|K_h\|_{L^1}\,(h^\theta + \varepsilon_h(k)) \tag{8.9}$$

for some $\theta > 0$ and $\varepsilon_h(k) = 0$ if $\alpha_k = 0$.

(III) *The term* A_0: *End of proof of Lemma* 8.4. The terms A_i are where lies the main difference between Lemmas 8.4 and 8.5 as P_k is not monotone in the first case and monotone after a certain threshold in the second. For this reason we now proceed separately for Lemmas 8.5 and 8.4. In the case with diffusion for Lemma 8.5, there also exist extra terms to handle, namely, $J+I$ with

$$J = -\frac{\lambda}{2}\int_0^s \int_{\mathbb{T}^{3d}} K_h(x-y)\chi(\delta\rho_k)\overline{K}_h(x-z)\, M|\nabla u_k|^z\, w_{0,k}^z\, dx\, dy\, dz\, dt$$

and

$$I = -\frac{1}{2}\int_0^s \int_{\mathbb{T}^{4d}} K_h(x-y)\,(\operatorname{div} u_k^x + \operatorname{div} u_k^y) \cdot (\chi'(\delta\rho_k)\,\delta\rho_k - 2\,\chi(\delta\rho_k))\; W_{0,k}^{x,y}\,\overline{K}_h(x-z)\,\overline{K}_h(y-w)\, dx\, dy\, dz\, dw\, dt.$$

We decompose this last term in a manner similar to what we have just done, first of all by introducing the truncation of ρ_k:

$$I \le C\,\|K_h\|_{L^1}\, L^{-\theta_0} \cdot (\chi'(\delta\rho_k)\,\delta\rho_k - 2\,\chi(\delta\rho_k))\, I_k^L\, W_{0,k}^{x,y}\,\overline{K}_h(x-z)\,\overline{K}_h(y-w)\, dx\, dy\, dz\, dw\, dt,$$

again with $\theta_0 = (p-2)/2$. Now introduce the μ_k:

$$\begin{aligned} I \le C\,\|K_h\|_{L^1}\, L^{\theta} &- \frac{1}{2}\int_0^s \int_{\mathbb{T}^{4d}} K_h(x-y)\,(\mu_k^x \operatorname{div}_x u_k^x + \mu_k^y \operatorname{div}_x u_k^y) \\ &\cdot \frac{I_k^L}{\mu_{k,\eta}(t,x)}\,(\chi'(\delta\rho_k)\,\delta\rho_k - 2\,\chi(\delta\rho_k))\; W_{0,k}^{x,y} \\ &\cdot \overline{K}_h(x-z)\,\overline{K}_h(y-w)\, dx\, dy\, dz\, dw\, dt \\ &+ \frac{1}{2}\int_0^s \int_{\mathbb{T}^{4d}} K_h(x-y) H_k^{x,y}(\chi'\,\delta\rho_k - 2\,\chi(\delta\rho_k))\, I_k^L\, W_{0,k}^{x,y}\,\overline{K}_h(x-z) \\ &\cdot \overline{K}_h(y-w)\, dx\, dy\, dw\, dz\, dt, \end{aligned}$$

where

$$H_k^{x,y} = \mu_k^x \operatorname{div}_x u_k^x \left(\frac{1}{\mu_{k,\eta}^x} - \frac{1}{\mu_k^x}\right) - \mu_k^y \operatorname{div}_y u_k^y \left(\frac{1}{\mu_k^y} - \frac{1}{\mu_{k,\eta}^x}\right).$$

By the compactness of μ_k, one has that

$$\begin{aligned} \int_0^s \int_{\mathbb{T}^{4d}} & K_h(x-y) H_k^{x,y}(\chi'\,\delta\rho_k - 2\,\chi(\delta\rho_k))\; W_{0,k}^{x,y}\,\overline{K}_h(x-z) \\ &\cdot \overline{K}_h(y-w)\, dx\, dy\, dw\, dz\, dt \\ &\le \|K_h\|_{L^1}\,\varepsilon_0(h)\,\|u_k\|_{L_t^2 H_x^1}\,\|\rho_k\|_{L^2_{t,x}} \le C\,\varepsilon_0(h)\,\|K_h\|_{L^1}. \end{aligned}$$

This implies that

$$\begin{aligned} I \le &-\frac{1}{2}\int_0^s \int_{\mathbb{T}^{4d}} K_h(x-y)\,(\mu_k^x \operatorname{div}_x u_k^x + \mu_k^y \operatorname{div}_y u_k^y) \\ &\cdot \frac{I_k^L}{\mu_{k,\eta}^x}\,(\chi'(\delta\rho_k)\,\delta\rho_k - 2\,\chi(\delta\rho_k))\; W_{0,k}^{x,y}\,\overline{K}_h(x-z)\,\overline{K}_h(y-w)\, dx\, dy\, dz\, dw\, dt \\ &+ C\,\|K_h\|_{L^1}(\,L^{-\theta} + \varepsilon_0(h)). \end{aligned}$$

Using (5.2) or, namely, that $\mu_k \mathrm{div}\, u_k = F_k + R_k + P_k(\rho_k)$, the quantity A_0+I+J may be written

$$A_0 + I + J \le C\, \|K_h\|_{L^1}(\, L^{-\theta} + \varepsilon_0(h)) + I_1 + I_2, \tag{8.10}$$

with

$$I_1 = A_0 - \frac{1}{2}\int_0^s \int_{\mathbb{T}^{4d}} K_h(x-y)\, (P_k^{x,\rho_k^x} + P_k^{y,\rho_k^y}) \frac{I_k^L}{\mu_{k,\eta}^x} \cdot (\chi'(\delta\rho_k)\, \delta\rho_k - 2\, \chi(\delta\rho_k))\; W_{0,k}^{x,y}\, \overline{K}_h(x-z)\, \overline{K}_h(y-w)\, dx\, dy\, dw\, dz\, dt$$

and

$$I_2 = -\frac{1}{2}\int_0^s \int_{\mathbb{T}^{4d}} K_h(x-y)\, (F_k^x + R_k^x + F_k^y + R_k^y) \frac{I_k^L}{\mu_{k,\eta}^x} \cdot (\chi'(\delta\rho_k)\, \delta\rho_k - 2\, \chi(\delta\rho_k))\; W_{0,k}^{x,y}\, \overline{K}_h(x-z)\, \overline{K}_h(y-w)\, dx\, dy\, dw\, dz\, dt - \frac{\lambda}{2}\int_0^s \int_{\mathbb{T}^{3d}} K_h(x-y)\chi(\delta\rho_k)\overline{K}_h(x-z)\, M|\nabla u_k|^z\, w_{0,k}^z\, dx\, dy\, dz\, dt.$$

In this case with diffusion, because P_k is essentially monotone, the term A_0 is mostly dissipative and helps control the rest. More precisely,

$$I_1 = -\frac{1}{2}\int_0^s \int_{\mathbb{T}^{2d}} \Big[K_h(x-y)\, \big[(P_k^{x,\rho_k^x} - P_k^{y,\rho_k^y})\chi'(\delta\rho_k)\bar\rho_k + (P_k^{x,\rho_k^x} + P_k^{y,\rho_k^y})(\chi'(\delta\rho_k)\delta\rho_k - 2\chi(\delta\rho_k))\big] \cdot \frac{I_k^L}{\mu_{k,\eta}^x} \int_{\mathbb{T}^{2d}} W_{0,k}^{x,y}\, \overline{K}_h(x-z)\, \overline{K}_h(y-w) dz\, dw\Big] dx\, dy\, dt.$$

As $P_k \ge 0$ and by (8.1), $\chi'(\delta\rho_k)\delta\rho_k - 2\chi(\delta\rho_k) \ge -\chi'(\delta\rho_k)\delta\rho_k$, thus

$$(P_k^{x,\rho_k^x}) - P_k^{y,\rho_k^y}))\chi'\bar\rho_k + (P_k^{x,\rho_k^x} + P_k^{y,\rho_k^y})(\chi'\delta\rho_k - 2\chi(\delta\rho_k)) \ge \chi'(\delta\rho_k)\, \Big[(P_k^{x,\rho_k^x} - P_k^{y,\rho_k^y})\, \bar\rho_k - (P_k^{x,\rho_k^x} + P_k^{y,\rho_k^y})\, \delta\rho_k\Big].$$

Without loss of generality, we may assume that $\rho_k(x) \ge \rho_k(y)$ and hence $\chi'(\delta\rho_k) \ge 0$. Develop

$$(P_k^{x,\rho_k^x} - P_k^{y,\rho_k^y})\, \bar\rho_k - (P_k^{x,\rho_k^x} + P_k^{y,\rho_k^y})\, \delta\rho_k = 2\, P_k^{x,\rho_k^x}\, \rho_k^y - 2\, P_k^{y,\rho_k^y}\, \rho_k^x.$$

We now use the quasi-monotonicity (5.8) of $P_k^{x,s}/s$. First of all, if $\rho_0 \le \rho_k^y \le \rho_k^x$, then necessarily P_k depends only on ρ_k^x or ρ_k^y plus $\tilde P_k$. Thus

$$P_k^{x,\rho_k^x}\, \rho_k^y - \; P_k^{y,\rho_k^y}\, \rho_k^x \ge -|\tilde P_k^x - \tilde P_k^y|. \tag{8.11}$$

If $\rho_k^y \le \rho_0$, then by (5.8), $P_k^{x,s} \to \infty$ as $s \to \infty$ while P_k^{y,ρ_k^y} is bounded. Hence there exists $\bar\rho$ large enough with respect to ρ_0, such that if $\rho_k^x \ge \bar\rho$, then again

$$P_k^{x,\rho_k^x}\,\rho_k^y - P_k^{y,\rho_k^y}\,\rho_k^x \ge 0.$$

The only case where one does not have the right sign is hence where both ρ_k^x and ρ_k^y are bounded by $\bar\rho$ and ρ_0. Therefore, using the local regularity of P_k given by (5.8),

$$(P_k^{x,\rho_k^x} - P_k^{y,\rho_k^y})\,\bar\rho_k - (P_k^{x,\rho_k^x} + P_k^{y,\rho_k^y})\,\delta\rho_k \ge -\bar P\,|\delta\rho_k| - Q_k(t,x,y). \tag{8.12}$$

Introducing estimates (8.11) and (8.12) in I_1 yields

(8.13)

$$\begin{aligned} I_1 \le \bar P \int_0^s\!\!\int_{\mathbb{T}^{2d}} & K_h(x-y)\,(|\tilde P_k^x - \tilde P_k^y| + Q_k + |\delta\rho_k|)\,\frac{|\chi'(\delta\rho_k)|\,I_k^L}{\mu_{k,\eta}^x} \\ & \cdot \left[\int_{\mathbb{T}^{2d}} W_{0,k}^{z,w}\,\overline{K}_h(x-z)\,\overline{K}_h(y-w)\,dz\,dq\right]\,dx\,dy\,dt \\ \le{} & \varepsilon_0(h)\,\|K_h\|_{L^1} \\ & + \bar P \int_0^s \int_{\mathbb{T}^{4d}} K_h(x-y)\chi(\delta\rho_k)\,W_{0,k}^{z,w}\,\overline{K}_h(x-z)\,\overline{K}_h(y-w)\,dx\,dy\,dz\,dw\,dt. \end{aligned}$$

Now turning to I_2, we observe that $\mu_k\,M|\nabla u_k| \ge \mu_k\,\mathrm{div}\,u_k \ge F_k + R_k$ and that $\chi(\delta_k) \ge (2\chi(\delta_k) - \chi'(\delta_k)\,\delta_k)/C$. Therefore for λ large enough, using $W_{0,k}^{z,w} = w_{0,k}^z + w_{0,k}^w$ and the symmetry, we find

$$\begin{aligned} I_2 \le -\frac{1}{2}\int_0^t \int_{\mathbb{T}^{4d}} & K_h(x-y) \\ & \cdot (F_k^x - F_k^z + R_k^x - R_k^z + F_k^y - F_k^z + R_k^y - R_k^z) \\ & \cdot \frac{I_k^L}{\mu_{k,\eta}^x}(\chi'(\delta\rho_k)\,\delta\rho_k - 2\,\chi(\delta\rho_k))W_{0,k}^{z,w}\,\overline{K}_h(x-z)\,\overline{K}_h(y-w). \end{aligned}$$

The differences in the R_k are controlled by the compactness of R_k and the differences in the effective viscous flux F_k by Lemma 8.3 as for the terms E_i. Hence, finally

$$I_2 \le C\,\|K_h\|_{L^1}\,(\varepsilon_0(h) + L\,\eta^{-1}\,h^\theta). \tag{8.14}$$

Conclusion of proof of Lemma 8.5. We sum up the contributions from B_0 in (8.3), E_0 in (8.9), $A_0 + I + J$ in (8.10) together with the bounds on I_1 in

(8.13) and I_2 in (8.14) to obtain

$$\begin{aligned}
&-\frac{1}{2}\int_0^s\int_{\mathbb{T}^{2d}} K_h(x-y)\,(\mathrm{div}_x u_k^x-\mathrm{div}_y u_k^y)\,G_{0,k}^{x,y}\,dx\,dy\,dt\\
&-\frac{1}{2}\int_0^s\int_{\mathbb{T}^{4d}} K_h(x-y)\,(\mathrm{div}_x u_k^x+\mathrm{div}_y u_k^y)\,(\chi'(\delta\rho_k)\,\delta\rho_k-2\chi(\delta\rho_k))\\
&\cdot W_{0,k}^{z,w}\,\overline{K}_h(x-z)\,\overline{K}_h(y-w)\,dx\,dy\,dw\,dz\,dt\\
&-\frac{\lambda}{2}\int_0^s\int_{\mathbb{T}^{3d}} K_h(x-y)\,\chi(\delta\rho_k)\,\overline{K}_h(x-z)\,M\,|\nabla u_k|^z\,w_0^z\,dx\,dy\,dz\,dt\\
\le{}& C\,\|K_h\|_{L^1}\left(L^{-\theta_0}+L\,(\varepsilon_0(h)+\varepsilon_0(\eta))+L\,\eta^{-1}\,h^\theta\right)\\
&+C\int_0^s\int_{\mathbb{T}^{4d}} K_h(x-y)\,\chi(\delta\rho_k)\,W_{0,k}^{z,w}\,\overline{K}_h(x-z)\,\overline{K}_h(y-w)\,dx\,dy\,dz\,dw\,dt.
\end{aligned}$$

Just optimizing in L and η leads to the desired $\varepsilon(h)$ and concludes the proof of Lemma 8.5. □

(IV) *The term A_i with $i = 1,\ 2$: End of proof of Lemma* 8.4. It now remains to analyze more precisely the terms $(P_k^{x,\rho_k^x}) - P_k^{y,\rho_k^y})\,G_{i,k}^{x,y}$ for $i = 1, 2$ concerning the case without diffusion but with non-monotone pressure. We will split the study into three cases but remark that now the possible dependence of P_k in terms of x affects the estimates. For this reason, we carefully write this dependence explicitly.

Case (1): *The case* $(P_k^{x,\rho_k^x} - P_k^{y,\rho_k^y})\delta\rho_k \ge 0$. Since G_2 obviously have the same sign as $\delta\rho_k$, one simply has

$$(P_k^{x,\rho_k^x} - P_k^{y,\rho_k^y})\,G_{2,k}^{x,y} \ge 0,$$

We can check this is the same for $G_{1,k}^{x,y}$, namely,

$$\begin{aligned}
(P_k^{x,\rho_k^x}&-P_k^{y,\rho_k^y})\,G_{1,k}^{x,y}\\
&= (P_k^{x,\rho_k^x} - P_k^{y,\rho_k^y})\left(\frac{1}{2}\chi'(\delta\rho_k)\,\bar\rho_k + \chi(\delta\rho_k) - \frac{1}{2}\chi'(\delta\rho_k)\,\delta\rho_k\right)w_{1,k}^x\\
&\ge |P_k^{x,\rho_k^x} - P_k^{y,\rho_k^y}|\left(\frac{1}{2}|\chi'(\delta\rho_k)|\,\bar\rho_k - \left|\chi(\delta\rho_k) - \frac{1}{2}\chi'(\delta\rho_k)\,\delta\rho_k\right|\right)w_{1,k}^x\\
&\ge 0,
\end{aligned}$$

by (8.1) as $\bar\rho_k \ge |\delta\rho_k|$. Therefore, in that case the terms have the right sign and can be dropped.

Case (2): *The case* $(P_k^{x,\rho_k^x} - P_k^{y,\rho_k^y})\delta\rho_k < 0$ *and* $\rho_k^y \le \rho_k^x/2$ *or* $\rho_k^y \ge 2\,\rho_k^x$ *for some constant* C. For $i = 1$, first assume that $P_k^{x,\rho_k^x} \ge P_k^{y,\rho_k^y}$ while $\rho_k^y \ge 2\,\rho_k^x$:

$$(P_k^{x,\rho_k^x} - P_k^{y,\rho_k^y})\,G_{1,k}^{x,y} \ge -\,P_k(\rho_k^x)\,(|\chi'(\delta\rho_k)|\,\bar\rho_k + \chi(\delta\rho_k))\,w_{1,k}^x.$$

Now observe that since $\rho_k^y \geq 2\,\rho_k^x$, then

$$|\chi'(\delta\rho_k)|\,\bar\rho_k \leq \frac{3}{2}\,|\chi'(\delta\rho_k)|\,\rho_k^y \leq 3\,|\chi'(\delta\rho_k)|\,|\delta\,\rho_k| \leq C\,\chi(\delta\rho_k),$$

by (8.1). Therefore in that case, by (5.9),

$$(P_k^{x,\rho_k^x} - P_k^{y,\rho_k^y})\,G_{1,k}^{x,y} \geq -C\,((\rho_k^x)^{\tilde\gamma} + \Theta_k^x)\;\chi(\delta\rho_k)\,w_{1,k}^x,$$

Note that the result is not symmetric in x and y. We also have to check that $P_k(x,\rho_k^x) \leq P_k(y,\rho_k^y)$ and $\rho_k^y \leq \rho_k^x/2$. Then simply bound since now $\rho_k^y \leq \rho_k^x$:

$$\begin{aligned}(P_k^{x,\rho_k^x} - P_k^{y,\rho_k^y})\,G_{1,k}^{x,y} &\geq -C\,((\rho_k^y)^{\tilde\gamma} + \Theta_k^y)\;\chi(\delta\rho_k)\,w_{1,k}^x\\ &\geq -C\,((\rho_k^x)^{\tilde\gamma} + \Theta_k^y)\;\chi(\delta\rho_k)\,w_{1,k}^x.\end{aligned}$$

In both cases, one finally obtains

$$(P_k^{x,\rho_k^x} - P_k^{y,\rho_k^y})\,G_{1,k}^{x,y} \geq -\,C\,((\rho_k^x)^{\tilde\gamma} + \Theta_k^x)\;\chi(\delta\rho_k)\,w_{1,k}^x - |\Theta_k^x - \Theta_k^y|\,\bar\rho_k\,w_{1,k}^x.$$

For $i = 2$, the calculations are similar (simpler in fact) for $G_{2,k}^{x,y}$, and this lets us deduce that if $P_k^{x,\rho_k^x} - P_k^{y,\rho_k^y}$ and $\rho_k^x - \rho_k^y$ have different signs but $\rho_k^y \leq \rho_k^x/2$ or $\rho_k^y \geq 2\,\rho_k^x$, then

$$(P_k^{x,\rho_k^x} - P_k^{y,\rho_k^y})\,G_{2,k}^{x,y} \geq -\,C\,((\rho_k^x)^{\tilde\gamma} + \tilde P_k^x\,\rho_k^x + (\rho_k^y)^{\tilde\gamma} + \tilde P_k^y\,\rho_k^y)\;\chi(\delta\rho_k)\,w_{1,k}^x\,w_{1,k}^y.$$

Case (3): *For $i = 1, 2$, the situation where $P_k^{x,\rho_k(x)} - P_k^{y,\rho_k(y)}$ and $\rho_k^x - \rho_k^y$ have different signs but $\rho_k^x/2 \leq \rho_k^y \leq 2\,\rho_k^x$.* Then using the Lipschitz bound on P_k given by (5.9), one bluntly estimates

$$\left|P_k^{x,\rho_k^x} - P_k^{y,\rho_k^y}\right| \leq C\,((\rho_k^x)^{\tilde\gamma-1} + \tilde P_k^x + (\rho_k^y)^{\tilde\gamma-1} + \tilde P_k^y)\,|\delta\rho_k| + Q_k.$$

Now bounding the G_i by (8.1),

$$\begin{aligned}&(P_k^{x,\rho_k^x} - P_k^{y,\rho_k^y})\,G_{2,k}^{x,y}\\ &\qquad\leq C\,((\rho_k^x)^{\tilde\gamma} + \tilde P_k^x\,\rho_k^x + (\rho_k^y)^{\tilde\gamma} + \tilde P_k^y\,\rho_k^y)\,\chi(\delta\rho_k)\,w_{1,k}^x\,w_{1,k}^y + Q_k\,\bar\rho_k\,w_{1,k}^x\end{aligned}$$

and

$$\begin{aligned}(P_k^{x,\rho_k^x} - P_k^{y,\rho_k^y})\,G_{1,k}^{x,y} &\leq C\,((\rho_k^x)^{\tilde\gamma} + \tilde P_k^x\,\rho_k^x + (\rho_k^y)^{\tilde\gamma}\\ &\qquad + \tilde P_k^y\,\rho_k^y)\,\chi(\delta\rho_k)\,w_{1,k}^x + Q_k\,\bar\rho_k\\ &\leq C\,(1 + (\rho_k^x)^{\tilde\gamma} + \tilde P_k^x\,\rho_k^x)\,\chi(\delta\rho_k)\,w_{1,k}^x\\ &\qquad + (Q_k + |P_k^x - P_k^y|\,\bar\rho_k)\,\bar\rho_k\,w_{1,k}^x,\end{aligned}$$

as ρ_k^x and ρ_k^y are of the same order. From Proposition 7.2 point (i), we know that $w_{1,k}^x \leq e^{-\lambda\,(\rho_k^x)^{p-1}}$. One the other hand, we are precisely in the case where

ρ_k^x and ρ_k^x are of the same order. Hence $\bar\rho_k^l\, w_{1,k}$ is uniformly bounded for any $l>0$. Hence in this case, we finally obtain

$$(P_k^{x,\rho_k^x} - P_k^{y,\rho_k^y})\, G_{2,k}^{x,y} \le C\,((\rho_k^x)^{\tilde\gamma} + \tilde P_k^x\,\rho_k^x + (\rho_k^y)^{\tilde\gamma} + \tilde P_k^y\,\rho_k^y)\,\chi(\delta\rho_k)\,w_{1,k}^x\,w_{1,k}^y + Q_k^{x,y}$$

and

$$(P_k^{x,\rho_k^x} - P_k^{y,\rho_k^y})\, G_{1,k}^{x,y} \le C\,(1 + (\rho_k^x)^{\tilde\gamma} + \tilde P_k^x\,\rho_k^x)\,\chi(\delta\rho_k)\,w_{1,k}^x + Q_k^{x,y} + |P_k^x - P_k^y|.$$

From the analysis of these three cases, one has that

$$A_1 \le C\int K_h(x-y)\,(1 + (\rho_k^x)^{\tilde\gamma} + \tilde P_k^x\,\rho_k^x + \Theta_k^x)\,\chi(\delta\rho_k)\,w_{1,k}^x\,dx\,dy\,dt + C\int K_h(x-y)\,(Q_k^{x,y} + |\tilde P_k^x - \tilde P_k^y| + |\Theta_k^x - \Theta_k^y|)\,dx\,dy\,dt.$$

Therefore by the compactness properties of P_k and the estimate on Q_k in the assumption (5.8),

(8.15)
$$A_1 \le C\int_0^s\int_{\mathbb{T}^{2d}} K_h(x-y)\,(1 + (\rho_k^x)^{\tilde\gamma} + \tilde P_k^x\,\rho_k^x + \Theta_k^x)\,\chi(\delta\rho_k)\,w_{1,k}^x\,dx\,dy\,dt + \|K_h\|_{L^1}\,\varepsilon_0(h),$$

and

$$A_2 \le C\int_0^s\int_{\mathbb{T}^{2d}} K_h(x-y)\,(1 + (\rho_k^x)^{\tilde\gamma} + \tilde P_k^x\,\rho_k^x + \Theta_k^x + (\rho_k^y)^{\tilde\gamma} + \tilde P_k^y\,\rho_k^y + \Theta_k^y)\,\chi(\delta\rho_k)\,w_{1,k}^x\,w_{1,k}^y\,dx\,dy\,dt + \|K_h\|_{L^1}\,\varepsilon_0(h). \tag{8.16}$$

Conclusion of the proof of Lemma 8.4. Summing up every term, namely, (8.3)–(8.9) and (8.15)–(8.16), we eventually find that

$$\begin{aligned}
&-\int_0^s\int_{\mathbb{T}^{2d}} K_h(x-y)\,(\operatorname{div}_x u_k^x - \operatorname{div}_y u_k^y)\,G_{1,k}^{x,y}\,dx\,dy\,dt\\
&\quad\le C\,\|K_h\|_{L^1}\left(L^{-\theta} + L\,(\varepsilon_0(h) + \varepsilon_0(\eta)) + L\,\eta^{-1}\,h^\theta\right)\\
&\qquad + C\int_0^s\int_{\mathbb{T}^{2d}} K_h(x-y)\,(1 + (\rho_k^x)^{\tilde\gamma} + \tilde P_k^x\,\rho_k^x + \Theta_k)\,\chi(\delta\rho_k)\,w_{1,k}^x\,dx\,dy\,dt
\end{aligned}$$

while

$$\begin{aligned}
&-\int_0^s\int_{\mathbb{T}^{2d}} K_h(x-y)\,(\operatorname{div}_x u_k^x - \operatorname{div}_y u_k^y)\,G_{2,k}^{x,y}\,dx\,dy\,dt\\
&\quad\le C\,\|K_h\|_{L^1}\left(L^{-\theta} + L\,(\varepsilon_0(h) + \varepsilon_0(\eta)) + L\,\eta^{-1}\,h^\theta\right)\\
&\qquad + C\int_0^s\int_{\mathbb{T}^{2d}} K_h(x-y)\,\Big(1 + (\rho_k^x)^{\tilde\gamma} + \tilde P_k^x\,\rho_k^x + \Theta_k^x + (\rho_k^y)^{\tilde\gamma}\\
&\qquad\qquad + \tilde P_k^y\,\rho_k^y + \Theta_k^y\Big)\,\chi(\delta\rho_k)\,w_{1,k}^x\,w_{1,k}^y\,dx\,dy\,dt.
\end{aligned}$$

To conclude the proof of Lemma 8.4, one optimizes in η and L. Just remark that since the inequalities depend polynomially in L and η, then the result depends on ε_0^θ for some θ.

8.4. *Conclusion of the proofs of Theorems* 5.1 *and* 5.2. Now we combine Lemma 8.1 with Lemma 8.5 or 8.4, and we finally use Proposition 7.2. Let us summarize the required assumptions. In all cases one assumes that ρ_k solves (5.1) and that $\operatorname{div} u_k$ is coupled with ρ_k through (5.2); bounds are assumed on the viscosity as per (5.3), on the time derivative of $\rho_k\, u_k$ per (5.6) and on u_k per (5.5). Finally the viscosity μ_k and the force term R_k are assumed to be compact in $L^1((0,T)\times\mathbb{T}^d)$. Moreover,

- In the case with diffusion, $\alpha_k > 0$, one assumes that the pressure term P_k satisfies (5.8) and the bounds (5.4) on ρ_k with $\gamma > d/2$ and $p > 2$.
- In the case without diffusion, $\alpha_k = 0$, one needs only (5.9) on the pressure P_k and the bounds (5.4) on ρ_k with $\gamma > d/2$ and $p > 2$. Moreover for Proposition 7.2, it is necessary that $p \geq \tilde\gamma$. (In general, $\tilde\gamma = \gamma < p$ so this is not a big issue.)

Then by taking λ large enough, using the properties of $\overline{K}_h$ and using a simple Gronwall lemma, one obtains

(8.17)
$$\begin{aligned}
&\int_{h_0}^1 \int_{\mathbb{T}^{4d}} \overline{K}_h(x-z)\,\overline{K}_h(y-w)\,(w_{0,k}^z + w_{0,k}^w)\,K_h(x-y)\,\chi(\delta\rho_k)\,dx\,dy\,dz\,dw\frac{dh}{h}\\
&\leq C\int_{\mathbb{T}^{4d}} \overline{K}_{h_0}(x-z)\,\overline{K}_{h_0}(y-w)\,(w_{0,k}^z + w_{0,k}^w)\,\mathcal{K}_{h_0}(x-y)\,\chi(\delta\rho_k)\,dx\,dy\,dz\,dw\\
&\leq C\left(|\log h_0|^{1/2} + \bar\varepsilon_{h_0}(k) + \int_{h_0}^1 \varepsilon(h)\frac{dh}{h}\right),
\end{aligned}$$

where we have used the monotonicity of $1/(h+|x|)^a$ to simplify the integration in h.

For the case without diffusion,

(8.18)
$$\begin{aligned}
&\int_{h_0}^1 \int_{\mathbb{T}^{2d}} (w_{1,k}^x + w_{1,k}^y)\,\overline{K}_h(x-y)\,\chi(\delta\rho_k)\,dx\,dy\frac{dh}{h}\\
&= \int_{\mathbb{T}^{2d}} (w_{1,k}^x + w_{1,k}^y)\,\mathcal{K}_{h_0}(x-y)\,\chi(\delta\rho_k)\,dx\,dy \leq C\left(|\log h_0|^{1/2} + \int_{h_0}^1 \varepsilon(h)\,\frac{dh}{h}\right),
\end{aligned}$$

finally with

$$\int_{\mathbb{T}^{2d}} w_{1,k}^x\, w_{1,k}^y\, K_h(x-y)\,\chi(\delta\rho_k)\,dx\,dy \leq C\,\|K_h\|_{L^1}\,\left(h^\theta + \varepsilon(h)\right), \tag{8.19}$$

where ε depends only on the smoothness of μ_k and R_k and $p > 2$.

The key point in all three cases is to be able to remove the weights from those estimates. For that, one uses point (ii) of Proposition 7.2.

The case with $w^x_{0,k}+w^y_{0,k}$. Let t be fixed. Denote $\omega_\eta=\{x:[\overline{K}_h\star w_{0,k}]^x\leq\eta\}\subset\mathbb{T}^d$ for some parameter η that we will only choose in $(0,\ 1)$. Remark that

$$\begin{aligned}\int_{\mathbb{T}^{2d}}\mathcal{K}_{h_0}(x-y)\,\chi(\delta\rho_k)\,dx\,dy&=\int_{h_0}^1\int_{\mathbb{T}^{2d}}\overline{K}_h(x-y)\,\chi(\delta\rho_k)\,dx\,dy\,\frac{dh}{h}\\&=\int_{h_0}^1\int_{x\in\omega_\eta^c\ or\ y\in\omega_\eta^c}\overline{K}_h(x-y)\,\chi(\delta\rho_k)\,dx\,dy\,\frac{dh}{h}\\&\quad+\int_{h_0}^1\int_{x\in\omega_\eta\ and\ y\in\omega_\eta}\overline{K}_h(x-y)\chi(\delta\rho_k)\,dx\,dy\,\frac{dh}{h}.\end{aligned}$$

Now

$$\begin{aligned}&\int_{h_0}^1\int_{x\in\omega_\eta^c\ or\ y\in\omega_\eta^c}\overline{K}_h(x-y)\,\chi(\delta\rho_k)\,dx\,dy\,\frac{dh}{h}\\&\qquad\leq\frac{1}{\eta}\int_{h_0}^1\int_{\mathbb{T}^{2d}}\overline{K}_h(x-y)(\overline{K}_h\star w_0(x)+\overline{K}_h\star w_0(y))\,\chi(\delta\rho_k)\,dx\,dy\,\frac{dh}{h},\end{aligned}$$

while by point (iii) in Proposition 7.2, using that $\rho\in L^p((0,T)\times\mathbb{T}^d)$ with $p>2$ and recalling that $\chi(\xi)\leq C\,|\xi|$,

$$\begin{aligned}&\int_{h_0}^1\int_{x\in\omega_\eta\ and\ y\in\omega_\eta}\overline{K}_h(x-y)\,\chi(\delta\rho_k)\,dx\,dy\,\frac{dh}{h}\\&\qquad\leq 2\int_{h_0}^1\int_{\mathbb{T}^{2d}}\overline{K}_h(x-y)\,\rho_k\,\mathbb{I}_{\overline{K}_h\star w_0\leq\eta}\,dx\,dy\,\frac{dh}{h}\leq\frac{C\,|\log h_0|}{|\log\eta|^{1/2}}.\end{aligned}$$

Therefore combining this with (8.17), one obtains

$$\begin{aligned}&\int_{\mathbb{T}^{2d}}\mathcal{K}_{h_0}(x-y)\,\chi(\delta\rho_k)\,dx\,dy\\&\qquad\leq C\left(\frac{\bar\varepsilon_{h_0}(k)+|\log h_0|^{1/2}+\int_{h_0}^1\varepsilon(h)\frac{dh}{h}}{\eta}+\frac{\|\mathcal{K}_{h_0}\|_{L^1}}{|\log\eta|^{1/2}}\right),\end{aligned}$$

recalling from equation (8.3) that in this case,

$$\bar\varepsilon_{h_0}(k)=\alpha_k\int_{h_0}^1 h^{-2}\frac{dh}{h}+\int_{h_0}^1\varepsilon_h(k)\,\frac{dh}{h},$$

with $\varepsilon_h(k)$ the function introduced in Lemma 8.5. We may freely assume that $\varepsilon_h(k)$ is decreasing in h (i.e., increasing as $h\to 0$) and hence

$$\bar\varepsilon_{h_0}(k)\leq\alpha_k\,h_0^{-2}\,|\log h_0|+\varepsilon_{h_0}(k)\,|\log h_0|.$$

We also denote

$$\bar\varepsilon(h_0)=\frac{1}{|\log h_0|}\int_{h_0}^1\varepsilon(h)\,\frac{dh}{h}.$$

Remark that $\bar\varepsilon(h_0)\to 0$ since $\varepsilon(h)\to 0$. For instance if $\varepsilon(h)=h^\theta$, then $\bar\varepsilon(h_0)\sim|\log h_0|^{-1}$. The estimate then reads

$$\int_{\mathbb{T}^{2d}} \mathcal{K}_{h_0}(x-y)\,\chi(\delta\rho_k)\,dx\,dy \leq C\left(|\log h_0|\frac{\alpha_k h_0^{-2}+\varepsilon_{h_0}(k)+|\log h_0|^{-1/2}+\bar\varepsilon(h_0)}{\eta}+\frac{\|\mathcal{K}_{h_0}\|_{L^1}}{|\log\eta|^{1/2}}\right).$$

As $\|\mathcal{K}_{h_0}\|_{L^1}\sim|\log h_0|$, we optimize in η by taking

$$\eta=\Big(\max(1/2,\alpha_k h_0^{-2}+\varepsilon_{h_0}(k))+|\log h_0|^{-1/2}+\bar\varepsilon(h_0)\Big)^{1/2},$$

and we observe that indeed $\eta<1$ if h_0 is small enough. The following estimate is obtained if k is large enough with respect to h_0 and hence $\alpha_k\,h_0^{-2}+\varepsilon_{h_0}(k)<1/2$:

$$\int_{\mathbb{T}^{2d}} \mathcal{K}_{h_0}(x-y)\,\chi(\delta\rho_k)\,dx\,dy \leq \frac{C\|\mathcal{K}_{h_0}\|_{L^1}}{|\log\big(\alpha_k h_0^{-2}+\varepsilon_{h_0}(k)+|\log h_0|^{-1/2}+\bar\varepsilon(h_0)\big)|^{1/2}}.$$

Per Proposition 4.1, this gives the compactness of ρ_k as

$$\limsup_k\left[\frac{1}{\|\mathcal{K}_{h_0}\|_{L^1}}\int_0^T\int_{\mathbb{T}^{2d}}\mathcal{K}_{h_0}(x-y)\,\chi(\delta\rho_k)\,dx\,dy\,dt\right] \leq \frac{C}{|\log\big(|\log h_0|^{-1/2}+\bar\varepsilon(h_0)\big)|^{1/2}}\longrightarrow 0,$$

as $h_0\to 0$. And it proves case (i) of Theorem 5.1.

The case with $w_{1,k}^x+w_{1,k}^y$. Similarly, from (8.18), one then proves that in the corresponding case,

$$\int_{\mathbb{T}^{2d}} \mathcal{K}_{h_0}(x-y)\,\chi(\delta\rho_k)\,dx\,dy \leq C\,|\log h_0|\left(\frac{|\log h_0|^{-1/2}+\bar\varepsilon(h_0)}{\eta}+\frac{1}{|\log\eta|^\theta}\right) \leq \frac{C\,\|\mathcal{K}_{h_0}\|_{L^1}}{|\log\big(|\log h_0|^{-1/2}+\bar\varepsilon(h_0)\big)|^\theta},$$

again using part (ii) of Proposition 7.2 to get rid of the weights w_1^x and w_1^y as shown in the previous case concerning the weight w_0^x and w_0^y. In both cases, using Proposition 4.1 together with Lemma 6.6 in the second case, one concludes that ρ_k is compact in x and then in t, x. Thus we have shown case (ii) and concluded the proof of Theorem 5.1.

The case with $w^y_{1,k} w^x_{1,k}$. The situation is more complicated for (8.19) and the product $w^y_{1,k} w^x_{1,k}$. Indeed $w^x_{0,k} + w^y_{0,k}$ or $w^x_{1,k} + w^y_{1,k}$ are small only if both $w^x_{0,k}$ and $w^y_{0,k}$ are small (or the corresponding terms for $w_{1,k}$). But $w^x_{1,k}\, w^y_{1,k}$ can be small if either w^x_1 or w^y_1 is small. This was previously an advantage, then with simpler computations, but not here, and (8.19) does not provide compactness.

This is due to the fact that one does not control the size of $\{w_\eta \leq \eta\}$ but only the mass of ρ_k over that set. The difference between the two is the famous vacuum problem for compressible fluid dynamics, which is still unsolved.

The best that can be done by part (ii) of Proposition 7.2 is for any η, η',

$$\begin{aligned}&\int_{\mathbb{T}^{2d}} \mathbb{I}_{\rho_k(x)\geq\eta}\, \mathbb{I}_{\rho_k(y)\geq\eta}\, K_h(x-y)\, \chi(\delta\rho_k)\, dx\, dy\\ &\qquad \leq \frac{1}{\eta'^2} \int w^x_{1,k}\, w^y_{1,k}\, K_h(x-y)\, \chi(\delta\rho_k)\, dx\, dy + C\, \frac{\|K_h\|_{L^1}}{\eta^{1/2}\, |\log \eta'|^{\theta/2}},\end{aligned}$$

using that $\rho_k \in L^2$ uniformly. Using (8.19) and optimizing in η', one finds for some $\theta > 0$,

$$\int_{\mathbb{T}^{2d}} \mathbb{I}_{\rho_k(x)\geq\eta}\, \mathbb{I}_{\rho_k(y)\geq\eta}\, K_h(x-y)\, \chi(\delta\rho_k)\, dx\, dy \leq C\, \frac{\|K_h\|_{L^1}}{\eta^{1/2}\, |\log(\varepsilon(h) + h^\theta)|^{\theta/2}}.$$

If μ_k and F_k are uniformly in $W^{s,1}$ for $s > 0$, then

$$\int_{\mathbb{T}^{2d}} \mathbb{I}_{\rho_k(x)\geq\eta}\, \mathbb{I}_{\rho_k(y)\geq\eta}\, K_h(x-y)\, \chi(\delta\rho_k)\, dx\, dy \leq C\, \frac{\|K_h\|_{L^1}}{\eta^{1/2}\, |\log h|^{\theta/2}},$$

which concludes the proof of Theorem 5.2. Note, however, that in many senses (8.19) is more precise than the final result.

8.5. *The coupling with the pressure in the anisotropic case.* In that case we need the weight w_a and its regularization $w_{a,h}$, defined by (7.2) with (7.5) in order to compensate some terms coming from the anisotropic non-local part of the stress tensor.

Lemma 8.6. *There exists* $C_* > 0$ *such that assuming that* ρ_k *solves* (5.1) *with* $\alpha_k = 0$, *that* (5.6), (5.5), (5.4) *with* $\gamma > d/2$ *and* $p > \gamma + 1 + \ell = \gamma^2/(\gamma-1)$ *hold where* $\ell = 1/(\gamma-1)$; *assuming moreover that* P_k *satisfying* (5.8) *and that* u_k *solves* (5.10) *with*

$$a_\mu \leq C_*, \tag{8.20}$$

then there exists $0 < \theta < 1$ *such that for* χ_a *verifying* (8.2) *for this choice of* ℓ,

$$\begin{aligned}&\Big[\int_{h_0}^1 \int_{\mathbb{T}^{2d}} \frac{\overline{K}_h(x-y)}{h} (w^x_{a,h} + w^y_{a,h})\, \chi_a(\delta\rho_k)\, dx\, dy\Big]\Big|_{t=s}\\ &\qquad \leq \Big[\int_{h_0}^1 \int_{\mathbb{T}^{2d}} \frac{\overline{K}_h(x-y)}{h} (w^x_{a,h} + w^y_{a,h})\, \chi_a(\delta\rho_k)\, dx\, dy\Big]\Big|_{t=0}\\ &\qquad + C\, (1+\ell)\, |\log h_0|^\theta.\end{aligned}$$

Proof. To simplify the estimate, we assume in this proof that $P_k(\rho_k) = \rho_k^\gamma$, the extension when P_k satisfies (5.8) instead being straightforward. We also recall that χ_a satisfies (8.2), meaning that for all practical purposes, $\chi_a(\xi) \sim |\xi|^{1+\ell}$. We use the formula written in Lemma 8.2, namely,

$$\begin{aligned}&\Big[\int_{h_0}^1\int_{\mathbb{T}^{2d}} \frac{\overline{K}_h(x-y)}{h}(w_{a,h}^x + w_{a,h}^y)\,\chi_a(\delta\rho_k)\,dx\,dy\,dh\Big]\Big|_{t=s}\\ &\quad - \Big[\int_{h_0}^1\int_{\mathbb{T}^{2d}} \frac{\overline{K}_h(x-y)}{h}(w_{a,h}^x + w_{a,h}^y)\,\chi_a(\delta\rho_k)\,dx\,dy\,dh\Big]\Big|_{t=0}\\ &\le C|\log h_0|^\theta + I + II - \mathbb{T}_a,\end{aligned}$$

where $0<\theta<1$ and with the dissipation term by symmetry,

$$\mathbb{T}_a = \lambda\int_0^s\int_{h_0}^1\int_{\mathbb{T}^{2d}} w_{a,h}^x\,\chi_a(\delta\rho_k)\,[\overline{K}_h \star (|\mathrm{div}\,u_k| + |A_\mu P_k(\rho_k)|)]^x\,\bar{K}_h\,dx\,dy\,\frac{dh}{h}dt,$$

while still by symmetry

$$I = -\frac{1}{2}\int_{h_0}^1\int_0^s\int_{\mathbb{T}^{2d}} \frac{\overline{K}_h(x-y)}{h}\,(\mathrm{div}_x u_k^x - \mathrm{div}_y u_k^y)\chi_a'(\delta\rho_k)\,\bar{\rho}_k\,w_{a,h}^x\,dx\,dy\,dt\,dh$$

and

$$\begin{aligned}II = -\frac{1}{2}\int_{h_0}^1\int_0^s\int_{\mathbb{T}^{2d}} &\frac{\overline{K}_h(x-y)}{h}\,(\mathrm{div}_x u_k^x + \mathrm{div}_y u_k^y)\\ &\cdot(\chi_a'(\delta\rho_k)\delta\rho_k - 2\chi_a(\delta\rho_k))\,w_{a,h}^x\,dx\,dy\,dt\,dh.\end{aligned}$$

(I) *The quantity I.* We recall that in this case one has the formula (5.10) on $\mathrm{div}\,u_k$,

$$\mathrm{div}\,u_k = \nu_k P_k(\rho_k) + \nu_k\,a_\mu A_\mu P_k(\rho_k) + F_{a,k}, \tag{8.21}$$

where

$$F_{a,k} = \nu_k(\Delta_\mu - a_\mu E_k)^{-1}\,\mathrm{div}\,(\partial_t(\rho_k\,u_k) + \mathrm{div}\,(\rho_k\,u_k\otimes u_k)).$$

Therefore, one may decompose

$$I = I_0 + I^D + I^R,$$

with

$$\begin{aligned}I_0 = -\frac{1}{2}\int_{h_0}^1\int_0^s\int_{\mathbb{T}^{2d}} &\frac{\overline{K}_h(x-y)}{h}\,(F_{a,k}^x - F_{a,k}^y)\\ &\cdot\chi_a'(\delta\rho_k)\,\bar{\rho}_k\,(w_{a,k,h}^x + w_{a,k,h}^y)\,dx\,dy\,dt\,dh,\end{aligned}$$

while

$$\begin{aligned}I^D = -\frac{\nu_k}{2}\int_{h_0}^1\int_0^s\int_{\mathbb{T}^{2d}} &\frac{\overline{K}_h(x-y)}{h}\,(P_k(\rho_k^x) - P_k(\rho_k^y))\\ &\cdot\chi_a'(\delta\rho_k)\,\bar{\rho}_k\,w_{a,k,h}^x\,dx\,dy\,dt\,dh\end{aligned}$$

and

$$I^R = -\frac{a_\mu\,\nu_k}{2}\int_{h_0}^1\int_0^s\int_{\mathbb{T}^{2d}}\frac{\overline{K}_h(x-y)}{h}\,(A_\mu P_k(\rho_k^x) - A_\mu P_k(\rho_k^y))\\ \cdot\,\chi_a'(\delta\rho_k)\,\bar\rho_k\,w_{a,k,h}^x\,dx\,dy\,dt\,dh.$$

(I-1) *The term* I_0. This term is handled just as in the proof of Lemmas 8.4 and 8.5 by using Lemma 8.3, and for this reason we do not fully detail all the steps here. First note that Lemma 8.3 applies to $F_{a,k}$ as well as for F_k as

$$F_{a,k} = (\nu_k(\Delta_\mu - a_\mu E_k)^{-1}\,\Delta)\,F_k.$$

Then as before, we first truncate by using some smooth function $I_k^L(t,x,y) = \phi(\rho_k^x/L)\phi(\rho_k^y/L)$ with some smooth and compactly supported function ϕ leading to $I_0 = I_0^L + I_0^{RL}$ with

$$I_0^L = -\frac{1}{2}\int_{h_0}^1\int_0^s\int_{\mathbb{T}^{2d}}\frac{\overline{K}_h(x-y)}{h}\,(F_{a,k}^x - F_{a,k}^y)\\ \cdot\,\chi_a'(\delta\rho_k)\,I_k^L\,\bar\rho_k\,(w_{a,h}^x + w_{a,h}^y)\,dx\,dy\,dt\,dh$$

and

$$I_0^{RL} = -\frac{1}{2}\int_{h_0}^1\int_0^s\int_{\mathbb{T}^{2d}}\frac{\overline{K}_h(x-y)}{h}\,(F_{a,k}^x - F_{a,k}^y)\\ \cdot\,\chi_a'(\delta\rho_k)\,(1-I_k^L)\,\bar\rho_k\,(w_{a,h}^x + w_{a,h}^y)\,dx\,dy\,dt\,dh.$$

Remark that $\operatorname{div} u_k \in L^2_{t,x}$, $P_k(\rho_k) \in L^{p/\gamma}_{t,x}$ and since A_μ is an operator of 0 order, $A_\mu P_k(\rho_k) \in L^{p/\gamma}_{t,x}$. Therefore by equation (8.21),

$$\sup_k \|F_{a,k}\|_{L^{\min(2,p/\gamma)}_{t,x}} < \infty.$$

On the other hand, $|\chi_a'(\delta\rho_k)| \le C\,(1+\ell)\,(|\rho_k(x)|^\ell + |\rho_k(y)|^\ell)$, and this lets us very simply bound I_0^{RL} by the Hölder estimates

$$\begin{aligned} I_0^{RL} &\le C\,(1+\ell)\,|\log h_0|\,\|F_{a,k}\|_{L^{\min(2,p/\gamma)}_{t,x}}\,\|(1-I_k^L)\,\rho_k^{1+\ell}\|_{L^{\max(2,q)}_{t,x}}\\ &\le C\,(1+\ell)\,|\log h_0|\,\|(1-I_k^L)\,\rho_k^{1+\ell}\|_{L^{\max(2,q)}_{t,x}}, \end{aligned}$$

with $1/q + \gamma/p = 1$. But $q\,(1+\ell) < p$ by the assumption $p > \gamma + 1 + \ell$ and similarly $2\,(1+\ell) < p$. As a consequence, for some exponent $\theta_1 > 0$,

$$I_0^{RL} \le C\,(1+\ell)\,|\log h_0|\,L^{-\theta_1}. \tag{8.22}$$

We now use Lemma 8.3 for $F_{a,k}$ and $\Phi = \chi_a'(\delta\rho_k)\,I_k^L\,\bar\rho_k\,w_{a,k,h}(x)$. We note that $\|\Phi\|_{L^\infty} \le C\,(1+\ell)\,L^{1+\ell}$. Moreover, just as in the proof of Lemmas 8.4 and

8.5, we can show that Φ satisfies a transport equation giving that

$$C_\Phi = \left\| \int_{\mathbb{T}^d} \overline{K}_h(x-y)\Phi(t,x,y)\,dy \right\|_{W_t^{1,1}W_x^{-1,1}} + \left\| \int_{\mathbb{T}^d} \overline{K}_h(x-y)\Phi(t,x,y)\,dx \right\|_{W_t^{1,1}W_x^{-1,1}} \leq C\,(1+\ell)\,L^{1+\ell}.$$

By Lemma 8.3, we obtain that for some $\theta_2 > 0$,

$$I_0^L \leq C\,(1+\ell)\,L^{1+\ell} \int_{h_0}^1 h^{\theta_2}\,\frac{dh}{h} \leq C\,(1+\ell)\,L^{1+\ell}. \tag{8.23}$$

By optimizing in L, this lets us conclude that again for some $0 < \theta < 1$ and provided that $p > \gamma + 1 + \ell$,

$$I_0 \leq C\,(1+\ell)\,|\log h_0|^\theta. \tag{8.24}$$

(1-2) *The term* I^D. This term has the right sign as

$$\int_{h_0}^1 \int_0^s \int_{\mathbb{T}^{2d}} \frac{\overline{K}_h(x-y)}{h}\,((\rho_k^x)^\gamma - (\rho_k^y)^\gamma)\,\chi_a'(\delta\rho_k)\,\bar{\rho}_k\,(w_{a,k,h}^x + w_{a,k,h}^y)\,dx\,dy\,dt\,dh$$
$$\geq C \int_{h_0}^1 \int_0^s \int_{\mathbb{T}^{2d}} \frac{\overline{K}_h(x-y)}{h}\,\chi_a'(\delta\rho_k)\,\delta\rho_k\,\bar{\rho}_k^\gamma\,(w_{ak,,h}^x + w_{a,k,h}^y)\,dx\,dy\,dt\,dh.$$

We will actually give a more precise control on $I^D + II^D$ later on when the corresponding decomposition of $II = II_0 + II^D + II^R$ will be introduced.

(I-3) *The term* I^R. The difficulty is thus in this quantity. From its definition, A_μ is a convolution operator. With a slight abuse of notation, we denote by A_μ as well its kernel or

$$A_\mu f = \int_{\mathbb{T}^d} A_\mu(x-y)\,f(y)\,dy,$$

and we note that A_μ corresponds to an operator of 0 order; i.e., for instance, it satisfies the property $\int_A A_\mu = 0$ for any annulus A centered at the origin, $|A_\mu(x)| \leq C\,|x|^{-d}$. Decompose

$$A_\mu = L_h + R_h, \qquad \operatorname{supp} L_h \subset \{|x| \leq \delta_h\}$$

such that both L_h and R_h remain bounded on any L^p space, $1 < p < \infty$, and moreover R_h is a regularization of A_μ, that is, $R_h = A_\mu \star N_{\delta_h}$ for some smooth kernel N_{δ_h}. The scale δ_h has to satisfy that

$$\delta_h << h, \quad \log\frac{h}{\delta_h} << |\log h|.$$

For simplicity, we choose here $\delta_h = h/|\log h|$.

Contribution of the R_h part. The first step is to decompose R_h into dyadic blocks in Fourier. Introduce a decomposition of identity Ψ_l as in Sections 6 and 11 such that $1 = \sum_l \hat{\Psi}_l$, and write

$$R_h = \sum_{l=|\log_2 h|}^{|\log_2 \delta_h|} \Psi_l \star R_h + \tilde{R}_h, \quad \tilde{R}_h = \sum_{l<|\log_2 h|} \Psi_l \star R_h = \tilde{N}_h \star N_{\delta_h} \star A_\mu. \tag{8.25}$$

Note that of course we require the Ψ_l to satisfy all the assumptions specified in Section 11 for the definition of Besov spaces. Now define $\overline{N}_h = \tilde{N}_h \star N_{\delta_h}$. This kernel $\overline{N}_h$ therefore satisfies that for any $s > 0$,

$$\|\overline{N}_h\|_{W^{s,1}} \leq C\, h^{-s}, \tag{8.26}$$

and moreover by the localization property of the Ψ_k, one has that for $s > 0$ and any $|\omega| \leq 1$,

$$\int_{\mathbb{T}^d} |z|^s \, |\overline{N}_h(z) + \overline{N}_h(z + \omega\, r)| \, dr \leq C\, h^s. \tag{8.27}$$

Fix t for the moment, and decompose accordingly

$$\begin{aligned}
&\int_{h_0}^1 \int_{\mathbb{T}^d} \frac{\overline{K}_h(z)}{h} \, \|(R_h \star \rho_k^\gamma)^\cdot - (R_h \star \rho_k^\gamma)^{\cdot + z}\|_{L^q_x} \\
&\quad \leq \int_{h_0}^1 \int_{\mathbb{T}^d} \frac{\overline{K}_h(z)}{h} \, \|(\tilde{R}_h \star \rho_k^\gamma)^\cdot - (\tilde{R}_h \star \rho_k^\gamma)^{\cdot + z}\|_{L^q_x} \\
&\qquad + \int_{h_0}^1 \sum_{l=|\log_2 h|}^{|\log_2 \delta_h|} \int_{\mathbb{T}^d} \frac{\overline{K}_h(z)}{h} \, \|(\Psi_l \star R_h \star \rho_k^\gamma)^\cdot - (\Psi_l \star R_h \star \rho_k^\gamma)^{\cdot + z}\|_{L^q_x}.
\end{aligned}$$

By (8.26) and (8.27), the kernel $\overline{N}_h$ satisfies the assumptions of Lemma 6.4. Thus with $U_k = A_\mu \star \rho_k^\gamma$, applying Lemma 6.4, for any $q > 1$,

$$\begin{aligned}
&\int_{h_0}^1 \int_{\mathbb{T}^d} \frac{\overline{K}_h(z)}{h} \, \|(\tilde{R}_h \star \rho_k^\gamma)^\cdot - (\tilde{R}_h \star \rho_k^\gamma)^{\cdot + z}\|_{L^q_x} \, dz \, dh \\
&\quad = \int_{h_0}^1 \int_{\mathbb{T}^d} \frac{\overline{K}_h(z)}{h} \, \|(\overline{N}_h \star U_k)^\cdot - (\overline{N}_h \star U_k)^{\cdot + z}\|_{L^q_x} \, dz \, dh \\
&\quad \leq C\, |\log h_0|^{1/2} \, \|U_k^\cdot\|_{L^q_x}.
\end{aligned}$$

Recalling that A_μ is continuous on every L^p space, one has that $\|U_k^\cdot\|_{L^p_x} \leq C\, \|(\rho_k^\gamma)^\cdot\|_{L^q_x}$, and hence

$$\int_{h_0}^1 \int_{\mathbb{T}^d} \frac{\overline{K}_h(z)}{h} \, \|(\tilde{R}_h \star \rho_k^\gamma)^\cdot - (\tilde{R}_h \star \rho_k^\gamma)^{\cdot + z}\|_{L^q_x} \, dz \, dh \leq C\, |\log h_0|^{1/2} \, \|\rho_k^\cdot\|^\gamma_{L^{q\gamma}_x}.$$

On the other hand, simply by bounding

$$|(\Psi_l \star R_h \star \rho_k^\gamma)^x - (\Psi_l \star R_h \star \rho_k^\gamma)^y|^q \leq C\, |(\Psi_l \star R_h \star \rho_k^\gamma)^x|^q + |(\Psi_l \star R_h \star \rho_k^\gamma)^y|^q,$$

we write

$$\int_{h_0}^1 \sum_{l=|\log_2 h|}^{|\log_2 \delta_h|} \int_{\mathbb{T}^d} \frac{\overline{K}_h(z)}{h} \|(\Psi_l \star R_h \star \rho_k^\gamma)^\cdot - (\Psi_l \star R_h \star \rho_k^\gamma)^{\cdot+z}\|_{L^q_x}\, dz\, dh$$
$$\leq C \sum_{l\leq |\log_2 h_0|+\log_2|\log_2 h_0|} \|(\Psi_l \star R_h \star \rho_k^\gamma)^\cdot\|_{L^q_x} \int_{2^{-l}}^{l\,2^{-l}} \frac{dh}{h},$$

recalling that $\delta_h = h/|\log_2 h|$. This leads to

$$\int_{h_0}^1 \sum_{l=|\log_2 h|}^{|\log_2 \delta_h|} \int_{\mathbb{T}^d} \frac{\overline{K}_h(z)}{h} \|(\Psi_l \star R_h \star \rho_k^\gamma)^\cdot - (\Psi_l \star R_h \star \rho_k^\gamma)^{\cdot+z}\|_{L^q_x}\, dz\, dh$$
$$\leq C \sum_{l\leq 2\,|\log_2 h_0|} \log l\, \|(\Psi_l \star R_h \star \rho_k^\gamma)^\cdot\|_{L^q_x}\, dz\, dh$$

and can in turn be directly bounded by

$$\leq C\, \log|\log h_0| \sum_{l\leq 2\,|\log_2 h_0|} \|(\Psi_l \star R_h \star \rho_k^\gamma)^\cdot\|_{L^q_x}$$
$$\leq C\, \log|\log h_0|\, |\log h_0|^{1/2} \|(R_h \star \rho_k^\gamma)^\cdot\|_{L^q_x} \leq C\, |\log h_0|^\theta\, \|\rho_k^\cdot\|^\gamma_{L^{q\gamma}_{tx}},$$

with $0<\theta<1$ by Lemma 11.3. Combining with the previous estimate, we deduce that

(8.28)
$$\int_{h_0}^1 \int_{\mathbb{T}^d} \frac{\overline{K}_h(z)}{h} \|(R_h \star \rho_k^\gamma)^\cdot - (R_h \star \rho_k^\gamma)^{\cdot+z}\|_{L^q_x}, dz\, dh \leq C\,|\log h_0|^\theta\, \|(\rho_k)^\cdot\|^\gamma_{L^{q\gamma}_x},$$

with $0<\theta<1$. Therefore since $\chi_a'(\xi) \leq (1+\ell)\,|\xi|^\ell$, by Hölder's inequality with the relation $1/q + (1+\ell)/(1+\gamma+\ell) = 1$, that is, $q = (1+\ell+\gamma)/\gamma$,

$$\int_{h_0}^1\int_0^s \int_{\mathbb{T}^{2d}} \frac{\overline{K}_h(x-y)}{h} \left((R_h \star \rho_k^\gamma)^x - (R_h \star \rho_k^\gamma)^y\right)$$
$$\cdot\, \chi_a'(\delta\rho_k)\, \bar\rho_k\, (w^x_{a,k,h} + w^y_{a,k,h})\, dx\, dy\, dt\, dh$$
$$\geq -C\,(1+\ell) \int_0^s \|\rho_k^\cdot\|^{1+\ell}_{L_x^{1+\ell+\gamma}} \int_{h_0}^1 \int_{\mathbb{T}^d}$$
$$\cdot\, \frac{\overline{K}_h(z)}{h} \|(R_h \star \rho_k^\gamma)^\cdot - (R_h \star \rho_k^\gamma)^{\cdot+z}\|_{L^q_x}\, dz\, dh\, dt.$$

Finally by (8.28) there exists $0<\theta<1$ such that

(8.29)
$$\int_{h_0}^1\int_0^s \int_{\mathbb{T}^{2d}} \frac{\overline{K}_h(x-y)}{h} \left((R_h \star \rho_k^\gamma)^x - (R_h \star \rho_k^\gamma)^y\right)$$
$$\cdot\, \chi_a'(\delta\rho_k)\, \bar\rho_k\, (w^x_{a,h} + w^y_{a,h})$$
$$\geq -C\,(1+\ell)\,|\log h_0|^\theta\, \|\rho_k\|^{\gamma+\ell+1}_{L^{\gamma+\ell+1}}.$$

Contribution of the L_h part. It remains to deal with the term involving L_h. First we symmetrize the position of the weight with respect to the convolution with L_h by

$$\int_{h_0}^1\int_0^s\int_{\mathbb{T}^{2d}}\frac{\overline{K}_h(x-y)}{h}\left((L_h\star\rho_k^\gamma)^x-(L_h\star\rho_k^\gamma)^y\right)\chi_a'(\delta\rho_k)\,\bar\rho_k\,w_{a,h}^x\,dx\,dy\,dt\,dh$$
$$=I_h^L-\mathrm{Diff},$$

with

$$I_h^L=\int_{h_0}^1\int_0^s\int_{\mathbb{T}^{3d}}\frac{\overline{K}_h(x-y)}{h}\,L_h(z)((\rho_k^\gamma)^{x-z}-(\rho_k^\gamma)^{y-z})$$
$$\cdot\,\chi_0'(\delta\rho_k)\,\bar\rho_k\,(w_{a,h}^x)^{1-\theta}\,(w_{a,h}^{x-z})^\theta\,dx\,dy\,dz\,dt\,dh$$

for $\theta=1-1/\gamma$. Recall that since $w_{a,h}=\overline{K}_h\star w_a$, then $|w_{a,h}^x-w_{a,h}^{x-z}|\le h^{-1}\,|z|$ while $|z|\sim\delta_h$ on the support of L_h. Thus using that $|\chi_a'(\xi)|\le C\,|\xi|^\ell$ from (8.2) and that $|L_h(z)|\le C\,|z|^{-d}$, one has

$$\mathrm{Diff}=\int_{h_0}^1\int_0^s\int_{\mathbb{T}^{3d}}\frac{\overline{K}_h(x-y)}{h}\,L_h(z)((\rho_k^\gamma)^{x-z}-(\rho_k^\gamma)^{y-z})$$
$$\cdot\,\chi_a'(\delta\rho_k)\,\bar\rho_k\,(w_{a,h}^x)^{1-\theta}\,((w_{a,h}^x)^\theta-(w_{a,h}^{x-z})^\theta)\,dx\,dy\,dz\,dt\,dh$$
$$\le C\,(1+\ell)\int_{h_0}^1\int_0^s\int_{\mathbb{T}^{3d}}\mathbb{I}_{|z|\le\delta_h}\,\frac{\overline{K}_h(w)}{h\,|z|^d}\,\left|(\rho_k^\gamma)^{x-z}+(\rho_k^\gamma)^{x-z+w}\right|\,\bar\rho^{\ell+1}\,h^{-\theta}$$
$$\cdot\,|z|^\theta\,dx\,dw\,dz\,dt\,dh$$
$$\le C\,(1+\ell)\int_{h_0}^1\int_0^t\int_{\mathbb{T}^{3d}}\mathbb{I}_{|z|\le\delta_h}\,\frac{\overline{K}_h(w)}{h^{1+\theta}\,|z|^{d-\theta}}\,\Big((\rho_k^{\gamma+\ell+1})^{x-z}+(\rho_k^{\gamma+\ell+1})^{x-z+w}$$
$$+\,(\rho_k^{\gamma+\ell+1})^x+(\rho_k^{\gamma+\ell+1})^{x-w}\Big)\,dx\,dw\,dz\,dt\,dh.$$

Using $|z|\le\delta_h=\frac{h}{|\log_2 h|}$, we obtain on the other hand that

$$\int_{h_0}^1\frac{dh}{h^{1+\theta}}\int_{|z|\le\delta_h}\frac{dz}{|z|^{d-\theta}}\le C\int_{h_0}^1\frac{\delta_h^\theta dh}{h^{1+\theta}}=C\int_{h_0}^1\frac{dh}{h\,|\log h|^\theta}$$
$$\le C\,|\log h_0|^{1-\theta}.$$

As $\theta=1-1/\gamma$, this leads to

$$\mathrm{Diff}\le C\,(1+\ell)\,\|\rho_k\|_{L^{\gamma+\ell+1}}^{\gamma+\ell+1}\,|\log h_0|^{1/\gamma}. \tag{8.30}$$

As for the first term by Hölder inequality, using again that $|\chi_a'(\xi)| \le C\,\ell\,|\xi|^\ell$,

$$I_h^L \le C\,(1+\ell) \int_{h_0}^1 \int_0^s \int_{\mathbb{T}^d} \left(\int_{\mathbb{T}^d} |\delta\rho_k|^{\ell+1}\, \bar\rho_k^{(\ell+1)/\ell}\, (w_{a,h}^{(1-\theta)\,(\ell+1)/\ell})^x)\, dx \right)^{\ell/(\ell+1)}$$
$$\cdot \left(\int_{\mathbb{T}^d} \left(L_h \star \left((\rho_k^\gamma)^{\cdot} - (\rho_k^\gamma)^{\cdot + w} \right) (w_{a,h}^\theta)^{\cdot} \right) \right)^{(\gamma+\ell+1)/\gamma}\, dx \right)^{\gamma/(\gamma+\ell+1)}$$
$$\cdot \frac{\overline{K}_h(w)}{h}\, dw\, dt\, dh,$$

provided that ℓ is chosen such that

$$\frac{\gamma}{\gamma+\ell+1} + \frac{\ell}{\ell+1} = 1, \quad \text{or} \qquad \ell+1 = \frac{\gamma}{\gamma-1},$$

implying, for instance, that

$$\frac{\ell+1}{\ell} = \gamma, \quad \frac{\gamma+\ell+1}{\gamma} = \ell+1 \cdots .$$

Given those algebraic relations and recalling that $L_h\star$ is continuous on every L^p for any $1 < p < \infty$,

$$I_h^L \le C\,(1+\ell) \int_{h_0}^1 \int_0^s \int_{\mathbb{T}^d} \frac{\overline{K}_h(w)}{h} \left(\int_{\mathbb{T}^d} |(\rho_k^\gamma)^x - (\rho_k^\gamma)^{x+w}|^{\frac{\gamma+\ell+1}{\gamma}}\, w_{a,k,h}^x\, dx \right)^{\frac{\gamma}{\gamma+\ell+1}}$$
$$\cdot \left(\int_{\mathbb{T}^d} |\chi_0'(\delta_k)|^\gamma\, \bar\rho_k^\gamma\, w_{a,h}^x\, dx \right)^{\ell/(\ell+1)} dw\, dt\, dh.$$

Since using the definition of ℓ,

$$|(\rho_k^\gamma)^x - (\rho_k^\gamma)^{x+w}|^{(\gamma+\ell+1)/\gamma} \le \gamma\, \bar\rho^\gamma\, |\delta\rho_k|^{(\gamma+\ell+1)/\gamma} = \gamma\, \bar\rho^\gamma\, |\delta\rho_k|^{\ell+1},$$

one has

$$I_h^L \le C\,(1+\ell)\,\gamma \int_{h_0}^1 \int_0^s \int_{\mathbb{T}^{2d}} \frac{\overline{K}_h(w)}{h}\, |\delta\rho_k|^{\ell+1}\, \bar\rho_k^\gamma\, w_{a,k,h}^x\, dx\, dw\, dt\, dh, \tag{8.31}$$

which multiplied by $-a_\mu\,\nu_k/2$ will be bounded by $I^D + II^D$ provided $|a_\mu|$ is small enough.

(II) *The quantity II.* Let us turn to II and decompose it as for I:

$$II = II_0 + II^D + II^R,$$

where

$$II_0 = -\frac{1}{2} \int_{h_0}^1 \int_0^s \int_{\mathbb{T}^{2d}} \frac{\overline{K}_h(x-y)}{h}\, (F_{a,k}^x + F_{a,k}^y)$$
$$\cdot\, (\chi_a'(\delta\rho_k)\delta\rho_k - 2\chi_a(\delta\rho_k))\, (w_{a,k,h}^x + w_{ak,,h}^y)\, dx\, dy\, dt\, dh,$$

while

$$II^D = -\frac{\nu_k}{2} \int_{h_0}^1 \int_0^s \int_{\mathbb{T}^{2d}} \frac{\overline{K}_h(x-y)}{h}\, (P_k(\rho_k^x) + P_k(\rho_k^y))$$
$$\cdot\, (\chi_a'(\delta\rho_k)\delta\rho_k - 2\chi_a(\delta\rho_k))\, (w_{a,k,h}^x + w_{a,k,h}^y)\, dx\, dy\, dt\, dh$$

and

$$II^R = -\frac{a_\mu\,\nu_k}{2}\int_{h_0}^1\int_0^s\int_{\mathbb{T}^{2d}}\frac{\overline{K}_h(x-y)}{h}\,(A_\mu P_k(\rho_k^x)+A_\mu P_k(\rho_k^y))$$
$$\cdot\,(\chi_a'(\delta\rho_k)\delta\rho_k-2\chi_a(\delta\rho_k))\,(w_{a,k,h}^x+w_{a,k,h}^x)\,dx\,dy\,dt\,dh.$$

(II-1) *The II_0 term.* For the term II_0, using Lemma 8.3 in a manner identical to I_0,

$$II_0 \le -\int_{h_0}^1\int_0^s\int_{\mathbb{T}^{2d}}\frac{\overline{K}_h(x-y)}{h}\,(\overline{K}_h\star F_{a,k})^x(\chi_a'(\delta\rho_k)\delta\rho_k-2\chi_a(\delta\rho_k))$$
$$\cdot\,w_{a,k,h}^x\,dx\,dy\,dt\,dh$$
$$+\,C\,(1+\ell)\,\|\rho_k\|_{L^{\gamma+\ell+1}}^{1+\ell}\,|\log h_0|^\theta$$

for some $0<\theta<1$. Using formula (5.10) or (8.21), one has that

$$\operatorname{div} u_k - a_\mu A_\mu P_k(\rho_k) \ge F_{a,k},$$

and hence since $-\chi_a'\,\xi+2\chi_a\ge -C\,(1+\ell)\,\chi_a$,

$$II_0 \le C\,\ell\,\|\rho_k\|_{L^{\gamma+\ell+1}}^{1+\ell}\,|\log h_0|^\theta + C\,(1+\ell)\int_{h_0}^1\int_0^s\int_{\mathbb{T}^{2d}}\frac{\overline{K}_h(x-y)}{h}$$
$$\cdot\,\overline{K}_h\star(|\operatorname{div} u_k|+a_\mu|A_\mu P_k(\rho_k)|)\,\chi_a(\delta\rho_k)\,w_{a,k,h}^x\,dx\,dy\,dt\,dh, \tag{8.32}$$

and the first integral will be bounded by $\mathbb{T}_a/2$ for λ large enough.

(II-2) *The II^D term.* The term II^D is controlled by I^D: For $a\ge b$, by (8.2),

$$(a^\gamma+b^\gamma)\,(-\chi_a'(a-b)(a-b)+2\chi_a(a-b)) \ge -(a^\gamma-b^\gamma)\,\frac{\ell-1}{\ell}\chi_a'(a-b)(a+b).$$

Therefore,

$$I^D+II^D \le -C\,\gamma\,\frac{\nu_k}{2}\int_{h_0}^1\int_0^s\int_{\mathbb{T}^{2d}}$$
$$\cdot\,\frac{\overline{K}_h(x-y)}{h}\,|\delta\rho_k|^{\ell+1}\,\bar\rho_k^\gamma\,(w_{a,k,h}^x+w_{a,k,h}^y)\,dx\,dy\,dt\,dh \tag{8.33}$$

for some C independent of ℓ and γ.

(II-3) *The II^R term.* The control on the last term, II^R, requires the use of the penalization $\mathbb{T}_a$:

$$II^R+\frac{1}{2}\mathbb{T}_a \le -a_\mu\,\nu_k\int_{h_0}^1\int_0^s\int_{\mathbb{T}^{2d}}\frac{\overline{K}_h(x-y)}{h}\,((A_\mu\rho_k^\gamma)^x-(A_\mu\overline{K}_h\star\rho_k^\gamma)^x)$$
$$\cdot\,(\chi_a'(\delta\rho_k)\delta\rho_k-2\chi_a(\delta\rho_k))\,w_{a,k,h}^x\,dx\,dy\,dt\,dh.$$

We use the same decomposition of $A_\mu=L_h+R_h$ as for I^R.

Contribution of the R_h part. Note that as $\chi_a(\xi) \leq C\,|\xi|^{1+\ell}$ and $|\chi_a'| \leq C\,(1+\ell)\,|\xi|^{\ell}$, for $q = (1+\ell+\gamma)/\gamma$ or $1/q + (1+\ell)/(1+\ell+\gamma) = 1$,

$$\begin{aligned}
&- a_\mu\,\nu_k \int_{h_0}^1 \int_0^s \int_{\mathbb{T}^{2d}} \frac{\overline{K}_h(x-y)}{h}\,((R_h \star \rho_k^\gamma)^x - (\overline{K}_h \star R_h \star \rho_k^\gamma)^x) \\
&\qquad \cdot (\chi_a'(\delta\rho_k)\delta\rho_k - 2\chi_a(\delta\rho_k))\, w_{a,kh}^x \,dx\,dy\,dt\,dh \\
&\leq C(1+\ell) \int_0^s \|\rho_k^\cdot\|_{L_x^{1+\ell+\gamma}}^{1+\ell} \int_{h_0}^1 \int_{\mathbb{T}^d} \frac{\overline{K}_h(z)}{h} \|(R_h \star \rho_k^\gamma)^\cdot - (R_h \star \rho_k^\gamma)^{\cdot+z}\|_{L_x^q}\,dz\,dh\,dt.
\end{aligned}$$

Now by estimate (8.28), we have that

$$\begin{aligned}
&- a_\mu\,\nu_k \int_{h_0}^1 \int_0^s \int_{\mathbb{T}^{2d}} \frac{\overline{K}_h(x-y)}{h}\,((R_h \star \rho_k^\gamma)^x - (\overline{K}_h \star R_h \star \rho_k^\gamma)^x) \\
&\qquad \cdot (\chi_a'(\delta\rho_k)\delta\rho_k - 2\chi_a(\delta\rho_k))\, w_{a,k,h}^x \,dx\,dy\,dt\,dh \\
&\leq C\,(1+\ell)\,|\log h_0|^{3/4} \int_0^s \|\rho_k^\cdot\|_{L_x^{1+\ell+\gamma}}^{1+\ell} \|\rho_k^\cdot\|_{L_x^{\gamma\,q}}^{\gamma}\,dt \\
&\leq C\,(1+\ell)\,|\log h_0|^{3/4}\,\|\rho_k\|_{L_{t,x}^{1+\ell+\gamma}}^{1+\ell+\gamma}.
\end{aligned} \tag{8.34}$$

Contribution of the L_h part. Similarly as for I^R, we symmetrize the weights leading to the following decomposition:

$$\begin{aligned}
&- a_\mu\,\nu_k \int_{h_0}^1 \int_0^s \int_{\mathbb{T}^{2d}} \frac{\overline{K}_h(x-y)}{h}\,((L_h \star \rho_k^\gamma)^x - (\overline{K}_h \star L_h \star \rho_k^\gamma)^x) \\
&\qquad \cdot (\chi_a'(\delta\rho_k)\delta\rho_k - 2\chi_a(\delta\rho_k))\, w_{a,k,h}^x \,dx\,dy\,dt\,dh \\
&\quad = II_h^L + \mathrm{Diff}_2,
\end{aligned}$$

where

$$\begin{aligned}
II_h^L = a_\mu\,\nu_k \int_{h_0}^1 \int_0^s \int_{\mathbb{T}^{2d}} \frac{\overline{K}_h(x-y)}{h}\, L_h \star \Big((w_{a,k,h}^x)^\theta\,((\rho_k^\gamma)^x - (\overline{K}_h \star \rho_k^\gamma)^x)\Big) \\
\cdot (-\chi_a'(\delta\rho_k)\delta\rho_k + 2\chi_a(\delta\rho_k))\,(w_{a,k,h}^x)^{1-\theta}\,dx\,dy\,dt\,dh,
\end{aligned}$$

still with $\theta = 1 - 1/\gamma$. The term Diff_2 is controlled as the term Diff in I^R using the regularity of $w_{a,h}$ and yielding

$$\mathrm{Diff}_2 \leq C\,(1+\ell)\,|\log h_0|^{1/\gamma}\,\|\rho_k\|_{L^{\gamma+1+\ell}}^{\gamma+1+\ell}. \tag{8.35}$$

We handle II_h^L with Hölder estimates quite similar to the ones used for the term I_h^L, recalling that $L_h\star$ is bounded on any L^q space for $1 < q < \infty$:

$$\begin{aligned}
II_h^L \leq\; & C\, a_\mu\, \nu_k\, (1+\ell) \int_{h_0}^1 \int_0^s \int_{\mathbb{T}^d} \frac{\overline{K}_h(w)}{h}\, \|w_{a,k,h}^\theta\, ((\rho_k^\gamma)^{\cdot} - (\overline{K}_h \star \rho_k^\gamma)^{\cdot}))\|_{L^{\ell+1}} \\
& \cdot \| |\delta\rho_k|^{\ell+1}\, (w_{a,k,h}^x)^{1-\theta} \|_{L^{(\ell+1)/\ell}}\, dw\, dt\, dh \\
\leq\; & C\, a_\mu\, \nu_k\, (1+\ell) \int_{h_0}^1 \int_0^s \int_{\mathbb{T}^d} \\
& \cdot \frac{\overline{K}_h(w)}{h} \int_{\mathbb{T}^d} w_{a,k,h}^x\, |(\rho_k^\gamma)^x - (\overline{K}_h \star \rho_k^\gamma)^x|^{\ell+1}\, dx\, dw\, dt\, dh \\
& + C\, a_\mu\, \nu_k\, (1+\ell) \int_{h_0}^1 \int_0^t \int_{\mathbb{T}^d} \frac{\overline{K}_h(w)}{h} \int_{\mathbb{T}^d} w_{a,k,h}^x\, |\delta\rho_k|^{(\ell+1)^2/\ell}\, dx\, dw\, dt\, dh.
\end{aligned}$$

One immediately has that

$$\begin{aligned}
& \int_{h_0}^1 \int_0^s \int_{\mathbb{T}^{2d}} \frac{\overline{K}_h(w)}{h}\, w_{a,k,h}^x\, |\delta\rho_k|^{(\ell+1)^2/\ell}\, dx\, dw\, dt\, dh \\
& \qquad \leq \int_{h_0}^1 \int_0^s \int_{\mathbb{T}^{2d}} \frac{\overline{K}_h(w)}{h}\, w_{a,k,h}^x\, |\delta\rho_k|^{\ell+1}\, \bar\rho_k^\gamma\, dx\, dw\, dt\, dh
\end{aligned}$$

as $|\delta\rho_k| \leq \bar\rho_k$ and again $(\ell+1)/\ell = \gamma$.

Finally as $(\ell+1)\,(\gamma-1) = \gamma$,

$$\begin{aligned}
& \int_{h_0}^1 \int_0^s \int_{\mathbb{T}^{2d}} \frac{\overline{K}_h(w)}{h}\, w_{a,k,h}^x\, |(\rho_k^\gamma)^x - (\overline{K}_h \star \rho_k^\gamma)^x|^{\ell+1}\, dx\, dw\, dt\, dh \\
& \quad \leq \gamma \int_{h_0}^1 \int_0^s \int_{\mathbb{T}^{3d}} \\
& \qquad \cdot \frac{\overline{K}_h(w)}{h}\, \overline{K}_h(z)\, w_{a,k,h}^x\, |\rho_k^x - \rho_k^{x+z}|^{\ell+1}\, (\rho_k^x + \rho_k^{x+z})^\gamma\, dx\, dw\, dz\, dt\, dh \\
& \quad \leq \gamma \int_{h_0}^1 \int_0^s \int_{\mathbb{T}^{2d}} \frac{\overline{K}_h(z)}{h}\, w_{a,k,h}^x\, |\rho_k^x - \rho_k^{x+z}|^{1+\ell}\, (\rho_k^x + \rho_k^{x+z})^\gamma\, dz\, dw\, dt\, dh,
\end{aligned}$$

as $\overline{K}_h(w)$ is the only term depending on w and is of integral 1. Therefore,

$$II_h^L \leq C\, a_\mu\, \nu_k\, \gamma\, (1+\ell) \int_{h_0}^1 \int_0^s \int_{\mathbb{T}^{2d}} \frac{\overline{K}_h(z)}{h}\, w_{a,k,h}^x\, |\delta\rho_k|^{1+\ell}\, \bar\rho_k^\gamma\, dz\, dx\, dt\, dh. \tag{8.36}$$

To conclude, we sum all the contributions, more precisely (8.24), (8.29), (8.30), (8.31), (8.32), (8.33), (8.34), (8.35), and (8.36), to find that for some

$0<\theta<1$ and provided $p>\gamma+1+\ell$,

$$\begin{aligned}&\Big[\int_{h_0}^1\int_{\mathbb{T}^{2d}}\frac{\overline{K}_h(x-y)}{h}(w_{a,k,h}^x+w_{a,k,h}^y)\,\chi_a(\delta\rho_k)\,dx,dy\,dh\Big]\Big|_{t=s}\\&\le\Big[\int_{h_0}^1\int_{\mathbb{T}^{2d}}\frac{\overline{K}_h(x-y)}{h}(w_{a,k,h}^x+w_{a,k,h}^y)\,\chi_a(\delta\rho_k)\,dx\,dy\,dh\Big]\Big|_{t=0}\\&\quad+C\,(1+\ell)\,|\log h_0|^\theta\\&\quad+C\,\left(a_\mu\,\nu_k(1+\ell)-C\frac{\nu_k}{2}\right)\gamma\int_{h_0}^1\int_0^s\int_{\mathbb{T}^{2d}}\frac{\overline{K}_h(z)}{h}\,w_{a,h}^x\,|\delta\rho_k|^{1+\ell}\,\bar\rho_k^\gamma\,dx\,dz\,dt\,dh.\end{aligned}$$

This finishes the proof of the lemma: As $1+\ell=\gamma/(\gamma-1)$ is bounded (we recall that $\gamma>d/2$), if $a_\mu\le C_*$ for $C_*>0$ well chosen, the last term in the right-hand side is non-positive. □

8.6. *Conclusion of the proof of Theorem* 5.3. We combine Lemmas 8.2 and 8.6 to get the following estimate:

$$\begin{aligned}\Big[\int_{h_0}^1\int_{\mathbb{T}^{2d}}\overline{K}_h(x-y)(w_{a,k,h}^x+w_{a,k,h}^y)\chi_a(\delta\rho_k)dxdy\frac{dh}{h}\Big]\Big|_{t=s}&\\ \le C\,|\log h_0|^\theta+\text{initial value},&\end{aligned}$$

with $0<\theta<1$. We now follow the same steps as in the proof of Theorem 5.1 with the weight $w_0^x+w_0^y$. We define $\omega_\eta=\{w_{a,k,h}^x\le\eta\}$ and note that

$$\begin{aligned}\int_{\mathbb{T}^{2d}}&\mathcal{K}_{h_0}(x-y)\chi_a(\delta\rho_k)\,dx\,dy\\&=\int_{h_0}^1\int_{\mathbb{T}^{2d}}\overline{K}_h(x-y)\,\chi_a(\delta\rho_k)\,dx\,dy\,\frac{dh}{h}\\&=\int_{h_0}^1\int_{x\in\omega_\eta^c\ or\ y\in\omega_\eta^c}\overline{K}_h(x-y)\,\chi_a(\delta\rho_k)\,dx\,dy\,\frac{dh}{h}\\&\quad+\int_{h_0}^1\int_{x\in\omega_\eta\ and\ y\in\omega_\eta}\overline{K}_h(x-y)\,\chi_a(\delta\rho_k)\,dx\,dy\,\frac{dh}{h}.\end{aligned}$$

Now

$$\begin{aligned}&\int_{h_0}^1\int_{x\in\omega_\eta^c\ or\ y\in\omega_\eta^c}\overline{K}_h(x-y)\,\chi_a(\delta\rho_k)\,dx\,dy\,\frac{dh}{h}\\&\quad\le\frac{1}{\eta}\int_{h_0}^1\int_{\mathbb{T}^{2d}}\overline{K}_h(x-y)(w_{a,k,h}^x+w_{a,k,h}^y)\chi_a(\delta\rho_k)\,dx\,dy\,\frac{dh}{h}\le C\,\frac{|\log h_0|^\theta}{\eta},\end{aligned}$$

while by point (iii) in Proposition 7.2 and using the L^p bound on ρ, for some $\theta > 0$,

$$\begin{aligned}\int_{h_0}^1 \int_{x\in\omega_\eta \ and \ y\in\omega_\eta} &\overline{K}_h(x-y)\,\chi_a(\delta\rho_k)\,dx\,dy\,\frac{dh}{h}\\ &\leq 2\int_{h_0}^1\int_{\mathbb{T}^{2d}} \overline{K}_h(x-y)\,\rho_k^{1+\ell}\,\mathbb{I}_{\overline{K}_h\star w_a\leq\eta}\,dx\,dy\,\frac{dh}{h}\\ &\leq \frac{C\,|\log h_0|}{|\log\eta|^\theta}.\end{aligned}$$

Hence we have

$$\begin{aligned}\Big[\int_{\mathbb{T}^{2d}} \mathcal{K}_{h_0}(x-y)\,\chi_a(\delta\rho_k\Big]\Big|_{t=s} &\leq C\,|\log h_0|\,\left(\frac{|\log h_0|^{\theta-1}}{\eta}+\frac{1}{|\log\eta|^\theta}\right)\\ &\leq \frac{C\,\|\mathcal{K}_{h_0}\|_{L^1}}{|\log|\log h_0||^\theta},\end{aligned}$$

by optimizing in η and recalling that $\|\mathcal{K}_{h_0}\|_{L^1} = |\log h_0|$. Using Proposition 4.1 together with Lemma 6.6, one concludes that ρ_k is compact in t, x. Thus we conclude the proof of Theorem 5.3.

9. Proof of Theorems 3.1 and 3.2: Approximate sequences

In this section, we construct approximate systems that allow us to use Theorems 5.1 and 5.3 to prove Theorems 3.1 and 3.2.

Here we do not need to use pressure laws P that depend explicitly on t or x, which simplifies the form of the assumptions on the behavior of P, either (5.8) or (5.9).

9.1. *From regularized systems with added viscosity to no viscosity.* Our starting point for global existence is the following regularized system:

$$(9.1)\qquad \begin{cases}\partial_t\rho_k+\mathrm{div}(\rho_k u_k)=\alpha_k\Delta\rho_k,\\ \partial_t(\rho_k u_k)+\mathrm{div}(\rho_k u_k\otimes u_k)-\mu\Delta u_k-(\lambda+\mu)\nabla\mathrm{div}u_k-\mathcal{A}_\varepsilon\star u_k\\ \qquad +\nabla P_\varepsilon(\rho_k)+\alpha_k\nabla\rho_k\cdot\nabla u_k=\rho_k f,\end{cases}$$

with the fixed initial data

$$(9.2)\qquad \rho_k|_{t=0}=\rho^0,\qquad \rho_k\,u_k|_{t=0}=\rho^0\,u^0.$$

The pressure P_ε satisfies the bound (3.2) with $\gamma > 3d/(d+2)$ uniformly in ε, that is,

$$C^{-1}\,\rho^\gamma - C \leq P_\varepsilon(\rho)\leq C\,\rho^\gamma + C,$$

implying that $e(\rho)\geq C^{-1}\,\rho^{\gamma-1}-C$. In addition we ask that P_ε satisfies the quasi-monotone property (5.8) but possibly depending on ε; i.e., there exists $\rho_{0,\varepsilon}$ such that

$$(P_\varepsilon(s)/s)'\geq 0 \text{ for all } s\geq\rho_{0,\varepsilon}.$$

And finally we impose an ε dependent bound (3.2) on P_ε for some $\gamma_\varepsilon > d$:

$$C_\varepsilon^{-1}\,\rho^{\gamma_\varepsilon} - C \le P_\varepsilon(\rho) \le C_\varepsilon\,\rho^{\gamma_\varepsilon} + C, \tag{9.3}$$

Similarly $\mathcal{A}_\varepsilon$ is assumed to be a given smooth function, possibly depending on ε but such that the operator defined by

$$\mathcal{D}_\varepsilon\, f = -\mu\Delta f - (\lambda+\mu)\nabla \mathrm{div} f - \mathcal{A}_\varepsilon \star f$$

satisfies (2.2) and (2.3) uniformly in ε.

As usual the equation of continuity is regularized by means of an artificial viscosity term and the momentum balance is replaced by a Faedo-Galerkin approximation to eventually reduce the problem on X_n, a finite-dimensional vector space of functions.

This approximate system can then be solved by a standard procedure. The velocity u of the approximate momentum equation is looked at as a fixed point of a suitable integral operator. Then given u, the approximate continuity equation is solved directly by means of the standard theory of linear parabolic equations. This methodology concerning the compressible Navier–Stokes equations is well explained and described in the reference books [32], [33], [50]. We omit the rest of this classical (but tedious) procedure and we assume that we have well-posed and global weak solutions to (9.1).

We now use the classical energy estimates detailed in Section 2.1. Note that they remain the same in spite of the added viscosity in the continuity equation because, in particular, of the added term $\alpha_k \nabla\rho_k \cdot \nabla u_k$ in the momentum equation. Let us summarize the a priori estimates that are obtained from (2.6):

$$\sup_{k,\varepsilon}\sup_t \int_{\mathbb{T}^d} (\rho_k\,|u_k|^2 + \rho_k^\gamma)\,dx < \infty, \qquad \sup_{k,\varepsilon}\int_0^T\int_{\mathbb{T}^d} |\nabla u_k|^2\,dx\,dt < \infty.$$

The estimate (2.9) may actually require the ε dependent bound from (9.3) with $\gamma_\varepsilon > d$ to control the additional term $\alpha_k\,\nabla\rho_k\cdot\nabla u_k$. It provides

$$\sup_k \int_0^T\int_{\mathbb{T}^d} \rho_k^p(t,x)\,dx\,dt < \infty,$$

with $p_\varepsilon = \gamma_\varepsilon + 2\,\gamma_\varepsilon/d - 1$ or $p = \gamma + 2\,\gamma/d - 1$, which means $p > 2$ as $\gamma > 3\,d/(d+2)$. This bound may not be independent of ε because it requires (9.3). However since α_k vanishes at the limit, it still implies that any weak limit ρ of ρ_k satisfies

$$\sup_\varepsilon \int_0^T\int_{\mathbb{T}^d} \rho_k^p(t,x)\,dx\,dt < \infty$$

for $p = \gamma + 2\,\gamma/d - 1$.

From these a priori estimates, we obtain (5.4) and (5.5). And from those bounds it is straightforward to deduce that $\rho_k\,u_k$ and $\rho_k\,|u_k|^2$ belong to $L^q_{t,x}$

for some $q > 1$, uniformly in k. Therefore using the continuity equation in (9.1), we deduce (5.7). Using the momentum equation, we obtain (5.6), but this bound (and only this bound) is not independent of ε because of $\mathcal{A}_\varepsilon$.

Finally taking the divergence of the momentum equation and inverting Δ,

$$(\lambda + 2\mu)\,\mathrm{div}\, u_k = P_\varepsilon(\rho_k) + \Delta^{-1}\mathrm{div}\,(\partial_t(\rho_k\, u_k) + \mathrm{div}\,(\rho_k\, u_k \otimes u_k)) - \Delta^{-1}\,\mathrm{div}\,(\rho_k\, f + \mathcal{A}_\varepsilon \star u_k) + \alpha_k \Delta^{-1}\,\mathrm{div}\,(\nabla\rho_k \cdot \nabla u_k),$$

which is exactly (5.2) with $\mu_k = \lambda + 2\,\mu$ satisfying (5.3) and compact, while

$$F_k = -\Delta^{-1}\,\mathrm{div}\,(\rho_k\, f + \mathcal{A}_\varepsilon \star u_k) + \alpha_k \Delta^{-1}\,\mathrm{div}\,(\nabla\rho_k \cdot \nabla u_k).$$

The first term in F_k is also compact in L^1 since $\mathcal{A}_\varepsilon$ is smooth for a fixed ε. On the other hand,

$$\alpha_k \Delta^{-1}\,\mathrm{div}\,(\nabla\rho_k \cdot \nabla u_k)$$

converges to 0 in L^1 since $\sqrt{\alpha_k}\nabla\rho_k$ is uniformly bounded in L^2 and ∇u_k is as well in L^2. Therefore F_k is compact in L^1. We may hence apply point (i) of Theorem 5.1 to obtain the compactness of ρ_k in L^1. Then extracting converging subsequences, we can pass to the limit in every term (see Section 2.2 for instance) and obtain the existence of weak solutions to

(9.4)

$$\begin{cases} \partial_t\rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) - \mu\Delta u - (\lambda+\mu)\nabla\mathrm{div} u - \mathcal{A}_\varepsilon \star u + \nabla P_\varepsilon(\rho) = \rho f. \end{cases}$$

9.2. *General pressure laws*: *End of proof of Theorem* 3.1. Now consider a non-monotone pressure P satisfying (3.2) and (3.3). Let us fix $c_{0,\varepsilon} = 1/\varepsilon$ and define

$$P_\varepsilon(\rho) = P(\rho) \quad \text{if } \rho \le c_{0,\varepsilon}, \qquad P_\varepsilon(\rho) = P(c_{0,\varepsilon}) + C(\rho - c_{0,\varepsilon})^\gamma \quad \text{if } \rho \ge c_{0,\varepsilon}.$$

If $\gamma \le d$, then we also add to P_ε a term in $\varepsilon\,\rho_\varepsilon^{\gamma_\varepsilon}$ to satisfy (9.3). Note that P_ε is Lipschitz and converges uniformly to P on any compact interval. Due to (3.2) there exists $\rho_{0,\varepsilon}$ with $\rho_{0,\varepsilon} \to +\infty$ as $\varepsilon \to +\infty$ such that for $\rho \ge \rho_{0,\varepsilon}$,

$$(P_\varepsilon(s)/s)' = (P_\varepsilon'(s)s - P(s))/s^2 \ge \big(C(\gamma-1)(\rho - c_{0,\varepsilon})^\gamma - P(c_{0,\varepsilon})\big)/s^2 \ge 0.$$

The approximate pressure P_ε still satisfies (3.2) with γ, and due to the previous inequality it satisfies (5.8) for $\rho \ge \rho_{0,\varepsilon}$ and (5.9) in the following sense: For all $s \ge 0$,

$$|P_\varepsilon'(s)| \le \overline{P} s^{\tilde\gamma - 1}\mathbb{I}_{s \le c_{0,\varepsilon}} + C(\gamma-1)s^{\gamma-1}\mathbb{I}_{s \ge c_{0,\varepsilon}}.$$

As a consequence, we have existence of weak solutions $(\rho,\ u)$ to (9.1) for this P_ε (assuming $\mathcal{A}_\varepsilon = 0$) for any $\varepsilon > 0$. Consider a sequence $\varepsilon_k \to 0$ and the corresponding sequence $(\rho_k,\ u_k)$ of weak solutions to (9.1).

Because the previous a priori estimates were uniform in ε for the limit ρ and u (including (5.6) since $\mathcal{A}_\varepsilon = 0$), then the sequence $(\rho_k,\ u_k)$ satisfies all the bounds (5.4), (5.5), (5.6), (5.7) and (5.9).

Moreover the representation (5.2) still holds with $\mu_k = 2\mu + \lambda$, compact in L^1 and satisfying (5.3). Finally the exponent p in (5.4) can be chosen up to $\gamma + 2\,\gamma/d - 1$. Since $\gamma > 3\,d/(d+2)$, then $p > 2$. Since $\gamma > (\tilde\gamma + 1)\,d/(d+2)$, then one has $p > \tilde\gamma$ as well.

Therefore all the assumptions of point (ii) of Theorem 5.1 are satisfied and one has the compactness of ρ_k. Extracting converging subsequences of ρ_k and u_k, one passes to the limit in every term. Note, in particular, that $P_{\varepsilon_k}(\rho_k)$ converges in L^1 to $P(\rho)$ by the compactness of ρ_k, the uniform convergence of P_{ε_k} to P on compact intervals and by truncating $P_{\varepsilon_k}(\rho_k)$ for large values of ρ_k since the exponent p in (5.4) is strictly larger than γ.

This proves the global existence in Theorem 3.1. The regularity of ρ follows from Theorem 5.2, which concludes the proof of Theorem 3.1.

9.3. *Anisotropic viscosities*: *End of proof of Theorem* 3.2. For simplicity, we take $f = 0$. Now consider a "quasi-monotone" pressure P satisfying (3.8). Observe that P then automatically satisfies (3.2) since $P(0) = 0$. To satisfy (5.8), we have to modify P on an interval $(c_{0,\varepsilon}, +\infty)$ with $c_{0,\varepsilon} \to +\infty$ when $\varepsilon \to +\infty$. More precisely, we consider P_ε as defined in the previous subsection:

$$P_\varepsilon(\rho) = P(\rho) \quad \text{if } \rho \le c_{0,\varepsilon}, \qquad P_\varepsilon(\rho) = P(c_{0,\varepsilon}) + C(\rho - c_{0,\varepsilon})^\gamma \quad \text{if } \rho \ge c_{0,\varepsilon}.$$

Remark that since $\gamma > d$ here, we never need to add a term with γ_ε.

Now given any smooth kernel, for instance $\overline{K}$, we define

$$\mathcal{A}_\varepsilon \star u = \operatorname{div}(\delta A(t)\, \nabla\, \overline{K}_\varepsilon \star u).$$

Because of the smallness assumption on $\delta A(t)$, the operator $\mathcal{D}_\varepsilon$ satisfies (2.2) and (2.3) uniformly in ε. Therefore we have existence of global weak solutions to (9.4) with this choice of P_ε and $\mathcal{A}_\varepsilon$. We again consider a sequence of such solutions $(\rho_k,\ u_k)$ corresponding to some sequence $\varepsilon_k \to 0$. Because the estimates are uniform in ε for (9.4), we again have that this sequence satisfies the bounds (5.4), (5.5), (5.7). We now assume that

$$\gamma > \frac{d}{2}\left[\left(1 + \frac{1}{d}\right) + \sqrt{1 + \frac{1}{d^2}}\right],$$

implying that p in (5.4) is strictly larger than $\gamma^2/(\gamma - 1)$. Moreover observe that

$$\|\mathcal{A}_{\varepsilon_k} u_k\|_{L^2_t\, H^{-1}_x} \le C\, \|\nabla u_k\|_{L^2_{t,x}},$$

such that (5.6) is also satisfied.

For simplicity, we assume that δA has a vanishing trace; otherwise just add the corresponding trace to μ. Denote $a_\mu = 2\,\|\delta A\|_{L^\infty}/(2\,\mu + \lambda)$ and the operator E_k

$$E_k\,u = -\mathrm{div}\left(\frac{\delta A(t)}{2\,\|\delta A\|_{L^\infty}}\,\nabla\,\overline{K}_\varepsilon \star u\right) = -\sum_{ij}\frac{\delta A_{ij}(t)}{2\,\|\delta A\|_{L^\infty}}\,\partial_{ij}\overline{K}_{\varepsilon_k}\star u.$$

For a_μ small enough, $\Delta - a_\mu\,E_k$ is a uniform elliptic operator so that $(\Delta - a_\mu\,E_k)^{-1}\,\Delta$ is bounded on every L^p space, uniformly in k. For the same reason, $A_\mu = (\Delta - a_\mu\,E_k)^{-1}\,E_k$ is bounded on every L^p space with norm less than 1 and can be represented by a convolution with a singular integral.

Taking the divergence of the momentum equation in (9.1), one has

$$\begin{aligned}(2\mu+\lambda)\,\big(\Delta\,\mathrm{div}\,u_k - a_\mu\,E_k\,\mathrm{div}\,u\big)&\\ = \Delta\,P(\rho_k) + \mathrm{div}\,&(\partial_t(\rho_k\,u_k) + \mathrm{div}\,(\rho_k\,u_k\otimes u_k)).\end{aligned}$$

Just write $\Delta\,P_\varepsilon(\rho_k) = (\Delta - a_\mu\,E_k)\,P_\varepsilon(\rho_k) + a_\mu\,E_k\,P_\varepsilon$ and take the inverse of $\Delta - a_\mu\,E_k$ to obtain

$$\begin{aligned}(2\mu+\lambda)\,\mathrm{div}\,u =& P(\rho_k) + a_\mu\,(\Delta - a_\mu\,E_k)^{-1}\,E_k\,P(\rho_k)\\ &+ (\Delta - a_\mu\,E_k)^{-1}\,\mathrm{div}\,(\partial_t(\rho_k\,u_k) + \mathrm{div}\,(\rho_k\,u_k\otimes u_k)),\end{aligned}$$

which is exactly (5.10) with $\nu_k = (2\,\mu+\lambda)^{-1}$. As a consequence, if $a_\mu \le C_*$, which is implied by $\|\delta A\|_{L^\infty}$ small enough, then Theorem 5.3 applies and ρ_k is compact. Passing to the limit again in every term proves Theorem 3.2. Note that $P_{\varepsilon_k}(\rho_k)$ converges in L^1 to $P(\rho)$ for the same reason as in the previous subsection.

The case with $D(u)$ instead of ∇u. Let us finish this proof by remarking on the different structure in the case with symmetric stress tensor $\mathrm{div}\,(A\,D\,u)$. In that case, one cannot find $\mathrm{div}\,u_k$ by taking the divergence of the momentum equation, but instead we have to consider the whole momentum equation. Let us write it as

$$\mathcal{E}\,u_k = \nabla P(\rho_k) + \partial_t(\rho_k\,u_k) + \mathrm{div}\,(\rho_k\,u_k\otimes u_k),$$

with $\mathcal{E}$ the elliptic vector-valued operator

$$\mathcal{E}\,u = \mu\,\Delta u + (\mu+\lambda)\,\nabla\mathrm{div}\,u + \mathrm{div}\,(\,\delta A\,D\,u).$$

The operator $\mathcal{E}$ is invertible for δA small enough as one can readily check in Fourier, for instance, where $-\hat{\mathcal{E}}$ becomes a perturbation of $\mu\,|\xi|^2\,I + (\mu + \lambda)\,\xi\otimes\xi$. Its inverse has most of the usual properties of inverses of a scalar elliptic operator (with the exception of the maximum principle for instance). Therefore, one may still write

$$\mathrm{div}\,u_k = \mathrm{div}\;\mathcal{E}^{-1}\,\nabla P(\rho_k) + \mathrm{div}\;\mathcal{E}^{-1}\,(\partial_t(\rho_k\,u_k) + \mathrm{div}\,(\rho_k\,u_k\otimes u_k)),$$

leading to the variant (5.11) of the simpler formula (5.10).

10. Appendix: Notation

For the reader's convenience, we repeat and summarize some of our main notation.

Physical quantities.

- ρ, or ρ_k denotes the density of the fluid.
- u, or u_k denotes the velocity field of the fluid.
- $P(.)$, or $P_k(.)$ denotes the pressure law.
- $e(\rho)$ is the internal energy density, which in the barotropic case, reads $e(\rho) = \int_{\rho_{ref}}^{\rho} P(s)/s^2\,ds$.
- $E(\rho, u) = \int \rho\,(|u|^2/2 + e(\rho))$ is the total energy of the fluid.
- μ, λ and μ_k denote various viscosity coefficients or combination thereof.
- $\mathcal{S}$ denotes the viscous stress tensor. In the simplest isotropic case, $\mathcal{S} = 2\mu D(u) + \lambda \mathrm{div}\, u \mathrm{Id}$.
- $\mathcal{D}$ is the diffusion term related to the viscous stress tensor by $\mathcal{D}u = \mathrm{div}\,\mathcal{S}$.

Technical notation.

- d is the dimension of space.
- g^X means $g(t, X)$, where X is any space variable in $\mathbb{T}^d$.
- $g^{X,Y}$ means $g(t, X, Y)$ where X and Y are space variables in $\mathbb{T}^d$.
- k as an index always denotes the index of a sequence.
- h and h_0 are scaling parameters used to measure oscillations of certain quantities such as the density.
- K_h is a convolution kernel on $\mathbb{T}^d$ and $K_h(x) = (h+|x|)^{-a}$ for x small enough and with $a > d$.
- $\overline{K}_h$ is equal to $K_h/\|K_h\|_{L^1}$.
- $\mathcal{K}_{h_0} = \int_{h_0}^{1} \overline{K}_h(x)\,\frac{dh}{h}$ is the weighted average of K_h. Note that $\|\mathcal{K}_{h_0}\|_{L^1} \sim |\log h_0|$.
- w_0, w_1 and w_a are the weights and $w_{i,h} = \overline{K}_h \star w_i$ their regularization with $i = 0, 1, a$.
- C is a constant whose exact value may change from one line to another but which is always independent of k, h or other scaling parameters.
- $\varepsilon(h)$ is a smooth function with $\varepsilon(0) = 0$.
- θ is an exponent whose exact value may change as for C but in $(0, 1)$.
- The exponent p is most of the time such that $\rho \in L^p_{t,x}$.
- q and r are other exponents for L^p type spaces that are used when needed.
- I, $II, \dots$ and A, B, D, $E, \dots$ denote some intermediary quantities used in the proofs. Their definitions may change from one proof to another.
- x, y, w, z are typically variables of integration over the space domain.
- $\delta\rho_k = \rho_k^x - \rho_k^y$ is the difference of densities.
- $\bar{\rho}_k = \rho_k^x + \rho_k^y$ is the sum of densities.

- $D\,\rho u_k = \Delta^{-1}\mathrm{div}\,(\partial_t(\rho_k\, u_k) + \mathrm{div}\,(\rho_k\, u_k \otimes u_k))$ denotes the effective viscous flux.
- The individual weights $w = w_0,\ w_1,\ w_a$ satisfy equation (7.2) or

$$\partial_t w + u_k \cdot \nabla w = -D\, w + \alpha_k\, \Delta w,$$

  where $D = D_0, D_1, D_a$ are respectively the penalizations in (7.3), (7.4), and (7.5).
- The weights w_0 or w_a may be convolved to give $w_h = \overline{K}_h \star w_0,\ w_{a,h} = \overline{K}_h \star w_a$.
- The weights are then added or multiplied to obtain the composed $W(t,x,y) = W_0, W_1,\ W_2, W_a$ with

$$W_0(t,x,y) = w_0(t,x) + w_0(t,y), \quad W_1(t,x,y) = w_1(t,x) + w_1(t,y),$$
$$W_2(t,x,y) = w_1(t,x)\, w_1(t,y), \qquad W_a(t,x,y) = w_a(t,x) + w_a(t,y).$$

  The main properties of the weights are given in Proposition 7.2.

11. **Appendix: Besov spaces and Littlewood–Paley decomposition**

We only recall some basic definitions and properties of Besov spaces for use in Lemma 6.3. We start with the classical Littlewood–Paley decomposition and refer to the readers, for instance, to [5], [1] and [7] for details and applications to fluid mechanic. Choose any family $\Psi_k \in \mathcal{S}(\mathbb{T}^d)$ such that

- its Fourier transform $\hat{\Psi}_k$ is positive and compactly supported in the annulus $\{2^{k-1} \leq |\xi| \leq 2^{k+1}\}$;
- it leads to a decomposition of the identity in the sense that there exists Φ with $\hat{\Phi}$ compactly supported in $\{|\xi| \leq 2\}$ such that for any ξ,

$$1 = \hat{\Phi}(\xi) + \sum_{k\geq 1} \hat{\Psi}_k(\xi);$$

- the family is localized in $\mathbb{T}^d$ in the sense that for all $s > 0$,

$$\sup_k \|\Psi_k\|_{L^1} < \infty, \quad \sup_k 2^{ks} \int_{\mathbb{T}^d} |z|^s\, |\Psi_k(z)|\, dz < \infty.$$

Note that in $\mathbb{R}^d$, one usually takes $\Psi_k(x) = 2^{kd}\, \Psi(2^k\, x)$ but in the torus, it can be advantageous to use a more general family. It is still necessary to take it smooth enough for the third assumption to be satisfied. (It is, for instance, the difference between the Dirichlet and Fejer kernels.)

For simplicity, we then denote $\Psi_0 = \Phi$ for $k = 0$ and for $k \geq 1$, $\Psi_k(x) = 2^{kd}\, \Psi(2^{-k}\, x)$. For any $f \in \mathcal{S}'(\mathbb{R}^d)$, we also write $f_k = \Psi_k \star f$ and then obtain the decomposition

$$f = \sum_{k=0}^{\infty} f_k. \tag{11.1}$$

From this decomposition one may easily define the Besov spaces:

Definition 11.1. The Besov space $B^s_{p,q}$ is the space of all $f \in L^1_{loc} \cap \mathcal{S}'(\mathbb{R}^d)$ for which

$$\|f\|_{B^s_{p,q}} = \left\|2^{s\,k}\,\|f_k\|_{L^p_x}\right\|_{l^q_k} = \left(\sum_{k=0}^{\infty} 2^{s\,k\,q}\,\|f_k\|^q_{L^p_x}\right)^{1/q} < \infty.$$

The main properties of the Littlewood–Paley decomposition that we use in this article can be summarized as

Proposition 11.2. *For any* $1 < p < \infty$ *and any* s, *there exists* $C > 0$ *such that for any* $f \in L^1_{loc} \cap \mathcal{S}'(\mathbb{R}^d)$,

$$\frac{2^{s\,k}}{C}\,\|f_k\|_{L^p} \leq \|\Delta^{s/2}\,f_k\|_{L^p} \leq C\,2^{s\,k}\,\|f_k\|_{L^p},$$

$$C^{-1}\left\|\left(\sum_{k=0}^{\infty} 2^{2\,k\,s}\,|f_k|^2\right)^{1/2}\right\|_{L^p} \leq \|f\|_{W^{s,p}} \leq C\left\|\left(\sum_{k=0}^{\infty} 2^{2\,k\,s}\,|f_k|^2\right)^{1/2}\right\|_{L^p}.$$

And as a consequence, *we have for* $1 < p \leq 2$,

$$C^{-1}\,\|f\|_{B^s_{p,2}} \leq \|f\|_{W^{s,p}} \leq C\,\|f\|_{B^s_{p,p}}.$$

Note that the norm $\left\|\left(\sum_{k=0}^{\infty} 2^{2\,k\,s}\,|f_k|^2\right)^{1/2}\right\|_{L^p}$ actually defines the $F^2_{p,2}$ spaces that for $1 < p < \infty$ are equivalent to the classical Sobolev spaces. In particular, a consequence of Proposition 11.2 is the following bound on truncated Besov norm:

Lemma 11.3. *For any* $1 < p \leq 2$, *there exists* $C > 0$ *such that for any* $f \in L^1_{loc} \cap \mathcal{S}'(\mathbb{R}^d)$ *and any* $K \in \mathbb{N}$,

$$\sum_{k=0}^{K} 2^{s\,k}\,\|f_k\|_{L^p_x} \leq C\,\sqrt{K}\,\|f\|_{W^{s,p}}.$$

Proof. By a simple Cauchy-Schwartz estimate,

$$\sum_{k=0}^{K} 2^{s\,k}\,\|f_k\|_{L^p_x} \leq \sqrt{K}\,\left(\sum_{k=0}^{\infty} 2^{2\,s\,k}\,\|f_k\|^2_{L^p_x}\right)^{1/2} = \sqrt{K}\,\|f\|_{B^s_{p,2}},$$

which can easily conclude the proof by applying Proposition 11.2. □

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References

[1] H. Abels, *Pseudodifferential and Singular Integral Operators*, *De Gruyter Graduate Lectures. An Introduction with Applications*, De Gruyter, Berlin, 2012. MR 2884718. Zbl 1235.35001.

[2] G. Alberti, S. Bianchini, and G. Crippa, A uniqueness result for the continuity equation in two dimensions, *J. Eur. Math. Soc.* (*JEMS*) **16** no. 2 (2014), 201–234. MR 3161282. Zbl 1286.35006. https://doi.org/10.4171/JEMS/431.

[3] L. Ambrosio, Transport equation and Cauchy problem for *BV* vector fields, *Invent. Math.* **158** no. 2 (2004), 227–260. MR 2096794. Zbl 1075.35087. https://doi.org/10.1007/s00222-004-0367-2.

[4] L. Ambrosio, C. De Lellis, and J. Malý, On the chain rule for the divergence of BV-like vector fields: applications, partial results, open problems, in *Perspectives in Nonlinear Partial Differential Equations*, *Contemp. Math.* **446**, Amer. Math. Soc., Providence, RI, 2007, pp. 31–67. MR 2373724. Zbl 1200.49043. https://doi.org/10.1090/conm/446/08625.

[5] H. Bahouri, J.-Y. Chemin, and R. Danchin, *Fourier Analysis and Nonlinear Partial Differential Equations*, *Grundlehren Math. Wiss.* **343**, Springer, Heidelberg, 2011. MR 2768550. Zbl 1227.35004. https://doi.org/10.1007/978-3-642-16830-7.

[6] F. B. Belgacem and P.-E. Jabin, Compactness for nonlinear continuity equations, *J. Funct. Anal.* **264** no. 1 (2013), 139–168. MR 2995703. Zbl 1262.35157. https://doi.org/10.1016/j.jfa.2012.10.005.

[7] S. Benzoni-Gavage and D. Serre, *Multi-dimensional Hyperbolic Partial Differential Equations. First-order Systems and Applications*, *Oxford Math. Monogr.*, The Clarendon Press, Oxford University Press, Oxford, 2007. MR 2284507. Zbl 1113.35001.

[8] F. Bouchut, Renormalized solutions to the Vlasov equation with coefficients of bounded variation, *Arch. Ration. Mech. Anal.* **157** no. 1 (2001), 75–90. MR 1822415. Zbl 0979.35032. https://doi.org/10.1007/PL00004237.

[9] F. Bouchut and G. Crippa, Lagrangian flows for vector fields with gradient given by a singular integral, *J. Hyperbolic Differ. Equ.* **10** no. 2 (2013), 235–282. MR 3078074. Zbl 1275.35076. https://doi.org/10.1142/S0219891613500100.

[10] F. Bouchut and L. Desvillettes, On two-dimensional Hamiltonian transport equations with continuous coefficients, *Differential Integral Equations* **14** no. 8 (2001), 1015–1024. MR 1827101. Zbl 1028.35042. Available at https://projecteuclid.org/euclid.die/1356123178.

[11] J. Bourgain, H. Brezis, and P. Mironescu, Another look at Sobolev spaces, in *Optimal Control and Partial Differential Equations*, IOS, Amsterdam, 2001, pp. 439–455. MR 3586796. Zbl 1103.46310.

[12] D. Bresch and B. Desjardins, On the existence of global weak solutions to the Navier-Stokes equations for viscous compressible and heat conducting fluids, *J. Math. Pures Appl.* (9) **87** no. 1 (2007), 57–90. MR 2297248. Zbl 1122.35092. https://doi.org/10.1016/j.matpur.2006.11.001.

[13] D. Bresch and B. Desjardins, Weak solutions with density-dependent viscosities, in *Handbook of Mathematical Analysis in Mechanics of Viscous Fluids*, 2018, Y. Giga and A. Novotny, editors; first online: 10 April 2017, pp. 1547–1599. Zbl 06382698. https://doi.org/10.1007/978-3-319-10151-4_44-1.

[14] D. Bresch, B. Desjardins, and D. Gérard-Varet, Rotating fluids in a cylinder, *Discrete Contin. Dyn. Syst.* **11** no. 1 (2004), 47–82. MR 2073946. Zbl 1138.76446. https://doi.org/10.3934/dcds.2004.11.47.

[15] D. Bresch, B. Desjardins, and E. Zatorska, Two-velocity hydrodynamics in fluid mechanics: Part II. Existence of global κ-entropy solutions to the compressible Navier-Stokes systems with degenerate viscosities, *J. Math. Pures Appl.* (9) **104** no. 4 (2015), 801–836. MR 3394617. Zbl 1359.35125. https://doi.org/10.1016/j.matpur.2015.05.004.

[16] N. Champagnat and P.-E. Jabin, Well posedness in any dimension for Hamiltonian flows with non BV force terms, *Comm. Partial Differential Equations* **35** no. 5 (2010), 786–816. MR 2753620. Zbl 1210.34010. https://doi.org/10.1080/03605301003646705.

[17] G.-Q. Chen, D. Hoff, and K. Trivisa, Global solutions of the compressible Navier-Stokes equations with large discontinuous initial data, *Comm. Partial Differential Equations* **25** no. 11-12 (2000), 2233–2257. MR 1789926. Zbl 0977.35104. https://doi.org/10.1080/03605300008821583.

[18] G.-Q. Chen and K. Trivisa, Analysis on models for exothermically reacting, compressible flows with large discontinous initial data, in *Nonlinear Partial Differential Equations and Related Analysis*, *Contemp. Math.* **371**, Amer. Math. Soc., Providence, RI, 2005, pp. 73–91. MR 2143860. Zbl 1075.35036. https://doi.org/10.1090/conm/371/06848.

[19] E. Chénier, R. Eymard, T. Gallouët, and R. Herbin, An extension of the MAC scheme to locally refined meshes: convergence analysis for the full tensor time-dependent Navier-Stokes equations, *Calcolo* **52** no. 1 (2015), 69–107. MR 3313589. Zbl 1310.76105. https://doi.org/10.1007/s10092-014-0108-x.

[20] F. Colombini, G. Crippa, and J. Rauch, A note on two-dimensional transport with bounded divergence, *Comm. Partial Differential Equations* **31** no. 7-9 (2006), 1109–1115. MR 2254607. Zbl 1122.35143. https://doi.org/10.1080/03605300500455933.

[21] F. Colombini and J. Rauch, Uniqueness in the Cauchy problem for transport in $\mathbb{R}^2$ and $\mathbb{R}^{1+2}$, *J. Differential Equations* **211** no. 1 (2005), 162–167. MR 2121112. Zbl 1134.35311. https://doi.org/10.1016/j.jde.2004.06.007.

[22] G. Crippa and C. De Lellis, Estimates and regularity results for the DiPerna-Lions flow, *J. Reine Angew. Math.* **616** (2008), 15–46. MR 2369485. Zbl 1160.34004. https://doi.org/10.1515/CRELLE.2008.016.

[23] C. De Lellis, Notes on hyperbolic systems of conservation laws and transport equations, in *Handbook of Differential Equations*: *Evolutionary Equations. Vol. III*, *Handb. Differ. Equ.*, Elsevier/North-Holland, Amsterdam, 2007, pp. 277–382. MR 2549371. Zbl 1195.35215. https://doi.org/10.1016/S1874-5717(07)80007-7.

[24] C. De Lellis, P. Gwiazda, and A. Świerczewska-Gwiazda, Transport equations with integral terms: existence, uniqueness and stability, *Calc. Var. Partial Differential Equations* **55** no. 5 (2016), Art. 128, 17. MR 3554338. Zbl 06654681. https://doi.org/10.1007/s00526-016-1049-9.

[25] N. Depauw, Non unicité des solutions bornées pour un champ de vecteurs BV en dehors d'un hyperplan, *C. R. Math. Acad. Sci. Paris* **337** no. 4 (2003), 249–252. MR 2009116. Zbl 1024.35029. https://doi.org/10.1016/S1631-073X(03)00330-3.

[26] R. J. DiPerna and P.-L. Lions, Ordinary differential equations, transport theory and Sobolev spaces, *Invent. Math.* **98** no. 3 (1989), 511–547. MR 1022305. Zbl 0696.34049. https://doi.org/10.1007/BF01393835.

[27] B. Ducomet, E. Feireisl, H. Petzeltová, and I. Straškraba, Global in time weak solutions for compressible barotropic self-gravitating fluids, *Discrete Contin. Dyn. Syst.* **11** no. 1 (2004), 113–130. MR 2073949. Zbl 1080.35068. https://doi.org/10.3934/dcds.2004.11.113.

[28] R. Eymard, T. Gallouët, R. Herbin, and J.-C. Latché, Convergence of the MAC scheme for the compressible Stokes equations, *SIAM J. Numer. Anal.* **48** no. 6 (2010), 2218–2246. MR 2763662. Zbl 05931236. https://doi.org/10.1137/090779863.

[29] R. Eymard, T. Gallouët, R. Herbin, and J. C. Latché, A convergent finite element-finite volume scheme for the compressible Stokes problem. II. The isentropic case, *Math. Comp.* **79** no. 270 (2010), 649–675. MR 2600538. Zbl 1197.35192. https://doi.org/10.1090/S0025-5718-09-02310-2.

[30] E. B. Fabes, C. E. Kenig, and R. P. Serapioni, The local regularity of solutions of degenerate elliptic equations, *Comm. Partial Differential Equations* **7** no. 1 (1982), 77–116. MR 0643158. Zbl 0498.35042. https://doi.org/10.1080/03605308208820218.

[31] E. Feireisl, Compressible Navier-Stokes equations with a non-monotone pressure law, *J. Differential Equations* **184** no. 1 (2002), 97–108. MR 1929148. Zbl 1012.76079. https://doi.org/10.1006/jdeq.2001.4137.

[32] E. Feireisl, *Dynamics of Viscous Compressible Fluids*, *Oxford Lecture Ser. Math. Appl.* **26**, Oxford University Press, Oxford, 2004. MR 2040667. Zbl 1080.76001. https://doi.org/10.1093/acprof:oso/9780198528388.001.0001.

[33] E. Feireisl and A. n. Novotný, *Singular Limits in Thermodynamics of Viscous Fluids*, *Adv. Math. Fluid Mech.*, Birkhäuser Verlag, Basel, 2009. MR 2499296. Zbl 1176.35126. https://doi.org/10.1007/978-3-7643-8843-0.

[34] E. Feireisl, A. Novotný, and H. Petzeltová, On the existence of globally defined weak solutions to the Navier-Stokes equations, *J. Math. Fluid Mech.* **3** no. 4 (2001), 358–392. MR 1867887. Zbl 0997.35043. https://doi.org/10.1007/PL00000976.

[35] T. Gallouët, L. Gastaldo, R. Herbin, and J.-C. Latché, An unconditionally stable pressure correction scheme for the compressible barotropic Navier-Stokes equations, *M2AN Math. Model. Numer. Anal.* **42** no. 2 (2008), 303–331. MR 2405150. Zbl 1132.35433. https://doi.org/10.1051/m2an:2008005.

[36] T. Gallouët, R. Herbin, and J.-C. Latché, A convergent finite element-finite volume scheme for the compressible Stokes problem. I. The isothermal case, *Math. Comp.* **78** no. 267 (2009), 1333–1352. MR 2501053. Zbl 1223.76041. https://doi.org/10.1090/S0025-5718-09-02216-9.

[37] T. Gallouët, R. Herbin, and J.-C. Latché, Kinetic energy control in explicit Finite Volume discretizations of the incompressible and compressible Navier-Stokes equations, *Int. J. Finite Vol.* **7** no. 2 (2010), 6 pp. MR 2753585. Available at https://hal.archives-ouvertes.fr/hal-01112391.

[38] M. Hauray, On two-dimensional Hamiltonian transport equations with $L^p_{\rm loc}$ coefficients, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **20** no. 4 (2003), 625–644. MR 1981402. Zbl 1028.35148. https://doi.org/10.1016/S0294-1449(02)00015-X.

[39] M. Hauray, On Liouville transport equation with force field in $BV_{\rm loc}$, *Comm. Partial Differential Equations* **29** no. 1-2 (2004), 207–217. MR 2038150. Zbl 1103.35003. https://doi.org/10.1081/PDE-120028850.

[40] P.-E. Jabin and N. Masmoudi, DiPerna-Lions flow for relativistic particles in an electromagnetic field, *Arch. Ration. Mech. Anal.* **217** no. 3 (2015), 1029–1067. MR 3356994. Zbl 1317.35249. https://doi.org/10.1007/s00205-015-0850-5.

[41] P.-E. Jabin, Differential equations with singular fields, *J. Math. Pures Appl.* (9) **94** no. 6 (2010), 597–621. MR 2737390. Zbl 1217.34015. https://doi.org/10.1016/j.matpur.2010.07.001.

[42] T. K. Karper, A convergent FEM-DG method for the compressible Navier-Stokes equations, *Numer. Math.* **125** no. 3 (2013), 441–510. MR 3117509. Zbl 1278.76059. https://doi.org/10.1007/s00211-013-0543-7.

[43] S. N. Kružkov, First order quasilinear equations with several independent variables, *Mat. Sb.* (*N.S.*) **81 (123)** (1970), 228–255. MR 0267257. Zbl 0202.11203. https://doi.org/10.1070/SM1970v010n02ABEH002156.

[44] C. Le Bris and P.-L. Lions, Renormalized solutions of some transport equations with partially $W^{1,1}$ velocities and applications, *Ann. Mat. Pura Appl.* (4) **183** no. 1 (2004), 97–130. MR 2044334. Zbl 1170.35364. https://doi.org/10.1007/s10231-003-0082-4.

[45] J. Leray, Sur le mouvement d'un liquide visqueux emplissant l'espace, *Acta Math.* **63** no. 1 (1934), 193–248. MR 1555394. JFM 60.0726.05. https://doi.org/10.1007/BF02547354.

[46] J. Li and Z. Xin, Global existence of weak solutions to the barotropic compressible Navier–Stokes flows with degenerate viscosities, 2015. arXiv 1504.06826.

[47] J.-L. Lions and J. Peetre, Sur une classe d'espaces d'interpolation, *Inst. Hautes Études Sci. Publ. Math.* **19** (1964), 5–68. MR 0165343. Zbl 0148.11403. https://doi.org/10.1007/BF02684796.

[48] P.-L. Lions, *Mathematical Topics in Fluid Mechanics. Vol. II*: *Compressible Models*, *Oxford Lecture Ser. Math. Appl.* **10**, The Clarendon Press, Oxford University Press, New York, 1998. MR 1637634. Zbl 1264.76003.

[49] A. Novotný, R. Danchin, and M. Perepelitsa, *Topics on Compressible Navier-Stokes Equations*, *Panor. Synthèses* **50**, Société Mathématique de France, Paris, 2016, papers from the session États de la Recherche held in Le Bourget du Lac, May 21–25, 2012; edited by D. Bresch. MR 3560280. Zbl 1355.35002.

[50] A. Novotný and I. Straškraba, *Introduction to the Mathematical Theory of Compressible Flow*, *Oxford Lecture Ser. Math. Appl.* **27**, Oxford University Press, Oxford, 2004. MR 2084891. Zbl 1088.35051.

[51] P. I. Plotnikov and W. Weigant, Isothermal Navier-Stokes equations and Radon transform, *SIAM J. Math. Anal.* **47** no. 1 (2015), 626–653. MR 3305369. Zbl 1323.35134. https://doi.org/10.1137/140960542.

[52] P. Plotnikov and J. Sokołowski, *Compressible Navier-Stokes Equations. Theory and Shape Optimization*, *Inst. Mat. Polskiej Akad. Nauk. Monogr. Mat.* **73**, Birkhäuser/Springer Basel AG, Basel, 2012. MR 2963679. Zbl 1260.35002. https://doi.org/10.1007/978-3-0348-0367-0.

[53] A. C. Ponce, An estimate in the spirit of Poincaré's inequality, *J. Eur. Math. Soc.* (*JEMS*) **6** no. 1 (2004), 1–15. MR 2041005. Zbl 1051.46019. https://doi.org/10.4171/JEMS/1.

[54] E. M. Stein, Maximal functions. I: Spherical means, *Proc. Nat. Acad. Sci. U.S.A.* **73** no. 7 (1976), 2174–2175. MR 0420116. Zbl 0332.42018. https://doi.org/10.1073/pnas.73.7.2174.

[55] E. M. Stein, *Harmonic Analysis*: *Real-Variable Methods*, *Orthogonality*, *and Oscillatory Integrals*, *Princeton Math. Ser.* **43**, Princeton Univ. Press, Princeton, NJ, 1993, with the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III. MR 1232192. Zbl 0821.42001. https://doi.org/10.1515/9781400883929.

[56] R. Temam and M. Ziane, Some mathematical problems in geophysical fluid dynamics, in *Handbook of Mathematical Fluid Dynamics. Vol. III*, North-Holland, Amsterdam, 2004, pp. 535–657. MR 2099038. Zbl 1222.35145. https://doi.org/10.1016/S1874-5792(05)80009-6.

[57] V. A. Vaĭgant and A. V. Kazhikhov, On the existence of global solutions of two-dimensional Navier-Stokes equations of a compressible viscous fluid, *Sibirsk. Mat. Zh.* **36** no. 6 (1995), 1283–1316. MR 1375428. Zbl 0860.35098. https://doi.org/10.1007/BF02106835.

[58] A. F. Vasseur and C. Yu, Existence of global weak solutions for 3D degenerate compressible Navier-Stokes equations, *Invent. Math.* **206** no. 3 (2016), 935–974. MR 3573976. Zbl 1354.35115. https://doi.org/10.1007/s00222-016-0666-4.

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