Annals of Mathematics **187** (2018), 65–156
https://doi.org/10.4007/annals.2018.187.1.2

A regime of linear stability for the Einstein-scalar field system with applications to nonlinear Big Bang formation

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Abstract

We linearize the Einstein-scalar field equations, expressed relative to constant mean curvature (CMC)-transported spatial coordinates gauge, around members of the well-known family of Kasner solutions on $(0,\infty)\times\mathbb{T}^3$. The Kasner solutions model a spatially uniform scalar field evolving in a (typically) spatially anisotropic spacetime that expands towards the future and that has a "Big Bang" singularity at $\{t=0\}$. We place initial data for the linearized system along $\{t=1\}\simeq\mathbb{T}^3$ and study the linear solution's behavior in the collapsing direction $t\downarrow 0$. Our first main result is the proof of an *approximate L^2 monotonicity identity* for the linear solutions. Using it, we prove a linear stability result that holds when the background Kasner solution is sufficiently close to the Friedmann-Lemaître-Robertson-Walker (FLRW) solution. In particular, we show that as $t\downarrow 0$, various time-rescaled components of the linear solution converge to regular functions defined along $\{t=0\}$. In addition, we motivate the preferred direction of the approximate monotonicity by showing that the CMC-transported spatial coordinates gauge can be viewed as a limiting version of a family of parabolic gauges for the lapse variable; an approximate monotonicity identity and corresponding linear stability results also hold in the parabolic gauges, but the corresponding parabolic PDEs are locally well posed only in the direction $t\downarrow 0$. Finally, based on the linear stability results, we outline a proof of the following result, whose complete proof will appear elsewhere: the FLRW solution is globally nonlinearly stable in the collapsing direction $t\downarrow 0$ under small perturbations of its data at $\{t=1\}$.

Keywords: BKL conjectures, constant mean curvature, FLRW, Kasner solution, monotonicity, parabolic gauge, quiescent cosmology, spatial harmonic coordinates, stable blowup, strong cosmic censorship, transported spatial coordinates

AMS Classification: Primary: 83C75; Secondary: 35A20, 35Q76, 83C05, 83F05.

Contents

1. **Introduction**

This is the first of two papers in which we derive a new approximate monotonicity identity for two Einstein-matter systems and use it to prove linear and nonlinear stability results for cosmological[1] solutions featuring Big Bang singularities. By a "Big Bang" singularity in a spacetime, we roughly mean a spacelike hypersurface such that the solution exhibits curvature blowup along the entire hypersurface. In particular, our nonlinear result constitutes a proof of stable curvature blowup along a spacelike hypersurface for an open set of solutions. We now briefly summarize the nonlinear result, which is proved[2] in our second paper [59]; see Theorem 8.1 for a precise statement and [59] for an even more detailed statement.

Theorem 1.1 (Stable Big Bang Formation for near-FLRW solutions; rough version). *Consider initial data for the Einstein-scalar field system given*

[1]By "cosmological," we mean that the spacetime manifold $\mathbf{M}$ has compact Cauchy hypersurfaces and that the Ricci curvature of the spacetime metric $\mathbf{g}_{\mu\nu}$ verifies $\mathbf{Ric}_{\alpha\beta}\mathbf{X}^{\alpha}\mathbf{X}^{\beta} \geq 0$ for all timelike vectors $\mathbf{X}^{\mu}$. For the Einstein-scalar field system, this Ricci curvature condition is always verified by solutions because Einstein's equations imply that $\mathbf{Ric}_{\mu\nu} = \mathbf{T}_{\mu\nu} - \frac{1}{2}(\mathbf{g}^{-1})^{\alpha\beta}\mathbf{T}_{\alpha\beta}\mathbf{g}_{\mu\nu}$ and because the energy-momentum tensor $\mathbf{T}_{\mu\nu}$ of a scalar field verifies the strong energy condition.

[2]More precisely, Theorem 1.1 is a special case of the results of [59]: in [59] we prove an analog of Theorem 1.1 for the stiff fluid matter model. Theorem 1.1 follows as a special case in which the fluid's vorticity is zero; see Section 1.1 for further clarification of this point.

on the manifold[3] $\Sigma_1 = \mathbb{T}^3$, *which we identify with a Cauchy hypersurface of constant time* $t = 1$, *i.e.*, $\Sigma_1 = \{1\} \times \mathbb{T}^3$. *If the data are close in a suitable Sobolev norm to the data of the Friedmann-Lemaître-Robertson-Walker* (FLRW) *solution* (*see Section* 1.3), *then there exists a system of constant mean curvature-transported spatial coordinates* (t, x^1, x^2, x^3) *such that the perturbed solution exists for* $(t, x) \in (0, 1] \times \mathbb{T}^3$. *Like the* FLRW *solution, the perturbed solution's Kretschmann scalar* $\mathbf{Riem}^{\alpha\beta\gamma\delta}\mathbf{Riem}_{\alpha\beta\gamma\delta}$ *blows up like* t^{-4} *as* $t \downarrow 0$. *Moreover, the solution exhibits asymptotically velocity term dominated* (AVTD) *behavior, which means that near* $t = 0$, *the dynamics are dominated by time derivative terms* (*that is, the spatial derivative terms in the equations become negligible*), *and certain* t*-rescaled components of the solution converge in a monotonic fashion to regular functions of* x *as* $t \downarrow 0$. *In particular, as* $t \downarrow 0$, *the solution is asymptotic to a solution of the* VTD *equations, which are obtained by setting all spatial derivative terms equal to* 0 *in the Einstein-scalar field equations* (*expressed relative to the* CMC-*transported spatial coordinates gauge*).

See Section 1.3 for further discussion of Theorem 1.1, Section 1.7 for a summary of our linear results, and Section 1.5 for a discussion of the relationship between the various results.

In addition to deriving stability results, we also identify a new one-parameter family of parabolic gauges for the lapse function, which, like the well-known constant mean curvature (CMC)-transported-spatial coordinates gauge, leads to a formulation of the equations exhibiting the key structural features that allow us to prove the main results. For our purposes here, none of the gauges that we employ are manifestly superior. The parabolic lapse gauges are more general/flexible in that one does not need to construct[4] a CMC hypersurface to employ them. However, in the present context, they are a bit more unwieldy to use. For this reason, most of our results here rely on CMC foliations of spacetime. However, it is conceivable that the parabolic gauges will be useful in future studies of cosmological spacetimes. For this reason, in Section 10, we provide these gauges in detail and re-derive our linear results relative to them. We stress that all of the gauges under consideration lead to a formulation of the equations exhibiting infinite speed[5] of propagation. The

[3]Throughout, $\mathbb{T}^n := [-\pi, \pi]^n$ (with the ends identified) is an n-dimensional torus.

[4]For a general spacetime, such a CMC hypersurface does not exist. However, CMC hypersurfaces do exist for the spacetime solutions studied here; see [59] for a proof of this fact.

[5]The fundamental (gauge-independent) dynamic variables in the Einstein-scalar field equations propagate at a finite speed. It is only our description of them that involves an infinite speed.

infinite speed is fundamental for our analysis since our approach is based on *synchronizing* the singularity across a spacelike hypersurface of constant time; in a purely hyperbolic gauge involving a time coordinate t, it is not generally possible to ensure that a blowup-hypersurface (should one exist) is of the form $\{t = \text{const}\}$.

1.1. *The Einstein-scalar field equations.* In the present article, we restrict our attention to the study of the Einstein-scalar field equations

$$\mathbf{Ric}_{\mu\nu} - \frac{1}{2}\mathbf{R}\mathbf{g}_{\mu\nu} = \mathbf{T}_{\mu\nu}, \tag{1.1a}$$

$$(\mathbf{g}^{-1})^{\alpha\beta}\mathbf{D}_\alpha\mathbf{D}_\beta\phi = 0 \tag{1.1b}$$

with data given on the Cauchy hypersurface $\Sigma_1 = \{1\} \times \mathbb{T}^3$. Above and throughout, $\mathbf{Ric}$ denotes the Ricci tensor of the spacetime[6] metric $\mathbf{g}$, $\mathbf{R} = (\mathbf{g}^{-1})^{\alpha\beta}\mathbf{Ric}_{\alpha\beta}$ denotes the scalar curvature of $\mathbf{g}$, $\mathbf{D}$ denotes the Levi–Civita connection of $\mathbf{g}$, and $\mathbf{T}$ denotes the energy-momentum tensor of the scalar field ϕ:

$$\mathbf{T}_{\mu\nu} = \mathbf{D}_\mu\phi\mathbf{D}_\nu\phi - \frac{1}{2}\mathbf{g}_{\mu\nu}(\mathbf{g}^{-1})^{\alpha\beta}\mathbf{D}_\alpha\phi\mathbf{D}_\beta\phi. \tag{1.2}$$

The scalar field is a simple matter model that has been well studied in mathematical general relativity in the context of asymptotically flat spacetimes; see [21], [23], [22], [20], [18], [19]. In our complementary article [59], we study the Einstein-stiff fluid system, where a stiff fluid has sound speed equal to unity (that is, equal to the speed of light). The stiff fluid model is more general in the sense that it reduces[7] to the scalar field model when the fluid's vorticity vanishes. Due to our gauge choices (which we explain in detail below), one should identify the "data hypersurface" Σ_1 with a surface of constant time 1. We will study the behavior of solutions as $t \downarrow 0$. The singular behavior that we will uncover occurs along Σ_0, which will be identified with a surface of constant time 0.

Although our results apply when the initial Cauchy hypersurface is $\mathbb{T}^3$, they can easily be generalized to the case of n spatial dimensions, that is, to the case of $\mathbb{T}^n$ for $n \geq 1$. We anticipate that similar results might also hold for some other matter models with special properties and, in the case of very high spatial dimensions, for the Einstein-vacuum equations; see the discussion in Section 1.9.

[6]By "spacetime," we mean a four-dimensional time-orientable manifold $\mathbf{M}$ equipped with a Lorentzian metric $\mathbf{g}$ of signature $(-, +, +, +)$.

[7]It reduces for scalar fields with a timelike gradient and under an exactness condition tied to the fluid velocity and enthalpy per particle; see [58] for further discussion on this point.

1.2. *Paper outline.*

- In the remainder of Section 1, we summarize our linear and nonlinear stability results, discuss their relationship, and provide context by discussing prior work.
- In Section 2, we introduce some notation and conventions that we use throughout the article.
- In Section 3, we provide the Einstein-scalar field equations in CMC-transported spatial coordinates. We then linearize the equations around members of the (generalized) Kasner family.
- In Section 4, we provide the norms and energies that we use in our analysis of linear solutions.
- In Section 5, we prove an approximate monotonicity identity for linear solutions. The identity lies at the heart of all of our results.
- In Section 6, we use the approximate monotonicity identity to derive mildly singular energy estimates for linear solutions in the case that the Kasner background is nearly spatially isotropic.
- In Section 7, we use the mildly singular energy estimates to prove a linear stability result for nearly spatially isotropic Kasner backgrounds.
- In Section 8, we use the results of the previous sections to outline a proof of the nonlinear stability of the FLRW solution near its Big Bang singularity; complete details are located in [59].
- In Section 10, we introduce a family of parabolic lapse gauges and re-derive our linear results in these gauges.

1.3. *The* FLRW *solution and preliminary context for the results.* A quintessential example of a Big Bang spacetime is the FLRW solution (referred to in Theorem 1.1) to the Einstein-scalar field system, which plays a prominent role in cosmology in view of its spatially isotropic nature. It can be expressed in the well-known form

$$\begin{aligned} \mathbf{g}_{\mathrm{FLRW}} &= -dt^2 + g_{\mathrm{FLRW}}, \qquad & g_{\mathrm{FLRW}} &= t^{2/3}\sum_{i=1}^{3}(dx^i)^2, \\ \phi_{\mathrm{FLRW}} &= \sqrt{\frac{2}{3}}\ln t, & (t,x) &\in (0,\infty)\times\mathbb{T}^3. \end{aligned} \tag{1.3}$$

One can compute that the Kretschmann scalar $\mathbf{Riem}^{\alpha\beta\gamma\delta}\mathbf{Riem}_{\alpha\beta\gamma\delta}$ of $\mathbf{g}_{\mathrm{FLRW}}$ blows up like t^{-4} as $t \downarrow 0$. That is, the FLRW solution has a Big Bang singularity at $t = 0$.

Theorem 1.1 shows that like the FLRW solution, perturbed solutions also exhibit the same kind of curvature blowup. We provide the complete proof of Theorem 1.1 in the companion article [59] (for the Einstein-stiff fluid system). The proof is part of a “five-step program” encompassing the results of both papers, which we summarize in Section 1.5. Some key steps in the program are of independent interest and hold in a more general context than the form in which they are used in the proof of Theorem 1.1. In this article, we identify such a more general context and give rigorous proofs of those key steps that remain valid. In particular, here we study a large family of linearized versions of the Einstein-scalar field equations, where the backgrounds around which we linearize have been well studied in the mathematical general relativity literature. For each linearized system, we derive the aforementioned *approximate monotonicity identity* for the linear solutions. Specifically, we linearize the equations around members of the family of *generalized Kasner solutions*, which are explicit spatially homogeneous (that is, non-x-dependent) solutions whose unique spatially isotropic member is the FLRW solution. For generalized Kasner solutions, the spacetime metric is of the form $\mathbf{g} = -dt \otimes dt + \sum_{I=1}^{3} t^{2q_I} \omega^I \otimes \omega^I$, where the q_I are constants verifying certain constraints and the $\omega^I := \omega^I_a dx^a$ are a set of three $\mathbf{g}$-orthogonal one-forms on $\mathbb{T}^3$. In particular, relative to standard coordinates $\{x^a\}_{a=1,2,3}$ on $\mathbb{T}^3$, we have $\omega^I_a = \omega^I_a dx^a$, where the ω^I_a are constants and $\det(\omega^I_a) \neq 0$. See Section 1.6 for more details regarding these generalized Kasner solutions. Here we only note that for brevity, we will often refer to these (nonlinear) Einstein-scalar field solutions as Kasner solutions. This breaks with the traditional convention, which reserves the label "Kasner solution" for Einstein-vacuum solutions. A fundamental aspect of the Kasner backgrounds (around which we linearize) is that, like the FLRW solution, they have Big Bang singularities at $t = 0$ (aside from some exceptional cases). In addition to deriving the approximate monotonicity identity, we also use it to prove a linear stability result for a subset of the Kasner backgrounds, specifically those that are nearly spatially isotropic (that is, for near-FLRW Kasner backgrounds). Before further describing the five-step program and how our linear/nonlinear stability results fit into it, we first provide context that clarifies the significance of Theorem 1.1.

- Although the data we consider fall under the scope of the Hawking-Penrose "singularity" theorems[8] [40], [50], Theorem 1.1 goes beyond the soft conclusion of geodesic incompleteness provided by those theorems in that it shows that the incompleteness is due to curvature blowup along the hypersurface $\{t = 0\}$. As such, the solutions of Theorem 1.1 exhibit Strong Cosmic Censorship-type behavior, by which we mean that the solution variables cannot be extended as C^2 tensorfields beyond the boundary portion $\{t = 0\}$ of the maximal development of the data. This is the first result of this type for Einstein's equations that does not involve symmetry or analyticity assumptions on the data.

[8] More precisely, see [70, Th. 9.5.1] for a version of "Hawking's theorem" that can be applied to the initial data considered in Theorem 1.1.

- The AVTD behavior proved in Theorem 1.1, though predicted via heuristic arguments for the scalar field model in [13] and for the stiff fluid in [11], had not previously been shown in solutions without symmetry, except under the assumption of spatial analyticity [7]; see Section 1.8 for further discussion on these works. Moreover, as we describe below, the solutions from Theorem 1.1 are such that at each fixed spatial point x, its asymptotic behavior is Kasner-like, by which we mean that its limiting behavior is well described by fields that are related to members of the aforementioned Kasner family. The belief that the "end states" should, at each fixed x, be Kasner-like was part of the heuristics given in [11], [13]. More precisely, the authors in [13] *assumed* that all spatial derivative terms in the evolution equations become negligible near the singularity $\{t = 0\}$. The authors then argued that the spacetime metric should asymptotically behave like $-dt \otimes dt + \sum_{I=1}^{3} t^{2q_I(x)} \omega^I(x) \otimes \omega^I(x)$ near the singularity, that is, like Kasner solutions in which the exponents and one-forms are x-dependent. See just below Theorem 1.4 for further comments on the asymptotic behavior of solutions to the linearized equations.
- The monotonic behavior of the solution as $t \downarrow 0$ was also predicted in [11], [13] and in fact is accounted for by the authors' posited asymptotic form of the metric $-dt^2 + \sum_{I=1}^{3} t^{2q_I(x)} \omega^I(x) \otimes \omega^I(x)$. This *existence* of an interesting set of *spatially analytic* solutions to the Einstein-scalar field and Einstein-stiff fluid systems exhibiting this kind of monotonic asymptotic behavior was rigorously shown in the aforementioned work [7]. Like the heuristic arguments given in [11], [13] and the rigorous results of [7], our proof of the monotonic behavior (via the approximate monotonicity identity and its consequences) relies on the particular structure of the scalar field and stiff fluid matter models; see Section 1.8 for further discussion on this point.

1.4. *Initial value problem formulation of the Einstein equations and gauges.* Before further discussing our results, we first discuss some basic issues concerning the initial value problem for the (nonlinear) Einstein-scalar field system (1.1a)–(1.1b) and our gauge choices. The fundamental results [35] and [17], which are respectively by Choquet–Bruhat and Choquet–Bruhat + Geroch, showed that the system (1.1a)–(1.1b) has an initial value problem formulation in which sufficiently regular data give rise to a unique maximal globally hyperbolic development.[9] The rest of our discussion here is adapted to the setup of the present article, where the initial Cauchy hypersurface is $\mathbb{T}^3$. The "geometric data" (for the nonlinear equations) consist of the following fields

[9] Roughly, this is the largest possible classical solution to the Einstein-scalar field equations that is uniquely determined by the data.

on $\mathbb{T}^3$: $({}^0g_{ij}, {}^0k_{ij}, {}^0\phi, {}^0\psi)$. Here, ${}^0g_{ij}$ is a Riemannian metric, ${}^0k_{ij}$ is a symmetric two-tensor, and ${}^0\phi$ and ${}^0\psi$ are a pair of functions. A solution launched by the data consists of a four-dimensional time-oriented spacetime $(\mathbf{M}, \mathbf{g})$, a scalar field ϕ on $\mathbf{M}$, and an embedding $\mathbb{T}^3 \overset{\iota}{\hookrightarrow} \mathbf{M}$ such that $\iota(\mathbb{T}^3)$ is a Cauchy hypersurface in $(\mathbf{M}, \mathbf{g})$. The spacetime fields must verify equations (1.1a)–(1.1b) and be such that $\iota^*\mathbf{g} = {}^0g$, $\iota^*\mathbf{k} = {}^0k$, $\iota^*\phi = {}^0\phi$, $\iota^*\hat{\mathbf{N}}\phi = {}^0\psi$, where $\mathbf{k}$ is the second fundamental form of $\iota(\mathbb{T}^3)$ (our sign convention is given in (3.1)), $\hat{\mathbf{N}}\phi$ is the derivative of ϕ in the direction of the future-directed normal $\hat{\mathbf{N}}$ to $\iota(\mathbb{T}^3)$, and ι^* denotes pullback by ι. Throughout the article, we will often suppress the embedding and identify $\mathbb{T}^3$ with $\iota(\mathbb{T}^3)$.

It is well known (see also Proposition 3.1) that the data are constrained by the *Gauss* and *Codazzi* equations, which take the following form for the Einstein-scalar field system:

$$\text{(1.4a)} \qquad {}^0R - {}^0k^a{}_b{}^0k^b{}_a + ({}^0k^a{}_a)^2 = 2\mathbf{T}(\hat{\mathbf{N}}, \hat{\mathbf{N}})|_{\mathbb{T}^3} = {}^0\psi^2 + \nabla^a{}^0\phi\nabla_a{}^0\phi,$$

$$\text{(1.4b)} \qquad \nabla_a{}^0k^a{}_j - {}^0\nabla_j{}^0k^a{}_a = -\mathbf{T}(\hat{\mathbf{N}}, \frac{\partial}{\partial x^j})|_{\mathbb{T}^3} = -{}^0\psi\nabla_j{}^0\phi.$$

Above, $\mathbf{T}(\hat{\mathbf{N}}, \hat{\mathbf{N}}) := \mathbf{T}_{\alpha\beta}\hat{\mathbf{N}}^\alpha\hat{\mathbf{N}}^\beta$, ∇ denotes the Levi–Civita connection of ${}^0g$, ${}^0R$ denotes the scalar curvature of ${}^0g$, and indices are lowered and raised with ${}^0g$ and its inverse. Equations (1.4a)–(1.4b) are known, respectively, as the *Hamiltonian* and *momentum* constraints.

As is well known, to obtain a hyperbolic formulation, an elliptic-hyperbolic formulation, or a parabolic-hyperbolic formulation of equations (1.1a)–(1.1b), suitable for studying the initial value problem, one must impose gauge choices. As we mentioned at the beginning, there are two gauges in which we are able to derive our main results. The first is the well-known CMC-transported-spatial-coordinates gauge, which we recall in detail in Section 3. In this gauge, the spacetime metric $\mathbf{g}$ is decomposed into the *lapse* n and the Riemannian 3-metric g on $\Sigma_t := \{(s,x) \in (0,1] \times \mathbb{T}^3 \mid s = t\}$ as follows:

$$\text{(1.5)} \qquad \mathbf{g} = -n^2dt^2 + g_{ab}dx^a dx^b.$$

The spatial coordinates[10] $\{x^a\}_{a=1,2,3}$ are called "transported" because they are constant along the integral curves of the vectorfield $\hat{\mathbf{N}} = n^{-1}\partial_t$, which is the future-directed unit normal to Σ_t. The basic variables to be solved for in the nonlinear equations are g_{ij}, $k_{ij} := -\frac{1}{2}n^{-1}\partial_t g_{ij}$, n, and ϕ. The hypersurfaces Σ_t have mean curvature $\frac{1}{3}k^a{}_a$ that is constant, that is, that depends only on t. To achieve this, n must verify an elliptic PDE on Σ_t. Hence, this gauge leads to an elliptic-hyperbolic formulation of the equations. Above and

[10]Technically, the spatial coordinates are only locally defined on $\mathbb{T}^3$, even though the coordinate partial derivative vectorfields $\partial_i := \frac{\partial}{\partial x^i}$ can be globally defined so as to be smooth.

throughout, $k^i{}_j = g^{ia}k_{aj}$ denotes the (mixed) second fundamental form of the constant-time hypersurface Σ_t. We normalize the time coordinate so that $k^a{}_a(t,x) = -t^{-1}$, and we identify Σ_1 with the initial Cauchy hypersurface. To be admissible under this setup, the initial mixed second fundamental form must verify ${}^0k^a{}_a = -1$. See Section 3 for a more detailed discussion of this gauge. In particular, we provide the corresponding constraint and evolution equations in Proposition 3.1. Until Section 10, we will work with CMC-transported spatial coordinates gauge.

The second gauge suitable for our purposes is a one-parameter family of gauges that is in many ways like the CMC-transported spatial coordinates gauge, except that the elliptic CMC lapse equation is replaced with a parabolic evolution equation for n that is well posed in the past direction; see Section 10 for the details. Gauges for Einstein's equations involving parabolic equations have been considered in the general relativity literature for several decades. For example, in the work [10], the authors introduced a family of gauges in which the lapse solves a parabolic equation, and they suggested that such gauges should lead to efficient and accurate numerical simulations. We also point out the work [68] on the Euler-Einstein equations under the equation of state $p = c_s^2\rho$, where p is the fluid pressure, ρ is its proper energy density, and the constant c_s verifies $0 < c_s \leq 1$. In [68], the authors introduced the *separable volume gauge*, which is a parabolic gauge that can be viewed as a Lorentzian version of inverse mean curvature flow. They posited that the separable volume gauge should be useful for proving rigorous theorems concerning the behavior of inhomogeneous cosmological solutions near a spacelike singularity. Their main result was geometric: they identified a set that is invariant under the flow of their equations and conjectured that it is the past attractor of the flow. Interestingly, well-posedness for the equations studied in [68] is not known because their principal part is not of any standard type. In the work [38], the authors slightly modified the equations of [68] to produce a system of transport-diffusion equations, which they showed to be well posed. Readers can also consult [39] for a discussion of local well-posedness for the Einstein equations under various gauge conditions involving a parabolic equation for the lapse.

1.5. *The five-step program.* We now summarize the five-step program mentioned in Section 1.3. In particular, we briefly introduce our linear results and explain in what sense they are tied to/constitute an extension of the proof of Theorem 1.1 given in [59].

(1) (*Approximate monotonicity identity*). In this article, for *all Kasner backgrounds*, we first establish an approximate monotonicity identity for solutions to the linearized equations. More precisely, we derive an integral

identity for solutions in which, due to some special cancellations, some unfavorable integrals are shown to be equal to favorably signed integrals, up to error terms. See Theorem 1.2 for a rough summary of the integral identity and Theorem 5.1 for the precise statement. The favorably signed integrals encourage some of the linear solution variables to decay as $t \downarrow 0$, that is, chronologically *towards* the Kasner background's Big Bang. The monotonicity is indeed only approximate in the sense that some of the unsigned error terms in the integral identity compete against the favorably signed integrals. It turns out that for nearly spatially isotropic backgrounds (that is, for near-FLRW backgrounds), the favorably signed integrals are sufficiently strong to *absorb* most of the unsigned error terms, which is crucial for the next step.

(2) (*Mildly singular energy estimates at the lowest order for near-*FLRW *backgrounds*). Next, for *nearly spatially isotropic* Kasner backgrounds, we use the approximate monotonicity identity from Step (1) to establish an energy estimate and elliptic estimates for solutions to the linearized equations. The elliptic estimates are needed to control the lapse, which verifies an elliptic equation in CMC gauge. If we were to instead use the parabolic lapse gauge mentioned above, then the elliptic estimates would be replaced with parabolic energy estimates; see Section 10. These estimates are at the level of the nondifferentiated linearized equations. A key aspect is that the energy can blow up at the *mild* rate $t^{-c\eta}$ as $t \downarrow 0$, where $c > 0$ is a universal constant and the constant $\eta \geq 0$ is a measure of how nonspatially-isotropic the Kasner background is. In particular, $\eta = 0$ for the FLRW background; see (1.9b) for the precise definition of η. Because of the energy blowup and because of the precise structure of the t-weights in the energies (see Definition 4.4), *the energy estimate is not in itself sufficient to establish linear stability results* that are consistent with the nonlinear stable blowup result provided by Theorem 1.1. Another key aspect of the energy estimate is that its proof crucially relies on the approximate monotonicity identity from Step (1). Without the combined strength of the cancellations and favorably signed integrals provided by the identity, we would have only been able to establish a more severe energy blowup-rate of t^{-C} as $t \downarrow 0$, where C is a large constant. Such a severe energy blowup-rate would not have been sufficient for establishing the linear stability of the solution (see Step (4) for clarification on this point), which in turn would have prevented us from controlling the nonlinear error terms that we encounter in the proof of Theorem 1.1. See Theorem 1.3 for a rough statement of the energy estimate and Theorem 6.1 for the precise statement.

(3) (*Mildly singular energy estimates up to top-order for near-*FLRW *backgrounds*). Next, we establish energy estimates and elliptic estimates for

the linear solution's higher spatial derivatives. Specifically, we show that the higher-order energies verify the same bounds as the base-level energy from Step (2). Since the Kasner backgrounds are spatially homogeneous, this step is analytically trivial though conceptually important, as will become clear in Step (4). Again, see Theorem 1.3 for a rough statement of the higher-order energy estimates and Theorem 6.1 for the precise statement. We emphasize that *these energy estimates do not incur any loss of derivatives*, which is of course crucial for closing the nonlinear problem.

(4) (*Linear stability and* AVTD *behavior*). Next, still within the class of nearly spatially isotropic Kasner backgrounds, we prove linear stability using the energy estimates and elliptic estimates from Step (3). In particular, we use the energy estimates for the linear solution and its higher-order spatial derivatives to establish *improved estimates for the linear solution at the lower derivative levels*, including convergence results consistent with the AVTD behavior stated in Theorem 1.1. In fact, this step constitutes a proof of the linear solution's AVTD behavior, which is a result that *does not directly follow from the singular energy estimates of the previous step*. This step incurs a loss of derivatives, roughly because in deriving the convergence results and proving the AVTD behavior, we "put all spatial derivative terms on the right-hand side" of the evolution equations. Thus, from the perspective of regularity, it is critically important that we have been able to independently establish the non-derivative-losing energy estimates from Step (3). It is also critically important that the energy blowup-rate $t^{-c\eta}$ from Step (3) is mild for nearly spatially isotropic Kasner backgrounds; the mild blowup-rate results in the following: many of the spatial-derivative-involving terms in the linearized equations are *integrable in time near the singularity*, which is the key to establishing linear stability. By integrable in time, we are roughly referring to the fact that $\int_{s=t}^{1} s^p \, ds < C_p$ whenever $p > -1$, uniformly for $t \in (0,1]$; the integrability in time of the error terms is one of the main analytical aspects of the solution's AVTD behavior.

(5) (*Control of nonlinear error terms*). To prove Theorem 1.1, we must similarly establish the following results for solutions to the nonlinear equations:
   (I) an approximate monotonicity identity;
   (II) a priori energy estimates and elliptic estimates up to top-order; and
   (III) improved/AVTD estimates at the lower derivative levels.

   In the usual fashion, we rely on a bootstrap argument to accomplish this. Most aspects of the proofs of (I)–(III) are similar to the linear analysis. The new feature is that we must also control the nonlinear error terms. It turns out that given the framework we have established in Steps (1)–(4), the nonlinear terms are not too difficult to control. The main thing that needs to be checked is that in all of the estimates, the "borderline" error terms (borderline in the sense of their blowup-rate as $t \downarrow 0$) generated by the nonlinear interactions can either

(i) be absorbed into the favorably-signed integrals generated by the approximate monotonicity identity from Step (1) or

(ii) are multiplied by a coefficient that remains L^∞-small as $t \downarrow 0$.

This allows us to prove that the energy blowup-rate in the nonlinear problem is also mild, roughly at worst $t^{-\delta}$, where $\delta > 0$ is small whenever the data are near the FLRW data. The detailed proofs are located in the companion article [59]. In Section 8, we outline all of the main ideas and show how to control several representative nonlinear error integrals, including a borderline one. All of the main ingredients needed to control the nonlinear terms and to prove the theorem are provided by Steps (1)–(4).

1.6. *The* (*generalized*) *Kasner solutions.* Before further discussing our results, we first formally introduce the Kasner solutions. They can be expressed as

$$\mathring{\mathbf{g}} = -dt^2 + \mathring{g}, \quad \mathring{g} = \sum_{i=1}^{3} t^{2q_i}(dx^i)^2, \quad \mathring{\phi} = A \ln t, \quad (t,x) \in (0,\infty) \times \mathbb{T}^3, \tag{1.6}$$

where the constants q_i are called the *Kasner exponents* and $A \geq 0$ is a constant denoting the value of $\partial_t \phi$ at $t = 1$. Note that we have the following identity (in a slight abuse of notation):

$$t\mathring{k}^i{}_j = -\mathrm{diag}(q_1, q_2, q_3). \tag{1.7}$$

The exponents q_i and A are constrained by the equations

$$\sum_{i=1}^{3} q_i = 1, \tag{1.8a}$$

$$\sum_{i=1}^{3} q_i^2 = 1 - A^2. \tag{1.8b}$$

(1.8a) corresponds to our gauge condition $k^a{}_a(t,x) = -t^{-1}$, while (1.8b) is a consequence of the gauge condition $k^a{}_a(t,x) = -t^{-1}$ and the Hamiltonian constraint equation (1.4a).

Remark 1.1. For convenience, in (1.6), we have written the Kasner metric in diagonal form. The diagonal form is a specific case of the more general form $-dt^2 + \sum_{I=1}^3 t^{2q_I}\omega^I \otimes \omega^I$ mentioned in Section 1.3 (where the ω^I are, by assumption, orthogonal with respect to the Kasner metric itself). The diagonal form can always be achieved by a change of spatial coordinates.

Exceptional cases aside, the Kasner solutions have Big Bang singularities along the past boundary $\{t = 0\}$ where their Kretschmann scalars blow up like[11] t^{-4}. In our study of solutions to the linearized equations, an important role is played by the constants $q_{\mathrm{Max}} > 0$ and $0 \leq \eta \leq \sqrt{\frac{2}{3}}$ defined by

$$q_{\mathrm{Max}} := \max\{q_1, q_2, q_3\}, \tag{1.9a}$$

$$\eta^2 := \sum_{i=1}^{3} q_i^2 - \frac{1}{3} = \sum_{i=1}^{3} \left(q_i - \frac{1}{3}\right)^2 = \frac{2}{3} - A^2. \tag{1.9b}$$

As we have mentioned, many of the results in this article hold only for nearly spatially isotropic Kasner backgrounds, that is, when all three q_i are near $1/3$. It is important to note that it is not even possible to have all three $q_i > 0$ in the absence of matter due to the Hamiltonian constraint. The nearly spatially isotropic assumption is equivalent to η being small. The analytic relevance of η is: for Kasner metrics (1.6), the trace-free part of the second fundamental form $\mathring{k}^i{}_j$ of Σ_t (see (3.1)), defined by $\hat{\mathring{k}}^i{}_j := \mathring{k}^i{}_j - \frac{1}{3}\mathring{k}^a{}_a I^i{}_j = \mathring{k}^i{}_j + \frac{1}{3}t^{-1} I^i{}_j$ (where $I^i{}_j = \mathrm{diag}(1,1,1)$ denotes the identity transformation), verifies (with $|\hat{\mathring{k}}|^2_{\mathring{g}} := \mathring{g}_{ab}\mathring{g}^{ij}\hat{\mathring{k}}^a{}_i\hat{\mathring{k}}^b{}_j$)

$$|\hat{\mathring{k}}|_{\mathring{g}} = \eta t^{-1}. \tag{1.10}$$

We again stress that the parameter η drives the blowup-rate of our L^2-based energies for the linear solutions as $t \downarrow 0$; see, for example, inequality (1.12)

1.7. *Rough statement of the main linear results and further discussion.* In this subsection, we summarize the main linear results of this paper. We start by summarizing the approximate monotonicity identity; see Theorem 5.1 for the precise statement. The proof is based on combining a collection of integration by parts identities in suitable proportions and judiciously using the constraint and lapse equations, which in total yields the cancellation of dangerous terms and the emergence of favorable ones.

THEOREM 1.2 (The approximate monotonicity identity; rough version). *Consider the Einstein-scalar field equations, written relative to* CMC*-transported-spatial coordinates* (*see Proposition* 3.1), *linearized* (*see Proposition* 3.2) *about any member of the Kasner family* (1.6), *and with initial data given at*

[11] One can compute that in terms of the Kasner exponents from (1.6), the Kretschmann scalar $\mathbf{Riem}^{\alpha\beta\gamma\delta}\mathbf{Riem}_{\alpha\beta\gamma\delta}$ is equal to

$$4t^{-4}\left\{\sum_{i=1}^{3} q_i^4 + \sum_{1\leq i<j\leq 3} q_i^2 q_j^2 + \sum_{i=1}^{3} q_i^2 - 2\sum_{i=1}^{3} q_i^3\right\} \geq 4t^{-4}\sum_{1\leq i<j\leq 3} q_i^2 q_j^2.$$

time 1. *Then with "Potential Terms" denoting the linearized lapse and its spatial derivatives, the spatial derivatives of the linearized scalar field, and the spatial derivatives of the linearized spatial metric; with "Solution" denoting the Potential Terms together with the linearized second fundamental form and the time derivative of the linearized scalar field; and with "Data" denoting quantities determined by the initial data, we have the following schematic identity, valid for* $t \in (0,1]$:

$$\begin{aligned}(1.11) \qquad \int_{\Sigma_t} |\text{Solution}|^2 \, dx = \int_{\Sigma_1} & \text{Data} \, dx \\ & - \int_{s=0}^{t} s^{-1} \int_{\Sigma_s} |\text{Potential Terms}|^2 \, dx \, ds \\ & + \int_{s=0}^{t} s^{-1} \int_{\Sigma_s} \text{Error terms} \, dx \, ds.\end{aligned}$$

Next, we roughly summarize the energy estimates that follow as a consequence of Theorem 1.2. See Theorem 6.1 for the precise statement of the energy estimates.

THEOREM 1.3 (Mildly singular energy estimates without derivative loss; rough version). *Consider the linearized equations from the statement of Theorem* 1.2. *Let* $\eta \geq 0$ *be as defined by* (1.9b). *Then there exists an energy* $\mathscr{E}_{(Total)}(t)$ *for the linear solution* (*see* (4.6e) *for the precise definition*), *whose* ***square*** *has the strength of the left-hand side of* (1.11), *and constants* $C > 0$ *and* $c > 0$ *such that the following estimate holds for* $t \in (0,1]$ *whenever the Kasner background is nearly spatially isotropic* (*that is, as long as* η *is sufficiently small*):

$$(1.12) \qquad \mathscr{E}_{(\text{Total})}(t) \leq C \mathscr{E}_{(\text{Total})}(t)(1) t^{-c\eta}.$$

Moreover, the higher-order spatial derivatives of the linear solution verify similar energy estimates featuring the same blowup-rate $t^{-c\eta}$.

Finally, we roughly summarize our linear stability results, whose proof relies on the energy estimates of Theorem 1.3. See Theorem 7.1 for the precise statement.

THEOREM 1.4 (Linear stability; rough version). *Let* $N \geq 2$ *be an integer. Consider a Kasner solution* $(1, \mathring{g}_{ij}, \mathring{k}^i_{\ j}, \mathring{\phi})$ (*where* 1 *is the Kasner lapse*), *and let* η *be as in Theorem* 1.3. *Consider data* (*at time* 1) *for the linearized* (*about the Kasner solution*) *system with enough regularity so that the norm* $\mathscr{S}_{(\text{Frame});N}$ (*see Definition* 4.3) *is initially finite, that is,* $\mathscr{S}_{(\text{Frame});N}(1) < \infty$. *Let* $(\nu, h_{ij}, \kappa^i_{\ j}, \varphi)$ *be a solution to the linearized* (*about the Kasner solution*) *equations of Proposition* 3.2*, where* ν *is the linearized variable corresponding*[12] *to* $n-1$*,* h_{ij} *is the linearized variable corresponding to* $g_{ij}-\mathring{g}_{ij}$*,* $\kappa^i{}_j$ *is the linearized variable corresponding to* $k^i{}_j-\mathring{k}^i{}_j$*, and* φ *is the linearized variable corresponding to* $\phi-\mathring{\phi}$*. Then there exists a* <u>*Kasner footprint state*</u> (*see below for further discussion*) *such that the linear solution converges towards it as* $t\downarrow 0$*. Specifically, there exist a symmetric type* $\binom{0}{2}$ *tensorfield* $h_{\mathrm{Regular}}\in H^{N-1}_{\mathrm{Frame}}(\mathbb{T}^3)$ (*the norms* $\|\cdot\|_{H^M_{\mathrm{Frame}}}$ *are defined by* (4.2))*, a type* $\binom{1}{1}$ *tensorfield* $K_{\mathrm{Bang}}\in H^{N-1}_{\mathrm{Frame}}(\mathbb{T}^3)$ *verifying* $(K_{\mathrm{Bang}})^a{}_a=0$*, and constants* $C>0$ *and* $c>0$ *such that if* η *is sufficiently small, then the following estimates hold*[13] *for* $t\in(0,1]$, $(i,j=1,2,3)$:

$$\|\nu\|_{H^{N-2}}\leq\frac{C}{\eta}\mathscr{S}_{(\mathrm{Frame});N}(1)t^{4/3-c\eta}, \tag{1.13a}$$

$$\left\|t^{-2q_j}h_{ij}+2\ln(t)(K_{\mathrm{Bang}})^i{}_j-(h_{\mathrm{Regular}})_{ij}\right\|_{H^{N-1}} \leq C\mathscr{S}_{(\mathrm{Frame});N}(1)t^{2/3-c\eta}\quad(\text{if } q_i=q_j), \tag{1.13b}$$

$$\left\|t^{-2q_j}h_{ij}+\frac{1}{q_i-q_j}t^{2(q_i-q_j)}(K_{\mathrm{Bang}})^i{}_j-(h_{\mathrm{Regular}})_{ij}\right\|_{H^{N-1}} \leq C\mathscr{S}_{(\mathrm{Frame});N}(1)t^{2/3-c\eta}\quad(\text{if } q_i\neq q_j), \tag{1.13c}$$

$$\left\|t\kappa^i{}_j-(K_{\mathrm{Bang}})^i{}_j\right\|_{H^{N-1}}\leq C\mathscr{S}_{(\mathrm{Frame});N}(1)t^{2/3-c\eta}, \tag{1.13d}$$

$$\|t\partial_t\varphi-\Psi_{\mathrm{Bang}}\|_{H^{N-1}}\leq C\mathscr{S}_{(\mathrm{Frame});N}(1)t^{2/3-c\eta}, \tag{1.13e}$$

$$\|\partial_i\varphi-\ln(t)\partial_i\Psi_{\mathrm{Bang}}\|_{H^{N-2}}\leq C\mathscr{S}_{(\mathrm{Frame});N}(1). \tag{1.13f}$$

We now explain the significance of the above convergence estimates, starting with (1.13a). We first recall that in studying the nonlinear solution, we decompose the spacetime metric as $\mathbf{g}=-n^2dt\otimes dt+g$ and that ν is the linearized variable corresponding to $n-1$. Hence, (1.13a) shows that at the linear level, the perturbation of the lapse converges to 0; that is, the lapse itself converges at the linear level to the Kasner state $n=1$. To further explain the convergence results stated in Theorem 1.4, we first explain what we mean by a "Kasner footprint state." Specifically, we mean a collection of variables $(\widetilde{\nu},\widetilde{h}_{ij},\widetilde{\kappa}^i{}_j,\widetilde{\varphi})$ defined by $\widetilde{\nu}=0$, $\widetilde{\kappa}^i{}_j=t^{-1}(K_{\mathrm{Bang}})^i{}_j$, $\widetilde{h}_{ij}=-2\ln(t)t^{2q_j}(K_{\mathrm{Bang}})^i{}_j+t^{2q_j}(h_{\mathrm{Regular}})_{ij}$ if $q_i=q_j$, $\widetilde{h}_{ij}=-\frac{1}{q_i+q_j}t^{2q_i}(K_{\mathrm{Bang}})^i{}_j+t^{2q_j}(h_{\mathrm{Regular}})_{ij}$ if $q_i\neq q_j$, $\partial_t\widetilde{\varphi}=t^{-1}\Psi_{\mathrm{Bang}}$, and $\partial_i\widetilde{\varphi}=\ln(t)\partial_i\Psi_{\mathrm{Bang}}$, where $(h_{\mathrm{Regular}})_{ij}$, $(K_{\mathrm{Bang}})^i{}_j$, and Ψ_{Bang} are

[12]See the beginning of Section 3.3 for further discussion on the linearization procedure and the linearized variables.

[13]On the left-hand sides of (1.13b)–(1.13c), we do not sum over i or j.

functions (of x) on $\mathbb{T}^3$. Note that the above definitions of the Kasner footprint states are obtained by setting the terms inside the norms on the left-hand sides of the estimates of Theorem 1.4 equal to 0. Roughly, Theorem 1.4 shows that the solutions to the linearized equations of Proposition 3.2 are asymptotic to a Kasner footprint state $(\widetilde{\nu}, \widetilde{h}_{ij}, \widetilde{\kappa}^i{}_j, \widetilde{\varphi})$ as $t \downarrow 0$. Note that the Kasner footprint states are generally *not solutions to the linear equations* of Proposition 3.2. For this reason, we will now explain why one might expect them to emerge as the "end states" of linear solutions and why the t-behaviors stated on the *left-hand* side of the estimates of Theorem 1.4 can be saturated. We will give two explanations, the first being completely heuristic and the second one rigorously illustrating the saturation of the t-behavior. First, one can easily check that given any (sufficiently regular) functions $(h_{\text{Regular}})_{ij}$, $(K_{\text{Bang}})^i{}_j$, and Ψ_{Bang} on $\mathbb{T}^3$, the corresponding Kasner footprint state is a solution to a *truncated* version of the linear equations of Proposition 3.2 in which all spatial derivative terms are set equal to 0. The truncated linear equations are linear analogs of the VTD equations mentioned at the end of the statement of Theorem 1.1. Thus, Theorem 1.4 shows that linear solutions converge towards solutions of the linear VTD system, which is quite natural since our proof of Theorem 1.4 relies on showing that spatial derivative terms become negligible as $t \downarrow 0$.

Our second explanation concerning the end state behavior of linear solutions is through the notion of variations of one-parameter families of Kasner solutions. For the sake of illustration, we only consider a one-parameter family of Kasner spatial metrics and mixed second fundamental forms. That is, for convenience, in this part of the discussion, we ignore the scalar field by setting it equal to 0; this will not have any substantial effect on the main ideas behind our discussion. Specifically, we consider the α-parametrized family (where $\alpha \in \mathbb{R}$) defined by $\mathring{g}_{ij}[\alpha] := \text{diag}(t^{2Q_1[\alpha]}, t^{2Q_2[\alpha]}, t^{2Q_3[\alpha]})$ and $\mathring{k}^i{}_j[\alpha] := -t^{-1}\text{diag}(Q_1[\alpha], Q_2[\alpha], Q_3[\alpha])$, where the $Q_i[\alpha]$ are a one-parameter family of Kasner exponents.[14] We assume that $Q_i[0] := q_i$, where the q_i are constants. For each fixed α, $(\mathring{g}_{ij}[\alpha], \mathring{k}^i{}_j[\alpha])$ is a solution to the nonlinear Einstein equations of Proposition 3.1 (where the lapse is identically 1 and the scalar field is identically 0). Thus, $(\mathring{g}_{ij}[\alpha], \mathring{k}^i{}_j[\alpha])$ can be viewed as a family of diagonal Kasner solutions that vary from "point to point," that is, that vary with α, in analogy with the x-dependent Kasner-type behavior of solutions to the nonlinear equations near singularities that was predicted in [11], [13]. To more fully explain the results of Theorem 1.4, we must also account for the following additional degrees of freedom: for each fixed α, we can perform a

[14]The Q_i must verify the constraint conditions (1.8a) and (1.8b), but this is not important for our discussion here.

change of spatial coordinates. We can account for this freedom by introducing a one-parameter family of invertible matrices $M^i{}_j[\alpha]$ (not depending on t) that represent a change of spatial coordinates at each fixed α. From these considerations, we see that a general picture of a family of Kasner solutions varying from point to point can be captured by a one-parameter family of Kasner solutions $(g_{ij}[\alpha], k^i{}_j[\alpha])$ of the form[15]

$$g_{ij}[\alpha] := M^a_i[\alpha] M^b_j[\alpha] \mathring{g}_{ab}[\alpha], \tag{1.14}$$

$$k^i{}_j[\alpha] := (M^{-1})^i{}_a[\alpha] M^b_j[\alpha] \mathring{k}^a{}_b[\alpha]. \tag{1.15}$$

In what follows, we will use the notation $Q_i'[0] := \frac{d}{d\alpha} Q_i[\alpha]|_{\alpha=0}$, and we use similar notation for other quantities that depend on α. We now compute that

$$g'[0] = M^\top[0] \cdot \mathring{g}'[0] \cdot M[0] + (M')^\top[0] \cdot \mathring{g}[0] \cdot M[0] + M^\top[0] \cdot \mathring{g}[0] \cdot M'[0], \tag{1.16}$$

$$\begin{aligned} k'[0] = M^{-1}[0] \cdot \mathring{k}'[0] \cdot M[0] - M^{-1}[0] \cdot M'[0] \cdot M^{-1}[0] \cdot \mathring{k}[0] \cdot M[0] \\ + M^{-1}[0] \cdot \mathring{k}[0] \cdot M'[0], \end{aligned} \tag{1.17}$$

where

$$\mathring{g}[0] = \mathrm{diag}(t^{2q_1}, t^{2q_2}, t^{2q_3}), \tag{1.18}$$

$$\mathring{g}'[0] = 2\ln(t) \mathrm{diag}(t^{2q_1} Q_1'[0], t^{2q_2} Q_2'[0], t^{2q_3} Q_3'[0]), \tag{1.19}$$

$$\mathring{k}[0] = -t^{-1} \mathrm{diag}(q_1, q_2, q_3), \tag{1.20}$$

$$\mathring{k}'[0] = -t^{-1} \mathrm{diag}(Q_1'[0], Q_2'[0], Q_3'[0]). \tag{1.21}$$

In (1.16)–(1.17), $\top$ denotes matrix transpose and $\cdot$ denotes matrix multiplication. We now compare the above computations with the results of Theorem 1.4. The key point is to observe that *the variations* $g'[0]$ *and* $k'[0]$ *solve the linearized Einstein equations*, where the background spatial metric and second fundamental form (about which the equations are linearized) are respectively $\mathring{g}[0]$ and $\mathring{k}[0]$. Indeed, one way to obtain the linearized equations is by differentiating a one-parameter family of nonlinear solutions with respect to the parameter; see Section 3.3 for further discussion on this point. Thus, to each one-parameter family $g_{ij}[\alpha]$, $k^i{}_j[\alpha]$ of the form (1.14)–(1.15), there exists an associated variation $g'[0]$ and $k'[0]$ that solves the corresponding linearized equations. Thus, the variations $g'[0]$ and $k'[0]$ are special (spatially homogeneous) examples of the Kasner footprint states stated in Theorem 1.4. To

[15] For fixed α, the form $g_{ij}[\alpha]$ of the Kasner spatial metric given by (1.14) is equivalent to the form $\sum_{I=1}^3 t^{2q_I} \omega^I \otimes \omega^I$ mentioned in Section 1.3.

further connect with the results of Theorem 1.4, we will investigate the structure of the variations. From (1.17), (1.20), and (1.21), it follows that $tk'[0]$ is a 3×3 matrix with constant entries. The key point is that this agrees with the fact that the limiting field K_{Bang} from Theorem 1.4 does not depend on t. Moreover, from (1.16), (1.18), and (1.19), we see that the entries of the matrix $g'[0]$ are sums of two kinds of terms: pure power-law terms proportional to factors of type t^p (where p is a constant), which come from the factors of $\mathring{g}[0]$ in (1.16), and similar power-law terms that are multiplied by a factor of $\ln(t)$, which come from the factor of $\mathring{g}'[0]$ in (1.16). This agrees with the limiting behavior of h_{ij} as $t \downarrow 0$ shown by the estimates (1.13b)–(1.13c). To summarize, our consideration of one-parameter Kasner families led us to conclude that all variations $g'[0](t)$ and $k'[0](t)$ are spatially homogeneous Kasner footprint states that are solutions to the linearization of the Einstein equations about the Kasner solution $(\mathring{g}[0](t), \mathring{k}[0](t))$. The results of Theorem 1.4 show that for near-FLRW backgrounds, all linear solutions are asymptotic to x-dependent Kasner footprint states whose time behavior at each fixed x is similar to the time behavior of one of the variations. Similar results hold for the scalar field, as is shown by the estimates (1.13e) and (1.13f). In total, the above picture is closely aligned with the vision of [13], [11], in which the end state of nonlinear solutions was posited to be a family Kasner-like solutions parametrized by the spatial point x.

Consistent with the nonlinear stable blowup result provided by Theorem 1.1, we could also extend the linear stability results of Theorem 1.4 to apply when the background solutions are near-FLRW as measured by a Sobolev norm (and hence are spatially dependent). We do not provide such an extension here because it would significantly lengthen the paper without contributing substantially to the main ideas. A related issue connected to the nonlinear problem is that in our proof of the existence and curvature blowup aspects Theorem 1.1, we do not rely on having precise knowledge of the solution's "end state" (that is, the asymptotics near $\{t = 0\}$) in advance; it suffices to control the difference between the perturbed solution and the FLRW solution. Put differently, in proving Theorem 1.1, we could derive the sharp asymptotics/convergence results as $t \downarrow 0$ as a separate argument, after we have already shown that the solution exists for $(t, x) \in (0, 1] \times \mathbb{T}^3$ and that the Kretschmann scalar blows up as $t \downarrow 0$. For this reason, our proof of Theorem 1.1 would allow for the following margin of error: the proof would go through if we controlled the difference between the perturbed solution and any near-FLRW Kasner solution rather than the perturbed solution and the FLRW solution.

1.8. *Previous work on singularities.* Previous work has provided related results showing the stability of singular solutions to the Einstein equations in various contexts, but only under under symmetry assumptions that reduce the problem to the study of $1+1$ dimensional PDEs[16] [25], [42], [56], [57]. There also is a body of work that provides the construction of (but not the stability of) singularity-containing solutions to select nonlinear Einstein-matter systems, but only under the assumption of symmetry [41], [51], [43], [8], [16], [63], [14], [1] and/or spatial analyticity [7], [27]. Readers can also consult [2] for a more general well-posedness result for singular initial value problems that applies to a class of symmetric hyperbolic quasilinear systems in more than one spatial dimension. More precisely, in [2], the authors prescribe Sobolev-class asymptotics featuring singular behavior. The main result of [2] is the existence of a Sobolev-class solution that realizes the singular asymptotics. We note, however, that [2] does not treat Einstein's equations. A related approach to studying Big Bang singularities involves devising a formulation of Einstein's equations that allows one to solve a Cauchy problem with initial data given on the singular hypersurface $\{t=0\}$ itself;[17] see, for example, [9], [26], [48], [49], [65], [66], [67]. In some cases, these works included a proof that the singular solutions exhibit AVTD behavior. Readers can consult [53] for a precise comparison of these results as well as an extension of them to prove the existence of singular solutions to the Einstein-vacuum equations with Gowdy symmetry.[18]

In contrast to the regular Cauchy problem studied here and in the companion article [59], the above works are based on prescribing the asymptotics as $t \downarrow 0$ and then constructing a solution that achieves those asymptotics. Most of those works are based on solving a Fuchsian PDE system that is singular at $\{t=0\}$. We now describe some aspects of the Fuchsian approach. A representative work is [1], in which the authors construct singular solutions to the Einstein-vacuum equations[19] with $\mathbb{T}^2$ symmetry under the polarized or half-polarized condition. In Section 9, we provide a simple model problem suggesting that results similar to those of [1] might also hold for the Einstein-scalar field system without symmetry assumptions. The Fuchsian PDEs[20] treated

[16]There also are stable singularity formation results in the class of spatially homogeneous solutions (in which case the equations reduce to ODEs); see [54] or [69] for an overview.

[17]This method is based on formulating the equations in terms of a rescaled metric, conformal to the physical spacetime metric, in such a way that the rescaled metric remains regular throughout the entire evolution. As such, this method can be viewed as an extension of Friedrich's *conformal method* [36], [37].

[18]Gowdy solutions are a subset of the $\mathbb{T}^2$-symmetric solutions characterized by the vanishing of the *twist constants* $(\mathbf{g}^{-1})^{\mu\mu'}\boldsymbol{\epsilon}_{\alpha\beta\mu\nu}\mathbf{X}^\alpha\mathbf{Y}^\beta\mathbf{D}_{\mu'}\mathbf{X}^\nu$ and $(\mathbf{g}^{-1})^{\mu\mu'}\boldsymbol{\epsilon}_{\alpha\beta\mu\nu}\mathbf{X}^\alpha\mathbf{Y}^\beta\mathbf{D}_{\mu'}\mathbf{Y}^\nu$, where $\boldsymbol{\epsilon}$ is the volume form of $\mathbf{g}$ and $\mathbf{X}$ and $\mathbf{Y}$ are the Killing fields corresponding to the two symmetries.

[19]More general Fuchsian systems in one spatial dimension are also treated in [1].

[20]Specifically, the PDEs are the $\mathbb{T}^2$-symmetric polarized or half-polarized Einstein-vacuum equations in areal coordinates with the singularity at $\{t=0\}$.

in [1] are of the form

$$A^0(t,x,u)t\partial_t u + A^1(t,x,u)t\partial_x u + B(t,x,u)u = f(t,x,u), \tag{1.22}$$

where u is the array of unknowns, A^α and B are symmetric matrices (the energy estimates rely on the symmetric hyperbolic framework), and f is an array, all of which verify a collection of technical assumptions. The analysis in [1] is based on splitting the solution as $u = u_0 + w$, where u_0 is the "leading order" part and w is an error term that one would like to show is small compared to u_0 as $t \downarrow 0$. An important technical assumption made in [1], which is used for deriving energy estimates, is that for small w, one can split $A^0(t,x,u_0+w) = A_0^0(x,u_0) + A_1^0(t,x,u_0+w)$, where $A_0^0(x,u_0)$ is symmetric positive definite, and the map $w \to A_1^0(t,x,u_0+w)$ maps certain time-weighted Sobolev spaces into other time-weighted Sobolev spaces. There are various methods for constructing u_0. The most relevant way in the context of the present article is to choose u_0 to be a prescribed solution to a truncated "VTD version" of (1.22) in which the spatial derivative terms are discarded. This approach is complementary to the one taken in the present article and [59], in which we show that AVTD behavior *dynamically emerges* in solutions to the nontruncated equations. From the VTD system and (1.22), one computes that the error term w solves an "error equation" depending on u_0. The main result of [1] is that under suitable additional assumptions, there exists a solution w to the error equation that becomes small relative to u_0 as $t \downarrow 0$ and that w is unique within appropriate time-weighted Sobolev spaces. The main idea of the proof is to derive uniform a priori symmetric hyperbolic energy estimates for a sequence $\{w_n\}_{n=1}^\infty$ of error equation solutions on intervals of the form $[t_n, \delta]$. More precisely, the w_n solve a standard symmetric hyperbolic Cauchy problem (to the future) with 0 initial data at time t_n. Here, $\delta > 0$ is a small constant and $\{t_n\}_{n=1}^\infty$ is a sequence of times decreasing to 0. A key aspect of the analysis in [1] is that the authors were able to close their estimates by inserting time weights by hand into the energies. More precisely, in the approach of [1], one derives energy estimates for $t^{-P}w_n$, where t^{-P} is a diagonal matrix whose nonzero entries are well-chosen negative powers of t that are allowed to depend on x (that is, $P = P(x)$). Another aspect of the approach of [1] is that the energies are weighted by an additional overall scalar factor of $e^{-\kappa t^\gamma}$, where κ and γ are positive constants. The time weights must be chosen to be compatible with the nonlinearities in the sense that the nonlinear error integrals arising in the energy estimates must be controllable. When successfully implemented, this leads to controlled energy growth towards the future (away from the singularity) in a neighborhood of the singularity. In particular, for well-chosen t-weights (as we illustrate in Remark 9.1, there is some freedom in choosing them), one can derive uniform estimates for the $\{w_n\}_{n=1}^\infty$ showing that the weighted energies cannot grow too fast towards the future; see Section 9 for a very simple linear model problem. Then through a standard limiting procedure, one can produce a solution w to the error equation that exists on the interval $(0, \delta]$, and it is unique within suitable time-weighted Sobolev spaces.

Although the Fuchsian approach furnishes the existence of a set of solutions with singularities, it is inadequate for treating the true stability problem of solving down towards $\{t = 0\}$ starting from Cauchy data for u given along a hypersurface $\{t = \text{const}\}$ with const > 0. One difficulty that we encounter in our study of the Einstein equations, which we stressed at the beginning, is that in order to synchronize the singularity across space, one cannot work with a purely hyperbolic formulation of the equations such as the one afforded by wave coordinates; gauges involving an infinite speed of propagation, such as the elliptic and parabolic ones for the lapse employed in the present article and in [59], seem essential. Hence, our approach to proving stability lies outside of the standard Fuchsian framework, which applies only to hyperbolic equations. Moreover, the Fuchsian strategy of inserting suitable time weights by hand into the energies is not sufficient for deriving our stability results because some of the terms in the equations are too singular to be treated in this fashion; see our discussion in Section 1.9 for further discussion on this point, where we highlight similar difficulties that would arise in an attempt to extend our approach to prove stability results for far-from-FLRW solutions. For near-FLRW solutions, our approach is viable only because of the cancellations that occur in our approximate monotonicity identity, which are tied to the special structure of the Einstein-scalar field system in our gauges.

The scalar field and stiff fluid matter models have some special properties that we exploit in deriving our results. We describe some of these properties in more detail in Section 1.9. In particular, we expect that our approximate monotonicity/stability results do not hold for general matter models. Actually, as we now explain, for certain fluid models, Ringström obtained rigorous results showing that solutions behave in a drastically nonmonotonic fashion. In [55], Ringström studied fluids verifying the equation of state $p = c_s^2 \rho$, where the constant c_s verifies $0 < c_s \leq 1$ and physically represents the speed of sound. For the Euler-Einstein equations with a sub-stiff equation of state (that is, with $0 < c_s < 1$), he showed that spatially homogeneous solutions with Bianchi IX symmetry[21] generically (that is, for non-Taub solutions) have limit points in the approach towards the singularity that must be either *vacuum* Bianchi type I (that is, vacuum Kasner), vacuum Bianchi type VII_0, or vacuum Bianchi

[21] Members of the Bianchi symmetry classes are spatially homogeneous, and hence the corresponding solutions depend on only a time variable. For a precise definition of these symmetry classes and the others that we mention, readers can consult [24].

type II. In particular, Ringström's work showed that a sub-stiff fluid has a negligible effect on Bianchi IX solutions near the singularity. Furthermore, he showed that almost all such solutions are *oscillatory* in the sense that there are at least three distinct limit points, *which stands in stark contrast to the approximately monotonic behavior of our linear solutions and the nonlinear solutions in* [59].

Ringström's work [55] also applied to the Einstein-vacuum equations in Bianchi IX symmetry and thus yielded the first examples of the oscillatory behavior conjectured in the work [44] of Belinsky, Khalatnikov, and Lifschitz (BKL). Specifically, in [44], the authors gave heuristic arguments suggesting that general solutions to the Einstein-vacuum equations containing incomplete timelike geodesics should exhibit highly oscillatory behavior near the boundary where the geodesics terminate. Moreover, their arguments suggested that the boundary should be a spacelike singularity. These so-called "BKL conjectures"[22] have been seminal in stimulating the investigation of solutions to Einstein's equations near singularities. However, as we now explain, despite Ringström's work, there is immense controversy surrounding the conjectures. First, they are false as stated because of, for example, the existence of Taub solutions, which develop a Cauchy horizon[23] rather than a true singularity. One might be tempted to weaken the conjectures by replacing the phrase "general solutions" with "generic solutions." However, Luk has constructed [45] a class of solutions to the Einstein-vacuum equations without symmetry assumptions such that the boundary of the maximal development contains *a null portion* along which the metric remains C^0 but its Christoffel symbols blow-up in L^2. His examples, which are stable in a certain sense, contradict the BKL vision of spacelike singularities. Moreover, outside of the class of spatially homogeneous solutions, there are currently no examples of Einstein-vacuum solutions that are rigorously known to exhibit the kind of oscillatory behavior near a singularity conjectured in [44]. In total, given the present-day state of knowledge, it is not clear to what extent the vision of BKL is realized in Einstein-vacuum solutions.

In the opposite direction, we recall the aforementioned work of Belinsky and Khalatnikov [13], who were the first to suggest the existence of nonspatially homogeneous *approximately monotonic* singular solutions to the Einstein-scalar field system. In a later article [11], Barrow argued that fluids verifying the

[22]The statements in [44] are somewhat vague, and thus it is imprecise to refer to them as "conjectures."

[23]Roughly, a Cauchy horizon is a boundary along which the solution remains regular but beyond which it cannot be continued uniquely as a solution due to lack of information for how to continue.

equation of state $p = c_s^2 \rho$ (where c_s is a nonnegative constant) should induce a similar effect if and only if $c_s = 1$; he referred to the mollifying effect of a stiff fluid as *quiescent cosmology*. The first rigorous construction of such solutions without symmetry was provided by the aforementioned work of Andersson and Rendall [7]. They constructed a family of spatially analytic solutions to the Einstein-scalar field and Einstein-stiff fluid systems that have Big Bang singularities and that exhibit approximately monotonic behavior near them. Their proof was based on a two-step process. In the first step, they constructed a family of spatially analytic solutions to VTD equations, which were obtained by throwing away the spatial derivative terms from the Einstein-matter equations.[24] In the second step, they constructed spatially analytic solutions to the Einstein-matter equations by writing the true solution as a solution to the VTD equations plus error terms that were shown, by Fuchsian analysis, to go to 0 as $t \downarrow 0$. The results of [7] were extended to higher dimensions and other matter models in [27]. The family of solutions constructed in this fashion is large in the sense that its number of degrees of freedom coincides with the number of free functions in the Einstein initial data. However, since the results are based on prescribing the asymptotics near the Big Bang within the class of spatially analytic solutions, they are not true stable singularity formation results. In particular, the work left open the possibility that the map from the set of spatially analytic asymptotic states realized in [7] to the set of Cauchy data (say at $t = 1$) might be highly degenerate in the sense that it cannot be extended as a map (with reasonable properties) between more physically relevant function spaces such as Sobolev spaces; see, however, the discussion in Section 9. The primary ingredient needed to upgrade the work of Andersson and Rendall to a true stable singularity formation result corresponding to solving a regular Cauchy problem is a suitable statement of linear stability, strong enough to control the nonlinear terms. Our linear stability result (Theorem 7.1) provides this missing ingredient in the near-FLRW case.

1.9. *Comments on other matter models, higher dimensions, and the analysis of far-from-*FLRW*-solutions.* The scalar field and stiff fluid matter models have two important properties, described in the next paragraph, that allow us to prove the stability results of the present paper and those of [59]. We anticipate that other matter models with similar properties might allow for proofs of similar results. Readers can consult [27] for a class of candidate matter models, where the authors used Fuchsian techniques to construct families of nonspatially homogeneous solutions with Big Bang singularities to various nonlinear

[24]In [7], the Einstein equations were formulated relative to a Gaussian coordinate system in which the spacetime metric takes the form $\mathbf{g} = -dt^2 + g_{ab}dx^a dx^b$.

Einstein-matter systems. We note that the authors' construction also applied to the Einstein-vacuum equations in ten or more spatial dimensions and thus yielded rigorous examples of the nonoscillatory and nonspatially-homogeneous solutions that were heuristically argued to exist in [28]. The existence of these spatially inhomogeneous Kasner-like vacuum solutions is relevant for the discussion three paragraphs below.

The first important property of the scalar field and stiff fluid matter models is simply that they allow for the existence of spatially isotropic and nearly spatially isotropic Kasner solutions to the Einstein-matter system. We recall that nearly spatially isotropic Kasner solutions have second fundamental forms with trace-free parts that blow up at the rate ηt^{-1}, where η is small (see (1.10)), and that this blowup-rate ultimately leads to the mild energy blowup-rate (1.12). We now contrast this against the case of the Einstein-vacuum equations in three spatial dimensions. In vacuum, we have $A = 0$ in (1.8b), and thus (1.9b) and (1.10) imply that the trace-free part of the Kasner second fundamental form blows up at the rate $\sqrt{\frac{2}{3}}t^{-1}$. Combining this blowup-rate with the methods of this paper, one would only be able to derive energy estimates in the spirit of (1.12) showing that the energy blows up like $t^{-c\sqrt{2/3}}$ as $t \downarrow 0$. Unfortunately, such a bound for the energy does not appear to be useful for controlling error terms in the nonlinear problem. In fact, an energy blowup-rate of $t^{-c\sqrt{2/3}}$ seems to be insufficient even for proving linear stability results of the type proved in Theorem 1.4; see the next paragraph for further discussion on this point. The second important property of the scalar field matter model is that *its time derivatives do not appear* in the evolution equations for the metric (equations (3.7a)–(3.7b)) nor in the elliptic PDE for the lapse (equation (3.10)). This property is closely tied to the fact that the characteristics of the scalar field agree with those of the Einstein field equations (that is, the characteristics for the Einstein-scalar field system are precisely the null hypersurfaces relative to $\mathbf{g}$). This property plays a critically important role in allowing us to prove our stability results because to close our estimates, we rely on the fact that spatial derivatives are small compared to time derivatives, at least at the lower derivative levels. Although the stiff fluid matter model exhibits similar good properties, fluids verifying the sub-stiff equation of state $p = c_s^2\rho$ with $0 < c_s < 1$ do not enjoy these properties, even if the fluid is irrotational (roughly because the sound cones are necessarily distinct from the gravitational null cones in the sub-stiff case). This is consistent with the oscillatory behavior for solutions to the Euler-Einstein system observed by Ringström [55] in the Bianchi IX symmetry class when $0 < c_s < 1$; see the discussion in Section 1.8.

We now further explain some of the obstacles to deriving stability results for the Einstein-scalar field system in the far-from-FLRW case (e.g., when η is no longer small in the linear problem). Although our methods could be used to obtain estimates for solutions to the linearized systems, they do not seem to be strong enough to allow for a proof of linear stability or stable blowup in the nonlinear problem. Our goal is to highlight why, for parameters corresponding to far-from-spatially isotropic Kasner backgrounds, our methods do *not* allow us to prove that $|t\kappa^i{}_j|$ remains uniformly bounded over the interval $t \in (0,1]$, where κ is the linearized second fundamental form variable. In the nonlinear problem, the same difficulty would arise, and it is tantamount to not even being able to recover (in the context of a bootstrap argument) the blowup-rate of t^{-1} exhibited by the trace-free part of the second fundamental form of a Kasner metric. In the nonlinear problem, such a bad estimate would lead (by a Gronwall estimate) to energy estimates that are drastically worse than (1.12): the top-order energies would be allowed to blow up faster than data $\times t^{-C}$ for all constants $C > 0$, where "data" denotes a term that is controlled by the initial data. Consequently, our entire approach to linear and nonlinear stability would break down, and in the nonlinear problem, we would not even be able to show that the solution exists near $\{t = 0\}$. To explain the source of the difficulty, we first explain how we prove the uniform boundedness of $|t\kappa^i{}_j|$ in the nearly spatially isotropic case. The main idea is that we can use the evolution equation for $\kappa^i{}_j$ (see (3.16b)) and the mildly singular energy estimates of Theorem 6.1 to prove (see (7.1a)) the estimate $|\partial_t(t\kappa^i{}_j)| \lesssim \text{data} \times t^{-1/3-c\eta}$. The key point is that the right-hand side is integrable in time over the time interval $(0,1]$ for η small. That is, if η is small, then we can express $t\kappa^i{}_j$ as an integral of $\partial_t(t\kappa^i{}_j)$ and use the time-integrability to obtain the desired bound $|t\kappa^i{}_j| \lesssim \text{data}$. In contrast, if η is large, then the bound $|\partial_t(t\kappa^i{}_j)| \lesssim \text{data} \times t^{-1/3-c\eta}$ does *not* imply the time-integrability of $|\partial_t(t\kappa^i{}_j)|$, and thus our approach does not work in its current form.

Finally, we make some comments on extending our stability results to higher dimensions. For brevity, we limit our discussion to the Einstein-scalar field and Einstein-vacuum systems. For the Einstein-scalar field system in any number of spatial dimensions, we expect that the proofs of our linear and nonlinear stability results (in the near-FLRW setting) would go through without any significant changes. Moreover, in the case of the Einstein-vacuum equations in n spatial dimensions with n sufficiently large, there exists a class of Kasner solutions for which it might be possible to prove sufficiently strong versions of linear stability (similar to the linear stability results of the present paper), suitable for deriving nonlinear stable blow-up results like those proved in [59]. As we mentioned above, the existence of (but not the stability of) nonspatially homogeneous solutions to the Einstein-vacuum equations with Big Bang singularities has already been shown in [27] when $n \geq 10$. We now provide some motivation for our speculation on the existence of stable

Einstein-vacuum singularities. First, we note that it is possible to derive an approximate monotonicity identity for the linearized (around a vacuum Kasner solution) Einstein-vacuum equations that parallels the results for the Einstein-scalar field model provided by Theorems 5.1 and 10.1. More precisely, the approximate monotonicity identities of Theorems 5.1 and 10.1 remain valid in the vacuum case; just set the scalar field and its amplitude A equal to 0 in the equations. However, one faces the difficulty that in vacuum, the trace-free part of the Kasner second fundamental form has large size $(1-1/n)^{1/2}t^{-1}$, a fact that follows from the vacuum Kasner exponent constraints:

$$\sum_{i=1}^{n} q_i = 1, \qquad \sum_{i=1}^{n} q_i^2 = 1. \tag{1.23}$$

The expression $(1-1/n)^{1/2}t^{-1}$ suggests that the energy blowup-rate for solutions to the linearized Einstein-vacuum equations becomes worse as $n \to \infty$, which seems to be an obstacle to proving stable blowup. Nonetheless, it might be possible to overcome this difficulty, at least in a certain regime. The main idea is the following observation: the proof of the energy blowup-rate can be somewhat sharpened compared to the proof that leads to inequality (1.12). More precisely, many of the error terms that contribute to the blowup-rate of the energy can be controlled by the *eigenvalues* of the second fundamental form[25] and its trace-free part. In particular, a more careful analysis, not carried out in this article,[26] shows that most error terms in the energy estimates that involve the second fundamental form can cause the energies to blow up at worst like $\lesssim$ data $\times\, t^{-c\alpha}$, where $c > 0$ is a universal constant independent of n and $\alpha := \max_{i=1}^{n}\{|q_i|\}$. Moreover, it is not difficult to see that there exists a family of vacuum Kasner solutions such that $\alpha \downarrow 0$ as $n \to \infty$. However, there are a few anomalous terms in the energy estimates that could in principle lead to a blowup-rate that is worse than data $\times\, t^{-c\alpha}$, and these terms are therefore a potential obstacle for proving stability. If one were able to sufficiently control the anomalous terms, then we expect that one would be able to prove that Kasner solutions with α sufficiently small[27] are linearly and nonlinearly stable in a neighborhood of the Big Bang by using the methods of the present article and those of [59]. We note that for fixed large n, only a small portion of the vacuum Kasner solutions could in principle be shown to be stable through this approach. If the argument goes through, then it would also be interesting

[25]Specifically, we mean the version of the second fundamental form with one index up and one down.

[26]See the proof of inequality (6.5) regarding the role that the eigenvalues play in deriving energy estimates.

[27]One can think of $1/n$ as a parameter that one would like to choose to be sufficiently small to close the estimates.

to discover the threshold value of n beyond which the stable Kasner solutions exist; it is conceivable that the threshold value $n \geq 10$ from [27], which is sufficient for the *existence* of nonspatially homogeneous solutions, is not large enough to imply their *stability*.

1.10. *A related instance in which monotonicity led to global results*. We now describe the work [5] by Andersson and Moncrief, in which they proved global existence results for the Einstein-vacuum equations using techniques that have some overlap with the ones used in the present article and in [59]. In the next paragraph, we compare and contrast the approach of [5] with that of the present work. We first describe their result in more detail. In [5], the authors proved a future-global existence theorem (that is, in the expanding direction) for perturbations of spatially compact versions of FLRW-like vacuum spacetimes in $1 + m$ dimensions for $m \geq 3$. The background solutions were of the "continuously self-similar" form $-dt^2 + \frac{t^2}{m^2}\gamma$, where the spatial metric γ verifies the *Einstein condition* $\mathrm{Ric} = -\frac{m-1}{m^2}\gamma$, where Ric is the Ricci curvature of γ. Readers can also consult [4], [52] for proofs of the results of [5] in the case $m = 3$, where unlike in [5] and the present article, the latter two works rely on curvature-based energies constructed from the Bel-Robinson tensor. Andersson and Moncrief made some technical assumptions on γ, notably one[28] that they called being "stable." This condition states that the eigenvalues of the operator $h_{ij} \to -\Delta_\gamma h_{ij} - 2R_{i\ j}^{\ a\ b} h_{ab}$, which appears in linearized versions of the evolution equations, are nonnegative. Here, h_{ij} is a symmetric type $\binom{0}{2}$ tensor and $R_{i\ j}^{\ a\ b}$ is the Riemann curvature tensor of γ. In our proof of nonlinear stable blowup [59], terms like $2R_{i\ j}^{\ a\ b} h_{ab}$ also appear, but we are able to treat them as lower-order nonlinear error terms. That is, we do not have to work with combinations such as $-\Delta_\gamma h_{ij} - 2R_{i\ j}^{\ a\ b} h_{ab}$; see Section 8.9 for an overview of how we handle nonlinear error terms. In [5], the authors also proved that a rescaled version of the perturbed spatial metric converges to an element of the moduli space of γ. In the case $m = 3$, the Einstein condition implies that γ has constant negative sectional curvature, and Mostow's rigidity theorem implies that the moduli space is trivial. Hence, the rescaled solution in fact converges to the background solution. In contrast, in our nonlinear results [59] and in the linear convergence results of Theorem 7.1, the family of possible end states (corresponding to the asymptotic behavior of the solution near the Big Bang) is much larger. In the nonlinear problem, the family of course includes members of the Kasner family (1.6). However, as we described below Theorem 1.4, even for

[28] The authors also made additional assumptions. Specifically, they assumed that either the moduli space of γ is trivial or that γ is contained in an integrable moduli space of Einstein structures.

the linear problem, it also includes[29] a much larger family of "x-dependent" Kasner-like states. As Andersson and Moncrief stated in [5], their work is closely related to the Fisher–Moncrief work [34], in which the authors carried out the linear stability analysis. Specifically, in [34], Fischer-Moncrief found a reduced Hamiltonian description of the Einstein-vacuum flow (see also the works [29], [31], [32], [33], [30], [34], [47], [46], [6] for related results) that applied to a family of spacetimes containing CMC hypersurfaces. Their Hamiltonian was the *volume* functional of constant-time hypersurfaces Σ_t, where, as in the present article, the Σ_t were CMC hypersurfaces. They showed that the Hamiltonian is monotonic along the flow of their reduced equations, that its critical points are precisely the continuously self-similar metrics $-dt^2 + \frac{t^2}{m^2}\gamma$ mentioned above (where γ verifies the Einstein condition), and, crucially for the linear stability analysis (on which global existence result [5] relied), that its second variation is positive definite when γ is stable in the sense described above.

The analysis in [5] has some features in common with the present work, including its reliance on CMC foliations to reveal monotonicity and its focus on studying the solution at the level of the metric. Moreover, the energies for the spatial metric and second fundamental form defined in [5, §7] are reminiscent of the metric energies that we use in the present article (see (4.6a) and (8.2a)). However, the energy identities of [5, §7] do not involve subtle cancellations of the type that we observe in deriving the approximate monotonicity identities of Propositions 5.2 and 10.3. A related fact is that in [5], Andersson and Moncrief were able to close their proof by bounding the lapse in terms of the second fundamental form via standard elliptic estimates. In contrast, to control the lapse, we rely on the approximate monotonicity identities and the AVTD-type estimates described in Step (4) of Section 1.5. Another notable difference is that unlike our work here, the results of [5] are based on spatial harmonic coordinates; see Remark 5.3 for additional comments about those coordinates.

2. **Notation and conventions**

In this section, we summarize some notation and conventions that we use throughout the article.

[29] More accurately, we do not rigorously prove that the family includes x-dependent end states. However, we recall here the work [7] described in Section 1.8, in which Andersson and Rendall constructed solutions with end states that are analytic in x (with nontrivial x dependence). Based on their work, our results here, and the results of [59], we expect that it might be possible to remove the analyticity assumption (perhaps only in the near-FLRW regime), which would yield new information about the set of *achievable* end states; see also Section 9.

2.1. *Indices.* Greek "spacetime" indices $\alpha, \beta, \cdots$ take on the values $0, 1, 2, 3$, while Latin "spatial" indices $a, b, \cdots$ take on the values $1, 2, 3$. Repeated indices are summed over (from 0 to 3 if they are Greek, and from 1 to 3 if they are Latin). We use the same conventions for primed indices such as a' as we do for their nonprimed counterparts. When working with the nonlinear equations in CMC-transported spatial coordinates gauge or the parabolic lapse gauges, spatial indices are lowered and raised with the Riemannian 3-metric g_{ij} and its inverse g^{ij}. When working with the linearized equations, we will always explicitly raise and lower indices with the background Kasner 3-metric $\mathring{g}_{ij}$ and its inverse $\mathring{g}^{ij}$.

2.2. *Spacetime tensorfields and Σ_t-tangent tensorfields.* We denote spacetime tensorfields $\mathbf{T}_{\nu_1 \cdots \nu_n}{}^{\mu_1 \cdots \mu_m}$ in bold font. In the nonlinear equations, we denote the $\mathbf{g}$-orthogonal projection of $\mathbf{T}_{\nu_1 \cdots \nu_n}{}^{\mu_1 \cdots \mu_m}$ onto the constant-time hypersurfaces $\Sigma_t := \{(s, x) \in \mathbb{R} \times \mathbb{T}^3 \mid s = t\}$ in nonbold font: $T_{b_1 \cdots b_n}{}^{a_1 \cdots a_m}$. We also denote general Σ_t-tangent tensorfields in nonbold font.

2.3. *Coordinate systems and differential operators.* We often work in a fixed standard local coordinate system (x^1, x^2, x^3) on $\mathbb{T}^3$. The vectorfields $\partial_j := \frac{\partial}{\partial x^j}$ are globally well defined even though the coordinates themselves are not. Hence, in a slight abuse of notation, we use $\{\partial_1, \partial_2, \partial_3\}$ to denote the globally defined vectorfield frame. We denote the corresponding dual frame by $\{dx^1, dx^2, dx^3\}$. As we described in Section 1.4, the spatial coordinates can be transported along the unit normal to Σ_t, thus producing a local coordinate system (x^0, x^1, x^2, x^3) on manifolds-with-boundary of the form $(T, 1] \times \mathbb{T}^3$, and we often write t instead of x^0. The corresponding vectorfield frame on $(T, 1] \times \mathbb{T}^3$ is $\{\partial_0, \partial_1, \partial_2, \partial_3\}$, and the corresponding dual frame is $\{dx^0, dx^1, dx^2, dx^3\}$. Relative to this frame, the Kasner metrics $\mathring{\mathbf{g}}$ are of the form (1.6). The symbol ∂_μ denotes the frame derivative $\frac{\partial}{\partial x^\mu}$, and we often write ∂_t instead of ∂_0 and dt instead of dx^0. Most of our equations and estimates are stated relative to the frame $\left\{\partial_\mu\right\}_{\mu=0,1,2,3}$ and dual frame $\left\{dx^\mu\right\}_{\mu=0,1,2,3}$.

We use the notation ∂f to denote the *spatial coordinate* gradient of the function f. Similarly, if Θ is a Σ_t-tangent one-form, then $\partial \Theta$ denotes the Σ_t-tangent type $\binom{0}{2}$ tensorfield with components $\partial_i \Theta_j$ relative to the frame described above.

If $\vec{I} = (n_1, n_2, n_3)$ is a triple of nonnegative integers, then we define the *spatial* multi-indexed differential operator $\partial_{\vec{I}}$ by $\partial_{\vec{I}} := \partial_1^{n_1} \partial_2^{n_2} \partial_3^{n_3}$. The notation $|\vec{I}| := n_1 + n_2 + n_3$ denotes the order of $\vec{I}$.

Throughout, $\mathbf{D}$ denotes the Levi–Civita connection of $\mathbf{g}$. We write

$$
\begin{aligned}
\text{(2.1)} \quad \mathbf{D}_\nu \mathbf{T}_{\nu_1 \cdots \nu_n}{}^{\mu_1 \cdots \mu_m} = \partial_\nu \mathbf{T}_{\nu_1 \cdots \nu_n}{}^{\mu_1 \cdots \mu_m} &+ \sum_{r=1}^{m} \mathbf{\Gamma}_{\nu\ \alpha}^{\ \mu_r} \mathbf{T}_{\nu_1 \cdots \nu_n}{}^{\mu_1 \cdots \mu_{r-1} \alpha \mu_{r+1} \cdots \mu_m} \\
&- \sum_{r=1}^{n} \mathbf{\Gamma}_{\nu\ \nu_r}^{\ \alpha} \mathbf{T}_{\nu_1 \cdots \nu_{r-1} \alpha \nu_{r+1} \cdots \nu_n}{}^{\mu_1 \cdots \mu_m}
\end{aligned}
$$

to denote a component of the covariant derivative of a tensorfield $\mathbf{T}$ (with components $\mathbf{T}_{\nu_1\cdots\nu_n}{}^{\mu_1\cdots\mu_m}$) defined on $(T,1]\times\mathbb{T}^3$. The Christoffel symbols of $\mathbf{g}$, which we denote by $\mathbf{\Gamma}_{\mu\ \nu}^{\ \lambda}$, are defined by

$$\mathbf{\Gamma}_{\mu\ \nu}^{\ \lambda} := \frac{1}{2}(\mathbf{g}^{-1})^{\lambda\sigma}\left\{\partial_\mu \mathbf{g}_{\sigma\nu} + \partial_\nu \mathbf{g}_{\mu\sigma} - \partial_\sigma \mathbf{g}_{\mu\nu}\right\}. \tag{2.2}$$

We use similar notation to denote the covariant derivative of a Σ_t-tangent tensorfield T (with components $T_{b_1\cdots b_n}{}^{a_1\cdots a_m}$) with respect to the Levi–Civita connection ∇ of the Riemannian metric g. The Christoffel symbols of g, which we denote by $\Gamma_{j\ k}^{\ i}$, are defined by

$$\Gamma_{j\ k}^{\ i} := \frac{1}{2}g^{ia}\left\{\partial_j g_{ak} + \partial_k g_{ja} - \partial_a g_{jk}\right\}. \tag{2.3}$$

2.4. *Integrals and L^2 norms.* Throughout this subsection, f denotes a scalar function defined on the hypersurface $\Sigma_t = \{(s,x)\in\mathbb{R}\times\mathbb{T}^3 \mid s=t\}$. We define

$$\int_{\Sigma_t} f\,dx := \int_{\mathbb{T}^3} f(t,x^1,x^2,x^3)\,dx. \tag{2.4}$$

Above, the notation "$\int_{\mathbb{T}^3} f\,dx$" denotes the integral of f over $\mathbb{T}^3$ with respect to the measure corresponding to the volume form of the *standard Euclidean metric* E on $\mathbb{T}^3$, which has the components $E_{ij} = \mathrm{diag}(1,1,1)$ relative to the coordinate frame described in Section 2.3. All of our Sobolev norms are built out of the (spatial) L^2 norms of scalar quantities (which may be the components of a tensorfield). We define the standard L^2 norm $\|\cdot\|_{L^2}$ over Σ_t as follows:

$$\|f\|_{L^2} = \|f\|_{L^2}(t) := \left(\int_{\Sigma_t} f^2\,dx\right)^{1/2}. \tag{2.5}$$

For integers $N\geq 0$, we define the standard H^N norm $\|\cdot\|_{H^N}$ over Σ_t as follows:

$$\|f\|_{H^N} = \|f\|_{H^N}(t) := \left(\sum_{|\vec{I}|\leq N}\left\|\partial_{\vec{I}} f\right\|_{L^2}^2(t)\right)^{1/2}. \tag{2.6}$$

2.5. *Constants.* We use C and c to denote positive numerical constants that are free to vary from line to line. If A and B are two quantities, then we often write

$$A \lesssim B \tag{2.7}$$

to indicate that "there exists a constant $C>0$ such that $A\leq CB$." We write $A=\mathcal{O}(B)$ to indicate that $|A|\leq CB$. Some of the constants C and c in our estimates are allowed to depend on the parameter N which, roughly speaking, represents the number of times that the equations have been differentiated with spatial derivatives.

3. **The Einstein-scalar field equations in CMC-transported spatial coordinates and the linearized equations**

In this section, we provide a standard formulation of the Einstein-scalar field equations relative to CMC-transported spatial coordinates. We then linearize the equations around a Kasner solution (1.6).

3.1. *Preliminary discussion.* We begin by stating some basic facts concerning the formulation of the equations. The fundamental unknowns are g, k, n, and ϕ, where g and n are as in (1.5), and k is the second fundamental form of the hypersurfaces Σ_t. More precisely, the Σ_t-tangent type $\binom{0}{2}$ tensorfield k is defined by requiring that following relation holds for all vectorfields X, Y tangent to Σ_t:

$$\mathbf{g}(\mathbf{D}_X \hat{\mathbf{N}}, Y) = -k(X, Y), \tag{3.1}$$

where $\mathbf{D}$ is the Levi–Civita connection of $\mathbf{g}$ and

$$\hat{\mathbf{N}} := n^{-1}\partial_t \tag{3.2}$$

is the future-directed normal to Σ_t. It is a standard fact that k is symmetric:

$$k(X, Y) = k(Y, X). \tag{3.3}$$

Let ∇ denote the Levi–Civita connection of g. The action of the Levi–Civita connection $\mathbf{D}$ of $\mathbf{g}$ can be decomposed into the action of ∇ and k as follows:

$$\mathbf{D}_X Y = \nabla_X Y - k(X, Y)\hat{\mathbf{N}}. \tag{3.4}$$

Remark 3.1. (The mixed form of k verifies equations with favorable structure and the meaning of $\partial_\alpha k^i{}_j$.) When working with the components of k, *we will always write it in the mixed form* $k^i{}_j := g^{ia}k_{aj}$ *with the first index upstairs and the second one downstairs.* The reason is that the nonlinear evolution and constraint equations verified by the components $k^i{}_j$ have a more favorable structure than the corresponding equations verified by k_{ij}. For this reason, throughout the article, we use the notation $\partial_\alpha k^i{}_j := \partial_\alpha(k^i{}_j)$.

3.2. *The Einstein-scalar field equations in* CMC-*transported spatial coordinates.* In the following proposition, we formulate the Einstein-scalar field equations (1.1a)–(1.1b) relative to CMC-transported spatial coordinates.

Proposition 3.1 (The Einstein-scalar field equations in CMC-transported spatial coordinates). *In* CMC-*transported spatial coordinates normalized by*

$$k^a{}_a(t, x) = -t^{-1}, \tag{3.5}$$

the Einstein-scalar field system comprises the following equations.

The ***Hamiltonian and momentum constraint equations*** *are respectively*

$$R - k^a_{\ b}k^b_{\ a} + \underbrace{(k^a_{\ a})^2}_{t^{-2}} = \overbrace{(n^{-1}\partial_t\phi)^2 + g^{ab}\nabla_a\phi\nabla_b\phi}^{2\mathbf{T}(\hat{\mathbf{N}},\hat{\mathbf{N}})}, \tag{3.6a}$$

$$\nabla_a k^a_{\ i} - \underbrace{\nabla_i k^a_{\ a}}_{0} = \underbrace{-n^{-1}\partial_t\phi\nabla_i\phi}_{-\mathbf{T}(\hat{\mathbf{N}},\partial_i)}, \tag{3.6b}$$

where R *denotes the scalar curvature of* g_{ij}.

The ***metric evolution equations*** *are*

$$\partial_t g_{ij} = -2ng_{ia}k^a_{\ j}, \tag{3.7a}$$

$$\partial_t k^i_{\ j} = -g^{ia}\nabla_a\nabla_j n + n\Big\{\mathrm{Ric}^i_{\ j} + \underbrace{k^a_{\ a}}_{-t^{-1}}\, k^i_{\ j} \underbrace{-g^{ia}\nabla_a\phi\nabla_j\phi}_{-T^i_{\ j}+(1/2)I^i_{\ j}\mathbf{T}}\Big\}, \tag{3.7b}$$

where $\mathrm{Ric}^i_{\ j}$ *denotes the Ricci curvature of* g_{ij} *(see* (3.22)*),* $I^i_{\ j} = \mathrm{diag}(1,1,1)$ *denotes the identity transformation, and* $\mathbf{T} := (\mathbf{g}^{-1})^{\alpha\beta}\mathbf{T}_{\alpha\beta}$ *denotes the trace of the energy-momentum tensor* (1.2).

The ***volume form factor*** $\sqrt{\mathrm{det}g}$ *verifies the auxiliary equation*[30]

$$\partial_t \ln\Big(t^{-1}\sqrt{\mathrm{det}g}\Big) = \frac{n-1}{t}. \tag{3.8}$$

The ***scalar field wave equation*** *is*

$$\overbrace{-n^{-1}\partial_t(n^{-1}\partial_t\phi)}^{-\mathbf{D}_{\hat{\mathbf{N}}}\mathbf{D}_{\hat{\mathbf{N}}}\phi} + g^{ab}\nabla_a\nabla_b\phi = \overbrace{\frac{1}{t}n^{-1}\partial_t\phi}^{-k^a_{\ a}\mathbf{D}_{\hat{\mathbf{N}}}\phi} - n^{-1}g^{ab}\nabla_a n\nabla_b\phi. \tag{3.9}$$

The ***elliptic lapse equation***[31] *is*

$$\begin{aligned} g^{ab}\nabla_a\nabla_b(n-1) = (n-1)\Big\{R + \underbrace{(k^a_{\ a})^2}_{t^{-2}} - g^{ab}\nabla_a\phi\nabla_b\phi\Big\} \\ + R - g^{ab}\nabla_a\phi\nabla_b\phi + \underbrace{(k^a_{\ a})^2 - \partial_t(k^a_{\ a})}_{0}. \end{aligned} \tag{3.10}$$

The gauge condition (3.5) *and the constraint equations* (3.6a)–(3.6b) *are preserved by the flow of the remaining equations if they are verified by the data.*

[30] This equation, which we do not use in the present article, is implied by (3.7a) and the CMC condition $k^a_{\ a} = -t^{-1}$.

[31] Below, when we linearize the equations, we will view $n-1$ as a linearly small quantity. Hence, we prefer to write (3.10) as an equation in $n-1$.

Proof of Proposition 3.1. It is well known that the constraint equations (3.6a)–(3.6b) follow from (1.1a); see, for example, [70, Ch. 10], and note that our k has the opposite sign convention of the one in [70]. It is also well known that equations (3.7a)–(3.9) follow from (1.1a)–(1.1b); see, for example, [62, §6.2] or [64, §10 of Ch. 18]. To derive (3.10), we take the trace of (3.7b) and use the CMC condition $k^a{}_a = -t^{-1}$. The preservation of the gauge condition and constraints is a standard result that can be derived from a straightforward modification of the argument presented in [3, Th. 4.2]. □

3.3. *The linearization procedure and the linearly small quantities.* In our linear analysis, we work with the "linearly small quantities" defined just below in Definition 3.1. In the definition, g denotes the (Riemannian) 3-metric from Proposition 3.1, $k^i{}_j$ denotes its mixed second fundamental form, $\mathring{g}$ denotes the 3-metric of the Kasner solution (see (1.6)), $\mathring{k}^i{}_j$ denotes its mixed second fundamental form (see (1.7)), and similarly for the other quantities. Before stating the definition of the linearly small quantities, we first make some remarks about how one can linearize the equations of Proposition 3.1 around a given solution. There are two ways that this can be achieved. Both approaches lead to the same system of linear PDEs but conceptually are somewhat different. The first way, which is manifestly invariant, is through the notion of one-parameter family of solutions to the equations, similar to our discussion below Theorem 1.4. That is, one can consider an α-parametrized family of solutions $(n[\alpha], g[\alpha], k[\alpha], \phi[\alpha])$ to the nonlinear equations of Proposition 3.1 such that $(n[0], g[0], k[0], \phi[0])$ is the background solution around which one would like to linearize. We set $n'[\alpha] := \frac{d}{d\alpha} n[\alpha]$ and similarly for the other variables. One can then differentiate the nonlinear equations with respect to α and set $\alpha = 0$ to deduce that the variations $(n'[0], g'[0], k'[0], \phi'[0])$ solve a system of linear PDEs whose coefficients depend on $(n[0], g[0], k[0], \phi[0])$. The system thus obtained is the linearization of the Einstein-scalar field equations in CMC-transported spatial coordinates gauge about the background solution $(n[0], g[0], k[0], \phi[0])$.

The second way to derive the linearized system is to perform a first-order Taylor expansion of the nonlinear equations of Proposition 3.1 about a given solution, in our case a Kasner solution $(1, \mathring{g}, \mathring{k}, \mathring{\phi})$, where 1 is the Kasner lapse. Equivalently, in the nonlinear equations, one decomposes the nonlinear spatial metric g_{ij} as $g_{ij} = \mathring{g}_{ij} + h_{ij}$ (where h_{ij} is the "linearly small" metric perturbation) and similarly for the other solution variables and then discards all terms that are quadratic or smaller in the perturbation variables (where the derivatives of the perturbation variables are also considered to be linearly small). After one discards the quadratic-or-higher-order small terms and accounts for the fact that the background Kasner solution is a solution to the nonlinear equations, what remains is a system of linear PDEs whose coefficients depend on the Kasner solution. This is the approach that we take in the proof of Proposition 3.2. Though seemingly less invariant than the first approach, it is straightforward to see that it yields the same linear PDE system.

Having made these remarks, we now define the linearly small "perturbation variables" that play a role in our derivation of the linearized equations.

Definition 3.1 (Linearly small quantities). We define (for $a, b, i, j = 1, 2, 3$)

$$h_{ij} := g_{ij} - \mathring{g}_{ij}, \tag{3.11a}$$

$$^{(h)}\Gamma_{a\ b}^{\ i} := \frac{1}{2}\mathring{g}^{ic}\left\{\partial_a h_{cb} + \partial_b h_{ac} - \partial_c h_{ab}\right\}, \tag{3.11b}$$

$$^{(h)}R := -\frac{1}{2}\mathring{g}^{ab}\mathring{g}^{ef}\partial_e\partial_f h_{ab} + \mathring{g}^{ef}\partial_a {}^{(h)}\Gamma_{e\ f}^{\ a}, \tag{3.11c}$$

$$^{(h)}\mathrm{Ric}^i_{\ j} := -\frac{1}{2}\mathring{g}^{ia}\mathring{g}^{ef}\partial_e\partial_f h_{ja} + \frac{1}{2}\mathring{g}^{ef}\partial_j {}^{(h)}\Gamma_{e\ f}^{\ i} + \frac{1}{2}\mathring{g}^{ia}\mathring{g}_{jb}\mathring{g}^{ef}\partial_a {}^{(h)}\Gamma_{e\ f}^{\ b}, \tag{3.11d}$$

$$\kappa^i_{\ j} := k^i_{\ j} - \mathring{k}^i_{\ j}, \tag{3.11e}$$

$$\varphi := \phi - \mathring{\phi}, \tag{3.11f}$$

$$\nu := n - 1. \tag{3.11g}$$

Remark 3.2 (Justification of Definition 3.1). The main point is that for solutions to the nonlinear equations that are near the Kasner solution (1.6), all of the quantities defined in Definition 3.1 are linearly small in the sense described above Definition 3.1.

Remark 3.3. Below and throughout, $\hat{T}$ denotes the trace-free part of the Σ_t-tangent tenor T.

Remark 3.4. Note that κ is trace-free, that is,

$$\kappa = \hat{\kappa}. \tag{3.12}$$

(3.12) follows from definition (3.11e), the CMC condition $k^a_{\ a}(t, x) = -t^{-1}$, and the fact that $\mathring{k}^a_{\ a}(t, x) = -t^{-1}$.

3.4. *The linearized Einstein-scalar field equations in* CMC-*transported spatial coordinates.* In the next proposition, we use the procedure described just above Definition 3.1 to linearize the equations of Proposition 3.1 around a given Kasner solution (1.6).

Proposition 3.2 (The linearized Einstein-scalar field equations in CMC-transported spatial coordinates). *Consider the equations of Proposition* 3.1 *linearized around a Kasner solution* (1.6). *The linearized equations in the unknowns* $(\nu, h, \kappa, \varphi)$, *which are functions of* $(t, x) \in (0, \infty) \times \mathbb{T}^3$, *take the following form* (*see Definition* 3.1 *for the definitions of some of the quantities*).

The ***linearized constant mean curvature condition*** *is*

$$\kappa^a{}_a = 0. \tag{3.13}$$

The ***linearized versions of the Hamiltonian and momentum constraint equations*** (3.6a)–(3.6b) *are*

$$t^{2(h)}R - 2(t\hat{\mathring{k}}^a{}_b)(t\kappa^b{}_a) - 2At\partial_t\varphi + 2A^2\nu = 0, \tag{3.14a}$$

$$\partial_a(t\kappa^a{}_i) = -A\partial_i\varphi - {}^{(h)}\Gamma_a{}^a{}_b(t\hat{\mathring{k}}^b{}_i) + {}^{(h)}\Gamma_a{}^b{}_i(t\hat{\mathring{k}}^a{}_b), \tag{3.14b}$$

$$\mathring{g}^{ab}\partial_a(t\kappa^i{}_b) = -A\mathring{g}^{ia}\partial_a\varphi - \mathring{g}^{ab(h)}\Gamma_a{}^i{}_c(t\hat{\mathring{k}}^c{}_b) + \mathring{g}^{ab(h)}\Gamma_a{}^c{}_b(t\hat{\mathring{k}}^i{}_c), \tag{3.14c}$$

where the constant $0 \leq A \leq \sqrt{2/3}$ *is defined by* (1.8b).

The ***linearized version of the lapse equation*** (3.10) *can be expressed in either of the following two forms*:

$$2A(t\partial_t\varphi) + 2(t\hat{\mathring{k}}^a{}_b)(t\kappa^b{}_a) = t^2\mathring{g}^{ab}\partial_a\partial_b\nu + (2A^2 - 1)\nu, \tag{3.15a}$$

$$t^2\mathring{g}^{ab}\partial_a\partial_b\nu - \nu = t^{2(h)}R. \tag{3.15b}$$

Equation (3.14a) *can be used to show that* (3.15a) *is equivalent to* (3.15b).

The ***linearized versions of the metric evolution equations*** (3.7a)–(3.7b) *are*

$$\partial_t h_{ij} = -2t^{-1}(t\mathring{k}^a{}_j)h_{ia} - 2t^{-1}\mathring{g}_{ia}(t\kappa^a{}_j) - 2t^{-1}\mathring{g}_{ia}(t\mathring{k}^a{}_j)\nu, \tag{3.16a}$$

$$\partial_t(t\kappa^i{}_j) = -t\mathring{g}^{ia}\partial_a\partial_j\nu - t^{-1}(t\mathring{k}^i{}_j)\nu + t^{(h)}\mathrm{Ric}^i{}_j. \tag{3.16b}$$

The ***linearized version of the scalar field wave equation*** (3.9) *is*

$$-\partial_t(t\partial_t\varphi) + t\mathring{g}^{ab}\partial_a\partial_b\varphi = -A\partial_t\nu + At^{-1}\nu. \tag{3.17}$$

Remark 3.5 (An alternate approach). One could adopt an alternate approach to the proof of our stability results in which the product $n^{-1}\partial_t\phi$ is treated as an independent quantity. In such an approach, one would not generate terms in the equations that depend on the time derivative of the lapse. This would simplify some aspects of the analysis. For example, upon linearizing the equations under the alternate approach, one would not generate the term $\partial_t\nu$, which appears on the right-hand side (3.17). The alternate approach would not have any substantial effect on our main results. For example, notice that $\partial_t\nu$ does not appear in the approximate monotonicity identity stated in Theorem 5.1 (though, under the approach of this paper, $\partial_t\nu$ does play a role in its *proof*). The alternate approach is closer in spirit to the approach that we take in [59] in our study of the Einstein-stiff fluid system, in which we avoid having to treat the time derivative of the lapse in the evolution equations.

Remark 3.6. Equation (3.14b) is the linearized version of the constraint $\nabla_a k^a{}_i = -n^{-1}\partial_t\phi\nabla_i\phi$, while equation (3.14c) is the linearized version of $\nabla^a k^i{}_a = -n^{-1}\partial_t\phi g^{ia}\nabla_a\phi$. We use both of these equations when deriving estimates.

Remark 3.7 (Propagation of L^2 regularity). In deriving the equations of Proposition 3.2, we have linearized a version of the Einstein-scalar field system written relative to a dynamic system of coordinates that is adapted to the nonlinear flow. It is for this reason that our approximate monotonicity identity for linear solutions, which we derive below in Proposition 5.2, should be viewed as providing relevant information about the L^2 regularity of the nonlinear solution. In particular, the proof of Proposition 5.2 can be modified in a straightforward fashion to yield a coercive integral identity for the nonlinear equations, consistent with well-posedness relative to the CMC-transported spatial coordinates gauge.

Proof of Proposition 3.2. We first note that (3.13) follows from (3.12). We will derive three more equations in detail. The remaining equations can be derived using similar arguments and we omit those details. The overall strategy is to consider the equations of Proposition 3.1 and to expand the Riemannian metric g as an order 0 "Kasner term" and a perturbation term as follows: $g_{ij} = \mathring{g}_{ij} + h_{ij}$, and similarly for $(tk^i{}_j, \phi, n)$. We then discard all terms that are quadratic or higher-order in the perturbations, which yields the proposition. Since this proof features the spatial metrics g and $\mathring{g}$, to avoid confusion, we will denote the components of the inverse Kasner spatial metric by $(\mathring{g}^{-1})^{ij}$ rather than $\mathring{g}^{ij}$.

As our first detailed example, we derive (3.17). We start by expanding the scalar field wave equation (3.9) as follows:

(3.18)
$$-\partial_t(t\partial_t\phi) + n^2 t g^{ab}\nabla_a\nabla_b\phi = \frac{(n-1)}{t}t\partial_t\phi - \frac{(\partial_t n)}{n}t\partial_t\phi - ntg^{ab}\nabla_a n\nabla_b\phi.$$

Using (3.18), we compute that

(3.19)
$$\begin{aligned}
&- \partial_t(t\partial_t\phi - A) + tg^{ab}\partial_a\partial_b\phi + t(n+1)(n-1)g^{ab}\nabla_a\nabla_b\phi \\
&= A\frac{(n-1)}{t} - A\partial_t n \\
&\quad + n^2 t g^{ab}\Gamma_{a\ b}^{\ j}\partial_j\phi + \frac{(n-1)}{t}(t\partial_t\phi - A) \\
&\quad - (\partial_t n)(t\partial_t\phi - A) + (\partial_t n)\frac{n-1}{n}t\partial_t\phi - tg^{ab}\partial_a n\partial_b\phi - (n-1)tg^{ab}\partial_a n\partial_b\phi.
\end{aligned}$$

We now discard the quadratically small terms, that is, the term

$$t(n+1)(n-1)g^{ab}\nabla_a\nabla_b\phi$$

and the terms on the last two lines of (3.19), which, in view of Definition 3.1, yields (3.17).

Next, we derive equation (3.16b). To this end, we expand the evolution equation (3.7b) for $k^i{}_j$ as follows:

$$\begin{aligned}\partial_t(tk^i{}_j) &= -tg^{ia}\partial_a\partial_j n + tg^{ia}\Gamma_a{}^b{}_j\partial_b n - \frac{n-1}{t}(tk^i{}_j)\\ &\quad + t\mathrm{Ric}^i{}_j + t(n-1)\mathrm{Ric}^i{}_j - tng^{ia}\partial_a\phi\partial_j\phi.\end{aligned}\tag{3.20}$$

From (3.20), we compute that

$$\begin{aligned}\partial_t\left\{tk^i{}_j - t\mathring{k}^i{}_j\right\} &= -t(\mathring{g}^{-1})^{ia}\partial_a\partial_j n - \frac{n-1}{t}(t\mathring{k}^i{}_j) + t\mathrm{Ric}^i{}_j\\ &\quad - t\left\{g^{ia} - (\mathring{g}^{-1})^{ia}\right\}\partial_a\partial_j n + tg^{ia}\Gamma_a{}^b{}_j\partial_b n\\ &\quad - \frac{n-1}{t}(tk^i{}_j - t\mathring{k}^i{}_j) + t(n-1)\mathrm{Ric}^i{}_j - tng^{ia}\partial_a\phi\partial_j\phi.\end{aligned}\tag{3.21}$$

Next, we note that it is straightforward to see that in Definition 3.1, ${}^{(h)}\Gamma_a{}^i{}_b$ is the linearization of the Christoffel symbol $\Gamma_a{}^i{}_b$ (see (2.3)) around the Kasner solution, and similarly for ${}^{(h)}\mathrm{Ric}^i{}_j$ and ${}^{(h)}R$. We have obtained the latter two linearizations from the standard expression

$$\mathrm{Ric}^i{}_j = g^{ic}\partial_a\Gamma_c{}^a{}_j - g^{ic}\partial_c\Gamma_j{}^a{}_a + g^{ic}\Gamma_a{}^a{}_b\Gamma_c{}^b{}_j - g^{ic}\Gamma_c{}^a{}_b\Gamma_a{}^b{}_j\tag{3.22}$$

for the Ricci curvature of g in terms of its Christoffel symbols (2.3) and the definition $R := \mathrm{Ric}^a{}_a$. From these facts and Definition 3.1, it follows that the linearly small terms in (3.21) are the term on the left-hand side, the first two terms on the right-hand side, and $t^{(h)}\mathrm{Ric}^i{}_j$, which we obtain from linearizing the third term $t\mathrm{Ric}^i{}_j$ on the right-hand side of (3.21). Discarding the remaining terms, we obtain the linearized equation (3.16b) as desired.

As our final example, we derive the linearized Hamiltonian constraint equation (3.14a). We first expand equation (3.6a) to deduce

$$\begin{aligned}t^2R - (tk^a{}_b)(tk^b{}_a) + 1 &= (t\partial_t\phi)^2 - 2(t\partial_t\phi)^2(n-1)\\ &\quad + \frac{(2n+1)}{n^2}(t\partial_t\phi)^2(n-1)^2 + t^2g^{ab}\partial_a\phi\partial_b\phi.\end{aligned}\tag{3.23}$$

Using (3.23), the CMC condition $tk^a{}_a = -1$, the identity $(t\mathring{k}^a{}_b)(t\mathring{k}^b{}_a) = \sum_{i=1}^3 q_i^2$, and the exponent constraints (1.8a)–(1.8b), we compute that

(3.24)

$$t^2 R - 2(t\hat{k}^a{}_b - t\hat{\mathring{k}}^a{}_b)(t\mathring{k}^b{}_a) + \underbrace{1 - \sum_{i=1}^{3} q_i^2 - A^2}_{=0}$$
$$= 2A(t\partial_t\phi - A) - 2A^2(n-1)$$
$$+ (t\partial_t\phi - A)^2 - 4A(n-1)(t\partial_t\phi - A)(t\partial_t\phi + A) - 2(n-1)(t\partial_t\phi - A)^2$$
$$+ \frac{(2n+1)}{n^2}(t\partial_t\phi)^2(n-1)^2 + t^2 g^{ab}\partial_a\phi\partial_b\phi.$$

With the help of Definition 3.1, we see that the linearly small terms in (3.24) are the two terms on the first line of the right-hand side, the term

$$2(t\hat{k}^a{}_b - t\hat{\mathring{k}}^a{}_b)(t\mathring{k}^b{}_a)$$

on the left-hand side, and the term $t^{2\,(h)}R$ obtained from linearizing the first term $t^2 R$ on the left-hand side. Discarding the remaining terms, we obtain the linearized equation (3.14a) as desired. This completes our proof of Proposition 3.2. □

4. Norms and energies

In this short section, we define the norms and energies that play a role in our analysis of linear solutions.

4.1. *Pointwise norms.* We will use the following two norms.

Definition 4.1 (Pointwise norms). Let T be a type $\binom{m}{n}$ Σ_t-tangent tensor with components $T_{b_1\cdots b_n}{}^{a_1\cdots a_m}$. Then $|T|_{\text{Frame}}$ denotes the following norm (involving the *components* of T relative to the transported coordinate frame):

$$|T|^2_{\text{Frame}} := \sum_{a_1=1}^{3} \cdots \sum_{a_m=1}^{3} \sum_{b_1=1}^{3} \cdots \sum_{b_n=1}^{3} \left| T_{b_1\cdots b_n}{}^{a_1\cdots a_m} \right|^2. \tag{4.1a}$$

$|T|_{\mathring{g}}$ denotes the $\mathring{g}$-norm of T, where $\mathring{g}$ is the background Kasner spatial metric from (1.6):

$$|T|^2_{\mathring{g}} := \mathring{g}_{a_1 a_1'} \cdots \mathring{g}_{a_m a_m'} (\mathring{g}^{-1})^{b_1 b_1'} \cdots (\mathring{g}^{-1})^{b_n b_n'} T_{b_1\cdots b_n}{}^{a_1\cdots a_m} T_{b_1'\cdots b_n'}{}^{a_1'\cdots a_m'}. \tag{4.1b}$$

4.2. *Sobolev and Lebesgue norms.* In our analysis, we will use the Sobolev norms $\|\cdot\|_{H^M_{\text{Frame}}}$ and the Lebesgue norm $\|\cdot\|_{L^2_{\mathring{g}}}$ defined below in Definition 4.2. The norms $\|\cdot\|_{H^M_{\text{Frame}}}$ are "less geometric" than the energies of Definition 4.4 because their definition involves the components of tensorfields relative to the transported coordinate frame rather than invariant quantities. The norms $\|\cdot\|_{H^M_{\text{Frame}}}$ are important for the proof of linear stability (see Theorem 7.1).

Definition 4.2 (Sobolev and Lebesgue norms). Let T be a type $\binom{m}{n}$ Σ_t-tangent tensorfield with components $T_{b_1\cdots b_n}{}^{a_1\cdots a_m}$. We define

$$\|T\|_{H^M_{\text{Frame}}} = \|T\|_{H^M_{\text{Frame}}}(t) := \sum_{|\vec{I}|\leq M} \left\| \left|\partial_{\vec{I}} T(t,\cdot)\right|_{\text{Frame}} \right\|_{L^2}, \tag{4.2}$$

where $\|f\|_{L^2}$ is defined in (2.5), $\vec{I}$ denotes a spatial coordinate derivative multi-index (see Section 2.3), and

$$(\partial_{\vec{I}} T)_{b_1\cdots b_n}{}^{a_1\cdots a_m} := \partial_{\vec{I}}(T_{b_1\cdots b_n}{}^{a_1\cdots a_m}). \tag{4.3}$$

We sometimes use the notation $\|T\|_{L^2_{\text{Frame}}}$ in place of $\|T\|_{H^0_{\text{Frame}}}$.

We also define the Lebesgue norm

$$\|T\|_{L^2_{\mathring{g}}} = \|T\|_{L^2_{\mathring{g}}}(t) := \left\| |T(t,\cdot)|_{\mathring{g}} \right\|_{L^2}, \tag{4.4}$$

where $|T(t,\cdot)|_{\mathring{g}}$ is defined in (4.1b).

Remark 4.1. If T is a scalar function, then we often write $|T|$ instead of $|T|_{\text{Frame}}$ or $|T|_{\mathring{g}}$, $\|T\|_{H^M}$ instead of $\|T\|_{H^M_{\text{Frame}}}$, and $\|T\|_{L^2}$ instead of $\|T\|_{L^2_{\text{Frame}}}$ or $\|T\|_{L^2_{\mathring{g}}}$ since for scalar functions, there is no danger of confusion over how to measure the size of T.

Definition 4.3 (Solution norms). The specific norms that are most relevant for the linear solutions under study are as follows:

$$\begin{aligned}\mathscr{S}_{(\text{Frame});M}(t) := {}& \|t\mathsf{k}\|_{H^M_{\text{Frame}}} + \|\partial h\|_{H^M_{\text{Frame}}} + \|t\partial_t\varphi\|_{H^M_{\text{Frame}}} \\ &+ t^{2/3}\|\partial\varphi\|_{H^M_{\text{Frame}}} + \sum_{p=0}^{2} t^{(2/3)p}\,\|\mathsf{v}\|_{H^{M+p}}.\end{aligned} \tag{4.5}$$

4.3. *Energies.* Our monotonicity identities and our energy estimates involve the following energies for the linearized variables.

Definition 4.4 (Energies). For $t \in (0,1]$, we define $\mathscr{E}_{(\text{Metric})}(t) \geq 0, \ldots, \mathscr{E}_{(\text{Total});\theta}(t) \geq 0$ as follows:

$$\mathscr{E}^2_{(\text{Metric})}(t) := \int_{\Sigma_t} |t\mathsf{k}|^2_{\mathring{g}} + \frac{1}{4}|t\partial h|^2_{\mathring{g}}\,dx, \tag{4.6a}$$

$$\mathscr{E}^2_{(\text{Scalar})}(t) := \int_{\Sigma_t} (t\partial_t\varphi)^2 + |t\partial\varphi|^2_{\mathring{g}}\,dx, \tag{4.6b}$$

$$\mathscr{E}^2_{(\partial\,\text{Lapse})}(t) := \int_{\Sigma_t} |t\partial\mathsf{v}|^2_{\mathring{g}}\,dx, \tag{4.6c}$$

$$\mathscr{E}^2_{(\text{Lapse})}(t) := \int_{\Sigma_t} \mathsf{v}^2\,dx, \tag{4.6d}$$

$$\begin{aligned}\mathscr{E}^2_{(\text{Total});\theta}(t) := {}& \mathscr{E}^2_{(\text{Scalar})}(t) + \mathscr{E}^2_{(\partial\,\text{Lapse})}(t) \\ &+ (1-A^2)\mathscr{E}^2_{(\text{Lapse})}(t) + \theta\mathscr{E}^2_{(\text{Metric})}(t),\end{aligned} \tag{4.6e}$$

where the constant $0 \leq A \leq \sqrt{2/3}$ is defined by (1.8b) and θ is a small positive constant that we choose below when we derive estimates for $\mathscr{E}^2_{(\mathrm{Total});\theta}(t)$.

We will also use up-to-order M energies. Specifically, we view the energy $\mathscr{E}^2_{(\mathrm{Total});\theta}$ defined in (4.6e) as a functional of $\kappa, \partial h, \partial_t \varphi, \partial\varphi, \partial\nu, \nu$ (that is, $\mathscr{E}^2_{(\mathrm{Total});\theta} = \mathscr{E}^2_{(\mathrm{Total});\theta}[\kappa, \partial h, \partial_t\varphi, \partial\varphi, \partial\nu, \nu]$), and we define

$$\mathscr{E}^2_{(\mathrm{Total});\theta;M}(t) := \sum_{|\vec{I}| \leq M} \mathscr{E}^2_{(\mathrm{Total});\theta}[\partial_{\vec{I}}\kappa, \partial\partial_{\vec{I}} h, \partial_t\partial_{\vec{I}}\varphi, \partial\partial_{\vec{I}}\varphi, \partial\partial_{\vec{I}}\nu, \partial_{\vec{I}}\nu](t). \tag{4.7}$$

In Lemma 4.3 below, we compare the strength of the energies to the strength of the norms. Its proof is straightforward and amounts to tracking powers of t. We first provide the following lemma, whose simple proof we omit.

LEMMA 4.1 (Basic properties of the spatial part of the Kasner metric). *Let $\eta \geq 0$ be as defined in* (1.9b). *The components $\mathring{g}_{ij}$ of the Kasner spatial metric* (*see* (1.6)) *and the components $\mathring{g}^{ij}$ of its inverse verify the following estimates for $(t,x) \in (0,1] \times \mathbb{T}^3$, $(i,j = 1,2,3)$*:

$$|\mathring{g}_{ij}| \leq t^{2/3 - 2\eta}, \tag{4.8a}$$

$$|\mathring{g}^{ij}| \leq t^{-2/3 - 2\eta}. \tag{4.8b}$$

Furthermore, the 3×3 matrices $\mathring{g}_{ij}$ and $\mathring{g}^{ij}$ have the following positive definiteness properties:

$$t^{2/3+2\eta}\delta_{ab}X^aX^b \leq \mathring{g}_{ab}X^aX^b \leq t^{2/3-2\eta}\delta_{ab}X^aX^b \qquad \forall X \in \mathbb{R}^3, \tag{4.9a}$$

$$t^{-2/3+2\eta}\delta^{ab}\xi_a\xi_b \leq \mathring{g}^{ab}\xi_a\xi_b \leq t^{-2/3-2\eta}\delta^{ab}\xi_a\xi_b \qquad \forall \xi \in \mathbb{R}^3, \tag{4.9b}$$

where δ_{ab} and δ^{ab} are standard Kronecker deltas.

Furthermore,

$$\partial_t\mathring{g}_{ij} = -2t^{-1}\mathring{g}_{ia}(t\mathring{k}^a{}_j), \qquad \partial_t\mathring{g}^{ij} = 2t^{-1}\mathring{g}^{ja}(t\mathring{k}^i{}_a), \tag{4.10}$$

where $t\mathring{k}^i{}_j = -\mathrm{diag}(q_1, q_2, q_3)$ (*see* (1.7)).

Before comparing the strength of the energies and the norms, we first provide the following simple elliptic estimate, which will allow us to derive estimates for the top-order derivatives of the linearized lapse.

LEMMA 4.2 (Top-order estimate for ν). *If ν verifies equation* (3.15a), *then the following*[32] *elliptic estimate holds*:

$$t^2\|\partial^2\nu\|_{L^2_{\mathring{g}}} \lesssim |2A^2 - 1|\|\nu\|_{L^2} + 2A\|t\partial_t\varphi\|_{L^2} + 2|t\hat{\mathring{k}}|_{\mathring{g}}\|t\kappa\|_{L^2_{\mathring{g}}}. \tag{4.11}$$

[32]We note that $\|\partial^2\nu\|^2_{L^2_{\mathring{g}}} = \int_{\Sigma_t} \mathring{g}^{ab}\mathring{g}^{ef}\partial_a\partial_e\nu\partial_b\partial_f\nu\, dx$.

Proof. We multiply equation (3.15a) by $t^2 \mathring{g}^{ef} \partial_e \partial_f \nu$, integrate by parts over Σ_t (relative to the Euclidean volume form on Σ_t), and use Cauchy-Schwarz and Young's inequality as well as the simple estimate $\|\mathring{g}^{ef} \partial_e \partial_f \nu\|_{L^2} \lesssim \|\partial^2 \nu\|_{L^2_{\mathring{g}}}$. □

LEMMA 4.3 (Energy-norm comparison lemma). *Let $N \geq 0$ be an integer, and let $\eta \geq 0$ be as defined in* (1.9b). *Under the assumptions of Lemma* 4.2, *there exist constants*[33] $C > 0$ *and* $c > 0$, *depending on* θ, *such that the following comparison estimates hold for the norm $\mathscr{S}_{(\mathrm{Frame});N}(t)$ defined in* (4.5) *and the total energy $\mathscr{E}_{(\mathrm{Total});\theta;N}(t)$ defined in* (4.7) *on the interval $t \in (0, 1]$*:

$$\mathscr{E}_{(\mathrm{Total});\theta;N}(t) \leq C t^{-c\eta} \mathscr{S}_{(\mathrm{Frame});N}(t), \tag{4.12a}$$

$$\mathscr{S}_{(\mathrm{Frame});N}(t) \leq C t^{-c\eta} \mathscr{E}_{(\mathrm{Total});\theta;N}(t). \tag{4.12b}$$

Proof. Lemma 4.3 follows easily from Lemma 4.1, Lemma 4.2 (which allows us to bound the top-order linearized lapse term $t^{4/3} \|\nu\|_{H^{N+2}}$ from (4.5) in terms of the other linear solution variables), and the definitions of the quantities involved. □

5. **The approximate monotonicity identity**

5.1. *Statement of the approximate monotonicity identity.* The next theorem provides the approximate monotonicity identity that lies at the heart of the linear stability of near-FLRW Kasner solutions. Unlike the results of Sections 6 and 7, *the identity is valid for all Kasner backgrounds.*

Remark 5.1 (Monotonicity-coaxing terms and error terms). The favorable "monotonicity-coaxing terms" are the negative definite spacetime integrals on the third through fifth lines of the right-hand side of (5.1). The last line of (5.1) features unsigned error integrals that compete against the negative definite integrals. In Theorem 6.1, we show that for near-FLRW Kasner backgrounds, the unsigned integrals can be absorbed into the negative definite integrals, except for one error integral whose coefficient is controlled by the parameter η.

THEOREM 5.1 (The approximate monotonicity identity). *For any constant $\theta > 0$, solutions to the linearized equations of Proposition* 3.2 *verify the following identity for $t \in (0, 1]$*:

$$\begin{aligned} (5.1)\qquad & \int_{\Sigma_t} (t\partial_t \varphi)^2 + |t\partial\varphi|^2_{\mathring{g}} \, dx + \int_{\Sigma_t} |t\partial\nu|^2_{\mathring{g}} \, dx + (1 - A^2) \int_{\Sigma_t} \nu^2 \, dx \\ & + \theta \int_{\Sigma_t} |t\kappa|^2_{\mathring{g}} + \frac{1}{4} |t\partial h|^2_{\mathring{g}} \, dx - \int_{\Sigma_t} \mathcal{N}_1 \, dx \end{aligned}$$

[33] As we have mentioned, C and c are free to vary from line to line and can depend on N.

$$
\begin{aligned}
&= \int_{\Sigma_1} (t\partial_t \varphi)^2 + |t\partial\varphi|_{\mathring{g}}^2 \, dx + \int_{\Sigma_1} |t\partial\nu|_{\mathring{g}}^2 \, dx + (1 - A^2) \int_{\Sigma_1} \nu^2 \, dx \\
&\quad + \theta \int_{\Sigma_1} |\kappa|_{\mathring{g}}^2 + \frac{1}{4} |t\partial h|_{\mathring{g}}^2 \, dx - \int_{\Sigma_1} \mathcal{N}_1 \, dx \\
&\quad - 2 \int_{s=t}^1 s^{-1} \int_{\Sigma_s} |s\partial\varphi|_{\mathring{g}}^2 \, dx \, ds - \int_{s=t}^1 s^{-1} \int_{\Sigma_s} |s\partial\nu|_{\mathring{g}}^2 \, dx \, ds \\
&\quad - \int_{s=t}^1 s^{-1} \int_{\Sigma_s} \nu^2 \, dx \, ds \\
&\quad - \frac{1}{2}\theta \int_{s=t}^1 s^{-1} \int_{\Sigma_s} |s\partial h|_{\mathring{g}}^2 \, dx \, ds \\
&\quad + \sum_{i=1}^3 \int_{s=t}^1 s^{-1} \int_{\Sigma_s} \mathcal{N}_i \, dx \, ds + \theta \sum_{i=4}^{10} \int_{s=t}^1 s^{-1} \int_{\Sigma_s} \mathcal{N}_i \, dx \, ds,
\end{aligned}
$$

where the constant $0 \le A \le \sqrt{2/3}$ *is defined by* (1.8b) *and along* Σ_s, *we have*

$$\mathcal{N}_1 = \mathcal{N}_1(s\hat{\mathring{k}}, s\kappa, \nu) := -2(s\hat{\mathring{k}}^a{}_b)(s\kappa^b{}_a)\nu, \tag{5.2a}$$

$$\mathcal{N}_2 = \mathcal{N}_2(s\mathring{k}, s\partial\varphi, s\partial\varphi) := -2s^2 \mathring{g}^{ab}(s\mathring{k}^c{}_b)\partial_a\varphi\partial_c\varphi, \tag{5.2b}$$

$$\mathcal{N}_3 = \mathcal{N}_3(s\partial\varphi, s\partial\nu) := -2As^2 \mathring{g}^{ab}\partial_a\varphi\partial_b\nu, \tag{5.2c}$$

$$\mathcal{N}_4 = \mathcal{N}_4(s\mathring{k}, s\partial h, s\partial h) := -\frac{1}{2}s^2 \mathring{g}^{ab}\mathring{g}^{ij}\mathring{g}^{cf}(s\mathring{k}^e{}_c)\partial_e h_{ai}\partial_f h_{bj}, \tag{5.2d}$$

$$
\begin{aligned}
\mathcal{N}_5 = \mathcal{N}_5(s\hat{\mathring{k}}, s\kappa, s\kappa) &:= 2\mathring{g}_{ic}\mathring{g}^{ab}(s\hat{\mathring{k}}^c{}_j)(s\kappa^i{}_a)(s\kappa^j{}_b) \\
&\quad - 2\mathring{g}_{ij}\mathring{g}^{ac}(s\hat{\mathring{k}}^b{}_c)(s\kappa^i{}_a)(s\kappa^j{}_b),
\end{aligned} \tag{5.2e}
$$

$$
\begin{aligned}
\mathcal{N}_6 = \mathcal{N}_6(s\hat{\mathring{k}}, s\partial h, s\partial h) &:= s^2 \mathring{g}_{ab}\mathring{g}^{ef}\mathring{g}^{ij}(s\hat{\mathring{k}}^a{}_c)^{(h)}\Gamma_i{}^c{}_j{}^{(h)}\Gamma_e{}^b{}_f \\
&\quad - s^2 \mathring{g}_{ab}\mathring{g}^{ef}\mathring{g}^{ij}(s\hat{\mathring{k}}^c{}_j)^{(h)}\Gamma_i{}^a{}_c{}^{(h)}\Gamma_e{}^b{}_f \\
&\quad + s^2 \mathring{g}^{ef}(s\hat{\mathring{k}}^a{}_c)^{(h)}\Gamma_a{}^c{}_b{}^{(h)}\Gamma_e{}^b{}_f - s^2 \mathring{g}^{ef}(s\hat{\mathring{k}}^c{}_b)^{(h)}\Gamma_a{}^a{}_c{}^{(h)}\Gamma_e{}^b{}_f,
\end{aligned} \tag{5.2f}
$$

$$
\begin{aligned}
\mathcal{N}_7 = \mathcal{N}_7(s\mathring{k}, s\partial h, s\partial\nu) &:= 2s^2 \mathring{g}^{ij}(s\hat{\mathring{k}}^b{}_i)^{(h)}\Gamma_a{}^a{}_b\partial_j\nu \\
&\quad - 2s^2 \mathring{g}^{ij}(s\hat{\mathring{k}}^a{}_b)^{(h)}\Gamma_a{}^b{}_i\partial_j\nu + s^2 \mathring{g}^{ij}\mathring{g}^{ef}(s\mathring{k}^a{}_j)\partial_e h_{ai}\partial_f\nu,
\end{aligned} \tag{5.2g}
$$

$$\mathcal{N}_8 = \mathcal{N}_8(s\hat{\mathring{k}}, s\kappa, \nu) := 2\mathring{g}_{ab}\mathring{g}^{ij}(s\hat{\mathring{k}}^a{}_i)(s\kappa^b{}_j)\nu, \tag{5.2h}$$

$$\mathcal{N}_9 = \mathcal{N}_9(s\partial\varphi, s\partial\nu) := 2As^2 \mathring{g}^{ij}\partial_i\varphi\partial_j\nu, \tag{5.2i}$$

$$\mathcal{N}_{10} = \mathcal{N}_{10}(s\partial h, s\partial\varphi) := -2As^2 \mathring{g}^{ef\,(h)}\Gamma_e{}^a{}_f\partial_a\varphi. \tag{5.2j}$$

Proof. Below we independently derive the identities (5.3) and (5.8). To obtain (5.1), we simply add (5.3) to θ times (5.8). $\square$

Corollary 5.1 (Approximate monotonicity identity for the solution's higher derivatives). *For any spatial derivative multi-index* $\vec{I}$ (*as defined in Section* 2.3), *the identity* (5.1) *holds with* $\kappa, \partial h, \varphi, \nu$ *replaced with, respectively,* $\partial_{\vec{I}}\kappa, \partial\partial_{\vec{I}}h, \partial_{\vec{I}}\varphi, \partial_{\vec{I}}\nu$.

Proof. Since the Kasner background metric is spatially homogeneous (that is, independent of $x \in \mathbb{T}^3$), the operators $\partial_{\vec{I}}$ commute through the linear equations of Proposition 3.2. Put differently, the differentiated quantities $\partial_{\vec{I}}\kappa, \partial\partial_{\vec{I}}h, \partial_{\vec{I}}\varphi, \partial_{\vec{I}}\nu$ verify the same equations satisfied by $\kappa, \partial h, \varphi, \nu$. Hence, Theorem 5.1 applies to the differentiated quantities as well. □

5.2. *The key integral identity for the linearized lapse and scalar field.* The most important ingredient in the proof of Theorem 5.1 is the following proposition, which provides an integral identity for the linearized scalar field and the linearized lapse. The proof of the proposition essentially involves combining several integration by parts-type identities in a manner that replaces dangerous error integrals with favorable ones.

Proposition 5.2 (The key integral identity for the linearized scalar field and the linearized lapse). *Solutions to the linearized equations of Proposition* 3.2 *verify the following identity for* $t \in (0,1]$:

(5.3)
$$\begin{aligned}
&\int_{\Sigma_t} (t\partial_t\varphi)^2 + |t\partial\varphi|^2_{\mathring{g}}\,dx + \int_{\Sigma_t} |t\partial\nu|^2_{\mathring{g}}\,dx + (1-A^2)\int_{\Sigma_t} \nu^2\,dx - \int_{\Sigma_t} \mathcal{N}_1\,dx \\
&\quad = \int_{\Sigma_1} (\partial_t\varphi)^2 + |\partial\varphi|^2_{\mathring{g}}\,dx + \int_{\Sigma_1} |\partial\nu|^2_{\mathring{g}}\,dx + (1-A^2)\int_{\Sigma_1} \nu^2\,dx - \int_{\Sigma_1} \mathcal{N}_1\,dx \\
&\qquad - 2\int_{s=t}^1 s^{-1}\int_{\Sigma_s} |s\partial\varphi|^2_{\mathring{g}}\,dx\,ds - \int_{s=t}^1 s^{-1}\int_{\Sigma_s} |s\partial\nu|^2_{\mathring{g}}\,dx\,ds \\
&\qquad - \int_{s=t}^1 s^{-1}\int_{\Sigma_s} \nu^2\,dx\,ds + \sum_{i=1}^3 \int_{s=t}^1 s^{-1}\int_{\Sigma_s} \mathcal{N}_i\,dx\,ds,
\end{aligned}$$

where the constant $0 \le A \le \sqrt{2/3}$ *is defined by* (1.8b) *and* $\mathcal{N}_1$, $\mathcal{N}_2$, *and* $\mathcal{N}_3$ *are defined in* (5.2a)–(5.2c).

Remark 5.2. *The surprising aspect of the identity* (5.3) *is the presence of the spacetime integrals that are negative definite in* ν *and* $\partial\nu$. In Section 10, we show that a version of (5.3) also holds when the CMC gauge is replaced with a parabolic lapse gauge.

Proof of Proposition 5.2. The proof involves combining three integration by parts identities. Throughout, we silently use the identities in (4.10). To obtain the first identity, we multiply both sides of the linearized lapse equation (3.15a) by ν and integrate by parts over Σ_t to deduce that

$$(5.4)\qquad 2A\int_{\Sigma_t}(t\partial_t\varphi)\nu\,dx = -\int_{\Sigma_t}|t\partial\nu|_{\mathring{g}}^2\,dx + (2A^2-1)\int_{\Sigma_t}\nu^2\,dx - 2\int_{\Sigma_t}(t\hat{\mathring{k}}^a{}_b)(t\kappa^b{}_a)\nu\,dx.$$

The second identity is an energy identity for the linearized scalar field wave equation. Specifically, we replace t with the integration variable s in equation (3.17), multiply by $-2s\partial_t\varphi$, and integrate by parts over $(s,x)\in[t,1]\times\mathbb{T}^3$ (we stress that $t\leq 1$) to deduce that the following identity holds for $t\in(0,1]$:

$$(5.5)\qquad \begin{aligned}\int_{\Sigma_t}&(t\partial_t\varphi)^2 + |t\partial\varphi|_{\mathring{g}}^2\,dx\\ &= \int_{\Sigma_1}(\partial_t\varphi)^2 + |\partial\varphi|_{\mathring{g}}^2\,dx\\ &\quad - 2\int_{s=t}^1 s^{-1}\int_{\Sigma_s}|s\partial\varphi|_{\mathring{g}}^2 + s^2\mathring{g}^{ab}(s\mathring{k}^c{}_b)\partial_a\varphi\partial_c\varphi\,dx\,ds\\ &\quad - 2A\int_{s=t}^1\int_{\Sigma_s}(s\partial_t\varphi)\partial_t\nu\,dx\,ds + 2A\int_{s=t}^1 s^{-1}\int_{\Sigma_s}(s\partial_t\varphi)\nu\,dx\,ds.\end{aligned}$$

Next, we multiply equation (3.17) by ν to obtain the following identity:

$$(5.6)\qquad (t\partial_t\varphi)\partial_t\nu = \partial_t(t\partial_t\varphi\nu) - \frac{1}{2}A\partial_t(\nu^2) - t\nu\mathring{g}^{ab}\partial_a\partial_b\varphi + At^{-1}\nu^2.$$

To obtain the third identity, we now replace t with the integration variable s in equation (5.6), multiply by $2A$, and integrate by parts over $(s,x)\in[t,1]\times\mathbb{T}^3$ to deduce that

$$(5.7)\qquad \begin{aligned}-2A&\int_{s=t}^1\int_{\Sigma_s}(s\partial_t\varphi)\partial_t\nu\,dx\,ds\\ &= -2A\int_{\Sigma_1}\partial_t\varphi\nu\,dx + A^2\int_{\Sigma_1}\nu^2\,dx\\ &\quad + 2A\int_{\Sigma_t}(t\partial_t\varphi)\nu\,dx - A^2\int_{\Sigma_t}\nu^2\,dx\\ &\quad - 2A\int_{s=t}^1 s^{-1}\int_{\Sigma_s}s^2\mathring{g}^{ab}\partial_a\varphi\partial_b\nu\,dx\,ds - 2A^2\int_{s=t}^1 s^{-1}\int_{\Sigma_s}\nu^2\,dx\,ds\\ &= \int_{\Sigma_1}|\partial\nu|_{\mathring{g}}^2\,dx + (1-A^2)\int_{\Sigma_1}\nu^2\,dx + 2\int_{\Sigma_1}\hat{\mathring{k}}^a{}_b\kappa^b{}_a\nu\,dx\\ &\quad - \int_{\Sigma_t}|t\partial\nu|_{\mathring{g}}^2\,dx - (1-A^2)\int_{\Sigma_t}\nu^2\,dx - 2\int_{\Sigma_t}(t\hat{\mathring{k}}^a{}_b)(t\kappa^b{}_a)\nu\,dx\\ &\quad - 2A\int_{s=t}^1 s^{-1}\int_{\Sigma_s}s^2\mathring{g}^{ab}\partial_a\varphi\partial_b\nu\,dx\,ds - 2A^2\int_{s=t}^1 s^{-1}\int_{\Sigma_s}\nu^2\,dx\,ds,\end{aligned}$$

where to obtain the second equality, we substituted the right-hand side of (5.4) for the integrals $2A\int_{\Sigma_1}\partial_t\varphi\nu\,dx$ and $2A\int_{\Sigma_t}(t\partial_t\varphi)\nu\,dx$. We now use the identity (5.4) with t replaced by s to substitute for the integral $2A\int_{\Sigma_s}(s\partial_t\varphi)\nu\,dx$ in the last spacetime integral on the right-hand side (5.5). Finally, we substitute the right-hand side of (5.7) for the next-to-last spacetime integral on the right-hand side of (5.5). In total, these steps lead to the identity (5.3). □

5.3. *An energy identity for the linearized metric variables.* In the next proposition, we derive an energy identity for the linearized metric solution variables.

PROPOSITION 5.3 (Energy identity for the linearized metric variables). *Solutions to the linearized equations of Proposition* 3.2 *verify the following identity for* $t\in(0,1]$:

$$\int_{\Sigma_t}|t\kappa|_{\mathring{g}}^2+\frac{1}{4}|t\partial h|_{\mathring{g}}^2\,dx=\int_{\Sigma_1}|\kappa|_{\mathring{g}}^2+\frac{1}{4}|\partial h|_{\mathring{g}}^2\,dx -\frac{1}{2}\int_{s=t}^1 s^{-1}\int_{\Sigma_s}|s\partial h|_{\mathring{g}}^2\,dx\,ds +\sum_{i=4}^{10}\int_{s=t}^1 s^{-1}\int_{\Sigma_s}\mathcal{N}_i\,dx\,ds, \tag{5.8}$$

where $\mathcal{N}_4,\ldots,\mathcal{N}_{10}$ *are defined in* (5.2d)–(5.2j).

Remark 5.3. (No need for spatial harmonic coordinates.) Proposition 5.3 shows, in particular, that we can derive energy estimates for solutions to Einstein's equations[34] directly in CMC-transported spatial coordinates. Remarkably, we have not seen this observation made in the literature. Previous authors (see, for example, [3], [5]) have instead chosen to impose the spatial harmonic coordinate condition $g^{ab}\nabla_a\nabla_b x^i=0$ to "reduce" the Ricci tensor R_{ij} of g to an elliptic operator acting on the components g_{ij}. That is, in spatial harmonic coordinates, we have $R_{ij}=-\frac{1}{2}g^{ab}\partial_a\partial_b g_{ij}+f_{ij}(g,\partial g)$, which eliminates the last two products on the right-hand side of (3.11d) and leads to a simpler proof of a basic L^2-type energy identity. In the proof of Proposition 5.3, we handle these two products through a procedure involving integration by parts and the constraint equations; see equations (5.16) and (5.17). The spatial harmonic coordinate condition, though it might have advantages in certain contexts, introduces additional complications into the analysis. The complications arise from the necessity of including a nonzero "shift vector" X^i in the spacetime metric $\mathbf{g}$: $\mathbf{g}=-n^2dt^2+g_{ab}(dx^a+X^a dt)(dx^b+X^b dt)$. To enforce the spatial

[34]Although the proposition addresses only the linearized equations, essentially the same argument can be used to derive a similar energy identity for the nonlinear equations.

harmonic coordinate condition, the components X^i must verify a system of elliptic PDEs that are coupled to the other solution variables.

Proof of Proposition 5.3. The proof involves combining a collection of integration by parts identities. Throughout, we silently use the identities in (4.10). To begin, we use the evolution equation (3.16b) to deduce that

$$\begin{aligned}(5.9)\quad \partial_t(|t\kappa|^2_{\mathring{g}}) = &-2t^{-1}\mathring{g}_{ic}\mathring{g}^{ab}(t\mathring{k}^c{}_j)(t\kappa^i{}_a)(t\kappa^j{}_b) + 2t^{-1}\mathring{g}_{ij}\mathring{g}^{ac}(t\mathring{k}^b{}_c)(t\kappa^i{}_a)(t\kappa^j{}_b)\\ &+ 2\mathring{g}_{ab}\mathring{g}^{ij}(t\kappa^a{}_i)\left\{-t\mathring{g}^{bc}\partial_c\partial_j\nu - t^{-1}\nu(t\mathring{k}^b{}_j) + t^{(h)}\mathrm{Ric}^b{}_j\right\}.\end{aligned}$$

Note that we can express the first line of the right-hand side of (5.9) as

$$(5.10)\qquad -2t^{-1}\mathring{g}_{ic}\mathring{g}^{ab}(t\hat{\mathring{k}}^c{}_j)(t\kappa^i{}_a)(t\kappa^j{}_b) + 2t^{-1}\mathring{g}_{ij}\mathring{g}^{ac}(t\hat{\mathring{k}}^b{}_c)(t\kappa^i{}_a)(t\kappa^j{}_b)$$

because the terms corresponding to the pure trace part of $\mathring{k}$ cancel. Furthermore, since equation (3.13) implies that $\kappa = \hat{\kappa}$, we can express the second product on the second line of the right-hand side of (5.9) as

$$(5.11)\qquad -2t^{-1}2\mathring{g}_{ab}\mathring{g}^{ij}(t\kappa^a{}_i)(t\mathring{k}^b{}_j)\nu = -2\mathring{g}_{ab}\mathring{g}^{ij}(s\hat{\mathring{k}}^a{}_i)(s\kappa^b{}_j)\nu.$$

Similarly, using the evolution equation (3.16a), we deduce that[35]

$$\begin{aligned}(5.12)\quad \frac{1}{4}\partial_t(t^2|\partial h|^2_{\mathring{g}}) = &\frac{1}{2}t|\partial h|^2_{\mathring{g}} + t\mathring{g}^{bc}\mathring{g}^{ij}\mathring{g}^{ef}(t\mathring{k}^a{}_c)\partial_e h_{ai}\partial_f h_{bj}\\ &+ \frac{1}{2}t\mathring{g}^{ab}\mathring{g}^{ij}\mathring{g}^{cf}(t\mathring{k}^e{}_c)\partial_e h_{ai}\partial_f h_{bj}\\ &+ \frac{1}{2}\mathring{g}^{ab}\mathring{g}^{ij}\mathring{g}^{ef}\partial_e h_{ai}\partial_f\left\{-2th_{bc}(t\mathring{k}^c{}_j) - 2t\mathring{g}_{bc}(t\kappa^c{}_j) - 2t\mathring{g}_{bc}(t\mathring{k}^c{}_j)\nu\right\}.\end{aligned}$$

For convenience, in the remainder of this proof, we denote terms that can be expressed as perfect spatial derivatives by "$\cdots$." These terms will vanish when we integrate the identities over $\mathbb{T}^3$. We now use equation (3.14b) and differentiation by parts to express the first product on the second line of the right-hand side of (5.9) as

$$\begin{aligned}(5.13)\quad -2t\mathring{g}_{ab}\mathring{g}^{bc}\mathring{g}^{ij}(t\kappa^a{}_i)\partial_c\partial_j\nu &= 2t\mathring{g}^{ij}\partial_a(t\kappa^a{}_i)\partial_j\nu + \cdots\\ &= -2At\mathring{g}^{ij}\partial_i\varphi\partial_j\nu\\ &\quad - 2t\mathring{g}^{ij\,(h)}\Gamma_a{}^a{}_b(t\hat{\mathring{k}}^b{}_i)\partial_j\nu + 2t\mathring{g}^{ij\,(h)}\Gamma_a{}^b{}_i(t\hat{\mathring{k}}^a{}_b)\partial_j\nu\\ &\quad + \cdots.\end{aligned}$$

[35] We recall that $|\partial h|^2_{\mathring{g}} = \mathring{g}^{ab}\mathring{g}^{ij}\mathring{g}^{ef}\partial_e h_{ai}\partial_f h_{bj}$.

Next, we use equation (3.11d) to express the third product on the second line of the right-hand side of (5.9) as

(5.14)
$$2t\mathring{g}_{ab}\mathring{g}^{ij}(t\kappa^{a}{}_{i})^{(h)}\mathrm{Ric}^{b}{}_{j} = -t\mathring{g}^{ij}\mathring{g}^{ef}(t\kappa^{a}{}_{i})\partial_e\partial_f h_{ja} + t\mathring{g}_{ab}\mathring{g}^{ij}\mathring{g}^{ef}(t\kappa^{a}{}_{i})\partial_j{}^{(h)}\Gamma_{e\ f}^{\ b} + t(t\kappa^{a}{}_{b})\mathring{g}^{ef}\partial_a{}^{(h)}\Gamma_{e\ f}^{\ b}.$$

Next, we use differentiation by parts to express the first product on the right-hand side of (5.14) as

$$-t\mathring{g}^{ij}\mathring{g}^{ef}(t\kappa^{a}{}_{i})\partial_e\partial_f h_{ja} = t\mathring{g}^{ij}\mathring{g}^{ef}\partial_e(t\kappa^{a}{}_{i})\partial_f h_{ja} + \cdots . \tag{5.15}$$

Next, we use equation (3.14c) and differentiation by parts to express the second product on the right-hand side of (5.14) as

$$\begin{aligned}(5.16)\quad & t\mathring{g}_{ab}\mathring{g}^{ij}\mathring{g}^{ef}(t\kappa^{a}{}_{i})\partial_j{}^{(h)}\Gamma_{e\ f}^{\ b} \\ &= -t\mathring{g}_{ab}\mathring{g}^{ij}\mathring{g}^{ef}\partial_j(t\kappa^{a}{}_{i})^{(h)}\Gamma_{e\ f}^{\ b} + \cdots \\ &= At\mathring{g}^{ef}\partial_a\varphi^{(h)}\Gamma_{e\ f}^{\ a} \\ &\quad + t\mathring{g}_{ab}\mathring{g}^{ef}\mathring{g}^{ij}(t\hat{\mathring{k}}^{c}{}_{j})^{(h)}\Gamma_{i\ c}^{\ a}{}^{(h)}\Gamma_{e\ f}^{\ b} - t\mathring{g}_{ab}\mathring{g}^{ef}\mathring{g}^{ij}(t\hat{\mathring{k}}^{a}{}_{c})^{(h)}\Gamma_{i\ j}^{\ c}{}^{(h)}\Gamma_{e\ f}^{\ b} \\ &\quad + \cdots .\end{aligned}$$

Next, we use equation (3.14b) and differentiation by parts to express the third product on the right-hand side of (5.14) as

$$\begin{aligned}(5.17)\qquad t(t\kappa^{a}{}_{b})\mathring{g}^{ef}\partial_a{}^{(h)}\Gamma_{e\ f}^{\ b} &= -t\mathring{g}^{ef}\partial_a(t\kappa^{a}{}_{b})^{(h)}\Gamma_{e\ f}^{\ b} + \cdots \\ &= At\mathring{g}^{ef}\partial_b\varphi^{(h)}\Gamma_{e\ f}^{\ b} + t\mathring{g}^{ef}(t\hat{\mathring{k}}^{c}{}_{b})^{(h)}\Gamma_{a\ c}^{\ a}{}^{(h)}\Gamma_{e\ f}^{\ b} \\ &\quad - t\mathring{g}^{ef}(t\hat{\mathring{k}}^{a}{}_{c})^{(h)}\Gamma_{a\ b}^{\ c}{}^{(h)}\Gamma_{e\ f}^{\ b} + \cdots .\end{aligned}$$

Combining (5.9)–(5.17) and carrying out straightforward computations, we deduce that

(5.18)
$$\begin{aligned}&\partial_t(|t\kappa|^2_{\mathring{g}}) + \frac{1}{4}\partial_t(t^2|\partial h|^2_{\mathring{g}}) \\ &\quad = \frac{1}{2}t|\partial h|^2_{\mathring{g}} + \frac{1}{2}t\mathring{g}^{ab}\mathring{g}^{ij}\mathring{g}^{cf}(t\mathring{k}^{e}{}_{c})\partial_e h_{ai}\partial_f h_{bj} \\ &\qquad - 2t^{-1}\mathring{g}_{ic}\mathring{g}^{ab}(t\hat{\mathring{k}}^{c}{}_{j})(t\kappa^{i}{}_{a})(t\kappa^{j}{}_{b}) + 2t^{-1}\mathring{g}_{ij}\mathring{g}^{ac}(t\hat{\mathring{k}}^{b}{}_{c})(t\kappa^{i}{}_{a})(t\kappa^{j}{}_{b}) \\ &\qquad + t\mathring{g}_{ab}\mathring{g}^{ef}\mathring{g}^{ij}(t\hat{\mathring{k}}^{c}{}_{j})^{(h)}\Gamma_{i\ c}^{\ a}{}^{(h)}\Gamma_{e\ f}^{\ b} - t\mathring{g}_{ab}\mathring{g}^{ef}\mathring{g}^{ij}(t\hat{\mathring{k}}^{a}{}_{c})^{(h)}\Gamma_{i\ j}^{\ c}{}^{(h)}\Gamma_{e\ f}^{\ b} \\ &\qquad + t\mathring{g}^{ef}(t\hat{\mathring{k}}^{c}{}_{b})^{(h)}\Gamma_{a\ c}^{\ a}{}^{(h)}\Gamma_{e\ f}^{\ b} - t\mathring{g}^{ef}(t\hat{\mathring{k}}^{a}{}_{c})^{(h)}\Gamma_{a\ b}^{\ c}{}^{(h)}\Gamma_{e\ f}^{\ b} \\ &\qquad - 2t\mathring{g}^{ij}(t\hat{\mathring{k}}^{b}{}_{i})^{(h)}\Gamma_{a\ b}^{\ a}\partial_j\nu + 2t\mathring{g}^{ij}(t\hat{\mathring{k}}^{a}{}_{b})^{(h)}\Gamma_{a\ i}^{\ b}\partial_j\nu - t\mathring{g}^{ij}\mathring{g}^{ef}(t\mathring{k}^{a}{}_{j})\partial_e h_{ai}\partial_f\nu \\ &\qquad - 2At\mathring{g}^{ij}\partial_i\varphi\partial_j\nu - 2\mathring{g}_{ab}\mathring{g}^{ij}(s\hat{\mathring{k}}^{a}{}_{i})(s\kappa^{b}{}_{j})\nu + 2At\mathring{g}^{ef}\partial_a\varphi^{(h)}\Gamma_{e\ f}^{\ a} + \cdots .\end{aligned}$$

To conclude (5.8), we have only to replace t with the integration variable s in the identity (5.18) and to integrate by parts over $(s,x) \in [t,1] \times \mathbb{T}^3$, where we stress that $t \leq 1$. □

6. Mildly singular energy estimates without derivative loss for the linearized equations

In the next result, Theorem 6.1, we use the approximate monotonicity identity of Theorem 5.1 to derive energy estimates for solutions to the linearized equations. A central aspect of the estimates is that the energies can blow up as $t \downarrow 0$. Consequently, the energy estimates by themselves do not yield a proof of linear stability. However, for near FLRW backgrounds, the blowup-rate is mild (see (6.2)), which is a key ingredient in our subsequent proof of linear stability (see Theorem 7.1). We stress that if our proof of Theorem 6.1 had relied on more standard energy identities rather than the approximate monotonicity identity of Theorem 5.1, then the outcome would have been a much worse energy blowup-rate, which in turn would have obstructed our proof of linear stability. In Theorem 6.1, we consider only the case of near-FLRW Kasner backgrounds, though it is possible to derive (perhaps very singular) energy estimates in the case of a general Kasner background.

Theorem 6.1 (Mildly singular energy estimates without derivative loss for solutions to the linearized equations). *Consider a solution to the linear equations of Proposition* 3.2 *corresponding to the data* $(\upkappa(1), h(1), \partial_t\varphi(1), \partial\varphi(1))$ (*given on* $\Sigma_1 = \{1\} \times \mathbb{T}^3$), *where* $\upnu(1)$ *is determined by the elliptic* PDE*s* (3.15a)–(3.15b). *There exist a small constant* $\uptheta_* > 0$ *and constants* $C > 0$ *and* $c > 0$ *such that if* $\upeta \geq 0$ *is sufficiently small* (*see definition* 1.9*b*) *and* $\mathscr{S}_{(\mathrm{Frame});0}(1) < \infty$ (*see definition* (4.5)), *then the base-level total energy* $\mathscr{E}_{(\mathrm{Total});\uptheta_*}(t)$ *defined in* (4.6e) *verifies the following inequality*[36] *for* $t \in (0,1]$:

$$\begin{aligned}
(6.1)\qquad \mathscr{E}^2_{(\mathrm{Total});\uptheta_*}(t) \leq C\mathscr{E}^2_{(\mathrm{Total});\uptheta_*}(1) & \underbrace{-\frac{1}{6}\uptheta_*\int_{s=t}^1 s^{-1}\int_{\Sigma_s}|s\partial h|^2_{\mathring{g}}\,dx\,ds}_{\text{Past-favorable sign}} \underbrace{-\frac{1}{6}\int_{s=t}^1 s^{-1}\int_{\Sigma_s}|s\partial\varphi|^2_{\mathring{g}}\,dx\,ds}_{\text{Past-favorable sign}} \\
& \underbrace{-\frac{1}{6}\int_{s=t}^1 s^{-1}\int_{\Sigma_s}|s\partial\upnu|^2_{\mathring{g}}\,dx\,ds}_{\text{Past-favorable sign}} \underbrace{-\frac{1}{2}\int_{s=t}^1 s^{-1}\int_{\Sigma_s}\upnu^2\,dx\,ds}_{\text{Past-favorable sign}}
\end{aligned}$$

[36]The explicit numerical constants on the right-hand side of (6.1) are not sharp, but that is not important when $\upeta$ is small.

$$+ \qquad \underbrace{c\eta \int_{s=t}^{1} s^{-1}\mathscr{E}^2_{(\mathrm{Total});\theta_*}(s)\,ds}_{\text{Error integral that can create blowup}} \qquad .$$

In addition, if $N \geq 0$ *is an integer and the solution norm* $\mathscr{S}_{(\mathrm{Frame});N}(t)$ *defined in* (4.5) *verifies* $\mathscr{S}_{(\mathrm{Frame});N}(1) < \infty$, *then the up-to-order* N *energy* $\mathscr{E}_{(\mathrm{Total});\theta_*;N}(t)$ *defined in* (4.7) *verifies the following inequality for* $t \in (0,1]$:

$$\mathscr{E}_{(\mathrm{Total});\theta_*;N}(t) \leq C\mathscr{E}_{(\mathrm{Total});\theta_*;N}(1)t^{-c\eta}. \tag{6.2}$$

Furthermore, if $N \geq 0$ *is an integer and* $\mathscr{S}_{(\mathrm{Frame});N}(1) < \infty$, *then there exist constants* $C > 0$ *and* $c > 0$ *such that the following inequality holds for* $t \in (0,1]$:

$$\mathscr{S}_{(\mathrm{Frame});N}(t) \leq C\mathscr{S}_{(\mathrm{Frame});N}(1)t^{-c\eta}. \tag{6.3}$$

Remark 6.1. Theorem 6.1 should be viewed as being relevant for estimating the solution's high-order derivatives in the nonlinear problem, while Theorem 7.1 below should be viewed as being relevant for estimating its low-order derivatives; see Section 8 for further discussion.

Remark 6.2. The proof of Theorem 6.1 essentially amounts to combining an intricate collection of integration by parts identities in the right way (this step was carried out in Theorem 5.1) and absorbing various integrals into favorably signed integrals. Certain aspects of our proof somewhat remind us of arguments used in [12], in which Bartnik gave a new proof of the positive mass theorem of Schoen–Yau [60], [61] and Witten [71]. His proof was simpler than the previous proofs but was valid only under the assumption that the metric is near-Euclidean and required the use of spatial harmonic coordinates. Like our proof, Bartnik's involved expressing the scalar curvature of the Riemannian 3-metric in terms of Christoffel symbols, integrating with respect to the measure corresponding to the Euclidean metric, and absorbing all of the unsigned quadratic terms into favorably signed quadratic terms (whose coefficients happened to be sufficiently large).

Proof of Theorem 6.1. To prove the theorem, we will use the following pointwise estimates for the integrand terms $\mathscr{N}_i$, $i = 1, 2, \ldots, 10$ defined in (5.2a)–(5.2j):

$$|\mathscr{N}_1| \leq 2|s\hat{\mathring{k}}|_{\mathring{g}}|s\kappa|_{\mathring{g}}|\nu| \leq \eta\theta|s\kappa|^2_{\mathring{g}} + \frac{\eta}{\theta}\nu^2, \tag{6.4}$$

$$\mathscr{N}_2 \leq \left(\frac{2}{3} + 2\eta\right)|\partial\varphi|^2_{\mathring{g}}, \tag{6.5}$$

$$|\mathscr{N}_3| \leq |s\partial\varphi|^2_{\mathring{g}} + A^2|s\partial\nu|^2_{\mathring{g}} \leq |s\partial\varphi|^2_{\mathring{g}} + \frac{2}{3}|s\partial\nu|^2_{\mathring{g}}, \tag{6.6}$$

$$\theta\mathcal{N}_4 \leq \left(\frac{1}{6} + \frac{1}{2}\eta\right)\theta|s\partial h|^2_{\mathring{g}}, \tag{6.7}$$

$$\theta|\mathcal{N}_5| \leq C\eta\theta|s\kappa|^2_{\mathring{g}}, \tag{6.8}$$

$$\theta|\mathcal{N}_6| \leq C\eta\theta|s\partial h|^2_{\mathring{g}}, \tag{6.9}$$

$$\theta|\mathcal{N}_7| \leq \frac{1}{12}\theta|s\partial h|^2_{\mathring{g}} + C\theta|s\partial\nu|^2_{\mathring{g}}, \tag{6.10}$$

$$\theta|\mathcal{N}_8| \leq \eta\theta|s\kappa|^2_{\mathring{g}} + C\eta\theta\nu^2, \tag{6.11}$$

$$\theta|\mathcal{N}_9| \leq C\theta|s\partial\varphi|^2_{\mathring{g}} + C\theta|s\partial\nu|^2_{\mathring{g}}, \tag{6.12}$$

$$\theta|\mathcal{N}_{10}| \leq \frac{1}{12}\theta|s\partial h|_{\mathring{g}} + C\theta|s\partial\varphi|^2. \tag{6.13}$$

All of the above estimates except for (6.5)–(6.7) are straightforward consequences of the Cauchy-Schwarz inequality relative to the metric $\mathring{g}$ and simple estimates of the form $|ab| \leq (1/2)\delta a^2 + (1/2)\delta^{-1}b^2$ for appropriately chosen constants $\delta > 0$. To derive (6.5) and (6.7), we use the fact that the eigenvalues of $t\mathring{k}^i{}_j$ are $\geq -q_{\text{Max}} \geq -\left\{\frac{1}{3}+\eta\right\}$, where q_{Max} is defined in (1.9a). To derive the second inequality in (6.6), we use the simple inequality $A \leq \sqrt{\frac{2}{3}}$.

We now claim that there exist constants $C > 0$ and $c > 0$ such that the following estimate holds when $\theta > 0$:

$$\begin{aligned}
&\int_{\Sigma_t} (t\partial_t\varphi)^2 + |t\partial\varphi|^2_{\mathring{g}}\,dx + \int_{\Sigma_t} |t\partial\nu|^2_{\mathring{g}}\,dx + \left\{1 - A^2 - \frac{\eta}{\theta}\right\}\int_{\Sigma_t} \nu^2\,dx \\
&\quad + \theta\left\{1-\eta\right\}\int_{\Sigma_t} |t\kappa|^2_{\mathring{g}}\,dx + \frac{1}{4}\theta\int_{\Sigma_t} |t\partial h|^2_{\mathring{g}}\,dx \\
&\leq \int_{\Sigma_1} (\partial_t\varphi)^2 + |\partial\varphi|^2_{\mathring{g}}\,dx + \int_{\Sigma_1} |\partial\nu|^2_{\mathring{g}}\,dx + \left\{1 - A^2 + \frac{\eta}{\theta}\right\}\int_{\Sigma_1} \nu^2\,dx \\
&\quad + \theta\left\{1+\eta\right\}\int_{\Sigma_1} |\kappa|^2_{\mathring{g}}\,dx + \frac{1}{4}\theta\int_{\Sigma_1} |\partial h|^2_{\mathring{g}}\,dx \\
&\quad - \left\{\frac{1}{3} - C\eta - C\theta\right\}\int_{s=t}^1 s^{-1}\int_{\Sigma_s} |s\partial\varphi|^2_{\mathring{g}}\,dx\,ds \\
&\quad - \left\{\frac{1}{3} - C\theta\right\}\int_{s=t}^1 s^{-1}\int_{\Sigma_s} |s\partial\nu|^2_{\mathring{g}}\,dx\,ds \\
&\quad - \left\{1 - C\frac{\eta}{\theta} - C\eta\theta\right\}\int_{s=t}^1 s^{-1}\int_{\Sigma_s} \nu^2\,dx\,ds \\
&\quad - \theta\left\{\frac{1}{6} - C\eta\right\}\int_{s=t}^1 s^{-1}\int_{\Sigma_s} |s\partial h|^2_{\mathring{g}}\,dx\,ds \\
&\quad + c\eta\theta\int_{s=t}^1 s^{-1}\int_{\Sigma_s} |s\kappa|^2_{\mathring{g}}\,dx\,ds.
\end{aligned} \tag{6.14}$$

To obtain (6.14), we simply substitute the estimates (6.4)–(6.13) into the approximate monotonicity identity (5.1) and keep careful track of the coefficients. For example, the coefficient $-\left\{\frac{1}{3} - C\eta - C\theta\right\}$ found in front of the integral $\int_{s=t}^{1} s^{-1} \int_{\Sigma_s} |s\partial\varphi|_{\mathring{g}}^2 \, dx \, ds$ on the right-hand side of (6.14) comes from adding the coefficient -2 on the third line of the right-hand side of (5.1) to the coefficient $\frac{2}{3} + 2\eta$ from (6.5), the coefficient 1 from (6.6), and the coefficients $C\theta$ from (6.12)–(6.13). Note that for $i = 4, \ldots, 10$, the terms $\mathcal{N}_i$ from (5.1) are multiplied by θ.

Next, from definition (4.6e), we deduce the following simple bound for the last integral in (6.14):

$$c\eta\theta \int_{s=t}^{1} s^{-1} \int_{\Sigma_s} |s\kappa|_{\mathring{g}}^2 \, dx \, ds \leq c\eta \int_{s=t}^{1} s^{-1} \mathscr{E}^2_{(\mathrm{Total});\theta}(s) \, ds. \tag{6.15}$$

The desired inequality (6.1) now follows from definition (4.6e), the identity (1.9b), and the estimates (6.14) and (6.15), and from first choosing $\theta := \theta_*$ to be sufficiently small and then choosing η to be sufficiently small in a manner that depends on the fixed choice of θ_*. We stress that the estimate (6.15) is precisely what generates the last error integral on the right-hand side of (6.1).

To deduce inequality (6.2), we first use (6.1) and Gronwall's inequality to deduce

$$\mathscr{E}^2_{(\mathrm{Total});\theta_*}(t) \leq C \mathscr{E}^2_{(\mathrm{Total});\theta_*}(1) t^{-c\eta}. \tag{6.16}$$

Next, we recall the following fact noted in the proof of Corollary 5.1: the $\partial_{\vec{I}}$-differentiated linear solution variables solve the same equations as the nondifferentiated linear solution variables. Thus, the energy of the $\partial_{\vec{I}}$-differentiated linear solution variables verifies an analog of the estimate (6.16). Summing these estimates for $|\vec{I}| \leq N$ and appealing to definition (4.7), we arrive at (6.2).

Inequality (6.3) then follows from (6.2) and Lemma 4.3. □

7. **Linear stability for near-FLRW Kasner backgrounds**

In this section, we state and prove Theorem 7.1, which is our main linear stability result. The theorem shows

(i) that for nearly spatially isotropic Kasner backgrounds, the lower-order derivatives of the linear solution enjoy improved estimates with respect to t (i.e., involving less singular powers of t) compared to the energy estimates of Theorem 6.1; and
(ii) that various time-rescaled components of the solution variables converge as $t \downarrow 0$.

As we outline in Section 8, the improved behavior is essential for proving the nonlinear stable blow-up results of [59]. The proof of the theorem is essentially based on revisiting the linearized equations and treating them as transport equations *with derivative-losing error terms* that we control with the energy estimates of Theorem 6.1. Elliptic estimates for the lapse also play a role. The main difficulty is finding a suitable order in which to prove the estimates. In essence, this amounts to finding effective dynamic decoupling.

THEOREM 7.1 (Linear stability). *Assume the hypotheses and conclusions of Theorem* 6.1. *Let* $N \geq 2$ *be an integer and assume that* $\|h\|_{L^2_{Frame}}(1) < \infty$ *and* $\mathscr{S}_{(\mathrm{Frame});N}(1) < \infty$ (*see definition* 4.5). *There exist constants* $C > 0$ *and* $c > 0$ *such that if* $\upeta > 0$ *is sufficiently small* (*see definition* 1.9*b*), *then the linear solution to the equations of Proposition* 3.2 *verifies the following estimates for* $t \in (0, 1]$:

$$\|\partial_t(t\upkappa)\|_{H^{N-1}_{\mathrm{Frame}}} \leq C\mathscr{S}_{(\mathrm{Frame});N}(1)t^{-1/3-c\upeta}, \tag{7.1a}$$

$$\|t\upkappa\|_{H^{N-1}_{\mathrm{Frame}}} \leq C\mathscr{S}_{(\mathrm{Frame});N}(1), \tag{7.1b}$$

$$\|h\|_{L^2_{\mathrm{Frame}}} \leq C\left\{\|h\|_{L^2_{\mathrm{Frame}}}(1) + \frac{1}{\upeta}\mathscr{S}_{(\mathrm{Frame});N}(1)\right\}t^{2/3-c\upeta}, \tag{7.1c}$$

$$\|\partial h\|_{H^{N-1}_{\mathrm{Frame}}} \leq \frac{C}{\upeta}\mathscr{S}_{(\mathrm{Frame});N}(1)t^{2/3-c\upeta}, \tag{7.1d}$$

$$\|\partial_t(t\partial_t\varphi - A\upnu)\|_{H^{N-1}} \leq C\mathscr{S}_{(\mathrm{Frame});N}(1)t^{-1/3-c\upeta}, \tag{7.1e}$$

$$\|t\partial_t\varphi\|_{H^{N-1}} \leq C\mathscr{S}_{(\mathrm{Frame});N}(1), \tag{7.1f}$$

$$\|\partial\varphi\|_{H^{N-2}_{\mathrm{Frame}}} \leq C\mathscr{S}_{(\mathrm{Frame});N}(1)\left\{1 + |\ln(t)|\right\}, \tag{7.1g}$$

$$\|\upnu\|_{H^N} \leq C\mathscr{S}_{(\mathrm{Frame});N}(1)t^{-c\upeta}, \tag{7.1h}$$

$$\|\upnu\|_{H^{N-1}} \leq C\mathscr{S}_{(\mathrm{Frame});N}(1)t^{2/3-c\upeta}, \tag{7.1i}$$

$$\|\upnu\|_{H^{N-2}} \leq \frac{C}{\upeta}\mathscr{S}_{(\mathrm{Frame});N}(1)t^{4/3-c\upeta}. \tag{7.1j}$$

Convergence. *There exist a symmetric type* $\binom{0}{2}$ *tensorfield* $h_{\mathrm{Regular}} \in H^{N-1}_{\mathrm{Frame}}(\mathbb{T}^3)$, *a type* $\binom{1}{1}$ *tensorfield* $K_{\mathrm{Bang}} \in H^{N-1}_{\mathrm{Frame}}(\mathbb{T}^3)$ *verifying* $(K_{\mathrm{Bang}})^a{}_a = 0$, *and a function* $\Psi_{\mathrm{Bang}} \in H^{N-1}(\mathbb{T}^3)$ *such that the following estimates hold*[37] *for* $t \in (0, 1]$:

$$\left\|t^{-2q_j}h_{ij} + 2\ln(t)(K_{\mathrm{Bang}})^i{}_j - (h_{\mathrm{Regular}})_{ij}\right\|_{H^{N-1}} \leq C\mathscr{S}_{(\mathrm{Frame});N}(1)t^{2/3-c\upeta} \quad (\text{if } q_i = q_j), \tag{7.2a}$$

[37] On the left-hand sides of (7.2a)–(7.2b), we do not sum over i or j.

$$\left\| t^{-2q_j} h_{ij} + \frac{1}{q_i - q_j} t^{2(q_i - q_j)} (K_{\text{Bang}})^i{}_j - (h_{\text{Regular}})_{ij} \right\|_{H^{N-1}} \leq C\mathscr{S}_{(\text{Frame});N}(1) t^{2/3 - c\eta} \quad (\text{if } q_i \neq q_j), \tag{7.2b}$$

$$\|t\kappa - K_{\text{Bang}}\|_{H^{N-1}_{\text{Frame}}} \leq C\mathscr{S}_{(\text{Frame});N}(1) t^{2/3-c\eta}, \tag{7.2c}$$

$$\|t\partial_t \varphi - \Psi_{\text{Bang}}\|_{H^{N-1}} \leq C\mathscr{S}_{(\text{Frame});N}(1) t^{2/3-c\eta}, \tag{7.2d}$$

$$\|\partial\varphi - \ln(t)\partial\Psi_{\text{Bang}}\|_{H^{N-2}_{\text{Frame}}} \leq C\mathscr{S}_{(\text{Frame});N}(1), \tag{7.2e}$$

and

$$\|h_{\text{Regular}} - h(1)\|_{H^{N-1}_{\text{Frame}}} \leq C\mathscr{S}_{(\text{Frame});N}(1), \tag{7.3a}$$

$$\|K_{\text{Bang}} - \kappa(1)\|_{H^{N-1}_{\text{Frame}}} \leq C\mathscr{S}_{(\text{Frame});N}(1), \tag{7.3b}$$

$$\|\Psi_{\text{Bang}} - \partial_t\varphi(1)\|_{H^{N-1}} \leq C\mathscr{S}_{(\text{Frame});N}(1). \tag{7.3c}$$

In addition, the same estimates hold in the case $\eta = 0$ *with all factors of* $\frac{1}{\eta}$ *replaced by* $1 + \ln t$.

Before proving the theorem, we first make some remarks.

- Just below the "rough" Theorem 1.4 (which is a recap of Theorem 7.1), we gave a detailed explanation of why the convergence results of Theorem 7.1 are natural. Furthermore, we highlighted the connection between the convergence results stated in the theorem and the heuristic statements made in [11], [13] concerning the asymptotic behavior of solutions to the nonlinear equations near singularities.
- The improved behavior in t provided by (7.1a)–(7.1j) is of critical importance in closing the nonlinear problem; see Section 8.

Proof of Theorem 7.1. We give the proof only in the case $\eta > 0$. The case $\eta = 0$ can be handled by straightforward modifications of the case $\eta > 0$. Throughout the proof, we silently use Lemma 4.1, Lemma 4.3, and the t-weights inherent in Definition 4.3.

Proof of (7.1h) *and* (7.1i). We commute equation (3.15b) with $\partial_{\vec{I}}$, multiply by $\partial_{\vec{I}}\nu$, and integrate by parts over Σ_t to deduce the elliptic estimate

$$t\|\partial\partial_{\vec{I}}\nu\|_{L^2_{\hat{g}}} + \|\partial_{\vec{I}}\nu\|_{L^2} \leq Ct^2\|\partial_{\vec{I}}{}^{(h)}R\|_{L^2}. \tag{7.4}$$

From (3.11c) and (6.3), we deduce that whenever $|\vec{I}| \leq N - 1$, we have $\|\partial_{\vec{I}}{}^{(h)}R\|_{L^2} \leq Ct^{-4/3-c\eta}\mathscr{S}_{(\text{Frame});N}(1)$. The estimates (7.1h) and (7.1i) now readily follow.

Proof of (7.1a). We first deduce from equation (3.16b) that

$$\|\partial_t(t\kappa)\|_{H^{N-1}_{\text{Frame}}} \leq Ct^{1/3-c\eta}\|\nu\|_{H^{N+1}} + Ct^{-1}\|\nu\|_{H^{N-1}} + Ct\|^{(h)}\text{Ric}\|_{H^{N-1}_{\text{Frame}}}. \tag{7.5}$$

From (3.11d), (6.3), and (7.1i), we conclude that the right-hand side of (7.5) is $\leq C\mathscr{S}_{(\text{Frame});N}(1)t^{-1/3-c\eta}$ as desired.

Proof of (7.1b), (7.2c), *and* (7.3b). We set $f(t) := t\kappa^i{}_j(t,\cdot)$, where we are viewing f as a scalar $H^{N-1}(\mathbb{T}^3)$-valued function of t. From (7.1a), we deduce that for $0 < s \leq t \leq 1$, we have

$$\|f(t) - f(s)\|_{H^{N-1}} \leq C\left(t^{2/3-c\eta} - s^{2/3-c\eta}\right)\mathscr{S}_{(\text{Frame});N}(1).$$

From this bound and the completeness of $H^{N-1}(\mathbb{T}^3)$, it follows that if η is sufficiently small, then $\lim_{t\downarrow 0} f(t)$ exists as an element of $H^{N-1}(\mathbb{T}^3)$. We denote the limit by $(K_{\text{Bang}})^i{}_j := f(0)$. Moreover, the previous estimate yields $\|f(t) - f(0)\|_{H^{N-1}} \leq C\mathscr{S}_{(\text{Frame});N}(1)t^{2/3-c\eta}$. The estimates (7.2c) and (7.3b) follow from this bound, while (7.1b) follows from (7.2c), (7.3b), and the bound $f(1) \leq \mathscr{S}_{(\text{Frame});N}(1)$.

Proof of (7.1c) *and* (7.1d). We give the details only for (7.1c) since the proof of (7.1d) is essentially the same. To proceed, we first split $t\mathring{k}^a{}_j$ into its pure trace and trace-free parts and use equation (3.16a) to deduce that

$$\partial_t(t^{-2/3}h_{ij}) = -2t^{-1}(t^{-2/3}h_{ia})(t\hat{\mathring{k}}^a{}_j) - 2t^{-5/3}\mathring{g}_{ia}(t\kappa^a{}_j) - 2t^{-5/3}\mathring{g}_{ia}(t\mathring{k}^a{}_j)\nu. \tag{7.6}$$

From equation (7.6), we deduce that

$$\begin{aligned}\|\partial_t(t^{-2/3}h)\|_{L^2_{\text{Frame}}} \leq Ct^{-1}|t\hat{\mathring{k}}|_{\text{Frame}}\|t^{-2/3}h\|_{L^2_{\text{Frame}}} + Ct^{-5/3}|\mathring{g}|_{\text{Frame}}\|t\kappa\|_{L^2_{\text{Frame}}}\\ + Ct^{-5/3}|\mathring{g}|_{\text{Frame}}|t\mathring{k}|_{\text{Frame}}\|\nu\|_{L^2}.\end{aligned} \tag{7.7}$$

From inequality (6.3), we deduce that the right-hand side of (7.7) is

$$\leq c\eta t^{-1}\|t^{-2/3}h\|_{L^2_{\text{Frame}}} + C\mathscr{S}_{(\text{Frame});N}(1)t^{-1-c\eta}.$$

Using this estimate and integrating (7.7) in time, we deduce that

$$\begin{aligned}t^{-2/3}\|h\|_{L^2_{\text{Frame}}}(t) \leq \|h\|_{L^2_{\text{Frame}}}(1) + \frac{C}{\eta}\mathscr{S}_{(\text{Frame});N}(1)t^{-c\eta}\\ + c\eta\int_{s=t}^1 s^{-1}\left\{s^{-2/3}\|h\|_{L^2_{\text{Frame}}}(s)\right\}ds.\end{aligned} \tag{7.8}$$

From (7.8) and Gronwall's inequality in the quantity $t^{-2/3}\|h\|_{L^2_{\text{Frame}}}(t)$, we conclude the desired inequality (7.1c).

Proof of (7.1j). We need only to revisit the proof of (7.1i) and use the fact that the improved estimate (7.1d) allows us to deduce that whenever $|\vec{I}| \leq N-2$, we have $\|\partial_{\vec{I}}{}^{(h)}R\|_{L^2} \leq \frac{C}{\eta}\mathscr{S}_{(\mathrm{Frame});N}(1)t^{-2/3-c\eta}$.

Proof of (7.1e). We first deduce from equation (3.17) that

$$\|\partial_t(t\partial_t\varphi - A\nu)\|_{H^{N-1}} \leq Ct\|\mathring{g}^{ab}\partial_a\partial_b\varphi\|_{H^{N-1}} + Ct^{-1}\|\nu\|_{H^{N-1}}. \tag{7.9}$$

From (6.3) and (7.1i), we deduce that the right-hand side of (7.9) is $\leq$ the right-hand side of (7.1e) as desired.

Proof of (7.1f), (7.2d), *and* (7.3c). The existence of the limiting tensorfield Ψ_{Bang} and the three estimates under consideration follow from inequalities (7.1e) and (7.1i) by the same reasoning we used to prove (7.1b), (7.2c), and (7.3b).

Proof of (7.1g) *and* (7.2e). From (7.2d), we deduce

$$\|\partial_t\left\{\partial\varphi - \ln t\partial\Psi_{\mathrm{Bang}}\right\}\|_{H^{N-2}_{\mathrm{Frame}}} \leq C\mathscr{S}_{(\mathrm{Frame});N}(1)t^{-1/3-c\eta}.$$

Integrating from time t to time 1, we find that

$$\begin{aligned}\|\partial\varphi - \ln t\partial\Psi_{\mathrm{Bang}}\|_{H^{N-2}_{\mathrm{Frame}}} &\leq C\mathscr{S}_{(\mathrm{Frame});N}(1)t^{2/3-c\eta} + \|\partial\varphi\|_{H^{N-2}_{\mathrm{Frame}}}(1)\\ &\leq C\mathscr{S}_{(\mathrm{Frame});N}(1),\end{aligned}$$

which yields (7.2e). (7.1g) then follows from (7.1f), (7.2e), and (7.3c).

Proof of (7.2a), (7.2b), *and* (7.3a). Throughout this paragraph, we do not use Einstein's summation convention for i or j. Recall that $\mathring{g}_{ii} = t^{2q_i}$, that $t(\mathring{k}^i{}_i) = -q_i$, and that the off-diagonal components of these tensorfields are 0. Multiplying equation (3.16a) by t^{-2q_j}, we deduce the equation $\partial_t(t^{-2q_j}h_{ij}) = -2t^{-1+2(q_i-q_j)}(t\kappa^i{}_j) + 2q_i\delta_{ij}t^{-1}\nu$. From this equation, the estimates (7.1i) and (7.2c), and the simple estimate $|q_i - q_j| \leq 2\eta$ (see (1.9b)), we deduce that for $i,j = 1,2,3$, we have

$$\begin{aligned}\left\|\partial_t\left\{t^{-2q_j}h_{ij} - 2\left(\int_{s=t}^1 s^{-1+2(q_i-q_j)}\,ds\right)(K_{\mathrm{Bang}})^i{}_j\right\}\right\|_{H^{N-1}}& \\ \leq C\mathscr{S}_{(\mathrm{Frame});N}(1)t^{-1/3-c\eta}&.\end{aligned} \tag{7.10}$$

When η is sufficiently small, the existence of the limiting tensorfield components $(h_{\mathrm{Regular}})_{ij}$ and the estimates (7.2a), (7.2b), and (7.3a) follow from (7.10) and (7.3b) by the same reasoning we used to prove (7.1b), (7.2c), and (7.3b). ∎

8. Summary of the proof of the nonlinear stability of the FLRW Big Bang singularity

Recall that the FLRW metric is $\mathbf{g}_{\mathrm{FLRW}} = -dt^2 + t^{2/3}\sum_{i=1}^{3}(dx^i)^2$ (see (1.3)). In this section, we outline the proof of Theorem 8.1, which yields the nonlinear stability of FLRW solution's Big Bang singularity. Our discussion will provide a detailed overview of the central role that the monotonicity identities and linear stability results play in the nonlinear problem. For complete details in the context of the Einstein-stiff fluid system, we refer the reader to [59].

Remark 8.1. In [59], we formulated the equations and estimates in terms of time-rescaled solution variables. Here, to keep the discussion short, we do not introduce such time-rescaled variables. This changes the appearance of various equations and estimates compared to [59], but not their content.

8.1. *Norms and energies.* We start by introducing the norms and energies that we use to control the nonlinear solution. In our analysis, we view the unknowns to be the solution variables $n, g_{ij}, k^i{}_j, \phi$ appearing in the nonlinear equations of Proposition 3.1. Note that the pure trace part of $k^i{}_j$ is controlled by the CMC condition $k^a{}_a = -t^{-1}$ and thus we only need to derive estimates for its trace-free part $\hat{k}^i{}_j$.

Definition 8.1 (The pointwise norm $|\cdot|_g$). Throughout this section, we use the pointwise norm $|\cdot|_g$, which is defined by replacing the background Kasner metric $\mathring{g}$ with the metric g on both sides of (4.1b).

Definition 8.2 (Solution norms). To control the nonlinear solution, we rely on norms[38] belonging to the following family:

$$
\begin{aligned}
(8.1)\qquad \mathscr{S}_{(\mathrm{Frame});M}(t) := {} & \left\|t\hat{k}\right\|_{H^M_{\mathrm{Frame}}} + \|\partial g\|_{H^M_{\mathrm{Frame}}} \\
& + t^{-2/3}\left\|g - g_{\mathrm{FLRW}}\right\|_{H^M_{\mathrm{Frame}}} + t^{2/3}\left\|g^{-1} - g^{-1}_{\mathrm{FLRW}}\right\|_{H^M_{\mathrm{Frame}}} \\
& + \|t\partial_t\phi\|_{H^M_{\mathrm{Frame}}} + t^{2/3}\|\partial\phi\|_{H^M_{\mathrm{Frame}}} + \sum_{p=0}^{2} t^{(2/3)p}\,\|n-1\|_{H^{M+p}}.
\end{aligned}
$$

To control the norms (8.1), we will use the energies provided by the next definition. The energies for the nonlinear solution are tied to approximate monotonicity identities for the nonlinear solution in the same way that the energies of Definition 4.4 for the linear solution are tied to the approximate monotonicity identity of Theorem 5.1.

[38] More precisely, in [59], our high-order solution norms do not directly control $\|g - g_{\mathrm{FLRW}}\|_{L^2_{\mathrm{Frame}}}$ or $\left\|g^{-1} - g^{-1}_{\mathrm{FLRW}}\right\|_{L^2_{\mathrm{Frame}}}$, but that detail is not important for the ensuing discussion.

Definition 8.3 (Energies). Let $\vec{I}$ be a spatial derivative multi-index (as defined in Section 2.3), let $M \geq 0$ be an integer, and let $\uptheta > 0$ be a constant. We define the energies $\mathscr{E}_{(\mathrm{Metric});\vec{I}}(t) \geq 0, \ldots, \mathscr{E}_{(\mathrm{Total});\uptheta;M}(t) \geq 0$ as follows:

$$\mathcal{E}^2_{(\mathrm{Metric});\vec{I}}(t) := \int_{\Sigma_t} \left| t\partial_{\vec{I}}\hat{k}\right|_g^2 + \frac{1}{4}\left|t\partial\partial_{\vec{I}}g\right|_g^2 \, dx, \tag{8.2a}$$

$$\mathscr{E}^2_{(\mathrm{Scalar});\vec{I}}(t) := \int_{\Sigma_t} (t\partial_t\partial_{\vec{I}}\phi)^2 + n^2|t\partial\partial_{\vec{I}}\phi|_g^2 \, dx, \tag{8.2b}$$

$$\mathscr{E}^2_{(\partial\mathrm{Lapse});\vec{I}}(t) := \int_{\Sigma_t} |t\partial\partial_{\vec{I}}n|_g^2 \, dx, \tag{8.2c}$$

$$\mathscr{E}^2_{(\mathrm{Lapse})}(t) := \int_{\Sigma_t} |\partial_{\vec{I}}(n-1)|^2 \, dx, \tag{8.2d}$$

$$\begin{aligned}\mathscr{E}^2_{(\mathrm{Total});\uptheta;\vec{I}}(t) := {} & \mathscr{E}^2_{(\mathrm{Scalar});\vec{I}}(t) + \mathscr{E}^2_{(\partial\mathrm{Lapse});\vec{I}}(t) \\ & + \frac{1}{3}\mathscr{E}^2_{(\mathrm{Lapse});\vec{I}}(t) + \uptheta\mathscr{E}^2_{(\mathrm{Metric});\vec{I}}(t),\end{aligned} \tag{8.2e}$$

$$\mathscr{E}^2_{(\mathrm{Total});\uptheta;M}(t) := \sum_{|\vec{I}|\leq M} \mathscr{E}^2_{(\mathrm{Scalar});\vec{I}}(t). \tag{8.2f}$$

We clarify that on the right-hand side of (8.2a),

$$\left|t\partial\partial_{\vec{I}}g\right|_g^2 = t^2 g^{ab}g^{ij}g^{ef}\partial_e\partial_{\vec{I}}g_{ai}\partial_f\partial_{\vec{I}}g_{bj}.$$

Remark 8.2. Note that our energies do not directly control the terms $t^{2/3}\|g - g_{\mathrm{FLRW}}\|_{H^M_{\mathrm{Frame}}}$ or $t^{-2/3}\left\|g^{-1} - g^{-1}_{\mathrm{FLRW}}\right\|_{H^M_{\mathrm{Frame}}}$, which are featured in the norm (8.1). Therefore, control of these terms does not directly follow from the energy estimates described below and instead requires a separate argument based on the metric evolution equation (3.7a); we will avoid further discussion of this issue here.

As in our proof of Theorem 6.1, in order to close the energy estimates for the nonlinear solutions, we have to make a suitable choice of $\uptheta > 0$. Here we note that the same choice of

$$\uptheta := \uptheta_* \tag{8.3}$$

that we made in the proof of Theorem 6.1 is also sufficient in our study of nonlinear solutions. The reason is that $\uptheta$ needs to be adapted only to handle various integrals in the approximate monotonicity identity that are generated by *linear* terms in the equations; quadratically small nonlinear terms do not affect the viability of the choice $\uptheta := \uptheta_*$, but rather generate error integrals that we explain how to control in Section 8.9.

8.2. *The nonlinear stability of the* FLRW *Big Bang singularity.* In this subsection, we state our main nonlinear result, namely Theorem 8.1. In the rest of Section 8, we will explain the main ideas behind the proof of the theorem. We refer the reader to [59] for complete details in the case of the Einstein-stiff fluid system. We note that in Remarks 3.5 and 8.1, we pointed out some minor differences between the approach outlined here and the approach taken in [59].

THEOREM 8.1 (Stable Big Bang Formation for near-FLRW solutions). *Consider initial data* (*as described in Section* 1.4) *for the Einstein-scalar field system given on the Cauchy hypersurface* $\Sigma_1 = \{1\} \times \mathbb{T}^3$ *that verify the* CMC *condition* ${}^0k^a{}_a = -1$, *where* ${}^0k^i{}_j := (k^i{}_j)|_{\Sigma_1}$ *is the mixed second fundamental form of* Σ_1. *Assume that* $N \geq 8$ *and that*[39] $\mathscr{S}_{(\mathrm{Frame});N}(1) \leq \varepsilon^2$, *where* $\mathscr{S}_{(\mathrm{Frame});N}$ *is defined by* (8.1) (*see also Remark* 8.3). *There exist constants* $C > 0$ *and* $c > 0$ *such that if* ε *is sufficiently small*, *then the perturbed solution to the Einstein-scalar field system in* CMC-*transported-spatial-coordinates gauge* (*that is*, *to equations* (3.6a)–(3.10)) *exists for* $(t, x) \in (0, 1] \times \mathbb{T}^3$ *and verifies the norm bound*

$$\mathscr{S}_{(\mathrm{Frame});N}(t) \leq \frac{1}{2}\varepsilon t^{-c\sqrt{\varepsilon}}. \tag{8.4}$$

Moreover, *the Kretschmann scalar verifies the pointwise bound*

$$\left| t^4 \mathbf{Riem}^{\alpha\beta\gamma\delta}\mathbf{Riem}_{\alpha\beta\gamma\delta}(t,x) - \frac{20}{27} \right| \leq C\varepsilon. \tag{8.5}$$

In particular, $\mathbf{Riem}^{\alpha\beta\gamma\delta}\mathbf{Riem}_{\alpha\beta\gamma\delta}$ *blows up like* t^{-4} *as* $t \downarrow 0$. *Moreover*, *there exists a type* $\binom{1}{1}$ *tensorfield* $K_{\mathrm{Bang}} \in H^{N-1}(\mathbb{T}^3)$ *such that the following convergence results for components holds for* $t \in (0, 1]$, $(i, j = 1, 2, 3)$:

$$\|n - 1\|_{H^{N-2}} \leq C\varepsilon t^{4/3 - c\sqrt{\varepsilon}}, \tag{8.6a}$$

$$\left\| t k^i{}_j - (K_{\mathrm{Bang}})^i{}_j \right\|_{H^{N-1}} \leq C\varepsilon t^{2/3 - c\sqrt{\varepsilon}}, \tag{8.6b}$$

$$\left\| (K_{\mathrm{Bang}})^i{}_j + \frac{1}{3} I^i{}_j \right\|_{H^{N-1}} \leq C\varepsilon, \tag{8.6c}$$

where $I^i{}_j := \mathrm{diag}(1, 1, 1)$ *is the identity transformation. Similar convergence results hold for other solution variables*, *in analogy with the convergence results for the linear solution proved in Theorem* 7.1; *see* [59] *for precise statements in the context of the Einstein-stiff fluid system.*

Remark 8.3. In Theorem 8.1, we formulated our near-FLRW data assumption as a smallness condition on the norm $\mathscr{S}_{(\mathrm{Frame});N}(1)$. We could have instead formulated a "more geometric" near-FLRW assumption by making

[39] Note that $\mathscr{S}_{(\mathrm{Frame});N}(1)$ is assumed to be quadratically small compared to the amplitude ε featured in the estimate (8.4). This assumption is nonoptimal and could be improved with further effort; we have aimed for a clean presentation rather than for optimizing powers of ε.

assumptions only on the "geometric data" from Section 1.4, which does not include the lapse. We could have then derived the smallness of $\mathscr{S}_{(\text{Frame});N}(1)$ as a consequence of the assumptions on the geometric data (essentially by deriving elliptic estimates for the lapse along Σ_1 by using equation (3.10)); for convenience, we have avoided doing this.

Remark 8.4. As stated, Theorem 8.1 applies only to data with constant mean curvature. However, this restriction is not necessary: in [59], we show that for perturbations of the FLRW solution, it is always possible to find a CMC hypersurface Σ_1' near Σ_1. One can then use CMC-transported coordinates gauge starting from the "data" induced on Σ_1'. Alternatively, one could employ the parabolic lapse gauges described in Section 10 starting from near-FLRW data on Σ_1; these gauges do not require the initial Cauchy hypersurface to have constant mean curvature.

8.3. *Outline of the proof.* We now outline the main steps in the proof of Theorem 8.1. In the remainder of Section 8, we will provide additional details about the most important aspects of the proof.

(1) (*Big picture*) The main step in the proof of the theorem is to derive the a priori estimate (8.4) for the "high-norm" $\mathscr{S}_{(\text{Frame});N}(t)$, which shows, in particular, that it remains finite for $t \in (0,1]$, even though it can blow up as $t \downarrow 0$. It then follows as a standard result for elliptic-hyperbolic systems (see [3]) that, as a consequence of the a priori norm estimate, the solution must exist for $(t,x) \in (0,1] \times \mathbb{T}^3$.
(2) (*High-norm bootstrap assumption*) We use a bootstrap argument to obtain the desired estimates for the norm $\mathscr{S}_{(\text{Frame});N}$. To this end, we let $(T,1]$ be any time interval on which the solution exists, where $0 < T < 1$. We make a bootstrap assumption for $\mathscr{S}_{(\text{Frame});N}(t)$ for $t \in (T,1]$; see Section 8.4. The bootstrap assumption weakly captures the fact that the perturbed solution is near-FLRW. By the remarks made in Step (1), to prove the existence result of Theorem 8.1, it suffices to derive the a priori estimate (8.4) for $\mathscr{S}_{(\text{Frame});N}(t)$ for $t \in (T,1]$, which is a strict improvement of the bootstrap assumption; by a standard continuity argument, this justifies the bootstrap assumption, shows that the solution exists for $(t,x) \in (0,1] \times \mathbb{T}^3$, and shows that in fact, the norm estimate (8.4) holds for $t \in (0,1]$. To obtain the desired a priori norm estimate, we will derive energy estimates via a nonlinear analog of Theorem 6.1, that is, a result showing that appropriately defined nonlinear energies can blow up at most in a very mild fashion as $t \downarrow 0$. We carry this out in Step (7). The intermediate steps stated below are mostly in service of Step (7).
(3) ("*Strong*" *low-norm bootstrap assumptions*) We make stronger bootstrap assumptions at the low-order derivative levels for $t \in (T,1]$, i.e., bootstrap

assumptions that involve less singular behavior in t than what is afforded by the bootstrap assumptions for the high-norm $\mathscr{S}_{(\mathrm{Frame});N}(t)$. These stronger bootstrap assumptions are key ingredients for controlling error terms in the energy estimates, for exhibiting the AVTD nature of the solution (that is, that the spatial derivative terms in the equations are negligible near the singularity), and for proving the convergence results such as (8.6a)–(8.6c). Although we do not explicitly state such stronger bootstrap assumptions in this paper, we note that they are essentially nonlinear analogs of the estimates that we proved in the linear stability results of Theorem 7.1. The existence of a $T \in (0,1)$ such that the solution exists and verifies the high-norm and low-norm bootstrap assumptions on $(T,1]$ follows from standard local well-posedness for elliptic-hyperbolic systems; see [3].

(4) (*Improvements of the low-norm bootstrap assumptions*) As an intermediate step, we derive improvements of the strong low-norm bootstrap assumptions from the previous step, thereby closing this portion of the bootstrap argument. By improvements, we mean estimates that are strictly stronger than the estimates afforded by the low-norm bootstrap assumptions. This step is tantamount to justifying the AVTD nature of the solution. We state several of the resulting estimates in Section 8.7. In this paper, we do not provide details behind this step since the desired estimates can be obtained by using arguments similar to the ones that we used in proving the linear stability results of Theorem 7.1, but with the added complication that one must control the nonlinear error terms. We will, however, explain how to bound some representative nonlinear error terms that arise in the energy estimates; see Steps (6)–(7). As in the proof of Theorem 7.1, the proofs in this step incur a loss of derivatives.

(5) (*Approximate monotonicity identity*) To obtain the desired energy estimates, the key starting point is an approximate monotonicity identity, that is, a nonlinear analog of Theorem 5.1; recall that for linear solutions, the approximate monotonicity identity provided by Theorem 5.1 is the main ingredient that we use to derive the mildly singular energy estimates of Theorem 6.1. In this article, we do not derive an approximate monotonicity identity for the nonlinear equations because the derivation would be very similar to the proof of Theorem 5.1 but would be rather lengthy due to the presence of many nonlinear error integrals. It turns out that these nonlinear error integrals have only a small effect on the dynamics in the sense that their presence is compatible with the proof of a mild blowup-rate for the nonlinear energies, similar to the (at most) mild blowup of the linear solution's energies guaranteed by Theorem 6.1. In the next two steps, we highlight some key representative nonlinear error integrals and overview how we can handle them.

(6) (*Bounds for nonlinear error integrals*) In Sections 8.8 and 8.9, we highlight three representative nonlinear error integrals, which appear in the approximate monotonicity identity described in the previous step, and bound the error integrals in terms of the energies. The improved estimates at the low derivative levels from Step (4) are crucial for this.

(7) (A priori *energy estimates*) Recall that by using the approximate monotonicity identity for linear solutions, we were able to show that they verify the estimate (6.1), which is the integral inequality for linear solutions' energies that we used to establish the mild energy blowup-rate (6.2). In the present nonlinear context, an analog of the integral inequality (6.1) also holds, but the right-hand side features all of the nonlinear error integrals generated by the previous two steps. In Section 8.5, we outline the derivation of the nonlinear energy integral inequalities that result from accounting for the nonlinear error integrals. We then use Gronwall's inequality to obtain the desired a priori energy estimates on the bootstrap interval $(T, 1]$ and sketch a proof of how the energy estimates allow one to derive strict improvements of the bootstrap assumption for the norm $\mathscr{S}_{(\mathrm{Frame});N}(t)$ made in Step (2). In particular, this yields the desired a priori estimate (8.4).

(8) (*Additional information*) Having derived improvements of both the low-norm and high-norm bootstrap assumptions, to complete the proof of the theorem, we need only to derive the curvature blowup result (8.5) and convergence results such as (8.6a)–(8.6c). We omit these details since they can be essentially be proved as part of Step (4), that is, by using derivative-losing arguments similar to the ones we gave in the proof of the linear stability results of Theorem 7.1.

8.4. *Bootstrap assumptions.* Let $T \in (0, 1)$ be a "bootstrap time" such that the solution classically exists on $(T, 1] \times \mathbb{T}^3$ and obeys the following bootstrap assumption, where the norm $\mathscr{S}_{(\mathrm{Frame});N}(t)$ is defined in (8.1):

$$\mathscr{S}_{(\mathrm{Frame});N}(t) \leq \varepsilon t^{-\sigma}, \qquad t \in (T, 1]. \tag{8.7}$$

In (8.7), ε and σ are two small positive bootstrap parameters that are constrained, in particular, by $0 < \sqrt{\varepsilon} \leq \sigma < 1$. We will adjust the allowable smallness of ε and σ throughout the course of the analysis. In particular, we will later impose a condition of the form $c\sqrt{\varepsilon} < \sigma$ for a large constant c; see just below inequality (8.11). One can think of σ as a rough bound for the maximum possible size of $t\hat{k}$, in analogy with the role that the parameter η played in driving the energy blowup-rates of Theorem 6.1. (Recall that η is equal to t times the norm of the trace-free part of the background Kasner metric's second fundamental form.) Note that our smallness assumption for σ is reasonable in the sense that $t\hat{k}$ is small for perturbations of the FLRW metric.

Note also that (8.7) allows for the possibility that $\mathscr{S}_{(\mathrm{Frame});N}(t)$ blows up as $t \downarrow 0$, consistent with the estimates for the linear solutions that we derived in Theorem 6.1. In Corollary 8.2, we sketch a proof that for near-FLRW data, the following bound holds:

$$\mathscr{S}_{(\mathrm{Frame});N}(t) \leq \frac{1}{2}\varepsilon t^{-c\sqrt{\varepsilon}}, \qquad t \in (T,1], \tag{8.8}$$

which is a strict improvement of the bootstrap assumption (8.7) for ε sufficiently small. Deriving (8.8) is the main technical step in the proof of Theorem 8.1.

Remark 8.5. Recall that in Theorem 8.1, we are assuming that $N \geq 8$. In [59], we show that this is sufficient to allow for closure of all nonlinear estimates at all orders.

8.5. *Statement of the main a priori energy and norm estimates.* In Proposition 8.1, we state the integral inequalities verified by the energies. In Corollary 8.2, we use use the integral inequalities to derive a Gronwall estimate for the energies, which leads to the improvement (8.8) of the norm bootstrap assumption and completes the main step in the proof of Theorem 8.1. Following this, we devote the rest of Section 8 to sketching the main ideas behind the proof of Proposition 8.1.

PROPOSITION 8.1 (Integral inequalities verified by the energies). *There exist constants $C > 0$ and $c > 0$ such that if the bootstrap assumption* (8.7) *holds for $t \in (T,1]$ and if ε and $\upsigma$ are sufficiently small, then the following analog of the linear energy inequality* (6.1) *holds for $0 \leq M \leq N$ and $t \in (T,1]$*:

$$\begin{aligned}
\mathcal{E}^2_{(\mathrm{Total});\uptheta_*;M}(t) \leq{} & \mathcal{E}^2_{(\mathrm{Total});\uptheta_*;M}(1) \\
& + \underbrace{c\varepsilon \int_{s=t}^{1} s^{-1}\mathcal{E}^2_{(\mathrm{Total});\uptheta_*;M}(s)\,ds}_{\text{Borderline term}} \\
& + \underbrace{C \int_{s=t}^{1} s^{-1/3-c\sqrt{\varepsilon}}\mathcal{E}^2_{(\mathrm{Total});\uptheta_*;M}(s)\,ds}_{\text{Nonborderline term}} \\
& + \text{Similar error integrals not treated here} \\
& + \text{Negative definite spacetime integrals, similar to those in (6.1).}
\end{aligned} \tag{8.9}$$

COROLLARY 8.2 (Main a priori energy estimates). *Assume that $N \geq 8$. Consider the energy $\mathcal{E}_{(\mathrm{Total});\uptheta_*;N}(t)$ defined in* (8.2f) *and the norm $\mathscr{S}_{(\mathrm{Frame});N}(t)$ defined in* (8.1). *Assume that $\mathscr{S}_{(\mathrm{Frame});N}(1) \leq \varepsilon^2$.* (*See footnote* 39 *regarding this assumption.*) *There exists a constant $C > 0$ such that under the bootstrap* *assumption* (8.7), *the following a priori estimate holds for* $t \in (T, 1]$ *whenever* ε *and* σ *are sufficiently small*:

$$\mathcal{E}_{(\text{Total});\theta_*;N}(t) \leq C\varepsilon^2 t^{-c\sqrt{\varepsilon}}. \tag{8.10}$$

Moreover, *the following estimate holds for* $t \in (T, 1]$:

$$\mathscr{S}_{(\text{Frame});N}(t) \leq \frac{1}{2}\varepsilon t^{-c\sqrt{\varepsilon}}, \tag{8.11}$$

which is an improvement of the bootstrap assumption (8.7) *whenever* $c\sqrt{\varepsilon} < \sigma$.

Discussion of the proof. We refer readers to [59, §13] for the complete details of the proof. Here we only sketch the main ideas. First, we note that it is straightforward to establish comparison estimates in the spirit of Lemma 4.3. The comparison estimates show, in particular, that (8.11) follows from combining the energy estimate (8.10) with estimates for the terms $t^{2/3} \|g - g_{\text{FLRW}}\|_{H^N_{\text{Frame}}}$ and $t^{-2/3} \left\|g^{-1} - g^{-1}_{\text{FLRW}}\right\|_{H^N_{\text{Frame}}}$, which are featured in the nonlinear norm (8.1) but which we do not discuss here.[40] These additional terms are also the reason that the amplitude on the right-hand side of (8.11) is $\mathcal{O}(\varepsilon)$ rather than $\mathcal{O}(\varepsilon^2)$. The proofs of the comparison estimates rely on estimates for the coordinate components g_{ij} and g^{ij}. Specifically, they rely on the estimates (8.18a) stated below, which do not follow directly from the bootstrap assumption (8.7) and thus require an independent proof (along the lines of the proof of the estimate (7.1c) for linear solutions).

To derive the energy estimates stated in (8.10), one can use inequality (8.9) to establish the following estimates by a straightforward argument based on Gronwall's inequality and induction in M for $0 \leq M \leq N$:

$$\mathcal{E}^2_{(\text{Total});\theta_*;M}(t) \leq C\varepsilon^4 t^{-c\sqrt{\varepsilon}}, \qquad t \in (T, 1]. \tag{8.12}$$

We now further comment on two aspects of the estimate (8.12). First, it is only the first "Borderline" term on the right-hand side of (8.9) that can cause $\mathcal{E}_{(\text{Total});\theta_*;M}(t)$ to blow up as $t \downarrow 0$; the "Nonborderline" term on the right-hand side of (8.9) is harmless in the sense that the function $s^{-1/3-c\sqrt{\varepsilon}}$ is integrable over the interval $s \in (0, 1]$ whenever ε is sufficiently small. Second, we note that the exponent on the right-hand side of (8.12) is $t^{-c\sqrt{\varepsilon}}$ due to some terms that we have not discussed here, that is, the terms "Similar error integrals not treated here" from (8.9); if not for the omitted terms, the exponent could be improved to $t^{-c\varepsilon}$. (This is a minor remark that has no substantial bearing on the main results.) $\square$

[40] Note that we did not include such terms in the norms (4.5) for linear solutions.

8.6. *A convenient frame and dual frame.* In the ensuing discussion, we will find it convenient to perform some computations relative to the frame[41] $\{e'_{(A)}\}_{A=1}^3$ and dual frame $\{\theta'^{(A)}\}_{A=1}^3$, whose elements are defined as follows:

$$e'_{(A)} := t^{-1/3}\partial_A, \qquad \theta'^{(A)} := t^{1/3}dx^A. \tag{8.13}$$

The appeal of the frame $\{e'_{(A)}\}_{A=1}^3$ is that it is orthonormal as measured by the background spatial metric $g_{\text{FLRW}} := t^{2/3}\sum_{i=1}^3(dx^i)^2$, and, as we explain in Section 8.7, it is approximately orthonormal for the perturbed metric g (in a sense that we make precise via the estimate (8.19)). The perturbed metric and its inverse can respectively be expanded[42] relative to the dual frame and frame as follows:

$$g = g_{AB}\theta'^{(A)} \otimes \theta'^{(B)}, \qquad g^{-1} = g^{AB}e'_{(A)} \otimes e'_{(B)}, \tag{8.14}$$

where $g_{AB} := g(e'_{(A)}, e'_{(B)})$ and $g^{AB} := g^{-1}(\theta'^{(A)}, \theta'^{(B)})$. We remark that in [59], instead of working with the "time-rescaled" frame and dual frame (8.13), we work with solution variables that are rescaled with respect to powers of t; see Remark 8.1.

The connection coefficients γ_{ACB} of the frame relative to g are determined by the equation[43]

$$\nabla_{e'_{(A)}} e'_{(B)} = g^{CD}\gamma_{ADB}e'_{(C)}, \tag{8.15}$$

where, since the vectorfield commutators $[e'_{(A)}, e'_{(B)}]$ vanish, we have[44]

$$\gamma_{ACB} = \frac{1}{2}\left\{e'_{(A)}(g_{CB}) + e'_{(B)}(g_{AC}) - e'_{(C)}(g_{AB})\right\}. \tag{8.16}$$

For use below, we note the following standard expression for the Ricci curvature of g (in type $\binom{1}{1}$ form):

$$\text{Ric} = \text{Ric}^A{}_B e'_{(A)} \otimes \theta'^{(B)},$$

where

$$\begin{aligned}\text{Ric}^A{}_B = {} & g^{AE}g^{CF}e'_{(C)}(\gamma_{EFB}) - g^{AE}g^{CF}e'_{(E)}(\gamma_{BCF}) \\ & + g^{AE}g^{CF}g^{DH}\gamma_{CFD}\gamma_{EHB} - g^{AE}g^{CF}g^{DH}\gamma_{EFD}\gamma_{CHB}.\end{aligned} \tag{8.17}$$

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[41] In (8.13) and the remainder of Section 8, $\partial_A := \frac{\partial}{\partial x^A}$, with $\{x^A\}_{A=1,2,3}$ denoting the transported spatial coordinates. Moreover, $\partial_{\vec{I}}$ is still the coordinate partial derivative multi-indexed operator defined in Section 2.3.

[42] Throughout this subsection and the next one, we use Einstein's summation convention for uppercase Latin indices.

[43] Recall that ∇ denotes the Levi–Civita connection of g.

[44] We are using here the standard notation $X(f)$ to denote the derivative of the scalar function f in the direction of the vectorfield X.

8.7. *Improved* AVTD*-type estimates at the lower derivative levels.* As we mentioned in Steps (3) and (4) of Section 8.3, to prove Proposition 8.1, we need to derive improved estimates at the lower derivative levels. By improved, we mean that they are less singular as $t \downarrow 0$ compared to the estimates afforded by the bootstrap assumption (8.7). In the context of the linear problem, we derived such estimates in Theorem 7.1. For brevity, we will take for granted here that we can derive similar estimates for the nonlinear solution, effectively postponing the discussion of the nonlinear error terms until Section 8.9, when we discuss them in the context of energy estimates. Specifically, we will take for granted that the following pointwise coordinate component estimates hold for $t \in (T, 1]$ whenever $|\vec{I}| \leq N - 3$ in (8.18a)–(8.18b), $1 \leq |\vec{I}| \leq N - 3$ in (8.18c), and $i, j = 1, 2, 3$:

$$\left|\partial_{\vec{I}}\left\{g_{ij} - (g_{\mathrm{FLRW}})_{ij}\right\}\right| \lesssim \sqrt{\varepsilon} t^{2/3 - c\sqrt{\varepsilon}}, \tag{8.18a}$$
$$\left|\partial_{\vec{I}}\left\{g^{ij} - (g^{-1}_{\mathrm{FLRW}})^{ij}\right\}\right| \lesssim \sqrt{\varepsilon} t^{-2/3 - c\sqrt{\varepsilon}},$$

$$\left|\partial_{\vec{I}}\left\{t\partial_t \phi - \sqrt{\frac{2}{3}}\right\}\right| \lesssim \varepsilon, \tag{8.18b}$$

$$\left|\partial_{\vec{I}}\phi\right| \lesssim \sqrt{\varepsilon} t^{-c\sqrt{\varepsilon}}. \tag{8.18c}$$

Note, for example, that (8.18b) is an improvement over the bootstrap assumption (8.7) in that (8.7) and Sobolev embedding would yield only the bound $\left|\partial_{\vec{I}}\left\{t\partial_t\phi - \sqrt{\frac{2}{3}}\right\}\right| \lesssim \varepsilon t^{-c\sigma}$, which, due to the singular behavior of the right-hand side as $t \downarrow 0$, is inadequate for treating the borderline integral that we control in (8.32). Similarly, for ε sufficiently small, the factors of $t^{2/3 - c\sqrt{\varepsilon}}$ and $t^{-2/3 - c\sqrt{\varepsilon}}$ in (8.18a) are improvements[45] over the factors of $t^{2/3 - c\sigma}$ and $t^{-2/3 - c\sigma}$ that would follow from (8.7) and Sobolev embedding. We refer readers to [59] for proofs of analogs of (8.18a)–(8.18c) in the context of the Einstein-stiff fluid system. The estimates stated in (8.18a) are analogs[46] of the estimates (7.1c) and (7.1d) from the linear problem while the estimates (8.18b) and (8.18c) are respectively analogs of (7.1f) and (7.1g).

Contracting inequalities (8.18a) against the frame/dual frame, we find that they are approximately orthonormal relative to the metric g in the following

[45]Note that the amplitude factors of $\sqrt{\varepsilon}$ in (8.18a) are worse than the amplitude factor of ε that would follow from (8.7) and Sobolev embedding. This is an artifact of some inefficiencies in our proof and is not important for our main results; the t-dependent factors of $t^{2/3 - c\sqrt{\varepsilon}}$ and $t^{-2/3 - c\sqrt{\varepsilon}}$ in (8.18a) are what matter.

[46]Note that the estimates stated in (8.18a)–(8.18c) are of pointwise type while the estimates of Theorem 7.1 are in terms of Sobolev norms. This is a minor point in the sense that we can obtain pointwise estimates from Sobolev estimates via Sobolev embedding (at the cost of a few derivatives).

weak sense (for $t \in (T, 1]$ and $|\vec{I}| \leq N - 3$):

$$\left|\partial_{\vec{I}} \left\{g_{AB} - \delta_{AB}\right\}\right| \lesssim t^{-c\sqrt{\varepsilon}}, \qquad \left|\partial_{\vec{I}} \left\{g^{AB} - \delta^{AB}\right\}\right| \lesssim t^{-c\sqrt{\varepsilon}}, \tag{8.19}$$

where δ_{AB} and δ^{AB} are standard Kronecker deltas.

In Section 8.9, when we bound some representative energy error integrals, we will use the following simple consequences of the above estimates:

$$\left\| |\gamma|_g \right\|_{L^\infty} (t) \lesssim t^{-1/3-c\sqrt{\varepsilon}}, \tag{8.20}$$

$$\left\| |\partial\phi|_g \right\|_{L^\infty} (t) \lesssim t^{-1/3-c\sqrt{\varepsilon}}. \tag{8.21}$$

To prove (8.20), we first note that (8.18a) with $|\vec{I}| = 1$ implies that

$$|\gamma_{ACE}| \lesssim t^{-1/3-c\sqrt{\varepsilon}}, \tag{8.22}$$

where in deriving (8.22), we have incurred three factors of $t^{-1/3}$ relative to the estimate (8.18a), one for each contraction against a frame vector belonging to $\{e'_{(A)}\}_{A=1}^3$. We therefore deduce from (8.19) and (8.22) that

$$|\gamma|_g^2 = g^{AB} g^{CD} (g^{-1})^{EF} \gamma_{ACE} \gamma_{BDF} \lesssim t^{-2/3-c\sqrt{\varepsilon}}, \tag{8.23}$$

which yields (8.20). To obtain (8.21), we note that (8.18c) with $|\vec{I}| = 1$ implies that

$$\left| e'_{(A)} \phi \right| \lesssim t^{-1/3-c\sqrt{\varepsilon}}, \tag{8.24}$$

where in deriving (8.24), we have incurred a factor of $t^{-1/3}$ relative to the estimate (8.18c) due to the contraction against the frame vector belonging to $\{e'_{(A)}\}_{A=1}^3$. We therefore deduce from (8.19) and (8.24) that

$$|\partial\phi|_g^2 = g^{AB} (e'_{(A)}\phi) e'_{(B)}\phi \lesssim t^{-2/3-c\sqrt{\varepsilon}}, \tag{8.25}$$

which yields (8.21).

For future use, we also note the following relations, which follow in a straightforward fashion from the definitions of the quantities involved:

$$\left| g^{AB} g^{CD} g^{EF} (s\partial_{\vec{I}} \gamma_{ACE})(s\partial_{\vec{I}} \gamma_{BDF}) \right| = \left| s\partial_{\vec{I}} \gamma \right|_g^2 \leq C \left| s\partial\partial_{\vec{I}} g \right|_g^2. \tag{8.26}$$

8.8. *Identifying some representative nonlinear error terms.* In Section 8.9, we will bound three representative nonlinear error integrals and explain how they contribute to the terms on the right-hand side of the energy integral inequality (8.9). In the present subsection, as a preliminary step, we commute some of the nonlinear Einstein-scalar field equations with the spatial derivative operator $\partial_{\vec{I}}$ (as defined in Section 2.3) and identify the representative nonlinear terms that lead to the error integrals.

First, we commute the evolution equation (3.7b) with $\partial_{\vec{I}}$. Using (8.17), we see that relative to the frame/dual frame (8.13), the commuted equation takes the form

$$\partial_t(t\partial_{\vec{I}} k^A_{\ B}) = g^{AE} g^{CF} g^{DH} \gamma_{CFD} \partial_{\vec{I}} \gamma_{EHB} + \cdots , \tag{8.27}$$

where, for illustration, we have kept only one representative nonlinear product generated by the right-hand side of (8.17).

Similarly, we commute the scalar field wave equation (3.19) with $\partial_{\vec{I}}$ (as defined in Section 2.3) and, for illustration, retain two products generated by terms on the last two lines of (3.19), which yields

(8.28)

$$\partial_t(t\partial_t\partial_{\vec{I}}\phi) + n^2 t g^{ab}\partial_a\partial_b\partial_{\vec{I}}\phi = \left(t\partial_t\phi - \sqrt{\frac{2}{3}}\right)\frac{(\partial_{\vec{I}} n)}{t} - t g^{ab}(\partial_a\partial_{\vec{I}} n)\partial_b\phi + \cdots .$$

Note that in writing down (8.27)–(8.28), we have ignored various *linearly* small products in the equations. Those terms are of crucial importance for deriving an analog of the approximate monotonicity identity from Theorem 5.1 and for this reason, they are not part of the nonlinear error term analysis that we are currently conducting.

8.9. *Bounds for some representative nonlinear error integrals and a proof sketch of Proposition* 8.1. Recall that in Theorem 5.1, we derived an approximate monotonicity identity for linear solutions, which was the main step in deriving the energy integral inequality for linear solutions stated in (6.1). In the nonlinear problem, the analog of inequality (6.1) is the energy integral inequality (8.9) provided by Proposition 8.1. The main difference between the linear estimate (6.1) and the nonlinear estimate (8.9) is, of course, the presence of nonlinear error integrals, which arise in the nonlinear analog of the approximate monotonicity identity. Ultimately, the nonlinear error integrals generate terms that appear on the right-hand side of the nonlinear energy integral inequality (8.9). In this subsection, to keep the discussion short, we consider only three representative error integrals generated by the quadratic nonlinear terms highlighted in Section 8.8. Our main goal is to show that the corresponding error integrals (which are cubically[47] small) are bounded by the right-hand side of (8.9). In view of the above remarks, it follows that the discussion in this subsection constitutes a proof sketch of Proposition 8.1. We note that the improved estimates at the lower derivative levels from Section 8.7 are essential for controlling the error integrals, especially the borderline one that we control in (8.32).

[47]Some of the error integrals that we treat here are similar to other error integrals that are generated by integration by parts. For example, cubic error integrals similar to the one in (8.33) arise from the nonlinear analog of (5.15).

8.9.1. *A nonborderline error integral involving the scalar field.* We start by explaining how the error term $tg^{ab}(\partial_a\partial_{\vec{I}}n)\partial_b\phi$ on the right-hand side of (8.28) contributes to the right-hand side of (8.9). Revisiting the *proof* of Proposition 5.2, we see that in the analog of the integral identity (5.3), the error term generates the following spacetime integral (where we are assuming that $1 \leq |\vec{I}| \leq M \leq N$):

$$\int_{s=t}^{1}\int_{\Sigma_s} sg^{ab}(\partial_a\partial_{\vec{I}}n)(\partial_b\phi)(s\partial_t\partial_{\vec{I}}\phi)\,dx\,ds. \tag{8.29}$$

Using (8.21), Definition 8.3, and Cauchy-Schwarz relative to g, we bound the magnitude of the integral in (8.29) as follows:

$$\begin{aligned}
&\leq \int_{s=t}^{1} \left\| |\partial\phi|_g \right\|_{L^\infty}(s)\int_{\Sigma_s} |s\partial\partial_{\vec{I}}n|_g|s\partial_t\partial_{\vec{I}}\phi|\,dx\,ds \\
&\lesssim \int_{s=t}^{1} s^{-1/3-c\sqrt{\varepsilon}}\mathscr{E}^2_{(\mathrm{Total});\theta_*;\vec{I}}(s)\,ds.
\end{aligned} \tag{8.30}$$

We now simply observe that the right-hand side of (8.30) is bounded by the nonborderline error integral on the right-hand side of (8.9), as desired.

8.9.2. *A borderline error integral involving the scalar field.* We now explain how the error term $\left(t\partial_t\phi - \sqrt{\frac{2}{3}}\right)\frac{(\partial_{\vec{I}}n)}{t}$ on the right-hand side of (8.28) contributes to the right-hand side of (8.9). For the same reasons given in Section 8.9.1, this error term generates the error integral

$$\int_{s=t}^{1} s^{-1}\int_{\Sigma_s}\left(s\partial_t\phi - \sqrt{\frac{2}{3}}\right)(\partial_{\vec{I}}n)(s\partial_t\partial_{\vec{I}}\phi)\,dx\,ds. \tag{8.31}$$

Using (8.18b) and Definition 8.3, we bound the magnitude of the integral in (8.31) as follows (where we are again assuming that $1 \leq |\vec{I}| \leq M \leq N$):

$$\begin{aligned}
&\leq \int_{s=t}^{1} s^{-1}\left\| s\partial_t\phi - \sqrt{\frac{2}{3}}\right\|_{L^\infty}(s)\int_{\Sigma_s}|\partial_{\vec{I}}n||s\partial_t\partial_{\vec{I}}\phi|\,dx\,ds \\
&\leq c\varepsilon\int_{s=t}^{1} s^{-1}\mathscr{E}^2_{(\mathrm{Total});\theta_*;\vec{I}}(s)\,ds.
\end{aligned} \tag{8.32}$$

Note that the right-hand side of (8.30) is bounded by the borderline error integral on the right-hand side of (8.9), as desired. We stress that the availability of the small coefficient ε is crucial since, in the Gronwall estimate for $\mathscr{E}^2_{(\mathrm{Total});\theta_*;\vec{I}}$, the right-hand side of (8.32) can cause $\mathscr{E}^2_{(\mathrm{Total});\theta_*;M}(t)$ to blow up like $t^{-c\varepsilon}$ as $t \downarrow 0$. Note also that for this argument, it is crucial that the singular integrand factor on the right-hand side of (8.32) is not worse than s^{-1}; a slightly worse factor of type $s^{-1-C\varepsilon}$ would radically alter the Gronwall estimate and would prevent us from deriving an improvement of the norm bootstrap assumption. For this reason, the "lossless" AVTD-type estimate (8.18b) is critically important for the proof of nonlinear stability.

8.9.3. *A nonborderline error integral involving the metric.* Finally, we will consider the effect of the error term $g^{AE}g^{CF}g^{DH}\gamma_{CFD}\partial_{\vec{I}}\gamma_{EHB}$ on the right-hand side of (8.27). Revisiting the proof of Proposition 5.3, we see that in the analog of the metric energy identity (5.8), the error term generates the following spacetime integral:

$$\int_{s=t}^{1}\int_{\Sigma_s} g^{AB}g^{CF}g^{DH}\gamma_{CFD}(s\partial_{\vec{I}}\gamma_{EHB})(s\partial_{\vec{I}}\hat{k}^{E}_{\ A})\,dx\,ds. \tag{8.33}$$

Using (8.20), (8.26), Definition 8.3, and Cauchy-Schwarz relative to g, we bound the magnitude of the integral in (8.33) as follows (where we are again assuming that $1 \leq |\vec{I}| \leq M \leq N$):

$$\begin{aligned} &\lesssim \int_{s=t}^{1}\int_{\Sigma_s} |\gamma|_g \left|s\partial_{\vec{I}}\gamma\right|_g \left|s\partial_{\vec{I}}\hat{k}\right|_g \,dx\,ds \\ &\lesssim \int_{s=t}^{1} \left\||\gamma|_g\right\|_{L^\infty}(s)\mathcal{E}^2_{(\mathrm{Total});\theta_*;\vec{I}}(s)\,ds \\ &\lesssim \int_{s=t}^{1} s^{-1/3-c\sqrt{\varepsilon}}\mathscr{E}^2_{(\mathrm{Total});\theta_*;\vec{I}}(s)\,ds. \end{aligned} \tag{8.34}$$

Like the right-hand side of (8.30), the right-hand side of (8.34) is bounded by the nonborderline error integral on the right-hand side of (8.9), as desired. This completes our discussion of the three representative nonlinear error integrals and finishes our proof sketch of Proposition 8.1.

9. Comments on realizing "end states"

The linear stability results of Theorem 7.1 show that for some time-rescaled versions of the linear solution variables, there is a well-defined map from their "initial state" along the data hypersurface Σ_1 to their "end state" along Σ_0. For example, the estimate (7.2d) exhibits this fact for $t\partial_t\varphi$, in which case the end state is Ψ_{Bang} and the map is from H^N to H^{N-1}. It is natural to inquire whether or not one can realize a given end state (more precisely, one in which time derivative terms in the equations are dominant) by finding suitable initial data that lead to it. Although we do not give a proof that one can "realize all end states in which time derivative terms dominate" in solutions to the linearized equations of Proposition 3.2, we do point to some evidence in this direction by discussing some relevant results in a simplified context. Our discussion here is closely connected to the work described in Section 1.8 in which authors used Fuchsian methods to construct singular solutions to various Einstein-matter systems under symmetry or analyticity assumptions. In this section, we consider a model equation in $1 + 1$ dimensions, obtained from the linearized scalar field equation (3.17) in the case $\mathring{g} = g_{\mathrm{FLRW}} = t^{2/3}\sum_{i=1}^{3}(dx^i)^2$ by dropping the linearized lapse terms and making the symmetry assumption that the solution depends only on t and a single spatial variable $x^1 \in \mathbb{T}$. We have made the symmetry assumption only to shorten the presentation; the arguments we sketch below remain valid without it. For convenience, in the rest of this section, we will write x instead of x^1. We caution that ignoring the lapse and its elliptic PDE is tantamount to sidestepping new difficulties not found in the standard Fuchsian framework, which applies to hyperbolic equations. Specifically, our model equation in $\varphi = \varphi(t,x)$ on the domain $(t,x) \in (0,1] \times \mathbb{T}$ is

$$-\partial_t(t\partial_t\varphi) + t^{1/3}\partial_x^2\varphi = 0. \tag{9.1}$$

The methods of [14], [15] (see also the many other related works cited in Section 1.8), can be used to show that given an asymptotic expansion for the end state of the form $\ln t\Psi_1(x) + \Psi_2(x)$ (where the Ψ_i have sufficient Sobolev regularity), one can construct a solution φ to (9.1) existing on a slab of the form $(0,1] \times \mathbb{T}$ such that

$$\varphi = \ln t\Psi_1(x) + \Psi_2(x) + \mathcal{R}(t,x). \tag{9.2}$$

Furthermore, there is a suitably strong t-dependent Sobolev norm on the time slices Σ_t such that the norm of the remainder term $\mathcal{R}$ vanishes as $t \downarrow 0$. In particular, $\mathcal{R}$ becomes negligible relative to $\ln t\Psi_1(x) + \Psi_2(x)$ as $t \downarrow 0$. We now sketch the proof of these phenomena by following the approach outlined in Section 1.8. We note that our analysis involves much simpler t weights in the energies compared to the weights of [14], [15] because we are treating a simple linear scalar equation. We recall that the overall strategy of the proof is to construct a sequence of standard initial value problems that approximate the "singular initial value problem with vanishing Cauchy data for $\mathcal{R}$ given along Σ_0." To begin our sketch of a proof, we use equation (9.1) and the ansatz (9.2) to deduce the following equation for $\mathcal{R}(t,x)$:

$$-\partial_t(t\partial_t\mathcal{R}) + t^{1/3}\partial_x^2\mathcal{R} = -t^{1/3}\ln t\partial_x^2\Psi_1(x) - t^{1/3}\partial_x^2\Psi_2(x). \tag{9.3}$$

We now derive an estimate for the energy $\mathscr{E}[\mathcal{R}](t) \geq 0$ defined by

$$\mathscr{E}^2[\mathcal{R}](t) := \int_{\Sigma_t}(t^{1/3}\partial_t\mathcal{R})^2 + (\partial_x\mathcal{R})^2\,dx. \tag{9.4}$$

A straightforward integration by parts argument, based on multiplying equation (9.3) by $t^{-1/3}\partial_t\mathcal{R}$, yields that for $0 < t_1 < t_2 \leq 1$, we have

$$\begin{aligned}\mathscr{E}[\mathcal{R}](t_2) &\leq \mathscr{E}[\mathcal{R}](t_1) + \left\{\left\|\partial_x^2\Psi_1\right\|_{L^2} + \left\|\partial_x^2\Psi_2\right\|_{L^2}\right\}\int_{s=t_1}^{t_2}(1+\ln s)s^{-1/3}\,ds \\ &\leq \mathscr{E}[\mathcal{R}](t_1) + C\left\{\left\|\partial_x^2\Psi_1\right\|_{L^2} + \left\|\partial_x^2\Psi_2\right\|_{L^2}\right\}\left\{t_2^p - t_1^p\right\},\end{aligned} \tag{9.5}$$

where p is a constant chosen to be slightly smaller than $2/3$. Inequality (9.5) is the main ingredient that one needs to deduce the desired existence result and estimates for $\mathcal{R}$. Note that the estimate (9.5) loses one derivative relative to Ψ_1 and Ψ_2. In a detailed proof of the desired results (see the methods of [14]), one considers a sequence $\{\mathcal{R}_n\}_{n=0}^{\infty}$ of solutions to (9.3), where $\mathcal{R}_n$ has 0 Cauchy data on Σ_{t_n} (and thus $\mathscr{E}[\mathcal{R}_n](t_n) = 0$) and is a classical solution on $[t_n, 1]$. Here, $\{t_n\}_{n=0}^{\infty}$ is a sequence of times in $(0, 1]$ that decreases to 0 as $n \to \infty$. An argument similar to the one used to prove (9.5) yields that for $m < n$, we have

$$\sup_{t\in[t_m,1]} \mathscr{E}[\mathcal{R}_n - \mathcal{R}_m](t) \leq C \left\{ \left\| \partial_x^2 \Psi_1 \right\|_{L^2} + \left\| \partial_x^2 \Psi_2 \right\|_{L^2} \right\} \{t_m^p - t_n^p\}. \tag{9.6}$$

It follows from (9.6) that for any $\epsilon > 0$, $\{\mathcal{R}_n\}_{n=0}^{\infty}$ is Cauchy in the norm[48]

$$f \to \sup_{t\in(\epsilon,1]} \left\{ \|t^{1/3} \partial_t f(t)\|_{L^2} + \|\partial_x f(t)\|_{L^2} \right\}$$

and thus converges[49] to the desired solution $\mathcal{R}$.

Remark 9.1. We could have instead derived energy estimates by multiplying equation (9.3) by $t^{-P}\partial_t \mathcal{R}$ for any choice of $P \in [1/3, 5/3)$, and a similar argument would yield a uniform bound for the energy $\int_{\Sigma_t} (t^{\frac{1-P}{2}} \partial_t \mathcal{R})^2 + (t^{\frac{1/3-P}{2}} \partial_x \mathcal{R})^2 \, dx$ for $t \in (0, 1]$. We could even have allowed P to mildly depend on x. This illustrates the freedom (mentioned in Section 1.8) in choosing viable t-weights in the Fuchsian approach.

It is not difficult to modify the above arguments so that they apply if one includes the semilinear term[50] $t^{1/3}(\partial_x \varphi)^2$ on the right-hand side of (9.1); this term is a model for the kinds of semilinear terms that one finds in the Einstein-scalar field system. It would be interesting to know to what extent the arguments can be extended to apply to the full linearized system of Proposition 3.2 and the full nonlinear Einstein-scalar field system in three spatial dimensions. The framework of [2] provides a possible starting point for establishing such an extension. However, that framework applies only to symmetric hyperbolic

[48]Higher-order energy estimates for the sequence $\{\mathcal{R}_n\}_{n=0}^{\infty}$ can be obtained in a similar fashion.

[49]In the fully detailed construction of the analog of $\mathcal{R}$ for the nonlinear problems treated in [14], [15], the authors extend the $\mathcal{R}_n$ to be 0 on $[0, t_n)$ and show that this extension implies that $\mathcal{R}_n$ is a weak solution on an interval $[0, \delta)$.

[50]More precisely, when the term $t^{1/3}(\partial_x \varphi)^2$ is present, one can show that the remainder term $\mathcal{R}$ exists and verifies estimates similar to the ones derived above on a small slab $(0, \delta] \times \mathbb{T}$, where $\delta > 0$ depends on a Sobolev norm of Ψ_1 and Ψ_2. This argument requires higher-order energy estimates because of the nonlinearity.

Fuchsian systems and thus it would need to be modified to treat the Einstein-scalar field system in gauges involving an elliptic or parabolic lapse PDE.

10. **Parabolic lapse gauges**

In this section, we introduce a new family of gauges for the Einstein-scalar field system. We show that a version of the approximate monotonicity identity also holds in solutions to linearized (around the Kasner backgrounds) versions of the corresponding equations; see Theorem 10.1. We also show that mildly singular energy estimates without derivative loss hold for the linear solutions when the Kasner backgrounds are nearly spatially isotropic; see Theorem 10.2. Using these results, one could also prove linear stability results when the Kasner backgrounds are nearly spatially isotropic, that is, an analog of Theorem 7.1. However, for brevity, we do not explicitly provide such a result here; given the results of Theorems 10.1 and 10.2, one could prove linear stability by making minor modifications to the proof of Theorem 7.1.

The gauge that we study in this section involves a parabolic equation for the lapse variable n that depends on a real parameter λ. The mildly singular energy estimates of Theorem 10.2 are valid for near-FLRW Kasner backgrounds when $2 < \lambda < \infty$. As we will see, for $\lambda > 0$, *the parabolic lapse* PDE*s are locally well posed only in the past direction*, that is, for t decreasing. Formally, $\lambda = \infty$ corresponds to the CMC lapse equation. However, our proofs in this section are somewhat different compared to our proofs in CMC gauge and do not allow us to directly recover the CMC gauge results by taking a limit $\lambda \to \infty$.

10.1. *Choice of a gauge and the corresponding formulation of the Einstein-scalar field equations.* In formulating the nonlinear Einstein-scalar field equations in the new gauge, we continue to use transported spatial coordinates and to decompose $\mathbf{g} = -n^2 dt^2 + g_{ab} dx^a dx^b$ as in (1.5).

10.1.1. *Fixing the gauge.* We now fix the lapse gauge.

Definition 10.1 (Choice of a parabolic lapse gauge). Let $\lambda \neq 0$ be a real number. We now impose the following relation, which fixes the lapse gauge:

$$\lambda^{-1}(n-1) = t k^a_{\ a} + 1. \tag{10.1}$$

Remark 10.1. Note that the CMC-transported spatial coordinates gauge of Section 3 corresponds to $\lambda = \infty$.

10.1.2. *Formulation of the Einstein-scalar field equations.* We now provide the (nonlinear) Einstein-scalar field equations relative to the gauge (10.1) with transported[51] spatial coordinates.

[51]By "transported," we mean in the sense described below equation (1.5).

PROPOSITION 10.1 (The Einstein-scalar field equations in the gauge (10.1) with transported spatial coordinates). *Under the gauge condition* (10.1) *and with transported spatial coordinates, the Einstein-scalar field system consists of the following equations.*

The ***Hamiltonian and momentum constraint equations*** *are respectively*

$$R - k^a{}_b k^b{}_a + \underbrace{(k^a{}_a)^2}_{t^{-2}\{\lambda^{-1}(n-1)-1\}^2} = \overbrace{(n^{-1}\partial_t\phi)^2 + g^{ab}\nabla_a\phi\nabla_b\phi}^{2\mathbf{T}(\hat{\mathbf{N}},\hat{\mathbf{N}})}, \tag{10.2a}$$

$$\nabla_a k^a{}_i - \underbrace{\nabla_i k^a{}_a}_{\lambda^{-1}t^{-1}\nabla_i n} = \underbrace{-n^{-1}\partial_t\phi\nabla_i\phi}_{-\mathbf{T}(\hat{\mathbf{N}},\partial_i)}, \tag{10.2b}$$

where R denotes the scalar curvature of g_{ij}.

The ***metric evolution equations*** *are*

$$\partial_t g_{ij} = -2n g_{ia} k^a{}_j, \tag{10.3a}$$

$$\partial_t k^i{}_j = -g^{ia}\nabla_a\nabla_j n + n\Big\{\mathrm{Ric}^i{}_j + \underbrace{k^a{}_a k^i{}_j}_{t^{-1}\{\lambda^{-1}(n-1)-1\}k^i{}_j} \underbrace{-g^{ia}\nabla_a\phi\nabla_j\phi}_{-T^i{}_j+(1/2)I^i{}_j\mathbf{T}}\Big\}, \tag{10.3b}$$

where $\mathrm{Ric}^i{}_j$ denotes the Ricci curvature of g_{ij}, $I^i{}_j = \mathrm{diag}(1,1,1)$ denotes the identity transformation, and $\mathbf{T} := (\mathbf{g}^{-1})^{\alpha\beta}\mathbf{T}_{\alpha\beta}$ denotes the trace of the energy-momentum tensor (1.2).

The ***volume form factor*** $\sqrt{\det g}$ *verifies the auxiliary equation*[52]

$$\partial_t \ln\left(t^{-1}\sqrt{\det g}\right) = (1-\lambda^{-1})\frac{n-1}{t}. \tag{10.4}$$

The ***scalar field wave equation*** *is*

$$\overbrace{-n^{-1}\partial_t(n^{-1}\partial_t\phi)}^{-\mathbf{D}_{\hat{\mathbf{N}}}\mathbf{D}_{\hat{\mathbf{N}}}\phi} + g^{ab}\nabla_a\nabla_b\phi = \overbrace{\frac{1}{t}n^{-1}\left\{1-\lambda^{-1}(n-1)\right\}\partial_t\phi}^{-k^a{}_a\mathbf{D}_{\hat{\mathbf{N}}}\phi} - n^{-1}g^{ab}\nabla_a n\nabla_b\phi. \tag{10.5}$$

[52]This equation, which we do not use in the present article, is implied by (3.7a) and the gauge condition (10.1).

The ***parabolic lapse equation*** *is*

$$(10.6)\qquad \lambda^{-1}\frac{1}{t}\partial_t(n-1) + g^{ab}\nabla_a\nabla_b(n-1)$$
$$= (n-1)\Big\{\frac{1}{t^2}(1-\lambda^{-1}) + R - g^{ab}\nabla_a\phi\nabla_b\phi\Big\}$$
$$+ \lambda^{-1}(\lambda^{-1}-2)\frac{1}{t^2}(n-1)^2 + \lambda^{-2}\frac{1}{t^2}(n-1)^3 + R - g^{ab}\nabla_a\phi\nabla_b\phi.$$

When $\lambda > 0$, *the gauge condition* (10.1) *and the constraint equations* (10.2a)–(10.2b) *are preserved by the* ***past*** *flow of the remaining equations if they are verified by the data.*

Remark 10.2. We are primarily interested in the gauge (10.1) when $\lambda > 2$ since our main results rely on this inequality. Note that when $\lambda > 0$, the parabolic equation (10.6) is locally well posed only in the *past* direction.

Remark 10.3 (Data for the lapse). In order to solve the equations of Proposition 10.1, we must prescribe the lapse along the initial Cauchy hypersurface Σ_1. That is, $n|_{\Sigma_1}$ is not determined by the geometric data; see Section 1.4 for discussion of the geometric data. This is in contrast to the CMC-transported spatial coordinates gauge, in which $n|_{\Sigma_1}$ is determined by the geometric data via the elliptic PDE (3.10). A natural choice in the context of proving the nonlinear stability of the FLRW solution's Big Bang singularity would be $n|_{\Sigma_1} = 1$.

Proof of Proposition 10.1. The proposition can be proved by making simple modifications to the standard arguments that yield Proposition 3.1. □

10.2. *Linearizing around the Kasner solutions.* In the next proposition, we linearize the equations of Proposition 10.1 around a Kasner solution (1.6). See Section 3.3 for some remarks on the linearization procedure.

PROPOSITION 10.2 (The linearized Einstein-scalar field equations in the gauge (10.1) with transported spatial coordinates). *Consider the equations of Proposition* 10.1 *linearized around a Kasner solution* (1.6). *The linearized equations in the unknowns* $(\nu, h, \kappa, \varphi)$, *which are functions of* $(t,x)\in(0,\infty)\times\mathbb{T}^3$, *take the following form* (*see Definition* 3.1 *for the definitions of some of the quantities*).

The ***linearized parabolic gauge condition*** (10.1) *is*

$$(10.7)\qquad t\kappa^a{}_a = \lambda^{-1}\nu.$$

The ***linearized versions of the Hamiltonian and momentum constraint equations*** (10.2a)–(10.2b) *are*

$$(10.8a)\qquad t^{2\,(h)}R - 2(t\hat{\mathring{k}}^a{}_b)(t\kappa^b{}_a) - 2A(t\partial_t\varphi) + 2(A^2-\lambda^{-1})\nu = 0,$$

$$(10.8b)\qquad \partial_a(t\kappa^a{}_i) = \lambda^{-1}\partial_i\nu - A\partial_i\varphi - {}^{(h)}\Gamma_a{}^a{}_b(t\hat{\mathring{k}}^b{}_i) + {}^{(h)}\Gamma_a{}^b{}_i(t\hat{\mathring{k}}^a{}_b),$$

(10.8c)
$$\mathring{g}^{ab}\partial_a(t\kappa^i{}_b) = \lambda^{-1}\mathring{g}^{ia}\partial_a\nu - A\mathring{g}^{ia}\partial_a\varphi - \mathring{g}^{ab(h)}\Gamma_a{}^i{}_c(t\hat{\mathring{k}}^c{}_b) + \mathring{g}^{ab(h)}\Gamma_a{}^c{}_b(t\hat{\mathring{k}}^i{}_c).$$

The ***linearized version of the lapse equation*** (10.6) *can be expressed in either of the following two forms*:

(10.9a)
$$2A(t\partial_t\varphi) + 2(t\hat{\mathring{k}}^a{}_b)(t\kappa^b{}_a) = \lambda^{-1}t\partial_t\nu + t^2\mathring{g}^{ab}\partial_a\partial_b\nu + (2A^2 - 1 - \lambda^{-1})\nu,$$

(10.9b)
$$\lambda^{-1}t\partial_t\nu + t^2\mathring{g}^{ab}\partial_a\partial_b\nu - (1-\lambda^{-1})\nu = t^{2(h)}R.$$

Equation (10.8a) *can be used to show that* (10.9a) *is equivalent to* (10.9b).

The ***linearized versions of the metric evolution equations*** (10.3a)–(10.3b) *are*

(10.10a)
$$\partial_t h_{ij} = -2t^{-1}(t\mathring{k}^a{}_j)h_{ia} - 2t^{-1}\mathring{g}_{ia}(t\kappa^a{}_j) - 2t^{-1}\mathring{g}_{ia}(t\mathring{k}^a{}_j)\nu,$$

(10.10b)
$$\partial_t(t\kappa^i{}_j) = -t\mathring{g}^{ia}\partial_a\partial_j\nu - (1-\lambda^{-1})t^{-1}(t\mathring{k}^i{}_j)\nu + t^{(h)}\mathrm{Ric}^i{}_j.$$

The ***linearized version of the scalar field wave equation*** (10.5) *is*

(10.11)
$$-\partial_t(t\partial_t\varphi) + t\mathring{g}^{ab}\partial_a\partial_b\varphi = -A\partial_t\nu + A(1-\lambda^{-1})t^{-1}\nu.$$

Proof. The proof is essentially the same as that of Proposition 3.2, and we therefore omit the details. We point out that in the gauge (10.1) (and therefore in Proposition 10.2 too), the linearly small quantities are the same as the ones from Definition 3.1, except that $t\kappa^a{}_a := tk^a{}_a - t\mathring{k}^a{}_a = \lambda^{-1}(n-1) = \lambda^{-1}\nu$ is now linearly small rather than completely vanishing as it did in Proposition 3.2. □

10.3. *Energies and norms.* In our analysis of solutions, we will use the energies and norms featured in the next two definitions. These controlling quantities lead to slightly different estimates for the lapse compared to the CMC gauge. The main point is that we are no longer able to obtain control of the highest-order analog of $\|\partial^2\nu\|_{L^2_{\mathring{g}}}$ because of the nature of parabolic energy estimates. We are, however, able to control a spacetime integral of the highest-order analog of $\partial^2\nu$, which is provided by the highest-order analog of the first term on the second line of the right-hand side of (10.28).

Definition 10.2 (Energies). In terms of the energies defined in Definition 4.4, we define the following energy $\mathscr{E}_{(\text{Almost Total});\theta}(t) \geq 0$ for $t \in (0,1]$:

(10.12)
$$\mathscr{E}^2_{(\text{Almost Total});\theta}(t) := \mathscr{E}^2_{(\text{Scalar})}(t) + \mathscr{E}^2_{(\text{Lapse})}(t) + \theta\mathscr{E}^2_{(\text{Metric})}(t).$$

As in Theorem 6.1, θ is a small positive constant that we will choose below in order in to obtain the desired energy estimates.

We will also use an up-to-order M energy. Specifically, we view the energy $\mathscr{E}_{(\text{Almost Total});\theta}$ defined in (10.12) as a functional of $\kappa, \partial h, \partial_t\varphi, \partial\varphi, \nu$ (that is,

$\mathscr{E}_{(\text{Almost Total});\theta} = \mathscr{E}_{(\text{Almost Total});\theta}[\kappa, \partial h, \partial_t \varphi, \partial \varphi, \nu])$, and we define

(10.13)
$$\mathscr{E}^2_{(\text{Almost Total});\theta;M}(t) := \sum_{|\vec{I}| \leq M} \mathscr{E}^2_{(\text{Almost Total});\theta}[\partial_{\vec{I}} \kappa, \partial \partial_{\vec{I}} h, \partial_t \partial_{\vec{I}} \varphi, \partial \partial_{\vec{I}} \varphi, \partial_{\vec{I}} \nu](t).$$

Definition 10.3 (Solution norms). In terms of the Sobolev norms of Definition 4.2, we define the solution norms

$$\begin{aligned} \mathscr{S}_{(\text{Parabolic Frame});M}(t) := {} & \|t\kappa\|_{H^M_{\text{Frame}}} + \|\partial h\|_{H^M_{\text{Frame}}} + \|t \partial_t \varphi\|_{H^M_{\text{Frame}}} \\ & + t^{2/3} \|\partial \varphi\|_{H^M_{\text{Frame}}} + \sum_{p=0}^{1} t^{(2/3)p} \|\nu\|_{H^{M+p}}. \end{aligned} \tag{10.14}$$

Remark 10.4. Note that $\mathscr{S}_{(\text{Parabolic Frame});0}$ controls one derivative of ν while $\mathscr{E}_{(\text{Almost Total});\theta}$ does not.

10.4. *The approximate monotonicity identity.* We now state our approximate monotonicity identity theorem for solutions to the linear equations of Proposition 10.2. The theorem is a direct analog of Theorem 5.1 in the CMC gauge.

THEOREM 10.1 (The approximate monotonicity identity in the parabolic lapse gauge). *Assume that the parabolic gauge parameter verifies $\lambda \neq 0$. Then for any constant $\theta > 0$, solutions to the linearized equations of Proposition* 10.2 *verify the following identity for $t \in (0,1]$:*

(10.15)
$$\begin{aligned} & \int_{\Sigma_t} (t\partial_t \varphi)^2 + |t \partial \varphi|^2_{\mathring{g}} \, dx + \left\{ A^2 + \frac{1}{2} \lambda^{-1} (1 - \lambda^{-1}) \right\} \int_{\Sigma_t} \nu^2 \, dx \\ & \quad + \theta \int_{\Sigma_t} |t\kappa|^2_{\mathring{g}} + \frac{1}{4} |t \partial h|^2_{\mathring{g}} \, dx + \int_{\Sigma_t} \mathcal{N}_1 \, dx \\ & = \int_{\Sigma_1} (\partial_t \varphi)^2 + t^2 |\partial \varphi|^2_{\mathring{g}} \, dx + \left\{ A^2 + \frac{1}{2} \lambda^{-1} (1 - \lambda^{-1}) \right\} \int_{\Sigma_1} \nu^2 \, dx \\ & \quad + \theta \int_{\Sigma_1} |\kappa|^2_{\mathring{g}} + \frac{1}{4} |t \partial h|^2_{\mathring{g}} \, dx + \int_{\Sigma_1} \mathcal{N}_1 \, dx \\ & \quad - 2 \int_{s=t}^{1} s^{-1} \int_{\Sigma_s} |s \partial \varphi|^2_{\mathring{g}} \, dx \, ds - (1 + 2\theta\lambda^{-1} - \lambda^{-1}) \int_{s=t}^{1} s^{-1} \int_{\Sigma_s} |s \partial \nu|^2_{\mathring{g}} \, dx \, ds \\ & \quad - \left\{ 1 - \lambda^{-2} \right\} \int_{s=t}^{1} s^{-1} \int_{\Sigma_s} \nu^2 \, dx \, ds \\ & \quad - \frac{1}{2} \theta \int_{s=t}^{1} s^{-1} \int_{\Sigma_s} |s \partial h|^2_{\mathring{g}} \, dx \, ds \\ & \quad + \sum_{i=2}^{4} \int_{s=t}^{1} s^{-1} \int_{\Sigma_s} \mathcal{N}_i \, dx \, ds + \theta \sum_{i=5}^{12} \int_{s=t}^{1} s^{-1} \int_{\Sigma_s} \mathcal{N}_i \, dx \, ds, \end{aligned}$$

where the constant $0 \leq A \leq \sqrt{2/3}$ *is defined by* (1.8b) *and along* Σ_s, *we have*

$$\mathcal{N}_1 = \mathcal{N}_1(s\partial_t\varphi, \nu) := -2A(s\partial_t\varphi)\nu, \tag{10.16a}$$

$$\mathcal{N}_2 = \mathcal{N}_2(s\hat{\mathring{k}}, s\kappa, \nu) := -2(1-\lambda^{-1})(s\hat{\mathring{k}}^a{}_b)(s\kappa^b{}_a)\nu, \tag{10.16b}$$

$$\mathcal{N}_3 = \mathcal{N}_3(s\mathring{k}, s\partial\varphi, s\partial\varphi) := -2s^2\mathring{g}^{ab}(s\mathring{k}^c{}_b)\partial_a\varphi\partial_c\varphi, \tag{10.16c}$$

$$\mathcal{N}_4 = \mathcal{N}_4(s\partial\varphi, s\partial\nu) := -2As^2\mathring{g}^{ab}\partial_a\varphi\partial_b\nu, \tag{10.16d}$$

$$\mathcal{N}_5 = \mathcal{N}_5(s\mathring{k}, s\partial h, s\partial h) := -\frac{1}{2}s^2\mathring{g}^{ab}\mathring{g}^{ij}\mathring{g}^{cf}(s\mathring{k}^e{}_c)\partial_e h_{ai}\partial_f h_{bj}, \tag{10.16e}$$

$$\begin{aligned}\mathcal{N}_6 &= \mathcal{N}_6(s\hat{\mathring{k}}, s\kappa, s\kappa) \\ &:= 2\mathring{g}_{ic}\mathring{g}^{ab}(s\hat{\mathring{k}}^c{}_j)(s\kappa^i{}_a)(s\kappa^j{}_b) - 2\mathring{g}_{ij}\mathring{g}^{ac}(s\hat{\mathring{k}}^b{}_c)(s\kappa^i{}_a)(s\kappa^j{}_b),\end{aligned} \tag{10.16f}$$

$$\begin{aligned}\mathcal{N}_7 &= \mathcal{N}_7(s\hat{\mathring{k}}, s\partial h, s\partial h) \\ &:= s^2\mathring{g}_{ab}\mathring{g}^{ef}\mathring{g}^{ij}(s\hat{\mathring{k}}^a{}_c)^{(h)}\Gamma_i{}^c{}_j{}^{(h)}\Gamma_e{}^b{}_f \\ &\quad - s^2\mathring{g}_{ab}\mathring{g}^{ef}\mathring{g}^{ij}(s\hat{\mathring{k}}^c{}_j)^{(h)}\Gamma_i{}^a{}_c{}^{(h)}\Gamma_e{}^b{}_f \\ &\quad + s^2\mathring{g}^{ef}(s\hat{\mathring{k}}^a{}_c)^{(h)}\Gamma_a{}^c{}_b{}^{(h)}\Gamma_e{}^b{}_f - s^2\mathring{g}^{ef}(s\hat{\mathring{k}}^c{}_b)^{(h)}\Gamma_a{}^a{}_c{}^{(h)}\Gamma_e{}^b{}_f,\end{aligned} \tag{10.16g}$$

$$\begin{aligned}\mathcal{N}_8 &= \mathcal{N}_8(s\hat{\mathring{k}}, s\partial h, s\partial\nu) \\ &:= 2s^2\mathring{g}^{ij}(s\hat{\mathring{k}}^b{}_i)^{(h)}\Gamma_a{}^a{}_b\partial_j\nu - 2s^2\mathring{g}^{ij}(s\hat{\mathring{k}}^a{}_b)^{(h)}\Gamma_a{}^b{}_i\partial_j\nu \\ &\quad + s^2\mathring{g}^{ij}\mathring{g}^{ef}(s\mathring{k}^a{}_j)\partial_e h_{ai}\partial_f\nu,\end{aligned} \tag{10.16h}$$

$$\mathcal{N}_9 = \mathcal{N}_9(s\hat{\mathring{k}}, s\kappa, \nu) := 2(1-\lambda^{-1})\mathring{g}_{ab}\mathring{g}^{ij}(s\hat{\mathring{k}}^a{}_i)(s\kappa^b{}_j)\nu, \tag{10.16i}$$

$$\mathcal{N}_{10} = \mathcal{N}_{10}(s\partial\varphi, s\partial\nu) := 2As^2\mathring{g}^{ij}\partial_i\varphi\partial_j\nu, \tag{10.16j}$$

$$\mathcal{N}_{11} = \mathcal{N}_{11}(s\partial h, s\partial\varphi) := -2As^2\mathring{g}^{ef\,(h)}\Gamma_e{}^a{}_f\partial_a\varphi, \tag{10.16k}$$

$$\mathcal{N}_{12} = \mathcal{N}_{12}(s\partial h, s\partial\nu) := 2\lambda^{-1}t\mathring{g}^{ef\,(h)}\Gamma_e{}^a{}_f\partial_a\nu. \tag{10.16l}$$

Remark 10.5. The terms $\mathcal{N}_i$ defined in (10.16a)–(10.16l) have different definitions than their counterparts from Sections 5 and 6.

Proof of Theorem 10.1. We will derive the identities (10.17) and (10.22) below using independent arguments. To obtain (10.1), we simply add (10.17) to θ times (10.22). □

As in the proof of Theorem 5.1, the most important step in the proof of Theorem 10.1 is an energy identity for the linearized scalar field and lapse that simultaneously yields favorably signed (to the past) integrals for both variables. We provide this identity in the next proposition.

Proposition 10.3 (The key integral identity for the linearized scalar field and linearized lapse in the parabolic lapse gauge). *Assume that the parabolic gauge parameter verifies $\lambda \neq 0$. Then solutions to the linearized equations of Proposition* 10.2 *verify the following identity for $t \in (0,1]$:*

(10.17)

$$\begin{aligned}
&\int_{\Sigma_t} (t\partial_t\varphi)^2 + |t\partial\varphi|_{\mathring{g}}^2\,dx + \left\{A^2 + \frac{1}{2}\lambda^{-1}(1-\lambda^{-1})\right\}\int_{\Sigma_t} \nu^2\,dx + \int_{\Sigma_t}\mathcal{N}_1\,dx \\
&= \int_{\Sigma_1} (\partial_t\varphi)^2 + t^2|\partial\varphi|_{\mathring{g}}^2\,dx + \left\{A^2 + \frac{1}{2}\lambda^{-1}(1-\lambda^{-1})\right\}\int_{\Sigma_1} \nu^2\,dx + \int_{\Sigma_1}\mathcal{N}_1\,dx \\
&\quad - 2\int_{s=t}^1 s^{-1}\int_{\Sigma_s} |s\partial\varphi|_{\mathring{g}}^2\,dx\,ds - (1-\lambda^{-1})\int_{s=t}^1 s^{-1}\int_{\Sigma_s} |s\partial\nu|_{\mathring{g}}^2\,dx\,ds \\
&\quad - (1-\lambda^{-2})\int_{s=t}^1 s^{-1}\int_{\Sigma_s}\nu^2\,dx\,ds \\
&\quad + \sum_{i=2}^4 \int_{s=t}^1 s^{-1}\int_{\Sigma_s}\mathcal{N}_i\,dx\,ds,
\end{aligned}$$

where the constant $0 \leq A \leq \sqrt{2/3}$ is defined by (1.8b) *and the terms $\mathcal{N}_1$, $\mathcal{N}_2$, $\mathcal{N}_3$, and $\mathcal{N}_4$ are defined in* (10.16a)–(10.16d).

Proof. The proof has some features in common with our proof of Proposition 5.2, but other aspects of it are different. Again, the main idea is to combine three integration by parts identities in the right way. Throughout, we silently use the identities in (4.10). To obtain the first identity, we divide equation (10.9a) by t and then replace t with the integration variable s, multiply by $(1-\lambda^{-1})\nu$, and integrate by parts over $(s,x) \in [t,1]\times\mathbb{T}^3$ (we stress that $t \leq 1$) to deduce that

$$\begin{aligned}
(10.18)\qquad &\frac{1}{2}\lambda^{-1}(1-\lambda^{-1})\int_{\Sigma_t}\nu^2\,dx \\
&= \frac{1}{2}\lambda^{-1}(1-\lambda^{-1})\int_{\Sigma_1}\nu^2\,dx \\
&\quad - (1-\lambda^{-1})\int_{s=t}^1 s^{-1}\int_{\Sigma_s}|s\partial\nu|_{\mathring{g}}^2\,dx\,ds \\
&\quad + (2A^2 - 1 - \lambda^{-1})(1-\lambda^{-1})\int_{s=t}^1 s^{-1}\int_{\Sigma_s}\nu^2\,dx\,ds \\
&\quad - 2A(1-\lambda^{-1})\int_{s=t}^1 s^{-1}\int_{\Sigma_s}(s\partial_t\varphi)\nu\,dx\,ds \\
&\quad - 2(1-\lambda^{-1})\int_{s=t}^1 s^{-1}\int_{\Sigma_s}(s\hat{\mathring{k}}^a{}_b)(s\kappa^b{}_a)\nu\,dx\,ds.
\end{aligned}$$

To obtain the second identity, we replace t with the integration variable s in equation (10.11), multiply by $-s\partial_t\varphi$, and integrate by parts over $(s,x) \in [t,1]\times\mathbb{T}^3$ (we again stress that $t \leq 1$) to deduce that

$$\begin{aligned}(10.19)\qquad & \int_{\Sigma_t} (t\partial_t\varphi)^2 + t^2|\partial\varphi|^2_{\mathring{g}}\,dx \\ = & \int_{\Sigma_1} (\partial_t\varphi)^2 + |\partial\varphi|^2_{\mathring{g}}\,dx \\ & - 2\int_{s=t}^1 s^{-1}\int_{\Sigma_s} |s\partial\varphi|^2_{\mathring{g}} + s^2\mathring{g}^{ab}(s\mathring{k}^c{}_b)\partial_a\varphi\partial_c\varphi\,dx\,ds \\ & - 2A\int_{s=t}^1\int_{\Sigma_s}(s\partial_t\varphi)\partial_t\nu\,dx\,ds \\ & + 2A(1-\lambda^{-1})\int_{s=t}^1 s^{-1}\int_{\Sigma_s}(s\partial_t\varphi)\nu\,dx\,ds.\end{aligned}$$

Next, we multiply equation (10.11) by ν to obtain the following identity:

$$(10.20)\quad (t\partial_t\varphi)\partial_t\nu = \partial_t(t\partial_t\varphi\nu) - \frac{1}{2}A\partial_t(\nu^2) - t\nu\mathring{g}^{ab}\partial_a\partial_b\varphi + A(1-\lambda^{-1})t^{-1}\nu^2.$$

To obtain the third identity, we replace t with the integration variable s in equation (10.20), multiply by $2A$, and integrate by parts over $(s,x)\in[t,1]\times\mathbb{T}^3$ to deduce that

$$\begin{aligned}(10.21)\qquad & -2A\int_{\Sigma_t}(t\partial_t\varphi)\nu\,dx + A^2\int_{\Sigma_t}\nu^2\,dx \\ = & -2A\int_{\Sigma_1}\partial_t\varphi\nu\,dx + A^2\int_{\Sigma_1}\nu^2\,dx \\ & + 2A\int_{s=t}^1\int_{\Sigma_s}(s\partial_t\varphi)\partial_t\nu\,dx\,ds \\ & - 2A\int_{s=t}^1 s^{-1}\int_{\Sigma_s} s^2\mathring{g}^{ab}\partial_a\varphi\partial_b\nu\,dx\,ds \\ & - 2A^2(1-\lambda^{-1})\int_{s=t}^1 s^{-1}\int_{\Sigma_s}\nu^2\,dx\,ds.\end{aligned}$$

Adding (10.18), (10.19), and (10.21), and noting the cancellation of the integrals $\pm 2A\int_{s=t}^1\int_{\Sigma_s}(s\partial_t\varphi)\partial_t\nu\,dx\,ds$ and $\pm 2A(1-\lambda^{-1})\int_{s=t}^1 s^{-1}\int_{\Sigma_s}(s\partial_t\varphi)\nu\,dx\,ds$, we arrive at the desired identity (10.17). □

In the next proposition, we derive an energy identity for the linearized metric solution variables. It is a direct analog of Proposition 5.3.

Proposition 10.4 (Energy identity for the linearized metric variables in the parabolic lapse gauge). *Assume that the parabolic gauge parameter verifies* $\lambda \neq 0$. *Then solutions to the linearized equations of Proposition* 10.2 *verify the* *following identity for* $t \in (0,1]$:

$$\begin{aligned}
(10.22) \qquad \int_{\Sigma_t} |t\kappa|_{\mathring{g}}^2 + \frac{1}{4}|t\partial h|_{\mathring{g}}^2 \, dx = & \int_{\Sigma_1} |\kappa|_{\mathring{g}}^2 + \frac{1}{4}|\partial h|_{\mathring{g}}^2 \, dx \\
& - \frac{1}{2}\int_{s=t}^{1} s^{-1} \int_{\Sigma_s} |s\partial h|_{\mathring{g}}^2 \, dx \, ds \\
& - 2\lambda^{-1}\int_{s=t}^{1} s^{-1} \int_{\Sigma_s} |s\partial \nu|_{\mathring{g}}^2 \, dx \, ds \\
& + \sum_{i=5}^{12} \int_{s=t}^{1} s^{-1} \int_{\Sigma_s} \mathcal{N}_i \, dx \, ds,
\end{aligned}$$

where the constant $0 \leq A \leq \sqrt{2/3}$ *is defined by* (1.8b) *and the terms* $\mathcal{N}_5, \ldots, \mathcal{N}_{12}$ *are defined in* (10.16e)–(10.16l).

Proof of Proposition 10.4. We repeat the proof of Proposition 5.3 and take into account the few differences between the linearized equations of Proposition 3.2 and the linearized equations of Proposition 10.2. In particular, the identity (5.18) holds in the present context, but with the next-to-last term $-2t^{-1}\mathring{g}_{ab}\mathring{g}^{ij}(t\hat{\mathring{k}}{}^{a}{}_{i})(t\kappa^{b}{}_{j})\nu$ multiplied by the factor $1-\lambda^{-1}$ (coming from the second term on the right-hand side of (10.10b)) and two additional terms: (i) the term $2\lambda^{-1}t|\partial\nu|_{\mathring{g}}^2$ coming from the analog of the step (5.13) and the presence of the term $\lambda^{-1}\partial_i\nu$ on the right-hand side of equation (10.8b), and (ii) the cross term $-2\lambda^{-1}t\mathring{g}^{ef}{}^{(h)}\Gamma_{e\ f}^{\ a}\partial_a\nu$ coming from the analog of steps (5.16) and (5.17) and the presence of the term $\lambda^{-1}\partial_i\nu$ on the right-hand side of equation (10.8b) and the term $\lambda^{-1}\mathring{g}^{ia}\partial_a\nu$ on the right-hand side of (10.8c). □

10.5. *Mildly singular energy estimates without derivative loss for the linearized equations in the parabolic lapse gauge.* In this subsection, we use the approximate monotonicity identity provided by Theorem 10.1 to derive mildly singular energy estimates for the linear solution when the Kasner background is nearly spatially isotropic. The results are contained in Theorem 10.2, which is a direct analog of Theorem 6.1. We provide the proof of Theorem 10.2 in Section 10.5.2.

THEOREM 10.2 (Mildly singular energy estimates without derivative loss for solutions to the linearized equations in the parabolic lapse gauge). *Consider a solution to the linear equations of Proposition* 10.2 *corresponding to the data* $(\kappa(1), h(1), \partial_t\varphi(1), \partial\varphi(1), \nu(1))$ (*given on* $\Sigma_1 = \{1\} \times \mathbb{T}^3$). *Assume that the parabolic gauge parameter verifies* $\lambda \geq \lambda_0$, *where* $\lambda_0 > 2$. *There exist constants* $\theta_{\lambda_0} > 0$, $\eta_{\lambda_0} > 0$, $C_{\lambda_0} > 0$, $c_{\lambda_0} > 0$, *and* $P_{\lambda_0} > 0$ (*depending on* λ_0) *such that if* $0 \leq \eta \leq \eta_{\lambda_0}$ *and if the solution norm* $\mathscr{S}_{(\text{Parabolic Frame});0}(t)$ *defined in* (10.14) *verifies* $\mathscr{S}_{(\text{Parabolic Frame});0}(1) < \infty$, *then the energy* $\mathscr{E}_{(\text{Almost Total});\theta_{\lambda_0}}(t)$ *defined* *in* (10.12) *verifies the following inequality for* $t \in (0,1]$:

(10.23)

$$\begin{aligned}\mathscr{E}^2_{(\text{Almost Total});\theta_{\lambda_0}}(t) \leq C_{\lambda_0}\mathscr{E}^2_{(\text{Almost Total});\theta_{\lambda_0}}(1)\\ &\underbrace{-P_{\lambda_0}\theta_{\lambda_0}\int_{s=t}^1 s^{-1}\int_{\Sigma_s}|s\partial h|^2_{\mathring{g}}\,dx\,ds}_{\text{Past-favorable sign}}\\ &\underbrace{-P_{\lambda_0}\int_{s=t}^1 s^{-1}\int_{\Sigma_s}|s\partial\varphi|^2_{\mathring{g}}\,dx\,ds}_{\text{Past-favorable sign}}\\ &\underbrace{-P_{\lambda_0}\int_{s=t}^1 s^{-1}\int_{\Sigma_s}|s\partial\nu|^2_{\mathring{g}}\,dx\,ds}_{\text{Past-favorable sign}}\underbrace{-P_{\lambda_0}\int_{s=t}^1 s^{-1}\int_{\Sigma_s}\nu^2\,dx\,ds}_{\text{Past-favorable sign}}\\ &+\underbrace{c_{\lambda_0}\eta\int_{s=t}^1 s^{-1}\mathscr{E}^2_{(\text{Almost Total});\theta_{\lambda_0}}(s)\,ds}_{\text{Error integral that can create energy blowup}}.\end{aligned}$$

Furthermore, the following estimate holds for $t \in (0,1]$:

$$\mathscr{E}_{(\text{Almost Total});\theta_{\lambda_0}}(t) \leq C_{\lambda_0}\mathscr{E}_{(\text{Almost Total});\theta_{\lambda_0}}(1)t^{-c_{\lambda_0}\eta}. \tag{10.24}$$

Also, if $N \geq 0$ *is an integer and the solution norm* $\mathscr{S}_{(\text{Parabolic Frame});N}(t)$ *defined in* (10.14) *verifies* $\mathscr{S}_{(\text{Parabolic Frame});N}(1) < \infty$, *then for* $t \in (0,1]$, *the energy* $\mathscr{E}_{(\text{Total});\theta_{\lambda_0};N}(t)$ *defined in* (4.7) *verifies the following estimate*:

$$\mathscr{E}_{(\text{Total});\theta_{\lambda_0};N}(t) \leq \begin{cases}\frac{C_{\lambda_0}}{\eta}\mathscr{S}_{(\text{Parabolic Frame});N}(1)t^{-c_{\lambda_0}\eta} & \text{if } \eta \neq 0,\\ C_{\lambda_0}\mathscr{S}_{(\text{Parabolic Frame});N}(1)(1+|\ln t|) & \text{if } \eta = 0.\end{cases} \tag{10.25}$$

In addition, if $N \geq 0$ *is an integer and* $\mathscr{S}_{(\text{Parabolic Frame});N}(1) < \infty$, *then the following inequality holds for* $t \in (0,1]$:

(10.26)

$$\mathscr{S}_{(\text{Parabolic Frame});N}(t) \leq \begin{cases}\frac{C_{\lambda_0}}{\eta}\mathscr{S}_{(\text{Parabolic Frame});N}(1)t^{-c_{\lambda_0}\eta} & \text{if } \eta \neq 0,\\ C_{\lambda_0}\mathscr{S}_{(\text{Parabolic Frame});N}(1)(1+|\ln t|) & \text{if } \eta = 0.\end{cases}$$

Remark 10.6. See the estimate (10.45) for more a more precise inequality that shows how the constants in the estimate (10.23) depend on λ_0 and on each other.

10.5.1. *Preliminary estimates and identities for the proof of Theorem* 10.2. In our proof of Theorem 10.2, we use the following comparison lemma, which can be proved by using arguments similar to the ones we used to prove Lemma

4.3 (except that clearly we no do not use the elliptic estimate provided by Lemma 4.2); we omit the simple proof.

LEMMA 10.5 (Parabolic energy-norm comparison lemma). *Let* $N \geq 0$ *be an integer, and let* $\eta \geq 0$ *be as defined in* (1.9b). *There exist constants* $C > 0$ *and* $c > 0$, *depending on* θ, *such that the following comparison estimates hold for the norm* $\mathscr{S}_{(\text{Parabolic Frame});N}(t)$ *defined in* (10.14) *and the energy* $\mathscr{E}_{(\text{Total});\theta;N}(t)$ *defined in* (4.7) *for* $t \in (0,1]$:

$$\mathscr{E}_{(\text{Total});\theta;N}(t) \leq Ct^{-c\eta}\mathscr{S}_{(\text{Parabolic Frame});N}(t), \tag{10.27a}$$

$$\mathscr{S}_{(\text{Parabolic Frame});N}(t) \leq Ct^{-c\eta}\mathscr{E}_{(\text{Total});\theta;N}(t). \tag{10.27b}$$

We will also use the following simple parabolic energy estimate, which can be used to derive top-order L^2 estimates for the linearized lapse variable.

LEMMA 10.6 (Parabolic energy estimate for ν). *There exists a constant* $C > 0$ *such that if* $\eta \geq 0$ (*see definition* 1.9*b*) *and if the parabolic gauge parameter verifies* $\lambda \geq 1$, *then solutions* ν *to the linear parabolic equation* (10.9a) *verify the following inequality for* $t \in (0,1]$:

(10.28)

$$\begin{aligned}
\lambda^{-1}\int_{\Sigma_t} |t\partial\nu|_{\mathring{g}}^2\,dx &\leq \lambda^{-1}\int_{\Sigma_1}|\partial\nu|_{\mathring{g}}^2\,dx \\
&\quad - \int_{s=t}^1 s^{-1}\int_{\Sigma_s}|s^2\partial^2\nu|_{\mathring{g}}^2\,dx\,ds - \lambda^{-1}\left(\frac{4}{3}-2\eta\right)\int_{s=t}^1 s^{-1}\int_{\Sigma_s}|s\partial\nu|_{\mathring{g}}^2\,dx\,ds \\
&\quad + C\int_{s=t}^1 s^{-1}\int_{\Sigma_s}\left|(s\hat{\mathring{k}}{}^a{}_b)(s\kappa^b{}_a)\right|^2\,dx\,ds + C\int_{s=t}^1 s^{-1}\int_{\Sigma_s}(s\partial_t\varphi)^2\,dx\,ds \\
&\quad + C\int_{s=t}^1 s^{-1}\int_{\Sigma_s}\nu^2\,dx\,ds.
\end{aligned}$$

Proof. Integrating by parts over $[t,1]\times\mathbb{T}^3$ (we stress that $t \leq 1$) we deduce (without using any equation)

(10.29)

$$\begin{aligned}
\lambda^{-1}\int_{\Sigma_t}|t\partial\nu|_{\mathring{g}}^2\,dx &= \lambda^{-1}\int_{\Sigma_1}|\partial\nu|_{\mathring{g}}^2\,dx \\
&\quad - 2\lambda^{-1}\int_{s=t}^1 s^{-1}\int_{\Sigma_s}|s\partial\nu|_{\mathring{g}}^2 + s^2\mathring{g}^{ab}(s\mathring{k}{}^c{}_b)\partial_a\nu\partial_c\nu\,dx\,ds \\
&\quad + 2\int_{s=t}^1\int_{\Sigma_s}s\mathring{g}^{ef}\partial_e\partial_f\nu(\lambda^{-1}s\partial_t\nu)\,dx\,ds.
\end{aligned}$$

Using equation (10.9a) to substitute for the product $\lambda^{-1}s\partial_t\nu$ in the last integrand on the right-hand side of (10.29) and integrating by parts over Σ_s on the resulting integrand product $\left\{\mathring{g}^{ef}\partial_e\partial_f\nu\right\}^2$, we deduce

(10.30)
$$\begin{aligned}
\lambda^{-1}\int_{\Sigma_t}|t\partial\nu|^2_{\mathring{g}}\,dx = \lambda^{-1}\int_{\Sigma_1}|\partial\nu|^2_{\mathring{g}}\,dx& \\
- 2\lambda^{-1}\int_{s=t}^{1}s^{-1}\int_{\Sigma_s}|s\partial\nu|^2_{\mathring{g}} + s^2\mathring{g}^{ab}(s\mathring{k}^c{}_b)\partial_a\nu\partial_c\nu\,dx\,ds& \\
- 2\int_{s=t}^{1}s^{-1}\int_{\Sigma_s}|s^2\partial^2\nu|^2_{\mathring{g}}\,dx\,ds& \\
- (2A^2 - 1 - \lambda^{-1})\int_{s=t}^{1}s^{-1}\int_{\Sigma_s}\nu(s^2\mathring{g}^{ef}\partial_e\partial_f\nu)\,dx\,ds& \\
+ 4A\int_{s=t}^{1}s^{-1}\int_{\Sigma_s}(s\partial_t\varphi)(s^2\mathring{g}^{ef}\partial_e\partial_f\nu)\,dx\,ds& \\
+ 4\int_{s=t}^{1}s^{-1}\int_{\Sigma_s}(s\hat{\mathring{k}}^a{}_b)(s\kappa^b{}_a)(s^2\mathring{g}^{ef}\partial_e\partial_f\nu)\,dx\,ds.&
\end{aligned}$$

Arguing as in the proof of (6.5) (in particular, using the fact that the eigenvalues of $t\mathring{k}^i{}_j$ are $\geq -q_{\mathrm{Max}} \geq -\left\{\frac{1}{3}+\eta\right\}$), we estimate the second integral on the right-hand side of (10.30) as follows:

$$\begin{aligned}
(10.31)\quad -2\lambda^{-1}\int_{s=t}^{1}s^{-1}\int_{\Sigma_s}|s\partial\nu|^2_{\mathring{g}} + s^2\mathring{g}^{ab}(s\mathring{k}^c{}_b)\partial_a\nu\partial_c\nu\,dx\,ds& \\
\leq -\lambda^{-1}\left(\frac{4}{3}-2\eta\right)\int_{s=t}^{1}s^{-1}\int_{\Sigma_s}|s\partial\nu|^2_{\mathring{g}}\,dx\,ds.&
\end{aligned}$$

Using the simple estimate $A \leq \sqrt{\frac{2}{3}}$, Young's inequality, and the simple estimate $\|\mathring{g}^{ef}\partial_e\partial_f\nu\|_{L^2} \lesssim \|\partial^2\nu\|_{L^2_{\mathring{g}}}$, we deduce that the four integrals on the third through sixth lines of the right-hand side of (10.30) are collectively bounded by

$$\begin{aligned}
(10.32)\qquad \leq -\int_{s=t}^{1}s^{-1}\int_{\Sigma_s}|s^2\partial^2\nu|^2_{\mathring{g}}\,dx\,ds& \\
+ C\int_{s=t}^{1}s^{-1}\int_{\Sigma_s}\nu^2\,dx\,ds + C\int_{s=t}^{1}s^{-1}\int_{\Sigma_s}(s\partial_t\varphi)^2\,dx\,ds& \\
+ C\int_{s=t}^{1}s^{-1}\int_{\Sigma_s}\left|(s\hat{\mathring{k}}^a{}_b)(s\kappa^b{}_a)\right|^2\,dx\,ds.&
\end{aligned}$$

The desired inequality (10.28) now follows easily from (1.9b) and (10.30) and inequalities (10.31) and (10.32). $\square$

10.5.2. *Proof of Theorem* 10.2. We first note that the following pointwise estimates hold for the integrand terms $\mathcal{N}_i$, $i = 1, 2, \ldots, 12$ defined in (10.16a)–(10.16l), where the constants $C > 0$ are independent of $\lambda \geq 1$ and θ:

$$|\mathcal{N}_1| \leq \frac{A^2}{A^2 + \frac{1}{4}\lambda^{-1}(1-\lambda^{-1})}(t\partial_t\varphi)^2 + \left\{A^2 + \frac{1}{4}\lambda^{-1}(1-\lambda^{-1})\right\}\nu^2, \tag{10.33}$$

$$|\mathcal{N}_2| \leq (1-\lambda^{-1})\eta\theta|s\kappa|_{\mathring{g}}^2 + (1-\lambda^{-1})\frac{\eta}{\theta}\nu^2, \tag{10.34}$$

$$\mathcal{N}_3 \leq \left(\frac{2}{3} + 2\eta\right)|\partial\varphi|_{\mathring{g}}^2, \tag{10.35}$$

$$|\mathcal{N}_4| \leq \left\{\frac{\lambda}{\lambda - 2 + \frac{1}{A^2}}\right\}|s\partial\varphi|_{\mathring{g}}^2 + \left\{\frac{A^2\lambda - 2A^2 + 1}{\lambda}\right\}|s\partial\nu|_{\mathring{g}}^2, \tag{10.36}$$

$$\theta\mathcal{N}_5 \leq \left(\frac{1}{6} + \frac{1}{2}\eta\right)\theta|s\partial h|_{\mathring{g}}^2, \tag{10.37}$$

$$\theta|\mathcal{N}_6| \leq C\eta\theta|s\kappa|_{\mathring{g}}^2, \tag{10.38}$$

$$\theta|\mathcal{N}_7| \leq C\eta\theta|s\partial h|_{\mathring{g}}^2, \tag{10.39}$$

$$\theta|\mathcal{N}_8| \leq \frac{1}{18}\theta|s\partial h|_{\mathring{g}}^2 + C\theta|s\partial\nu|_{\mathring{g}}^2, \tag{10.40}$$

$$\theta|\mathcal{N}_9| \leq C(1-\lambda^{-1})\eta\theta|s\kappa|_{\mathring{g}}^2 + C(1-\lambda^{-1})\eta\theta\nu^2, \tag{10.41}$$

$$\theta|\mathcal{N}_{10}| \leq C\theta|s\partial\varphi|_{\mathring{g}}^2 + C\theta|s\partial\nu|_{\mathring{g}}^2, \tag{10.42}$$

$$\theta|\mathcal{N}_{11}| \leq \frac{1}{18}\theta|s\partial h|_{\mathring{g}}^2 + C\theta|s\partial\varphi|_{\mathring{g}}^2, \tag{10.43}$$

$$\theta|\mathcal{N}_{12}| \leq \frac{1}{18}\lambda^{-1}\theta|s\partial h|_{\mathring{g}}^2 + C\lambda^{-1}\theta|s\partial\varphi|_{\mathring{g}}^2. \tag{10.44}$$

The estimates (10.33)–(10.44) can be derived by using essentially the same reasoning that we used to prove (6.4)–(6.13), and we therefore omit the details. Note, however, that the $\mathcal{N}_i$ have different definitions in (10.33)–(10.44) than they do in (6.4)–(6.13).

We now claim that there exist constants $C > 0$ and $c > 0$ such that the following estimate holds when $\theta > 0$ and $\lambda \geq 1$:

$$\begin{aligned}
&\left\{\frac{\frac{1}{4}\lambda^{-1}(1-\lambda^{-1})}{A^2 + \frac{1}{4}\lambda^{-1}(1-\lambda^{-1})}\right\}\int_{\Sigma_t}(t\partial_t\varphi)^2\,dx + \int_{\Sigma_t}|t\partial\varphi|_{\mathring{g}}^2\,dx \\
&\quad + \frac{1}{4}\lambda^{-1}(1-\lambda^{-1})\int_{\Sigma_t}\nu^2\,dx + \theta\int_{\Sigma_t}|t\kappa|_{\mathring{g}}^2\,dx + \frac{1}{4}\theta\int_{\Sigma_t}|t\partial h|_{\mathring{g}}^2\,dx \\
&\leq \left\{1 + \frac{A^2}{A^2 + \frac{1}{4}\lambda^{-1}(1-\lambda^{-1})}\right\}\int_{\Sigma_1}(\partial_t\varphi)^2\,dx \\
&\quad + \int_{\Sigma_1}|\partial\varphi|_{\mathring{g}}^2\,dx + \left\{2A^2 + \frac{3}{4}\lambda^{-1}(1-\lambda^{-1})\right\}\int_{\Sigma_1}\nu^2\,dx
\end{aligned} \tag{10.45}$$

$$
\begin{aligned}
&+ \theta \int_{\Sigma_1} |\kappa|^2_{\mathring{g}}\, dx + \frac{1}{4}\theta \int_{\Sigma_1} |\partial h|^2_{\mathring{g}}\, dx \\
&- \left\{ \frac{A^2\lambda - 8A^2 + 4}{3\left[A^2(\lambda-2)+1\right]} - C\eta - C\theta \right\} \int_{s=t}^1 s^{-1} \int_{\Sigma_s} |s\partial\varphi|^2_{\mathring{g}}\, dx\, ds \\
&- \left\{ (\lambda-2)\frac{1-A^2}{\lambda} + 2\theta\lambda^{-1} - C\theta \right\} \int_{s=t}^1 s^{-1} \int_{\Sigma_s} |s\partial\nu|^2_{\mathring{g}}\, dx\, ds \\
&- \left\{ 1 - \lambda^{-2} - C(1-\lambda^{-1})\eta\theta - (1-\lambda^{-1})\frac{\eta}{\theta} \right\} \int_{s=t}^1 s^{-1} \int_{\Sigma_s} \nu^2\, dx\, ds \\
&- \theta \left\{ \frac{2}{9} - \lambda^{-1}\frac{1}{18} - C\eta \right\} \int_{s=t}^1 s^{-1} \int_{\Sigma_s} |s\partial h|^2_{\mathring{g}}\, dx\, ds \\
&+ \left\{ C(1-\lambda^{-1})\eta\theta + C\eta\theta \right\} \int_{s=t}^1 s^{-1} \int_{\Sigma_s} |s\kappa|^2_{\mathring{g}}\, dx\, ds.
\end{aligned}
$$

To obtain (10.45), we simply substitute the estimates (10.33)–(10.44) into the approximate monotonicity identity (10.15) and keep careful track of the coefficients.

Next, we note that by (1.9b), if $2 < \lambda_0 \leq \lambda$ and η is sufficiently small in a manner that is independent of λ_0, then the factor $\frac{A^2\lambda - 8A^2+4}{3[A^2(\lambda-2)+1]}$ in front of the integral $\int_{s=t}^1 s^{-1} \int_{\Sigma_s} |s\partial\varphi|^2_{\mathring{g}}\, dx\, ds$ on the right-hand side of (10.45) is uniformly positive (with a lower bound that *does* depend on λ_0) and increases to $\frac{1}{3}$ as $\lambda \to \infty$. From this observation and definition (10.12), we see that if $2 < \lambda_0 \leq \lambda$, then the desired estimate (10.23) follows from (10.45) by first choosing $\theta := \theta_{\lambda_0}$ to be sufficiently small in a manner that depends on λ_0 and then choosing η to be sufficiently small in a manner that depends on λ_0 and θ_{λ_0}. The estimate (10.24) then follows from (10.23) and Gronwall's inequality.

Our next goal is to prove the estimate (10.25) for $\mathscr{E}_{(\mathrm{Total});\theta_{\lambda_0};N}(t)$. As a first step, we will use the estimate (10.24) to control the top-order terms in $\mathscr{E}_{(\mathrm{Total});\theta_{\lambda_0};0}$ (see definition (4.7)) that are not present in the definition (10.13) of $\mathscr{E}_{(\mathrm{Almost\ Total});\theta_{\lambda_0};0}$, namely, the term $\mathscr{E}^2_{(\partial\mathrm{Lapse})}(t)$ defined in (4.6c). To this end, we insert the estimates implied by (10.24) into the last three integrals on the right-hand side of (10.28), carry out straightforward computations, and use Lemma 10.5 at $t = 1$, thereby deducing that

$$
\mathscr{E}_{(\mathrm{Total});\theta_{\lambda_0};0}(t) \leq \begin{cases} \frac{C_{\lambda_0}}{\eta} \mathscr{S}_{(\mathrm{Parabolic\ Frame});0}(1) t^{-c_{\lambda_0}\eta} & \text{if } \eta \neq 0, \\ C_{\lambda_0} \mathscr{S}_{(\mathrm{Parabolic\ Frame});0}(1)(1 + |\ln t|) & \text{if } \eta = 0. \end{cases} \tag{10.46}
$$

Next, we note that since the $\partial_{\vec{I}}$-differentiated quantities $\partial_{\vec{I}}\kappa, \partial\partial_{\vec{I}}h, \partial_{\vec{I}}\varphi, \partial_{\vec{I}}\nu$ verify the same linear equations as their nondifferentiated counterparts (for reasons similar to the ones given in the proof of Corollary 5.1), it follows that the energy of the $\partial_{\vec{I}}$-differentiated linear solution variables verifies an analog of the estimate (10.46). Summing these estimates for $|\vec{I}\,| \leq N$ and appealing to the definition (4.7) of $\mathscr{E}_{(\mathrm{Total});\theta_{\lambda_0};N}(t)$, we arrive at the desired estimate (10.25). Finally, we note that inequality (10.26) follows from inequality (10.25) and Lemma 10.5. This completes the proof of Theorem 10.2. □

Acknowledgments

The authors thank Mihalis Dafermos for offering enlightening comments on an earlier version of this work. They also thank the anonymous referees for providing valuable feedback that helped improve the exposition. IR gratefully acknowledges support from NSF grant # DMS-1001500. JS gratefully acknowledges support from NSF grant # DMS-1162211, from NSF CAREER grant # DMS-1454419, from a Sloan Research Fellowship provided by the Alfred P. Sloan foundation, and from a Solomon Buchsbaum grant administered by the Massachusetts Institute of Technology.

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(Received: February 19, 2015)
(Revised: July 5, 2017)

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