

# Dynamical spectral rigidity among $\mathbb{Z}_2$ -symmetric strictly convex domains close to a circle

(Appendix B coauthored with H. Hezari)

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## Abstract

We show that any sufficiently (finitely) smooth  $\mathbb{Z}_2$ -symmetric strictly convex domain sufficiently close to a circle is dynamically spectrally rigid; i.e., all deformations among domains in the same class that preserve the length of all periodic orbits of the associated billiard flow must necessarily be isometric deformations. This gives a partial answer to a question of P. Sarnak.

## 1. Introduction

In this paper we study a problem motivated by the famous question of M. Kac [12]: “Can one hear the shape of a drum?” More formally: let  $\Omega \subset \mathbb{R}^2$  be a planar domain, and denote by  $\text{Sp}(\Omega) = \{0 < \lambda_0 \leq \lambda_1 \leq \dots \leq \lambda_k \leq \dots\}$  the *Laplace Spectrum* of  $\Omega$  with some specified boundary conditions (e.g., one can consider Dirichlet boundary conditions<sup>1</sup>). In other words,  $\text{Sp}(\Omega)$  is the set (with multiplicities) of positive real numbers  $\lambda$  that satisfy the eigenvalue problem

$$\Delta u + \lambda^2 u = 0, \quad u = 0 \text{ on } \partial\Omega.$$

Given a class  $\mathcal{M}$  of domains and a domain  $\Omega \in \mathcal{M}$ , we say that  $\Omega$  is *spectrally determined in  $\mathcal{M}$*  if it is the unique element (modulo isometries) of  $\mathcal{M}$  with its Laplace Spectrum: if  $\Omega, \Omega' \in \mathcal{M}$  are *isospectral*, i.e.,  $\text{Sp}(\Omega') = \text{Sp}(\Omega)$ , then  $\Omega'$  is the image of  $\Omega$  by an isometry (i.e., a composition of translations and rotations).

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Keywords: inverse problem, Laplace spectrum, length spectrum, isospectrality, spectral rigidity, convex billiards, Lazutkin coordinates

AMS Classification: Primary: 37D50; Secondary: 35P05, 58J53, 35R30.

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<sup>1</sup>From the physical point of view, the Dirichlet eigenvalues  $\lambda$  correspond to the eigenfrequencies of a vibrating membrane of shape  $\Omega$  that is fixed along its boundary.

The question of Kac can be thus formulated as follows, assuming we have fixed a class of domains  $\mathcal{M}$ :

INVERSE SPECTRAL PROBLEM. *Is every  $\Omega \in \mathcal{M}$  spectrally determined?*

If  $\mathcal{M}$  is the space of all planar domains, the answer is well known to be negative; see, e.g., [7], which generalizes some results previously obtained for compact manifolds without boundary (see [23], [24]).<sup>2</sup> However, all known examples of domains that are not spectrally determined are not convex; moreover, they are bounded by curves that are only piecewise analytic (e.g., plane domains with corners). On the other hand, Zelditch proved in [26] that the inverse spectral problem has a positive answer when  $\mathcal{M}$  is a generic class of analytic  $\mathbb{Z}_2$ -symmetric convex domains (i.e., symmetric with respect to reflection about a given axis).

The problem for nonanalytic domains is substantially more challenging. In the  $C^\infty$  category, Osgood–Phillips–Sarnak [16], [15], [17] showed that isospectral sets are necessarily compact in the  $C^\infty$  topology. Sarnak (see [22]) also conjectured that an isospectral set consists of isolated domains. In other words,  $C^\infty$ -close to a  $C^\infty$  domain there should be no isospectral domains, except those that can be obtained by an isometry. A weaker version of this conjecture can be stated as follows: a domain  $\Omega$  is said to be *spectrally rigid in  $\mathcal{M}$*  if any  $C^1$ -smooth one-parameter isospectral family  $(\Omega_\tau)_{|\tau| \leq 1} \subset \mathcal{M}$  with  $\Omega_0 = \Omega$  is necessarily an isometric family. We can then ask: “Are all  $C^\infty$  domains spectrally rigid?”

The problem of spectral rigidity is in principle much simpler than the inverse spectral problem; yet it turns out to be extremely challenging. Hezari–Zelditch (see [11]) provided a result in the affirmative direction: let  $\Omega_0$  be bounded by an ellipse  $\mathcal{E}$ ; then any one-parameter isospectral  $C^\infty$ -deformation  $(\Omega_\tau)_{|\tau| \leq 1}$  that additionally preserves the  $\mathbb{Z}_2 \times \mathbb{Z}_2$  symmetry group of the ellipse is necessarily flat (i.e., all derivatives have to vanish for  $\tau = 0$ ).<sup>3</sup> Popov–Topalov [21] recently extended these results (see also [20]).

Further historical remarks on the inverse spectral problem can also be found in [11] and in the surveys [25] and [27].

1.1. *The length spectrum and its relation with the Laplace spectrum.* There is a remarkable relation between the Laplace spectrum of a domain and a dynamically defined object that we now proceed to define. The *length spectrum*

<sup>2</sup>Remarkably, Sunada (see [23]) exhibits isospectral sets (i.e., sets of isospectral manifolds) of arbitrarily large cardinality.

<sup>3</sup>Results of this kind are usually referred to as *infinitesimal spectral rigidity*.

of  $\Omega$  is defined as the set

$$\mathcal{L}(\Omega) = \mathbb{N} \{\text{length of all closed geodesics of } \Omega\} \cup \mathbb{N} \{l_{\partial\Omega}\},$$

where  $l_{\partial\Omega}$  denotes the length of the boundary  $\partial\Omega$  and  $\mathbb{N} = \{1, 2, \dots\}$ . By *closed geodesic of  $\Omega$*  above we mean a periodic trajectory of the billiard flow (i.e., geodesic flow in the interior of  $\Omega$  with optical reflections on  $\partial\Omega$ ).

Andersson–Melrose (see [1, Th. (0.5)], which generalized earlier results in [3], [6]) showed that, for strictly convex  $C^\infty$  domains, the following relation between the singular support of the *wave trace* and the length spectrum holds:

$$(1.1) \quad \text{sing supp} \left( t \mapsto \sum_{\lambda_j \in \text{Sp}(\Omega)} \exp(i\lambda_j t) \right) \subset \pm\mathcal{L}(\Omega) \cup \{0\}.$$

Indeed, the above inclusion holds for nonconvex  $C^\infty$  domains in arbitrary dimension (see [19, Th. 5.4.6]). Moreover, under generic conditions (see Remark 2.10 for more details) it can be shown that the above inclusion is indeed an equality and the Laplace Spectrum determines the length spectrum.

It is natural to pose the same questions as above in this dynamical setting. We say that  $\Omega$  is *dynamically spectrally determined in  $\mathcal{M}$*  if it is the unique element (modulo isometries) of  $\mathcal{M}$  with its length spectrum.

INVERSE DYNAMICAL PROBLEM. *Is every  $\Omega \in \mathcal{M}$  dynamically spectrally determined?*

All counterexamples to the inverse spectral problem mentioned earlier also constitute counterexamples to the inverse dynamical problem. Likewise, at present, there is no known counterexample realized by convex domains. Moreover, the above mentioned result by Zelditch (in [26]) also holds in the dynamical context. In the case of sufficiently smooth convex domain, the problem is open and presents the same challenges as the inverse spectral problem. Let us now define the dynamical notion corresponding to spectral rigidity: we say that a domain  $\Omega_0 \in \mathcal{M}$  is *dynamically spectrally rigid in  $\mathcal{M}$*  if any  $C^1$ -smooth one-parameter dynamically isospectral family  $(\Omega_\tau)_{|\tau| \leq 1} \subset \mathcal{M}$  is necessarily an isometric family. We can now present our result, which will be more precisely stated in Section 2.

MAIN RESULT. *Let  $\mathcal{M}$  be the set of strictly convex domains with sufficiently (finitely) smooth boundary, axial symmetry and that are sufficiently close to a circle. Then any  $\Omega \in \mathcal{M}$  is dynamically spectrally rigid in  $\mathcal{M}$ .*

1.2. *Related prior results.* The problem of isospectral deformations of manifolds without boundary were considered in some early works on variations of the spectral functions and wave invariants.

Let  $(M, g)$  be a compact boundaryless Riemannian manifold. A family  $(g_\tau)_{|\tau| \leq 1}$  of Riemannian metrics on  $M$  depending smoothly on the parameter  $|\tau| \leq 1$  is called a *deformation of the metric  $g$*  if  $g_0 = g$ . A deformation is called *trivial* if there exists a one-parameter family of diffeomorphisms  $\varphi_\tau : M \rightarrow M$  such that  $\varphi_0 = \text{Id}$ , and  $g_\tau = (\varphi_\tau)^* g_0$ . For each homotopy class of closed curves in  $M$ , consider the infimum of  $g$ -lengths of curves belonging to the given homotopy class. The *length spectrum*  $\mathcal{L}(M, g)$  is defined as the union of these lengths over all homotopy classes. The *inverse spectral problem* in this setting is to show that two metrics with the same length spectrum are isometric.

Likewise, a deformation  $(g_\tau)_{|\tau| \leq 1}$  is said to be *isospectral* if  $\mathcal{L}(M, g_\tau) = \mathcal{L}(M, g)$ . We say that a Riemannian manifold  $(M, g)$  is *spectrally rigid* if it does not admit nontrivial isospectral deformations.

Guillemin–Kazhdan in [8] showed that any negatively curved surface is spectrally rigid among negatively curved surfaces. This result has been later extended to compact manifolds of negative curvature in [5]. Remark that an open question is whether one can generalize the result of [8] to hyperbolic billiards.

Our result is *an analog of [8] for  $\mathbb{Z}_2$ -symmetric convex domains close to a disk*.

It is also worth mentioning that for such systems there is a partial solution of the inverse spectral problem due independently to Croke [4] and Otal [18], which can be stated as follows: any negatively curved manifold is uniquely determined by its *marked length spectrum*.<sup>4</sup>

Another example of deformational spectral rigidity appears in de la Llave, Marco and Moriyón [14]. Recall that one can associate to a symplectic map a generating function. Then, for each periodic orbit, one can define the corresponding *action* by summing the generating function along the orbit. This value of the action is invariant under symplectic coordinate changes. The union of the values all these actions over all periodic orbits is called the action spectrum of the symplectic map. In [14], it is shown that there are no nontrivial deformations of exact symplectic mappings  $B_\tau$ ,  $\tau \in [-1, 1]$ , leaving the action spectrum fixed, when  $B_\tau$  are Anosov’s mappings on a symplectic manifold. One of the reasons for symplectic rigidity in [14] is that all periodic points of  $B_\tau$  are hyperbolic and form a dense set.

*Outline of our paper.* In Section 2, after introducing the necessary objects, we give a formal statement of the main result. In Section 3, we reduce any

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<sup>4</sup>The *marked length spectrum* is given by the collection of lengths of closed geodesics paired with their homotopy type.

family  $(\Omega_\tau)_{|\tau|\leq 1}$  of axially symmetric domains to a *normalized* family  $(\tilde{\Omega}_\tau)_{|\tau|\leq 1}$  by rotations and translations, so that they share the same symmetry axis and their boundaries share a common point on this axis; we then restate our main result for normalized families (see Theorem 3.2). In Section 4, we prove the existence of maximal symmetric periodic orbits of period  $q$  and rotation number  $1/q$  for any  $q > 1$ , i.e., axially symmetric  $q$ -gons of maximal perimeter. If a family  $(\Omega_\tau)_{|\tau|\leq 1}$  is isospectral, then to each orbit of this type we can associate an isoperimetric functional  $\ell_{\Omega,q}$  that vanishes in the direction of the perturbation. Using these linear functionals  $(\ell_{\Omega,q})_{q>1}$  we define a *linearized isospectral operator*  $\mathcal{T}_\Omega$  (see (4.6)) and reduce the main result to the claim that this operator  $\mathcal{T}_\Omega$  is injective (Theorem 4.9). In Section 5, we introduce a modification of Lazutkin coordinates designed to study the behaviour of  $\mathcal{T}_\Omega$  and we prove Theorem 4.9 using the modified Lazutkin coordinates and some explicit computations that we obtain in Appendix B, which is joint with H. Hezari. In Section 6, we outline the proof of a generalization of our result to domains that are not necessarily close to a circle. In Section 7, we add some remarks on the challenges that we expect when trying to prove our result in a more general setting. In Appendix A, we derive required properties of the modified Lazutkin coordinates.

## 2. Definitions and statement of results

We now provide more precise definitions of the objects introduced in the previous introductory section in the billiard table setting.

Denote by  $\mathcal{D}^r$  the set of strictly convex open planar domains  $\Omega$  whose boundary is  $C^{r+1}$  smooth.<sup>5</sup> For each domain  $\Omega \in \mathcal{D}^r$  denote by  $\rho_\Omega$  the radius of the curvature of the boundary  $\partial\Omega$ . We will always consider the underlying class of domains  $\mathcal{M} \subset \mathcal{D}^r$  for<sup>6</sup>  $r \geq 2$ . By convention, we set the *positive* orientation of  $\partial\Omega$  to be counterclockwise.

By definition, geodesics in a bounded planar domain are geodesics (straight lines) that get reflected at the boundary according to the optical law “angle of reflection = angle of incidence.” Such geodesics are often called *broken geodesics*. In particular,

*Definition 2.1.* A *closed geodesic* in  $\Omega$  is a (not necessarily convex) polygon inscribed in  $\partial\Omega$  such that at each vertex, the angles formed by each of the two sides joining at the vertex with the tangent line to  $\partial\Omega$  are equal. The *perimeter* of the polygon is called the *length of the geodesic*.

<sup>5</sup>We use the superscript with  $r$  because the associated billiard map is  $C^r$  (see Section 4).

<sup>6</sup>The fact that the boundary is at least  $C^3$  guarantees that the (broken) geodesic flow is complete (see, e.g., [9]).

*Definition 2.2.* The *length spectrum* of a domain  $\Omega$  is the set of positive real numbers

$$\mathcal{L}(\Omega) = \mathbb{N} \{\text{length of all closed geodesics of } \Omega\} \cup \mathbb{N} \{l_{\partial\Omega}\},$$

where  $l_{\partial\Omega}$  is the length of the boundary  $\partial\Omega$ .

Let us introduce the notion of a deformation of a domain. Recall the standard notation  $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ .

*Definition 2.3.* We say that  $(\Omega_\tau)_{|\tau| \leq 1}$  is a  $C^1$  *one-parameter family of domains* in  $\mathcal{M}$  if  $\Omega_\tau \in \mathcal{M}$  for any  $|\tau| \leq 1$  and there exists

$$\gamma : [-1, 1] \times \mathbb{T}^1 \rightarrow \mathbb{R}^2$$

such that  $\gamma(\tau, \xi)$  is continuously differentiable in  $\tau$  and, for any  $\tau \in [-1, 1]$ , the map  $\gamma(\tau, \cdot)$  is a  $C^{r+1}$  diffeomorphism of  $\mathbb{T}^1$  onto  $\partial\Omega_\tau$ . The function  $\gamma$  is said to be a *parametrization* of the family.

*Notational Remark.* We adopt the following typographical conventions for parametrizations. The symbol  $\tau$  is always used to denote different elements of the family  $\Omega_\tau$ . The symbol  $\xi$  denotes an arbitrary parametrization of the boundary of some domain  $\Omega$ . The symbol  $s$  always denotes *arc-length* parametrization of the boundary of some domain  $\Omega$ . In Section 5 we introduce the *Lazutkin* parametrization of the boundary of a domain  $\Omega$ : it is always denoted by the symbol  $x$ .

*Definition 2.4.* A family  $(\Omega_\tau)_{|\tau| \leq 1}$  is said to be *isometric* (or *trivial*) if there exists a family  $(J_\tau)_{|\tau| \leq 1}$  of isometries  $J_\tau : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  (i.e., composition of a rotation and a translation) such that  $\Omega_\tau = J_\tau \Omega_0$ . A family  $(\Omega_\tau)_{|\tau| \leq 1}$  is said to be *constant* if  $\Omega_\tau = \Omega_0$  for all  $|\tau| \leq 1$ .

*Remark 2.5.* For a given family  $(\Omega_\tau)_{|\tau| \leq 1}$ , the parametrization  $\gamma$  is, of course, not unique. In fact,  $\gamma$  and  $\tilde{\gamma}$  parametrize the same family  $(\Omega_\tau)_{|\tau| \leq 1}$  if and only if there exists a  $C^1$  family of  $C^{r+1}$  circle diffeomorphisms  $\xi : [-1, 1] \times \mathbb{T}^1 \rightarrow \mathbb{T}^1$ , such that  $\tilde{\gamma}(\tau, \tilde{\xi}) = \gamma(\tau, \xi(\tau, \tilde{\xi}))$ . We call two parametrizations *equivalent* if they correspond to the same family of domains. Furthermore, notice that we do not consider families that differ by a *time re-parametrization* to be equivalent.

We now proceed to define the main object of our work: families of isospectral domains.

*Definition 2.6.* A family  $(\Omega_\tau)_{|\tau| \leq 1}$  is said to be *dynamically isospectral*<sup>7</sup> if  $\mathcal{L}(\Omega_\tau) = \mathcal{L}(\Omega_{\tau'})$  for any  $\tau, \tau' \in [-1, 1]$ .

<sup>7</sup>In the literature this notion is also known as *length-isospectrality*.

Equipped with the above definition, we can define the *dynamical spectral rigidity* of a domain  $\Omega$ .

*Definition 2.7.* A domain  $\Omega \in \mathcal{M}$  is said to be *dynamically spectrally rigid* in  $\mathcal{M}$  if any dynamically isospectral family of domains  $(\Omega_\tau)_{|\tau| \leq 1}$  in  $\mathcal{M}$  with  $\Omega_0 = \Omega$  is an isometric family.

We are going to show that if  $r$  is sufficiently large, any domain that is sufficiently close to a circle and axially symmetric is dynamically spectrally rigid in this class of domains. We now proceed to define the class.

*Definition 2.8.* We say that  $\Omega$  is  $\mathbb{Z}_2$ -*symmetric* (or *axially symmetric*) if there exists a reflection of the plane  $\mathcal{R} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  such that  $\mathcal{R}\Omega = \Omega$ . Denote by  $\mathcal{S}^r$  the set of all  $\mathbb{Z}_2$ -symmetric domains in  $\mathcal{D}^r$ .

To introduce the notion of *closeness to a circle*, recall that a closed curve is a circle if and only if its curvature is constant.

*Definition 2.9.* Let  $\Omega \in \mathcal{D}^r$  of perimeter 1 parametrized in arc-length by  $\gamma$ , and let  $\mathbb{D}_\Omega$  be a disk of perimeter 1 that is tangent to  $\Omega$  at the point  $s = 0$ , parametrized in arc-length by  $\gamma_{\mathbb{D}}$ . For  $\delta > 0$ ,  $\Omega$  is said to be  $\delta$ -*close to a circle* if

$$\|\gamma - \gamma_{\mathbb{D}}\|_{C^{r+1}} \leq \delta.$$

A domain  $\Omega \in \mathcal{D}^r$  of arbitrary perimeter is said to be  $\delta$ -*close to a circle* if its rescaling of perimeter 1 is  $\delta$ -close to a circle.

We denote by  $\mathcal{D}_\delta^r$  (resp.  $\mathcal{S}_\delta^r$ ) the set of domains in  $\mathcal{D}^r$  (resp.  $\mathcal{S}^r$ ) that are  $\delta$ -close to a circle. We are finally able to state the main result of this paper.

**MAIN THEOREM.** *Let  $r = 8$ ; there exists  $\delta > 0$  such that any domain  $\Omega \in \mathcal{S}_\delta^r$  is dynamically spectrally rigid in  $\mathcal{S}_\delta^r$ .*

*Remark 2.10.* It turns out that the Laplace spectrum generically determines the length spectrum. More precisely, assume that the following generic conditions are met:

- (a) no two distinct periodic orbits have the same length;
- (b) the Poincaré map of any periodic orbit of the associated billiard ball map (see (4.1)) is nondegenerate.

Then we can replace the “ $\subset$ ” symbol in (1.1) with an “ $=$ ” sign. (See [19, Ch. 7]; indeed the same result holds in arbitrary dimension, and one can even drop the convexity assumption in the case of planar domains.)

In view of the above remark, our Main Theorem has an immediate rephrasing in terms of the spectral rigidity problem.

COROLLARY. *Let  $r = 8$ , and let  $\tilde{\mathcal{S}}_\delta^r \subset \mathcal{S}_\delta^r$  be the set of domains in  $\mathcal{S}_\delta^r$  that satisfy the generic assumptions listed in Remark 2.10. There exists  $\delta > 0$  such that any  $\Omega \in \tilde{\mathcal{S}}_\delta^r$  is spectrally rigid in  $\tilde{\mathcal{S}}_\delta^r$ .*

In other words, finitely smooth  $\mathbb{Z}_2$ -symmetric convex domains close to a circle are generically spectrally rigid.

*Remark 2.11.* Hezari, in a recent preprint (see [10]), using the method of this paper combined with wave trace invariants of Guillemin–Melrose and the heat trace invariants of Zayed for the Laplacian with Robin boundary conditions, shows that one can generalize the Dirichlet/Neumann spectral rigidity claimed in the above corollary to the case of Robin boundary conditions. (See [10] for the references.)

### 3. A preliminary reduction

It is natural to introduce a notion of normalization, which allows us to restate our result in a simpler manner; this will be accomplished in Theorem 3.2. Let  $\Omega \in \mathcal{S}^r$ ; in the case that  $\Omega$  admits more than one axes of symmetry, let us choose (arbitrarily) one of such axes and refer to it as *the symmetry axis* of  $\Omega$ . Since the domain  $\Omega$  is convex, its symmetry axis intersects  $\partial\Omega$  in two points. Choose (arbitrarily) one of such points: we refer to it as *the marked point* of  $\partial\Omega$ ; the other point will be referred to as *the auxiliary point* of  $\partial\Omega$ . From now on, whenever we consider a domain  $\Omega$ , we assume that a choice for the symmetry axis, the marked point and the auxiliary point has been made. Observe that once they have been chosen for  $\Omega_0$ , then, by continuity, they are unambiguously determined for any element of the family  $(\Omega_\tau)_{|\tau| \leq 1}$ . Furthermore, we also assume that the parametrization  $\gamma$  of the family  $(\Omega_\tau)_{|\tau| \leq 1}$  is such that  $\gamma(\tau, 0)$  is the marked point of  $\Omega_\tau$ .

*Definition 3.1.* A domain  $\Omega \in \mathcal{S}^r$  is said to be *normalized* if the marked point of  $\partial\Omega$  is at the origin of  $\mathbb{R}^2$  and the auxiliary point lies on the positive  $x$ -semi-axis. A family  $(\Omega_\tau)_{|\tau| \leq 1}$  is said to be *normalized* if  $\Omega_\tau$  is normalized for any  $|\tau| \leq 1$ .

Naturally, given a family  $(\Omega_\tau)_{|\tau| \leq 1}$ , we can always use isometries to construct an associated normalized family  $(\tilde{\Omega}_\tau)_{|\tau| \leq 1}$  as follows:

- translate each domain so that the marked point of  $\partial\Omega_\tau$  is at the origin of  $\mathbb{R}^2$ .
- rotate the domain around the origin so that the auxiliary point of  $\partial\Omega_\tau$  lies on the positive horizontal semi-axis.

We call  $(\tilde{\Omega}_\tau)_{|\tau| \leq 1}$  the *normalization* of the family  $(\Omega_\tau)_{|\tau| \leq 1}$ . Since  $(\Omega_\tau)_{|\tau| \leq 1}$  is a  $C^1$  family, we gather that  $(\tilde{\Omega}_\tau)_{|\tau| \leq 1}$  is also a  $C^1$  family. Observe that, as each

$\tilde{\Omega}_\tau$  is obtained from  $\Omega_\tau$  via an isometry, we have  $\mathcal{L}(\Omega_\tau) = \mathcal{L}(\tilde{\Omega}_\tau)$ . In particular,  $(\Omega_\tau)_{|\tau| \leq 1}$  is a dynamically isospectral family if and only if so is  $(\tilde{\Omega}_\tau)_{|\tau| \leq 1}$ .

We can now give an equivalent statement of our Main Theorem as follows:

**THEOREM 3.2.** *Let  $r = 8$ ; there exists  $\delta > 0$  such that if  $(\Omega_\tau)_{|\tau| \leq 1}$  is a normalized, dynamically isospectral  $C^1$  family of domains in  $S_\delta^r$ , then  $(\Omega_\tau)_{|\tau| \leq 1}$  is a constant family.*

We now proceed to set up yet another equivalent statement of our Main Theorem, which will be stated as Theorem 3.4.

Given a parametrization  $\gamma$  of a family  $(\Omega_\tau)_{|\tau| \leq 1}$  in  $S^r$ , we define the *infinitesimal deformation function*

$$n_\gamma(\tau, \xi) = \langle \partial_\tau \gamma(\tau, \xi), N_\gamma(\tau, \xi) \rangle,$$

where  $\langle \cdot, \cdot \rangle$  is the usual scalar product in  $\mathbb{R}^2$  and  $N_\gamma(\tau, \xi)$  is the outgoing unit normal vector to  $\partial\Omega_\tau$  at the point  $\gamma(\tau, \xi)$ . Observe that  $n$  is continuous in  $\tau$  and  $n(\tau, \cdot) \in C^r(\mathbb{T}^1, \mathbb{R})$  for any  $\tau \in [-1, 1]$ . By the normalization condition of  $(\Omega_\tau)_{|\tau| \leq 1}$ , we conclude that  $n_\gamma(\tau, \cdot)$  is an even function, i.e.,  $n_\gamma(\tau, \xi) = n_\gamma(\tau, -\xi)$ , and  $n_\gamma(\tau, 0) = 0$  for any  $\tau \in [-1, 1]$ .

**LEMMA 3.3.** *Let  $(\Omega_\tau)_{|\tau| \leq 1}$  be a family of domains in  $\mathcal{D}^r$ , and let  $\gamma(\tau, \xi)$  be a parametrization of the family. Then*

- (a) *for any other parametrization  $\tilde{\gamma}(\tau, \tilde{\xi}) = \gamma(\tau, \xi(\tau, \tilde{\xi}))$ , we have  $n_{\tilde{\gamma}}(\tau, \tilde{\xi}) = n_\gamma(\tau, \xi(\tau, \tilde{\xi}))$ . In particular, if for some  $\tau \in [-1, 1]$  we have that  $n_\gamma(\tau, \cdot)$  is identically 0, then  $n_{\tilde{\gamma}}(\tau, \cdot)$  is identically 0.*
- (b)  *$n_\gamma(\tau, \xi) = 0$  for all  $(\tau, \xi) \in [-1, 1] \times \mathbb{T}^1$  if and only if  $(\Omega_\tau)_{|\tau| \leq 1}$  is a constant family.*

*Proof.* Let us fix  $\tau$  and assume that  $\gamma(\tau, \cdot)$  and  $\tilde{\gamma}(\tau, \cdot)$  are two parametrizations of  $\Omega_\tau$ , i.e.,  $\gamma(\tau, \xi) = \tilde{\gamma}(\tau, \tilde{\xi}(\tau, \xi))$ . Differentiation with respect to  $\tau$  reads

$$\partial_\tau \gamma(\tau, \xi) = \partial_\tau \tilde{\gamma}(\tau, \tilde{\xi}(\tau, \xi)) + \partial_\xi \tilde{\gamma}(\tau, \tilde{\xi}(\tau, \xi)) \partial_\xi \tilde{\xi}(\tau, \xi).$$

By taking the scalar product with  $N_\gamma(\tau, \xi) = N_{\tilde{\gamma}}(\tau, \tilde{\xi}(\tau, \xi))$  we conclude that  $n_\gamma(\tau, \xi) = n_{\tilde{\gamma}}(\tau, \tilde{\xi}(\tau, \xi))$ , which proves item (a).

In order to prove item (b), observe that if  $n_\gamma(\tau, \xi) = 0$ , then the vector  $\partial_\tau \gamma(\tau, \xi)$  is necessarily a multiple of  $\partial_\xi \gamma(\tau, \xi)$ ; since  $\gamma(\tau, \cdot)$  is assumed to be a diffeomorphism, we conclude that  $d\gamma(\tau, \xi)$  has rank 1 everywhere. It follows from the Constant Rank Theorem that the image of  $\gamma$  is a manifold of dimension 1 that can be parametrized by  $\gamma(0, \cdot)$ . This implies that  $\Omega_\tau = \Omega_0$  for any  $|\tau| \leq 1$ . The converse holds trivially by item (a). □

We can thus further restate Theorem 3.2 (and thus our Main Theorem) as follows:

**THEOREM 3.4.** *Let  $r = 8$ ; then there exists  $\delta > 0$  such that if  $(\Omega_\tau)_{|\tau| \leq 1}$  is a normalized dynamically isospectral  $C^1$  family of domains in  $\mathcal{S}_\delta^r$ , then  $n_\gamma = 0$  for all parametrizations  $\gamma$ .*

#### 4. Billiard dynamics of $\mathbb{Z}_2$ -symmetric domains

Let  $\Omega \in \mathcal{S}^r$ ; for definiteness we fix the perimeter of its boundary to be 1. Recall that  $s$  denotes the arc-length parametrization and that we conventionally assume that the marked point has coordinate  $s = 0$ ; moreover, since  $\Omega$  has perimeter 1, the auxiliary point has coordinate  $s = 1/2$ . We consider the billiard dynamics on  $\Omega$ , which is described as follows: a point particle travels with constant velocity in the interior of  $\Omega$ ; when the particle hits  $\partial\Omega$ , it bounces according to the law of optical reflection: angle of incidence = angle of reflection. Periodic trajectories of the billiard dynamics are thus, essentially<sup>8</sup> in 2-to-1 correspondence to closed geodesics of  $\Omega$ . It is customary to study the billiard dynamics by passing to a discrete-time version of it, i.e., to a map on the canonical Poincaré section  $M = \partial\Omega \times [-1, 1]$ . The first coordinate (parametrized<sup>9</sup> by  $\gamma(s)$ ) identifies the point at which the particle has collided with  $\partial\Omega$  and the second coordinate  $y$  equals  $\cos \varphi$ , where  $\varphi$  is the angle that the outgoing trajectory forms with the positively oriented tangent to  $\partial\Omega$ . The billiard ball map  $f$  on  $M$  is then defined as

$$(4.1) \quad \begin{aligned} f : \partial\Omega \times [-1, 1] &\rightarrow \partial\Omega \times [-1, 1], \\ (s, y) &\mapsto (s', y'), \end{aligned}$$

where  $s'$  is the coordinate of the point at which the trajectory emanating from  $s$  with angle  $\varphi$  collides once again with  $\partial\Omega$  and  $y' = \cos \varphi'$ , where  $\varphi'$  is the angle of incidence of the trajectory with the negatively oriented tangent to  $\partial\Omega$  at  $s'$ . The map  $f$  is an exact twist diffeomorphism which preserves the area form  $ds \wedge dy$ . Let us denote by

$$L(s, s') = \|\gamma(s) - \gamma(s')\|$$

the Euclidean distance between the two points on  $\partial\Omega$  parametrized by  $s$  and  $s'$ . Notice that  $L$  is a generating function of the billiard ball map, i.e., we have

$$\begin{cases} \frac{\partial L}{\partial s}(s, s') = -y, \\ \frac{\partial L}{\partial s'}(s, s') = y'. \end{cases}$$

<sup>8</sup>A periodic trajectory and its time-reversal trace are in fact the same closed geodesic; the exceptions are *bouncing ball trajectories*, which are invariant for time-reversal and thus correspond 1-to-1 to their closed geodesics.

<sup>9</sup>We abuse notation and also denote by  $\gamma : \mathbb{T}^1 \rightarrow \mathbb{R}^2$  the parametrization of the single domain  $\Omega$ .

Given  $\Omega_0, \Omega_1 \in \mathcal{S}^r$ , both normalized and both of length 1, let  $\gamma_0$  and  $\gamma_1$  be the corresponding arc-length parametrizations of their boundaries; let us define  $\text{dist}(\Omega_0, \Omega_1) = \|\gamma_0 - \gamma_1\|_{C^{r+1}}$ ; then by the above considerations we gather that for any  $\delta' > 0$ , there exists a  $\delta$  such that if  $\text{dist}(\Omega_0, \Omega_1) < \delta$ , then the corresponding generating functions will also be  $C^{r+1}$ -close to each other on the set  $\{s \neq s'\}$ ; hence  $\|f_{\Omega_0} - f_{\Omega_1}\|_{C^r} < \delta'$ .

Once we have defined the billiard map  $f$ , we can prove a simple but important property of the length spectrum.

LEMMA 4.1. *For any  $\Omega \in \mathcal{D}^r$  with  $r \geq 2$ ,  $\mathcal{L}(\Omega)$  has zero Lebesgue measure.*

*Proof.* Recall that Sard's Lemma implies that the set of critical values of a real valued  $C^r$ -function defined on an  $n$ -dimensional manifold has zero Lebesgue measure provided that  $r \geq n$ . For any  $q$ , let us define the function

$$(4.2) \quad \tilde{L}_q(s, y) = L(s_0, s_1) + L(s_1, s_2) + \cdots + L(s_{q-1}, s_0),$$

where  $(s_k, y_k) = f^k(s, y)$ . Periodic orbits of period  $q$  of the billiard map correspond to critical points of  $\tilde{L}_q$ . Indeed, if the  $q$ -tuple  $(s_0, s_1, \dots, s_{q-1})$  identifies the vertices of a periodic orbit, equality of the angle of reflection and the angle of incidence of the trajectory at any given  $s_k$  implies that partial derivative of the right-hand side of (4.2) with respect to  $s_k$  equals zero. Since we can express  $s_k = s_k(s, y)$ , using the chain rule we conclude that if  $(s, y)$  is a periodic point, then it is a critical point of  $\tilde{L}_q(s, y)$ ; the set of lengths of such orbits thus corresponds to the set of critical values of  $\tilde{L}_q$ . Since the billiard map  $f$  is  $C^r$  with  $r \geq 2$ , we conclude that the set of lengths of periodic orbits of period  $q$  has zero Lebesgue measure; by taking the (countable) union over  $q$  we conclude that  $\mathcal{L}(\Omega)$  has zero Lebesgue measure.  $\square$

*Remark.* Note that it is possible to construct (nongeneric) examples of smooth domains  $\Omega$  whose length spectrum has positive Hausdorff dimension.

The above lemma will be used to impose constraints on isospectral families by means of the following immediate corollary:

COROLLARY 4.2. *Let  $(\Omega_\tau)_{|\tau| \leq 1}$  be a family of domains in  $\mathcal{D}^r$ , and let  $\Delta : [-1, 1] \rightarrow \mathbb{R}$  be a Darboux function<sup>10</sup> so that  $\Delta(\tau) \in \mathcal{L}(\Omega_\tau)$ . If  $(\Omega_\tau)_{|\tau| \leq 1}$  is isospectral, then  $\Delta$  is constant.*

*Proof.* Assume that there exists  $\tau \in [-1, 1]$  such that  $\Delta(\tau) \neq \Delta(0)$ ; since  $\Delta$  is Darboux, we conclude that  $\Delta([0, \tau])$  contains an open set. Since the family

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<sup>10</sup>Recall that a function is said to be *Darboux* if it has the intermediate value property. In particular, the corollary applies to continuous functions.

is isospectral,  $\Delta(\tau) \in \mathcal{L}(\Omega_\tau) = \mathcal{L}(\Omega_0)$ ; we conclude that  $\Delta([0, \tau]) \subset \mathcal{L}(\Omega_0)$ , but this contradicts Lemma 4.1.  $\square$

In the sequel we assume  $(\Omega_\tau)_{|\tau| \leq 1}$  to be fixed together with a parametrization  $\gamma$ ; without risk of confusion we thus drop all subscripts  $\gamma$ . The symbol  $\Omega$  simply denotes an arbitrary element of the family. Let us start with a simple case: let  $\Delta_0(\tau)$  denote the perimeter of  $\Omega_\tau$ , that is,

$$\Delta_0(\tau) = \int_0^1 \|\partial_s \gamma(\tau, \xi)\| d\xi.$$

By definition,  $\Delta_0$  is continuous and  $\Delta_0(\tau) \in \mathcal{L}(\Omega_\tau)$ ; we conclude by Corollary 4.2 that  $\Delta_0$  is constant; hence,

$$0 = \Delta'_0(\tau) = \int_0^1 \langle \partial_{\tau\xi} \gamma(\tau, \xi), T(\tau, \xi) \rangle d\xi,$$

where  $T(\tau, \xi) = \partial_s \gamma(\tau, \xi) / \|\partial_s \gamma(\tau, \xi)\|$  is the positively oriented unit tangent vector to  $\Omega_\tau$  at the point  $\gamma(\tau, \xi)$ . Integrating by parts we obtain

$$\begin{aligned} 0 &= - \int_0^1 \langle \partial_\tau \gamma(\tau, \xi), \partial_s T(\tau, \xi) \rangle d\xi \\ &= \int_0^1 \frac{n(\tau, \xi)}{\rho_{\Omega_\tau}(\xi)} \frac{ds}{d\xi} d\xi, \end{aligned}$$

where  $\frac{ds}{d\xi}$  accounts for the change of variable from arc-length  $s$  to  $\xi$  and, recall,  $\rho_{\Omega_\tau}$  is the radius of the curvature of  $\partial\Omega_\tau$ .

For any  $\Omega$  ( parametrized by  $\xi$ ), we define the linear functional

$$(4.3) \quad \ell_{\Omega,0}(\nu) = \int_0^1 \frac{\nu(\xi)}{\rho_\Omega(\xi)} \frac{ds}{d\xi} d\xi.$$

By our above discussion we conclude that if  $(\Omega_\tau)_{|\tau| \leq 1}$  is isospectral, then for any  $\tau \in [-1, 1]$ , we have  $\ell_{\Omega_\tau,0}(n(\tau, \cdot)) = 0$ .

We will now proceed to define a sequence of functionals that are related to the variation of lengths of a special class of periodic orbits of the billiard map. Consider a periodic orbit of period  $q$ , and let  $p \in \mathbb{Z}$  denote its winding number.<sup>11</sup> Then we define the *rotation number* of the orbits as the ratio  $p/q$ . The following lemma is a simple consequence of the fact that  $\Omega$  has  $\mathbb{Z}_2$ -symmetry:

LEMMA 4.3. *Let  $\Omega \in \mathcal{S}^r$ ; for any  $q \geq 2$ , there exists a periodic orbit of rotation number  $1/q$  passing through the marked point of  $\partial\Omega$  and having maximal length among other periodic orbits passing through the marked point.*

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<sup>11</sup>We can define the winding number as the number of times that the associated polygon wraps around the boundary  $\partial\Omega$ ; alternatively, by considering a lift  $\hat{f}$  of  $f$  to the universal cover  $\mathbb{R} \times [-1, 1]$ , we have that  $\hat{f}^q(\hat{s}, y) = (\hat{s} + p, y)$ , where  $p \in \mathbb{Z}$  defines the winding number.

We call such an orbit marked symmetric maximal periodic orbit and denote it by  $S^q(\Omega)$ .

*Proof.* Let us recall that  $s$  denotes the parametrization in arc-length; we distinguish the cases of even and odd period.

*Case 1:  $q = 2k$  is even.* We claim there exists a  $q$ -periodic orbit passing through the marked point and the auxiliary point. Indeed, let us fix  $s_0 = 0$  and  $s_k = 1/2$  and consider the problem of maximizing the function

$$(4.4a) \quad L_q(\mathbf{s}) := 2 \sum_{i=0}^{k-1} L(s_i, s_{i+1}),$$

where  $\mathbf{s} = (s_1, \dots, s_{k-1})$  belongs to the compact set  $0 = s_0 \leq s_1 \leq \dots \leq s_{k-1} \leq s_k = 1/2$ . Let  $\bar{\mathbf{s}} = (\bar{s}_1, \dots, \bar{s}_{k-1})$  be a maximum of  $L_q$ ; observe that by the triangle inequality and strict convexity, we have  $\bar{s}_0 < \bar{s}_1 < \dots < \bar{s}_{k-1} < \bar{s}_k$ . If we fix conventionally  $\bar{s}_0 = 0, \bar{s}_k = 1/2$ , then

$$\partial_1 L(\bar{s}_i, \bar{s}_{i+1}) = -\partial_2 L(\bar{s}_{i-1}, \bar{s}_i), \quad i = 1, \dots, k - 1.$$

Completing  $\bar{s}_{2k-i} = -\bar{s}_i, i = 1, \dots, k - 1$ , we obtain a periodic orbit of period  $2k = q$ , which is of maximal length among symmetric orbits.

*Case 2:  $q = 2k + 1$  is odd.* We claim there exists a periodic orbit passing through the marked point and so that the segment  $\gamma(s_k)\gamma(s_{k+1})$  is perpendicular to the symmetry axis. Indeed, let us fix  $s_0 = 0$  and consider the problem of maximizing the function

$$(4.4b) \quad L_q(\mathbf{s}) := \sum_{i=0}^{k-1} 2L(s_i, s_{i+1}) + L(s_k, -s_k),$$

where  $\mathbf{s} = (s_1, \dots, s_k)$  belongs to the compact set  $0 = s_0 \leq s_1 \leq \dots \leq s_k \leq 1/2$ . Once again by the triangle inequality and strict convexity, the maximum is attained at a critical point  $\bar{\mathbf{s}} = (\bar{s}_1, \dots, \bar{s}_k)$  so that  $0 < \bar{s}_1 < \dots < \bar{s}_{k-1} < 1/2$ . Moreover, if we conventionally fix  $\bar{s}_0 = 0$ , we have

$$\begin{aligned} 0 &= \partial_1 L(\bar{s}_i, \bar{s}_{i+1}) + \partial_2 L(\bar{s}_{i-1}, \bar{s}_i), \quad i = 1, \dots, k - 1, \\ 0 &= \partial_2 L(\bar{s}_{k-1}, \bar{s}_k) + \frac{1}{2} \partial_1 L(\bar{s}_k, -\bar{s}_k) - \frac{1}{2} \partial_2 L(\bar{s}_k, -\bar{s}_k) \\ &= \partial_2 L(\bar{s}_{k-1}, \bar{s}_k) + \partial_1 L(\bar{s}_k, -\bar{s}_k). \end{aligned}$$

Completing  $\bar{s}_{2k+1-i} = -\bar{s}_i, i = 1, \dots, k - 1$ , we obtain a periodic orbit of period  $2k + 1 = q$  that is of maximal length amongst all symmetric orbits.  $\square$

Let us define  $L(\tau, s, s') = \|\gamma(\tau, s) - \gamma(\tau, s')\|$ . For  $q \geq 2$ , let  $L_q(\tau; \mathbf{s})$  denote the function defined in (4.4) for  $\Omega = \Omega_\tau$ . Correspondingly, let  $\Delta_q(\tau)$  denote the

length of the marked symmetric maximal periodic orbits of rotation number  $1/q$  for the domain  $\Omega_\tau$ ; that is,

$$\Delta_q(\tau) = \max_{\mathbf{s}} L_q(\tau; \mathbf{s}).$$

LEMMA 4.4. *For any  $q \geq 2$ , the function  $\Delta_q(\tau) : [-1, 1] \rightarrow \mathbb{R}$  is a Lipschitz function.*

*Proof.* Define

$$K_q := \max_{\tau \in [-1, 1]} \max_{\mathbf{s}} \|\partial_\tau L_q(\tau; \mathbf{s})\|.$$

Let us arbitrarily fix  $\tau \in [-1, 1]$  and let  $\bar{\mathbf{s}}$  realize the maximum of  $L_q(\tau; \mathbf{s})$ ; that is,  $L_q(\tau, \bar{\mathbf{s}}) = \Delta_q(\tau)$ . Then  $L_q(\tau'; \bar{\mathbf{s}}) \leq \Delta_q(\tau')$  for any  $\tau' \in \bar{U}'$ , and hence

$$\Delta_q(\tau) - \Delta_q(\tau') \leq L_q(\tau; \bar{\mathbf{s}}) - L_q(\tau'; \bar{\mathbf{s}}) \leq K_q |\tau - \tau'|.$$

Exchanging  $\tau$  and  $\tau'$  in the above inequality, we can thus conclude that

$$|\Delta_q(\tau) - \Delta_q(\tau')| \leq K_q |\tau - \tau'|. \quad \square$$

Observe that, by definition,  $\Delta_q(\tau) \in \mathcal{L}(\Omega_\tau)$  and thus, by Corollary 4.2, if  $(\Omega_\tau)_{|\tau| \leq 1}$  is isospectral, then  $\Delta_q$  is constant. There is an obvious obstacle that arises when one tries to apply the strategy that we employed above with  $\Delta_0$ : *a priori*  $\Delta_q$  is not differentiable. We therefore need to consider a slightly more general approach. Given a continuous function  $\Phi : [-1, 1] \subset \mathbb{R} \rightarrow \mathbb{R}$ , one can define its upper (resp. lower) differential  $D^+\Phi(\tau)$  (resp.  $D^-\Phi(\tau)$ ), which is characterized as follows:  $p \in D^+\Phi(\tau)$  (resp.  $p \in D^-\Phi(\tau)$ ) if and only if there exists a function  $G \in C^1([-1, 1], \mathbb{R})$ , such that  $G'(\tau) = p$  and  $G \geq \Phi$  (resp.  $G \leq \Phi$ ) in a neighborhood of  $\tau$ , with equality at  $\tau$ . Both  $D^+\Phi(\tau)$  and  $D^-\Phi(\tau)$  are convex subset of  $\mathbb{R}$ ; they are both nonempty if and only if  $\Phi$  is differentiable, and in this case  $D^+\Phi(\tau) = D^-\Phi(\tau) = \{\Phi'(\tau)\}$ .

LEMMA 4.5. *If  $\bar{\mathbf{s}}$  is a point realizing the maximum, i.e.,*

$$L_q(\tau; \bar{\mathbf{s}}) = \max_{\mathbf{s}} L_q(\tau; \mathbf{s}) = \Delta_q(\tau),$$

*then we have  $\partial_\tau L_q(\tau; \bar{\mathbf{s}}) \in D^-\Delta_q(\tau)$ .*

*Proof.* Since  $\Delta_q(\cdot)$  is the maximum of  $L_q(\cdot, \mathbf{s})$ , if  $\bar{\mathbf{s}}$  realizes the maximum at  $\tau$ , we have  $L_q(\tau'; \bar{\mathbf{s}}) \leq \Delta_q(\tau')$  for any  $\tau' \in [-1, 1]$ . Noticing that  $L_q(\cdot; \bar{\mathbf{s}})$  is a  $C^1$  function, by definition of lower differential, we conclude that  $p = \partial_\tau L_q(\tau; \bar{\mathbf{s}}) \in D^-\Delta_q(\tau)$ .  $\square$

*Remark.* Indeed, one could show that  $\Delta_q(t)$  is a semi-convex function and

$$D^-\Delta_q(\tau) = \text{co}\{\partial_\tau L_q(\tau; \bar{\mathbf{s}}) \mid L_q(\tau; \bar{\mathbf{s}}) = \Delta_q(\tau)\}.$$

Now let  $\Omega \in \mathcal{S}^r$  parametrized by  $\xi$ , and assume we fixed  $S^q(\Omega) = (\xi_q^k, \varphi_q^k)_{k=0}^{q-1}$  as a maximal marked symmetric periodic orbit of rotation number  $1/q$ ; then we define the functional  $\ell_{\Omega,q}$  as follows: for any continuous function  $\nu : \mathbb{T}^1 \rightarrow \mathbb{R}$ , we let

$$(4.5) \quad \ell_{\Omega,q}(\nu) := \sum_{k=0}^{q-1} \nu(\xi_q^k) \sin \varphi_q^k.$$

*Remark.* These functionals can, of course, be defined for any periodic orbit (rather than only for marked symmetric maximal orbits). Since we will not use nonsymmetric orbits for the proof of our Main Theorem, we find it simpler to use the above definition.

**PROPOSITION 4.6.** *Let  $(\Omega_\tau)_{|\tau| \leq 1}$  be an isospectral family; then for any  $\tau \in [-1, 1]$ ,  $q \geq 2$ , and having fixed arbitrarily  $\bar{S}_\tau^q$  as a maximal marked symmetric periodic orbit for  $\Omega_\tau$ , we have  $\ell_{\Omega_\tau,q}(n(\tau, \cdot)) = 0$ .*

*Proof.* Let us fix  $\tau \in [-1, 1]$  arbitrarily. To ease our notation let us write  $S^q = S_\tau^q$ . By assumption we have that the point  $\bar{\mathbf{s}}$  that corresponds to  $\bar{S}_\tau^q$  is a maximum, i.e.,  $\Delta_q(\tau) = \max_{\mathbf{s}} L_q(\tau; \mathbf{s}) = L^q(\tau, \bar{\mathbf{s}})$ . Then by Lemma 4.5,

$$\partial_\tau L_q(\tau; \bar{\mathbf{s}}) \in D^- \Delta_q(\tau) = \{0\}.$$

In particular, for  $\bar{S}^q = (\bar{s}_q^k, \bar{\varphi}_q^k)_{k=0}^{q-1}$ , since  $L_q(\tau; \bar{\mathbf{s}}) = \sum_{k=0}^{q-1} L(\tau, \bar{s}_q^k, \bar{s}_q^{k+1})$  and observing that

$$\begin{aligned} \partial_\tau L(\tau, s, s') &= \partial_\tau \|\gamma(\tau, s) - \gamma(\tau, s')\| \\ &= \frac{\gamma(\tau, s) - \gamma(\tau, s')}{\|\gamma(\tau, s) - \gamma(\tau, s')\|} \cdot [\partial_\tau \gamma(\tau, s) - \partial_\tau \gamma(\tau, s')], \end{aligned}$$

we get

$$\begin{aligned} 0 &= \partial_\tau L_q(\tau; \bar{\mathbf{s}}) = \partial_\tau \sum_{k=0}^{q-1} L(\tau, \bar{s}_q^k, \bar{s}_q^{k+1}) \\ &= \sum_{k=0}^{q-1} \left[ \frac{\gamma(\tau, \bar{s}_q^k) - \gamma(\tau, \bar{s}_q^{k-1})}{\|\gamma(\tau, \bar{s}_q^k) - \gamma(\tau, \bar{s}_q^{k-1})\|} - \frac{\gamma(\tau, \bar{s}_q^{k+1}) - \gamma(\tau, \bar{s}_q^k)}{\|\gamma(\tau, \bar{s}_q^{k+1}) - \gamma(\tau, \bar{s}_q^k)\|} \right] \cdot \partial_\tau \gamma(\tau, \bar{s}_q^k) \\ &= \sum_{k=0}^{q-1} 2 \sin \bar{\varphi}_q^k N(\tau, \bar{s}_q^k) \cdot \partial_\tau \gamma(\tau, \bar{s}_q^k) \\ &= 2 \sum_{k=0}^{q-1} n(\tau, \bar{s}_q^k) \sin \bar{\varphi}_q^k, \end{aligned}$$

which concludes the proof.  $\square$

*Remark 4.7.* Let  $S = \{(\tau, \partial_\tau L_q(\tau; \mathbf{s})) \mid \partial_{\mathbf{s}} L_q(\tau, \mathbf{s}) = 0\}$ ,  $U = [-1, 1]$ ; then  $S \subset T^*U$  is a Lagrangian submanifold. Let  $\pi : T^*U \rightarrow U$  denote the natural

projection; then  $L_q(t; \cdot)$  is a Morse function (critical points are nondegenerate) if and only if  $\tau$  is a regular value of the map  $\pi|_S$ . The set  $U_1$  of such values is an open set, and by Sard's theorem, it has full measure. Furthermore, the set  $U_0 \subset U_1$  of  $\tau$  such that  $L_q(\tau; \cdot)$  is an excellent Morse function (Morse function whose critical points have pairwise distinct critical values) is also an open subset of full measure. For any  $\tau_0 \in X_0$ , the critical points of  $L_q(\tau; \cdot)$  depends smoothly on  $\tau$  within a sufficiently small neighborhood of  $\tau_0$ . Hence we have

$$\Delta'_q(\tau) = 2\ell_{S^q}(n(\tau, \cdot)), \quad \tau \in U_0$$

for an arbitrary (i.e., not necessarily isospectral)  $C^2$  deformation  $\gamma$ .

We now conventionally define the additional functional  $\ell_{\Omega,1}(\nu)$  as the evaluation of the function  $\nu$  at the marked point  $s = 0$ ; that is, we simply let

$$\ell_{\Omega,1}(\nu) = \nu(0).$$

Observe that if  $(\Omega_\tau)_{|\tau| \leq 1}$  is a normalized family, then the marked point is fixed at the origin and, therefore,  $\ell_{\Omega_\tau,1}(n(\tau, \cdot)) = 0$ . We summarize our findings in the following statement:

**COROLLARY 4.8.** *Let  $(\Omega_\tau)_{|\tau| \leq 1}$  be a normalized isospectral family; then for any  $q \geq 0$ , we have  $\ell_{\Omega_\tau,q}(n_\gamma(\tau, \cdot)) = 0$  for any  $\tau \in [-1, 1]$ .*

Let us now define the space of  $C^r$ -smooth even functions

$$C_{\text{sym}}^r = \{\nu \in C^r(\mathbb{T}^1) \text{ s.t. } \nu(\xi) = \nu(-\xi)\}.$$

We then define the *linearized isospectral operator*  $\mathcal{T} : C_{\text{sym}}^r \rightarrow \mathbb{R}^{\mathbb{N}}$ :

$$(4.6) \quad \mathcal{T}_\Omega \nu = (\ell_{\Omega,0}(\nu), \ell_{\Omega,1}(\nu), \dots, \ell_{\Omega,q}(\nu), \dots).$$

In fact,  $\mathcal{T}$  has range in  $\ell^\infty$ , by definition of the functionals  $\ell_{\Omega,q}$ , since by [2, Lemma 8], there exists some  $C > 0$  so that for any  $q \geq 2$ , we have  $\sin \varphi_q^k \leq C/q$ .

We now prove that our Main Theorem is implied by the following statement:

**THEOREM 4.9.** *Let  $r = 8$ ; there exists  $\delta > 0$  so that the operator  $\mathcal{T}_\Omega : C_{\text{sym}}^r \rightarrow \ell^\infty$  is injective for any  $\Omega \in \mathcal{S}_\delta^r$ .*

*Proof of Theorem 3.4.* Assume that  $\delta$  is sufficiently small so that Theorem 4.9 holds. Suppose by contradiction that for some  $\tau \in [-1, 1]$ , we have that  $n_\gamma(\tau, \cdot)$  is not identically zero; hence, by Theorem 4.9 we conclude that there exists  $q$  so that  $\ell_{\Omega,q}(n_\gamma(\tau, \cdot)) \neq 0$ ; this contradicts Corollary 4.8.  $\square$

The rest of this paper is devoted to the proof of Theorem 4.9.

5. Proof of Theorem 4.9

Let us introduce some useful notation.

5.1. *Lazutkin coordinates.* We first define a convenient parametrization of  $\Omega$ , which is known as the *Lazutkin parametrization* (see [13]). Recall that the symbol  $s$  denotes parametrization by arc-length; then we define the Lazutkin parametrization, which will always be denoted by the symbol  $x$ , as follows:

$$(5.1) \quad x(s) = C_{\mathbb{I}} \int_0^s \rho(s')^{-2/3} ds', \quad \text{where } C_{\mathbb{I}} = \left[ \int_{\partial\Omega} \rho(s')^{-2/3} ds' \right]^{-1}.$$

We also introduce *Lazutkin weight* as the positive function:

$$(5.2) \quad \mu(x) = \frac{1}{2C_{\mathbb{I}}\rho(x)^{1/3}}.$$

The main advantage of this parametrization is that dynamical quantities related to marked symmetric (maximal) orbits have a particularly simple form with respect to the variable  $x$ .

LEMMA 5.1. *Assume  $r \geq 8$ ; for any  $\varepsilon > 0$  sufficiently small, there exists  $\delta > 0$  so that for any  $\Omega \in S_\delta^r$ , there exist  $C^{r-4}$  real-valued functions  $\alpha(x)$  and  $\beta(x)$  so that  $\alpha$  is an odd function,  $\beta$  is even,  $\|\alpha\|_{C^{r-4}}, \|\beta\|_{C^{r-4}} < \varepsilon$  and, for any marked symmetric (maximal)  $q$ -periodic orbit  $(x_q^0, \dots, x_q^{q-1})$ ,*

$$(5.3a) \quad x_q^k = k/q + \frac{\alpha(k/q)}{q^2} + \varepsilon O(q^{-4}).$$

Moreover, if  $\varphi_q^k$  denotes the angle of reflection of the trajectory at the  $k$ -th collision, we have

$$(5.3b) \quad \varphi_q^k = \frac{\mu(x_q^k)}{q} \left( 1 + \frac{\beta(k/q)}{q^2} + \varepsilon O(q^{-4}) \right).$$

The proof of the above lemma is given in Appendix A.2; it suggests that the Lazutkin parametrization is particularly well suited to study the functionals  $\ell_q$ 's.

5.2. *The linearized map modified by the Lazutkin weight.* It is more natural to define the auxiliary sequence of functionals

$$\tilde{\ell}_q(u) = \ell_q(\mu^{-1}u)$$

and correspondingly define

$$(5.4) \quad \tilde{\mathcal{T}}u = (\tilde{\ell}_0(u), \tilde{\ell}_1(u), \dots, \tilde{\ell}_q(u), \dots).$$

Observe that since  $\mu$  does not vanish, the injectivity of  $\tilde{\mathcal{T}}$  is equivalent to the injectivity of  $\mathcal{T}$ . However, the operator  $\tilde{\mathcal{T}}$  turns out to be more convenient

to study. It is, in fact, immediate to check (using the explicit formula (5.1) and (4.3)) that

$$\tilde{\ell}_0(u) = 2 \int_0^1 u(x) dx;$$

i.e.,  $\tilde{\ell}_0$  is proportional to the averaging functional with respect to Lebesgue measure. On the other hand,  $\tilde{\ell}_1(u) = \mu^{-1}(0)u(0)$  is the evaluation of  $\mu^{-1}u$  at the marked point. In Appendix B (joint with H. Hezari) we study the properties of the functionals  $\tilde{\ell}_k$  that will be used in the rest of this section.

5.3. *Mapping structure of the linearized map  $\tilde{\mathcal{J}}$ .* Recall that  $C_{\text{sym}}^r$  denotes the space of even  $C^r$ -functions of  $\mathbb{T}^1$ ; define the projector  $P_* : C_{\text{sym}}^r \rightarrow C_{*,\text{sym}}^r$ , where  $C_{*,\text{sym}}^r$  is the space of even, zero average  $C^r$ -functions of  $\mathbb{T}^1$ ; i.e.,

$$P_*u = u - \int_0^1 u(x) dx,$$

where  $dx$  is the Lebesgue measure with respect to the Lazutkin parameter  $x$ . Let  $L_{*,\text{sym}}^1$  be the space of even, zero average,  $L^1$  functions of  $\mathbb{T}^1$ ; i.e.,

$$\begin{aligned} L_{*,\text{sym}}^1 &= \left\{ u \in L^1(\mathbb{T}^1) \text{ s.t. } u(x) = u(-x), \int u(x) dx = 0 \right\} \\ &= \left\{ u \in L^1(\mathbb{T}^1) \text{ s.t. } u(x) = \sum_{j \geq 1} \hat{u}_j e_j \right\}, \end{aligned}$$

where  $\hat{u}_j$  denotes its Fourier coefficients in the basis  $B$ . We now proceed to define a space of *admissible functions*: for  $3 < \gamma < 4$ , define the subspace

$$X_{*,\gamma} = \{u \in L_{*,\text{sym}}^1 \text{ s.t. } \lim_{j \rightarrow \infty} j^\gamma |\hat{u}_j| = 0\},$$

equipped with the norm

$$\|u\|_\gamma = \max_{j \geq 1} j^\gamma |\hat{u}_j|.$$

The space  $(X_{*,\gamma}, \|\cdot\|_\gamma)$  is a (separable) Banach space.

*Remark 5.2.* Because of our constraints on  $\gamma$ , we conclude that  $C_{*,\text{sym}}^3 \subset X_{*,\gamma} \subset C_{*,\text{sym}}^2$ , whence the functionals

$$\tilde{\ell}_q(n) = \tilde{\ell}_q \left( \sum_{j=1}^\infty n_j e_j \right) = \sum_{j=1}^\infty n_j \tilde{\ell}_q(e_j)$$

are well defined on  $X_{*,\gamma}$ , since the Fourier series converges uniformly.

Notice that with our choice for the parameter  $r$ , we have  $r > \gamma$ .

Let  $\ell_*^\infty = \{b = (a_i)_{i \geq 0} \in \ell^\infty \text{ s.t. } a_0 = 0\}$ , and let us introduce the subspace

$$h_{*,\gamma} = \{b = (a_i)_{i \geq 0} \in \ell_*^\infty \text{ s.t. } \lim_{j \rightarrow \infty} j^\gamma a_j = 0\}$$

equipped with the norm  $\|b\|_\gamma = \max_{j \geq 0} j^\gamma |a_j|$ .

We now state the main technical result of this section; then we show how Theorem 4.9 follows from this result and finally provide its proof.

LEMMA 5.3. *There exist linearly independent vectors  $b_l, b_\bullet \in \ell^\infty \setminus h_{*,\gamma}$  so that, for any  $\Omega \in \mathcal{S}_\delta^r$  with  $r = 8$ , the operator  $\tilde{\mathcal{T}} : C_{\text{sym}}^r \rightarrow \ell^\infty$  can be decomposed as follows:*

$$\tilde{\mathcal{T}} = b_l \tilde{\ell}_0 + [b_\bullet \tilde{\ell}_\bullet + \tilde{\mathcal{T}}_{*,R}] P_*,$$

where  $\tilde{\mathcal{T}}_{*,R} : X_{*,\gamma} \rightarrow h_{*,\gamma}$  is an invertible operator provided that  $\delta$  is sufficiently small.

*Proof of Theorem 4.9.* By assumptions, the vectors  $b_l, b_\bullet$  and  $\tilde{\mathcal{T}}_{*,R} P_* u$  are linearly independent for any  $u \in C_{\text{sym}}^r$ . Hence, if  $u \in \ker \tilde{\mathcal{T}}$ , then necessarily  $\tilde{\ell}_0(u) = \tilde{\ell}_\bullet(u) = 0$  and  $\tilde{\mathcal{T}}_{*,R} P_*(u) = 0$ . Now, by definition, if  $\tilde{\ell}_0(u) = 0$ , then  $u = P_* u \in C_{*,\text{sym}}^r$ . Since  $\tilde{\mathcal{T}}_{*,R}$  is injective and  $\tilde{\mathcal{T}}_{*,R} u = 0$ , we thus conclude that  $u = 0$ .  $\square$

*Proof of Lemma 5.3.* Let us first decompose

$$\tilde{\mathcal{T}} = \tilde{\mathcal{T}}((1 - P_*) + P_*) = \tilde{\mathcal{T}}(1 - P_*) + \tilde{\mathcal{T}}_* P_*,$$

where  $\tilde{\mathcal{T}}_*$  is the restriction of  $\tilde{\mathcal{T}}$  on  $C_{*,\text{sym}}^r$ . Observe that, by definition,  $(1 - P_*)u$  is the constant function equal to  $\tilde{\ell}_0(u)$ ; we can thus set

$$b_l := \tilde{\mathcal{T}}(1) = (1, 1, \dots, 1, \dots) + \varepsilon(0, 0, O(1), \dots, O(q^{-2}), \dots) \in \ell^\infty.$$

We thus conclude that  $\tilde{\mathcal{T}} = b_l \tilde{\ell}_0 + \tilde{\mathcal{T}}_* P_*$ . Let us now define

$$b_\bullet = (0, 0, 1/4, \dots, 1/q^2, \dots).$$

Let  $\tilde{\mathcal{T}}_{*,R} = \tilde{\mathcal{T}}_* - b_\bullet \tilde{\ell}_\bullet$ , where  $\tilde{\ell}_\bullet$  is defined in (B.9) so that

$$\tilde{\ell}_\bullet(u) = \tilde{\ell}_\bullet \left( \sum_{j \geq 1} \hat{u}_j e_j \right) = \sum_{j \geq 1} \hat{u}_j (\tilde{\sigma}_j + \beta_j - 2\pi j \alpha_j),$$

where  $\alpha_j$  and  $\beta_j$  are the Fourier coefficients of  $\alpha$  and  $\beta$ ,  $\tilde{\sigma}_j$  is defined in (B.9) and  $|\alpha_j|, |\beta_j|, |\tilde{\sigma}_j| = \varepsilon O(j^{-r'})$ , where  $r' = r - 4$  is the smoothness of  $\alpha$  and  $\beta$ , provided  $\delta$  is sufficiently small. Clearly,  $b_\bullet$  and  $b_l$  are linearly independent and neither of them belongs to  $h_{*,\gamma}$  since  $\gamma > 3$ . We now consider the operator  $\tilde{\mathcal{T}}_{*,R}$ ; let  $(\tilde{T}_{qj})_{q,j} = (\tilde{\ell}_q(e_j))_{q,j}$  denote the matrix representation of  $\tilde{\mathcal{T}}_{*,R}$  in the canonical basis. We will in fact show that

$$(5.5) \quad \|\tilde{\mathcal{T}}_{*,R} - \text{Id}\|_\gamma < 1 \quad \text{if } \varepsilon \text{ is sufficiently small,}$$

where  $\|\cdot\|_\gamma$  is the operator norm from  $(X_{*,\gamma}, \|\cdot\|_\gamma)$  to  $(h_{*,\gamma}, |\cdot|_\gamma)$ . For any linear operator  $\mathcal{L} : X_{*,\gamma} \rightarrow h_{*,\gamma}$  identified by the matrix  $(L_{qj})_{q,j}$ , we have

$$\|\mathcal{L}\|_\gamma = \sup_q \sum_{j>0} q^\gamma j^{-\gamma} |L_{qj}|.$$

For  $q = 1$ , we have by definition  $\tilde{T}_{1j} = \tilde{\ell}_1(e_j) = 1$ ; on the other hand, for  $q \geq 2$ , Lemma B.1 yields the expression

$$(5.6) \quad (\tilde{\mathcal{T}}_{*,R})_{qj} = \left(1 + \sigma_{q,0} + \frac{\beta_0}{q^2}\right) \delta_{q|j} + \mathcal{R}_{qj},$$

where  $\mathcal{R}_{qj}$  is the (matrix representation of the) remainder term. First, we claim that the operator  $\Delta : X_{*,\gamma} \rightarrow h_{*,\gamma}$ , identified by the matrix  $\delta_{q|j}$ , satisfies the following bound:

$$(5.7) \quad \|\Delta - \text{Id}\|_\gamma < \zeta(3) - 1 < 0.21,$$

where  $\zeta$  is the Riemann zeta function. In particular,<sup>12</sup>  $\|\Delta\|_\gamma < \zeta(3) < 1.21$ , and  $\Delta$  has bounded inverse. In fact, by definition, the norm  $\|\Delta - \text{Id}\|_\gamma$  is given<sup>13</sup> by

$$\begin{aligned} \|\Delta - \text{Id}\|_\gamma &= \sup_q q^\gamma \sum_{j>0} j^{-\gamma} (\delta_{q|j} - \delta_{qj}) = \sup_q \left[ q^\gamma \sum_{j>0} j^{-\gamma} \delta_{q|j} \right] - 1 \\ &= \sup_q q^\gamma \sum_{s>0} (sq)^{-\gamma} - 1 \leq \sum_{s>0} s^{-3} - 1 = \zeta(3) - 1 \end{aligned}$$

since  $\gamma > 2$ . In particular,  $\|\Delta\| < \zeta(3)$ . This shows that if  $\Delta'$  is the operator, defined by

$$\begin{aligned} (\Delta')_{1j} &= 0 \\ q \geq 2: \quad (\Delta')_{qj} &= \left(\sigma_{q,0} + \frac{\beta_0}{q^2}\right) \Delta_{qj}, \end{aligned}$$

then by (B.8) we conclude that

$$\|\Delta'\|_\gamma \leq \left(\frac{(\pi + \varepsilon)^2}{24} + \frac{\varepsilon}{4}\right) \zeta(3).$$

By choosing  $\varepsilon > 0$  small enough we can make sure that

$$\|\Delta'\|_\gamma \leq 0.51.$$

We thus conclude that

$$\|\Delta + \Delta' - \text{Id}\|_\gamma < \|\Delta + \Delta' - \text{Id}\|_\gamma < 0.8.$$

Now using the above expression, we prove (5.5) by showing that if  $\varepsilon$  is sufficiently small,  $\|\mathcal{R}\|_\gamma < C\varepsilon$ . Recall that

$$\mathcal{R}_{qj} = \frac{1}{q^2} \sum_{\substack{s \in \mathbb{Z} \setminus \{0\} \\ sq \neq j}} (q^2 \sigma_{q, sq-j} + \beta_{sq-j} + 2\pi i j \alpha_{sq-j}) + \varepsilon O(j^2 q^{-4}).$$

<sup>12</sup>The value  $\zeta(3)$  is also known as the *Apéry's constant*.

<sup>13</sup>Here  $\delta_{qj}$  is the usual Kronecker delta notation.

Let us first check the contribution to the norm of the term  $\varepsilon O(j^2 q^{-4})$ :

$$\varepsilon \sup_q \sum_{j>0} O(j^{2-\gamma} q^{\gamma-4}).$$

Since  $\gamma > 3$ , the sum on  $j$  converges, and since  $\gamma < 4$ ,

$$\lim_{q \rightarrow \infty} \sum_{j>0} O(j^{2-\gamma} q^{\gamma-4}) = 0.$$

We conclude that this term can be made as small as needed by choosing  $\varepsilon$  sufficiently small. Next, we deal with the sum: since  $\alpha, \beta$  are  $C^{r'}$ -smooth functions where  $r' = r - 4 = 4$  and by (B.8), we gather

$$\begin{aligned} & q^\gamma j^{-\gamma} \frac{1}{q^2} \left| \sum_{\substack{s \in \mathbb{Z} \setminus \{0\} \\ sq \neq j}} (q^2 \sigma_{q, sq-j} + \beta_{sq-j} + 2\pi i j \alpha_{sq-j}) \right| \\ & < \varepsilon O(q^{\gamma-2}) \sum_{\substack{s \in \mathbb{Z} \setminus \{0\} \\ sq \neq j}} \frac{1}{|sq - j|^{r'} j^{\gamma-1}}. \end{aligned}$$

Let us now estimate the sum over  $j$  of the above expression:

$$\begin{aligned} & q^{\gamma-2} \sum_s \sum_{j>0} \frac{1}{|sq - j|^{r'} j^{\gamma-1}} \\ & = q^{\gamma-2} \sum_s \left[ \sum_{0 < j < [sq]^+} + \sum_{j > [sq]^+} \right] \frac{1}{|sq - j|^{r'} j^{\gamma-1}} = \text{I} + \text{II}, \end{aligned}$$

where  $[sq]^+ = \max\{0, sq\}$ . Let us first consider the term II; we have

$$\text{II} \leq q^{\gamma-2} \sum_s \frac{1}{|sq|^{\gamma-1}} \sum_{j > sq} \frac{1}{|sq - j|^{r'}} = \frac{C}{q} \sum_s |s|^{1-\gamma} < \frac{C}{q}.$$

Then we consider term I; let us write

$$\text{I} = q^{\gamma-2} \sum_s \left[ \sum_{0 < j < sq/2} + \sum_{sq/2 \leq j < sq} \right] \frac{1}{|sq - j|^{r'} j^{\gamma-1}} = \text{I}' + \text{I}''.$$

Then

$$\begin{aligned} \text{I}' & \leq q^{\gamma-2} \sum_{s>0} \sum_{0 < j < sq/2} \frac{2^{r'}}{|sq|^{r'} j^{\gamma-1}} < \frac{C}{q^{r'-\gamma}} \sum_{s>0} \sum_{j>0} \frac{1}{s^{r'} j^{1+\gamma}} < \frac{C}{q^{r'-\gamma}}, \\ \text{I}'' & \leq q^{\gamma-2} \sum_{s>0} \sum_{sq/2 < j < sq} \frac{1}{|sq - j|^{r'}} \frac{2^{\gamma-1}}{(sq)^{\gamma-1}} < \frac{C}{q} \sum_{s>0} \sum_{p>0} \frac{1}{p^{r'} s^{\gamma-1}} < \frac{C}{q}, \end{aligned}$$

which allows us to conclude since  $\gamma < r' = 4$ . □

**6. A finite dimensional reduction for arbitrary symmetric domains**

In this section, we outline the proof of a more general result, which holds for  $\mathbb{Z}_2$ -symmetric domains that are not necessarily close to a circle. For any  $\nu \in C_{\text{sym}}^r$ , let

$$\nu(x) = \sum_{k \geq 0} \hat{\nu}_k \cos(2\pi kx)$$

be its Fourier expansion. Define the finite co-dimension space

$$C_{q_0, \text{sym}}^r = \{\nu \in C_{\text{sym}}^r \text{ s.t. } \hat{\nu}_0 = \dots = \hat{\nu}_{q_0-1} = 0\}.$$

Moreover, define the space of admissible function

$$L_{q_0, \text{sym}}^1 = \{\nu \in L^1(\mathbb{T}^1), \nu(x) = \sum_{j \geq q_0} \hat{\nu}_j \cos 2\pi jx\}$$

and

$$X_{q_0, \gamma} = \{\nu \in L_{q_0, \text{sym}}^1 \text{ s.t. } \lim_{j \rightarrow \infty} j^\gamma |\hat{\nu}_j| = 0\}$$

equipped with the norm  $\|\nu\|_{q_0, \gamma} = \max_{j \geq q_0} j^\gamma |\hat{\nu}_j|$ .

Define the operator  $\tilde{\mathcal{T}}_{q_0} : C_{q_0, \text{sym}}^r \rightarrow \ell^\infty$  as

$$\tilde{\mathcal{T}}_{q_0}(u) = (\tilde{\ell}_{q_0}(u), \tilde{\ell}_{q_0+1}(u), \dots, \tilde{\ell}_q(u), \dots).$$

Then we can show the following:

**THEOREM 6.1.** *Let  $r = 8$ ; then for any domain  $\Omega \in S^r$ , there exists  $q_0 = q_0(\Omega)$  such that the operator  $\tilde{\mathcal{T}}_{q_0} : C_{\text{sym}}^r(q_0) \rightarrow \ell^\infty$  is injective.*

*Proof.* The proof follows the one of Theorem 4.9: first we need to establish the existence of good Lazutkin coordinates; in the main case, since  $\delta$  was sufficiently small, we could find Lazutkin coordinates of order 5 on the whole phase space; in our present context this is not guaranteed, and one can only find them in a neighborhood of  $y = 0$ , that is, for sufficiently large  $q$ . Then, similar to Lemma 5.3, we have a decomposition for the operator  $\tilde{T}_{q_0}$  as follows:

$$\tilde{T}_{q_0} = b_{q_0} \tilde{\ell}_* + \tilde{T}_{q_0, R}$$

with  $b_{q_0} = (1/q^2)_{q \geq q_0}$ , and  $\tilde{T}_{q_0, R} : X_{q_0, \gamma} \rightarrow h_\gamma$  is an invertible operator, where  $h_\gamma = \{a = (a_i)_{i \geq 1} \in \ell^\infty, \lim_{j \rightarrow \infty} j^\gamma |a_j| = 0\}$  equipped with the norm  $\|a\|_\gamma = \max_{j \geq 1} j^\gamma |a_j|$ . The proof of the invertibility of  $\tilde{T}_{q_0, R}$  follows that of  $\tilde{T}_{*, R}$  where, without the condition of being close to a circle,  $O(\varepsilon)$  is replaced by  $O(1)$  everywhere, as defined in (5.6), instead we have

$$\|(R_{q, j})_{q, j \geq q_0}\|_\gamma < C/q_0,$$

which hence can be made arbitrarily small when  $q_0$  is large enough. □

In Definition 2.7 we define dynamically spectrally rigid domains. In our setting the space  $\mathcal{M} = \mathcal{S}^r$  consists of  $\mathbb{Z}_2$ -symmetric domains.

**COROLLARY 6.2.** *For  $r \geq 8$  and any nondynamically spectrally rigid domain  $\Omega \in \mathcal{S}^r$ , there is a linear subspace  $\mathcal{N}(\Omega) \ni \Omega$  of dimension at most  $q_0(\Omega)$  such that any isospectral family  $(\Omega_\tau)_{|\tau| \leq 1}$ ,  $\Omega_0 = \Omega$  is tangent to  $\mathcal{N}(\Omega)$  for  $\tau = 0$ .*

*Proof.* Consider a family of isospectral deformations  $(\Omega_\tau)_{|\tau| \leq 1}$ . Let  $\nu \in C_{\text{sym}}^r$  be the associated function at  $\tau = 0$ . By Corollary 4.8 we have that  $\tilde{\mathcal{J}}_{q_0}(\nu) = 0$ . Decompose  $\nu$  into  $\nu = \nu_{q_0} + \nu_{q_0}^\perp$ , where  $\nu_{q_0}^\perp \in C_{q_0, \text{sym}}^r$  is the natural projection of  $\nu$  in  $C_{\text{sym}}^r$  onto this subspace and  $\nu_{q_0}$  is the complement given by a trigonometric polynomial of degree  $< q_0$ . Since  $\tilde{\mathcal{J}}_{q_0}(\nu) = 0$ , we have

$$\tilde{\mathcal{J}}_{q_0}(\nu_{q_0}^\perp) = -\tilde{\mathcal{J}}_{q_0}(\nu_{q_0}).$$

Therefore, by Theorem 6.1, for each  $\nu_{q_0}$  there is at most one  $\nu_{q_0}^\perp$  solving this equation. If the linear spaces  $\tilde{\mathcal{J}}_{q_0}(C_{q_0, \text{sym}}^r)$  and the image of its orthogonal complement under  $\tilde{\mathcal{J}}_{q_0}$  intersect, then any isospectral family  $(\Omega_\tau)_{|\tau| \leq 1}$  is tangent to it. □

### 7. Concluding remarks

In this paper we proved dynamical spectral rigidity of convex domains that are  $\mathbb{Z}_2$ -symmetric and close to a circle; it is indeed natural to ask if the same result holds if one drops some of our assumptions

*$\mathbb{Z}_2$ -symmetry.* the main challenge in removing the symmetry assumption with our strategy is that one would need to find another sequence of periodic orbit that generates linear functionals that are linearly independent of the ones corresponding to maximal periodic orbits. This appears to be a nontrivial task, since, as  $q$  increases (by the results of Appendix A), the dynamics is closer and closer (to any order) to the dynamics of a billiard in a disk. On the other hand, due to the symmetries of the disk, all linear functionals corresponding to orbits of the same rotation number would be linearly dependent.

*Closeness to a circle.* In Section 6 we showed that to prove our result for domains that are not necessarily close to a circle, it would suffice to find a finite number of linearly independent functionals for small periods. However, we do not have *a priori* any control on such orbits, and the general strategy is unclear.

*Convexity.* As we mentioned in the introduction, our result is an analog of Guillemin–Kazhdan (see [8]) for  $\mathbb{Z}_2$ -symmetric domains close to the circle. However, from the dynamical point of view, more natural analogs of geodesic

flows on surfaces of negative curvature are dispersing billiards: “*Are dispersing billiards spectrally rigid?*” One possible approach to prove this statement would be to introduce and study the linearized isospectral operator analogous to (4.6).

### Appendix A. Lazutkin coordinates

A.1. *An abstract setting.* Let  $\mathbb{A} = \mathbb{T}^1 \times I$ , where  $I \subset \mathbb{R}$  is a compact interval; let us assume without loss of generality that  $I = [0, 1]$ . We denote by  $(x, y)$  the natural coordinates in  $\mathbb{T}^1 \times I$ . Let  $F \in C^s(\mathbb{A}, \mathbb{A})$  be a monotone orientation preserving twist diffeomorphism that leaves invariant both boundary components of  $\mathbb{A}$ . We further assume that the circle  $\{y = 0\}$  is the union of *fixed* points, i.e.,

$$F(x, 0) = (x, 0) \quad \text{for all } x \in \mathbb{T}.$$

To avoid complications in the exposition, let us assume that the other boundary component  $\{y = 1\}$  is also fixed by  $F$ . Furthermore, denote with  $\hat{F}$  a lift of  $F$  to  $\hat{\mathbb{A}} = \mathbb{R} \times I$  so that

$$\hat{F}(X, 0) = (X, 0) \quad \text{for all } X \in \mathbb{R}.$$

To fix ideas, we further assume that for any  $X \in \mathbb{R}$ , we have  $\hat{F}(X, 1) = (X + 1, 1)$ .

*Definition A.1.* Let  $N \in \mathbb{N}$ ; a function  $\mathbb{J} \in C^s(\hat{\mathbb{A}}, \mathbb{R})$  is called a  *$C^s$ -Lazutkin function of order  $N$*  if

- (a) for any  $y \in I$ , the map  $\mathbb{J}(\cdot, y) : \mathbb{R} \rightarrow \mathbb{R}$  is a lift of an orientation preserving diffeomorphism of  $\mathbb{T}^1$  so that  $\mathbb{J}(0, y) = 0$ ;
- (b) there exists  $\mathcal{R} \in C^s(\hat{\mathbb{A}}, \mathbb{R})$  with  $\mathcal{R}(x, y) = \mathcal{R}_*(x, y)y^{N+1}$  and  $\mathcal{R}_* \in C^s(\hat{\mathbb{A}}, \mathbb{R})$  so that

$$(A.1) \quad \mathbb{J} \circ \hat{F} - 2\mathbb{J} + \mathbb{J} \circ \hat{F}^{-1} = \mathcal{R}.$$

Observe that, by definition,  $\mathcal{R}(x, 0) = 0$  and our assumptions on  $\hat{F}$  and  $\mathbb{J}$  imply that  $\mathcal{R}(x, 1) = 0$  for any  $x \in \mathbb{T}^1$ .

Given a Lazutkin function  $\mathbb{J}$ , define the real function  $y_{\mathbb{J}} : \hat{\mathbb{A}} \rightarrow \mathbb{R}$ :

$$(A.2) \quad y_{\mathbb{J}} = \frac{1}{2} [\mathbb{J} \circ \hat{F} - \mathbb{J} \circ \hat{F}^{-1}].$$

Since  $F$  is a twist map,  $y_{\mathbb{J}}(x, \cdot)$  is strictly increasing for any fixed  $x$ . Moreover, by our assumptions on  $F$  we gather that  $y_{\mathbb{J}}(x, 0) = 0$  and  $y_{\mathbb{J}}(x, 1) = 1$ . We conclude that, for any  $x \in \mathbb{R}$ , the function  $y_{\mathbb{J}}(x, \cdot)$  is a diffeomorphism of  $I$  onto itself. Hence,  $(X_{\mathbb{J}}, Y_{\mathbb{J}})$  (where we set  $X_{\mathbb{J}} = \mathbb{J}(x, y)$ ) are good coordinates on  $\hat{\mathbb{A}} = \mathbb{R} \times I$ , which factor to  $\mathbb{A}$  as  $(x_{\mathbb{J}}, y_{\mathbb{J}})$ . Let us denote with  $\Psi_{\mathbb{J}}$  the change of variables  $\Psi_{\mathbb{J}} : (x, y) \mapsto (x_{\mathbb{J}}, y_{\mathbb{J}})$ ; notice that by design,  $\Psi_{\mathbb{J}}$  leaves invariant the

boundary components of  $\mathbb{A}$ . A simple computation shows that  $\Psi_{\mathbb{I}}$  conjugates the map  $F$  to  $\tilde{F} : (\xi, \eta) \mapsto (\xi^+, \eta^+)$  where

$$\begin{cases} \xi^+ &= \xi + \eta + \mathcal{R}_{\mathbb{I}}(\xi, \eta) \pmod{1}, \\ \eta^+ &= \eta + \mathcal{R}_{\mathbb{I}}(\xi, \eta) + \mathcal{R}_{\mathbb{I}}(\xi^+, \eta^+), \end{cases} \quad \text{where } \mathcal{R}_{\mathbb{I}} = \frac{1}{2}\mathcal{R} \circ \Psi_{\mathbb{I}}^{-1}.$$

In particular, for any  $\xi \in \mathbb{T}$ , we can write  $\mathcal{R}_{\mathbb{I}}(\xi, \eta) = \mathcal{R}_{\mathbb{I},*}(\xi, \eta)\eta^{N+1}$  and  $\mathcal{R}_{\mathbb{I}}(\xi, 0) = \mathcal{R}_{\mathbb{I}}(\xi, 1) = 0$ .

LEMMA A.2 (Properties of the Normal Form). *Assume  $s \geq 2$  and let  $\mathcal{R} \in C^s(\mathbb{A}, \mathbb{R})$  be so that  $\mathcal{R}(\xi, 0) = \mathcal{R}(\xi, 1) = 0$  with  $\|\mathcal{R}\|_{C^s}$  sufficiently small. Then there is a unique map  $F_{\mathcal{R}} \in C^s(\mathbb{A}, \mathbb{A})$  that fixes  $\{\eta = 0\}$  and so that  $F_{\mathcal{R}}(\xi, \eta) = (\xi^+, \eta^+)$ , where*

$$(A.3a) \quad \begin{cases} \xi^+(\xi, \eta) = \xi + \eta + \mathcal{R}(\xi, \eta) \pmod{1}, \\ \eta^+(\xi, \eta) = \eta + \mathcal{R}(\xi, \eta) + \mathcal{R}(\xi^+, \eta^+). \end{cases}$$

Moreover,  $F_{\mathcal{R}}(\xi, 1) = (\xi, 1)$  and  $F_{\mathcal{R}}^{-1}(\xi, \eta) = (\xi^-, \eta^-)$ , where

$$(A.3b) \quad \begin{cases} \xi^-(\xi, \eta) = \xi - \eta + \mathcal{R}(\xi, \eta) \pmod{1}, \\ \eta^-(\xi, \eta) = \eta - \mathcal{R}(\xi, \eta) - \mathcal{R}(\xi^-, \eta^-). \end{cases}$$

*Proof.* Since  $\mathcal{R}$  is fixed, observe that  $\xi^+$  is an explicit well-defined function; we thus only need to show that there exists a unique  $\eta^+(\xi, \eta)$  satisfying the required relation with initial condition  $\eta^+(\xi, \eta = 0) = 0$ . It follows from the Implicit Function Theorem applying to the relation

$$\eta + \mathcal{R}(\xi, \eta) + \mathcal{R}(\xi^+(\xi, \eta), \eta^+) - \eta^+ = 0$$

since  $\partial_2 \mathcal{R} - 1$  is not zero. Then observe that  $\eta^+(\xi, 1) = 1$  satisfies the functional equation and thus, by uniqueness, we conclude that  $F_{\mathcal{R}}$  fixes the boundaries  $\{\eta = 0\}$  and  $\{\eta = 1\}$ .

The expression (A.3b) follows from simple algebraic manipulations of (A.3a) and is left to the reader.  $\square$

We call (A.3a) the *Lazutkin normal form with remainder  $\mathcal{R}$  of order  $N$  and class  $C^s$*  if  $\mathcal{R}(x, y) = \mathcal{R}_*(x, y)y^{N+1}$  and  $\mathcal{R}_* \in C^s(\mathbb{A}^\dagger, \mathbb{R})$ . Coordinates  $(\xi, \eta)$  are said to be *Lazutkin coordinates of order  $N$  for  $F$*  if they conjugate  $F$  to a Lazutkin normal form of order  $N$ .

The next lemma constitutes the main result of this section: it gives sufficient conditions to find Lazutkin coordinates of any order.

LEMMA A.3. *Let  $s \geq 3$ , and assume that for some  $N \geq 1$  the dynamics  $F$  is described in the coordinates  $(x, y)$  by the Lazutkin Normal Form (A.3a) with remainder  $\mathcal{R}$  of order  $N$  and class  $C^s$ , i.e.,  $\mathcal{R}(x, y) = \mathcal{R}_*(x, y)y^{N+1}$  with  $\mathcal{R}_* \in C^s(\hat{\mathbb{A}}, \mathbb{R})$ . Then, if  $\|\mathcal{R}_*\|_{C^s}$  is sufficiently small, there exists a  $C^s$  change*

of variables  $\Psi : (x, y) \mapsto (\bar{x}, \bar{y})$  so that  $(\bar{x}, \bar{y})$  are Lazutkin coordinates of order  $N + 1$  with remainder  $\bar{\mathcal{R}}(\bar{x}, \bar{y}) = \bar{\mathcal{R}}_*(x, y)y^{N+2}$  with  $\bar{\mathcal{R}}_* \in C^{s-1}(\hat{\mathbb{A}}, \mathbb{R})$  and, moreover, for some universal  $C_*$ , we have

- (a)  $\Psi(x, y) = (x + \Psi_0(x, y)y^{N-1}, y + \Psi_1(x, y)y^N)$ , where  $\|\Psi_i\|_{C^s} < C_*\|\mathcal{R}_*\|_{C^s}$  for  $i = 0, 1$ ;
- (b)  $\|\bar{\mathcal{R}}_*\|_{C^{s-1}} \leq C_*\|\mathcal{R}_*\|_{C^s}$ .

*Proof.* The key to the proof is to find suitable Lazutkin functions; to simplify the exposition it is convenient to treat separately the case  $N = 1$  and  $N > 1$ . First, let us set some convenient notation: let

$$\mathcal{R}(x, y) = \mathcal{R}_*(x, y)y^{N+1} = \mathcal{R}_0(x)y^{N+1} + \tilde{\mathcal{R}}(x, y)y^{N+2},$$

where  $\mathcal{R}_0(x) = \mathcal{R}_*(x, 0)$  is  $C^s$  and, by definition,  $\tilde{\mathcal{R}}$  is a  $C^{s-1}$  function so that  $\|\tilde{\mathcal{R}}\|_{C^{s-1}} \leq \|\mathcal{R}_*\|_{C^s}$ .

*Case  $N = 1$ :* We proceed to construct a Lazutkin function  $\mathcal{J}$  of order 2: let us make the *ansatz*  $\mathcal{J}(x, y) = \pi(x)$ , where  $\pi$  solves the differential equation

$$(A.4) \quad 2\pi'(x)\mathcal{R}_0(x) + \pi''(x) = 0$$

with boundary conditions  $\pi(0) = 0$  and  $\pi(1) = 1$ . Let us prove that  $\mathcal{J}$  is a Lazutkin function. First of all, elementary ODE considerations imply that  $\pi$  is  $C^{s+2}$  and that  $\pi' > 0$  which, together with the boundary conditions, implies that  $x \mapsto \pi(x)$  is the lift of a circle diffeomorphism. Moreover, by construction we have that  $\|\pi - \text{Id}\|_{C^s} \leq \|\mathcal{R}_*\|_{C^s}$ . We thus need to show that  $\mathcal{J}$  satisfies (A.1) with a remainder of order 2. Fix  $x \in \mathbb{T}$ ; then for any  $x' \in \mathbb{T}$ , we write

$$\pi(x') = \pi(x) + \pi'(x)(x' - x) + \pi''(x)(x' - x)^2 + \tilde{\pi}(x, x')(x' - x)^3,$$

where  $\tilde{\pi}$  is a  $C^{s-1}$  function so that  $\|\tilde{\pi}\|_{C^{s-1}} \leq \|\pi'''\|_{C^{s-1}}$  and by definition of  $\pi$ , we have  $\|\pi'''\|_{C^{s-1}} < C_*\|\mathcal{R}_*\|_{C^s}$ . First, we apply the above expansion to  $x' = x^+$ ; notice that

$$\begin{aligned} (x^+ - x) &= y + \mathcal{R}_0(x)y^2 + \tilde{\mathcal{R}}_1(x, y)y^3, \\ (x^+ - x)^2 &= y^2 + \tilde{\mathcal{R}}_2(x, y)y^3, \\ (x^+ - x)^3 &= \tilde{\mathcal{R}}_3(x, y)y^3, \end{aligned}$$

where  $\tilde{\mathcal{R}}_1, \tilde{\mathcal{R}}_2, \tilde{\mathcal{R}}_3$  are  $C^{s-1}$  functions so that  $\|\tilde{\mathcal{R}}_i\|_{C^{s-1}} \leq C_*\|\mathcal{R}_*\|_{C^s}$ , provided that  $C_*$  is sufficiently large. Then we conclude that

$$\begin{aligned} \pi(x^+) &= \pi(x) + \pi'(x)[y + \mathcal{R}_0(x)y^2] + \frac{\pi''(x)}{2}y^2 + \bar{\pi}^+(x, y)y^3 \\ &= \pi(x) + \pi'(x)y + \left( \pi'(x)\mathcal{R}_0(x) + \frac{\pi''(x)}{2} \right) y^2 + \bar{\pi}^+(x, y)y^3, \end{aligned}$$

and by (A.4) we conclude

$$= \pi(x) + \pi'(x)y + \bar{\pi}^+(x, y)y^3,$$

where  $\bar{\pi}^+$  is  $C^{s-1}$  and  $\|\bar{\pi}^+\|_{C^{s-1}} \leq \|\mathcal{R}_*\|_{C^s}$ , Applying the same argument to  $x' = x^-$  we similarly obtain

$$\pi(x^-) = \pi(x) - \pi'(x)y + \bar{\pi}^-(x, y)y^3,$$

where  $\bar{\pi}^-$  has the same properties as  $\bar{\pi}^+$ . We thus found that  $\mathbb{J}\Pi(x, y) = \pi(x)$  satisfies (A.1) with the  $C^{s-1}$  remainder

$$\bar{\mathcal{R}}(x, y) = (\bar{\pi}^+(x, y) + \bar{\pi}^-(x, y))y^3$$

of order 2, which concludes the proof of the case  $N = 1$ .

*Case  $N > 1$ :* This case is similar to the previous one, but simpler. In this case we make the ansatz  $\mathbb{J}\Pi(x, y) = x + \pi(x)y^{N-1}$ , where we assume that

$$(A.5) \quad \pi''(x) = -2\mathcal{R}_0(x)$$

with boundary conditions  $\pi(0) = \pi(1) = 0$ . Then, if  $\|\mathcal{R}_*\|_{C^0}$  is sufficiently small,  $\partial_x \mathbb{J}\Pi(x, y) = 1 + \pi'(x)y^{N-1} > 0$ , thus  $\mathbb{J}\Pi(\cdot, y)$  is the lift of a circle diffeomorphism. Moreover, it is immediate to observe that  $\|\mathbb{J}\Pi - \text{Id}\|_{C^s} = \|\pi\|_{C^s}y^{N-1} \leq \|\mathcal{R}_*\|_{C^s}y^{N-1}$ . Moreover, we can write

$$\begin{aligned} \mathbb{J}\Pi(x^+, y^+) &= x + y + \mathcal{R}_0(x)y^{N+1} + \pi(x)y^{N-1} + \pi'(x)y^N \\ &\quad + \frac{\pi''(x)}{2}y^{N+1} + \bar{\pi}^+(x, y)y^{N+2} \\ &= x + y + \pi(x)y^{N-1} + \pi'(x)y^N \\ &\quad + \left( \mathcal{R}_0(x) + \frac{\pi''(x)}{2} \right) y^{N+1} + \bar{\pi}^+(x, y)y^{N+2}, \end{aligned}$$

and again using our ansatz

$$, = x + y + \pi(x)y^{N-1} + \pi'(x)y^N + \bar{\pi}^+(x, y)y^{N+2}.$$

Correspondingly,

$$(A.6) \quad \mathbb{J}\Pi(x^-, y^-) = x - y + \pi(x)y^{N-1} - \pi'(x)y^N + \bar{\pi}^-(x, y)y^{N+2}.$$

We thus conclude that  $\mathbb{J}\Pi(x, y) = x + \pi(x)y^{N-1}$  is a Lazutkin function of order  $N + 1$  with a  $C^{s-1}$  remainder that we call  $\bar{\mathcal{R}}(x, y)$ . The estimates on the norms follow from arguments that are similar to the case  $N = 1$  and are left to the reader.  $\square$

Let us define the *symmetrized annulus*  $A^\dagger = \mathbb{T} \times [-1, 1]$  together with its universal cover  $\hat{A}^\dagger = \mathbb{R} \times [-1, 1]$  and define the idempotent map  $\mathcal{J} : (x, y) \mapsto (x, -y)$ .

We now proceed to state a refinement of the above lemma, which holds under some additional assumptions. With a little abuse of terminology, let us say that a function  $h \in C^s([0, 1], \mathbb{R})$  is even (resp. odd) if its even (resp. odd) extension  $h^\dagger$  to  $[-1, 1]$  is of class  $C^s$ . We say that the remainder  $\mathcal{R}$  is *even* if for any fixed  $x \in \mathbb{T}$ , the function  $\mathcal{R}(x, \cdot) : I \rightarrow \mathbb{R}$  is even. As we will see in the following section, this assumption will be satisfied in our setting. The following lemma is an immediate consequence of the definitions and of Lemma A.2.

LEMMA A.4. *Assume  $s \geq 2$ , and let  $\mathcal{R} \in C^s(\mathbb{A}, \mathbb{R})$  be an even function satisfying the hypotheses of Lemma A.2; then there exists a diffeomorphism  $F_{\mathcal{R}}^\dagger : A^\dagger \rightarrow A^\dagger$  so that  $F_{\mathcal{R}}^\dagger|_{\mathbb{A}} = F_{\mathcal{R}}$  and  $F^\dagger \circ \mathcal{J} = \mathcal{J} \circ (F^\dagger)^{-1}$ .  $\square$*

In other terms, the map  $F^\dagger$  admits an involution, which is given by  $\mathcal{J}$ . Observe, moreover, that if  $\mathcal{R}$  is an even function and  $\mathcal{R}(x, y) = \mathcal{R}_*(x, y)y^{N+1}$ , then we can always assume that  $N$  is odd. This leads to the following version of Lemma A.3:

LEMMA A.5. *Let  $s \geq 4$ , and assume that for some  $N = 2K + 1$  with  $K \geq 0$ , the dynamics  $F$  is described in the coordinates  $(x, y)$  by the Lazutkin Normal Form (A.3a) with even remainder  $\mathcal{R}$  of order  $N$  and class  $C^s$ . Then, if  $\|\mathcal{R}_*\|_{C^s}$  is sufficiently small, there exists a  $C^s$  change of variables  $\Psi : (x, y) \mapsto (\bar{x}, \bar{y})$  so that  $(\bar{x}, \bar{y})$  are Lazutkin coordinates of order  $N + 2$  with even remainder  $\bar{\mathcal{R}}$  of class  $C^{s-2}$  so that, for some universal  $C_*$ ,*

- (a)  $\Psi(x, y) = (x + \Psi_0(x, y)y^{N-1}, y + \Psi_1(x, y)y^N)$  where  $\|\Psi_i\|_{C^s} < C_*\|\mathcal{R}_*\|_{C^s}$  for  $i = 0, 1$ ;
- (b)  $\|\bar{\mathcal{R}}_*\|_{C^{s-2}} \leq C_*\|\mathcal{R}_*\|_{C^s}$ .

*Proof.* The proof is analogous to the one given for Lemma A.3; in fact following the same steps, we obtain the needed result, except for the fact that we only know that the new remainder  $\bar{\mathcal{R}}$  is an even function only when expressed in the old coordinates  $(x, y)$ . Hence, the only thing that we need to show is that  $\bar{\mathcal{R}}$  is an even function when also expressed in the new coordinates  $(\bar{x}, \bar{y}) = \Psi(x, y)$ . This, however, follows by Lemma A.4 and the definition (A.2), which in turn imply that the coordinate change  $\Psi$  commutes with the involution  $\mathcal{J}$ , i.e.,  $\Psi \circ \mathcal{J} = \mathcal{J} \circ \Psi$ . This concludes the proof.  $\square$

Let us summarize in words the results of this section: for any  $N > 1$ , if we can find Lazutkin coordinates of order 1 with remainder that is both sufficiently smooth and sufficiently small, then we can find Lazutkin coordinates of order  $N$  and we have good control on the change of variables. The following lemma gives sufficient conditions for existence of Lazutkin coordinates of order 1 with even remainder.

LEMMA A.6. *Assume that there exists  $F^\dagger : A^\dagger \rightarrow A^\dagger$  of class  $C^r$  so that  $F^\dagger \circ \mathcal{J} = \mathcal{J} \circ (F^\dagger)^{-1}$  and  $F = F^\dagger|_{\mathbb{A}}$ . Then there exist Lazutkin coordinates of order 1 with even remainder of class  $C^{r-2}$ .*

*Proof.* Let us write  $F^\dagger(x, y) = (x^\dagger, y^\dagger)$  so that  $x^+(x, y) = x^\dagger(x, y)$  and  $x^-(x, y) = x^\dagger(x, -y)$ . Let us then choose  $\mathbb{I}(x, y) = x$ ; hence,

$$\begin{aligned} \mathbb{I} \circ F(x, y) - 2\mathbb{I}(x, y) + \mathbb{I} \circ F^{-1}(x, y) \\ = x^\dagger(x, y) - 2x + x^\dagger(x, -y) = \mathcal{R}(x, y), \end{aligned}$$

which is by construction an even function with  $\mathcal{R}(x, 0) = 0$ . We conclude that  $\mathcal{R}$  is an even remainder of order 1, which allows us to construct Lazutkin coordinates of order 1 with even remainder.  $\square$

A.2. *Application to billiard dynamics.* In this section we apply the results of the previous section to the billiard map  $f$  corresponding to some domain  $\Omega \in \mathcal{D}^r$  and prove Lemma 5.1

Let us assume  $\Omega$  to be of perimeter 1. As we mentioned in Section 4, if  $s \in \mathbb{T}$  denotes the arc-length parametrization of the boundary and  $y = \cos \varphi \in [-1, 1]$ , where  $\varphi \in [0, \pi]$  is the angle of the outgoing trajectory with the positively oriented tangent vector, then the map  $f_\Omega = f : \mathbb{T} \times [-1, 1] \rightarrow \mathbb{T} \times [-1, 1]$  is a monotone twist map of class  $C^r$ . Moreover, it is clear by the definition that  $f$  fixes the boundary components  $y = -1$  and  $y = 1$  and that  $f(s, [-1, 1])$  twists only once around the annulus. In summary, we can apply to the map  $f$  the results described in an abstract setting in the previous section; the first step is to find Lazutkin coordinates of order 1.

Let  $s$  identify a point on the boundary of  $\Omega$ , and let  $L_\varphi$  be the oriented line passing through the point  $s$  with angle  $\varphi \in [-\pi, \pi]$  measured counterclockwise from the positively oriented tangent to  $\Omega$ . Since  $\Omega$  is strictly convex, for any  $\varphi \in (-\pi, \pi) \setminus \{0\}$ , there exists a unique other point of intersection of  $L_\varphi$  with  $\Omega$ ; let us denote this point by  $s^\dagger(s, \varphi)$  (and extend by continuity  $s^\dagger(s, 0) = s^\dagger(s, \pi) = s^\dagger(s, -\pi) = s$ ). Moreover, let us denote with  $\varphi^\dagger(s, \varphi)$  the angle between  $L_\varphi$  and the positively oriented tangent vector to  $\Omega$  at  $s'$ , also measured counterclockwise. Notice that if  $\varphi \in [0, \pi]$ , by our construction, we have  $s^\dagger(s, \varphi) = s^+(s, \varphi)$  and  $\varphi^\dagger(s, \varphi) = \varphi^+(s, \varphi)$ . Observe moreover that, since  $L_\varphi = L_{\pi+\varphi}$  (with reversed orientation), we also have  $s^-(s, \varphi) = s^\dagger(s, -\varphi)$  and  $\varphi^-(s, \varphi) = -\varphi^\dagger(s, -\varphi)$ . In other words, if we define  $F^\dagger(s, \varphi) = (s^\dagger, \varphi^\dagger)$ , this map satisfies the hypotheses of Lemma A.6; hence we conclude that there exist Lazutkin coordinates of order 1 for the billiard with even remainder  $\mathcal{R}$  of class  $C^{r-2}$ .

As noticed earlier, if two domains  $\Omega$  and  $\Omega'$  of length 1 are  $C^{r+1}$ -close, then the corresponding billiard maps  $f_\Omega$  and  $f_{\Omega'}$  are  $C^r$ -close; since  $\mathcal{R}(s, \varphi) = s^\dagger(s, \varphi) - 2s + s^\dagger(s, -\varphi)$ , we gather that the corresponding remainders will be

$C^r$ -close. Since for the circle we have that  $\mathcal{R} = 0$ , we conclude that  $\mathcal{R}$  can be made arbitrarily small in the  $C^r$  topology provided that  $\Omega$  is sufficiently  $C^{r+1}$ -close to a circle.

We are now in the position to give the main result of this appendix.

*Proof of Lemma 5.1.* An explicit computation (but see also [13, (1.1–2)]) allows us to write

$$s^\dagger(s, \varphi) = s + 2\rho(s)\varphi + a(s, \varphi)\varphi^2,$$

where  $a(s, \varphi)$  is a  $C^{r-2}$ -smooth function. As pointed out in Lemma A.6, we can always find Lazutkin coordinates of order 1 with even remainder; moreover, for any  $\varepsilon > 0$ , we can choose  $\delta > 0$  so that if  $\Omega \in \mathcal{D}_\delta^r$ , then  $\|\mathcal{R}(s, \varphi)\|_{C^{r-2}} < \varepsilon$ .

We can then apply Lemma A.5 and obtain Lazutkin coordinates of order 3, which we denote with  $(x, y)$ . It is easy to check that  $x$  is (unsurprisingly) given by the Lazutkin parametrization defined in (5.1).

Using the explicit formula (A.2) we conclude that<sup>14</sup>

$$y(s, \varphi) = 2C_\Omega \rho^{1/3}(s)\varphi [1 + \beta_0(s)\varphi^2 + c],$$

where  $\beta_0 \in C^{r-2}$  with  $\|\beta_0\|_{C^{r-2}} < \varepsilon$  and  $O_{C^{r-4}}(\varphi^k)$  denotes a  $C^{r-4}$  function of  $(s, \varphi)$  whose  $C^{r-4}$  norm is bounded by  $C\varphi^k$  for some  $C > 0$ . Inverting the above expression we obtain

$$(A.7) \quad \varphi(x, y) = \mu(x)y [1 + \beta_1(x)y^2 + \varepsilon O_{C^{r-4}}(y^4)],$$

where recall that  $\mu$  was defined in (5.2) and  $\beta_1$  satisfies the same estimates as  $\beta_0$ . Finally, observe that the remainder in the coordinates  $(x, y)$  is  $O_{C^{r-4}}(\varepsilon)$ . By once again applying Lemma A.5 we can now obtain Lazutkin coordinates of order 5, which we denote with  $(\bar{x}, \bar{y})$ . By construction such coordinates conjugate the dynamics to

$$(A.8) \quad \begin{cases} \bar{x}^+ = \bar{x} + \bar{y} + \varepsilon O_{C^{r-6}}(y^6), \\ \bar{y}^+ = \bar{y} + \varepsilon O_{C^{r-6}}(y^6), \end{cases}$$

and moreover there exist  $C^{r-4}$  functions  $\alpha(\bar{x})$  and  $\beta_2(\bar{x})$  so that

$$(A.9) \quad x = \bar{x} + \alpha(\bar{x})\bar{y}^2, \quad y = \bar{y} [1 + \beta_2(\bar{x})\bar{y}^2 + \varepsilon O_{C^{r-6}}(\bar{y}^4)].$$

Moreover, since the remainder in the coordinates  $(x, y)$  is  $O_{C^{r-4}}(\varepsilon)$ , we conclude that  $\alpha$  and  $\beta_2$  are  $O_{C^{r-4}}(\varepsilon)$ . Let us now consider a marked symmetric periodic orbit of rotation number  $1/q$ , which we denote with  $((\bar{x}_q^k, \bar{y}_q^k))_{k \in \{0, \dots, q-1\}}$ .

<sup>14</sup>Notice that although the  $x$  coordinate is given by the standard Lazutkin parametrization, the  $y$  variable is not the usual Lazutkin coordinate  $y$  defined in [13, (1.3)]

Recall that by convention,  $\bar{x}_q^0 = 0$ . Then (A.8) yields

$$\bar{x}_q^k = \frac{k}{q} + \varepsilon O(q^{-4}), \quad \bar{y}_q^k = \frac{1}{q} + \varepsilon O(q^{-5}).$$

Combining the above with (A.9) and (A.8) yields

$$x_q^k = \frac{k}{q} + \frac{\alpha(k/q)}{q^2} + \varepsilon O(q^{-4}),$$

which is (5.3a). We also see that by (5.3a) and estimates (A.7) and (A.9), we gather that for any  $\varepsilon > 0$  there exists  $\delta > 0$  so that for any  $\Omega \in \mathcal{S}_\delta^r$ , there exists  $\alpha \in C^{r-4}$  odd and  $\beta_3 \in C^{r-4}$  even such that

$$\begin{aligned} x_q^k &= \frac{k}{q} + \frac{\alpha(k/q)}{q^2} + \varepsilon O(q^{-4}), \\ \varphi_q^k &= \frac{\mu(x_q^k)}{q} \left( 1 + \frac{\beta_3(k/q)}{q^2} + \varepsilon O(q^{-4}) \right), \end{aligned}$$

where  $\alpha = O_{r-4}(\varepsilon)$  and  $\beta_3 = O_{r-4}(\varepsilon)$ . □

### Appendix B. Representation of the linearized problem in the Fourier basis

By JACOPO DE SIMOI, HAMID HEZARI, VADIM KALOSHIN, and QIAOLING WEI

The original version of this paper (arXiv:1606.00230v1) contained an error that was found and corrected by H. Hezari (see also [10]). This resulted in modifying the form of the operator  $\tilde{\mathcal{T}}$ , statement and proof of Lemma B.1, estimates in Section 5 and in the proof of Theorem 4.9. These modifications naturally introduce a new term  $S_q(x)$  (B.2) that one has to keep track of in the proof of the Main Theorem. In this joint appendix, the corrections are combined with the original argument to produce a complete proof of Lemma B.1. This lemma is used in the proof of Theorem 4.9.

To study the linear functionals  $\tilde{\ell}_q$  we first need to understand the expressions  $\frac{\sin \varphi_q^k}{\mu(x_q^k)}$ . We recall that  $\mu(x) = \frac{1}{2C_{\mathbb{H}}\rho(x)^{1/3}}$ . One can easily check that if  $\partial\Omega$  is a circle, then  $\mu(x)$  is a constant equal to  $\pi$ . Observe, moreover, that for any  $\varepsilon > 0$ , there exists  $\delta > 0$  so that for any  $\Omega \in \mathcal{S}_\delta^r$ , we have  $\|\mu(x) - \pi\|_{C^{r-1}} < \varepsilon$ . First, using Lemma 5.1, we will show that

$$(B.1) \quad \frac{\sin \varphi_q^k}{\mu(x_q^k)} = \frac{1}{q} \left[ 1 + \frac{\beta(k/q)}{q^2} + S_q\left(\frac{k}{q}\right) + \varepsilon O(q^{-4}) \right],$$

where

$$(B.2) \quad S_q(x) = \frac{\sin(\mu(x)/q)}{\mu(x)/q} - 1.$$

To do this, we first simplify  $\mu(x_q^k)$  using the asymptotic of  $x_q^k$  provided in (5.3a), the mean value theorem, and the fact that  $\mu(x)$  is uniformly bounded from below to obtain

$$(B.3) \quad \mu(x_q^k) = \mu\left(\frac{k}{q} + \frac{\alpha(k/q)}{q^2}\right) (1 + \varepsilon O(q^{-4})).$$

Plugging the above expression into the equation of  $\varphi_q^k$  provided in (5.3b), we get

$$\varphi_q^k = \frac{1}{q} \mu\left(\frac{k}{q} + \frac{\alpha(k/q)}{q^2}\right) \left(1 + \frac{\beta(k/q)}{q^2} + \varepsilon O(q^{-4})\right).$$

Next, we take  $\sin$  of both sides and use the mean value theorem again and also the lower bound  $\sin(\mu(x)/q) \geq C/q$  to get

$$\sin \varphi_q^k = \sin\left(\frac{1}{q} \mu\left(\frac{k}{q} + \frac{\alpha(k/q)}{q^2}\right) \left(1 + \frac{\beta(k/q)}{q^2}\right)\right) (1 + \varepsilon O(q^{-4})).$$

Dividing by  $\mu(x_q^k)$ , using (B.3), and also multiplying and dividing by  $1 + \frac{\beta(k/q)}{q^2}$ , we obtain

$$\frac{\sin \varphi_q^k}{\mu(x_q^k)} = \frac{1}{q} \frac{\sin\left(\frac{1}{q} \mu\left(\frac{k}{q} + \frac{\alpha(k/q)}{q^2}\right) \left(1 + \frac{\beta(k/q)}{q^2}\right)\right)}{\frac{1}{q} \mu\left(\frac{k}{q} + \frac{\alpha(k/q)}{q^2}\right) \left(1 + \frac{\beta(k/q)}{q^2}\right)} \left(1 + \frac{\beta(k/q)}{q^2} + \varepsilon O(q^{-4})\right).$$

On the other hand by the mean value theorem and the properties  $\|\alpha\|_{C^0}, \|\beta\|_{C^0}, \|\mu'\|_{C^0} \leq C\varepsilon$ , we have

$$\frac{1}{q} \mu\left(\frac{k}{q} + \frac{\alpha(k/q)}{q^2}\right) \left(1 + \frac{\beta(k/q)}{q^2}\right) = \frac{1}{q} \mu\left(\frac{k}{q}\right) + \varepsilon O(q^{-3}).$$

Using this and the mean value theorem once more and also using  $\frac{d}{dy} \left(\frac{\sin y}{y}\right) = O(y)$ , we finally get

$$\frac{\sin \varphi_q^k}{\mu(x_q^k)} = \frac{1}{q} \left( \frac{\sin\left(\frac{\mu(k/q)}{q}\right)}{\frac{\mu(k/q)}{q}} + \frac{\beta(k/q)}{q^2} + \varepsilon O(q^{-4}) \right).$$

We now conclude by writing this expression in the form

$$(B.4) \quad \frac{\sin \varphi_q^k}{\mu(x_q^k)} = \frac{1}{q} \left(1 + \frac{\beta(k/q)}{q^2} + \varepsilon O(q^{-4})\right) + \frac{1}{q} S_q(k/q),$$

where  $S_q(x)$  is defined in (B.2); this implies (B.1) as claimed.

Next, in order to obtain a useful expression for  $\tilde{\ell}_q$  for  $q \geq 2$ , recall the definition

$$\ell_q(\nu) = \frac{1}{q} \sum_{k=0}^{q-1} \nu(x_q^k) \sin(\varphi_q^k),$$

where  $(x_q^k)_{k \in \{0, \dots, q-1\}}$  is a marked symmetric maximal periodic orbit and the corresponding reflection angles are  $(\varphi_q^k)_{k \in \{0, \dots, q-1\}}$ . Using (5.3), we conclude

$$\begin{aligned} \text{(B.5)} \quad \tilde{\ell}_q(u) &= \ell_q(\mu^{-1}u) = \frac{1}{q} \sum_{k=0}^{q-1} u(x_q^k) \frac{\sin(\varphi_q^k)}{\mu(x_q^k)} \\ &= \frac{1}{q} \sum_{k=0}^{q-1} u \left( k/q + \frac{\alpha(k/q)}{q^2} \right) \left( 1 + \frac{\beta(k/q)}{q^2} \right) + \Sigma_q(u) + \varepsilon O(q^{-4}), \end{aligned}$$

where  $\Sigma_q$  is the following functional:

$$\Sigma_q(u) = \frac{1}{q} \sum_{k=0}^{q-1} S_q(k/q) u \left( k/q + \frac{\alpha(k/q)}{q^2} \right),$$

which is obtained by substituting in the expression for  $\tilde{\ell}_q(u)$  the expression (B.1) and summing over  $k$ .

Let us now introduce the Fourier basis  $B = (e_j)_{j \geq 0}$  of even real functions of the circle in the Lazutkin parametrization  $(e_j = \cos 2\pi jx)_{j \geq 0}$  together with the convenient notation

$$\delta_{q|j} = \begin{cases} 1 & \text{if } q \mid j, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, let us write  $\alpha(x) = \sum_{k \in \mathbb{Z}} \alpha_k \exp(ikx)$  (and similarly for  $\beta$ ); by the parity properties of  $\alpha$  and  $\beta$ , we conclude that

$$\alpha(x) = \sum_{k \geq 1} 2i\alpha_k \sin 2\pi kx, \quad \beta(x) = \sum_{k \geq 0} 2\beta_k \cos 2\pi kx,$$

where  $\alpha_k = -\alpha_{-k}$  is purely imaginary and  $\beta_k = \beta_{-k}$  is real.

We now analyze the functional  $\Sigma_q(e_j)$ ; let us first record the following properties of the function  $S_q$ . For any  $x$ , we have

$$\text{(B.6a)} \quad |S_q(x)| \leq \frac{\mu^2(x)}{6q^2} \leq \frac{(\pi + \varepsilon)^2}{6q^2},$$

$$\text{(B.6b)} \quad |S_q^{(r)}(x)| = \varepsilon O(q^{-2}), \quad r \geq 1.$$

Next, by definition,

$$\Sigma_q(e_j) = \frac{1}{q} \sum_{k=0}^{q-1} S_q(k/q) \cos \left( 2\pi j \left( k/q + \frac{\alpha(k/q)}{q^2} \right) \right).$$

By the mean value theorem and (B.6a), we can write

$$\Sigma_q(e_j) = \frac{1}{q} \sum_{k=0}^{q-1} \cos \left( \frac{2\pi jk}{q} \right) S_q \left( \frac{k}{q} \right) + \varepsilon O \left( \frac{j}{q^4} \right).$$

We then plug in the Fourier series of  $S_q(x)$ , given by

$$S_q(x) = \sum_{p \in \mathbb{Z}} \sigma_{q,p} e^{2\pi i p x}$$

and obtain the expression

$$(B.7) \quad \Sigma_q(e_j) = \sum_{s \in \mathbb{Z}} \sigma_{q, sq-j} + \varepsilon O\left(\frac{j^2}{q^4}\right).$$

Notice that, since  $S_q$  is even, we have  $\sigma_{q,p} = \sigma_{q,-p}$ ; moreover, by the properties of  $\Sigma_q(x)$ , we have

$$(B.8a) \quad |\sigma_{q,0}| \leq \frac{(\pi + \varepsilon)^2}{6q^2},$$

$$(B.8b) \quad |\sigma_p(q)| = \varepsilon O\left(\frac{1}{p^r q^2}\right) \quad \text{for } p \neq 0,$$

where the second equation follows from integration by parts.

We can now prove the following convenient expansion:

LEMMA B.1. *For all  $q \geq 2$  and  $j \geq 1$ , one has*

$$\tilde{\ell}_q(e_j) = \left(1 + \sigma_{q,0} + \frac{\beta_0}{q^2}\right) \delta_{q|j} + \frac{\tilde{\ell}_\bullet(e_j)}{q^2} + \mathcal{R}_q(e_j),$$

where

$$(B.9) \quad \tilde{\ell}_\bullet(e_j) = \tilde{\sigma}_j + \beta_j - 2\pi j \alpha_j, \quad \text{with } \tilde{\sigma}_j = - \int_0^1 \frac{\mu^2(x)}{6} e^{2\pi i j x} dx$$

and

$$\mathcal{R}_q(e_j) = \frac{1}{q^2} \sum_{\substack{s \in \mathbb{Z} \setminus \{0\} \\ sq \neq j}} q^2 \sigma_{q, sq-j} + \beta_{sq-j} + 2\pi i j \alpha_{sq-j} + \varepsilon O\left(\frac{j^2}{q^4}\right).$$

*Proof.* First, we claim that

$$(B.10) \quad \begin{aligned} \tilde{\ell}_q(e_j) &= \left(1 + \sigma_{q,0} + \frac{\beta_0}{q^2}\right) \delta_{q|j} + \frac{q^2 \sigma_{q,j} + \beta_j - 2\pi i j \alpha_j}{q^2} \\ &\quad + \frac{1}{q^2} \sum_{\substack{s \in \mathbb{Z} \setminus \{0\} \\ sq \neq j}} (q^2 \sigma_{q, sq-j} + \beta_{sq-j} + 2\pi i j \alpha_{sq-j}) + \varepsilon O\left(\frac{j^2}{q^4}\right). \end{aligned}$$

The expression (B.10) is the result of the following simple computation: by plugging in  $u = e_j$  in (B.5), we obtain

$$\begin{aligned}
 \tilde{\ell}_q(e_j) &= \frac{1}{q} \sum_{k=0}^{q-1} \cos \left( 2\pi j \left( k/q + \frac{\alpha(k/q)}{q^2} \right) \right) \left( 1 + \frac{\beta(k/q)}{q^2} \right) + \Sigma_q(e_j) + O \left( \frac{\varepsilon}{q^4} \right) \\
 &= \frac{1}{q} \sum_{k=0}^{q-1} \left( \cos(2\pi j k/q) - 2\pi j \sin(2\pi j k/q) \frac{\alpha(k/q)}{q^2} \right) \left( 1 + \frac{\beta(k/q)}{q^2} \right) \\
 &\quad + \Sigma_q(e_j) + O \left( \frac{\varepsilon j^2}{q^4} \right) \\
 &= \delta_{q|j} \\
 &\quad + \frac{1}{2q^3} \sum_{k=0}^{q-1} \sum_{p \in \mathbb{Z}} \left[ 2\pi i j (\exp(2\pi i j k/q) - \exp(-2\pi i j k/q)) \alpha_p \exp(2\pi i p k/q) \right. \\
 &\quad \quad \left. + (\exp(2\pi i j k/q) + \exp(-2\pi i j k/q)) \beta_p \exp(2\pi i p k/q) \right] \\
 &\quad + \Sigma_q(e_j) + O \left( \frac{\varepsilon j^2}{q^4} \right) \\
 &= \delta_{q|j} + \frac{1}{2q^2} \sum_{s \in \mathbb{Z}} [\beta_{sq-j} + 2\pi i j \alpha_{sq-j} + \beta_{sq+j} - 2\pi i j \alpha_{sq+j}] \\
 &\quad + \Sigma_q(e_j) + O \left( \frac{\varepsilon j^2}{q^4} \right),
 \end{aligned}$$

and using the fact that  $\alpha_p = -\alpha_{-p}$ ,  $\beta_p = \beta_{-p}$  and by (B.7),

$$= \delta_{q|j} + \frac{1}{q^2} \sum_{s \in \mathbb{Z}} [q^2 \sigma_{q, sq-j} + \beta_{sq-j} + 2\pi i j \alpha_{sq-j}] + O \left( \frac{\varepsilon j^2}{q^4} \right),$$

which immediately implies (B.10). In order to conclude the proof, we need to control the term  $\sigma_{q,j}$  appearing on the second term in (B.10) in a way that is independent of  $q$ . To perform this task we note that

$$\begin{aligned}
 \sigma_{q,j} &= \int_0^1 S_q(x) e^{2\pi i j x} dx = \int_0^1 \left( \frac{\sin \left( \frac{1}{q} \mu(x) \right)}{\frac{1}{q} \mu(x)} - 1 \right) e^{2\pi i j x} dx \\
 &= - \int_0^1 \frac{\mu^2(x)}{6q^2} e^{2\pi i j x} dx + \int_0^1 R \left( \frac{\mu(x)}{q} \right) e^{2\pi i j x} dx,
 \end{aligned}$$

where  $R$  is defined by  $\frac{\sin(y)}{y} - 1 = -\frac{y^2}{6} + R(y)$ . Since  $R(y) = O(y^4)$  and  $R'(y) = O(y^3)$ , by performing integration by parts once to the second integral

and the fact  $|\mu'(x)| = O(\varepsilon)$ , we get

$$\int_0^1 R\left(\frac{\mu(x)}{q}\right) e^{2\pi i j x} dx = O\left(\frac{\varepsilon j}{q^4}\right).$$

Therefore, we can absorb this term in the remainder term  $\mathcal{R}_q(e_j)$ . We conclude that we can write

$$\tilde{\ell}_q(e_j) = \left(1 + \sigma_{q,0} + \frac{\beta_0}{q^2}\right) \delta_{q|j} + \frac{\tilde{\ell}_\bullet(e_j)}{q^2} + \mathcal{R}_q(e_j),$$

with  $\tilde{\ell}_\bullet(e_j)$  and  $\tilde{\sigma}_j$  as in the statement of the lemma.  $\square$

*Acknowledgments.* We thank H. Eliasson, B. Fayad, A. Figalli, J. Mather, I. Polterovich, P. Sarnak, S. Zeldich for their most useful comments, which allowed us to vastly improve the exposition of our result. JDS acknowledges partial NSERC support. VK acknowledges partial support of the NSF grant DMS-1402164. QW acknowledges support of University of Maryland during her stay in College Park, where part of the work was done.

## References

- [1] K. G. ANDERSSON and R. B. MELROSE, The propagation of singularities along gliding rays, *Invent. Math.* **41** no. 3 (1977), 197–232. MR 0494322. Zbl 0373.35053. <https://doi.org/10.1007/BF01403048>.
- [2] A. AVILA, J. DE SIMOI, and V. KALOSHIN, An integrable deformation of an ellipse of small eccentricity is an ellipse, *Ann. of Math.* **184** no. 2 (2016), 527–558. MR 3548532. Zbl 06662219. <https://doi.org/10.4007/annals.2016.184.2.5>.
- [3] J. CHAZARAIN, Formule de Poisson pour les variétés riemanniennes, *Invent. Math.* **24** (1974), 65–82. MR 0343320. Zbl 0281.35028. <https://doi.org/10.1007/BF01418788>.
- [4] C. B. CROKE, Rigidity for surfaces of nonpositive curvature, *Comment. Math. Helv.* **65** no. 1 (1990), 150–169. MR 1036134. Zbl 0704.53035. <https://doi.org/10.1007/BF02566599>.
- [5] C. B. CROKE and V. A. SHARAFUTDINOV, Spectral rigidity of a compact negatively curved manifold, *Topology* **37** no. 6 (1998), 1265–1273. MR 1632920. Zbl 0936.58013. [https://doi.org/10.1016/S0040-9383\(97\)00086-4](https://doi.org/10.1016/S0040-9383(97)00086-4).
- [6] J. J. DUISTERMAAT and V. W. GUILLEMIN, The spectrum of positive elliptic operators and periodic bicharacteristics, *Invent. Math.* **29** no. 1 (1975), 39–79. MR 0405514. Zbl 0307.35071. <https://doi.org/10.1007/BF01405172>.
- [7] C. GORDON, D. L. WEBB, and S. WOLPERT, One cannot hear the shape of a drum, *Bull. Amer. Math. Soc.* **27** no. 1 (1992), 134–138. MR 1136137. Zbl 0756.58049. <https://doi.org/10.1090/S0273-0979-1992-00289-6>.
- [8] V. GUILLEMIN and D. KAZHDAN, Some inverse spectral results for negatively curved 2-manifolds, *Topology* **19** no. 3 (1980), 301–312. MR 0579579. Zbl 0465.58027. [https://doi.org/10.1016/0040-9383\(80\)90015-4](https://doi.org/10.1016/0040-9383(80)90015-4).

- [9] B. HALPERN, Strange billiard tables, *Trans. Amer. Math. Soc.* **232** (1977), 297–305. MR 0451308. Zbl 0374.53001. <https://doi.org/10.2307/1998942>.
- [10] H. HEZARI, Robin spectral rigidity of strictly convex domains with a reflectional symmetry, 2016. arXiv 1609.01011.
- [11] H. HEZARI and S. ZELDITCH,  $C^\infty$  spectral rigidity of the ellipse, *Anal. PDE* **5** no. 5 (2012), 1105–1132. MR 3022850. Zbl 1264.35150. <https://doi.org/10.2140/apde.2012.5.1105>.
- [12] M. KAC, Can one hear the shape of a drum?, *Amer. Math. Monthly* **73** no. 4, part II (1966), 1–23. MR 0201237. Zbl 0139.05603. <https://doi.org/10.2307/2313748>.
- [13] V. F. LAZUTKIN, Existence of caustics for the billiard problem in a convex domain, *Izv. Akad. Nauk SSSR Ser. Mat.* **37** (1973), 186–216. MR 0328219. Zbl 0256.52001. <https://doi.org/10.1070/IM1973v007n01ABEH001932>.
- [14] R. DE LA LLAVE, J. M. MARCO, and R. MORIYÓN, Canonical perturbation theory of Anosov systems and regularity results for the Livšic cohomology equation, *Ann. of Math.* **123** no. 3 (1986), 537–611. MR 0840722. Zbl 0603.58016. <https://doi.org/10.2307/1971334>.
- [15] B. OSGOOD, R. PHILLIPS, and P. SARNAK, Compact isospectral sets of surfaces, *J. Funct. Anal.* **80** no. 1 (1988), 212–234. MR 0960229. Zbl 0653.53021. [https://doi.org/10.1016/0022-1236\(88\)90071-7](https://doi.org/10.1016/0022-1236(88)90071-7).
- [16] B. OSGOOD, R. PHILLIPS, and P. SARNAK, Extremals of determinants of Laplacians, *J. Funct. Anal.* **80** no. 1 (1988), 148–211. MR 0960228. Zbl 0653.53022. [https://doi.org/10.1016/0022-1236\(88\)90070-5](https://doi.org/10.1016/0022-1236(88)90070-5).
- [17] B. OSGOOD, R. PHILLIPS, and P. SARNAK, Moduli space, heights and isospectral sets of plane domains, *Ann. of Math.* **129** no. 2 (1989), 293–362. MR 0986795. Zbl 0677.58045. <https://doi.org/10.2307/1971449>.
- [18] J.-P. OTAL, Le spectre marqué des longueurs des surfaces à courbure négative, *Ann. of Math.* **131** no. 1 (1990), 151–162. MR 1038361. Zbl 0699.58018. <https://doi.org/10.2307/1971511>.
- [19] V. M. PETKOV and L. N. STOYANOV, *Geometry of Reflecting Rays and Inverse Spectral Problems, Pure Appl. Math.*, John Wiley & Sons, Ltd., New York, 1992. MR 1172998. Zbl 0761.35077.
- [20] G. POPOV and P. TOPALOV, Invariants of isospectral deformations and spectral rigidity, *Comm. Partial Differential Equations* **37** no. 3 (2012), 369–446. MR 2889558. Zbl 1245.58015. <https://doi.org/10.1080/03605302.2011.641051>.
- [21] G. POPOV and P. TOPALOV, From K.A.M. Tori to isospectral invariants and spectral rigidity of billiard tables, 2016, preprint. arXiv 1602.03155.
- [22] P. SARNAK, Determinants of Laplacians; heights and finiteness, in *Analysis, et Cetera*, Academic Press, Boston, 1990, pp. 601–622. MR 1039364. Zbl 0703.53037. <https://doi.org/10.1016/B978-0-12-574249-8.50033-X>.
- [23] T. SUNADA, Riemannian coverings and isospectral manifolds, *Ann. of Math.* **121** no. 1 (1985), 169–186. MR 0782558. Zbl 0585.58047. <https://doi.org/10.2307/1971195>.

- [24] M.-F. VIGNÉRAS, Variétés riemanniennes isospectrales et non isométriques, *Ann. of Math.* **112** no. 1 (1980), 21–32. MR 0584073. Zbl 0445.53026. <https://doi.org/10.2307/1971319>.
- [25] S. ZELDITCH, The inverse spectral problem, in *Surveys in Differential Geometry. Vol. IX, Surv. Differ. Geom.* **9**, Int. Press, Somerville, MA, 2004, pp. 401–467. MR 2195415. Zbl 1061.58029. <https://doi.org/10.4310/SDG.2004.v9.n1.a12>.
- [26] S. ZELDITCH, Inverse spectral problem for analytic domains. II.  $\mathbb{Z}_2$ -symmetric domains, *Ann. of Math.* **170** no. 1 (2009), 205–269. MR 2521115. Zbl 1196.58016. <https://doi.org/10.4007/annals.2009.170.205>.
- [27] S. ZELDITCH, Survey on the inverse spectral problem, *ICCM Not.* **2** no. 2 (2014), 1–20. MR 3314780. <https://doi.org/10.4310/ICCM.2014.v2.n2.a1>.

(Received: June 28, 2016)

(Revised: March 2, 2017)

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