

Affine Grassmannians and the geometric Satake in mixed characteristic

By XINWEN ZHU

Abstract

We endow the set of lattices in \mathbb{Q}_p^n with a reasonable algebro-geometric structure. As a result, we prove the representability of affine Grassmannians and establish the geometric Satake equivalence in mixed characteristic. We also give an application of our theory to the study of Rapoport-Zink spaces.

Contents

Introduction	404
0.1. Mixed characteristic affine Grassmannians	404
0.2. The geometric Satake	407
0.3. Dimension of affine Deligne-Lusztig varieties	409
0.4. Plan of the paper	411
0.5. Notation	411
0.6. Acknowledgement	412
1. Affine Grassmannians	412
1.1. p -adic loop groups and affine Grassmannians	412
1.2. The affine Grassmannian for GL_n	415
1.3. Demazure resolution	421
1.4. Affine Grassmannians and affine flag varieties	425
2. The geometric Satake	429
2.1. The Satake category Sat_G	430
2.2. Semi-infinite orbits	433
2.3. The monoidal structure on H^*	440
2.4. The commutativity constraints	443
2.5. Identification with the dual group	454
3. Dimension of affine Deligne-Lusztig varieties	455
3.1. Dimension of affine Deligne-Lusztig varieties	455
3.2. Affine Deligne-Lusztig varieties and Rapoport-Zink spaces	460

Appendix A. Generalities on perfect schemes	464
A.1. Perfect schemes and perfect algebraic spaces	464
A.2. Perfect algebraic spaces perfectly of finite presentation	471
A.3. ℓ -adic sheaves	478
Appendix B. More on mixed characteristic affine Grassmannians	482
B.1. The determinant line bundle	482
B.2. Deperfection	484
B.3. Example: $\widetilde{\mathrm{Gr}}_2 \rightarrow \overline{\mathrm{Gr}}_2$.	486
References	488

Introduction

0.1. *Mixed characteristic affine Grassmannians.*

0.1.1. Let F be a non-archimedean local field; i.e., F is either $\mathbb{F}_q((t))$ or a finite extension of \mathbb{Q}_p , with $\mathcal{O} \subset F$ its ring of integers. Let $V = F^n$ denote an n -dimensional F -vector space. A lattice of V is a finitely generated \mathcal{O} -submodule Λ of V such that $\Lambda \otimes F = V$. For example, $\Lambda_0 = \mathcal{O}^n$ is a lattice in V , and every other lattice in V can be translated to Λ_0 by a linear automorphism of V . Therefore, the set of lattices in V can be identified with the set $\mathrm{GL}_n(F)/\mathrm{GL}_n(\mathcal{O})$.

For various applications in number theory, representation theory and algebraic geometry, it is highly desirable to realize this set as the set of (k) -points of some (reasonable, infinite dimensional) algebraic variety defined over k , where k denotes the residue field of F . If $F = k((t))$, such an algebro-geometric object, called the affine Grassmannian, does exist, and it plays a fundamental role in geometric representation theory and in the study of moduli spaces of vector bundles on algebraic curves. However, a reasonable algebro-geometric structure on the set $\mathrm{GL}_n(\mathbb{Q}_p)/\mathrm{GL}_n(\mathbb{Z}_p)$ has not been available for many years, although some attempts have been made ([Hab05], [Kre14], [CKV15]). The first goal of this paper is to give a solution of this problem to some extent. We will call this new algebro-geometric object the mixed characteristic affine Grassmannian.

0.1.2. To explain the ideas, let us first recall the equal characteristic story. (See, e.g., [BL94] for details.) First one can make sense of the notion of a family of lattices in $k((t))^n$: for a k -algebra R , a lattice in $R((t))^n$ is a finitely generated projective $R[[t]]$ -submodule Λ of $R((t))^n$ such that $\Lambda \otimes_{R[[t]]} R((t)) = R((t))^n$. The affine Grassmannian $\mathrm{Gr}^{\flat 1}$ is defined as the moduli space that

¹In this paper, objects defined in the equal characteristic setting are usually written with \flat in their superscripts. But this does not mean that they are the tilts of the corresponding mixed characteristic objects as in Scholze's theory of perfectoid spaces.

assigns every k -algebra R the set of lattices in $R((t))^n$. In particular, the set of k -points of Gr^b is just $\text{GL}_n(k((t)))/\text{GL}_n(k[[t]])$.

Given a lattice Λ in $R((t))^n$, there always exists some big integer N such that

$$(0.1.1) \quad t^N R[[t]]^n \subset \Lambda \subset t^{-N} R[[t]]^n.$$

So Gr^b is the union of subfunctors $\text{Gr}^{b,(N)}$ consisting of those lattices satisfying (0.1.1). The key fact is that via the map

$$\Lambda \mapsto t^{-N} R[[t]]^n / \Lambda,$$

$\text{Gr}^{b,(N)}(R)$ is identified with the set of quotient $R[[t]]$ -modules of

$$t^{-N} R[[t]]^n / t^N R[[t]]^n$$

that are projective as R -modules. Then it is not hard to see that $\text{Gr}^{b,(N)}$ is represented by a closed subscheme of the usual Grassmannian variety.

0.1.3. There is an obvious guess of the moduli problem that the mixed characteristic affine Grassmannian Gr should represent: it should associate to every k -algebra R the set

$$(0.1.2) \quad \left\{ \begin{array}{l} \text{finite projective } W(R)\text{-submodules } \Lambda \text{ of } W(R)[1/p]^n \\ \text{such that } \Lambda[1/p] = W(R)[1/p]^n \end{array} \right\},$$

where $W(R)$ is the ring of Witt vectors for R . Unfortunately, this definition is unreasonable,² as the ring of the Witt vectors for a nonperfect ring R is not well behaved. For example, p could be a zero divisor of $W(R)$ if R is nonreduced, so $\Lambda_{0R} = W(R)^n$ may not be a submodule of $W(R)[1/p]^n$. On the other hand, note that

- (1) if R is a perfect k -algebra, then $W(R)$ is well behaved;
- (2) the values of a scheme X at perfect rings R determine X up to perfection;³
- (3) the (étale) topology of a scheme (e.g., the ℓ -adic cohomology) does not change when passing to the perfection.

Therefore, we restrict the naive moduli problem (0.1.2) to the category of perfect k -algebras. This defines a presheaf on this category,⁴ and the best question one can ask is whether this functor is represented by a(n inductive

²Alternatively, one could try to define $\text{Gr}(R)$ as the set of pairs (Λ, β) , where Λ is a finite projective $W(R)$ -module and $\beta : \Lambda \otimes W(R)[1/p] \simeq W(R)[1/p]^n$ is an isomorphism. But we still do not know whether this is reasonable.

³The category of perfect k -schemes is a full subcategory of the category of presheaves on the category of perfect k -algebras. See Lemma A.12.

⁴In fact, Kreidl [Kre14] proved that this is an fpqc sheaf.

limit of) perfect k -scheme(s). Our first main theorem gives a positive answer to a slightly weaker version of the question.

THEOREM 0.1. *The above functor can be written as an increasing union of subfunctors $\mathrm{Gr} = \varinjlim X_i$, where each X_i is the perfection of some proper algebraic space defined over k and $X_i \rightarrow X_{i+1}$ is a closed embedding.*

Perfect k -schemes/algebraic spaces are almost never of finite type over k . But as stated in Theorem 0.1, each X_i appearing above is in fact the perfection of some proper algebraic space over k .⁵ We do not know how to canonically construct these algebraic spaces without passing to the perfection. But this does not bother us. We can still study their topological properties. In particular, we can define the ℓ -adic derived category on X_i , the notion of perverse sheaves, etc.

As soon as the representability of Gr is known, the representability of mixed characteristic affine Grassmannians and affine flag varieties for general reductive groups follows by the same argument as in equal characteristic situation. See Section 1.4.

0.1.4. Now we explain some ideas behind the proof of Theorem 0.1. As in the equal characteristic situation, it is enough to prove the representability of the subfunctor $\mathrm{Gr}^{(N)}$ of Gr defined by a condition similar to (0.1.1). With a little further work, one then reduces to prove the representability of the following functor:

$$\overline{\mathrm{Gr}}_N = \{ \Lambda \in \mathrm{Gr} \mid \Lambda \subset \Lambda_{0R} \text{ such that } \wedge^n \Lambda = p^N \wedge^n \Lambda_{0R} \},$$

where $\Lambda_{0R} = W(R)^n$, and $\wedge^n(-)$ denotes the n -th wedge product. Now, the essential difficulty is that the quotient Λ_{0R}/Λ is not an R -module so the previous strategy to embed this functor into the Grassmannian fails. In fact, a basic question is whether there exists a nontrivial line bundle on $\overline{\mathrm{Gr}}_N$. Note that in equal characteristic, such a line bundle exists, known as the determinant line bundle $\mathcal{L}_{\mathrm{det}}^b$ on $\overline{\mathrm{Gr}}_N^b$. Its fiber over a lattice $\Lambda \subset R[[t]]^n$ is the top exterior wedge of $R[[t]]^n/\Lambda$ regarded as an R -module. This construction certainly fails in mixed characteristic. (See Appendix B.1 for more discussions.)

Therefore, we proceed in another way. Our observation is that after adding level structures, $\overline{\mathrm{Gr}}_N$ is represented by an affine scheme defined by matrix equations. More precisely, for each h , let $W_h(R)$ denote the ring of truncated

⁵It is expected that these X_i 's are the perfections of projective varieties over k . See Appendix B for further discussions. But knowing that they are algebraic spaces is sufficient for all the applications we have in mind.

That X_i 's are the perfections of projective varieties was proved very recently by Bhatt-Scholze [BS].

Witt vectors of length h , and define

$$\overline{\text{Gr}}_{N,h} = \{(\Lambda, \bar{\epsilon}) \mid \Lambda \in \overline{\text{Gr}}_N, \bar{\epsilon} : W_h(R)^n \simeq \Lambda/p^h\Lambda\}.$$

This is an $L^h\text{GL}_n$ -torsor over $\overline{\text{Gr}}_N$, where $L^h\text{GL}_n$ denotes the affine k -group scheme that is the perfection of the Greenberg realization of GL_n over \mathcal{O}/p^h and that acts on $\overline{\text{Gr}}_{N,h}$ by changing the isomorphism $\bar{\epsilon}$. We will show that when $h > N$, $\overline{\text{Gr}}_{N,h}$ can be (noncanonically) identified with the following scheme of pairs of matrices:

$$\{(A, \gamma) \mid \gamma \in L^h\text{GL}_n, A \in L^hM_n, \det A \in p^N(\mathcal{O}/p^h\mathcal{O})^\times, A\gamma = A\},$$

where M_n denotes the scheme of all $n \times n$ matrices and L^hM_n denotes the perfection of its Greenberg realization over \mathcal{O}/p^h . In fact, A is the matrix representing the map

$$W_h(R)^n \xrightarrow{\bar{\epsilon}} \Lambda/p^h\Lambda \rightarrow \Lambda_{0R}/p^h\Lambda_{0R} = W_h(R)^n.$$

Therefore, $\overline{\text{Gr}}_N$ can be expressed as a quotient of an affine scheme by a free action of an affine group scheme. One can expect that such a quotient should exist, at least as an algebraic space over k . This is indeed the case here, but it cannot follow directly from the general theory, because neither $\overline{\text{Gr}}_{N,h}$ nor $L^h\text{GL}_n$ is of finite type. However, we manage to prove the following result, which is enough to conclude Theorem 0.1.

THEOREM 0.2 (see Theorem A.29). *Let G be the perfection of an algebraic group over k , and let X be the perfection of an affine scheme of finite type over k . Assume that G acts on X and that the action map $G \times X \rightarrow X \times X$, $(g, x) \mapsto (gx, x)$ is a closed embedding. Then the quotient $[X/G]$ (as a stack) is represented by the perfection of an algebraic space separated and of finite type over k .*

0.2. The geometric Satake. There are a lot of applications of mixed characteristic affine Grassmannians. The most fundamental one is to establish the geometric Satake equivalence in mixed characteristic. Its equal characteristic counterpart is a result of works of Lusztig, Drinfeld, Ginzburg, Mirković-Vilonen (cf. [Lus83], [Gin90], [BD], [MV07]). In a forthcoming joint work with Liang Xiao [XZ], we will apply the mixed characteristic geometric Satake to the study of some arithmetic geometry of Shimura varieties.⁶

Let G be a reductive group over \mathcal{O} , the ring of integers of a p -adic field F , and let Gr_G denote its affine Grassmannian. As explain above, it makes sense to define the category of L^+G -equivariant perverse sheaves (with coefficients in $\overline{\mathbb{Q}}_\ell$) on Gr_G , denoted by $\text{P}_{L^+G}(\text{Gr}_G)$. As in the equal characteristic situation, this is a semisimple monoidal category with the monoidal structure given by

⁶We refer to [Laf12] for some amazing applications of the equal characteristic geometric Satake to the Langlands correspondence over function fields.

Lusztig's convolution product of sheaves. In addition, one can still endow the hypercohomology functor $H^*(\mathrm{Gr}_G, -) : P_{L+G}(\mathrm{Gr}_G) \rightarrow \mathrm{Vect}_{\overline{\mathbb{Q}}_\ell}$ with a canonical monoidal structure (although the methods of [Gin90], [BD], [MV07] do not work directly in our setting). Our second main theorem is the geometric Satake equivalence in this setting.

THEOREM 0.3. *The monoidal functor H^* factors as the composition of an equivalence of monoidal categories from $P_{L+G}(\mathrm{Gr}_G)$ to the category $\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\hat{G})$ of finite dimensional representations of the Langlands dual group \hat{G} over $\overline{\mathbb{Q}}_\ell$ and the forgetful functor from $\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\hat{G})$ to the category $\mathrm{Vect}_{\overline{\mathbb{Q}}_\ell}$ of finite dimensional $\overline{\mathbb{Q}}_\ell$ -vector spaces.*

In particular, the theorem implies that there exist the commutativity constraints of $P_{L+G}(\mathrm{Gr}_G)$ such that H^* is a tensor functor. In equal characteristic, such constraints come from the interpretation of the convolution product as the fusion product (cf. [MV07, §5] or [BD, §5.3.17]). As the fusion product currently does not exist in mixed characteristic,⁷ it is probably surprising that we can still establish these constraints in the current setting.

In fact, a construction of the commutativity constraints using a categorical version of the classical Gelfand's trick already appeared in [Gin90]. It was then claimed in [BD, §§5.3.8, 5.3.9] (but without proof) that (a modification of) Ginzburg's construction coincides with the construction via the fusion product. Therefore, we do have candidates of the commutativity constraints even in mixed characteristic. The problem is that it is not clear how to verify the properties they suppose to satisfy (e.g., the hexagon axiom) without using the fusion interpretation.

Our new observation is that the validity of these properties is equivalent to a numerical result for the affine Hecke algebra. Namely, in [LV12], [Lus12] Lusztig and Vogan introduced, for a Coxeter system (W, S) with an involution, certain polynomials $P_{y,w}^\sigma(q)$ similar to the usual Kazhdan-Lusztig polynomials $P_{y,w}(q)$ ([KL79]). Then it was conjectured in [Lus12] that if (W, S) is an affine Weyl group and y, w are certain elements in W ,

$$P_{y,w}^\sigma(q) = P_{y,w}(-q);$$

see loc. cit. or Section 2.4.5 for more details. This conjecture is purely combinatoric, but its proof by Lusztig and Yun [LY13] is geometric, which in fact uses the equal characteristic geometric Satake! We then go in the opposite direction by showing that this formula implies that the above mentioned commutativity constraints are the correct ones.

⁷But we note that the recent work Scholze on diamonds opens a door to this direction; see [SW].

So our proof of Theorem 0.3 uses the geometric Satake in equal characteristic. It is an interesting question to find a direct proof of the above combinatoric formula, which will yield a purely local proof of the geometric Satake, in both equal and mixed characteristic.

Along the way of our proof, we also establish the Mirković-Vilonen theory in mixed characteristic. This is very useful to the study of affine Deligne-Lusztig varieties; see below.

0.3. *Dimension of affine Deligne-Lusztig varieties.* One motivation to introduce mixed characteristic affine Grassmannians is their relation to Rapoport-Zink spaces. Let G be a connected reductive group over \mathbb{Q}_p , and assume (for simplicity) that there is an extension of G to a reductive group scheme over \mathbb{Z}_p , still denoted by G . Let $k = \overline{\mathbb{F}}_p$, and let $L = W(k)[1/p]$ denote the completion of the maximal unramified extension of \mathbb{Q}_p . Let σ denote the Frobenius automorphism of L . Let b be a σ -conjugacy class of $G(L)$ and μ a geometric conjugacy class of one parameter subgroups of G (also known as a dominant coweight of G with respect to some chosen Borel). When the triple (G, b, μ) (sometimes called a Rapoport-Zink datum) comes from a PEL-datum, Rapoport-Zink (cf. [RZ96]) defined a formal scheme $\check{\mathcal{M}}(G, b, \mu)$, locally of finite type over $W = W(k)$, as certain moduli problem of p -divisible groups. Recently, Kim (cf. [Kim13]) and Howard-Pappas (cf. [HP15]) generalized the definition of $\check{\mathcal{M}}(G, b, \mu)$ to include those Rapoport-Zink data of Hodge type and proved the representability of $\check{\mathcal{M}}(G, b, \mu)$ in the case $p > 2$. In any case, a serious restriction is that μ must be *minuscule*. Under this assumption, by the Dieudonné theory one identifies the set of k -points of $\check{\mathcal{M}}(G, b, \mu)$ with the set

$$(0.3.1) \quad X_\mu(b) = \{g \in G(L)/G(W) \mid g^{-1}b\sigma(g) \in G(W)p^\mu G(W)\}.$$

This identification endows $X_\mu(b)$ with an algebro-geometric structure; therefore $X_\mu(b)$ is sometimes called an affine Deligne-Lusztig variety (cf. [Rap05]). Note that the definition of $X_\mu(b)$ as a set makes sense for any triple (G, b, μ) , but only for minuscule μ , $X_\mu(b)$ may relate to the moduli of p -divisible groups.

It has been hoped to endow $X_\mu(b)$ with an algebro-geometric structure without using p -divisible groups and the Dieudonné theory. Now, the existence of the mixed characteristic affine Grassmannian Gr_G (for G over a p -adic field F) allows us to realize $X_\mu(b)$ as a (locally) closed subset of Gr_G ⁸ and therefore to give $X_\mu(b)$ a structure as an ind-perfect algebraic space. Note that in this new definition, there is no restriction on the cocharacter μ . But when (G, b, μ) arises as a(n unramified) Rapoport-Zink datum of Hodge type as above, we have the following proposition, as a simple application of the

⁸Note that (3.1.2) shows that $X_\mu(b)$ itself is a kind of moduli space of mixed characteristic local Shtukas over k .

equivalence of categories between p -divisible groups and F -crystals over a perfect ring in characteristic $p > 0$ (a theorem of Gabber; see also [Lau13, §6]).

PROPOSITION 0.4. *Let $\overline{\mathcal{M}}_\mu^{p-\infty}(b)$ denote the perfection of the special fiber of $\check{\mathcal{M}}(G, b, \mu)$. Then there is a canonical isomorphism of spaces $\overline{\mathcal{M}}_\mu^{p-\infty}(b) \simeq X_\mu(b)$.*

Even if the primary interests are the study of the Rapoport-Zink spaces, having another definition of $X_\mu(b)$ gives us extra flexibility. For example, the new definition is group theoretical, so it allows us to study $\check{\mathcal{M}}(G, b, \mu)$ by using root subgroups or Levi subgroups G , or passing to central isogenies of G .

In a forthcoming work [XZ], these extra flexibilities allow us to understand the irreducible components of certain RZ spaces. Here, we illustrate this idea by one simple example:⁹ we prove the dimension formula of $X_\mu(b)$ as conjectured by Rapoport.

THEOREM 0.5. *The ind-perfect algebraic space $X_\mu(b)$ is finite dimensional, and*

$$\dim X_\mu(b) = \langle \rho, \mu - \nu_b \rangle - \frac{1}{2} \operatorname{def}_G(b).$$

We refer to Section 3.1 for unexplained notation. Thanks to Proposition 0.4, when (G, b, μ) is of Hodge type, we obtain the dimension formula of the corresponding Rapoport-Zink space.¹⁰

Not surprisingly, after the machinery is set up, we can imitate the methods used in the equal characteristic situation (with one innovation): we can apply the arguments of [GHKR06] in the current setting and reduce to prove Theorem 0.5 for those b that are the so-called superbasic σ -conjugacy classes. It was shown in [GHKR06], [CKV15] that if G is of adjoint type, superbasic σ -conjugacy classes exist only when $G = \operatorname{PGL}_n$ or $G = \operatorname{Res}_{E/F} \operatorname{PGL}_n$, where E/F is an unramified extension. The case when $G = \operatorname{PGL}_n$ was treated in [Vie06] (in equal characteristic, but the same arguments apply to mixed characteristic as well). The case when $G = \operatorname{Res}_{E/F} \operatorname{PGL}_n$ was treated in [Ham15a] in the equal characteristic situation and then was adapted in [Ham15b] to deal with the corresponding Rapoport-Zink spaces. Our innovation here is a reduction of the $\operatorname{Res}_{E/F} \operatorname{PGL}_n$ case to the PGL_n case so one can invoke the results of [Vie06] directly (see Proposition 3.3). In particular, it gives a much shorter proof of the main result of [Ham15a] (assuming [Vie06]). We note that

⁹An earlier example is the study of the connected components of $X_\mu(b)$ in [CKV15], although the notion of connected components of $X_\mu(b)$ was defined in loc. cit. in an ad hoc way, due to the lack of the representability of $X_\mu(b)$ at the time.

¹⁰When the first draft version of the paper was completed, Hamacher released his preprint [Ham15b] where Rapoport's conjecture for PEL type Rapoport-Zink spaces was solved.

this reduction step uses the representability of $X_\mu(b)$ for nonminuscule μ (even we are just interested in proving Theorem 0.5 for minuscule μ) and also the semismallness of convolution maps of affine Grassmannians.

0.4. *Plan of the paper.* We quickly discuss the organization of this paper. In Section 1, we prove the representability of affine Grassmannians and affine flag varieties and discuss their first properties, first for GL_n in Sections 1.2 and 1.3, and then for general groups in Section 1.4. We establish the geometric Satake equivalence in Section 2. In particular, the semi-infinite geometry of the affine Grassmannian is discussed in Section 2.2 and the Tannakian structure on the category is established in Sections 2.3 and 2.4. In Section 3, we prove the dimension formula for Rapoport-Zink spaces conjectured by Rapoport, as an application of our theory. The paper contains two appendices. Appendix A discusses perfect schemes and perfect algebraic spaces in some generality, which is the setting we will work with in the paper. Appendix B discusses some further questions on affine Grassmannians, including conjectures related to the representability of affine Grassmannians as *schemes* and the deperfection of “Schubert varieties” inside them. Finally, we discuss an example of these “Schubert varieties” in our setting.

0.5. *Notation.* We fix a perfect field k of characteristic $p > 0$. For a k -algebra R , let

$$W(R) = \{(a_0, a_1, \dots) \mid a_i \in R\}$$

denote its ring of Witt vectors, and let $R \rightarrow W(R)$, $x \mapsto [x] = (x, 0, 0, \dots)$ be the Teichmüller lifting. We denote by $W_h(R)$ the ring of truncated Witt vectors of length h . If R is perfect, $W_h(R) = W(R)/p^h W(R)$.

Let us write $\mathcal{O}_0 = W(k)$, $F_0 = W(k)[1/p]$. Except in Section 3.2, F denotes a totally ramified finite extension of F_0 and \mathcal{O} denotes the ring of integers of F . Let ϖ denote a uniformizer of \mathcal{O} . For a perfect k -algebra R , let

$$W_{\mathcal{O}}(R) = W(R) \otimes_{W(k)} \mathcal{O}$$

and

$$W_{\mathcal{O},n}(R) = W(R) \otimes_{W(k)} \mathcal{O}/\varpi^n.$$

We write $W_{\mathcal{O}}(\bar{k}) = \mathcal{O}_L$ and $L = \mathcal{O}_L[1/p]$, which is the completion of the maximal unramified extension of F . We write

$$D_{F,R} := \text{Spec } W_{\mathcal{O}}(R), \quad D_{F,R}^* := \text{Spec } W_{\mathcal{O}}(R)[1/p].$$

Informally, $D_{F,R}$ (resp. $D_{F,R}^*$) can be thought as a family of discs (resp. punctured discs) parametrized by $\text{Spec } R$. In the above notation, we will omit the subscript F if $F = F_0$, and we will omit the subscript R if $R = k$.

Unless otherwise stated, we will assume that G is a smooth affine group scheme over \mathcal{O} with connected geometric fibers. We will denote by \mathcal{E}_0 the trivial G -torsor.

When G is a split reductive group, we will choose a Borel subgroup $B \subset G$ over \mathcal{O} and a split maximal torus $T \subset B$. Let $U \subset B$ denote the unipotent radical of B . Sometimes, we denote by $\bar{G}, \bar{B}, \bar{U}, \bar{T}$ their reductions modulo ϖ .

Let \mathbb{X}_\bullet denote the coweight lattice of T and \mathbb{X}^\bullet the weight lattice. Let \mathbb{X}_\bullet^+ denote the semi-group of dominant co-weights with respect to the chosen B . We denote by $2\rho \in \mathbb{X}^\bullet$ the sum of positive roots. Let “ \leq ” be the following partial order on \mathbb{X}_\bullet : $\lambda \leq \mu$ if $\mu - \lambda$ is a linear combination of positive roots with coefficients in $\mathbb{Z}_{\geq 0}$. For $\lambda \in \mathbb{X}_\bullet$, the image of $\varpi \in F^\times = \mathbb{G}_m(F)$ under the map $\lambda : \mathbb{G}_m \rightarrow T \rightarrow G$ is denoted by ϖ^λ .

The dual group of G (over a field of characteristic zero) is denoted by \hat{G} . We equip it with a dual Borel \hat{B} and a maximal torus \hat{T} dual to T .¹¹ For $\mu \in \mathbb{X}_\bullet^+$, let V_μ denote the irreducible representation of \hat{G} with the highest weight μ . For $\lambda \in \mathbb{X}_\bullet$, let $V_\mu(\lambda)$ denote the λ -weight subspace of V_μ .

0.6. *Acknowledgement.* An ongoing project with L. Xiao is the main motivation of the current paper. The author thanks him cordially for the collaboration. The author also thanks B. Bhatt, B. Conrad, V. Drinfeld, B. Elias, A. de Jong, X. He, J. Kamnitzer, L. Moret-Bailly, G. Pappas, P. Scholze and Z. Yun for useful discussions and comments. He in particular would like to thank J. Kamnitzer for pointing out a serious mistake in an early draft of the paper. The author is partially supported by NSF grant DMS-1001280/1313894 and DMS-1303296/1535464 and the AMS Centennial Fellowship.

1. Affine Grassmannians

In this section, we construct affine Grassmannians and affine flag varieties in mixed characteristic. We will work with a class of geometric objects called perfect algebraic spaces. We refer to Appendix A for the necessary background for these objects.

1.1. *p-adic loop groups and affine Grassmannians.* In this subsection, we will define affine Grassmannians and state our first main theorem. We refer to Section 0.5 for the notation.

1.1.1. Let \mathcal{X} be a finite type \mathcal{O} -scheme. According to Greenberg ([Gre61]), there are the following two presheaves on the category of affine k -schemes defined as

$$L_p^+ \mathcal{X}(R) = \mathcal{X}(W_{\mathcal{O}}(R)), \quad L_p^h \mathcal{X}(R) = \mathcal{X}(W_{\mathcal{O},h}(R)),$$

¹¹In fact, if \hat{G} is the Tannakian group constructed from the geometric Satake, it is automatically equipped with \hat{B} and \hat{T} .

which are represented by schemes over k . In addition, $L_p^h \mathcal{X}$ is of finite type over k , and $L_p^+ \mathcal{X} = \varprojlim L_p^h \mathcal{X}$. If $\mathcal{X} \subset \mathcal{Y}$ is open, then $L_p^+ \mathcal{X} \subset L_p^+ \mathcal{Y}$ is open. We denote their perfection by

$$L^+ \mathcal{X} = (L_p^+ \mathcal{X})^{p^{-\infty}}, \quad L^h \mathcal{X} = (L_p^h \mathcal{X})^{p^{-\infty}},$$

and we call them p -adic jet spaces. The justification of the choice of the notation is that perfect objects behave better and are more similar to their equal characteristic analogues. If $f : \mathcal{X} \rightarrow \mathcal{Y}$ is an \mathcal{O} -morphism, we denote by $L_p^+ f : L_p^+ \mathcal{X} \rightarrow L_p^+ \mathcal{Y}$ and $L^+ f : L^+ \mathcal{X} \rightarrow L^+ \mathcal{Y}$ the induced maps.

Let X be an affine scheme over F . We define the p -adic loop space LX of X as a perfect space by assigning a perfect k -algebra R the set

$$LX(R) = X(W_{\mathcal{O}}(R)[1/p]).$$

Every F -morphism $f : X \rightarrow Y$ induces a morphism $Lf : LX \rightarrow LY$. We do not define the object $L_p X$ as a presheaf on the category of all affine k -schemes. According to Lemma A.12, the following statement makes sense.

PROPOSITION 1.1. *If X is affine of finite type, then LX is represented by an ind-perfect schemes.*

Proof. As soon as we go to the perfection, the proof is similar to the representability of the usual loop groups in the equal characteristic setting.

First, it is enough to consider the case $F = F_0 = W(k)[1/p]$. If $X = \mathbb{A}^1$, then $LX = \varprojlim (\mathbb{A}^\infty)^{p^{-\infty}}$. This follows from the fact that every element in $W(R)[1/p]$ can be uniquely written as

$$x = \sum_{i \geq -N} p^i [x_i].$$

Second, $X = X_1 \times X_2$, then $L(X_1 \times X_2) = LX_1 \times LX_2$ so $L\mathbb{A}^n$ is representable. Finally, if $Z \subset \mathbb{A}^n$ is a closed embedding, then $LZ \rightarrow L\mathbb{A}^n$ is a closed embedding. Indeed we can write

$$[x] + [y] = \sum p^j [\Sigma_j(x, y)^{1/p^j}],$$

where $\Sigma_j(X, Y)$'s are certain polynomials with two variables X, Y , of homogeneous degree p^j . Now assume that

$$\mathcal{O}_Z = F[t_1, \dots, t_n]/(f_1, \dots, f_m).$$

It is easy to see that if $f(t_1, \dots, t_n)$ is a polynomial with coefficients in F , then

$$f\left(\sum p^i [x_{1i}], \dots, \sum p^i [x_{ni}]\right) = \sum p^j [f^{(j)}(x_{ml})^{1/p^j}].$$

where $f^{(j)}$'s are some polynomials in terms of the variables $X_{ml}, m = 1, \dots, n, l \in \mathbb{Z}$. Then $L_p Z$ is defined in $L_p \mathbb{A}^n$ by $f_r^{(j)}, (f_r^{(j)})^{1/p}, (f_r^{(j)})^{1/p^2}, \dots$. \square

The above arguments also give the following lemma.

LEMMA 1.2.

- (i) *Let \mathcal{X} be an affine scheme of finite type over \mathcal{O} , and let $X = \mathcal{X} \otimes_{\mathcal{O}} F$. Then $L^+ \mathcal{X} \subset LX$ is a closed subscheme.*
- (ii) *If $X \rightarrow Y$ is a closed embedding, then $LX \rightarrow LY$ is a closed embedding.*

Now, let $\mathcal{X} = G$ be a smooth affine group scheme over \mathcal{O} . We write $G^{(0)} = G$ and define the h -th congruence group scheme of G over \mathcal{O} , denoted by $G^{(h)}$, as the dilatation of $G^{(h-1)}$ along the unit. There is a natural map $G^{(h)} \rightarrow G$ that identifies

$$(1.1.1) \quad L^+G^{(h)} = \ker(L^+G \rightarrow L^hG).$$

Note, however, that $L_p^+G^{(h)} \neq \ker(L_p^+G \rightarrow L_p^hG)$.

1.1.2. Let G be a smooth affine group scheme over \mathcal{O} . We define the affine Grassmannian of G as the perfect space

$$\mathrm{Gr}_G := [LG/L^+G];$$

see Section A.1.4 for the notation. Explicitly, for a perfect k -algebra R , $\mathrm{Gr}_G(R)$ is the set of pairs (P, ϕ) , where P is an L^+G -torsor on $\mathrm{Spec} R$ and $\phi : P \rightarrow LG$ is an L^+G -equivariant morphism. Similar to the equal characteristic situation, there is the following interpretation of Gr_G . Recall that we denote by \mathcal{E}_0 the trivial G -torsor on \mathcal{O} .

LEMMA 1.3. *We have*

$$\mathrm{Gr}_G(R) = \left\{ (\mathcal{E}, \beta) \left| \begin{array}{l} \mathcal{E} \text{ is a } G\text{-torsor on } D_{F,R}, \text{ and} \\ \beta : \mathcal{E}|_{D_{F,R}^*} \simeq \mathcal{E}_0|_{D_{F,R}^*} \text{ is a trivialization} \end{array} \right. \right\}.$$

Proof. Let us temporarily denote the functor defined by the right-hand side by Gr'_G . We define a new presheaf $L'G$ as

$$L'G(R) \simeq \left\{ (\mathcal{E}, \beta, \epsilon) \left| \begin{array}{l} (\mathcal{E}, \beta) \in \mathrm{Gr}'_G(R) \\ \epsilon : \mathcal{E}_0 \simeq \mathcal{E} \text{ is a trivialization} \end{array} \right. \right\}.$$

We claim that

- (a) $L'G$ is an L^+G -torsor over Gr'_G ; and
- (b) there is an L^+G -equivariant isomorphism $LG = L'G$.

Then it follows that $\mathrm{Gr}_G = \mathrm{Gr}'_G$.

Let \mathcal{E} be a G -torsor on $D_{F,R}$. Since G is smooth, after replacing R by its an étale cover, we may assume that \mathcal{E} is trivial when restricted to $\mathrm{Spec} W_{\mathcal{O}}(R)/\varpi$. Then it is trivial on $D_{F,R}$, again by the smoothness of G . In other words, étale locally on R a trivialization ϵ as in the definition of $L'G$ always exists. Claim (a) follows. The isomorphism in Claim (b) is given by $A \mapsto (\mathcal{E}_0, A, \mathrm{id})$ with the inverse map given by $(\mathcal{E}, \beta, \epsilon) \mapsto A := \beta\epsilon$. \square

According to Lemma A.12, one can ask whether Gr_G is represented by a (n ind)-perfect scheme. Our main theorem of this section gives a positive answer to a slightly weaker version of this question.

THEOREM 1.4. *The affine Grassmannian Gr_G is represented by a separated ind-pfp ind-perfect algebraic space. If G is reductive over \mathcal{O} , then Gr_G is ind-perfectly proper.*

Again, similar to the equal characteristic situation, one can reduce the proof of this theorem to the case $G = \text{GL}_n$ and $F = F_0$ (see Proposition 1.20). So in the next two subsections, we will focus on the GL_n case first.

1.2. *The affine Grassmannian for GL_n .* We denote Gr_{GL_n} by Gr in this subsection for simplicity. We will introduce some closed subspace $\overline{\text{Gr}}_N \subset \text{Gr}$, which can be realized as a quotient of a finite dimensional affine scheme by an action of a finite dimensional affine group scheme. It then follows from Theorem A.29 that $\overline{\text{Gr}}_N$ (and therefore Gr) is representable.

1.2.1. Let R be a perfect k -algebra. As usual we will identify GL_n -torsors on $D_R = \text{Spec } W(R)$ with finite projective $W(R)$ -modules. So we can rewrite the moduli problem Gr as follows. Let $\mathcal{E}_0 = W(R)^n$ denote the rank n free $W(R)$ -module. Then for a perfect k -algebra R , we have

$$\text{Gr}(R) = \left\{ (\mathcal{E}, \beta) \mid \begin{array}{l} \mathcal{E} \text{ is a rank } n \text{ projective } W(R)\text{-module,} \\ \beta : \mathcal{E}[1/p] \rightarrow \mathcal{E}_0[1/p] \text{ is an isomorphism} \end{array} \right\}.$$

Note that via the inclusion $\mathcal{E} \subset \mathcal{E}[1/p] \xrightarrow{\beta} \mathcal{E}_0[1/p] = W(R)[1/p]^n$, we can think of \mathcal{E} as a lattice in $W(R)[1/p]^n$. Therefore, the above moduli interpretation coincides with the one given by (0.1.2). We will use these two points of views interchangeably.

Recall that a finite projective $W(R)$ -module is the same as a finite rank locally free crystal on $\text{Spec } R$. Due to this reason, for two finite projective $W(R)$ -modules \mathcal{E}_1 and \mathcal{E}_2 , an isomorphism $\beta : \mathcal{E}_1[1/p] \simeq \mathcal{E}_2[1/p]$ will be called a quasi-isogeny. Sometimes, we write it as

$$\beta : \mathcal{E}_1 \dashrightarrow \mathcal{E}_2$$

for simplicity. If β extends to a genuine map $\mathcal{E}_1 \rightarrow \mathcal{E}_2$, it is called an isogeny. Now we recall some basic facts of quasi-isogenies.

Let k be a perfect field, and let $\beta : E_1 \dashrightarrow E_2$ be a quasi-isogeny of finite projective $W(k)$ -modules.¹² The relative position of β , denoted by $\text{Inv}(\beta)$, is

¹²Finite projective modules over $W(k)$ are usually denoted by E_0, E_1, \dots instead of $\mathcal{E}_0, \mathcal{E}_1, \dots$ in the sequel.

defined as an element in

$$\mathbb{X}_\bullet(D_n)^+ = \{\mu = (m_1, \dots, m_n) \in \mathbb{Z}^n \mid m_1 \geq m_2 \geq \dots \geq m_n\}$$

as follows. There always exist a basis (e_1, \dots, e_n) of E_1 and a basis (f_1, \dots, f_n) of E_2 such that β is given by

$$\beta(e_i) = p^{m_i} f_i$$

and $m_1 \geq m_2 \geq \dots \geq m_n$. In addition, this sequence (m_1, \dots, m_n) is independent of the choice of the basis. Then we define

$$(1.2.1) \quad \text{Inv}(\beta) = (m_1, \dots, m_n).$$

Note that β is an isogeny if and only if $m_n \geq 0$.¹³

For $0 \leq i \leq n$, we denote by $\omega_i = (1, \dots, 1, 0, \dots, 0)$ with the first i entries 1 and the last $n-i$ entries 0. Let $\omega_i^* = \omega_{n-i} - \omega_n$. Note that $\text{Inv}(\beta) = \omega_i$ if and only if β extends to a genuine map $E_1 \rightarrow E_2$ such that $pE_2 \subset E_1$ and E_2/E_1 is a k -vector space of dimension i . Similarly, $\text{Inv}(\beta) = \omega_i^*$ if and only if β^{-1} induces the inclusions $pE_1 \subset E_2 \subset E_1$ such that E_1/E_2 is of dimension i .

Note that $\mathbb{X}_\bullet(D_n)^+$ can be identified with the set of dominant co-weights of GL_n in the usual way. The partial order “ \leq ” from Section 0.5 then can be explicitly described as follows: $\mu_1 = (m_1, \dots, m_n) \leq \mu_2 = (l_1, \dots, l_n)$ if

$$m_1 + \dots + m_j \leq l_1 + \dots + l_j, \quad j = 1, \dots, n, \quad \text{and} \quad m_1 + \dots + m_n = l_1 + \dots + l_n.$$

Note that ω_0 is a minimal element.

Now let R be a perfect k -algebra, and let $\beta : \mathcal{E}_1 \dashrightarrow \mathcal{E}_2$ be a quasi-isogeny of finite projective $W(R)$ -modules. For $x \in \text{Spec } R$, we denote by

$$\beta_x : \mathcal{E}_1 \otimes_{W(R)} W(k(x))[1/p] \rightarrow \mathcal{E}_2 \otimes_{W(R)} W(k(x))[1/p]$$

the base change of β to x . Let

$$(\text{Spec } R)_\mu = \{x \in \text{Spec } R \mid \text{Inv}(\beta_x) = \mu\}$$

$$\subset (\text{Spec } R)_{\leq \mu} = \{x \in \text{Spec } R \mid \text{Inv}(\beta_x) \leq \mu\}.$$

If $(\text{Spec } R)_\mu = \text{Spec } R$, we say that the quasi-isogeny β is of relative position μ .

LEMMA 1.5. *Let R be perfect k -algebra, and let $\beta : \mathcal{E}_1 \dashrightarrow \mathcal{E}_2$ be a quasi-isogeny as above. If it is of relative position ω_i , then β induces a chain of inclusions $p\mathcal{E}_2 \subset \mathcal{E}_1 \subset \mathcal{E}_2$ and the quotient $\mathcal{E}_2/\mathcal{E}_1$ is a finite projective $W(R)/p = R$ module of rank i . A similar statement holds for ω_i^* .*

Proof. We claim that if β_x is a genuine map at every point $x \in \text{Spec } R$, then β is a genuine map. Indeed, there is an open cover $\text{Spec } W(R) = \cup \text{Spec } W(R)_{f_i}$ such that both \mathcal{E}_1 and \mathcal{E}_2 are free so that we can represent β

¹³In this case, the relative position of β is sometimes also called the Hodge polygon of β and denoted by $HP(\beta)$ in the literature.

by an element in $A_i \in M_{n \times n}(W(R)_{f_i}[1/p])$. We need to show that $A_i \in M_{n \times n}(W(R)_{f_i})$. Let $\bar{f}_i = f_i \bmod p$. Then there is a natural map $j : W(R)_{f_i} \rightarrow W(R_{\bar{f}_i})$ and it is enough to show that $j(A_i) \in M_{n \times n}(W(R_{\bar{f}_i}))$. But this can be checked at every point of $\text{Spec } R_{\bar{f}_i}$. This proves that β induces an inclusion $\mathcal{E}_1 \subset \mathcal{E}_2$. By applying the same argument to the quasi-isogeny

$$\frac{1}{p}\beta^{-1} : p\mathcal{E}_2 \dashrightarrow \mathcal{E}_1,$$

one shows that $p\mathcal{E}_2 \subset \mathcal{E}_1$.

To show that $\mathcal{E}_2/\mathcal{E}_1$ is locally free, first note that for every homomorphism $R \rightarrow R'$ of perfect k -algebras, there is a natural isomorphism

$$(1.2.2) \quad (\mathcal{E}_2/\mathcal{E}_1) \otimes_R R' \simeq (\mathcal{E}_2 \otimes_{W(R)} W(R'))/(\mathcal{E}_1 \otimes_{W(R)} W(R')).$$

So we can assume that both $\mathcal{E}_1, \mathcal{E}_2$ are free, and the dimension of the fibers of $\mathcal{E}_2/\mathcal{E}_1$ is constant on $\text{Spec } R$. Note that $\mathcal{E}_1/\mathcal{E}_2 = \text{coker}(\mathcal{E}_1/p \rightarrow \mathcal{E}_2/p)$. So we reduce to show that on a reduced affine scheme $\text{Spec } R$, if N is finitely presented, and the fiber dimension of N is constant, then N is locally free. But as N is finitely presented, it is isomorphic to an R -module of the form $(\text{coker}(A^m \rightarrow A^n)) \otimes_A R$, where $A \subset R$ is a subring, of finite type over \mathbb{Z} . Then we reduce to the noetherian situation, in which case the statement is well known. \square

Recall the following basic fact ([Kat79, §2.3]).

LEMMA 1.6. $(\text{Spec } R)_{\leq \mu}$ is closed in $\text{Spec } R$, and $(\text{Spec } R)_\mu$ is open in $(\text{Spec } R)_{\leq \mu}$. In particular, $(\text{Spec } R)_{\omega_0}$ is closed.

Here is a direct corollary of this lemma. See Section A.1.4 for the definition of closed embedding between two perfect spaces.

COROLLARY 1.7. The diagonal map $\Delta : \text{Gr} \rightarrow \text{Gr} \times \text{Gr}$ is a closed embedding.

Proof. Let $\text{Spec } R \rightarrow \text{Gr} \times \text{Gr}$ be a map, given by two pairs (\mathcal{E}, β) and (\mathcal{E}', β') . We consider the quasi-isogeny $(\beta')^{-1}\beta : \mathcal{E} \dashrightarrow \mathcal{E}'$. Then $\text{Spec } R \times_{\text{Gr} \times \text{Gr}, \Delta} \text{Gr}$ is represented by $(\text{Spec } R)_{\omega_0}$. \square

For every $\mu \in \mathbb{X}_\bullet(D_n)^+$, let

$$\text{Gr}_{\leq \mu}(R) = \{(\mathcal{E}, \beta) \in \text{Gr}(R) \mid (\text{Spec } R)_{\leq \mu} = \text{Spec } R\}.$$

If $\lambda \leq \mu$, we have the closed embedding $\text{Gr}_{\leq \lambda} \subset \text{Gr}_{\leq \mu}$ by Lemma 1.6. Define

$$\text{Gr}_\mu = \text{Gr}_{\leq \mu} - \cup_{\lambda < \mu} \text{Gr}_{\leq \lambda},$$

which is an open subspace of $\text{Gr}_{\leq \mu}$.

We record the following fact for later use. For $\mu = (m_1, \dots, m_n) \in \mathbb{X}_\bullet(D_n)^+$, let

$$(1.2.3) \quad \Lambda_\mu = W(k)\{p^{m_1}e_1, \dots, p^{m_n}e_n\} \subset W(k)[1/p]^n$$

be the lattice generated by $\{p^{m_1}e_1, \dots, p^{m_n}e_n\}$. Then Λ_μ defines a k -point Gr , denoted by p^μ . The following lemma is a reformulation of the Cartan decomposition.

LEMMA 1.8.

- (i) *The group $\text{GL}_n(W(k))$ acts transitively on the set $\text{Gr}_\mu(k)$. In fact, $\text{Gr}_\mu(k) = \text{GL}_n(W(k))p^\mu$.*
- (ii) $\text{Gr}(k) = \sqcup_{\mu \in \mathbb{X}_\bullet(D_n)} \text{Gr}_\mu(k)$.

The above discussions can be generalized to general split reductive groups over \mathcal{O} ; see Section 1.4.3.

1.2.2. We write $\overline{\text{Gr}}_N$ instead of $\text{Gr}_{\leq N\omega_1}$ and Gr_N instead of $\text{Gr}_{N\omega_1}$. Note that the group $L\text{GL}_n$ acts on Gr , and every $\text{Gr}_{\leq \lambda}$ is contained in $g\overline{\text{Gr}}_N$ as a closed subspace for some $g \in \text{GL}_n(F)$ and some N big enough. Therefore, it enough to prove the representability of $\overline{\text{Gr}}_N$. We make the moduli interpretation of $\overline{\text{Gr}}_N$ more explicit. Namely, for a perfect k -algebra R ,

$$\overline{\text{Gr}}_N(R) = \left\{ (\mathcal{E}, \beta) \left| \begin{array}{l} \mathcal{E} \text{ is a rank } n \text{ projective } W(R)\text{-module,} \\ \text{and } \beta : \mathcal{E} \rightarrow \mathcal{E}_0 \text{ is an isogeny, which} \\ \text{induces } \wedge^n \beta : \wedge^n \mathcal{E} \simeq p^N W(R) \subset \wedge^n \mathcal{E}_0 \end{array} \right. \right\}.$$

Let M_n denote the scheme of $n \times n$ matrices. Define the following morphisms of \mathbb{Z}_p -schemes:

$$\begin{aligned} \pi : M &:= M_n \times \mathbb{G}_m \rightarrow \mathbb{A}^1, & \pi(A, t) &= t \det A, \\ i_N : \text{Spec } \mathbb{Z}_p &\rightarrow \mathbb{A}^1 = \text{Spec } \mathbb{Z}_p[u], & u &\mapsto p^N. \end{aligned}$$

Define a scheme of finite type over \mathbb{Z}_p as

$$V_N = \mathbb{Z}_p \times_{i_N, \mathbb{A}^1, \pi} M.$$

By definition, $L_p^+ V_N(R)$ is the set of pairs (A, t) consisting of an $n \times n$ -matrix A with entries in $W(R)$ and $t \in W(R)^\times$ such that $t \det A = p^N$. Note that $L_p^+ \text{GL}_n$ acts on $L_p^+ V_N$ by left and right multiplications. Passing to the perfection, both actions become free. By the same proof of Lemma 1.3, we obtain the following statement.

LEMMA 1.9. *There is a canonical isomorphism*

$$L^+ V_N / L^+ \text{GL}_n = \overline{\text{Gr}}_N.$$

This lemma expresses $\overline{\text{Gr}}_N$ as a quotient of an affine scheme by an affine group scheme. But it is not very useful since both $L^+ V_N$ and $L^+ \text{GL}_n$ are infinite dimensional. We need to work at the finite level.

Recall that we have the affine group scheme $L^+GL_n^{(h)} = \ker(L^+GL_n \rightarrow L^hGL_n)$. Define

$$\overline{Gr}_{N,h} = L^+V_N/L^+GL_n^{(h)}.$$

In terms of the moduli interpretation,

$$\overline{Gr}_{N,h}(R) = \left\{ (\mathcal{E}, \beta, \bar{\epsilon}) \mid \begin{array}{l} (\mathcal{E}, \beta) \in \overline{Gr}_N(R), \\ \bar{\epsilon} : \mathcal{E}_0|_{W_h(R)} \simeq \mathcal{E}|_{W_h(R)} \end{array} \right\}.$$

This is an L^hGL_n -torsor over \overline{Gr}_N on which L^hGL_n acts by changing the isomorphism $\bar{\epsilon}$. Our main observation is that $\overline{Gr}_{N,h}$ is already represented by an affine scheme when h is large. To prove this, we need to introduce certain affine schemes defined by matrix equations.

We assume that $h > N$. Via the Greenberg realization, the determinant map $\det : M_n \rightarrow \mathbb{A}^1$ induces a morphism

$$(\det_0, \dots, \det_{h-1}) : L_p^h M_n = \mathbb{A}^{n^2h} \rightarrow L_p^h \mathbb{A}^1 = \mathbb{A}^h.$$

Define

$$(1.2.4) \quad \begin{aligned} V'_{N,h} &:= \{A \in L_p^h M_n \mid \det_0 A = \dots = \det_{N-1} A = 0, \det_N A \in \mathbb{G}_m\}, \\ V_{N,h} &:= (V'_{N,h})^{p^{-\infty}}. \end{aligned}$$

Note that there is an $L_p^hGL_n \times L_p^hGL_n$ -action on $V'_{N,h}$ by left and right multiplications. Passing to the perfection, we obtain an action of $L^hGL_n \times L^hGL_n$ on $V_{N,h}$. Let J be the stabilizer group scheme over $V_{N,h}$ with respect to the right multiplication by L^hGL_n ; i.e., J is defined by the Cartesian product

$$(1.2.5) \quad \begin{array}{ccc} J & \longrightarrow & V_{N,h} \times L^hGL_n \\ \downarrow & & \downarrow \\ V_{N,h} & \xrightarrow{\Delta} & V_{N,h} \times V_{N,h}. \end{array}$$

Or explicitly,

$$J = \{(A, \gamma) \in V_{N,h} \times L^hGL_n \mid A\gamma = A\}.$$

Likewise, let J' denote the stabilizer group scheme over $V'_{N,h}$ with respect to the right multiplication by $L_p^hGL_n$. Then J' is an affine scheme of finite type over k , which is a deperfection of J .

There is a natural map

$$\overline{Gr}_{N,h} \rightarrow V_{N,h},$$

given by $(\mathcal{E}, \beta, \bar{\epsilon}) \mapsto (\beta|_{W_h(R)})\bar{\epsilon}$.

The key lemma is the following.

LEMMA 1.10. *Assume that $h > N$. There is an isomorphism*

$$J \simeq \overline{Gr}_{N,h}.$$

In particular, $\overline{Gr}_{N,h}$ is represented by a perfect affine scheme, perfectly of finite type.

Proof. Recall that $J = (J')^{p^{-\infty}}$. Therefore, the second statement follows from the first, which we now prove.

Let R be a perfect k -algebra, and let $A \in V_{N,h}(R)$. Then J_R classifies those $\gamma \in L^h\mathrm{GL}_n(R)$ that make the following diagram commutative:

$$\begin{array}{ccc} \mathcal{E}_0|_{W_h} & \xrightarrow{A} & \mathcal{E}_0|_{W_h} \\ \gamma \downarrow & & \parallel \\ \mathcal{E}_0|_{W_h} & \xrightarrow{A} & \mathcal{E}_0|_{W_h}. \end{array}$$

On the other hand, $(\overline{\mathrm{Gr}}_{N,h})_R$ classifies those $(\mathcal{E}, \beta, \bar{\epsilon})$ such that the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\beta} & \mathcal{E}_0 \\ \bar{\epsilon}^{-1} \downarrow & & \downarrow \\ \mathcal{E}_0|_{W_h} & \xrightarrow{A} & \mathcal{E}_0|_{W_h}, \end{array}$$

where the notation $\bar{\epsilon}^{-1}$ is understood as the composition $\mathcal{E} \rightarrow \mathcal{E}|_{W_h(R)} \xrightarrow{\bar{\epsilon}^{-1}} \mathcal{E}_0|_{W_h(R)}$. Therefore, there is a natural action

$$\overline{\mathrm{Gr}}_{N,h} \times_{V_{N,h}} J \rightarrow \overline{\mathrm{Gr}}_{N,h}, \quad ((\mathcal{E}, \beta, \bar{\epsilon}), \gamma) \mapsto (\mathcal{E}, \beta, \bar{\epsilon}\gamma).$$

Note that the natural map $L^+V_N \rightarrow V_{N,h}$ is surjective on R -points if $h > N$. Indeed, if $A \in V_{N,h}(R)$, then $\det A \in p^N W_h(R)^\times$. Regard A as a matrix in $M_n(W_h(R))$, and let $\tilde{A} \in M_n(W(R))$ denote a lifting. Then $\det \tilde{A} \in p^N W(R)^\times$, and there is a unique $t \in W(R)^\times$ such that $t \det \tilde{A} = p^N$. Then $(\tilde{A}, t) \in L^+V_N(R)$ is a lifting of A .

As a consequence, the map $\overline{\mathrm{Gr}}_{N,h} \rightarrow V_{N,h}$ admits a section. Indeed, if $(\tilde{A}, t) \in L^+V_N$ is a lifting of A , then $(\mathcal{E}_0, \tilde{A}, \mathrm{id}) \in \overline{\mathrm{Gr}}_{N,h}$.

Let us fix such a section:

$$s : V_{N,h} \rightarrow \overline{\mathrm{Gr}}_{N,h}, \quad A \mapsto (\mathcal{E}_A, \beta_A, \bar{\epsilon}_A).$$

It induces a map

$$s : J \rightarrow \overline{\mathrm{Gr}}_{N,h}, \quad \gamma \mapsto (\mathcal{E}_A, \beta_A, \bar{\epsilon}_A \gamma),$$

which is injective on R -points since the action of $L^h\mathrm{GL}_n$ on $\overline{\mathrm{Gr}}_{N,h}$ is free. To show that it is also surjective on R -points, let $(\mathcal{E}, \beta, \bar{\epsilon})$ be a point of $\overline{\mathrm{Gr}}_{N,h}$ such that $(\beta|_{W_h(R)})\epsilon = A$. Then there exists a unique isomorphism $\alpha : \mathcal{E}_A \simeq \mathcal{E}$ such

that the following diagram is commutative:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathcal{E}_A & \xrightarrow{\beta_A} & \mathcal{E}_0 & \longrightarrow & \text{coker } A \longrightarrow 0 \\
 & & \alpha \downarrow & & \parallel & & \parallel \\
 0 & \longrightarrow & \mathcal{E} & \xrightarrow{\beta} & \mathcal{E}_0 & \longrightarrow & \text{coker } A \longrightarrow 0.
 \end{array}$$

Let $\gamma = \bar{\epsilon}_A(\alpha|_{W_h(R)})^{-1}\bar{\epsilon}^{-1}$. Then $(A, \gamma) \in J$ is the preimage of $(\mathcal{E}, \beta, \bar{\epsilon})$ under the above map $s : J \rightarrow \overline{\text{Gr}}_{N,h}$. Therefore, the first claim of the lemma follows. \square

Remark 1.11. The above isomorphism depends on a choice of lifting of the projection $L^+V_N \rightarrow V_{N,h}$. To fix the ambiguity, we will use the obvious lifting given by

$$W_h(R) \rightarrow W(R), \quad \left(\sum_{0 \leq i < h} p^i[r_i] \bmod p^h \right) \mapsto \sum_{0 \leq i < h} p^i[r_i].$$

As a corollary of the above lemma and Theorem A.29, we have

PROPOSITION 1.12. *The functor $\overline{\text{Gr}}_N$ is represented by a separated perfect algebraic space, perfectly of finite type over k . In particular, Gr is representable.*

Proof. Let $G = L^h\text{GL}_n$, which is the perfection of the smooth algebraic group $G_0 = L_p^h\text{GL}_n$. To apply Theorem A.29, it remains to check that $G \times \overline{\text{Gr}}_{N,h} \rightarrow \overline{\text{Gr}}_{N,h} \times \overline{\text{Gr}}_{N,h}$ is a closed embedding. But this follows from Corollary 1.7. \square

1.3. Demazure resolution. The perfect algebraic space $\overline{\text{Gr}}_N$ is in general “singular.” In this subsection, we construct a morphism $\widetilde{\text{Gr}}_N \rightarrow \overline{\text{Gr}}_N$, which can be regarded as the “Demazure resolution” in the current setting. Using it, we show that $\overline{\text{Gr}}_N$ is irreducible and perfectly proper. Therefore, Gr is ind-perfectly proper.

1.3.1. As before, let $\mathcal{E}_0 = W(R)^n$ denote the rank n free $W(R)$ -module. Let $\mu_\bullet = (\mu_1, \dots, \mu_N)$ be a sequence, where each $\mu_i \in \{\omega_1, \dots, \omega_n, \omega_1^*, \dots, \omega_n^*\}$. We consider the following space Gr_{μ_\bullet} on Aff_k^{pf} : for a perfect k -algebra R , $\text{Gr}_{\mu_\bullet}(R)$ classifies chains of quasi-isogenies

$$(1.3.1) \quad \mathcal{E}_N \overset{\beta_N}{\dashrightarrow} \mathcal{E}_{N-1} \overset{\beta_{N-1}}{\dashrightarrow} \cdots \overset{\beta_2}{\dashrightarrow} \mathcal{E}_1 \overset{\beta_1}{\dashrightarrow} \mathcal{E}_0,$$

where all \mathcal{E}_i 's are rank n finite projective $W(R)$ -modules and $\mathcal{E}_i \dashrightarrow \mathcal{E}_{i-1}$ is of relative position μ_i .

PROPOSITION 1.13. *The space Gr_{μ_\bullet} is represented by a perfect k -scheme, perfectly proper over k .*

Proof. We will prove the proposition by induction on N . First, we show that Gr_{ω_i} is represented by $\mathrm{Gr}^{p^{-\infty}}(i, n)$, the perfection of the usual Grassmannian variety that classifies i -dimensional *quotients* of k^n . In fact, we will prove a slightly more general statement.

We make use of the following notation. Let X be a perfect k -scheme, and let \mathcal{E} be a rank n locally free crystal on X . Let R be a perfect k -algebra and $x \in X(R)$ an R -point. We denote the value of \mathcal{E} at the universal PD thickening $W(R) \rightarrow R$ by $x^*\mathcal{E}$, which is a finite projective $W(R)$ -module of rank n , and we denote the value at the trivial thickening $R \xrightarrow{\mathrm{id}} R$ by $x^*\mathcal{E}/p$. By varying x , these $x^*\mathcal{E}/p$ glue together to form a vector bundle of rank n on X , denoted by \mathcal{E}/p .

LEMMA 1.14. *Let X be a perfect k -scheme and \mathcal{E} a locally free crystal of rank n on X . Let Y be the perfect space over X that assigns to every $x : \mathrm{Spec} R \rightarrow X$ the set of isogenies $\mathcal{F}_x \rightarrow x^*\mathcal{E}$ of finite projective $W(R)$ -modules such that $x^*\mathcal{E}/\mathcal{F}_x$ is a locally free $W(R)/p$ -module of rank i . Then Y is represented by the perfect scheme $\mathrm{Gr}^{p^{-\infty}}(i, \mathcal{E}/p)$ introduced in Corollary A.23.*

Proof. The map $Y \rightarrow \mathrm{Gr}^{p^{-\infty}}(i, \mathcal{E}_0/p)$ sends an R -point of Y represented by $\mathcal{F}_x \rightarrow x^*\mathcal{E}$ to an R -point of $\mathrm{Gr}^{p^{-\infty}}(i, \mathcal{E}_0/p)$ represented by $x^*\mathcal{E}/p \rightarrow x^*\mathcal{E}/\mathcal{F}_x \rightarrow 0$. Conversely, given an R -point of $\mathrm{Gr}^{p^{-\infty}}(i, \mathcal{E}_0/p)$ represented by $x^*\mathcal{E}/p \rightarrow \mathcal{Q} \rightarrow 0$, where \mathcal{Q} is a finite projective R -module of rank i , we define $\mathcal{F} = \ker(x^*\mathcal{E} \rightarrow x^*\mathcal{E}/p \rightarrow \mathcal{Q})$. We need to show that it is a finite projective $W(R)$ -module of rank n . Then $\mathcal{F} \rightarrow x^*\mathcal{E}$ is an isogeny and therefore defines a point of Y .

It is enough to show that $\mathcal{F}/p^i\mathcal{F}$ is a finite projective $W(R)/p^i$ -module of rank n for every i . First by definition,

$$p(x^*\mathcal{E}) \subset \mathcal{F} \subset x^*\mathcal{E},$$

and $\mathcal{F}/p(x^*\mathcal{E})$ is a direct summand of $x^*\mathcal{E}/p$. Therefore, $\mathcal{F}/p(x^*\mathcal{E})$ is a finite projective R -module. Now, by tensoring the short exact sequence

$$0 \rightarrow \mathcal{F} \rightarrow x^*\mathcal{E} \rightarrow \mathcal{Q} \rightarrow 0$$

with $- \otimes_{W(R)} R$, we obtain an exact sequence

$$0 \rightarrow \mathrm{Tor}^{W(R)}(\mathcal{Q}, W(R)/p) \rightarrow \mathcal{F}/p \rightarrow x^*\mathcal{E}/p \rightarrow \mathcal{Q} \rightarrow 0.$$

In addition, there is a canonical isomorphism

$$(1.3.2) \quad \mathcal{Q} = \mathrm{Tor}^{W(R)}(\mathcal{Q}, W(R)/p).$$

Therefore, \mathcal{F}/p , which is an extension of $\mathcal{F}/p(x^*\mathcal{E})$ by \mathcal{Q} , is a finite projective R -module of rank n . From the exact sequence

$$0 \rightarrow p(x^*\mathcal{E})/p\mathcal{F} \rightarrow \mathcal{F}/p \rightarrow \mathcal{F}/p(x^*\mathcal{E}) \rightarrow 0,$$

we conclude that $p(x^*\mathcal{E})/p\mathcal{F}$ is a direct summand of \mathcal{F}/p and therefore is a finite projective R -module. Now by induction, we deduce that each $p^i\mathcal{F}/p^{i+1}\mathcal{F}$

is a finite projective R -module of rank n and that $p^{i+1}(x^*\mathcal{E})/p^{i+1}\mathcal{F}$ is a direct summand of $p^i\mathcal{F}/p^{i+1}\mathcal{F}$. Finally, using the exact sequence

$$0 \rightarrow p^i\mathcal{F}/p^{i+1}\mathcal{F} \rightarrow \mathcal{F}/p^{i+1}\mathcal{F} \rightarrow \mathcal{F}/p^i\mathcal{F} \rightarrow 0,$$

and by induction again, we conclude that each $\mathcal{F}/p^i\mathcal{F}$ is a finite projective $W(R)/p^i$ -module. This finishes the proof of the lemma. \square

Combining with Lemma 1.5, we see that Gr_{ω_i} is represented by $\text{Gr}^{p^{-\infty}}(i, n)$. Now assume that Gr_{μ_\bullet} is represented by a perfectly projective perfect k -scheme. Let μ_{N+1} be an additional element in $\mathbb{X}_\bullet(D_n)^+$. Let $U = \text{Spec } R$ be an affine open of Gr_{μ_\bullet} . Then by definition, there is the tautological chain of isogenies $\mathcal{E}_N \rightarrow \mathcal{E}_{N-1} \rightarrow \dots \rightarrow \mathcal{E}_0$ of finite projective $W(R)$ -modules over U , and \mathcal{E}_N/p is a finite projective R -module of rank n . Clearly, by varying U , we obtain a locally free crystal \mathcal{E}_N on Gr_{μ_\bullet} . By Lemmas 1.5 and 1.14 again,

$$\text{Gr}_{\mu_\bullet, \mu_{N+1}} \simeq \begin{cases} \text{Gr}^{p^{-\infty}}(i, \mathcal{E}_N/p), & \mu_{N+1} = \omega_i, \\ \text{Gr}^{p^{-\infty}}(i, (\mathcal{E}_N/p)^*), & \mu_{N+1} = \omega_i^*. \end{cases}$$

By Corollary A.23, Gr_{μ_\bullet} is perfectly proper. \square

Remark 1.15. One can show that

$$\widetilde{\text{Gr}}_1 = \mathbb{P}^{n-1, p^{-\infty}}, \quad \widetilde{\text{Gr}}_2 = \mathbb{P}^{p^{-\infty}}(\Omega_{\mathbb{P}^{n-1}} \oplus \mathcal{O}_{\mathbb{P}^{n-1}}).$$

See Section B.3 for a sample calculation. On the other hand, one can define the equal characteristic Demazure variety $\widetilde{\text{Gr}}_N^{\flat}$ that assigns every (not necessarily perfect) k -algebra R the set of chains

$$\mathcal{E}_N \subset \mathcal{E}_{N-1} \subset \dots \subset \mathcal{E}_0 = R[[t]]^n$$

of finite projective $R[[t]]$ -modules of rank n such that each $\mathcal{E}_i/\mathcal{E}_{i+1}$ is an invertible $R[[t]]/t$ -module. Then

$$\widetilde{\text{Gr}}_N = (\widetilde{\text{Gr}}_N^{\flat})^{p^{-\infty}}, \quad N = 1, 2.$$

We do *not* think this is true for general N .

Likewise, one can define the equal characteristic analogue $\overline{\text{Gr}}_N^{\flat}$ of $\widetilde{\text{Gr}}_N$ as the moduli space of pairs (\mathcal{E}, β) , where \mathcal{E} is a finite projective $R[[t]]$ -module of rank n and $\beta : \mathcal{E} \rightarrow \mathcal{E}_0$ is a map of $R[[t]]$ -modules such that $\wedge^n \beta$ induces $\wedge^n \mathcal{E} \simeq t^N R[[t]] \subset R[[t]]$. From the example given in Section B.3, when $n = 2$ and $N = 2$, $\overline{\text{Gr}}_2 \simeq (\overline{\text{Gr}}_2^{\flat})^{p^{-\infty}}$. But we do not think this is true for general N .

Remark 1.16. There is a canonical deperfection of $\widetilde{\text{Gr}}_N$, which can be regarded as certain moduli space related to p -divisible groups; see Proposition B.8.

1.3.2. Let $\lambda_1 = (m_1, \dots, m_n)$ and $\lambda_2 = (l_1, \dots, l_n)$. We define their sum as

$$\lambda_1 + \lambda_2 = (m_1 + l_1, \dots, m_n + l_n).$$

If we identify $\mathbb{X}_\bullet(D_n)^+$ with the semi-group of dominant co-weights of GL_n , this coincides with the usual addition. Let $\mu_\bullet = (\mu_1, \dots, \mu_N)$ be a sequence with $\mu_i \in \{\omega_1, \dots, \omega_n, \omega_1^*, \dots, \omega_n^*\}$ as before, and let $|\mu_\bullet| = \sum \mu_i$. There is a natural map

$$(1.3.3) \quad \pi : \mathrm{Gr}_{\mu_\bullet} \rightarrow \mathrm{Gr}_{\leq |\mu_\bullet|},$$

which sends $(\mathcal{E}_\bullet, \beta_\bullet) \in \mathrm{Gr}_{\mu_\bullet}$ to $(\mathcal{E}_N, \beta_1 \cdots \beta_N)$.

LEMMA 1.17. *The morphism $\pi : \mathrm{Gr}_{\mu_\bullet} \rightarrow \mathrm{Gr}_{\leq |\mu_\bullet|}$ is representable. It is perfectly proper, and fibers are perfectly proper perfect schemes.*

Proof. Let $\mathrm{Spec} R \rightarrow \mathrm{Gr}_{\leq |\mu_\bullet|}$ be a morphism represented by $(\mathcal{E}, \beta : \mathcal{E} \dashrightarrow \mathcal{E}_0)$. Then the fiber product

$$(\mathrm{Gr}_{\mu_\bullet})_R = \mathrm{Spec} R \times_{\mathrm{Gr}_{\leq |\mu_\bullet|}, \pi} \mathrm{Gr}_{\mu_\bullet}$$

classifies all possible chains of quasi-isogenies as in (1.3.1) such that $\mathcal{E}_N = \mathcal{E}$ and $\beta_1 \cdots \beta_N = \beta$. We consider another moduli problem X over $\mathrm{Spec} R$ that assigns to every homomorphism $R \rightarrow R'$ of perfect k -algebras the set

$$X(R') = \{\mathcal{F}_N \dashleftarrow \mathcal{F}_{N-1} \dashleftarrow \cdots \dashleftarrow \mathcal{F}_0 = \mathcal{E} \mid \mathrm{Inv}(\mathcal{F}_i \dashrightarrow \mathcal{F}_{i+1}) = \mu_i^*\}.$$

By Lemma 1.14, X is represented by a perfect scheme over R , perfectly proper over R . Over X we consider the quasi-isogeny $\mathcal{F}_N \dashrightarrow \mathcal{F}_0 = \mathcal{E} \dashrightarrow \mathcal{E}_0$. Then $(\mathrm{Gr}_{\mu_\bullet})_R$ is represented by X_{ω_0} , which is closed in X by Lemma 1.6. This finishes the proof of the lemma. \square

Now we assume $\mu_i = \omega_1$ for all i . We denote $\mathrm{Gr}_{\mu_\bullet}$ by $\widetilde{\mathrm{Gr}}_N$, which classifies those chains in (1.3.1) such that all β_i 's are isogenies and all $\mathcal{E}_{i-1}/\mathcal{E}_i$ are invertible $W(R)/p$ -modules. Then (1.4.2) specializes to a map $\pi : \widetilde{\mathrm{Gr}}_N \rightarrow \overline{\mathrm{Gr}}_N$.

LEMMA 1.18. *The restriction of the map $\pi : \widetilde{\mathrm{Gr}}_N \rightarrow \overline{\mathrm{Gr}}_N$ to $\pi^{-1}\mathrm{Gr}_N \rightarrow \mathrm{Gr}_N$ is an isomorphism. The fiber of π over every point $x \in \overline{\mathrm{Gr}}_N - \mathrm{Gr}_N$ is nonempty, geometrically connected, and has positive dimension.*

Proof. First, we show that $\pi : \pi^{-1}(\mathrm{Gr}_N) \rightarrow \mathrm{Gr}_N$ is an isomorphism by exhibiting an inverse morphism. Indeed, given $(\mathcal{E}, \beta) \in \mathrm{Gr}_N(R)$, there is a chain of finitely generated $W(R)$ -modules

$$\mathcal{E} = \mathcal{E}_N \subset \mathcal{E}_{N-1} \subset \cdots \subset \mathcal{E}_0 = W(R)^n, \quad \mathcal{E}_i = \mathcal{E} + p^i \mathcal{E}_0.$$

It is enough to show that each \mathcal{E}_i is a projective $W(R)$ -module and $\mathcal{E}_i/\mathcal{E}_{i+1}$ is an invertible $W(R)/p$ -module. Indeed, at each point $x \in \mathrm{Spec} R$, the dimension of the stalk of $\mathcal{E}_i/\mathcal{E}_{i+1}$ is one. Then by the same argument as in the last part of the proof of Lemma 1.5, $\mathcal{E}_i/\mathcal{E}_{i+1}$ is invertible. In addition, by the same

argument as in Lemma 1.14, and by induction on i , each \mathcal{E}_i is a projective $W(R)$ -module.

Next, let K be a perfect field over k . Every isogeny $\mathcal{E} \rightarrow \mathcal{E}_0 = W(K)^n$ of finite projective $W(K)$ -modules can be factored as a sequence of maps $\mathcal{E} = \mathcal{E}_N \rightarrow \mathcal{E}_{N-1} \rightarrow \cdots \rightarrow \mathcal{E}_1 \rightarrow \mathcal{E}_0$ such that $\mathcal{E}_i/\mathcal{E}_{i+1}$ is a one-dimensional vector space over K . This proves that the fibers of π are nonempty.

Next, let $x \in \overline{\text{Gr}}_N$ be a geometric point. We show that $\pi^{-1}(x)$ is connected. Let C_1, \dots, C_r denote its connected components. We factor $\widetilde{\text{Gr}}_N \rightarrow \overline{\text{Gr}}_N$ as

$$(1.3.4) \quad \widetilde{\text{Gr}}_N \xrightarrow{\pi_1} \mathbb{P}^{p-\infty}(\mathcal{E}/p) \xrightarrow{\pi_2} \overline{\text{Gr}}_N,$$

where $\mathcal{E} \rightarrow \mathcal{E}_0$ denotes the tautological isogeny over $\overline{\text{Gr}}_{N-1}$ so $\mathbb{P}^{p-\infty}(\mathcal{E}/p)$ is a $\mathbb{P}^{n-1, p-\infty}$ -bundle over $\overline{\text{Gr}}_{N-1}$. Given $(\mathcal{E}_\bullet, \beta_\bullet) \in \widetilde{\text{Gr}}_N$, the first map forgets $\mathcal{E}_{N-2}, \dots, \mathcal{E}_1$, and the second map further forgets \mathcal{E}_{N-1} . By induction, the first map has geometrically connected fibers. This implies that $\{\pi_1(C_i)\}$ are disjoint subsets of $\pi_2^{-1}(x)$. In addition, since by Lemma 1.17, $\pi^{-1}(x)$ is proper, so each $\pi_1(C_i)$ is closed in $\pi_2^{-1}(x)$. Therefore, to show that $\pi^{-1}(x)$ is connected, it is enough to show that $\pi_2^{-1}(x)$ is connected. Let K be the residue field of x . Let us regard K -points of Gr as lattices $W(K)[1/p]^n$ and switch the notation to represent x by a lattice Λ . Then the fiber of π_2 over this point is given by $\mathbb{P}^{p-\infty}((p^{-1}\Lambda \cap \Lambda_0)/\Lambda)$ (recall that $\Lambda_0 = W(K)^n$ denotes the standard lattice), which is the perfection of a projective space and therefore is connected.

Finally, we show that the fiber over every point in $\overline{\text{Gr}}_N - \text{Gr}_N$ has positive dimension. First note that $\widetilde{\text{Gr}}_N \rightarrow \overline{\text{Gr}}_N$ is $L^+\text{GL}_n$ -equivariant, where $L^+\text{GL}_n$ acts on both spaces via automorphisms of \mathcal{E}_0 . Let $p^\lambda \in \text{Gr}(k)$ be the point defined by the lattice Λ_λ as in (1.2.3). Then by Lemma 1.8, it is enough to show that for $\lambda < N\omega_1$, the fiber over $p^\lambda \in \overline{\text{Gr}}_N(k)$ has positive dimension. If $\lambda < N\omega_1$, there exists some i such that

$$\dim_k(\Lambda_\lambda \cap p^i \Lambda_0 / \Lambda_\lambda \cap p^{i+1} \Lambda_0) > 1.$$

Therefore, the fiber $\pi^{-1}(p^\lambda)$ contains at least a $\mathbb{P}^{1, p-\infty}$, corresponding to possible choices of a line in $(\Lambda_\lambda \cap p^i \Lambda_0 / \Lambda_\lambda \cap p^{i+1} \Lambda_0)$. \square

We have the following consequence.

COROLLARY 1.19. *The separated pfp perfect algebraic space $\overline{\text{Gr}}_N$ is irreducible and perfectly proper. In particular, $\text{Gr} = \text{Gr}_{\text{GL}_n}$ is ind-perfectly proper.*

1.4. *Affine Grassmannians and affine flag varieties.*

1.4.1. Once the representability of Gr_{GL_n} is established, it is not hard to show that the affine Grassmannian $\text{Gr}_G = LG/L^+G$ for a general smooth affine group scheme G over \mathcal{O} is representable.

PROPOSITION 1.20. *Let $\rho : G \rightarrow \mathrm{GL}_n$ be a linear representation such that GL_n/G is quasi-affine. Then $\mathrm{Gr}_G \rightarrow \mathrm{Gr}_{\mathrm{GL}_n}$ is a locally closed embedding. In addition, if GL_n/G is affine, this is a closed embedding.*

Proof. The proof as in [BD, Th. 4.5.1] or [PR08, Th. 1.4] extends *verbatim* to the present situation. \square

For a smooth affine group scheme G over a Dedekind domain, it is known that there exists a linear representation $\rho : G \rightarrow \mathrm{GL}_n$ such that GL_n/G is quasi-affine (cf. [PR08, §1.b]). In addition, if G is reductive, one can choose ρ such that GL_n/G is affine (cf. [Alp14, Cor. 9.7.7]). Therefore, it follows from the representability of $\mathrm{Gr}_{\mathrm{GL}_n}$ and the above proposition that Theorem 1.4 holds for group schemes over $\mathcal{O}_0 = W(k)$. To finish the proof for the general case, it is enough to note that if \mathcal{O} is a totally ramified extension of \mathcal{O}_0 and G is an affine group scheme over \mathcal{O} , then the affine Grassmannian Gr_G of G is isomorphic to the affine Grassmannian $\mathrm{Gr}_{\mathrm{Res}_{\mathcal{O}/\mathcal{O}_0}G}$ of the Weil restriction $\mathrm{Res}_{\mathcal{O}/\mathcal{O}_0}G$ (which is a group scheme over \mathcal{O}_0).

1.4.2. Now we study affine Grassmannians for an important a class of group schemes over \mathcal{O} , namely, parahoric group schemes in the sense of Bruhat-Tits. Following the standard terminology in the literature, we call the affine Grassmannian of a parahoric group scheme a (partial) affine flag variety. As the theory is completely parallel to the equal characteristic situation (after passing to the perfection), we will be sketchy here and refer to [PR08] for details. We will assume that k is algebraically closed.

We temporarily use notation different from Section 0.5. Namely, we start with a connected reductive group over F , denoted by G . Let $B(G, F)$ denote the Bruhat-Tits building of G . We fix an apartment $A(G, F) \subset B(G, F)$ and an alcove $\mathbf{a} \subset A(G, F)$. They determine a maximal split torus $A \subset G$ and an Iwahori group scheme $\mathcal{G}_{\mathbf{a}}$ of G over \mathcal{O} . Let $T = Z_G(A)$ be the centralizer of A in G , which is a maximal torus of G . Its connected Néron model, denoted by \mathcal{T} , is a closed subgroup scheme of $\mathcal{G}_{\mathbf{a}}$. Let \widetilde{W} denote the Iwahori-Weyl group, which is the quotient of the normalizer $N(F)$ of $T(F)$ by $\mathcal{T}(\mathcal{O})$, and let $W_{\mathbf{a}} \subset \widetilde{W}$ denote the affine Weyl group. Let $\{s_i, i \in \mathbb{S}\}$ denote the set of simple reflections, corresponding to the codimension one walls \mathbf{a}_i of the closure $\bar{\mathbf{a}}$ of \mathbf{a} in $A(G, F)$, and let “ \leq ” denote the Bruhat order on \widetilde{W} . We refer to [PR08] (and, in particular, [HR]) for detailed discussions of the above notions.

For $i \in \mathbb{S}$, let \mathcal{G}_i denote the corresponding parahoric group scheme. There is a natural map $\mathcal{G}_{\mathbf{a}} \rightarrow \mathcal{G}_i$. Let $I = L^+\mathcal{G}_{\mathbf{a}}$ and $P_i = L^+\mathcal{G}_i$. Let us write $\mathcal{F}\ell = LG/I$ and call it the affine flag variety of G . By Theorem 1.4, it is representable. For $w \in \widetilde{W}$, let S_w denote the closure of the I -orbit through \dot{w} , where \dot{w} is a lifting of w to $G(F)$. This is the “Schubert variety,” which in the current setting is a separated pfp perfect algebraic space. As in the equal

characteristic situation,

$$S_w = \bigsqcup_{v \leq w} I\dot{v}I/I$$

is a decomposition of S_w into locally closed subsets and each $I\dot{v}I/I$ is isomorphic to the perfection of an affine space of dimension $\ell(v)$. We show that S_w is perfectly proper so that $\mathcal{F}\ell = \varinjlim S_w$ is ind-perfectly proper. The idea is similar to the proof of Corollary 1.19.

Note that I is a subgroup of P_i . (However, $L_p^+ \mathcal{G}_C \rightarrow L_p^+ \mathcal{G}_i$ is *not* a closed embedding.) It is easy to see that $P_i/I \simeq \mathbb{P}^{1,p^{-\infty}}$. Then to any sequence $\tilde{w} = (s_{j_1}, \dots, s_{j_m}), j_1, \dots, j_m \in \mathbb{S}$ (sometimes called a word), one can associate the ‘‘Demazure’’ variety

$$(1.4.1) \quad D_{\tilde{w}} = P_{j_1} \times^I P_{j_2} \times^I \dots \times P_{j_m}/I.$$

Similar to $\widetilde{\text{Gr}}_N$, this is an iterated $\mathbb{P}^{1,p^{-\infty}}$ -bundle. In particular, it is perfectly proper. Now assume that \tilde{w} is a reduced word, i.e., the length $\ell(w) = m$, where $w = s_{j_1} \cdots s_{j_m}$. Then as in [PR08, §8], there is a surjective map

$$(1.4.2) \quad \pi_{\tilde{w}} : D_{\tilde{w}} \rightarrow S_w,$$

with geometrically connected fibers. This shows that S_w is perfectly proper.

In addition, we have the following proposition as in the equal characteristic situation.

PROPOSITION 1.21. *There is a canonical isomorphism $\pi_1(G)_{\text{Gal}(\bar{F}/F)} \simeq \pi_0(LG) \simeq \pi_0(\text{Gr}_G)$.*

Proof. One can argue as in [PR08, §5]: By the standard argument (using the z -extension), it reduces to consider the case when $G = T$ is a torus or when $G = G_{\text{sc}}$ is semisimple and simply connected. Note that the functor $T \mapsto \pi_0(LT)$ from the category of F -tori to the category of abelian groups satisfies the condition of [Kot97, §2]. Therefore, it follows from loc. cit. that the proposition holds for $G = T$. Using the ‘‘Demazure resolution’’ and the Cartan decomposition, one shows that LG is connected if G is simply-connected. \square

1.4.3. Now we switch back to the notation as in Section 0.5. So G denotes an affine group scheme over \mathcal{O} . In addition, we assume that G is split reductive.

We first discuss some generalizations of Section 1.2.1. Let \mathcal{E}_1 and \mathcal{E}_2 be two G -torsors over \mathcal{O} , and let $\beta : \mathcal{E}_1|_{D_F^*} \simeq \mathcal{E}_2|_{D_F^*}$ be an isomorphism between them over F . One can generalize (1.2.1) to define the relative position $\text{Inv}(\beta)$ of β as an element in \mathbb{X}_{\bullet}^+ . In addition, Lemma 1.6 also admits a natural generalization.

LEMMA 1.22. *Let \mathcal{E}_1 and \mathcal{E}_2 be two G -torsors over $D_{F,R} = \text{Spec } W_{\mathcal{O}}(R)$, and let $\beta : \mathcal{E}_1|_{D_{F,R}^*} \simeq \mathcal{E}_2|_{D_{F,R}^*}$ be an isomorphism. Then the set*

$$(\text{Spec } R)_{\leq \mu} = \{x \in \text{Spec } R \mid \text{Inv}_x(\beta) \leq \mu\}$$

is a closed subset.

In equal characteristic, these facts are well known (e.g., see [Zhu16, § 2.1]), and exactly the same arguments apply here.

Then we define $\text{Gr}_{\leq \mu} \subset \text{Gr}$ as

$$\text{Gr}_{\leq \mu} = \{(\mathcal{E}, \beta) \in \text{Gr} \mid \text{Inv}(\beta) \leq \mu\},$$

which is a closed subspace of Gr by Lemma 1.22. It contains

$$\text{Gr}_{\mu} = \{(\mathcal{E}, \beta) \in \text{Gr} \mid \text{Inv}(\beta) = \mu\}$$

as an open subspace. We call $\text{Gr}_{\leq \mu}$ the (spherical) ‘‘Schubert variety’’ corresponding to μ and Gr_{μ} the corresponding ‘‘Schubert cell.’’ The terminology is justified by the following proposition.

PROPOSITION 1.23.

(1) *Let $\mu \in \mathbb{X}_{\bullet}^+$, and let $\varpi^{\mu} \in \text{Gr}$ be the corresponding point (see Section 0.5).*

Then the map

$$(1.4.3) \quad i_{\mu} : L^+G / (L^+G \cap \varpi^{\mu} L^+G \varpi^{-\mu}) \rightarrow LG / L^+G, \quad g \mapsto g \varpi^{\mu}$$

induces an isomorphism $L^+G / (L^+G \cap \varpi^{\mu} L^+G \varpi^{-\mu}) \simeq \text{Gr}_{\mu}$.

(2) *Gr_{μ} is the perfection of a quasi-projective smooth variety of dimension $(2\rho, \mu)$.*

(3) *$\text{Gr}_{\leq \mu}$ is the Zariski closure of Gr_{μ} in Gr and therefore is perfectly proper of dimension $(2\rho, \mu)$.*

Proof. Note that for $h \gg 0$, the Greenberg realization $L_p^h G$ of $G \otimes \mathcal{O} / \varpi^h$ is a canonical model of $L^h G = L^+G / L^+G^{(h)}$, and there is a unique reduced closed subgroup $K \subset L_p^h G$ whose perfection is $(L^+G \cap \varpi^{\mu} L^+G \varpi^{-\mu}) / L^+G^{(h)}$. Then the quotient $L_p^h G / K$ is represented by a smooth quasi-projective variety Gr'_{μ} whose perfection is $L^+G / (L^+G \cap \varpi^{\mu} L^+G \varpi^{-\mu})$. In addition, similar to the equal characteristic situation, it is not hard to show that $\dim \text{Gr}'_{\mu} = (2\rho, \mu)$. By Proposition A.32 and the Cartan decomposition, the inclusion $\text{Gr}'_{\mu}{}^{p^{-\infty}} \subset \text{Gr}_{\mu}$ is a bijective locally closed embedding and therefore is an isomorphism. This implies (1) and (2).

Finally (3) follows from Lemma 1.22 and (2) by the same argument as in the equal characteristic situation; e.g., see [Zhu16, Prop. 2.1.4] for details. \square

For a coweight μ , let P_{μ} denote the parabolic subgroup of G generated by the root subgroups U_{α} of G corresponding to those roots α satisfying $\langle \alpha, \mu \rangle \leq 0$. Let \bar{G} and \bar{P}_{μ} be the special fibers of G and P . Let us denote the natural

projection $L^+G \rightarrow \bar{G}^{p^{-\infty}}$ defined by reduction mod ϖ by $g \mapsto \bar{g}$. Then there is a natural projection

$$(1.4.4) \quad \begin{aligned} \pi_\mu : L^+G / (L^+G \cap \varpi^\mu L^+G \varpi^{-\mu}) &\rightarrow (\bar{G} / \bar{P}_\mu)^{p^{-\infty}}, \\ (gt^\mu \bmod L^+G) &\mapsto (\bar{g} \bmod \bar{P}_\mu^{p^{-\infty}}). \end{aligned}$$

The fibers are isomorphic to the perfection of affine spaces.

We have the following generalization of the isomorphism $\mathrm{Gr}_{\omega_i} \simeq \mathrm{Gr}^{p^{-\infty}}(i, n)$ from Proposition 1.13. Recall that μ is called minuscule if $\langle \mu, \alpha \rangle \leq 1$ for every positive root.

COROLLARY 1.24. *If μ is minuscule, then $\mathrm{Gr}_\mu = \mathrm{Gr}_{\leq \mu}$, and therefore π_μ induces an isomorphism $\mathrm{Gr}_\mu = (\bar{G} / \bar{P}_\mu)^{p^{-\infty}}$.*

In particular, for minuscule μ , $\mathrm{Gr}_{\leq \mu}$ is isomorphic to the perfection of its equal characteristic counterpart. But as mentioned in Remark 1.15, we do *not* think this is true for general ‘‘Schubert varieties.’’

There is a map $\mathbb{X}_\bullet \rightarrow \mathbb{Z}/2$, $\mu \mapsto (-1)^{(2\rho, \mu)}$, which factors through $\mathbb{X}_\bullet(T) \rightarrow \pi_1(G) \rightarrow \mathbb{Z}/2$ and therefore induces a map

$$(1.4.5) \quad p : \mathrm{Gr}_G \rightarrow \pi_0(\mathrm{Gr}_G) \rightarrow \mathbb{Z}/2$$

by Proposition 1.21.

LEMMA 1.25. *The Schubert cell Gr_μ is in the even (resp. odd) components, i.e., $p(\mathrm{Gr}_\mu) = 1$ (resp. $p(\mathrm{Gr}_\mu) = -1$) if and only if $\dim \mathrm{Gr}_\mu$ is even (resp. odd).*

To finish this section, we remark that although affine Grassmannians in mixed and equal characteristic share many similar properties, there are some essential difference. The first difference is that since there is no analogue of the Birkhoff decomposition for p -adic groups, it is not clear whether one can construct the ‘‘big open cell’’ in the mixed characteristic affine Grassmannian. This is also related to the lack of a Beauville-Laszlo style description of the mixed characteristic affine Grassmannian via a ‘‘global curve.’’ The second difference is that there is no ‘‘rotation’’ torus acting on the mixed characteristic affine Grassmannian. As a result, there is no natural section of the projection π_μ defined in (1.4.4).

2. The geometric Satake

We establish the geometric Satake equivalence in this setting. We use notation from Section 0.5. In addition, we assume that k is algebraically closed and that G is a connected reductive group scheme over \mathcal{O} in this section. As explained in the introduction, one can define the category of L^+G -equivariant perverse sheaves on Gr_G , denoted by $\mathrm{P}_{L^+G}(\mathrm{Gr}_G)$. As will be explained below,

there is a convolution product that makes it a semisimple monoidal category. In addition, the global cohomology functor is a natural monoidal functor. Then we establish the commutativity constraints using some numerical results of the affine Hecke algebra. In the course, we will also develop the Mirković-Vilonen's theory in this setting.

For simplicity, we will write Gr for Gr_G if the group G is clear. Proofs are sketchy or omitted if they are similar to their equal characteristic counterparts.

2.1. *The Satake category* Sat_G . In this subsection, we define the Satake category Sat_G as a monoidal category.

2.1.1. Recall that Gr can be written as an inductive limit of L^+G -invariant closed subsets $\mathrm{Gr}_{\leq\mu}$, which are perfectly proper, and that the action of L^+G on $\mathrm{Gr}_{\leq\mu}$ factors through some L^hG that is perfectly of finite type. Therefore, it makes sense to talk about the category of L^+G -equivariant perverse sheaves on $\mathrm{Gr}_{\leq\mu}$ (see Section A.3.5), denoted by $\mathrm{P}_{L^+G}(\mathrm{Gr}_{\leq\mu})$. Then we denote by

$$\mathrm{P}_{L^+G}(\mathrm{Gr}_G) = \varinjlim \mathrm{P}_{L^+G}(\mathrm{Gr}_{\leq\mu})$$

the category of L^+G -equivariant perverse sheaves on Gr_G . We denote by IC_μ the intersection cohomology sheaf on $\mathrm{Gr}_{\leq\mu}$. Then $\mathrm{IC}_\mu|_{\mathrm{Gr}_\mu} = \overline{\mathbb{Q}}_\ell[(2\rho, \mu)]$, and its restriction to each stratum Gr_λ is constant. As

$$\mathrm{Gr}_\mu = L^+G / (L^+G \cap \varpi^\mu L^+G \varpi^{-\mu})$$

and $L^+G \cap \varpi^\mu L^+G \varpi^{-\mu}$ is connected, the irreducible objects of $\mathrm{P}_{L^+G}(\mathrm{Gr}_G)$ are exactly these IC_μ 's.

LEMMA 2.1. *The category $\mathrm{P}_{L^+G}(\mathrm{Gr}_G)$ is semisimple.*

Proof. The proof is literally the same as the equal characteristic situation (see [Lus83] and [Gai01, Prop. 1] for details): The existence of the ‘‘Demazure resolution’’ (see (1.4.2)) whose fibers have pavings by (perfect) affine spaces implies the parity property of the stalk cohomology of IC_μ 's. Together with Lemma 1.25, one concludes that there is no extension between two irreducible objects. \square

2.1.2. We refer to Section A.1.3 for the definition of the twisted product, which will also be called the convolution product in the current setting. Note that there are the L^+G -torsor $LG \rightarrow \mathrm{Gr}$ and the L^+G -space Gr . Then one can form the convolution affine Grassmannian

$$\mathrm{Gr} \tilde{\times} \mathrm{Gr} := LG \times^{L^+G} \mathrm{Gr}.$$

As in the equal characteristic situation (e.g., [MV07]), one can interpret $\mathrm{Gr} \tilde{\times} \mathrm{Gr}$ as

$$\mathrm{Gr} \tilde{\times} \mathrm{Gr}(R) = \left\{ (\mathcal{E}_1, \mathcal{E}_2, \beta_1, \beta_2) \mid \begin{array}{l} \mathcal{E}_1, \mathcal{E}_2 \text{ are } G\text{-torsors on } D_{F,R}, \\ \beta_1 : \mathcal{E}_1|_{D_{F,R}^*} \simeq \mathcal{E}_0|_{D_{F,R}^*}, \beta_2 : \mathcal{E}_2|_{D_{F,R}^*} \simeq \mathcal{E}_1|_{D_{F,R}^*} \end{array} \right\}.$$

Note that the convolution map

$$m : \mathrm{Gr} \tilde{\times} \mathrm{Gr} \rightarrow \mathrm{Gr}, \quad (\mathcal{E}_1, \mathcal{E}_2, \beta_1, \beta_2) \mapsto (\mathcal{E}_2, \beta_1 \beta_2)$$

and the natural projection

$$\mathrm{pr}_1 : \mathrm{Gr} \tilde{\times} \mathrm{Gr} \rightarrow \mathrm{Gr}, \quad (\mathcal{E}_1, \mathcal{E}_2, \beta_1, \beta_2) \mapsto (\mathcal{E}_1, \beta_1)$$

induce $(\mathrm{pr}_1, m) : \mathrm{Gr} \tilde{\times} \mathrm{Gr} \simeq \mathrm{Gr} \times \mathrm{Gr}$. In particular, the convolution Grassmannian is representable as an ind-perfect algebraic space, ind-perfectly proper. Given $\mu_1, \mu_2 \in \mathbb{X}_\bullet^+$ of G , one can form the convolution product of $\mathrm{Gr}_{\leq \mu_1}$ and $\mathrm{Gr}_{\leq \mu_2}$,

$$\mathrm{Gr}_{\leq \mu_1} \tilde{\times} \mathrm{Gr}_{\leq \mu_2} = \{(\mathcal{E}_1, \mathcal{E}_2, \beta_1, \beta_2) \in \mathrm{Gr} \tilde{\times} \mathrm{Gr} \mid \mathrm{Inv}(\beta_1) \leq \mu_1, \mathrm{Inv}(\beta_2) \leq \mu_2\},$$

which is closed in $\mathrm{Gr} \tilde{\times} \mathrm{Gr}$ and therefore is representable. Similarly, one can form the n -fold convolution Grassmannian $\mathrm{Gr} \tilde{\times} \cdots \tilde{\times} \mathrm{Gr}$, classifying $\{(\mathcal{E}_i, \beta_i), i = 1, \dots, n\}$ where \mathcal{E}_i is a G -torsor on $D_{F,R}$ and $\beta_i : \mathcal{E}_i|_{D_{F,R}^*} \simeq \mathcal{E}_{i-1}|_{D_{F,R}^*}$ is an isomorphism. By sending $\{(\mathcal{E}_i, \beta_i), i = 1, \dots, n\}$ to $\beta_1 \cdots \beta_i : \mathcal{E}_i|_{D_{F,R}^*} \simeq \mathcal{E}_0|_{D_{F,R}^*}$, we obtain a map $m_i : \mathrm{Gr} \tilde{\times} \cdots \tilde{\times} \mathrm{Gr} \rightarrow \mathrm{Gr}$. They together induce an isomorphism

$$(2.1.1) \quad (m_1, \dots, m_n) : \mathrm{Gr} \tilde{\times} \cdots \tilde{\times} \mathrm{Gr} \simeq \mathrm{Gr}^n.$$

We call $m_n = m : \mathrm{Gr} \tilde{\times} \cdots \tilde{\times} \mathrm{Gr} \rightarrow \mathrm{Gr}$ the convolution map. Given a sequence of dominant co-weights $\mu_\bullet = (\mu_1, \dots, \mu_n)$ of G , one can define the closed subspace $\mathrm{Gr}_{\leq \mu_\bullet} = \mathrm{Gr}_{\leq \mu_1} \tilde{\times} \cdots \tilde{\times} \mathrm{Gr}_{\leq \mu_n}$ that classifies those $\{(\mathcal{E}_i, \beta_i), i = 1, \dots, n\}$ satisfying $\mathrm{Inv}(\beta_i) \leq \mu_i$. Let $|\mu_\bullet| = \sum \mu_i$; then the convolution map m induces

$$(2.1.2) \quad m : \mathrm{Gr}_{\leq \mu_\bullet} \rightarrow \mathrm{Gr}_{\leq |\mu_\bullet|}, \quad (\mathcal{E}_\bullet, \beta_\bullet) \mapsto (\mathcal{E}_n, \beta_1 \cdots \beta_n).$$

There are variants of the above construction. Namely, one can replace $\mathrm{Gr}_{\leq \mu_i}$ by Gr_{μ_i} and define $\mathrm{Gr}_{\mu_\bullet} = \mathrm{Gr}_{\mu_1} \tilde{\times} \cdots \tilde{\times} \mathrm{Gr}_{\mu_n}$. In particular,

$$(2.1.3) \quad \mathrm{Gr}_{\leq \mu_\bullet} = \bigcup_{\mu'_\bullet \leq \mu_\bullet} \mathrm{Gr}_{\mu'_\bullet}$$

form a stratification of $\mathrm{Gr}_{\leq \mu_\bullet}$, where $\mu'_\bullet \leq \mu_\bullet$ means $\mu'_i \leq \mu_i$ for each i .

Now, as in the equal characteristic situation, one can define a monoidal structure on $\mathrm{P}_{L+G}(\mathrm{Gr})$, using Lusztig’s convolution of sheaves; e.g., see [MV07, §4] for more details. For $\mathcal{A}_1, \mathcal{A}_2 \in \mathrm{P}_{L+G}(\mathrm{Gr})$, we denote by $\mathcal{A}_1 \tilde{\boxtimes} \mathcal{A}_2$ the “external twisted product” of \mathcal{A}_1 and \mathcal{A}_2 on $\mathrm{Gr} \tilde{\times} \mathrm{Gr}$; i.e., the pullback of $\mathcal{A}_1 \tilde{\boxtimes} \mathcal{A}_2$

along $LG \times \text{Gr} \rightarrow \text{Gr} \tilde{\times} \text{Gr}$ is equal to the pullback the external product $\mathcal{A}_1 \boxtimes \mathcal{A}_2$ along $LG \times \text{Gr} \rightarrow \text{Gr} \times \text{Gr}$. Define

$$\mathcal{A}_1 \star \mathcal{A}_2 := m_!(\mathcal{A}_1 \tilde{\boxtimes} \mathcal{A}_2)$$

to be their convolution product, which is an L^+G -equivariant ℓ -adic complex on Gr . Similarly, one can define the n -fold convolution $\mathcal{A}_1 \star \cdots \star \mathcal{A}_n = m_!(\mathcal{A}_1 \tilde{\boxtimes} \cdots \tilde{\boxtimes} \mathcal{A}_n)$.

PROPOSITION 2.2. *The convolution $\mathcal{A}_1 \star \mathcal{A}_2$ is perverse.*

This can be proved using the numerical results of the affine Hecke algebra [Lus83]; see [Gin90] for details. We will outline a direct proof in the next subsection (see Section 2.2.3) following [NP01, §9], after we introduce the semi-infinite orbits.

There is an equivalent formulation of this proposition.

PROPOSITION 2.3. *The convolution product $m : \text{Gr}_{\leq \mu_\bullet} \rightarrow \text{Gr}$ is semismall. That is, the dimension of $\text{Gr}_{\leq \mu_\bullet}^\lambda := m^{-1}(\text{Gr}_{\leq \lambda})$ is at most $(\rho, |\mu_\bullet| + \lambda)$.*

Proof. The direction from Proposition 2.3 to Proposition 2.2 is [MV07, Lemma 4.3]. The inverse direction is mentioned in [MV07, Rem. 4.5]. As we will make use of this statement in Proposition 3.3, we include a sketch of the proof.

Let $d = \dim \text{Gr}_{\mu_\bullet} \cap m^{-1}(\varpi^\lambda)$. By Lemma 2.1 and Proposition 2.2, we can write

$$\text{IC}_{\mu_\bullet} := \text{IC}_{\mu_1} \star \cdots \star \text{IC}_{\mu_n} = \bigoplus_{\lambda} V_{\mu_\bullet}^\lambda \otimes \text{IC}_\lambda,$$

where $V_{\mu_\bullet}^\lambda = \text{Hom}(\text{IC}_\lambda, \text{IC}_{\mu_\bullet})$. The spectral sequence induced by the stratification (2.1.3) implies that the degree $2d - (2\rho, |\mu_\bullet|)$ stalk cohomology of the left-hand side at ϖ^λ is given by

$$H_c^{2d}(\text{Gr}_{\mu_\bullet} \cap m^{-1}(\varpi^\lambda), \overline{\mathbb{Q}}_\ell).$$

The perversity of the right-hand side then implies that $2d - (2\rho, |\mu_\bullet|) \leq -(2\rho, \lambda)$. This implies that $d \leq (\rho, |\mu_\bullet| - \lambda)$. By induction on λ , we conclude that

$$\dim \text{Gr}_{\leq \mu_\bullet}^\lambda \leq d + \dim \text{Gr}_\lambda = (\rho, |\mu_\bullet| + \lambda). \quad \square$$

Remark 2.4. This argument also gives a canonical isomorphism

$$V_{\mu_\bullet}^\lambda = H_c^{(2\rho, |\mu_\bullet| - \lambda)}(\text{Gr}_{\mu_\bullet} \cap m^{-1}(\varpi^\lambda), \overline{\mathbb{Q}}_\ell).$$

Together with Section A.3.3, we see that there is a canonical basis of $V_{\mu_\bullet}^\lambda$ given by the set $\mathbb{B}_{\mu_\bullet}^\lambda$ of irreducible components of $\text{Gr}_{\mu_\bullet} \cap m^{-1}(\varpi^\lambda)$ of dimension $(\rho, |\mu_\bullet| - \lambda)$.

By identifying $(\mathcal{A}_1 \star \mathcal{A}_2) \star \mathcal{A}_3$ and $\mathcal{A}_1 \star (\mathcal{A}_2 \star \mathcal{A}_3)$ with $\mathcal{A}_1 \star \mathcal{A}_2 \star \mathcal{A}_3$, one equips $P_{L^+G}(\text{Gr})$ with a natural monoidal structure. The monoidal category $(P_{L^+G}(\text{Gr}), \star)$ is sometimes also denoted by Sat_G for simplicity.

2.2. *Semi-infinite orbits.* In this subsection, we discuss the geometry of semi-infinite orbits and establish the corresponding Mirković-Vilonen theory in our setting. We will use the notation from Section 0.5.

2.2.1. The affine Grassmannian Gr_U of U is clearly represented by an inductive limit of perfect affine spaces. Since $U \setminus G$ is quasi-affine, Proposition 1.20 implies that $\text{Gr}_U \subset \text{Gr}_G$ is a locally closed embedding. Write $S_0 = \text{Gr}_U \subset \text{Gr}_G$. For $\lambda \in \mathbb{X}_\bullet$, let $S_\lambda = LU\varpi^\lambda$ be the orbit through ϖ^λ . Then S_λ equals $\varpi^\lambda \text{Gr}_U$ and therefore is locally closed in Gr_G . By the Iwasawa decomposition,

$$\text{Gr}_G = \bigcup_{\lambda \in \mathbb{X}_\bullet} S_\lambda.$$

As in the equal characteristic situation, one can also regard semi-infinite orbits as the attractor locus of certain torus-action on Gr_G . Namely, let $2\rho^\vee$ denote the sum of positive coroots of G (with respect to B), regarded as a cocharacter of G . Note that the projection $L^+_p \mathbb{G}_m \rightarrow \mathbb{G}_m$ admits a unique section $\mathbb{G}_m \rightarrow L^+_p \mathbb{G}_m$ (as the maximal torus of $L^+_p \mathbb{G}_m$). Then we have a cocharacter

$$\mathbb{G}_m^{p^{-\infty}} \rightarrow L^+ \mathbb{G}_m \xrightarrow{L^+(2\rho^\vee)} L^+T \subset L^+G.$$

The action of L^+G on Gr induces a $\mathbb{G}_m^{p^{-\infty}}$ -action on Gr_G . The set of fixed points are $\{\varpi^\lambda \mid \lambda \in \mathbb{X}_\bullet\}$ and the action contracts S_λ to ϖ^λ . On the other hand, let $B^- \subset G$ be the Borel opposite to B with U^- its unipotent radical, and let $\{S^-_\lambda = LU^-\varpi^\lambda, \lambda \in \mathbb{X}_\bullet\}$ be the opposite semi-infinite orbits. These orbits can be regarded as the repeller locus of the above $\mathbb{G}_m^{p^{-\infty}}$ -action on Gr_G .

As in the equal characteristic situation, we have the following closure relation for semi-infinite orbits.

PROPOSITION 2.5. *The closure $\bar{S}_\lambda = \cup_{\lambda' \leq \lambda} S_{\lambda'}$. More precisely, $\overline{S_\lambda \cap \text{Gr}_{\leq \mu}} = \cup_{\lambda' \leq \lambda} S_{\lambda'} \cap \text{Gr}_{\leq \mu}$. Similarly, $\bar{S}^-_\lambda = \cup_{\lambda' \geq \lambda} S^-_{\lambda'}$.*

Proof. It is enough to prove the statement for S_λ . The argument of [MV07, Prop. 3.1] does not apply directly because in mix characteristic one cannot attach to an affine root a map $\text{SL}_2 \rightarrow LG$ and (currently) there is no Kac-Moody theory available. However, the alternative argument given in [Zhu16, Prop. 5.3.6] applies to the current situation. \square

Note that the restriction of the L^+G -torsor $LG \rightarrow \text{Gr}_G$ over S_λ has a canonical reduction as an L^+U -torsor given by

$$LU \rightarrow S_\lambda, n \mapsto \varpi^\lambda n \text{ mod } L^+G.$$

Then it makes sense to talk about the twisted product of these semi-infinite orbits. Let ν_\bullet be a sequence of (not necessarily dominant) co-weights of G . One can define

$$S_{\nu_\bullet} := S_{\nu_1} \tilde{\times} S_{\nu_2} \tilde{\times} \cdots \tilde{\times} S_{\nu_n} \subset \text{Gr} \tilde{\times} \text{Gr} \tilde{\times} \cdots \tilde{\times} \text{Gr}.$$

The formula

$$(2.2.1) \quad (\varpi^{\nu_1} x_1, \dots, \varpi^{\nu_n} x_n) \mapsto (\varpi^{\nu_1} x_1, \varpi^{\nu_1 + \nu_2} (\varpi^{-\nu_2} x_1 \varpi^{\nu_2}) x_2, \dots)$$

defines an isomorphism

$$(2.2.2) \quad m : S_{\nu_\bullet} \simeq S_{\nu_1} \times S_{\nu_1 + \nu_2} \times \cdots \times S_{|\nu_\bullet|}$$

as locally closed subsets of $\text{Gr} \tilde{\times} \text{Gr} \tilde{\times} \cdots \tilde{\times} \text{Gr} \simeq \text{Gr}^n$; see (2.1.1) for this isomorphism.

Note that each $S_{\nu_i} \cap \text{Gr}_{\leq \mu_i}$ is L^+U -invariant. So it also makes sense to define the twisted product of these L^+U -spaces $S_{\nu_i} \cap \text{Gr}_{\leq \mu_i}$. In addition, there is the canonical isomorphism

$$(2.2.3) \quad (S_{\nu_1} \cap \text{Gr}_{\leq \mu_1}) \tilde{\times} \cdots \tilde{\times} (S_{\nu_n} \cap \text{Gr}_{\leq \mu_n}) \simeq S_{\nu_\bullet} \cap \text{Gr}_{\leq \mu_\bullet}.$$

Remark 2.6. (i) Note that $S_\nu \cap \text{Gr}_{\leq \mu}$ is closed in S_ν and therefore is a scheme. (ii) Unlike [NP01, Lemma 9.1], it is not clear whether the twisted product on the left-hand side of (2.2.3) splits as a product.

The Mirković-Vilonen theory exists in our situation. The key statement is the following.

PROPOSITION 2.7. *For any $\mathcal{A} \in \text{P}_{L^+G}(\text{Gr}_G)$, $H_c^i(S_\lambda, \mathcal{A}) = 0$ unless $i = (2\rho, \lambda)$.*

The proof of this proposition will be sketched in Section 2.2.3. Note that the proof in [MV07, Th. 3.5] does not work in mixed characteristic. Following [MV07], we define the weight functor

$$(2.2.4) \quad \text{CT}_\lambda : \text{Sat}_G \rightarrow \text{Vect}_{\overline{\mathbb{Q}}_\ell}, \quad \text{CT}_\lambda(\mathcal{A}) = H_c^{2(\rho, \lambda)}(S_\lambda, \mathcal{A}).$$

COROLLARY 2.8. *The perfect scheme $S_\lambda \cap \text{Gr}_{\leq \mu}$ is equidimensional, of dimension $(\rho, \lambda + \mu)$. The number of its irreducible components equals the dimension of the λ -weight space $V_\mu(\lambda)$ of the irreducible representation V_μ of \hat{G} of highest weight μ .*

Proof. First, we show that $S_\lambda \cap \text{Gr}_{\leq \mu}$ is of dimension $(\rho, \lambda + \mu)$, and the number of irreducible components of maximal dimension equals the dimension of $V_\mu(\lambda)$. The proof is a special case of [GHKR06, Prop. 5.4.2]: first note that the group G is in fact already defined over the ring of integers of a p -adic field

so Gr is in fact defined over some finite field. Then it is enough to show that

$$(2.2.5) \quad \lim_{q \rightarrow \infty} \frac{|(S_\lambda \cap \text{Gr}_{\leq \mu})(\mathbb{F}_q)|}{q^{(\rho, \lambda + \mu)}} = \dim V_\mu(\lambda).$$

To prove this, one can replace $\text{Gr}_{\leq \mu}$ in the above formula by the open cell Gr_μ . Now we regard $U(F)$ as a locally compact topological group, and we normalize the measure on $U(F)$ so that the volume of $U(\mathcal{O})$ is one. Then one can express

$$(2.2.6) \quad |(S_\lambda \cap \text{Gr}_\mu)(\mathbb{F}_q)| = \int_{U(F)} 1_{G(\mathcal{O})\varpi^\mu G(\mathcal{O})}(\varpi^\lambda u) du.$$

Recall that the Satake isomorphism

$$\text{Sat} : C_c^\infty(G(\mathcal{O}) \backslash G(F) / G(\mathcal{O})) \simeq C_c^\infty(T(F) / T(\mathcal{O}))^W = C[\mathbb{X}_\bullet(T)]^W$$

is given by

$$\text{Sat}(f)(\varpi^\lambda) = q^{-(\rho, \lambda)} \int_{U(F)} f(\varpi^\lambda u) du.$$

Let H_μ denote the function on $C_c^\infty(G(\mathcal{O}) \backslash G(F) / G(\mathcal{O}))$ such that

$$(2.2.7) \quad \text{Sat}(H_\mu)(\varpi^\lambda) = \dim V_\mu(\lambda).$$

The theory of Lusztig-Kato polynomials implies that

$$(2.2.8) \quad q^{-(\rho, \mu)} 1_{G(\mathcal{O})\varpi^\mu G(\mathcal{O})} = H_\mu + \sum_{\nu < \mu} P_{\mu\nu}(q^{-1}) H_\nu,$$

where $P_{\mu\nu}(v)$ is some polynomial of v without the constant coefficient. Combining (2.2.6), (2.2.7) and (2.2.8),

$$\frac{|(S_\lambda \cap \text{Gr}_\mu)(\mathbb{F}_q)|}{q^{(\rho, \lambda + \mu)}} = \dim V_\mu(\lambda) + \sum_{\nu < \mu} c_{\mu\nu}(q^{-1}) \dim V_\nu(\lambda).$$

As $q \rightarrow \infty$, the error term goes to zero and the dominant term becomes $\dim V_\mu(\lambda)$.

Next, one can follow [GHKR06, Lemma 2.17.4] to deduce the equidimensionality of $S_\lambda \cap \text{Gr}_{\leq \mu}$ from the the upper bounds of the dimension of $S_\lambda \cap \text{Gr}_{\leq \mu}$ and Proposition 2.7. □

We have another two corollaries of Proposition 2.7. Let $\mathbb{B}_\mu(\lambda)$ denote the set of irreducible components of $S_\lambda \cap \text{Gr}_{\leq \mu}$. More generally, let $\mathbb{B}_{\mu_\bullet}(\lambda)$ denote the set of irreducible components of $m^{-1}(S_\lambda) \cap \text{Gr}_{\leq \mu_\bullet}$.

COROLLARY 2.9. *There is a canonical isomorphism $\text{CT}_\lambda(\text{IC}_\mu) \simeq \overline{\mathbb{Q}}_\ell[\mathbb{B}_\mu(\lambda)]$. More precisely, the cycle classes of irreducible components of $S_\lambda \cap \text{Gr}_{\leq \mu}$ form a basis of $\text{H}_c^i(S_\lambda, \text{IC}_\mu)$.*

Proof. One can use the same argument as in [MV07, Prop. 3.10]. Namely, the stratification of S_λ by $\{S_\lambda \cap \text{Gr}_\mu, \mu \in \mathbb{X}_\bullet(T)^+\}$ induces a spectral sequence

with the E_1 -term $H_c^*(S_\lambda \cap \text{Gr}_\mu, \mathcal{A})$ and the abutment $H_c^*(S_\lambda, \mathcal{A})$. Combining Proposition 2.7 and Corollary 2.8, one obtains that $H_c^{2(\rho, \lambda)}(S_\lambda, \text{IC}_\mu) \simeq H_c^{(2\rho, \lambda)}(S_\lambda \cap \text{Gr}_\mu, \overline{\mathbb{Q}}_\ell)$. The claim follows. \square

We define the total weight functor (which is the categorical analogue of the Satake transform) as

$$(2.2.9) \quad \text{CT} := \bigoplus_{\lambda} \text{CT}_\lambda : P_{L+G}(\text{Gr}_G) \rightarrow \text{Vect}_{\overline{\mathbb{Q}}_\ell}.$$

COROLLARY 2.10. *There is a canonical isomorphism*

$$H^*(\text{Gr}_G, -) \simeq \text{CT} : P_{L+G}(\text{Gr}_G) \rightarrow \text{Vect}_{\overline{\mathbb{Q}}_\ell}.$$

The functor $H^(\text{Gr}_G, -)$ is faithful.*

Proof. The argument is the same as in [MV07, Th. 3.6]. Namely, according to Proposition 2.5, there are two stratifications of Gr , one by semi-infinite orbits $\{S_\lambda, \lambda \in \mathbb{X}_\bullet\}$, and the other by opposite semi-infinite orbits $\{S_\lambda^-, \lambda \in \mathbb{X}_\bullet\}$. The first stratification induces a spectral sequence with the E_1 -term $H_c^*(S_\lambda, -)$ and the abutment $H^*(-)$. It degenerates at the E_1 -term for degree reasons by virtue of Proposition 2.7. So there is a natural filtration on H^* with the associated graded being $\bigoplus_{\lambda} H_c^*(S_\lambda, -)$. Explicitly, this is a filtration indexed by $(\mathbb{X}_\bullet, \leq)$ defined as

$$\text{Fil}_{\geq \mu} H^*(\mathcal{A}) = \ker(H^*(\mathcal{A}) \rightarrow H^*(S_{< \lambda}, \mathcal{A})),$$

where $S_{< \lambda} = \overline{S}_\lambda - S_\lambda$.

For $\mathcal{A} \in \text{Sat}_G$ and $Z \subset \text{Gr}$ a closed subset, let $H_Z^*(\mathcal{A})$ denote the cohomology of the !-restriction of \mathcal{A} to Z . Applying Braden’s theorem for algebraic spaces (see [DG14]) to some model, there is a canonical isomorphism

$$(2.2.10) \quad H_c^*(S_\lambda, \mathcal{A}) \simeq H_{S_\lambda^-}^*(\mathcal{A}).$$

Then the second stratification of Gr also induces a filtration of H^* as

$$\text{Fil}'_{< \lambda} H^*(\mathcal{A}) = \text{Im}(H_{S_{< \lambda}^-}^*(\mathcal{A}) \rightarrow H^*(\mathcal{A})),$$

where $S_{< \lambda}^- = \overline{S}_\lambda^- - S_\lambda^-$. These two filtrations are complimentary to each other by (2.2.10) and together define the decomposition $H^* = \bigoplus_{\lambda} H_c^*(S_\lambda, -)$.

Since $P_{L+G}(\text{Gr}_G)$ is semisimple and $H^*(\text{Gr}_G, \text{IC}_\mu)$ is nonzero for every μ , H^* is faithful. \square

2.2.2. We discuss the geometry of $\text{Gr}_{\leq \mu}$ when μ is a (quasi-)minuscule cocharacter, similar to [NP01, §§6–8], but with a few justifications. Denote by $\overline{G} = G \otimes_{\mathcal{O}} k$ the special fiber of G , which is a reductive group over k , with $\overline{U} \subset \overline{B} \subset \overline{G}$.

Recall that a dominant coweight μ of G is called (quasi-)minuscule if all (nonzero weights) of the irreducible representation V_μ of \overline{G} are in a single orbit

under the Weyl group. Recall that if G is a simple group not of type A , then the unique quasi-minuscule (but nonminuscule) coweight is the unique *short* dominant coroot.

LEMMA 2.11. *Proposition 2.7 holds for $\mathcal{A} = \text{IC}_\mu$ when μ is (quasi-)minuscule.*

If μ is a minuscule coweight of G , then $\text{Gr}_{\leq\mu} = \text{Gr}_\mu = (\bar{G}/\bar{P}_\mu)^{p^{-\infty}}$ by Corollary 1.24. In this case,

$$S_\lambda \cap \text{Gr}_\mu = \begin{cases} \emptyset, & \lambda \notin W\mu, \\ (\bar{U}w\bar{P}_\mu/\bar{P}_\mu)^{p^{-\infty}}, & \lambda = w\mu \end{cases}$$

is irreducible, isomorphic to the perfection of an affine space. Then Lemma 2.11 is clear.

Next we assume that μ is quasi-minuscule but nonminuscule. In this case $\mu = \theta$ is a coroot and we denote the corresponding root by θ^\vee . In this case $\text{Gr}_{\leq\mu} = \text{Gr}_\mu \sqcup \text{Gr}_0$. Several discussions in [NP01] need justification. We first construct a “resolution” of $\text{Gr}_{\leq\mu}$. The one given in loc. cit. does not work in mixed characteristic. Our construction is different and arises as a discussion with X. He.

Recall that we fix a maximal torus $T \subset G$ over \mathcal{O} . For a root α , let U_α denote the corresponding root subgroup of G over \mathcal{O} . We identify $U_\alpha(F) = F$ such that $U_\alpha(\mathcal{O}) = \mathcal{O}$. For a real number $r \in [0, 1]$, we consider the parahoric subgroup of $G(F)$ generated by $T(\mathcal{O})$ and the subgroups $\varpi^{[\langle r\mu, \alpha \rangle]} \mathcal{O} \subset F = U_\alpha(F)$ for all roots α . It determines the parahoric group scheme \mathcal{G}_r over \mathcal{O} . Let $Q_r = L^+ \mathcal{G}_r$ denote the corresponding p -adic jet group. Note that

- (1) $Q_0 = L^+G$ and $Q_1 = \varpi^\mu L^+G\varpi^{-\mu}$;
- (2) $Q_{\frac{1}{2}}$ is a maximal parahoric;
- (3) $Q_{\frac{1}{4}} = Q_0 \cap Q_{\frac{1}{2}}$ and $Q_{\frac{3}{4}} = Q_{\frac{1}{2}} \cap Q_1$.

LEMMA 2.12.

- (i) *The quotient $Q_{\frac{1}{2}}/Q_{\frac{3}{4}}$ is isomorphic to $\mathbb{P}^{1,p^{-\infty}}$.*
- (ii) *The map*

$$\pi : \widetilde{\text{Gr}}_{\leq\mu} := Q_0 \times^{Q_{\frac{1}{4}}} Q_{\frac{1}{2}}/Q_{\frac{3}{4}} \rightarrow \text{Gr}_{\leq\mu}, \quad (g, g') \mapsto gg'\varpi^\mu$$

restricts to an isomorphism

$$\mathring{\pi} : Q_0 \times^{Q_{\frac{1}{4}}} (Q_{\frac{1}{4}}Q_{\frac{3}{4}})/Q_{\frac{3}{4}} \simeq \text{Gr}_\mu$$

and to a contraction

$$\pi_0 : (\bar{G}/\bar{P}_\mu)^{p^{-\infty}} \simeq Q_0 \times^{Q_{\frac{1}{4}}} Q_{\frac{1}{4}}s_{1+\theta}Q_{\frac{3}{4}}/Q_{\frac{3}{4}} \rightarrow \text{Gr}_0 = \{1\},$$

where $s_{1+\theta}$ is the affine reflection corresponding to $1 + \theta$.

Proof. For (i), it is enough to observe that the only affine root appearing in $Q_{\frac{1}{2}}$ but not in $Q_{\frac{1}{4}}$ is $-\theta - 1$. For (ii), note that $Q_0 \cap Q_{\frac{3}{4}} = Q_0 \cap Q_1$. Therefore, the statement for $\tilde{\pi}$ holds. The statement for π_0 is clear. \square

Let us write

$$\begin{aligned} \phi : \widetilde{\text{Gr}}_{\leq \mu} &\rightarrow Q_0/Q_{\frac{1}{4}} = (\bar{G}/\bar{P}_\mu)^{p^{-\infty}}, \\ \mathring{\phi} : \text{Gr}_\mu &\xrightarrow{\tilde{\pi}^{-1}} Q_0 \times^{Q_{\frac{1}{4}}} (Q_{\frac{1}{4}}Q_{\frac{3}{4}})/Q_{\frac{3}{4}} \rightarrow Q_0/Q_{\frac{1}{4}}, \end{aligned}$$

where \bar{P}_μ as before is the parabolic of \bar{G} whose roots are those α with $\langle \alpha, \mu \rangle \leq 0$. Note that $\mathring{\phi}$ is nothing but the projection π_μ from (1.4.4). Let Δ_θ denote the subset of simple coroots that are conjugate to θ under the action of the Weyl group. If G is a simple group, then Δ_θ is the set of short simple coroots.

Now we study $S_\lambda \cap \text{Gr}_{\leq \mu}$. If $\lambda = w\mu$ for some $w \in W$, then $S_\lambda \cap \text{Gr}_{\leq \mu} = L^+U\varpi^\lambda$, from which one deduces: if $\lambda = w\mu$ is a positive coroot, then

$$S_\lambda \cap \text{Gr}_{\leq \mu} = S_\lambda \cap \text{Gr}_\mu = \mathring{\phi}^{-1}(\bar{U}w\bar{P}_\mu/\bar{P}_\mu)^{p^{-\infty}};$$

if $\lambda = w\mu$ is a negative coroot, then still $S_\lambda \cap \text{Gr}_{\leq \mu} = S_\lambda \cap \text{Gr}_\mu$ and

$$\mathring{\phi} : S_\lambda \cap \text{Gr}_{\leq \mu} \simeq (\bar{U}w\bar{P}_\mu/\bar{P}_\mu)^{p^{-\infty}}.$$

Finally,

$$S_0 \cap \text{Gr}_{\leq \mu} = \pi \left(\phi^{-1} \left(\bigcup_{w\mu < 0} \bar{U}w\bar{P}_\mu/\bar{P}_\mu \right)^{p^{-\infty}} \right) \setminus \bigcup_{w\mu < 0} (S_{w\mu} \cap \text{Gr}_{\leq \mu}).$$

There is a canonical bijection between Δ_θ and the set of irreducible components of $S_0 \cap \text{Gr}_{\leq \mu}$ given as follows: $\alpha \in \Delta_\theta$ corresponds to the unique irreducible component of $S_0 \cap \text{Gr}_{\leq \mu}$ given by

$$(2.2.11) \quad (S_0 \cap \text{Gr}_{\leq \mu})^\alpha := \text{Gr}_0 \bigcup \pi(\phi^{-1}(\bar{U}w\bar{P}_\mu/\bar{P}_\mu)^{p^{-\infty}}) \setminus (S_{w\mu} \cap \text{Gr}_{\leq \mu}),$$

where $w\mu = -\alpha$.

Now we prove Lemma 2.11 for $\mu = \theta$. This is clear for $\lambda = w\mu$. It remains to consider the case $S_0 \cap \text{Gr}_{\leq \mu}$. Let $d = (2\rho, \mu)$. We will ignore the Tate twist in the sequel. By the decomposition theorem (applying to certain model of $\pi : \widetilde{\text{Gr}}_{\leq \mu} \rightarrow \text{Gr}_{\leq \mu}$), we have

$$\pi_* \overline{\mathcal{Q}}_\ell[d] = \text{IC}_\mu \oplus \mathcal{C},$$

where \mathcal{C} is a certain complex of vector spaces supported at Gr_0 . One has

$$(2.2.12) \quad H^i(\mathcal{C}) = \begin{cases} H^{i+d}(\bar{G}/\bar{P}_\mu), & i \geq 0, \\ H^{i+d-2}(\bar{G}/\bar{P}_\mu), & i < 0. \end{cases}$$

Indeed, the first equality follows from the fact that the stalk cohomology of IC_μ is concentrated in negative degrees, and the second equality follows from the first by duality.

On the other hand, we have

$$R\Gamma_c(\pi^{-1}(S_0 \cap \mathrm{Gr}_{\leq \mu}), \overline{\mathbb{Q}}_\ell[d]) = R\Gamma_c(S_0, \mathrm{IC}_\mu) \oplus \mathcal{C}.$$

Note that

$$\pi^{-1}(S_0 \cap \mathrm{Gr}_{\leq \mu}) = \phi^{-1}\left(\bigcup_{w\mu < 0} \bar{U}w\bar{P}_\mu/\bar{P}_\mu\right)^{p^{-\infty}} \setminus \pi^{-1}\left(\bigcup_{w\mu < 0} (S_{w\mu} \cap \mathrm{Gr}_{\leq \mu})\right)$$

and that the map

$$\phi : \phi^{-1}\left(\bigcup_{w\mu < 0} \bar{U}w\bar{P}_\mu/\bar{P}_\mu\right)^{p^{-\infty}} \rightarrow \left(\bigcup_{w\mu < 0} \bar{U}w\bar{P}_\mu/\bar{P}_\mu\right)^{p^{-\infty}}$$

is a $\mathbb{P}^{1,p^{-\infty}}$ -fibration. In addition, $\pi^{-1}(\bigcup_{w\mu < 0} (S_{w\mu} \cap \mathrm{Gr}_{\leq \mu}))$ can be regarded as a section of this map. Therefore,

$$(2.2.13) \quad R\Gamma_c(\pi^{-1}(S_0 \cap \mathrm{Gr}_{\leq \mu}), \overline{\mathbb{Q}}_\ell[d]) = R\Gamma_c\left(\bigcup_{w\mu < 0} \bar{U}w\bar{P}_\mu/\bar{P}_\mu, \overline{\mathbb{Q}}_\ell[d-2]\right).$$

To prove Lemma 2.11 for $S_0 \cap \mathrm{Gr}_{\leq \mu}$, it remains to compare (2.2.12) and (2.2.13). However, note that the right-hand sides of both equalities only involve the group \bar{G} , defined over k . Therefore, one can directly apply the computation in [NP01, §8] to conclude that $H^i(\mathcal{C}) = H_c^i(\pi^{-1}(S_0 \cap \mathrm{Gr}_{\leq \mu}))$ for $i \neq 0$ and if $i = 0$,

$$H^0(S_0, \mathrm{IC}_\mu) \simeq \overline{\mathbb{Q}}_\ell^{|\Delta_\theta|}.$$

This finishes the proof of Lemma 2.11.

Remark 2.13. In fact, we also showed that Corollary 2.9 holds in these cases.

2.2.3. Now combining the proof of Lemma 2.11 with (2.2.3), we have the following corollaries, whose proofs are as in [NP01, 9.2–9.4]. Let M be the set of minimal elements in $\mathbb{X}_\bullet^+ \setminus \{0\}$. This is exactly the set of nonzero quasisimple co-weights.

COROLLARY 2.14. *Let $\mu_\bullet = (\mu_1, \dots, \mu_m) \in M$. Then for any $\lambda_\bullet = (\lambda_1, \dots, \lambda_m)$, $S_{\lambda_\bullet} \cap \mathrm{Gr}_{\leq \mu_\bullet}$ is equidimensional, and*

$$\dim(S_{\lambda_\bullet} \cap \mathrm{Gr}_{\leq \mu_\bullet}) = (\rho, |\lambda_\bullet| + |\mu_\bullet|).$$

COROLLARY 2.15. *Let $\mu_\bullet = (\mu_1, \dots, \mu_m) \in M$. Then the map $\pi : \mathrm{Gr}_{\leq \mu_\bullet} \rightarrow \mathrm{Gr}_{\leq |\mu_\bullet|}$ is semi-small, and therefore $\mathrm{IC}_{\mu_1} \star \dots \star \mathrm{IC}_{\mu_m}$ is perverse.*

This corollary allows us to define a full additive subcategory of Sat_G , spanned by objects isomorphic to $\text{IC}_{\mu_1} \star \cdots \star \text{IC}_{\mu_m}$ for $\mu_\bullet \subset M$. Let us denote this subcategory by Sat_G^0 . Note that Sat_G^0 is in fact a monoidal subcategory of Sat_G under the convolution.

LEMMA 2.16. *As a monoidal abelian category, Sat_G is the idempotent completion of Sat_G^0 . Concretely, every IC_μ appears as a direct summand of $\text{IC}_{\mu_1} \star \cdots \star \text{IC}_{\mu_m}$ for $\mu_\bullet \in M$.*

This is a geometric version of the so-called PRV conjecture. The argument as in [NP01, Prop. 9.6] applies here. Note that this lemma and Corollary 2.15 together imply Proposition 2.2.

In addition, we have the following corollary. The argument is similar to the proof of [NP01, Th. 3.1] given at the beginning of Section 11 of *ibid.* But due to Remark 2.6, one justification is needed.

COROLLARY 2.17. *Let $\mu_\bullet \subset M$. For any λ , $R\Gamma_c(S_\lambda, \text{IC}_{\mu_1} \star \cdots \star \text{IC}_{\mu_m})$ is concentrated in degree $(2\rho, \lambda)$.*

Proof. Note that $\pi^{-1}S_\lambda = \bigsqcup_{\nu_\bullet, |\nu_\bullet|=\lambda} S_{\nu_\bullet}$. It is enough to show that $R\Gamma_c(S_{\nu_\bullet}, \text{IC}_{\mu_1} \star \cdots \star \text{IC}_{\mu_m})$ is concentrated in degree $(2\rho, \lambda)$.

For an integer n , let $S_\nu^{(n)}$ denote the pushout of the L^+U -torsor $LU \rightarrow S_\nu$ along $L^+U \rightarrow L^nU$, and let $(S_\nu \cap \text{Gr}_{\leq \mu})^{(n)}$ denote the restriction of $S_\nu^{(n)}$ to $S_\nu \cap \text{Gr}_{\leq \mu} \subset S_\nu$. This is an L^nU -torsor over $S_\nu \cap \text{Gr}_{\leq \mu}$. Note that the action of L^+U on every $S_{\nu_i} \cap \text{Gr}_{\leq \mu_i}$ factors through some $L^{r_i}U$. Now we can choose $\{r_i, i = 1, \dots, m\}$ such that $r_m = 0$ and that the action of L^+U on $(S_{\nu_i} \cap \text{Gr}_{\leq \mu_i})^{(r_i)}$ factors through $L^{r_i-1}U$. Let $\text{pr}^*\text{IC}_{\mu_i}$ denote the pullback the sheaf along the projection $(S_{\nu_i} \cap \text{Gr}_{\leq \mu_i})^{(r_i)} \rightarrow (S_{\nu_i} \cap \text{Gr}_{\leq \mu_i})$. Then $\prod (S_{\nu_i} \cap \text{Gr}_{\leq \mu_i})^{(r_i)}$ is an $\prod L^{r_i}U$ -torsor. Since $L^{r_i}U$ is isomorphic to the perfection of an affine space of dimension $r_i \dim U$, we have

$$\begin{aligned} R\Gamma_c((S_{\nu_1} \cap \text{Gr}_{\leq \mu_1}) \tilde{\times} \cdots \tilde{\times} (S_{\nu_m} \cap \text{Gr}_{\leq \mu_m}), \text{IC}_{\mu_1} \tilde{\boxtimes} \cdots \tilde{\boxtimes} \text{IC}_{\mu_m}) \\ = R\Gamma_c((S_{\nu_1} \cap \text{Gr}_{\leq \mu_1})^{(r_1)}, \text{pr}^*\text{IC}_{\mu_1}) \\ \otimes \cdots \otimes R\Gamma_c((S_{\nu_m} \cap \text{Gr}_{\leq \mu_m})^{(r_m)}, \text{pr}^*\text{IC}_{\mu_m}) \left[2 \dim U \sum r_i \right]. \end{aligned}$$

The corollary now follows from (2.2.3) and Lemma 2.11. \square

Note that this corollary and Lemma 2.16 together imply Proposition 2.7.

2.3. The monoidal structure on H^* .

2.3.1. We endow the hypercohomology functor

$$H^*(-) := H^*(\text{Gr}, -) : \text{Sat}_G \rightarrow \text{Vect}_{\overline{\mathbb{Q}}_l}$$

with a monoidal structure. In equal characteristic, this is achieved by identifying the convolution product with the fusion product defined using a global

curve (cf. [MV07] and [BD, §5.3]). If in addition, $k = \mathbb{C}$, one can endow \mathbb{H}^* with another monoidal structure by identifying Gr_G^b with the based loop space of a maximal compact subgroup of G and identifying the convolution map of the affine Grassmannian with the multiplication of the loop group (cf. [Gin90]). Neither method applies directly in our setting so we need a third construction. It is not hard to check that in equal characteristic, all three monoidal structures coincide.

Recall that for $\mathcal{A} \in \mathrm{Sat}_G$, it makes sense to consider its L^+G -equivariant cohomology $\mathbb{H}_{L^+G}^*(\mathcal{A})$, which is an $R_{\bar{G},\ell}$ -module (see Section A.3.5). But as is well known, there is another $R_{\bar{G},\ell}$ -module structure on $\mathbb{H}_{L^+G}^*(\mathcal{A})$ so it is an $R_{\bar{G},\ell}$ -bimodule. In fact, let $L^+G^{(m)} \subset L^+G$ denote the m -th congruence subgroup, and let

$$\mathrm{Gr}^{(m)} = LG/L^+G^{(m)}$$

denote the universal L^mG -torsor on Gr . Then $\mathrm{Gr}^{(m)}$ admits an action of $L^+G \times L^mG$ and the projection $\pi_m : \mathrm{Gr}^{(m)} \rightarrow \mathrm{Gr}$ is L^+G -equivariant. Then by (A.3.5),

$$\mathbb{H}_{L^+G}^*(\mathcal{A}) \simeq \mathbb{H}_{L^+G \times L^mG}^*(\pi_m^*\mathcal{A}),$$

giving an $R_{\bar{G},\ell}$ -bimodule structure on $\mathbb{H}_{L^+G}^*(\mathcal{A})$. This structure is independent of m , as soon as $m > 0$. Recall that the category of $R_{\bar{G},\ell}$ -bimodules has a natural monoidal structure.

LEMMA 2.18. *There is a natural monoidal structure on $\mathbb{H}_{L^+G}^*(-) : \mathrm{Sat}_G \rightarrow (R_{\bar{G},\ell} \otimes R_{\bar{G},\ell})\text{-mod}$. That is, for every $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$, there is a canonical isomorphism of $R_{\bar{G},\ell}$ -bimodules*

$$\mathbb{H}_{L^+G}^*(\mathcal{A}_1 \star \dots \star \mathcal{A}_n) \simeq \mathbb{H}_{L^+G}^*(\mathcal{A}_1) \otimes_{R_{\bar{G},\ell}} \dots \otimes_{R_{\bar{G},\ell}} \mathbb{H}_{L^+G}^*(\mathcal{A}_n)$$

satisfying the natural compatibility conditions.

Proof. This is standard (in light of Soergel’s bimodules), and we sketch a proof. In fact, the idea already appears in the proof of Corollary 2.17.

For a closed subset $Z \subset \mathrm{Gr}$, let $Z^{(m)}$ denote its preimage in $\mathrm{Gr}^{(m)}$. We choose a sequence of positive integers (m_1, \dots, m_n) , such that L^+G acts on $\mathrm{Supp}(\mathcal{A}_i)^{(m_i)}$ via $L^+G \rightarrow L^{m_i-1}G$. Then there is an $L^+G \times \prod_i L^{m_i}G$ -equivariant projection

$$\prod_i \mathrm{Supp}(\mathcal{A}_i)^{(m_i)} \rightarrow \mathrm{Supp}(\mathcal{A}_1) \tilde{\times} \dots \tilde{\times} \mathrm{Supp}(\mathcal{A}_n),$$

where L^+G acts by left multiplication, and $L^{m_i}G$ acts on $\mathrm{Supp}(\mathcal{A}_i)^{(m_i)} \times \mathrm{Supp}(\mathcal{A}_{i+1})^{(m_{i+1})}$ diagonally from the middle. It induces a canonical isomorphism

$$(2.3.1) \quad \mathbb{H}_{L^+G}^*(\mathcal{A}_1 \star \dots \star \mathcal{A}_n) \simeq \mathbb{H}_{L^+G \times \prod L^{m_i}G}^*(\boxtimes_i \pi_{m_i}^* \mathcal{A}_i).$$

On the other hand, the $L^+G \times \prod_i L^{m_i}G$ -equivariant projection

$$\prod_i \text{Supp}(\mathcal{A}_i)^{(m_i)} \rightarrow \prod_i \text{Supp}(\mathcal{A}_i),$$

where L^+G acts on $\text{Supp}(\mathcal{A}_1)$ and $L^{m_i}G$ acts on $\text{Supp}(\mathcal{A}_{i+1})$ by left multiplication, induces a map

$$(2.3.2) \quad H_{L^+G}^*(\mathcal{A}_1) \otimes_{R_{\bar{G},\ell}} \cdots \otimes_{R_{\bar{G},\ell}} H_{L^+G}^*(\mathcal{A}_n) \rightarrow H_{L^+G \times \prod L^{m_i}G}^*(\boxtimes_i \pi_{m_i}^* \mathcal{A}_i).$$

The composition of (2.3.1) and (2.3.2) gives a map

$$H_{L^+G}^*(\mathcal{A}_1) \otimes_{R_{\bar{G},\ell}} \cdots \otimes_{R_{\bar{G},\ell}} H_{L^+G}^*(\mathcal{A}_n) \rightarrow H_{L^+G}^*(\mathcal{A}_1 \star \cdots \star \mathcal{A}_n),$$

which is an isomorphism by an easy spectral sequence argument. Its inverse then gives the desired isomorphism, which is clearly compatible with the associativity constraints. \square

2.3.2. To continue, we make the following observation. Recall that we fix $T \subset B \subset G$, and denote by $\bar{T} \subset \bar{B} \subset \bar{G}$ their fibers over \mathcal{O}/ϖ . Let $\widetilde{W} = N_G(T)(F)/T(\mathcal{O})$ denote the Iwahori-Weyl group of $G(F)$, where $N_G(T)$ is the normalizer of T in G . Let $\overline{W} = \widetilde{W}/\mathbb{X}_\bullet$ denote the finite Weyl group, i.e., the Weyl group of G . For (the p -adic jet group of) a parahoric P that contains T , let \bar{L}_P denote the reductive quotient of P . (We ignore the perfection.) Then $\bar{T} \subset \bar{L}_P$. Let $W_P \subset \widetilde{W}$ denote the Weyl group of \bar{L}_P , and let \overline{W}_P denote its image in \overline{W} . Then

$$(2.3.3) \quad R_{\bar{L}_P,\ell} = R_{\bar{T},\ell}^{\overline{W}_P}.$$

In particular, we see that for every P , $R_{\bar{G},\ell} = R_{\bar{T},\ell}^{\overline{W}} \subset R_{\bar{L}_P,\ell} \subset R_{\bar{T},\ell}$.

LEMMA 2.19. *The two $R_{\bar{G},\ell}$ -structures on $H_{L^+G}^*(\mathcal{A})$ coincide.*

Proof. According to Lemmas 2.16 and 2.18, it is enough to prove this for $\mathcal{A} = \text{IC}_\mu$ when μ is quasi-minuscule. We first consider the case when μ is quasi-minuscule but nonminuscule. Recall the definition of $\widetilde{\text{Gr}}_{\leq \mu}$ from Lemma 2.12,

$$\widetilde{\text{Gr}}_{\leq \mu} = Q_0 \times^{Q_{\frac{1}{4}}} Q_{\frac{1}{2}} \times^{Q_{\frac{3}{4}}} Q_1/Q_1.$$

Then by the same argument as in Lemma 2.18,

$$H_{L^+G}^*(\widetilde{\text{Gr}}_{\leq \mu}) = R_{\bar{L}_{Q_{1/4},\ell}} \otimes_{R_{\bar{L}_{Q_{1/2},\ell}}} R_{\bar{L}_{Q_{3/4},\ell}}.$$

The first $R_{\bar{G},\ell}$ -structure comes from the inclusion $R_{\bar{G},\ell} = R_{\bar{L}_{Q_0,\ell}} \subset R_{\bar{L}_{Q_{1/4},\ell}}$, and the second comes from the map $R_{\bar{G},\ell} = R_{\bar{L}_{Q_1,\ell}} \subset R_{\bar{L}_{Q_{3/4},\ell}}$. But as $R_{\bar{G},\ell}$ is a subring of $R_{\bar{L}_{Q_{1/2},\ell}}$, these two $R_{\bar{G},\ell}$ structures coincide. It follows that the two $R_{\bar{G},\ell}$ -structures on $\text{IH}_{L^+G}(\text{Gr}_{\leq \mu}) = H_{L^+G}^*(\text{IC}_\mu[-(2\rho, \mu)])$ also coincide, as it is direct summand of $H_{L^+G}^*(\widetilde{\text{Gr}}_{\leq \mu})$.

Next we consider the case when μ is minuscule. Note that the definition of Q_r before Lemma 2.12 in fact makes sense for every μ . In particular, $Q_0 = L^+G$, $Q_1 = \varpi^\mu L^+G\varpi^{-\mu}$ and $Q_{\frac{1}{2}} = Q_1 \cap Q_0$. Then one can argue similarly to conclude that

$$H_{L^+G}^*(\text{Gr}_\mu) = R_{\bar{L}_{Q_{1/2}, \ell}},$$

with the two $R_{\bar{G}, \ell}$ structures given by $R_{\bar{G}, \ell} = R_{\bar{L}_{Q_0, \ell}} \subset R_{\bar{L}_{Q_{1/2}, \ell}}$ and $R_{\bar{G}, \ell} = R_{\bar{L}_{Q_1, \ell}} \subset R_{\bar{L}_{Q_{1/2}, \ell}}$, which clearly coincide. \square

Note that there is a canonical isomorphism $H^*(\mathcal{A}) = \bar{\mathbb{Q}}_\ell \otimes_{R_{\bar{G}, \ell}} H_{L^+G}^*(\mathcal{A})$, where $R_{\bar{G}, \ell} \rightarrow \bar{\mathbb{Q}}_\ell$ is via the augmentation map, again by an easy spectral sequence argument. Then combining the above two lemmas, we obtain the following statement.

PROPOSITION 2.20. *The L^+G -equivariant hypercohomology functor*

$$H_{L^+G}^*(-) := H_{L^+G}^*(\text{Gr}_G, -) : \text{Sat}_G \rightarrow \text{Proj}_{R_{\bar{G}, \ell}}$$

has a canonical monoidal structure, where $\text{Proj}_{R_{\bar{G}, \ell}}$ denotes the tensor category of finite projective $R_{\bar{G}, \ell}$ -modules. After base change along the augmentation map $R_{\bar{G}, \ell} \rightarrow \bar{\mathbb{Q}}_\ell$, the usual hypercohomology functor

$$H^*(-) := H^*(\text{Gr}_G, -) : \text{Sat}_G \rightarrow \text{Vect}_{\bar{\mathbb{Q}}_\ell}$$

is a natural monoidal functor.

2.4. The commutativity constraints.

2.4.1. In this subsection, we endow Sat_G with the commutativity constraints. The main statement is

PROPOSITION 2.21. *For every $\mathcal{A}_1, \mathcal{A}_2 \in \text{Sat}_G$, there exists a unique isomorphism $c_{\mathcal{A}_1, \mathcal{A}_2} : \mathcal{A}_1 \star \mathcal{A}_2 \simeq \mathcal{A}_2 \star \mathcal{A}_1$ such that the following diagram is commutative:*

$$\begin{CD} H^*(\mathcal{A}_1 \star \mathcal{A}_2) @>H^*(c_{\mathcal{A}_1, \mathcal{A}_2})>> H^*(\mathcal{A}_2 \star \mathcal{A}_1) \\ @V \simeq VV @VV \simeq V \\ H^*(\mathcal{A}_1) \otimes H^*(\mathcal{A}_2) @>{c_{\text{vect}}}>> H^*(\mathcal{A}_2) \otimes H^*(\mathcal{A}_1), \end{CD}$$

where the vertical isomorphisms come from Proposition 2.20, and the isomorphism c_{vect} in the bottom row is the usual flip isomorphism between vector spaces.

As $H^* : \text{Sat}_G \rightarrow \text{Vect}_{\bar{\mathbb{Q}}_\ell}$ is faithful, the uniqueness of $c_{\mathcal{A}_1, \mathcal{A}_2}$ is clear. The content is its existence. This proposition will be proved in the rest of the subsection. We first give its consequence.

COROLLARY 2.22. *The monoidal category Sat_G , equipped with the above constraints $c_{\mathcal{A}_1, \mathcal{A}_2}$, form a symmetric monoidal category. The hypercohomology functor H^* is a tensor functor.*

Proof. The proof of the first statement follows the idea of Ginzburg (cf. [Gin90]). Namely, we need to check $c_{\mathcal{A}_2, \mathcal{A}_1} c_{\mathcal{A}_1, \mathcal{A}_2} = \text{id}$ and the hexagon axiom. Using the faithfulness of H^* , it is enough to prove these statements after taking the cohomology. Using Proposition 2.21, and the fact $c_{\text{vect}}^2 = \text{id}$, we conclude that $H^*(c_{\mathcal{A}_2, \mathcal{A}_1} c_{\mathcal{A}_1, \mathcal{A}_2}) = \text{id}$, and therefore $c_{\mathcal{A}_2, \mathcal{A}_1} c_{\mathcal{A}_1, \mathcal{A}_2} = \text{id}$. The hexagon axiom can be proved similarly. The second statement is clear. \square

2.4.2. In order to construct $c_{\mathcal{A}_1, \mathcal{A}_2}$, we need some preparations. Define

$$\text{Gr}_G^{\text{op}} := L^+G \backslash LG,$$

on which L^+G acts by right multiplication. As before, sometimes we denote Gr_G^{op} by Gr^{op} for simplicity. Let $\text{P}_{L^+G}(\text{Gr}_G^{\text{op}})$ denote the corresponding category of equivariant perverse sheaves. Note that $\text{P}_{L^+G}(\text{Gr}_G^{\text{op}})$ also has a monoidal structure: There is the convolution Grassmannian

$$\text{Gr}^{\text{op}} \tilde{\times} \text{Gr}^{\text{op}} := L^+G \backslash LG \times^{L^+G} LG$$

equipped with $(m, \text{pr}_2) : \text{Gr}^{\text{op}} \tilde{\times} \text{Gr}^{\text{op}} \rightarrow \text{Gr}^{\text{op}} \times \text{Gr}^{\text{op}}$. Then for $\mathcal{A}_1, \mathcal{A}_2 \in \text{P}_{L^+G}(\text{Gr}_G^{\text{op}})$, one forms the twisted product $\mathcal{A}_1 \tilde{\boxtimes} \mathcal{A}_2$ whose pullback along $\text{Gr}^{\text{op}} \times LG \rightarrow \text{Gr}^{\text{op}} \tilde{\times} \text{Gr}^{\text{op}}$ is the pullback of $\mathcal{A}_1 \boxtimes \mathcal{A}_2$ along $\text{Gr}^{\text{op}} \times LG \rightarrow \text{Gr}^{\text{op}} \times \text{Gr}^{\text{op}}$, and forms the convolution product $\mathcal{A}_1 \star \mathcal{A}_2 = m_!(\mathcal{A}_1 \tilde{\boxtimes} \mathcal{A}_2)$. For simplicity, we sometimes denote $(\text{P}_{L^+G}(\text{Gr}_G^{\text{op}}), \star)$ by Sat_G^{op} .

We have the following statements, Lemmas 2.23 and 2.24.

LEMMA 2.23. *There is an equivalence of the monoidal categories*

$$\text{Id}' : \text{Sat}_G^{\text{op}} \simeq \text{Sat}_G,$$

sending the intersection cohomology sheaf $\text{IC}_\mu^{\text{op}}$ of $\text{Gr}_{\leq \mu}^{\text{op}}$ to IC_μ , where $\text{Gr}_{\leq \mu}^{\text{op}}$ is the closure of $\text{Gr}_\mu^{\text{op}} = L^+G \backslash L^+G \varpi^\mu L^+G$.

Let $(LG)_{\leq \mu}$ denote the preimage of $\text{Gr}_{\leq \mu}$ under the projection $LG \rightarrow \text{Gr}$. Let m be an integer large enough such that the m -th congruence subgroup $L^+G^{(m)}$ is contained in $L^+G \cap \varpi^\mu L^+G \varpi^{-\mu}$. Then we obtain the following diagram of surjective maps:

$$(2.4.1) \quad \text{Gr}_{\leq \mu} \xrightarrow{\bar{\pi}^m} \text{Gr}_{\leq \mu}^{(m)} = (LG)_{\leq \mu} / L^+G^{(m)} \xrightarrow{\phi_m} L^+G \backslash (LG)_{\leq \mu} = \text{Gr}_{\leq \mu}^{\text{op}}.$$

LEMMA 2.24. *There exists a unique isomorphism*

$$\text{id}'_\mu : \phi_m^* \text{IC}_\mu^{\text{op}} \simeq \pi_m^* \text{IC}_\mu$$

of sheaves on $\mathrm{Gr}_{\leq \mu}^{(m)}$, whose restriction to $\mathrm{Gr}_{\mu}^{(m)}$ is given by

$$\phi_m^* \mathrm{IC}_{\mu}^{\mathrm{op}}|_{\mathrm{Gr}_{\mu}^{(m)}} = \overline{\mathbb{Q}}_{\ell}[(2\rho, \mu)] = \pi_m^* \mathrm{IC}_{\mu}|_{\mathrm{Gr}_{\mu}^{(m)}}.$$

In particular, $\phi_m^* \mathrm{IC}_{\mu}^{\mathrm{op}}[m \dim G]$ is perverse.

Remark 2.25. (i) Informally, one can think of both categories as a certain category of $(L^+G \times L^+G)$ -equivariant sheaves on LG . As we did not introduce sheaves on infinite-dimensional spaces, we give a concrete approach here.

(ii) As π_m is an L^mG -torsor, $\pi_m^*[m \dim G]$ preserves perversity. However, as we do not know whether ϕ_m is perfectly smooth, *a priori* it is not obvious that $\phi_m^* \mathrm{IC}_{\mu}^{\mathrm{op}}[m \dim G]$ is perverse. On the other hand, as soon as the perversity of $\phi_m^* \mathrm{IC}_{\mu}^{\mathrm{op}}[m \dim G]$ is known, the existence and the uniqueness of id'_{μ} are clear.

Proof. We will prove these two lemmas simultaneously. First, if μ is minuscule, then $\mathrm{Gr}_{\leq \mu}$ and $\mathrm{Gr}_{\leq \mu}^{\mathrm{op}}$ are perfectly smooth so there is a unique isomorphism $\mathrm{id}'_{\mu} : \phi_m^* \mathrm{IC}_{\mu}^{\mathrm{op}} = \overline{\mathbb{Q}}_{\ell}[(2\rho, \mu)] = \pi_m^* \mathrm{IC}_{\mu}$ as required by Lemma 2.24.

Next, if μ is quasi-minuscule but nonminuscule, let $\widetilde{\mathrm{Gr}}_{\leq \mu} \rightarrow \mathrm{Gr}_{\leq \mu}$ denote the “resolution” as constructed in Lemma 2.12. We can define

$$\widetilde{\mathrm{Gr}}_{\leq \mu}^{\mathrm{op}} = \mathbb{Q}_0 \setminus \mathbb{Q}_0 \times^{\mathbb{Q}_{1/4}} \mathbb{Q}_{1/2} \times^{\mathbb{Q}_{3/4}} \mathbb{Q}_1 = \mathbb{Q}_{1/4} \setminus \mathbb{Q}_{1/2} \times^{\mathbb{Q}_{3/4}} \mathbb{Q}_1$$

and the map

$$\pi_{\mu}^{\mathrm{op}} : \widetilde{\mathrm{Gr}}_{\leq \mu}^{\mathrm{op}} \rightarrow \mathrm{Gr}_{\leq \mu}^{\mathrm{op}}, (g, g') \mapsto gg' \varpi^{\mu},$$

which is a “resolution” of $\mathrm{Gr}_{\leq \mu}^{\mathrm{op}}$. We define $\widetilde{\mathrm{Gr}}_{\leq \mu}^{(m)}$ by requiring that both squares in the following diagram are Cartesian:

$$\begin{array}{ccccc} \widetilde{\mathrm{Gr}}_{\leq \mu}^{\mathrm{op}} & \xleftarrow{\tilde{\phi}_m} & \widetilde{\mathrm{Gr}}_{\leq \mu}^{(m)} & \xrightarrow{\tilde{\pi}_m} & \widetilde{\mathrm{Gr}}_{\leq \mu} \\ \pi_{\mu}^{\mathrm{op}} \downarrow & & \downarrow \pi_{\mu}^{(m)} & & \downarrow \pi_{\mu} \\ \mathrm{Gr}_{\leq \mu}^{\mathrm{op}} & \xleftarrow{\phi_m} & \mathrm{Gr}_{\leq \mu}^{(m)} & \xrightarrow{\pi_m} & \mathrm{Gr}_{\leq \mu} \end{array}$$

Then we obtain the canonical isomorphisms

$$\begin{aligned} \phi_m^* \mathrm{IC}_{\mu}^{\mathrm{op}} \oplus \phi_m^* \mathcal{C}^{\mathrm{op}} &\simeq \phi_m^* (\pi_{\mu}^{\mathrm{op}})_* \overline{\mathbb{Q}}_{\ell}[d] \\ &\simeq (\pi_{\mu}^{(m)})_* \overline{\mathbb{Q}}_{\ell}[d] \simeq \pi_m^* (\pi_{\mu})_* \overline{\mathbb{Q}}_{\ell}[d] \simeq \pi_m^* \mathrm{IC}_{\mu} \oplus \pi_m^* \mathcal{C}, \end{aligned}$$

where $d = (2\rho, \mu)$, and \mathcal{C} and $\mathcal{C}^{\mathrm{op}}$ are as in the proof of Lemma 2.11. We therefore obtain id'_{μ} as in Lemma 2.24.

Now, let $\mu_{\bullet} \subset M$ as in Section 2.2.3. Let (m_1, \dots, m_n) be a sequence of positive integers, such that L^+G acts on $\mathrm{Gr}_{\leq \mu_i}^{(m_i)}$ via $L^+G \rightarrow L^{m_i-1}G$ and that

$\phi_{m_i} : \text{Gr}_{\leq \mu_i}^{(m_i)} \rightarrow \text{Gr}_{\leq \mu_i}^{\text{op}}$ is defined. Then from the diagram

$$\begin{array}{ccccc}
 \prod \text{Gr}_{\leq \mu_i}^{\text{op}} & \xleftarrow{\prod \phi_{m_i}} & \prod \text{Gr}_{\leq \mu_i}^{(m_i)} & \xrightarrow{\prod \pi_{m_i}} & \prod \text{Gr}_{\leq \mu_i}, \\
 & & \downarrow & & \\
 \text{Gr}_{\leq \mu_\bullet}^{\text{op}} & \longleftarrow & \text{Gr}_{\leq \mu_1} \tilde{\times} \cdots \tilde{\times} \text{Gr}_{\leq \mu_{n-1}} \tilde{\times} \text{Gr}_{\leq \mu_n}^{(m_n)} & \longrightarrow & \text{Gr}_{\leq \mu_\bullet}. \\
 \downarrow & & \downarrow & & \downarrow \\
 \text{Gr}_{\leq |\mu_\bullet|}^{\text{op}} & \xleftarrow{\phi_m} & \text{Gr}_{\leq |\mu_\bullet|}^{(m)} & \xrightarrow{\pi_m} & \text{Gr}_{\leq |\mu_\bullet|}
 \end{array}$$

and the canonical isomorphism $\prod_i \text{id}'_{\mu_i} : (\prod \phi_{m_i})^*(\boxtimes \text{IC}_{\mu_i}^{\text{op}}) \simeq (\prod \pi_{m_i})^*(\boxtimes \text{IC}_{\mu_i})$, we obtain a canonical isomorphism

$$\text{id}'_{\mu_\bullet} : \phi_m^*(\text{IC}_{\mu_1}^{\text{op}} \star \cdots \star \text{IC}_{\mu_n}^{\text{op}}) \simeq \pi_m^*(\text{IC}_{\mu_1} \star \cdots \star \text{IC}_{\mu_n}).$$

By Lemma 2.16, we conclude that for every μ , the isomorphism id'_μ as required in Lemma 2.24 exists.

In addition, the isomorphism id'_{μ_\bullet} also provides us the desired monoidal structure on Id' . Again, by Lemma 2.16, it is enough to exhibit the monoidal structure of Id' when restricted to the subcategories $\text{Id}' : \text{Sat}_G^{0,\text{op}} \simeq \text{Sat}_G^0$, where Sat_G^0 is defined before Lemma 2.16 and $\text{Sat}_G^{0,\text{op}}$ is defined similarly. For $\lambda_\bullet, \mu_\bullet \subset M$, we write $\text{IC}_{\lambda_\bullet} = \text{IC}_{\lambda_1} \star \cdots \star \text{IC}_{\lambda_n}$, etc. Then there are canonical isomorphisms

$$\begin{aligned}
 \text{Hom}(\text{IC}_{\lambda_\bullet}, \text{IC}_{\mu_\bullet}) &\simeq \text{Hom}(\pi_m^* \text{IC}_{\lambda_\bullet}, \pi_m^* \text{IC}_{\mu_\bullet}) \\
 &\simeq \text{Hom}(\phi_m^* \text{IC}_{\lambda_\bullet}^{\text{op}}, \phi_m^* \text{IC}_{\mu_\bullet}^{\text{op}}) \simeq \text{Hom}(\text{IC}_{\lambda_\bullet}^{\text{op}}, \text{IC}_{\mu_\bullet}^{\text{op}}),
 \end{aligned}$$

which are clearly independent of m (as soon as m large enough). This isomorphism provides the monoidal structure on Id' as it is compatible with the union of sequences of co-weights in M . \square

We have the following corollary of Lemma 2.24. For $\lambda \in \mathbb{X}_\bullet$, let $\mathcal{H}_\lambda^j \mathcal{A}$ (resp. $\mathcal{H}_{\lambda,!}^j \mathcal{A}$) denote the degree j stalk (resp. costalk) cohomology of \mathcal{A} at ϖ^λ .

COROLLARY 2.26. *There is a canonical isomorphism $\mathcal{H}_\lambda^j \text{id}' : \mathcal{H}_\lambda^j \mathcal{A} \simeq \mathcal{H}_\lambda^j \text{Id}' \mathcal{A}$ for $\mathcal{A} \in \text{P}_{L+G}(\text{Gr}^{\text{op}})$, and similarly for $\mathcal{H}_{\lambda,!}^j$.*

Proof. We prove the first statement, as the second is obtained by the Verdier duality. It is enough to assume that $\mathcal{A} = \text{IC}_\mu^{\text{op}}$. Then the isomorphism $\mathcal{H}_\lambda^j \text{id}'_\mu$ is given by the composition

$$\mathcal{H}_\lambda^j \text{IC}_\mu^{\text{op}} = \mathcal{H}_\lambda^j \phi_m^* \text{IC}_\mu^{\text{op}} \xrightarrow{\mathcal{H}_\lambda^j \text{id}'_\mu} \mathcal{H}_\lambda^j \pi_m^* \text{IC}_\mu = \mathcal{H}_\lambda^j \text{IC}_\mu,$$

which is clearly independent of the choice of m . \square

2.4.3. Now, we construct $c_{\mathcal{A}_1, \mathcal{A}_2}$ as in Theorem 2.21. In the equal characteristic situation, this is obtained from the fusion product interpretation of the convolution product; see [MV07] and [BD, §5.3]. Currently, the fusion product does not exist in mixed characteristic. Our method is a kind of categorification of classical Gelfand’s trick. (See also [BD, §5.3.8], which modifies the construction of [Gin90].)

Fix a pinning (G, B, T, X) of G , and let θ' be the involution that sends a dominant coweight λ to its dual $\lambda^* = -w_0(\lambda)$, where w_0 is the longest element in the finite Weyl group \overline{W} of G . We define the anti-involution θ of G as $\theta(g) = \theta'(g)^{-1}$. It induces an anti-involution of LG preserving L^+G , which are still denoted by θ (rather than $L\theta$ if no confusion will arise). Note that θ induces an isomorphism

$$\theta : \mathrm{Gr}_G^{\mathrm{op}} = L^+G \backslash LG \simeq LG / L^+G = \mathrm{Gr}_G$$

and therefore an equivalence of categories

$$\theta^* : \mathrm{P}_{L^+G}(\mathrm{Gr}_G) \simeq \mathrm{P}_{L^+G}(\mathrm{Gr}_G^{\mathrm{op}}).$$

Now θ also induces

$$\theta \tilde{\times} \theta : \mathrm{Gr}^{\mathrm{op}} \tilde{\times} \mathrm{Gr}^{\mathrm{op}} \rightarrow \mathrm{Gr} \tilde{\times} \mathrm{Gr}, \quad (g_1, g_2) \mapsto (\theta(g_2), \theta(g_1)),$$

and there is a canonical isomorphism $(\theta \tilde{\times} \theta)^*(\mathcal{A}_1 \tilde{\boxtimes} \mathcal{A}_2) \simeq \theta^* \mathcal{A}_2 \tilde{\boxtimes} \theta^* \mathcal{A}_1$. Using $m(\theta \tilde{\times} \theta) = \theta m$, and the proper base change, we obtain a canonical isomorphism

$$\theta^*(\mathcal{A}_1 \star \mathcal{A}_2) \simeq \theta^* \mathcal{A}_2 \star \theta^* \mathcal{A}_1.$$

Considering the 3-fold convolutions, we concludes that θ^* is an anti-equivalence of monoidal categories.

Therefore, we obtain an anti-autoequivalence $\mathrm{Id}' \circ \theta^*$ of Sat_G as a monoidal category. Now we define an isomorphism of (plain) functors

$$e : \mathrm{Id}' \circ \theta^* \rightarrow \mathrm{Id}.$$

We will fix a square root $\sqrt{-1}$ in $\overline{\mathbb{Q}}_\ell$ in the sequel and define $(-1)^{(\rho, \mu)} := \sqrt{-1}^{(2\rho, \mu)}$ for any coweight μ . By Lemma 2.1, it is enough to give an isomorphism $e_\mu : \mathrm{Id}' \circ \theta^* \mathrm{IC}_\mu \rightarrow \mathrm{IC}_\mu$ for every μ . Note that $\theta^* \mathrm{IC}_\mu$ is (noncanonically) isomorphic to $\mathrm{IC}_\mu^{\mathrm{op}}$. We define the isomorphism

$$(2.4.2) \quad N_\mu : \theta^* \mathrm{IC}_\mu \rightarrow \mathrm{IC}_\mu^{\mathrm{op}}$$

by requiring that its restriction to $\mathrm{Gr}_\mu^{\mathrm{op}}$ is given by

$$\theta^* \mathrm{IC}_\mu|_{\mathrm{Gr}_\mu^{\mathrm{op}}} = \mathrm{IC}_\mu|_{\mathrm{Gr}_\mu} = \overline{\mathbb{Q}}_\ell[(2\rho, \mu)] = \overline{\mathbb{Q}}_\ell[(2\rho, \mu)] = \mathrm{IC}_\mu^{\mathrm{op}}|_{\mathrm{Gr}_\mu^{\mathrm{op}}}.$$

We define $M_\mu = (-1)^{-(\rho, \mu)} N_\mu$ and let $e_\mu = \mathrm{Id}'(M_\mu)$. Let us emphasize that the factor $(-1)^{-(\rho, \mu)}$ is crucial.

Now, we define the isomorphism $c'_{\mathcal{A}_1, \mathcal{A}_2}$ as

(2.4.3)

$$\begin{aligned} c'_{\mathcal{A}_1, \mathcal{A}_2} : \mathcal{A}_1 \star \mathcal{A}_2 &\xleftarrow{e_{\mathcal{A}_1 \star \mathcal{A}_2}} \text{Id}'\theta^*(\mathcal{A}_1 \star \mathcal{A}_2) \simeq \text{Id}'(\theta^* \mathcal{A}_2 \star \theta^* \mathcal{A}_1) \\ &\simeq \text{Id}'\theta^* \mathcal{A}_2 \star \text{Id}'\theta^* \mathcal{A}_1 \xrightarrow{e_{\mathcal{A}_2 \star \mathcal{A}_1}} \mathcal{A}_2 \star \mathcal{A}_1. \end{aligned}$$

Note that $c'_{\mathcal{A}_1, \mathcal{A}_2}$ is independent of the choice of $\sqrt{-1}$.

Finally, the isomorphism $c_{\mathcal{A}_1, \mathcal{A}_2}$ is obtained from $c'_{\mathcal{A}_1, \mathcal{A}_2}$ by a Koszul sign change. Namely, the category $\text{P}_{L+G}(\text{Gr})$ admits a $\mathbb{Z}/2$ -grading induced by (1.4.5). We say \mathcal{A} has pure parity if $p(\text{Supp}(\mathcal{A}))$ is 1 or -1 , in which case we define $p(\mathcal{A}) = p(\text{Supp}(\mathcal{A}))$. Then

$$(2.4.4) \quad c_{\mathcal{A}_1, \mathcal{A}_2} := (-1)^{p(\mathcal{A}_1)p(\mathcal{A}_2)} c'_{\mathcal{A}_1, \mathcal{A}_2}$$

if \mathcal{A}_1 and \mathcal{A}_2 have the pure parity $p(\mathcal{A}_1)$ and $p(\mathcal{A}_2)$. See also [BD, §5.3.21] or [MV07] after Remark 6.2 for a more elegant formulation.

2.4.4. We prove that $c_{\mathcal{A}_1, \mathcal{A}_2}$ constructed as above satisfies the requirement as in Proposition 2.21. From the definition, this will be the consequence of the following three statements, Lemmas 2.27–2.29. Recall that we set $\text{IH}_{L+G}(\text{Gr}_{\leq \mu}) = \text{H}_{L+G}^*(\text{Gr}, \text{IC}_{\mu}[-(2\rho, \mu)])$.

To state the first lemma, note that Lemmas 2.18 and 2.19 hold for Sat_G^{op} , and therefore $\text{H}^* : \text{Sat}^{\text{op}} \rightarrow \text{Vect}_{\overline{\mathbb{Q}}_{\ell}}$ has a natural monoidal structure.

LEMMA 2.27. *There is a natural isomorphism of monoidal functors $\gamma : \text{H}^* \simeq \text{H}^* \circ \text{Id}' : \text{Sat}_G^{\text{op}} \rightarrow \text{Vect}_{\overline{\mathbb{Q}}_{\ell}}$.*

Proof. It is enough to construct the canonical isomorphism $\gamma_{\mu} : \text{IH}^*(\text{Gr}_{\leq \mu}^{\text{op}}) \simeq \text{IH}^*(\text{Gr}_{\leq \mu})$ for every μ . From the diagram (2.4.1), we obtain a canonical isomorphism

$$\text{IH}_{L+G}^*(\text{Gr}_{\leq \mu}^{\text{op}}) \simeq \text{IH}_{L+G \times L+G}^*(\text{Gr}_{\leq \mu}^{(m)}) \simeq \text{IH}_{L+G}^*(\text{Gr}_{\leq \mu}).$$

as $(R_{\overline{G}, \ell} \otimes R_{\overline{G}, \ell})$ -bimodules. Note that this is independent of the choice of m (as soon as it is large). As

$$\text{IH}^*(\text{Gr}_{\leq \mu}) = \overline{\mathbb{Q}}_{\ell} \otimes_{R_{\overline{G}, \ell}} \text{IH}_{L+G}^*(\text{Gr}_{\leq \mu}), \quad \text{IH}^*(\text{Gr}_{\leq \mu}^{\text{op}}) = \text{IH}_{L+G}^*(\text{Gr}_{\leq \mu}^{\text{op}}) \otimes_{R_{\overline{G}, \ell}} \overline{\mathbb{Q}}_{\ell},$$

we obtain the desired isomorphism γ_{μ} by Lemma 2.19. It follows from the construction of the monoidal structure of H^* given by Lemmas 2.18 and 2.19 and the construction of the monoidal structure on Id' as in Section 2.4.2 that γ is an isomorphism of monoidal functors. \square

The second lemma is as follows.

LEMMA 2.28. *There is a canonical isomorphism of functors $\delta : H^* \simeq H^* \circ \theta^*$, such that for every $\mathcal{A}_1, \mathcal{A}_2 \in P_{L+G}(\text{Gr})$, the following diagram is commutative:*

$$\begin{CD} H^*(\mathcal{A}_1 \star \mathcal{A}_2) @>\delta>> H^*(\theta^*(\mathcal{A}_1 \star \mathcal{A}_2)) @>\simeq>> H^*(\theta^* \mathcal{A}_2 \star \theta^* \mathcal{A}_1) \\ @VVV @VVV @VVV \\ H^*(\mathcal{A}_1) \otimes H^*(\mathcal{A}_2) @>c_{\text{vect}}>> H^*(\mathcal{A}_2) \otimes H^*(\mathcal{A}_1) @>\delta \otimes \delta>> H^*(\theta^* \mathcal{A}_2) \otimes H^*(\theta^* \mathcal{A}_1). \end{CD}$$

Proof. If $f : X \rightarrow Y$ is a morphism and \mathcal{F} a complex of sheaves on Y , there is a canonical map $f^* : H^*(Y, \mathcal{F}) \rightarrow H^*(Y, f_* f^* \mathcal{F}) \simeq H^*(X, f^* \mathcal{F})$. Applying this construction to $\theta : \text{Gr}^{\text{op}} \simeq \text{Gr}$ gives the isomorphism δ . It remains to check the commutativity of the diagram.

We will use notation as in the proof of Lemma 2.18. So for a closed subset $Z \subset \text{Gr}^{\text{op}}$, let $Z^{(m)}$ denote its preimage in $L^+G^{(m)} \setminus LG \rightarrow \text{Gr}^{\text{op}}$. Note that the following diagram is commutative:

$$\begin{CD} \text{Supp}(\theta^* \mathcal{A}_n)^{(m_n)} \times \cdots \times \text{Supp}(\theta^* \mathcal{A}_1)^{(m_1)} @>\theta>> \text{Supp}(\mathcal{A})^{(m_1)} \times \cdots \times \text{Supp}(\mathcal{A}_n)^{(m_n)} \\ @VVV @VVV \\ \text{Supp}(\theta^* \mathcal{A}_i) @>\theta>> \text{Supp}(\mathcal{A}_i). \end{CD}$$

In addition, from the construction of the isomorphism in Lemma 2.18, the following diagram is also commutative:

$$\begin{CD} H_{L+G}^*(\mathcal{A}_1 \star \cdots \star \mathcal{A}_n) @>\simeq>> H_{L+G}^*(\theta^* \mathcal{A}_n \star \cdots \star \theta^* \mathcal{A}_1) \\ @V \simeq VV @VV \simeq V \\ H_{L+G}^*(\mathcal{A}_1) \otimes \cdots \otimes H_{L+G}^*(\mathcal{A}_n) @>\simeq>> H_{L+G}^*(\theta^* \mathcal{A}_n)^{\text{op}} \otimes \cdots \otimes H_{L+G}^*(\theta^* \mathcal{A}_1)^{\text{op}}, \end{CD}$$

where the tensor products are taken over $R_{\bar{G}, \ell}$, and where for an $R_{\bar{G}, \ell}$ -bimodule M , M^{op} denotes the new $R_{\bar{G}, \ell}$ -bimodule structure on M by switching the two actions. Specializing along $R_{\bar{G}, \ell} \rightarrow \bar{\mathbb{Q}}_\ell$ shows that δ is an isomorphism of monoidal functors. \square

Now, for every $\mathcal{A} \in P_{L+G}(\text{Gr})$, we can define an automorphism of its cohomology

$$\Theta : H^*(\mathcal{A}) \xrightarrow{\delta} H^*(\theta^* \mathcal{A}) \xrightarrow{\gamma} H^*(\text{Id}' \circ \theta^* \mathcal{A}) \xrightarrow{H^*(\epsilon)} H^*(\mathcal{A}).$$

The above two lemmas imply that the following diagram is commutative:

$$\begin{CD} H^*(\mathcal{A}_1 \star \mathcal{A}_2) @>\simeq>> H^*(\mathcal{A}_1) \otimes H^*(\mathcal{A}_2) @>c_{\text{vect}}>> H^*(\mathcal{A}_2) \otimes H^*(\mathcal{A}_1) \\ @V \Theta VV @VVV @VV \Theta \otimes \Theta V \\ H^*(\mathcal{A}_1 \star \mathcal{A}_2) @>H^*(c'_{\mathcal{A}_1 \star \mathcal{A}_2})>> H^*(\mathcal{A}_2 \star \mathcal{A}_1) @>\simeq>> H^*(\mathcal{A}_2) \otimes H^*(\mathcal{A}_1). \end{CD}$$

Now it is easy to see that Proposition 2.21 is a consequence of the following lemma.

LEMMA 2.29. *For every j , $\Theta = \sqrt{-1}^j$ on $H^j(\mathcal{A})$.*

Note that by the definition of e , and our normalization $\mathrm{IH}^*(\mathrm{Gr}_{\leq \mu}) = H^*(\mathrm{IC}_\mu[-(2\rho, \mu)])$, it is enough to show the following lemma.

LEMMA 2.30. *The map*

$$\Theta_\mu : \mathrm{IH}^{2j}(\mathrm{Gr}_{\leq \mu}) \xrightarrow{\mathrm{H}(N_\mu)\delta} \mathrm{IH}^{2j}(\mathrm{Gr}_{\leq \mu}^{\mathrm{op}}) \xrightarrow{\gamma} \mathrm{IH}^{2j}(\mathrm{Gr}_{\leq \mu})$$

is given by the multiplication by $(-1)^j$, where $N_\mu : \theta^\mathrm{IC}_\mu \rightarrow \mathrm{IC}_\mu^{\mathrm{op}}$ is the canonical isomorphism in (2.4.2).*

We do not know a direct proof of this lemma. In [LY13], its equal characteristic analogue was deduced from the equal characteristic geometric Satake. They use this formula to deduce a numerical result for the affine Hecke algebra, as conjectured by Lusztig [Lus12]. We will reverse their steps to deduce this lemma from this numerical result. In the sequel, we follow the convention in literature to write $\mathrm{H}(N_\mu)\delta$ as θ^* . It should not be confused with the pullback of sheaves. First note the following.

LEMMA 2.31. *The map Θ_μ is an involution.*

Proof. Choose some m, m' such that the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{Gr}_{\leq \mu}^{(m')} & \xrightarrow{\theta^{-1}} & (\mathrm{Gr}_{\leq \mu}^{\mathrm{op}})^{(m')} \\ \downarrow & & \downarrow \\ (\mathrm{Gr}_{\leq \mu}^{\mathrm{op}})^{(m)} & \xrightarrow{\theta} & \mathrm{Gr}_{\leq \mu}^{(m)}. \end{array}$$

Then taking the $(L^+G \times L^+G)$ -equivariant intersection cohomology and specializing along $R_{\bar{G}, \ell} \rightarrow \bar{\mathbb{Q}}_\ell$, we obtain the following commutative diagram:

$$\begin{array}{ccc} \mathrm{IH}^*(\mathrm{Gr}_{\leq \mu}^{\mathrm{op}}) & \xrightarrow{(\theta^{-1})^*} & \mathrm{IH}^*(\mathrm{Gr}_{\leq \mu}) \\ \gamma \downarrow & & \uparrow \gamma \\ \mathrm{IH}^*(\mathrm{Gr}_{\leq \mu}) & \xrightarrow{\theta^*} & \mathrm{IH}^*(\mathrm{Gr}_{\leq \mu}^{\mathrm{op}}). \end{array}$$

The lemma follows. □

To continue, let us understand a toy case. Note that θ induces an isomorphism between L^+G -orbits $\mathrm{Gr}_\mu^{\mathrm{op}} \simeq \mathrm{Gr}_\mu$, and therefore we have a canonical isomorphism

$$\mathring{\Theta}_\mu : H^*(\mathrm{Gr}_\mu) \xrightarrow{\theta^*} H^*(\mathrm{Gr}_\mu^{\mathrm{op}}) \xrightarrow{\gamma} H^*(\mathrm{Gr}_\mu),$$

where the isomorphisms γ is constructed by the same way as in Lemma 2.27.

LEMMA 2.32. For every j , $\mathring{\Theta}_\mu = (-1)^j$ on $H^{2j}(\text{Gr}_\mu)$.

Proof. The argument is essentially the same as [LY13, Lemma 3.3], although the set-up is different. (The authors of loc. cit. work over \mathbb{C} and with the based loop group of a compact Lie group rather than the affine Grassmannian.) Recall the projection $\pi_\mu : \text{Gr}_\mu \rightarrow (\bar{G}/\bar{P}_\mu)^{p^{-\infty}}$ from (1.4.4). As the fibers are the perfection of affine spaces of the same dimension, the pullback induces a canonical ring isomorphism $H^*(\bar{G}/\bar{P}_\mu) \simeq H^*(\text{Gr}_\mu)$. Therefore, $H^*(\text{Gr}_\mu)$ is generated by H^2 . In addition, it is clear that $\mathring{\Theta}_\mu$ is a ring homomorphism so it is enough to prove that $\mathring{\Theta}_\mu = -1$ on H^2 .

On the other hand, $\text{Gr}_\mu^{\text{op}}$ projects to $(\bar{G}/\bar{P}_{-\mu})^{p^{-\infty}}$ given by

$$\varpi^\mu g \mapsto g^{-1} \text{ mod } \varpi.$$

A direct computation shows that the following diagram is commutative:

$$\begin{array}{ccc} \text{Gr}_\mu^{\text{op}} & \xrightarrow{\theta} & \text{Gr}_\mu \\ \downarrow & & \downarrow \\ (\bar{G}/\bar{P}_{-\mu})^{p^{-\infty}} & \xrightarrow{g \mapsto \theta'(g)\dot{w}_0} & (\bar{G}/\bar{P}_\mu)^{p^{-\infty}}, \end{array}$$

where \dot{w}_0 is a lifting of w_0 to \bar{G} . Taking the equivariant cohomology, we obtain the commutativity of the following diagram:

$$\begin{array}{ccc} H_G^*(\bar{G}/\bar{P}_\mu) = R_{(\bar{P}_\mu)_{\text{red}}, \ell} & \xrightarrow{\theta^*} & R_{(\bar{P}_{-\mu})_{\text{red}}, \ell} = H_G^*(\bar{G}/\bar{P}_{-\mu}) \\ \downarrow & & \downarrow \\ R_{\bar{T}, \ell} & \xrightarrow{\chi \mapsto -\chi} & R_{\bar{T}, \ell}, \end{array}$$

where $\chi \in \mathbb{X}^\bullet(\bar{T})$, regarded as elements in $R_{\bar{T}, \ell}$ of degree two.

On the other hand, the isomorphism $H_G^*(\bar{G}/\bar{P}_{-\mu}) \simeq H_G^*(\bar{G}/\bar{P}_\mu)$, given by $\gamma : H_{L+G}^*(\text{Gr}_\mu^{\text{op}}) \simeq H_{L+G \times L+G}^*(\text{Gr}_\mu^{(m)}) \simeq H_{L+G}^*(\text{Gr}_\mu)$, is the restriction of the identity map on $R_{\bar{T}, \ell}$ by definition. Therefore, the equivariant version of $\mathring{\Theta}_\mu$ acts as (-1) on degree two parts. Specializing gives the lemma. \square

Remark 2.33. This lemma in particular proves Lemma 2.30 in the case when μ is minuscule. The difficulty to prove Lemma 2.30 for general μ is that the intersection cohomology ring is not generated by Chern classes, but we do not know more cohomology classes in it.¹⁴

To continue, it is convenient to set $\mathbf{C}_\mu = \text{IC}_\mu[(2\rho, \mu)]$, as in [LY13]. For each $\lambda \leq \mu$, let $i_\lambda : \text{Gr}_\lambda \rightarrow \text{Gr}_{\leq \mu}$ denote the corresponding locally closed

¹⁴Although there are MV basis in $\text{IH}^*(\text{Gr}_{\leq \mu})$, it seems hard to understand the map γ in terms of them.

embedding. For j , let $\mathcal{H}_\lambda^j \mathbf{C}_\mu$ denote the degree j -th sheaf cohomology of $i_\lambda^* \mathbf{C}_\mu$, which is constant along Gr_λ . Then there is a canonical isomorphism

$$\mathcal{H}_\lambda^j \Psi_\mu : \mathcal{H}_\lambda^j \mathbf{C}_\mu = \mathcal{H}_\lambda^j \theta^* \mathbf{C}_\mu \simeq \mathcal{H}_\lambda^j \mathbf{C}_\mu^{\text{op}} \simeq \mathcal{H}_\lambda^j \mathbf{C}_\mu,$$

where the second isomorphism is $N_\mu : \theta^* \text{IC}_\mu \simeq \text{IC}_\mu^{\text{op}}$ from (2.4.2), and the last isomorphism is from Corollary 2.26. Clearly, $\mathcal{H}_\lambda^j \Psi_\mu$ is an involution. Recall that the existence of the Demazure “resolution” in our setting (see (1.4.2)) implies that all the stalk cohomology of \mathbf{C}_μ concentrate in even degrees.

LEMMA 2.34. *For every j , $\mathcal{H}_\lambda^{2j} \Psi_\mu = (-1)^j$.*

Now, we prove Lemma 2.30, assuming Lemma 2.34. In [LY13, 3.4, 6.4], it was shown that the equal characteristic analogue of Lemma 2.30 implies the equal characteristic analogue of Lemma 2.34. But their argument can be reversed. We sketch it here and refer to loc. cit. for details. (But note that their set-up is different.) We extend the partial order “ \leq ” on \mathbb{X}_\bullet^+ to a total order, still denoted by \leq . We consider the stratification of $\text{Gr}_{\leq \mu}$ given by $\{\text{Gr}_\lambda, \lambda \leq \mu\}$. Let $\text{Gr}_{< \lambda} = \sqcup_{\lambda' < \lambda} \text{Gr}_{\lambda'}$, and let $i_{< \lambda}$ and $i_{\leq \lambda}$ denote the corresponding closed embeddings from $\text{Gr}_{< \lambda}$ and $\text{Gr}_{\leq \lambda}$ to $\text{Gr}_{\leq \mu}$. Then there is a long exact sequence of cohomology

$$\cdots \rightarrow H^i(\text{Gr}_{< \lambda}, i_{< \lambda}^! \mathbf{C}_\mu) \rightarrow H^i(\text{Gr}_{\leq \lambda}, i_{\leq \lambda}^! \mathbf{C}_\mu) \rightarrow H^i(\text{Gr}_\lambda, i_\lambda^! \mathbf{C}_\mu) \rightarrow \cdots,$$

which splits into short exact sequences as all the cohomology in odd degree vanish. Therefore, we obtain a filtration on $\text{IH}^*(\text{Gr}_{\leq \mu})$, given by

$$\text{Im}(H^i(\text{Gr}_{\leq \lambda}, i_{\leq \lambda}^! \mathbf{C}_\mu) \rightarrow \text{IH}^*(\text{Gr}_{\leq \mu})).$$

The associated graded is $\bigoplus_{\lambda \leq \mu} H^i(\text{Gr}_\lambda, i_\lambda^! \mathbf{C}_\mu)$. There is a similar picture on $\text{Gr}_{\leq \mu}^{\text{op}}$.

The isomorphisms $\theta^* : \text{IH}^*(\text{Gr}_{\leq \mu}) \simeq \text{IH}^*(\text{Gr}_{\leq \mu}^{\text{op}})$ and $\gamma : \text{IH}^*(\text{Gr}_{\leq \mu}) \simeq \text{IH}^*(\text{Gr}_{\leq \mu}^{\text{op}})$ preserve the filtrations on $\text{IH}^*(\text{Gr}_{\leq \mu})$ and on $\text{IH}^*(\text{Gr}_{\leq \mu}^{\text{op}})$, and therefore give rise to isomorphisms

$$\text{gr } \Theta : \text{gr } \text{IH}^*(\text{Gr}_{\leq \mu}) \xrightarrow{\text{gr } \theta^*} \text{gr } \text{IH}^*(\text{Gr}_{\leq \mu}^{\text{op}}) \xrightarrow{\text{gr } \gamma} \text{gr } \text{IH}^*(\text{Gr}_{\leq \mu}).$$

Note that $i_\lambda^! \mathbf{C}_\mu = (i_\lambda^* \mathbf{C}_\mu[2(2\rho, \lambda - \mu)])^*$. In addition, it is easy to identify $\text{gr } \Theta$ with the direct sum over λ of the maps

$$\mathring{\Theta}_\lambda \otimes \mathcal{H}_{\lambda, !}^* \Psi_\mu : H^*(\text{Gr}_\lambda) \otimes i_\lambda^! \mathbf{C}_\mu \simeq H^*(\text{Gr}_\lambda) \otimes i_\lambda^! \mathbf{C}_\mu,$$

where $\mathcal{H}_{\lambda, !}^* \Psi_\mu$ is the inverse of the dual of $\mathcal{H}_\lambda^* \Psi_\mu$. So the action of $\text{gr } \Theta$ on the degree $2j$ piece of $H^*(\text{Gr}_\lambda) \otimes \mathcal{H}_\lambda^* \mathbf{C}_\mu$ is given by $(-1)^j$. But as Θ itself is an involution, it acts on $\text{IH}^{2j}(\text{Gr}_{\leq \mu})$ by $(-1)^j$.

2.4.5. It remains to prove Lemma 2.34. Let I be the (p -adic jet group of) the standard Iwahori (i.e., whose reduction mod ϖ is $\bar{B} \subset \bar{G}$). Let \widetilde{W} denote the Iwahori-Weyl group of $G(F)$ as before, and let $W_a \subset \widetilde{W}$ denote the corresponding affine Weyl group, with the set of simple reflections $\{s_i, i \in \mathbb{S}\}$ determined by I . We identify \mathbb{S} with the set of vertices of the affine Dynkin diagram of $G(F)$. Let $0 \in \mathbb{S}$ denote the vertex corresponding to the hyperspecial parahoric $G(\mathcal{O})$. Let $J = \mathbb{S} - \{0\}$, and let $W_J \subset \widetilde{W}_a$ denote subgroup generated by $\{s_i, i \in J\}$. Let w_J denote the longest element in W_J . Then W_J is isomorphic to the finite Weyl group $\overline{W} = \widetilde{W}/\mathbb{X}_\bullet$ of $G(F)$, and w_J maps to the longest element w_0 in \overline{W} mentioned before. Let $\Omega \subset \widetilde{W}$ denote the subgroup of length zero elements, i.e., those that fix I . It acts on W_a by conjugation. Then $\widetilde{W} = W_a \rtimes \Omega$. Let $*$: $\widetilde{W} \rightarrow \widetilde{W}$ be the involution given by $w^* := w_J w w_J$ for $w \in W_J$ and $\lambda^* = -w_0(\lambda)$ for $\lambda \in \mathbb{X}_\bullet$. This is an involution of \widetilde{W} that stabilizes $\{s_i, i \in \mathbb{S}\}$ and fixes s_0 .

Let $\omega \in \Omega$. Then by [LY13, Lemma 6.2] $\omega^* = \omega^{-1}$, and the map

$$\diamond : W_a \rightarrow W_a, \quad w \mapsto w^\diamond := \omega w^* \omega^{-1}$$

is an involution of W_a , which stabilizes $\{s_i, i \in \mathbb{S}\}$. Let

$$I_\diamond = \{w \in W_a \mid w^\diamond = w^{-1}\} \text{ and } W_J^\diamond = \{w^\diamond \mid w \in W_J\} = \omega W_J \omega^{-1}.$$

Then as argued in [Lus12, Prop. 8.2] and [LY13, Th. 6.3 (1)], the longest element in every $(W_J \times W_J^\diamond)$ -double coset belongs to I_\diamond .

Applying the results of [LV12, Lus12] to $(W_a, \{s_i, i \in \mathbb{S}\}, \diamond)$, one attaches a polynomial $P_{y,w}^{\sigma, \diamond}(q) \in \mathbb{Z}[q]$ to every pair $(y, w) \in I_\diamond$, with $y \leq w$. On the other hand, there is the usual Kazhdan-Lusztig polynomial $P_{y,w}(q)$ attached to (y, w) [KL79]. The following theorem was conjectured in [Lus12, Conj. 8.4], and was proved in [LY13, Th. 6.3].

THEOREM 2.35. *Let d_1 and d_2 be longest elements of (W_J, W_J^\diamond) -double cosets in W_a . Then*

$$P_{d_1, d_2}^{\sigma, \diamond}(q) = P_{d_1, d_2}(-q).$$

Let us note that this theorem was deduced in [LY13] from the equal characteristic analogue of Lemma 2.32.

Finally, we explain why Lemma 2.34 follows from this theorem. Let $\mu \in \mathbb{X}_\bullet$, and let ω denote the unique element in Ω such that $\varpi^\mu \in W_a \omega$. Let d_μ be the longest element in $W_J \varpi^\mu W_J \omega^{-1} = W_J (\varpi^\mu \omega^{-1}) W_J^\diamond$. Then for $\lambda \leq \mu$, $\varpi^\lambda \omega^{-1} \in W_a$. Let d_λ denote the corresponding longest element in $W_J \varpi^\lambda \omega^{-1} W_J^\diamond$. The usual Kazhdan-Lusztig theory [KL79], [KL80] works in our situation. So $P_{y,w}(q)$ is the Poincare polynomial for the stalk cohomology at y of the intersection cohomology sheaf IC_w of the Schubert variety S_w on

the affine flag variety $\mathcal{F}\ell = LG/I$. Then, in particular, (see [Lus83]),

$$P_{d_\lambda, d_\mu}(q) = \sum (\dim \mathcal{H}_\lambda^{2j} \mathbf{C}_\mu) q^j.$$

On the other hand, in [LV12, §3], similar interpretations were given to the polynomials $P_{y,w}^{\sigma, \diamond}$. Such interpretations in particular imply that

$$P_{d_\lambda, d_\mu}^{\sigma, \diamond}(q) = \sum \text{tr}(\mathcal{H}_\lambda^{2j} \Psi_\mu | \mathcal{H}_\lambda^{2j} \mathbf{C}_\mu) q^j.$$

Since $\mathcal{H}_\lambda^{2j} \Psi_\mu$ is an involution, Theorem 2.35 implies Lemma 2.34.

2.5. Identification with the dual group. We have endowed $\mathbf{P}_{L+G}(\text{Gr})$ with a symmetric monoidal category structure and the hypercohomology functor $\mathbf{H}^* : \mathbf{P}_{L+G}(\text{Gr}) \rightarrow \text{Vect}_{\overline{\mathbb{Q}}_\ell}$ a tensor functor structure. It is clear that IC_0 is a unit object in $\mathbf{P}_{L+G}(\text{Gr})$. Now we proceed as in [MV07, §7] to conclude that $(\mathbf{P}_{L+G}, \star, \mathbf{H}^*)$ is a Tannakian category with the fiber functor \mathbf{H}^* . Let $\tilde{G} = \text{Aut}^\otimes \mathbf{H}^*$ denote the Tannakian group. It is a connected reductive group, by the same argument as in [MV07, §7]. Our next goal is to identify \tilde{G} with the dual group \hat{G} of G .

First, if $G = T$ is a torus, Gr_T is a discrete set of points canonically isomorphic to $\mathbb{X}_\bullet(T)$. Then it is easy to see that Sat_T is equivalent to the category of $\mathbb{X}_\bullet(T)$ -graded finite dimensional $\overline{\mathbb{Q}}_\ell$ -vector spaces and \mathbf{H}^* is just the functor that forgets the grading. Therefore, $\tilde{T} = \hat{T}$ is the dual torus of T .

Now consider the general case. We can regard the weight functor CT as a functor from $\text{Sat}_G \rightarrow \text{Sat}_T$, and the isomorphism in Corollary 2.10 as an isomorphism $\mathbf{H}^* \circ \text{CT} \simeq \mathbf{H}^* : \text{Sat}_G \rightarrow \text{Vect}_{\overline{\mathbb{Q}}_\ell}$.

PROPOSITION 2.36. *There is a unique monoidal structure on CT such that the isomorphism $\mathbf{H}^* \circ \text{CT} \simeq \mathbf{H}^* : \text{Sat}_G \rightarrow \text{Vect}_{\overline{\mathbb{Q}}_\ell}$ in Corollary 2.10 is monoidal.*

In equal characteristic, this was proved in [MV07, Prop. 6.4] using the fusion product interpretation of the convolution product. However, there is another purely local approach using equivariant cohomology, given in [Zhu16, Prop. 5.3.14]. The latter approach works in mixed characteristic as well. Namely, S_λ is stable under the action of the torus $\bar{T}^{p^{-\infty}} \subset L^+T \subset L^+G$, and therefore one can use the \bar{T} -equivariant cohomology $\mathbf{H}_{\bar{T}}^*$ as in loc. cit. for the arguments.

Applying Proposition 2.21, we see that the weight functor CT in fact is a tensor functor between two Tannakian categories. It thus induces a homomorphism

$$\hat{T} \simeq \tilde{T} \rightarrow \tilde{G}.$$

This defines a subtorus of \tilde{G} . By the same argument as in [MV07, § 7], this is in fact a maximal torus. In addition, the filtration on $\mathbf{H}^*(\text{Gr}_G, -)$ defines a Borel subgroup $\hat{B} \subset \tilde{G}$ that contains \hat{T} . Then it follows by the same argument

as the end of [MV07, § 7] that \tilde{G} is isomorphic to \hat{G} . We refer to [Zhu16, § 5.3] for more details.

Remark 2.37. Our methods can also be applied to establish the mixed characteristic geometric Satake for ramified groups (cf. [Zhu15]).

3. Dimension of affine Deligne-Lusztig varieties

In this section, we give an application of mixed characteristic affine Grassmannians to the study of the Rapoport-Zink (RZ) spaces. More applications will appear in [XZ].

3.1. Dimension of affine Deligne-Lusztig varieties.

3.1.1. We use the notation as in Section 0.5. So F is a totally ramified extension of $F_0 = W(k)[1/p]$ with \mathcal{O} its ring of integers. Let L be the completion of its maximal unramified extension, with \mathcal{O}_L its ring of integers. Let $\sigma \in \text{Gal}(L/F)$ denote the Frobenius element. Let G be a reductive group scheme over \mathcal{O} . For $b \in G(L)$ and $\mu \in \mathbb{X}_\bullet^+$, we define the (closed) affine Deligne-Lusztig “variety” as

$$(3.1.1) \quad X_{\leq \mu}(b) = \{g \bmod L^+G \in \text{Gr}_G \mid g^{-1}b\sigma(g) \in \overline{L^+G\varpi^\mu L^+G}\}.$$

More precisely, one can interpret $X_{\leq \mu}(b)$ as the following moduli functor: let \mathcal{E}_0 be the trivial G -torsor on $D_F = \text{Spec } \mathcal{O}$, with an isomorphism $b : \sigma^*\mathcal{E}_0|_{D_F^*} \rightarrow \mathcal{E}_0|_{D_F^*}$. Then for a perfect k -algebra R ,

$$(3.1.2) \quad X_{\leq \mu}(b)(R) = \{(\mathcal{E}, \beta) \in \text{Gr}_G(R) \mid \text{Inv}_x(\beta^{-1}b\sigma(\beta)) \leq \mu, \forall x \in \text{Spec } R\}.$$

By Lemma 1.22, $X_{\leq \mu}(b)$ is a closed subset of Gr_G . One can replace “ \leq ” in the above definition by “ $=$,” which defines an open subset of $X_{\leq \mu}(b)$, denoted by $X_\mu(b)$. If we denote $\Phi = \beta^{-1}b\sigma(\beta)$, then (\mathcal{E}, Φ) is an F -crystal with G -structure on $\text{Spec } R$, whose Hodge polygon is bounded by μ (resp. equal to μ).

It turns out that the dimension of $X_{\leq \mu}(b)$ is finite, and Rapoport gave a conjectural formula of its dimension ([Rap05]) with a reformulation given by Kottwitz ([GHKR06]):

$$(3.1.3) \quad \dim X_{\leq \mu}(b) = \langle \rho, \mu - \nu_b \rangle - \frac{1}{2} \text{def}_G(b).$$

Here ν_b is the Newton point of b and $\text{def}_G(b)$ is the defect of b . We refer to [GHKR06] for the precise definitions. This dimension formula has been proved in equal characteristic by combining the works [GHKR06], [Vie06], [Ham15a], but it remains open in general in mixed characteristic. In fact, before our work, it was not clear how to define the dimension of $X_{\leq \mu}(b)$ in mixed characteristic in general, and this formula only makes sense for some special triples (G, b, μ) when (3.1.1) can be interpreted as the $\overline{\mathbb{F}}_p$ -points of some moduli spaces of p -divisible groups (also known as RZ spaces). In the case when the RZ spaces are of PEL type, this dimension formula was proved recently by Hamacher

([Ham15b]), and some special cases were proved earlier by Viehmann ([Vie08b], [Vie08a]).

THEOREM 3.1. *Rapoport's conjecture (3.1.3) holds in general.*

Not surprisingly, the machinery developed so far in the paper allows us to imitate the the arguments in equal characteristic with only a few justifications. First, one can argue as in [GHKR06], [Ham15a] to reduce the general Rapoport conjecture to the case when b is superbasic. It was shown in [GHKR06], [CKV15] that if G is of adjoint type, superbasic σ -conjugacy classes exist only when $G_F = \mathrm{PGL}_n$ or $G_F = \mathrm{Res}_{E/F} \mathrm{PGL}_n$, where E/F is an unramified extension. The PGL_n case was treated by Viehmann [Vie06] (in equal characteristic, but the same arguments apply here). We will reduce the $\mathrm{Res}_{E/F} \mathrm{PGL}_n$ case to the PGL_n case and then apply [Vie06]. This, in particular, gives a shorter proof of the main result of [Ham15a] (but it uses [Vie06]). We sketch the arguments in the sequel.

Remark 3.2. This is a side remark arising as a comment by G. Pappas. Although the algebro-geometric structure on $X_{\leq \mu}(b)$ was not known before, the authors of [CKV15] defined a notion of the set of connected components $\pi_0(X_{\leq \mu}(b))$ of $X_{\leq \mu}(b)$. One can check that if two points in $X_{\leq \mu}(b)(\bar{k})$ are in the same connected component in the sense of loc. cit., they are in the same connected component under the Zariski topology. The converse will also hold if in their definition arbitrary test rings (rather than just smooth rings) are allowed.¹⁵ On the other hand, it seems that one can directly adapt their arguments to our setting to prove that the structure of connected components of $X_{\leq \mu}(b)$ in our sense is also given by the statement of [CKV15, Th. 1.1]. Then it would follow *a posteriori* that the two notions are the same. In any case, when $X_{\leq \mu}(b)$ is the set of $\bar{\mathbb{F}}_p$ -points of a Rapoport-Zink space, their π_0 coincides with the π_0 of the RZ space, and by Proposition 3.11 below, also coincides with π_0 of $X_{\leq \mu}(b)$ as the perfection of an algebraic space.

3.1.2. Now one can argue as in [GHKR06, Prop. 5.6.1, Th. 5.8.1] to reduce the Rapoport conjecture for general (G, μ, b) to the case when b is basic. First, the Newton point ν_b is defined over F , whose centralizer in G is a rational Levi M . One can find a representative in the σ -conjugacy class of b that is contained in $M(L)$. We rename this representative by b . So b is basic in $M(L)$. Then their arguments reduce Rapoport's conjecture for (G, b, μ) to (M, b, μ_M) (for various μ_M). These arguments rely on their Propositions 5.3.1 and 5.4.3. The proof of Proposition 5.3.1 in loc. cit. applies to the current setting. Note that the arguments involve an M -equivariant isomorphism $N \simeq \mathfrak{n}$. In the equal

¹⁵In loc. cit. it was conjectured that these two definitions coincide.

characteristic situation, this isomorphism makes sense either as F -schemes or as k -ind-schemes. In our setting, it only makes sense as an isomorphism of F -schemes. But it still makes sense to talk about the p -adic loop space of \mathfrak{n} so the arguments in §4 of *ibid.* apply. The proof of Proposition 5.4.3 in *ibid.* extends verbatim in mixed characteristic, by taking account of the Lefschetz trace formula for separated pfp perfect algebraic spaces (see Section A.3.4). A special case of this type of argument has appeared in the proof of Proposition 2.9 (where $M = T$).

As explained in *loc. cit.*, even b is basic for G ; it still might happen that b is contained in a proper Levi subgroup of G . A basic σ -conjugacy class that does not meet in proper Levi subgroups of G defined over F is called a *superbasic* σ -conjugacy class. Therefore, it is enough to prove Rapoport’s conjecture for superbasic b . In addition, one can assume that $G = G_{\text{ad}}$ is simple of adjoint type. Then it follows from [GHKR06], [CKV15] that superbasic b exists only when $G_F = \text{Res}_{E/F} \text{PGL}_n$ for some unramified extension E/F .

3.1.3. It remains to prove the following.

PROPOSITION 3.3. *Formula (3.1.3) holds for $G_F = \text{Res}_{E/F} \text{GL}_n$ and b superbasic.*

Remark 3.4. This proposition was proved by Hamacher when $F = \mathbb{Q}_p$ and μ is minuscule. Our method is different and is simpler, but it uses [Vie06].

Proof. We first reduce the $\text{Res}_{E/F} \text{GL}_n$ case to GL_n case. We start with a generalization of affine Deligne-Lusztig varieties. Let H be a connected reductive group over \mathcal{O}_E . First observe that $X_{\leq \mu}(b)$ can be defined as the following Cartesian pullback:

$$(3.1.4) \quad \begin{array}{ccc} X_{\leq \mu}(b) & \longrightarrow & \text{Gr}_H \tilde{\times} \text{Gr}_{\leq \mu} \\ \downarrow & & \downarrow \text{pr} \times m \\ \text{Gr} & \xrightarrow{1 \times b\sigma} & \text{Gr}_H \times \text{Gr}_H. \end{array}$$

Now by replacing μ by a sequence of dominant co-weights μ_\bullet , we can define a convolution version of the affine Deligne-Lusztig variety,

$$(3.1.5) \quad \begin{array}{ccc} X_{\leq \mu_\bullet}(b) & \longrightarrow & \text{Gr}_H \tilde{\times} \text{Gr}_{\leq \mu_\bullet} \\ \downarrow & & \downarrow \text{pr}_1 \times m \\ \text{Gr} & \xrightarrow{1 \times b\sigma} & \text{Gr}_H \times \text{Gr}_H. \end{array}$$

Concretely, $X_{\leq \mu_\bullet}(b)$ classifies the following commutative diagram of maps such that $\text{Inv}_x(\Phi_i) \leq \mu_i$ for every $x \in \text{Spec } R$:

$$\begin{array}{ccccccc}
 \sigma^* \mathcal{E}_1 & \xrightarrow{\Phi_d} & \mathcal{E}_d & \xrightarrow{\Phi_{d-1}} & \cdots & \xrightarrow{\Phi_1} & \mathcal{E}_1 \\
 \sigma^*(\beta) \downarrow & & & & & & \downarrow \beta \\
 \sigma^* \mathcal{E}_0 & \xrightarrow{\quad b \quad} & & & & & \mathcal{E}_0.
 \end{array}$$

Note that in equal characteristic, this is the local version of the moduli space of iterated Shtukas.

In the sequel, we write $X_{\leq \mu_\bullet}^H(b\sigma_E)$ for $X_{\leq \mu_\bullet}(b)$ if we want to emphasize that the underlying group H is define over \mathcal{O}_E , and that the Frobenius $\sigma_E \in \text{Gal}(L/E)$.

Now we start our reduction step. Assume that E/F is unramified of degree d . Let Σ denote the set of embeddings $\tau : E \rightarrow L$ over F . Then $\text{Gal}(L/F)$ acts transitively on Σ . We fix $\tau_0 \in \Sigma$, and let $\tau_i = \sigma^i(\tau_0)$, $i = 0, 1, 2, \dots, d - 1$. Let $\sigma_E := \sigma^d \in \text{Gal}(L/\tau_0(E))$.

Now assume that $G = \text{Res}_{\mathcal{O}_E/\mathcal{O}_F} H$ for some unramified group H over E . The canonical isomorphism $E \otimes_F L \simeq \prod_{\tau \in \Sigma} L$, $a \otimes b \mapsto (\tau_i(a)b, \tau_i \in \Sigma)$ induces a canonical isomorphism

$$G \otimes L \simeq \prod_{\tau \in \Sigma} H \otimes_{E,\tau} L.$$

Let μ be a dominant coweight of G_L . Then under the above isomorphism, it gives a sequence $\mu_\bullet = (\mu_{\tau_0}, \dots, \mu_{\tau_{d-1}})$, where μ_{τ_i} is a dominant coweight of $H \otimes_{E,\tau_i} L$. Similarly, $b \in G(L)$ gives $(b_\tau) \in \prod_{\tau \in \Sigma} (H \otimes_{E,\tau} L)(L)$.

Note that $\sigma^i \in \text{Gal}(L/F)$ induces an isomorphism $H \otimes_{E,\tau_i} L \simeq H \otimes_{E,\tau_0} L$. By abuse of notation, the induced map on the cocharacters and on the L -points are still denoted by σ^i . (This coincides with the standard notation if $H = (H_0)_E$ for some group H_0 defined over F .)

For an $\overline{\mathbb{F}}_p$ -algebra R , we identify $(\mathcal{E}, \beta) \in \text{Gr}_G$ with $(\mathcal{E}_\tau, \beta_\tau) \in \prod_{\tau \in \Sigma} \text{Gr}_H$ in an obvious way. Then the condition (3.1.2) is equivalent to the commutativity of the following diagram:

$$\begin{array}{ccccccc}
 (\sigma^d)^* \mathcal{E}_{\tau_0} & \longrightarrow & (\sigma^{d-1})^* \mathcal{E}_{\tau_{d-1}} & \longrightarrow & \cdots & \longrightarrow & \mathcal{E}_{\tau_0} \\
 (\sigma^d)^* \beta_{\tau_0} \downarrow & & (\sigma^{d-1})^* \beta_{\tau_{d-1}} \downarrow & & & & \downarrow \beta_{\tau_0} \\
 (\sigma^d)^* \mathcal{E}_0 & \xrightarrow{\sigma^{d-1}(b_{\tau_{d-1}})} & (\sigma^{d-1})^* \mathcal{E}_0 & \xrightarrow{\sigma^{d-2}(b_{\tau_{d-2}})} & \cdots & \xrightarrow{b_{\tau_0}} & \mathcal{E}_0.
 \end{array}$$

Let

$$\text{Nm } b = b_{\tau_0} \sigma(b_{\tau_1}) \cdots \sigma^{d-1}(b_{\tau_{d-1}}) \in (H \otimes_{E,\tau_0} L)(L).$$

Then the above discussions imply the following lemma.

LEMMA 3.5. *If $G = \text{Res}_{\mathcal{O}_E/\mathcal{O}_F} H$ for some unramified group H over \mathcal{O}_E , then*

$$X_{\leq \mu}^G(b\sigma) \simeq X_{\leq \mu_\bullet}^H((\text{Nm } b)\sigma_E),$$

where $\mu_\bullet = (\mu_{\tau_0}, \sigma(\mu_{\tau_1}), \dots, \sigma^{d-1}(\mu_{\tau_{d-1}}))$.

Remark 3.6. The map $b \mapsto \text{Nm } b$ defines a map from the σ -conjugacy class of $G(L)$ to the σ_E -conjugacy class $H(L)$ (where E embeds into L via τ_0).

We also need the following purely group theoretical lemma, whose proof is by chasing the definitions.

LEMMA 3.7. *Let μ and b be as above.*

- (1) *Let ρ_G be the half sum of positive roots of $G \otimes L$ and let ρ_H be the half sum of positive roots of $H \otimes_{E, \tau_0} L$. Then*

$$(\rho_G, \mu) = (\rho_H, \sum_i \sigma^i(\mu_{\tau_i})).$$

- (2) *Let ν_b be the Newton point of b and let $\nu_{\text{Nm } b}$ be the Newton point of $\text{Nm } b$. Then*

$$(\rho_G, \nu_b) = (\rho_H, \nu_{\text{Nm } b}).$$

- (3) *Let J_b^G be the σ -twisted centralizer of $b \in G(L)$, i.e.,*

$$J_b^G(R) = \{g \in G(R \otimes_F L) \mid g^{-1}b\sigma(g) = b\}$$

for any F -algebra R . This is an F -group. Similarly, let $J_{\text{Nm } b}^H$ be the σ_E -twisted centralizer of $\text{Nm } b$, which is an E -group. Then $J_b^G = \text{Res}_{E/F} J_{\text{Nm } b}^H$. In particular,

$$\text{def}_G(b) = \text{def}_H(\text{Nm } b).$$

Now, assuming that the dimension formula for affine Deligne-Lusztig varieties of H has been established, we calculate the dimension of $X_{\leq \mu}^G(b\sigma) = X_{\leq \mu_\bullet}^H((\text{Nm } b)\sigma_E)$. Recall the convolution map of affine Grassmannians (2.2) (for H),

$$m : \text{Gr}_{\leq \mu_\bullet} \rightarrow \text{Gr}_{\leq |\mu_\bullet|}.$$

By (3.1.4) and (3.1.5), the following diagram is Cartesian:

$$\begin{CD} X_{\leq \mu_\bullet}^H((\text{Nm } b)\sigma_E) @>>> \text{Gr}_H \tilde{\times} \text{Gr}_{\leq \mu_\bullet} \\ @VVV @VVV \\ X_{\leq |\mu_\bullet|}^H((\text{Nm } b)\sigma_E) @>>> \text{Gr}_H \tilde{\times} \text{Gr}_{\leq |\mu_\bullet|}. \end{CD}$$

By (the proof of) Proposition 2.3, for λ , the dimension of the fiber $m^{-1}(\tau^\lambda)$ is $\leq (\rho_H, |\mu_\bullet| - \lambda)$. Therefore, by Lemma 3.7, the preimage of $X_{\leq \lambda}^H((\text{Nm } b)\sigma_E) \subset$

$$\begin{aligned}
 X_{\leq|\mu_\bullet|}^H((\mathrm{Nm} b)\sigma_E) \text{ in } X_{\leq\mu_\bullet}^H((\mathrm{Nm} b)\sigma_E) & \text{ has dimension} \\
 & \leq (\rho_H, \lambda - \nu_{\mathrm{Nm} b}) - \frac{1}{2} \mathrm{def}_H(\mathrm{Nm} b) + (\rho_H, |\mu_\bullet| - \lambda) \\
 & = (\rho_H, |\mu_\bullet| - \nu_{\mathrm{Nm} b}) - \frac{1}{2} \mathrm{def}_H(\mathrm{Nm} b) \\
 & = (\rho_G, \mu - \nu_b) - \frac{1}{2} \mathrm{def}_G(b).
 \end{aligned}$$

In addition, if $\lambda = |\mu_\bullet|$, the equality achieves. It follows that

$$\dim X_{\leq\mu}^G(b\sigma) = \dim X_{\leq\mu_\bullet}^H((\mathrm{Nm} b)\sigma_E) = (\rho_G, \mu - \nu_b) - \frac{1}{2} \mathrm{def}_G(b).$$

Therefore, it remains to prove the case when $G = \mathrm{GL}_n$ and b superbasic. Now one can argue exactly the same as [Vie06] to complete the proof. \square

3.2. Affine Deligne-Lusztig varieties and Rapoport-Zink spaces. Let us recall the definition of Rapoport-Zink (RZ) spaces. In the PEL case, they were defined by Rapoport-Zink in their original work [RZ96]. In a more general situation but under the assumption that the group is unramified, they are recently defined by Kim [Kim13] and Howard-Pappas [HP15]. We assume (for simplicity) that $k = \overline{\mathbb{F}}_p$ is algebraically closed. To follow the standard notation, we write $W = W(k)$ (which was usually denoted by \mathcal{O} in previous sections). Let $L = W \otimes \mathbb{Q}_p$. We use F to denote σ -linear maps between vector spaces over L (unlike the rest part of the paper where F denotes a local field). Let Nilp_W denote the category of W -algebras in which p is nilpotent.

First we recall the following fundamental result of Rapoport-Zink. Let \mathbb{X}_0 be a p -divisible group over k . We consider the functor $\check{\mathcal{M}}_{\mathbb{X}_0}$ that assigns to every $R \in \mathrm{Nilp}_W$ the groupoid of pairs (\mathbb{X}, ι) , where \mathbb{X} is a p -divisible group over $\mathrm{Spec} R$, and $\iota : \mathbb{X}_0 \otimes_k R/p \rightarrow \mathbb{X} \otimes_R R/p$ is a quasi-isogeny. Rapoport-Zink proved that $\check{\mathcal{M}}_{\mathbb{X}_0}$ is represented by a separated formal scheme, formally smooth and formally locally of finite type over W .

Now we start with a reductive group G over \mathbb{Z}_p , a geometric conjugacy class of cocharacters $\mu : \mathbb{G}_m \rightarrow G$, and a σ -conjugacy class b of $G(L)$ with a representative in $G(W)p^\mu G(W)$, still denoted by b . We assume that there exists a free \mathbb{Z}_p -lattice Λ ¹⁶ and a faithful representation

$$\rho : G \rightarrow \mathrm{GL}(\Lambda),$$

such that the cocharacter $\rho\mu : \mathbb{G}_m \rightarrow \mathrm{GL}(\Lambda \otimes W)$ has weights 0, 1. We fix a representative of μ , still denoted by the same notation. Let

$$\Lambda \otimes W = \Lambda^0 \oplus \Lambda^1$$

denote the decomposition of $\Lambda \otimes W$ according to the weights of $\rho\mu$, which in turn induces a filtration $\mathrm{Fil}^0(\Lambda \otimes W) = \Lambda \otimes W \supset \mathrm{Fil}^1(\Lambda \otimes W) = \Lambda^1$. We

¹⁶Our Λ corresponds to Λ^* , and μ corresponds to $-\mu$ in [Kim13].

assume that $\text{rk } \Lambda^1 = n$ and $\text{rk } \Lambda = h$. This is equivalent to assuming that $\rho\mu$ is the n -th fundamental coweight of $\text{GL}(\Lambda \otimes W)$.

Let Λ^\otimes denote the tensor algebra of $\Lambda \oplus \Lambda^*$. Note that $\Lambda^\otimes = (\Lambda^*)^\otimes$. Elements in Λ^\otimes are called tensors. We choose a finite collection of tensors $\{s_i \in \Lambda^\otimes, i \in I\}$ such that $G \subset \text{GL}(\Lambda)$ is the schematic stabilizer of this collection. That is,

$$G = \text{Aut}(\Lambda, \{s_i, i \in I\}).$$

For example, if $G = \text{GL}_h$, we can choose $\{s_i\}$ to be the empty set. Note that

$$P_\mu := \text{Aut}(\Lambda, \{s_i, i \in I\}, \text{Fil}^*(\Lambda \otimes W))$$

is a parabolic subgroup of G_W determined by μ .

Note that by our assumption and the classical Dieudonné theory, there exists a p -divisible group \mathbb{X}_0 of dimension n and height h over $\overline{\mathbb{F}}_p$, together with an isomorphism

$$\varepsilon : \mathbb{D}(\mathbb{X}_0) \simeq \Lambda \otimes_{\mathbb{Z}_p} W,$$

where $\mathbb{D}(\mathbb{X}_0)$ is the *contravariant* Dieudonné module of \mathbb{X}_0 , equipped with (F, V) , such that

- (1) $\varepsilon F = \rho(b)(\text{id}_\Lambda \otimes \sigma)\varepsilon$,
- (2) $\varepsilon(\text{Lie}\mathbb{X}_0)^* = \text{Fil}^1 \Lambda \otimes \overline{\mathbb{F}}_p$.

The pair $(\mathbb{X}_0, \varepsilon)$ is unique up to a unique isomorphism, and we fix it in the sequel.

Finally [Kim13], we define (crystalline)-Tate tensors for p -divisible groups. Let $R \in \text{Nilp}_W$, and let \mathbb{X} be a p -divisible group on $\text{Spec } R$. Let $\mathbb{D}(\mathbb{X})$ denote its *contravariant* Dieudonné crystal. This is an F -crystal on $\text{Spec } R$, by which we mean a locally free crystal \mathcal{E} on the big crystalline site $\text{CRIS}(R/W)$, with a map (the Frobenius map)

$$F : \sigma^* \mathcal{E} \rightarrow \mathcal{E},$$

such that there exist an integer $i \geq 0$ and $V : \mathcal{E} \rightarrow \sigma^* \mathcal{E}$ satisfying $VF = p^i$. In addition, there is a decreasing filtration $\text{Fil}^\bullet \mathbb{D}(\mathbb{X})_R$ on $\mathbb{D}(\mathbb{X})_R$ (the value of $\mathbb{D}(\mathbb{X})$ at the trivial PD-thickening $R \xrightarrow{\text{id}} R$) whose associated graded is locally free over R . Namely,

$$\text{Fil}^0 \mathbb{D}(\mathbb{X})_R = \mathbb{D}(\mathbb{X})_R, \quad \text{Fil}^1 \mathbb{D}(\mathbb{X})_R = (\text{Lie}\mathbb{X})^*, \quad \text{and } \text{Fil}^2 \mathbb{D}(\mathbb{X})_R = 0.$$

Note that $\mathbb{D}(\mathbb{X})^\otimes$ is also an F -crystal with a filtration $\text{Fil}^\bullet \mathbb{D}(\mathbb{X})^\otimes_R$. For example, let

$$1 := \mathbb{D}(\mathbb{Q}_p/\mathbb{Z}_p)$$

be the filtered F -crystal given by the Dieudonné module of the constant p -divisible group $\mathbb{Q}_p/\mathbb{Z}_p$. Then $1_{R'} = R'$ for every PD-thickening $R' \rightarrow R$ and $F : 1_{R'} \rightarrow 1_{R'}$ sending $F(1) = 1$. In addition, $\text{Fil}^1 1_R = 0$. Then we call a (crystalline)-Tate tensor of \mathbb{X} a morphism $t : 1 \rightarrow \mathbb{D}(\mathbb{X})^\otimes$ of crystals, such that

$t_R : 1_R \rightarrow \mathbb{D}(\mathbb{X})_R^\otimes$ is compatible with the filtrations, and such that the induced map $t : 1 \rightarrow \mathbb{D}(\mathbb{X})^\otimes[\frac{1}{p}]$ of isocrystals is Frobenius-invariant.

For example, we can interpret $\{s_i, i \in I\} \subset \Lambda^\otimes$ as Tate tensors of the above fixed p -divisible group \mathbb{X}_0 as follows. First, via ε , we can regard $\{s_i\}$ as tensors in $\mathbb{D}(\mathbb{X}_0)^\otimes$. Since G fixes $\{s_i\}$, $b\sigma$ fixes $\{s_i\}$. So $\{s_i\}$ are F -invariant in $\mathbb{D}(\mathbb{X}_0)^\otimes[\frac{1}{p}]$. In addition, the cocharacter $\rho\mu : \mathbb{G}_m \rightarrow \mathrm{GL}(\Lambda \otimes W)$ also fixes $\{s_i\}$. Therefore, $\{s_i\}$ are in $\mathrm{Fil}^0(\mathbb{D}(\mathbb{X}_0)_{\mathbb{F}_p}^\otimes)$. Then we can define $s_i : 1 \rightarrow \mathbb{D}(\mathbb{X}_0)^\otimes$ by sending 1 to s_i .

For a p -divisible group over a general base R , the notion of Tate tensors may not be well behaved. Following [Kim13], let $\mathrm{Nilp}_W^{\mathrm{sm}}$ denote the full subcategory of Nilp_W consisting of formally smooth formally finitely generated W/p^m -algebras for some $m > 0$.

Definition 3.8. The Rapoport-Zink space associated to (G, b, μ) is the functor $\check{\mathcal{M}}(G, b, \mu)$ on $\mathrm{Nilp}_W^{\mathrm{sm}}$ classifying, for every $R \in \mathrm{Nilp}_W^{\mathrm{sm}}$,

- (1) a p -divisible group \mathbb{X} on $\mathrm{Spec} R$;
- (2) a collection of crystalline-Tate tensors $\{t_i\}, i \in I$ of \mathbb{X} ;
- (3) a quasi-isogeny $\iota : \mathbb{X}_0 \otimes_k R/J \rightarrow \mathbb{X} \otimes_R R/J$ that sends t_i to $s_i \otimes 1$ for $i \in I$, where J is some (and therefore any) ideal of definition of R that contains p such that
- (*) the R -scheme

$$\mathrm{Isom}((\mathbb{D}(\mathbb{X})_R, \{t_i\}, \mathrm{Fil}^\bullet(\mathbb{D}(\mathbb{X})_R)), (\Lambda \otimes_{\mathbb{Z}_p} R, \{s_i \otimes 1\}, \mathrm{Fil}^\bullet \Lambda \otimes_{\mathbb{Z}_p} R))$$

that classifies the isomorphisms between locally free sheaves $\mathbb{D}(\mathbb{X})_R$ and $\Lambda \otimes_{\mathbb{Z}_p} R$ on $\mathrm{Spec} R$ preserving the tensors and the filtrations is a $(P_\mu \otimes_W R)$ -torsor.¹⁷

Remark 3.9. (i) Our definition is slightly different from the original definition given in [Kim13]. But it is not hard to see that Condition (*) combines items (2) and (3) in [Kim13, Def. 4.6].

(ii) As explained in [Kim13], $\check{\mathcal{M}}(G, b, \mu)$ is independent of the choice of $\rho : G \rightarrow \mathrm{GL}(\Lambda)$ up to isomorphism.

The main theorem of Kim [Kim13] (see also [HP15, Th. 3.2.1, §2.4]) is as follows.

THEOREM 3.10. *Assume that $p > 2$. Then*

- (1) $\check{\mathcal{M}}(G, b, \mu)$ is represented by a closed formal subscheme $\check{\mathcal{M}}(G, b, \mu) \subset \check{\mathcal{M}}_{\mathbb{X}_0}$, formally smoothly over W ;
- (2) if $G = \mathrm{GL}_h$, $\mu = \omega_n$ and $\rho = \mathrm{id}$, then $\check{\mathcal{M}}(G, b, \mu) = \check{\mathcal{M}}_{\mathbb{X}_0}$;

¹⁷Recall that P_μ is the automorphism group scheme of $(\Lambda, \{s_i\}, \mathrm{Fil}^\bullet \Lambda)$.

(3) *there is a canonical bijection $\check{\mathcal{M}}(G, b, \mu)(k) \simeq X_\mu(b)(k)$ compatible with the embeddings $\check{\mathcal{M}}(G, b, \mu) \rightarrow \check{\mathcal{M}}(\mathrm{GL}_h, \rho(b), \omega_n)$ and $X_\mu(b) \rightarrow X_{\rho\mu}(\rho(b))$.*

We write $\overline{\mathcal{M}}_\mu(b)$ for the special fiber of $\check{\mathcal{M}}(G, b, \mu)$.

PROPOSITION 3.11. *Fix $(\mathbb{X}_0, \varepsilon)$ as above. There is a canonical isomorphism $X_\mu(b) \simeq \overline{\mathcal{M}}_\mu(b)^{p^{-\infty}}$. In particular, $\dim \overline{\mathcal{M}}_\mu(b)_{\mathrm{red}} = \dim X_\mu(b)$.*

Note that this proposition in particular describes the values of $\check{\mathcal{M}}(G, b, \mu)$ on a perfect ring R (which is not obvious from Definition 3.8).

Proof. We first prove the proposition for $(G, b, \mu) = (\mathrm{GL}_h, b, \omega_n)$. The key input is a theorem of Gabber (see also [Lau13, §6]) on p -divisible groups over perfect rings. We write $\overline{\mathcal{M}}^{p^{-\infty}}$ instead of $\overline{\mathcal{M}}_\mu(b)^{p^{-\infty}}$ for simplicity.

We first construct $X_\mu(b) \rightarrow \overline{\mathcal{M}}^{p^{-\infty}}$ as follows. Let R be a perfect $\overline{\mathbb{F}}_p$ -algebra. We write $\sigma : R \rightarrow R$ for the Frobenius automorphism. Let $(\mathcal{E}, \beta) \in X_\mu(b)(R)$. We obtain a crystal $\mathbb{D} := \mathcal{E} \times^{G, \rho} \Lambda$ over R (= a locally free sheaf on $W(R)$), with $F = \beta^{-1}(b\sigma)\beta$. That is, the following diagram is commutative:

$$(3.2.1) \quad \begin{array}{ccc} \mathbb{D}[\frac{1}{p}] & \xrightarrow{F} & \mathbb{D}[\frac{1}{p}] \\ \beta \downarrow & & \beta \downarrow \\ W(R) \otimes_{\mathbb{Z}_p} \Lambda[\frac{1}{p}] & \xrightarrow{b\sigma} & W(R) \otimes_{\mathbb{Z}_p} \Lambda[\frac{1}{p}]. \end{array}$$

Note that $\omega_n : \mathbb{G}_m \rightarrow \mathrm{GL}(\Lambda)$ is the n -th fundamental coweight of $\mathrm{GL}(\Lambda)$, and therefore the condition in (3.1.2) together with Lemma 1.5 implies that $F : \sigma^*\mathbb{D} \rightarrow \mathbb{D}$ is regular and the quotient is a locally free module of rank n over $W(R)/p = R$. Applying Lemma 1.5 again to the quasi-isogeny (of crystals) $V = pF^{-1} : \mathbb{D} \rightarrow \sigma^*\mathbb{D}$, we see that V is regular. Therefore, there is a σ^{-1} -linear map $V : \mathbb{D} \rightarrow \mathbb{D}$ such that $FV = VF = p$. By Gabber’s theorem (see also [Lau13, §6]), there is a p -divisible group \mathbb{X} on $\mathrm{Spec} R$ together with a quasi-isogeny $\iota : \mathbb{X} \rightarrow (\mathbb{X}_0)_R$ such that $\mathbb{D} = \mathbb{D}(\mathbb{X})$ and the induced map $\mathbb{D}(\iota) : \mathbb{D}[\frac{1}{p}] \rightarrow W(R)[\frac{1}{p}] \otimes \Lambda$ is β .

Conversely, we construct $\overline{\mathcal{M}}^{p^{-\infty}} \rightarrow X_\mu(b)$ as follows. Let R be a perfect $\overline{\mathbb{F}}_p$ -algebra. Let (\mathbb{X}, ι) be an object in $\overline{\mathcal{M}}(R)$. Then we have the $\mathrm{GL}(\Lambda)$ -torsor

$$\mathcal{E} = \mathrm{Isom}(\mathbb{D}(\mathbb{X}), \Lambda \otimes_{\mathbb{Z}_p} W(R)).$$

The quasi-isogeny ι defines an isomorphism

$$\mathbb{D}(\mathbb{X}) \left[\frac{1}{p} \right] \simeq \Lambda \otimes_{\mathbb{Z}_p} W(R) \left[\frac{1}{p} \right],$$

and therefore defines a quasi-isogeny $\beta : \mathcal{E} \dashrightarrow \mathcal{E}_0$. The map $(\mathbb{X}, \iota) \mapsto (\mathcal{E}, \beta)$ defines a map $\overline{\mathcal{M}}^{p^{-\infty}} \rightarrow \mathrm{Gr}_{\mathrm{GL}_h}$. It is clear that $\mathrm{Inv}_x(F) \leq \omega_n$ for every $x \in \mathrm{Spec} R$, and therefore $(\mathcal{E}, \beta) \in X_\mu(b)$. We thus define a map $\overline{\mathcal{M}}^{p^{-\infty}} \rightarrow X_\mu(b)$.

Now for general (G, b, μ) , we have $\overline{\mathcal{M}}_\mu(b)^{p^{-\infty}} \subset \overline{\mathcal{M}}_{\rho\mu}(\rho(b))^{p^{-\infty}}$ and $X_\mu(b) \subset X_{\rho\mu}(\rho(b))$. To prove that $X_\mu(b) \simeq \overline{\mathcal{M}}_\mu(b)^{p^{-\infty}}$, it is enough to show that the above isomorphism sends $X_\mu(b)(k) \subset X_{\rho\mu}(\rho(b))(k)$ to $\overline{\mathcal{M}}_\mu(b)^{p^{-\infty}}(k) \subset \overline{\mathcal{M}}_{\rho\mu}(\rho(b))^{p^{-\infty}}(k)$. But this follows from Theorem 3.10(3). \square

COROLLARY 3.12. *Rapoport’s conjecture of the dimension formula holds for the reduced schemes of the Rapoport-Zink spaces.*

Appendix A. Generalities on perfect schemes

This section can be regarded as a user’s guide to the algebraic geometry of perfect schemes and perfect algebraic spaces, which is the setting we work with in the paper. We include some discussions more general than needed in the paper. The main result is Theorem A.29, which explains the construction of the quotients in this setting.

A.1. Perfect schemes and perfect algebraic spaces.

A.1.1. We fix a field k . Let Aff_k denote the category of affine k -schemes, i.e., the category opposite to the category $k\text{-alg}$ of k -algebras. Following [LMB00], [BL94], we call a sheaf on Aff_k with respect to the fpqc topology a k -space. Explicitly, a space \mathcal{F} is a covariant functor $k\text{-alg} \rightarrow \mathrm{Set}$ that respects finite products, and such that if $R \rightarrow R'$ is faithfully flat, then the sequence

$$(A.1.1) \quad \mathcal{F}(R) \rightarrow \mathcal{F}(R') \rightrightarrows \mathcal{F}(R' \otimes_R R')$$

is an equalizer. Morphisms between two spaces are natural transformations of functors. The category of k -spaces is denoted by Sp_k . It contains the category Sch_k of k -schemes as a full subcategory. Recall that a map $f : \mathcal{F} \rightarrow \mathcal{G}$ in Sp_k is called schematic if for every scheme T , the fiber product $\mathcal{F} \times_{\mathcal{G}} T$ is a scheme.

In this paper, we need to consider a subcategory of Sp_k larger than Sch_k . Recall that an algebraic space is an étale sheaf X on Aff_k such that

- (i) $X \rightarrow X \times X$ is schematic;
- (ii) there exists an étale surjective map $U \rightarrow X$, where U is a scheme.

We denote by AlgSp_k the category of algebraic spaces. We have full embeddings $\mathrm{Sch}_k \subset \mathrm{AlgSp}_k \subset \mathrm{Sp}_k$, where the second inclusion follows from a theorem of Gabber which says algebraic spaces are fpqc sheaves (see [St, Tag03W8]).

Remark A.1. In literature such as [Knu71], [LMB00], it sometimes requires that X is quasi-separated; i.e., the diagonal $X \rightarrow X \times X$ is quasi-compact. We prefer not to make this additional assumption.

A map $f : \mathcal{F} \rightarrow \mathcal{G}$ of k -spaces is called representable if for every affine scheme T , $\mathcal{F} \times_{\mathcal{G}} T$ is represented by an algebraic space. It is called *fpqc* if in addition $\mathcal{F} \times_{\mathcal{G}} T$ is faithfully flat over T , and there is a quasi-compact open subset U of $\mathcal{F} \times_{\mathcal{G}} T$ that maps surjectively to T . Recall that fpqc maps are effective epimorphisms in Sp_k . That is, if $U \rightarrow X$ is an fpqc map of spaces, then for every space \mathcal{F} , the following diagram is an equalizer:

$$(A.1.2) \quad \mathrm{Hom}_{\mathrm{Sp}_k}(X, \mathcal{F}) \rightarrow \mathrm{Hom}_{\mathrm{Sp}_k}(U, \mathcal{F}) \rightrightarrows \mathrm{Hom}_{\mathrm{Sp}_k}(U \times_X U, \mathcal{F}).$$

In particular, any fpqc sheaf on Aff_k extends uniquely to an fpqc sheaf on Sch_k (although we do not use the latter in this paper).

A.1.2. From now on we assume that k is a perfect field of characteristic $p > 0$. For a k -algebra R , let $\sigma : R \rightarrow R$, $r \mapsto r^p$ denote the Frobenius map. Recall that R is called *perfect* if σ is an isomorphism. The forgetful functor from the category of perfect k -algebras to the category of all k -algebras admits a left adjoint

$$R \mapsto R^{p^{-\infty}} = \varinjlim_{\sigma} R.$$

Sometime, $R^{p^{-\infty}}$ is called the perfection (or the perfect closure) of R .

These facts admit the following globalization. A k -scheme (resp. algebraic space) X is called *perfect* if its Frobenius endomorphism $\sigma_X : X \rightarrow X$ is an isomorphism. We write σ for σ_X if no confusion will likely arise. The category of perfect schemes (resp. perfect algebraic spaces) over k is denoted by $\mathrm{Sch}_k^{\mathrm{pf}}$ (resp. $\mathrm{AlgSp}_k^{\mathrm{pf}}$). Then the embedding $\mathrm{AlgSp}_k^{\mathrm{pf}} \rightarrow \mathrm{AlgSp}_k$ admits a right adjoint. To see this, first note that Frobenius endomorphisms commute with étale localizations.

LEMMA A.2. *For any étale morphism of algebraic spaces $X \rightarrow Y$, the relative Frobenius morphism $X \rightarrow X \times_{Y, \sigma_Y} Y$ induced by σ_X is an isomorphism.*

Proof. We first assume that X is a scheme. Then $X \rightarrow X \times_{Y, \sigma_Y} Y$ is a schematic radical étale surjective map and therefore is an isomorphism by [Gro67, Th. 17.9.1]. For general X , choose an étale cover $U \rightarrow X$ by a scheme U . Then we have $U \rightarrow U \times_{X, \sigma_X} X \rightarrow U \times_{Y, \sigma_Y} Y$, with the first map and the composition map being isomorphisms. Therefore, the second map is an isomorphism as well. Note that $U \times_{X, \sigma_X} X \rightarrow U \times_{Y, \sigma_Y} Y$ is nothing but the base change of $X \rightarrow X \times_{Y, \sigma_Y} Y$ along the étale cover $U \times_{Y, \sigma_Y} Y \rightarrow Y$. Therefore, $X \rightarrow X \times_{Y, \sigma_Y} Y$ is also an isomorphism. \square

COROLLARY A.3. *The embedding $\text{AlgSp}_k^{\text{pf}} \rightarrow \text{AlgSp}_k$ admits a right adjoint functor, given by $X \rightarrow X^{p^{-\infty}} = \varprojlim_{\sigma} X$.*

We call $X^{p^{-\infty}}$ the *perfection* of X .

Proof. Applying Lemma A.2 to the étale cover $U \rightarrow X$, we see that $\sigma : X \rightarrow X$ is an affine morphism. Then the diagonal of $X^{p^{-\infty}} = \varprojlim_{\sigma} X$ is representable, and $U^{p^{-\infty}} \simeq U \times_X X^{p^{-\infty}} \rightarrow X^{p^{-\infty}}$ is an étale cover. Therefore, $X^{p^{-\infty}}$ is a perfect algebraic space.

It remains to show that the tautological map $\varepsilon : X^{p^{-\infty}} \rightarrow X$ induces an isomorphism

$$(A.1.3) \quad \text{Hom}(Y, X^{p^{-\infty}}) = \text{Hom}(Y, X)$$

for every perfect k -algebraic space Y . But by Lemma A.2 and (A.1.2), we reduce to the known case where X, Y are affine. \square

Remark A.4. Recall that σ_X is a universal homeomorphism if X is a scheme and therefore is a universal homeomorphism if X is an algebraic space by Lemma A.2. Therefore, $\varepsilon : X^{p^{-\infty}} \rightarrow X$ is a universal homeomorphism. It also follows that X is a scheme if and only if $X^{p^{-\infty}}$ is a scheme. See Lemma A.7 below.

The following statement is crucial for later applications. For an algebraic space X , we denote by $X_{\text{ét}}$ its small étale site with objects being algebraic spaces étale over X .

PROPOSITION A.5. *Let X be an algebraic space over k , and let $X^{p^{-\infty}}$ denote its perfection. Then the functor*

$$(U \rightarrow X) \mapsto (U^{p^{-\infty}} \simeq U \times_X X^{p^{-\infty}} \rightarrow X^{p^{-\infty}})$$

induces an equivalence of étale sites $X_{\text{ét}} \simeq X_{\text{ét}}^{p^{-\infty}}$ and therefore the étale topoi $\varepsilon^ : \widetilde{X}_{\text{ét}} \simeq \widetilde{X}_{\text{ét}}^{p^{-\infty}} : \varepsilon_*$.*

Proof. First, assume that X is a scheme. Then $X^{p^{-\infty}} \rightarrow X$ is a universal homeomorphism. Therefore, $(U \rightarrow X) \mapsto (U^{p^{-\infty}} \simeq U \times_X X^{p^{-\infty}} \rightarrow X^{p^{-\infty}})$ induces an equivalence of subcategories of scheme objects in $X_{\text{ét}}$ and $X_{\text{ét}}^{p^{-\infty}}$ (cf. [St, Tag04DZ]). Then an argument similar to [CLO12, Prop. A.1.3] shows that it induces a full equivalence $X_{\text{ét}} \simeq X_{\text{ét}}^{p^{-\infty}}$. Again, by a similar argument as [CLO12, Prop. A.1.3], the case when X is an algebraic space also follows. \square

Remark A.6. It follows from Remark A.4 that the equivalence preserves the subcategories of scheme objects in $X_{\text{ét}}$ and $X_{\text{ét}}^{p^{-\infty}}$.

We list a few properties of morphisms that are preserved after passing to the perfection.

LEMMA A.7. *Let $f : X \rightarrow Y$ be a morphism of algebraic spaces over k , and let $f^{p^{-\infty}} : X^{p^{-\infty}} \rightarrow Y^{p^{-\infty}}$ denote its perfection. The following properties hold for f if and only if the same hold for $f^{p^{-\infty}}$:*

- (1) *quasi-compact;*
- (2) *quasi-separated;*
- (3) *(universally) homeomorphic;*
- (4) *(universally) closed;*
- (5) *separated;*
- (6) *affine;*
- (7) *integral.*

In addition, if f is either:

- (8) *étale; or*
- (9) *(faithfully) flat;*
- (10) *fpqc, so is $f^{p^{-\infty}}$.*

Proof. (1)–(4) are clear since $\varepsilon : X^{p^{-\infty}} \rightarrow X$ is a universal homeomorphism.

For (5), first note that if $X^{p^{-\infty}} \rightarrow X^{p^{-\infty}} \times_{Y^{p^{-\infty}}} X^{p^{-\infty}}$ is a closed embedding, it is universally closed, and therefore $\Delta_{X/Y} : X \rightarrow X \times_Y X$ is also universally closed. But $\Delta_{X/Y}$ is always a separated, locally of finite type monomorphism. Therefore, it is a closed embedding. The inverse direction is clear.

For (6), we can assume that Y is an affine scheme by Lemma A.2. Then if X is affine, so is $X^{p^{-\infty}}$. Conversely, if $X^{p^{-\infty}}$ is affine, then it is quasi-compact and separated, and so is X . As $X^{p^{-\infty}} = \varprojlim_{\sigma} X$, X is affine by [St, Tag07SE, Lemma 5.8].

For (7), first note that if $f^{p^{-\infty}}$ is integral, it is affine, and therefore f is affine by (6). We reduce to show that if $A^{p^{-\infty}} \rightarrow B^{p^{-\infty}}$ is integral, so is $A \rightarrow B$. Given $b \in B$, there is a monic polynomial $g(x) \in A^{p^{-\infty}}[x]$ such that $g(b) = 0$. Then there is some n large enough such that all coefficients of $h(x) := g(x)^{p^n}$ are in A . Since $h(b) = 0$, b is integral over A . The inverse direction is clear.

(8) follows from Lemma A.2.

For (9), we may assume that $f : X \rightarrow Y$ is a morphism between affine schemes and therefore is given by a ring homomorphism $f : R \rightarrow R'$. In addition, we may assume that R is perfect. As $(R')^{p^{-\infty}} = \varprojlim_{\sigma} R'$, it is enough to show that the composition $R \rightarrow R' \xrightarrow{\sigma^n} R'$ is flat. But this map is the same as $R \xrightarrow{\sigma^n} R \rightarrow R'$ and therefore is flat. Finally, (10) follows from (1) and (9). \square

We will also consider the perfection of certain pro-algebraic spaces. Let $\{X_i\}$ be a projective system of algebraic spaces, with the transition maps $X_{i+1} \rightarrow X_i$ being affine. Then the pro-algebraic space $X = \varprojlim X_i$ is also an algebraic space, and it is easy to show that

$$(A.1.4) \quad X^{p^{-\infty}} \simeq \varprojlim X_i^{p^{-\infty}};$$

i.e., “the perfection commute with inverse limits.”

A.1.3. Let H be an affine group scheme over k , regarded as a group object in Sp_k . An H -torsor over a space X is a space E with a free H -action and an H -equivariant fpqc map $\pi : E \rightarrow X$ (where X is endowed with the trivial H -action), such that the natural map $E \times H \rightarrow E \times_X E$ is an isomorphism. If X is an algebraic space over k , then E is represented by an algebraic space, affine over X . In addition, we have the following lemma.

LEMMA A.8. *If X and H are perfect, then E is perfect.*

Proof. Note that $E^{p^{-\infty}} \rightarrow X^{p^{-\infty}}$ is also an fpqc map by Lemma A.7 and therefore is an $H^{p^{-\infty}}$ -torsor over $X^{p^{-\infty}}$. In addition, $E^{p^{-\infty}} \rightarrow E \times_{X,\varepsilon} X^{p^{-\infty}}$ is a morphism of $H^{p^{-\infty}}$ -torsors, where $H^{p^{-\infty}}$ acts on E through the morphism $\varepsilon : H^{p^{-\infty}} \rightarrow H$. Therefore, $E^{p^{-\infty}} \simeq E \times_X X^{p^{-\infty}} \simeq E$ is an isomorphism. \square

Now, let H' be a smooth affine group scheme over k , and let $H = H'^{p^{-\infty}}$ denote its perfection. Then H is an affine group scheme.

LEMMA A.9. *Let X be a perfect algebraic space. Then the functor $E' \rightarrow E'^{p^{-\infty}}$ is an equivalence of categories between the groupoid of H' -torsors on X and the groupoid of H -torsors on X . The quasi-inverse functor is given by push-out of an H -torsor along $\varepsilon : H \rightarrow H'$, denoted by $E \mapsto E \times^{H,\varepsilon} H'$.*

Proof. Given an H -torsor E , the natural map $E \rightarrow E \times^{H,\varepsilon} H'$ gives a morphism $E \rightarrow (E \times^{H,\varepsilon} H')^{p^{-\infty}}$ of H -torsors and therefore is an isomorphism. Conversely, let E' be an H' -torsor on X , and let $E = E'^{p^{-\infty}}$ be the corresponding H -torsor. We want to show that $E \times^{H,\varepsilon} H' \simeq E'$. As H' is smooth, we can trivialize E' by an étale cover $U \rightarrow X$, and therefore E' can be represented by a cocycle $c' : U \times_X U \rightarrow H'$. As $U \times_X U$ is perfect by Lemma A.2, the cocycle c' gives a cocycle $c : U \times_X U \rightarrow H$ by (A.1.3), which is nothing but the cocycle representing E . Then $E \times^{H,\varepsilon} H'$ is represented by the cocycle $U \times_X U \xrightarrow{c} H \xrightarrow{\varepsilon} H'$, which is exactly c' . \square

For a space X with an action by an affine group scheme H , we denote by $[X/H]$ the quotient stack (in fpqc topology) whose R -points are the groupoid of pairs (E, ϕ) , where E is an H -torsor on $\mathrm{Spec} R$, and $\phi : E \rightarrow Y$ is an H -equivariant morphism. Note that if the action is free, then $[X/H]$ is a k -space

and the natural morphism $X \rightarrow [X/H]$ is an H -torsor. In this case, we write $[X/H]$ by X/H for simplicity.

We also recall the construction of the twisted product. Let H be an affine group scheme and $E \rightarrow X$ an H -torsor, and let T be a space with an H -action. Then one can form the twisted product

$$(A.1.5) \quad X \tilde{\times} T := E \times^H T = E \times T/H,$$

which is a space over k . Now assume that H is (the perfection of) an affine group scheme of finite type over k . We claim that if X, T are (perfect) algebraic spaces, so is $X \tilde{\times} T$. Indeed, we can find an fppf cover $U \rightarrow X$ that trivializes E .¹⁸ Then $U \times T$ is an fppf cover of $X \tilde{\times} T$. Therefore, $X \tilde{\times} T$ is an algebraic space by [St, Tag04S5].

A.1.4. In fact in the paper we will only consider presheaves on the category of perfect k -algebras.

Definition A.10. Let Aff_k^{pf} be the opposite category of perfect k -algebras. A perfect space is a sheaf on Aff_k^{pf} with respect to the fpqc topology. The category of perfect k -spaces is denoted by Sp_k^{pf} .

There is a natural functor $\text{Sp}_k \rightarrow \text{Sp}_k^{\text{pf}}$ by restricting a sheaf \mathcal{F} on Aff_k to a sheaf on Aff_k^{pf} . We denote the induced perfect space by \mathcal{F}^{pf} . Note that if X is an algebraic space, then $(X^{p^{-\infty}})^{\text{pf}} = X^{\text{pf}}$. More generally, we have the following lemma.

LEMMA A.11. *Let $V \rightrightarrows U$ be a flat groupoid of algebraic spaces over k , and let $V^{p^{-\infty}} \rightrightarrows U^{p^{-\infty}}$ denote its perfection. Let $[U/V]$ and $[U^{p^{-\infty}}/V^{p^{-\infty}}]$ denote the quotient stack (in fpqc topology). Then $[U^{p^{-\infty}}/V^{p^{-\infty}}]^{\text{pf}} \simeq [U/V]^{\text{pf}}$. That is, $[U^{p^{-\infty}}/V^{p^{-\infty}}](R) \simeq [U/V](R)$ for every perfect k -algebra R .*

Proof. Clearly there is a morphism $[U^{p^{-\infty}}/V^{p^{-\infty}}]^{\text{pf}} \rightarrow [U/V]^{\text{pf}}$. Conversely, let x be an R -point of $[U/V]$ where R is perfect. Then there is a faithfully flat map $R \rightarrow R'$ and a lifting $\tilde{x} : \text{Spec } R' \rightarrow U$ of x . Passing to the perfection gives $\text{Spec } R'^{p^{-\infty}} \rightarrow U^{p^{-\infty}}$. Since $R \rightarrow R'^{p^{-\infty}}$ is faithfully flat by Lemma A.7, it gives an R -point of $[U^{p^{-\infty}}/V^{p^{-\infty}}]$. \square

The functor $\text{Sp}_k \rightarrow \text{Sp}_k^{\text{pf}}$ is far from being faithful. However, the following statement is true.

LEMMA A.12. *The the composition*

$$\text{AlgSp}_k^{\text{pf}} \subset \text{AlgSp}_k \subset \text{Sp}_k \rightarrow \text{Sp}_k^{\text{pf}}$$

is a full embedding.

¹⁸If H is of finite type, this is clear. Otherwise, use Lemma A.27.

Proof. Let X and Y be two perfect algebraic spaces. We need to show that

$$\mathrm{Hom}_{\mathrm{AlgSp}_k^{\mathrm{pf}}}(X, Y) = \mathrm{Hom}_{\mathrm{Sp}_k^{\mathrm{pf}}}(X^{\mathrm{pf}}, Y^{\mathrm{pf}}).$$

Let $\{U_i \rightarrow X\}$ be a family of étale covers of X by affine schemes, and let $\{V_{ijh} \rightarrow U_i \times_X U_j\}$ be a family of étale covers of $U_i \times_X U_j$ by affine schemes. By Lemma A.2, all U_i and V_{ijh} are perfect schemes. Therefore, by definition $\mathrm{Hom}_{\mathrm{AlgSp}_k^{\mathrm{pf}}}(U_i, Y) = \mathrm{Hom}_{\mathrm{Sp}_k^{\mathrm{pf}}}(U_i^{\mathrm{pf}}, Y^{\mathrm{pf}})$, etc.

Note that (A.1.2) implies that the following sequence is an equalizer:

$$\mathrm{Hom}_{\mathrm{AlgSp}_k^{\mathrm{pf}}}(X, Y) \rightarrow \prod_i \mathrm{Hom}_{\mathrm{AlgSp}_k^{\mathrm{pf}}}(U_i, Y) \rightarrow \prod_{ijh} \mathrm{Hom}_{\mathrm{AlgSp}_k^{\mathrm{pf}}}(V_{ijh}, Y).$$

Likewise, the sequence

$$\mathrm{Hom}_{\mathrm{Sp}_k^{\mathrm{pf}}}(X^{\mathrm{pf}}, Y^{\mathrm{pf}}) \rightarrow \prod_i \mathrm{Hom}_{\mathrm{Sp}_k^{\mathrm{pf}}}(U_i^{\mathrm{pf}}, Y^{\mathrm{pf}}) \rightarrow \prod_{ijh} \mathrm{Hom}_{\mathrm{Sp}_k^{\mathrm{pf}}}(V_{ijh}^{\mathrm{pf}}, Y^{\mathrm{pf}})$$

is also an equalizer. (In fact, it is enough to use the injectivity of the first map.) The lemma follows by comparing these two sequences. \square

Therefore, given a presheaf \mathcal{F} on $\mathrm{Aff}_k^{\mathrm{pf}}$, it makes sense to ask whether it is represented by a perfect algebraic space, and given a map $f : \mathcal{F} \rightarrow \mathcal{G}$ of presheaves, it makes sense to ask whether it is representable by perfect algebraic spaces. If a property (P) of morphisms between algebraic spaces is stable under base change and is étale local on the source and target, then it makes sense to ask whether a representable morphism $f : \mathcal{F} \rightarrow \mathcal{G}$ of perfect spaces has Property (P). For example, we can define open/closed immersions, étale morphisms, fpqc maps in $\mathrm{Sp}_k^{\mathrm{pf}}$, etc.

We can also define the notion of torsors in $\mathrm{Sp}_k^{\mathrm{pf}}$, just as Sp_k . Let H be a perfect affine group scheme. It gives an object H^{pf} in $\mathrm{Sp}_k^{\mathrm{pf}}$. If X is a perfect space with an action of H^{pf} , then we can define a stack $[X/H^{\mathrm{pf}}]$ on $\mathrm{Aff}_k^{\mathrm{pf}}$ as before. If the action is free, then $[X/H^{\mathrm{pf}}]$ is also a perfect space and the natural map $X \rightarrow [X/H^{\mathrm{pf}}]$ is an H^{pf} -torsor. As before, in this case we write $[X/H^{\mathrm{pf}}]$ by X/H^{pf} for simplicity. Note that if X is a perfect algebraic space, which gives X^{pf} in $\mathrm{Sp}_k^{\mathrm{pf}}$, then by Lemma A.12 giving an action of H^{pf} on X^{pf} is the same as giving an action of H on X , and if the action is free, then $X^{\mathrm{pf}}/H^{\mathrm{pf}} = (X/H)^{\mathrm{pf}}$.

We define an *ind-perfect algebraic space* as a perfect k -space that can be represented as an inductive limit $\{X_i\}$ of perfect algebraic spaces, such that every transition map $X_i \rightarrow X_{i+1}$ is a closed embedding.

In the sequel and the main body of the paper, the image of a perfect algebraic space X in $\mathrm{Sp}_k^{\mathrm{pf}}$ is still denoted by X , as opposed to X^{pf} as above. However, for a general space \mathcal{F} , its image in $\mathrm{Sp}_k^{\mathrm{pf}}$ will be denoted by $\mathcal{F}^{\mathrm{pf}}$.

A.2. *Perfect algebraic spaces perfectly of finite presentation.*

A.2.1. Perfect schemes/algebraic spaces of positive dimension are never of finite type over k . But as we shall see below, the “infinity” here is really mild.

Definition A.13. A perfect k -algebraic space X is said to be locally perfectly of finite type¹⁹ if there exist an étale affine cover $\{U_i\}$ of X such that each U_i is the perfection of an affine scheme of finite type over k . A perfect k -algebraic space X is said perfectly of finite type if it is locally perfectly of finite type and quasi-compact. A perfect k -algebraic space is said to be perfectly of finite presentation (*pfp* for short) if it is perfectly of finite type and quasi-separated.

Remark A.14. In [Ser60], a separated and perfectly of finite type perfect k -scheme is called a perfect variety.

Clearly, if there exists an algebraic space X' of finite presentation over k such that $X = X'^{p^\infty}$, then X is perfectly of finite presentation. We call such X' a “model” or a “deperfection” of X . We will show a model of a pfp perfect algebraic space always exists. In fact, we will prove a slightly stronger result. For this purpose, we need some preparations.

For an algebraic space S , let $|S|$ denote its underlying topological space. Recall that for a quasi-compact and quasi-separated algebraic space S , there is an open dense subspace $U \subset S$ that is a scheme (e.g., [St, Tag03JG]). The generic points of $|S|$ are in $|U|$ and are the generic points of U . So given a generic point η of S , its residue field $k(\eta)$ makes sense. Recall that a reduced scheme X of finite type over k is called weakly normal if every finite birational universal homeomorphism $f : Y \rightarrow X$ is an isomorphism. By [Man80], weak normality is local under the étale topology (even under the fppf topology). Therefore, this notion makes sense for algebraic spaces of finite presentation over k . We have the following result, generalizing [Ser60, §1.4, Prop. 9].

PROPOSITION A.15. *Let X be a pfp perfect algebraic space over k , with $\{\eta_1, \dots, \eta_n\}$ the set of its generic points. For every i , let $K_i \subset k(\eta_i)$ be a subfield, which is finitely generated over k and whose perfection is $k(\eta_i)$. Then there exists a unique weakly normal algebraic space X' , of finite presentation over k , such that $X = X'^{p^\infty}$ and the residue fields of the generic points of X' are these K_i .*

Proof. We first assume that X is a scheme. We define a sheaf of rings on $|X|$ by

$$(A.2.1) \quad \mathcal{O}_{X'} = \{f \in \mathcal{O}_X \mid f(\eta_i) \in K_i\}.$$

¹⁹The terminology is suggested by B. Conrad.

It is easy to check that the ringed space $(|X|, \mathcal{O}_{X'})$ is a scheme, of finite type over k , and $X = X'^{p^{-\infty}}$. In addition, X' is weakly normal. In fact, let $U' = \text{Spec } A$ be an affine open subscheme of X' . Then the ring A is p -closed in the sense that for every a in its quotient ring, if $a^p \in A$, then $a \in A$. By the remark after Proposition 1 in [Ito83], this condition is equivalent to weak normality of A , as proved by Yanagihara.

Note that X' is just the push out of the diagram $\bigsqcup \text{Spec } K_i \leftarrow \bigsqcup \eta_i \rightarrow X$ in the category of locally ringed spaces. In particular, for any scheme Y , the natural map

$$(A.2.2) \quad \text{Hom}(X', Y) \rightarrow \text{Hom}(X, Y) \times_{\text{Hom}(\bigsqcup \eta_i, Y)} \text{Hom}(\bigsqcup \text{Spec } K_i, Y)$$

is an isomorphism.

Now for an algebraic space X , we can choose a presentation $V \rightrightarrows U \rightarrow X$ of X where U, V are schemes. The collection $\{K_i \subset k(\eta_i)\}$ determine a collection of subfields in the residue fields of the generic points of $|V|$ and $|U|$. Then the above construction gives U' and V' . Let $\text{pr}_i : V \rightarrow U$ be one of the two projections, which descends to an étale map $V'_i \rightarrow U'$ for some scheme V'_i . As U' is weakly normal, so is V'_i . Note that the quotient ring of V'_i and V' are the same. By (A.2.2), there is a canonical map $V' \rightarrow V'_i$, which is finite birational, and bijective, and therefore is an isomorphism. In other words, the étale equivalence relation $V \rightrightarrows U$ descends to an étale equivalence relation $V' \rightrightarrows U'$. Then $X' = U'/V'$ is the sought-after algebraic space. \square

COROLLARY A.16. *Let $f : X \rightarrow Y$ be a separated universal homeomorphism between two pfp perfect algebraic spaces over k . Then f is an isomorphism.*

Proof. Let η be a generic point of X . Since f is a universal homeomorphism, $k(\eta)$ is purely inseparable over $k(f(\eta))$ and therefore is an isomorphism since both fields are perfect. Now, we can choose for each generic point η_i a finitely generated subfield $K_i \subset k(\eta_i)$ as above. Let X' and Y' be the corresponding weakly normal models. It follows from the construction that f descends to a finite birational universal homeomorphism $f' : X' \rightarrow Y'$ and therefore is an isomorphism. The corollary then follows by passing to the perfection. \square

The following statement generalizes [Ser60, §1.4, Prop. 8].

PROPOSITION A.17. *Let $f : X \rightarrow Y$ be a morphism between pfp perfect algebraic spaces over k . Then there exists a morphism $f' : X' \rightarrow Y'$ between algebraic spaces of finite presentation over k such that $f = f'^{p^{-\infty}}$.*

Proof. Let X', Y' be models of X, Y . Then there is a canonical map $\varepsilon : Y \rightarrow Y'$. Recall that $\sigma : X' \rightarrow X$ is affine by Lemma A.2. Then by a

criterion of locally of finite presentation morphisms ([Gro66, §8.14], generalized in [CLO12, Prop. A.3.1]; see also [St, Tag049I]), the map εf factors as $X \rightarrow X^{(m)} \rightarrow Y'$, where $X^{(m)} = X'$ with the k -structure given by $X' \xrightarrow{\sigma^m} X' \rightarrow \text{Spec } k$. Rename $X^{(m)}$ as X' , and we are done. \square

Definition A.18. Let $f : X \rightarrow Y$ be a morphism between two pfp perfect algebraic spaces over k . We say f is perfectly proper if it is separated and is universally closed. We say X is perfectly proper if $X \rightarrow \text{Spec } k$ is perfectly proper.

LEMMA A.19. *For a morphism $f : X \rightarrow Y$ between two pfp perfect algebraic spaces over k , f is perfectly proper if and only if for every $f' : X' \rightarrow Y'$ is as in Proposition A.17, f' is proper.*

Proof. By Lemma A.7, f is separated and universally closed if and only if so is f' . \square

PROPOSITION A.20. *Let $f : X \rightarrow Y$ be a morphism between two pfp perfect algebraic spaces. Then f is perfectly proper if and only if the valuative criterion holds for every perfect valuation ring R over k .*

We note that the perfection of a valuation ring is a valuation ring.

Proof. In fact, if f is perfectly proper, then it is the perfection of a proper morphism $f' : X' \rightarrow Y'$. Therefore, the map $\text{Spec } R \rightarrow Y \rightarrow Y'$ lifts to $\text{Spec } R \rightarrow X'$. As R is perfect, it factors through $\text{Spec } R \rightarrow X$ by (A.1.3). To prove the converse, note that every perfect local ring A in a perfect field K is dominated by a perfect valuation ring. In addition, to check that $f : X \rightarrow Y$ is universally closed, it is enough to check that for every perfect ring R , the base change $X_R \rightarrow Y_R$ is closed. Then the usual arguments of valuative criterion for properness go through with obvious modifications. \square

The following lemma is not used in the paper.

LEMMA A.21. *Let $f : X \rightarrow Y$ be a perfectly proper morphism between two pfp perfect algebraic spaces, with geometrically connected fibers. Then the natural map $\mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$ is an isomorphism.*

Proof. Let $f' : X' \rightarrow Y'$ be a model of f , and let Z' be the relative spectrum of $f'_*\mathcal{O}_{X'}$ over Y' . Since f' has geometrically connected fibers, $Z' \rightarrow Y'$ is a universal homeomorphism. By Corollary A.16, $Z'^{p^{-\infty}} \rightarrow Y'$ is an isomorphism. But since $Z'^{p^{-\infty}}$ is the relative spectrum of $f_*\mathcal{O}_X$, the lemma follows. \square

LEMMA A.22. *Let \mathcal{E} be a locally free sheaf of finite rank on a pfp perfect algebraic X over k . Then there exists a model (X', \mathcal{E}') of (X, \mathcal{E}) , i.e., algebraic*

space X' , of finite presentation over k , and a locally free sheaf \mathcal{E}' of finite rank such that $(X, \mathcal{E}) = (X'^{p^{-\infty}}, \varepsilon^* \mathcal{E}')$, where $\varepsilon : X \rightarrow X'$ is the tautological map.

Proof. Let $\varepsilon : X \rightarrow X'$ be a model. As X is quasi-compact, we can find a finite étale cover $\{U_i\}$ of X such that $\mathcal{E}|_{U_i} \simeq \mathcal{O}_{U_i}^r$. Then we obtain a Čech cocycle $f_{ij} : U_{ij} := U_i \times_X U_j \rightarrow \mathrm{GL}_r$. By Proposition A.5, the étale cover $\{U_i\}$ descend to an étale cover $\{U'_i\}$ of X' . Let $U'_{ij} = U'_i \times_{X'} U'_j$, and $U'_{ijk} = U'_i \times_{X'} U'_j \times_{X'} U'_k$. The map f_{ij} factors as $f'_{ij} : U'_{ij}{}^{(m)} \rightarrow \mathrm{GL}_r$ for some m large enough, where as in Proposition A.17, $U'_{ij}{}^{(m)} = U'_{ij}$ with the k -structure given by $U'_{ij} \xrightarrow{\sigma^m} U'_{ij} \rightarrow \mathrm{Spec} k$. Let $h_{ijk} = f'_{ij} f'_{jk} f'_{ki} : U'_{ijk}{}^{(m)} \rightarrow \mathrm{GL}_r$. Then $\varepsilon^* h_{ijk} = 1 : U_{ijk} \rightarrow \mathrm{GL}_r$. Then there is some n big enough such that $(\sigma^n)^* h_{ijk} = 1$ for all i, j, k . Therefore, we can define a locally free sheaf \mathcal{E} on $X'^{(m+n)}$ by the Čech cocycle $(\sigma^n)^* f'_{ij}$ (with respect to the étale cover $\{U'_i{}^{(m+n)}\}$). By construction, $(X'^{(m+n)}, \mathcal{E}')$ is a desired model. \square

COROLLARY A.23. *Let X be a pfp perfect algebraic space over k . Let \mathcal{E} be a locally free sheaf of rank n over X . Then the perfect space that assigns every $f : \mathrm{Spec} R \rightarrow X$ the set of rank i quotients \mathcal{Q} of $f^* \mathcal{E}$ is represented by a perfect algebraic space $\mathrm{Gr}^{p^{-\infty}}(i, \mathcal{E})$ perfectly proper over X . In particular, if X is perfectly proper, so is $\mathrm{Gr}^{p^{-\infty}}(i, \mathcal{E})$.*

Proof. Let (X', \mathcal{E}') be as in Lemma A.22. Then $\mathrm{Gr}^{p^{-\infty}}(i, \mathcal{E})$ is the perfection of the usual Grassmannian $\mathrm{Gr}(i, \mathcal{E}')$ of rank i quotients of \mathcal{E}' . \square

In the sequel, we denote $\mathrm{Gr}^{p^{-\infty}}(i, \mathcal{E})$ by $\mathrm{Gr}^{p^{-\infty}}(i, n)$ if $X = \mathrm{Spec} k$ and $\mathcal{E} = k^n$ is the standard n -dimensional vector space. We denote $\mathrm{Gr}^{p^{-\infty}}(1, \mathcal{E})$ by $\mathbb{P}^{p^{-\infty}}(\mathcal{E})$ and $\mathrm{Gr}^{p^{-\infty}}(1, n+1)$ by $\mathbb{P}^{n, p^{-\infty}}$.

Remark A.24. On $\mathbb{P}^{n, p^{-\infty}}$, there is the following tautological rank one quotient:

$$\mathcal{O}_{\mathbb{P}^{n, p^{-\infty}}}^{n+1} \rightarrow \mathcal{O}_{\mathbb{P}^{n, p^{-\infty}}}(1).$$

Therefore there is a distinguished element $\mathcal{O}(1) := \mathcal{O}_{\mathbb{P}^{n, p^{-\infty}}}(1)$ in $\mathrm{Pic}(\mathbb{P}^{n, p^{-\infty}})$. However, $\mathcal{O}(1)$ is not the generator of the Picard group. Namely there exists the invertible sheaf $\mathcal{O}(1/p) = (\sigma^{-1})^* \mathcal{O}(1)$, and the Picard group is isomorphic to $\mathbb{Z}[1/p]$.

We will also need the following definition.

Definition A.25. Let $f : X \rightarrow Y$ be a map between two pfp perfect algebraic spaces. We say that f is perfectly smooth at $x \in X$ if there exists an étale atlas $U \rightarrow X$ at x and an étale atlas $V \rightarrow Y$ at $f(x)$, such that the map $U \rightarrow Y$ factors as $U \xrightarrow{h} V \rightarrow Y$ and h factors as $h = \mathrm{pr} \circ h'$, where $h' : U \rightarrow V \times (\mathbb{A}^n)^{p^{-\infty}}$ is étale and $\mathrm{pr} : V \times (\mathbb{A}^n)^{p^{-\infty}} \rightarrow V$ is the projection.

We say that f is perfectly smooth if it is perfectly smooth at every point of X . We say that X is perfectly smooth (at x) if $X \rightarrow \text{Spec } k$ is perfectly smooth (at x).

Note that every pfp perfect algebraic space X contains a perfectly smooth open dense subspace.

A.2.2. We need to construct the quotient for an action of a perfect group scheme on a perfect scheme in some cases. In [Ser60], a perfect group variety is defined as a group object in the category of perfect varieties.

LEMMA A.26. *Let H be a pfp perfect group scheme over k . Then there exists a smooth algebraic group H_0 over k such that $H = H_0^{p^{-\infty}}$.*

Proof. A priori, H is the perfection of a scheme of finite type over k . But as was shown in [Ser60, §1.4, Prop. 10], H is the perfection of a group scheme H' of finite type over k . Let $H_0 = H'_{\text{red}}$ be the reduced subscheme. As k is a perfect field, H_0 is closed subgroup scheme of H' and is smooth. In addition, $H_0^{p^{-\infty}} = H$. □

COROLLARY A.27. *Let H be an affine pfp perfect group scheme over k . Then every H -torsor on a perfect algebraic space X can be trivialized étale locally on X .*

Proof. This is the combination of Lemmas A.26 and A.9. □

LEMMA A.28. *Let H be a pfp perfect affine group scheme over k acting on a pfp perfect affine scheme X over k . Then this action arises as the perfection of an action of a smooth affine algebraic group H' over k on an affine scheme X' of finite type over k .*

Proof. Recall the following basic fact: let H be an affine group scheme over k , with A its ring of functions. Let $\rho : V \rightarrow A \otimes_k V$ be a representation of H . Then V is a union of finite dimensional H -modules.

Now let B denote the ring of regular functions on X . Let $B_0 \subset B$ be a finitely generated subalgebra whose perfection is B , and let $V_0 \subset B_0$ be a finite dimensional subspace containing a set of generators of B_0 . Let V' be a finite dimensional H -invariant subspace of B that contains V_0 . Let $B' \subset B$ be the subalgebra generated by V' . Then B' is an H -invariant subalgebra of B and $B'^{p^{-\infty}} = B$ (since $B'^{p^{-\infty}} \supset B_0^{p^{-\infty}} = B$). Let $X' = \text{Spec } B'$. Then X' is of finite type over k , and it admits an action of H , which induces the action of H on X . In addition, as $H = \varprojlim H_0$, the action of H on V' is induced from some action of H' on V' with H' being smooth. Then the action $H \times X' \rightarrow X'$ is induced from some action $H' \times X' \rightarrow X'$ as required. □

Now the main result of this appendix is as follows.

THEOREM A.29. *Let H, X be as above. Furthermore, we assume that the action is free, i.e., the map $(\text{act}, \text{pr}_2) : H \times X \rightarrow X \times X$, $(g, x) \mapsto (gx, x)$ is a monomorphism. Then X/H is represented by a pfp perfect algebraic space over k . In addition, if $(\text{act}, \text{pr}_2)$ is a closed embedding, then X/H is separated as well.*

Proof. First note that the diagonal $X/H \rightarrow X/H \times X/H$ is always representable. So it is enough to show that X/H admits an étale cover by a scheme. Therefore, in order to prove the theorem, we are free to replace X by an H -equivariant étale cover $Y \rightarrow X$ by a perfect affine scheme Y perfectly of finite type. Indeed, if Y/H is representable by a perfect pfp algebraic space over k , then by Lemma A.27, we can find an étale cover of Y/H by an affine scheme that trivializes the H -torsor $Y \rightarrow Y/H$. That is, after a further étale localization, we can assume $Y = H \times U$ where U is a scheme. Then $U \rightarrow X/H$ is an étale atlas of X/H .

By Lemma A.28, there is an action $\text{act}' : H' \times X' \rightarrow X'$ that induces $\text{act} : H \times X \rightarrow X$. Now the action may not be free, but it is quasi-finite. Therefore, the stack $\mathfrak{X}' = [X'/H']$ is of finite presentation over k and has quasi-finite diagonal. It follows that there is an étale cover $\mathfrak{Y}' \rightarrow \mathfrak{X}'$ by an Artin stack \mathfrak{Y}' that is separated and of finite presentation over k , such that there exists a finite flat cover $\mathfrak{Z}' \rightarrow \mathfrak{Y}'$ with \mathfrak{Z}' being a quasi-projective scheme over k (see [KM97, Lemma 3.3, Prop. 4.2] or [Con, Lemma 2.1, 2.2]). Let $Y' = X' \times_{\mathfrak{X}'} \mathfrak{Y}'$. Then $Y' \rightarrow X'$ is an H' -equivariant étale cover, and after passing to the perfection, the action of H on $Y = Y'^{p^{-\infty}}$ is free. As explained above, it is then enough to show that Y/H is representable. Let $V' = \mathfrak{Z}' \times_{\mathfrak{Y}'} \mathfrak{Z}'$. Then we have a finite flat groupoid $V' \rightrightarrows \mathfrak{Z}'$ such that $\mathfrak{Y}' = [\mathfrak{Z}'/V']$. Let $V \rightrightarrows \mathfrak{Z}$ denote the perfection of this groupoid. It gives rise to an equivalence relation of \mathfrak{Z} ; i.e., $V \rightarrow \mathfrak{Z} \times \mathfrak{Z}$ is a monomorphism. By Lemma A.11, $Y/H = \mathfrak{Z}/V$. The theorem then follows from the next statement. \square

THEOREM A.30. *Let $V \rightrightarrows U$ be an equivalence relation of pfp perfect schemes. Assume that it is the perfection of a finite locally free groupoid $V' \rightrightarrows U'$ of quasi-projective schemes over k . Then U/V is represented by a pfp perfect scheme.*

Proof. By the standard reduction ([St, Tag03JE]), we can assume that $U' = \text{Spec } A$ and $V' = \text{Spec } B$ are affine. Let $s, t : A \rightrightarrows B$ be the source and target map.

Let C be the equalizer of $A \rightrightarrows B$. The theorem follows if we can show that $U/V \simeq \text{Spec } C^{p^{-\infty}}$. Let us write $X' = \text{Spec } C$ and $X = X'^{p^{-\infty}}$. We claim that

$$V \rightarrow U \times_X U$$

is an isomorphism. First it is a monomorphism since $V \rightarrow U \times U$ is an equivalence relation. It is also an integral morphism since it is the perfection of

a finite morphism (cf. Lemma A.7). In addition, it follows from the classical theory on quotients by finite locally free groupoids that the map is surjective (cf. [St, Tag03BL]). Therefore, $V \rightarrow U \times_X U$ is a universal homeomorphism. By Corollary A.16, it is an isomorphism.

Therefore, it remains to show that $A^{p^{-\infty}}$ is flat over $C^{p^{-\infty}}$. It follows by applying the following lemma to the $C^{p^{-\infty}}$ -module $A^{p^{-\infty}}$ and the injective integral map $C^{p^{-\infty}} \rightarrow A^{p^{-\infty}}$. \square

LEMMA A.31. *Let $R \rightarrow S$ be an injective integral map of perfect rings, which is the perfection of a finite map. Let M be an R -module. If $M \otimes_R S$ is flat over S , then M is flat over R .*

Proof. This is a perfect version of a result of Ferrand. For the completeness, we repeat the argument as in [St, Tag0533]. Assume that $R \rightarrow S$ is the perfection of a finite map $R' \rightarrow S'$. By [St, Tag0531], there is a finite free ring extension $R' \rightarrow R''$ such that $S'' = S' \otimes_{R'} R'' = R''[T_1, \dots, T_n]/J$, where J contains

$$(P_1(T_1), \dots, P_n(T_n)), P_i(T) = \prod_{j=1, \dots, d_i} (T - a_{ij}), a_{ij} \in R''.$$

Then $R \rightarrow R''^{p^{-\infty}}$ is faithfully flat, and it is enough to prove the flatness of $M \otimes_R R''^{p^{-\infty}}$. Therefore, we may replace R', S' by R'', S'' and rename them as R' and S' . Now for $\underline{k} = (k_1, \dots, k_n)$, $1 \leq k_i \leq d_i$, we define the ideal $J_{\underline{k}} \subset R'$ as the image of J under the map $R''[T_1, \dots, T_n] \rightarrow R'', T_i \mapsto a_{ik_i}$. Then the quotient map $R' \rightarrow R'/J_{\underline{k}}$ factors through $R' \rightarrow S' \rightarrow R'/J_{\underline{k}}$. Therefore, $M/J_{\underline{k}}^{p^{-\infty}} M$ is flat over $R/J_{\underline{k}}^{p^{-\infty}}$.

Since $R \rightarrow S$ is injective integral, $\text{Spec } S \rightarrow \text{Spec } R$ is surjective. Therefore, $\text{Spec } S' \rightarrow \text{Spec } R'$ is surjective. Then by [St, Tag0532], $I = \cap J_{\underline{k}} \subset R'$ is in the nilradical. Passing to the perfection, $\cap J_{\underline{k}}^{p^{-\infty}} = (0)$. It then follows from [St, Tag0522] that M is flat over R . \square

Finally, let us discuss the orbits for the group action.

PROPOSITION A.32. *Let H be a connected pfp perfect affine group scheme acting on a separated pfp perfect algebraic space X . Let $x \in X$ be a closed point, and let H_x be the stabilizer of x in H . Then the induced map $i : H/H_x \rightarrow X$ is a locally closed embedding.*

One can regard H/H_x as the H -orbit through x .

Proof. Let us denote H/H_x by Y for simplicity. By definition, $i : Y \rightarrow X$ is a monomorphism. In the classical theory, since the induced map of tangent spaces is injective, it follows easily that i is a locally closed embedding. In our setting, we need an alternative argument. First, as in the classical situation,

$|i(Y)| \subset |X|$ is locally closed. Namely, $|i(Y)|$ contains an open subset of its closure in $|X|$. Then by the group action, $|i(Y)|$ is open in its closure. Therefore, we may replace X by a locally closed subspace and assume that $i : Y \rightarrow X$ is a bijective monomorphism. Let us choose some model $i' : Y' \rightarrow X'$. Note that i' is quasi-finite, and therefore by Zariski's main theorem ([St, Tag05W7]), it factors as $Y' \xrightarrow{j'} Z' \xrightarrow{q'} X'$, where $j' : Y' \rightarrow Z'$ is open and $q' : Z' \rightarrow X'$ is finite. By replacing Z' by the closure of Y' in it, we can assume that Z' is irreducible. Then $\dim(Z' \setminus Y') < \dim X'$, and therefore, there is an open subspace $U' \subset X'$ such that $q'^{-1}(U') \subset j'(Y')$. In other words, $i' : i'^{-1}(U') \rightarrow U'$ is finite. Finite morphisms are integral, and therefore, after passing to the perfection, $i : i^{-1}(U) \rightarrow U$ is integral by Lemma A.7. Since H acts transitively on points, we see that $i : Y \rightarrow X$ is integral. Then, i is an integral, bijective monomorphism and therefore is a universal homeomorphism. By Corollary A.16, i is an isomorphism. \square

A.3. ℓ -adic sheaves.

A.3.1. The notion of (constructible) étale sheaves makes sense for separated pfp perfect algebraic spaces. Indeed, let X be such an algebraic space, and let $\varepsilon : X \rightarrow X'$ be a model of X . Note that ε is a universal homeomorphism. Therefore, by Lemma A.5, for an étale sheaf \mathcal{F} on X and an étale sheaf \mathcal{F}' on X' , the natural maps

$$(A.3.1) \quad \varepsilon^* \varepsilon_* \mathcal{F} \rightarrow \mathcal{F}, \quad \mathcal{F}' \rightarrow \varepsilon_* \varepsilon^* \mathcal{F}'$$

are isomorphisms. In particular, for such X , one can define the corresponding ℓ -adic derived category $D_c^b(X, \overline{\mathbb{Q}}_\ell)$ ($\ell \neq p$) as usual, with a pair of adjoint functors that are equivalences

$$\varepsilon^* : D_c^b(X', \overline{\mathbb{Q}}_\ell) \simeq D_c^b(X, \overline{\mathbb{Q}}_\ell) : \varepsilon_*.$$

One can define six operations between these categories, thanks to Proposition A.17. The usual proper base change or smooth base change holds for perfectly proper or perfectly smooth maps. The definition of perverse sheaves works in this setting without change. In particular, we have the notion of the Goresky-Macpherson intermediate extension, and for every X , the intersection cohomology sheaf IC_X . The restriction of IC_X to any perfectly smooth open subset U is canonically isomorphic to $\overline{\mathbb{Q}}_\ell[2 \dim X](\dim X)$. We will denote by $\mathrm{P}(X)$ the category of perverse sheaves on X .

A.3.2. Let X be a separated pfp perfect algebraic space over k . One can define Chern classes for locally free sheaves on X as usual. For an invertible sheaf \mathcal{L} on X , corresponding to a class $[\mathcal{L}] \in H^1(X, \mathbb{G}_m)$, we define its Chern

class $c_1(\mathcal{L})$ as the image of $[\mathcal{L}]$ under

$$H^1(X, \mathbb{G}_m) \rightarrow H^1(X_{\bar{k}}, \mathbb{G}_m) \rightarrow H^2(X_{\bar{k}}, \mu_{\ell^m}).$$

In general, for a locally free sheaf \mathcal{E} of rank n over X , let $\mathcal{O}(1) = \mathcal{O}_{\mathbb{P}^{p-\infty}(\mathcal{E})}(1)$ denote the tautological line bundle on $\mathbb{P}^{p-\infty}(\mathcal{E})$, and let

$$\xi = c_1(\mathcal{O}(1)) \in H^2(\mathbb{P}^{p-\infty}(\mathcal{E})_{\bar{k}}, \mu_{\ell^m}).$$

Then there are unique cohomology classes $c_i(\mathcal{E}) \in H^{2i}(X_{\bar{k}}, \mu_{\ell^m}^{\otimes i})$ such that

$$\xi^n - c_1(\mathcal{E})\xi^{n-1} + \dots + (-1)^n c_n(\mathcal{E}) = 0.$$

Passing to the inverse limit and inverting ℓ , we obtain $\overline{\mathbb{Q}}_\ell$ -coefficient Chern classes. The usual properties of Chern classes hold in this setting.

One can also define characteristic classes for general principal homogeneous spaces. Let G be a (connected) reductive group over k , and let $G_{\overline{\mathbb{Q}}_\ell}$ be the corresponding split group over $\overline{\mathbb{Q}}_\ell$. Let $R_{G,\ell} = \text{Sym}(\mathfrak{g}_{\overline{\mathbb{Q}}_\ell}^*(-1))^{G_{\overline{\mathbb{Q}}_\ell}}$ denote the algebra of invariant polynomials on the Lie algebra $\mathfrak{g}_{\overline{\mathbb{Q}}_\ell}(1)$. Then given a G -torsor E on X (equivalently a $G^{p-\infty}$ -torsor on X by Section A.1.3), its characteristic classes can be regarded as a ring homomorphism

$$(A.3.2) \quad c(E) : R_{G,\ell} \rightarrow H^*(X_{\bar{k}}, \overline{\mathbb{Q}}_\ell),$$

which can be constructed as follows. Let B be a Borel subgroup of $G_{\bar{k}}$, with the unipotent radical U and $T = B/U$. Let W be the Weyl group. Then the T -torsor $E_{\bar{k}}/U \rightarrow E_{\bar{k}}/B$ induces a map $c(E_{\bar{k}}/U) : R_{T,\ell} \rightarrow H^*(E_{\bar{k}}/B, \overline{\mathbb{Q}}_\ell)$ via the above construction of the Chern classes. Passing to some models, we see that $R_{G,\ell}(= R_{T,\ell}^W)$ maps to $H^*(X_{\bar{k}}, \overline{\mathbb{Q}}_\ell) \subset H^*(E_{\bar{k}}/B, \overline{\mathbb{Q}}_\ell)$, giving the desired map (A.3.2). For a general connected algebraic group G , let G^{red} denote its reductive quotient. Then a G -torsor E induces a G^{red} -torsor E^{red} , and we define $c(E)$ as $c(E^{\text{red}})$.

Remark A.33. Alternatively, one can define the Chern classes (or general characteristic classes) of $E \rightarrow X$ by first passing to some model $E' \rightarrow X'$ (Lemma A.22) and then set $c(E) = c(E')$ using the identification $H^*(X_{\bar{k}}, \overline{\mathbb{Q}}_\ell) = H^*(X'_{\bar{k}}, \overline{\mathbb{Q}}_\ell)$. Then one shows that this definition is independent of the choice of the model.

A.3.3. We can also define the cycle class map in the current setting. First, if X is irreducible of dimension d , there is a canonical isomorphism

$$c_X : H_c^{2d}(X_{\bar{k}}, \overline{\mathbb{Q}}_\ell(d)) \simeq \overline{\mathbb{Q}}_\ell,$$

given as follows. Choose a model $\varepsilon : X \rightarrow X'$, which induces the canonical isomorphism

$$\varepsilon^* : H_c^{2d}(X'_{\bar{k}}, \overline{\mathbb{Q}}_\ell(d)) \simeq H_c^{2d}(X_{\bar{k}}, \overline{\mathbb{Q}}_\ell(d)).$$

Then we define $c_X = c_{X'} \circ (\varepsilon^*)^{-1}$. Note that if $f : X' \rightarrow X''$ is a morphism of two d -dimensional irreducible algebraic spaces of finite presentation, the canonical isomorphism $f^* : H_c^{2d}(X''_{\bar{k}}, \overline{\mathbb{Q}}_\ell(d)) \simeq H_c^{2d}(X'_{\bar{k}}, \overline{\mathbb{Q}}_\ell(d))$ is compatible with $c_{X'}$ and $c_{X''}$. Therefore, c_X is well defined. Alternatively, one can build c_X directly, starting from the canonical isomorphism

$$c_{\mathbb{P}^{1,p-\infty}} : H^2(\mathbb{P}_{\bar{k}}^{1,p-\infty}, \mu_{\ell^n}) \simeq \text{coker}(\text{Pic}(\mathbb{P}^{1,p-\infty}) \xrightarrow{\ell^n} \text{Pic}(\mathbb{P}^{1,p-\infty})) \simeq \mathbb{Z}/\ell^n$$

and then using the functoriality of the six operations.

Now let X be a separated pfp perfect algebraic space. Let $\omega_X = f^! \overline{\mathbb{Q}}_\ell$ denote the dualizing sheaf. We define the Borel-Moore homology of X as

$$H_i^{\text{BM}}(X_{\bar{k}}) = H^{-i}(X_{\bar{k}}, \omega_X(-i/2)).$$

The usual properties of the Borel-Moore homology hold (by the functoriality of the six operations). We list a few.

- If $f : X \rightarrow Y$ is a perfectly proper morphism, there is a canonical map

$$f_* : H_*^{\text{BM}}(X_{\bar{k}}) \rightarrow H_*^{\text{BM}}(Y_{\bar{k}}).$$

- There is a canonical isomorphism $H_i^{\text{BM}}(X_{\bar{k}}) \simeq H_c^i(X_{\bar{k}}, \overline{\mathbb{Q}}_\ell(i/2))^*$. Therefore, if X is irreducible of dimension d , there exists the fundamental class

$$[X] := c_X \in H_{2d}^{\text{BM}}(X_{\bar{k}}).$$

In general, if X is d -dimensional, with X_1, \dots, X_n its irreducible components of dimension d , then the natural map $\bigoplus_i H_{2d}^{\text{BM}}((X_i)_{\bar{k}}) \simeq H_{2d}^{\text{BM}}(X_{\bar{k}})$ is an isomorphism. We set $[X] = \sum_i [X_i]$.

- If X is irreducible and perfectly smooth, the fundamental class $[X]$, regarded as a map of sheaves $\overline{\mathbb{Q}}_\ell \rightarrow \omega_X[-2d](-d)$ is an isomorphism. Therefore, $H_i^{\text{BM}}(X_{\bar{k}}) \simeq H^{2d-i}(X_{\bar{k}}, \overline{\mathbb{Q}}_\ell(d-i/2))$.

Finally, let Z be a closed subset of codimension r . We define the cycle class $\text{cl}(Z)$ of Z as the image of $[Z]$ in $H_{2(d-r)}^{\text{BM}}(X_{\bar{k}})$. If X is perfectly smooth and perfectly proper, we can regard $\text{cl}(Z)$ as a class in $H^{2r}(X_{\bar{k}}, \overline{\mathbb{Q}}_\ell(r))$.

A.3.4. We have the Lefschetz trace formula in this setting. Let \mathcal{F} be an ℓ -adic complex with constructible cohomology on a separated pfp perfect algebraic space X over \mathbb{F}_q . As usual, one can attach a function

$$f_{\mathcal{F}} : X(\mathbb{F}_q) \rightarrow \overline{\mathbb{Q}}_\ell, \quad x \mapsto \text{tr}(\sigma_x, \mathcal{F}_{\bar{x}}) = \sum_i (-1)^i \text{tr}(\sigma_x, (\mathcal{H}^i \mathcal{F})_{\bar{x}}),$$

where $x \in X(\mathbb{F}_{q^r})$, \bar{x} a geometric point over x , $(\mathcal{H}^i \mathcal{F})_{\bar{x}}$ the stalk cohomology of \mathcal{F} at \bar{x} , and σ_x is the *geometric* Frobenius at x .

Let $f : X \rightarrow Y$ be a morphism of separated pfp perfect algebraic spaces over \mathbb{F}_q . Let \mathcal{F} be an ℓ -adic complex with constructible cohomology on X .

Then the usual trace formula holds in this setting. Namely,

$$f_{f,\mathcal{F}}(y) = \sum_{x \in f^{-1}(y)(\mathbb{F}_q)} f_{\mathcal{F}}(x).$$

To prove this, one can replace $f : X \rightarrow Y$ by a model $f' : X' \rightarrow Y'$ and replace \mathcal{F} by $\varepsilon_*\mathcal{F}$.

A.3.5. In the paper, we also need some basic facts about equivariant category and equivariant cohomology. Let J be an affine pfp perfect group scheme over k . Recall that by Lemma A.26, J is perfectly smooth. Let X be a separated pfp perfect algebraic space over k with an action of J . Then it makes sense to talk about the category of J -equivariant perverse sheaves on X , denoted by $P_J(X)$.²⁰ That is, an object in $P_J(X)$ is a perverse sheaf on X together with an isomorphism of the pullbacks along the two maps $J \times X \rightrightarrows X$, satisfying the usual compatibility conditions.

We have the following two properties of the equivariant category. Let $J_1 \subset J$ be a closed normal subgroup.

- (i) If the action of J_1 on X is free and $[X/J_1]$ is represented by an algebraic space \bar{X} , then the pullback along $q : X \rightarrow \bar{X}$ induces an equivalence of categories

$$(A.3.3) \quad q^*[\dim J_1] : P_{J/J_1}(\bar{X}) \simeq P_J(X).$$

- (ii) Assume that J_1 is connected and that the action of J_1 on X is trivial. Then the forgetful functor

$$(A.3.4) \quad P_J(X) \rightarrow P_{J/J_1}(X)$$

is an equivalence of categories.

The proof is the same as the classical (i.e., nonperfect) situation (e.g., see [Zhu16, Lemma A.1.4]).

For $\mathcal{A} \in P_J(X)$ (or more generally a J -equivariant complex), it makes sense to talk about the J -equivariant cohomology $H_J^*(X_{\bar{k}}, \mathcal{A})$. Namely, let J_0 be a smooth model of J as in Lemma A.26. Let $\{E_n \rightarrow B_n\}$ denote a sequence of J_0 -torsors over $\{B_n\}$, which approximate of the classifying space of J_0 in the sense that $B_n \subset B_{n+1}$ is a closed embedding, and $H^*(\varinjlim_n B_n) = H^*(BJ_0)$. For example, we can embed J_0 into some GL_r such that GL_r/J_0 is quasi-affine. Then for n large, let $E_n := S_{n,r}$ be the Stiefel variety, i.e., the tautological GL_r -torsor over $Gr(r, n)$. Then $B_n := E_n/J_0$ is represented by a scheme, and the ind-scheme $\varinjlim_n B_n$ satisfies the required property. Then the sheaf $\overline{Q}_\ell \boxtimes \mathcal{A}$ on $E_n^{p-\infty} \times X$ is J -equivariant with respect to the diagonal action, and therefore

²⁰One can also define the equivariant derived category.

by (A.3.3), it descends to a sheaf $\overline{\mathbb{Q}}_\ell \boxtimes \mathcal{A}$ on $E_n \widetilde{\times} X$, which is a separated pfp perfect algebraic space by the discussion in Section A.1.3. Then

$$H_J^*(X_{\bar{k}}, \mathcal{A}) := H^*(\varinjlim_n (E_n^{p-\infty} \widetilde{\times} X)_{\bar{k}}, \overline{\mathbb{Q}}_\ell \boxtimes \mathcal{A}).$$

From the construction $H_J^*(X_{\bar{k}}, \mathcal{A})$ is a module over $H^*(\varinjlim_n B_n) = H^*(BJ_0)$. Let us also recall the Lie theoretical description of $H^*(BJ_0)$ in the case when J_0 is connected: if we denote by $G = J_0^{\text{red}}$ the reductive quotient of J_0 over k , then $H^*(BJ_0) = R_{G,\ell}$.

It is clear from the definition that if $J_1 \subset J$ acts freely on X with \bar{X} the quotient as above, then

$$(A.3.5) \quad H_J^*(X_{\bar{k}}, q^* \mathcal{A}) = H_{J/J_1}^*(\bar{X}_{\bar{k}}, \mathcal{A}).$$

On the other hand, if J_1 is the perfection of a unipotent group and acts trivially on X , then

$$(A.3.6) \quad H_J^*(X_{\bar{k}}, \mathcal{A}) = H_{J/J_1}^*(X_{\bar{k}}, \mathcal{A}).$$

Let J be a perfect affine group scheme acting on a pfp perfect algebraic space X satisfying the following condition (which is always the case in the paper). There exists a closed normal subgroup $J_1 \subset J$ of finite codimension acting trivially on X , and J_1 is the perfection of a pro-unipotent pro-algebraic group. Then we can define $P_J(X)$ as $P_{J/J_1}(X)$ and define $H_J^*(X, \mathcal{A})$ for $\mathcal{A} \in P_J(X)$ as $H_{J/J_1}^*(X, \mathcal{A})$. By (A.3.4) and (A.3.6), both definitions are independent of the choice of $J_1 \subset J$.

Appendix B. More on mixed characteristic affine Grassmannians

In this section, we discuss some unsolved questions related to mixed characteristic affine Grassmannians.²¹ We also give an example of our construction. Proofs are generally omitted in this section.

B.1. The determinant line bundle. We continue to use the notation as in Section 1. Recall that in equal characteristic, there is the important determinant line bundle $\mathcal{L}_{\text{det}}^b$ on Gr^b . Its fiber at a point $(\mathcal{E}, \beta) \in \overline{\text{Gr}}_N^b$ (the equal characteristic analogue of $\overline{\text{Gr}}_N$) is $\wedge^N(\mathcal{E}_0/\mathcal{E})$. (Recall that $\mathcal{E}_0/\mathcal{E}$ is a projective R -module of rank N .) In mixed characteristic, $\mathcal{E}_0/\mathcal{E}$ is not an R -module, and therefore $\wedge^N(\mathcal{E}_0/\mathcal{E})$ does not make sense *a priori*. Alternatively, one can try to define this line bundle by choosing a filtration of $\mathcal{E}/\mathcal{E}_0$ such that the associated graded is a projective R -module and then taking its top exterior power. This idea leads to a line bundle on $\widetilde{\text{Gr}}_N$. Let us formulate it more generally for

²¹Conjectures I and II have been recently proved by Bhatt-Scholze [BS]. Conjecture III is still open.

$\widetilde{\text{Gr}}_{\mu_\bullet}$. Given a quasi-isogenies $\mathcal{E} \xrightarrow{\beta} \mathcal{E}'$ with $\text{Inv}(\beta) \in \{\omega_1, \dots, \omega_n, \omega_1^*, \dots, \omega_n^*\}$, we can define a line bundle

$$\mathcal{L}_i = \begin{cases} \wedge^j \mathcal{E}' / \mathcal{E}, & \mu_i = \omega_j, \\ \wedge^j \mathcal{E} / \mathcal{E}', & \mu_i = \omega_j^*. \end{cases}$$

As Gr_{μ_\bullet} classifies all chains of quasi isogenies $\mathcal{E}_n \dashrightarrow \mathcal{E}_{n-1} \dashrightarrow \dots \dashrightarrow \mathcal{E}_0$ with each step being of the above form, we can define line bundles \mathcal{L}_i as above and set

$$(B.1.1) \quad \tilde{\mathcal{L}}_{\det} = \bigotimes_i \mathcal{L}_i.$$

CONJECTURE I. *There is a unique line bundle \mathcal{L}_{\det} on $\overline{\text{Gr}}_N$ such that its pullback along $\widetilde{\text{Gr}}_N \rightarrow \overline{\text{Gr}}_N$ is $\tilde{\mathcal{L}}_{\det}$ in (B.1.1).*

An evidence of Conjecture I is the following; we ignore the Tate twist:

PROPOSITION B.1. *The map $H^*((\overline{\text{Gr}}_N)_{\bar{k}}, \overline{\mathbb{Q}}_\ell) \rightarrow H^*(\widetilde{\text{Gr}}_N)_{\bar{k}}, \overline{\mathbb{Q}}_\ell)$ is injective. There exists a class $c \in H^2((\overline{\text{Gr}}_N)_{\bar{k}}, \overline{\mathbb{Q}}_\ell)$, whose image in $H^2((\widetilde{\text{Gr}}_N)_{\bar{k}}, \overline{\mathbb{Q}}_\ell)$ is the Chern class $c(\tilde{\mathcal{L}}_{\det})$.*

We can reduce Conjecture I to Conjecture IV stated in the sequel. It is based on the following assertion.

PROPOSITION B.2. *If $\Gamma(\widetilde{\text{Gr}}_N, \tilde{\mathcal{L}}_{\det})$ is base point free, i.e., for every closed point $x \in \widetilde{\text{Gr}}_N$, there exists a section $s \in \Gamma(\widetilde{\text{Gr}}_N, \tilde{\mathcal{L}}_{\det})$ such that $s(x) \neq 0$, then Conjecture I holds.*

Indeed, if the assumption in the above proposition holds, then the push-forward of $\tilde{\mathcal{L}}_{\det}$ along $\widetilde{\text{Gr}}_N \rightarrow \overline{\text{Gr}}_N$ will give \mathcal{L}_{\det} .

Remark B.3. More generally, given a perfect ring R , one can define the following category \mathcal{C}_R of triples $(\mathcal{E}_1, \mathcal{E}_2, \beta)$, where \mathcal{E}_1 and \mathcal{E}_2 are two finite projective $W(R)$ -modules and $\beta : \mathcal{E}_1 \dashrightarrow \mathcal{E}_2$ is a quasi-isogeny. This is an exact category. It is an interesting question to see whether the algebraic K-theory of \mathcal{C}_R is related to the K-theory of R .

CONJECTURE II. *There exists a model of the line bundle \mathcal{L}_{\det} (in the sense of Lemma A.22) induces an embedding of a model of $\overline{\text{Gr}}_N$ into some projective space. In particular, $\overline{\text{Gr}}_N$ is the perfection of a projective variety.*

Note that assuming Conjecture I, Conjecture II holds if the space of global sections of \mathcal{L}_{\det} separate points. That is, for every two different points $x, y \in \overline{\text{Gr}}_N$, there exists $s \in \Gamma(\overline{\text{Gr}}_N, \mathcal{L}_{\det})$ such that $s(x) = 0$ and $s(y) \neq 0$. As evidence, we see in Remark 1.15 and, in particular, in the sequel Section B.3, that in some cases when μ is (very) small, $\text{Gr}_{\leq \mu}$ is the perfection of some projective variety.

B.2. *Deperfection.* Recall that from the proof of Proposition 1.23, the perfect scheme $\text{Gr}_\mu = L^+G/L^+G \cap \varpi^\mu L^+G\varpi^{-\mu}$ has a canonical model Gr'_μ , which is a smooth quasi-projective variety. As Gr_μ is open dense in $\text{Gr}_{\leq\mu}$, it gives rise to a weakly normal model $\text{Gr}'_{\leq\mu}$ of $\text{Gr}_{\leq\mu}$, which is a proper algebraic space over k , containing Gr'_μ as an open subset (see Proposition A.15).

CONJECTURE III. *The proper algebraic space $\text{Gr}'_{\leq\mu}$ is normal and Cohen-Macaulay.*

Evidence of this conjecture is the following lemma.

LEMMA B.4. *The affine scheme $V'_{N,h}$ defined via (1.2.4) is normal and a locally complete intersection.*

We give a hint of the proof. First, note that $V'_{N,h}$ is defined by N -equations in an affine space of dimension n^2h . On the other hand, it is not hard to show that $\dim V'_{N,h} = \dim V_{N,h} = n^2h - N$ (e.g., by calculating the dimension of the fiber of J at $\text{diag}\{p^{m_1}, p^{m_2}, \dots, p^{m_n}\}$). So it is a complete intersection.

To show that it is normal, it is then enough to show the smoothness in codimension one. First, one shows $V'_{N,h}$ is nonsingular at $A = \text{diag}\{p^N, 1, 1, \dots, 1\}$ by calculating the dimension of the tangent space at A . Next using the action of $L_p^h\text{GL}_n \times L_p^h\text{GL}_n$ by left and right multiplication, $V'_{N,h}$ is nonsingular at every point of $L_p^h\text{GL}_n \cdot A \cdot L_p^h\text{GL}_n$. On the other hand, topologically, this double coset is exactly the preimage of Gr_N under $V_{N,h} \rightarrow \overline{\text{Gr}}_N$. By Lemma 1.18, $\overline{\text{Gr}}_N$ is of dimension $N(n-1)$, and the codimension of $\overline{\text{Gr}}_N - \text{Gr}_N$ is at least two. Then $V'_{N,h}$ is nonsingular away from a codimension two closed subset and therefore is normal.

Remark B.5. As mentioned in Remark 1.15, $\text{Gr}'_{\leq\mu}$ is probably not isomorphic to its equal characteristic counterpart $\text{Gr}^b_{\leq\mu}$ for general μ . However, according to the geometric Satake, their intersection cohomology (or more generally, their motives) are isomorphic.

Remark B.6. If $\lambda < \mu$, we have the closed embedding $i : \text{Gr}_{\leq\lambda} \subset \text{Gr}_{\leq\mu}$, which by Proposition A.17, is induced from some map $i' : \text{Gr}'_{\leq\lambda}^{(m)} \rightarrow \text{Gr}'_{\leq\mu}$, where $\text{Gr}'_{\leq\lambda}^{(m)} = \text{Gr}'_{\leq\lambda}$ but with the k -structure given by $\text{Gr}'_{\leq\lambda} \xrightarrow{\sigma^m} \text{Gr}'_{\leq\lambda} \rightarrow \text{Spec } k$. However, i' is *not* a closed embedding, and therefore we do not have a deperfection of the whole affine Grassmannian.

A natural question is whether $\text{Gr}'_{\leq\mu}$ has a natural moduli interpretation. We have no idea how to answer this question. The following discussion provides some hints that this might be an interesting question.

We consider $G = \text{GL}_n$. As mentioned above, we do not know a moduli interpretation of $\overline{\text{Gr}}'_N := \text{Gr}'_{\leq N\omega_1}$. On the other hand, recall that there is the

“Demazure resolution” $\widetilde{\text{Gr}}_N \rightarrow \overline{\text{Gr}}_N$. By Lemma 1.18, $\text{Gr}_N \subset \widetilde{\text{Gr}}_N$ is open dense, so by Proposition A.15 we also have a canonical model $\widetilde{\text{Gr}}'_N$.

We do have a moduli interpretation of $\widetilde{\text{Gr}}'_N$, as suggested by L. Xiao. For simplicity, we assume that $k = \overline{\mathbb{F}}_p$. Fix $h \geq N$. Let E/\mathbb{Q}_p be an unramified extension of degree $2h$, with ring of integers $\mathcal{O}_E = \mathbb{Z}_{p^{2h}}$. We fix an embedding $\tau_0 : E \rightarrow \overline{\mathbb{Q}}_p$ and let $\tau_i = \sigma^i \tau_0$, where $\sigma : \overline{\mathbb{Q}}_p \rightarrow \overline{\mathbb{Q}}_p$ is (a lift of) the Frobenius automorphism. Then $\tau_{i+2h} = \tau_i$.

Let \mathbb{X}_0 be a p -divisible group of height $2hn$ and dimension hn over $\overline{\mathbb{F}}_p$, with an action $\iota : \mathcal{O}_E \rightarrow \text{End}\mathbb{X}_0$. We assume that the signature of (\mathbb{X}_0, ι) is $(0, \dots, 0, n, \dots, n)$. That is, $\text{rk}(\text{Lie}\mathbb{X}_0)_{\tau_i} = 0$ for $i = 0, \dots, h - 1$, where $(\text{Lie}\mathbb{X}_0)_{\tau_i} = \text{Lie}\mathbb{X}_0 \otimes_{\mathcal{O}_E, \tau_i} \overline{\mathbb{F}}_p$.

We define a space $M_{N,h} \in \text{Sp}_{\overline{\mathbb{F}}_p}$ as follows. For an $\overline{\mathbb{F}}_p$ -algebra R , the set $M_{N,h}(R)$ classifies chains of isogenies of p -divisible groups on R with an \mathcal{O}_E -action,

$$(\mathbb{X}_0)_R \xrightarrow{\phi_1} \mathbb{X}_1 \rightarrow \dots \xrightarrow{\phi_N} \mathbb{X}_N$$

satisfying

- (1) (\mathbb{X}_i, ι) has signature $(1, \dots, 1, 0, \dots, 0, n, \dots, n, n - 1, \dots, n - 1)$, where the first i positions are 1's and the last i positions are $(n - 1)$'s;
- (2) $\deg \phi_i = p^{2i-1}$ for $i = 1, \dots, N$;
- (3) the differential $(d\phi_i)_j : (\text{Lie}\mathbb{X}_{i-1})_{\tau_j} \rightarrow (\text{Lie}\mathbb{X}_i)_{\tau_j}$ is zero for $j = 0, \dots, h - 1$; and
- (4) the dual of the differential $(d\phi_i^*)_j : (\text{Lie}\mathbb{X}_i^*)_{\tau_j} \rightarrow (\text{Lie}\mathbb{X}_{i-1}^*)_{\tau_j}$ is zero for $j = h, \dots, 2h - 1$.

Note that the first two conditions imply that $\ker \phi_i \subset \mathbb{X}_{i-1}[p]$.

The first two conditions define a moduli scheme closely related to RZ spaces. However, it is not irreducible in general (not even equidimensional), and the last two conditions cut out $M_{N,h}$ inside it as an irreducible component. More precisely, by induction we have

LEMMA B.7. *The space $M_{i+1,h}$ is a \mathbb{P}^{n-1} -bundle over $M_{i,h}$ for $i = 1, \dots, h$. Therefore, $M_{N,h}$ is represented by a smooth projective variety.*

Indeed, let $\mathbb{X}_i^{\text{univ}}$ denote the universal p -divisible group on $M_{i,h}$ appearing at the end of the chain. Let $M(\mathbb{X}_i^{\text{univ}})^*$ denote the dual of the Lie algebra of the universal extension of $\mathbb{X}_i^{\text{univ}}$ by vector groups. Let $M(\mathbb{X}_i^{\text{univ}})_{\tau_j}^* = M(\mathbb{X}_i^{\text{univ}})^* \otimes_{\mathcal{O}_E, \tau_j} \overline{\mathbb{F}}_p$. Then we have a rank n vector bundle $E_i = M(\mathbb{X}_i^{\text{univ}})_{\tau_0}^*$ on $M_{i,h}$ and $M_{i+1,h} = \mathbb{P}(E_i)$.

PROPOSITION B.8. *There is an isomorphism $\widetilde{\text{Gr}}'_N \simeq M_{N,h}$.*

Indeed, choosing a trivialization $\mathbb{D}(\mathbb{X}_0) \simeq W(k)^n = \mathcal{E}_0$, and using the argument as in Proposition 3.11, one shows that there is an isomorphism

$M_{N,h}^{p^{-\infty}} \simeq \widetilde{\text{Gr}}_N$ given by sending $\mathbb{X}_0 \rightarrow \cdots \rightarrow \mathbb{X}_N$ to $\mathbb{D}(\mathbb{X}_N)_{\tau_0} \rightarrow \cdots \rightarrow \mathbb{D}(\mathbb{X}_0)_{\tau_0}$. On the other hand, there exists an open subscheme $\mathring{M}_{N,h} \subset M_{N,h}$ parametrizing those chains such that the kernel of the map $\phi_N \cdots \phi_1 : \mathbb{X}_0 \rightarrow \mathbb{X}_N$ is *not* contained in $\mathbb{X}_0[p^{N-1}]$, and one can show that $\mathring{M}_{N,h} \simeq \text{Gr}'_N$, compatible with $\widetilde{\text{Gr}}_N \simeq M_{N,h}^{p^{-\infty}}$. Then by (A.2.2), we have a projective birational universal homeomorphism $\widetilde{\text{Gr}}'_N \rightarrow M_{N,h}$. As $M_{N,h}$ is a smooth projective variety, we have $\widetilde{\text{Gr}}'_N \simeq M_{N,h}$. Details are left to readers.

There is a line bundle on $M_{N,h}$ given by

$$\tilde{\mathcal{L}}'_{\text{det}} := \bigotimes_{i=1}^N \omega_{\mathbb{X}_N, \tau_{-i}}^{-p^i},$$

where $\omega_{\mathbb{X}_N, \tau_j} = \wedge^{\text{top}}(\text{Lie}\mathbb{X}_N)_{\tau_j}^*$.

LEMMA B.9. *Under the map $\widetilde{\text{Gr}}_N \simeq M_{N,h}^{p^{-\infty}} \xrightarrow{\varepsilon} M_{N,h}$, there is a canonical isomorphism $\tilde{\mathcal{L}}_{\text{det}} \simeq \varepsilon^* \tilde{\mathcal{L}}'_{\text{det}}$.*

Now, in view of Proposition B.2, Conjecture I would be a consequence of the following conjecture.

CONJECTURE IV. *The line bundle $\tilde{\mathcal{L}}'_{\text{det}}$ on $M_{N,h}$ is semi-ample.*

B.3. *Example:* $\widetilde{\text{Gr}}_2 \rightarrow \overline{\text{Gr}}_2$. We assume that $p > 2$ so that the Teichmüller lifting of -1 is -1 . The purpose here is to illustrate the construction of Sections 1.2 and 1.3 in the simplest but nontrivial case: $G = \text{GL}_2$, and $N = 2$. We hope to convince the readers that the bizarre-looking construction is indeed reasonable. We follow the notation there. We will see that $\overline{\text{Gr}}_2$ is a scheme, and a model of $\widetilde{\text{Gr}}_2 \rightarrow \overline{\text{Gr}}_2$ can be regarded as a resolution of the singularities of $\overline{\text{Gr}}_2$.

First, $\widetilde{\text{Gr}}_1 = \mathbb{P}^{1,p^{-\infty}}$, over which there is a sequence of maps of locally free sheaves (notation from the proof of Proposition 1.13)

$$\mathcal{E}/p \rightarrow \mathcal{O}_{\mathbb{P}^{1,p^{-\infty}}}^2 \rightarrow \mathcal{O}_{\mathbb{P}^{1,p^{-\infty}}}(1).$$

Then $\widetilde{\text{Gr}}_2 = \mathbb{P}^{p^{-\infty}}(\mathcal{E}/p)$. From (1.3.2), we know that \mathcal{E}/p fits into the following exact sequence:

$$0 \rightarrow \mathcal{O}(1) \rightarrow \mathcal{E}/p \rightarrow \mathcal{O}(-1) \rightarrow 0.$$

Therefore, $\mathcal{E}/p \simeq \mathcal{O}(1) \oplus \mathcal{O}(-1)$ (but this isomorphism is noncanonical), and $\widetilde{\text{Gr}}_2$ is isomorphic to $\mathbb{P}^{p^{-\infty}}(\mathcal{O}(1) \oplus \mathcal{O}(-1))$.

Next we consider $\overline{\text{Gr}}_2$. According to Section 1.2.2,

$$\overline{\text{Gr}}_2 = \overline{\text{Gr}}_{2,3}/L^3\text{GL}_2.$$

Consider the scheme $U = \text{Spec } R$ where $R = k[x, y, z]/x^2 - yz$. There is a natural map $U^{p^{-\infty}} \rightarrow \overline{\text{Gr}}_2$ given by

$$(x, y, z) \mapsto (\mathcal{E}_0, A), \quad \text{where } A = \begin{pmatrix} p + [x] & -[y] \\ [z] & p - [x] \end{pmatrix} \in \text{GL}_2(W(R^{p^{-\infty}})[1/p]).$$

LEMMA B.10. *This is an open embedding.*

Note that, in particular, $\overline{\text{Gr}}_2$ has an open cover by Gr_2 and $U^{p^{-\infty}}$ and therefore is a scheme.

Proof. We write G for $L^3\text{GL}_2$ for simplicity. We lift $U^{p^{-\infty}} \rightarrow \overline{\text{Gr}}_2$ to a map $U^{p^{-\infty}} \rightarrow \overline{\text{Gr}}_{2,3}$ by sending $A \mapsto (\mathcal{E}_0, A, \text{id})$. Then we need to show that the action map $U^{p^{-\infty}} \times G \rightarrow \overline{\text{Gr}}_{2,3}$ is an open embedding.

We need the following lemma, whose proof is based some linear algebra calculation. Recall that $V_{2,3}$ is the scheme classifying pairs $(X, \lambda) \in M_2(W_3(R)) \times R^\times$ satisfying $[\lambda] \det X = p^2$. We define a subfunctor $W \subset V_{2,3}$, whose values at a perfect k -algebra R consist of those $X \in V_{2,3}(R)$ such that there exist a decomposition $X = Ag$ with

$$g \in G(R), \quad A = \begin{pmatrix} p + [x] & -[y] \\ [z] & p - [x] \end{pmatrix} \in M_2(W_3(R)), \quad \det A = p^2.$$

LEMMA B.11. *The functor W is represented by an open subscheme of $V_{2,3}$. In addition, given $X \in W(R)$, the matrix A is unique in the decomposition $X = Ag$ as above.*

Proof. Let $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in V_{2,3}$, and let $X^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ be its adjugate matrix. We write $a = [a_0] + p[a_1] + p^2[a_2]$ and similarly expand b, c, d . Note that $\det X$ is divisible by p^2 if and only if

$$(B.3.1) \quad a_0d_0 = b_0c_0, \quad \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} d_0 & -b_0 \\ -c_0 & a_0 \end{pmatrix} + \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} \begin{pmatrix} d_1 & -b_1 \\ -c_1 & a_1 \end{pmatrix} = 0.$$

In other words, $V_{2,3}$ is represented by an open subscheme of the affine scheme defined by equations in (B.3.1).

Let $\widetilde{W} \subset V_{2,3}$ be the open subscheme defined by the condition $a_1d_1 - b_1c_1 \in R^*$. We claim that every $X \in \widetilde{W}$ admits a decomposition $X = Ag$ with (A, g) as in the definition of W . In addition, such A is unique. Assuming this claim, we first finish the proof of the lemma. Let $\widetilde{W}G$ be the minimal G -invariant open subset of $V_{2,3}$ containing \widetilde{W} . Then it follows from the claim that every $X \in \widetilde{W}G$ admits such a decomposition $X = Ag$, i.e., $\widetilde{W}G \subset W$. Conversely, let $X \in W$, with a decomposition $X = Ag$ as required. Since $A \in \widetilde{W}$, we have $X \in \widetilde{W}G$. Therefore, $W = \widetilde{W}G$, and in the decomposition $X = Ag$, the matrix A is unique. The lemma follows.

It remains to prove the above claim. Let $X \in \tilde{W}$, and suppose that $X = Ag$ is a required decomposition. Multiplying this equation by X^* on the left and g^{-1} on the right gives

$$p^2[\lambda]g^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} p + [x] & -[y] \\ [z] & p - [x] \end{pmatrix}.$$

It follows that the triple (x, y, z) must satisfy

$$(B.3.2) \quad \begin{pmatrix} d_0 & -b_0 \\ -c_0 & a_0 \end{pmatrix} \begin{pmatrix} x & -y \\ z & -x \end{pmatrix} = 0, \quad \begin{pmatrix} d_1 & -b_1 \\ -c_1 & a_1 \end{pmatrix} \begin{pmatrix} x & -y \\ z & -x \end{pmatrix} = - \begin{pmatrix} d_0 & -b_0 \\ -c_0 & a_0 \end{pmatrix}.$$

When $X \in \tilde{W}$, it is easy to see that there is a unique triple (x, y, z) satisfying these two equations. Namely,

$$(B.3.3) \quad \begin{pmatrix} x & -y \\ z & -x \end{pmatrix} = \frac{1}{b_1c_1 - a_1d_1} \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} d_0 & -b_0 \\ -c_0 & a_0 \end{pmatrix}.$$

Therefore, such A in the decomposition $X = Ag$ is unique.

Conversely, for $X \in \tilde{W}$, let $A = \begin{pmatrix} p+[x] & -[y] \\ [z] & p-[x] \end{pmatrix}$, where (x, y, z) is given by (B.3.3). Using the lifting $V_{2,3} \rightarrow L^+V_2$ as in Remark 1.11, we regard X and A as elements in LGL_2 , denoted by \tilde{X} and \tilde{A} . Then the determinant of

$$\tilde{g} := \tilde{X}\tilde{A}^{-1} = p^{-2}\tilde{X}\tilde{A}^* \in LGL_2$$

is an element in $W(R)^*$. But since (B.3.2) holds, the entries of \tilde{g} are in fact in $W(R)$. Therefore, $\tilde{g} \in L^+GL_2$. If we set $g = (\tilde{g} \bmod p^3)$, then $X = Ag$. The claim follows. \square

Now let V be the preimage of W under the projection $\overline{Gr}_{2,3} \rightarrow V_{2,3}$. Then the action map $U^{p^{-\infty}} \times G \rightarrow V$ is an isomorphism. In fact, the uniqueness of A as in the above lemma implies that this map is a monomorphism. On the other hand, the definition of W together with an argument as in Lemma 1.10 implies the surjectivity of R -points. Therefore, the lemma holds. \square

Finally, one can also verify Conjecture I in this case. Namely, let $j : Gr_2 \rightarrow \overline{Gr}_2$ be the open inclusion. One can restrict $\tilde{\mathcal{L}}_{\det}$ to $Gr_2 \subset \overline{Gr}_2$. Then $\mathcal{L}_{\det} = j_*(\tilde{\mathcal{L}}_{\det}|_{Gr_2})$.

References

[Alp14] J. ALPER, Adequate moduli spaces and geometrically reductive group schemes, *Algebr. Geom.* **1** (2014), 489–531. MR 3272912. Zbl 1322.14026. <https://doi.org/10.14231/AG-2014-022>.

[BL94] A. BEAUVILLE and Y. LASZLO, Conformal blocks and generalized theta functions, *Comm. Math. Phys.* **164** (1994), 385–419. MR 1289330. Zbl 0815.14015. <https://doi.org/10.1007/BF02101707>.

- [BD] A. BEILINSON and V. DRINFELD, Quantization of Hitchin's integrable system and Hecke eigensheaves, preprint. Available at <http://www.math.uchicago.edu/~mitya/langlands/QuantizationHitchin.pdf>.
- [BS] B. BHATT and P. SCHOLZE, Projectivity of Witt vector Grassmannians, preprint. arXiv 1507.06490.
- [CKV15] M. CHEN, M. KISIN, and E. VIEHMANN, Connected components of affine Deligne-Lusztig varieties in mixed characteristic, *Compos. Math.* **151** (2015), 1697–1762. MR 3406443. Zbl 1334.14017. <https://doi.org/10.1112/S0010437X15007253>.
- [Con] B. CONRAD, The Keel-Mori theorem via stacks. Available at <http://math.stanford.edu/~Econrad/papers/coarsespace.pdf>.
- [CLO12] B. CONRAD, M. LIEBLICH, and M. OLSSON, Nagata compactification for algebraic spaces, *J. Inst. Math. Jussieu* **11** (2012), 747–814. MR 2979821. Zbl 1255.14003. <https://doi.org/10.1017/S1474748011000223>.
- [DG14] V. DRINFELD and D. GAITSGORY, On a theorem of Braden, *Transform. Groups* **19** (2014), 313–358. MR 3200429. Zbl 1314.14039. <https://doi.org/10.1007/s00031-014-9267-8>.
- [Gai01] D. GAITSGORY, Construction of central elements in the affine Hecke algebra via nearby cycles, *Invent. Math.* **144** (2001), 253–280. MR 1826370. Zbl 1072.14055. <https://doi.org/10.1007/s002220100122>.
- [Gin90] V. A. GINZBURG, Sheaves on a loop group, and Langlands duality, *Funktsional. Anal. i Prilozhen.* **24** (1990), 76–77. MR 1092806. Zbl 0736.22009. <https://doi.org/10.1007/BF01077338>.
- [GHKR06] U. GÖRTZ, T. J. HAINES, R. E. KOTTWITZ, and D. C. REUMAN, Dimensions of some affine Deligne-Lusztig varieties, *Ann. Sci. École Norm. Sup.* **39** (2006), 467–511. MR 2265676. Zbl 1108.14035. <https://doi.org/10.1016/j.ansens.2005.12.004>.
- [Gre61] M. J. GREENBERG, Schemata over local rings, *Ann. of Math.* **73** (1961), 624–648. MR 0126449. Zbl 0115.39004. <https://doi.org/10.2307/1970321>.
- [Gro66] A. GROTHENDIECK, Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. III, *Inst. Hautes Études Sci. Publ. Math.* **28** (1966), 255. MR 0217086. Zbl 0144.19904. <https://doi.org/10.1007/BF02684343>.
- [Gro67] A. GROTHENDIECK, Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV, *Inst. Hautes Études Sci. Publ. Math.* **32** (1967), 361. MR 0238860. Zbl 0153.22301.
- [Hab05] W. J. HABOUSH, Infinite dimensional algebraic geometry: algebraic structures on p -adic groups and their homogeneous spaces, *Tohoku Math. J.* **57** (2005), 65–117. MR 2113991. Zbl 1119.14004. <https://doi.org/10.2748/tmj/1113234835>.
- [HR] T. HAINES and M. RAPOPORT, On parahoric subgroups, appendix to [PR].

- [Ham15a] P. HAMACHER, The dimension of affine Deligne-Lusztig varieties in the affine Grassmannian, *Int. Math. Res. Not.* **2015** (2015), 12804–12839. MR 3431637. Zbl 06527279. <https://doi.org/10.1093/imrn/rnv081>.
- [Ham15b] P. HAMACHER, The geometry of Newton strata in the reduction modulo p of Shimura varieties of PEL type, *Duke Math. J.* **164** (2015), 2809–2895. MR 3430453. Zbl 1335.14008. <https://doi.org/10.1215/00127094-3328137>.
- [HP15] B. HOWARD and G. PAPPAS, Rapoport-Zink spaces for spinor groups, 2015. arXiv 1509.03914.
- [Ito83] S. ITOH, On weak normality and symmetric algebras, *J. Algebra* **85** (1983), 40–50. MR 0723066. Zbl 0534.13002. [https://doi.org/10.1016/0021-8693\(83\)90117-5](https://doi.org/10.1016/0021-8693(83)90117-5).
- [Kat79] N. M. KATZ, Slope filtration of F -crystals, in *Journées de Géométrie Algébrique de Rennes, Vol. I* (Rennes, 1978), *Astérisque* **63**, Soc. Math. France, Paris, 1979, pp. 113–163. MR 0563463. Zbl 0426.14007.
- [KL79] D. KAZHDAN and G. LUSZTIG, Representations of Coxeter groups and Hecke algebras, *Invent. Math.* **53** (1979), 165–184. MR 0560412. Zbl 0499.20035. <https://doi.org/10.1007/BF01390031>.
- [KL80] D. KAZHDAN and G. LUSZTIG, Schubert varieties and Poincaré duality, in *Geometry of the Laplace Operator* (Proc. Sympos. Pure Math., Univ. Hawaii, Honolulu, Hawaii, 1979), *Proc. Sympos. Pure Math.* **XXXVI**, Amer. Math. Soc., Providence, RI, 1980, pp. 185–203. MR 0573434. Zbl 0461.14015.
- [KM97] S. KEEL and S. MORI, Quotients by groupoids, *Ann. of Math.* **145** (1997), 193–213. MR 1432041. Zbl 0881.14018. <https://doi.org/10.2307/2951828>.
- [Kim13] W. KIM, Rapoport-Zink spaces of Hodge type, 2013. arXiv 1308.5537.
- [Knu71] D. KNUTSON, *Algebraic Spaces, Lecture Notes in Math.* **203**, Springer-Verlag, New York, 1971. MR 0302647. Zbl 0221.14001. <https://doi.org/10.1007/BFb0059750>.
- [Kot97] R. E. KOTTWITZ, Isocrystals with additional structure. II, *Compositio Math.* **109** (1997), 255–339. MR 1485921. Zbl 0966.20022. <https://doi.org/10.1023/A:1000102604688>.
- [Kre14] M. KREIDL, On p -adic lattices and Grassmannians, *Math. Z.* **276** (2014), 859–888. MR 3175163. Zbl 1304.13010. <https://doi.org/10.1007/s00209-013-1225-y>.
- [Laf12] V. LAFFORGUE, Chtoucas pour les groupes réductifs et paramétrisation de Langlands globale, 2012. arXiv 1209.5352.
- [Lau13] E. LAU, Smoothness of the truncated display functor, *J. Amer. Math. Soc.* **26** (2013), 129–165. MR 2983008. Zbl 1273.14040. <https://doi.org/10.1090/S0894-0347-2012-00744-9>.
- [LMB00] G. LAUMON and L. MORET-BAILLY, *Champs Algébriques, Ergeb. Math. Grenzgeb.* **39**, Springer-Verlag, New York, 2000. MR 1771927. Zbl 0945.14005.

- [Lus83] G. LUSZTIG, Singularities, character formulas, and a q -analog of weight multiplicities, in *Analysis and Topology on Singular Spaces, II, III* (Luminy, 1981), *Astérisque* **101**, Soc. Math. France, Paris, 1983, pp. 208–229. MR 0737932. Zbl 0561.22013.
- [Lus12] G. LUSZTIG, A bar operator for involutions in a Coxeter group, *Bull. Inst. Math. Acad. Sin.* **7** (2012), 355–404. MR 3051318. Zbl 1283.20045.
- [LV12] G. LUSZTIG and D. A. VOGAN, JR., Hecke algebras and involutions in Weyl groups, *Bull. Inst. Math. Acad. Sin.* **7** (2012), 323–354. MR 3051317. Zbl 1288.20006. Available at http://web.math.sinica.edu.tw/bulletin/archives_articlecontent16.jsp?bid=MjAxMjMwMQ==.
- [LY13] G. LUSZTIG and Z. YUN, A $(-q)$ -analogue of weight multiplicities, *J. Ramanujan Math. Soc.* **28A** (2013), 311–340. MR 3115198. Zbl 1295.20046.
- [Man80] M. MANARESI, Some properties of weakly normal varieties, *Nagoya Math. J.* **77** (1980), 61–74. MR 0556308. Zbl 0403.14001. <https://doi.org/10.1017/S0027763000018663>.
- [MV07] I. MIRKOVIĆ and K. VILONEN, Geometric Langlands duality and representations of algebraic groups over commutative rings, *Ann. of Math.* **166** (2007), 95–143. MR 2342692. Zbl 1138.22013. <https://doi.org/10.4007/annals.2007.166.95>.
- [NP01] B. C. NGÔ and P. POLO, Résolutions de Demazure affines et formule de Casselman-Shalika géométrique, *J. Algebraic Geom.* **10** (2001), 515–547. MR 1832331. Zbl 1041.14002.
- [PR08] G. PAPPAS and M. RAPOPORT, Twisted loop groups and their affine flag varieties, *Adv. Math.* **219** (2008), 118–198. MR 2435422. Zbl 1159.22010. <https://doi.org/10.1016/j.aim.2008.04.006>.
- [Rap05] M. RAPOPORT, A guide to the reduction modulo p of Shimura varieties, in *Automorphic Forms. I*, *Astérisque* **298**, Soc. Math. France, Paris, 2005, pp. 271–318. MR 2141705. Zbl 1084.11029.
- [RZ96] M. RAPOPORT and T. ZINK, *Period Spaces for p -Divisible Groups*, *Annals of Math. Stud.* **141**, Princeton Univ. Press, Princeton, NJ, 1996. MR 1393439. Zbl 0873.14039. <https://doi.org/10.1515/9781400882601>.
- [SW] P. SCHOLZE and J. WEINSTEIN, p -adic geometry, Lecture notes from course at UC Berkeley in Fall 2014. Available at <http://math.berkeley.edu/people/jsweinst/Math274/ScholzeLectures.pdf>.
- [Ser60] J-P. SERRE, Groupes proalgébriques, *Inst. Hautes Études Sci. Publ. Math.* **7** (1960), 567. MR 0118722. Zbl 0097.35901. Available at http://www.numdam.org/item?id=PMIHES_1960__7__5_0.
- [St] Stack project authors, Stack project. Available at <http://stacks.math.columbia.edu>.
- [Vie06] E. VIEHMANN, The dimension of some affine Deligne-Lusztig varieties, *Ann. Sci. École Norm. Sup.* **39** (2006), 513–526. MR 2265677. Zbl 1108.14036. <https://doi.org/10.1016/j.ansens.2006.04.001>.

- [Vie08a] E. VIEHMANN, The global structure of moduli spaces of polarized p -divisible groups, *Doc. Math.* **13** (2008), 825–852. MR 2466182. Zbl 1162.14033. Available at <http://www.math.uiuc.edu/documenta/vol-13/24.pdf>.
- [Vie08b] E. VIEHMANN, Moduli spaces of p -divisible groups, *J. Algebraic Geom.* **17** (2008), 341–374. MR 2369090. Zbl 1144.14040. <https://doi.org/10.1090/S1056-3911-07-00480-8>.
- [XZ] L. XIAO and X. ZHU, Cycles on Shimura varieties via geometric Satake, in preparation.
- [Zhu15] X. ZHU, The geometric Satake correspondence for ramified groups, *Ann. Sci. Éc. Norm. Supér.* **48** (2015), 409–451. MR 3346175. Zbl 06456002.
- [Zhu16] X. ZHU, An introduction to affine Grassmannians and the geometric Satake equivalence, 2016. arXiv 1603.05593.

(Received: October 24, 2014)

(Revised: June 24, 2016)

CALIFORNIA INSTITUTE OF TECHNOLOGY, PASADENA, CA

E-mail: xzhu@caltech.edu