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Global smooth and topological rigidity of hyperbolic lattice actions

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Abstract

In this article we prove global rigidity results for hyperbolic actions of higher-rank lattices.

Suppose Γ is a lattice in a semisimple Lie group, all of whose factors have rank 2 or higher. Let α be a smooth Γ-action on a compact nilmanifold M that lifts to an action on the universal cover. If the linear data ρ of α contains a hyperbolic element, then there is a continuous semiconjugacy intertwining the actions of α and ρ on a finite-index subgroup of Γ. If α is a C^∞ action and contains an Anosov element, then the semiconjugacy is a C^∞ conjugacy.

As a corollary, we obtain C^∞ global rigidity for Anosov actions by cocompact lattices in semisimple Lie groups with all factors rank 2 or higher. We also obtain global rigidity of Anosov actions of $\mathrm{SL}(n,\mathbb{Z})$ on $\mathbb{T}^n$ for $n \geq 5$ and probability-preserving Anosov actions of arbitrary higher-rank lattices on nilmanifolds.

Contents

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1. **Introduction and statement of main results**

1.1. *Background and motivation.* Let G be a connected, semisimple Lie group with finite center, no compact factors, and all almost-simple factors of real-rank at least 2. Let $\Gamma \subset G$ be a lattice; that is, Γ is a discrete subgroup of G such that G/Γ has finite Haar volume. The celebrated superrigidity theorem of Margulis states that, for G and Γ as above, any linear representation

$\psi\colon \Gamma \to \mathrm{PSL}(d,\mathbb{R})$ is of algebraic nature; that is, ψ extends to a continuous representation $\psi'\colon G \to \mathrm{PSL}(d,\mathbb{R})$ up to a compact error. See Theorem 6.2 and Proposition 6.3 below for more formal statements.

Shortly after, based on the analogy between linear groups and diffeomorphism groups $\mathrm{Diff}^\infty(M)$ of compact manifolds, Zimmer proposed a number of conjectures for representations of Γ into $\mathrm{Diff}^\infty(M)$. These and related conjectures are referred to as the *Zimmer program*, which aims to understand and classify smooth actions by higher-rank lattices. We refer the reader to the excellent survey [Fis11] by Fisher for a detailed account of the Zimmer program.

A major direction of research in the Zimmer program is the classification of actions containing some degree of hyperbolicity; see [Hur94] and [Fis11, §7] for further discussion. For instance, the following conjecture is motivated by works of Feres-Labourie [FL98] and Goetze-Spatzier [GS99].

CONJECTURE 1.1 ([Fis11, Conj. 1.3]). *If Γ is a lattice in $\mathrm{SL}(n,\mathbb{R})$ where $n \geq 3$, then all C^∞ actions by Γ on a compact manifold that both preserves a volume form and contains an Anosov diffeomorphism are algebraically defined.*

Here, being algebraically defined means the action is smoothly conjugate to an action on an infranilmanifold by affine automorphisms. See also [Hur94, Conj. 1.1] and [KL96, Conj. 1.1] for related conjectures. We recall that it is conjectured that infranilmanifolds are the only manifolds supporting Anosov diffeomorphisms.

The assumption in Conjecture 1.1 that the action preserves a volume is a standard assumption in results on the rigidity of group actions. The majority of advances in the Zimmer program, including most predecessors of the results discussed in this paper ([KLZ96], [GS99], [MQ01]), assume the action $\Gamma \to \mathrm{Diff}^\infty(M)$ preserves a Borel probability measure on M. In such settings, Zimmer's superrigidity theorem for cocycles (generalizing Margulis's superrigidity theorem for linear representations; see [Zim84], [FM03]) gives that the derivative cocycle is measurably cohomologous to a linear representation of G up to a compact correction. This provides evidence for the conjectures behind the Zimmer program and is the starting point for many of the local and global rigidity results preceding this paper.

For the remainder, we consider representations $\alpha\colon \Gamma \to \mathrm{Diff}^\infty(M)$ where M is either a torus $\mathbb{T}^d$ or a compact nilmanifold N/Λ. If M is a torus (or if M is a nilmanifold and the action α lifts to an action $\tilde{\alpha}\colon \Gamma \to \mathrm{Diff}^\infty(N)$), one can define a linear representation $\rho\colon \Gamma \to \mathrm{GL}(d,\mathbb{Z})$ (or $\rho\colon \Gamma \to \mathrm{Aut}(\Lambda)$) associated to the action α called the *linear data of α*. We then obtain an action $\rho\colon \Gamma \to \mathrm{Aut}(\mathbb{T}^d)$ (or $\rho\colon \Gamma \to \mathrm{Aut}(N/\Lambda)$.) See Section 2.1 for more details. We assume throughout that ρ is hyperbolic, that is, that $\rho(\gamma)$ is a hyperbolic linear transformation for some $\gamma \in \Gamma$.

As a primary example of such an action consider any homomorphism $\rho\colon \Gamma \to \mathrm{Aut}(\Lambda)$ where Λ is a lattice in a nilpotent, simply connected Lie group N. Then N/Λ is compact, and ρ induces an action by automorphisms $\rho\colon \Gamma \to \mathrm{Diff}^\infty(N/\Lambda)$ and hence coincides with its linear data. Similarly, one can build model algebraic actions of Γ by affine transformations of N/Λ; see [Hur93] for constructions and discussion. Early rigidity results in this setting focused on various notions of rigidity for nonlinear perturbations of affine actions. For instance, in [Hur92] Hurder proved a number of *deformation rigidity* results for certain standard affine actions; that is, under certain hypotheses, a 1-parameter family of perturbations of an affine action ρ are smoothly conjugate to ρ. A related rigidity phenomenon, the *infinitesimal rigidity*, has been studied for affine actions in [Hur92], [Hur95], [Lew91], [Qia96].

The primary rigidity phenomenon studied for perturbations is *local rigidity*; that is, given an affine action $\rho\colon \Gamma \to \mathrm{Diff}^\infty(N/\Lambda)$ and $\alpha\colon \Gamma \to \mathrm{Diff}^\infty(N/\Lambda)$ with $\alpha(\gamma_i)$ sufficiently C^1-close to $\rho(\gamma_i)$ for a finite generating set $\{\gamma_i\} \subset \Gamma$, one wishes to find a C^∞ change of coordinates $h\colon N/\Lambda \to N/\Lambda$ with $h \circ \alpha(\gamma) = \rho(\gamma) \circ h$ for all $\gamma \in \Gamma$. For isometric actions, local rigidity has been shown to hold for cocompact lattices considered above [Ben00] and for property (T) groups [FM05]. For hyperbolic affine actions on tori and nilmanifolds, local rigidity has been established for a number of specific actions or under additional dynamical hypotheses in [Hur92], [QY98], [KL91], [KLZ96], [GS99], [Qia95].

For the general case of actions by higher-rank lattices on nilmanifolds, the local rigidity problem for affine Anosov actions was settled by Katok and Spatzier in [KS97]. In [MQ01] Margulis and Qian extended local rigidity to weakly hyperbolic affine actions. Fisher and Margulis [FM09] established local rigidity in full generality for quasi-affine actions by higher-rank lattices which, in particular, includes actions by nilmanifold automorphisms without assuming any hyperbolicity. We note that the local rigidity results discussed above require property (T); in particular, they do not hold for irreducible lattices in products of rank-oneLie groups.

We turn our attention for the remainder to the question of *global rigidity* of actions on tori and nilmanifolds. That is, given an action $\alpha\colon \Gamma \to \mathrm{Diff}^\infty(N/\Lambda)$ with linear data $\rho\colon \Gamma \to \mathrm{Aut}(N/\Lambda)$, we ask

(1) *topological rigidity*: is there a continuous $h\colon N/\Lambda \to N/\Lambda$ with $h \circ \alpha(\gamma) = \rho(\gamma) \circ h$ for all γ in a finite-index subgroup?
(2) *smooth rigidity*: if so, is h a C^∞ diffeomorphism?

Note that for a general finitely generated discrete group Γ and an action $\alpha\colon \Gamma \to \mathrm{Diff}^\infty(N/\Lambda)$, there is no expectation that such an h would exist. Indeed when Γ is a finitely generated free group, examples of actions α (including actions containing Anosov elements) exist for which no h as above exists. On the other hand, for C^∞ actions on nilmanifolds of higher-rank lattices Γ as introduced above, one may expect such a continuous h to exist. However, examples constructed in [KL96] by blowing up fixed points show that (even when α is real analytic, volume preserving, and ergodic) such h need not be invertible. However, if the nonlinear action possess an Anosov element $\alpha(\gamma_0)$, then any h as above is necessarily invertible. In this setting, one may expect such h to be C^∞.

Global rigidity results under strong dynamical hypotheses appear already in [Hur92]. Global rigidity for Anosov actions by $\mathrm{SL}(n,\mathbb{Z})$ on $\mathbb{T}^n$, $n \geq 3$, were obtained in [KL96], [KLZ96]. Other global rigidity results appear in [Qia97]. We also remark that Feres-Labourie [FL98] and Goetze-Spatzier [GS99] established very strong global rigidity properties for Anosov actions, in which no assumptions on the topology of M are made. In both works, under strong dynamical hypotheses including that the dimension of M is small relative to G, it is shown that M is necessarily an infranilmanifold and the action is algebraically defined.

Global topological rigidity results for Anosov actions by higher-rank lattices on general nilmanifolds were proven in [MQ01, Th. 1.3]. Here, a C^0-conjugacy is obtained assuming the existence of a fully supported invariant measure for the nonlinear action. Topological conjugacies between actions on more general manifolds M whose action on $\pi_1(M)$ factors through an action of a finitely-generated, torsion-free, nilpotent group are studied in [FW01]. (See Section 2.3 and Theorem 3.2 for related results in this direction.)

In this paper we study the global rigidity problem for actions of higher-rank lattices on nilmanifolds with hyperbolic linear data. See Theorems 1.3 and 1.7 below. We provide complete solutions to the global rigidity questions above under the mild assumption that the action lifts to an action on the universal cover. (See Remark 1.5 and Section 9 for discussion on when the lifting is guaranteed to hold.) In particular, for such actions α with hyperbolic linear data, we construct a continuous semiconjugacy to the linear data when restricted to a finite-index subgroup. Moreover, if the action contains an Anosov element, we show that the semiconjugacy (which is necessarily a homeomorphism in this case) is, in fact, a C^∞ diffeomorphism.

We remark that the majority of global rigidity results discussed above assume the existence of an (often smooth or fully supported) invariant measure for the action. We emphasize that we do not assume the existence of an invariant measure in Theorems 1.3 and 1.7.

1.2. *Topological rigidity for maps.* Consider a homeomorphism $f\colon \mathbb{T}^n \to \mathbb{T}^n$. Recall that there exists a unique $A \in \mathrm{GL}(n,\mathbb{Z})$ such that any lift $\tilde{f}\colon \mathbb{R}^n \to \mathbb{R}^n$ is of the form

$$\tilde{f}(x) = Ax + u(x), \tag{1.1}$$

where $u\colon \mathbb{R}^n \to \mathbb{R}^n$ is $\mathbb{Z}^n$-periodic (see [KH95, p. 87]). We call A the linear data of f. As A preserves the lattice $\mathbb{Z}^n$ in $\mathbb{R}^n$, we have an induced map $L_A\colon \mathbb{T}^n \to \mathbb{T}^n$. It follows that f is homotopic to L_A. A similar construction holds for diffeomorphisms of nilmanifolds.

The starting point for the global rigidity problem we study is the following classical theorem of Franks.

Franks' Theorem [Fra70]. *Assume A has no eigenvalues of modulus 1. Then there is a continuous $h\colon \mathbb{T}^d \to \mathbb{T}^d$, homotopic to the identity, such that*

$$L_A \circ h = h \circ f. \tag{1.2}$$

Moreover, fixing a lift $\tilde f$ of f, the map $h\colon \mathbb{T}^n \to \mathbb{T}^n$ is unique among the continuous maps having a lift $\tilde h\colon \mathbb{R}^n \to \mathbb{R}^n$ satisfying $\tilde h \circ \tilde f = A \circ \tilde h$.

A map h satisfying (1.2) is called a *semiconjugacy* between f and L_A.

Recall that a diffeomorphism f of a manifold M is *Anosov* if TM admits a continuous decomposition $E^u \oplus E^s$ that is preserved by Df such that E^u and E^s are, respectively, uniformly expanded and contracted by Df. The fundamental examples of Anosov diffeomorphisms are affine automorphisms of nilmanifolds and tori — that is, diffeomorphisms of the form $x \mapsto b \cdot A(x)$ on a nilmanifold $M = N/\Lambda$ where $b \in N$ and $A \in \mathrm{Aut}(M)$ such that $DA\restriction_{T_eN}$ is a hyperbolic linear transformation of the Lie algebra $\mathfrak{n}$ of N. It is conjectured that the only manifolds admitting Anosov diffeomorphisms are finite quotients of tori or nilmanifolds.

In the case that f is an Anosov diffeomorphism of a torus or nilmanifold, it is well known that the linear data of f is hyperbolic. Moreover, it follows from the work of Franks [Fra70] and Manning [Man74] that the map h satisfying (1.2) is a homeomorphism; in this case we call such an h a *conjugacy* between f and L_A. Moreover, one can show in this case that h is bi-Hölder. However, in general one cannot obtain any additional regularity of h even when f is Anosov.

1.3. *Setting for main results.* Let Γ be a discrete group and α an action of Γ by homeomorphisms on a compact nilmanifold N/Λ. In Section 2.1 we define the *linear data* $\rho\colon \Gamma \to \mathrm{Aut}(N/\Lambda)$ for such actions under the assumption either that N is abelian or that the action α lifts to an action by homeomorphisms of N. We assume for the time being that the linear data ρ associated to α is defined. For individual elements $\alpha(\gamma)$ of the action such that $\rho(\gamma)$ is hyperbolic, one can build a semiconjugacy between the elements $\alpha(\gamma)$ and $\rho(\gamma)$. However, even assuming the action α lifts, one rarely expects to be able to build a single map $h\colon N/\Lambda \to N/\Lambda$ such that

$$\rho(\gamma) \circ h = h \circ \alpha(\gamma)$$

holds for *every element* of Γ, or even for every element in a finite-index subgroup of Γ.

We focus in this paper on discrete groups Γ exhibiting certain rigidity properties — namely, lattices in higher-rank, semisimple Lie groups. For such Γ, we can exploit certain properties of Γ to study the rigidity of actions of Γ. For most results, we will assume the following hypothesis.

Hypothesis 1.2. *Suppose G is a connected semisimple Lie group with finite center, all of whose noncompact almost-simple factors have $\mathbb{R}$-rank 2 or higher, and suppose Γ is a lattice in G.*

We allow G to have compact factors for full generality. In general, the quotient of G by the maximal compact group of G is a new semisimple Lie group G' without compact factors and contains a finite quotient Γ' of Γ as a lattice. However, an action by Γ typically does not factor through an action of Γ' if Γ' does not coincide with Γ.

1.4. *Topological rigidity for actions with hyperbolic linear data.* Our first main theorem is a solution to the topological global rigidity problem assuming the action lifts and the linear data $\rho(\gamma_0)$ is hyperbolic for some $\gamma_0 \in \Gamma$.

Theorem 1.3. *Let G and Γ be as in Hypothesis 1.2. Let α be a C^0 action of Γ on a compact nilmanifold $M = N/\Lambda$. Suppose α can be lifted to an action on the universal cover N of M, and let ρ be the associated linear data of α. If $D\rho(\gamma)$ is hyperbolic for some element $\gamma \in \Gamma$, then there are a finite-index subgroup $\Gamma_1 < \Gamma$ and a surjective continuous map $h\colon M \to M$, homotopic to identity, such that $\rho(\gamma) \circ h = h \circ \alpha(\gamma)$ for all $\gamma \in \Gamma_1$. If α acts by Lipschitz homeomorphisms, then h is Hölder continuous.*

For the definition of the linear data ρ, see Section 2.1.2 below.

Remark 1.4. It is known that genuinely affine actions exist, i.e., actions by Γ that act by affine automorphisms, but cannot be conjugated to a Γ action by linear automorphisms. Such actions can still be conjugated to a linear action after restricting to a finite-index subgroup. This is demonstrated by an example of Hurder [Hur93, Th. 2]. Hence the restriction to a finite-index subgroup Γ_1 is necessary.

In the case that $N = \mathbb{R}^d$ and $M = \mathbb{T}^d$, the obstruction to lifting the action α of Γ on $\mathbb{T}^d$ to an action of Γ on $\mathbb{R}^d$ is represented by an element in the group cohomology $H^2_\rho(\Gamma, \mathbb{Z}^d)$. If this element vanishes in $H^2_\rho(\Gamma, \mathbb{R}^d)$, then it vanishes in $H^2_\rho(\Gamma, \mathbb{Z}^d)$ after possibly passing to a finite-index subgroup of Γ and the lifting of the action is automatic. For actions on nilmanifolds, the action lifts assuming the vanishing of certain obstructions in the group cohomology associated to a finite number of induced representations. Sufficient conditions for the vanishing of the cohomological obstructions are given by [GH68, Th. 3.1] and [Bor81, Th. 4.4]. In particular, the lifting hypotheses can be verified using only knowledge about the linear data ρ (in the case $M = \mathbb{T}^d$) or the induced $\alpha_{\#} \colon \Gamma \to \mathrm{Out}(\Lambda)$ (in the case $M = N/\Lambda$). See Section 9 for more details.

In particular,

Remark 1.5. The restriction of α to a finite-index subgroup of Γ lifts to an action on N assuming any of the following condition holds:

(1) $\Gamma = \mathrm{SL}(d, \mathbb{Z})$ acting on $\mathbb{T}^d$, $d \geq 5$;
(2) Γ is a cocompact lattice in G;
(3) α is an action of Γ on a torus $\mathbb{T}^d$ that preserves a probability measure μ;
(4) Γ is as in Hypothesis 1.2, α is an action of Γ on a compact nilmanifold N/Λ that preserves a probability measure μ, and $\alpha(\gamma_0)$ is Anosov for some γ_0.

Note that the lifting property is easy to verify when the Γ-action has a finite orbit; criteria (3) above is a generalization of this fact in the torus case. This property will be proved as Proposition 9.7.

The main advantage of our method of proof is that, unlike the majority of previous results discussed above, we do not assume the existence of an invariant measure for the action α in order to construct a semiconjugacy. Note that given a conjugacy between the linear and nonlinear actions, one can obtain a Γ-invariant measure for the nonlinear action. However, in the case of a semiconjugacy, the existence of an invariant measure for the nonlinear actions is more subtle. As a corollary of Theorem 1.3, we present certain conditions under which a nonlinear non-Anosov action of Γ has a “large” invariant measure.

THEOREM 1.6. *Let $\Gamma \subset \mathrm{SL}(n+1, \mathbb{Z})$, $n \geq 2$ be of finite index, and let α be an action of Γ on $\mathbb{T}^{n+1}$ by $C^{1+\beta}$ diffeomorphisms. Suppose the action α lifts to an action on $\mathbb{R}^{n+1}$, and let h denote the semiconjugating map guaranteed by Theorem* 3.1. *Moreover, suppose that the linear data $\rho \colon \Gamma \to \mathrm{GL}(n+1, \mathbb{Z})$ is the identity representation $\rho(\gamma) = \gamma$.*

Then there exists a unique, α-invariant, absolutely continuous probability measure μ on $\mathbb{T}^{n+1}$ such that $h_*\mu$ is the Haar measure on $\mathbb{T}^{n+1}$. Moreover, μ is the unique ergodic α-invariant measure on $\mathbb{T}^n$ such that $h_*\mu$ is not atomic.

We remark that, up to restricting to finite-index subgroups and conjugating by an element of $\mathrm{GL}(n+1, \mathbb{Q})$, the only nontrivial representations $\rho \colon \Gamma \to \mathrm{GL}(n+1, \mathbb{Z})$ are the identity $\rho(\gamma) = \gamma$ and the inverse transpose $\rho(\gamma) = (\gamma^{\mathrm{t}})^{-1}$. Moreover, replacing Γ with its image under $\gamma \mapsto (\gamma^{\mathrm{t}})^{-1}$ if necessary, we may assume ρ is the identity. In particular, the conclusion of Theorem 1.6 holds for any action α such that the linear data $\rho \colon \Gamma \to \mathrm{GL}(n+1, \mathbb{Z})$ has infinite image.

Proof of Theorem 1.6. We may find a copy of $\mathbb{Z}^n$ inside Γ so that, in the terminology of [KK07], the corresponding linear action $\rho\restriction_{\mathbb{Z}^n}$ is a linear

Cartan action. It follows from the results of [KS96] that any ergodic, $\rho\restriction_{\mathbb{Z}^n}$-invariant measure with positive Hausdorff dimension is Haar measure. As $\mathbb{Z}^d$ is amenable, there exists an ergodic, $\alpha\restriction_{\mathbb{Z}^n}$-invariant, measure on $\mathbb{T}^{n+1}$ projecting to the Haar measure under h. From the main theorem [KK07], it follows that any such μ is absolutely continuous.

Moreover, from the main result of [KRH07], it follows that there is a unique measure μ on $\mathbb{T}^{n+1}$ such that $h_*\mu$ is the Haar measure. It follows from this uniqueness criterion that μ is invariant under the entire action α. Finally, if ν is α-invariant and if $h_*\nu$ is not atomic then, as $h_*\nu$ is ρ-invariant, a Fourier analysis argument shows it must be Haar measure. □

1.5. *Smooth rigidity for Anosov actions.* The main result of the paper is the following solution to the global smooth rigidity problem. We show that, in the setting of Theorem 1.3, if α is an action by C^∞ diffeomorphisms and if $\alpha(\gamma)$ is Anosov for some element γ of Γ, then the semiconjugacy h (which is necessarily invertible) is a diffeomorphism.

THEOREM 1.7. *Let G and Γ be as in Hypothesis* 1.2. *Let α be a C^∞ action of Γ on a compact nilmanifold $M = N/\Lambda$. Suppose α can be lifted to an action on the universal cover N of M, and let ρ be the associated linear data of α. If $\alpha(\gamma)$ is Anosov for some element $\gamma \in \Gamma$, then there are a finite-index subgroup $\Gamma' < \Gamma$ and a C^∞ diffeomorphism $h\colon M \to M$, homotopic to identity, such that $h \circ \alpha(\gamma) = \rho(\gamma) \circ h$ for all $\gamma \in \Gamma'$.*

The analogue of Theorem 1.7 for $\mathbb{Z}^r$-actions without rank-one factors is part of a conjecture of Katok and Spatzier that has been established in the works of Fisher-Kalinin-Spatzier [FKS13] and Rodriguez Hertz-Wang [RHW14]. Our proof of Theorem 1.7 works by finding a large abelian subgroup of Γ on which the restriction of α has no rank-one factors; we then apply the aforementioned analogous theorem for $\mathbb{Z}^r$-actions to this subgroup.

Remark 1.8. In Theorem 1.7, the map h actually conjugates the entire action by Γ to an action by affine nilmanifold automorphisms. This can be easily deduced from the well-known fact that the centralizer of an hyperbolic automorphism is affine; see, e.g., [Wit94].

In light of Remark 1.5, from Theorem 1.7 we immediately obtain the following.

COROLLARY 1.9. *Suppose any one of the following holds*:

(1) *$n \geq 5$, and α is a C^∞ action by $\Gamma = \mathrm{SL}(n, \mathbb{Z})$ on $M = \mathbb{T}^n$.*
(2) *G is as in Hypothesis* 1.2, *Γ is a cocompact lattice in G, and α is a C^∞ action by Γ on any nilmanifold M.*
(3) *Γ is as in Hypothesis* 1.2 *and α is an action of Γ on a compact nilmanifold N/Λ that preserves a probability measure μ.*

(4) Γ *is as in Hypothesis* 1.2, ρ *is a linear action by* Γ *on a compact nilmanifold* N/Λ *by linear automorphisms*, *and* α *is a* C^∞ *action* Γ *on* N/Λ. *Suppose* α *is sufficiently close to* ρ *in* C^1*-norm*; *namely*, $d_{C^1}(\alpha(\gamma), \rho(\gamma)) < \varepsilon$ *for all elements* γ *from a set of generators* $S \subset \Gamma$, *where the constant* $\varepsilon > 0$ *depends on* Γ, ρ, *and* S.

If $\alpha(\gamma)$ *is Anosov for some* $\gamma \in \Gamma$, *then there is a finite-index subgroup* $\Gamma' < \Gamma$, *a* Γ'*-action* ρ *on* M *by linear automorphisms and a* C^∞ *diffeomorphism* $h\colon M \to M$, *homotopic to identity*, *such that* $h \circ \alpha(\gamma) = \rho(\gamma) \circ h$ *for all* $\gamma \in \Gamma'$.

We remark that in case (4), the subgroup Γ' is actually Γ. This case recovers all previously known C^∞ local rigidity results for affine Anosov actions including [Hur92], [QY98], [KL91], [KLZ96], [GS99], [Qia95].

1.6. *Organization of paper*. In Section 2 we present the major technical background and definitions for the paper. In Section 3, we present the main technical theorems, Theorems 3.1 and 3.2, which assert the existence of a semiconjugacy between a nonlinear action and its linear data under a number of technical hypotheses. In Section 4 we introduce suspension spaces that convert the problem of building a semiconjugacy between Γ-actions into a problem of building a semiconjugacy between G-actions. In Section 5, we obtain this semiconjugacy for G-actions by first constructing it for a single element of a Cartan subgroup in G, and then extending to a semiconjugacy between entire G-actions. In Section 6 we present a number of classical results that will be used in the following sections. In Section 7, we show that the technical assumptions required by Theorem 3.1 are satisfied in the setting of Theorem 1.3. In Section 8, we prove Theorem 1.7 by finding a large abelian subgroup of Γ whose action contains an Anosov diffeomorphism. Finally, the lifting hypothesis and Remark 1.5 are discussed in Section 9.

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2. **Preliminaries**

In this section we present the main definitions and constructions that will be used for our main technical theorems in Section 3. We also recall and prove some related facts. A key technical observation that is new in this paper appears in Section 2.5.

2.1. *Linear data associated to torus and nilmanifold actions.* To extend Franks' Theorem to continuous actions of discrete groups on tori and nilmanifolds we need an appropriate notion of the linearization of such an action.

2.1.1. *Linear data associated to torus actions.* Given $B \in \mathrm{GL}(d, \mathbb{Z})$, we write $L_B \colon \mathbb{T}^d \to \mathbb{T}^d$ for the map on $\mathbb{T}^d$ induced by B. Given $f \in \mathrm{Homeo}(\mathbb{T}^d)$, recall the unique $A_f \in \mathrm{GL}(d, \mathbb{Z})$ as in (1.1) with f homotopic to L_{A_f}. For $f, g \in \mathrm{Homeo}(\mathbb{T}^d)$, we verify from the characterization (1.1) that $A_{f \circ g} = A_f A_g$.

Consider a discrete group Γ and an action $\alpha \colon \Gamma \to \mathrm{Homeo}(\mathbb{T}^d)$. It follows that there exists an induced homomorphism

$$\rho \colon \Gamma \to \mathrm{GL}(d, \mathbb{R}), \quad \rho \colon \gamma \to A_{\alpha(\gamma)}. \tag{2.1}$$

Moreover, for each $\gamma \in \Gamma$, $\alpha(\gamma)$ is homotopic to $L_{\rho(\gamma)}$. Below, we abandon the notation $L_{\rho(\gamma)}$ and simply write $\rho(\gamma) \colon \mathbb{T}^d \to \mathbb{T}^d$; whether $\rho(\gamma)$ is an element of $\mathrm{GL}(d, \mathbb{R})$ or $\mathrm{Homeo}(\mathbb{T}^d)$ will be clear from context. The representation $\rho \colon \Gamma \to \mathrm{GL}(d, \mathbb{R})$ is called the *linear data* of α.

2.1.2. *Linear data associated to nilmanifold actions.* In the case of actions on nilmanifolds, the above situation is more complicated. Indeed, let $M = N/\Lambda$, where N is a simply connected, nilpotent Lie group and Λ is a finite volume discrete subgroup. Consider an action $\alpha \colon \Gamma \to \mathrm{Homeo}(M)$. Then for each base point $x \in M$, every γ induces an automorphism $\alpha(\gamma)_*$ of $\pi_1(M, x) \cong \Lambda$. (Here, for every other point $x' - \alpha(\gamma)(x) \in M$, we fix a path from x' to x in order to identify $\pi_1(M, x')$ with $\pi_1(M, x)$.) As the map $\alpha(\gamma) \colon N/\Lambda \to N/\Lambda$ need not fix a base point, the map $\Gamma \to \mathrm{Aut}(\Lambda)$ sending γ to $\alpha(\gamma)_*$ is defined only up to conjugation. Thus, one has an induced homomorphism $\alpha_\# \colon \Gamma \to \mathrm{Out}(\Lambda) = \mathrm{Aut}(\Lambda)/\mathrm{Inn}(\Lambda)$.

However, under the additional assumption that $\alpha \colon \Gamma \to \mathrm{Homeo}(M)$ lifts to an action $\tilde{\alpha} \colon \Gamma \to \mathrm{Homeo}(N)$ by the canonical identification of Λ with the group of deck transformations for the cover $N \to N/\Lambda$, we obtain a well-defined action $\rho \colon \Gamma \to \mathrm{Aut}(\Lambda)$. As it is necessary in our method of proof to assume the lift $\tilde{\alpha}$ of the action exists, this is not a very restrictive assumption. By [Mal51, Th. 5], every element of $\mathrm{Aut}(\Lambda)$ extends uniquely to an element of $\mathrm{Aut}(N)$; in particular, we may extend ρ to a homomorphism $\rho \colon \Gamma \to \mathrm{Aut}(N)$. Moreover, for each $\gamma \in \Gamma$, we have that $\alpha(\gamma)$ is homotopic to $\rho(\gamma) \colon N/\Lambda \to N/\Lambda$. Here, as above, we use $\rho(\gamma)$ to indicate both an element of $\mathrm{Aut}(N)$ and the induced element of $\mathrm{Aut}(N/\Lambda)$.

Definition 2.1. If $\alpha \colon \Gamma \to \mathrm{Homeo}(N/\Lambda)$ either acts on a torus or lifts to the universal cover N, we call $\rho \colon \Gamma \to \mathrm{Aut}(N)$ (or the induced $\rho \colon \Gamma \to \mathrm{Aut}(N/\Lambda)$) the *linear data* associated to α.

We remark that in [RHW14] the linear data was referred to as the "homotopy data."

2.2. *Structure of compact nilmanifolds.* We collect some standard facts about nilpotent Lie groups and their lattices. A standard reference is [Rag72].

Let N be a simply connected, nilpotent Lie group, and let $\Lambda \subset N$ be a lattice. Write $M = N/\Lambda$ for the quotient nilmanifold. We have that M is compact. The set $\exp^{-1}(\Lambda)$ generates a lattice in the Lie algebra $\mathfrak{n}$. This, together with the coordinate system $\exp\colon \mathfrak{n} \to N$, determines a $\mathbb{Q}$-structure on N. For a connected closed normal subgroup $N' \lhd N$, the following are equivalent:

(1) N' is defined over $\mathbb{Q}$;

(2) $N' \cap \Lambda$ is a lattice in N';

(3) $\Lambda/(N' \cap \Lambda)$ is a lattice in N/N';

(4) $N/N'\Lambda = (N/N')\Big/\big(\Lambda/(N' \cap \Lambda)\big)$ defines a compact nilmanifold that is naturally a quotient of N/Λ.

In any of the above cases, we say N' is *rational* and $M' = N/N'\Lambda$ is an *algebraic factor* of M.

Recall that for an automorphism $f \in \mathrm{Aut}(N)$, $D_e f$ is an automorphism of $\mathfrak{n}$, and $f \circ \exp = \exp \circ D_e f$. Moreover, f preserves Λ if and only if $D_e f$ preserves $\exp^{-1}(\Lambda)$. In this case, f descends to an automorphism of M. Hence $\mathrm{Aut}(M) = \mathrm{Aut}(N)$ can be regarded as a subgroup of $\mathrm{GL}(d, \mathbb{Z})$ if we identify the subgroup generated by $\exp^{-1}(\Lambda) \subset \mathfrak{n}$ with $\mathbb{Z}^d \subset \mathbb{R}^d$.

If, in addition, f preserves a rational normal subgroup N', then it further descends an automorphism of M'.

Let $Z(N)$ denote the center of N. Then

(1) $Z(N)$ is normal and rational;

(2) any element in $\mathrm{Aut}(M)$ preserves $Z(N)$ and thus descends to an element of $\mathrm{Aut}(N/Z(N)\Lambda)$.

It follows that we have a series of central extensions

$$N = N_0 \to N_1 \to N_2 \to \cdots \to N_{r-1} \to N_r = \{e\}. \tag{2.2}$$

Here r is the degree of nilpotency. Each N_i is a simply connected, nilpotent Lie group, and the kernel of the map $N_i \to N_{i+1}$ is the center of N_i. We have a corresponding series of central extensions

$$\Lambda = \Lambda_0 \to \Lambda_1 \to \Lambda_2 \to \cdots \to \Lambda_r = \{e\}, \tag{2.3}$$

where $\Lambda_{i+1} = \Lambda_i/(\Lambda_i \cap Z(N_i))$ and Λ_i is a lattice in N_i. As automorphisms preserve the center of a group, an automorphism f of N preserving Λ descends inductively to an automorphism of $M_i = N_i/\Lambda_i$ for each i.

Suppose Γ is a discrete group, and suppose we have an action $\rho\colon \Gamma \to \mathrm{Aut}(\Lambda)$. ρ extends uniquely to an action $\rho\colon \Gamma \to \mathrm{Aut}(N)$ and induces an action by homeomorphisms on N/Λ. Moreover, ρ descends to a Λ_i-preserving action $\rho_i\colon \Gamma \to \mathrm{Aut}(N_i)$ for every element of the central series (2.2) and (2.3). As above, we write $\rho_i(\gamma)$ to denote both an element of $\mathrm{Aut}(N_i)$ and the induced element of $\mathrm{Aut}(M_i)$.

2.3. *π_1-factors.* In the sequel, we construct a semiconjugacy between actions in a more general setting than in the introduction. This follows [FW01]. Consider M to be any connected finite CW-complex. Let $\tilde{M}$ be any normal covering of M, and let Λ_M denote the corresponding group of deck transformations. We denote the action of Λ_M on $\tilde{M}$ on the right. Let Γ be a discrete group and $\alpha\colon \Gamma \to \mathrm{Homeo}(M)$ an action. We assume α lifts to an action $\tilde{\alpha}\colon \Gamma \to \mathrm{Homeo}(\tilde{M})$; we then obtain an induced action $\alpha_*\colon \Gamma \to \mathrm{Aut}(\Lambda_M)$ defined by

$$\tilde{\alpha}(\gamma)(x\lambda) = \tilde{\alpha}(\gamma)(x)\alpha_*(\gamma)(\lambda).$$

Let N be a simply connected, nilpotent Lie group and let $\Lambda \subset N$ be a lattice. Suppose there is a surjective homomorphism $P_*\colon \Lambda_M \to \Lambda$. We moreover assume that $\alpha_*(\gamma)(\ker P_*) = \ker P_*$ for all $\gamma \in \Gamma$. Then α_* induces an action

$$\rho\colon \Gamma \to \mathrm{Aut}(\Lambda), \quad \rho(\gamma)(P_*(\lambda)) = P_*\alpha_*(\gamma)(\lambda).$$

We extend ρ to $\rho\colon \Gamma \to \mathrm{Aut}(N)$. As N/Λ is a $K(\Lambda, 1)$, there is a continuous $P\colon M \to N/\Lambda$ such that P lifts to $\tilde{P}\colon \tilde{M} \to N$ and the map between deck transformation groups Λ_M and Λ induced by $\tilde{P}$ coincides with P_*; that is, $\tilde{P}(x\lambda) = \tilde{P}(x) \cdot P_*(\lambda)$. (See [FW01, Th. 3.2]). In particular, for each $\gamma \in \Gamma$, we have $P \circ \alpha(\gamma)\colon M \to N/\Lambda$ is homotopic to $\rho(\gamma) \circ P\colon M \to N/\Lambda$.

Definition 2.2. Under the above hypotheses, we say that the action $\rho\colon \Gamma \to \mathrm{Aut}(\Lambda)$ (or $\rho\colon \Gamma \to \mathrm{Aut}(N/\Lambda)$) is a *$\pi_1$-factor of* α induced by the map P_*.

Let $P_0 = P$, and let P_i be the composition of P with the natural map $N/\Lambda \to N_i/\Lambda_i$, where N_i and Λ_i are as in (2.2) and (2.3). We similarly obtain maps $\tilde{P}_i\colon \tilde{M} \to N_i$ and $P_{*,i}\colon \Lambda_M \to \Lambda_i$. Let ρ_i be the action of Γ on N_i induced by ρ.

We have the following.

Claim 2.3. *If ρ is a π_1-factor of α induced by P_*, then for every i, ρ_i is a π_1-factor of α induced by $P_{*,i}$.*

Remark 2.4. The requirement that P_* is surjective is not very restrictive. In fact, if the image of P_* is a proper subgroup Λ' of Λ, then Λ' will be a lattice in its Zariski closure N' ([Rag72, Th. II.2.3]). After replacing N and Λ with N' and Λ', we have a π_1-factor.

2.4. *Coarse geometry of lattices.* Let G be a connected semisimple Lie group equipped with a right-invariant metric, and let $\Gamma \subset G$ be a finitely generated discrete subgroup. We may equip Γ with two metrics: the word metric d_{word} induced by a fixed choice of generators and a right-invariant metric d_G on Γ inherited as a subset of G from a right-invariant Riemannian metric on G.

We say these two metrics are *quasi-isometric* (or that Γ is *quasi-isometrically embedded in* G) if there are $A > 1$ and $B > 0$ such that for any $\gamma_1, \gamma_2 \in \Gamma$, we have

$$A^{-1} \cdot d_{\text{word}}(\gamma_1, \gamma_2) - B \leq d_G(\gamma_1, \gamma_2) \leq A \cdot d_{\text{word}}(\gamma_1, \gamma_2) + B. \tag{2.4}$$

Note that all word metrics are quasi-isometric. For the remainder we fix a finite generating set $F = \{\gamma_\ell\}$ for Γ with induced word metric $d_{\text{word}}(\cdot, \cdot)$.

Note that if $\Gamma \subset G$ is a cocompact lattice, then Γ is automatically quasi-isometrically embedded in G. For nonuniform lattices, we have the following result.

THEOREM 2.5 (Lubotzky-Mozes-Raghunathan [LMR00]). *A lattice* Γ *is quasi-isometrically embedded in* G *if the projection of* Γ *to any* $\mathbb{R}$*-rank* 1 *factor is dense. In particular*, Γ *is quasi-isometrically embedded in* G *under Hypothesis* 1.2.

2.5. *Nonresonant linear representations.* In this section, we introduce the main new technical idea in this paper. Let G be a semisimple Lie group. Let $\mathfrak{g}$ be the Lie algebra of G. We fix a Cartan involution θ of $\mathfrak{g}$ and write $\mathfrak{k}$ and $\mathfrak{p}$, respectively, for the $+1$ and -1 eigenspaces of θ. Denote by $\mathfrak{a}$ the maximal abelian subalgebra of $\mathfrak{p}$ and by $\mathfrak{m}$ the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. Recall that $\dim_{\mathbb{R}}(\mathfrak{a})$ is the $\mathbb{R}$-rank of G.

Consider a linear representation $\tau\colon G \to \mathrm{GL}(n, \mathbb{R})$ of G. Then τ induces a representation $d\tau\colon \mathfrak{g} \to \mathfrak{gl}(n, \mathbb{R})$. Let $\{\chi_i\}$ denote the restricted weights of $d\tau$ relative to $\mathfrak{a}$, and let $\Sigma := \{\zeta_j\}$ denote the restricted roots of $\mathfrak{g}$ relative to $\mathfrak{a}$; that is, Σ is the restricted weights of the adjoint representation. Given a simple factor $\mathfrak{g}' \subset \mathfrak{g}$ we denote by $\Sigma(\mathfrak{g}')$ the irreducible restricted root system. Recall that each χ_i and ζ_j is a real linear functional on $\mathfrak{a}$. Given $\zeta \in \Sigma$, let $\mathfrak{g}^\zeta$ denote the corresponding subspace. Recall that $\mathfrak{g}^0 = \mathfrak{m} \oplus \mathfrak{a}$.

Definition 2.6. Let $\psi\colon \mathfrak{g} \to \mathfrak{gl}(n, \mathbb{R})$ be a linear representation. We say a restricted root ζ_j of $\mathfrak{g}$ is *resonant* (with ψ) if there is a $c > 0$ and a restricted weight χ_i of ψ such that $\zeta_j = c\chi_i$; otherwise we say ζ_j is *nonresonant*. Let Σ_{NR} denote the set of nonresonant restricted roots.

We say the representation ψ is *strongly nonresonant* (with respect to Ad) if every nonzero restricted root of $\mathfrak{g}$ is nonresonant with ψ. We say the representation ψ is *weakly nonresonant* (with respect to Ad) if the set

$$\mathfrak{g}^0 \cup \bigcup_{\zeta \in \Sigma_{NR}} \mathfrak{g}^\zeta$$

generates $\mathfrak{g}$ as a Lie algebra.

If $\psi = d\tau$ for $\tau \colon G \to \mathrm{GL}(n, \mathbb{R})$, we say τ is weakly nonresonant if $d\tau$ is. Note (if ψ is nontrivial) that the existence of a nonresonant root implies that the $\mathbb{R}$-rank of G is at least 2.

Nonresonance of roots will be used later in Section 5.3.

In the remainder, we will be interested only in representations with all weights *nontrivial* — that is, representations for which the weight space corresponding to the zero weight is trivial. We note that for many classical simple Lie groups, particularly for $G = \mathrm{SL}(n, \mathbb{R})$, there are infinitely many irreducible representations with all weights nontrivial.

Given a semisimple Lie algebra $\mathfrak{g}$ containing rank-one factors, a representation $\psi \colon \mathfrak{g} \to \mathfrak{gl}(n, \mathbb{R})$ with all weights nontrivial may or may not be weakly nonresonant. However, if all noncompact factors have $\mathbb{R}$-rank at least 2, the following lemma guarantees that all representations we consider in the sequel are weakly nonresonant. Note, however, that there are representations with all weights nontrivial for which there are resonant restricted roots.

The following lemma is a crucial new observation introduced in this article.

Lemma 2.7. *Suppose $\mathfrak{g}$ is a semisimple real Lie algebra such that every noncompact factor has $\mathbb{R}$-rank 2 or higher. Let ψ be a finite-dimensional, real representation of $\mathfrak{g}$ such that all restricted weights of ψ are nontrivial. Then ψ is weakly nonresonant.*

Moreover, if no noncompact simple factors of $\mathfrak{g}$ have restricted root system of type C_ℓ, then ψ is strongly nonresonant.

As an example showing that we must consider weakly nonresonant representations, consider the standard action of $\mathrm{Sp}(4, \mathbb{R})$ on $\mathbb{R}^4$. The Lie algebra $\mathfrak{g}$ of $\mathrm{Sp}(4, \mathbb{R})$ is of type C_2 (and is moreover a split real form). Relative to a certain basis $\{e_1, e_2\}$ of $\mathfrak{a}$, the restricted roots of $\mathfrak{g}$ are $\{\pm\varepsilon_1 \pm \varepsilon_2\} \cup \{\pm 2\varepsilon_1, \pm 2\varepsilon_2\}$ where $\varepsilon_i(e_j) = \delta_{ij}$. Take a representation whose highest weight is given by $\lambda = \varepsilon_1$. Then the weights of the representation are $\{\pm\varepsilon_1, \pm\varepsilon_2\}$, and hence are all nontrivial. The resonant restricted roots are $\{\pm 2\varepsilon_1, \pm 2\varepsilon_2\}$; however, the Lie algebra is generated by $\mathfrak{g}^0$ and the root spaces corresponding to the set of nonresonant roots $\{\pm\varepsilon_1 \pm \varepsilon_2\}$.

Proof of Lemma 2.7. We recall some facts from the representation theory of semisimple Lie algebras that can be found, for instance, in the book [Kna02]. Though usually stated for complex representations of complex Lie algebras, all facts used here hold for real representations of real Lie algebras. Consider first the case that ψ is irreducible. Then there is a restricted weight λ, called the *highest weight*, such that every other weight χ of ψ is of the form

$$\chi = \lambda - \sum n_i\beta_i,$$

where $\{\beta_i\}$ is a set of simple positive roots and n_i are positive integers. We have that every weight χ is *algebraically integral*: that is, $2\frac{\langle\chi,\xi\rangle}{\langle\xi,\xi\rangle} \in \mathbb{Z}$ for every root $\xi \in \Sigma$, where $\langle\cdot,\cdot\rangle$ is the inner product on $\mathfrak{a}^*$ induced from the Killing form. Given a simple positive root β_j, there is a distinguished *fundamental weight* ϖ_j defined by $2\frac{\langle\varpi_i,\beta_j\rangle}{\langle\beta_j,\beta_j\rangle} = \delta_{ij}$. We have that every highest weight λ is a positive integer combination of the fundamental weights ϖ_i.

If $\mathfrak{g}$ is simple and the root system $\Sigma(\mathfrak{g})$ is reduced with Cartan matrix $C = [C_{ij}]$, then the simple roots and fundamental weights are related by $\beta_i = \sum C_{ij}\varpi_j$. The only simple nonreduced root system is of type BC_ℓ; the roots of BC_ℓ are the union of the roots of B_ℓ and C_ℓ, and the fundamental weights are those from C_ℓ.

We proceed with the proof of the lemma. Suppose that a weight χ and a root ξ are positively proportional. We can assume χ is a weight of an irreducible component of ψ and that ξ is a root of a noncompact simple factor $\mathfrak{g}_k$ of $\mathfrak{g} = \oplus\mathfrak{g}_k$. Moreover, as the fundamental weights for distinct simple factors of $\mathfrak{g}$ are linearly independent, we may assume $\chi = \lambda - \sum n_i\beta_i$, where β_i are simple roots for $\Sigma(\mathfrak{g}_k)$, and $\lambda = \sum k_j\varpi_j$, where ϖ_j are the fundamental weights of the root system $\Sigma(\mathfrak{g}_k)$ and k_j are nonnegative integers. We may also take ξ so that $\frac{1}{2}\xi$ is not a root. Then there is an element of the Weyl group of $\mathfrak{g}_k$ that sends ξ to a simple root β_{i_0} of $\Sigma(\mathfrak{g}_k)$ [Kna02, Prop. 2.62]. Moreover, the Weyl group preserves weights of ψ, hence we may assume that χ is positively proportional to a simple positive root β_{i_0} of $\mathfrak{g}_k$.

First consider the case that $\Sigma(\mathfrak{g}_k)$ is not of type B_ℓ, C_ℓ or BC_ℓ. Suppose $\chi = t\beta_{i_0}$. We have $\chi = \sum_j m_j\varpi_j$ for some integers m_j. Since the functionals ϖ_j are linearly independent, it follows that $m_j = tC_{i_0 j}$ for every j. Then t is rational and $t = \frac{p}{q}$ where $q \in \mathbb{N}$ is smaller than the greatest common factor of all entries in the i_0-th row of $[C_{ij}]$. For $\Sigma(\mathfrak{g}_k)$ not of type B_ℓ, C_ℓ or BC_ℓ, the entries of every row of the corresponding Cartan matrix $[C_{ij}]$ have greatest common factor of 1. Thus t is an integer. Then, the restricted weights of ψ include the chain $-t\beta_{i_0}, -(t-1)\beta_{i_0}, \dots, (t-1)\beta_{i_0}, t\beta_{i_0}$. (This is deduced from the fact that the simple root β_{i_0} appears in a $\mathfrak{sl}(2)$ triple and the representation theory of $\mathfrak{sl}(2,\mathbb{R})$.) As we assume 0 is not a (nontrivial) weight, it follows that no such positively proportional pair χ and β_{i_0} exists. It follows that if all simple factors $\mathfrak{g}_k$ are not of type B_ℓ, C_ℓ or BC_ℓ, then ψ is strongly nonresonant.

In the case that $\Sigma(\mathfrak{g}_k)$ is of type C_ℓ, the Cartan matrix contains one row whose entries have greatest common factor 2; all other rows have greatest common factor 1. Then there is at most one simple root that is resonant with ψ. The orbits of the remaining simple roots under the Weyl group generate all of $\mathfrak{g}_k$. In the case that $\Sigma(\mathfrak{g}_k)$ is of type B_ℓ, from the tables of root and fundamental weight data (cf. [Kna02, App. C]), the only root system of type B_ℓ admitting a representation with resonant roots and all weights nontrivial occurs for B_2. However, B_2 is isomorphic to C_2.

Finally, if $\Sigma(\mathfrak{g}_k)$ is of type BC_ℓ, then the fundamental weights of $\Sigma(\mathfrak{g}_k)$ coincide with those of type C_ℓ and, comparing tables of root data (cf. [Kna02, App. C]), it follows that the fundamental weights of $\Sigma(\mathfrak{g}_k)$ are linear combinations of restricted roots of $\Sigma(\mathfrak{g}_k)$. In particular, if χ is resonant with a root of $\mathfrak{g}_k$ of type BC_ℓ, then $\chi = k\beta_i$ for some positive integer k. It follows that 0 is a nontrivial weight of ψ.

In the case that $\psi = \oplus\psi_i$ is reducible, the above shows that all restricted roots corresponding to simple factors $\mathfrak{g}_k$ of $\mathfrak{g}$ are nonresonant with ψ for all $\mathfrak{g}_k$ with root systems of type other than C_ℓ. If $\mathfrak{g}_k$ is of type C_ℓ, the above shows that all roots of $\mathfrak{g}_k$ that are resonant with ψ are long roots. As the short roots generate $\mathfrak{g}_k$, the result follows. □

3. **The main technical theorems**

To state the main technical theorems, fix G to be a connected semisimple Lie group with finite center.

3.1. *Main theorem*: *actions on nilmanifolds.*

THEOREM 3.1. *Let $\Gamma \subset G$ be a lattice. Let N be a simply connected, nilpotent Lie group with Lie algebra $\mathfrak{n}$, and let $\Lambda \subset N$ be a lattice. Let $M = N/\Lambda$, and let $\alpha\colon \Gamma \to \mathrm{Homeo}(M)$ be an action. Assume $\alpha\colon \Gamma \to \mathrm{Homeo}(M)$ lifts to an action $\tilde{\alpha}\colon \Gamma \to \mathrm{Homeo}(N)$, and let $\rho\colon \Gamma \to \mathrm{Aut}(N)$ denote the associated linear data.*

Assume the following technical hypotheses are satisfied:

(1) *the linear data $\rho\colon \Gamma \to \mathrm{Aut}(N)$ is the restriction to Γ of a continuous morphism $\rho\colon G \to \mathrm{Aut}(N)$*;
(2) *Γ is quasi-isometrically embedded in G*;
(3) *the representation $D\rho\colon G \to \mathrm{Aut}(\mathfrak{n})$ is weakly nonresonant with* $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathfrak{g})$;
(4) *all restricted weights of the representation $D\rho$ with respect to $\mathfrak{a}$ are nontrivial*, *where $\mathfrak{a}$ is as in Section* 2.5.

Then there exists a surjective continuous map $h\colon M \to M$, homotopic to the identity, *such that*
$$h \circ \alpha(\gamma) = \rho(\gamma) \circ h$$
for every $\gamma \in \Gamma$.

The surjectivity of the map h in Theorem 3.1 follows from elementary degree arguments.

3.2. *Main theorem*: π_1-*factors.* Under the setup introduced in Section 2.3, we have the following generalization of Theorem 3.1.

THEOREM 3.2. *Let* $\Gamma \subset G$ *be a lattice. Let* M *be a connected finite* CW-*complex and* $\alpha\colon \Gamma \to \mathrm{Homeo}(M)$ *an action. Suppose for some normal cover* $\tilde{M}$ *of* M *with deck group* Λ_M, *we have that* α *lifts to an action* $\tilde{\alpha}\colon \Gamma \to \mathrm{Homeo}(\tilde{M})$.

Let N *be a nilmanifold and* $\Lambda \subset N$ *a lattice. Assume there is a surjective homomorphism* $P_*\colon \Lambda_M \to \Lambda$ *inducing a* π_1-*factor* $\rho\colon \Gamma \to \mathrm{Aut}(N)$. *Assume the following technical hypotheses are satisfied*:

(1) *the linear representation* $\rho\colon \Gamma \to \mathrm{Aut}(N)$ *is the restriction to* Γ *of a continuous morphism* $\rho\colon G \to \mathrm{Aut}(N)$;
(2) Γ *is quasi-isometrically embedded in* G;
(3) *the representation* $D\rho\colon G \to \mathrm{Aut}(\mathfrak{n})$ *is weakly nonresonant with* $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathfrak{g})$;
(4) *all restricted weights of the representation* $D\rho$ *with respect to* $\mathfrak{a}$ *are nontrivial, where* $\mathfrak{a}$ *is as in Section* 2.5.

Then there exists a continuous map $h\colon M \to N/\Lambda$, *homotopic to* $P\colon M \to N/\Lambda$, *such that*

$$h \circ \alpha(\gamma) = \rho(\gamma) \circ h$$

for every $\gamma \in \Gamma$.

The map P in Theorem 3.2 was defined in Section 2.3. Theorem 3.1 follows immediately from Theorem 3.2 taking P_* and P to be the identity maps.

We prove Theorem 3.2 inductively on the step of nilpotency. As in Section 2.3, given $\rho\colon G \to \mathrm{Aut}(N)$, let $\rho_i\colon G \to \mathrm{Aut}(N_i)$ denote the induced action on the factor N_i of (2.2). Recall that for $0 \le i \le r$, we let $P_i\colon M \to N_i/\Lambda_i$, $\tilde{P}_i\colon \tilde{M} \to N_i$, and $P_{*,i}\colon \Lambda_M \to \Lambda_i$ be the compositions of P, $\tilde{P}$, and P_* followed by the natural projections $N/\Lambda \to N_i/\Lambda_i$, $N \to N_i$, and $\Lambda \to \Lambda_i$. Note that $\tilde{P}_i(x \cdot \lambda) = \tilde{P}_i(x) \cdot P_{*,i}(\lambda)$. If $h\colon M \to N_i/\Lambda_i$ is homotopic to P_i, we say a lift $\tilde{h}\colon \tilde{M}_i \to N_i$ is Λ_M-equivariantly homotopic to $\tilde{P}_i$ if there is a homotopy from $\tilde{h}$ to $\tilde{P}_i$ that factors over a homotopy from h to P_i. If $\tilde{h}$ is Λ_M-equivariantly homotopic to $\tilde{P}_i$, we have $\tilde{h}(x \cdot \lambda) = \tilde{h}(x) \cdot P_{*,i}(\lambda)$.

Note that N_i/Λ_i has a natural structure of a fiber bundle over N_{i+1}/Λ_{i+1}. Note that if conditions (1), (3), and (4) of Theorem 3.2 hold, then they hold for the action $\rho_i\colon G \to \mathrm{Aut}(N_i)$.

THEOREM 3.3. *Let* M, G, Γ, α *and* ρ *be as in Theorem* 3.2. *Let* N_i/Λ_i *be one of the factors appearing in* (2.2) *and* (2.3). *Assume there exists a map* $h_{i+1}\colon M \to N_{i+1}/\Lambda_{i+1}$, *homotopic to* $P_{i+1}\colon M \to N_{i+1}/\Lambda_{i+1}$, *such that*

$$h_{i+1} \circ \alpha(\gamma) = \rho_{i+1}(\gamma) \circ h_{i+1}$$

for all $\gamma \in \Gamma$. *Moreover, assume* $h_{i+1}\colon M \to N_{i+1}/\Lambda_{i+1}$ *lifts to* $\tilde{h}_{i+1}\colon \tilde{M} \to N_{i+1}$ *with* $\tilde{h}_{i+1} \circ \tilde{\alpha}(\gamma) = \rho_{i+1}(\gamma) \circ \tilde{h}_{i+1}$ *and* $\tilde{h}_{i+1}$ Λ_M*-equivariantly homotopic to* $\tilde{P}_{i+1}$.

Then there exists a continuous map $h_i\colon M \to N_i/\Lambda_i$ *such that* h_i *is homotopic to* $P_i\colon M \to N_i/\Lambda_i$, $h_i\colon M \to N_i/\Lambda_i$ *lifts* h_{i+1}, *and*

$$h_i \circ \alpha(\gamma) = \rho_i(\gamma) \circ h_i$$

for all $\gamma \in \Gamma$. *Moreover,* h_i *is the unique map having a lift* $\tilde{h}_i\colon \tilde{M} \to N_i$ *with* $\tilde{h}_i \circ \tilde{\alpha}(\gamma) = \rho_i(\gamma) \circ \tilde{h}_i$ *and* $\tilde{h}_i$ Λ_M*-equivariantly homotopic to* $\tilde{P}_i$.

Theorem 3.2 follows immediately from backwards induction by Theorem 3.3 (with base case $i = r$ and $N_r = \{e\}$).

4. **Preparatory constructions for the proof of Theorem 3.3**

4.1. *Lifting property.* We retain all notation appearing in Theorem 3.3. Recall that N_i/Λ_i has the structure of a fiber bundle over N_{i+1}/Λ_{i+1} (with fiber $Z_i/(\Lambda_i \cap Z_i)$ isomorphic to $\mathbb{T}^{d_i}$). By construction, $P_{i+1}\colon M \to N_{i+1}/\Lambda_{i+1}$ lifts to $P_i\colon M \to N_i/\Lambda_i$. Write $p_{i,i+1}\colon N_i/\Lambda_i \to N_{i+1}/\Lambda_{i+1}$ and $\tilde{p}_{i,i+1}\colon N_i \to N_{i+1}$ for the natural projection maps. As we assume h_{i+1} is homotopic to P_{i+1}, by the lifting property of fiber bundles we may find a continuous $\phi\colon M \to N_i/\Lambda_i$ such that

(1) ϕ is homotopic to P_i;
(2) ϕ is a lift of h_{i+1};
(3) the homotopy from ϕ to P_i factors through $p_{i,i+1}$ to the homotopy from h_{i+1} to P_{i+1}.

In particular, as h_{i+1} intertwines the linear and nonlinear Γ-actions, we have equality of maps from $M \to N_{i+1}/\Lambda_{i+1}$,

$$p_{i,i+1}(\phi(\alpha(\gamma)(x))) = \rho_{i+1}(\gamma)(p_{i,i+1} \circ \phi(x)) \tag{4.1}$$

for all $\gamma \in \Gamma$. Our goal in proving Theorem 3.3 will be to correct ϕ so that (4.1) remains valid without the projection factor $p_{i,i+1}$.

Applying the homotopy lifting property to the bundle $\tilde{M} \to M$ we may select a distinguished lift $\tilde{\phi}\colon M \to N_i$ such that $\tilde{\phi}$ is Λ_M-equivariantly homotopic to $\tilde{P}_i$. Note that $\tilde{p}_{i,i+1} \circ \tilde{\phi}$ is a lift $p_{i,i+1} \circ \phi = h_{i+1}$. Moreover, the image of the homotopy from $\tilde{\phi}$ to $\tilde{P}_i$ under $\tilde{p}_{i,i+1}$ is a lift of the homotopy from h_{i+1} to P_{i+1}. Since $\tilde{p}_{i,i+1} \circ \tilde{P}_i = \tilde{P}_{i+1}$, it follows that $\tilde{p}_{i,i+1} \circ \tilde{\phi} = \tilde{h}_{i+1}$. In particular, for $\tilde{\phi}$, we have

$$\tilde{p}_{i,i+1} \circ \tilde{\phi} \circ \tilde{\alpha}(\gamma) = \rho_{i+1}(\gamma) \circ \tilde{p}_{i,i+1} \circ \tilde{\phi} \tag{4.2}$$

and $\tilde{\phi}(x \cdot \lambda) = \tilde{\phi}(x) \cdot P_{*,i}(\lambda)$.

4.2. *Suspension spaces.* Recall that, as α is assumed to lift to $\tilde{\alpha}\colon \Gamma \to \mathrm{Homeo}(\tilde{M})$, we have an action $\alpha_*\colon \Gamma \to \mathrm{Aut}(\Lambda_M)$. We define the (right) semi-direct product $\Gamma \ltimes_{\alpha_*} \Lambda_M$ by

$$(\gamma, \lambda) \cdot (\bar{\gamma}, \bar{\lambda}) = (\gamma\bar{\gamma}, \alpha_*(\bar{\gamma}^{-1})(\lambda)\bar{\lambda}).$$

We let $\Gamma \ltimes_{\alpha_*} \Lambda_M$ act on $G \times \tilde{M}$ on the right by

$$(g, x) \cdot (\gamma, \lambda) = (g\gamma, [\alpha(\gamma^{-1})(x)]\lambda).$$

We similarly define the (right) semi-direct product $\Gamma \ltimes_\rho \Lambda_i$ by

$$(\gamma, \lambda) \cdot (\bar{\gamma}, \bar{\lambda}) = (\gamma\bar{\gamma}, \rho_i(\bar{\gamma}^{-1})(\lambda)\bar{\lambda})$$

acting on $G \times N_i$ by

$$(g, n) \cdot (\gamma, \lambda) = (g\gamma, n\rho_i(g\gamma)(\lambda)).$$

We remark that the asymmetry in the actions is intentional.

We have right Γ- and Λ_M-actions (respectively Γ- and Λ_i-actions) on $G \times \tilde{M}$ (resp. $G \times N_i$) induced by the natural embeddings of Γ and Λ_M into $\Gamma \ltimes_{\alpha_*} \Lambda_M$ (resp. Γ and Λ_i into $\Gamma \ltimes_\rho \Lambda_i$).

$P_{*,i}\colon \Lambda_M \to \Lambda_i$ can be extended to

$$\Psi\colon \Gamma \ltimes_{\alpha_*} \Lambda_M \to \Gamma \ltimes_\rho \Lambda_i$$

by

$$\Psi(\gamma, \lambda) = (\gamma, P_{*,i}(\lambda)). \tag{4.3}$$

We check that Ψ defines a homomorphism.

Recall that we have a continuous representation $\rho\colon G \to \mathrm{Aut}(N)$ that in turn descends to $\rho_i\colon G \to \mathrm{Aut}(N_i)$. We define left G-actions on $G \times \tilde{M}$ and $G \times N_i$ by

$$a \cdot (g, x) = (ag, x), \qquad a \cdot (g, n) = (ag, \rho_i(a)n)$$

for all $a \in G, g \in G, x \in M$ and $n \in N_i$. (Again the asymmetry in the definitions is intentional.)

As the left and right actions defined above commute, we obtain left G-actions on the quotient spaces:

(1) $M^\alpha := G \times M/\Gamma \ltimes_{\alpha_*} \Lambda_M$;
(2) $(N_i/\Lambda_i)_\rho := G \times N_i/\Gamma \ltimes_\rho \Lambda_i$.

Here, the upper subscript denotes the standard suspension space construction. The lower subscript denotes a twisted *Lyapunov suspension* space.

Remark 4.1. We use the twisted Lyapunov suspension $(N_i/\Lambda_i)_\rho$ in this and the next section as the hyperbolicity of the left G-action on the fibers is best observed through this construction.

However, the standard suspension of ρ acting on N_i/Λ_i has the advantage that it can be viewed as a homogeneous space, which we will use in Proposition 6.5 below. Indeed, consider the semi-direct product $G \ltimes_\rho N_i$ given by

$$(g, n) \cdot (\bar{g}, \bar{n}) = (g\bar{g}, \rho_i(\bar{g}^{-1})(n)\bar{n}).$$

Then $\Gamma \ltimes_\rho \Lambda_i$ is a subgroup of $G \ltimes_\rho N_i$ and acts on the right as

$$(g, n) \cdot (\gamma, \lambda) = (g\gamma, \rho_i(\gamma^{-1})(n)\lambda).$$

For $a \in G$, we have

$$(a, e) \cdot (g, n) = (ag, n)$$

inducing a left G-action that commutes with the right action of $\Gamma \ltimes_\rho \Lambda_i$. We then obtain a natural G-action on the homogeneous space $(N_i/\Lambda_i)^\rho := (G \ltimes_\rho N_i)/(\Gamma \ltimes_\rho \Lambda_i)$.

Let $\tilde{\Upsilon}\colon G \ltimes_\rho N_i \to G \times N_i$ be given by

$$\tilde{\Upsilon}(g, n) = (g, \rho_i(g)(n)).$$

We claim that Υ intertwines left G-actions and right $(\Gamma \ltimes_\rho \Lambda_i)$-actions and hence induces a continuous

$$\Upsilon\colon (N_i/\Lambda_i)^\rho \to (N_i/\Lambda_i)_\rho$$

intertwining G-actions. Thus the two suspension spaces are equivalent.

We remark that the use of suspension spaces and the equivalence between them has a long history; see, e.g., [Zim84].

4.3. *Approximate conjugacy.* We extend the map $\tilde{\phi}$ constructed above to a Ψ-equivariant map $\Phi\colon G \times \tilde{M} \to G \times N_i$ that intertwines the G-actions up to a defect that we will later correct. This will in turn induce a semiconjugacy between the G-actions on M^α and $(N_i/\Lambda_i)_\rho$.

Fix a right-invariant Riemannian metric d_G on G. This induces a metric $d_{G/\Gamma}$ on G/Γ. For the remainder, we fix a *Dirichlet fundamental domain* for Γ; that is, let $D \subset G$ be a fundamental domain for Γ such that

(1) D contains an open neighborhood of the identity e;

(2) D contains an open dense subset of full Haar measure;

(3) if $g \in D$, then $d(g, e) = \min_{\gamma\in\Gamma} d(g, \Gamma) = d_{G/\Gamma}(g\Gamma, \Gamma)$.

We will frequently use the following standard fact.

LEMMA 4.2. *If Γ is a lattice in a semisimple Lie group G, then*

$$d_{G/\Gamma}(g\Gamma, \Gamma) \in L^1(G/\Gamma, \mathrm{m}_G).$$

Note that this is equivalent to saying that $d(g, e)$ is in $L^1(D, \mathrm{m}_G)$. Indeed, this can be achieved by choosing a fundamental domain D that is contained in a Siegel set. See, for instance, [FM09, Prop. 3.17].

Let D be the Dirichlet fundamental domain fixed above. Given $g \in G$, let $\gamma_g \in \Gamma$ be the unique element with $g\gamma_g^{-1} \in D$; that is, $g \in D\gamma_g$. For $g \in G$, define $\tilde{\phi}_g \colon \tilde{M} \to N_i$ by

$$\tilde{\phi}_g(x) = \rho_i(g\gamma_g^{-1})\tilde{\phi}(\tilde{\alpha}(\gamma_g)(x)). \tag{4.4}$$

Define $\Phi \colon G \times \tilde{M} \to G \times N_i$ by $\Phi(g,x) = (g, \tilde{\phi}_g(x))$. Note that the kernel of $\tilde{p}_{i,i+1}$ is the center of N_i and is necessarily preserved by $\rho_i(g)$ for every g. It follows that

$$\rho_{i+1}(g) \circ \tilde{p}_{i,i+1} = \tilde{p}_{i,i+1} \circ \rho_i(g).$$

In particular, for $g \in G$, we have

$$\tilde{p}_{i,i+1} \circ \tilde{\phi}_g = \rho_{i+1}(g) \circ \tilde{h}_{i+1}. \tag{4.5}$$

Indeed,

$$\begin{aligned}
\tilde{p}_{i,i+1} \circ \tilde{\phi}_g(x) &= \tilde{p}_{i,i+1} \circ \rho_i(g\gamma_g^{-1})\tilde{\phi}(\tilde{\alpha}(\gamma_g)(x)) \\
&= \rho_{i+1}(g\gamma_g^{-1})\tilde{h}_{i+1}(\tilde{\alpha}(\gamma_g)(x)) \\
&= \rho_{i+1}(g\gamma_g^{-1})\rho_{i+1}(\gamma_g)\tilde{h}_{i+1}(x) \\
&= \rho_{i+1}(g) \circ \tilde{h}_{i+1}(x).
\end{aligned}$$

We then have for any $a \in G$ that

$$\tilde{p}_{i,i+1} \circ \tilde{\phi}_{ag}(x) = \rho_{i+1}(a)(\tilde{p}_{i,i+1} \circ \tilde{\phi}_g(x)) \tag{4.6}$$

as

$$\tilde{p}_{i,i+1} \circ \tilde{\phi}_{ag} = \rho_{i+1}(a)\rho_{i+1}(g) \circ \tilde{h}_{i+1} = \rho_{i+1}(a)(\tilde{p}_{i,i+1} \circ \tilde{\phi}_g).$$

Our goal below will be to modify the family $\tilde{\phi}_g$ so that (4.6) holds without the projection term.

We claim that, for the map Ψ defined in (4.3),

LEMMA 4.3. *Φ is Ψ-equivariant*:

$$\Phi((g,x)\cdot(\gamma,\lambda)) = \Phi(g,x)\cdot\Psi(\gamma,\lambda).$$

In particular,

(1) $\tilde{\phi}_{g\gamma}(\tilde{\alpha}(\gamma^{-1})(x)) = \tilde{\phi}_g(x)$;
(2) $\tilde{\phi}_g(x\lambda) = \tilde{\phi}_g(x)\rho_i(g)(P_{*,i}(\lambda))$.

Proof. Note that $\gamma_{g\gamma} = \gamma_g\gamma$. We then have

$$\begin{aligned}
\Phi\left((g,x)\cdot(\gamma,\lambda)\right) &= \Big(g\gamma, \tilde{\phi}_{g\gamma}\Big(\tilde{\alpha}(\gamma^{-1})(x)\lambda\Big)\Big) \\
&= \Big(g\gamma, \rho_i((g\gamma)\gamma_{g\gamma}^{-1})\tilde{\phi}\left(\tilde{\alpha}(\gamma_{g\gamma})\Big(\tilde{\alpha}(\gamma^{-1})(x)\lambda\Big)\right)\Big) \\
&= \Big(g\gamma, \rho_i(g\gamma_g^{-1})\tilde{\phi}\left(\tilde{\alpha}(\gamma_{g\gamma})(\tilde{\alpha}(\gamma^{-1})(x))\alpha_*(\gamma_{g\gamma})(\lambda)\right)\Big) \\
&= \Big(g\gamma, \Big[\rho_i(g\gamma_g^{-1})\tilde{\phi}\Big(\tilde{\alpha}(\gamma_g)(x)\Big)\Big]\,\rho_i(g\gamma_g^{-1})P_{*,i}\left(\alpha_*(\gamma_{g\gamma})(\lambda)\right)\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \big(g\gamma, \big[\rho_i(g\gamma_g^{-1})\tilde{\phi}\big(\tilde{\alpha}(\gamma_g)(x)\big)\big]\, \rho_i(g\gamma_g^{-1})\rho_i(\gamma_{g\gamma})\,(P_{*,i}(\lambda)))\big) \\
&= \big(g\gamma, \big[\rho_i(g\gamma_g^{-1})\tilde{\phi}\big(\tilde{\alpha}(\gamma_g)(x)\big)\big]\, \rho_i(g\gamma)\,(P_{*,i}(\lambda))\big) \\
&= \big(g, \tilde{\phi}_g(x)\big) \cdot \Psi(\gamma, \lambda). \qquad \square
\end{aligned}
$$

Note that Φ is only a Borel measurable function. However, $\tilde{\phi}_g \colon \tilde{M} \to N_i$ is defined and continuous for *every* $g \in G$. Moreover, from Lemma 4.3, for each $g \in G$ the map $\tilde{\phi}_g$ factors to a map $\phi_g \colon M \to N_i/\,(\rho_i(g)\Lambda)$. In particular, $\tilde{\phi}_g$ is uniformly continuous for each $g \in G$.

4.3.1. *Central defect.* Recall that the center Z_i of N_i is the kernel of $\tilde{p}_{i,i+1} \colon N_i \to N_{i+1}$. Let $\mathfrak{z}_i$ denote the Lie algebra of Z_i.

Recall the Cartan subalgebra $\mathfrak{a} \subset \mathfrak{g}$ defined in Section 2.5, and let A be the analytic subgroup of G associated with $\mathfrak{a}$. By condition (4) of Theorem 3.2, for some $a \in A$, $S = D\rho(a)\restriction_{T_eN} \in \mathrm{Aut}(\mathfrak{n}_i)$ is a hyperbolic matrix. Therefore, the restriction of S to $\mathfrak{z}_i$ is hyperbolic.

We fix such a distinguished element a from now on. Let E^s and E^u be the stable and unstable subspaces for the restriction of S to $\mathfrak{z}_i$.

From (4.6) it follows that, given $g \in G$ and $x \in \tilde{M}$, there are unique vectors $\psi^s(g,x) \in E^s$ and $\psi^u(g,x) \in E^u$ such that

$$
\rho_i(a)\tilde{\phi}_g(x) = \tilde{\phi}_{ag}(x)\exp(\psi^s(g,x))\exp(\psi^u(g,x)), \tag{4.7}
$$

where $\exp \colon \mathfrak{n}_i \to N_i$ is the Lie-exponential map.

LEMMA 4.4. *For $\sigma = s, u$, the map $G \times \tilde{M} \to E^\sigma$, given by $(g,x) \to \psi^\sigma(g,x)$, is $\Gamma \ltimes_{\alpha_*} \Lambda_M$-invariant.*

Proof. We have

$$
a \cdot \Phi(g,x) = (ag, \rho_i(a)\tilde{\phi}_g(x)) = (ag, \tilde{\phi}_{ag}(x)\exp(\psi^s(g,x))\exp(\psi^u(g,x))).
$$

Moreover, since the left and right actions commute, repeatedly using Lemma 4.3 we have

$$
\begin{aligned}
a \cdot \Phi&\left(g\gamma, \tilde{\alpha}(\gamma^{-1})(x)\lambda\right) \\
&= a \cdot (\Phi(g,x) \cdot \Psi(\gamma,\lambda)) \\
&= \big(ag, \tilde{\phi}_{ag}(x)\exp(\psi^s(g,x))\exp(\psi^u(g,x))\big) \cdot \Psi(\gamma,\lambda) \\
&= \big(ag\gamma, \tilde{\phi}_{ag}(x)\exp(\psi^s(g,x))\exp(\psi^u(g,x)))\rho_i(ag\gamma)(P_{*,i}(\lambda))\big) \\
&= \big(ag\gamma, \tilde{\phi}_{ag}(x)\rho_i(ag\gamma)(P_{*,i}(\lambda))\exp(\psi^s(g,x))\exp(\psi^u(g,x)))\big) \\
&= \big(ag\gamma, \tilde{\phi}_{ag}(x)\rho_i(ag)(P_{*,i}(\alpha_*(\gamma)(\lambda)))\exp(\psi^s(g,x))\exp(\psi^u(g,x)))\big) \\
&= \big(ag\gamma, \tilde{\phi}_{ag}(x\alpha_*(\gamma)(\lambda))\exp(\psi^s(g,x))\exp(\psi^u(g,x)))\big)
\end{aligned}
$$

$$= \left(ag\gamma, \tilde{\phi}_{ag\gamma}(\tilde{\alpha}(\gamma^{-1})(x\alpha_*(\gamma)(\lambda))) \exp(\psi^s(g,x)) \exp(\psi^u(g,x)))\right)$$
$$= \left(ag\gamma, \tilde{\phi}_{ag\gamma}([\tilde{\alpha}(\gamma^{-1})(x)]\lambda) \exp(\psi^s(g,x)) \exp(\psi^u(g,x))\right).$$

It follows that

$$\rho_i(a)\tilde{\phi}_{g\gamma}([\tilde{\alpha}(\gamma^{-1})(x)]\lambda) = \tilde{\phi}_{ag\gamma}([\tilde{\alpha}(\gamma^{-1})(x)]\lambda) \exp(\psi^s(g,x)) \exp(\psi^u(g,x)). \quad \square$$

4.3.2. *Subexponential growth of central defects.* In this part we overcome the possible nonboundedness of the functions ψ^σ near the cusp of G/Γ, where $\sigma = s, u$, by showing they grow subexponentially along orbits which is sufficient for our construction. Readers who primarily think of cocompact lattices may ignore the technical discussion below.

Fix any norm on $\mathfrak{n}$. By the invariance in Lemma 4.4, the maps $(g,x) \to \psi^\sigma(g,x)$ descend to maps on the suspension space $M^\alpha \to E^\sigma$. In particular, as $M = \tilde{M}/\Lambda_M$ is compact, for every $g \in G$, the functions $\|\psi^\sigma(g,x)\|$ are bounded in x . The main technical obstruction to building the conjugacy is that (as G/Γ is not assumed compact) the functions $\|\psi^\sigma(g,x)\|$ need not be bounded in (g,x).

Let

$$C^\sigma(g) := \max_{x\in\tilde{M}} \|\psi^\sigma(g,x)\|.$$

For $\gamma \in \Gamma$, the above invariance gives $C^\sigma(g) = C^\sigma(g\gamma)$. In particular, the functions $C^\sigma(g)$ descend to functions on G/Γ.

LEMMA 4.5. *For $\sigma \in \{s,u\}$. we have*

$$\int_{G/\Gamma} \log^+(C^\sigma(g\Gamma))\ d(g\Gamma) < \infty.$$

Proof. Recall our fundamental domain D. We show $\int_D \log^+(C^\sigma(g))\, dg < \infty$. Let $\psi(g,x) = \psi^s(g,x) + \psi^u(g,x)$. For $g \in D$, we have $\gamma_g = e$ and

$$\begin{aligned}
\exp(-\psi(g,x)) :&= (\rho_i(a)(\tilde{\phi}_g(x)))^{-1}(\tilde{\phi}_{ag}(x)) \\
&= \rho_i(a)(\rho_i(g\gamma_g^{-1})\tilde{\phi}(\alpha(\gamma_g)(x))^{-1}(\rho_i(ag\gamma_{ag}^{-1})\tilde{\phi}(\tilde{\alpha}(\gamma_{ag})(x))) \\
&= \rho_i(ag)\left(\rho_i(\gamma_g^{-1})(\tilde{\phi}(\alpha(\gamma_g)(x))^{-1})(\rho_i(\gamma_{ag}^{-1})\tilde{\phi}(\tilde{\alpha}(\gamma_{ag})(x)))\right) \\
&= \rho_i(ag)\left(\tilde{\phi}(x)^{-1}(\rho_i(\gamma_{ag}^{-1})\tilde{\phi}(\tilde{\alpha}(\gamma_{ag})(x)))\right).
\end{aligned}$$

Let $F = \{\gamma_\ell\}$ be a the finite set of generators for Γ fixed above, and write

$$\gamma_{ag} = \gamma_{\ell(1)}\gamma_{\ell(2)}\cdots\gamma_{\ell(n(g))},$$

where $n(g)$ is the word-length of γ_{ag} relative to the generators $\{\gamma_\ell\}$. From (4.2), for each $x \in \tilde{M}$, we have

$$\rho_i(\gamma_\ell)^{-1}\tilde{\phi}(\tilde{\alpha}(\gamma_\ell)(x))) = \tilde{\phi}(x)z$$

for some $z = z_{\gamma_\ell}(x) \in Z_i$. Moreover, the function z_{γ_ℓ} is Λ_M-invariant, hence there is a uniform constant $D_\ell > 0$ such that

$$\|\exp^{-1}(z_{\gamma_\ell})\| \leq D_\ell.$$

Then for each x, we have a sequence $z_j \in Z_i$ for $1 \leq j \leq n(g)$ with $\|\exp^{-1}(z_j)\| \leq D_{\ell(j)}$ and

$$\begin{aligned}
&\rho_i(\gamma_{ag}^{-1})\tilde{\phi}(\tilde{\alpha}(\gamma_{ag})(x))) \\
&\quad := \rho_i(\gamma_{\ell(n(g))})^{-1}\rho_i(\gamma_{\ell(n(g)-1)})^{-1}\cdots\rho_i(\gamma_{\ell(1)})^{-1}\left(\tilde{\phi}(\tilde{\alpha}(\gamma_{\ell(1)})\cdots\tilde{\alpha}(\gamma_{\ell(n(g))})(x))\right) \\
&\quad = \rho_i(\gamma_{\ell(n(g))})^{-1}\rho_i(\gamma_{\ell(n(g)-1)})^{-1}\cdots\rho_i(\gamma_{\ell(2)})^{-1}\left(\tilde{\phi}(\tilde{\alpha}(\gamma_{\ell(2)})\cdots\tilde{\alpha}(\gamma_{\ell(n(g))})(x))z_1\right) \\
&\quad = \rho_i(\gamma_{\ell(n(g))})^{-1}\rho_i(\gamma_{\ell(n(g)-1)})^{-1}\cdots\rho_i(\gamma_{\ell(3)})^{-1} \\
&\qquad\quad \left(\tilde{\phi}(\tilde{\alpha}(\gamma_{\ell(3)})\cdots\tilde{\alpha}(\gamma_{\ell(n(g))})(x))\rho_i(\gamma_{\ell(2)})^{-1}(z_1)z_2\right) \\
&\qquad \vdots \\
&\quad = \tilde{\phi}(x)\prod_{j=1}^{n(g)}\rho_i(\gamma_{\ell(n(g))})^{-1}\rho_i(\gamma_{\ell(n(g)-1)})^{-1}\cdots\rho_i(\gamma_{\ell(j+1)})^{-1}(z_j).
\end{aligned}$$

Let

- $S_\ell = D\rho_i(\gamma_\ell)\restriction_{T_eN_i}$;
- $S = D\rho_i(a)\restriction_{T_eN_i}$;
- $S_g = D\rho_i(g)\restriction_{T_eN_i}$;
- $C = \max\|S_\ell\|$;
- $D = \max D_\ell$.

Then, as

$$\begin{aligned}
\exp(-\psi(g,x)) &= \rho_i(ag)\left(\tilde{\phi}(x)^{-1}(\rho_i(\gamma_{ag}^{-1})\tilde{\phi}(\tilde{\alpha}(\gamma_{ag})(x)))\right) \\
&= \rho_i(ag)\left(\prod_{j=1}^{n(g)}\rho_i(\gamma_{\ell(n(g))})^{-1}\rho_i(\gamma_{\ell(n(g)-1)})^{-1}\cdots\rho_i(\gamma_{\ell(j+1)})^{-1}(z_j)\right),
\end{aligned}$$

we have

$$\|\psi(g,x)\| \leq \|S\|\|S_g\|n(g)C^{n(g)}D.$$

Note that (as $D\rho\colon G \to \mathrm{Aut}(\mathfrak{n})$ is a continuous representation) there is a constant C_1 with

$$\log\|D\rho(g)\| \leq C_1 d(g,e),$$

hence we have

$$\int_D \log\|S_g\| \leq C_1\int_D d_G(g\Gamma,\Gamma)\ dg\Gamma.$$

By Lemma 4.2, $\int_D \log\|S_g\| < \infty$.

Moreover, from (2.4),

$$\begin{aligned}\int_D n(g)\ dg &\leq \int_D Ad_G(e,\gamma_{ag}) + B\ dg \\ &\leq \int_D A\left[d_G(e,g) + d_G(g,ag) + d_G(ag,\gamma_{ag})\right] + B\ dg \\ &\leq \int_D A\left[d_G(e,g) + d_G(e,a) + d_G(ag\gamma_{ag}^{-1},e)\right] + B\ dg.\end{aligned} \tag{4.8}$$

From the choice of fundamental domain, we have

$$d_G(e,g) = d_G(\Gamma, g\Gamma)$$

and

$$d_G(ag\gamma_{ag}^{-1},e) = d_G(ag\gamma_{ag}^{-1}\Gamma,\Gamma) \leq d_G(g\Gamma,\Gamma) + d_G(ag\Gamma,g\Gamma),$$

hence

$$\int_D n(g)\ dg \leq \int_D 2A\left[d_G(g\Gamma,\Gamma) + d_G(ag\Gamma,g\Gamma)\right] + B\ dg,$$

and it follows that again from Lemma 4.2 that $\int_D n(g)\ dg$ is finite. The claim then follows. ☐

From Lemma 4.5 and standard tempering kernel arguments ([BP07, Lemma 3.5.7]) we immediately obtain the following.

Proposition 4.6. *For any $\varepsilon > 0$, there is a measurable, Γ-invariant function $L = L_\varepsilon\colon G \to [0,\infty)$ so that for almost every $g \in G$ and every $k \in \mathbb{Z}$,*
(1) $C^\sigma(g) \leq L(g)$;
(2) $L(a^k g) \leq e^{\varepsilon|k|}L(g)$.

5. Construction of semiconjugacy: proof of Theorem 3.3

In this section, we build a continuous semiconjugacy H between the left G-actions on $G\times\tilde{M}$ and $G\times N_i$. Moreover, the conjugacy will be Ψ-equivariant and hence descend to a semiconjugacy between left G-actions on M^α and $(N_i/\Lambda_i)_\rho$.

We first construct a measurable (with respect to Haar) function H intertwining the action of our distinguished $a \in A$. We then extend H to intertwine the actions of the centralizer of a and finally all of G. That H agrees almost everywhere with a continuous function will follow from construction and the fact that H intertwines the left G-actions.

5.1. *Semiconjugating the action of a.* In this section we first construct a measurable semiconjugacy for the action of a. The proof follows the same ideas as that of Franks' Theorem on Anosov diffeomorphism on tori. Readers who wish to get a quick understanding of the main idea without considering the more complicated setting we are dealing with may refer to Theorem 2.6.1 and the discussion on page 588 of [KH95] for a discussion of Franks' Theorem.

Recall that we write $S \in \operatorname{Aut}(\mathfrak{n})$ for $D\rho_i(a)$, where a is the distinguished element of A fixed in 4.3.1. Given $g \in G$, define a *formal conjugacy* by adding (central) correction terms to the family $\tilde{\phi}_g$:

(5.1)

$$h_g(x) := \tilde{\phi}_g(x) \exp\left(\sum_{k=1}^{\infty} S^{k-1}\Big(\psi^s(a^{-k}g,x)\Big)\right) \exp\left(-\sum_{k=0}^{\infty} S^{-k-1}\Big(\psi^u(a^k g,x)\Big)\right).$$

We check formally that

$$\rho_i(a)(h_g(x)) = h_{ag}(x).$$

Indeed,

$$\begin{aligned}
&\rho_i(a)(h_g(x))\\
=&\rho_i(a)\left[\tilde{\phi}_g(x) \exp\left(\sum_{k=1}^{\infty} S^{k-1}\Big(\psi^s(a^{-k}g,x)\Big)\right) \exp\left(-\sum_{k=0}^{\infty} S^{-k-1}\Big(\psi^u(a^k g,x)\Big)\right)\right]\\
=&\tilde{\phi}_{ag}(x)\exp(\psi^s(g,x))\exp(\psi^u(g,x))\\
&\qquad\cdot \exp\left(\sum_{k=1}^{\infty} S^{k}\Big(\psi^s(s^{-k}g,x)\Big)\right) \exp\left(-\sum_{k=0}^{\infty} S^{-k}\Big(\psi^u(a^k g,x)\Big)\right)\\
=&\tilde{\phi}_{ag}(x) \exp\left(\sum_{k=0}^{\infty} S^{k}\Big(\psi^s(a^{-k}g,x)\Big)\right) \exp\left(-\sum_{k=1}^{\infty} S^{-k}\Big(\psi^u(a^k g,x)\Big)\right)\\
=&\tilde{\phi}_{ag}(x) \exp\left(\sum_{\ell=1}^{\infty} S^{\ell-1}\Big(\psi^s(a^{-\ell}(ag),x)\Big)\right) \exp\left(-\sum_{\ell=0}^{\infty} S^{-\ell-1}\Big(\psi^u(a^\ell (ag),x)\Big)\right)\\
=&h_{ag}(x).
\end{aligned}$$

We say a family of maps $g \to h_g \colon \tilde{M} \to N_i$ parametrized by $g \in G$ is Ψ-*equivariant* if the map $G \times \tilde{M} \to G \times N_i$ defined by $(g,x) \to (g, h_g(x))$ is Ψ-equivariant.

Lemma 5.1. *There is a full measure set of $g \in G$ such that $h_g \colon \tilde{M} \to N_i$ is well defined, continuous, and Ψ-equivariant. Moreover, for such g,*

$$\tilde{p}_{i,i+1} \circ h_g = \rho_{i+1}(g)\tilde{h}_{i+1}.$$

Proof. Note that S has no eigenvalues of modulus 1. Taking $0 < \varepsilon < \min_\lambda \frac{|\log|\lambda||}{100}$ where λ runs over all eigenvalues of S, the claim holds for all g such that Proposition 4.6 holds. □

Define $H \colon G \times \tilde{M} \to G \times N_i$ by $H(g,x) = (g, h_g(x))$. Then H defines a measurable conjugacy between the actions of a: for almost every $g \in G$,

$$H(a \cdot (g,x)) = a \cdot H(g,x).$$

Here the action on the left-hand side is the one on $G \times \tilde{M}$ and the action on the right-hand side is the one on $G \times N_i$.

LEMMA 5.2. *The family of maps $g \to h_g$ is unique among the measurable family of Ψ-equivariant, continuous functions $\tilde{M} \to N_i$ with the property that $h_{ag}(x) = \rho_i(a)h_g(x)$ and $\tilde{p}_{i,i+1} \circ h_g = \rho_{i+1}(g)\tilde{h}_{i+1}$.*

Proof. Suppose $g \to \bar{h}_g$ is another such family. As $\tilde{p}_{i,i+1} \circ \bar{h}_g = \tilde{p}_{i,i+1} \circ h_g$, it follows that $\bar{h}_g(x) = h_g(x)\exp(\hat{\psi}(g,x))$ for some $\hat{\psi}\colon G \times \tilde{M} \to \mathfrak{z}_i$. Write $\hat{\psi}(g,x) = \hat{\psi}^s(g,x) + \hat{\psi}^u(g,x)$.

By the Ψ-equivariance of h_g and $\bar{h}_g$ and as $\hat{\psi}^s(g,x) \in \mathfrak{z}_i$ it follows that

$$\hat{\psi}^s((g,x)\cdot(\gamma,\lambda)) = \hat{\psi}^s(g,x).$$

Similarly, $\hat{\psi}^u((g,x)\cdot(\gamma,\lambda)) = \hat{\psi}^u(g,x)$. In particular, for almost every $g \in G$, the function $\hat{\psi}^\sigma$ descends to a continuous function from M to $\mathfrak{z}_i$. It follow that $\|\hat{\psi}^\sigma(g,x)\|$ is bounded uniformly in x for almost every g.

As the families h_g and $\bar{h}_g$ intertwine the dynamics we have, moreover, that

- $\hat{\psi}^s(g,x) = S^k\hat{\psi}^s\left(a^{-k}\cdot(g,x)\right)$,
- $\hat{\psi}^u(g,x) = S^{-k}\hat{\psi}^u\left(a^{k}\cdot(g,x)\right)$.

By Poincarè recurrence to sets on which $g \mapsto \max_{x\in\tilde{M}} \|\hat{\psi}^\sigma(g,x)\|$ is uniformly bounded, it follows that $\hat{\psi}^s(g,x) = 0 = \hat{\psi}^u(g,x)$ for almost every g and every $x \in M$. Hence $h_g = \bar{h}_g$ almost everywhere. □

5.2. *Extending the semiconjugacy to the centralizer of a.* Recall our distinguished $a \in A$ where $A \subset G$ is a maximal split Cartan subgroup. Let $C_G(a) \subset G$ denote the centralizer of a in G. Note, in particular, that $A \subset C_G(a)$. Moreover, every compact almost-simple factor of G is contained in $C_G(a)$.

PROPOSITION 5.3. *Let $\bar{a} \in C_G(a)$. Then for m_G-almost every $g \in G$,*

$$h_{\bar{a}g} = \rho_i(\bar{a})h_g.$$

Proof. Let $\bar{a} \in C_G(a)$. For $g \in G$, define $\bar{h}_g := \rho_i(\bar{a}^{-1})h_{\bar{a}g}$. We check that $g \to \bar{h}_g$ defines a measurable family of Ψ-equivariant, continuous functions $\tilde{M} \to N_i$ with the property that $\tilde{p}_{i,i+1}(\bar{h}_g) = \rho_i(g)\tilde{h}_{i+1}$. Moreover,

$$\begin{aligned}\bar{h}_{ag}(x) :&= \rho_i(\bar{a}^{-1})h_{\bar{a}ag} = \rho_i(\bar{a}^{-1})h_{a\bar{a}g} = \rho_i(\bar{a}^{-1})\rho_i(a)h_{\bar{a}g}\\ &= \rho_i(a)\rho_i(\bar{a}^{-1})h_{\bar{a}g} = \rho_i(a)\bar{h}_g(x).\end{aligned}$$

By Lemma 5.2, $\bar{h}_g = h_g$ for almost every g. □

It follows from standard ergodic theoretic constructions that there is a full measure subset X_0 of G such that — after modifying the family $g \to h_g$ on a set of measure zero — we have

$$\rho_i(a')h_g = h_{a'}(g)$$

for all $g \in X_0$ and all $a' \in C_G(a)$. In particular, the map $H\colon G \times \tilde{M} \to G \times N_i$ defines a measurable conjugacy between the left actions of $C_G(a)$ on $G \times \tilde{M}$ and on $G \times N_i$.

5.3. *Extension of the semiconjugacy to G.* Using that there are sufficiently many nonresonant roots (as defined in Definition 2.6), we show that H intertwines the full G-actions on $G \times \tilde{M}$ and $G \times N_i$ via the following proposition.

PROPOSITION 5.4. *Let ζ be a restricted root of $\mathfrak{g}$ that is nonresonant with the representation $D\rho_i$. Let $X \in \mathfrak{g}^\zeta$, and let $v = \exp(X)$. Then for almost every $g \in G$,*

$$h_{vg} = \rho_i(v)h_g.$$

Proof. As ζ is not positively proportional to any weight of ρ_i, we may find $a_1, a_2 \in A$ and a splitting $\mathfrak{z}_i = E \times F$ so that writing $S_j = D\rho(a_j)$,

(1) $\zeta(a_1) = \zeta(a_2) = 0$,
(2) $\|S_1\restriction_E\| < 1$,
(3) $\|S_2\restriction_F\| < 1$.

Note then that $a_i^{-1}va_i = v$ for $i \in \{1, 2\}$.

Recall that for almost every g and any $x \in \tilde{M}$, we have

$$\tilde{p}_{i,i+1} \circ \rho_i(v)h_g(x) = \tilde{p}_{i,i+1}h_{vg}(x).$$

Thus, given almost every $g \in G$ and any $x \in \tilde{M}$, there is a unique $\eta(g, x) \in \mathfrak{z}_i$ with

$$\rho_i(v)h_g(x) = h_{vg}(x)\exp(\eta(g, x)).$$

For almost every $g \in G$, h_g is continuous and descends to a function defined on M. It follows that $\|\eta(g, x)\|$ is bounded uniformly in x for almost every g. As the family h_g is Ψ-equivariant, we have $\eta(g, x) = \eta(g\gamma, \tilde{\alpha}(\gamma^{-1})(x))$, and hence the function $g \mapsto \max_{x \in \tilde{M}} \|\eta(g, x)\|$ is Γ-invariant. Fix $C > 0$, and let $B \subset X_0 \subset G/\Gamma$ be such that for $g \in B$, $\|\eta(g, x)\| < C$ for all x. Taking C sufficiently large we may ensure B has measure arbitrarily close to 1.

Now, consider $g \in B$ such that $a_j^{-k}g\Gamma \in B$ for $j = \{1, 2\}$ and infinitely many $k \in \mathbb{N}$. Note that as

$$\begin{aligned}
\rho_i(v)h_{a_jg}(x) &= \rho_i(v)\rho_i(a_j)h_g(x)\\
&= \rho_i(a_j)\left(\rho_i(v)h_g(x)\right)\\
&= \rho_i(a_j)\left(h_{vg}(x)\exp(\eta(g, x))\right)\\
&= h_{a_jvg}(x)\exp(S_j\eta(g, x))\\
&= h_{va_jg}(x)\exp(S_j\eta(g, x)),
\end{aligned}$$

we have $\eta(a_j g, x) = S_j \eta(g, x)$. Write $\eta(g, x) = \eta^E(g, x) + \eta^F(g, x)$. We then have

$$\eta^E(g, x) = S_1^k(\eta^E(a_1^{-k} g, x)), \qquad \eta^F(g, x) = S_2^k(\eta^F(a_2^{-k} g, x)).$$

It then follows that

$$\eta^E(g, x) = \eta^F(g, x) = 0,$$

proving the proposition. □

Recall that we assume that the representation $D\rho\colon G \to \mathrm{Aut}(\mathfrak{n})$ is weakly nonresonant with the adjoint representation. In particular, every $g \in G$ can be written as

$$g = \exp(X_1)\exp(X_2)\cdots\exp(X_\ell),$$

where each X_j is either a vector in $\mathfrak{g}^0$, or a vector in $\mathfrak{g}^\xi$ for some restricted root ξ that is nonresonant with the representation $D\rho_i$. Recall that for $X_j \in \mathfrak{g}^0$, $\exp(X_j)$ lies in $C_G(s)$.

It follows from Propositions 5.3 and 5.4 that, after modifying H on a set of measure zero, we have

$$g' \cdot H(g, x) = H(g' \cdot (g, x))$$

for almost every $g \in G$, every $x \in \tilde{M}$ and every $g' \in G$. As G acts transitively on itself,

$$g' \cdot H(g, x) = H(g' \cdot (g, x))$$

holds for *every* $g \in G$, every $x \in \tilde{M}$, and every $a \in G$. Now, fix g so that $h_g\colon \tilde{M} \to N_i$ is continuous. As $h_{g'g} = \rho(g')h_g$, it follows that h_g is continuous for every $g \in G$ and, moreover, the family $g \to h_g$ varies continuously in the parameter g whence $H\colon G \times \tilde{M} \to G \times N_i$ is continuous.

Finally, recall that the family h_g is Ψ-equivariant (whence the map $H\colon G \times \tilde{M} \to G \times N_i$ is Ψ-equivariant) and $\tilde{p}_{i,i+1} \circ h_g = \rho_{i+1}(g)\tilde{h}_{i+1}$. Indeed, these properties hold for almost every g and extend to every g by continuity. In particular, this shows

Corollary 5.5. *There is a continuous, Ψ-equivariant function $H\colon G \times \tilde{M} \to G \times N_i$ of the form $H(g, x) = (g, h_g(x))$ with $g' \cdot H(g, x) = H(g' \cdot (g, x))$ for any $g' \in G$.*

To complete the proof of Theorem 3.3 define $\tilde{h}_i\colon \tilde{M} \to N_i$ by

$$\tilde{h}_i := h_e.$$

Then, $\tilde{h}_i$ satisfies

(1) $\tilde{h}_i(x\lambda) = \tilde{h}_i(x)P_{*,i}(\lambda)$ for $\lambda \in \Lambda_M$;
(2) $\tilde{p}_{i,i+1} \circ \tilde{h}_i = \tilde{h}_{i+1}$;
(3) $\tilde{h}_i(\alpha(\gamma)(x)) = h_e(\alpha(\gamma)(x)) = h_\gamma(x) = \rho(\gamma)h_e(x) = \rho(\gamma)\tilde{h}_i(x)$ for all $\gamma \in \Gamma$.

Moreover, writing $\tilde{h}_i = \rho(g^{-1})h_g$ for some $g \in G$ such that h_g coincides with the family defined by (5.1), it follows that $\tilde{h}_i$ is Λ_M-equivariantly homotopic to $\tilde{\phi}$. Consequently, $\tilde{h}_i \colon \tilde{M} \to N_i$ descends to a function $h_i \colon M \to N_i/\Lambda_i$ satisfying the conclusions of the theorem.

Moreover, if $h \colon M \to N_i/\Lambda_i$ is any continuous function having a lift $\tilde{h} \colon \tilde{M} \to N_i$ as in the conclusion of the theorem, it follows that $\tilde{h}(x\lambda) = \tilde{h}(x)P_{*,i}(\lambda)$ and $\tilde{p}_{i,i+1}\tilde{h} = \tilde{h}_{i+1}$. That $h = h_i$ then follows from the uniqueness given in Lemma 5.2.

6. **Superrigidity, arithmeticity, and orbit closures**

In this section we collect a number of classical facts that will be used in the sequel to prove Theorems 1.3 and 1.7.

Theorem 1.3 follows from verification of the hypotheses of Theorem 3.1. To show (1) of Theorem 3.1, we use the superrigidity theorem of Margulis. Note that for a lattice Γ as in Hypothesis 1.2 and a linear representation ψ of Γ, the standard superrigidity theorem of Margulis ([Mar91, Th. IX.6.16], see also [Mor15, 16.1.4]) guarantees (if G is a simply connected real semisimple algebraic group or if the Zariski closure of $\rho(\Gamma)$ is center-free) that the linear representation $\rho\colon \Gamma \to \mathrm{GL}(d, \mathbb{R})$ extends on a finite-index subgroup to a continuous representation $\rho\colon G \to \mathrm{GL}(d, \mathbb{R})$ up to a compact correction. We use the arithmeticity theorem of Margulis and the arithmetic lattice version of superrigidity below to ignore the compact correction by replacing G with a compact extension.

The proof of Theorem 1.7 will require two additional facts that we also present here: the superrigidity theorem for cocycles due to Zimmer and the classification of orbit closures for the action of the linear data ρ associated to an action α. Such classification follows from the orbit closure classification theorem of Ratner.

6.1. *Arithmeticity and superrigidity.* Let G be a connected, semisimple Lie group with finite center, and let $\Gamma \subset G$ be a lattice. For this section, we make the following standing assumption:

(6.1) Γ has dense image in every $\mathbb{R}$-rank 1, almost-simple factor of G.

In particular, (6.1) holds under Hypothesis 1.2.

Let G' denote the quotient of G by the maximal compact normal subgroup, and let Γ' be the image of Γ in G'. Then the projection to Γ' has finite kernel in Γ. Moreover, (G', Γ') still satisfies (6.1).

Let $\mathfrak{g}'$ denote the Lie algebra of G'. Note that G' acts on $\mathfrak{g}'$ via the adjoint representation. Let $G^* = \mathrm{Ad}\, G' \subset \mathrm{GL}(\mathfrak{g}')$ denote the image of G'. Let Γ^* denote the image of Γ' in G^*. Γ^* is a lattice in G^*.

Let $\psi\colon \Gamma \to \mathrm{GL}(d,\mathbb{Q})$ be a linear representation. As $\psi(\Gamma)$ is a finitely generated subgroup of $\mathrm{GL}(d,\mathbb{R})$, it contains a finite-index, torsion-free, normal subgroup ([Rag72, Th. 6.11]). Restricting to a finite-index subgroup of Γ, we may assume $\psi(\Gamma)$ has no torsion. On the other hand, note that the kernel of $\Gamma \to \Gamma^*$ is finite. It then follows that $\psi(\gamma)$ is a torsion element for every γ in the kernel of $\Gamma \to \Gamma^*$. Thus we may assume the representation $\psi\colon \Gamma \to \mathrm{GL}(d,\mathbb{Q})$ factors through a representation $\psi^*\colon \Gamma^* \to \mathrm{GL}(d,\mathbb{Q})$.

Note that G^* is a semisimple Lie group without compact factors and with trivial center. In this case $G^* = \mathbf{G}^*(\mathbb{R})^\circ$ for a semisimple algebraic group $\mathbf{G}^*$. Moreover, $\Gamma^* \subset G^*$ is a lattice whose projection to every rank-one factor is dense.

Theorem 6.1 (Margulis Arithmeticity Theorem [Mar91]). *Let $\mathbf{G}^*$ be a semisimple algebraic Lie group defined over $\mathbb{R}$ and Γ^* a lattice in $G^* = \mathbf{G}^*(\mathbb{R})^\circ$ that satisfies hypothesis* (6.1). *Then Γ^* is arithmetic*: *there exist a connected semisimple algebraic group $\mathbf{H}$ defined over $\mathbb{Q}$ and a surjective algebraic morphism $\phi\colon \mathbf{H} \to \mathbf{G}^*$ defined over $\mathbb{R}$, such that*

(1) *ϕ is a quotient morphism between algebraic groups, and the surjective morphism $\phi : \mathbf{H}(\mathbb{R})^\circ \to G^*$ is continuous with compact kernel*;
(2) *$\phi(\mathbf{H}(\mathbb{Z}) \cap \mathbf{H}(\mathbb{R})^\circ)$ is commensurable with Γ^*.*

The case where Γ^* is irreducible follows from [Mar91, introduction, Theorem 1′], where irreducible lattices is defined as in [Mar91, p. 133]. In general, $\mathbf{G}$ decomposes as an almost direct product $\prod_i \mathbf{G}_i^*$ of normal subgroups defined over $\mathbb{R}$, and there are irreducible lattices $\Gamma_i^* < G_i^* = \mathbf{G}_i^*(\mathbb{R})^\circ$ such that $\prod \Gamma_i$ has finite index in Γ. Then each pair (G_i, Γ_i) satisfies (6.1). This reduces to the irreducible case.

By passing to a finite cover we may assume the group $\mathbf{H}$ in Theorem 6.1 is simply connected as an algebraic group. Let $H = \mathbf{H}(\mathbb{R})^\circ$. Let $\Delta = \phi^{-1}(\Gamma^*) \cap \mathbf{H}(\mathbb{Z})$. Then Δ is commensurable with $\mathbf{H}(\mathbb{Z}) \cap H$ and hence $\mathbf{H}(\mathbb{Z})$. The map ϕ induces a linear representation $\bar\psi\colon \Delta \to \mathrm{GL}(d,\mathbb{Q})$ given by $\bar\psi = \psi \circ \phi$.

As we assume $\mathbf{H}$ is simply connected, $\mathbf{H}$ decomposes uniquely as the direct product of simply connected almost $\mathbb{Q}$-simple, $\mathbb{Q}$-groups $\mathbf{H} = \prod \mathbf{H}_i$. Moreover, since ϕ has compact kernel and G^* has no compact simple factors, our assumption (6.1) implies every $\mathbb{Q}$-simple factor $\mathbf{H}_i$ has $\mathbb{R}$-rank at least 2.

We have the following version of Margulis Superrigidity.

Theorem 6.2 (Margulis Superrigidity Theorem; arithmetic lattice case [Mar91]). *Let $\mathbf{H}$ be a simply connected semisimple algebraic group defined over $\mathbb{Q}$, all of whose almost $\mathbb{Q}$-simple factors have $\mathbb{R}$-rank* 2 *or higher, and let $\Delta \subset \mathbf{H}(\mathbb{R})$ be a subgroup commensurable to $\mathbf{H}(\mathbb{Z})$. Suppose k is a field of characteristic* 0, *$\mathbf{J}$ is an algebraic group defined over k, and $\psi\colon \Delta \to \mathbf{J}(k)$ is* *a group morphism. Then there are a k-rational morphism $\psi'\colon \mathbf{H}\to\mathbf{J}$ and a finite-index normal subgroup $\Delta'\lhd\Delta$ such that $\psi(\delta)=\psi'(\delta)$ for all $\delta\in\Delta'$.*

When $\mathbf{H}$ is almost $\mathbb{Q}$-simple, this is a direct consequence of [Mar91, VIII, Th. B]. See also [Mor15, Cor. 16.4.1]. In general, $\mathbf{H}$ is the direct product of its almost $\mathbb{Q}$-simple factors $\mathbf{H}=\prod\mathbf{H}_i$ and $\prod\Delta_i$ has finite index in Δ where each Δ_i is defined by $\Delta_i=\Delta\cap\mathbf{H}_i(\mathbb{R})$ and is commensurable with $\mathbf{H}_i(\mathbb{Z})$. On each Δ_i, ψ coincides with a $\mathbb{Q}$-representation ψ'_i of $\mathbf{H}_i$ on a finite-index subgroup. Via projection to $\mathbb{Q}$-simple factors, the ψ'_i assemble into a coherent representation ψ'; moreover, ψ' coincides with ψ on a finite-index subgroup of Δ.

Replace Δ with Δ' coming from Theorem 6.2. Recall that we have projections $p_1\colon G\to G^*$ and $p_2\colon H\to G^*$. Consider the Lie group

$$\bar{L}:=\{(g,h)\in G\times H : p_1(g)=p_2(h)\}.$$

Let L be the identity component of $\bar{L}$, and let $\hat\Gamma\subset L$ be

$$\hat\Gamma:=\{(\gamma,\delta)\in\Gamma\times\Delta : p_1(\gamma)=p_2(\delta)\}\cap L.$$

We have natural maps $L\to G$ and $L\to H$ given by coordinate projections. Because $G\to G^*$ and $H\to G^*$ are surjective, so are $L\to G$ and $L\to H$. Moreover, the kernel of $L\to G$ is the set $\{(e,h) : p_2(h)=e\}$. As the kernel of p_2 is compact, L is a compact extension of G. Moreover, the image of $\hat\Gamma$ in G has finite index in Γ. Thus $\hat\Gamma$ is a lattice in L. Restricting to a finite-index subgroup, we may assume $\hat\Gamma$ maps into Γ and Δ.

Summarizing the above we have the following.

Proposition 6.3. *For G a connected, semisimple Lie group with finite center, $\Gamma\subset G$ a lattice satisfying* (6.1), *an algebraic group $\mathbf{J}$ defined over a field k with* $\mathrm{char}(k)=0$, *and any group morphism $\psi\colon\Gamma\to\mathbf{J}(k)$, there are*

(1) *semisimple Lie groups G^*, H and L, such that G^* has trivial center, and $H=\mathbf{H}(\mathbb{R})^\circ$ for some simply connected semisimple algebraic group $\mathbf{H}$ defined over $\mathbb{Q}$;*
(2) *a finite-index subgroup $\bar\Gamma\subset\Gamma$;*
(3) *lattices $\hat\Gamma\subset L, \Delta\subset H$, and $\Gamma^*\subset G^*$;*
(4) *surjective homomorphisms $\pi_1\colon L\to G, \pi_2\colon L\to H, p_1\colon G\to G^*$, and $p_2\colon H\to G^*$, all of which have compact kernels;*
(5) *a representation $\psi^*\colon\Gamma^*\to\mathbf{J}(k)$ such that $\psi\restriction_{\bar\Gamma}=\psi^*\circ p_1$;*
(6) *a k-rational morphism $\psi'\colon\mathbf{H}\to\mathbf{J}$*

such that the following diagrams commute:

$$\begin{tikzcd} L \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & H \arrow[r, "\psi'"] \arrow[d, "p_2"] & \mathbf{J} \\ G \arrow[r, "p_1"'] & G^* & \end{tikzcd} \qquad \begin{tikzcd} \hat{\Gamma} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & \Delta \arrow[r, "\psi'"] \arrow[d, "p_2"] & \mathbf{J}(k). \\ \bar{\Gamma} \arrow[r, "p_1"'] & \Gamma^* \arrow[ur, "\psi^*"'] & \end{tikzcd}$$

6.2. *Cocycle superrigidity.* The superrigidity theorem (with compact error) for representations admits a generalization due to Zimmer for linear cocycles over measure-preserving actions of higher-rank groups. We state here a version that will be sufficient for our later purposes.

Theorem 6.4 (Zimmer Cocycle Superrigidity Theorem [Zim84], [FM03]). *Let $\mathbf{G}$ be a simply connected semisimple algebraic group defined over $\mathbb{R}$ all of whose almost-simple factors have $\mathbb{R}$-rank 2 or higher. Let $G = \mathbf{G}(\mathbb{R})^\circ$, and let $\Gamma \subset G$ be a lattice. Let (X, μ) be a standard probability space and let $\alpha\colon \Gamma \to \mathrm{Aut}(X, \mu)$ be an ergodic action by measure-preserving transformations. Let $\psi\colon \Gamma \times X \to \mathrm{GL}(d, \mathbb{R})$ be a measurable cocycle over α such that for any γ, $\log^+ \|\psi(\gamma, x)\|$ is in $L^1(X, \mu)$. Then there exist a representation $\psi'\colon G \to \mathrm{GL}(d, \mathbb{R})$, a cocycle $\beta\colon \Gamma \times X \to \mathrm{SO}(d)$, and a measurable map $\theta\colon X \to \mathrm{GL}(d, \mathbb{R})$ such that*

(1) $\psi(\gamma, x) = \theta(\alpha(\gamma)(x))\psi'(\gamma)\beta(\gamma, x)(\theta(x))^{-1}$ *for every γ and μ-almost every x;*
(2) *for all $g \in G$, $\gamma \in \Gamma$ and μ-almost every x, $\psi'(g)$ and $\beta(\gamma, x)$ commute.*

The version of Zimmer Cocycle Superrigidity given in Theorem 6.4 was proved by Fisher and Margulis in [FM03, Th. 1.4]. Note that in the statement of Theorem 1.7, we do not assume that the action α preserves a measure. However, because in this setting the semiconjugacy h between α and the linear data ρ is a conjugacy, we are able to induce α-invariant measures from ρ-invariant measures.

6.3. *Orbit closures for linear data.* We present here a fact that will be important in Section 8.3. We assume that G and Γ are as in Hypothesis 1.2, $M = N/\Lambda$ is a compact nilmanifold, $\alpha\colon \Gamma \to \mathrm{Homeo}(M)$ is an action that lifts to an action on N, and $\rho\colon \Gamma \to \mathrm{Aut}(N/\Lambda)$ is the linear data. We recall all of the notation from Proposition 6.3. In particular, there are a Lie group L, a lattice $\hat{\Gamma} \subset L$, a surjective homomorphism $\pi\colon L \to G$ with $\pi(\hat{\Gamma}) \subset \Gamma$ with finite index, and a continuous representation $\psi\colon L \to \mathrm{Aut}(\mathfrak{n})$ with $D\rho(\pi(\hat{\gamma})) = \psi(\hat{\gamma})$ for all $\hat{\gamma} \in \Gamma$. Then ψ extends to a representation $\hat{\rho}\colon L \to \mathrm{Aut}(N)$ such that $\hat{\rho}(\hat{\gamma}) = \rho(\pi(\hat{\gamma}))$ for $\hat{\gamma} \in \hat{\Gamma}$.

Let $M^{\hat{\rho}}$ denote the suspension space $(L \ltimes_{\hat{\rho}} N)/(\hat{\Gamma} \ltimes_{\hat{\rho}} \Lambda)$ discussed in Remark 4.1 of in Section 4.2. As remarked there, $\hat{\Gamma} \ltimes_{\hat{\rho}} \Lambda$ is a lattice in $L \ltimes_{\hat{\rho}} N$. Let

$L' \subset L$ be the connected subgroup generated by all noncompact, almost-simple factors. Note that L' is generated by unipotent elements of L. Moreover, if $u \in L$ is a unipotent element of L, then (u, e) is unipotent in $(L \ltimes_{\hat\rho} N)$. It follows that the natural embedding $L' \subset (L \ltimes_{\hat\rho} N)$ is generated by unipotent elements.

From the orbit classification theorem of Ratner [Rat95, Th. 11], it follows that the L'-orbit closure of any point in $M^{\hat\rho}$ is a homogeneous submanifold. As L is a compact extension of L', it similarly follows that the L-orbit closure of any point in $M^{\hat\rho}$ is a homogeneous submanifold. Note that the closures of L-orbits on $M^{\hat\rho}$ are in one-to-one correspondence with closures of $\hat\rho(\hat\Gamma)$-orbits on N/Λ. It follows that all $\hat\rho(\hat\Gamma)$-orbit closures on N/Λ are $\hat\rho(\hat\Gamma)$-invariant, homogeneous submanifolds. Note that the Haar measure on every $\hat\rho(\hat\Gamma)$-invariant, homogeneous submanifold is $\hat\rho(\hat\Gamma)$-invariant. Moreover, if $\hat\rho(\hat\gamma_0)$ is hyperbolic for some $\hat\gamma_0$, then each of these measures are ergodic.

Since $\pi(\hat\Gamma)$ is of finite index in Γ, we have the following.

Proposition 6.5. *For Γ and ρ as above and any $x \in N/\Lambda$, the orbit closure $\overline{\rho(\Gamma)(x)}$ is the finite union of homogeneous sub-nilmanifolds of the same dimension. In particular, every orbit closure $\overline{\rho(\Gamma)(x)}$ coincides with the support of a ρ-invariant probability measure μ_x on N/Λ. Moreover, if $\rho(\gamma_0)$ is hyperbolic for some γ_0, then the measures μ_x are ergodic.*

7. **Topological rigidity for actions with hyperbolic linear data**

In this section we prove Theorem 1.3 by verifying the hypotheses of Theorem 3.1. Note that the Hölder continuity of h in Theorem 1.3 follows from standard arguments once the action α is by Lipschitz homomorphisms and the linear data is hyperbolic ([KH95, §19.1]).

7.1. *Verification of* (1) *of Theorem* 3.1. Let G and Γ be as in Hypothesis 1.2. Given a nilmanifold $M = N/\Lambda$, let $\alpha\colon \Gamma \to \mathrm{Homeo}(M)$ be an action. We assume α lifts to an action $\tilde\alpha\colon \Gamma \to \mathrm{Homeo}(N)$ and let $\rho\colon \Gamma \to \mathrm{Aut}(M) \subset \mathrm{Aut}(N)$ be the induced linear data. Identifying $\mathfrak{n}$ with $\mathbb{R}^d$, the derivative of ρ induces a linear representation $\psi = D\rho\colon \Gamma \to \mathrm{Aut}(\mathfrak{n}) \subset \mathrm{GL}(d, \mathbb{R})$. We remark that $\mathrm{Aut}(\mathfrak{n})$ is a real algebraic group.

Let $\bar\Gamma, L, \hat\Gamma, \Delta, \pi_1, \pi_2, \psi', \psi^*$ be as in Proposition 6.3, with $k = \mathbb{R}$ and $\mathbf{J} = \mathrm{Aut}(\mathfrak{n})$. As $\pi_1(\hat\Gamma) \subset \bar\Gamma$, we have an induced action $\hat\alpha\colon \hat\Gamma \to \mathrm{Homeo}(M)$, by $\hat\alpha(\hat\gamma) = \alpha(\pi_1(\hat\gamma))$. The action $\hat\alpha$ lifts to an action on N, and the induced linear data is $\hat\psi(\hat\gamma) = \psi^*(p_1(\pi_1(\hat\gamma))) = \psi'(\pi_2(\hat\gamma))$. It follows that the linear data $\hat\psi$ of $\hat\alpha$ extends to a continuous representation $\hat\psi'\colon L \to \mathrm{Aut}(\mathfrak{n})$ given by $\hat\psi'(\ell) = \psi'(\pi_2(\ell))$. Via the exponential map, we have that $\hat\rho\colon \hat\Gamma \to \mathrm{Aut}(N)$ extends to $\hat\rho\colon L \to \mathrm{Aut}(N)$ by $\hat\rho(\ell) = \exp(\hat\psi'(\ell))$. Thus, after replacing $(G, \Gamma, \alpha, \rho)$ with $(L, \hat\Gamma, \hat\alpha, \hat\rho)$, (1) of Theorem 3.1 holds.

7.2. *All weights are nontrivial.* Since, in the previous section, the linear data $\hat{\rho}\colon \hat{\Gamma} \to \mathrm{Aut}(N)$ extends to a continuous $\hat{\rho}'\colon L \to \mathrm{Aut}(N)$ and $\hat{\rho}'$ factors through a $\rho'\colon H \to \mathrm{Aut}(N)$, to establish (4) of Theorem 3.1 for $(L, \hat{\rho})$, it is sufficient to show nontriviality of the weights of $D\rho' = \psi'$.

As $H = \mathbf{H}(\mathbb{R})^\circ$ is real algebraic, the Lie subalgebra $\mathfrak{a}$ is the lie algebra of a maximal $\mathbb{R}$-split torus, so the result follows from the following basic fact.

LEMMA 7.1. *Suppose $G = \mathbf{H}(\mathbb{R})^\circ$ for a semisimple algebraic group $\mathbf{H}$ defined over $\mathbb{R}$ and $\psi : \mathbf{H} \to \mathbf{GL}(d)$ is a $\mathbb{R}$-rational representation. Let $\mathbf{A}$ be a maximal $\mathbb{R}$-split torus in $\mathbf{H}$. Suppose $\psi(g)$ is hyperbolic for some $g \in G$. Then all restricted weights of χ of ψ with respect to $\mathbf{A}$ are nontrivial.*

Proof. The element g has a unique Jordan decomposition $g = g_s g_u = g_u g_s$, where g_s is semisimple and g_u is unipotent, and $g_s, g_u \in \mathbf{H}(\mathbb{R})$. Then $\psi(g) = \psi(g_s)\psi(g_u)$ is the Jordan decomposition of $\psi(g)$. Since $\psi(g)$ and $\psi(g_s)$ have the same eigenvalues, $\psi(g_s)$ is a hyperbolic matrix.

The semisimple element $g_s \in \mathbf{H}(\mathbb{R})$ belongs to a $\mathbb{R}$-torus $\mathbf{T}$. It follows that for all weights $\lambda \in X^*(\mathbf{T})$ of ψ, $|\lambda(g_s)| \neq 1$, where $X^*(\mathbf{T})$ denotes the group of characters of $\mathbf{T}$.

There is a unique decomposition $\mathbf{T} = \mathbf{T}_s \mathbf{T}_a$ into an $\mathbb{R}$-split torus $\mathbf{T}_s$ and an $\mathbb{R}$-anisotropic torus $\mathbf{T}_a$. We further decompose $g_s = g_{s,s} g_{s,a}$ with $g_{s,s} \in \mathbf{T}_s(\mathbb{R})$ and $g_{s,a} \in \mathbf{T}_a(\mathbb{R})$.

Write $\tilde{\lambda}(t) = \lambda(t)\overline{\lambda(\bar{t})}$; then $\tilde{\lambda}$ is a character of $\mathbf{T}$ defined over $\mathbb{R}$ and is hence trivial on $\mathbf{T}_a$. Then $\tilde{\lambda}(g_{s,s}) = \tilde{\lambda}(g_s) = |\lambda(g_s)|^2 \neq 1$. Moreover, $\lambda|_{\mathbf{T}_s}$ is defined over $\mathbb{R}$ because $\mathbf{T}_s$ is split, and thus $\tilde{\lambda}(g_{s,s}) = \lambda^2(g_{s,s})$. So $\lambda(g_{s,s})$, which is real, is not equal to ± 1. This shows $\psi(g_{s,s})$ is a hyperbolic matrix.

On the other hand, $\mathbf{T}_s$ is contained in some maximal $\mathbb{R}$-split torus $\mathbf{A}'$. It follows that all restricted weights χ of $\mathbf{A}'$ are nontrivial. As all maximal $\mathbb{R}$-split tori are $\mathbf{H}(\mathbb{R})$-conjugate, the lemma follows. $\square$

7.3. *Proof of Theorem* 1.3. As discussed above, (1) and (4) of Theorem 3.1 hold for $(L, \hat{\Gamma}, \hat{\alpha}, \hat{\rho})$. (2) of Theorem 3.1 follows immediately from Theorem 2.5. Once we know that all weights of the representation given by the linear data are nontrivial, (3) of Theorem 3.1 follows immediately from Lemma 2.7. Theorem 3.1 then gives that a map h intertwining the actions $\hat{\alpha}$ and $\hat{\rho}$. Since $\hat{\alpha}$ and $\hat{\rho}$ factor through the restriction of the actions α and ρ to a finite-index subgroup $\bar{\Gamma} \subset \Gamma \subset G$, the same h intertwines $\alpha\restriction_{\bar{\Gamma}}$ and $\rho\restriction_{\bar{\Gamma}}$, and Theorem 1.3 follows.

8. **Smooth rigidity for Anosov actions**

In this section we prove Theorem 1.7. Our approach uses many of the same ideas as [KLZ96].

8.1. *Reductions and proof of Theorem* 1.7. Let G and Γ satisfy Hypothesis 1.2, and let α be as in Theorem 1.7. Let ρ be the linear data of α. Note that if $\alpha(\gamma)$ is Anosov, then $D\rho(\gamma)$ is hyperbolic. Replacing Γ with a finite-index subgroup $\Gamma_1 \subset \Gamma$, we may assume from Theorem 1.3 that there is a (Hölder) continuous $h\colon M \to M$ intertwining the actions α and ρ. Recall that we assume $\alpha(\gamma_0)$ is Anosov for some $\gamma_0 \in \Gamma$. Taking a power, it follows that $\alpha(\gamma_1)$ is Anosov for some $\gamma_1 \in \Gamma_1$. By Manning's Theorem [Man74], h is a bi-Hölder homeomorphism (see also [KH95, §18.6]); in particular, from the linear data ρ and h, we recover the nonlinear action α of Γ_1.

We recall the notation and constructions from Section 7.1. In particular, there are a center-free, semisimple Lie group G^* without compact factors, a continuous surjective morphism $p_1\colon G \to G^*$, finite-index subgroup $\bar\Gamma \subset \Gamma_1$, and representation $\rho^*\colon \Gamma^* \to \mathrm{Aut}(N)$ where $\Gamma^* = p_1(\bar\Gamma)$ such that $\rho\restriction_{\bar\Gamma} = \rho^* \circ p_1$. Since the linear data $\rho\restriction_{\bar\Gamma}$ uniquely determines the nonlinear action $\alpha\restriction_{\bar\Gamma}$, it follows that α factors through an action $\tilde\alpha^*$ of Γ^*: $\tilde\alpha^*(\gamma^*) = \alpha(p_1^{-1}(\gamma^*))$.

Recall that the Lie group G^* is a real algebraic group; that is, $G^* = \mathbf{G}^*(\mathbb{R})^\circ$ for a semisimple algebraic group $\mathbf{G}^*$ defined over $\mathbb{R}$. Let $\tilde{\mathbf{G}}$ be the algebraically simply connected cover of $\mathbf{G}^*$. Then $\tilde G = \tilde{\mathbf{G}}(\mathbb{R})^\circ$ is a finite cover of G^*, whence $\tilde G$ has finite center and no compact factors. Let $\tilde\Gamma$ be the lift of Γ^* to $\tilde G$. The projection $\tilde\Gamma \to \Gamma^*$ induces an action $\tilde\alpha$ of $\tilde\Gamma$ by C^∞ diffeomorphisms of M; moreover, the action $\tilde\alpha$ lifts to an action by diffeomorphisms of N and induces linear data $\tilde\rho$ that factors through ρ^*.

Note that the map h guaranteed by Theorem 1.3 intertwining the actions of α and ρ also intertwines the actions $\tilde\rho$ and $\tilde\alpha$. It is therefore sufficient to prove Theorem 1.7 under the following stronger hypotheses.

Hypothesis 8.1. *Suppose* $\mathbf{G}$ *is a simply connected semisimple algebraic group defined over* $\mathbb{R}$, *all of whose* $\mathbb{R}$*-simple factors have* $\mathbb{R}$*-rank* 2 *or higher*, $G = \mathbf{G}(\mathbb{R})^\circ$, *and* $\Gamma \subset G$ *is a lattice. Suppose* α *is an action of* Γ *by* C^∞ *diffeomorphisms of a nilmanifold* $M = N/\Lambda$ *that lifts to an action by diffeomorphisms of* N, *and* $\rho\colon \Gamma \to \mathrm{Aut}(M)$ *is the associated linear data. Suppose* $h\colon M \to M$ *is a homeomorphism such that* $h \circ \alpha(\gamma) = \rho(\gamma) \circ h$ *for all* $\gamma \in \Gamma$.

Assuming Hypothesis 8.1, the proof of Theorem 1.7 proceeds by studying the restriction of α and ρ to an appropriately chosen finitely generated discrete higher-rank abelian subgroup $\Sigma \subset \Gamma$.

We recall the following definition.

Definition 8.2. For an abelian group Σ and two actions $\rho\colon \Sigma \to \mathrm{Aut}(M)$, $\rho'\colon \Sigma \to \mathrm{Aut}(M')$ by nilmanifold automorphisms, we say ρ' is an *algebraic factor action* of ρ, if there is an algebraic factor map $\pi\colon M \to M'$ such that $\pi \circ \rho(g) = \rho'(g) \circ \pi$ for all $g \in \Sigma$. We further say ρ' is a *rank-one algebraic* *factor action* of ρ if in addition there is a finite-index subgroup $\Sigma' < \Sigma$ such that the image group $\rho'(\Sigma') < \mathrm{Aut}(M')$ is cyclic.

We remark that it follows from [RHW14, Lemma 2.9] that if there exists a finite-index subgroup $\Sigma' \subset \Sigma$ such that $\rho\restriction_{\Sigma'}$ has a rank-one algebraic factor, then ρ has a rank-one factor.

The following proposition is the main result of this section.

PROPOSITION 8.3. *Let G, Γ, α, and ρ be as in Hypothesis* 8.1. *Suppose $\alpha(\gamma_0)$ is an Anosov diffeomorphism for some $\gamma_0 \in \Gamma$. Then there exists a free abelian subgroup $\Sigma \subset \Gamma$ such that $\rho|_\Sigma$ has no rank-one algebraic factor actions and $\alpha(\gamma_1)$ is Anosov for some $\gamma_1 \in \Sigma$.*

Having found an appropriate $\Sigma \subset \Gamma$, Theorem 1.7 follows from the following proposition. We recall that for an action α of a discrete abelian group Σ by diffeomorphisms of a nilmanifold with an Anosov element, there is always an abelian action, called the *linearization* of α, by affine nilmanifold transformations. Moreover, these two actions are conjugate. The main result of [RHW14] shows this conjugacy is smooth.

THEOREM 8.4 ([RHW14]). *Let α be a C^∞ action by a discrete abelian group Σ on a nilmanifold M, and let ρ be its linearization. Suppose that ρ has no rank-one algebraic factor action and $\alpha(\gamma_1)$ is Anosov for some $\gamma_1 \in \Sigma$. Then α is conjugate to ρ by a C^∞ diffeomorphism that is homotopic to the identity.*

Let G, Γ, α, and ρ be as in Hypothesis 8.1. Suppose $\alpha(\gamma_0)$ is an Anosov diffeomorphism for some $\gamma_0 \in \Gamma$. Let $\Sigma \subset \Gamma$ and $\gamma_1 \in \Sigma$ be as in Proposition 8.3. Note that the conjugacy h in Hypothesis 8.1 guarantees that the linearization of $\alpha\restriction_\Sigma$ coincides with the restriction of the linear data ρ of α to Σ. By Theorem 8.4, there is a C^∞ diffeomorphism $h' \colon M \to M$ homotopic to the identity that intertwines $\alpha\restriction_\Sigma$ and the linearization of $\alpha\restriction_\Sigma$. Furthermore, there is a lift of h' intertwining the lifts of $\alpha(\gamma_1)$ and $\rho(\gamma_1) \in \mathrm{Aut}(N)$. From the uniqueness criterion of semiconjugacies, it follows that h coincides with h' and hence is C^∞. Theorem 1.7 follows immediately.

In the remainder of this section, we prove Proposition 8.3. First, starting from one Anosov element, we produce a large Zariski dense semigroup of Γ that acts by Anosov diffeomorphisms. By works of Prasad and Rapinchuk [PR03], [PR05], generic elements of this semigroup will have centralizers of rank equal to $\mathrm{rank}_{\mathbb{R}}(G)$. Finally, we show that the restriction of ρ to such generic centralizers will not have rank-one factors using arithmeticity of the representation $\rho : \Gamma \mapsto \mathrm{Aut}(\mathfrak{n})$. This reduces the problem to the global smooth rigidity property of Anosov actions without rank-one factors by higher-rank abelian groups, which is known by [RHW14].

8.2. *The semigroup of Anosov elements.* We recall the setting of Proposition 8.3. Let $\gamma_0 \in \Gamma$ be such that $\alpha(\gamma_0)$ is an Anosov diffeomorphism. Let $E^{\sigma}_{\gamma_0}(x)$, $x \in M$, $\sigma = s, u$, be the stable and unstable bundles for $\alpha(\gamma_0)$. Given $\varepsilon > 0$ and $\sigma = s, u$, let $C^{\sigma}_{\varepsilon}(x)$ be the ε-cone around $E^{\sigma}_{\gamma_0}(x)$; that is, for $\sigma = u$, decomposing $v = v^s + v^u$ with respect to the splitting $E^s_{\gamma_0}(x) \oplus E^u_{\gamma_0}(x)$ we have $v \in C^u_{\varepsilon}(x)$ if and only if $|v^s| \leq \varepsilon |v^u|$.

Fix any $0 < \varepsilon \leq 1$. Let S be the set of all $\gamma \in \Gamma$ such that for every $x \in M$,

$$D_x \alpha(\gamma) C^u_{\varepsilon}(x) \subset C^u_{\frac{1}{2}\varepsilon}(\alpha(\gamma)(x))$$

and

$$D_x (\alpha(\gamma))^{-1} C^s_{\varepsilon}(x) \subset C^s_{\frac{1}{2}\varepsilon}((\alpha(\gamma))^{-1}(x))$$

(that is, $\alpha(\gamma)$ preserves the ε-stable and unstable cones) and, moreover, for every vector $v \in C^u_{\varepsilon}(x)$, $|D_x \alpha(\gamma) v| \geq 2|v|$ and for $v \in C^s_{\varepsilon}(x)$, $|D_x (\alpha(\gamma))^{-1} v| \geq 2|v|$.

We claim that S is Zariski dense:

PROPOSITION 8.5. *Suppose α, Γ, and γ_0 are as in the assumptions of Proposition* 8.3. *For every $0 < \varepsilon \leq 1$, the set S defined above is Zariski dense in G.*

Proof of Proposition 8.5. By Lemma 8.6 below, $\gamma_0^{N_0} \in S$ for some N_0. In order to show Zariski density, we may assume without loss of generality that $N_0 = 1$.

Let $\bar{S}$ be the Zariski closure of S. Then $\bar{S}$ is a group. By Proposition 8.7 below, for every $\eta \in \Gamma \cap W$, there is an N such that $\gamma_0^N \eta \gamma_0^N \in S \subset \bar{S}$. As $\gamma_0 \in S \subset \bar{S}$ and as $\bar{S}$ is a group, it follows that $\eta \in \bar{S}$. Hence $\Gamma \cap W \subset \bar{S}$. Since W is Zariski open in G and Γ is Zariski dense in G by Borel density theorem, $\Gamma \cap W \subset \bar{S}$ is Zariski dense in G. Since $\bar{S}$ is Zariski closed, $\bar{S} = G$ and thus S is Zariski dense. □

LEMMA 8.6. *In the setting of Proposition* 8.5, *the set S satisfies the following conditions*:

(1) *S is a semigroup*;
(2) *for every $\gamma \in S$, $\alpha(\gamma)$ is an Anosov diffeomorphism*;
(3) *for some $N_0 > 0$, $\gamma_0^{N_0} \in S$.*

Proof. That S is a semigroup is clear from definition. That $\gamma_0^{N_0} \in S$ is straightforward by choosing $N_0 > 0$ large enough. Condition (2) follows from standard cone estimates (see [KH95, §6.4]). □

PROPOSITION 8.7. *In the setting of Proposition* 8.5, *there is a Zariski open set $W \subset G$ such that for every $\eta \in \Gamma \cap W$, there is $N > 0$ such that $\gamma_0^N \eta \gamma_0^N \in S$.*

The proof of this proposition will occupy the next section.

8.3. *Proof of Proposition* 8.7. In this section, we follow and substantially extend the main argument in [KLZ96]. Recall that the derivative of the action $\alpha\colon \Gamma \to \mathrm{Diff}(M)$ induces a linear cocycle $D_x\alpha(\gamma)$ over the action α. Recall also that, as $\alpha(\gamma_0)$ is Anosov for some γ_0, the semiconjugacy $h\colon M \to N/\Lambda$ is a homeomorphism. The push-forward of the Haar measure on N/Λ under h^{-1} is then α-invariant and, moreover, coincides with the measure of maximal entropy for $\alpha(\gamma_0)$. By Theorem 6.4 (applied to the Jacobian-determinant cocycle) and Livsic's Theorem it follows that this measure is smooth. Denote this smooth α-invariant measure by m. Note that, as the linear data associated to $\alpha(\gamma_0)$ is hyperbolic, the Haar measure on N/Λ is ergodic whence m is ergodic for the action α. Fix a trivialization of $TM = M \times \mathfrak{n}$ and an identification $\mathfrak{n} = \mathbb{R}^d$. Identify $\mathfrak{n} = \mathbb{R}^d = V$ and $\mathrm{GL}(d,\mathbb{R}) = \mathrm{GL}(V) = \mathrm{GL}(\mathfrak{n})$ unless some confusion arises.

By Theorem 6.4, for each ergodic α-invariant measure μ, there are a measurable map $C^\mu\colon (M,\mu) \to \mathrm{GL}(d,\mathbb{R})$, a linear representation $D^\mu\colon G \to \mathrm{GL}(d,\mathbb{R})$, and a compact-group valued, measurable cocycle $K^\mu\colon \Gamma \times M \to \mathrm{SO}(d)$ such that

$$D_x\alpha(\gamma) = C^\mu(\alpha(\gamma)(x)) D^\mu(\gamma) K^\mu(\gamma,x)(C^\mu(x))^{-1} \tag{8.1}$$

and K^μ commutes with D^μ: for every $g \in G$, $\eta \in \Gamma$, and $x \in M$, $D^\mu(g)K^\mu = K^\mu(\eta,x)D^\mu(g)$.

Recall that γ_0 is the distinguished element with $\alpha(\gamma_0)$ Anosov. Fix an enumeration $\Gamma \smallsetminus \{\gamma_0\} = \{\gamma_1, \gamma_2, \dots\}$. For $j = 0,1,2,\dots$ let

$$\eta_j := \gamma_j \gamma_0 \gamma_j^{-1}.$$

Observe that $\alpha(\eta_j)$ is Anosov for every j. For $x \in M$, let $E^s_{\eta_j}(x)$ and $E^u_{\eta_j}(x)$ denote, respectively, the stable and unstable bundles for $\alpha(\eta_j)$ at the point x.

For $\sigma = s,u$, let $d^\sigma = \dim E^\sigma_{\gamma_0}(x)$. Note that $E^\sigma_{\eta_j}(x) = \left(D\alpha(\gamma_j)E^\sigma_{\gamma_0}\right)(x)$. For $g \in G$, let $E^{s,\mu}_g$ and $E^{u,\mu}_g$ denote, respectively, the stable and unstable spaces of the linear map $D^\mu(g)$. Note that $D^\mu(g)$ need not be hyperbolic whence the subspaces $E^{\sigma,\mu}_g$ may not be transverse. However, (as $K^\mu(\gamma,x)$ is compact-valued and commutes with D^μ) for μ-a.e. x, we have

$$E^\sigma_{\eta_i}(x) = C^\mu(x) E^{\sigma,\mu}_{\eta_i}.$$

For $r = 0,1,2,\dots$, let $S^{r,\mu} \subset \mathrm{GL}(V)$ be given by

$$S^{r,\mu} = \bigcap_{\substack{i=0,\dots,r-1\\ \sigma=s,u}} \mathrm{Stab}(E^{\sigma,\mu}_{\eta_i}) = \bigcap_{\substack{i=0,\dots,r-1\\ \sigma=s,u}} \mathrm{Stab}(D^\mu(\gamma_i)E^{\sigma,\mu}_{\gamma_0}).$$

Note that each $S^{r,\mu} = \mathbf{S}^{r,\mu}(\mathbb{R})$, where $\mathbf{S}^{r,\mu}$ is an algebraic group defined over $\mathbb{R}$. Moreover, $\mathbf{S}^{r+1,\mu} \subset \mathbf{S}^{r,\mu}$ and each $\mathbf{S}^{r,\mu}$ has finitely many components. Counting dimension and connected components it follows that there is a $r(\mu)$ so that $\mathbf{S}^{r,\mu} = \mathbf{S}^{r(\mu),\mu}$ for all $r \geq r(\mu)$. Let $\mathbf{S}^{\mu} = \mathbf{S}^{r(\mu),\mu}$ and $S^{\mu} = \mathbf{S}^{\mu}(\mathbb{R})$. We then have

$$S^{\mu} = \bigcap_{\substack{\gamma\in\Gamma\\ \sigma=s,u}} D^{\mu}(\gamma)\operatorname{Stab}(E^{\sigma,\mu}_{\gamma_0})D^{\mu}(\gamma)^{-1}.$$

It follows that $D^{\mu}(\Gamma)$ normalizes S^{μ} whence by Zariski density of Γ in G,

$$S^{\mu} = \bigcap_{\substack{g\in G\\ \sigma=s,u}} D^{\mu}(g)\operatorname{Stab}(E^{\sigma,\mu}_{\gamma_0})D^{\mu}(g)^{-1}.$$

For $\sigma = s,u$, denote by $\operatorname{Gr}(V,d^{\sigma})$ the Grassmannian of subspaces in V of dimension $\dim E^{\sigma,\mu}_{\gamma_0}$. Let

$$\Phi\colon \operatorname{GL}(V)\times\Big((\operatorname{Gr}(V,d^s))^{r(\mu)}\times(\operatorname{Gr}(V,d^u))^{r(\mu)}\Big) \to (\operatorname{Gr}(V,d^s))^{r(\mu)}\times(\operatorname{Gr}(V,d^u))^{r(\mu)}$$

be the natural action. As Φ is an algebraic action,

LEMMA 8.8. *Let $E \in (\operatorname{Gr}(V,d^s))^{r(\mu)}\times(\operatorname{Gr}(V,d^u))^{r(\mu)}$. Then* $\operatorname{Orb}_{\Phi}(E)$, *the orbit of E under Φ, is open in its Zariski closure* $\overline{\operatorname{Orb}_{\Phi}(E)}$.

Let $\tau\colon M \to (\operatorname{Gr}(V,d^s))^{r(\mu)}\times(\operatorname{Gr}(V,d^u))^{r(\mu)}$ be defined by

$$\tau(x) = \Big((E^s_{\eta_i}(x))_i,(E^u_{\eta_i}(x))_i\Big)_{i=0,\dots,r(\mu)}.$$

The map τ is continuous since the bundles $E^s_{\eta_j}(x)$ and $E^u_{\eta_j}(x)$ are the stable and unstable bundles of Anosov diffeomorphisms and are therefore continuous in x. As observed above, for μ-a.e. x

$$\tau(x) = \Phi\left(C^{\mu}(x),\Big((E^{s,\mu}_{\eta_i})_i,(E^{u,\mu}_{\eta_i})_i\Big)\right).$$

Let Orb^{μ} be the orbit under Φ of $\Big((E^{s,\mu}_{\eta_i})_i,(E^{u,\mu}_{\eta_i})_i\Big)_{i=0,\dots,r(\mu)}$. Then Φ induces a smooth parametrization

$$\phi^{\mu}\colon \operatorname{GL}(V)/S^{\mu} \to \operatorname{Orb}^{\mu}.$$

Since τ is continuous, we have that $\tau(\operatorname{supp}(\mu))$ is compact. By the Lemma 8.8, for a μ-.a.e. x, the orbit $\operatorname{Orb}_{\Phi}(\tau(x)) = \operatorname{Orb}^{\mu}$ is open in $\overline{\operatorname{Orb}^{\mu}} \supset \tau(\operatorname{supp}(\mu))$. Hence

$$U^{\mu} := \tau^{-1}(\operatorname{Orb}^{\mu})$$

is (relatively) open and dense in $\operatorname{supp}(\mu)$. Moreover, $\mu(U^{\mu}) = 1$. Via the parametrization ϕ^{μ}, we have a continuous map $\hat{C}^{\mu}\colon U^{\mu} \to \operatorname{GL}(V)/S^{\mu}$ given by

$$\hat{C}^{\mu}(x) := (\phi^{\mu})^{-1}\circ\tau(x).$$

Moreover, $C^{\mu}(x)S^{\mu} = \hat{C}^{\mu}(x)$ for μ-a.e. x.

LEMMA 8.9. *Let μ be an ergodic α-invariant measure. The set U^μ is α-invariant, and for every $\gamma \in \Gamma$, $x \in U^\mu$, and $\sigma = s, u$, we have*

$$D_x\alpha(\gamma)E^\sigma_{\gamma_0}(x) = D_x\alpha(\gamma)\hat{C}^\mu(x)E^{\sigma,\mu}_{\gamma_0} = \hat{C}^\mu(\alpha(\gamma)(x))D^\mu(\gamma)E^{\sigma,\mu}_{\gamma_0}.$$

Proof of Lemma 8.9. For μ-a.e. $x \in U^\mu$, we have that

$$\tau(\alpha(\gamma)(x)) = \Phi\left(D^\mu(\gamma), \tau(x)\right)$$

whence, for such x,

$$\alpha(\gamma)(x) \in \tau^{-1}\left(\Phi\left(D^\mu(\gamma), \tau(x)\right)\right). \tag{8.2}$$

Since $\tau \circ \alpha(\gamma)$ and $x \mapsto \Phi\left(D^\mu(\gamma), \tau(x)\right)$ are continuous and since U^μ is open and dense in $\operatorname{supp}(\mu)$, (8.2) holds for all $x \in U^\mu$. It follows that $\alpha(\gamma)(x) \in U^\mu$ for all $x \in U^\mu$.

For any $x \in M$ and $\gamma \in \Gamma$, we have that

$$D_x\alpha(\gamma)E^\sigma_{\gamma_0}(x) = E^\sigma_{\gamma\gamma_0\gamma^{-1}}(\alpha(\gamma)(x)).$$

Also,

$$D^\mu(\gamma)E^{\sigma,\mu}_{\gamma_0} = E^{\sigma,\mu}_{\gamma\gamma_0\gamma^{-1}}.$$

Recall that S^μ stabilizes each of the spaces $E^{\sigma,\mu}_{\gamma\gamma_0\gamma^{-1}}$ and that there is a measurable $C^\mu(x)$ with

$$C^\mu(x)E^{\sigma,\mu}_{\gamma\gamma_0\gamma^{-1}} = E^\sigma_{\gamma\gamma_0\gamma^{-1}}(x)$$

for μ-a.e. x. As the function $\hat{C}^\mu(x)$ and the bundles $D_x\alpha(\gamma)E^\sigma_{\gamma_0}(x)$ are continuous on U^μ, the result follows. □

We summarize the above with the following lemma.

LEMMA 8.10. *There are countably many ergodic, α-invariant probability measures μ_i, and relatively-open, relatively-dense, α-invariant sets $U_i \subset \operatorname{supp}(\mu_i)$, and continuous maps $\hat{C}^{\mu_i}\colon U_i \to \mathrm{GL}(V)/S^{\mu_i}$ such that*

(1) *M is the union $M = \bigcup_{i=0}^\infty U_i$;*
(2) *$C^{\mu_i}(x)S^{\mu_i} = \hat{C}^{\mu_i}(x)$ for μ_i-a.e. $x \in U_i$.*

Proof. We start with the smooth measure $\mu_0 = m$ and the corresponding open set $U_0 = U^m$ as constructed above. The image of U_0 under the conjugacy h is a ρ-invariant, open dense subset of the nilmanifold M. It follows that the complement of $h(U_0)$ coincides with the boundary of $h(U_0)$ and is a closed, ρ-invariant set. In particular, the complement of $h(U_0)$ is saturated by orbit closures. From Proposition 6.5 and using that U_0 is dense in M, the complement of $h(U_0)$ is a countable union

$$N/\Lambda \smallsetminus h(U_0) = \bigcup V_i,$$

where each V_i is a finite union of ρ-invariant sub-nilmanifolds of N/Λ of dimension at most $d-1$ where $d = \dim N$. Moreover, each V_i coincides with the support of an ergodic, ρ-invariant ν^{V_i}. We note (as $\rho(\gamma)$ is hyperbolic for some γ) that there are a countable number of such V_i.

Let $\mu^{V_i} := (h^{-1})_*\nu^{V_i}$. For each μ^{V_i}, we may repeat the above procedure and obtain sets U_{V_i} such that $h(U_{V_i})$ is open and dense in V_i. As U_{V_i} is α-invariant and V_i is the finite union of submanifolds of dimension at most $d-1$, it follows that $V_i \smallsetminus h(U_{V_i})$ is a countable union

$$V_i \smallsetminus h(U_{V_i}) = \bigcup W_j,$$

where each W_j is a finite union of ρ-invariant sub-nilmanifolds of N/Λ of dimension at most $d-2$.

Proceeding recursively, we define a countable collection of ergodic α-invariant measures μ_i with corresponding sets U_i. That every $x \in M$ is contained in a U_i follows as the dimension of the complement decreases at each step of recursion. $\square$

The set W appearing in Proposition 8.7 is defined as the set W in the following lemma.

Lemma 8.11. *Let μ_i be the measures in Lemma* 8.10. *Let W be the set of all $g \in G$ such that $D^{\mu_i}(g)E^{s,\mu_i}_{\gamma_0}$ is transverse to $E^{u,\mu_i}_{\gamma_0}$ and $D^{\mu_i}(g)E^{u,\mu_i}_{\gamma_0}$ is transverse to $E^{s,\mu_i}_{\gamma_0}$ for every i. Then W is a nonempty Zariski open set in G.*

Proof. Up to conjugation, there are only finitely many representations of G into $\mathrm{GL}(d, \mathbb{R})$. In particular, up to conjugation there are only finitely many values of $D^{\mu_i}(g)$ and $E^{s,\mu_i}_{\gamma_0}$. Since $\alpha(\gamma_0)$ is Anosov, for every μ_i, $E^{s,\mu_i}_{\gamma_0}$ is transverse to $E^{u,\mu_i}_{\gamma_0}$. Then W is the finite intersection of Zariski open sets indexed by conjugacy classes of representations $D^{\mu_i}(g)$. $\square$

With the above lemmas we show that the set W satisfies Proposition 8.7 via the following proposition.

Proposition 8.12. *Let f be an Anosov diffeomorphism with splitting $T_xM = E^s(x) \oplus E^u(x)$, and let $0 < \varepsilon \le 1$. Let g be a diffeomorphism, and assume for every $x \in M$ that $D_xgE^s(x)$ intersects $E^u(g(x))$ transversally and that $D_xgE^u(x)$ intersects $E^s(g(x))$ transversally. Then there is $N > 0$ such that for every $n \ge N$, writing $F = f^n \circ g \circ f^n$, for every $x \in M$, we have*

$$D_xFC^u_\varepsilon(x) \subset C^u_{\frac{1}{2}\varepsilon}(F(x))$$

and

$$D_xF^{-1}C^s_\varepsilon(x) \subset C^s_{\frac{1}{2}\varepsilon}(F^{-1}(x)).$$

Moreover, for every vector $v \in C^u_\varepsilon(x)$, we have $|D_xFv| \ge 2|v|$, and for $v \in C^s_\varepsilon(x)$, we have $|D_xF^{-1}v| \ge 2|v|$.

Recall that here $C^\sigma_\varepsilon(x)$ denotes the ε-cone around $E^\sigma(x)$. The proof is a standard argument. We include it here for completeness.

Proof of Proposition 8.12. By symmetry it is sufficient to prove the result for the unstable cone. We write all linear transformations with respect to the continuous splitting $T_xM = E^s(x) \oplus E^u(x)$. Then,

$$D_xf = \begin{pmatrix} A(x) & 0 \\ 0 & B(x) \end{pmatrix}$$

and

$$D_xg = \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix},$$

where $a(x)\colon E^s(x) \to E^s(g(x))$, $b(x)\colon E^u(x) \to E^s(g(x))$, $c(x)\colon E^s(x) \to E^u(g(x))$, and $d(x)\colon E^u(x) \to E^u(g(x))$. As $D_xgE^u(x)$ and $E^s(g(x))$ are transverse, it follows that $d(x)$ is invertible for every x. By continuity of the bundle $E^u(x)$, continuity of the derivative D_xg, and compactness of M, there is $r > 0$ such that $m(d(x)) \geq r$ for every $x \in M$. Here $m(L)$ denotes the co-norm of a linear map $m(L) = \|L^{-1}\|^{-1}$. Let $C = \max_{x\in m} \|D_xg\|$. Observe that

$$D_xf^n = \begin{pmatrix} A^{(n)}(x) & 0 \\ 0 & B^{(n)}(x) \end{pmatrix},$$

where $A^{(n)}(x) = A(f^{n-1}(x)) \cdots A(x)$. Choose a norm on TM adapted to f; that is, decomposing $v = v^s + v^u$ according to the splitting $T_xM = E^s(x) \oplus E^u(x)$ we have $|v| = \max\{|v^u|, |v^s|\}$, and there is a constant $\lambda < 1$ such that for every $x \in M$ and $n \geq 0$, $\|A^{(n)}(x)\| \leq \lambda^n$ and $m(B^{(n)}(x)) \geq \lambda^{-n}$.

Let $\varepsilon > 0$. The N in the proposition will depend on r, C, λ, and ε. We first show for N sufficiently large that DF preserves the ε-unstable cone. First observe that for every positive real number $t > 0$, natural number $n \geq 0$, and $x \in M$, we have that

$$Df^nC^u_t(x) \subset C^u_{\lambda^{2n}t}(f^n(x)).$$

In the next lemma, the transversality between DgE^u and E^s is used; in particular, we use that for r as defined above, $r > 0$.

Lemma 8.13. *There are $\delta_0 > 0$ and $T > 0$ such that for every $0 < \delta \leq \delta_0$, we have that*

$$D_xgC^u_\delta(x) \subset C^u_T(g(x)).$$

Given Lemma 8.13, for N sufficiently large and $n \geq N$,

$$\begin{aligned} D_xFC^u_\varepsilon(x) &= D_x(f^n \circ g \circ f^n)C^u_\varepsilon(x) = D_{f^n(x)}(f^n \circ g)D_xf^nC^u_\varepsilon(x) \\ &\subset D_{f^n(x)}(f^n \circ g)C^u_{\lambda^{2n}\varepsilon}(f^n(x)) \\ &= D_{g(f^n(x))}f^nD_{f^n(x)}gC^u_{\lambda^{2n}\varepsilon}(f^n(x)) \\ &\subset D_{g(f^n(x))}f^nC^u_T(g(f^n(x))) \subset C^u_{\lambda^{2n}T}(f^n(g(f^n(x)))) \\ &= C^u_{\lambda^{2n}T}(F(x)). \end{aligned}$$

Choosing N large enough so that $\lambda^{2N}\varepsilon \le \delta_0$ and $\lambda^{2N}T \le \frac{1}{2}\varepsilon$, it follows that F preserves the ε-unstable cones.

We now consider the growth of the vectors. Let $v \in C^u_\varepsilon(x)$. Recall that $|v| = \max\{|v^s|, |v^u|\}$, hence $|v| = |v^u|$. Since DF preserves the ε-unstable cone, we have that $|(D_F v)^s| \le \frac{\varepsilon}{2}|(D_F v)^u|$ and, since $\varepsilon \le 1$, we have that $|D_F v| = |(D_F v)^u|$. Now,

$$(D_F v)^u = B^{(n)}(g(f^n(x)))[c(f^n(x))A^{(n)}(x)v^s + d(f^n(x))B^{(n)}(x)v^u]$$

so

$$\begin{aligned}|D_F v| &= |(D_F v)^u| \ge \lambda^{-n}|c(f^n(x))A^{(n)}(x)v^s + d(f^n(x))B^{(n)}(x)v^u| \\ &\ge \lambda^{-n}[r\lambda^{-n}|v^u| - C\lambda^n\varepsilon|v^u|] \\ &= \lambda^{-n}(r\lambda^{-n} - C\lambda^n\varepsilon)|v|.\end{aligned}$$

Take N large enough such that $\lambda^{-n}(r\lambda^{-n} - C\lambda^n\varepsilon) \ge 2$ for all $n \ge N$. □

Proof of Lemma 8.13. Take a vector $v = v^s + v^u$ in $C^u_{\delta_0}(x)$ (where δ_0 will be determined later). Then

$$D_x g v = (a(x)v^s + b(x)v^u) + (c(x)v^s + d(x)v^u).$$

We prove that

$$|a(x)v^s + b(x)v^u| \le T|c(x)v^s + d(x)v^u|$$

for some $T > 0$. Note that

$$|a(x)v^s + b(x)v^u| \le (C\delta_0 + C)|v^u| = C(\delta_0 + 1)|v^u|$$

and

$$|c(x)v^s + d(x)v^u| \ge r|v^u| - C\delta_0|v^u| = (r - C\delta_0)|v^u|.$$

Take δ_0 such that $r - C\delta_0 > 0$, and let $T = \frac{C(\delta_0+1)}{(r-C\delta_0)}$. □

For W as in Lemma 8.11, from Lemma 8.9 we verify for $\eta \in W \cap \Gamma$ that, with $f = \alpha(\gamma_0)$ and $g = \alpha(\eta)$, the transversality in Proposition 8.12 holds. Proposition 8.7 then follows immediately from Proposition 8.12.

8.4. *Abelian subactions without rank-one factors.* In this part we assume

HYPOTHESIS 8.14. *Suppose* $\mathbf{H}$ *is a simply connected semisimple algebraic group defined over* $\mathbb{Q}$ *for which all* $\mathbb{R}$*-simple factors are either anisotropic or of* $\mathbb{R}$*-rank* 2 *or higher. Let* $H = \mathbf{H}(\mathbb{R})^\circ$, *and let* $\Gamma \subset \mathbf{H}(\mathbb{Z}) \cap H$ *be an arithmetic lattice in* H. *Suppose* $\rho\colon \Gamma \to \mathrm{Aut}(M)$ *is a* Γ*-action by linear automorphisms on a compact nilmanifold* $M = N/\Lambda$. *Let us also denote* $\rho : \Gamma \to \mathrm{Aut}(N)$ *as the lift and assume that* $D\rho : \Gamma \to \mathrm{Aut}(\mathfrak{n})$ *extends to a* $\mathbb{Q}$*-rational representation* $D\rho : \mathbf{H} \to \mathbf{GL}(d)$.

Proposition 8.15. *Let* $\mathbf{H}$, Γ, M, ρ *be as in Hypothesis* 8.14, *and let* $S < \Gamma$ *be a Zariski dense semigroup such that* $D\rho(\eta)$ *is hyperbolic for all* $\eta \in S$. *Then there is an element* $\gamma \in S$ *such that*

(1) *the identity component* $Z_{\mathbf{H}}(\gamma)^\circ$ *of the centralizer of* γ *in* $\mathbf{H}(\mathbb{R})$ *is a maximal* $\mathbb{Q}$*-torus, contains a maximal* $\mathbb{R}$*-split torus of* $\mathbf{H}$, *and* $\gamma \in Z_{\mathbf{H}}(\gamma)^\circ$;

(2) $Z_{\mathbf{H}}(\gamma)^\circ \cap \Gamma$ *contains a free abelian group* $\Sigma \cong \mathbb{Z}^{\mathrm{rank}_{\mathbb{R}} \mathbf{H}}$ *of finite index*;

(3) *the restricted action* $\rho|_\Sigma$ *has no algebraic factor action of rank* 1.

Proof. The proof is based on the works of Prasad and Rapinchuk [PR03], [PR05].

We recall that if $\mathbf{T} \subset \mathbf{H}$ is an algebraic torus defined over $\mathbb{Q}$, then there is a canonical $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-action on the character group $X^*(\mathbf{T})$ defined over $\bar{\mathbb{Q}}$, given by $(\sigma.\chi)(t) = \sigma^{-1}(\chi(\sigma(t)))$. Moreover, if $\mathbf{T}$ is a maximal $\mathbb{Q}$-torus in a semisimple algebraic group $\mathbf{H}$ defined over $\mathbb{Q}$, then the Galois action permutes the roots.

An element $\gamma \in \mathbf{H}(\mathbb{R})$ is called *regular* (resp. $\mathbb{R}$*-regular*) if the number of eigenvalues of Ad_γ that are equal to 1 (resp. on the unit circle), counted with multiplicity, is minimum possible. It is called *hyper-regular*, if the number of eigenvalues of $\bigwedge \mathrm{Ad}_\gamma$ that are equal to 1, again counted with multiplicity, is minimum possible. Here $\bigwedge \mathrm{Ad}_\gamma$ denotes the action on the exterior powers of $\mathfrak{h}$, $\bigwedge \mathfrak{h}$. It is known that hyper-regular elements are regular, and that both hyper-regular and regular are Zariski open conditions [PR72, Rem. 1.2]. Furthermore, for a regular and $\mathbb{R}$-regular element γ, $Z_{\mathbf{H}}(\gamma)^\circ$ is a maximal torus that contains γ (see [PR03, introduction]).

By the discussion in [PR05, pp. 240–241], there is a Zariski-dense subset $S' \subset S$ such that for every $\gamma \in S'$,

(1) Property (1) holds;

(2′) γ is regular and $\mathbb{R}$-regular;

(3′) For $\mathbf{T} = Z_{\mathbf{H}}(\gamma)^\circ$, the Galois action contains all elements from the Weyl group $W(\mathbf{H}, \mathbf{T})$, and the cyclic group $\langle \gamma \rangle$ is Zariski dense in $\mathbf{T}$.

By Zariski openness of hyper-regular elements, we can find a $\gamma \in S'$ that is hyper-regular. Prasad-Raghunathan proved in [PR72, Lemma 1.15] that hyper-regularity and $\mathbb{R}$-regularity together implies (2) for γ. It remains to prove (3).

We assume $\rho_1 \colon \Sigma \to \mathrm{Aut}(M_1)$ is a rank-one algebraic factor of $\rho|_\Sigma$ and obtain a contradiction. Write $M = N/\Lambda$ and $M_1 = N_1/\Lambda_1$. By passing to a subgroup if necessary, we may assume $\rho_1(\Sigma)$ is a cyclic group.

Since M_1 is a compact quotient nilmanifold, $N_1 = N/N_0$, where N_0 is a $\rho|_\Sigma$-invariant subgroup of N defined over $\mathbb{Q}$. The Lie algebra $\mathfrak{n}_0$ is hence $D\rho|_\Sigma$-invariant rational subspace of $\mathfrak{n}$.

Notice that the cyclic group $\langle\gamma\rangle$ is Zariski dense in $\mathbf{T}$, $\langle\gamma\rangle\cap\Sigma$ has finite index in $\langle\gamma\rangle$, and $\mathbf{T}$ is Zariski connected. It follows that $\langle\gamma\rangle\cap\Sigma$ is also Zariski dense in $\mathbf{T}$, therefore $\mathfrak{n}_0\otimes_{\mathbb{R}}\mathbb{C}$ is $D\rho|_{\mathbf{T}}$-invariant. So $\rho|_{\mathbf{T}}$ projects to a $\mathbb{Q}$-representation $D\rho_1$ of the $\mathbb{Q}$-torus $\mathbf{T}$ on $\mathfrak{n}_1\otimes_{\mathbb{R}}\mathbb{C}$, where $\mathfrak{n}_1$ is the Lie algebra of N_1.

By property (1), $\mathbf{T}$ contains a maximal $\mathbb{R}$-split $\mathbb{Q}$-torus $\mathbf{T}_s$ of $\mathbf{H}$. Denote $r=\dim\mathbf{T}_s=\operatorname{rank}_{\mathbb{R}}\mathbf{H}$. The restriction to $\mathbf{T}_s$ is a morphism $X^*(\mathbf{T})\to X^*(\mathbf{T}_s)$.

The $\mathbb{R}$-torus $\mathbf{T}$ decomposes as an almost direct product $\mathbf{T}_a\cdot\mathbf{T}_s$, where $\mathbf{T}_a$ is a maximal $\mathbb{R}$-anisotropic torus in $\mathbf{T}$. The intersection $\mathbf{T}_a\cap\mathbf{T}_s$ is a finite subgroup of torsion elements in $\mathbf{T}_s\cong\mathbf{H}_m^r$. In particular, $\mathbf{T}_a(\mathbb{R})\cap\mathbf{T}_s(\mathbb{R})$ is a finite subgroup all of whose elements have order 2. Therefore for $t\in\mathbf{T}(\mathbb{R})$ and $g=t^2$, there is a unique decomposition $g=g_ag_s$ with $g_a\in\mathbf{T}_a(\mathbb{R})$ and $g_s\in\mathbf{T}_s(\mathbb{R})$. Moreover, g_s is in $\mathbf{T}_s(\mathbb{R})^\circ\cong(\mathbb{R}_{>0})^r$.

Without loss of generality, one may replace Σ with $\{\sigma^2:\sigma\in\Sigma\}$. Then all elements $\sigma\in\Sigma$ can be decomposed as above. Moreover, $\sigma\to\sigma_a$ and $\sigma\to\sigma_s$ are group morphisms on Σ.

For any nontrivial $\sigma\in\Sigma$, σ_s is not trivial. Otherwise $\sigma=\sigma_a$ lie in the compact $\mathbf{T}_a(\mathbb{R})$. Since $\Sigma\subset\Lambda$ is discrete, σ must be a torsion element, contradicting our assumption that Σ is free abelian. It follows that $\sigma\to\sigma_s$ is an isomorphism from Σ to $\Sigma_s=\{\sigma_s:\sigma\in\Sigma\}$. Furthermore, again because $\mathbf{T}_a(\mathbb{R})$ is compact and Σ is discrete, Σ_s is discrete and hence is a lattice in $\mathbf{T}_s(\mathbb{R})^\circ$.

Fix a basis $\sigma_1,\dots,\sigma_r$ of Σ and write $(\sigma_i)_s$ as $(e^{\theta_{i1}},\dots,e^{\theta_{ir}})$ in $\mathbf{T}_s(\mathbb{R})^\circ\cong(\mathbb{R}_{>0})^r$. Then (θ_{ij}) is a nondegenerate matrix. Define a group morphism $\mathcal{L}\colon X^*(\mathbf{T}_s)\to\mathbb{R}^r$ by

$$\mathcal{L}(\chi)=\Big(\log|\chi((\sigma_1)_s)|,\dots,\log|\chi((\sigma_r)_s)|\Big).$$

Recall that with $\mathbf{T}_s$ identified with $\mathbf{H}_m^r$, the coordinate maps $\pi_j\colon(t_1,\dots,t_r)\to t_j$ form a basis of $X^*(\mathbf{T}_s)\cong\mathbb{Z}^r$. Note that $\mathcal{L}(\pi_j)=(\theta_{1j},\dots,\theta_{rj})$. Thus by nondegeneracy of (θ_{ij}), $\mathcal{L}$ embeds $X^*(\mathbf{T}_s)$ as a lattice into $\mathbb{R}^r$.

For any character $\chi\in X^*(\mathbf{T})$, $\tilde\chi\colon t\to\chi(t)\overline{\chi(\bar t)}$ is defined over $\mathbb{R}$ and its restriction to $\mathbf{T}_a$ is trivial. In particular, for $\sigma\in\Sigma$, $\tilde\chi(\sigma)=\tilde\chi(\sigma_s)$. Notice that $\chi|_{\mathbf{T}_s}$ is defined over $\mathbb{R}$, and hence $\tilde\chi|_{\mathbf{T}_s}=(\chi|_{\mathbf{T}_s})^2$.

Denote by $\Lambda\subset X^*(\mathbf{T})$ the sets of weights of the $\mathbb{Q}$-representation $D\rho_1$ of $\mathbf{T}$, which is invariant under the canonical Galois action. For $\gamma\in S$, by [Man74], $D\rho(\gamma)$ is a hyperbolic matrix, and hence so is $D\rho_1(\gamma)$. It follows, because $\gamma\in\mathbf{T}(\mathbb{R})$, that $\tilde\lambda(\gamma)=|\lambda(\gamma)|^2\neq1$ for $\lambda\in\Lambda$. Thus $\tilde\lambda((\gamma^2)_s)=\tilde\lambda(\gamma^2)=\tilde\lambda(\gamma)^2\neq1$.

In particular, $\tilde\lambda|_{\mathbf{T}_s}$ is nontrivial. Because Λ is invariant under the Galois action, it is also invariant under the action of $W(\mathbf{H},\mathbf{T})$. Furthermore, by

[Bor91, Cor. 21.4], every element $\omega \in W(\mathbf{H}, \mathbf{T}_s)$ is represented by the restriction of some element $\tilde{\omega} \in W(\mathbf{H}, \mathbf{T})$ to $\mathbf{T}_s$. Therefore, the set

$$\tilde{\Lambda}_s := \{\tilde{\lambda}|_{\mathbf{T}_s} : \lambda \in \Lambda\} = \{(\lambda|_{\mathbf{T}_s})^2 : \lambda \in \Lambda\}$$

is nonempty and $W(\mathbf{H}, \mathbf{T}_s)$-invariant.

One can always decompose $\mathbf{H}$ into a direct product $\prod \mathbf{H}_i$ of almost $\mathbb{R}$-simple factors in such a way that $\mathbf{T}_s = \prod \mathbf{T}_{s,i}$, where $\mathbf{T}_{s,i}$ is a maximal $\mathbb{R}$-split torus in $\mathbf{H}_i$. Fix a $\lambda \in \Lambda$; its restriction $\tilde{\lambda}|_{\mathbf{T}_{s,i}}$ to some $\mathbf{T}_{s,i}$ is nontrivial. The action by $W(\mathbf{H}_i, \mathbf{T}_{s,i}) \subset W(\mathbf{H}, \mathbf{T}_s)$ on $X^*(\mathbf{T}_{s,i})$ preserves no proper rank-one subgroup. Hence as $\dim \mathbf{T}_{s,i} \geq 2$ by assumption, the $W(\mathbf{H}, \mathbf{T}_{s,i})$-orbit of $\tilde{\lambda}|_{\mathbf{T}_{s,i}}$ is not contained in any cyclic subgroup. Thus, the same is true for the $W(\mathbf{H}, \mathbf{T}_s)$-orbit of $\tilde{\lambda}|_{\mathbf{T}_s}$ and for the $W(\mathbf{H}, \mathbf{T})$-orbit of $\tilde{\lambda}$. In other words, there is no cyclic subgroup of $X^*(\mathbf{T}_s)$ that contains $\tilde{\Lambda}_s$, which is the same as that there is no one-dimensional subgroup of $\mathbb{R}^r$ that contains $\mathcal{L}(\tilde{\Lambda}_s)$.

However, on the other hand, as $\rho_1(\Sigma)$ is assumed to be a cyclic group $\{A^n\}$, there are integers $n_1, \ldots, n_r$ such that $\rho_1(\sigma_i) = A^{n_i}$. Then for each $\lambda \in \Lambda$, there is $a_\lambda \in \mathbb{C}$ such that $\lambda(\sigma_i) = a_\lambda^{n_i}$. Therefore,

$$\begin{aligned}\mathcal{L}(\tilde{\lambda}|_{\mathbf{T}_s}) =& \big(\log|\tilde{\lambda}((\sigma_1)_s)|, \ldots, \log|\tilde{\lambda}((\sigma_r)_s|\big) = \big(\log \tilde{\lambda}(\sigma_1), \ldots, \log \tilde{\lambda}(\sigma_r)\big)\\ =& \big(2\log|\lambda(\sigma_1)|, \ldots, 2\log|\lambda(\sigma_r)|\big)\\ =& 2\log|a_\lambda|(n_1, \ldots, n_r)\end{aligned}$$

belongs to a given one-dimensional subspace for every $\lambda \in \Lambda$. This produces the desired contradiction and completes the proof. □

8.5. *Proof of Proposition* 8.3. We deduce Proposition 8.3 from Propositions 8.5 and 8.15.

Proof of Proposition 8.3. Under Hypothesis 8.1, Theorem 6.1 applies to $\mathbf{G}$ and Γ. Let $\mathbf{H}$ and $\phi : \mathbf{H} \mapsto \mathbf{G}$ be as in Theorem 6.1. Denote $H = \mathbf{H}(\mathbb{R})^\circ$.

Let $\hat{\Gamma} = \phi^{-1}(\Gamma) \cap \mathbf{H}(\mathbb{Z})$. Then $\hat{\Gamma}$ is a lattice in H and is of finite index in $\mathbf{H}(\mathbb{Z})$. Define $\hat{\Gamma}$-actions $\hat{\alpha} = \alpha \circ \phi$ and $\hat{\rho} = \rho \circ \phi$ that act through Γ. Note that $\hat{\rho}$ is the linear data of $\hat{\alpha}$.

By the discussion in Section 2.2, there is a $\mathbb{Q}$-structure of $\mathfrak{n}$ such that $D\rho$ sends Γ into $\mathrm{GL}(d, \mathbb{Q})$. Hence the image of $\hat{\Gamma}$ under $D\hat{\rho}$ is in $\mathrm{GL}(d, \mathbb{Q})$ as well. Applying Theorem 6.2 with $k = \mathbb{Q}$ and $\mathbf{J} = \mathrm{GL}(d)$, we know that, after restricting to a finite-index subgroup $\hat{\Gamma}' \subset \hat{\Gamma}$, $D\hat{\rho}$ extends to a representation $\mathbf{H} \to \mathbf{GL}(d)$ defined over $\mathbb{Q}$. We replace $\hat{\Gamma}$ with $\hat{\Gamma}'$ in the sequel.

We apply Proposition 8.5 to obtain a Zariski open set $W \subset G$ and a semigroup $S \subset \Gamma$. Let $\hat{W} = \phi^{-1}(W)$, which is a Zariski open set in H. Note that because no $\mathbb{Q}$-simple factor of $\mathbf{H}$ is $\mathbb{R}$-anisotropic, $\mathbf{H}(\mathbb{Z})$, and hence $\hat{\Gamma}$ as well, are Zariski dense in $\mathbf{H}$ by Borel density theorem (see [Mor15, Cor. 4.5.6]).

In addition, define $\hat{S} = \phi^{-1}(S) \cap \hat{\Gamma}$, and fix a preimage $\hat{\gamma}_0 \in \phi^{-1}(\gamma_0)$.

Note that $\hat{\alpha} = \alpha \circ \phi$ defines a C^∞ action by $\hat{\Gamma}$. We claim that the $(H, \hat{\Gamma}, \hat{S}, \hat{W}, \hat{\alpha}, \hat{\gamma}_0)$, instead of $(G, \Gamma, S, W, \alpha, \gamma_0)$, also satisfies the conclusion of Proposition 8.5. In fact, the analogous conclusions of Lemma 8.6 and Proposition 8.7 in the setting of $(H, \hat{\Gamma}, \hat{S}, \hat{W}, \hat{\alpha}, \hat{\gamma}_0)$ follow directly from the corresponding properties for $(G, \Gamma, S, W, \alpha, \gamma_0)$. The conclusion of Proposition 8.5 follows for $\hat{S}$ exactly as in the proof of Proposition 8.5, using now that $\hat{\Gamma}$ is Zariski dense in H and $\hat{W}$ is Zariski open.

Recall that if $\hat{\alpha}(\gamma)$ is Anosov, then $D\hat{\rho}(\gamma)$ is a hyperbolic matrix. Hence $(H, \hat{\Gamma}, \hat{S}, \hat{\rho})$ satisfies Hypothesis 8.14 as well as the conditions in Proposition 8.15. Let $\hat{\gamma} \in \hat{S}$ and $\hat{\Sigma} \subset Z_{\mathbf{H}}(\hat{\gamma})^\circ \cap \Lambda$ be given by Proposition 8.15. Then $\hat{\rho}|_{\hat{\Sigma}}$ has no rank-one algebraic factor. Since $\hat{\gamma} \in Z_{\mathbf{H}}(\hat{\gamma})^\circ$ and as $\hat{\Sigma}$ is of finite index in $Z_{\mathbf{H}}(\hat{\gamma})^\circ \cap \Lambda$, a nontrivial power $\hat{\gamma}^k$ is in $\hat{\Sigma}$. Then $\hat{\alpha}(\hat{\gamma}^k)$ is an Anosov diffeomorphism as $\hat{\alpha}(\hat{\gamma})$ is.

Consider $\Sigma = \phi(\hat{\Sigma}) \subset \Gamma$. Then as $\hat{\rho}|_{\hat{\Sigma}}$ acts through $\rho|_\Sigma$, it has no rank-one algebraic factor actions. Moreover, $\alpha(\gamma)$ is Anosov for $\gamma = \phi(\hat{\gamma}) \in \Sigma$. □

9. Cohomological obstructions to lifting actions on nilmanifolds

In this section, we justify Remark 1.5, which in turn gives Corollary 1.9.

Let M be a finite connected CW-complex. Let $\Lambda_M = \pi_1(M)/K$ be a quotient of the fundamental group of M. Let $\tilde{M}$ be the normal cover of M with deck transformation group Λ_M. Let Γ be a finitely generated discrete group, and let $\alpha\colon \Gamma \to \mathrm{Homeo}(M)$ be an action. Given $\gamma \in \Gamma$, select an arbitrary lift $\tilde{\alpha}(\gamma)\colon \tilde{M} \to \tilde{M}$ of $\alpha(\gamma)\colon M \to M$. Given $\lambda \in \Lambda_M$, select any $x \in \tilde{M}$ and let $\tilde{\alpha}(\gamma)_*\colon \Lambda_M \to \Lambda_M$ be such that

$$\tilde{\alpha}(x\lambda) = \tilde{\alpha}(x)\tilde{\alpha}(\gamma)_*(\lambda). \tag{9.1}$$

Note that $\alpha_*(\lambda)$ is independent of the choice of x by continuity.

Given a second lift $\tilde{\alpha}'(\gamma)$, there is some $\lambda' \in \Lambda_M$ so that for all x,

$$\tilde{\alpha}'(\gamma)(x) = \tilde{\alpha}(\gamma)(x)\lambda'.$$

Then

$$\tilde{\alpha}'(x\lambda) = \tilde{\alpha}(\gamma)(x\lambda)\lambda' = \tilde{\alpha}(\gamma)(x)\tilde{\alpha}(\gamma)_*(\lambda)\lambda' = \tilde{\alpha}'(\gamma)(x)(\lambda')^{-1}\tilde{\alpha}(\gamma)_*(\lambda)\lambda'. \tag{9.2}$$

It follows that $\tilde{\alpha}(\gamma)_*$ is defined up to $\mathrm{Inn}(\Lambda_M)$. We thus obtain a well-defined homomorphism $\alpha_\#\colon \Gamma \to \mathrm{Out}(\Lambda_M)$.

Let N be a simply connected, m-dimensional, nilpotent Lie group, and let $\Lambda \subset N$ be a lattice. Let $P_*\colon \Lambda_M \to \Lambda$ be a surjective homomorphism. As the kernel of P_* is normal in Λ_M, whether or not $\alpha_\#$ preserves the kernel of P_* is well defined. We assume for the remainder that $\alpha_\#$ preserves this kernel. Replacing Λ_M with $\Lambda_M/(\mathrm{Ker}(P_*))$ if necessary, we may further assume that P_* is an isomorphism. We then identify Λ_M and Λ for the remainder and continue to write $\tilde{M}$ for the normal cover of M with deck group Λ. Recall that $P\colon M \to N/\Lambda$ is the continuous map induced by P_*.

Recall that we have a series of central extensions in (2.2) and (2.3). Let Z_i denote the kernel of $N_i \to N_{i+1}$. Then $Z_i \cap \Lambda_i \simeq \mathbb{Z}^{d_i}$ is the center of Λ_i and is the kernel of each map $\Lambda_i \to \Lambda_{i+1}$. As each $Z_i \cap \Lambda_i$ is the center of Λ_i, an element $\psi \in \operatorname{Out}(\Lambda_i)$ restricts to an automorphism $\psi\restriction_{Z_i\cap\Lambda_i} \in \operatorname{Aut}(Z_i \cap \Lambda_i)$. Moreover, as automorphisms fix centers, we have natural maps

$$\operatorname{Out}(\Lambda_0) \to \operatorname{Out}(\Lambda_1) \to \cdots \to \operatorname{Out}(\Lambda_{r-1}).$$

In particular, $\alpha_\#$ induces representations $\alpha_{\#,i}\colon \Gamma \to \operatorname{Aut}(Z_i \cap \Lambda_i) = \operatorname{GL}(d_i, \mathbb{Z})$.

We have the following proposition, which guarantees the action α lifts to an action of $\operatorname{Homeo}(\tilde{M})$ given the vanishing of certain cohomological obstructions.

Proposition 9.1. *Suppose that the cohomology group $H^2_{\alpha_{\#,i}}(\Gamma, \mathbb{R}^{d_i})$ is trivial for every representation $\alpha_{\#,i}\colon \Gamma \to \operatorname{Aut}(Z_i \cap \Lambda_i)$. Then, there is a finite-index subgroup $\Gamma' \subset \Gamma$ such the restricted action $\alpha\colon \Gamma' \to \operatorname{Homeo}(M)$ lifts to an action $\tilde{\alpha}\colon \Gamma' \to \operatorname{Homeo}(\tilde{M})$.*

In particular, whether or not the action $\alpha\colon \Gamma \to \operatorname{Homeo}(M)$ lifts (when restricted to a finite-index subgroup) is determined only by the data of the linear representations $\alpha_{\#,i}$ associated to α.

The vanishing of $H^2_\rho(\Gamma; \mathbb{R}^d)$ has been studied in [Bor81] and [GH68]. In particular, it is known to vanish in case (2) of Remark 1.5; case (1) follows from computations in [FW01]. Cases (3) and (4) will be discussed in Section 9.3.

9.1. *Candidate liftings and defect functional.* Recall that we identify $\Lambda \subset N$ with the deck group of the cover $\tilde{M} \to M$. For $\gamma \in \Gamma$, consider an arbitrary lift $\tilde{\alpha}(\gamma)\colon \Gamma \to \operatorname{Homeo}(\tilde{M})$. The collection of lifts $\{\tilde{\alpha}(\gamma) : \gamma \in \Gamma\}$ need not assemble into an action. The defect of the lifts $\{\tilde{\alpha}(\gamma)\}$ forming a coherent action is measured by the associated defect functional $\beta\colon \Gamma \times \Gamma \to \Lambda$ defined by

$$\tilde{\alpha}(\gamma_1)(\tilde{\alpha}(\gamma_2)(x))\beta(\gamma_1, \gamma_2) = \tilde{\alpha}(\gamma_1\gamma_2)(x). \tag{9.3}$$

Note that the value $\beta(\gamma_1, \gamma_2)$ is independent of the choice of x by continuity. Clearly,

Claim 9.2. *The action $\alpha\colon \Gamma \to \operatorname{Homeo}(M)$ lifts to an action $\tilde{\alpha}\colon \Gamma \to \operatorname{Homeo}(N)$ if and only if the lifts $\tilde{\alpha}(\gamma)$ above can be chosen so that $\beta \equiv e$.*

We consider the range of β modulo the kernel of the map $\Lambda \to \Lambda_i$. Write Δ_i for the kernel of this projection. Let $\beta_i\colon \Gamma \times \Gamma \to \Lambda_i$ denote the induced map $\beta_i(\gamma_1, \gamma_2) = \beta(\gamma_1, \gamma_2) \bmod \Delta_i$. Below, we will have the inductive hypothesis that $\beta_{i+1} \equiv e$. Note that this ensures that β_i takes values in $(Z \cap \Lambda_i) \simeq \mathbb{Z}^{d_i}$.

In particular, if $\beta_{i+1} \equiv e$, then the representation $\alpha_{\#,i}\colon \Gamma \to \mathrm{Out}(\Lambda_i)$ induces a linear representation $\alpha_{\#,i}\colon \Gamma \to \mathrm{GL}(d_i, \mathbb{Z})$ on the range of β_i.

We recall the definition of group cohomology for nontrivial representations. Let V be an abelian group, and let $\psi\colon \Gamma \to \mathrm{Aut}(V)$ be a Γ-action on V. Denote by $C^k(\Gamma, V)$ the space of functions from Γ^k to V. Define a map $d_{\psi,k}\colon C^k(\Gamma, V) \to C^{k+1}(\Gamma, V)$ by

$$\begin{aligned} d_{\psi,k} f(\gamma_1, \dots, \gamma_{k+1}) = \psi(\gamma_1).f(\gamma_2, \dots, \gamma_{k+1}) \\ + \sum_{j=1}^{k} (-1)^j f(\gamma_1, \dots, \gamma_{j-1}, \gamma_j\gamma_{j+1}, \gamma_{j+2}, \dots, \gamma_{k+1}) \\ + (-1)^{k+1} f(\gamma_1, \dots, \gamma_k). \end{aligned}$$

It is a standard fact that $d_{\psi,k+1} \circ d_{\psi,k} = 0$, and thus

$$0 \to C(\Gamma, V) \xrightarrow{d_{\psi,1}} C(\Gamma^2, V) \xrightarrow{d_{\psi,2}} C(\Gamma^3, V) \xrightarrow{d_{\psi,3}} \cdots$$

forms a cochain complex, denoted by $X_\psi(\Gamma, V)$.

The group cohomology $H^\bullet_\psi(\Gamma, V)$ is defined by the homology groups of $X_\psi(\Gamma, V)$. The map $f \in C^k(\Gamma, V)$ is called a k-cocycle if f is in the kernel of $d_{\psi,k}$ and a k-coboundary if f is in the image of $d_{\psi,k-1}$.

Claim 9.3. *Assume $\beta_{i+1} \equiv e$. Then β_i is 2-cocycle over the representation $\alpha_{\#,i}$.*

Proof. For $\gamma_1, \gamma_2, \gamma_3 \in \Gamma$, we have that

$$\begin{aligned} &\tilde{\alpha}(\gamma_1) \circ \tilde{\alpha}(\gamma_2) \circ \tilde{\alpha}(\gamma_3)(x) \tilde{\alpha}(\gamma_1)_* \beta(\gamma_2, \gamma_3) \beta(\gamma_1, \gamma_2\gamma_3) \beta(\gamma_1\gamma_2, \gamma_3)^{-1} \beta(\gamma_1, \gamma_2)^{-1} \\ &\quad = \tilde{\alpha}(\gamma_1) \left(\tilde{\alpha}(\gamma_2) \circ \tilde{\alpha}(\gamma_3)(x) \beta(\gamma_2, \gamma_3) \right) \beta(\gamma_1, \gamma_2\gamma_3) \beta(\gamma_1\gamma_2, \gamma_3)^{-1} \beta(\gamma_1, \gamma_2)^{-1} \\ &\quad = \tilde{\alpha}(\gamma_1) \left(\tilde{\alpha}(\gamma_2\gamma_3)(x) \right) \beta(\gamma_1, \gamma_2\gamma_3) \beta(\gamma_1\gamma_2, \gamma_3)^{-1} \beta(\gamma_1, \gamma_2)^{-1} \\ &\quad = \tilde{\alpha}(\gamma_1\gamma_2\gamma_3)(x) \beta(\gamma_1\gamma_2, \gamma_3)^{-1} \beta(\gamma_1, \gamma_2)^{-1} \\ &\quad = \tilde{\alpha}(\gamma_1\gamma_2) \circ \tilde{\alpha}(\gamma_3)(x) \beta(\gamma_1, \gamma_2)^{-1} \\ &\quad = \tilde{\alpha}(\gamma_1) \circ \tilde{\alpha}(\gamma_2) \circ \tilde{\alpha}(\gamma_3)(x), \end{aligned}$$

whence

$$\tilde{\alpha}(\gamma_1)_* \beta(\gamma_2, \gamma_3) \beta(\gamma_1, \gamma_2\gamma_3) \beta(\gamma_1\gamma_2, \gamma_3)^{-1} \beta(\gamma_1, \gamma_2)^{-1} \equiv e.$$

Taken modulo Δ_i, the terms commute and

$$\begin{aligned} &d_{(\alpha_{\#,i}),2} \beta_i(\gamma_1, \gamma_2, \gamma_3) \\ &\qquad = \alpha_{\#,i}(\gamma_1) \beta_i(\gamma_2, \gamma_3) \left[\beta_i(\gamma_1\gamma_2, \gamma_3) \right]^{-1} \beta_i(\gamma_1, \gamma_2\gamma_3) \left[\beta_i(\gamma_1, \gamma_2) \right]^{-1} \\ &\qquad = \left(\tilde{\alpha}(\gamma_1)_* \beta(\gamma_2, \gamma_3) \beta(\gamma_1, \gamma_2\gamma_3) \beta(\gamma_1\gamma_2, \gamma_3)^{-1} \beta(\gamma_1, \gamma_2)^{-1} \right) \mod \Delta_i \\ &\qquad = e. \end{aligned}$$

□

9.2. *Vanishing of defect given vanishing of cohomology.* Proposition 9.1 follows from the following lemma.

LEMMA 9.4. *Suppose for some $1 \leq i \leq r$ that the lifts $\{\tilde{\alpha}(\gamma) : \gamma \in \Gamma\}$ are chosen so that $\beta_{i+1} \equiv e$. Then, under the hypotheses of Proposition 9.1, there is a finite-index subgroup $\hat{\Gamma} \subset \Gamma$ and choice of lifts $\{\hat{\alpha}(\gamma) : \gamma \in \hat{\Gamma}\}$ so that, for the new defect functional defined by*

$$\hat{\alpha}(\gamma_1)(\hat{\alpha}(\gamma_2)(x))\hat{\beta}(\gamma_1, \gamma_2) = [\hat{\alpha}(\gamma_1\gamma_2)(x)],$$

$\hat{\beta}_i \equiv e$.

Clearly $\beta_r \equiv e$ for any choice of lifts $\{\tilde{\alpha}(\gamma) : \gamma \in \Gamma\}$. Proposition 9.1 then follows from finite induction using Lemma 9.4.

In order to prove Lemma 9.4, we recall some elementary properties of group cohomology. First, note that if ψ is a morphism into $\mathrm{Aut}(\mathbb{Z}^d) = \mathrm{GL}(d, \mathbb{Z})$, then for any abelian group A, ψ induces a morphism $\Gamma \to \mathrm{Aut}(A^d)$, which we still denote by ψ. Moreover, we have the relation

$$X_\psi(\Gamma, A^d) = X_\psi(\Gamma, \mathbb{Z}^d) \otimes_{\mathbb{Z}} A$$

between cochain complexes. Any short exact sequence $0 \to A \to B \to C \to 0$ of abelian groups induces a short exact sequence

$$0 \to X_\psi(\Gamma, A^d) \to X_\psi(\Gamma, B^d) \to X_\psi(\Gamma, C^d) \to 0$$

of cochain complexes. It follows that there is a long exact sequence

$$\cdots \to H^k_\psi(\Gamma, C^d) \to H^{k+1}_\psi(\Gamma, A^d) \to H^{k+1}_\psi(\Gamma, B^d) \to H^{k+1}_\psi(\Gamma, C^d) \to \cdots$$

between group cohomologies. Finally, we recall that by the universal coefficients theorem, there is a short exact sequence

$$0 \to H^k_\psi(\Gamma, \mathbb{Z}^d) \otimes_{\mathbb{Z}} A \to H^k_\psi(\Gamma, A^d) \to \mathrm{Tor}(H^{k+1}_\psi(\Gamma, \mathbb{Z}^d), A) \to 0.$$

In particular, when A is a flat $\mathbb{Z}$-module, or equivalently when A is torsion-free, $\mathrm{Tor}(H^{k+1}_\psi(\Gamma, \mathbb{Z}^d), A)$ vanishes and $H^k_\psi(\Gamma, \mathbb{Z}^d) \otimes_{\mathbb{Z}} A \cong H^k_\psi(\Gamma, A^d)$.

CLAIM 9.5. *Under the assumption that $H^2_{\alpha_{\#,i}}(\Gamma; \mathbb{R}^{d_i}) = 0$, after restricting to a finite-index subgroup $\hat{\Gamma} \subset \Gamma$, we have that β_i vanishes in $H^2_{\alpha_{\#,i}}(\hat{\Gamma}; \mathbb{Z}^{d_i})$.*

Proof. By the universal coefficients theorem,

$$H^2_{\alpha_{\#,i}}(\Gamma; \mathbb{Z}^{d_i}) \otimes_{\mathbb{Z}} \mathbb{R} = H^2_{\alpha_{\#,i}}(\Gamma; \mathbb{R}^{d_i}) = 0.$$

Hence all elements in $H^2_{\alpha_{\#,i}}(\Gamma; \mathbb{Z}^{d_i})$ are torsion elements, and again by the universal coefficients theorem, $H^2_{\alpha_{\#,i}}(\Gamma; \mathbb{Q}^{d_i}) = H^2_{\alpha_{\#,i}}(\Gamma; \mathbb{Z}^{d_i}) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$. Moreover, as Γ is finitely generated, $H^2_{\alpha_{\#,i}}(\Gamma; \mathbb{Z}^{d_i})$ is a finitely generated abelian group. Thus $H^2_{\alpha_{\#,i}}(\Gamma; \mathbb{Z}^{d_i})$ is finite.

On the other hand, take the long exact sequence

$$\cdots \to H^1_{\alpha_{\#,i}}(\Gamma; (\mathbb{Q}/\mathbb{Z})^{d_i}) \to H^2_{\alpha_{\#,i}}(\Gamma; \mathbb{Z}^{d_i}) \to H^2_{\alpha_{\#,i}}(\Gamma; \mathbb{Q}^{d_i}) \to \cdots .$$

We see that β_i is the image of some $\eta \in H^1_{\alpha_{\#,i}}(\Gamma; (\mathbb{Q}/\mathbb{Z})^{d_i})$. Since Γ is finitely generated, there is a denominator q such that η can be chosen to take values in the finite abelian group $(\frac{1}{q}\mathbb{Z}/\mathbb{Z})^{d_i}$. The zero set $\eta^{-1}(0)$ is a finite-index subgroup $\hat{\Gamma}$ of Γ, which establishes the claim. □

Identifying $Z_i \cap \Lambda_i$ with $\mathbb{Z}^{d_i}$, from Claim 9.5 it follows under the hypotheses of Proposition 9.1 that by restricting to a finite-index subgroup $\hat{\Gamma} \subset \Gamma$, we have that β_i is 2-coboundary over $\alpha_{\#,i}$. That is, there is a function $\eta\colon \hat{\Gamma} \to Z_i \cap \Lambda_i$ with

$$d_{(\alpha_{\#,i}),1}\eta(\gamma_1, \gamma_2) := \alpha_{\#,i}(\gamma_1)\eta(\gamma_2) \cdot [\eta(\gamma_1\gamma_2)]^{-1} \cdot \eta(\gamma_1) = \beta_i(\gamma_1, \gamma_2)$$

for all $\gamma_1, \gamma_2 \in \hat{\Gamma}$.

Note that η takes vales in Λ/Δ_i. Given $\gamma \in \hat{\Gamma}$, let $\tilde{\eta}(\gamma) \in \Lambda$ be any choice of representative. We use $\tilde{\eta}$ to correct the original choice of lifts $\tilde{\alpha}(\gamma)$: given $\gamma \in \hat{\Gamma}$, let $\hat{\alpha}(\gamma) = \tilde{\alpha}(\gamma)\tilde{\eta}(\gamma)$. With the new family of lifts $\{\hat{\alpha}(\gamma) : \gamma \in \hat{\Gamma}\}$ define a new defect functional $\hat{\beta}\colon \hat{\Gamma} \times \hat{\Gamma} \to \Lambda$ as in the lemma and similarly define induced functionals $\hat{\beta}_k$.

We have

Claim 9.6. *The defect $\hat{\beta}_i$ vanishes.*

Proof. By definition, we have

$$\tilde{\alpha}(\gamma_1)\left(\tilde{\alpha}(\gamma_2)(x)\tilde{\eta}(\gamma_2)\right)\tilde{\eta}(\gamma_1)\hat{\beta}(\gamma_1, \gamma_2) = \tilde{\alpha}(\gamma_1\gamma_2)(x)\tilde{\eta}(\gamma_1\gamma_2).$$

With β the defect of $\tilde{\alpha}$, we have

$$\tilde{\alpha}(\gamma_1\gamma_2)(x)\beta(\gamma_1, \gamma_2)^{-1}\tilde{\alpha}(\gamma_1)_*(\tilde{\eta}(\gamma_2))\tilde{\eta}(\gamma_1)\hat{\beta}(\gamma_1, \gamma_2) = \tilde{\alpha}(\gamma_1\gamma_2)(x)\tilde{\eta}(\gamma_1\gamma_2)$$

and

$$\beta(\gamma_1, \gamma_2)^{-1}\tilde{\alpha}(\gamma_1)_*(\tilde{\eta}(\gamma_2))\tilde{\eta}(\gamma_1)\hat{\beta}(\gamma_1, \gamma_2) = \tilde{\eta}(\gamma_1\gamma_2).$$

Modulo Δ_i, we have

$$\begin{aligned}\hat{\beta}(\gamma_1, \gamma_2) \bmod \Delta_i =&\beta_i(\gamma_1, \gamma_2) \cdot \left(\alpha_{\#,i}(\gamma_1)(\eta(\gamma_2))^{-1} \cdot \eta(\gamma_1\gamma_2) \cdot \eta(\gamma_1)^{-1}\right)\\ =&\beta_i(\gamma_1, \gamma_2) \cdot [d\eta(\gamma_1, \gamma_2)]^{-1}\\ =&e.\end{aligned}$$

□

Lemma 9.4 follows from the above claims.

9.3. *Vanishing of defect in the case of an invariant measure.* Consider first an action $\alpha\colon \Gamma \to \mathrm{Homeo}(\mathbb{T}^d)$ on a torus preserving a probability measure μ. It will follow from the proof of the more general Proposition 9.8 below that the action α lifts establishing (3) of Remark 1.5.

PROPOSITION 9.7. *Suppose the action $\alpha\colon \Gamma \to \mathrm{Homeo}(\mathbb{T}^d)$ preserves a Borel probability measure μ. Then α lifts to an action $\tilde{\alpha}\colon \hat{\Gamma} \to \mathrm{Homeo}(\mathbb{R}^d)$ when restricted to a finite-index subgroup $\hat{\Gamma} \subset \Gamma$.*

In the case of actions on nilmanifolds or, more generally, actions on CW-complexes admitting π_1-factors, the corresponding result is more complicated. Recall that we fix a connected finite CW-complex M and an action $\alpha\colon \Gamma \to \mathrm{Homeo}(M)$. We also fix a simply connected, nilpotent Lie group N, a lattice $\Lambda \subset N$, and a normal cover $\tilde{M}$ of M whose deck group is identified with Λ.

Recall the sequences of N_i and Λ_i in (2.2) and (2.3). For each i, let $\tilde{M}_i$ denote the intermediate normal cover of M with deck group Λ_i. The natural identification of Λ_i with the deck groups of $\tilde{M}_i \to M$ and $N_i \to N_i/\Lambda_i$ induces a map $P_i\colon M \to N_i/\Lambda_i$. Recall (as we identify Λ_M and Λ) that we have distinguished lifts $\tilde{P}_i\colon \tilde{M}_i \to N_i$ with $\tilde{P}_i(x\lambda) = \tilde{P}_i(x)\lambda$. Suppose for some $1 \le i \le r-1$ that the action $\alpha\colon \Gamma \to \mathrm{Homeo}(M)$ lifts to an action $\tilde{\alpha}\colon \Gamma \to \mathrm{Homeo}(\tilde{M}_{i+1})$. We then obtain an action $\rho_{i+1}\colon \Gamma \to \mathrm{Aut}(\Lambda_{i+1})$ that uniquely extends to an action $\rho_{i+1}\colon \Gamma \to \mathrm{Aut}(N_{i+1})$; in particular, $\rho_{i+1}\colon \Gamma \to \mathrm{Aut}(N_{i+1}/\Lambda_{i+1})$ defines a π_1 factor of α.

Below is the general proposition guaranteeing the lifting of an action given an invariant measure. Note that we use the existence of a semiconjugacy to guarantee the lifting.

PROPOSITION 9.8. *With the above setup, suppose there is a continuous $h\colon M \to N_{i+1}/\Lambda_{i+1}$ homotopic to P_{i+1} that lifts to a map $\tilde{h}\colon \tilde{M}_{i+1} \to N_{i+1}$ and that intertwines the actions $\tilde{\alpha}_{i+1}\colon \Gamma \to \mathrm{Homeo}(\tilde{M}_{i+1})$ and $\rho_{i+1}\colon \Gamma \to \mathrm{Aut}(N_{i+1})$ and is Λ_{i+1}-equivariantly homotopic to P_{i+1}.*

Then, if the action $\alpha\colon \Gamma \to \mathrm{Homeo}(M)$ preserves a Borel probability measure μ, the action α lifts to an action $\tilde{\alpha}_i\colon \hat{\Gamma} \to \mathrm{Homeo}(\tilde{M}_i)$ when restricted to a finite-index subgroup $\hat{\Gamma} \subset \Gamma$.

Proposition 9.7 follows from Proposition 9.8 with $h\colon \mathbb{T}^d \to \{e\}$. For measure preserving actions $\alpha\colon \Gamma \to \mathrm{Homeo}(N/\Lambda)$ on nilmanifolds, we automatically obtain the lifting of α to $\tilde{\alpha}_{r-1}\colon \hat{\Gamma} \to \mathrm{Homeo}(\tilde{M}_{r-1})$. We can then define the π_1-factor $\rho_{r-1}\colon \hat{\Gamma} \to \mathrm{Aut}(N_{r-1}/\Lambda_{r-1})$. If ρ_{r-1} satisfies the hypotheses of Theorem 3.2, the semiconjugacy satisfying the hypotheses of Proposition 9.8 exists and we may lift α to $\tilde{\alpha}_{r-2}\colon \bar{\Gamma} \to \mathrm{Homeo}(\tilde{M}_{r-2})$. We then recursively verify whether or not the induced π_1-factors $\rho_i\colon \hat{\Gamma} \to \mathrm{Aut}(N_i/\Lambda_i)$ satisfy the hypotheses of Theorem 3.2 in order to continue to lift the action. Under Hypothesis 1.2, if $\rho(\gamma_0)$ is hyperbolic for some $\gamma_0 \in \Gamma$, then the same arguments as in Section 7 show that the representations ρ_i satisfy the hypotheses of Theorem 3.2 (after restricting to finite-index subgroups and extending from G to L as in Section 7.1.) It follows that at each step an h satisfying the hypotheses of Proposition 9.8 can be found. This establishes (4) of Remark 1.5.

Proof of Proposition 9.8. Recall that we assume $\alpha\colon \Gamma \to \mathrm{Homeo}(M)$ lifts to $\tilde\alpha_{i+1}\colon \Gamma \to \mathrm{Homeo}(\tilde M_{i+1})$. For every $\gamma \in \Gamma$, choose an arbitrary lift $\tilde\alpha_i(\gamma) \in \mathrm{Homeo}(\tilde M_i)$ of $\tilde\alpha_{i+1}(\gamma)$. For $\gamma \in \Gamma$, let $\tilde\alpha_i(\gamma)_*\colon \Lambda_i \to \Lambda_i$ be defined as in (9.1). Note that given a second lift $\tilde\alpha_i'(\gamma)$, we have $\tilde\alpha_i'(\gamma) = \tilde\alpha_i(\gamma)\lambda'$ for a central $\lambda' \in Z_i \cap \Lambda_i$. In particular, from (9.2) we have for $\lambda \in \Lambda_i$ that

$$\tilde\alpha_i'(\gamma)_*(\lambda) = (\lambda')^{-1}\tilde\alpha_i(\gamma)_*(\lambda)\lambda' = \tilde\alpha_i(\gamma)_*(\lambda).$$

Thus any choice of lifts $\{\tilde\alpha_i(\gamma) : \gamma \in \Gamma\}$ of the action $\tilde\alpha_{i+1}$ induces a representation $\alpha_{i,*}\colon \Gamma \to \mathrm{Aut}(\Lambda_i)$. This in turn induces a representation $\rho_i\colon \Gamma \to \mathrm{Aut}(N_i)$, which in turn induces a π_1-factor of $\tilde\alpha$ on N_i/Λ_i.

Fix an arbitrary family of lifts $\{\tilde\alpha_i(\gamma) : \gamma \in \Gamma\}$ of the action $\tilde\alpha_{i+1}$. We define the defect functional $\beta_i(\gamma_1, \gamma_2)$ as in (9.3). As we assume α_{i+1} lifts, we have that β_i has range $Z_i \cap \Lambda_i$.

Recall that we have $\tilde h\colon M \to N_{i+1}/\Lambda_{i+1}$ homotopic to P_{i+1} and lifting to a map $\tilde h\colon \tilde M_{i+1} \to N_{i+1}$ that intertwines the actions of $\tilde\alpha_{i+1}$ and ρ_{i+1}. Let $H\colon M \to N_i/\Lambda_i$ be any continuous map, homotopic to P_i and lifting h. As discussed in Section 4.1 we may find a lift $\tilde H\colon \tilde M_i \to N_i$ of H that is Λ_i-equivariantly homotopic to $\tilde P_i$ and also lifts $\tilde h\colon \tilde M_{i+1} \to N_{i+1}$.

Given $\gamma \in \Gamma$ and $x \in \tilde M_{i+1}$, let

$$\tilde\omega_\gamma(x) = H(\tilde\alpha_i(\gamma)(x))^{-1}\rho_i(\gamma)(H(x)).$$

Using that $\tilde h$ intertwines the actions of $\tilde\alpha_{i+1}$ and ρ_{i+1} and that $\tilde H(x\lambda) = \tilde H(x)\lambda$ for $\lambda \in \Lambda_i$, we verify for every γ that

(1) $\tilde\omega_\gamma$ is Λ_i-invariant, and
(2) $\tilde\omega_\gamma(x) \in Z_i$ for every x.

It follows that $\tilde\omega_\gamma$ induces a function $\omega_\gamma\colon M \to Z_i$. Recall that μ is the invariant measure for the action α on M. Identifying $Z_i \simeq \mathbb{R}^{d_i}$, define $\eta\colon \Gamma \to Z_i$ by

$$\eta(\gamma) = \int_M \omega_\gamma \, d\mu.$$

Viewing $\rho_i\restriction_{Z^i} \in \mathrm{Aut}(Z_i) \simeq \mathrm{GL}(\mathbb{R}^{d_i})$, we claim

Claim 9.9. *We have $d_{\rho_i,1}\eta = \beta_i$.*

Proof. We have for any $x \in \tilde M_i$ that

$$\tilde\alpha_i(\gamma_1\gamma_2)(x) = \tilde\alpha_i(\gamma_1)(\tilde\alpha_i(\gamma_2)(x))\beta_i(\gamma_1, \gamma_2).$$

Applying the map H to both sides, we have

$$\begin{aligned}\rho_i(\gamma_1\gamma_2)(H(x))\cdot\tilde{\omega}_{\gamma_1\gamma_2}(x)^{-1} &= H(\tilde{\alpha}_i(\gamma_1\gamma_2)(x))\\ &= H\left(\tilde{\alpha}_i(\gamma_1)(\tilde{\alpha}_i(\gamma_2)(x))\right)\cdot\beta_i(\gamma_1,\gamma_2)\\ &= \rho_i(\gamma_1)(H(\tilde{\alpha}_i(\gamma_2)(x)))\cdot\tilde{\omega}_{\gamma_1}(\tilde{\alpha}_i(\gamma_2)(x))^{-1}\cdot\beta_i(\gamma_1,\gamma_2)\\ &= \rho_i(\gamma_1)(\rho_i(\gamma_2)(H(x)))\cdot\rho_i(\gamma_1)(\tilde{\omega}_{\gamma_2}(x))^{-1}\cdot\tilde{\omega}_{\gamma_1}(\tilde{\alpha}_i(\gamma_2)(x))^{-1}\cdot\beta_i(\gamma_1,\gamma_2)\\ &= \rho_i(\gamma_1\gamma_2)(H(x)))\cdot\rho_i(\gamma_1)(\tilde{\omega}_{\gamma_2}(x))^{-1}\cdot\tilde{\omega}_{\gamma_1}(\tilde{\alpha}_i(\gamma_2)(x))^{-1}\cdot\beta_i(\gamma_1,\gamma_2).\end{aligned}$$

It follows that for any $x\in\tilde{M}_i$,

$$\beta_i(\gamma_1,\gamma_2)=\tilde{\omega}_{\gamma_1\gamma_2}(x)^{-1}\cdot\rho_i(\gamma_1)(\tilde{\omega}_{\gamma_2}(x))\cdot\tilde{\omega}_{\gamma_1}(\tilde{\alpha}_i(\gamma_2)(x)).$$

Using that the measure μ is $\alpha(\gamma_2)$-invariant, it follows for any $x\in\tilde{M}$ that

$$\begin{aligned}\beta_i(\gamma_1,\gamma_2) &= \omega_{\gamma_1\gamma_2}(x)^{-1}\cdot\rho_i(\gamma_1)(\omega_{\gamma_2}(x))\cdot\omega_{\gamma_1}(\alpha(\gamma_2)(x))\\ &= \rho_i(\gamma_1)(\eta(\gamma_2))\cdot\eta(\gamma_1\gamma_2)^{-1}\cdot\eta(\gamma_1)\\ &= d_{\rho_i,1}\eta(\gamma_1,\gamma_2).\end{aligned}$$

□

It follows that β_i vanishes as an element of $H^2_{\rho_i}(\Gamma;\mathbb{R}^{d_i})$. As discussed above, by passing to a finite-index subgroup $\hat{\Gamma}\subset\Gamma$, it follows that β_i vanishes as an element of $H^2_{\rho_i}(\Gamma;\mathbb{Z}^{d_i})$, where the lattice $\mathbb{Z}^{d_i}$ is identified with $Z_i\cap\Lambda_i$. Then, as in the proof of Lemma 9.4, we may correct the choice of lifts $\tilde{\alpha}_i(\gamma)$ for $\gamma\in\hat{\Gamma}$ into a coherent action that lifts the action α. The proposition follows. □

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