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Hyperbolic triangles without embedded eigenvalues

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Abstract

We consider the Neumann Laplacian acting on square-integrable functions on a triangle in the hyperbolic plane that has one cusp. We show that the generic such triangle has no eigenvalues embedded in its continuous spectrum. To prove this result we study the behavior of the real-analytic eigenvalue branches of a degenerating family of triangles. In particular, we use a careful analysis of spectral projections near the crossings of these eigenvalue branches with the eigenvalue branches of a model operator.

1. **Introduction**

Though well studied for over fifty years, the spectral theory of hyperbolic surfaces still presents basic unresolved questions [Sar03]. For example, does there exist a noncompact, finite area hyperbolic surface whose Laplacian has no nonconstant square-integrable eigenfunctions? This question has been the subject of many investigations including [Col83], [PS85], [DIPS85], [PS92b], [Wol92a], [Wol92b], [Wol94], and [PS94].

As a model problem, Phillips and Sarnak [PS92a] suggested studying the Neumann eigenvalue problem on the domain $\mathcal{T}_t \subset \mathbb{R} \times \mathbb{R}^+$ pictured in Figure 1. In this paper, we prove the following theorem.

Theorem 1.1. *For all but at most countably many $t \in\ ]0,1[$, the Neumann Laplacian on the geodesic triangle $\mathcal{T}_t$ in the hyperbolic plane has no nonconstant* (*square-integrable*) *eigenfunction.*

Keywords: Laplacian, hyperbolic geometry, cusp form, analytic perturbation theory, eigenvalue crossing

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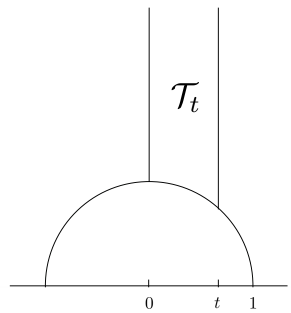

Figure 1. The triangle $\mathcal{T}_t$ defined by $x^2 + y^2 \geqslant 1$ and $0 \leqslant x \leqslant t$.

The group G_t of hyperbolic isometries generated by reflections in the geodesic arcs that bound $\mathcal{T}_t$ is discrete if and only if $t = \cos(\pi/n)$ for $n \geqslant 3$ an integer. For example, if $t = 1/2$, then G_t contains an index two subgroup that is naturally isomorphic to $\mathrm{PSL}_2(\mathbb{Z})$. It follows from the seminal work of A. Selberg [Sel89] that if $n = 3, 4$, or 6, then the Neumann Laplacian has infinitely many nonconstant eigenfunctions. In particular, for some special t, there do exist square-integrable solutions to the Neumann problem.[1]

In [Jud95], Theorem 1.1 was verified under an additional — and as of yet unjustified — assumption concerning the spectral multiplicities of the Neumann Laplacian acting on $L^2(\mathcal{T}_1)$. The proof consisted of studying the singular perturbation problem associated with letting t tend to 1. Similar singular perturbations were studied in the context of degenerating hyperbolic surfaces [Wol94] and unitary characters over a fixed hyperbolic surface [PS94]. In [Wol94], [PS94], [Jud95], and all prior work on this problem, it was necessary to make assumptions about the multiplicities of the spectrum of the limiting surface.

The angles of a geodesic triangle in the hyperbolic plane determine the isometry class of the triangle. The angles of $\mathcal{T}_t$ are $(\pi/2, \arccos(t), 0)$. It is not difficult to extend Theorem 1.1 to triangles with angles $(\theta_1, \theta_2, 0)$; see Section 4.

Theorem 1.2. *The set of (θ_1, θ_2) for which the hyperbolic triangle with angles $(\theta_1, \theta_2, 0)$ admits a nonconstant Neumann Laplace eigenfunction has Lebesgue measure zero and is contained in a countable union of nowhere dense sets.*

[1]In the case where $n \geqslant 3$ is an integer not equal to $3, 4$, or 6, Phillips and Sarnak asked whether the domain $\mathcal{T}_{\cos(\pi/n)}$ has no nonconstant Neumann eigenfunctions [PS92a]. We should point out that since Theorem 1.1 allows for countably many exceptional t, it does not directly answer their question.

In other words, the generic hyperbolic triangle with one cusp has no nonconstant Neumann eigenvalue where "generic" can be taken in both a topological and a measurable sense. Theorem 1.1 gives the existence of a triple of angles $(\theta_1, \theta_2, 0)$ for which there are no nonconstant Neumann eigenfunctions. Theorem 1.2 then results from applying a general and well-understood principle concerning analytic perturbations (See, for example, [HJ09]). On the other hand, the proof of Theorem 1.1 is much more involved. In particular, the proof will rely upon a refined analysis of "crossings" of eigenvalue branches.

To prove Theorem 1.1, we further develop the method of asymptotic separation of variables that we introduced in [HlrJdg11] to study generic simplicity of eigenvalues. This method facilitates the study of real-analytic eigenvalue branches in situations where a geometric domain degenerates onto a lower dimensional domain. There is a vast literature — for instance, [BF10], [GJ96], [FS09] — concerning perturbations involving degeneration onto lower dimensional domains, but most of these studies do not address analytic eigenvalue branches. In contrast, our results depend crucially on a study of real-analytic eigenbranches and their crossings.

1.1. *An outline of this paper.* We now describe the content of each section. In Section 2, we establish notation and recall basic features of the spectral theory of the Laplacian acting on functions on a domain in the hyperbolic plane having one cusp. We describe the Fourier decomposition associated to the cusp. The zeroth Fourier mode is responsible for an essential spectrum of $[\frac{1}{4}, \infty[$. Following [LP76] and [Col82], we will replace the Dirichlet quadratic form $\mathcal{E}(u) = \langle \Delta u, u \rangle$ with a modification $\mathcal{E}_\beta$ obtained by "truncating" the zeroth Fourier coefficient above $y = \beta$. An eigenfunction u of $\mathcal{E}_\beta$ corresponds to an eigenfunction of $\mathcal{E}$ if and only if the zeroth Fourier coefficient of u vanishes identically. We will call such an eigenfunction a *cusp form.*[2] The operator associated to $\mathcal{E}_\beta$ has compact resolvent and hence a discrete spectrum. This makes $\mathcal{E}_\beta$ a much better candidate for the application of methods from spectral perturbation theory.

In Section 3, we recall and make precise some ideas familiar in the perturbational study of cusp form existence. We consider a real-analytic family, $t \mapsto q_t$, of quadratic forms that have the same domain as $\mathcal{E}_\beta$. We say that a real-analytic family $t \mapsto u_t$ of eigenfunctions of q_t is a *cusp form eigenbranch* if and only u_t is a cusp form for each t. We demonstrate a dichotomy: Either

[2]For $t = \cos(\pi/n)$, these are "cusp forms" in the sense of the theory of automorphic forms, but otherwise there is no discrete group, and hence they are not cusp forms in the traditional sense. In this paper we will always be considering "even" cusp forms, that is, eigenfunctions satisfying Neumann conditions.

the family $t \mapsto q_t$ has a real-analytic cusp form eigenbranch or the set of t such that q_t has a cusp form is countable.

In Section 4, we consider arbitrary real-analytic paths in the space of hyperbolic triangles with one cusp. We apply the results of Section 3 to deduce Theorem 1.2 under the assumption that there exists a triangle with no nonconstant Neumann eigenfunction. The remainder of the paper is devoted to proving Theorem 1.1, which will give the existence of such a triangle.

In Section 5 we specialize to the family $\mathcal{T}_t$. After renormalizing by a factor of t^2, we find that for each u, the function $t \mapsto q_t(u)$ has a Taylor expansion at $t = 0$. We compute the leading order terms of this expansion.

In Section 6 we show that the method of the asymptotic separation of variables introduced in [HJ11] may be used to analyze the family of quadratic forms $q_{\beta,t}$. In particular, we define a reference quadratic form a_t to which separation of variables applies and that is asymptotic to $q_{\beta,t}$ at "first order." By separation of variables we mean that each eigenfunction of a_t is of the form $v_t^\ell(y) \cdot \cos(\pi \ell x)$ with $\ell \in \mathbb{Z}$ and v_t^ℓ a solution to

$$-t^2 \cdot u'' + \left((k\pi)^2 - \frac{\lambda}{y^2}\right) \cdot u = 0, \tag{1}$$

a renormalized form of the equation for a modified Bessel function with imaginary parameter. The potential $(k\pi)^2 - \lambda \cdot y^{-2}$ is positive for y large and has a unique zero at $y = \sqrt{\lambda}/(k\pi)$. To analyze the solutions to (1), we will relate them to the Airy functions, solutions to the ordinary differential equation $\partial_x^2 A = x \cdot A$. The remainder of the paper depends crucially on the analysis of (1) using Airy functions, which has been placed in the appendices.

In Section 7 we prove a nonconcentration estimate — Proposition 7.2 — and use this estimate to derive information concerning the real-analytic eigenbranches (E_t, u_t) of $q_{\beta,t}$. First, we show that there exists an integer k so that E_t limits to $(\pi k)^2$ as t tends to zero. Second, we find that if the spectral projection of u_t onto the space V_k spanned by functions of the form the $\psi(y) \cdot \cos(\pi k x)$ is relatively small, then the derivative $\partial_t E_t$ is of order $1/t$. Finally, we show that if (E_t, u_t) is a cusp form eigenbranch, then E_t cannot limit to zero.

In Section 8 we prove Theorem 1.1. By the dichotomy of Section 3, it suffices to show that real-analytic cusp form eigenbranches do not exist. We suppose to the contrary that the real-analytic family $q_{\beta,t}$ has a cusp form eigenbranch (E_t, u_t). By the results of Section 7, we have that E_t limits to $(\pi k)^2$ where $k \in \mathbb{Z}^+$. By improving the analysis of [HJ11], we show that there exists an eigenbranch, λ_t^*, of a_t that "tracks" E_t at order t in the sense that

$$\limsup_{t \to 0} \frac{1}{t} \cdot |E_t - \lambda_t^*| < \infty. \tag{2}$$

We will obtain a contradiction to this estimate by estimating $f(t) := \frac{d}{dt}\,(E_t - \lambda_t^*)$ from below.

Indeed, we show that when the norm of the projection w_t^k of u_t onto V_k is relatively large with respect to $\|u_t\|$, the function $f(t)$ is of order $t^{-\frac{1}{3}}$, whereas when $\|w_t^k\|$ is relatively small, the function $f(t)$ is of order t^{-1}. By controlling the sizes of the sets where $\|w_t^k\|$ is respectively small and large relative to $\|u_t\|$ and by integrating, we will contradict (2).

The key observation is the following: Since E_t limits to $(k\pi)^2$, it has to "cross" each of the eigenbranches of a_t that limit to zero. We show that near such a crossing, the branch u_t must "interact" with the functions in V_0 to such an extent that the projection onto V_k cannot be too large. The effect of each interaction is made precise by careful estimates of the off-diagonal terms in the quadratic form $q_t - a_t$ (Appendix A). By summing the effects of these interactions, we eventually prove that there exists $c > 0$ so that

$$E_t - \lambda_t^* \geqslant c \cdot t^{\frac{2}{3}},$$

thus contradicting (2).

2. The spectrum of a domain in the hyperbolic plane with a cusp

In this section, we describe some basic spectral theory of the Neumann Laplace operator acting on the square-integrable functions on domains in the hyperbolic plane with a cusp. We define the Laplacian and associated Dirichlet quadratic form, describe the Fourier decomposition of eigenfunctions along horocycles, and construct a modification of the Dirichlet form whose eigenfunctions include the eigenfunctions (cusp forms) of the standard Laplacian but whose spectrum is discrete.

2.1. *The quadratic forms associated to the Neumann Laplacian.* The half plane $\{(x, y),\ y > 0\}$ equipped with the Riemannian metric $y^{-2}(dx^2 + dy^2)$ is the Poincaré-Lobachevsky model for the 2-dimensional hyperbolic space $\mathbb{H}^2$. The measure associated to the Riemannian metric $g = y^{-2}(dx^2 + dy^2)$ is given by integrating

$$dm = \frac{dxdy}{y^2}.$$

In the present context, a *cusp of width w and height y_0* is the subset $S_{w,y_0} := [0, w] \times [y_0, \infty[$ of the upper half plane. A domain $\Omega \subset \mathbb{H}^2$ is said to *have one cusp* if Ω is the union of a cusp and a compact set. We will assume that the boundary of Ω is the union of finitely many geodesic arcs and that the interior of Ω is connected.

Let $\mathcal{D}(\overline{\Omega})$ denote the set of functions $u : \Omega \to \mathbb{C}$ such that u is the restriction to Ω of a compactly supported smooth function defined on some neighbourhood of Ω.

The L^2-inner product of two functions u and v in $\mathcal{D}(\overline{\Omega})$ is defined by

$$\mathcal{N}(u,v) := \int_\Omega u(x,y) \cdot \overline{v(x,y)}\ dm. \tag{3}$$

Abusing notation slightly, we will often write $\mathcal{N}(u)$ in place of $\mathcal{N}(u,u)$. Let $L^2(\Omega, dm)$ denote the completion of $\mathcal{D}(\overline{\Omega})$ with respect to the norm $\mathcal{N}(u)^{\frac{1}{2}}$.

To define the Neumann Laplacian we consider the bilinear form defined on $\mathcal{D}(\overline{\Omega})$ by

$$\mathcal{E}(u,v) := \int_\Omega g(\nabla u, \nabla v)\ dm,$$

where ∇ satisfies $g(\nabla f, X) = Xf$ for all vector fields X and smooth functions f. Let $\mathcal{E}(u)$ denote the value of the quadratic form $u \mapsto \mathcal{E}(u,u)$. One computes that

$$\mathcal{E}(u) = \int_\Omega |\partial_x u(x,y)|^2 + |\partial_y u(x,y)|^2\ dxdy.$$

Let $H^1(\Omega)$ denote the completion of $\mathcal{D}(\overline{\Omega})$ with respect to the norm $(\mathcal{E}(u) + \mathcal{N}(u))^{\frac{1}{2}}$. We will consider $\mathcal{E}$ as a nonnegative symmetric bilinear form on $L^2(\Omega)$ with domain $H^1(\Omega)$. As such it is densely defined and closed, and hence there exists a unique, densely defined, self-adjoint operator Δ on $L^2(\Omega)$ such that for each $v \in H^1(\Omega)$ and u in the domain of Δ, we have[3]

$$\mathcal{N}(\Delta u, v) = \mathcal{E}(u,v). \tag{4}$$

The operator Δ is called the *Neumann Laplacian*. It can be shown that $u \in H^1(\Omega)$ if and only if $u \in L^2(\Omega, dm)$ and $\mathcal{E}(u) < +\infty$ where in the definition of $\mathcal{E}$ the partial derivatives are to be taken in the distributional sense.

It is well known that Δ has an essential spectrum equal to $[1/4, \infty[$. (For example, see [LP76] or [Col82]). Apart from this essential spectrum, Δ may also have eigenvalues either smaller than $1/4$ or embedded in the continuous spectrum.

From (4) we see that u is an eigenfunction of Δ with eigenvalue E if and only if $u \in H^1(\Omega)$ and for each $v \in H^1(\Omega)$,

$$\mathcal{E}(u,v) = E \cdot \mathcal{N}(u,v). \tag{5}$$

2.2. *Fourier decomposition in the cusp.* For each positive integer k, define

$$e_k(x) := 2^{\frac{1}{2}} \cdot \cos(k\pi \cdot x)$$

and define $e_0 \equiv 1$. The collection $\{e_k \mid k \geqslant 0\}$ is an orthonormal basis for $L^2([0,1])$. Hence, the functions

$$x \mapsto \frac{1}{\sqrt{w}} \cdot e_k\left(\frac{x}{w}\right)$$

[3] See, for example, [Kat95, Th. VI.2.1].

provide an orthonormal basis of $L^2([0,w])$.

For positive $w,\ y_0$, let $S_{w,y_0} = [0,w] \times [y_0, \infty[$ be a cusp of width w and height y_0.

For each u in $L^2(S_{w,y_0}, dm)$ and almost every $y \geqslant y_0$, the function $x \mapsto u(x,y)$ belongs to $L^2([0,w])$. Thus we can write

$$u(x,y) = \sum_{k \geqslant 0} u^k(y) \cdot e_k\left(\frac{x}{w}\right), \tag{6}$$

where

$$u^k(y) := \frac{1}{w} \int_0^w u(x,y) \cdot e_k\left(\frac{x}{w}\right)\, dx \tag{7}$$

belongs to $L^2\left([y_0, \infty[, y^{-2}dy\right)$. We refer to u^k as the k^{th} Fourier coefficient of u. More generally, if Ω is a domain with a cusp S_{w,y_0}, then we define the k^{th} Fourier coefficient of a function $v : \Omega \to \mathbb{C}$ to be the k^{th} Fourier coefficient restriction of v to S_{w,y_0}. Parseval's theorem gives

$$\mathcal{N}\left(u \cdot \mathbb{1}_{[y_0,\infty[}\right) = \sum_{k \geqslant 0} \int_{y_0}^{\infty} \left|u^k(y)\right|^2 \frac{dy}{y^2},$$

where $\mathbb{1}_X$ denotes the characteristic function of a set X.

Lemma 2.1. *If $u \in H^1(\Omega, dm)$ is a Neumann eigenfunction of $\mathcal{E}$ with eigenvalue E, then for each $k \in \mathbb{N}$ and each $y > y_0$, the Fourier coefficient u^k satisfies*

$$-\left(u^k\right)'' + \left(\frac{(k\pi)^2}{w^2} - \frac{E}{y^2}\right) u^k = 0. \tag{8}$$

Proof. If v is a smooth function on Ω, then since u is a Neumann eigenfunction of $\mathcal{E}$, integration by parts gives

$$-\int_\Omega \left(u \cdot \partial_x^2 v + u \cdot \partial_y^2 v\right)\, dxdy = E \int_\Omega u \cdot v\, \frac{dxdy}{y^2}.$$

By letting $v = \phi(y) \cdot e_k(x/w)$, where ϕ is a smooth function with compact support in $]y_0, \infty[$, we find that

$$\frac{(k\pi)^2}{w^2} \int_{y_0}^{\infty} u^k(y) \cdot \phi(y)\, dy - \int_{y_0}^{\infty} u^k(y) \cdot \phi''(y)\, dy = E \int_{y_0}^{\infty} u^k(y) \cdot \frac{dy}{y^2}.$$

It follows that u^k satisfies (8) in the distributional sense in $\mathcal{D}'((y_0,\infty))$. By elliptic regularity, u^k is actually smooth and satisfies (8) in the strong sense. $\square$

In particular, $u^0 = Ay^s + By^{1-s}$ for some constants A and B with $E = s(1-s)$. If $E \geqslant 1/4$, then the real part of s equals $1/2$, and hence u^0 does not belong to $L^2\left([y_0,\infty[, y^{-2}dy\right)$ unless both A and B equal zero. Therefore, we have the following.

COROLLARY 2.2. *If u is a Neumann eigenfunction with eigenvalue $E \geqslant \frac{1}{4}$, then the zeroth Fourier coefficient u^0 vanishes identically on $[y_0, \infty[$.*

In the classical spectral theory of a quotient of $\mathbb{H}$ by a lattice in $\mathrm{SL}_2(\mathbb{R})$, a Laplace eigenfunction u with vanishing zeroth Fourier coefficient in each cusp is called a (*weight zero*) *Maass cusp form.* Even though most of the domains that we will consider are not fundamental domains for discrete groups of isometries, we will adapt this terminology.

Definition 2.3. If u is an eigenfunction for the Neumann Laplacian on a domain with a cusp, and $\left(u \cdot \mathbb{1}_{[y_0,\infty[}\right)^0 \equiv 0$, then we will call u a *cusp form.*

Traditionally, the Neumann eigenfunctions for $\mathcal{T}_{\cos(\pi/n)}$ are called *even cusp forms* whereas the solutions to the Dirichlet eigenvalue problem are called *odd cusp forms.* We will not consider odd cusp forms in this paper.

2.3. *A related quadratic form.* We wish to apply analytic perturbation theory to study the behavior of eigenfunctions of $\mathcal{E}$ on $\mathcal{T}_t$ as we vary t. Because the eigenvalues of $\mathcal{E}$ might lie inside the essential spectrum, standard perturbation theory does not apply directly. Following [Col82] and [PS85], we will use a modification of $\mathcal{E}$ first constructed by P. Lax and R. Phillips [LP76] [LP80].[4] In this section, we recall the construction, show that the eigenvalues of the modification are isolated, and relate the eigenfunctions of the modification to those of $\mathcal{E}$.

For $\beta > y_0$, let Z_β denote the set of $u \in \mathcal{D}(\overline{\Omega})$ such that for each $y \geqslant \beta$, we have $u^0(y) = 0$. Let $L^2_\beta(\Omega, dm)$ denote the Hilbert space completion of Z_β with respect to $u \mapsto \mathcal{N}(u)^{\frac{1}{2}}$. Let $\mathcal{N}_\beta$ denote the restriction of $\mathcal{N}$ to $L^2_\beta(\Omega)$.

Let $H^1_\beta(\Omega)$ denote the Hilbert space completion of Z_β with respect to the norm $u \mapsto (\mathcal{E}(u) + \mathcal{N}(u))^{\frac{1}{2}}$. The restriction, $\mathcal{E}_\beta$, of $\mathcal{E}$ to $H^1_\beta(\Omega)$ is a closed, densely defined quadratic form on $L^2_\beta(\Omega)$. A simple argument shows that

$$L^2_\beta(\Omega) = \left\{u \in L^2(\Omega, dm),\ \mid \forall y > \beta,\ u^0(y) = 0\right\}$$

and

$$H^1_\beta(\Omega) = \left\{u \in H^1(\Omega),\ \mid \forall y > \beta,\ u^0(y) = 0\right\}.$$

In the sequel it will be more convenient to replace Z_β by the following other set.

Definition 2.4. Define W_β to be the set of functions u in H^1_β such that

- u extends to a continuous function on the closure $\overline{\Omega}$ of Ω,
- u is smooth on $\Omega \setminus \{y = \beta\}$.

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[4] See page 206 of [LP76] under the heading "A related quadratic form."

Observe that since $Z_\beta \subset W_\beta$, the closure of W_β with respect to the norm $u \mapsto (\mathcal{E}(u) + \mathcal{N}(u))^{\frac{1}{2}}$ is H^1_β. The latter assertion says that W_β is a core of the quadratic form $\mathcal{E}_\beta$

Let Δ_β denote the unique operator such that $\mathrm{dom}(\Delta_\beta) \subset H^1_\beta$ and that satisfies $\mathcal{N}_\beta(\Delta_\beta u, v) = \mathcal{E}_\beta(u, v)$ for each $u \in \mathrm{dom}(\Delta_\beta),\ v \in H^1_\beta(\Omega)$.

Lemma 2.5 ([LP76]). *For each $\beta > y_0$, the resolvent of Δ_β is compact. Hence, the spectrum of $\mathcal{E}_\beta$ with respect to $\mathcal{N}_\beta$ is discrete and each eigenspace is finite dimensional.*

Proof. Using the Fourier decomposition, one shows that for each $b > 0$, the set of $v \in H^1_\beta(\Omega)$ such that $\mathcal{N}(v) \leqslant 1$ and $\mathcal{E}(v) \leqslant b$ is compact in $L^2_\beta(\Omega)$; see [LP76, Lemma 8.7]. It follows that Δ_β has compact resolvent. Hence, by standard spectral theory, the spectrum is discrete and the eigenspaces are finite dimensional. □

Definition 2.6 (cusp form). We will say that an eigenfunction u of $\mathcal{E}_\beta$ with respect to $\mathcal{N}_\beta$ is a *cusp form* if and only for each $y > y_0$, we have $u^0(y) = 0$.

Lemma 2.7. *The following assertions are equivalent*:

(1) *u is a cusp form of $\mathcal{E}$ with respect to $\mathcal{N}$;*

(2) *there exists $\beta > y_0$ such that u is a cusp form of $\mathcal{E}_\beta$ with respect to $\mathcal{N}_\beta$;*

(3) *for each $\beta > y_0$, the function u is a cusp form for $\mathcal{E}_\beta$ with respect to $\mathcal{N}_\beta$.*

Proof. (1) ⇒ (2): If u is an eigenfunction of $\mathcal{E}$ with eigenvalue E, then by Lemma 2.1 the zeroth Fourier coefficient u^0 satisfies the differential equation $0 = (u^0)'' + (E/y^2) \cdot u^0$. Since $u^0(y)$ vanishes for $y > y_0$, it must vanish for $y > \beta$.

(2) ⇒ (1): Fix a smooth function χ such that $\chi(y) = 0$ for $y \leqslant \frac{2y_0+\beta}{3}$ and $\chi(y) = 1$ for $y \geqslant \frac{y_0+2\beta}{3}$. If $u^0(y) = 0$ for each $y > y_0$, then

$$\mathcal{N}_\beta(u, v - \chi \cdot v^0) = \mathcal{N}(u, v)$$

and

$$\mathcal{E}(u, v) = \mathcal{E}(u, v - \chi \cdot v^0) = \mathcal{E}_\beta(u, v - \chi \cdot v^0)$$

for each $v \in H^1(\Omega)$. Thus, if $\mathcal{E}_\beta(u, v) = E \cdot \mathcal{N}_\beta(u, v)$, then $\mathcal{E}(u, v) = E \cdot \mathcal{N}(u, v)$.

(2) ⇔ (3): Follows from the equivalence of (1) and (2). □

Not every eigenfunction u of $\mathcal{E}_\beta$ is an eigenfunction of $\mathcal{E}$. For example, if $w = \cos(\pi/n)$ and E_s is an Eisenstein series whose zeroth Fourier coefficient vanishes at $y = \beta$, then $E_s(x, y) - E^0_s(y) \cdot \chi_{[\beta,\infty[}(y)$ is an eigenfunction of $\mathcal{E}_\beta$. This function is not smooth across $y = \beta$, but elliptic regularity implies that each eigenfunction of $\mathcal{E}$ is smooth.

3. Real-analyticity and generic properties of eigenfunctions

Let $S = [0,1] \times [y_0, \infty)$, and let $\beta > \underline{\alpha} > y_0$. In this section, we consider a fixed domain Ω that contains the cusp S and a real-analytic family $t \mapsto q_t$ of quadratic forms defined on $H^1_\beta(\Omega) \subset L^2_\beta(S, dm)$ that represents the cusp of width w_t for $y > y_0$ and some real-analytic function $t \mapsto w_t$ (see Definition 3.1 below). We prove the following dichotomy: Either there exists a real-analytic eigenfunction branch consisting of "cusp forms" or the set of t such that q_t has a "cusp form" eigenfunction is countable. This fact is fundamental to the proofs of both Theorems 1.1 and 1.2.

Let $S = [0,1] \times [y_0, \infty[$, and let $\beta > \underline{\alpha} > y_0$. For any $w > 0$, we can define a transformation $\widehat{\Phi}_w$ between $L^2_\beta(S)$ and $L^2_\beta(S_{w,y_0})$ by asking that, for any $u \in L^2_\beta(S)$, the function $v := \widehat{\Phi}_w(u)$ is defined by $v(x,y) = \frac{1}{\sqrt{w}} u(\frac{x}{w}, y)$. Since, in the sequel y_0 will be fixed, we will drop the index y_0 and denote by S_w the cusp of width w.

It is straightforward that $\widehat{\Phi}_w$ is an isometry between $L^2_\beta(S)$ and $L^2_\beta(S_w)$. Moreover, $\widehat{\Phi}_w$ also preserves H^1_β in the sense that $\widehat{\Phi}_w(u) \in H^1_\beta(S_w)$ if and only if $u \in H^1_\beta(S)$.

We may thus define $\mathcal{E}_{\beta,w}$ the quadratic form obtained by pulling back $\mathcal{E}_\beta$ on S_w using $\widehat{\Phi}_w$. This quadratic form is then closed on the domain $H^1_\beta(S)$, and for each $u \in H^1_\beta(S)$,

$$\mathcal{E}_{\beta,w}(u) = \int_S w^{-2} \cdot |\partial_x u(x,y)|^2 + |\partial_y u(x,y)|^2 \, dxdy.$$

Definition 3.1. Let Ω be a domain that has S as a cusp and $\beta > \underline{\alpha} > y_0$. Let q be a quadratic form closed over the domain $H^1_\beta(\Omega)$. We will say that q represents the cusp of width w for $y \geqslant \underline{\alpha}$ if, for any $u \in H^1_\beta$ that is supported in $\{y > \underline{\alpha}\}$, we have

$$q(u) = \mathcal{E}_{\beta,w}(u).$$

For such a quadratic form, we will say that an eigenfunction u is a cuspform if $u^0(y)$ vanishes on $\{y_0 \leqslant y \leqslant \beta\}$.

The aim of this section is to prove that being a cuspform is a real-analytic condition. To make this statement precise we have to consider a family q_t of quadratic forms that satisfies the following assumptions.

ASSUMPTION 3.2. *Let $t_- < t_+$ and $\beta > \underline{\alpha} > y_0$. For each $t \in I :=]t_-, t_+[$, let w_t be a positive real-analytic function on I. Let q_t denote a nonnegative, closed quadratic form with domain $H^1_\beta(S)$ that represents the cusp of width w_t for $y \geqslant \underline{\alpha}$.*

Lastly, we assume that the family $t \mapsto q_t$ is real-analytic of type (a) *in the sense of* [Kat95]. *That is, for each $u \in H^1_\beta(S)$, the map $t \mapsto q_t(u)$ is real-analytic.*

A straightforward application of analytic perturbation theory — [Kat95, §VII] — gives the following.

THEOREM 3.3 (Existence of a real-analytic eigenbasis). *Let $t \mapsto q_t$ satisfy the assumptions above* 3.2. *Then there exist a collection of real-analytic paths $\{t \mapsto u_{j,t} \in L^2(\Omega, dm) \mid j \in \mathbb{Z}^+\}$ and a collection of real-analytic functions $\{t \mapsto \lambda_{j,t} \in \mathbb{R} \mid j \in \mathbb{Z}^+\}$ so that for each t, the set $\{u_{j,t} \mid j \in \mathbb{Z}\}$ is an orthonormal basis for $L^2_\beta(\Omega, dm)$, and for each (j,t), the function $u_{j,t}$ is an eigenfunction of q_t with eigenvalue $\lambda_{j,t}$.*

Proof. Since the embedding from H^1_β into L^2_β is compact, for any t, the spectrum of q_t consists only in eigenvalues. The proof is then similar to the proof of Theorem 3.9 in [Kat95, §VIII.3.5]. □

For $u \in W_\beta$, define

$$L(u) = \lim_{y \to \beta^-} \frac{u^0(y)}{y - \beta}.$$

LEMMA 3.4. *An eigenfunction u of q_t is a cusp form if and only if $L(u) = 0$.*

Proof. $L(u)$ is the left-sided derivative of u^0 at β. Since u is an eigenfunction, $u \in W_\beta$, and u^0 is a solution to a second order ordinary differential equation on $[y_0, \beta]$ with $u_0(\beta) = 0$. Thus, u^0 vanishes identically on $[y_0, \beta]$ if and only if $L(u) = 0$. □

For real-analytic eigenbranches we have the following.

LEMMA 3.5. *Let (u_t, λ_t) be an analytic eigenbranch of q_t. Then the mapping $t \mapsto L(u_t)$ is analytic on $]t_-, t_+[$.*

Proof. The zeroth mode u^0_t of u_t is a solution to the ODE

$$-u'' - \frac{\lambda_t}{y^2} \cdot u = 0$$

on $[\underline{\alpha}, \beta]$ with Dirichlet boundary condition at β. Denote by G_λ the unique solution to this ordinary differential equation that satisfies $G_\lambda(\beta) = 0$, $G'_\lambda(\beta) = 1$. Since the coefficients of the ordinary differential equation depend analytically on the parameter λ, the mapping $\lambda \mapsto G_\lambda$ is analytic (for instance, with values in $\mathcal{C}^2([\underline{\alpha}, \beta])$). Moreover, for each compact set $K \subset]t_-, t_+[$, we can find $\underline{\alpha} < \alpha_K < \beta$ such that, for each $t \in K$, $\int_{\alpha_K}^\beta G_\lambda > 0$. Since u_t is a multiple of G_{λ_t}, we then have

$$\begin{aligned} L(u_t) &= \left(\int_{\alpha_K}^\beta G_{\lambda_t}(y)\, dy \right)^{-1} \cdot \int_{\alpha_K}^\beta u^0_t(y)\, dy \\ &= \left(\int_{\alpha_K}^\beta G_{\lambda_t}(y)\, dy \right)^{-1} \cdot \int_0^1 \int_{\alpha_K}^\beta u_t(x,y)\, dx dy. \end{aligned}$$

Analyticity on K then follows from the analyticity of $t \mapsto u_t$ and $t \mapsto G_{\lambda_t}$ and the choice of α_K. $\square$

We will say that real-analytic eigenfunction branch u_t of q_t is *a cusp form eigenbranch* if and only if for each $t \in I$, the eigenfunction u_t is a cusp form; see Definition 3.1. Using the real-analyticity proved in Lemma 3.4 we obtain the following.

COROLLARY 3.6. *If u_t is a real-analytic eigenfunction branch that is not a cusp form eigenbranch, then the set of $t \in I$ such that u_t is a cusp form is discrete.*

We now proceed to prove that if q_t has no real-analytic cusp form eigenbranch, then for a generic t, the form q_t has no cusp form. As it turns out, we will actually first prove that the spectrum of q_t is generically simple.

Let I_{mult} denote the set of $t \in I$ such that q_t has an eigenspace of dimension at least two.

PROPOSITION 3.7. *If q_t does not have a real-analytic cusp form eigenbranch, then I_{mult} is countable.*

Proof. Let $\{u_{j,t} \mid j \in \mathbb{Z}^+,\ t \in I\}$ and $\{\lambda_{j,t} \mid j \in \mathbb{Z}^+,\ t \in I\}$ be as in Theorem 3.3. For each $j, k \in \mathbb{Z}^+$, let $Z_{j,k} = \{t \mid \lambda_{j,t} = \lambda_{k,t}\}$. Since each eigenspace of q_t is spanned by a finite collection of $\{u_{j,t}\}$, the union $\bigcup_{j,k} Z_{j,k}$ equals I_{mult}.

The function $t \mapsto \lambda_{j,t} - \lambda_{k,t}$ is analytic, and hence $Z_{j,k} = \{t \mid \lambda_{j,t} = \lambda_{k,t}\}$ is either countable or equals I. Thus to prove the claim, it suffices to show that it is not possible for $Z_{j,k}$ to equal I.

Suppose that there exists j and k so that $u_{j,t}$ and $u_{k,t}$ are real-analytic eigenbranches so that $\lambda_{j,t} = \lambda_{k,t}$ for each $t \in I$. To prove the proposition, it suffices to produce a linear combination u_t of $u_{j,t}$ and $u_{k,t}$ so that for each $t \in I$, the function u_t is a real-analytic cusp form eigenbranch.

By hypothesis, neither $u_{j,t}$ nor $u_{k,t}$ are cusp form eigenbranches. By Corollary 3.6, the set J of t such that either $u_{j,t}$ or $u_{k,t}$ is a cusp form is discrete.

For each $t \notin J$, define

$$u_t = \frac{L(u_{k,t}) \cdot u_{j,t} - L(u_{j,t}) \cdot u_{k,t}}{\sqrt{L(u_{j,t})^2 + L(u_{k,t})^2}}.$$

It suffices to show that $t \to u_t$ extends to a real-analytic function on I. Indeed, since L is linear, we have $L(u_t) = 0$ for each $t \notin J$. By Corollary 3.5, the real-analytic extension would satisfy $L(u_t) \equiv 0$.

The order of vanishing of $t \mapsto L(u_{j,t})$ (resp. $t \mapsto L(u_{k,t})$) is finite at each $t \in J$. If the order of vanishing of $L(u_{k,t})$ at $t_0 \in J$ is at least the order of vanishing of $L(u_{j,t})$ at t_0, then the ratio $L(u_{k,t})/L(u_{j,t})$ has a real-analytic extension near t_0. Hence, factorizing $L(u_{j,t})$ we obtain that

$$\frac{L(u_{k,t})}{\sqrt{L(u_{j,t})^2 + L(u_{k,t})^2}} = \frac{L(u_{k,t})}{L(u_{j,t})} \cdot \left(1 + \left(\frac{L(u_{k,t})}{L(u_{j,t})}\right)^2\right)^{-\frac{1}{2}}$$

and

$$\frac{L(u_{j,t})}{\sqrt{L(u_{j,t})^2 + L(u_{k,t})^2}} = \left(1 + \left(\frac{L(u_{k,t})}{L(u_{j,t})}\right)^2\right)^{-\frac{1}{2}}$$

have real-analytic extensions near t_0. If the order of vanishing of $L(u_{k,t})$ at t_0 is at most the order of vanishing of $L(u_{j,t})$ at t_0, then a similar argument applies by factorizing $L(u_{k,t})$ everywhere. Thus, u_t extends to a real-analytic cusp form eigenbranch. $\square$

Let $I_{\rm cf}$ denote the set of $t \in I$ such that q_t has at least one cusp form eigenfunction.

PROPOSITION 3.8. *If q_t has no real-analytic cusp form eigenbranch, then $I_{\rm cf}$ is countable.*

Remark 3.9 (Dichotomy). If there exists a cusp form eigenbranch, then $I_{\rm cf} = I$. Therefore, we have the following dichotomy: Either the set $I_{\rm cf}$ is countable or the family $t \mapsto q_t$ has a real-analytic cusp form eigenbranch.

Proof of Proposition 3.8. Let $\{u_{j,t} |\ j \in \mathbb{Z},\ t \in I\}$ be as in Theorem 3.3. By Corollary 3.6, the zero set $Z_j = \{t \mid L(u_{j,t} = 0\,)\}$ is countable.

If each eigenspace E of q_t is 1-dimensional, then there exists a unique j such that E equals the span of $u_{j,t}$. Thus, if t does not belong to $I_{\rm mult}$ or to any Z_j, then t does not belong to $I_{\rm cf}$. In other words, $I_{\rm cf} \subset (\bigcup Z_j) \cup I_{\rm mult}$. By Proposition 3.7, the set $I_{\rm mult}$ is countable, and hence so is $I_{\rm cf}$. $\square$

4. **Perturbation theory for hyperbolic triangles with one cusp**

In this section we use the results of the previous section to explain how Theorem 1.2 can be deduced from the existence of a triangle with a cusp that has no nonconstant Neumann eigenfunctions. This fact might be considered to be folklore as it follows from the general philosophy of using analyticity to prove generic spectral results; see [HJ09]. The main task here is to construct a real-analytic family of quadratic forms that is associated with each real-analytic path in the moduli space of triangles.

4.1. *The moduli space of triangles.* First, we discuss the parametrization of the set of triangles with one cusp. The statement of Theorem 1.2 makes use of the fact that hyperbolic triangles with one cusp are parametrized by the two nonzero vertex angles. But in order to prove Theorem 1.2, it will be more convenient to use an alternate set of parameters.

For each geodesic triangle T in the hyperbolic upper half plane $\mathbb{H}^2$ having (exactly) one cusp, there exist a unique $c \in \,]0,1[$ and $w \in [2c, 1+c[$ so that T is isometric to the domain

$$\mathcal{T}_{c,w} = \left\{(x,y) \mid 0 \leqslant x \leqslant w, \quad (x-c)^2 + y^2 > 1\right\}; \tag{9}$$

see Figure 2.

In this way, the set of hyperbolic triangles may be identified with the Euclidean triangle

$$\mathcal{M} = \{(c,w) \mid 0 \leqslant c < 1,\ 2c < w < c+1\}.$$

We say that a subset of the set of triangles has measure zero if and only if the corresponding subset of $\mathcal{M}$ has measure zero. Similarly, a subset of the set of triangles is said to be a real-analytic curve if and only if the corresponding subset of $\mathcal{M}$ is a real-analytic curve.

These notions are equivalent to those used in the statement of Theorem 1.2 because the relationship between the angles (θ_1, θ_2) and the parameters (c,w) is real-analytic. Indeed, we have $c = \cos(\theta_1)$ and $\cos(\theta_2) = w - c$; see Figure 2.

To prove Theorem 1.2, we will apply perturbation theory. The following fact makes this approach feasible.

Proposition 4.1. *Each nonconstant Neumann eigenfunction on $\mathcal{T}_{c,w}$ is a cusp form and hence an eigenfunction of the modified quadratic form $\mathcal{E}_\beta$.*

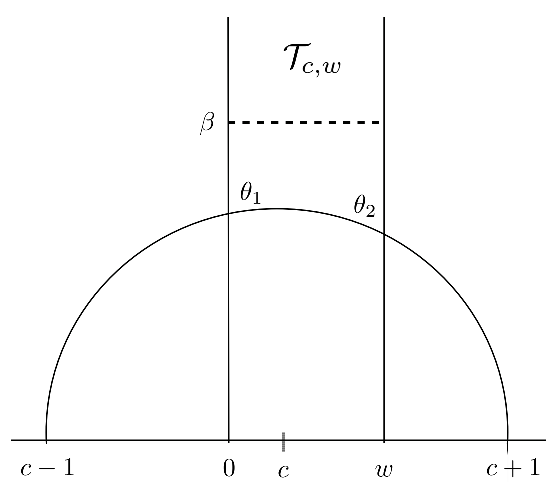

Figure 2. The triangle $\mathcal{T}_{c,w}$ in the upper half plane.

Proof. The eigenvalue of a nonconstant Neumann eigenfunction on T is at least $1/4$ [Jud07].[5] Thus, the claim follows from Corollary 2.2 and Lemma 2.5. □

Let $\mathcal{M}_{\mathrm{cf}}$ denote the set of $(c, w) \in \mathcal{M}$ such that there exists $\beta > 1$ so that the modified quadratic from $\mathcal{E}_\beta$ has a cusp form. To prove Theorem 1.2 it will suffice to show that $\mathcal{M}_{\mathrm{cf}}$ has measure zero and is a countable collection of nowhere dense sets.

4.2. *A family of diffeomorphisms.* To show that $\mathcal{M}_{\mathrm{cf}}$ is nongeneric, we will use analytic perturbation theory and Proposition 3.8. In order to use analytic perturbation theory we will have to normalize the Hilbert space and the domains of the quadratic forms. To accomplish this, we let $S = [0,1] \times [1, \infty[$, and for each (c, w), we define a C^1 diffeomorphism $\varphi_{c,w} : \mathcal{T}_{c,w} \to S$ such that[6]

(1) the restriction of $\varphi_{c,w}$ is the identity for $y > \underline{\alpha} = (\beta+1)/2$;
(2) for each path $\gamma : I \to \mathcal{M}$, the family $t \mapsto \varphi_{c,w}$ is a real-analytic path.

To construct $\varphi_{c,w}$, we use the fact that the map $(x, y) \mapsto x$ defines a fibration of $\mathcal{T}_{c,w}$ over $[0, w]$ and a fibration of S over $[0, 1]$. We define $\varphi_{c,w}$ by sending the fiber over $\{x\}$ onto the fiber over $\{x/w\}$.

LEMMA 4.2. *For each $\alpha \in \,]0, \underline{\alpha}[$, there exists a unique cubic polynomial B_α so that*

- $B_\alpha(\alpha) = 1$,
- $B'_\alpha(0) = \alpha$.
- $B_\alpha(\underline{\alpha}) = \underline{\alpha}$,
- $B'_\alpha(\underline{\alpha}) = 1$

The coefficients of B_α are real-analytic for $\alpha \in]0, \underline{\alpha}[$.

Moreover, if $\underline{\alpha} > 2 + \sqrt{3}$, then for all $\alpha \in (0, 1]$ and $y \in [0, \underline{\alpha}]$, we have $B'_\alpha(y) > 0$.

Proof. Since $\underline{\alpha} \neq 0$, satisfying the two conditions on B'_α is equivalent to the existence of some A such that

$$B'_\alpha(y) \;=\; A \cdot y(\underline{\alpha} - y) + \frac{1}{\underline{\alpha}} \cdot y + \frac{\alpha}{\underline{\alpha}} \cdot (\underline{\alpha} - y).$$

[5] See also [Sar80] for the case of triangles that are fundamental domains for the Hecke groups.

[6] In [HJ11], we considered a simpler mapping from $\mathcal{T}_{0,t}$ onto S. The mapping that we define here is more complicated because it must preserve the notion of zeroth Fourier coefficient for all y above some point. In particular, the vertical displacement of vertical lines should not depend on x for large y. In [HJ11], we considered Dirichlet boundary conditions, and in that context there is no need to truncate the zeroth Fourier coefficient.

Denote by Q_α the cubic polynomial defined by

$$Q_\alpha(y) \;=\; \int_\alpha^y z(\underline{\alpha}-z)\,dz.$$

By integration, there exists some C such that

$$B_\alpha(y) \;=\; A\cdot Q_\alpha(y) + \frac{1}{\underline{\alpha}}\cdot\frac{y^2}{2} - \frac{\alpha}{\underline{\alpha}}\cdot\frac{(\underline{\alpha}-y)^2}{2} + C.$$

Evaluating at α and using the condition on $B_\alpha(\alpha)$ we find

$$C = 1 - \frac{\alpha^2}{2\underline{\alpha}} - \frac{\alpha(\underline{\alpha}-\alpha)^2}{2\underline{\alpha}}.$$

Observe that $Q_\alpha(\underline{\alpha}) \neq 0$ if $\alpha \in [0,\underline{\alpha}[$ and, under this condition, we can solve the last equation on B_α to find A. We obtain

$$\begin{aligned} Q(\underline{\alpha})A &= \underline{\alpha} - \frac{\underline{\alpha}}{2} - C \\ &= \frac{1}{2\underline{\alpha}}\left[\underline{\alpha}^2 + \alpha^2 - \alpha(\underline{\alpha}-\alpha)^2 - 2\underline{\alpha}\right] \\ &= \frac{1-\alpha}{2\underline{\alpha}}\left[\alpha^2 + \underline{\alpha}^2 - 2\underline{\alpha}(\alpha+1)\right] \\ &= \frac{(1-\alpha)\left[\alpha^2 - 2\underline{\alpha}\alpha + \underline{\alpha}^2 - 2\underline{\alpha}\right]}{2\underline{\alpha}}. \end{aligned}$$

It follows that we have a unique solution provided $\underline{\alpha} \neq 0$ and $0 < \alpha < \underline{\alpha}$, and that the coefficients are real-analytic in α.

We now check the last statement. For $\alpha = 1$, we have $B_\alpha(y) = y$ so that the claim follows. For $\alpha < 1$, we observe that the numerator of A is a cubic polynomial that has three roots at $1, \underline{\alpha} \pm \sqrt{2\underline{\alpha}}$. Thus, if $\underline{\alpha} > 2+\sqrt{3}$, then 1 is the smallest root. Since this cubic polynomial is positive for large negative α and the denominator also is positive, it follows that A is positive for $0 < \alpha < 1$. So B'_α is a concave function, and by construction $B'_\alpha(0) > 0$ and $B'_\alpha(\underline{\alpha}) > 0$. The claim follows. □

Notation. We will use the notation $B_\alpha(y)$ as well as the notation $B(\alpha, y)$.

Define $F_{c,x} : \mathbb{R} \to \mathbb{R}$ by

$$F_c(x,y) = \begin{cases} B\left(f_c(x), y\right) & \text{if } y \leqslant \underline{\alpha}, \\ y & \text{if } y \geqslant \underline{\alpha}, \end{cases}$$

where

$$f_c(x) = \sqrt{1-(x-c)^2}.$$

Define $\varphi_{c,w} : \mathcal{T}_{c,w} \to S$ by

$$\varphi_{c,w}(x,y) = (x/w, F_c(x,y)).$$

Observe that the conditions on B imply that F, $\partial_x F_c$ and $\partial_y F_c$ are continuous on $\mathcal{T}_{c,w}$ so that $\varphi_{c,w}$ is C^1.

We will use this function $\varphi_{c,w}$ to normalize the triangle $\mathcal{T}_{c,w}$. This is made possible by the following lemma.

LEMMA 4.3. *Suppose that* $\partial_y B(f_c(x), y) > 0$ *for each* $(x,y) \in \mathcal{T}_{c,w} \cap \{y \leqslant \underline{\alpha}\}$. *Then the map* $\varphi_{c,w}$ *is a* C^1 *diffeomorphism from* $\mathcal{T}_{c,w}$ *onto* S.
In particular, for each $\underline{\alpha} > 2 + \sqrt{3}$ *and each* $(c,w) \in \mathcal{M}$, *the mapping* $\varphi_{c,w}$ *is a* C^1 *diffeomorphism from* $\mathcal{T}_{c,w}$ *onto* S.

Proof. It suffices to show that the map $F_{c,x}$ is a C^1 diffeomorphism from $[f_c(x), \infty[$ onto $[1, \infty[$. By assumption, $\partial_y B\,(f_c(x), y) > 0$ for each x. We have $B(f_c(x), f_c(x)) = 1$, $B(f_c(x), \underline{\alpha}) = \underline{\alpha}$, and $\partial_y B(f_c(x), \underline{\alpha}) = 1$. Since $F_{c,x}$ is the identity for $y > \underline{\alpha}$, we find that $F_{c,x}$ is a C^1 diffeomorphism from $[f_c(x), \infty[$ onto $[1, \infty]$. □

For each $\underline{\alpha}$ and each $M \subset \mathcal{M}$, we define $X_{\underline{\alpha},M}$ and $A_{\underline{\alpha},M}$ by

$$X_{\underline{\alpha},M} := \{(x,y,c,w) \mid (c,w) \in M,\ (x,y) \in \mathcal{T}_{c,w},\ y \leqslant \underline{\alpha}\},$$
$$A_{\underline{\alpha},M} := \{(a,b,c,w) \mid (c,w) \in M,\ (a,b) \in S,\ b \leqslant \underline{\alpha}\}.$$

We then have

LEMMA 4.4. *For each* $\underline{\alpha}, M$, *each of the following maps is analytic on* $X_{\underline{\alpha},M}$:
(1) $(x,y,c,w) \mapsto \varphi_{c,w}(x,y)$;
(2) $(x,y,c,w) \mapsto \partial_x \varphi_{c,w}(x,y)$;
(3) $(x,y,c,w) \mapsto \partial_y \varphi_{c,w}(x,y)$.
If, for each $(c,w) \in M$, *the assumption of Lemma* 4.3 *holds, then the map* $(a,b,c,w) \mapsto \varphi_{c,w}^{-1}(a,b)$ *is also analytic on* $A_{\underline{\alpha},M}$.
Moreover, each restriction extends analytically to an open neighbourhood.

Proof. The coefficients of the cubic polynomial B_α depend analytically on α, and hence $(\alpha, y) \mapsto B(\alpha, y)$ is analytic. The map $(c,x) \mapsto f_c(x)$ is analytic, and hence it follows that map (1) is analytic. Maps (2) and (3) are therefore analytic.

Since $(\alpha, y) \mapsto B(\alpha, y)$ is analytic and $\partial_y B(\alpha, y) > 0$ for $y > 0$, the implicit function theorem (Theorem 2.1.2 in [Ḧ90]) implies that there exists a function $(\alpha, b) \mapsto Y_\alpha(b)$ that is analytic and a solution to

$$B_\alpha(Y_\alpha(b)) - b = 0.$$

We then have

$$\varphi_{c,w}^{-1}(a,b) = \big(w \cdot a, Y_{f_c(w \cdot a)}(b)\big)$$

and, since $(c,x) \mapsto f_c(x)$ is analytic, the claim follows. □

In the rest of the section, $\underline{\alpha} > 2 + \sqrt{3}$ will be fixed so that we can use Lemmas 4.3 and 4.4.

4.3. *The quadratic form with fixed domain.* We use the family of diffeomorphisms $\varphi_{c,w}$ to define a quadratic form q_t with domain $H^1_\beta(S) \subset L^2_\beta(S)$ that is unitarily equivalent to $\mathcal{E}_\beta$ on $H^1_\beta(\mathcal{T}_{c,w}) \subset L^2_\beta(\mathcal{T}_{c,w})$.

Define $\Phi_{c,w} : L^2(S, da\, db/b^2) \to L^2(\mathcal{T}_{c,w}, dx\, dy/y^2)$ by

$$\Phi_{c,w}(u) = y \cdot \sqrt{|\det(\mathrm{Jac}(\varphi_{c,w}))|} \cdot \left(\frac{u}{b} \circ \varphi_{c,w}\right),$$

where Jac is the operator that returns the Jacobian matrix of a map.

LEMMA 4.5. *The isometry $\Phi_{c,w}$ is a unitary isomorphism from $L^2_\beta(S)$ onto $L^2_\beta(\mathcal{T}_{c,w})$, and it maps $H^1_\beta(S)$ onto $H^1_\beta(\mathcal{T}_{c,w})$. On functions that are supported in $b \geqslant \underline{\alpha}$, $\Phi_{c,w}$ coincides with $\widehat{\Phi}_w$.*

Proof. We have

$$\int_{\mathcal{T}_{c,w}} |\Phi_{c,w}(u)|^2 \, \frac{dx\ dy}{y^2} = \int_{\mathcal{T}_{c,w}} \left(\frac{u}{b} \circ \varphi_{c,w}\right)^2 |\det(\mathrm{Jac}(\varphi_{c,w}))| \cdot y^2 \cdot \frac{dx\ dy}{y^2}$$
$$= \int_S \left(\frac{u}{b}\right)^2 \ da\ db.$$

It follows that $\Phi_{c,w}$ is a unitary isomorphism from $L^2_\beta(S)$ onto $L^2_\beta(\mathcal{T}_{c,w})$.

Let $u \in H^1_\beta(S)$. Since $\varphi_{c,w}$ is a $\mathcal{C}^1$ diffeomorphism and $\sqrt{|\det(\mathrm{Jac}(\varphi_{c,w}))|}$ is continuous on $\mathcal{T}_{c,w}$ and smooth away from $y = \beta$, then $\Phi_{c,w}(u)$ is continuous and in $H^1_\beta(\mathcal{T}_{c,w} \setminus \{y = \beta\})$. The jump formula implies that $\Phi_{c,w}(u) \in H^1_\beta(\mathcal{T}_{c,w})$. Since for $y > \underline{\alpha}$, $\varphi_{c,w}(x, y) = (\frac{x}{w}, y)$, the last statement is a direct verification. □

Definition 4.6. Define the quadratic form $q_{c,w}$ on $H^1_\beta(S) \subset L^2(S, da\, db/b^2)$ by

$$q_{c,w}(u) := \mathcal{E}_\beta \circ \Phi_{c,w}(u).$$

LEMMA 4.7. *The function u is a cusp form for $q_{c,w}$ if and only if $v = \Phi_{c,w} \circ u$ is a cusp form for $\mathcal{E}$ on $\mathcal{T}_{c,w}$.*

Proof. If $y \geqslant \underline{\alpha}$, then $\varphi_{c,w}(x, y) = (x/w, y)$. It follows that if $y \geqslant \underline{\alpha}$, then $u^0(y) = 0$ if and only if $v^0(y) = 0$. For $y \geqslant 1$, the function v^0 is a solution to a second order ordinary differential equation, and hence $v^0(y) = 0$ for $y \geqslant \underline{\alpha}$ if and only if $v^0(y) = 0$ for $y \geqslant 1$. □

It will be convenient to have the following alternate form for q_t.

PROPOSITION 4.8. *We have*

$$q_{c,w}(u) = \int_S \nabla(\rho_{c,w} \cdot u) \cdot Q_{c,w} \cdot \overline{\nabla(\rho_{c,w} \cdot u)}^* \ da\, db, \tag{10}$$

where $\rho_{c,w} : S \mapsto \mathbb{R}$ *is defined by*

$$\rho_{c,w} = \frac{\left(y \cdot \sqrt{|\det(\mathrm{Jac}(\varphi_{c,w}))|}\right) \circ \varphi_{c,w}^{-1}}{b},$$

and $Q_{c,w} : S \to \mathrm{GL}_2(\mathbb{R})$ *is defined by*

$$Q_{c,w} \circ \varphi_{c,w} = \frac{1}{\det\left(\mathrm{Jac}(\varphi_{c,w})\right)} \cdot \mathrm{Jac}(\varphi_{c,w}) \cdot \mathrm{Jac}(\varphi_{c,w})^{*}. \tag{11}$$

Moreover, $q_{c,w}$ *represents the cusp of width* w *for* $y \geqslant \underline{\alpha}$.

Proof. This is a straightforward calculation using the chain rule and the change of variables formula. □

4.4. *Analytic paths in* $\mathcal{M}$. Let $I =]t_-, t_+[$, and let $\gamma : I \to \mathcal{M}$ be a real-analytic path.

THEOREM 4.9. *The family of quadratic forms* $t \mapsto q_{\gamma(t)}$ *is analytic of type* (a) *in the sense of* [Kat95].

Proof. For each t, the quadratic form $q_{\gamma(t)} = \mathcal{E}_\beta \circ \Phi_{\gamma(t)}$ is a closed form with domain $H^1_\beta(S)$. It suffices to show that for each $u \in H^1_\beta(S)$, the function $t \mapsto q_{\gamma(t)}(u)$ is real-analytic.

By Proposition 4.8, we have

$$q_{c,w}(u) = \int_1^{\underline{\alpha}} \int_0^1 \; I_t \; da\, db \; + \int_{\underline{\alpha}}^{\infty} \int_0^1 \; I_t \; da\, db, \tag{12}$$

where

$$I_t = \nabla(\rho_{\gamma(t)} \cdot u) \cdot Q_{\gamma(t)} \cdot \overline{\nabla(\rho_{\gamma(t)} \cdot u)}^{*}.$$

If $(a,b) \in [0,1] \times [\underline{\alpha}, \infty[$, then the matrix $Q(a,b)$ is given by

$$Q \; = \begin{pmatrix} \frac{1}{w_t^2} & 0 \\ 0 & 1 \end{pmatrix}$$

and $\rho_{\gamma(t)}(a,b) = 1$. Thus, the second integral on the right of (12) depends analytically on t.

It remains to consider the integral over $[0,1] \times [1, \underline{\alpha}]$. The integrand I_t can be expanded into a finite sum of terms of the form

$$\int_1^{\underline{\alpha}} \int_0^1 w(a,b) \cdot H(t,a,b) da db, \tag{13}$$

where H is a function that is obtained by multiplying ρ, or its derivatives and the entries of Q and w is one of the L^1 functions obtained by making the product $v_1 v_2$ where both v_i are either u or one of its partial derivatives.

By Lemma 4.4, the coordinates of $\varphi_{c,w}$ and $\varphi_{c,w}^{-1}$ are analytic functions of (c,w). It follows that $(t,a,b) \mapsto \rho_{\gamma(t)}(a,b)$ and $(t,a,b) \mapsto Q_{ij}(t)(a,b)$ are analytic (in a neighborhood of $I \times [0,1] \times [1,\underline{\alpha}]$). In all possible choices, the function H then is analytic.

The analyticity of $t \mapsto q_{\gamma(t)}(u)$ follows from Lemma 4.10 below. $\square$

LEMMA 4.10. *If $H : I \times [0,1] \times [1,\underline{\alpha}]$ is analytic, then for each $p \in L^1([0,1] \times [1,\underline{\alpha}])$, the function*

$$t \longmapsto \int_1^{\underline{\alpha}} \int_0^1 p(a,b) \cdot H(t,a,b)\, dadb \tag{14}$$

is analytic on I.

Proof. There exists an open neighborhood $U \subset \mathbb{C}^3$ of $I \times [0,1] \times [1,\underline{\alpha}]$ such that the map h extends to a holomorphic function on U. Since $[0,1] \times [1,\underline{\alpha}]$ is compact,

$$\frac{H(t,a,b) - H(s,a,b)}{t-s}$$

converges uniformly to $\frac{d}{dt}H(s,a,b)$ as t approaches s. It follows that the (complex) t-derivative of the map in (14) exists at each $t \in U$. $\square$

4.5. *Generic absence of cusp forms.* Given Theorem 4.9, we now explain why the generic triangle $\mathcal{T}_{c,w}$ has no cusp forms provided that one triangle has none.

THEOREM 4.11. *If there exists a point $(c_0, w_0) \in \mathcal{M}$ such that $\mathcal{E}$ on $L^2_\beta(\mathcal{T}_{c_0,w_0}, dm)$ has no nonconstant eigenfunction, then $\mathcal{M}_{\rm cf}$ has measure zero and is a countable union of nowhere dense sets.*

Proof. By Proposition 4.1, the quadratic form $\mathcal{E}_\beta$ on $L^2_\beta(\mathcal{T}_{c_0,w_0}, dm)$ has no cusp form, and hence by Lemma 4.7, the quadratic form q_{c_0,w_0} has no cusp form.

To show that $\mathcal{M}_{\rm cf}$ has measure zero, we apply Fubini's theorem in a fashion similar to [HJ09]: Let $\gamma_{c_0}(t) = (c_0, w_0 + t)$, and apply Lemma 3.8 to find that the set B of w such that $(c_0, w) \in \mathcal{M}_{\rm cf}$ is countable. For each $w \notin B$, let $\gamma_w(s) = (c_0 + s, w)$, and apply Lemma 3.8 to find that the intersection I_w of the line $\{(c,w) \mid c \in \mathbb{R}\}$ with $\mathcal{M}_{\rm cf}$ is countable. Hence for each $w \notin B$, the set I_w has measure zero with respect to the linear measure da. Hence, the measure of $\mathcal{M}_{\rm cf}$ equals the measure of $\bigcup_{w \in B} I_w$. Since B is countable, the measure equals zero.

For $N \in \mathbb{Z}$, let $\mathcal{M}^N_{\rm cf}$ be the set of $(c,w) \in \mathcal{M}$ such that $\mathcal{E}$ on $L^2(\mathcal{T}_{c,w}, dm)$ has a cusp form with eigenvalue at most N. Using the continuity of $(c,w) \to q_{c,w}$ and the continuity of linear functional L, one can show that $\mathcal{M}^N_{\rm cf}$ is closed. Thus, it suffices to show that $\mathcal{M}^N_{\rm cf}$ is nowhere dense.

Given a point $(c,w) \in \mathcal{M}^N_{\rm cf}$, let $\gamma :\ [0,1] \to \mathcal{M}$ be a real-analytic path joining (c_0, w_0) to (c,w). Since $\mathcal{E}_\beta$ on $L^2_\beta(\mathcal{T}_{c_0,w_0}, dm)$ has no cusp forms, the family

$t \mapsto q_{\gamma(t)}$ has no cusp form eigenfunction branch. It follows from Lemma 3.8 that for each open neighborhood U of (c, w), there exists $t \in [0,1]$ such that $\gamma(t) \in U$ and $q_{\gamma(t)}$ has no cusp forms. Hence $\mathcal{M}_{\mathrm{cf}}^N$ is nowhere dense. $\square$

5. The family $\mathcal{T}_t$

In the remainder of this paper we consider the specific family of triangles $\mathcal{T}_t = \mathcal{T}_{0,t}$ defined in the introduction. In particular, we will study the spectral properties of $q_{0,t}$ for small t. The family $q_{0,t}$ of quadratic forms does not converge as t tends to zero nor do its real-analytic eigenbranches. But a simple renormalization will give convergence.

Fix $\beta > 1$ and $\underline{\alpha}$ such that $1 < \underline{\alpha} < \beta$. Let B be the function defined in Lemma 4.2. When α tends to 1, the function $y \mapsto \partial_y B(\alpha, y)$ converges to 1 uniformly for $y \in [0, \underline{\alpha}]$. Thus, there exists $\eta > 0$ such that if $1 - \eta \leqslant \alpha \leqslant 1$ and $0 \leqslant y \leqslant \underline{\alpha}$, then $\partial_y B(\alpha, y) \geqslant \frac{1}{2}$. Choose $t_0 = \sqrt{1-(1-\eta)^2}$. Then, for each $t < t_0$ and each $(x, y) \in \mathcal{T}_t \cap \{y \leqslant \underline{\alpha}\}$, we have $f_0(x) < \eta$ so that $\partial_y B(f_0(x), y) > 0$. We may thus use Lemmas 4.3 and 4.4. The methods and results of Section 4.4 then apply, and we define the quadratic form $q_{0,t}$ as previously.

For each $t \in \ [0, t_0[$, define the renormalized quadratic form by

$$q_t := t^2 \cdot q_{0,t}$$

with domain $H^1_\beta(S)$. By Theorem 4.9, the family $t \mapsto q_t$ is real-analytic of type (a) for $t \in \,]0,1[$. In particular, the results of Section 4.5 apply.

To study the limiting properties of the family q_t, we re-express q_t in a more convenient form: For each C^1 function $w : S \to \mathbb{C}$, define

$$\widetilde{\nabla}_t\, w = (\partial_x w,\ t \cdot \partial_y w). \tag{15}$$

Recall that Y_α is the inverse of B_α, and set $f(x) = f_0(x) = \sqrt{1-x^2}$. Define

$$\widetilde{\rho}_t(a,b) = \frac{Y(f(ta)b)}{b} \cdot (\partial_y Y(f(at),b))^{\frac{1}{2}} \tag{16}$$

and

$$\begin{aligned}\widetilde{Q}_t(a,b) &= (\partial_y(Y(f(at),b))^{-1} \\ &\cdot \begin{pmatrix} 1 & (\partial_\alpha B \circ Y) \cdot f'(ta) \\ (\partial_\alpha B \circ Y) \cdot f'(ta) & ((\partial_\alpha B \circ Y) \cdot f'(ta))^2 + (\partial_y B)^2 \end{pmatrix},\end{aligned} \tag{17}$$

where the subscript (or first argument) in each Y and B is $f(t \cdot a)$. When comparing ρ and $\tilde{\rho}$ (Q and $\tilde{Q}$) we see that we only miss some powers of t that eventually cancel in the computation leading to Proposition 4.8.

This shows that for each $u \in H^1_\beta(S)$,

$$q_t(u) = \int_{S_-} \widetilde{\nabla}(\widetilde{\rho}_t \cdot u) \cdot \widetilde{Q}_t \cdot \widetilde{\nabla}(\widetilde{\rho}_t \cdot \overline{u})^*\ da\ db + \int_{S_+} \widetilde{\nabla} u \cdot \left(\widetilde{\nabla}\overline{u}\right)^* da\, db, \tag{18}$$

where $S_{-} = [0,1] \times [1, \underline{\alpha}]$ and $S^{+} = [0,1] \times [\underline{\alpha}, \infty]$.

By arguing as in the proof of Theorem 4.9, one can show that $t \mapsto q_t(u)$ is analytic at $t = 0$.[7]

We will now compute the first few terms in the Taylor series in t for $\widetilde{\rho}$ and $\widetilde{Q}$. These functions are analytic on a neighbourhood of $[0, t_0[\times S_{-}$. In particular, in the following, the expressions like $O(t^2)$ are uniform with respect to $(a, b) \in S^{-}$ and may be differentiated with respect to t, a and b.

We first compute

$$f(ta) = 1 - \frac{1}{2} \cdot t^2 \cdot a^2 + O(t^4) \tag{19}$$

and

$$f'(ta) = -t \cdot a + O(t^3). \tag{20}$$

Since $\alpha \mapsto Y_\alpha$ is analytic and $Y_1(b) = b$, it follows from (19) that

$$Y_{f(ta)}(b) = b + O(t^2). \tag{21}$$

Moreover, using analyticity, this asymptotic expansion may be differentiated with respect to (a, b). We thus obtain

$$Y'_{f(ta)}(b) = 1 + O(t^2). \tag{22}$$

Substituting these into (16) and differentiating, we find that

$$\widetilde{\rho}_t(a, b) = 1 + O(t^2), \ \nabla_{a,b}\widetilde{\rho}_t(a, b) \ = \ O(t^2). \tag{23}$$

Using (19), (20), (21), and (22) we find that

$$\widetilde{Q}_t(a, b) = I + t \cdot a \cdot p(b) \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + O(t^2), \tag{24}$$

where I is the identity matrix, $O(t^2)$ is a matrix whose operator norm is bounded by a constant times t^2 as t tends to zero, and p is the polynomial

$$p(b) = -\left. \partial_\alpha B_\alpha(b) \right|_{\alpha=1}. \tag{25}$$

To prove Theorem 1.1 we will need to know that $p(1) \neq 0$.

LEMMA 5.1. $p(1) = 1$.

Proof. By construction we have $B(\alpha, \alpha) = 1$. By differentiating with respect to α and setting $\alpha = 1$ we get

$$\partial_\alpha B(1, 1) + \partial_y B(1, 1) = 0.$$

Since $\partial_y B(\alpha, y) = 1 + O((\alpha - 1)^2)$, the claim follows. $\square$

[7] However, because q_0 is not closed on the domain $H^1_\beta(S)$, the family q_t is *not* analytic at $t = 0$ in the sense of [Kat95].

6. Asymptotic separation of variables

In this section we apply the method of asymptotic separation of variables developed in [HJ11] (see also [HJ12]) to the family q_t. Using the small t asymptotics derived in Section 5, we approximate q_t to first order with a family of quadratic forms a_t for which separation of variables apply. We also derive a nonconcentration estimate for eigenfunctions of q_t.

Notation. In this section and the following sections, we will use (x, y) in place of (a, b) as coordinates for $S = [0,1] \times [1, \infty[$, and unless it is specified otherwise $\|\cdot\|$ is the norm in $L^2(S, y^{-2}dxdy)$.

6.1. *Asymptotic approximation.* We begin by using the expansions obtained in Section 5 to determine the forms used to approximate q_t. In particular, by substituting the expansions (23) and (24) into (18) we are led to define

$$a_t(u,v) = \int_S \widetilde{\nabla} u \cdot \widetilde{\nabla} \overline{v} \; dx\,dy = \int_S \left(u_x \cdot \overline{v}_x + t^2 \cdot u_y \cdot \overline{v}_y \right) \; dx\,dy \tag{26}$$

and

$$b_t(u,v) = \int_{S_-} \widetilde{\nabla} u \cdot \mathcal{B}(x,y) \cdot \widetilde{\nabla} \overline{v} \; dx\,dy, \tag{27}$$

where the operator $\widetilde{\nabla}$ is defined by (15), $S_- = [0,1] \times [1, \underline{\alpha}]$, and

$$\mathcal{B}(x,y) = x \cdot p(y) \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We wish to show that q_t is asymptotic to a_t at first order in the sense of [HJ11]. It will also be used to help derive a key estimate for crossing eigenbranches. However, although a_t is a positive quadratic form, the bottom of its spectrum tends to 0 so that it is more convenient to use the quadratic form $\widetilde{a_t}$ that we now define to control quantities.

Definition 6.1. The quadratic form $\widetilde{a_t}$ is defined on $\mathrm{dom}(a_t)$ by

$$\widetilde{a_t}(v) \;=\; a_t(v) + \|v\|^2.$$

The following proposition can be seen as the beginning of an asymptotic expansion for q_t.

PROPOSITION 6.2. *There exists C such that for each $u, v \in H^1_\beta(S)$,*

$$|q_t(u,v) - a_t(u,v) - t \cdot b_t(u,v)| \;\leqslant\; C \cdot t^2 \cdot \widetilde{a}_t(u)^{\frac{1}{2}} \widetilde{a}_t(v)^{\frac{1}{2}}. \tag{28}$$

Proof. We have

$$\widetilde{\nabla}_t \; \widetilde{\rho}_t \cdot u = \widetilde{\rho}_t \cdot \widetilde{\nabla}_t u + u \cdot \widetilde{\nabla}_t \, \widetilde{\rho}_t. \tag{29}$$

If $y \geqslant \underline{\alpha}$, then $\widetilde{\rho}_t$ is identically equal to 1 and $\widetilde{Q}_t$ is identically equal to I. Hence, by substituting (29) into (18), we find that $q_t(u,v) - a_t(u,v) - t \cdot b_t(u,v)$ is the sum of the following four terms:

$$\int_{S_-} \widetilde{\nabla}_t u \cdot (\tilde{\rho}^2 \cdot \widetilde{Q}_t - I - t \cdot \mathcal{B}) \cdot \widetilde{\nabla}_t v \; dx \; dy, \tag{30}$$

$$\int_{S_-} \widetilde{\rho}_t \cdot v \cdot (\widetilde{\nabla}_t \widetilde{\rho}_t \cdot \widetilde{Q}_t \cdot \widetilde{\nabla}_t u) \; dx \; dy, \tag{31}$$

$$\int_{S_-} \widetilde{\rho}_t \cdot u \cdot \left(\widetilde{\nabla}_t \widetilde{\rho}_t \cdot \widetilde{Q}_t \cdot \widetilde{\nabla}_t v\right) \; dx \; dy, \tag{32}$$

$$\int_{S_-} \left(\widetilde{\nabla}_t \widetilde{\rho}_t \cdot \widetilde{Q}_t \cdot \widetilde{\nabla}_t \widetilde{\rho}_t\right) \cdot u \cdot v \; dx \; dy. \tag{33}$$

In order to estimate these four terms, we use the asymptotic expansions of Section 5. For example, by (24) we have that (30) is equal to

$$\int_S \widetilde{\nabla}_t u \cdot O(t^2) \cdot \widetilde{\nabla}_t v \; dx \; dy.$$

Since the operator norm of the matrix $O(t^2)$ is bounded by a constant C times t^2, we can apply the Cauchy-Schwarz inequality to find that the norm of (30) is bounded by $C \cdot t^2 \cdot a_t(u)^{\frac{1}{2}} \cdot a_t(v)^{\frac{1}{2}}$.

Similar arguments show that there is a constant C so that

- (31) is bounded above by $C \cdot t^2 \cdot a_t(u)^{\frac{1}{2}} \cdot \|v\|^{\frac{1}{2}}$;
- (32) is bounded above by $C \cdot t^2 \|u\|^{\frac{1}{2}} \cdot a_t(v)^{\frac{1}{2}}$;
- (33) is bounded above by $C \cdot t^2 \cdot \|u\|^{\frac{1}{2}} \cdot \|v\|^{\frac{1}{2}}$.

The claim follows. □

6.2. *The spectrum of a_t via separation of variables.* We recall the Fourier decomposition of Section 2.2. Since now $w = 1$, we thus have for each $u \in L^2(S, \frac{dxdy}{y^2})$,

$$u^k(y) = \int_0^1 u(x,y) \cdot e_k(x) \; dx,$$

where the latter makes sense for almost every y and defines an element of $L^2((1,\infty), \frac{dy}{y^2})$.

As above, let $\mathcal{D}(\overline{S})$ denote the set of functions $v : S \to \mathbb{C}$ such that v is the restriction of a compactly supported, smooth function defined in a neighborhood of S. If $u \in \mathcal{D}$, then each u^k is smooth, and a straightforward computation shows that[8]

$$\begin{aligned} a_t(u) &= \sum_{k \in \mathbb{N}} a_t\left(u^k \otimes e_k\right) \\ &= \sum_{k \in \mathbb{N}} \int_1^\infty \left(t^2 \cdot \partial_y u^k(y)^2 + (k\pi)^2 \cdot u^k(y)^2\right) \; dy. \end{aligned} \tag{34}$$

[8]Here $\otimes$ is the operation defined by $(v \otimes w)(x,y) = v(y) \cdot w(x)$.

We define $\mathcal{D}([1,\infty[)$ to be the set of compactly supported, smooth functions defined on $[1,\infty[$. For $v \in \mathcal{D}([1,\infty))$, each integer k, and each $t > 0$, we define

$$a_t^k(v) = \int_1^\infty \left(t^2 \cdot v'(y)^2 + (k\pi)^2 \cdot v(y)^2\right)\, dy. \tag{35}$$

For v, w in $L^2([1,\infty[\ , y^{-2}\,dy)$, the inner product is defined by

$$\langle u, v\rangle_y = \int_1^\infty u(y)\cdot v(y)\ \frac{dy}{y^2}.$$

Let L^2_β denote the subspace consisting of those functions whose support lies in $[1,\beta]$.

For each $k \in \mathbb{N}$, the quadratic form a_t^k extends to a closed, densely defined form on the completion of $\mathcal{D}([1,\infty[)$ with respect to $v \mapsto a_t^k(v)^{\frac{1}{2}} + \langle v, v\rangle_y^{\frac{1}{2}}$. For $k = 0$, we will restrict the domain of a_t^k to be the completion of those smooth functions whose support lies in $[1,\beta]$.

If u is an eigenfunction of a_t with eigenvalue λ, then for each v in the domain of a_t^k, we have

$$a_t^k(u^k, v) = a_t(u, v\otimes e_k) = \lambda\cdot\langle u, v\otimes e_k\rangle = \lambda\cdot\langle u^k, v\rangle_y,$$

and hence u^k is an eigenfunction of a_t^k with eigenvalue λ with respect to $\langle\cdot,\cdot\rangle_y$. Thus, each eigenfunction u of a_t may be written uniquely as

$$u = \sum_{k\in\mathbf{N}} u^k \otimes e_k,$$

where the k^{th} Fourier coefficient u^k is an eigenfunction of a_t^k. Moreover, the spectrum of a_t with respect to $\langle\cdot,\cdot\rangle$ is the union of the spectra of a_t^k with respect to $\langle\cdot,\cdot\rangle_y$. In what follows we will often suppress the subscript y from the notation.

The next two lemmas identify the eigenfunctions of a_t^k for each k. The zero modes are given by the following lemma.

Lemma 6.3. *The spectrum of a_t^0 with respect to $\langle\cdot,\cdot\rangle_y$ is the set*

$$\left\{t^2\cdot\left(\frac{1}{4}+r^2\right)\ \middle|\ r>0 \text{ and } 2r = \tan(r\cdot\ln(\beta))\right\}. \tag{36}$$

The eigenspace associated to $t^2\cdot(\frac{1}{4}+r^2)$ is spanned by the eigenvector

$$\psi(y) = y^{\frac{1}{2}}\cdot\cos\left(r\ln(y)\right)\ -\ \frac{y^{\frac{1}{2}}}{2r}\cdot\sin(r\ln(y)). \tag{37}$$

Proof. Suppose that v is an eigenfunction, that is $a_t^0(v,w) = \lambda\cdot\langle v, w\rangle$ for all w. This implies first that

$$-t^2\cdot v''(y) = -\frac{\lambda}{y^2}\cdot v(y)$$

holds in the distributional sense. Ellipticity then yields that v is smooth. Moreover, by integrating by parts against a smooth function that is identically equal to 1 near $y = 1$, we also find that $v'(0)=0$. Let s be such that $s \cdot (1-s) = \lambda/t^2$. Then two linearly independent solutions are given by y^s and y^{1-s} if $s \neq \frac{1}{2}$ and by $y^{\frac{1}{2}}$ and $y^{\frac{1}{2}} \cdot \ln(y)$ if $s = \frac{1}{2}$. The condition that λ/t^2 is real and nonnegative implies either that $s = \frac{1}{2} + ir$ with $r > 0$, that $s \in [0, 1/2)$, or that $s = \frac{1}{2}$. If Re(s) $= 1/2$, then the boundary conditions $v'(1) = 0$ and $v(\beta) = 0$ imply that the solutions take the form given in (37) with $2r = \tan(r \cdot \ln(\beta))$. If $s \in [0, 1/2]$, then there are no solutions that satisfy the boundary conditions. $\square$

The nonzero modes are given by the following lemma.

LEMMA 6.4. *For each t and k and eigenvalue λ of a_t^k with respect to $\langle \cdot, \cdot \rangle_y$, the associated eigenspace consists of functions of the form $y \mapsto f(\pi k \cdot y/t)$ such that*

(i) $f''(z) = \left(1 - \frac{\lambda}{(t \cdot z)^2}\right) \cdot f(z)$;
(ii) $f \in L^2\left([1, \infty), \frac{dy}{y^2}\right)$;
(iii) $f'(\pi \cdot k/t) = 0$.

Moreover, when t varies, the spectrum is organized into eigenvalues branches $\lambda_i(t)$. For each i, the function $t \mapsto \lambda_i(t)$ is increasing, and

$$\lim_{t \to 0} \lambda_i(t) = (\pi \cdot k)^2.$$

Proof. Integrate by parts as in the proof of Lemma 6.3, make the change of variables $z \mapsto \pi k \cdot y/2t$, and use the boundary conditions. The existence of a complete set of real-analytic branches of eigenvalues and eigenvectors follows from Theorem 3.3.

For each real-analytic eigenbranch $t \mapsto \lambda_i(t)$, the first order variation is obtained by the classical formula

$$\dot{\lambda}_i(t) \;=\; \dot{a}_t^k(v_t) \;=\; 2t \int_1^\infty v_t'(y)^2 \, dy,$$

where v_t is the normalized eigenfunction associated with $\lambda_i(t)$. Since the integrand is nonnegative, the eigenvalue branches are increasing.

The fact that each eigenbranch $t \mapsto \lambda_i(t)$ limits to $(\pi \cdot k)^2$ follows from the methods of [HJ11]. Alternatively, one can transform the eigenvalue problem into a problem involving a Schrödinger operator with a potential whose minimum value equals $(\pi \cdot k)^2$. $\square$

Remark 6.5. The eigenfunctions of a_t^k for positive k may be regarded as Bessel functions since the corresponding differential equation can be transformed into a Bessel equation on the half-line $[i, i \cdot \infty)$.

As a consequence of the identification of the eigenfunctions, we have the following Poincaré type inequality.

LEMMA 6.6. *For each $t \leqslant 2\pi$ and each $u \in H^1_\beta(S)$, we have*

$$a_t(u) \geqslant \frac{t^2}{4} \cdot n(u). \tag{38}$$

Proof. We have

$$a_t(u) = \sum_k a_t^k(u^k) \quad \text{and} \quad n(u) = \sum_K \int_1^\infty \left|u^k\right|^2 \frac{dy}{y^2}.$$

Lemmas 6.4 and 6.3 imply

$$a_t^0(u^0) \geqslant \frac{t^2}{4} \cdot \int_1^\beta \left|u^0\right|^2 \frac{dy}{y^2}$$

and

$$a_t^k(u^k) \geqslant (\pi \cdot k)^2 \cdot \int_1^\infty \left|u^k\right|^2 \frac{dy}{y^2} \tag{39}$$

for $k > 0$. □

In the sequel we will use different kind of projections associated either with the Fourier decomposition $u = \sum u^k \otimes e_k$ or with the spectral decomposition of a_t.

More precisely, for each $\ell \in \mathbb{N}$, define the orthogonal projection $\Pi_\ell : L^2_\beta(S) \to L^2_\beta(S)$ by

$$\Pi_\ell(v) = v^\ell(y) \cdot e_\ell(x), \tag{40}$$

and let V_ℓ denote the image of Π_ℓ. For each $k \in \mathbb{Z}$, we define

$$\Pi_{\ell<k}(v) = \sum_{\ell<k} v^\ell(y) \cdot e_\ell(x) \tag{41}$$

to be the projection onto $\bigoplus_{0\leqslant\ell<k} V_\ell$.

We also define $P^\lambda_{a_t}$ to be the orthogonal projection onto the eigenspace of a_t associated to the eigenvalue λ of a_t. For a fixed interval I, define the a_t-spectral projection in the energy interval I to be

$$P^I_{a_t}(v) := \sum_{\lambda\in\operatorname{spec}(a_t)\cap I} P^\lambda_{a_t}(v).$$

For each eigenvalue λ of a_t, the associated a_t-eigenspace W_λ is the orthogonal direct sum $\oplus_\ell(W^\ell_\lambda \otimes \operatorname{vect}(e_\ell))$, where W^ℓ_λ is the λ-eigenspace of a^ℓ_t and $\operatorname{vect}(e_\ell)$ is the span of e_ℓ. It follows that

$$\Pi_\ell\left(P^I_{a_t}(v)\right) := \sum_{\lambda\in\operatorname{spec}(a^\ell_t)\cap I} P^\lambda_{a_t}(\Pi_\ell(v)). \tag{42}$$

More generally, for a quadratic form b, the notation P_b^I will always denote the spectral projection onto the interval I.

6.3. *Asymptotic at first order.* In the following, we let $\dot{q}_t$ (resp. $\dot{a}_t$) denote the derivative of q_t (resp. a_t) in t. The following proposition will allow us to compare a_t and q_t in the limit $t \to 0$.

PROPOSITION 6.7 (Asymptotic at first order). *There exist a constant C and t_0 such that, for all $u, v \in H^1_\beta$ and all $t \leqslant t_0$,*

$$|q_t(u,v) - a_t(u,v)| \leqslant C \cdot t \cdot a_t(u)^{\frac{1}{2}} \cdot a_t(v)^{\frac{1}{2}}, \tag{43}$$

$$|\dot{q}_t(u) - \dot{a}_t(u)| \leqslant C \cdot a_t(u). \tag{44}$$

In [HJ11], two real-analytic families of quadratic forms a_t and q_t satisfying (43) and (44) were said to be asymptotic at first order. We will use the same terminology here.

Proof. One argues as in the proof of Proposition 6.2, paying a little more attention to the terms (31), (32), and (33). For example, to estimate (31), use the Cauchy-Schwarz inequality and (23) to obtain

$$|\widetilde{\nabla}_t \widetilde{\rho}_t \cdot \widetilde{\nabla}_t u| \leqslant |\widetilde{\nabla}_t \widetilde{\rho}_t| \cdot |\widetilde{\nabla}_t u| = O(t^2) \cdot \left|\widetilde{\nabla}_t u\right|. \tag{45}$$

The Cauchy-Schwarz inequality and Lemma 6.3 give

$$\int_S |v| \cdot |\widetilde{\nabla}_t u| \, dx\, dy \leqslant \frac{1}{t \cdot \sqrt{1/4 + r_0^2}} \cdot a_t(u)^{\frac{1}{2}} \cdot a_t(v)^{\frac{1}{2}}. \tag{46}$$

By combining (45) and (46) and using (23), we find that

$$\int_S |\widetilde{\rho}_t| \cdot |v| \cdot |\widetilde{\nabla}_t \widetilde{\rho}_t \cdot \widetilde{Q}_t \cdot \widetilde{\nabla}_t u| \, dx\, dy = O(t) \cdot a_t(u)^{\frac{1}{2}} \cdot a_t(v)^{\frac{1}{2}}.$$

Switching the roles of u and v, we obtain the same bound for the expression in (32). Similar methods apply to bound the other terms.

The estimate for $\dot{q} - \dot{a}$ is obtained in a similar way. □

7. **Limits of eigenvalue branches**

Since q_t is asymptotic to a_t at first order and a_t and $\dot{a}_t$ are nonnegative quadratic forms, each real-analytic eigenvalue branch E_t of q_t converges to a finite limit E_0 as t tends to zero (Theorem 3.4 of [HJ11]). For the Dirichlet eigenvalue problem on $\mathcal{T}_t$, we showed in [HJ11], [HJ12] that each limit E_0 has the form $(\pi k)^2$ where k is an integer. The methods of [HJ11] can be applied to show that the same fact is true in the present context. In this section we highlight the necessary modifications. We also show that if the eigenvalue branch is associated to a cusp form, then k must be positive. This latter fact will be used crucially in the proof of Theorem 1.1.

7.1. *Non-concentration and first variation.* The proof of convergence depends crucially on the following "nonconcentration" result proved for the Dirichlet problem in [HJ11].

We will let $\mathcal{D}_\ell$ denote the domain of the quadratic form a_t^ℓ. We define a quadratic form $\widetilde{a}_t^\ell : \mathcal{D}_\ell \to \mathbb{C}$ by setting

$$\widetilde{a}_t^\ell(v) \;=\; a_t^\ell(v) + \|v\|^2$$

for each $v \in \mathcal{D}_\ell$.

Proposition 7.1 (Compare Proposition 9.1 of [HJ11]). *Let $\ell \in \mathbb{N}$, let K be a compact subset of $](\pi\ell)^2, \infty[$, and let $C > 0$. There exist positive constants t_0 and κ (that only depend on ℓ, K and C) so that if $E \in K$, if $t < t_0$, and if for each $w \in \mathcal{D}_\ell$, the function $v \in \mathcal{D}_\ell$ satisfies*

$$\left| a_t^\ell(v,w) - E \cdot \langle v, w\rangle \right| \;\leqslant C \cdot t \cdot \|w\| \cdot \|v\|,$$

then

$$\int_1^\infty \left(\frac{E}{y^2} - (\ell\pi)^2 \right) \cdot |v(y)|^2 \; dy \;\geqslant\; \kappa \cdot \|v\|^2. \tag{47}$$

Proof. If $\ell = 0$ and we let $\kappa = \inf(K) > 0$, then (47) holds. If $\ell > 0$, this follows from Proposition 9.1 of [HJ11] with $\mu = (\pi\ell)^2$ and $\sigma(y) = y^{-2}$. See the end of [HJ12] for a proof of Proposition 9.1 of [HJ11]. $\square$

In the language of semi-classical analysis, Proposition 7.1 asserts that a quasimode v of order t at energy E does not concentrate at $y = \sqrt{E}/(\ell\pi)$ if $\ell \neq 0$. In Section 12 of [HJ11], we used nonconcentration to derive indirect estimates for $\dot{a}_t$. The following proposition and corollary make these estimates more transparent and simpler to apply.

Proposition 7.2. *Let $\ell \in \mathbb{N}$, and let $K \subset](\pi\ell)^2, \infty[$ be compact. For each $\epsilon > 0$, there exist $\kappa' > 0$ and $t_0 > 0$ such that for each $v \in \mathcal{D}_\ell$ and $t < t_0$,*

$$\|v\|^2 \;\leqslant\; \frac{t}{\kappa'} \cdot \dot{a}_t^\ell(v) + \frac{\epsilon}{t^2} \cdot N_\ell(v,E)^2, \tag{48}$$

where

$$N_\ell(v,E) = \sup_{w \in \mathcal{D}} \; \frac{|a_t^\ell(v,w) - E\langle v, w\rangle|}{\widetilde{a}_t^\ell(w)^{\frac{1}{2}}}.$$

Proof. From (35) we find that $\dot{a}_t^\ell(v) = 2t \displaystyle\int_1^\infty v'(y)^2$ and hence

$$t \cdot \dot{a}_t^\ell(v) = 2 \cdot \int_1^\infty \left(\frac{E}{y^2} - (\pi\ell)^2 \right) \cdot v^2 + 2 \left(a_t^\ell(v) - E \cdot \|v\|^2 \right). \tag{49}$$

If the claim is not true, then for each $\kappa' > 0$, there exist a sequence $(t_n)_{n\geqslant 1}$ tending to zero and sequences $(\tilde{v}_n)_{n\geqslant 1},\ \tilde{v}_n \in \mathcal{D}$, $(E_n)_{n\geqslant 1},\ E_n \in K$ such that

$$\|\tilde{v}_n\|^2 \geqslant \frac{t_n}{\kappa'} \cdot \dot{a}_\ell(\tilde{v}_n) + \frac{\epsilon \cdot N_\ell(\tilde{v}_n, E_n)^2}{t_n^2}. \tag{50}$$

In particular, since $\dot{a} \geqslant 0$, we have $N(\tilde{v}_n, E_n)^2 \leqslant (t_n^2/\epsilon) \cdot \|\tilde{v}_n\|^2$. It follows that for each $w \in \mathcal{D}$,

$$|\widetilde{a}_t^\ell(\tilde{v}_n, w) - (E_n + 1) \cdot \langle \tilde{v}_n, w\rangle| \leqslant \frac{t_n}{\sqrt{\epsilon}} \cdot \|\tilde{v}_n\| \cdot \widetilde{a}_t^\ell(w)^{\frac{1}{2}}. \tag{51}$$

Fix $\delta > 0$ such that $[-\delta, \delta] + K \subset ((\ell\pi)^2, \infty)$. Set $I_n = [E_n - \delta, E_n + \delta]$ and $v_n = P_{a_t}^{I_n}(\tilde{v}_n)$. Note that $v_n = P_{\widetilde{a}_t^\ell}^{I_n+1}(\tilde{v}_n)$, where $I_n+1 := [E_n-\delta+1, E_n+\delta+1]$.

We now argue as in the proof of Lemma 2.3 in [HJ11]: The estimate (51) implies that

$$\sum_{i\geqslant 0} \frac{(\lambda_i - E_n + 1)^2}{\lambda_i} \left|\langle \tilde{v}_n, \psi_i^\ell\rangle\right|^2 \leqslant \frac{t_n^2}{\epsilon}\|\tilde{v}_n\|^2,$$

where $(\psi_i^\ell)_{i\geqslant 0}$ is a complete orthonormal set of eigenfunctions of a_t^ℓ (with corresponding eigenvalues λ_i). By retaining in the sum only the terms for which $\lambda_i \notin I_n' = [E_n + 1 - \delta, E_N + 1 + \delta]$, we find that

$$\widetilde{a}_t^\ell(\tilde{v}_n - v_n) \leqslant \frac{t_n^2}{\epsilon} \cdot \|\tilde{v}_n\|^2 \cdot \left(1 + \frac{E_n}{\delta}\right).$$

Observe that the sequences $(v_n)_{n\geqslant 1},\ (t_n)_{n\geqslant 1}$ and $(E_n)_{n\geqslant 1}$ depend on the initial choice of κ' but the preceding estimate gives a constant C that is independent of κ' such that

$$\widetilde{a}_t^\ell(\tilde{v}_n - v_n) \leqslant C \cdot t_n^2 \cdot \|\tilde{v}_n\|^2.$$

This implies, in particular, that $\|\tilde{v}_n - v_n\|^2 \leqslant C \cdot t_n^2 \cdot \|\tilde{v}_n\|^2$ so that, for n sufficiently large, we have $\|v_n\| \leqslant \|\tilde{v}_n\| \leqslant 2\|v_n\|$.

In equation (51) we replace the test function w by $P_{a_t^\ell}^{I_n}(w)$ and use that the spectral projector is self-adjoint and commutes with $\widetilde{a}_t^\ell$. We obtain that for each $w \in \mathcal{D}_\ell$,

$$\begin{aligned} |\widetilde{a}_t^\ell(v_n, w) - (E_n + 1) \cdot \langle v_n, w\rangle| &\leqslant \frac{t_n}{\sqrt{\epsilon}} \cdot \|\tilde{v}_n\| \cdot \widetilde{a}_t^\ell\left(P_{a_t^\ell}^{I_n}(w)\right)^{\frac{1}{2}} \\ &\leqslant C \cdot t_n \cdot \|v_n\|\|w\|, \end{aligned} \tag{52}$$

where we have used that $\|v_n\|$ is controlling $\|\tilde{v}_n\|$ and that

$$\widetilde{a}_t^\ell\left(P_{a_t^\ell}^{I_n}(w)\right) \leqslant (\sup(K) + \delta) \cdot \|w\|^2$$

by definition of a spectral projector.

Since $\dot{a}_t^\ell \leqslant \frac{2}{t} \cdot a_t^\ell$ and $\dot{a}_t^\ell$ is a nonnegative quadratic form, we also have

$$\begin{aligned} \left| \dot{a}_t^\ell(\tilde{v}_n)^{\frac{1}{2}} - \dot{a}_t^\ell(v_n)^{\frac{1}{2}} \right| &\leqslant \dot{a}_t^\ell(\tilde{v}_n - v_n)^{\frac{1}{2}} \\ &\leqslant \left(\frac{C}{t} a_t^\ell(\tilde{v}_n - v_n) \right)^{\frac{1}{2}} \\ &\leqslant C \cdot \sqrt{t} \cdot \|v_n\|. \end{aligned}$$

Equation (52) implies that we may use Proposition 7.1 to find

$$\int_1^\infty \left(\frac{E_n}{y^2} \ - \ (\ell\pi)^2 \right) |v_n(y)|^2 \ dy \geqslant \kappa \cdot \|v_n\|^2.$$

Since (52) also implies $\left| a_t^\ell(v_n) - E_n \|v_n\|^2 \right| \leqslant C \cdot t_n \cdot \|v_n\|^2$, using (49) we find that

$$t_n \cdot \dot{a}_t^\ell(v_n) \geqslant (\kappa - C \cdot t_n) \cdot \|v_n\|^2. \tag{53}$$

On the other hand, the contradiction assumption implies that

$$\begin{aligned} \kappa' \cdot \|v_n\|^2 &\geqslant t_n \cdot \dot{a}_t^\ell(\tilde{v}_n) \\ &\geqslant \left(\sqrt{t_n} \dot{a}_t^\ell(v_n)^{\frac{1}{2}} - C t_n \|v_n\|^2 \right)^2 \\ &\geqslant \left((\kappa - C \cdot t_n)^{\frac{1}{2}} - C t_n \right)^2 \cdot \|v_n\|^2 \\ &\geqslant (\kappa - C \cdot t_n) \, \|v_n\|^2. \end{aligned} \tag{54}$$

The implied constant C does not depend on κ', so if we take $\kappa' < \kappa$, then choosing t_n small enough yields the contradiction. □

This proposition yields an estimate for $\dot{a}(w)$ from below in terms of the projection $\Pi_{\ell<k} w$.

Corollary 7.3. *Let $k \in \mathbb{Z}^+$, and let $K \subset \mathbf{R}^+$ be a compact subset of $](\pi k)^2, \infty[$. For each $\varepsilon' > 0$, there exists $\kappa > 0$ $t_0 > 0$ such that if $E \in K$, $w \in \mathrm{dom}(a_t)$, and $t < t_0$, then*

$$\dot{a}_t(w) \geqslant \frac{\kappa}{t} \cdot \left(\|\Pi_{\ell<k}(w)\|^2 \ - \ \frac{\varepsilon'}{t^2} \cdot N(w, E)^2 \right), \tag{55}$$

where

$$N(w, E) = \sup_{v \in \mathrm{dom}(a_t)} \frac{|a_t(w, v) - E \cdot \langle w, v \rangle|}{\widetilde{a}_t(v)^{\frac{1}{2}}}.$$

Remark 7.4. The functional $v \mapsto N(v, E)$ is equivalent to the H^{-1}-norm of $(A_t - E)(v)$ where here A_t is the operator such that $\langle A_t u, v \rangle = a_t(u, v)$ for each $u, v \in \mathrm{dom}(a_t)$.

Proof of Corollary 7.3. Since $\dot{a}_t$ is block diagonal with respect to the sum $\bigoplus_\ell V_\ell$ and $\dot{a}_t^\ell \geqslant 0$, we have

$$\dot{a}_t(w) = \dot{a}_t\left(\sum_{\ell=0}^{\infty} w^\ell \otimes e_\ell\right) = \sum_\ell \dot{a}_t^\ell(w^\ell) \geqslant \sum_{\ell=0}^{k-1} \dot{a}_t^\ell(w^\ell).$$

We may apply Proposition 7.1 with $\varepsilon = \varepsilon'/k$ to each term on the right-hand side to find that

$$\dot{a}_t(w) \geqslant \frac{\kappa}{t} \cdot \left(\sum_{\ell=0}^{k-1} \|w^\ell\|^2 - \frac{\varepsilon'}{k \cdot t^2} \cdot \sum_{\ell=0}^{k-1} N_\ell(w^\ell, E)^2\right),$$

where κ is the minimum of the κ' coming from Proposition 47. For each ℓ and $v \in \mathcal{D}_\ell$, we have

$$\begin{aligned}\frac{|a_t^\ell(w^\ell, v) - E \cdot \langle w^\ell, v\rangle|}{\widetilde{a}_t^\ell(v)^{\frac{1}{2}}} &= \frac{|a_t(w^\ell \otimes e_\ell, v \otimes e_\ell) - E \cdot \langle w^\ell \otimes e_\ell, v \otimes e_\ell\rangle|.}{\widetilde{a}_t(v \otimes e_\ell)^{\frac{1}{2}}} \\ &= \frac{|a_t(w, v \otimes e_\ell) - E \cdot \langle w, v \otimes e_\ell\rangle|.}{\widetilde{a}_t(v \otimes e_\ell)^{\frac{1}{2}}},\end{aligned}$$

and hence $N_\ell(w^\ell, E) \leqslant N(w, E)$. We also have $\sum_{\ell<k} \|w^\ell\|^2 = \|\Pi_{\ell<k}(w)\|^2$, and the claim follows. □

7.2. *The spectral projection w_t.* The bounds proved in Section 7.1 depend on a bound on $N(w, E)$. In this subsection, we show that if w is an a_t-spectral projection of a q_t-eigenfunction in an interval containing the eigenvalue E, then $N(w, E)$ is of order t.

We start with a real-analytic eigenfunction branch u_t for q_t with associated real-analytic eigenvalue branch E_t. We let

$$w_t^I := P_{a_t}^I(u_t), \tag{56}$$

where we recall that $P_{a_t}^I$ denotes the spectral projector on the interval I that is associated to a_t. In the sequel, in arguments for which the interval I is fixed, this notation will be often abbreviated to w_t.

Let E_0 denote the limit of E_t as t tends to zero. The following two lemmas express the fact that the projection w_t^I is an order t quasimode for a_t at energy E_t.

The following lemma is similar to Lemma 2.3 in [HJ11].

LEMMA 7.5. *If I is a compact interval whose interior contains E_0, then there exist $t_0 > 0$ and C such that if $t < t_0$, then*

$$a_t\left(u_t - w_t^I\right) + \left\|u_t - w_t^I\right\|^2 \leqslant C \cdot t^2 \cdot \|u_t\|^2.$$

Proof. Using the fact that u_t is an eigenfunction of q_t and that a_t and q_t are asymptotic at first order, for each $w \in H_\beta^1$,

$$|a_t(u_t, w) - E_t\langle u_t, w\rangle| \leqslant C \cdot t \cdot \widetilde{a}_t(u_t)^{\frac{1}{2}} \widetilde{a}_t(w)^{\frac{1}{2}}. \tag{57}$$

Observe that letting $w = u_t$ yields that $a_t(u_t) \leqslant \frac{E_t}{1-Ct^2} \cdot \|u_t\|^2$. Moreover, the former equation can be rewritten as

$$\left|\widetilde{a}_t(u_t, w) - \tilde{E}_t\langle u_t, w\rangle\right| \leqslant C \cdot t \cdot \widetilde{a}_t(u_t)^{\frac{1}{2}} \cdot \widetilde{a}_t(w)^{\frac{1}{2}},$$

where $\tilde{E}_t := E_t + 1$. We may now follow the proof of Lemma 2.3 in [HJ11] observing that $P^I_{a_t} = P^{I+\{1\}}_{\widetilde{a}_t}$. This yields a constant C such that

$$\begin{aligned} a_t(u_t - w_t^I) + \|u_t - w_t^I\|^2 &\leqslant C \cdot t^2 \cdot \widetilde{a}_t(u_t) \\ &\leqslant \frac{C \cdot t^2}{1 - C \cdot t^2} \cdot \|u_t\|^2 \\ &\leqslant C' \cdot t^2 \cdot \|u_t\|^2. \end{aligned}$$

The claim follows. □

Remark 7.6. Lemma 7.5 implies that most of the mass of u_t lies in its projection, w_t^I, onto the energy interval I. More precisely, for t small, we have

$$(1 - C \cdot t) \cdot \|u_t\| \leqslant \left\|w_t^I\right\| \leqslant \|u_t\|, \tag{58}$$

where C is the constant in Lemma 7.5.

LEMMA 7.7. *If I is a compact interval whose interior contains E_0, then there exist $t_0 > 0$ and C such that if $t < t_0$, then*

$$N\left(w_t^I, E_t\right) \leqslant C \cdot t \cdot \left\|w_t^I\right\|.$$

Proof. For each $w \in H^1_\beta$, we have

$$a_t(w_t, w) = a_t(u_t, w) - a_t(u_t - w_t, w)$$

so that the Cauchy-Schwarz inequality and the preceding lemma imply

$$|a_t(w_t, w) - a_t(u_t, w)| \leqslant C \cdot t\|u_t\| a_t(w)^{\frac{1}{2}}.$$

We also have using Cauchy-Schwarz and the preceding lemma

$$\left|\langle u_t, w\rangle - \langle w_t^I, w\rangle\right| \leqslant C \cdot t \cdot \|u_t\|\|w\|.$$

We now start again from (57). First, in the bounding term, we have already seen that we could replace $\tilde{a}_t(u_t)^{\frac{1}{2}}$ by $C\|u_t\|$. Thus from the triangle inequality, (57), and the two preceding estimates, we obtain

$$\begin{aligned} \left|a_t(w_t^I, w) - E\langle w_t^I, w\rangle\right| &\leqslant C \cdot t\|u_t\| \cdot \left(a_t(w)^{\frac{1}{2}} + \|w\|\right) \\ &\leqslant C \cdot t \cdot \|u_t\| \cdot \widetilde{a}_t(w)^{\frac{1}{2}}. \end{aligned}$$

The claim follows using (58). □

Lemma 7.7 has the following corollary that expresses, in the language of semiclassical analysis, that w_t^I is an order t quasimode.

COROLLARY 7.8. *If I is a compact interval that contains E_0, there exist C and $t_0 > 0$ such that, for $t < t_0$ and each $v \in \mathrm{dom}(a_t)$, we have*

$$\left|a_t(w_t^I, v) - E_t\langle w_t^I, v\rangle\right| \leqslant C \cdot t \cdot \|w_t^I\| \cdot \|v\|.$$

Proof. Since $P_{a_t}^I$ is a spectral projector, we have

$$\left|a_t(w_t^I, v) - E_t\langle w_t^I, v\rangle\right| = \left|a_t(w_t^I, P_{a_t}^I v) - E_t\langle w_t^I, P_{a_t}^I v\rangle\right|,$$

and hence Lemma 7.7 implies

$$\left|a_t(w_t^I, v) - E_t\langle w_t^I, v\rangle\right| \leqslant C \cdot \|w_t^I\| \cdot \widetilde{a}_t(P_{a_t}^I(v))^{\frac{1}{2}}.$$

Since $\widetilde{a}_t(P_{a_t}^I(v))^{\frac{1}{2}} \leqslant (1 + \sup(I))^{\frac{1}{2}} \cdot \|v\|$, the claim follows. □

7.3. *Limits of eigenvalue branches.* By combining Lemma 7.7 with Corollary 7.3 we prove the following.

THEOREM 7.9 (Compare Theorem 13.1 [HJ11]). *Let (E_t, u_t) be an eigenbranch of q_t. Then there exists $k \in \mathbb{N}$ such that*

$$\lim_{t\to 0} E_t = (k \cdot \pi)^2. \tag{59}$$

Proof. Suppose to the contrary that E_0 is not of the form $(k \cdot \pi)^2$, where k is an integer. Let $n = \inf\{\ell \in \mathbb{N} \mid (\pi\ell)^2 > E_0\}$. Choose a compact interval $I \subset](n-1)^2\pi^2, n^2\pi^2[$ whose interior contains E_0.

Let u_t be a real-analytic eigenfunction branch of q_t associated to E_t. As before, let $w_t^I = P_{a_t}^I(u_t)$ be the projection of u_t onto the modes of a_t that have energy lying in I. Since I is fixed in the rest of this argument, we abbreviate the notation and simply write $w_t := w_t^I$.

If $\ell \geqslant n$, according to Lemma 6.4, the eigenvalues branches of a_t^n are increasing and limit to $n^2\pi^2$. It follows that for each t, each eigenvalue of a_t^ℓ is at least $(\pi n)^2$. Thus, since $\sup(I) < (\pi n)^2$, equation (42) implies that

$$\Pi_{\ell<n}(w_t) = w_t.$$

Let C be as in Lemma 7.7, and apply Lemma 7.3 with $\varepsilon' = 1/(2C^2)$ to obtain κ so that

$$\dot{a}_t(w_t) \geqslant \frac{\kappa}{2t} \cdot \|w_t\|^2.$$

It follows that $\dot{a}_t(w_t)/\|w_t\|^2$ is not integrable. This contradicts Theorem 4.2 of [HJ11], which we state below as Theorem 7.10. □

THEOREM 7.10 ([HJ11, Th. 4.2]). *Let q_t be asymptotic to a_t at first order, and suppose that for each $t > 0$, we have*

$$0 \leqslant \dot{a}_t(v) \leqslant t^{-1} \cdot a_t(v). \tag{60}$$

Let $t \mapsto E_t$ be a real-analytic eigenbranch of q_t that converges to a limit E_0 as t tends to 0, *and let V_t be the associated eigenspace.*

If $t \mapsto u_t \in V_t$ is continuous on the complement of a countable set, then the function

$$t \mapsto \frac{\dot{a}_t\left(P^I_{a_t}(u_t)\right)}{\left\|P^I_{a_t}(u_t)\right\|^2} \tag{61}$$

is integrable on each interval of the form $(0, t^*]$.

The next proposition will be the starting point of the contradiction argument in the following sections. It says that a cusp form eigenbranch cannot limit to 0. Heuristically, the zeroth Fourier coefficient of a cusp form vanishes identically whereas an eigenvalue branch that limits to 0 must eventually have nontrivial zeroth Fourier mode. However, because we have made a nontrivial change of variables, this fact requires an argument.

PROPOSITION 7.11. *If E_t is a real-analytic cusp form eigenvalue branch of q_t, then the integer k appearing in* (59) *is positive.*

Proof. Suppose to the contrary that $\lim_{t\to 0} E_t = 0$. Set $I = [0,1]$, and consider $w_t = w_t^I$ defined as in (56). If $\ell > 0$, the restriction of a to V_ℓ is bounded below by $\pi^2 > 1$, and thus we have $\Pi_0(w_t) = w_t$. On the other hand, the projection of u_t onto $\bigoplus_{\ell>0} V_\ell$ equals $u_t - u_t^0 \otimes 1$. Let v_t^0 denote the projection of $u_t - w_t$ onto V_0.

Since each V_ℓ is a direct sum of eigenspaces of a, we have

$$a_t(u_t - w_t) = a_t(v_t^0) + a_t(u - u_t^0 \otimes 1).$$

The quadratic form a is nonnegative, and the restriction of a_t to $\bigoplus_{\ell>0} V_\ell$ is bounded below by π^2. Hence $a(u_t - w_t) \geqslant \pi^2 \cdot \|u_t - u_t^0 \otimes 1\|^2$. By Lemma 7.5, we have

$$a_t(u_t - w_t) \leqslant C \cdot t^2 \cdot \|u_t\|^2.$$

Therefore, $\|u_t - u_t^0 \otimes 1\|^2 \leqslant C' \cdot t^2 \cdot \|u_t\|^2$, and hence

$$\|u_t^0\|^2_{L^2([1,\beta])} \geqslant (1 - C't^2) \cdot \|u_t\|^2 \tag{62}$$

for small t.

To obtain a contradiction, we will bound $\|u_t^0\|$ from above. Towards this end, we will compare u_t^0 with $\Phi_{0,t}(u_t)$; see Section 4.3. In particular, Lemma 4.7 implies that $\Phi_{0,t}(u_t)$ is a cusp form for $\mathcal{E}$, and hence for each $y \in [1, \beta]$,

$$\int_0^1 \eta(t \cdot x, y) \cdot u_t\left(x, b(t \cdot x, y)\right)\, dx = 0, \tag{63}$$

where $b(x,y) = B(\sqrt{1-x^2}, y)$ and $\eta(x,y) = (y/b(x,y)) \cdot \sqrt{\partial_y B(\sqrt{1-x^2}, y)}$. (See Section 4.2 for the definition of B.) Thus, for all $y \in [1,\beta]$, we have

$$\begin{aligned} u_t^0(y) &= \int_0^1 u_t(x,y)\,dx \\ &= \int_0^1 u_t(x,y) - \eta(t\cdot x,y) u_t(x, b(t\cdot x, y))\,dx \\ &= \int_0^1 (1-\eta(t\cdot x,y))\, u_t(x,y) dx \\ &\quad + \int_0^1 \eta(t\cdot x,y)\,[u_t(x,y) - u_t(x,b(t\cdot x,y))]\;dx. \end{aligned}$$

Let r_1 and r_2 denote, respectively, the two integrals on the right-hand side of this equation. Thus $u_t^0 = r_1 + r_2$, where we regard both r_1 and r_2 as functions of $y \in [1,\beta]$.

To bound $\|r_1\|$ and $\|r_2\|$, we will repeatedly use the following well-known fact: For each $f \in L^2([0,1])$, we have

$$\left| \int_0^1 f(x)\,dx \right|^2 \leqslant \int_0^1 |f(x)|^2\,dx.$$

Using the properties of B, one finds that there exists C such that

$$\sup\left\{ |\eta(t\cdot x, y) - 1|,\ (x,y) \in [0,1]\times[1,\beta] \right\} \leqslant C\cdot t$$

for small t. It follows that there exists a (perhaps different) constant C so that

$$\|r_1\|^2_{L^2([1,\beta])} \leqslant \int_1^\beta \int_0^1 |(1-\eta(t\cdot x,y))\,u_t(x,y)|^2\,dxdy \leqslant C\cdot t^2\cdot \|u_t\|^2. \tag{64}$$

Using the fundamental theorem of calculus and the Cauchy-Schwarz inequality we find that

$$|u_t\,(x, b(t\cdot x,y)) \ - \ u_t(x,y)|^2 \leqslant |b(t\cdot x,y) - y| \cdot \int_y^{b(t\cdot x,y)} |\partial_2 u_t\,(x,z)|^2\ dz,$$

where here ∂_2 denotes the derivative with respect to the second variable. Using the properties of B, we find that there exists C' so that

$$\sup\left\{ |b(t\cdot x,y) - y|\,,\ (x,y)\in[0,1]\times[1,\beta]\right\} \leqslant C'\cdot t$$

for small t. It follows that the interval $[y, b(t\cdot x,y)]$ is always a subset of the interval $[y - C'\cdot t, y + C'\cdot t]$, so that we obtain the bound

$$|u_t\,(x, b(t\cdot x,y)) \ - \ u_t(x,y)|^2 \leqslant C'\cdot t\cdot \int_{|y-z|\leqslant C'\cdot t} |\partial_2 u_t\,(x,z)|^2\ dz.$$

Thus we may estimate $\|r_2\|$ as follows:

$$\begin{aligned}\|r_2\|^2_{L^2([1,\beta])} &\leqslant \int_1^\beta \int_0^1 |\eta(t\cdot x,y)\,[u_t(x,y)-u_t(x,b(t\cdot x,y))]|^2\,dxdy \\ &\leqslant C\cdot t\int_1^\beta\int_0^1\int_{|y-z|\leqslant C'\cdot t}|\partial_2 u_t\,(x,z)|^2\;dzdxdy \\ &\leqslant C\cdot t^2\cdot\int_1^\beta\int_0^1|\partial_2 u_t\,(x,z)|^2\;dzdx.\end{aligned}$$

Here the constant C may vary from line to line. Since a_t and q_t are asymptotic at first order,

$$t^2\int_S|\partial_2 u_t\,(x,z)|^2\;dx\,dz\leqslant a_t(u_t)\leqslant(E_t+C\cdot t)\cdot\|u_t\|^2$$

for sufficiently small t. This yields

$$\|r_2\|_{L^2([1,\beta])}\leqslant C\,(E_t+C\cdot t)^{\frac12}\cdot\|u_t\|. \tag{65}$$

By combining estimates (64) and (65), we find that

$$\|u_t^0\|_{L^2([1,\beta])}\leqslant C\cdot(E_t+t)^{\frac12}\cdot\|u_t\| \tag{66}$$

for t small and some constant C. If E_t were to tend to zero as t tends to zero, then we would have a contradiction to (62) for small t. □

7.4. *Bounds on the first variation of the eigenvalue.* We can also use the nonconcentration of the spectral projection to give an $O(t^{-1})$ lower bound on the first variation, $\dot E_t$, when the projection of the eigenfunction onto the "small" modes is significant:

PROPOSITION 7.12. *Let I be a compact interval whose interior contains E_0 and $I\subset((k-1)^2\pi^2,(k+1)^2\pi^2)$, and let $w_t=P^I_{a_t}(u_t)$ (see (56)). For each $\delta>0$, there exists $\kappa'>0$ and $t_0>0$ so that if $t<t_0$ and*

$$\|\Pi_{\ell<k}(w_t)\|\geqslant\delta\cdot\|u_t\|,$$

then

$$\dot E_t\geqslant\frac{\kappa'}{t}. \tag{67}$$

Proof. Using the Cauchy-Schwarz inequality and the nonnegativity of $\dot a_t$, we have

$$\dot a_t(u_t)\geqslant\dot a_t(w_t)-\dot a_t(w_t)^{\frac12}\cdot\dot a_t(u_t-w_t)^{\frac12}. \tag{68}$$

It follows from (26) that for all $v\in\mathrm{Dom}(a_t)$,

$$\dot a_t(v)\leqslant 2t^{-1}\cdot a_t(v), \tag{69}$$

and hence $\dot a_t(w_t)^{\frac12}\leqslant\sqrt2\cdot t^{-\frac12}\cdot a(w_t)^{\frac12}\leqslant t^{-\frac12}\cdot(2\sup(I))^{\frac12}\cdot\|w_t\|$. Moreover, by combining this with Lemma 7.5 we find $\dot a_t(u_t-w_t)\leqslant Ct\|u_t\|^2$.

Thus, from (68) we obtain

$$\dot{a}_t(u_t) \geqslant \dot{a}_t(w_t) - C \cdot \|u_t\|^2.$$

Hence by applying Lemma 7.5, we have

$$\begin{aligned}\dot{E}_t \cdot \|u_t\|^2 &= \dot{q}(u_t) \\ &\geqslant \dot{a}_t(u_t) - C \cdot a(u_t) \\ &\geqslant \dot{a}_t(w_t) - C \cdot \|u_t\|^2 - C \cdot q(u_t) \\ &\geqslant \dot{a}_t(w_t) - C \cdot \|u_t\|^2\end{aligned} \tag{70}$$

for t sufficiently small.

As in the proof of Theorem 7.9, we have $\Pi_{\ell<k+1}(w_t) = w_t = \Pi_{\ell<k}(w_t) + \Pi_k(w_t)$. Since $\dot{a}_t$ is nonnegative and "block-diagonal," we have

$$\dot{a}_t(w_t) \geqslant \dot{a}_t(\Pi_{\ell<k}(w_t)).$$

Let C be as in Lemma 7.7, and apply Lemma 7.3 with $\varepsilon' = \delta^2/(2C^2)$ to obtain κ so that

$$\begin{aligned}\dot{a}_t(w_t) &\geqslant \frac{\kappa}{t}\left(\|\Pi_{\ell<k}(w_t)\|^2 - \frac{\delta^2}{2}\|w_t\|^2\right) \\ &\geqslant \frac{\kappa \cdot \delta^2}{2t} \cdot \|w_t\|^2\end{aligned}$$

for t sufficiently small. Estimate (58) implies that $\|w_t\|^2 \geqslant \frac{1}{2}\|u_t\|^2$ for t small, and therefore by combining the above inequalities, we prove the claim. □

In contrast to Proposition 7.12, we have the following.

LEMMA 7.13. *There exists $t_0 > 0$ and C such that if $t < t_0$, then*

$$\frac{\dot{E}_t}{E_t} \leqslant \frac{2}{t} + 3C.$$

Proof. It follows from (26) that for all $v \in \mathrm{Dom}(a_t)$,

$$\dot{a}_t(v) \leqslant 2t^{-1} \cdot a_t(v). \tag{71}$$

From (43), there exists C so that for sufficiently small t,

$$a_t(v) \leqslant (1 + C \cdot t) \cdot q_t(v)$$

and

$$\dot{q}_t(v) \leqslant \dot{a}_t(v) + C \cdot a_t(v).$$

Thus, if u_t is the real-analytic eigenfunction branch of q_t associated to E_t, then

$$\begin{aligned}\dot{E}_t \cdot \|u_t\|^2 &= \dot{q}_t(u_t)\\ &\leqslant \dot{a}_t(u_t) + C \cdot a_t(u_t)\\ &\leqslant (2 \cdot t^{-1} + C) \cdot a_t(u_t)\\ &\leqslant (2 \cdot t^{-1} + C) \cdot (1 + C \cdot t) \cdot q_t(u_t)\\ &= \left(2 \cdot t^{-1} + C\right) \cdot (1 + C \cdot t) \cdot E_t \cdot \|u_t\|^2.\end{aligned}$$

By choosing t_0 sufficiently small, we obtain the claim. □

8. Proof of the main theorem

Proposition 3.8 reduces the proof of Theorem 1.1 to the following.

THEOREM 8.1. *The family $t \mapsto q_t$ does not have a real-analytic cusp form eigenbranch.*

The proof of Theorem 8.1 will be by contradiction. We will assume that there exists a real-analytic cusp form eigenvalue branch, E_t. By the results of Section 7, we have

$$\lim_{t\to 0} E_t = (\pi \cdot k)^2, \tag{72}$$

where k is positive. We aim to contradict the positivity of k.

8.1. *Choosing β.* We recall that our construction of the quadratic form q_t depends on a parameter β, by forcing the zeroth Fourier coefficient of the elements of $\mathrm{dom}(q_t)$ to vanish for $y > \beta$.

The proof of Theorem 8.1 will rely on the estimates of solutions to ordinary differential equations made in Appendix B. To make the estimates less tedious, we will choose β to be sufficiently close to 1, where "sufficiently close" will be determined by the integer k that appears in (72).

However, the construction of the quadratic form q_t depends on β.[9] Therefore, the integer k that appears in (72) depends a priori on β. In order to avoid circularity of reasoning, we will prove the following.

PROPOSITION 8.2. *Let E_t be a real-analytic eigenvalue branch associated to a real-analytic cusp form eigenfunction branch of $t \mapsto q_t^{\beta}$. For each $\beta' > 1$, the family $t \mapsto q_t^{\beta'}$ has a real-analytic cusp form eigenfunction branch with associated eigenvalue branch E_t.*[10]

[9] We have suppressed this dependence from notation until now.

[10] The respective eigenfunction branches will not be the same if $\beta \neq \beta'$.

Proof. For each fixed t, since E_t corresponds to a cusp form, it belongs to the spectrum of $q_t^{\beta'}$ for all β'; see Lemma 2.7. By Theorem 3.3, there exists a real-analytic eigenvalue branch $s \mapsto \bar{E}_s^t$ of $s \mapsto q_s^{\beta'}$ such that $\bar{E}_t^t = E_t$. Since $s \mapsto q_s^{\beta'}$ has only countably many real-analytic eigenvalue branches, there exists some branch $s \mapsto \bar{E}_s^t$ such that the set of t' with $\bar{E}_{t'}^t = E_{t'}$ has an accumulation point. Thus, by real-analyticity, we have $\bar{E}_{t'}^t = E_{t'}$ for all t'.

If for each t, the dimension of the eigenspace V_t of $q_t^{\beta'}$ associated to E_t is greater than one, then one can argue as in the proof of Proposition 3.7 to obtain a real-analytic cusp form eigenfunction branch of $q_t^{\beta'}$ associated to E_t.

Otherwise, by real-analyticity, for each t in the complement of a discrete set A of t, we have $\dim(V_t) = 1$. Let $t \mapsto u_t^{\beta'}$ be a real-analytic eigenfunction branch of $q_t^{\beta'}$ associated to E_t.

Let $t \mapsto u_t^{\beta}$ denote a real-analytic eigenfunction branch of q_t^{β} associated to E_t. For each t, the pull-back of u_t by the diffeomorphism $\varphi_{0,t}^{\beta}$ is a cusp form of $\mathcal{E}$ on $\mathcal{T}_t$. In turn, for each t, the pull-back of $u_t \circ \varphi_{0,t}^{\beta}$ by $(\varphi_{0,t}^{\beta'})^{-1}$ is a cusp form eigenfunction of $q_t^{\beta'}$. Hence, the eigenfunction $u_t^{\beta'}$ is a cusp form for $t \notin A$. Thus, by Corollary 3.6, the branch $t \mapsto u_t^{\beta'}$ is a real-analytic cusp form eigenbranch of $q_t^{\beta'}$. $\square$

As a consequence of Proposition 8.2, we may fix β to satisfy[11]

$$1 < \beta < \frac{k}{k-1}. \tag{73}$$

It follows that for each $\ell < k$ and $y \in [1, \beta]$, we have

$$(\pi \cdot \ell)^2 - \frac{E_t}{y^2} < 0, \tag{74}$$

as soon as t is small enough.

In what follows, we will drop β from the notation for q_t^{β}.

8.2. *Tracking.* In this section we show that there exists a real-analytic eigenvalue branch of a_t^k, which we denote by λ_t^* such that $|\lambda_t^* - E_t|$ is at most of order t. In Section 8.4, we will show to the contrary that $|\lambda_t^* - E_t|$ is at least of order $t^{\frac{2}{3}}$. This will provide the desired contradiction.

THEOREM 8.3 (Tracking). *If E_t is a cusp form eigenvalue branch of q_t with positive limit $(k\pi)^2$, then there exist $t_0 > 0$, $C > 0$, and a real-analytic*

[11] This choice of β is most probably not necessary, but it will simplify the arguments in the appendix. In particular, it implies that on $[1, \beta]$ and for $\ell < k$, the Sturm-Liouville equations associated with a_t^{ℓ} have no turning point.

eigenvalue branch λ_t^* of a_t^k so that for each $t < t_0$,

$$\operatorname{spec}\left(a_t^k\right) \bigcap \ [E - Ct, E + Ct] = \{\lambda_t^*\}. \tag{75}$$

Proof. Let u_t denote a real-analytic eigenfunction branch of q_t associated to the eigenvalue branch E_t. Let $I \subset ((k-1)^2\pi^2, (k+1)^2\pi^2)$ be a compact interval that contains $k^2\pi^2$ in its interior. Let $w_t = P^I_{a_t}(u_t)$ be the associated spectral projection.

By Corollary 7.8 and the fact that Π_k is an orthogonal projection that commutes with a_t, there exist positive constants C_{qm} and t_1 so that if $t < t_1$ and $v \in \mathcal{D}$, then

$$|a_t\left(\Pi_k(w_t), \Pi_k(v)\right) - E_t \cdot \langle \Pi_k(w_t), \Pi_k(v)\rangle| \leqslant \frac{C_{qm}}{2} \cdot t \cdot \|w_t\| \cdot \|\Pi_k(v)\|. \tag{76}$$

Let G be the set of $t > 0$ such that $\operatorname{spec}(a_t^k) \cap [E_t - C_{qm} \cdot t, E_t + C_{qm} \cdot t]$ is nonempty. To prove the claim, it suffices to show that there exists $t_0 > 0$ so that $G \cap]0, t_0[=]0, t_0[$ and for each $t < t_0$, the intersection

$$\operatorname{spec}(a_t^k) \cap [E_t - C_{qm} \cdot t, E_t + C_{qm} \cdot t]$$

is a single point.

Let B denote closure of the complement of G, namely, the set of $t > 0$ so that the distance from E_t to the spectrum of a_t^k is at least $C_{qm} \cdot t$. For each $t \in B \cap]0, t_1[$, we apply a resolvent estimate to (76) and obtain $\|\Pi_k(w_t)\| \leqslant \|w_t\|/2$. Orthogonality then implies that for each $t \in B \cap]0, t_1[$, we have

$$\|\Pi_{\ell<k}\ w_t\| \geqslant \frac{\sqrt{3}}{2} \cdot \|w_t\|\,. \tag{77}$$

By estimate (58) we have $\|w_t\| \sim \|u_t\|$, and so we can apply Proposition 7.12 to find $\kappa > 0$ so that

$$\dot{E}_t \geqslant \kappa \cdot \frac{\mathbb{1}_B(t)}{t}, \tag{78}$$

where $\mathbb{1}_B$ is the indicator function for B.

We first observe that (78) implies that 0 is a limit point of G. Indeed, since $\lim_{\epsilon\to 0} E_\epsilon = E_0$, the fundamental theorem of calculus implies that for each positive integer n, the limit

$$\lim_{\epsilon \to 0} \int_\epsilon^{\frac{1}{n}} \dot{E}_t\, dt$$

exists and is finite. Thus, by (78), the limit

$$\limsup_{\epsilon\to 0} \int_\epsilon^{\frac{1}{n}} \frac{\mathbb{1}_B(t)}{t}\, dt$$

is finite. Therefore, since $1/t$ is not integrable on $]0, 1/n]$, the set $]0, 1/n] \cap G$ is nonempty for each n.

Next, we note that there exists a positive $t_2 \leqslant t_1$ such that if $t < t_2$ and $t \in G$, then $\operatorname{spec}(a_t^k) \cap [E_t - C_{qm} \cdot t,\, E_t + C_{qm} \cdot t]$ consists of a single point. This is a consequence of the *super-separation* phenomenon described in Theorem 10.4 of [HJ11]. We recall the exact statement here: *Let $(s_n)_{n=1}^{\infty}$ be a sequence of positive numbers that tends to zero as n tends to infinity. For each positive integer n, let λ_n^+ and λ_n^- be distinct eigenvalues of $a_{s_n}^k$. If $\lambda_n^{\pm}$ tends to $(\pi k)^2$ as n tends to ∞, then*

$$\lim_{n\to\infty} s_n^{-1} \cdot \left|\lambda_n^+ - \lambda_n^-\right| \; = +\infty.$$

The set $G \cap]0, t_2]$ is a disjoint union of intervals. Since 0 is a limit point of G, to prove the theorem it suffices to show that the number of intervals is finite. Since the eigenbranches of a_t^k are real-analytic for $t > 0$, the intervals can be enumerated $I_1, I_2, I_3, \ldots$ so that $\sup I_{j+1} < \inf I_j$ for each positive integer j. Because the spectrum of each a_t^k is discrete, simple, and nonnegative, the eigenvalue branches of a_t^k can be enumerated $\lambda^1, \lambda^2, \lambda^3, \ldots$ so that $\lambda_t^{\ell} < \lambda_t^{\ell+1}$ for each $t > 0$. By super-separation, for each interval I_j, there exists a unique positive integer $\ell(j)$ so that for each $t \in I_j$, we have

$$\operatorname{spec}(a_t^k) \cap [E_t - C_{qm} \cdot t,\, E_t + C_{qm} \cdot t] = \left\{\lambda_t^{\ell(j)}\right\}.$$

To finish the proof of the theorem, it suffices to show that the function $j \mapsto \ell(j)$ is strictly decreasing.

We claim that there exists $t_3 \leqslant t_2$ such that if $t \in \partial G$, $t < t_3$, and $\lambda_t = E_t \pm C_{qm} \cdot t$, then

$$\dot{\lambda}_t < \dot{E}_t - C_{qm} < \dot{E}_t + C_{qm}. \tag{79}$$

Indeed, if $t \in \partial G$, then $t \in B$, and so (78) gives that $\dot{E}_t \geqslant \kappa \cdot t$. Thus, if the claim were not true, then we would have a sequence t_n tending to zero such that $t_n \cdot \dot{\lambda}_{t_n}$ would be bounded below by a positive constant κ. But this would contradict Lemma 8.4 below.

Let $a < t_3$ be the left endpoint of an interval I_j. Then $a \in \partial G$, and hence estimate (79) implies that there exists $\epsilon > 0$ so that if $a - \epsilon < t < a$, then $\lambda_t^{\ell(j)} > E_t + C_{qm} \cdot t$. Let b be the infimum of $s < a$ such that for all $t \in]b, a[$, we have $\lambda^{\ell(j)} > E_t + C_{qm} \cdot t$. We cannot have $b > 0$, because then $b \in \partial G$ and estimate (79) would give a contradiction. Therefore, $\lambda_t^{\ell(j)} > E_t + C_{qm} \cdot t$ for all $t < a$; see Figure 3. Thus, the branch $\lambda^{\ell(j+1)}$ must lie below the branch $\lambda^{\ell(j)}$. That is, $j \mapsto \ell(j)$ is strictly decreasing as desired. □

LEMMA 8.4. *Let $k > 0$, and let t_n be a sequence converging to zero. For each $n \in \mathbf{Z}^+$, let $t \mapsto \lambda_n(t)$ be a real-analytic eigenbranch of the family a_t^k. If $\lim_{n\to\infty} \lambda_n(t_n) = (\pi k)^2$, then*

$$\lim_{n\to\infty} t_n \cdot \dot{\lambda}_n(t_n) \; = 0.$$

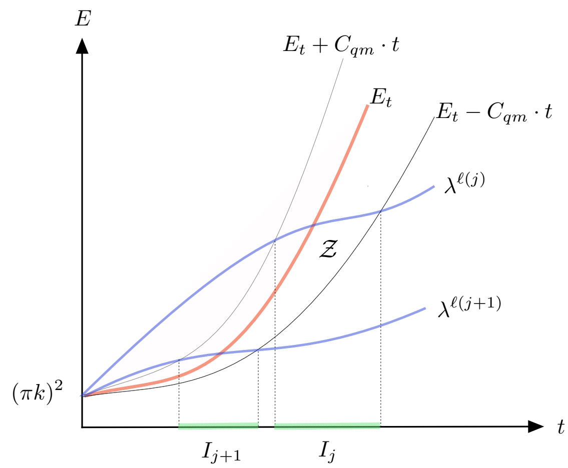

Figure 3. Let $\mathcal{Z}$ denote the set of (t, E) such that $E \in [E_t - C_{qm} \cdot t, E_t + C_{qm}]$. As t decreases from t_3 to 0, each eigenvalue branch a_t^k that enters $\mathcal{Z}$ must enter from below, and if it exits $\mathcal{Z}$, then it must exit from above.

Remark 8.5. One may replace the assumption in Lemma 8.4 that $t \mapsto \lambda_n(t)$ is real-analytic with the assumption that $t \mapsto \lambda_n(t)$ continuous. Indeed, for each $t > 0$, the eigenvalue problem for a_t^k corresponds to an eigenvalue problem for an ordinary differential equation, and hence the eigenvalues are simple. It follows that each continuous eigenvalue branch $t \mapsto \lambda_n(t)$ of the real-analytic family a_t^k is necessarily real-analytic.

Remark 8.6. Lemma 8.4 is not a direct consequence of Lemma A.1 because the eigenvalue branch λ_n may vary with n. By keeping track of the constants in the proof of Lemma A.1, one can produce a version that directly implies Lemma 8.4. We prefer to give a direct proof here.

Proof. For each n, we denote by $y \mapsto \psi_n(y)$ a unit norm eigenfunction of $a_{t_n}^k$ with eigenvalue $\lambda_n(t_n)$. By the standard variational formula and (35),

$$\dot{\lambda}_n(t_n) = \dot{a}_{t_n}(\psi_n(y)) \; = 2t_n \cdot \int_1^\infty \left|\psi_n'(y)\right|^2 \, dy.$$

Since ψ_n is an eigenfunction of $a_{t_n}^k$ with eigenvalue $\lambda_n(t_n)$, using (35), we have

$$t_n^2 \cdot \int_1^\infty \left|\psi_n'(y)\right|^2 \, dy = \int_1^\infty \left(\frac{\lambda_n(t_n)}{y^2} - (\pi k)^2\right) |\psi_n(y)|^2 \, dy. \tag{80}$$

It suffices to show that the right-hand side of (80) tends to zero as n tends to infinity.

Let $\epsilon > 0$. Since $\lambda_n(t_n)$ tends to $(\pi k)^2$, there exists $\delta > 0$ so that if $|y-1| < \delta$, then $\left(\lambda_n(t_n) \cdot y^{-2} - (\pi k)^2\right) < \epsilon/2$ and thus

$$\int_1^{1+\delta} \left(\frac{\lambda}{y^2} - (\pi k)^2\right) |\psi_n(y)|^2 \, dy \leqslant \frac{\epsilon}{2} \cdot \int_1^{1+\delta} |\psi_n(y)|^2 \, dy. \tag{81}$$

To estimate the remaining integral over $[1+\delta, \infty)$, we will apply a standard convexity estimate from the theory of ordinary differential equations.[12] If n is sufficiently large, $\lambda_n(t_n)/(\pi k)^2 < (1+\delta/4)^2$, and hence there exists $\eta > 0$ so that if $y > z > 1+\delta/2$, then

$$|\psi_n(y)|^2 \leqslant |\psi_n(z)|^2 \cdot \exp\left(-\frac{\eta}{t} \cdot (y-z)\right).$$

It follows that there exists t_0 so that if $t < t_0$, then

$$\int_{1+\delta}^{\infty} |\psi_n(y)|^2 \, dy \leqslant \frac{\epsilon}{4(\pi k)^2} \cdot \int_1^{\infty} |\psi_n(y)|^2 \, dy. \tag{82}$$

Since $\left|\lambda_n(t_n)/y^2 - (\pi k)^2\right| \leqslant (\pi k)^2$ for sufficiently large n, we may combine (82) with (81) to show that (80) is less than the given ϵ for sufficiently large n. □

Notation. For the convenience of the reader, we recall the notation that is being used in the proof of the main theorem. We are considering an eigenbranch (u_t, E_t) of q_t such that $\lim_{t\to 0} E_t = k^2\pi^2 > 0$. We have chosen a compact interval $I \subset](k-1)^2\pi^2, (k+1)^2\pi^2[$ that contains $k^2\pi^2$ in its interior. We have set $w_t = P_t^I u_t$, where P_t^I is the spectral projector on I associated with a_t. Finally, w_t^k is the orthogonal projection of w_t onto V_k so that there exists $v_t^k \in L^2((1,+\infty), y^{-2}dy)$ such that $w_t^k = v_t^k \otimes e_k$.

8.3. *Crossings.* In this subsection, we show that $\|w_s^k\|$ is smaller than $\|u_s\|$ for s near a *crossing time*, a value of the parameter t such that E_t belongs to the spectrum of a_t^0. Then we show that there exists a sequence of crossing times t_n and intervals of width $O(t^{\frac{8}{3}})$ about the crossing times on which $\|w_t^k\|$ is smaller than $\rho\|u_t\|$ for some fixed ρ.

The proof of the first result depends on the analysis contained in Appendix B that provides estimates on the *off-diagonal part* of the quadratic form b_t which, we recall, has been defined in (27) in such a way that it is the leading part of $q_t - a_t$; see Proposition 6.2.

PROPOSITION 8.7. *Given $\rho < 1$, there exists $\eta > 0$ and $t_0 > 0$ such that if $t < t_0$ and*

$$\operatorname{dist}\left(E_t, \operatorname{spec}\left(a_t^0\right)\right) \leqslant \eta \cdot t^{\frac{5}{3}}, \tag{83}$$

[12] See, for example, Lemma 6.3 in [HJ11].

then

$$\left\|w_t^k\right\| \leqslant \rho \cdot \|u_t\|.$$

Proof. Let ψ_t^0 be an eigenfunction of a_t^0 with eigenvalue λ_t^0 satisfying $|E_t - \lambda_t^0| < \eta \cdot t^{\frac{5}{3}}$. We have

$$(E_t - \lambda_t^0) \cdot \langle u_t, \psi_t^0 \rangle = (a_t - q_t)(u_t, \psi_t^0) = t \cdot b_t(u_t, \psi_t^0) + O(t^2) \cdot \|u_t\| \cdot \|\psi_t^0\|$$

and hence

$$\left|E_t - \lambda_t^0\right| \cdot \left|\langle u_t, \psi_t^0 \rangle\right| \;\geqslant\; t \cdot \left|b_t(u_t, \psi_t^0)\right| \;-\; O(t^2) \cdot \|u_t\| \cdot \|\psi_t^0\|. \tag{84}$$

In Appendix B, Proposition B.1, we prove that there exists $\kappa > 0$ so that

$$\left|b_t(u_t, \psi_t^0)\right| \;\geqslant\; \kappa \cdot t^{\frac{2}{3}} \cdot \left(\|w_t^k\| - t^\delta \cdot \|u_t\|\right) \cdot \|\psi_t^0\|.$$

Hence by applying the Cauchy-Schwarz inequality to the left-hand side of (84), we find that

$$\left|E_t - \lambda_t^0\right| \|u_t\| \cdot \|\psi_t^0\| \;\geqslant\; \left(\kappa \cdot t^{\frac{5}{3}} \cdot \left(\|w_t^k\| - t^\delta \cdot \|u_t\|\right) - O(t^2)\|u_t\|\right) \cdot \|\psi_t^0\|.$$

Let $\eta = \rho \cdot \kappa/2$, and use (83) to find that

$$\frac{\rho}{2} \cdot \|u_t\| \geqslant \|w_t^k\| \;-\; O(t^\delta) \cdot \|u_t\| \;-\; O(t^{\frac{1}{3}}) \cdot \|u_t\|.$$

The claim follows by choosing t_0 sufficiently small. $\square$

PROPOSITION 8.8. *For all $\eta > 0$, let $\delta = \frac{\eta}{12(\pi k)^2}$. There exists $s_0 > 0$ such that if $s < s_0$, $E_s \in \operatorname{spec}(a_s^0)$, and $t \in \left[s, s + \delta \cdot s^{\frac{8}{3}}\right]$, then*

$$\operatorname{dist}\left(E_t, \operatorname{spec}(a_t^0)\right) \leqslant \eta \cdot s^{\frac{5}{3}}.$$

Proof. Let λ_t^0 be the eigenvalue branch of a_t^0 such that $E_s = \lambda_s^0$. By Lemma 6.3, we have $\lambda_t^0 = c \cdot t^2$ for some $c > 0$, and hence $\dot{\lambda}_t^0 = 2 \cdot t^{-1} \cdot \lambda_t^0$. Using the fact that a_t and q_t are asymptotic and the fact that $\dot{a}_t$ is non negative, there exists a constant C such that $\dot{E}_t \geqslant -CE_t$ for all sufficiently small t. Thus, for even smaller t we obtain

$$\frac{d}{dt} \ln\left(\frac{\lambda_t^0}{E_t}\right) \leqslant 3 \cdot t^{-1}.$$

Since $E_s = \lambda_s$, integration over $[s, t]$ and exponentiation gives

$$\frac{\lambda_t^0}{E_t} \leqslant \left(\frac{t}{s}\right)^3. \tag{85}$$

If $t \leqslant s + \delta \cdot s^{\frac{8}{3}}$, then

$$\left(\frac{t}{s}\right)^3 \leqslant \left(1 + \delta \cdot s^{\frac{5}{3}}\right)^3 \;\leqslant 1 + 4 \cdot \delta \cdot s^{\frac{5}{3}},$$

where the last inequality holds for $s \leqslant s_1 = (2\delta)^{-\frac{3}{5}}$. By combining this with (85), one finds that for $t \in \left[s, s + \delta \cdot s^{\frac{8}{3}}\right]$, we have

$$\lambda_t^0 - E_t \leqslant E_t \cdot 4 \cdot \delta \cdot s^{\frac{5}{3}}. \tag{86}$$

Using Lemma 7.13 and $\dot{\lambda} \geqslant 0$, we have that

$$\frac{d}{dt} \ln\left(\frac{E_t}{\lambda_t^0}\right) \leqslant 3 \cdot t^{-1}. \tag{87}$$

An argument similar to the one above gives that

$$E_t - \lambda_t^0 \leqslant \lambda_t^0 \cdot 4 \cdot \delta \cdot s^{\frac{5}{3}}$$

for $t \in \left[s, s + \delta \cdot s^{\frac{8}{3}}\right]$.

Since by assumption, $\lim_{t \to 0} E_t = (\pi k)^2$, there exists s_2 so that if $t < s_2 + \delta \cdot s_2^{\frac{8}{3}}$, then $E_t \leqslant 2 \cdot (\pi k)^2$. Thus, by (86), we have that $\left\{\lambda_t^0 \mid s \leqslant t \leqslant s + \delta \cdot s^{\frac{8}{3}}\right\}$ is bounded above by $3(\pi k)^2$ for $s < s_0 = \min\{s_1, s_2\}$. In sum, if $s < s_0$ and $t \in \left[s, s + \delta \cdot s^{\frac{8}{3}}\right]$, then

$$\left|\lambda_t^0 - E_t\right| \leqslant 12(\pi k)^2 \cdot \delta \cdot s^{\frac{5}{3}} \ \leqslant \ \eta \cdot s^{\frac{5}{3}}$$

by the choice of δ. □

We wish to estimate from below the size of the set of t for which (83) holds true. This is accomplished by the following proposition.

PROPOSITION 8.9. *Let $\delta > 0$. There is a sequence t_n of crossing times such that*

$$\lim_{n \to \infty} n \cdot t_n = k \cdot \ln(\beta). \tag{88}$$

If $n \neq m$ are large enough, then the intervals

$$\left[t_n, t_n + \delta \cdot t_n^{\frac{8}{3}}\right] \ \text{and} \ \left[t_m, t_m + \delta \cdot t_m^{\frac{8}{3}}\right]$$

are disjoint.

Proof. By Lemma A.1, there exists, $\nu^* > 0$ so that $\lambda_t^* = (\pi k)^2 + \nu^* \cdot t^{\frac{2}{3}} + O\left(t^{\frac{4}{3}}\right)$. It also follows from Proposition 8.3 that there exists M so that

$$(\pi k)^2 + \nu^* \cdot t^{\frac{2}{3}} - M \cdot t < E_t < (\pi k)^2 + \nu^* \cdot t^{\frac{2}{3}} + M \cdot t \tag{89}$$

for sufficiently small t. By Lemma 6.3, the eigenvalues of a_t^0 have the form $c_n \cdot t^2$, where $c_n = (1/4 + r_n^2)$ and r_n is the increasing sequence of positive solutions to the equation $2r = \tan(r \ln(\beta))$. Standard asymptotic analysis shows that

$$r_n = \frac{n\pi + \frac{\pi}{2}}{\ln(\beta)} + o(1). \tag{90}$$

Fix $0 < \nu_0^- < \nu^* < \nu_0^+$. For each $\nu \in [\nu_0^-, \nu_0^+]$ and each $n \in \mathbb{Z}$, there exists a unique $t_n^\nu \in \mathbb{R}^+$ so that

$$c_n \cdot (t_n^\nu)^2 = (\pi k)^2 + \nu \cdot (t_n^\nu)^{\frac{2}{3}} \,. \tag{91}$$

We drop the dependence in ν from the notation for a moment. If we set $x_n = c_n^{-\frac{1}{6}}$ and $y_n = c_n^{\frac{1}{6}} \cdot t_n^{\frac{1}{3}}$, then (91) becomes

$$y_n^6 = (\pi \cdot k)^2 + \nu \cdot x_n^2 \cdot y_n^2.$$

The analytic (polynomial) function F defined by $F(x, y) = y^6 - (\pi \cdot k)^2 - \nu \cdot x^2 \cdot y^2$ satisfies $F(0, (\pi k)^{\frac{2}{3}}) = 0$ and $\partial_y F(0, (\pi k)^{\frac{2}{3}}) \neq 0$. Thus, by the analytic implicit function theorem, for x near 0, there exists a unique analytic function $Y(x)$ so that

$$Y(x)^6 = (\pi \cdot k)^2 + \nu \cdot x^2 \cdot Y(x)^2.$$

By inspecting the first few coefficients in the Taylor expansion of Y^3, we find that

$$Y(x)^3 = \pi \cdot k + \frac{\nu}{2 \cdot (\pi k)^{\frac{1}{3}}} \cdot x^2 + O(x^3).$$

Thus, since $\lim_{n\to\infty} x_n = 0$ and $t_n = c_n^{-\frac{1}{2}} \cdot Y^3(c_n^{-\frac{1}{2}})$, we find that

$$t_n^\nu = (\pi k) \cdot c_n^{-\frac{1}{2}} + \tau \cdot c_n^{-\frac{5}{6}} + O(c_n^{-1}), \tag{92}$$

where $\tau = \nu \cdot (\pi k)^{\frac{1}{3}}/2$.

Choose $\epsilon > 0$ so that if $\nu^\pm = \nu^* \pm \epsilon$, then $\nu_0^- \leqslant \nu^- < \nu^* < \nu^+ \leqslant \nu_0^+$. Define $t_n^\pm = t_n^{\pm\nu}$. By applying the intermediate value theorem to $\lambda_t - E_t$, there exists $t_n \in \,]t_n^-, t_n^+[$ so that $E_{t_n} = \lambda_{t_n}$; see Figure 4. Since c_n is increasing to infinity, the sequence t_n is decreasing to zero.

Moreover, since $\nu^\pm = \nu^* \pm \varepsilon$,

$$t_n = (\pi k) \cdot c_n^{-\frac{1}{2}} + \tau^* \cdot c_n^{-\frac{5}{6}} + o(c_n^{-\frac{5}{6}}), \tag{93}$$

where $\tau^* = \nu^* \cdot (\pi k)^{\frac{1}{3}}/2$. From (90) we have

$$c_n^{-1} = \frac{\sigma^2}{n^2} \cdot \left(1 + O\left(\frac{1}{n}\right)\right),$$

where $\sigma = \ln(\beta)/\pi$. By substituting this into (92) and (93) we find that

$$t_n^\pm = (\pi k) \cdot n^{-1} + \tau \cdot \sigma^{\frac{5}{3}} \cdot n^{-\frac{5}{3}} + O_\pm(n^{-2})$$

and

$$t_n = (\pi k) \cdot n^{-1} + \tau^* \cdot \sigma^{\frac{5}{3}} \cdot n^{-\frac{5}{3}} + o(n^{-\frac{5}{3}}).$$

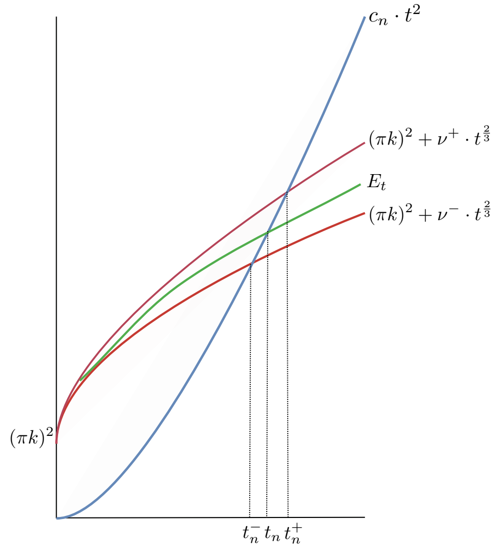

Figure 4. The crossing t_n.

The first claim follows. Moreover, since $\nu^{\pm} = \nu^* \pm \varepsilon$, we have

$$
\begin{aligned}
t_n^+ - t_n &\sim \varepsilon \cdot \frac{(\pi k)^{\frac{1}{3}}}{2} \cdot n^{-\frac{5}{3}}, \\
t_n - t_n^- &\sim \varepsilon \cdot \frac{(\pi k)^{\frac{1}{3}}}{2} \cdot n^{-\frac{5}{3}}, \\
t_n^- - t_{n+1}^+ &\sim \varepsilon \cdot (\pi k)^{\frac{1}{3}} \cdot n^{-\frac{5}{3}}, \\
t_n^{\frac{8}{3}} &= O(n^{-\frac{8}{3}}).
\end{aligned}
$$

It follows that, for all sufficiently large n, we have $[t_n, t_n + \delta \cdot t_n^{\frac{8}{3}}] \subset [t_n^-, t_n^+]$. Since the intervals $\{[t_n^-, t_n^+]\}$ are disjoint, the claim is proven. $\square$

8.4. *Relative variation and the contradiction.* In this section we derive the desired contradiction. In particular, we prove the following.

THEOREM 8.10. *Suppose that E_t is a cusp form eigenvalue branch with a positive limit. If λ_t^* is the eigenvalue branch of a_t that satisfies* (75), *then there exists $t_0 > 0$ and $c > 0$ so that if $t < t_0$, then*

$$E_t - \lambda_t^* > c \cdot t^{\frac{2}{3}}.$$

The proof will consist of two types of lower estimates. The first depends on the fact that near each crossing the "relative variation" $\dot{E}_t - \dot{\lambda}_t^*$ is at least of order $O(t^{-1})$. The second shows that away from the crossings the relative variation is not too negative.

Define

$$K(t,\rho) = \left\{ s \in\,]0,t] \;\middle|\; \left\| w_s^k \right\| \leqslant \rho \cdot \|u_s\| \right\}.$$

If $\rho < 1$, then it follows from Proposition 7.12 that there exists $\kappa > 0$ so that for $s \in K(t,\rho)$, we have

$$\dot{E}_s \geqslant \kappa \cdot s^{-1}. \tag{94}$$

Hence, since $\dot{\lambda}_t^* = O(t^{-\frac{1}{3}})$, there exists $t^* > 0$ so that if $t < t^*$ and $\rho < 1$, then

$$\dot{E}_s - \dot{\lambda}_s^* \geqslant \frac{\kappa}{2} \cdot s^{-1} \tag{95}$$

for each $s \in K(t,\rho)$. We will integrate this estimate near the crossings to obtain the following.

LEMMA 8.11. *For each $\rho < 1$, there exists $t_0 > 0$ and $\gamma(\rho) > 0$ so that for each $t < t_0$, we have*

$$\int_{K(t,\rho)} \left(\dot{E}_s - \dot{\lambda}_s \right) \, ds \geqslant \gamma(\rho) \cdot t^{\frac{2}{3}}.$$

Proof. By (95), the integrand is positive on $K(t,\rho)$; it suffices to show the same estimate holds for a subset G of $K(t,\rho)$.

To define this subset, we first combine Propositions 8.7, 8.8, and 8.9 to find $\delta > 0$, $N' \geqslant 2$, and a monotone sequence $\{t_n\}$ so that for each n,

$$I_{n,\delta} = \left[t_n, t_n + \delta \cdot t_n^{\frac{8}{3}} \right]$$

belongs to $K(1/2,\rho)$, the intervals I_n and I_{n+1} are disjoint, and for each $n \geqslant N'$,

$$\frac{\tau}{2n} \leqslant t_n \leqslant \frac{2\tau}{n}, \tag{96}$$

where $\tau = k \cdot \ln(\beta)$. The subset G will be defined as a union of I_n over sufficiently large n.

We have $\int_{I_{n,\delta}} s^{-1} ds = \ln(1 + \delta \cdot t_n^{\frac{5}{3}})$, and hence there exists $N^* \geqslant N'$ so that if $n \geqslant N^*$, we have

$$\int_{I_{n,\delta}} s^{-1} \, ds \geqslant \frac{\delta}{2} \cdot t_n^{\frac{5}{3}}. \tag{97}$$

Thus, from (96) we find that if $N \geqslant N^*$, then

$$\left(\frac{2}{\tau}\right)^{\frac{5}{3}} \sum_{n \geqslant N+2} t_n^{\frac{5}{3}} \geqslant \sum_{n \geqslant N+2} n^{-\frac{5}{3}} \geqslant \int_{N+2}^{\infty} x^{-\frac{5}{3}} \, dx = (N+2)^{-\frac{2}{3}} \geqslant \left(\frac{t_N}{4\tau}\right)^{\frac{2}{3}}. \tag{98}$$

Since the intervals $I_{n,\delta}$ are disjoint, by combining (95), (97), and (98), we find that

$$\int_{G_N} \left(\dot{E}_s - \dot{\lambda}_s^* \right) \, ds \geqslant \gamma \cdot t_N^{\frac{2}{3}}, \tag{99}$$

where $\gamma = \kappa \cdot \delta \cdot \tau \cdot 2^{-\frac{16}{3}}$ and

$$G_N := \bigcup_{n \geqslant N+2} I_{n,\delta}.$$

Let $t_0 = t_{N^*}$. If $t < t_0$, then $t \in [t_{N+1}, t_N]$ for some $N \geqslant N^*$. We have $t_{N+2} + t_{N+2}^{\frac{8}{3}} \leqslant t_{N+1} \leqslant t$, and hence $G_N \subset K(t, \rho)$ and $t_N^{\frac{2}{3}} \geqslant t^{\frac{2}{3}}$. Therefore, (99) implies the claim. $\square$

To bound the relative variation on the complement of $K(t, \rho)$, we will use the following.

PROPOSITION 8.12. *There exists C and $t_0 > 0$ such that, if $t < t_0$, then*

$$\dot{E}_t \ \geqslant \ \frac{\left\|w_t^k\right\|^2}{\|u_t\|^2} \cdot \dot{\lambda}_t^* - C \cdot t^{-\frac{1}{9}}. \tag{100}$$

Proof. By arguing as in (70), we have

$$\dot{E}_t \cdot \|u_t\|^2 \geqslant \dot{a}_t\left(w_t^k\right) \ - \ O\left(\|u_t\|^2\right). \tag{101}$$

Let w_t^* denote the orthogonal projection of w_t^k onto the eigenfunction branch of a_t that corresponds to λ_t^* from Theorem 8.3. Let $w_t^\perp := w_t^k - w_t^*$. For $\diamond = k, *, \perp$, we define $v_t^\diamond$ so that $w_t^\diamond = v_t^\diamond \otimes e_k$. Observe that by definition, v_t^* is an eigenfunction of a_t^k with eigenvalue λ_t^*.

Using the Cauchy-Schwarz inequality and the nonnegativity of $\dot{a}_t$, we have

$$\dot{a}_t(w_t^k) \ = \ \dot{a}_t^k\left(v_t^k\right) \geqslant \dot{a}_t^k\left(v_t^*\right) - 2 \cdot \left|\dot{a}_t^k\left(v_t^*, v_t^\perp\right)\right|. \tag{102}$$

Since v_t^* is an eigenfunction, we have $\dot{a}_t^k\left(v_t^*\right) = \dot{\lambda}^* \cdot \|v_t^*\|^2$. Using (35) and the fact that v_t^* is an eigenfunction that is orthogonal to $v_t^\perp$, we find that

$$\begin{aligned} \dot{a}_t^k(v_t^*, v_t^\perp) &= 2t^{-1} \cdot \left(a_t^k(v_t^*, v_t^\perp) - (\pi k)^2 \int_1^\infty v_t^*(y) \cdot v_t^\perp(y)\, dy\right) \\ &= -2(\pi k)^2 \cdot t^{-1} \int_1^\infty v_t^*(y) \cdot v_t^\perp(y)\, dy. \end{aligned} \tag{103}$$

Since $\langle v_t^*, v_t^\perp \rangle = 0$, we have

$$\int_1^\infty v_t^* \cdot v_t^\perp \, dy \ = \int_1^\infty v_t^* \cdot v_t^\perp \cdot (1 - y^{-2})\, dy.$$

The large y asymptotics of v_t^* and $v_t^\perp$ can be analyzed using the same methods as in Appendix B for v_t^k. We thus define $r_t^\diamond$, for $\diamond = k, *, \perp$, by

$$r_t^\diamond \ = \ t^2 (v_t^\diamond)'' + \left(\frac{E_t}{y^2} - k^2\pi^2\right) v_t^\diamond,$$

so that Proposition B.7 gives

$$\int_{1+2t^\alpha}^{\infty} |v_t^\diamond(y)|^2 \, dy \leqslant C \cdot \left(t^{-2\alpha} \cdot \int_1^\infty |r_t^\diamond|^2 \, + \exp\left(-t^{\frac{3\alpha-2}{2}}\right) \cdot \int_{1+t^\alpha}^\infty |v_t^\diamond(y)|^2 \, y^{-2} dy \right). \tag{104}$$

We can now estimate $r_t^\diamond$ with the same techniques as in Lemma B.5 : we test again a smooth function ϕ to obtain

$$\int_1^\infty r_t^\diamond(y)\phi(y) \; = \; -a_t^k(v_t^\diamond, \phi) + E_t \cdot \langle v_t^\diamond, \phi \rangle.$$

We now observe that $v_t^\diamond \otimes e_k = P_t^\diamond w_t$, where $P_t^\diamond$ is some spectral projector associated with a_t. Arguing as in Corollary 7.8, we thus obtain

$$\left| \int_1^\infty r_t^\diamond(y)\phi(y) \, dy \right| \; \leqslant \; C' \cdot t \cdot \|w_t\| \cdot \|\phi\|.$$

This now implies (see the proof of Lemma B.5)

$$\int_1^\infty |r_t^\diamond(y)|^2 \; dy \; \leqslant C \cdot t^2 \cdot \|w_t\|^2.$$

We plug this estimate into (104) (see also the proof of Corollary B.8) to obtain that, for each $\alpha < \frac{2}{3}$, there exists a constant C so that for sufficiently small t,

$$\int_{1+2t^\alpha}^\infty |v_t^*|^2 \, dy \leqslant C \cdot t^{2-2\alpha} \cdot \|w_t\|^2$$

and

$$\int_{1+2t^\alpha}^\infty |v_t^\perp|^2 \, dy \leqslant C \cdot t^{2-2\alpha} \cdot \|w_t\|^2.$$

If $y \leqslant 1 + 2t^\alpha$, then $(1 - y^{-2}) \leqslant 4t^\alpha$ for sufficiently small t. Therefore, by splitting the domain of integration into $[1, 1 + 2t^\alpha]$ and $[1 + 2t^\alpha, \infty[$ and using the Cauchy-Schwarz inequality, we find that

$$\left| \int_1^\infty v_t^* \cdot v_t^\perp \, dy \right| \; \leqslant 5 \cdot t^\alpha \cdot \|v_t^*\| \cdot \left\| v_t^\perp \right\| + C \cdot t^{2-2\alpha} \cdot \|w_t\|^2 \tag{105}$$

for sufficiently small t.

We claim that $\|v_t^\perp\| = O(t^{\frac{1}{3}}) \cdot \|w_t\|$. Indeed, by applying Lemma 7.7 with $v \in V_k$, we have

$$|a_t^k(v_t^k, v) - E \cdot \langle v_t^k, v \rangle| \leqslant C \cdot t \cdot \|w_t\| \cdot \|v\|$$

for some constant C. Thus, since the eigenvalue λ^* satisfies $|E_t - \lambda^*| < C' \cdot t$, we find that

$$|a_t^k(v_t^\perp, v) \; - E \; \cdot \langle v_t^\perp, v \rangle| \leqslant 2C \cdot t \cdot \|w_t\| \cdot \|v\|. \tag{106}$$

By definition, $v_t^\perp$ is a spectral projection onto eigenspaces of a_t^k whose associated eigenvalues are distinct from λ^*. By Lemma A.1, there exists $\delta > 0$ so that such eigenvalues differ from λ^* by at least $\delta \cdot t^{\frac{2}{3}}$. Because of (106), we can thus apply a resolvent estimate (e.g., [HJ11, Lemma 2.1]) to find that

$$\|v_t^{\perp}\| \leqslant \frac{2C}{\delta} \cdot t^{\frac{1}{3}} \cdot \|w_t\|. \tag{107}$$

By substituting (107) into (105) and setting $\alpha = 5/9$, we find a constant C' so that

$$\left| \int_1^{\infty} v_t^* \cdot v_t^{\perp} \, dy \right| \leqslant C' \cdot t^{\frac{8}{9}} \cdot \|w_t\|^2.$$

By combining this estimate with (103), (102), and (101), we obtain a constant C'' so that

$$\dot{E}_t \cdot \|u_t\|^2 \geqslant \dot{\lambda}^* \cdot \|w_t^*\|^2 \ - \ 2C'' \cdot t^{-\frac{1}{9}} \cdot \|w_t\|^2.$$

By orthogonality, $\|w_t^*\|^2 = \|w_t^k\|^2 - \|w_t^{\perp}\|^2$, and hence by (107) and Lemma A.1, we have a constant C''' so that

$$\dot{\lambda}^* \cdot \|w_t^*\|^2 = \dot{\lambda}^* \cdot \|w_t^k\|^2 - C''' \cdot t^{\frac{1}{3}} \cdot \|w_t\|^2.$$

The desired result follows. □

COROLLARY 8.13. *There exists C' such that for each $\rho \in \,]0, 1[$, there exists $t_0 > 0$ such that if $0 < t < t_0$, then*

$$\int_{[0,t]\setminus K(t,\rho)} \left(\dot{E}_s - \dot{\lambda}_s^* \right) \, ds \ \geqslant \ C' \cdot \left(\rho^2 - 1 \right) \cdot t^{\frac{2}{3}}. \tag{108}$$

Proof. By definition, if $s \in [0, t] \setminus K(t, \rho)$, then $\|w_t^k\|^2 / \|u_t\|^2 \geqslant \rho^2$, and hence from Proposition 8.12, we find that

$$\dot{E}_t - \dot{\lambda}_t^* \geqslant \left(\rho^2 - 1 \right) \cdot \dot{\lambda}_t^* - C \cdot t^{-\frac{1}{9}}.$$

By using Lemma A.1, we find C' and t_0 so that for $t < t_0$,

$$\dot{E}_t - \dot{\lambda}_t^* \geqslant \frac{2}{3} \cdot C' \cdot \left(\rho^2 - 1 \right) \cdot t^{-\frac{1}{3}}.$$

The claim follows from integration. □

Finally, we use Lemma 8.11 and Corollary 8.13 to prove Theorem 8.10. This will complete the proof of the main theorem.

Proof of Theorem 8.10. Apply Lemma 8.11 with $\rho = 1/2$. Then apply Corollary 8.13 with $\rho = \rho_0 \geqslant 1/2$ such that

$$C' \cdot \frac{\rho_0^2 - 1}{2} \geqslant -\frac{1}{2} \cdot \gamma \left(\frac{1}{2} \right).$$

Since $s \mapsto \dot{E}_s - \dot{\lambda}_s^*$ is positive on $K(t, \rho_0) \supset K(t, \frac{1}{2})$, we find that

$$\int_0^t \left(\dot{E}_s - \dot{\lambda}_s^* \right) \, ds \ \geqslant \ \frac{1}{2} \cdot \gamma \left(\frac{1}{2} \right) \cdot t^{\frac{2}{3}}.$$

Since $\lim_{t \to 0} \ E_t - \lambda_t^* = 0$, we have the desired conclusion. □

Appendix A. Eigenvalue branches of a_t^ℓ

In this appendix, we compute the asymptotics of each real-analytic eigenvalue branch of a_t^ℓ for each $\ell \in \mathbb{Z}^+$.

PROPOSITION A.1. *Let $\ell \in \mathbb{Z}^+$, and let $t \mapsto \lambda_t$ be a real-analytic eigenvalue branch of a_t^ℓ for $t > 0$. Then*

$$\lambda_t = (\ell\pi)^2 + a \cdot t^{\frac{2}{3}} + O\left(t^{\frac{4}{3}}\right), \tag{109}$$

where $a = \left(2(\pi\ell)^2\right)^{\frac{2}{3}} \cdot (-\zeta)$ and ζ is a zero of the derivative of the Airy function A_- defined in (161). *Moreover,*

$$\lim_{t \to 0^+} \dot{\lambda}_t \cdot t^{\frac{1}{3}} = \frac{2}{3} \cdot a.$$

To prove Proposition (A.1), we will first transform the eigenvalue problem into an eigenvalue problem that is easier to analyze. If v is an eigenfunction of a_t^ℓ with respect to $\|\cdot\|$ with eigenvalue λ, then for each $w \in C_0^\infty([0,\infty[)$ and $t > 0$, we have

$$t^2 \int_1^\infty v' \cdot w' \, dy + \mu \int_1^\infty v \cdot w \, dy = \lambda \int_1^\infty \frac{v \cdot w}{y^2} \, dy,$$

where we have set $\mu = \ell^2\pi^2$.

Hence

$$t^2 \int_1^\infty v' \cdot w' \, dy + \mu \int_1^\infty \frac{(y-1)\cdot(y+1)}{y^2} \cdot v \cdot w \, dy = (\lambda - \mu) \int_1^\infty \frac{v \cdot w}{y^2} \, dy.$$

By making the change of variable $y = t^{\frac{2}{3}} \cdot x + 1$, letting $\overline{v}(x) = v\left(t^{\frac{2}{3}} \cdot x + 1\right)$ and $\overline{w}(x) = w\left(t^{\frac{2}{3}} \cdot x + 1\right)$, and dividing by $t^{\frac{4}{3}}$, we find that

$$\int_0^\infty \overline{v}_t' \cdot \overline{w}' \, dx + \mu \int_0^\infty x \cdot g\left(t^{\frac{2}{3}} \cdot x\right) \cdot \overline{v}_t \cdot \overline{w} \, dx = t^{-\frac{2}{3}} \cdot (\lambda - \mu) \int_0^\infty f(t^{\frac{2}{3}} \cdot x) \cdot \overline{v}_t \cdot \overline{w} \, dx,$$

where

$$f(z) = \frac{1}{(z+1)^2}$$

and

$$g(z) = \frac{z+2}{(z+1)^2}.$$

This leads us to set $s = t^{\frac{1}{3}}$ and define for each $w \in C_0^\infty([0,\infty[)$ the quadratic forms

$$\mathcal{A}_s(w) = \int_0^\infty (w')^2 \, dx + \mu \int_0^\infty x \cdot g(s^2 \cdot x) \cdot w^2 \, dx$$

and

$$\mathcal{N}_s(w) = \int_0^\infty f(s^2 \cdot x) \cdot w^2 \, dx.$$

Define $\mathcal{H} := L^2([0,\infty),\ \frac{1}{x^2+1}\,dx)$, and $\mathcal{D} := H^1([0,\infty))$ (i.e., the set of functions $u \in L^2([0,\infty))$ such that the distributional derivative also is in L^2).

Then for each $s > 0$, the form $\mathcal{N}_s$ is a bounded quadratic form on $\mathcal{H}$ and $\mathcal{A}_s$ is a closed quadratic form on $\mathcal{H}$ with domain $\mathcal{D}$.

Since $w \mapsto \overline{w}$ maps bijectively $C_0^\infty([1,\infty])$ onto $C_0^\infty([0,\infty])$, the function $\overline{v}$ is an eigenfunction of $\mathcal{A}_s$ with respect to $\mathcal{N}_s$ with eigenvalue $\nu = s^{-2}\cdot(\lambda-\mu)$.

It follows from the perturbation theory of generalized eigenvalue problems (see [Kat95, §VII.6]) that the eigenvalues of $\mathcal{A}_s$ with respect to $\mathcal{N}_s$ can be organized into real-analytic eigenvalue branches for $s > 0$.[13]

Since the generalized eigenvalue problem $\mathcal{A}_s(u,v) = \nu\cdot\mathcal{N}_s(u,v)$ corresponds to a Sturm-Liouville problem with Neumann condition at $x = 0$, the eigenspaces are 1-dimensional. Hence, we may enumerate the real-analytic eigenvalue branches ν_s^i so that for each $i \geqslant 0$ and $s > 0$, we have

$$\nu_s^i < \nu_s^{i+1}. \tag{110}$$

Lemma A.2. *For each i, there exists $s_0 > 0$ and C so that if $s < s_0$, then*

$$\left|\dot{\nu}_s^i\right| \leqslant C\cdot s. \tag{111}$$

In particular, there exists a so that for small $s > 0$,

$$\nu_s^i = a + O(s^2). \tag{112}$$

Moreover, $-a/(2\mu)^{\frac{2}{3}}$ is a zero of the derivative of the Airy function A_-.

Proof. First, we show that each ν_i^s is bounded. To this end, define

$$\mathcal{B}(v) = \int_0^\infty \left(v'(x)\right)^2 + 2\mu\cdot x\cdot v(x)^2\,dx.$$

Since g is bounded above by 2, we have $\mathcal{A}_s(v) \leqslant \mathcal{B}(v)$ for each $s > 0$ and $v \in C_0^\infty([0,\infty[)$. Note that for each $s \leqslant 1$, we have $\mathcal{N}_s(v) \geqslant \mathcal{N}_1(x)$, and hence

$$\frac{\mathcal{A}_s(v)}{\mathcal{N}_s(v)} \leqslant \frac{\mathcal{B}(v)}{\mathcal{N}_1(v)}. \tag{113}$$

Integration by parts shows that the eigenfunctions of $\mathcal{B}$ with respect to $\mathcal{N}_1$ are solutions to the Sturm-Liouville problem

$$-v''(x) + 2\mu\cdot x\cdot v(x) = \frac{v(x)}{(1+x)^2}.$$

Standard convexity estimates on solutions to ordinary differential equations imply that each eigenfunction belongs to the domain $\mathcal{D}$ of $\mathcal{A}_s$ for each $s > 0$. In particular, the sum of the first i eigenspaces of $\mathcal{B}$ with respect to $\mathcal{N}_1$ belongs to $\mathcal{D}$.

[13] At $s = 0$, the domains of $\mathcal{A}_s$ and $\mathcal{N}_s$ change, and hence analytic perturbation theory cannot be applied.

Therefore, using (113), the minimax principle, and (110), we find that ν_s^i is bounded by the i^{th} eigenvalue of $\mathcal{B}$ with respect to $\mathcal{N}_1$.

In the remainder of the argument we drop the superscript i and focus on an individual real-analytic eigenfunction branch u_s with eigenvalue ν_s. For $s > 0$, we have

$$\dot{\nu}_s = \frac{\dot{\mathcal{A}}_s(u_s)}{\mathcal{N}_s(u_s)} - \nu_s \cdot \frac{\dot{\mathcal{N}}_s(u_s)}{\mathcal{N}_s(u_s)},$$

where $\cdot$ indicates differentiation with respect to s. A computation gives that for each w,

$$\dot{\mathcal{A}}_s(w) = 2s \cdot \mu \int_0^\infty x^2 \cdot g'(s^2 \cdot x) \cdot w(x)^2 \, dx$$

and

$$\dot{\mathcal{N}}_s(w) = 2s \int_0^\infty x \cdot f'(s^2 \cdot x) \cdot w(x)^2 \, dx.$$

Let u_s be a real-analytic eigenfunction branch of $\mathcal{A}_s$ with respect to $\mathcal{N}_s$ associated to the real-analytic eigenvalue branch ν_s. Integration by parts gives

$$-u_s''(x) + \mu \cdot x \cdot g\left(s^2 \cdot x\right) \cdot u_s(x) = \nu_s \cdot f(s^2 \cdot x) \cdot u_s(x). \tag{114}$$

Let M be the upper bound on ν_s proven above. If $s \leqslant 1$ and $x > x_0 := \max\{1, M/\mu\}$, then

$$\mu \cdot x \cdot g(s^2 \cdot x) - \nu_s \cdot f(s^2 \cdot x) \geqslant \frac{\mu}{2},$$

and hence $u_s'' u_s(x) \geqslant \frac{\mu}{2} \cdot u_s^2(x)$ for $s \leqslant 1$. It follows that $(u_s^2)''(x) \geqslant \mu \cdot u_s^2(x)$ for $x \geqslant x_0$. Thus, since $\mathcal{N}(u_s)$ is finite, we find that for $x_0 \leqslant x \leqslant y$,

$$\frac{u_s(y)^2}{u_s(x)^2} \leqslant \frac{\exp\left(-\sqrt{\mu} \cdot y\right)}{\exp\left(-\sqrt{\mu} \cdot x\right)}. \tag{115}$$

Integrating from x_0 to $2x_0$, we find a constant C (which depends on x_0) such that, for $y > 2x_0$, we have

$$y^2 \cdot u(y)^2 \leqslant C \cdot y^2 \exp(-\sqrt{\mu} y) \cdot \int_{x^0}^{2x^0} \frac{u(x)^2}{1+x^2} \, dx.$$

From this we find constants C such that

$$\left|\dot{\mathcal{A}}_s(u_s)\right| \leqslant C \cdot s \cdot \mathcal{N}_s(u_s).$$

A similar argument shows that

$$\left|\dot{\mathcal{N}}_s(u_s)\right| \leqslant C \cdot s \cdot \mathcal{N}_s(u_s).$$

Therefore, (111) holds, and via integration we find a so that (112) holds true.

Continuity of solutions to ordinary differential equations with respect to coefficients applies to (114) with fixed initial conditions $u_s'(0) = 0$ and $u_s(0) = 1$. In particular, we have a solution u_0 to

$$-u_0''(x) + 2\mu \cdot x \cdot u_0(x) = a \cdot u_0(x).$$

It follows that

$$v(z) := u_0\left((2\mu)^{-\frac{1}{3}} \cdot z + (2\mu)^{-1} \cdot a)\right)$$

is a solution to $v''(z) = z \cdot v(z)$. Estimate (115) applies to u_0, and hence it follows from (161) that v is a multiple of A_-. The function u satisfies the Neumann condition $u'(0) = 0$ and hence $v'(-(2\mu)^{-\frac{2}{3}} \cdot a) = 0$ as desired. □

Proof of Proposition A.1. If v_t is a real-analytic eigenfunction branch of of a_t^ℓ associated to the eigenvalue branch λ_t, then $\overline{v}_{s^3}$ is a real-analytic eigenfunction branch of $\mathcal{A}_s$ with eigenvalue branch $\nu_s = s^{-2}(\lambda_{s^3} - \mu)$. Lemma A.2 implies that

$$\lambda_t = \mu + a \cdot t^{\frac{2}{3}} + O(t^{\frac{4}{3}}).$$

By differentiating $\lambda_{s^3} = \mu + s^2 \cdot \nu_s$, we find that

$$3\dot{\lambda}_{s^3} = \dot{\nu}_s + 2 \cdot \nu_s \cdot s^{-1}.$$

By Lemma A.2, both $\dot{\nu}_s$ and ν_s are bounded. Therefore, $\dot{\lambda}_{s^3} = O(s^{-1})$ and hence $\dot{\lambda}_t = O(t^{-\frac{1}{3}})$. □

Appendix B. The off-diagonal estimates

Let (E_t, u_t) be a real-analytic eigenbranch of q_t such that $\lim_{t\to 0} E_t = E_0 = (\pi \cdot k)^2$ for some positive integer k. For a fixed constant $C > 0$, let

$$I = [E_0 - C, E_0 + C].$$

As in Section 7.2, let w_t denote the orthogonal projection of u_t onto the sum of the eigenspaces of a_t whose eigenvalues lie in I.

The purpose of this appendix is to prove the following fact, which is crucially used in the proof of Proposition 8.8. We recall that b_t is the quadratic form defined in (27).

PROPOSITION B.1. *Let $\eta > 0$. There exist $\kappa > 0$, $\delta > 0$, and $t_0 > 0$ such that, if $t < t_0$ and if ψ^0 is an eigenfunction of a_t^0 with eigenvalue λ^0 satisfying*

$$|\lambda^0 - E_t| \leqslant \eta \cdot t^{\frac{5}{3}}, \tag{116}$$

then

$$\left|b_t(u_t, \psi^0 \otimes 1)\right| \geqslant \kappa \cdot t^{\frac{2}{3}} \cdot \left(\|w_t^k\| - t^\delta \cdot \|u_t\|\right) \cdot \|\psi^0\|. \tag{117}$$

Remark B.2. The condition on λ^0 is only used to ensure that, when t tends to 0, λ^0 tends to $k^2\pi^2$.

Proof. Proposition 6.2 says that the quadratic form b_t is controlled by $\widetilde{a}_t$. There exists a constant C such that for $u, v \in \mathrm{dom}(a_t)$,

$$|b_t(u, v)| \leqslant C \cdot \widetilde{a}_t(u)^{\frac{1}{2}} \cdot \widetilde{a}_t(v)^{\frac{1}{2}}.$$

Thus, Lemma 7.5 and the fact that $\widetilde{a}_t(\psi^0) = O(\|\psi_0\|^2)$ imply that

$$b_t(u_t - w_t, \psi^0 \otimes 1) = O(t) \cdot \|u_t\| \cdot \|\psi^0\|,$$

and hence it suffices to bound $b_t(w_t, \psi^0)$ from below.

Observe also that Lemma 7.5 also implies that $\|u_t\| \sim \|w_t\|$ in the limit $t \to 0$ so that we can freely replace $\|u_t\|$ by $\|w_t\|$ and vice-versa in each (multiplicative) estimate.

By the discussion in Sections 6.2 and 7.2, for each t, we can uniquely write

$$w_t(x,y) = \sum_{\ell \leqslant k} \sum_{\lambda \in \mathrm{spec}(a_t^\ell) \cap I_t} \psi_\lambda^\ell(y) \cdot e_\ell(x),$$

where each $\psi_\lambda^\ell(y)$ is an eigenfunction of a_t^ℓ with eigenvalue $\lambda \in I_t$. Set

$$v_t^\ell(y) \;=\; \sum_{\lambda \in \mathrm{spec}(a_t^\ell) \cap I_t} \psi_\lambda^\ell(y) \tag{118}$$

and

$$w_t^\ell(x,y) = v_t^\ell(y) \cdot e_\ell(x).$$

By linearity,

$$b_t(w_t, \psi^0 \otimes 1) = \sum_{\ell \leqslant k} b_t\left(w_t^\ell, \psi^0 \otimes 1\right). \tag{119}$$

From (27) we have

$$b_t\left(w_t^\ell, \psi^0 \otimes 1\right) = \int_1^\alpha \int_0^1 \widetilde{\nabla}_t w_t^\ell \cdot \begin{pmatrix} 0 & x \cdot p(y) \\ x \cdot p(y) & 0 \end{pmatrix} \cdot \left(\widetilde{\nabla}_t \psi^0(y)\right)^* \, dxdy,$$

where $\widetilde{\nabla}_t f = [\partial_x f, t\partial_y f]$ and $p(y)$ is defined in (25). Since $\partial_x \psi(y) \equiv 0$ and $e_\ell(x) = 2^{-\frac{1}{2}} \cos(\ell \pi x)$, we find that

$$\begin{aligned} b_t\left(w_t^\ell, \psi^0 \otimes 1\right) = &\left(-2^{-\frac{1}{2}} \ell\pi \cdot \int_0^1 x \cdot \sin(\ell \pi x) dx\right) \\ &\cdot \left(\int_1^\alpha p(y) \cdot v_t^\ell(y) \cdot \left(t \cdot (\psi^0)'(y)\right) dy\right). \end{aligned}$$

If $\ell = 0$, then $\sin(\ell\pi x) \equiv 0$, and so $b_t(w_t^\ell, \psi^0 \otimes 1) = 0$. For $0 < \ell < k$, apply Lemma B.3 below to find that

$$\left| b_t(w_t^\ell, \psi^0 \otimes 1) \right| = O_\ell(t) \cdot \|v_t^\ell\| \cdot \|\psi^0\|. \tag{120}$$

Since w_t^ℓ and $w_t^{\ell'}$ are orthogonal if $\ell \neq \ell'$, we have

$$\sum_{\ell=1}^{k-1} \|v_t^\ell\|^2 = 2^{-\frac{1}{2}} \sum_{\ell=1}^{k-1} \|w_t^\ell\|^2 \;\leqslant \|w_t\|^2.$$

Thus, by summing (120) over $\ell \in \{0, \dots, k-1\}$, we obtain

$$
\begin{aligned}
\left| \sum_{\ell=0}^{k-1} b_t\left(w_t^\ell, \psi^0 \otimes 1\right) \right| &\leqslant O(t) \cdot \left(\sum_{1}^{k-1} \|v_t^\ell\| \right) \cdot \|\psi_t^0\| \\
&\leqslant O(t) \left(\sum_{1}^{k-1} \|v_t^\ell\|^2 \right)^{\frac{1}{2}} \cdot \|\psi_t^0\| \\
&\leqslant O(t) \cdot \|w_t\| \cdot \|\psi_t^0\|.
\end{aligned}
\tag{121}
$$

For $\ell = k$, we have

$$
k\pi \int_0^1 x \sin(k\pi x)\, dx = (-1)^k \neq 0.
$$

Thus, from Lemma B.4 and Lemma 5.1, there exists $\kappa' > 0$ so that

$$
\left| b_t(w_t^k, \psi^0 \otimes 1) \right| \geqslant \kappa' \cdot t^{\frac{2}{3}} \cdot \left(\|w_t^k\| - t^\delta \|u_t\| \right) \cdot \|\psi^0\|
$$

for some $\kappa' > 0$. The latter estimate, combined with (119), (121), and the triangle inequality, yield the claim. $\square$

Lemma B.3. *For each smooth function $g : [1, \underline{\alpha}] \to \mathbb{R}$, there exist $C > 0$ and $t_0 > 0$ such that if $t \leqslant t_0$ and $0 < \ell < k$, then*

$$
\left| \int_1^{\underline{\alpha}} g(y) \cdot v_t^\ell(y) \cdot \left(t \cdot \left(\psi^0\right)'(y) \right)\, dy \right| \leqslant C \cdot t \cdot \|u_t\| \cdot \left\| \psi^0 \right\|. \tag{122}
$$

Lemma B.4. *For each smooth function $g : [1, \underline{\alpha}] \to \mathbb{R}$ with $g(1) \neq 0$, there exist $\kappa, \delta, t_0 > 0$ such that for each $t < t_0$,*

$$
\left| \int_1^{\underline{\alpha}} g(y) \cdot v_t^k(y) \cdot \left(t \cdot \left(\psi^0\right)'(y) \right)\, dy \right| \geqslant \kappa \cdot t^{\frac{2}{3}} \cdot \left(\|w_t^k\| - t^\delta \|u_t\| \right) \cdot \|\psi^0\|.
$$

The remainder of this appendix is devoted to proving the preceding lemmas.

B.1. *The proof of Lemma* B.3. Define $r_t^\ell : [1, \infty[\to \mathbb{R}$ by

$$
r_t^\ell(y) = t^2 \cdot (v_t^\ell)''(y) + \left(\frac{E_t}{y^2} - (\ell \cdot \pi)^2 \right) \cdot v_t^\ell(y). \tag{123}
$$

Lemma B.5. *There exist $t_0 > 0$ and C so that if $t < t_0$, then for each $\ell \in \mathbb{N}$,*

$$
\int_1^\infty \left| r_t^\ell(y) \right|^2 \, dy \leqslant C \cdot t^2 \cdot \|u_t\|^2.
$$

Proof. Multiply both sides of (123) by a smooth function with compact support ϕ^ℓ and integrate over $y \in [1, \infty[$, then integrate by parts to obtain

$$
\int_1^\infty r_t^\ell(y) \phi^\ell(y)\, dy = -a_t^\ell(v_t^\ell, \phi) + E_t \cdot \langle v_t^\ell, \phi \rangle.
$$

Observe that $a_t^\ell(v_t^\ell, \phi^\ell) - E_t \cdot \langle v_t^\ell, \phi \rangle = a_t(w_t, \phi^\ell \otimes e_\ell) - E_t \langle w_t, \phi^\ell \otimes e_\ell \rangle$ so that by applying Lemma 7.8 to the test function $\phi^\ell \otimes e_\ell$, there exist $t_0 > 0$ and C' such that for $t < t_0$, we have

$$\left| \int_1^\infty r_t^\ell(y) \phi^\ell(y)\, dy \right| \leqslant C' \cdot t \cdot \|u_t\| \cdot \|\phi^\ell\|.$$

Recalling that the L^2-norm on the right-hand side has the weight y^{-2}, this implies that

$$\int_1^\infty y^2 \left| r_t^\ell(y) \right|^2 dy \leqslant \left(C' \cdot t \cdot \|u_t\| \right)^2.$$

The claim follows since $y^2 \geqslant 1$ on the interval over which we integrate. $\square$

The strategy of the proof of Lemma B.3 is as follows. By (123), the function v_t^ℓ is a solution to the inhomogeneous equation

$$t^2 \cdot v'' + f_\mu^\ell \cdot v = r, \tag{124}$$

where $\mu = E_t$ and

$$f_\mu^\ell(y) := \frac{\mu}{y^2} - (\ell \cdot \pi)^2. \tag{125}$$

The function ψ_t^0 is a solution to the homogeneous equation

$$v'' + t^{-2} \cdot f_\mu^\ell \cdot v = 0, \tag{126}$$

where $\mu = \lambda^0$. Our choice of β in (73) implies that f_μ^ℓ is bounded below by a constant $\delta_1 > 0$ for all small t, $\ell < k$, and $\mu \in I$. Hence we can use WKB type estimates to find a basis $v_\pm$ of solutions to the homogeneous equation (126). We will then use "variation of parameters" to express each solution to (124) in terms of this basis, and we use Lemma B.5 to provide control of the inhomogeneous term r. Finally, we will estimate the integral in (122) using a Riemann-Lebesgue type estimate.

Proof of Lemma B.3. For $\ell < k$ and $\mu \in I$, we have $f_\mu^\ell \geqslant \delta_1 > 0$, and hence we can apply Theorem 6.2.1 in [Olv74] to obtain a basis $(v_{\mu,+}^\ell, v_{\mu,-}^\ell)$ of solutions to the homogeneous equation (126) that satisfy

$$v_{\mu,\pm}^\ell(y) = \left| f_\mu^\ell(y) \right|^{-\frac{1}{4}} \exp\left(\pm \frac{i}{t} \int_1^y \left| f_\mu^\ell(z) \right|^{\frac{1}{2}} dz \right) (1 + \varepsilon(y)) \tag{127}$$

and

$$t \cdot \left(v_{\mu,\pm}^\ell \right)'(y) = \pm i \cdot \left| f_\mu^\ell(y) \right|^{\frac{1}{4}} \cdot \exp\left(\pm \frac{i}{t} \int_1^y \left| f_\mu^\ell(z) \right|^{\frac{1}{2}} dz \right) (1 + \overline{\varepsilon}(y)) \tag{128}$$

where, for $\mu \in I$ and $\ell < k$, the smooth functions ε and $\overline{\varepsilon}$ have C^0-norm that is uniformly $O(t)$.

Observe that $v_{\mu,\pm}^\ell$ have $L^2([1,\beta])$-norms that are uniformly bounded above and away from 0. Moreover, since $v_{\mu,+}^\ell$ and $v_{\mu,-}^\ell$ are highly oscillatory for small t, an integration by parts argument shows that the $L^2([1,\beta])$-inner product $\langle v^{\ell}_{\mu,+}, v^{\ell}_{\mu,-}\rangle$ is $O(t)$. It follows that there exists $m > 0$ such that if $(a_+, a_-) \in \mathbb{C}$, then

$$m \cdot \left(|a_+|^2 + |a_-|^2\right)^{\frac{1}{2}} \leqslant \left\| a_+ \cdot v^{\ell}_{\mu,+} + a_- \cdot v^{\ell}_{\mu,-} \right\| \leqslant m^{-1} \cdot \left(|a_+|^2 + |a_-|^2\right)^{\frac{1}{2}} \tag{129}$$

for all sufficiently small t. Here $\|\cdot\|$ denotes the $L^2([1,\beta])$-norm.

By the method of "variation of constants," each solution to

$$v'' + t^{-2} \cdot f^{\ell}_{\mu} \cdot v = t^{-2} \cdot r \tag{130}$$

is of the form

$$v \ = \left(a_+ + h^{\ell,r}_{\mu,+}\right) \cdot v^{\ell}_{\mu,+} \ + \left(a_- + h^{\ell,r}_{\mu,-}\right) \cdot v^{\ell}_{\mu,-}, \tag{131}$$

where $(a_+, a_-) \in \mathbb{C}^2$,

$$h^{\ell,r}_{\mu,\pm}(y) \ = \pm\, t^{-2} \cdot \mathcal{W}^{-1} \int_1^y r(z) \cdot v^{\ell}_{\mu,\mp}(z)\, dz,$$

and $\mathcal{W} = v'_{\mu,+} \cdot v_{\mu,-} - v'_{\mu,-} \cdot v_{\mu,+}$ is the Wronskian.

In particular, for each ℓ and each t, there exists $(a^{\ell}_{t,+}, a^{\ell}_{t,-}) \in \mathbb{C}^2$ so that the function v^{ℓ}_t of (118) satisfies

$$v^{\ell}_{E_t} \ = \left(a^{\ell}_{t,+} + h^{\ell,r^{\ell}_t}_{E_t,+}\right) \cdot v^{\ell}_{E_t,+} + \left(a^{\ell}_{t,-} + h^{\ell,r^{\ell}_t}_{E_t,-}\right) \cdot v^{\ell}_{E_t,-}.$$

The eigenfunction ψ^0 of a^0_t satisfies (130) with $r = 0$, and hence there exists $(c_+, c_-) \in \mathbb{C}^2$ so that

$$\psi^0 = c_+ \cdot v^0_{\lambda^0,+} + c_- \cdot v^0_{\lambda^0,-}.$$

The integral in (122) is equal to

$$\int_1^{\alpha} g \cdot \left(\sum_{\pm} \left(a^{\ell}_{t,\pm} + h^{\ell,r^{\ell}_t}_{E_t,\pm}\right) \cdot v^{\ell}_{E_t\pm}\right) \cdot \left(\sum_{\pm} c_{\pm} \cdot \left(t \cdot v^0_{\lambda^0,\pm}\right)'\right)\, dy.$$

By expanding the product of sums, one obtains a sum of 2^3 integrals. By substituting the expressions (128) and (127), integration by parts, and applying standard estimates, we find that each integral is $O(t)$.

For example, consider the terms of the form

$$a^{\ell}_{t,\pm} \cdot c_{\pm} \int_1^{\alpha} g \cdot \left(\frac{f^0_{\lambda^0}}{f^{\ell}_{E_t}}\right)^{\frac{1}{4}} \exp\left(\frac{i}{t} \int_1^y \pm \left(f^{\ell}_{E_t}\right)^{\frac{1}{2}} \mp \left(f^0_{\lambda_0}\right)^{\frac{1}{2}}\right) \cdot (1 + \varepsilon^*)\, dy. \tag{132}$$

Since $\ell > 0$, an elementary computation shows that there exists $\delta > 0$ so that if $z \in [1, \beta]$ and t is sufficiently small, then

$$\delta \leqslant \left| \left(f^{\ell}_{E_t}(z)\right)^{\frac{1}{2}} \pm \left(f^0_{\lambda^0}(z)\right)^{\frac{1}{2}} \right|. \tag{133}$$

Thus, we may integrate by parts to find a constant C so that the integral in (132) is at most $C \cdot t \cdot \|g\|_{C^1}$. It follows that all the terms of this form are bounded above by

$$C' \cdot t \cdot \|g\|_{C^1} \cdot \|\psi^0\| \cdot \left(\sum_{\pm} |a^{\ell}_{t,\pm}| \right). \tag{134}$$

We also have terms of the form

$$c_{\pm} \int_1^{\alpha} g \cdot h^{\ell, r^{\ell}_t}_{E_t,\pm} \cdot \left(\frac{f^0_{\lambda^0}}{f^{\ell}_{E_t}} \right)^{\frac{1}{4}} \exp\left(\frac{i}{t} \int_1^y \pm \left(f^{\ell}_{\mu}\right)^{\frac{1}{2}} \mp \left(f^0_{\mu}\right)^{\frac{1}{2}} \right) \cdot (1 + \varepsilon^*) \, dy. \tag{135}$$

We integrate by parts as above, but this time we need to also bound $h_{\pm} = h^{\ell, r^{\ell}_t}_{E_t,\pm}$ and its derivative.

From (127) and (128) we find that there exists t_1 so that if $t < t_1$, then $|t \cdot \mathcal{W}| \geqslant 1$. From (127) and (125) we find that for each ℓ,

$$\sup_{y \in [1,\beta]} \left| v^{\ell}_{E_t,\mp}(y) \right| \leqslant \frac{2}{\sqrt{\delta_1}} \tag{136}$$

for all sufficiently small t. Hence, using the Cauchy-Schwarz inequality and Lemma B.5 we have, for all $y \in [1, \beta]$,

$$\begin{aligned} |h_{\pm}(y)| &\leqslant \frac{1}{t} \cdot \left| \int_1^y r(y) \cdot v^{\ell}_{E_t,\mp}(y) \, dy \right| \\ &\leqslant \frac{1}{t} \cdot \left(\int_1^y r(y)^2 \, dy \right)^{\frac{1}{2}} \cdot \left(\int_1^y v^{\ell}_{E_t,\mp}(y)^2 \, dy \right)^{\frac{1}{2}} \\ &\leqslant \frac{1}{t} \cdot C \cdot t \cdot \|u_t\| \cdot \sqrt{\beta - 1} \cdot 2 \cdot \delta_1^{-\frac{1}{2}} \end{aligned}$$

for all sufficiently small t. For the derivative of $h^{\ell,r}_{\mu,\pm}$, we have

$$|h'_{\pm}(y)| \leqslant \frac{3}{t} \cdot |r(y)| \cdot 2\delta^{-\frac{1}{2}}.$$

Applying Cauchy-Schwarz and Lemma B.5 gives

$$\begin{aligned} \int_1^{\alpha} |h'_{\pm}(y)| \, dy &\leqslant \frac{6 \cdot \delta^{-\frac{1}{2}}}{t} \sqrt{\beta - 1} \cdot \left(\int_1^{\beta} |r(y)|^2 \, dy \right)^{\frac{1}{2}} \\ &\leqslant \frac{2 \cdot \delta^{-\frac{1}{2}}}{t} \sqrt{\beta - 1} \cdot C \cdot t \cdot \|u_t\|. \end{aligned} \tag{137}$$

Finally, we apply integration by parts to (135). The resulting terms that do not contain $h'_{\pm}$ have uniformly bounded C^0-norm. The term that contains $h'_{\pm}$ can be bounded using (137). It follows that all the terms of this form are bounded by

$$C \cdot t \cdot \|u_t\| \cdot \|\psi^0\|. \tag{138}$$

The final step consists in bounding $\sum |a_\pm|$ by $\|v_t^\ell\|$ to control the terms of equation (134).

Using (129) and (131), we have

$$m\left(|a_+|^2+|a_-|^2\right)^{\frac{1}{2}} \leqslant \|v_t^\ell\| + (\beta-1)\cdot\frac{2}{\sqrt{\delta_1}}\sup_{[1,\beta]}\{|h_+(y)|+|h_-(y)|\}.$$

By orthogonality we have $\|v_t^\ell\| \leqslant \|u_t\|$ and, using the bound on $|h_\pm(y)|$, we finally obtain

$$\left(|a_+|^2+|a_-|^2\right)^{\frac{1}{2}} \leqslant C\cdot\|u_t\|.$$

This finishes the proof. □

B.2. *The proof of Lemma* B.4. As in the previous subsection, the function v_t^k is a solution to the inhomogeneous equation (124) with $\mu = E_t$ and r defined by (123). However, for $\ell = k$, the function

$$f_t^k(y) = \frac{E_t}{y^2} - k^2\pi^2$$

is negative for large y. In fact, since E_t decreases to $(\pi k)^2$, the function f_t^k changes sign nearer and nearer to $y = 1$. Since the solution v_t^k belongs to $L^2(\mathbb{R}, y^{-2}dxdy)$, we expect it to decay exponentially as soon as y moves away from 1. For y near 1, we will approximate v_t^ℓ using Airy functions. In this subsection we will make these approximations precise and use them to give a proof of Lemma B.4

B.2.1. *Normalization of* ψ^0. By Lemma 6.3, ψ^0 is a constant multiple of ψ defined in (37). Because both sides of the estimate in Lemma B.4 are homogeneous functions of degree 1 in ψ^0, it suffices to assume that $\psi^0 = \psi$.

Let $|f|_0$ denote the supremum norm of f over $[1,\beta]$.

Lemma B.6. *There exists $t_0 > 0$ such that if $t < t_0$ and $\lambda^0 \in I$, then*

$$\frac{1}{2} \leqslant |\psi|_0 \leqslant 2\sqrt{\beta}, \tag{139}$$

$$\frac{\sqrt{\ln(\beta)}}{2} \leqslant \|\psi\| \leqslant \sqrt{\ln(\beta)}, \tag{140}$$

and

$$|t\cdot\psi'|_0 \leqslant 2\sqrt{\sup(I)}. \tag{141}$$

Proof. We have

$$\psi(y) = \omega^+(r,y) - (2r)^{-1}\cdot\omega^-(r,y), \tag{142}$$

where

$$\omega^+(r,y) = y^{\frac{1}{2}}\cdot\cos(r\cdot\ln(y)),$$

$$\omega^-(r,y) = y^{\frac{1}{2}}\cdot\sin(r\cdot\ln(y)),$$

and

$$\lambda^0 = t^2 \cdot (1/4 + r^2).$$

In particular, for sufficiently small t,

$$\frac{\sqrt{\inf(I)}}{2t} \leqslant r \leqslant \frac{\sqrt{\sup(I)}}{t}. \tag{143}$$

Thus, since $|\omega^{\pm}|_0 \leqslant \sqrt{\beta}$ and $|\omega^{+}|_0 \geqslant 1$, the triangle inequality applied to (142) implies that (139) holds for sufficiently small t:

$$\begin{aligned} \int_1^{\beta} |\omega_+(y)|^2\, y^{-2} dy &= \int_1^{\beta} |\cos(r\ln(y)|^2 \frac{dy}{y} \\ &= \int_0^{\ln\beta} |\cos(rz)|^2 dz \\ &= \frac{1}{2}\ln\beta + O\left(\frac{1}{r}\right). \end{aligned} \tag{144}$$

The same estimate applies for ω_-. Hence the triangle inequality and (143) imply that (140) holds for sufficiently small t.

The bound on $t \cdot \psi'$ is proven in a similar fashion using the fact that

$$\psi'(y) \; = -\left(r + \frac{1}{4r}\right) \cdot y^{-1} \cdot \omega^{-}(r, y) \tag{145}$$

together with (143) and the fact that r is of order t^{-1}. □

B.2.2. *Localization near* $y = 1$. The following proposition provides a quantitative description of the concentration of solutions to $t^2 \cdot v'' + f_t^k \cdot v = r$ near $y = 1$.

Proposition B.7. *Let* $k \in \mathbb{Z}^+$. *For each* $\alpha \in \left]0, \frac{2}{3}\right[$, *there exist* $t_0 > 0$ *and* C *such that if* v *is a solution to* $t^2 \cdot v'' + f_t^k \cdot v = r$ *and* $t < t_0$, *then*

$$\int_{1+2t^{\alpha}}^{\infty} |v(y)|^2 \; dy \leqslant C \cdot \left(t^{-2\alpha} \cdot \int_1^{\infty} |r|^2 \; + \exp\left(-t^{\frac{3\alpha-2}{2}}\right) \cdot \int_{1+t^{\alpha}}^{\infty} v^2(y)\, y^{-2} dy\right). \tag{146}$$

Proof. By Proposition 8.3 and Lemma A.1, there exists C so that if t is sufficiently small, then

$$E_t \leqslant (k\pi)^2 + C \cdot t^{\frac{2}{3}}. \tag{147}$$

Hence, for $y \geqslant 1 + t^{\alpha}$, one finds that

$$(k\pi)^2 - \frac{E_t}{y^2} \geqslant (k\pi)^2 \cdot \left(\frac{2 \cdot t^{\alpha} \, + \, t^{2\alpha}}{1 + 2t^{\alpha} + t^{2\alpha}}\right) - C \cdot t^{\frac{2}{3}}.$$

Thus, since $\alpha < 2/3$, there exists $t_1 > 0$ such that if $t \leqslant t_1$, then for all $y \geqslant 1 + t^{\alpha}$, we have

$$(k\pi)^2 - \frac{E_t}{y^2} \geqslant (k\pi)^2 \cdot t^{\alpha}. \tag{148}$$

For each smooth function φ with support in $]1+t^\alpha,\infty[$, define

$$L_t(\varphi) = -t^2 \cdot \varphi'' \; + \left((k\pi)^2 - \frac{E_t}{y^2}\right) \cdot \varphi.$$

Extend L_t to a self-adjoint operator on $L^2([1+t^\alpha,\infty[,dy)$. It follows from (148) that the spectrum of L_t lies in $[(k\pi)^2 \cdot t^\alpha,\infty[$. Hence L_t is invertible and the operator norm of $\|L_t^{-1}\|$ is bounded above by $\frac{t^{-\alpha}}{k^2\pi^2}$. Therefore,

$$\int_{1+t^\alpha}^\infty \left|L_t^{-1}(r)\right|^2 \frac{dy}{y^2} \;\leqslant\; \int_{1+t^\alpha}^\infty \left|L_t^{-1}(r)\right|^2 dy \;\leqslant\; \frac{t^{-2\alpha}}{(k\pi)^4} \int_{1+t^\alpha}^\infty |r(y)|^2 \, dy. \tag{149}$$

The function $L_t^{-1}(r)$ is a solution to (124) on $[1+t^\alpha,\infty)$, and hence $w := v - L_t^{-1}(r)$ is a solution to the homogeneous equation (126). It follows from (148) that

$$\left(w^2\right)''(y) \;\geqslant\; t^{\alpha-2} \cdot w^2(y) \tag{150}$$

if $y \in [1+t^\alpha,\infty[$. In particular, w^2 is convex and, moreover, w^2 is nonnegative and in $L^2([1+t^\alpha,\infty), y^{-2}dy)$ since $v \in L^2([1+t^\alpha,\infty), y^{-2}dy)$ and $L_t^{-1}r \in L^2([1+t^\alpha,\infty),dy) \subset L^2([1+t^\alpha,\infty), y^{-2}dy)$. This implies $\lim_{y\to\infty} w^2(y) = 0$. Indeed, since w^2 is convex, $(w^2)'$ has a limit m in $\mathbf{R} \cup \{+\infty\}$. If this limit is positive, then it implies that $w^2(y) \geqslant \frac{m}{2} y$ for large y, and this contradicts the fact that $w^2(y)y^{-2}$ is integrable. In particular, $(w^2)'$ is bounded so that by integrating (150) we find that $w^2 \in L^1([1+t^\alpha,\infty),dy)$. The argument also shows that $(w^2)'$ is nonpositive for large y so that w^2 has a limit when $y \to \infty$. Since w^2 is integrable, this limit is 0.

For each $y \in [1+t^\alpha,\infty)$, the function e_y, which is defined by

$$e_y(z) = w^2(y) \cdot \exp\left(-t^{\frac{\alpha-2}{2}} \cdot (z-y)\right),$$

satisfies $e_y''(z) = t^{\alpha-2} \cdot e_y(z)$ with $e_y(y) = w^2(y)$ and $\lim_{z\to\infty} e_y(z) = 0$. Therefore, by comparison with (150), and using the maximum principle, we find that if $z \geqslant y$, then $w^2(z) \leqslant e_z(y)$. Applying this to $z = y + t^\alpha$, we find that for each $y \geqslant 1+t^\alpha$,

$$w^2(y+t^\alpha) \leqslant \exp\left(-t^{\frac{3\alpha-2}{2}}\right) w^2(y).$$

By integration, we obtain

$$\int_{1+2t^\alpha}^\infty w^2(y)\,dy \leqslant \exp\left(-t^{\frac{3\alpha-2}{2}}\right) \cdot \int_{1+t^\alpha}^\infty w^2(y)\,dy.$$

This implies

$$\int_{1+2t^\alpha}^\infty w^2(y)\,dy \leqslant \frac{\exp\left(-t^{\frac{3\alpha-2}{2}}\right)}{1-\exp\left(-t^{\frac{3\alpha-2}{2}}\right)} \cdot \int_{1+t^\alpha}^{1+2t^\alpha} w^2(y)\,dy.$$

It follows that for t small enough, we have

$$\begin{aligned}\int_{1+2t^\alpha}^{\infty} w^2(y)\,dy &\leqslant \exp\left(-t^{\frac{3\alpha-2}{2}}\right)\int_{1+t^\alpha}^{1+2t^\alpha} w^2(y)\,dy\\ &\leqslant \frac{1}{2}\exp\left(-t^{\frac{3\alpha-2}{2}}\right)\int_{1+t^\alpha}^{1+2t^\alpha} w^2(y)\,y^{-2}dy \qquad (151)\\ &\leqslant \frac{1}{2}\exp\left(-t^{\frac{3\alpha-2}{2}}\right)\int_{1+t^\alpha}^{\infty} w^2(y)\,y^{-2}dy.\end{aligned}$$

We now use (149) and the triangle inequality to obtain

$$\begin{aligned}\|v\|_{[1+2t^\alpha,+\infty)} &\leqslant \|w\|_{[1+2t^\alpha,+\infty)} + \|L_t^{-1}(r)\|_{[1+2t^\alpha,+\infty)}\\ &\leqslant \exp\left(-t^{\frac{3\alpha-2}{2}}/2\right)\|w\|_{[1+t^\alpha,+\infty),y^{-2}} + Ct^{-\alpha}\|r\|_{[1;+\infty)}\\ &\leqslant \exp\left(-t^{\frac{3\alpha-2}{2}}/2\right)\left(\|v\|_{[1+t^\alpha,+\infty),y^{-2}} + \|L_t^{-1}(r)\|_{[1+t^\alpha,+\infty),y^{-2}}\right)\\ &\quad + Ct^{-\alpha}\|r\|_{[1,+\infty)}.\end{aligned}$$

The claim follows. □

Corollary B.8. *For each $\alpha \in]0,\frac{2}{3}[$, there exist C and t_0 such that, for each $t < t_0$,*

$$\int_{1+2t^\alpha}^{\infty}\left|v_t^k(y)\right|^2\frac{dy}{y^2} \leqslant C\cdot t^{2-2\alpha}\|u_t\|^2.$$

Proof. Using orthogonality, we have that $\|v_t^k\|_{L^2\left(\frac{dy}{y^2}\right)} \leqslant \|w_t\| \sim \|u_t\|$. Since $y^{-2} \leqslant 1$ on the interval $[1+t^\alpha,\infty)$, the integral with v_t^k on the right-hand side is bounded by $C\|u_t\|^2$. The integral with r is controlled via Lemma B.5. □

Corollary B.9. *There exist C and $t_0 > 0$ so that if $t < t_0$, then*

$$\left|\int_{1+2t^\alpha}^{\underline{\alpha}} g(y)\cdot v_t^k(y)\cdot(t\psi)'(y)\,dy\right| \leqslant C\cdot t^{1-\alpha}\cdot\|u_t\|\cdot\|\psi\|. \qquad (152)$$

Proof. Use the boundedness of g, the Cauchy-Schwarz inequality, the preceding corollary, and Lemmas B.5 and B.6. □

This corollary holds for each $\alpha \in]0,\frac{2}{3}[$. However, we will want this contribution to be $o(t^{\frac{2}{3}})$ so that we will need to take $\alpha \in]0,\frac{1}{3}[$.

B.2.3. *The Airy approximation.* For small t, the function f_t^k has a simple zero near $y = 1$. Thus, to approximate solutions of $t^2\cdot v'' + f_{E_t}^k\cdot v = r$ near $y = 1$, we will use solutions to Airy's differential equation $w''(x) - x\cdot w = 0$ where $x = y - 1$.

We first describe the link to Airy's equation. If we define $W(x) := v_t^k(x+1)$, then we have

$$-t^2\cdot W''(x) + \left((k\pi)^2 - \frac{E}{(x+1)^2}\right)\cdot W(x) = \tilde{r}(x), \qquad (153)$$

where $\tilde{r}(x) = r(x+1)$. Let ρ be the smooth function that satisfies

$$\frac{1}{(x+1)^2} = 1 - 2x + x^2 \cdot \rho(x).$$

By substituting the latter expression into (153) and by dividing by $2E_t$, we find that

$$-\frac{t^2}{2E_t} \cdot W'' + x \cdot W + \frac{1}{2} \cdot \left(\frac{(k\pi)^2}{E_t} - 1 \right) \cdot W = \frac{\widetilde{r}}{2E_t} - \frac{x^2}{2} \cdot \rho \cdot W.$$

Setting

$$s = \frac{t}{\sqrt{2E_t}} \tag{154}$$

and

$$z_s = \frac{1}{2} \cdot \left(1 - \frac{(k\pi)^2}{E_t} \right), \tag{155}$$

we have

$$- s^2 \cdot W''(x) + (x - z_s) \cdot W(x) = R_t(x), \tag{156}$$

where

$$R_t(x) = (2E_t)^{-1} \cdot \widetilde{r}(x) - 2^{-1} \cdot x^2 \cdot \rho(x) \cdot W(x).$$

In the next few subsections, we will analyze the solutions to (156). But first, we provide an estimate of the $L^2([0, 3t^\alpha])$-norm of R_t.

LEMMA B.10. *For each $\alpha < \frac{1}{3}$, there exist $C > 0$ and $t_0 > 0$ such that for each $t < t_0$,*

$$\int_0^{3t^\alpha} |R_t(x)|^2 \, dx \leqslant C \cdot t^{2(2\alpha + \frac{1}{3})} \cdot \|u_t\|^2. \tag{157}$$

Proof. We have $E \geqslant (k\pi)^2 \geqslant 1$, and hence by Lemma B.5,

$$\left\| \frac{r}{E} \right\|^2 \leqslant \|r_t\|^2 \leqslant C \cdot t^2 \cdot \|u_t\|^2.$$

Let $\psi^* = \psi_t^*$ be the tracking eigenvalue branch associated to $E = E_t$. The eigenvalue λ_t^* corresponding to ψ^* is in a $O(t)$ neighbourhood of E_t. Moreover, when t tends to 0, Proposition A.1 implies that λ_t is at a distance of order $t^{\frac{2}{3}}$ of the rest of the spectrum. Using (109), (123), Lemma B.5 and a resolvent estimate, we have $\|v_t^k - \psi_t^*\| = O(t^{\frac{1}{3}})\|u_t\|$ and hence

$$\int_0^{3t^\alpha} \left| x^2 \cdot \rho(x) \cdot (v_t - \psi_t^*)(x+1) \right|^2 \, dx \leqslant C \cdot t^{4\alpha} \cdot t^{\frac{2}{3}} \|u_t\|^2. \tag{158}$$

The tracking eigenfunction ψ^* satisfies (124) with $r = 0$. Hence, by Proposition B.7, if $\alpha \leqslant \widetilde{\alpha} < 2/3$, we obtain

$$\int_{2t^{\widetilde{\alpha}}}^{3t^\alpha} \left| x^2 \cdot \rho(x) \cdot \psi^*(x+1) \right|^2 \, dx \leqslant C \cdot t^{4\alpha} \cdot \exp\left(-t^{\frac{3\widetilde{\alpha} - 2}{2}} \right) \|\psi^*\|^2.$$

Observe that, by orthogonality, $\|\psi^*\| \leqslant \|u_t\|$; it follows that

$$\int_0^{3t^\alpha} \left|x^2 \cdot \rho(x) \cdot \psi^*(x+1)\right|^2 \, dx \leqslant C' \cdot \left(t^{4\widetilde{\alpha}} + t^{4\alpha} \exp\left(-t^{\frac{3\widetilde{\alpha}-2}{2}}\right)\right) \cdot \|u_t\|^2.$$

Since $2(2\alpha + \frac{1}{3}) < 2$ (because $\alpha < \frac{1}{3}$), we may thus take $\tilde{\alpha} = \frac{1}{2}$, and the biggest term is then of order $t^{2(2\alpha+\frac{1}{3})}$. The claim follows. □

B.2.4. *The inhomogeneous, semi-classical Airy equation.* By (156), an estimate of v will result from estimating the solutions to

$$-s^2 \cdot W'' + (x - z_s) \cdot W = R \tag{159}$$

for $s^{-\frac{2}{3}} \cdot z_s$ in a fixed compact set.

We first construct solutions to the associated homogeneous equation

$$-s^2 \cdot W'' + (x - z_s) \cdot W = 0 \tag{160}$$

using Airy functions. In particular, it is well known that there exists a basis $\{A_+, A_-\}$ of solutions to $-A''(x) + x \cdot A(x) = 0$ such that

$$\lim_{x\to\infty} \frac{A_\pm(x)}{x^{-\frac{1}{4}} \cdot \exp\left(\pm\frac{2}{3} \cdot x^{\frac{3}{2}}\right)} = 1 \tag{161}$$

and

$$\lim_{x\to\infty} \frac{A'_\pm(x)}{x^{\frac{1}{4}} \cdot \exp\left(\pm\frac{2}{3} \cdot x^{\frac{3}{2}}\right)} = \pm 1. \tag{162}$$

One checks that

$$W_\pm(x) = A_\pm\left(s^{-\frac{2}{3}}(x - z_s)\right) \tag{163}$$

defines a basis of solutions to (160).

It follows from well-known identities that the Wronskian, $A'_+A_- - A_+A'_-$, of $\{A_+, A_-\}$ is 2. Hence the Wronskian of $\{W_+, W_-\}$ is $2s^{-\frac{2}{3}}$. Therefore, by the method of variation of constants, for each $\overline{x} > 0$, the function

$$W_{\overline{x}}(x) = \frac{1}{2} \cdot s^{-\frac{4}{3}} \left(W_+(x) \int_x^{\overline{x}} R \cdot W_- + W_-(x) \int_0^x R \cdot W_+\right) \tag{164}$$

is a solution to (159).

Lemma B.11. *For each compact set $K \subset \mathbb{R}$, there exists C such that for each $\overline{x} > 0$, if $s^{-\frac{2}{3}} \cdot z_s \in K$ and $x \in [0, \overline{x}]$, then*

$$|W_{\overline{x}}(x)| \leqslant C \cdot s^{-\frac{4}{3}} \cdot s^{\frac{1}{3}} \cdot \|R\| \tag{165}$$

and

$$|W'_{\overline{x}}(x)| \leqslant C \cdot s^{-\frac{4}{3}} \cdot s^{-\frac{1}{3}} \cdot \|R\|, \tag{166}$$

where $\|R\|$ denotes the L^2-norm of R over $[0,\overline{x}]$. Moreover, there exists M so that if $x \in \left[M \cdot s^{\frac{2}{3}}, \overline{x}\right]$, then

$$|W_{\overline{x}}(x)| \leqslant 2 \cdot s^{-\frac{4}{3}} \cdot s^{\frac{1}{3}} \cdot s^{\frac{1}{2}} \cdot \|R\| \cdot x^{-\frac{3}{4}} \tag{167}$$

and

$$\left|W'_{\overline{x}}(x)\right| \leqslant 2 \cdot s^{-\frac{4}{3}} \cdot s^{-\frac{1}{3}} \cdot s^{\frac{1}{6}} \cdot \|R\| \cdot x^{-\frac{1}{4}}. \tag{168}$$

Proof. Using the Cauchy-Schwarz inequality, for $x \in [0,\overline{x}]$ we have

$$\left|\int_x^{\overline{x}} R(z) \cdot W_-(z)\ dz\right| \leqslant \left(\int_0^{\overline{x}} R(z)^2\ dz\right)^{\frac{1}{2}} \cdot \left(\int_x^{\infty} W_-(z)^2\ dz\right)^{\frac{1}{2}}$$

and

$$\left|\int_0^{x} R(z) \cdot W_+(z)\ dz\right| \leqslant \left(\int_0^{\overline{x}} R(z)^2\ dz\right)^{\frac{1}{2}} \cdot \left(\int_0^{x} W_+(z)^2\ dz\right)^{\frac{1}{2}}.$$

Thus, from (164) and the triangle inequality, we have

$$\begin{aligned} |W_{\overline{x}}(x)| \leqslant \frac{1}{2} s^{-\frac{4}{3}} \cdot \Bigg(& |W_+(x)| \cdot \|R\| \cdot \left(\int_x^{\infty} W_-^2\right)^{\frac{1}{2}} \\ & + |W_-(x)| \cdot \|R\| \cdot \left(\int_0^{x} W_+^2\right)^{\frac{1}{2}}\Bigg). \end{aligned} \tag{169}$$

Estimate (165) then follows from Lemma B.12 below.

To prove (166) we apply a similar argument to

$$W'_{\overline{x}}(x) = 2 \cdot s^{-\frac{4}{3}} \cdot \left(W'_+(x) \int_x^{\overline{x}} R \cdot W_- \ + W'_-(x) \int_0^{x} R \cdot W_+\right). \qquad \square$$

Define $I_+(x) = [0,x]$ and $I_-(x) = [x,\infty]$.

Lemma B.12. *There exists C so that if $x \geqslant 0$ and $s^{-\frac{2}{3}} \cdot z_s \in K$, then*

$$W_\pm(x)^2 \int_{I_\mp(x)} W_\mp(y)^2\ dy \leqslant C \cdot s^{\frac{2}{3}} \tag{170}$$

and

$$W'_\pm(x)^2 \int_{I_\mp(x)} W_\mp(y)^2\ dy \leqslant C \cdot s^{-\frac{2}{3}}. \tag{171}$$

Moreover, there exists a constant M so that if $x > M \cdot s^{\frac{2}{3}}$, then

$$W_\pm(x)^2 \int_{I_\mp(x)} W_\mp(y)^2\ dy \leqslant 4 \cdot \sqrt{2} \cdot s^{\frac{2}{3}} \cdot s \cdot x^{-\frac{3}{2}} \tag{172}$$

and

$$W'_\pm(x)^2 \int_{I_\mp(x)} W_\mp(y)^2\ dy \leqslant 2\sqrt{2} \cdot s^{-\frac{2}{3}} \cdot s^{\frac{1}{3}} \cdot x^{-\frac{1}{2}}. \tag{173}$$

Proof. The proof is a straightforward consequence of the continuity and known asymptotics of $A_\pm$ and $A'_\pm$. From (161) and integration by parts we find that

$$\int_{I_\mp(u)} |A_\pm(r)|^2 \, dr \ \sim \ \frac{1}{2} \cdot u^{-1} \cdot \exp\left(\pm\frac{4}{3} \cdot u^{\frac{3}{2}}\right), \tag{174}$$

as u tends to ∞.

Thus there exists u^* so that if $u \geqslant u^*$, then

$$\int_u^\infty A_-(r)^2 \, dr \leqslant u^{-1} \exp\left(-\frac{4}{3} \cdot u^{\frac{3}{2}}\right).$$

Therefore, for $u \geqslant u^*$,

$$A_+(u)^2 \int_u^\infty A_-(r)^2 \, dr \leqslant 2 \cdot u^{-\frac{3}{2}} \tag{175}$$

and, using (162),

$$A'_+(u)^2 \int_u^\infty A_-(r)^2 \, dr \leqslant 2 \cdot u^{-\frac{1}{2}}. \tag{176}$$

The expressions on the left-hand sides of (175) and (176) are continuous in u, and hence they are bounded by a constant C for $u \in \check{K} \cup [0, \infty[$ where $u \in \check{K} \Leftrightarrow -u \in K$.

By (163) and the change of variable $r = s^{-\frac{2}{3}} \cdot (y - z_s)$, we have

$$\int_x^\infty W_-(y)^2 \, dy = s^{\frac{2}{3}} \cdot \int_{u_s(x)}^\infty A_-(r)^2 \, dr, \tag{177}$$

where $u_s(x) = s^{-\frac{2}{3}} \cdot (x - z_s)$. For each $x \geqslant 0$, we have $u_s(x) \geqslant -\sup K$, and hence estimate (170) follows from (175).

Moreover, from (175) and (176) we have

$$W_+(x)^2 \int_{u_s(x)}^\infty W_-(y)^2 \, dy \leqslant 2 \cdot s^{\frac{2}{3}} \cdot u_s(x)^{-\frac{3}{2}} \tag{178}$$

and

$$W'_+(x)^2 \int_{u_s(x)}^\infty W_-(y)^2 \, dy \leqslant 2 \cdot s^{-\frac{2}{3}} \cdot u_s(x)^{-\frac{1}{2}}, \tag{179}$$

provided $u_s \geqslant u^*$. Let $M = u^* + 2 \sup K$. If $x > M \cdot s^{\frac{2}{3}}$, then $x - z_s > x/2$ and $u_s(x) > u_*$. The desired estimates in the $+/-$ case follow. The estimates in the $-/+$ case are proved in a similar fashion. $\square$

B.2.5. *The end of the proof of Lemma* B.4. By (152) it suffices to estimate

$$\int_1^{1+3t^\alpha} g(y)\cdot v_t^k(y)\cdot (t\psi)'(y)\,dy = \int_0^{3t^\alpha} \widetilde{g}(x)\cdot W_t(x)\cdot \left(t\widetilde{\psi}\right)'(x)\,dx, \tag{180}$$

where $W_t = v_t^k(x+1)$, $\widetilde{g}(x) = g(x+1)$, and $\widetilde{\psi}(x) = \psi(x+1)$. By assumption, the C^1 norm of g, and hence of $\widetilde{g}$, is uniformly bounded.

The function W_t satisfies the inhomogeneous equation (159) with $s = t/\sqrt{E_t}$, and the inhomogeneity R_t satisfies (157). In order to estimate W_t and hence (180), we write

$$W_t = W_{p,t} + W_{h,t},$$

where $W_{p,t}$ is taken to be the particular solution $W_{\overline{x}}$ to (159) defined by (164), where we set $\overline{x} = 3t^\alpha$. The function $W_{h,t}$ is then a solution to the associated homogeneous equation.

LEMMA B.13. *For each $\alpha \in \left]\frac{13}{42}, \frac{1}{3}\right[$, there exists $\delta > 0$ such that*

$$\left|\int_0^{2t^\alpha} \widetilde{g}\cdot W_{p,t}\cdot \left(t\widetilde{\psi}\right)'\ dx\ \right| \leqslant C\cdot t^{\frac{2}{3}+\delta}\cdot \|\psi\|\cdot\|u\|$$

for all t sufficiently small.

Proof. Integration by parts gives

$$\int_0^{2t^\alpha} \widetilde{g}\cdot W_{p,t}\cdot \left(t\widetilde{\psi}\right)' = \widetilde{g}\cdot W_{p,t}\cdot t\widetilde{\psi}\,\Big|_0^{2t^\alpha} - \int_0^{2t^\alpha} \partial_x\left(\widetilde{g}\cdot W_{p,t}\right)\cdot t\widetilde{\psi}. \tag{181}$$

Using Lemmas B.10 and B.11, one finds that there exists C such that, for $x\in[0,3t^\alpha]$,

$$|W_{p,t}(x)| \leqslant C\cdot t^{-1}\cdot t^{2\alpha+\frac{1}{3}}\cdot\|u_t\|.$$

Thus, using (B.6), we conclude that the first term on the right-hand side of (181) is $O(t^{2\alpha+\frac{1}{3}})\cdot\|u_t\|\cdot\|\psi\|$.

To bound the second term on the right-hand side of (181), we separately consider the integral of $(\partial_x\widetilde{g})\cdot W_{p,t}\cdot t\widetilde{\psi}$ and the integral of $\widetilde{g}\cdot(\partial_x W_{p,t})\cdot t\widetilde{\psi}$. For the first integral we can use the same uniform bound on $W_{p,t}$ as above to obtain a contribution that is $t^\alpha\cdot O(t^{2\alpha+\frac{1}{3}})\cdot\|u_t\|\cdot\|\psi\|$.

To estimate the second integral, we choose $\widetilde{\alpha}$ so that $\frac{2}{3} > \widetilde{\alpha} > \alpha$, and we separately estimate the integral over $[0, t^{\widetilde{\alpha}}]$ and the integral over $[t^{\widetilde{\alpha}}, 3t^\alpha]$. Observe that since $\widetilde{\alpha} < \frac{2}{3}$, then $t^{\widetilde{\alpha}} > Ms^{\frac{2}{3}}$. Using Lemmas B.10 and B.11, we find C so that for all sufficiently small t,

$$\int_{t^{\widetilde{\alpha}}}^{2t^\alpha} \left|\widetilde{g}\cdot W'_{p,t}(x)\right|\ dx \leqslant C\cdot t^{-\frac{3}{2}}\cdot t^{-\frac{1}{4}\cdot\widetilde{\alpha}}\cdot t^{2\alpha+\frac{1}{3}}\cdot t^\alpha\cdot\|u_t\|$$

and

$$\int_0^{t^{\widetilde{\alpha}}} \left|\widetilde{g}\cdot W'_{p,t}(x)\right|\ dx \leqslant C\cdot t^{-\frac{5}{3}}\cdot t^{2\alpha+\frac{1}{3}}\cdot t^{\widetilde{\alpha}}\cdot\|u_t\|.$$

By combining these estimates and using (141), we find that

$$\begin{aligned}\int_0^{2t^\alpha} \widetilde{g}\cdot W_{p,t}\cdot\left(t\cdot\widetilde{\psi}\right)' &\leqslant C\cdot t^{2\alpha+\frac{1}{3}}\,\|u_t\| + C\cdot t\cdot\int_0^{2t^\alpha}\left|\widetilde{g}\cdot W_{p,t}'(x)\right|\,dx\\ &\leqslant C\cdot\left(t^{2\alpha+\frac{1}{3}}+t^{-\frac{1}{6}+3\alpha-\frac{1}{4}\widetilde{\alpha}}+t^{-\frac{1}{3}+2\alpha+\widetilde{\alpha}}\right)\|u_t\|\,.\end{aligned}$$

The claim will follow provided we can choose $(\alpha,\widetilde{\alpha})$ so that $\alpha<\frac{1}{3}$, $\alpha<\widetilde{\alpha}$, and each power of t appearing on the right-hand side is greater than $2/3$. The solution set to this problem is the open triangle in $\mathbb{R}^2$ bounded by the lines $\alpha<1/3$, $2\alpha+\widetilde{\alpha}=1$, and $3\alpha-\widetilde{\alpha}/4=5/6$. The two latter lines intersect for $\alpha=\frac{13}{42}$. The claim follows. □

The same kind of argument allows us to estimate the norm of $W_{p,t}$.

Lemma B.14. *For all* $\alpha\in\,]\frac{7}{33},\frac{1}{3}[$, *there exist* $\delta>0$ *and* $C>0$ *such that*

$$\|W_{p,t}\|_{[0,3t^\alpha]}\leqslant C\cdot t^{\delta}\cdot\|u_t\|. \tag{182}$$

Proof. As above we consider $\alpha<\frac{1}{3}$ and take some $\widetilde{\alpha}>\alpha$. Using Lemmas B.10 and B.11, one finds that

$$\|W_{p,t}\|_{[t^{\widetilde{\alpha}},3t^\alpha]}\leqslant C\cdot t^{\frac{5\alpha}{2}-\frac{3\widetilde{\alpha}}{4}-\frac{1}{6}}\|u_t\|$$

and

$$\|W_{p,t}\|_{[0,t^{\widetilde{\alpha}}]}\leqslant C\cdot t^{2\alpha+\frac{\widetilde{\alpha}}{2}-\frac{2}{3}}\|u_t\|.$$

The claim will follow provided we can find $\alpha<\widetilde{\alpha}$ and $\alpha<\frac{1}{3}$ such that $\frac{5\alpha}{2}-\frac{3\widetilde{\alpha}}{4}-\frac{1}{6}>0$ and $2\alpha+\frac{\widetilde{\alpha}}{2}-\frac{2}{3}>0$. Here the solution set is a quadrilateral whose projection onto the α-axis is the interval $]\frac{7}{33},\frac{1}{3}[$. □

Finally, we consider the integral corresponding to the homogeneous part $W_{h,t}$ of W_t:

$$\int_0^{2t^\alpha}\widetilde{g}(x)\cdot W_{h,t}(x)\cdot\left(t\cdot\widetilde{\psi}\right)'(x)\,dx. \tag{183}$$

There exist constants a_+, a_-, depending on t, such that

$$W_{h,t}=a_+\cdot W_+ + a_-\cdot W_-,$$

where W_+ and W_- are as defined in (163) with the parameter s and z_s defined in (154).

We first prove a lemma that roughly says that in the decomposition $W=W_{p,t}+a_+W_++a_-W_-$ the L^2-norm is mainly supported by a_-W_-.

LEMMA B.15. *For all $\alpha \in]\frac{7}{33}, \frac{1}{3}[$, there exists $\delta > 0$ such that*

$$\|a_+W_+\|_{[0,2t^\alpha]} = O(t^\infty) \cdot \|u_t\|, \tag{184}$$

where $O(t^\infty)$ is a function that is of order t^n for each n, and

$$\|a_-W_-\|_{[0,2t^\alpha]} \geqslant \frac{1}{2} \cdot \left(\|w_t^k\| - C \cdot t^\delta \|u_t\|\right). \tag{185}$$

Proof. Using the behavior of the norm of $A_\pm$ we find that

$$\|a_+W_+\|_{[0,2t^\alpha]} = O(t^\infty) \cdot \|a_+W_+\|_{[2t^\alpha,3t^\alpha]}$$

and

$$\|a_-W_-\|_{[2t^\alpha,3t^\alpha]} \leqslant C \cdot \|a_-W_-\|_{[0,2t^\alpha]}.$$

We thus have

$$\begin{aligned}
\|a_+W_+\|_{[0,2t^\alpha]} &= O(t^\infty) \cdot \|a_+W_+\|_{[2t^\alpha,3t^\alpha]} \\
&\leqslant O(t^\infty) \cdot \left(\|W\|_{[2t^\alpha,3t^\alpha]} + \|a_-W_-\|_{[2t^\alpha,3t^\alpha]} + \|W_{p,t}\|_{[2t^\alpha,3t^\alpha]}\right) \\
&\leqslant O(t^\infty) \cdot \left(\|u_t\| + \|W_-\|_{[0,2t^\alpha]} + t^\delta \|u_t\|\right) \\
&\leqslant O(t^\infty) \cdot \left(\|u_t\| + \|W\|_{[0,2t^\alpha]} + \|a_+W_+\|_{[0,2t^\alpha]} + \|W_{p,t}\|_{[0,2t^\alpha]}\right) \\
&\leqslant O(t^\infty) \left(\|a_+W_+\|_{[0,2t^\alpha]} + \|u_t\|\right).
\end{aligned}$$

Estimate (184) then follows by absorbing the norm of a_+W_+ into the left-hand side.

To prove estimate (185), we first observe that by using the triangle inequality we find that

$$\|W\|_{[0,2t^\alpha]} \leqslant \|W_{p,t}\|_{[0,2t^\alpha]} + \|a_-W_-\|_{[0,2t^\alpha]} + \|a_+W_+\|_{[0,2t^\alpha]}.$$

The first term on the right-hand side is $O(t^\delta)\|u_t\|$ and the last one is $O(t^\infty)\|u_t\|$ so that we obtain

$$\|a_-W_-\|_{[0,2t^\alpha]} \geqslant \left(\|W\|_{[0,2t^\alpha]} - O(t^\delta)\|u_t\|\right).$$

The claim then follows by observing that Corollary B.8 implies that

$$\|W\|_{[0,2t^\alpha]} \geqslant \frac{1}{2} \left(\|w_t^k\| - O(t^{1-\alpha})\|u_t\|\right). \qquad \square$$

LEMMA B.16. *We have*

$$\int_0^{2t^\alpha} \widetilde{g} \cdot W_- \cdot \left(t \cdot \widetilde{\psi}\right)' \, dx = (\pi k) \cdot t \cdot g(1) \cdot A_- \left(-s^{-\frac{2}{3}} z_s\right) + O\left(t^{\frac{4}{3}}\right)$$

for t small.

Proof. From (145), we have

$$\widetilde{\psi}'(x) = -(x+1)^{-\frac{1}{2}} \cdot \left(r + \frac{1}{2r}\right) \cdot \sin(r \cdot \ln(x+1)).$$

Thus, the integral we want to estimate can be written as

$$t\left(r + \frac{1}{2r}\right) \int_0^{2t^\alpha} a_0(x) A_-(s^{-\frac{2}{3}}(x - z_s)) \sin(r \cdot \ln(x+1))\, dx,$$

where we have set $a_0(x) := -(x+1)^{-\frac{1}{2}}\widetilde{g}$. Denote by $I(t)$ the integral

$$I(t) = r \int_0^{2t^\alpha} a_0(x) A_-(s^{-\frac{2}{3}}(x - z_s)) \exp(ir \cdot \ln(x+1)\, dx.$$

Integration by parts shows that

$$\begin{aligned} I(t) = & -ia_1(x) A_-(s^{-\frac{2}{3}}(x - z_s)) \exp(ir \cdot \ln(x+1)\Big|_0^{2t^\alpha} \\ & - \frac{1}{ir} \int_0^{2t^\alpha} \partial_x \left(a_1(x) A_-(s^{-\frac{2}{3}}(x - z_s)\right) \left(r \exp(ir \cdot \ln(x+1))\right)\, dx, \end{aligned}$$

where we have set $a_1(x) = a_0(x)(x+1)$.

Since $\alpha < \frac{1}{3} < \frac{2}{3}$ and s is of order t, and since A_- is rapidly decreasing, the boundary term at $2t^\alpha$ is $O(t^\infty)$. Observe that we have a global $\frac{1}{r}$ prefactor in front of the integral term. Thus, when the ∂_x is applied to a_1, we gain $1/r$, that is, something of order t. When ∂_x hits the Airy function, we lose a $s^{-\frac{2}{3}}$ so that the global prefactor is of order $\frac{s^{-\frac{2}{3}}}{r}$, which is $O(t^{\frac{1}{3}})$. Summarizing, integrating by parts gains at least a prefactor $t^{\frac{1}{3}}$.

By repeated integration by parts we thus observe that for each N, we can write

$$I(t) = \sum_{n=0}^{N-1} \sum_{k+\ell=n} r^{-k} \left(\frac{s^{-\frac{2}{3}}}{r}\right)^\ell a_{k,\ell} A^{(\ell)}(-s^{-\frac{2}{3}} z_s) + R_N + O(t^\infty),$$

where the $a_{k,\ell}$ are some constants and the remainder term R_N can be written

$$\begin{aligned} R_N(t) := \sum_{k+\ell=N} r^{-k} \left(\frac{s^{-\frac{2}{3}}}{r}\right)^\ell \int_0^{2t^\alpha} & a_{k,\ell}(x) A_-^\ell(s^{-\frac{2}{3}}(x - z_s)) \\ & \cdot \left(r \exp(ir \cdot \ln(x+1))\right) dx \end{aligned}$$

for some smooth functions $a_{k,\ell}$. If we fix some order t^M then, using that A_- and all its derivatives are rapidly decreasing, we can find N such that the remainder R_N is $O(t^N)$. This tells us that $I(t)$ admits a complete asymptotic expansion of the form

$$I(t) \sim a_{00} r^{-1} A(-s^{-\frac{2}{3}} z_s) + \sum_{k,l \geqslant 1} a_{k,l} r^{-k} \left(\frac{s^{-\frac{2}{3}}}{r}\right)^\ell a_{k\ell} A^{(\ell)}(-s^{-\frac{2}{3}} z_s).$$

From the first integration by parts we see that

$$a_{00} = i\widetilde{g}(0),$$

and the second term is then of order $t^{\frac{1}{3}}$. The claim follows by taking the imaginary part. $\square$

We will use the following to verify that the leading order term does not vanish.

LEMMA B.17. *We have*

$$\lim_{s\to 0} s^{-\frac{2}{3}} \cdot z_s = -\zeta,$$

where $-\zeta$ *is a zero of the derivative of* A_-.

Proof. From (154) we have

$$s^{-\frac{2}{3}} \cdot z_s = \frac{E_t - (\pi k)^2}{2^{\frac{2}{3}} \cdot t^{\frac{2}{3}} \cdot E_t^{\frac{2}{3}}}.$$

By combining Lemmas 8.3 and A.1, we have

$$E_t - (\pi k)^2 = 2^{\frac{2}{3}} \cdot (\pi k)^{\frac{4}{3}} \cdot (-\zeta) \cdot t^{\frac{2}{3}} + O\left(t\right),$$

where ζ is a zero of A_-'. Since $\lim_{t\to 0} E_t = (\pi k)^2$, the claim follows. $\square$

COROLLARY B.18. *There exist* $\kappa' > 0$ *and* $t_0 > 0$ *so that if* $t < t_0$, *then*

$$\left| \int_0^{3t^\alpha} \widetilde{g}(x) \cdot W_-(x) \cdot \left(t \cdot \widetilde{\psi}\right)'(x)\ dx \right| \geqslant \kappa' \cdot t^{\frac{2}{3}} \cdot \|W_-\|$$

for t *sufficiently small.*

Proof. Let ζ be the zero of A_- that comes from Lemma B.17. Since A_- is a nontrivial solution to a second order differential equation, A_- cannot vanish at a zero of the derivative A_-'. Hence, for sufficiently for small t, we have $|A_-(-s^{\frac{2}{3}} \cdot z_s)| > \frac{1}{2}|A_-(\zeta)| > 0$.

By arguing as in the proof of Lemmas B.17 and B.12 and using $s \sim t$, we find $c_1 > 0$ so that

$$\int_0^{\beta-1} W_-(x)^2\ dx \geqslant \frac{1}{4} \cdot c_1 \cdot t^{\frac{2}{3}},$$

where $k_1 = \int_{-\sup(K)}^{\infty} |A_-(u)|^2$ and t is sufficiently small. In particular,

$$1 \leqslant \frac{2}{\sqrt{c_1}} \cdot t^{-\frac{1}{3}} \cdot \|W_-\|. \tag{186}$$

Hence the claim follows from Lemma B.16. $\square$

The estimate in the latter corollary is homogeneous so that we can multiply W_- by a_-.

Using Lemma B.15 we then have

$$\begin{aligned}\left|\int_0^{2t^\alpha} \widetilde{g}(x)\cdot a_- W_-(x)\cdot \left(t\cdot\widetilde{\psi}\right)'(x)\ dx\right|& \\ \geqslant \kappa'\cdot t^{\frac{2}{3}}\cdot\|a_- W_-\| \geqslant \frac{\kappa'}{2}\cdot t^{\frac{2}{3}}\left(\|w_t^k\| - t^\delta\|u_t\|\right)&\end{aligned}$$

and

$$\left|\int_0^{2t^\alpha} \widetilde{g}(x)\cdot a_+ W_+(x)\cdot \left(t\cdot\widetilde{\psi}\right)'(x)\ dx\right| \leqslant O(t^\infty)\cdot\|u_t\|.$$

Putting all the different pieces together yields the estimate.

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