

# Characters of odd degree

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*Dedicated to Gabriel Navarro, for his fundamental contributions*

## Abstract

We prove the McKay conjecture on characters of odd degree. A major step in the proof is the verification of the inductive McKay condition for groups of Lie type and primes  $\ell$  such that a Sylow  $\ell$ -subgroup or its maximal normal abelian subgroup is contained in a maximally split torus by means of a new equivariant version of Harish-Chandra induction. Specifics of characters of odd degree, namely, that most of them lie in the principal Harish-Chandra series, then allow us to deduce from it the McKay conjecture for the prime 2, hence for characters of odd degree.

## 1. Introduction

In his 1972 note [McK72] dedicated to Richard Brauer on the occasion of his 70th birthday, John McKay put forward the following conjecture, based on observations on the known character tables of finite simple groups and of symmetric groups:

*For a finite simple group  $G$ ,  $m_2(G) = m_2(N_G(S_2))$ , where  $S_2$  is a Sylow 2-group of  $G$ .*

Here, for a finite group  $H$ ,  $m_2(H)$  denotes the number of complex irreducible characters of  $H$  of odd degree. Soon after the appearance of [McK72], this observation was generalised to arbitrary finite groups and primes. The *McKay conjecture* thus claims that for every finite group  $G$  and every prime  $\ell$ , the number of ordinary irreducible characters  $\chi \in \text{Irr}(G)$  with  $\ell \nmid \chi(1)$  is locally determined, namely,

$$|\text{Irr}_\ell(G)| = |\text{Irr}_\ell(N_G(P))|,$$

where  $P$  is a Sylow  $\ell$ -subgroup of  $G$  and  $N_G(P)$  denotes its normaliser in  $G$ . The main result of our paper is the proof of that conjecture for *all* finite groups and the prime 2.

**THEOREM 1.** *Let  $G$  be a finite group. Then the numbers of odd degree irreducible characters of  $G$  and of the normaliser of a Sylow 2-subgroup of  $G$  agree.*

McKay’s conjecture had a decisive influence on the development of modern representation theory of finite groups. Its prediction of how local structures like the normaliser of a Sylow  $\ell$ -subgroup should influence the representation theory of a group gave rise to a whole array of stronger and farther reaching conjectures, like those of Alperin, of Broué and of Dade. Simultaneously, functors relating the representation theory of certain families of finite (nearly simple) groups with those of suitable subgroups were introduced. For example, Harish-Chandra induction and its generalisation by Deligne–Lusztig are key to the parametrisation of characters of groups of Lie type.

It was Gabriel Navarro who, in his work and in his talks, insisted that the McKay conjecture ‘lies at the heart of everything.’ His insights led to the proof of the fundamental result [IMN07] that the McKay conjecture holds for all finite groups at a prime  $\ell$  if every finite non-abelian simple group satisfies a set of properties, the now so-called *inductive McKay condition*, for  $\ell$ . (A streamlined version of this reduction was presented in [Spä13] while a novel approach to groups with self-normalising Sylow 2-subgroups was recently devised by Navarro and Tiep [NT16].) This opens the possibility to solve the conjecture through the classification of finite simple groups. Thanks to the work of several authors this inductive condition has been shown for all but seven infinite series of simple groups of Lie type  $S$  at primes  $\ell$  different from the defining characteristic of  $S$ ; see [CS16], [Mal08b], [Spä12].

The second main result of our paper is meant to provide an important step towards verifying the McKay conjecture in the case of odd primes, showing that the inductive McKay condition holds for most simple groups of Lie type in the maximally split case:

**THEOREM 2.** *Let  $\mathbf{G}$  be a simple linear algebraic group of simply connected type defined over  $\mathbb{F}_q$  with respect to the Frobenius endomorphism  $F : \mathbf{G} \rightarrow \mathbf{G}$  such that  $S := \mathbf{G}^F / Z(\mathbf{G}^F)$  is simple. Assume that  $\mathbf{G}^F \notin \{D_{l,\text{sc}}(q), E_{6,\text{sc}}(q)\}$  for any prime power  $q$ . Then the inductive McKay condition from [IMN07, §10] holds for  $S$  and all primes  $\ell$  dividing  $q - 1$ .*

For many simple groups  $S$  of Lie type and primes  $\ell$  different from the defining characteristic of  $S$ , the authors had constructed a bijection satisfying some (but not all) of the required properties from the inductive McKay condition. Moreover, in the cases where the associated algebraic group has connected centre, Cabanes and the second author [CS13] could then verify the inductive McKay condition. It thus remains to deal with simple groups of Lie type coming from algebraic groups of simply connected type with disconnected centre.

One decisive ingredient in our proof is a criterion for the inductive McKay condition tailored to groups of Lie type, see [Spä12, Thm. 2.12], which we recall

here in [Theorem 2.1](#). It had already been used for groups of type  $A_l$  as well as in the defining characteristic; see [\[CS16\]](#) and [\[Spä12\]](#). The main assumption of that theorem on the universal covering group  $G$  of a simple group  $S$  of Lie type and the prime  $\ell$  consists of three requirements:

- the global part concerns the stabilisers in the automorphism group and the extendibility of elements in  $\text{Irr}_{\ell'}(G)$  (see [assumption 2.1\(ii\)](#));
- for a suitably chosen subgroup  $N$  that has properties similar to the normaliser of a Sylow  $\ell$ -subgroup of  $G$ , the elements of  $\text{Irr}_{\ell'}(N)$  have only stabilisers of specific structures and have an analogous property with respect to extendibility (see [assumption 2.1\(iii\)](#));
- there exists an equivariant global-local bijection between the relevant characters of certain groups containing  $G$  and  $N$ , respectively (see [assumption 2.1\(iv\)](#)).

We successively establish those assumptions in the cases relevant to our [Theorems 1](#) and [2](#). In accordance with [\[Mal07\]](#) we choose  $N$  to be the normaliser of a suitable Sylow  $d$ -torus. Extending earlier results of the second author we derive the required statement about the stabilisers of local characters; see [Section 3](#). Afterwards we study the parametrisation of irreducible characters in terms of Harish-Chandra induction, control how automorphisms act on these characters and express this in terms of their labels; see [Theorem 4.6](#). The proof requires an equivariant version of Howlett–Lehrer theory describing the decomposition of Harish-Chandra induced cuspidal characters and relies on an extendibility result of Howlett–Lehrer and Lusztig. We then prove that many characters of  $G$  have stabilisers of the structure required in the criterion.

*Structure of the paper.* After introducing some notation in [Section 2](#), we start by recalling the parametrisation of characters of normalisers of Sylow  $d$ -tori for  $d \in \{1, 2\}$  and describe how automorphisms act on the characters and the associated labels in [Section 3](#).

In [Section 4](#), after recalling the basic results on the endomorphism algebra of Harish-Chandra induced modules of  $G$ , we describe the action of outer automorphisms  $\sigma \in \text{Aut}(G)$  on such modules. This enables us in [Theorem 5.2](#) to construct an equivariant local-global bijection given by Harish-Chandra induction. The aforementioned results on stabilisers of characters of the normaliser of a maximally split torus leads to a description of the stabilisers of some characters of  $G$  in a similar way; see [Corollary 5.3](#).

The remaining part of the paper is devoted to the completion of the proof of our main [Theorems 1](#) and [2](#). First we show that all necessary assumptions of [Theorem 2.1](#) are satisfied for proving [Theorem 2](#) and clarify which additional properties need to be proved for obtaining an even more general statement.

Then, after the classification of odd degree characters of quasi-simple groups of Lie type in [Theorem 7.7](#) that may be of independent interest, we complete the proof of [Theorem 1](#).

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## 2. Background

In this section we recall the criterion from [[Spä12](#)] for the inductive McKay condition that is the main tool in the proof of our main result. Afterwards we introduce the groups of Lie type that play a central role in the paper and describe their automorphisms.

*2.A. A criterion for the inductive McKay condition.* We first introduce some notation. If a group  $A$  acts on a finite set  $X$ , we denote by  $A_x$  the stabiliser of  $x \in X$  in  $A$ ; analogously we denote by  $A_{X'}$  the setwise stabiliser of  $X' \subseteq X$ . For an element  $a \in A$ , we denote by  $o(a)$  the order of  $a$ . If  $A$  acts on a group  $G$  by automorphisms, there is a natural action of  $A$  on  $\text{Irr}(G)$  given by

$$a^{-1}\chi(g) = \chi^a(g) = \chi(g^{a^{-1}}) \quad \text{for every } g \in G, a \in A \text{ and } \chi \in \text{Irr}(G).$$

For  $P \leq G$  and  $\chi \in \text{Irr}(H)$  for some  $A_P$ -stable subgroup  $H \leq G$ , we denote by  $A_{P,\chi}$  the stabiliser of  $\chi$  in  $A_P$ .

We denote the restriction of  $\chi \in \text{Irr}(G)$  to a subgroup  $H \leq G$  by  $\chi|_H$ , while  $\psi^G$  denotes the character induced from  $\psi \in \text{Irr}(H)$  to  $G$ . For  $N \triangleleft G$  and  $\chi \in \text{Irr}(G)$ , we denote by  $\text{Irr}(N | \chi)$  the set of irreducible constituents of the restricted character  $\chi|_N$ , and for  $\psi \in \text{Irr}(N)$ , the set of irreducible constituents of the induced character  $\psi^G$  is denoted by  $\text{Irr}(G | \psi)$ . For a subset  $\mathcal{N} \subseteq \text{Irr}(N)$ , we define

$$\text{Irr}(G | \mathcal{N}) := \bigcup_{\chi \in \mathcal{N}} \text{Irr}(G | \chi).$$

Additionally, for  $N \triangleleft G$ , we sometimes identify the characters of  $G/N$  with the characters of  $G$  whose kernel contains  $N$ . For a prime  $\ell$ , we let  $\text{Irr}_\ell(G) := \{\chi \in \text{Irr}(G) \mid \ell \nmid \chi(1)\}$ .

The following criterion was proved in Späth [[Spä12](#), Thm. 2.12]:

**THEOREM 2.1.** *Let  $S$  be a finite non-abelian simple group and  $\ell$  a prime dividing  $|S|$ . Let  $G$  be the maximal perfect central extension of  $S$  and  $Q$  a Sylow  $\ell$ -subgroup of  $G$ . Assume there exist groups  $A, \tilde{G} \leq A, D \leq A$  and  $N \leq G$ , such that with  $\tilde{N} := NN_{\tilde{G}}(Q)$  the following conditions hold:*

- (i) (1)  $G \triangleleft A, G \leq \tilde{G}$  and  $A = \tilde{G} \rtimes D$ ,
- (2)  $\tilde{G}/G$  is abelian,

- (3)  $C_{\widetilde{G} \rtimes D}(G) = Z(\widetilde{G})$  and  $A/Z(\widetilde{G}) \cong \text{Aut}(G)$  by the natural map,
  - (4)  $N$  is  $\text{Aut}(G)_Q$ -stable,
  - (5)  $N_G(Q) \leq N$ ,
  - (6) every  $\chi \in \text{Irr}_{\ell'}(G)$  extends to its stabiliser  $\widetilde{G}_\chi$ ,
  - (7) every  $\psi \in \text{Irr}_{\ell'}(N)$  extends to its stabiliser  $\widetilde{N}_\psi$ .
- (ii) Let  $\mathcal{G} := \text{Irr}(\widetilde{G} \mid \text{Irr}_{\ell'}(G))$ . For every  $\chi \in \mathcal{G}$ , there exists some  $\chi_0 \in \text{Irr}(G \mid \chi)$  such that
- (1)  $(\widetilde{G} \rtimes D)_{\chi_0} = \widetilde{G}_{\chi_0} \rtimes D_{\chi_0}$  and
  - (2)  $\chi_0$  extends to  $(G \rtimes D)_{\chi_0}$ .
- (iii) Let  $\mathcal{N} := \text{Irr}(\widetilde{N} \mid \text{Irr}_{\ell'}(N))$ . For every  $\psi \in \mathcal{N}$ , there exists some  $\psi_0 \in \text{Irr}(N \mid \psi)$  such that  $O := G(\widetilde{G} \rtimes D)_{N, \psi_0}$  satisfies
- (1)  $O = (\widetilde{G} \cap O) \rtimes (D \cap O)$  and
  - (2)  $\psi_0$  extends to  $(G \rtimes D)_{N, \psi_0}$ .
- (iv) There exists a  $(\widetilde{G} \rtimes D)_Q$ -equivariant bijection  $\widetilde{\Omega} : \mathcal{G} \rightarrow \mathcal{N}$  with
- (1)  $\widetilde{\Omega}(\mathcal{G} \cap \text{Irr}(\widetilde{G} \mid \nu)) = \mathcal{N} \cap \text{Irr}(\widetilde{N} \mid \nu)$  for every  $\nu \in \text{Irr}(Z(\widetilde{G}))$ ,
  - (2)  $\widetilde{\Omega}(\chi\delta) = \widetilde{\Omega}(\chi) \delta \upharpoonright_{\widetilde{N}}$  for every  $\chi \in \mathcal{G}$  and every  $\delta \in \text{Irr}(\widetilde{G} \mid 1_G)$ .

Then the inductive McKay condition from [IMN07, §10] holds for  $S$  and  $\ell$ .

2.B. *Simple groups of Lie type.* We now introduce the most relevant groups and automorphisms. For the later detailed calculations, it is relevant to fix them in a rather precise way. Let  $\mathbf{G}$  be a simple linear algebraic group of simply connected type over an algebraic closure of  $\mathbb{F}_q$ . Let  $\mathbf{B}$  be a Borel subgroup of  $\mathbf{G}$  with maximal torus  $\mathbf{T}$ . Let  $\Phi, \Phi^+$  and  $\Delta$  denote the set of roots, positive roots and simple roots of  $\mathbf{G}$  that are determined by  $\mathbf{T}$  and  $\mathbf{B}$ . Let  $\mathbf{N} := N_{\mathbf{G}}(\mathbf{T})$ . We denote by  $W$  the Weyl group of  $\mathbf{G}$  and by  $\pi : N_{\mathbf{G}}(\mathbf{T}) \rightarrow W$  the defining epimorphism. For calculations with elements of  $\mathbf{G}$ , we use the Chevalley generators subject to the Steinberg relations as in [GLS94, Thm. 1.12.1], i.e., the elements  $x_\alpha(t), n_\alpha(t)$  and  $h_\alpha(t)$  ( $t \in \overline{\mathbb{F}}_q$  and  $\alpha \in \Phi$ ) defined therein.

In the following we describe automorphisms of  $\mathbf{G}$ . Let  $p$  be the prime with  $p \mid q$  and  $F_0 : \mathbf{G} \rightarrow \mathbf{G}$  the field endomorphism of  $\mathbf{G}$  given by

$$F_0(x_\alpha(t)) = x_\alpha(t^p) \quad \text{for every } t \in \overline{\mathbb{F}}_q \text{ and } \alpha \in \Phi.$$

Any length-preserving automorphism  $\tau$  of the Dynkin diagram associated to  $\Delta$  and hence automorphism of  $\Phi$  determines a graph automorphism  $\gamma$  of  $\mathbf{G}$  given by

$$\gamma(x_\alpha(t)) = x_{\tau(\alpha)}(t) \quad \text{for every } t \in \overline{\mathbb{F}}_q \text{ and } \alpha \in \pm\Delta.$$

Note that any such  $\gamma$  commutes with  $F_0$ .

For the construction of diagonal automorphisms of the associated finite groups of Lie type, we introduce further groups: Let  $r$  be the rank of  $Z(\mathbf{G})$  (as abelian group) and  $\mathbf{Z} \cong (\overline{\mathbb{F}}_q^\times)^r$  a torus of that rank with an embedding

of  $Z(\mathbf{G})$ . We set

$$\widetilde{\mathbf{G}} := \mathbf{G} \times_{Z(\mathbf{G})} \mathbf{Z},$$

the central product of  $\mathbf{G}$  with  $\mathbf{Z}$  over  $Z(\mathbf{G})$ . Then  $\widetilde{\mathbf{G}}$  is a connected reductive group with connected centre and the natural map  $\mathbf{G} \rightarrow \widetilde{\mathbf{G}}$  is a regular embedding; see [CE04, 15.1]. Note that  $\widetilde{\mathbf{B}} := \mathbf{B}\mathbf{Z}$  is a Borel subgroup of  $\widetilde{\mathbf{G}}$  and  $\widetilde{\mathbf{T}} := \mathbf{T}\mathbf{Z}$  is a maximal torus therein. Furthermore, let  $\widetilde{\mathbf{N}} := N_{\widetilde{\mathbf{G}}}(\widetilde{\mathbf{T}}) = \mathbf{N}\mathbf{Z}$ .

As  $F_0$  acts on  $Z(\mathbf{G})$  via  $x \mapsto x^p$  for every  $x \in Z(\mathbf{G})$ , we can extend it to a Frobenius endomorphism  $F_0 : \widetilde{\mathbf{G}} \rightarrow \widetilde{\mathbf{G}}$  via

$$F_0(g, x) := (F_0(g), x^p) \quad \text{for every } g \in \mathbf{G} \text{ and } x \in \mathbf{Z}.$$

Now assume that  $\gamma$  is a graph automorphism of  $\mathbf{G}$ . If  $\gamma$  acts trivially on  $Z(\mathbf{G})$ , then it extends to an automorphism of  $\widetilde{\mathbf{G}}$ , which we also denote by  $\gamma$ , via

$$\gamma(g, x) := (\gamma(g), x) \quad \text{for every } g \in \mathbf{G} \text{ and } x \in \mathbf{Z}.$$

If  $\gamma$  acts on  $Z(\mathbf{G})$  by inversion, then it can be extended via

$$\gamma(g, x) := (\gamma(g), x^{-1}) \quad \text{for every } g \in \mathbf{G} \text{ and } x \in \mathbf{Z}.$$

A similar extension of  $\gamma$  is possible in the remaining cases. In any case  $F_0$  and  $\gamma$  stabilise  $\widetilde{\mathbf{B}}$  and  $\widetilde{\mathbf{T}}$ .

Now consider a Steinberg endomorphism  $F := F_0^m \gamma$ , with  $\gamma$  a (possibly trivial) graph automorphism of  $\mathbf{G}$ . Then  $F$  defines an  $\mathbb{F}_q$ -structure on  $\widetilde{\mathbf{G}}$ , where  $q = p^m$ , and  $\mathbf{B}, \mathbf{T}, \widetilde{\mathbf{B}}, \widetilde{\mathbf{T}}$  are  $F$ -stable so, in particular,  $\mathbf{T}, \widetilde{\mathbf{T}}$  are maximally split tori in  $\mathbf{G}, \widetilde{\mathbf{G}}$  respectively. We let  $G := \mathbf{G}^F$ . By construction the order of  $F_0$  as automorphism of  $\widetilde{G} := \widetilde{\mathbf{G}}^F$  coincides with that of  $F_0$  as automorphism of  $G$ . The analogous statement also holds for any graph automorphism  $\gamma$  and the automorphisms of  $\widetilde{G}$  associated with it.

Let  $D$  be the subgroup of  $\text{Aut}(G)$  generated by  $F_0$  and the graph automorphisms commuting with  $F$ . Then  $\widetilde{G} \rtimes D$  is well defined and induces all automorphisms of  $G$ ; see [GLS94, Thm. 2.5.1]. Moreover,  $D$  acts naturally on the set of  $F$ -stable subgroups of  $\mathbf{G}$ .

2.C. *An embedding of the group  $D_{l,sc}(q)$  into  $B_{l,sc}(q)$ .* We recall an embedding of  $D_{l,sc}(q)$  into  $B_{l,sc}(q)$  given explicitly in [Spä10, 10.1] in terms of the aforementioned Chevalley generators. Let  $\overline{\Phi}$  be a root system of type  $B_l$  with base  $\Delta = \{\overline{\alpha}_1, \alpha_2, \dots, \alpha_l\}$ , where  $\overline{\alpha}_1 = e_1$  and  $\alpha_i = e_i - e_{i-1}$  ( $i \geq 2$ ) as in [GLS94, Rem. 1.8.8]. Let  $\overline{\mathbf{G}}$  be the associated simple algebraic group of simply connected type over  $\overline{\mathbb{F}}_q$ . In analogy to our previous terminology we denote its Chevalley generators by  $\overline{x}_\alpha(t_1)$ ,  $\overline{n}_\alpha(t_2)$  and  $\overline{h}_\alpha(t_2)$  with  $(\alpha \in \overline{\Phi}, t_1 \in \overline{\mathbb{F}}_q \text{ and } t_2 \in \overline{\mathbb{F}}_q^\times)$ .

Let  $\Phi \subseteq \overline{\Phi}$  be the root system consisting of all long roots of  $\overline{\Phi}$ . Then the group  $\langle x_\alpha(t) \mid \alpha \in \Phi, t \in \overline{\mathbb{F}}_q \rangle$  is a simply connected simple group over  $\overline{\mathbb{F}}_q$  with the root system  $\Phi$  of type  $D_l$ .

Whenever  $\Phi$  is of type  $D_l$ , we identify  $\mathbf{G}$  with  $\langle \bar{x}_\alpha(t) \mid \alpha \in \Phi, t \in \overline{\mathbb{F}}_q \rangle$  via  $\iota_D : \mathbf{G} \rightarrow \overline{\mathbf{G}}, x_\alpha(t) \mapsto \bar{x}_\alpha(t)$  and choose the notation of elements in  $\mathbf{G}$  such that this defines a monomorphism. Let  $\zeta \in \overline{\mathbb{F}}_q$  be a primitive  $(2, q - 1)^2$ -th root of unity. The graph automorphism of  $\mathbf{G}$  of order 2 coincides with the map  $x \mapsto x^{\bar{n}_{e_1(1)}} \prod_{i=2}^l \bar{h}_{e_i(\zeta)}$ , see [Spä10, Lemma 11.2], which because of  $Z(\mathbf{G}) = \langle \bar{h}_{e_1}(-1), \prod_{i=1}^l \bar{h}_{e_i}(\zeta) \rangle$  (by [GLS94, Tab. 1.12.6 and Thm. 1.12.1(e)]) coincides with  $x \mapsto x^{\bar{n}_{e_1(1)} \bar{h}_{e_1}(\zeta)}$ .

### 3. Parametrisation of some local characters

In this section we prove a result on stabilisers of characters that leads to the verification of [condition 2.1\(iii\)](#) in the cases considered in this paper. These results enable us later in [Theorem 6.3](#) to construct a bijection  $\widetilde{\Omega} : \mathcal{G} \rightarrow \mathcal{N}$  as required in [Theorem 2.1\(iv\)](#).

The aim of this section is the proof of the following statement, which concerns normalisers of Sylow  $d$ -tori, sometimes also called Sylow  $d$ -normalisers. Sylow  $d$ -tori were introduced in [BM92] under the name of Sylow  $\Phi_d$ -tori (with  $\Phi_d$  denoting the  $d$ -th cyclotomic polynomial), and they play an important role in the study of height 0 characters, see [Mal07].

**THEOREM 3.1.** *Let  $d \in \{1, 2\}$ ,  $\mathbf{S}_0$  be a Sylow  $d$ -torus of  $(\mathbf{G}, F)$ ,  $N_0 := N_{\mathbf{G}}(\mathbf{S}_0)^F$ ,  $\widetilde{N}_0 := N_{\widetilde{\mathbf{G}}}(\mathbf{S}_0)^F$  and  $\psi \in \text{Irr}(\widetilde{N}_0)$ . There exists some  $\psi_0 \in \text{Irr}(N_0 \mid \psi)$  such that*

- (1)  $O_0 = (\widetilde{\mathbf{G}}^F \cap O_0) \rtimes (D \cap O_0)$  for  $O_0 := \mathbf{G}^F (\widetilde{\mathbf{G}}^F \rtimes D)_{\mathbf{S}_0, \psi_0}$ ; and
- (2)  $\psi_0$  extends to  $(\mathbf{G}^F \rtimes D)_{\mathbf{S}_0, \psi_0}$ .

This statement is related to [Theorem 5.1](#) of [CS16], where the same assertion was proved for all positive integers  $d$  in the case that the root system of  $\mathbf{G}$  is of type  $A_l$ . Accordingly we may and will assume in the following that  $\Phi$  is not of type  $A_l$ .

We verify the statement in five steps mimicking the strategy applied in [CS16, §5]. First, in [3.A](#) we replace  $\mathbf{G}^F$  by an isomorphic group, then for subgroups of this group we construct in [3.B](#) an extension map that is compatible with certain automorphisms of  $\mathbf{G}^F$ , which gives in [3.C](#) a parametrisation of  $\text{Irr}(N_0)$ . In the end, [condition 2.1\(iii.1\)](#) on the structure of stabilisers is deduced from properties of characters of relative inertia groups.

By what we said before we may and will also assume throughout this section that  $D$  is nontrivial and that  $\widetilde{\mathbf{G}}$  induces noninner automorphisms on  $\mathbf{G}$ . Accordingly the root system  $\Phi$  of  $\mathbf{G}$  is of type  $B_l, C_l, D_l, E_6$  or  $E_7$  and  $Z(\mathbf{G}^F) \neq 1$ , hence, in particular,  $\mathbf{G}^F \neq {}^3D_{4,sc}(q)$ .

3.A. *Transfer to twisted groups.* Recall the notation from Section 2. We set  $V := \langle n_\alpha(\pm 1) \mid \alpha \in \Phi \rangle \leq N_{\mathbf{G}}(\mathbf{T})$  and  $H := V \cap \mathbf{T}$ . Since a set of generators of  $V$  satisfies the braid relations, computations with the Steinberg relations show that  $H$  is an elementary abelian 2-group of rank  $|\Delta|$ . We define  $v \in \mathbf{G}$  as

$$v := \begin{cases} \text{id}_{\mathbf{G}} & \text{if } d = 1, \\ \widetilde{\mathbf{w}}_0 & \text{if } d = 2, \end{cases}$$

where  $\widetilde{\mathbf{w}}_0$  is the canonical representative in  $V$  of the longest element of  $W$  defined as in [Spä10, Def. 3.2].

LEMMA 3.2. *The torus  $\mathbf{T}$  contains a Sylow  $d$ -torus  $\mathbf{S}$  of  $(\mathbf{G}, vF)$ . Moreover,  $\mathbf{T} = C_{\mathbf{G}}(\mathbf{S})$  and  $N = TV_1$ , where  $N := N_{\mathbf{G}}(\mathbf{S})^{vF}$ ,  $T := \mathbf{T}^{vF}$  and  $V_1 := V^{vF}$ .*

*Proof.* Let  $\phi$  denote the automorphism induced by  $F$  on  $W$ . Comparing with the tables in [Spr74, §§5 and 6] one sees that  $\pi(v)\phi$  is a  $d$ -regular element of  $W\phi$  in the sense of Springer; see [Spr74, §4 and 6]. Hence the centraliser of any Sylow  $d$ -torus in  $\mathbf{G}$  is a torus.

According to [Spä10, Rem. 3.3 and Lemma 3.4] there exists some Sylow  $d$ -torus  $\mathbf{S} \leq \mathbf{T}$  of  $(\mathbf{G}, vF)$ . If  $\Phi$  is of classical type and  $F = F_0^m$ , then  $TV_1 = N$  by [Spä10, Rem. 3.3(c)]. For exceptional types, this was proven in [Spä09, Props. 6.3 and 6.4].

It remains to consider the case where  $\mathbf{G}^F = {}^2D_{l,\text{sc}}(q)$ . Here for  $d = 1$ , one uses [Spä10, Lemma 11.2] and computes using the Steinberg relations that  $H^F$  is an elementary abelian 2-group of rank  $l - 1$  and that  $\pi(V_1)$  is isomorphic to a Coxeter group of type  $B_{l-1}$  and hence to  $C_W(\phi)$ . One can see analogously for  $d = 2$  and hence  $v = \widetilde{\mathbf{w}}_0$  that  $H^{vF}$  is an elementary abelian 2-group of rank  $l - 1$ , and  $\pi(V^{vF}) = C_W(\pi(v)\phi)$  if  $w_0 \in Z(W)$  and hence  $v \in Z(V)$ . If  $w_0 \notin Z(W)$ , computations in the braid group show that  $H^{vF} = H$  and  $V^{vF} = V$ . □

Notation 3.3. Let  $e := o(v)$ , the order of  $v$ . In the following we denote by  $C_i$  the cyclic group of order  $i$ . Write  $E_1$  for the subgroup  $\Gamma_{\widetilde{K}}$  of  $\text{Aut}(\mathbf{G})$  from [GLS94, Def. 1.15.5], a group generated by certain graph automorphisms. Let  $E := C_{2em} \times E_1$  act on  $\widetilde{\mathbf{G}}^{F_0^{2em}}$  such that the first summand  $C_{2em}$  of  $E$  acts by  $\langle F_0 \rangle$  and the second by the group generated by graph automorphisms. Note that this action is faithful. Let  $\widehat{F}_0, \widehat{\gamma}, \widehat{F} \in E$  be the elements that act on  $\widetilde{\mathbf{G}}^{F_0^{2em}}$  by  $F_0, \gamma$  and  $F$ , respectively.

Note that  $E$  stabilises  $N, T, V, v$  and hence  $H, V_1$  and  $H^{vF}$ .

PROPOSITION 3.4. *Let  $\mathbf{S}$  and  $N$  be as in Lemma 3.2, and  $\widetilde{N} := N_{\widetilde{\mathbf{G}}}(\mathbf{S})^{vF}$ . Suppose that for every  $\chi \in \text{Irr}(\widetilde{N})$ , there exists some  $\chi_0 \in \text{Irr}(N \mid \chi)$  such that*



- (1)  $(\widetilde{N} \rtimes E)_{\chi_0} = \widetilde{N}_{\chi_0} \rtimes E_{\chi_0}$ , and
- (2)  $\chi_0$  has an extension  $\widetilde{\chi}_0 \in \text{Irr}(N \rtimes E_{\chi_0})$  with  $v\widehat{F} \in \ker(\widetilde{\chi}_0)$ .

Then the conclusion of [Theorem 3.1](#) holds for  $(\mathbf{G}, F)$  and  $d$ .

*Proof.* The statement is an analogue of [[CS16](#), Prop. 5.3]. The proof given there is independent of the underlying type and is based on the application of Lang’s theorem using that  $v$  is  $D$ - and hence  $E$ -invariant. It relies on the fact that conjugation by a suitable element of  $\mathbf{G}$  gives an isomorphism  $\iota : \mathbf{G} \rightarrow \mathbf{G}$  with  $\iota(\mathbf{G}^F) = \mathbf{G}^{vF}$ . Via  $\iota$  the automorphisms of  $\mathbf{G}^F$  induced by  $\widetilde{\mathbf{G}}^F \rtimes D$  coincide with the ones of  $\mathbf{G}^{vF}$  induced by  $\widetilde{\mathbf{G}}^{vF} E / \langle v\widehat{F} \rangle$ , and  $\widetilde{\mathbf{G}}^F \rtimes D \cong \widetilde{\mathbf{G}}^{vF} E / \langle v\widehat{F} \rangle$ . □

3.B. *Extension maps with respect to  $H_1 \triangleleft V_1$ .* In order to verify the assumptions of [Proposition 3.4](#) on the characters of  $N$  we label them via some so-called extension map.

*Definition 3.5* (Definition 5.7 of [[CS16](#)]). Let  $Y \triangleleft X$  and  $\mathcal{Y} \subseteq \text{Irr}(Y)$ . We say that *maximal extendibility holds for  $\mathcal{Y}$  with respect to  $Y \triangleleft X$*  if every  $\chi \in \mathcal{Y}$  extends (as irreducible character) to  $X_\chi$ . Then, an *extension map for  $\mathcal{Y}$  with respect to  $Y \triangleleft X$*  is a map

$$\Lambda : \mathcal{Y} \rightarrow \bigcup_{Y \leq I \leq X} \text{Irr}(I),$$

such that for every  $\chi \in \mathcal{Y}$ , the character  $\Lambda(\chi) \in \text{Irr}(X_\chi)$  is an extension of  $\chi$ . If  $\mathcal{Y} = \text{Irr}(Y)$ , we also say that there exists *an extension map with respect to  $Y \triangleleft X$* .

The following is easily verified:

**LEMMA 3.6.** *Let  $X$  be a finite group,  $Y \triangleleft X$  and  $\mathcal{Y} \subseteq \text{Irr}(Y)$  an  $X$ -stable subset. Assume there exists an extension map for  $\mathcal{Y}$  with respect to  $Y \triangleleft X$ . Then there exists an  $X$ -equivariant extension map for  $\mathcal{Y}$  with respect to  $Y \triangleleft X$ .*

In order to prove [Theorem 3.1](#) in the form suggested by [Proposition 3.4](#), our next goal is to establish the following intermediate step. Recall  $V_1 = V^{vF}$ , and set  $H_1 := H^{vF}$ .

**THEOREM 3.7.** *There exists a  $V_1 E$ -equivariant extension map with respect to  $H_1 \triangleleft V_1$ .*

The proof will be given in several steps. We first consider the case when  $F$  is untwisted and  $d = 1$ .

**PROPOSITION 3.8.** *For  $\Phi$  not of type  $D_1$ , there exists an extension map with respect to  $H \triangleleft V$ .*

*Proof.* According to [Spä09, Prop. 5.1] we can assume that  $\Phi$  is of type  $B_l$  or  $C_l$ . Assume that  $q = 3$ . Then  $V = \mathbf{N}^F$  and  $H = \mathbf{T}^F$ . Maximal extendibility holds with respect to  $H = \mathbf{T}^F \triangleleft \mathbf{N}^F$  according to [HL80, Cor. 6.11] or [Spä10, Thm. 1.1].

By assumption  $q$  is odd. Then the isomorphism types of  $H$  and  $V$  are independent of  $q$  since  $V$  and  $H$  can be described as finitely presented groups whose relations are independent of  $q$ ; see [Tit66] and [Spä, Lemma 2.3.1(b)]. Hence the considerations for  $q = 3$  already imply the statement.  $\square$

**PROPOSITION 3.9.** *For  $\Phi$  not of type  $D_l$ , there exists a  $VE_1$ -equivariant extension map with respect to  $H \triangleleft V$ .*

*Proof.* If  $\Phi$  has no graph automorphism, Proposition 3.8 together with Lemma 3.6 proves that a  $V$ -equivariant extension map with respect to  $H \triangleleft V$  exists.

If  $\Phi$  is of type  $E_6$ , the generator  $\hat{\gamma}$  of  $E_1$  corresponds to an automorphism of the associated braid group  $\mathbf{B}$ , which acts by permuting the generators. The epimorphism  $\tau : \mathbf{B} \rightarrow V$  is  $\hat{\gamma}$ -equivariant. Let  $r : W \rightarrow \mathbf{B}$  be the map from [GP00, 4.1.1] and  $w_0$  the longest element in  $W$ . Conjugation with  $r(w_0) = w_0$  acts on  $\mathbf{B}$  like  $\hat{\gamma}$  by [GP00, Lemma 4.1.9], analogously conjugating by  $\widetilde{w}_0$ , which is the image of  $r(w_0)$  under the natural epimorphism from  $\mathbf{B}$  to  $V$ , acts on  $V$  like  $\hat{\gamma}$ . Hence the automorphism induced by  $\hat{\gamma}$  on  $V$  is an inner automorphism, and hence any  $V$ -equivariant extension map is also  $VE_1$ -equivariant.  $\square$

**PROPOSITION 3.10.** *If  $\Phi$  is of type  $D_l$ , there exists a  $VE_1$ -equivariant extension map with respect to  $H \triangleleft V$ .*

*Proof.* First let us consider the case where  $\Phi$  is of type  $D_l \neq D_4$ . Let  $\iota_D : \mathbf{G} \rightarrow \overline{\mathbf{G}}$  be the embedding from 2.C. Then  $\iota_D(V) \leq \overline{V} := \langle \overline{n}_\alpha(\pm 1) \mid \alpha \in \overline{\Phi} \rangle$  and  $\iota_D(H) = \overline{H} := \langle \overline{h}_\alpha(\pm 1) \mid \alpha \in \overline{\Phi} \rangle$ . Note that  $\overline{H} = H$  and hence  $V_\lambda \leq \overline{V}_\lambda$  for every  $\lambda \in \text{Irr}(H) = \text{Irr}(\overline{H})$ . Let  $\Lambda_B$  be the  $\overline{V}$ -equivariant extension map with respect to  $\overline{H} \triangleleft \overline{V}$  from Proposition 3.8.

As explained in 2.C,  $\gamma(x) = x^{\overline{n}_{e_1}(1)\overline{h}_{e_1}(\zeta)}$  for every  $x \in \mathbf{G}$ , where  $\zeta$  is some primitive 8-th root of unity. (Note that because of our initial reductions we can assume that  $2 \nmid q$ .) Let  $\zeta' \in \overline{\mathbb{F}}_q$  be a primitive 8-th root of unity and  $t := \prod_{i=1}^l h_{e_i}(\zeta')$ . For  $n \in V$ , we have

$$[t, n] = \prod_{j \in J} h_{e_j}(\zeta'^2)$$

for a set  $J \subseteq \{1, \dots, l\}$  with  $2 \mid |J|$ . This proves  $[t, V] \subseteq H$  and hence  $V^t = V$ . Hence there is a well-defined extension map  $\Lambda_0$  given by

$$\Lambda_0(\lambda) = (\Lambda_B(\lambda)]_{V_\lambda})^t \quad \text{for all } \lambda \in \text{Irr}(H).$$

Since  $\Lambda_B$  is  $\bar{V}$ -equivariant,  $\Lambda_0$  is  $\bar{V}^t$ -equivariant. The element  $\bar{n}_{\alpha_1}(1)^t = \bar{n}_{\alpha_1}(1)h_{\alpha_1}(\zeta'^2)$  and  $\gamma$  induce the same automorphism on  $V$ , according to 2.C. Hence  $\Lambda_0$  is  $V \langle \gamma \rangle$ -equivariant.

Now assume that  $\Phi$  is of type  $D_4$ . According to the above considerations there exists some  $V \langle \hat{\gamma}_2 \rangle$ -equivariant extension map with respect to  $H \triangleleft V$  for some  $\hat{\gamma}_2 \in E_1$  of order 2. For the proof, it is sufficient to show maximal extendibility for some  $VE_1$ -transversal  $\mathbb{T} \subset \text{Irr}(H)$  with respect to  $H \triangleleft VE_1$ . We may choose  $\mathbb{T}$  such that for each  $\lambda \in \mathbb{T}$ , some Sylow 2-subgroup of  $(VE_1)_\lambda$  is contained in  $(V \langle \hat{\gamma}_2 \rangle)_\lambda$ .

Recall that  $H$  is a 2-group. According to [Isa76, Thm. 6.26], every  $\lambda \in \mathbb{T}$  extends to  $(VE_1)_\lambda$  if  $\lambda$  extends to the full preimage of a Sylow 2-subgroup of  $(VE_1)_\lambda/H$ , which is itself a Sylow 2-subgroup of  $(VE_1)_\lambda$ . By the choice of  $\mathbb{T}$  a Sylow 2-subgroup of  $(VE_1)_\lambda$  is contained in  $(V \langle \hat{\gamma}_2 \rangle)_\lambda$  for every  $\lambda \in \mathbb{T}$ . By the above,  $\lambda$  has a  $(V \langle \hat{\gamma}_2 \rangle)$ -invariant extension to  $V_\lambda$ . Since  $(V \langle \hat{\gamma}_2 \rangle)_\lambda/V_\lambda$  is cyclic,  $\lambda$  extends to  $(V \langle \gamma_2 \rangle)_\lambda$  by [Isa76, Cor. 11.22], and hence to a Sylow 2-subgroup of  $(VE_1)_\lambda$ . This proves the claim.  $\square$

In the next step we construct extension maps in the case where the Frobenius endomorphism is twisted. Recall  $V_1 = V^{vF}$  and  $H_1 = H^{vF}$ .

LEMMA 3.11. *Let  $F_0$ ,  $m$  and  $\gamma$  be defined as in 2.B. Assume that  $\Phi$  is of type  $D_l$ ,  $v = \text{id}_G$  and  $F = \gamma F_0^m$ . Then there exists a  $V_1E_1$ -equivariant extension map with respect to  $H_1 \triangleleft V_1$ .*

*Proof.* By the proof of Lemma 3.2,  $V_1/H_1$  is isomorphic to  $C_W(\gamma)$ .

Let  $\Delta = \{\alpha_1, \dots, \alpha_l\}$  be a base of  $\Phi$ . For a positive integer  $i$  and elements  $x, y \in V$ , let  $\text{prod}(x, y, i)$  be defined by

$$\text{prod}(x, y, i) = \underbrace{x \cdot y \cdot x \cdot y \cdots}_i.$$

Following [Tit66] the group  $V$  coincides with the extended Weyl group of  $G$  that is the finitely presented group generated by  $n_i = n_{\alpha_i}(1)$  and  $h_i = n_i^2$ , subject to the relations

$$h_i h_j = h_j h_i, \quad h_i^2 = 1,$$

$$\text{prod}(n_i, n_j, m_{ij}) = \text{prod}(n_j, n_i, m_{ij}), \quad h_i^{n_j} = h_j^{A_{i,j}} h_i \text{ for all } 1 \leq i, j \leq l,$$

where  $m_{ij}$  is the order of  $s_{\alpha_i} s_{\alpha_j}$  in  $W$  and  $(A_{i,j})$  is the associated Cartan matrix; see [Spä, Lemma 2.3.1(b)] for more details.

Assume that  $\Delta$  is chosen such that the graph automorphism  $\gamma$  of order 2 permutes  $\alpha_1$  and  $\alpha_2$ . Using straightforward calculations one sees that the elements  $n'_2 := n_{\alpha_1}(-1)n_{\alpha_2}(-1)$  and  $n'_i := n_{\alpha_i}(-1)$  for  $i > 2$  satisfy the defining relations of an extended Weyl group of type  $B_{l-1}$ . As the orders of the groups

coincide, they are isomorphic. Together with Proposition 3.8 this implies the existence of the required extension map.

Note that according to Lemma 3.6 the extension map can be chosen to be  $V_1$ -equivariant. Since by definition  $\gamma$  acts trivially on  $V_1$ , the extension map is also  $V_1E_1$ -equivariant. □

*Proof of Theorem 3.7.* The statement follows from the existence of a  $V_1E_1$ -equivariant extension map since  $\widehat{F}_0$  acts trivially on  $V$ .

If  $\Phi$  is of type  $E_6$ , the claim is implied by [Spä09, Lemma 8.2]. In the remaining cases Propositions 3.9 and 3.10 and Lemma 3.11 imply the statement if  $d = 1$ .

If  $d = 2$  and  $\Phi$  is of type  $B_l, C_l$  or  $E_7$ , the proof of [Spä09, Lemma 6.1] shows that  $v \in Z(V)$ . Hence  $H = H^{vF} = H_1$  and  $V = V^{vF} = V_1$ . Then Proposition 3.8 yields the claim.

The only remaining case is when  $\Phi$  is of type  $D_l$  and  $d = 2$ . Computations in  $V$  show that either  $V_1 = V$  or  $V_1 = C_V(\gamma)$ , and then the statement about the maximal extendibility follows from the observations made for  $d = 1$  in Proposition 3.10 and Lemma 3.11. Hence there exists a  $V_1E_1$ -equivariant extension map with respect to  $H_1 \triangleleft V_1$  in all cases. □

We next state a lemma that helps to construct extensions with specific properties.

LEMMA 3.12. *Let  $\lambda \in \text{Irr}(H_1)$  and  $\tilde{\lambda} \in \text{Irr}(V_{1,\lambda})$  a  $(V_1E)_\lambda$ -invariant extension of  $\lambda$ . Then  $\tilde{\lambda}$  has an extension  $\hat{\lambda} \in \text{Irr}((V_1E)_\lambda)$  with  $\hat{\lambda}(v\widehat{F}) = 1$ .*

*Proof.* Recall  $E_1 = \langle \widehat{\gamma} \rangle \leq E$  when  $\mathbf{G}^F \neq D_{4,\text{sc}}(q)$  and  $E_1 = \langle \widehat{\gamma}_2, \widehat{\gamma}_3 \rangle$  otherwise, with  $\widehat{\gamma}_i$  of order  $i$ . Note that  $\widehat{F}_0$  is central in  $V_1 \rtimes E$ , i.e.,  $V_1E = (V_1 \rtimes E_1) \times \langle \widehat{F}_0 \rangle$ . Since all Sylow subgroups of  $E_1$  are cyclic,  $\tilde{\lambda}$  extends to a character  $\psi$  of  $(V_1E_1)_{\tilde{\lambda}}$ . Note that  $(V_1E_1)_{\tilde{\lambda}} = (V_1E_1)_\lambda$ .

Recall that  $\mathbf{G}^F \neq {}^3D_{4,\text{sc}}(q)$  and  $v\widehat{F} = v\kappa\widehat{F}_0^m$  for some  $\kappa \in E_1$ . We have  $o(\widehat{F}_0^m) = 2o(v)$  by the definition of  $E$  and  $o(v\kappa) \mid (2o(v))$  since  $v$  and  $\kappa$  commute. Accordingly there exists some character  $\varepsilon \in \text{Irr}(\langle \widehat{F}_0 \rangle)$  with  $\psi(1)\varepsilon(\widehat{F}_0^m) = \psi(v\kappa)^{-1}$ . The character  $\hat{\lambda} = \psi \times \varepsilon$  is an extension of  $\tilde{\lambda}$  with the required properties. □

3.C. *Parametrisation of  $\text{Irr}(N)$ .* For the later understanding of the characters of  $\text{Irr}(N)$ , we construct an extension map with respect to  $T \triangleleft N$ . Recall  $N := N_{\mathbf{G}}(\mathbf{S})^{vF}$  and  $T := C_{\mathbf{G}}(\mathbf{S})^{vF} = \mathbf{T}^{vF}$ .

COROLLARY 3.13. *There exists an extension map  $\Lambda$  with respect to  $T \triangleleft N$  such that*

- (1)  $\Lambda$  is  $N \rtimes E$ -equivariant; and
- (2) for every  $\lambda \in \text{Irr}(T)$ , there exists some linear  $\tilde{\lambda} \in \text{Irr}((N \rtimes E)_\lambda \mid \Lambda(\lambda))$  with  $\tilde{\lambda}(v\widehat{F}) = 1$ .

Note that the existence of  $\Lambda$  (without the properties required here) is known from [HL80, Cor. 6.11] for  $d = 1$  and from [Spä09] and [Spä12].

*Proof.* According to Lemma 3.2 we have  $N = TV_1$ . Let  $\Lambda_0$  be the  $V_1E$ -equivariant extension map with respect to  $H_1 \triangleleft V_1$  from Theorem 3.7. We obtain an  $NE$ -equivariant extension map  $\Lambda$  by sending  $\lambda \in \text{Irr}(T)$  to the common extension of  $\lambda$  and  $\Lambda_0(\lambda)|_{H_1}$   $\Big|_{V_1, \lambda}$ . According to the proof of [Spä09, Lemma 4.3],  $\Lambda$  is then well defined.

For proving (2), let  $\lambda \in \text{Irr}(T)$ . Then  $\lambda_0 := \lambda|_{H_1}$  extends to some  $\tilde{\lambda}_0 \in \text{Irr}((V_1E)_{\lambda_0})$  with  $\tilde{\lambda}_0(v\widehat{F}) = 1$  by Lemma 3.12. According to the proof of [Spä09, Lemma 4.3] there exists a unique common extension  $\tilde{\lambda}$  of  $\Lambda(\lambda)$  and  $\tilde{\lambda}_0|_{(V_1E)_\lambda}$  to  $(NE)_\lambda$ . Then  $\tilde{\lambda}(v\widehat{F}) = 1$ .  $\square$

For later use, we describe the action of  $\widetilde{NE}$  on the extension map  $\Lambda$  from Corollary 3.13. Recall  $\widetilde{T} := \widetilde{\mathbf{T}}^{vF}$  and  $\widetilde{N} := N_{\widetilde{\mathbf{G}}}(\mathbf{S})^{vF}$ . In the following we set  $W(\lambda) := N_\lambda/T$  for  $\lambda \in \text{Irr}(T)$  and  $W(\tilde{\lambda}) := N_{\tilde{\lambda}}/T$  for  $\tilde{\lambda} \in \text{Irr}(\widetilde{T})$ .

PROPOSITION 3.14. *Let  $\lambda \in \text{Irr}(T)$ ,  $\tilde{\lambda} \in \text{Irr}(\widetilde{T}|\lambda)$ ,  $x \in \widetilde{NE}$ , and let  $\Lambda$  be the extension map from Corollary 3.13. Then the character  $\delta \in \text{Irr}(W(\lambda)^x)$  with  $\delta\Lambda(\lambda^x) = \Lambda(\lambda)^x$  satisfies  $\ker(\delta) \geq W(\tilde{\lambda}^x)$ .*

*Proof.* Observe that  $\delta$  is well defined by [Isa76, Cor. 6.17]. Since  $\Lambda$  is  $NE$ -equivariant,  $\delta$  associated with  $x$  is trivial whenever  $x \in NE$ . For  $x \in \widetilde{T}$ , we have  $\lambda^x = \lambda$  so  $\delta$  has the stated property. Taking those two results together we obtain the claim.  $\square$

As mentioned earlier the extension map constructed above is key to a labelling and understanding of the characters of  $\text{Irr}(N)$ .

PROPOSITION 3.15. *Let  $\Lambda$  be the extension map from Corollary 3.13 with respect to  $T \triangleleft N$ . Then the map*

$$\Pi : \mathcal{P} = \{(\lambda, \eta) \mid \lambda \in \text{Irr}(T), \eta \in \text{Irr}(W(\lambda))\} \longrightarrow \text{Irr}(N), \quad (\lambda, \eta) \longmapsto (\Lambda(\lambda)\eta)^N,$$

*is surjective and satisfies*

- (1)  $\Pi(\lambda, \eta) = \Pi(\lambda', \eta')$  if and only if there exists some  $n \in N$  such that  ${}^n\lambda = \lambda'$  and  ${}^n\eta = \eta'$ .
- (2)  $\sigma\Pi(\lambda, \eta) = \Pi(\sigma\lambda, \sigma\eta)$  for every  $\sigma \in E$ .
- (3) Let  $t \in \widetilde{T}$ , and let  $\nu_t \in \text{Irr}(N_\lambda)$  be the linear character given by  ${}^t\Lambda(\lambda) = \Lambda(\lambda)\nu_t$ . Then  $N_{\tilde{\lambda}} = \ker(\nu_t)$  for any  $\tilde{\lambda} \in \text{Irr}(\langle T, t \rangle|\lambda)$ . For  $\tilde{\lambda}_0 \in \text{Irr}(\widetilde{T}|\lambda)$ , the map  $\widetilde{T} \rightarrow \text{Irr}(N_\lambda/N_{\tilde{\lambda}_0})$  given by  $t \mapsto \nu_t$  is surjective, and  ${}^t\Pi(\lambda, \eta) = \Pi(\lambda, \eta\nu_t)$ .

*Proof.* The arguments from [CS16, Prop. 5.11] can be transferred to prove the statement. Straightforward considerations show that the map in 3.15(3) is surjective.  $\square$

3.D. *Maximal extendibility with respect to  $W(\tilde{\lambda}) \triangleleft W(\lambda)$ .* Our aim in this subsection is to show that maximal extendibility holds with respect to  $W(\tilde{\lambda}) \triangleleft N_{W_1E}(W(\tilde{\lambda}))$  for every  $\tilde{\lambda} \in \text{Irr}(\tilde{T})$  with  $W_1 := \pi(N)$ , where  $W(\tilde{\lambda}) := N_{\tilde{\lambda}}/T$ . Two less general results are known in particular cases: For  $d = 1$ , maximal extendibility is known to hold with respect to  $W(\tilde{\lambda}) \triangleleft W(\lambda)$ , where  $\lambda = \tilde{\lambda}|_T$ ; see Proposition 3.16 below. Proposition 5.12 of [CS16] shows the analogue for arbitrary positive integers  $d$  assuming that the underlying root system is of type  $A_l$ .

The statement in Theorem 3.17 plays a crucial role in proving Theorem 3.18 via the parametrisation of characters of  $N$  given above.

We start by rephrasing the old result known for  $d = 1$ .

PROPOSITION 3.16. *Assume that  $d = 1$ . Let  $\lambda \in \text{Irr}(T)$  and  $\tilde{\lambda} \in \text{Irr}(\tilde{T}|\lambda)$ . Then maximal extendibility holds with respect to  $W(\tilde{\lambda}) \triangleleft W(\lambda)$ .*

*Proof.* The quotient  $W(\lambda)/W(\tilde{\lambda})$  is abelian and for every  $\eta \in \text{Irr}(W(\lambda))$  every character  $\eta_0 \in \text{Irr}(W(\tilde{\lambda}) \mid \eta)$  has multiplicity one in the restriction  $\eta|_{W(\tilde{\lambda})}$ , see [Bon06, 13.13(a)]. This implies the statement.  $\square$

THEOREM 3.17. *Let  $\lambda \in \text{Irr}(T)$ ,  $\tilde{\lambda} \in \text{Irr}(\tilde{T}|\lambda)$  and  $W_1 := \pi(N)$ . Then every  $\eta_0 \in \text{Irr}(W(\tilde{\lambda}))$  has an extension  $\kappa \in \text{Irr}(N_{W_1E}(W(\tilde{\lambda}))_{\eta_0})$  with  $v\widehat{F} \in \ker(\kappa)$ .*

*Proof.* We first prove that maximal extendibility holds with respect to  $W(\tilde{\lambda}) \triangleleft N_{W_1E}(W(\tilde{\lambda}))$ . Let  $(\tilde{\mathbf{G}}^*, \tilde{\mathbf{T}}^*, v'F^*)$  be the dual to  $(\tilde{\mathbf{G}}, \tilde{\mathbf{T}}, vF)$  constructed as in [DM91, Def. 13.10].

Note that because of our particular choice of  $v$ , the automorphism on  $W$  induced by  $vF$  coincides with a graph automorphism  $\phi'$  on  $W$ . The character  $\tilde{\lambda}$  corresponds to a semisimple element  $s \in (\tilde{\mathbf{T}}^*)^{v'F^*}$  of the dual group  $(\tilde{\mathbf{G}}^*, v'F^*)$ . Let  $R(\tilde{\lambda})$  be the Weyl group of  $C_{\tilde{\mathbf{G}}^*}(s)$ . Since  $C_{\tilde{\mathbf{G}}^*}(s)$  is connected,  $R(\tilde{\lambda})$  is a reflection group. We have  $W(\tilde{\lambda}) = C_{R(\tilde{\lambda})}(v'F^*) = C_{R(\tilde{\lambda})}(\phi')$ . Accordingly  $W(\tilde{\lambda})$  is a reflection group and the  $\phi'$ -orbits on the roots of  $R(\tilde{\lambda})$  form a root system, which we denote by  $\Phi(\lambda)$ ; see [MT11, Thm. C.5]. Straightforward calculations show that  $\Phi(\lambda)$  is already determined by  $\lambda$ .

The group  $K := N_{W_1E}(W(\tilde{\lambda}))$  acts on  $W(\tilde{\lambda})$ ,  $R(\tilde{\lambda})$  and  $\Phi(\lambda)$  by conjugation. Let  $\Delta$  be a base of  $\Phi(\lambda)$ . Then by the properties of root systems  $K = W(\tilde{\lambda}) \text{Stab}_K(\Delta)$ , even  $K = W(\tilde{\lambda}) \rtimes \text{Stab}_K(\Delta)$ , where  $\text{Stab}_K(\Delta)$  denotes the stabiliser of  $\Delta$  in  $K$ .

First let us prove that maximal extendibility holds with respect to  $W(\tilde{\lambda}) \rtimes \text{Aut}(\Delta)$ , where  $\text{Aut}(\Delta)$  is the group of length-preserving automorphisms of  $\Delta$ .

Whenever  $\Delta$  is indecomposable the statement is true, since then all Sylow subgroups of  $\text{Aut}(\Delta)$  are cyclic. If  $\Delta = \Delta_1 \sqcup \dots \sqcup \Delta_r$  with isomorphic indecomposable systems  $\Delta_i$ , the group  $\text{Aut}(\Delta)$  is isomorphic to the wreath product  $\text{Aut}(\Delta_1) \wr S_r$ . Since maximal extendibility holds with respect to  $H^r \triangleleft H \wr S_r$  for any group  $H$  according to [Hup98, Thm. 25.6], we see that maximal extendibility holds with respect to  $W(\tilde{\lambda}) \rtimes \text{Aut}(\Delta)$  in that case. Since maximal extendibility holds for  $H_1 \times H_2 \triangleleft G_1 \times G_2$  whenever it holds for  $H_1 \triangleleft G_1$  and  $H_2 \triangleleft G_2$ , the above implies that maximal extendibility holds with respect to  $W(\tilde{\lambda}) \triangleleft W(\tilde{\lambda}) \rtimes \text{Aut}(\Delta)$ .

Now let  $C := C_K(\Delta)$ . By definition  $C \triangleleft K$ . Let  $\bar{K} := K/C$ . Then maximal extendibility holds with respect to  $W(\tilde{\lambda}) \triangleleft K$  if it holds with respect to  $\bar{R} := W(\tilde{\lambda})C/C \triangleleft \bar{K}$ . We see that  $\bar{S} := \text{Stab}_K(\Delta)/C$  is a subgroup of  $\text{Aut}(\Delta)$  and by the above maximal extendibility holds with respect to  $W(\tilde{\lambda}) \triangleleft W(\tilde{\lambda}) \rtimes \bar{S}$ . But this implies maximal extendibility with respect to  $W(\tilde{\lambda}) \triangleleft K$ . This proves the first part of the claim.

We finish by constructing the required extension  $\kappa$ . Let  $\eta_0 \in \text{Irr}(W(\tilde{\lambda}))$ . Recall  $E_1$  from 3.3. By what we showed above,  $\eta_0$  extends to some  $\widehat{F}_0$ -stable  $\kappa_1 \in \text{Irr}(N_{W_1E_1}(W(\tilde{\lambda}))_{\eta_0})$ . Since  $\pi(v\widehat{F}\widehat{F}_0^{-m}) \in \mathbf{Z}(W_1E)$  and  $N_{W_1E}(W(\tilde{\lambda}))_{\eta_0} = N_{W_1E_1}(W(\tilde{\lambda}))_{\eta_0} \times \langle \widehat{F}_0 \rangle$ , the considerations from the proof of Lemma 3.12 ensure the existence of  $\kappa$ , as required. □

3.E. *Consequences.* The previous considerations allow us also to conclude that the considered characters of  $N$  have the structure stated in Proposition 3.4.

**THEOREM 3.18.** *For every  $\chi \in \text{Irr}(\widetilde{N})$ , there exists some  $\chi_0 \in \text{Irr}(N|\chi)$  with the following properties:*

- (1)  $(\widetilde{N} \rtimes E)_{\chi_0} = \widetilde{N}_{\chi_0} \rtimes E_{\chi_0}$ , and
- (2)  $\chi_0$  has an extension  $\tilde{\chi}_0 \in \text{Irr}(N \rtimes E_{\chi_0})$  with  $v\widehat{F} \in \ker(\tilde{\chi}_0)$ .

*Proof.* Let  $\chi_1 \in \text{Irr}(N|\chi)$  and  $(\lambda, \eta) \in \mathcal{P}$  with  $\chi_1 = \Pi(\lambda, \eta)$  for the map  $\Pi$  from Proposition 3.15. Let  $\tilde{\lambda} \in \text{Irr}(\widetilde{T}|\lambda)$  and  $\eta_0 \in \text{Irr}(W(\tilde{\lambda}))$  such that  $\eta \in \text{Irr}(W(\lambda)|\eta_0)$ . By Clifford correspondence there exists a unique character  $\eta_1 \in \text{Irr}(W(\lambda)_{\eta_0}|\eta_0)$  such that  $\eta = \eta_1^{W(\lambda)}$ . Now since  $W(\lambda)/W(\tilde{\lambda})$  is abelian and as by Proposition 3.16 maximal extendibility holds with respect to  $W(\tilde{\lambda}) \triangleleft W(\lambda)$ , the character  $\eta_1$  is an extension of  $\eta_0$ .

Let  $W_1 := \pi(N)$ . According to Theorem 3.17 there exists an extension  $\tilde{\eta}_0 \in \text{Irr}(W(\lambda)_{\eta_0})$  of  $\eta_0$  that is  $N_{W_1E}(W(\tilde{\lambda}))_{\eta_0}$ -invariant. The character  $\eta' := (\tilde{\eta}_0)^{W(\lambda)}$  is irreducible. Hence  $\chi_0 := \Pi(\lambda, \eta')$  is a well-defined character of  $N$ .

We show that  $\chi_0$  is  $\tilde{N}$ -conjugate to  $\chi_1$ : The map  $\tilde{T} \rightarrow \text{Irr}(W(\lambda)/W(\tilde{\lambda}))$ ,  $t \mapsto \nu_t$ , from Proposition 3.15(3) is surjective, so there exists  $t \in \tilde{T}$  such that  $\eta' = \eta\nu_t$ . This proves  ${}^t\chi = {}^t\Pi(\lambda, \eta) = \Pi(\lambda, \eta\nu_t) = \chi_0$ .

For analysing the stabiliser of  $\chi_0$ , let  $t \in \tilde{T}$  and  $e \in E$  such that  $\chi_0^{te} = \chi_0$ . Then there exists some  $n \in N$  such that  $(\lambda, \eta') = (\lambda^{ne}, (\eta')^{ne}\nu_t)$ . Without loss of generality  $n$  can be chosen such that  $\pi(n)e \in N_{W_1E}(W(\tilde{\lambda}))_{\eta_0}$ . By the choice of  $\tilde{\eta}_0$ ,  $\pi(n)e$  stabilises  $\tilde{\eta}_0$ , hence  $(\lambda^{ne}, (\eta')^{ne}) = (\lambda, \eta')$  and  $\chi_0^e = \chi_0$ . This proves the equation in (1).

By Corollary 3.13 the character  $\Lambda(\lambda)$  has an extension  $\tilde{\lambda}$  to  $(NE)_{\Lambda(\lambda)\eta'}$  with  $\tilde{\lambda}(v\hat{F}) = 1$ . On the other hand, the character  $\tilde{\eta}_0$  can be chosen to have an extension  $\hat{\eta}_0$  to  $N_{W_1E}(W(\tilde{\lambda}))_{\eta_0}$  with  $v\hat{F} \in \ker(\hat{\eta}_0)$ ; see Theorem 3.17. We denote by  $\kappa_1$  the lift of  $\hat{\eta}_0|_{N_{W_1E}(W(\tilde{\lambda}))_{\eta_0,\lambda}}$  to  $(NE)_{\eta_0,\lambda}$ . Then  $\kappa_2 := (\kappa_1)^{(NE)\lambda,\eta}$  is irreducible with  $v\hat{F} \in \ker(\kappa_2)$ . The character  $(\tilde{\lambda}\kappa_2)^{(NE)\chi_0}$  is an extension of  $\chi_0$  with the required properties. □

Via Proposition 3.4 the above proves Theorem 3.1. For later, two further consequences of our considerations are important. First we give the following interpretation of Theorem 3.1:

LEMMA 3.19. *Let  $\mathbf{S}_0$ ,  $N_0$ ,  $\tilde{N}_0$  and  $O_0$  be defined as in Theorem 3.1. For  $\psi_0 \in \text{Irr}(N_0)$ , the following are equivalent:*

- (i)  $O_0 = (\tilde{\mathbf{G}}^F \cap O_0) \rtimes (D \cap O_0)$ ,
- (ii)  $(\tilde{\mathbf{G}}^F D)_{\mathbf{S}_0,\psi_0} = \tilde{N}_{0,\psi_0}(\mathbf{G}^F \rtimes D)_{\mathbf{S}_0,\psi_0}$ .

*Proof.* By the definition of  $O_0$  one deduces (ii) from (i) by considering the stabiliser of  $\mathbf{S}_0$ :

$$\begin{aligned} (\tilde{\mathbf{G}}^F D)_{\mathbf{S}_0,\psi_0} &= (O_0)_{\mathbf{S}_0} = ((\tilde{\mathbf{G}}^F \cap O_0) \rtimes (D \cap O_0))_{\mathbf{S}_0} \\ &= ((\tilde{\mathbf{G}}^F \cap O_0)(\mathbf{G}^F D \cap O_0))_{\mathbf{S}_0} = (\tilde{\mathbf{G}}^F \cap O_0)_{\mathbf{S}_0}(\mathbf{G}^F D \cap O_0)_{\mathbf{S}_0} \\ &= \tilde{N}_{0,\psi_0}(\mathbf{G}^F D)_{\mathbf{S}_0,\psi_0}. \end{aligned}$$

Here, recall that by [BM92, Thm. 3.4] all Sylow  $d$ -tori of  $(\mathbf{G}, F)$  are  $\mathbf{G}^F$ -conjugate and the  $(D \cap O_0)$ -conjugates of  $\mathbf{S}_0$  are Sylow  $d$ -tori.

Multiplying the equation in (ii) with  $\mathbf{G}^F$  gives

$$O_0 = (\tilde{\mathbf{G}}^F \cap O_0)((\mathbf{G}^F \rtimes D) \cap O_0).$$

Since  $\mathbf{G}^F \leq (\mathbf{G}^F \rtimes D) \cap O_0$ , this gives (i). □

PROPOSITION 3.20. *Let  $\mathbf{S}_0$ ,  $N_0$  and  $\tilde{N}_0$  be defined as in Theorem 3.1. Let  $\tilde{C}_0 := C_{\tilde{\mathbf{G}}^F}(\mathbf{S}_0)$ . Then there exists some  $N_{\tilde{\mathbf{G}}^F D}(\mathbf{S}_0)$ -equivariant extension map  $\tilde{\Lambda}$  with respect to  $\tilde{C}_0 \triangleleft \tilde{N}_0$ , such that in addition  $\tilde{\Lambda}(\tilde{\lambda} \delta|_{\tilde{C}_0}) = \tilde{\Lambda}(\tilde{\lambda}) \delta|_{\tilde{N}_0}$  for every  $\tilde{\lambda} \in \text{Irr}(\tilde{C}_0)$  and  $\delta \in \text{Irr}(\tilde{\mathbf{G}}^F|_{1_{\mathbf{G}^F}})$ .*



*Proof.* The considerations from the proof of [CS16, Cor. 5.14] can be transferred: Applying the isomorphism  $\iota$  from Proposition 3.4 shows that it is sufficient to verify that there exists some  $NE$ -equivariant extension map  $\tilde{\Lambda}$  with respect to  $\tilde{T} \triangleleft \tilde{N}$ , such that in addition  $\tilde{\Lambda}(\tilde{\lambda} \delta|_T) = \tilde{\Lambda}(\tilde{\lambda}) \delta|_{\tilde{N}}$  for every  $\tilde{\lambda} \in \text{Irr}(\tilde{T})$  and  $\delta \in \text{Irr}(\tilde{\mathbf{G}}^{vF}|_{\mathbf{1}_{\mathbf{G}^{vF}}})$ . Let  $\Lambda$  be the  $NE$ -equivariant extension map with respect to  $T \triangleleft N$  from Corollary 3.13 and

$$\tilde{\Lambda} : \text{Irr}(T) \rightarrow \bigcup_{T \leq I \leq N} \text{Irr}(I)$$

be the map sending  $\tilde{\lambda} \in \text{Irr}(\tilde{T})$  to the unique common extension of  $\tilde{\lambda}$  and  $\Lambda(\lambda)|_{N_{\tilde{\lambda}}}$ , where  $\lambda := \tilde{\lambda}|_T$ . Then according to [Spä09, Lemma 4.3],  $\tilde{\Lambda}$  is well defined and has the required properties.  $\square$

The next statement is later applied to verify assumption 2.1(i.7) in the considered cases.

**COROLLARY 3.21.** *For the groups  $N_0$  and  $\tilde{N}_0$ , from Theorem 3.1 maximal extendibility holds with respect to  $N_0 \triangleleft \tilde{N}_0$ .*

*Proof.* Like in the proof of the preceding proposition the isomorphism  $\iota$  from the proof of Proposition 3.4 allows us to prove the statement by establishing that maximal extendibility holds with respect to  $N \triangleleft \tilde{N}$ . Let  $\tilde{\Lambda}$  be the extension map with respect to  $\tilde{T} \triangleleft \tilde{N}$  from (the proof of) Proposition 3.20. Then every character  $\tilde{\psi} \in \text{Irr}(\tilde{N})$  is of the form  $(\tilde{\Lambda}(\tilde{\lambda})\eta_0)^{\tilde{N}}$  for some  $\tilde{\lambda} \in \text{Irr}(\tilde{T})$  and  $\eta_0 \in \text{Irr}(W(\tilde{\lambda}))$ . Thus

$$\begin{aligned} \tilde{\psi}|_N &= (\tilde{\Lambda}(\tilde{\lambda})\eta_0)^{\tilde{N}}|_N = \left( (\tilde{\Lambda}(\tilde{\lambda})\eta_0)|_{N_{\tilde{\lambda}}} \right)^N = \left( \Lambda(\lambda)|_{N_{\tilde{\lambda}}} \eta_0 \right)^N \\ &= \left( (\Lambda(\lambda)|_{N_{\tilde{\lambda}}} \eta_0)^{N_{\tilde{\lambda}}} \right)^N = \left( \Lambda(\lambda)(\eta_0^{N_{\tilde{\lambda}}}) \right)^N, \end{aligned}$$

where  $\lambda := \tilde{\lambda}|_T$ . According to Theorem 3.17 maximal extendibility holds with respect to  $W(\tilde{\lambda}) \triangleleft W(\lambda)$ . Since  $W(\lambda)/W(\tilde{\lambda})$  is abelian,  $\eta_0^{N_{\tilde{\lambda}}}$  and hence  $\tilde{\psi}|_N$  is multiplicity-free. This proves the statement.  $\square$

#### 4. The action of $\text{Aut}(G)$ on Harish-Chandra induced characters

The aim of this section is to verify that assumptions of Theorem 2.1 concerning the characters of  $G$  are satisfied. For this, we describe the action of  $\text{Aut}(G)$  on Harish-Chandra induced characters in terms of their parameters. Thus we first have to recall how one obtains the parametrisation of those characters. We follow here the treatment of the subject given in [Car85, Chap. 10], which is based on the results of [HL80] and [HL83].

We consider the following slightly more general setting. Let  $G$  be a finite group with a split  $BN$ -pair of characteristic  $p$ . We write  $W = N/(N \cap B)$  for the Weyl group of  $G$ , which we assume to be of crystallographic type. Then there is a root system  $\Phi$  attached to  $W$  and we let  $\Delta$  denote a base of  $\Phi$  corresponding to the simple reflections of  $W$ . We write  $s_\alpha \in W$  for the reflection along the root  $\alpha \in \Phi$ .

Let  $P \leq G$  be a standard parabolic subgroup with standard Levi subgroup  $L$  and Levi decomposition  $P = U \rtimes L$ . Let  $N(L) := (N_G(L) \cap N)L$ . We choose and fix once and for all an  $N(L)$ -equivariant extension map  $\Lambda$  for  $L \triangleleft N(L)$ , which exists according to [Gec93] and [Lus84, Thm. 8.6].

Let  $\lambda$  be an irreducible cuspidal character of  $L$ . Via the Levi decomposition  $\lambda$  can be inflated to a character of  $P$ . Let  $M$  be a left  $\mathbb{C}P$ -module affording  $\lambda$ , and denote by  $\rho$  the corresponding representation. Let  $\mathfrak{F}(\rho)$  be the vector space of  $\mathbb{C}$ -linear maps  $f : \mathbb{C}G \rightarrow M$  satisfying

$$f(px) = \rho(p)f(x) \quad \text{for all } p \in P \text{ and } x \in \mathbb{C}G.$$

This vector space becomes a  $\mathbb{C}G$ -module via

$$(4.1) \quad (g \star f)(x) = f(xg) \quad \text{for all } g \in G, f \in \mathfrak{F}(\rho) \text{ and } x \in \mathbb{C}G.$$

We denote by  $R_L^G(\lambda)$  the character of  $G$  afforded by this module. It is known that  $R_L^G(\lambda)$  only depends on  $\lambda$ , not on the choice of  $P$  or of  $\rho$  (see [DD93, HL94]). The set of constituents of  $R_L^G(\lambda)$  is called the Harish-Chandra series above  $(L, \lambda)$  and will be denoted by  $\mathcal{E}(G, (L, \lambda))$ . The union of Harish-Chandra series associated with  $N \cap B$  and its characters is called the *principal series of  $G$* .

4.A. *Actions of automorphisms on the standard basis.* Let  $\sigma$  be an automorphism of  $G$  stabilising  $P, L$ , and the  $BN$ -pair. Recall that  $\sigma$  acts on the class functions on  $G$  via  $\chi \mapsto \sigma\chi$ , where  $\sigma\chi(g) = \chi(\sigma^{-1}(g))$  for all  $g \in G$ . It is immediate from the definitions that  $(L, \sigma\lambda)$  is again a cuspidal pair of  $G$ .

The character  $\sigma R_L^G(\lambda)$  is afforded by the  $\mathbb{C}G$ -module  $\sigma\mathfrak{F}(\rho)$  obtained from the vector space  $\mathfrak{F}(\rho)$  together with the  $G$ -action

$$(4.2) \quad (g \star_\sigma f)(x) = f(x\sigma^{-1}(g)) \quad \text{for all } g \in G, f \in \mathfrak{F}(\rho), \text{ and } x \in \mathbb{C}G.$$

One easily sees that  $\text{End}_{\mathbb{C}G}(\mathfrak{F}(\rho))$  and  $\text{End}_{\mathbb{C}G}(\sigma\mathfrak{F}(\rho))$  can be canonically identified via  ${}^\sigma B(f) := B(f)$  for  $B \in \text{End}_{\mathbb{C}G}(\mathfrak{F}(\rho))$  and  $f \in \mathfrak{F}(\rho)$ . Let  $\mathfrak{F}(\sigma\rho)$  be the coinduced module associated to  $\sigma\rho$  defined as above. Then  $\iota : \sigma\mathfrak{F}(\rho) \rightarrow \mathfrak{F}(\sigma\rho)$  given by

$$f \mapsto \sigma f \text{ with } \sigma f(x) = f(\sigma^{-1}(x)) \text{ for all } x \in \mathbb{C}G$$

defines a  $\mathbb{C}G$ -module isomorphism. Moreover  $B \mapsto \iota \circ B \circ \iota^{-1}$  for  $B \in \text{End}_{\mathbb{C}G}(\sigma\mathfrak{F}(\rho))$  induces an isomorphism from  $\text{End}_{\mathbb{C}G}(\sigma\mathfrak{F}(\rho))$  to  $\text{End}_{\mathbb{C}G}(\mathfrak{F}(\sigma\rho))$ .

We denote by  $\hat{\iota} : \text{End}_{\mathbb{C}G}(\mathfrak{F}(\rho)) \rightarrow \text{End}_{\mathbb{C}G}(\sigma\mathfrak{F}(\rho)) \rightarrow \text{End}_{\mathbb{C}G}(\mathfrak{F}(\sigma\rho))$  the composed isomorphism.

Since we are interested in the irreducible constituents of  $R_L^G(\lambda)$  and  $R_L^G(\sigma\lambda)$ , which are parametrised by the isomorphism classes of irreducible modules of  $\text{End}_{\mathbb{C}G}(\mathfrak{F}(\rho))$  and of  $\text{End}_{\mathbb{C}G}(\mathfrak{F}(\sigma\rho))$  respectively (see [Car85, Prop. 10.1.2]), we will need to compute  $\hat{\iota}(B)$  for some elements  $B \in \text{End}_{\mathbb{C}G}(\mathfrak{F}(\rho))$ . We start by determining  $\hat{\iota}$  on a natural basis of  $\text{End}_{\mathbb{C}G}(\mathfrak{F}(\rho))$ .

With  $N(L) = (N_G(L) \cap N)L$  let  $W_G(L) := N(L)/L$ , the *relative Weyl group* of  $L$  in  $G$ , and set  $W(\lambda) := N(L)_\lambda/L$ . For  $w \in W(\lambda)$ , we denote by  $\dot{w} \in N(L)$  a once and for all chosen preimage under the natural map. We let  $\Phi_L \subseteq \Phi$  denote the root system of  $W_L$ , with simple system  $\Delta_L \subseteq \Delta$ .

Let  $\tilde{\rho}$  be an extension of  $\rho$  to  $N(L)_\lambda$  affording the extension  $\Lambda(\lambda)$  from our chosen equivariant extension map  $\Lambda$ . For  $w \in W_G(L)$ , let  $B_{w,\rho} \in \text{End}_{\mathbb{C}G}(\mathfrak{F}(\rho))$  be defined by

$$(B_{w,\rho}f)(x) = \tilde{\rho}(\dot{w})f(\dot{w}^{-1}e_Ux) \quad \text{for all } f \in \mathfrak{F}(\rho) \text{ and } x \in \mathbb{C}G,$$

where  $e_U := \frac{1}{|U|} \sum_{u \in U} u$  is the idempotent associated to the unipotent radical  $U$  of  $P$ . Note that  $B_{w,\rho}$  is independent of the actual choice of  $\dot{w}$ .

Analogously we define  $B_{w,\sigma\rho}$  by using the extension  $\tilde{\rho}'$  of  $\sigma\rho$  affording  $\Lambda(\sigma\lambda)$ . Note that  $\tilde{\rho}'$  and  $\sigma(\tilde{\rho})$  then may differ. We denote by  $\delta_{\lambda,\sigma} \in \text{Irr}(W(\sigma\lambda))$  the character of  $N(L)_{\sigma\lambda}$  with  $\delta_{\lambda,\sigma}\Lambda(\sigma\lambda) = \sigma\Lambda(\lambda)$ . This character is well defined by [Isa76, Cor. 6.17].

For  $w \in W(\lambda)$ , let  $B_{w,\sigma\rho} \in \text{End}_{\mathbb{C}G}(\mathfrak{F}(\sigma\rho))$  be defined via

$$(B_{w,\sigma\rho}f)(x) = \tilde{\rho}'(\dot{w})f(\dot{w}^{-1}e_Ux) \quad \text{for all } f \in \mathfrak{F}(\sigma\rho) \text{ and } x \in \mathbb{C}G.$$

LEMMA 4.1. *For all  $w \in W(\lambda)$ , we have  $\hat{\iota}(B_{w,\rho}) = \delta_{\lambda,\sigma}(\sigma(w))B_{\sigma(w),\sigma\rho}$ .*

*Proof.* Indeed, for  $f \in \mathfrak{F}(\sigma\rho)$  and  $x \in \mathbb{C}G$ , we have

$$\hat{\iota}(B_{w,\rho})(f)(x) = \tilde{\rho}(\dot{w})f(\sigma(w)^{-1}e_U\sigma^{-1}(x)),$$

which agrees with

$$\delta_{\lambda,\sigma}(\sigma(w))B_{\sigma(w),\sigma\rho}(f)(x) = \tilde{\rho}(\dot{w})f(\sigma(\dot{w})^{-1}e_Ux)$$

as  $\sigma(e_U) = e_U$ . □

4.B. *The decomposition of  $W(\lambda)$ .* In order to transfer our results to the  $T_{w,\rho}$ -basis of the endomorphism algebra we need to recall the semi-direct product decomposition of  $W(\lambda)$ ; see [HL80, §§2 and 4]. Define

$$\hat{\Omega} := \{\alpha \in \Phi \setminus \Phi_L \mid w(\Delta_L \cup \{\alpha\}) \subseteq \Delta \text{ for some } w \in W\},$$

and for  $\alpha \in \hat{\Omega}$ , set  $v(\alpha) := w_0^L w_0^\alpha$ , where  $w_0^L, w_0^\alpha$  are the longest elements in  $W_L, \langle W_L, s_\alpha \rangle$  respectively. Then let  $\Omega := \{\alpha \in \hat{\Omega} \mid v(\alpha)^2 = 1\}$ . Note that  $\Omega$  is  $\sigma$ -invariant. For  $\alpha \in \Omega$ , let  $L_\alpha$ , denote the standard Levi subgroup of  $G$

corresponding to the simple system  $\Delta_L \cup \{\alpha\}$ . Then  $L$  is a standard Levi subgroup of  $L_\alpha$ . We write  $p_{\alpha,\lambda} \geq 1$  for the ratio between the degrees of the two different constituents of  $R_L^{L_\alpha}(\lambda)$ . Let

$$\Phi_\lambda := \{\alpha \in \Omega \mid s_\alpha \in W(\lambda), p_{\alpha,\lambda} \neq 1\},$$

a root system with set of simple roots  $\Delta_\lambda \subseteq \Phi_\lambda \cap \Phi^+$ . Let  $R(\lambda) := \langle s_\alpha \mid \alpha \in \Phi_\lambda \rangle$  be its Weyl group. Then  $W(\lambda)$  satisfies  $W(\lambda) = R(\lambda) \rtimes C(\lambda)$ , where the group  $C(\lambda)$  is the stabiliser of  $\Delta_\lambda$  in  $W(\lambda)$ ; see [Car85, Prop. 10.6.3].

LEMMA 4.2. *We have  $p_{\alpha,\lambda} = p_{\sigma(\alpha),\sigma\lambda}$  for all  $\alpha \in \Phi_\lambda$  and hence  $R(\sigma\lambda) = \sigma(R(\lambda))$  and  $C(\sigma\lambda) = \sigma(C(\lambda))$ .*

*Proof.* By definition we have  ${}^\sigma R_L^{L_\alpha}(\lambda) = R_L^{L_{\sigma(\alpha)}}(\sigma\lambda)$  since  $\sigma$  stabilises  $U$ . This implies  $p_{\alpha,\lambda} = p_{\sigma(\alpha),\sigma\lambda}$  by its definition. □

For  $w \in W$ , we set  $\text{ind}(w) := |U_0 \cap (U_0)^{w_0 w}|$ , where  $U_0$  is the unipotent radical of  $B$  and  $w_0 \in W$  is the longest element. Also, for  $\alpha \in \Delta_\lambda$  a simple root of  $\Phi_\lambda$ , we define  $\varepsilon_{\alpha,\lambda} \in \{\pm 1\}$  by

$$(4.3) \quad B_{s_\alpha,\rho}^2 = \frac{1}{\text{ind}(s_\alpha)} \text{id} + \varepsilon_{\alpha,\lambda} \frac{p_{\alpha,\lambda} - 1}{\sqrt{\text{ind}(s_\alpha)p_{\alpha,\lambda}}} B_{s_\alpha,\rho}$$

(see [Car85, Prop. 10.7.9]). Here, the square root is always taken positive.

LEMMA 4.3. *If  $R(\sigma\lambda) \leq \ker(\delta_{\lambda,\sigma})$ , then  $\text{ind}(\sigma(s_\alpha)) = \text{ind}(s_\alpha)$  and  $\varepsilon_{\sigma(\alpha),\sigma\lambda} = \varepsilon_{\alpha,\lambda}$  for all  $\alpha \in \Delta_\lambda$ .*

*Proof.* Let  $\alpha \in \Delta_\lambda$ , and set  $s := s_\alpha$ ,  $\alpha' := \sigma(\alpha)$ ,  $s' := s_{\alpha'}$ ,  $\lambda' := \sigma\lambda$ ,  $\rho' := \sigma\rho$ . Applying  $\hat{\iota}$  to Equation (4.3) we obtain

$$\hat{\iota}(B_{s,\rho}^2) = \frac{1}{\text{ind}(s)} \text{id} + \varepsilon_{\alpha,\lambda} \frac{p_{\alpha,\lambda} - 1}{\sqrt{\text{ind}(s)p_{\alpha,\lambda}}} \hat{\iota}(B_{s,\rho}).$$

Now  $p_{\alpha',\lambda'} = p_{\alpha,\lambda}$  by Lemma 4.2, and since  $\sigma$  stabilises  $U_0$  and  $w_0$ , we also have  $\text{ind}(s') = \text{ind}(s)$ . Then Lemma 4.1 yields

$$\delta_{\lambda,\sigma}(s')^2 B_{s',\rho'}^2 = \frac{1}{\text{ind}(s')} \text{id} + \varepsilon_{\alpha,\lambda} \frac{p_{\alpha',\lambda'} - 1}{\sqrt{\text{ind}(s')p_{\alpha',\lambda'}}} \delta_{\lambda,\sigma}(s') B_{s',\rho'}.$$

Since  $s' \in \sigma(R(\lambda)) = R(\sigma\lambda)$ , the assumption  $R(\lambda') \leq \ker(\delta_{\lambda,\sigma})$  allows us to simplify this to

$$B_{s',\rho'}^2 = \frac{1}{\text{ind}(s')} \text{id} + \varepsilon_{\alpha,\lambda} \frac{p_{\alpha',\lambda'} - 1}{\sqrt{\text{ind}(s')p_{\alpha',\lambda'}}} B_{s',\rho'}.$$

The claim follows by comparison with (4.3) for  $B_{s',\rho'}$ . □

Now for  $\alpha \in \Delta_\lambda$ , set  $T_{s_\alpha,\rho} := \varepsilon_{\alpha,\lambda} \sqrt{\text{ind}(s_\alpha)p_{\alpha,\lambda}} B_{s_\alpha,\rho}$ ; for  $w \in R(\lambda)$  with a reduced expression  $w = s_1 \cdots s_r$  with  $s_i = s_{\alpha_i}$  simple reflections (so

$\alpha_i \in \Delta_\lambda$ , let  $T_{w,\rho} := T_{s_{1,\rho}} \cdots T_{s_r,\rho}$ ; for  $w \in C(\lambda)$  define  $T_{w,\rho} := \sqrt{\text{ind}(w)} B_{w,\rho}$ , and then for  $w \in W(\lambda)$  with  $w = w_1 w_2$ , where  $w_1 \in C(\lambda)$ ,  $w_2 \in R(\lambda)$ , let  $T_{w,\rho} := T_{w_1,\rho} T_{w_2,\rho}$ . This does not depend on the choice of reduced expressions; see [Car85, Prop. 10.8.2]. Then we have

PROPOSITION 4.4. *If  $R(\sigma\lambda) \leq \ker(\delta_{\lambda,\sigma})$ , then for all  $w \in W(\lambda)$  we have*

$$\widehat{\iota}(T_{w,\rho}) = \delta_{\lambda,\sigma}(\sigma(w)) T_{\sigma(w),\sigma\rho}.$$

*Proof.* First assume that  $w = s_\alpha =: s$  for some  $\alpha \in \Delta_\lambda$ . Then

$$\begin{aligned} \widehat{\iota}(T_{s,\rho}) &= \widehat{\iota}(\varepsilon_{\alpha,\lambda} \sqrt{\text{ind}(s)} p_{\alpha,\lambda} B_{s,\rho}) \\ &= \varepsilon_{\alpha,\lambda} \sqrt{\text{ind}(s)} p_{\alpha,\lambda} \widehat{\iota}(B_{s,\rho}) = \varepsilon_{\alpha,\lambda} \sqrt{\text{ind}(s)} p_{\alpha,\lambda} \delta_{\lambda,\sigma}(s') B_{s',\sigma\rho} \end{aligned}$$

by Lemma 4.1, where  $s' = \sigma(s)$ ,  $\alpha' = \sigma(\alpha)$ . From Lemmas 4.2 and 4.3 we know  $p_{\alpha',\lambda'} = p_{\alpha,\lambda}$ ,  $\text{ind}(s') = \text{ind}(s)$  and  $\varepsilon_{\alpha',\lambda'} = \varepsilon_{\alpha,\lambda}$ . So indeed

$$\widehat{\iota}(T_{s,\rho}) = \delta_{\lambda,\sigma}(s') \varepsilon_{\alpha',\lambda'} \sqrt{\text{ind}(s')} p_{\alpha',\lambda'} B_{s',\sigma\rho} = \delta_{\lambda,\sigma}(s') T_{s',\sigma\rho}.$$

Next, if  $w \in C(\lambda)$ , then

$$\begin{aligned} \widehat{\iota}(T_{w,\rho}) &= \sqrt{\text{ind}(w)} \widehat{\iota}(B_{w,\rho}) \\ &= \delta_{\lambda,\sigma}(\sigma(w)) \sqrt{\text{ind}(w)} B_{\sigma(w),\sigma\rho} = \delta_{\lambda,\sigma}(\sigma(w)) T_{\sigma(w),\sigma\rho}. \end{aligned}$$

In the general case, let  $w \in W(\lambda)$  with  $w = w_1 w_2$ , where  $w_1 \in C(\lambda)$ , and  $w_2 \in R(\lambda)$  has a reduced expression  $w_2 = s_1 \cdots s_r$ . Then by the above we get

$$\begin{aligned} \widehat{\iota}(T_{w,\rho}) &= \widehat{\iota}(T_{w_1,\rho}) \widehat{\iota}(T_{s_1,\rho}) \cdots \widehat{\iota}(T_{s_r,\rho}) \\ &= \delta_{\lambda,\sigma}(\sigma(w_1)) \left( \prod_{i=1}^r \delta_{\lambda,\sigma}(\sigma(s_i)) \right) T_{\sigma(w_1),\sigma\rho} T_{\sigma(s_1),\sigma\rho} \cdots T_{\sigma(s_r),\sigma\rho} \\ &= \delta_{\lambda,\sigma}(\sigma(w_1 s_1 \cdots s_r)) T_{\sigma(w_1),\sigma\rho} T_{\sigma(s_1 \cdots s_r),\sigma\rho} = \delta_{\lambda,\sigma}(\sigma(w)) T_{\sigma(w),\sigma\rho} \end{aligned}$$

as claimed. □

4.C. *Central-primitive idempotents of  $\text{End}_{\mathbb{C}G}(\mathfrak{F}(\rho))$ .* Next we describe the central primitive idempotents of  $\text{End}_{\mathbb{C}G}(\mathfrak{F}(\rho))$ . It is well known (see, e.g., [Car85, Prop. 10.9.2]) that  $\text{End}_{\mathbb{C}G}(\mathfrak{F}(\rho))$  is a symmetric algebra with symmetrising trace defined by the linear map  $\tau_\rho : \text{End}_{\mathbb{C}G}(\mathfrak{F}(\rho)) \rightarrow \mathbb{C}$  with

$$\tau_\rho(T_{w,\rho}) = \begin{cases} 1 & w = 1, \\ 0 & w \neq 1. \end{cases}$$

Let us denote by  $\{T_{w,\rho}^\vee\}$  the basis dual to  $\{T_{w,\rho}\}$  with respect to the bilinear form associated with  $\tau_\rho$ . Thus

$$T_{w,\rho}^\vee = p_{w,\lambda}^{-1} T_{w^{-1},\rho} \quad \text{for } w \in W(\lambda)$$

where  $p_{w,\lambda} := \prod_{\alpha \in \Phi_\lambda^+, w(\alpha) < 0} p_{\alpha,\lambda}$  (see [Car85, pp. 348–349] or [GP00, 8.1.1]). Note that via  $\hat{\iota}$ ,  $\tau_\rho$  defines a symmetrising trace  $\tau_{\sigma\rho}$  on  $\text{End}_{\mathbb{C}G}(\mathfrak{F}(\sigma\rho))$  with

$$\tau_{\sigma\rho}(T_{w,\sigma\rho}) = \begin{cases} 1 & w = 1, \\ 0 & w \neq 1. \end{cases}$$

Let  $M$  be a simple  $\text{End}_{\mathbb{C}G}(\mathfrak{F}(\rho))$ -module and  $\eta$  its character. It can be considered as a submodule of  $e_{\eta,\rho}\mathfrak{F}(\rho)$ , where

$$e_{\eta,\rho} := \frac{1}{c_{\eta,\rho}} \sum_{w \in W(\lambda)} \eta(T_{w,\rho}) T_{w,\rho}^\vee$$

denotes the central-primitive idempotent of  $\text{End}_{\mathbb{C}G}(\mathfrak{F}(\rho))$  corresponding to  $M$  (see [GP00, 7.2.7(c)]). Here,  $c_{\eta,\rho}$  is the Schur element associated to  $\eta$  as in [GP00, Thm. 7.2.1].

PROPOSITION 4.5. *Assume that  $R(\sigma\lambda) \leq \ker(\delta_{\lambda,\sigma})$ . There exists a simple  $\text{End}_{\mathbb{C}G}(\mathfrak{F}(\sigma\rho))$ -module with character  $\eta'$  such that*

$$\eta'(T_{\sigma(w),\sigma\rho}) = \delta_{\lambda,\sigma}^{-1}(w) \eta(T_{w,\rho}) \quad \text{for all } w \in W(\lambda).$$

The  $\mathbb{C}G$ -modules  $e_{\eta,\rho}\mathfrak{F}(\rho)$  and  $(e_{\eta',\sigma\rho}\mathfrak{F}(\sigma\rho))^\sigma$  are isomorphic.

*Proof.* Since  $\hat{\iota}$  is an isomorphism of algebras, we see from Proposition 4.4 that if  $R(\sigma\lambda) \leq \ker(\delta_{\lambda,\sigma})$ , then

$$\hat{\iota}(e_{\eta,\rho}) = \frac{1}{c_{\eta,\rho}} \sum_{w \in W(\lambda)} \eta(T_{w,\rho}) \delta_{\lambda,\sigma}(\sigma(w^{-1})) T_{\sigma(w),\sigma\rho}^\vee$$

is a central-primitive idempotent of  $\text{End}_{\mathbb{C}G}(\mathfrak{F}(\rho^\sigma))$ . Let  $\eta'$  be the character of the associated simple  $\text{End}_{\mathbb{C}G}(\mathfrak{F}(\sigma\rho))$ -module. Then comparison of coefficients between  $\hat{\iota}(e_{\eta,\rho})$  and

$$e_{\eta',\sigma\rho} = \frac{1}{c_{\eta',\sigma\rho}} \sum_{w \in W(\sigma\lambda)} \eta'(T_{w,\sigma\rho}) T_{w,\sigma\rho}^\vee$$

at  $w = 1$  gives

$$\frac{\eta(T_{1,\rho})}{c_{\eta,\rho}} = \frac{\eta'(T_{1,\sigma\rho})}{c_{\eta',\sigma\rho}}.$$

Since  $T_{1,\rho}$  is the identity element of  $\text{End}_{\mathbb{C}G}(\mathfrak{F}(\rho))$ , and  $\eta, \eta'$  have the same degree, this implies  $c_{\eta,\rho} = c_{\eta',\sigma\rho}$ . Then comparison of coefficients at arbitrary  $w \in W(\lambda)$  gives the first statement. The second is also clear as  $\hat{\iota}$  is a  $\mathbb{C}G$ -module isomorphism. □

4.D. *The generic algebra  $\mathcal{H}$ .* We next analyse in more detail the bijection between  $\text{Irr}(\text{End}_{\mathbb{C}G}(\mathfrak{F}(\rho)))$  and  $\text{Irr}(W(\lambda))$  using the approach presented in [HL83, §4]. The main idea is to introduce a generic algebra over a polynomial ring  $\mathbb{C}[u_\alpha \mid \alpha \in \Delta_\lambda]$ . One specialisation then gives the endomorphism algebra  $\text{End}_{\mathbb{C}G}(\mathfrak{F}(\rho))$  and another specialisation gives the group algebra

$\mathbb{C}W(\lambda)$ . Application of these specialisations to the irreducible characters defines a parametrisation of the constituents of  $R_L^G(\lambda)$  by  $\text{Irr}(W(\lambda))$ .

Let  $\mathbf{u} = (u_\alpha \mid \alpha \in \Delta_\lambda)$  be indeterminates with  $u_\alpha = u_\beta$  if and only if  $\alpha$  and  $\beta$  are conjugate under  $W(\lambda)$ . Let  $K$  be an algebraic closure of the quotient field of the Laurent series ring  $A_0 = \mathbb{C}[\mathbf{u}^{\pm 1}]$ , and let  $A$  be the integral closure of  $A_0$  in  $K$ . Let  $\mathcal{H}$  be the free  $A$ -module with basis  $\{a_w \mid w \in W(\lambda)\}$ . According to [HL83, 4.1] one can define a unique  $A$ -bilinear associative multiplication on  $\mathcal{H}$  such that for all  $x \in C(\lambda)$ ,  $w \in W(\lambda)$  and  $\alpha \in \Delta_\lambda$ , one has

$$\begin{aligned} a_x a_w &= a_{xw} \text{ and } a_w a_x = a_{wx}, \\ a_{s_\alpha} a_w &= \begin{cases} a_{s_\alpha w} & \text{if } w^{-1}\alpha \in \Phi_\lambda^+, \\ u_\alpha a_{s_\alpha w} + (u_\alpha - 1)a_w & \text{if } w^{-1}\alpha \notin \Phi_\lambda^+, \end{cases} \\ a_w a_{s_\alpha} &= \begin{cases} a_{ws_\alpha} & \text{if } w\alpha \in \Phi_\lambda^+, \\ u_\alpha a_{ws_\alpha} + (u_\alpha - 1)a_w & \text{if } w\alpha \notin \Phi_\lambda^+. \end{cases} \end{aligned}$$

Any homomorphism  $f : A \rightarrow \mathbb{C}$  induces a right  $A$ -module structure on the field  $\mathbb{C}$ , so we obtain from  $\mathcal{H}$  a  $\mathbb{C}$ -algebra  $\mathcal{H}^f := \mathbb{C} \otimes_A \mathcal{H}$  with  $\mathbb{C}$ -vector space basis  $\{1 \otimes a_w \mid w \in W(\lambda)\}$ . The structure constants of  $\mathcal{H}^f$  are obtained from the ones of  $\mathcal{H}$  by applying  $f$ .

By [HL83, 4.2] the morphisms  $f_0, g_0 : A_0 \rightarrow \mathbb{C}$  defined by  $f_0(u_\alpha) = p_{\alpha,\lambda}$  and  $g_0(u_\alpha) = 1$  for  $\alpha \in \Delta_\lambda$  can be extended to morphisms  $f, g : A \rightarrow \mathbb{C}$ . Then  $\mathcal{H}^f$  is isomorphic to  $\text{End}_{\mathbb{C}G}(\mathfrak{F}(\rho))$  via  $1 \otimes a_w \mapsto T_{w,\rho}$  and  $\mathcal{H}^g$  is isomorphic to  $\mathbb{C}W(\lambda)$  via  $1 \otimes a_w \mapsto w$ .

By [HL83, 4.7] the map  $\eta \mapsto \eta^f$  with  $\eta^f(1 \otimes a_w) := f(\eta(a_w))$  defines a bijection between the set of  $K$ -characters associated to simple  $K \otimes_A \mathcal{H}$ -modules and the characters associated to simple  $\mathcal{H}^f$ -modules. The analogous result holds for  $\mathcal{H}^g$ . This combines to give a bijection between  $\text{Irr}(W(\lambda))$  and  $\text{Irr}(\text{End}_{\mathbb{C}G}(\mathfrak{F}(\rho)))$  and thus provides a labelling of the irreducible constituents of  $R_L^G(\lambda)$  by  $\text{Irr}(W(\lambda))$ : for  $\eta \in \text{Irr}(W(\lambda))$ , we denote by  $R_L^G(\lambda)_\eta$  the irreducible character of  $G$  occurring in  $e_{\eta',\rho^f} \mathfrak{F}(\rho)$ , where  $\eta'$  is the  $K$ -character of  $\mathcal{H}$  with  $\eta'^g = \eta$ .

Together with Proposition 4.5 this proves

**THEOREM 4.6.** *If  $R(\sigma\lambda) \leq \ker(\delta_{\lambda,\sigma})$ , then for  $\eta \in \text{Irr}(W(\lambda))$ , we have*

$$\sigma(R_L^G(\lambda)_\eta) = R_L^G(\sigma\lambda)_{\eta'}$$

with  $\eta' := \sigma\eta\delta_{\lambda,\sigma}^{-1}$ .

4.E. *Uniqueness of parametrisation.* So far, our parametrisation of constituents of  $R_L^G(\lambda)$  and hence also the assertion of Theorem 4.6 both depend on the choice of the parabolic subgroup  $P$  containing  $L$ , although the character  $R_L^G(\lambda)$  itself is independent of that choice; see [DD93], [HL94]. The following result extends [McG82, Thm. 2.12] from the case of a torus to an arbitrary

Levi subgroup. It allows us to compare the different parametrisations of the constituents of  $R_L^G(\lambda)$  by  $\text{Irr}(W(\lambda))$  for different parabolic subgroups  $P$  containing  $L$ .

**THEOREM 4.7.** *Let  $n \in N(L)$ . Assume that the parametrisation of the constituents of  $R_L^G(\lambda)$  and  $R_L^G({}^n\lambda)$  is obtained using the same parabolic subgroup  $P$  of  $G$  with  $L \leq P$  and extensions of  $\lambda$  and  ${}^n\lambda$  given by an  $N(L)$ -equivariant extension map. Then*

$$R_L^G(\lambda)_\eta = R_L^G({}^n\lambda)_{n\eta},$$

where  ${}^n\eta \in \text{Irr}(W({}^n\lambda))$  is the character with  ${}^n\eta({}^n x) = \eta(x)$  for  $x \in W(\lambda)$ .

*Proof.* Write  $w$  for the image of  $n$  in  $W$ . Note that by multiplying  $n$  by elements of  $L$  we may assume that  $w$  fixes  $\Delta_L$ , and also that  $w$  preserves the set of positive roots  $\Phi_\lambda^+$  (by multiplying with a suitable element from  $R(\lambda)$ ). By [Car85, 10.1.3], for  $v \in W$ , the map

$$\theta_v : \mathfrak{F}(\rho) \rightarrow \mathfrak{F}({}^v\rho), \quad \theta_v(f)(x) := f(vex) \quad \text{for } f \in \mathfrak{F}(\rho), x \in G,$$

is a homomorphism of  $G$ -modules. Moreover, it is invertible by [Car85, 10.5.1, 10.5.3]. Now let  $v \in W(\lambda)$  and set  $v' := wvw^{-1}$ . It then follows by [Car85, 10.7.5] that

$$\theta_w\theta_v = \sqrt{\frac{\text{ind}(wv)}{\text{ind}(w)\text{ind}(v)}}\theta_{wv} \quad \text{and} \quad \theta_{v'}\theta_w = \sqrt{\frac{\text{ind}(v'w)}{\text{ind}(w)\text{ind}(v')}}\theta_{v'w},$$

so that

$$\theta_w\theta_v\theta_w^{-1} = \sqrt{\frac{\text{ind}(v')}{\text{ind}(v)}}\theta_{v'}.$$

Now we have that  $B_{v,\rho} = \tilde{\rho}(v) \circ \theta_v$ , and that  $\theta_w \circ \tilde{\rho}(v) = \tilde{\rho}(v) \circ \theta_w$  by the argument given in the proof of [Car85, Prop. 10.2.4]. Thus

$$\begin{aligned} \theta_w \circ B_{v,\rho} \circ \theta_w^{-1} &= \theta_w \circ \tilde{\rho}(v) \circ \theta_v \circ \theta_w^{-1} = \tilde{\rho}(v) \circ \theta_w \circ \theta_v \circ \theta_w^{-1} \\ &= \tilde{\rho}(v) \circ \sqrt{\frac{\text{ind}(v')}{\text{ind}(v)}}\theta_{v'} = \sqrt{\frac{\text{ind}(v')}{\text{ind}(v)}}{}^n\tilde{\rho}(v') \circ \theta_{v'} = \sqrt{\frac{\text{ind}(v')}{\text{ind}(v)}}B_{v',{}^n\rho}. \end{aligned}$$

Comparing the quadratic polynomials satisfied by  $B_{s_\alpha,\rho}$  and  $B_{w s_\alpha, {}^n\rho}$  we see that  $\varepsilon_{\alpha,\lambda}$  and  $\varepsilon_{w(\alpha),w\lambda}$  agree for  $\alpha \in \Delta_\lambda$ . Also, as conjugation by  $n$  does not change the degrees of the two constituents of  $R_L^{L_\alpha}(\lambda)$ , we have  $p_{\alpha,\lambda} = p_{w(\alpha),w\lambda}$ . Thus, the isomorphism of  $G$ -modules  $\theta_w$  sends the standard generators  $T_{v,\rho}$  of  $\text{End}_{\mathbb{C}G}(\mathfrak{F}(\rho))$  to the generators  $T_{v',{}^n\rho}$  of  $\text{End}_{\mathbb{C}G}(\mathfrak{F}({}^n\rho))$ . (This can also be seen by use of the explicit isomorphism given by [HL94, Thm. 2.3].) It then follows from our construction of the central primitive idempotents and the specialisation argument as in the proof of Theorem 4.6 that conjugation by  $n$  sends the character of  $\text{End}_{\mathbb{C}G}(\mathfrak{F}(\rho))$  parametrised by  $\eta \in \text{Irr}(W(\lambda))$  to the character parametrised by  ${}^n\eta \in \text{Irr}(W({}^n\lambda))$ . □



### 5. The stabilisers of some Harish-Chandra induced characters

Using Proposition 3.15 for characters lying in the principal Harish-Chandra series together with the results from Section 4 we can show that Harish-Chandra induction induces an equivariant map between certain local characters and suitable characters of  $G = \mathbf{G}^F$ . For this we determine the stabilisers of characters  $\chi \in \text{Irr}(G)$  lying in a Harish-Chandra series  $\mathcal{E}(G, (L, \lambda))$  where  $L$  is a maximally split torus.

In order to assure the assumptions made in Section 4 we first describe the groups  $R(\lambda)$ . Recall that  $\mathbf{B}^F$  and  $N_{\mathbf{G}}(\mathbf{T})^F$  form a split  $BN$ -pair in  $G$ ; for example, see [MT11, Thm. 24.10]. Let  $\Phi_1$  be the associated root system of  $G$  and  $X_\alpha \leq G$  ( $\alpha \in \Phi_1$ ) the associated root subgroups. In the following we use freely the notation around Harish-Chandra induction introduced in the previous section.

LEMMA 5.1. *Assume that  $\mathbf{G}$  is not of type  $A_l$ . Let  $\lambda \in \text{Irr}(\mathbf{T}^F)$ ,  $\alpha \in \Phi_1$ , and let  $\Phi_\lambda$  be defined as in 4.B. Then  $\alpha \in \Phi_\lambda$  if and only if  $\lambda(\mathbf{T}^F \cap \langle X_\alpha, X_{-\alpha} \rangle) = 1$ . Moreover,  $R(\lambda) \leq W(\tilde{\lambda})$  for any  $\tilde{\lambda} \in \text{Irr}(\tilde{\mathbf{T}}^F|\lambda)$ .*

*Proof.* Assume first that  $F$  is a standard Frobenius endomorphism. Since  $T = \mathbf{T}^F$  is a torus, the integer  $p_{\alpha,\lambda}$  from Section 4.B is determined inside the standard Levi subgroup  $L_\alpha := \langle T, X_{\pm\alpha} \rangle$ . As  $\mathbf{G}$  is of simply connected type, the group  $K := \langle X_{\pm\alpha} \rangle$  is isomorphic to  $\text{SL}_2(q)$ . Moreover,  $T_0 := \mathbf{T} \cap \langle X_{\pm\alpha} \rangle = \langle h_\alpha(t) \mid t \in \mathbb{F}_q^\times \rangle$ . Let  $\lambda_0 := \lambda|_{T_0}$ .

From the situation in  $\text{SL}_2(q)$  we know that  $R_{T_0}^K(\lambda_0)$  splits into two constituents of different degrees if  $\lambda_0$  is trivial. Hence in that case  $p_{\alpha,\lambda} \neq 1$ . If  $\lambda_0$  is not trivial, then either the character  $R_{T_0}^K(\lambda_0)$ , and hence  $R_T^{L_\alpha}(\lambda)$ , is irreducible or it is the sum of two characters of the same degree.

The above argument remains valid for twisted Frobenius endomorphisms, possibly replacing  $\text{SL}_2(q)$  by  $\text{SL}_2(q^m)$  with  $m$  the order of the graph automorphism induced by  $F$ .

Let  $\tilde{\lambda} \in \text{Irr}(\tilde{\mathbf{T}}^F|\lambda)$ . If  $\alpha \in \Phi_\lambda$  and  $s_\alpha \in \pi(N_{\mathbf{G}}(\mathbf{T}))$  is the reflection associated with  $\alpha$ , the Steinberg relations imply

$$[\tilde{\mathbf{T}}^F, s_\alpha] \subseteq \mathbf{T}^F \cap \langle X_\alpha, X_{-\alpha} \rangle.$$

Together with the above we conclude that  $s_\alpha \in W(\tilde{\lambda})$  if  $\alpha \in \Phi_\lambda$ . Since  $R(\lambda)$  is generated by the elements  $s_\alpha$  ( $\alpha \in \Phi_\lambda$ ), this proves the statement.  $\square$

THEOREM 5.2. *Let  $T$  be a maximally split torus of  $(\mathbf{G}, F)$ . Let  $N := N_G(\mathbf{S})$  and  $\tilde{N} := N_{\tilde{\mathbf{G}}}(\tilde{\mathbf{S}})$ , where  $\mathbf{S}$  is a torus of  $\mathbf{G}$  such that  $C_G(\mathbf{S}) = T$ . Then there exists an  $\tilde{N}D$ -equivariant bijection*

$$\text{Irr}_{\text{cusp}}(N) \longrightarrow \bigcup_{\lambda \in \text{Irr}_{\text{cusp}}(T)} \mathcal{E}(G, (T, \lambda)),$$

where  $\mathcal{E}(G, (T, \lambda))$  denotes the set of constituents of  $R_T^G(\lambda)$ ,  $\text{Irr}_{\text{cusp}}(T)$  is the set of cuspidal characters of  $T$  and  $\text{Irr}_{\text{cusp}}(N) := \text{Irr}(N | \text{Irr}_{\text{cusp}}(T))$ .

*Proof.* In [Proposition 3.15](#) we gave a parametrisation of the set  $\text{Irr}_{\text{cusp}}(N)$ . As  $T$  is a maximally split torus, we have  $N = \mathbf{N}^F$ . Then  $\text{Irr}_{\text{cusp}}(T) = \text{Irr}(T)$  and hence  $\text{Irr}_{\text{cusp}}(N) = \text{Irr}(N)$ . Let  $\Lambda$  be the  $ND$ -equivariant extension map from [Corollary 3.13](#) applied with  $d = 1$  and  $v = 1$ . Then [Proposition 3.15](#) yields a map

$$\Pi : \mathcal{P} \longrightarrow \text{Irr}(N), \quad (\lambda, \eta) \longmapsto (\Lambda(\lambda)\eta)^{\mathbf{N}^F},$$

with  $\mathcal{P} = \{(\lambda, \eta) \mid \lambda \in \text{Irr}(T), \eta \in \text{Irr}(W(\lambda))\}$ . On the other hand let

$$\Pi' : \mathcal{P} \longrightarrow \bigcup_{\lambda \in \text{Irr}_{\text{cusp}}(T)} \mathcal{E}(G, (T, \lambda)), \quad (\lambda, \eta) \longmapsto R_T^G(\lambda)_\eta,$$

where  $R_T^G(\lambda)_\eta$  is defined using  $\Lambda$ . The maps  $\Pi$  and  $\Pi'$  induce bijections between the set of  $N$ -orbits in  $\mathcal{P}$  and  $\text{Irr}_{\text{cusp}}(N)$  and  $\bigcup_{\lambda \in \text{Irr}_{\text{cusp}}(T)} \mathcal{E}(G, (T, \lambda))$  respectively; see [Proposition 3.15\(1\)](#) for  $\Pi$  and [Theorem 4.7](#) for the statement about  $\Pi'$ . Accordingly the concatenation  $\Pi' \circ \Pi^{-1}$  gives the required bijection

$$\text{Irr}_{\text{cusp}}(N) \rightarrow \bigcup_{\lambda \in \text{Irr}_{\text{cusp}}(T)} \mathcal{E}(G, (T, \lambda)), \quad \Pi(\lambda, \eta) \mapsto \Pi'(\lambda, \eta).$$

Now the action of  $\widetilde{ND}$  on  $\text{Irr}(N)$  has been described in [Proposition 3.15](#) in terms of the associated labels. Analogously [Theorem 4.6](#) determines the action of  $\widetilde{TD}$  on the sets  $\mathcal{E}(G, (T, \lambda))$  in terms of the associated labels. (Note that by [Proposition 3.14](#) the map  $\Lambda$  satisfies the requirements made in [Theorem 4.6](#).) Comparing the induced actions on  $\mathcal{P}$  we see that the bijection is  $\widetilde{ND}$ -equivariant. □

**COROLLARY 5.3.** *Let  $(T, \lambda)$  be a cuspidal pair as in [Theorem 5.2](#). Then for every character  $\chi_0 \in \mathcal{E}(G, (T, \lambda))$ , there exists some  $\widetilde{G}$ -conjugate  $\chi$  such that*

$$(\widetilde{GD})_\chi = \widetilde{G}_\chi D_\chi.$$

*Proof.* Via the bijection from [Theorem 5.2](#) the character  $\chi_0$  corresponds to a character  $\psi_0 \in \text{Irr}_{\text{cusp}}(N)$ . Some  $\widetilde{N}$ -conjugate  $\psi$  of  $\psi_0$  satisfies  $(\widetilde{ND})_\psi = \widetilde{N}_\psi (ND)_\psi$ ; see [Theorem 3.1](#) together with [Lemma 3.19](#). Since the bijection from [Theorem 5.2](#) is  $\widetilde{ND}$ -equivariant, this implies

$$(\widetilde{GD})_\chi = G(\widetilde{ND})_\chi = G(\widetilde{ND})_\psi = G(\widetilde{N}_\psi (ND)_\psi) = \widetilde{G}_\chi (ND)_\chi = \widetilde{G}_\chi D_\chi,$$

where  $\chi$  corresponds to  $\psi$  via [Theorem 5.2](#), so is  $\widetilde{G}$ -conjugate to  $\chi_0$ . □

### 6. Towards the inductive McKay condition

In this section we collect the previous results to prove [Theorem 2](#) on the inductive McKay condition for primes  $\ell$  with  $d_\ell(q) = 1$ , whenever  $\mathbf{G}^F \notin \{D_{l,sc}(q), E_{6,sc}(q)\}$ . Here  $d_\ell(q)$  denotes the order of  $q$  modulo  $\ell$  if  $\ell > 2$ , respectively the order of  $q$  modulo 4 if  $\ell = 2$ . We describe under which additional assumptions this result can be extended to primes with  $d_\ell(q) = 2$  and to the missing types.

We first collect some cases in which the assertion of [Theorem 2](#) had already been proven.

**PROPOSITION 6.1.** *Assume that  $S := \mathbf{G}^F / Z(\mathbf{G}^F)$  is simple, and let  $\ell$  be a prime dividing  $|S|$ . The inductive McKay condition holds for  $S$  and  $\ell$  if one of the following is satisfied:*

- $\ell = p$ ;
- $Z(\mathbf{G}^F) = 1$ ;
- $\Phi$  is of type  $A_l$ ; or
- $\ell = 2$  and  $\mathbf{G}^F = C_{l,sc}(q)$ , where  $q$  is an odd power of an odd prime.

*Proof.* The case where  $\ell$  is the defining characteristic has been settled in [[Spä12](#), Thm. 1.1]. The case  $Z(\mathbf{G}^F) = 1$  has been considered in [[CS13](#), Thm. A, Prop. 5.2]. If  $\Phi$  is of type  $A_l$ , the statement follows from [[CS16](#), Thm. A]. According to [[Mal08a](#), Thm. 4.11] the inductive McKay condition is satisfied for  $S$  and  $\ell = 2$  whenever  $\mathbf{G}^F = C_{l,sc}(q)$  for an odd power  $q$  of an odd prime. □

**PROPOSITION 6.2.** *Let  $(\mathbf{G}, F)$  be as in [Section 2](#) and  $\ell$  a prime different from the defining characteristic with  $d = d_\ell(q) \in \{1, 2\}$ . Assume that  $\mathbf{G}^F$  and  $\ell$  are not as in [Proposition 6.1](#). Let  $\mathbf{S}$  be a Sylow  $d$ -torus of  $(\mathbf{G}, F)$ . Then [Assumption 2.1\(i\)](#) is satisfied for  $G := \mathbf{G}^F$ ,  $\tilde{G} := \tilde{\mathbf{G}}^F$ ,  $D, N := N_{\mathbf{G}^F}(\mathbf{S})$  and some Sylow  $\ell$ -subgroup  $Q$  of  $G$  such that  $N_{\tilde{\mathbf{G}}^F}(\mathbf{S}) = N_{\tilde{\mathbf{G}}^F}(Q)N$ .*

*Proof.* We can argue as in the proof of Lemma 7.1 of [[CS16](#)]. According to [[Mal07](#), Thms. 5.14 and 5.19], since  $\mathbf{G}^F$  and  $\ell$  are not as in [Proposition 6.1](#), there exists some Sylow  $\ell$ -subgroup  $Q$  of  $\mathbf{G}^F$  with  $N_{\mathbf{G}^F}(Q) \leq N_{\mathbf{G}^F}(\mathbf{S}) \leq \mathbf{G}^F$ . Since all Sylow  $d$ -tori of  $(\mathbf{G}, F)$  are  $\mathbf{G}^F$ -conjugate, we can conclude that  $N_{\mathbf{G}^F}(\mathbf{S})$  is  $\text{Aut}(\mathbf{G}^F)_Q$ -stable; see also [[CS13](#), §2.5].

Maximal extendibility for  $\mathbf{G}^F \triangleleft \tilde{\mathbf{G}}^F$  as required in [2.1\(i.6\)](#) was shown by Lusztig; see [[Lus88](#), Prop. 10] or [[CE04](#), Thm. 15.11]. The maximal extendibility with respect to  $N \triangleleft \tilde{N}$  as required in [2.1\(i.7\)](#) has been proven in [Corollary 3.21](#). □

In our next step we establish the existence of a bijection as required in [2.1\(iv\)](#).

**THEOREM 6.3.** *Let  $\ell$  be a prime such that  $d = d_\ell(q) \in \{1, 2\}$ . Moreover, let  $\mathbf{S}$  be a Sylow  $d$ -torus of  $(\mathbf{G}, F)$ ,  $N := N_G(\mathbf{S})$  and  $\tilde{N} := N_{\tilde{G}}(\mathbf{S})$ . Let  $\mathcal{G} := \text{Irr}(\tilde{G} \mid \text{Irr}_{\ell'}(G))$  and  $\mathcal{N} := \text{Irr}(\tilde{N} \mid \text{Irr}_{\ell'}(N))$ . Then there is a  $(\tilde{G} \rtimes D)_{\mathbf{S}}$ -equivariant bijection*

$$\tilde{\Omega} : \mathcal{G} \longrightarrow \mathcal{N}$$

*with  $\tilde{\Omega}(\mathcal{G} \cap \text{Irr}(\tilde{G} \mid \nu)) = \mathcal{N} \cap \text{Irr}(\tilde{N} \mid \nu)$  for every  $\nu \in \text{Irr}(\mathbf{Z}(\tilde{G}))$ , and  $\tilde{\Omega}(\chi\delta) = \tilde{\Omega}(\chi) \delta \upharpoonright_{\tilde{N}}$  for every  $\delta \in \text{Irr}(\tilde{G} \mid 1_G)$  and  $\chi \in \mathcal{G}$ .*

*Proof.* According to [Corollary 3.13](#) there exists an  $N_{\tilde{G}D}(\mathbf{S})$ -equivariant extension map  $\Lambda$  with respect to  $C_{\tilde{G}}(\mathbf{S}) \triangleleft \tilde{N}$  that is compatible with multiplication by linear characters of  $\tilde{G}$ . The considerations made in Section 6 of [\[CS16\]](#) for groups of type  $A_l$  apply in our more general situation as well. Using our map  $\Lambda$  the construction presented there gives the required bijection.  $\square$

**THEOREM 6.4.** *Let  $(\mathbf{G}, F)$  be as in [Section 2](#) and  $\ell$  a prime different from the defining characteristic of  $\mathbf{G}$  with  $d = d_\ell(q) \in \{1, 2\}$ . Assume that  $\mathbf{G}^F$  and  $\ell$  are not as in [Proposition 6.1](#) and that  $\mathbf{G}^F$  is the universal covering group of  $S = \mathbf{G}^F/\mathbf{Z}(\mathbf{G}^F)$ . Then the inductive McKay condition holds for  $S$  and  $\ell$  in any of the following cases:*

- (a)  $(\mathbf{G}, F)$  is of type  $B_l, C_l, {}^2D_l, {}^2E_6$  or  $E_7$  and  $d = 1$ ;
- (b)  $(\mathbf{G}, F)$  is of type  $D_l$  or  $E_6$ ,  $d = 1$ , and [2.1\(ii.2\)](#) holds;
- (c)  $(\mathbf{G}, F)$  is of type  $B_l, C_l, {}^2D_l, {}^2E_6$  or  $E_7$ ,  $d = 2$  and [2.1\(ii.1\)](#) holds; or
- (d)  $(\mathbf{G}, F)$  is of type  $D_l$  or  $E_6$ ,  $d = 2$ , and [2.1\(ii.1\)](#) and [2.1\(ii.2\)](#) hold.

*Proof.* This is proven by an application of [Theorem 2.1](#). We successively ensure that the necessary assumptions are satisfied.

The groups  $G := \mathbf{G}^F$ ,  $\tilde{G} := \tilde{\mathbf{G}}^F$ ,  $D$ ,  $N$  and  $Q$  are chosen as in [Proposition 6.2](#) and satisfy accordingly the assumptions made in [2.1\(i\)](#). For this group  $N$ , the characters satisfy [assumption 2.1\(iii\)](#) according to [Theorem 3.1](#).

Let  $\chi \in \text{Irr}(\mathbf{G}^F)$  lie in a Harish-Chandra series  $\mathcal{E}(\mathbf{G}^F, (\mathbf{T}^F, \lambda))$  for some character  $\lambda \in \text{Irr}(\mathbf{T}^F)$ . Then [assumption 2.1\(ii.1\)](#) holds for  $\chi$  (after suitable  $\tilde{\mathbf{G}}^F$ -conjugation) according to [Corollary 5.3](#).

Now assume that  $d = 1$ . Then according to [\[Mal07, Prop. 7.3\]](#) each character in  $\text{Irr}_{\ell'}(\mathbf{G}^F)$  lies in a Harish-Chandra series  $\mathcal{E}(\mathbf{G}^F, (\mathbf{T}^F, \lambda))$  for some character  $\lambda \in \text{Irr}(\mathbf{T}^F)$ . So [assumption 2.1\(ii.1\)](#) holds again by [Corollary 5.3](#). On the other hand, [2.1\(ii.2\)](#) clearly holds whenever  $D$  is cyclic or by assumption. If  $(\mathbf{G}, F)$  of type  $B_l, C_l, {}^2D_l, {}^2E_6$  or  $E_7$ , then  $D$  is cyclic.

Whenever  $d \in \{1, 2\}$ , the bijection from [Theorem 6.3](#) has the properties required in [2.1\(iv\)](#). Altogether this proves the above statements.  $\square$

*Remark 6.5.* Note that the equation given in 2.1(ii.1) has only to be checked for characters of  $\ell'$ -degree of  $G$  that are not  $\widetilde{G}$ -invariant since every  $\chi \in \text{Irr}(G)$  with  $\widetilde{G}_\chi = \widetilde{G}$  satisfies  $(\widetilde{G} \rtimes D)_\chi = \widetilde{G} \rtimes D_\chi$ . In particular, only characters in Lusztig rational series  $\mathcal{E}(G, s)$  have to be considered where the centraliser of  $s$  in the dual group  $\mathbf{G}^*$  is not connected, since characters in Lusztig series corresponding to elements with connected centralisers are  $\widetilde{G}$ -invariant according to [Lus88, Prop. 5.1]; see also [CE04, Cor. 15.14]. Furthermore, 2.1(ii) holds whenever  $\text{Aut}(S)/S$  is cyclic since then  $(\widetilde{G} \rtimes D)_\chi / (GZ(\widetilde{G}))$  is cyclic and hence coincides with  $(\widetilde{G}_\chi \rtimes D_\chi) / (GZ(\widetilde{G}))$ .

**Theorem 2** in the case where  $\mathbf{G}^F$  is a universal covering group and  $d_\ell(q) = 1$  is now part (a) of the preceding theorem. For odd primes  $\ell$ , the following completes the proof of **Theorem 2**. The cases where the universal covering group of a simple group  $S$  is not of the form  $\mathbf{G}^F$  can be determined by Table 6.1.4 of [GLS94]. Then the Schur multiplier of  $S$  is said to have a nontrivial exceptional part; see [GLS94, §6.1].

**LEMMA 6.6.** *Let  $S$  be a simple group of Lie type with a nontrivial exceptional part of the Schur multiplier, and let  $\ell$  be a prime dividing  $|S|$ . Assume that  $\ell$  is the defining characteristic or  $d_\ell(q) \in \{1, 2\}$ . Then the inductive McKay condition holds for  $S$  and  $\ell$ .*

*Proof.* If  $S$  is a Suzuki or Ree group, the result is known from [IMN07, Thm. 16.1] and [CS13, Thm. A]. Otherwise  $S \cong \mathbf{G}^F / Z(\mathbf{G}^F)$  for some pair  $(\mathbf{G}, F)$  as in Section 2. If  $\ell$  is the defining characteristic of  $\mathbf{G}$ , the statement follows from [Spä12, Thm. 1.11]. If  $(\mathbf{G}, F)$  is of type  $A_l, {}^2A_l, F_4,$  or  $G_2$ , then the claim is known by [CS13, Thm. A] and [CS16, Thm. A].

In the other cases the considerations from [CS16, §7] can be transferred: the inductive McKay condition holds for  $S$  if it holds for any pair  $(S, Z)$  in the sense of [CS16, Def. 7.3], where  $Z$  is a cyclic  $\ell'$ -quotient of the Schur multiplier of  $S$ ; see [CS16, Lemma 7.4(a)]. Taking into account [Mal08b, Thm. 1.1] it is sufficient to prove the claim in the cases where  $Z$  is a quotient of the nonexceptional Schur multiplier of  $S$ .

According to [GLS94, Table 6.1.3] all Sylow subgroups of  $D$  are cyclic and every automorphism of  $\mathbf{G}^F$  is induced by  $\widetilde{\mathbf{G}}^F D$  for groups  $\widetilde{\mathbf{G}}^F$  and  $D$  defined as in Section 2. Further, in those cases any Sylow subgroup of the outer automorphism group of  $S$  is cyclic and hence 2.1(ii) holds. The proofs of **Theorem 6.4** and [Spä12, Thm. 2.12] imply that the inductive McKay condition holds for  $(S, Z)$  in those missing cases if  $d = d_\ell(q) \in \{1, 2\}$ . □

### 7. The McKay conjecture for $\ell = 2$

In this section we prove **Theorem 1** from the introduction. Let  $\mathbf{G}$  be a connected reductive linear algebraic group over an algebraically closed field

of characteristic  $p$  and  $F : \mathbf{G} \rightarrow \mathbf{G}$  a Steinberg endomorphism defining an  $\mathbb{F}_q$ -structure on  $\mathbf{G}$  such that  $G := \mathbf{G}^F$  has no component of Suzuki or Ree type.

7.A. *Degree polynomials.* We begin by defining degree polynomials for the irreducible characters of  $G = \mathbf{G}^F$ , which play a crucial role in our arguments. These are probably known to (some) experts, but we have not been able to find a convenient reference. Let  $\mathbb{G}$  be the complete root datum of  $(\mathbf{G}, F)$ . Then there is a monic integral polynomial  $|\mathbb{G}| \in \mathbb{Z}[X]$  such that  $|\mathbf{G}^{F^m}| = |\mathbb{G}|(q^m)$  for all natural numbers  $m$  prime to the order of the automorphism induced by  $F$  on the Weyl group  $W$  of  $\mathbf{G}$  (see [BM92, 1C]). The same then also holds for any connected reductive  $F$ -stable subgroup of  $\mathbf{G}$ , like maximal tori or connected components of centralisers. Furthermore, by work of Lusztig the unipotent characters of any finite reductive group with complete root datum  $\mathbb{G}$  are parameterised uniformly, and the degree of a unipotent character  $\chi$  of  $\mathbf{G}^F$  is given by  $f_\chi(q)$  for a suitable polynomial  $f_\chi \in \mathbb{Q}[X]$  depending only on the parameter of  $\chi$ ; see [BMM93, §1B]. Now let  $\chi \in \text{Irr}(G)$  be arbitrary. Then  $\chi$  lies in the Lusztig series  $\mathcal{E}(G, s)$  of a semisimple element  $s$  of the dual group  $G^* = \mathbf{G}^{*F}$ , and Lusztig’s Jordan decomposition of characters gives a bijection

$$(7.1) \quad \Psi : \mathcal{E}(G, s) \rightarrow \mathcal{E}(C_{G^*}(s), 1) \quad \text{such that} \quad \chi(1) = |G^* : C_{G^*}(s)|_{p'} \Psi(\chi)(1),$$

where  $\mathcal{E}(C_{G^*}(s), 1)$  denotes the unipotent characters of  $C_{G^*}(s)$ ; see [DM91, Thm. 13.25]. While this bijection is not defined canonically, the formula in loc. cit. for scalar products with Deligne–Lusztig characters shows that its uniform projection is and hence, in particular, so is the correspondence of degrees. Moreover, by the description in [Lus88, Prop. 5.1], the multiplicities of unipotent characters of  $C_{G^*}^\circ(s)$  in those of  $C_{G^*}(s)$  are determined by the complete root datum of  $(\mathbf{G}, F)$ , hence generic, so there exist well-defined degree polynomials  $f_\psi$  for the unipotent characters  $\psi$  of the possibly disconnected group  $C_{G^*}(s)$ . Thus, denoting by  $|\mathbb{G}_s|$  the order polynomial of  $C_{G^*}(s)$ , we can define from (7.1) the *degree polynomial*  $f_\chi := (|\mathbb{G}|/|\mathbb{G}_s|)_{X'} f_{\Psi(\chi)} \in \mathbb{Q}[X]$  of  $\chi$ .

LEMMA 7.1. *Let  $G = \mathbf{G}^F$  be as above. If  $\chi \in \text{Irr}(G)$  lies in the Harish-Chandra series of a cuspidal character of a Levi subgroup of  $G$  of semisimple  $\mathbb{F}_q$ -rank  $r$ , then its degree polynomial  $f_\chi$  is divisible by  $(X - 1)^r$ .*

*Proof.* Let  $\mathbf{L} \leq \mathbf{G}$  be an  $F$ -stable Levi subgroup such that  $\chi$  lies in the Harish-Chandra series  $\mathcal{E}(G, (L, \lambda))$ , where  $L = \mathbf{L}^F$ . Let  $\mathbf{T} \leq \mathbf{L}$  be an  $F$ -stable maximal torus of  $\mathbf{G}$ , with Sylow 1-torus  $\mathbf{T}_1 \leq \mathbf{T}$ . Then  $\mathbf{M} = C_{\mathbf{G}}(\mathbf{T}_1)$  is a (1-split) Levi subgroup of  $\mathbf{G}$ , and

$$\langle R_{\mathbf{T}}^G(\theta), \chi \rangle = \langle R_{\mathbf{M}}^G(\mu), \chi \rangle$$

with  $\mu = R_T^M(\theta)$  a (virtual) character of  $M = \mathbf{M}^F$ , where  $T = \mathbf{T}^F$ . Thus, if  $\langle R_T^G(\theta), \chi \rangle \neq 0$ , then by disjointness of Harish-Chandra series we must have  $\mathbf{M} \geq \mathbf{L}$  up to conjugation, whence  $\mathbf{T}_1 \leq Z(\mathbf{M})_{\Phi_1} \leq Z(\mathbf{L})_{\Phi_1}$ , where  $Z(\mathbf{M})_{\Phi_1}$  and  $Z(\mathbf{L})_{\Phi_1}$  denote the Sylow 1-torus of the groups  $Z(\mathbf{M})$  and  $Z(\mathbf{L})$ .

Now we have  $\chi(1) = \langle \text{reg}_G, \chi \rangle$ , where the regular character  $\text{reg}_G$  of  $G$  is given by

$$\text{reg}_G = \frac{1}{|W|} \sum_{w \in W} |G: T_w|_{p'} R_{T_w}^G(\text{reg}_{T_w}),$$

where  $W$  is the Weyl group of  $\mathbf{G}$ ,  $T_w = \mathbf{T}_w^F$  is a maximal torus of  $G$  of type  $w$ , and  $\text{reg}_{T_w}$  denotes the regular character of  $T_w$  (see [DM91, Cor. 12.14]). Let  $W_0$  denote the set of elements  $w \in W$  satisfying  $\dim(\mathbf{T}_w)_{\Phi_1} \leq \dim(Z(\mathbf{L})_{\Phi_1})$ . Our above considerations then yield

$$\chi(1) = \frac{1}{|W|} \sum_{w \in W_0} |G: T_w|_{p'} \langle R_{T_w}^G(\text{reg}_{T_w}), \chi \rangle.$$

(This is generic, as by [Lus84, Thm. 4.23] the multiplicities  $\langle R_{T_w}^G(\text{reg}_{T_w}), \chi \rangle$  only depend on the unipotent Jordan correspondent of  $\chi$ .) For any  $F$ -stable reductive subgroup  $\mathbf{H}$  of  $\mathbf{G}$ , in the following we write  $\mathbf{H}_{\Phi_1}$  for a Sylow 1-torus of  $(\mathbf{H}, F)$ . For  $w \in W_0$ , we have

$$\dim(\mathbf{G}_{\Phi_1}) - \dim((\mathbf{T}_w)_{\Phi_1}) \geq \dim(\mathbf{L}_{\Phi_1}) - \dim(Z(\mathbf{L})_{\Phi_1}) = r,$$

where  $r$  is the rank of a Sylow 1-torus of the semisimplification of  $\mathbf{L}$ , hence the semisimple  $\mathbb{F}_q$ -rank of  $\mathbf{L}$  (see [DM91, Def. 8.6]). So the degree polynomial  $f_\chi$  is divisible by  $(X - 1)^r$ . □

We also recall the following facts from ordinary Harish-Chandra theory (see, e.g., [Car85, Thm. 10.11.5]). Let  $L \leq G$  be a Levi subgroup with a cuspidal character  $\lambda \in \text{Irr}(L)$ , and let  $W(\lambda)$  denote the relative Weyl group of this cuspidal pair. Assume that  $\chi \in \text{Irr}(G)$  lies in the Harish-Chandra series above  $(L, \lambda)$ . Let  $\eta \in \text{Irr}(W(\lambda))$  be the character associated to  $\chi$  and  $D_\chi \in \mathbb{Q}(X)$  the inverse of the Schur element of  $\eta$  of the corresponding generic Hecke algebra, so numerator and denominator of  $D_\chi$  are prime to  $X - 1$ . Then

$$(7.2) \quad \chi(1) = |G: L|_{p'} D_\chi(q) \lambda(1) \quad \text{and} \quad D_\chi(1) = \eta(1)/|W(\lambda)|.$$

With the degree polynomial  $f_\lambda$  of  $\lambda$  we define a *degree function*  $f'_\chi \in \mathbb{Q}(X)$  for  $\chi$  as  $f'_\chi = (|G|/|\mathbb{L}|)_X D_\chi f_\lambda$ , where  $\mathbb{L}$  denotes the complete root datum associated to the standard Levi subgroup  $(\mathbf{L}, F)$  with  $\mathbf{L}^F = L$ . The following is shown in [BMM93, Thm. 3.2]:

LEMMA 7.2. *Let  $G = \mathbf{G}^F$  be as above. If  $\chi \in \text{Irr}(G)$  is unipotent, then  $f_\chi = f'_\chi$ .*

7.B. *Unipotent characters of odd degree.* From now on and for the rest of this section assume that  $p$  and hence  $q$  is odd.

**PROPOSITION 7.3.** *Let  $G = \mathbf{G}^F$  be as above. Then every nontrivial cuspidal unipotent character  $\chi$  of  $G$  has even degree. More precisely, if  $\chi$  has degree polynomial  $a(X-1)^m f$ , with  $a \in \mathbb{Q}$ ,  $m \geq 0$  and  $f \in \mathbb{Z}[X]$  monic and prime to  $X-1$ , then  $a(q-1)^m$  is even.*

*Proof.* First note that unipotent characters of  $G$  restrict irreducibly to unipotent characters of  $[\mathbf{G}, \mathbf{G}]^F$ , so we may assume that  $\mathbf{G}$  is semisimple. Furthermore, degrees of unipotent characters are insensitive to the isogeny type of  $\mathbf{G}$ , whence we may assume that  $\mathbf{G}$  is of simply connected type and hence a direct product of simple algebraic groups. As unipotent characters of a direct product are the exterior products of the unipotent characters of the factors, we may reduce to the case that  $\mathbf{G}$  is a direct product of  $r$  isomorphic simple groups  $\mathbf{H}_i \cong \mathbf{H}$ ,  $1 \leq i \leq r$ , transitively permuted by  $F$ . But then  $\mathbf{G}^F \cong \mathbf{H}^{Fr}$ , and  $f(X^r)$  is divisible by the same power of  $X-1$  as  $f(X)$ , so that finally we may assume that  $\mathbf{G}$  is simple.

We then use Lusztig's classification of cuspidal unipotent characters. In fact, when  $q \equiv 1 \pmod{4}$  then the first claim is already proved in [Mal07, Prop. 6.5]. But a quick check of that argument shows that it only relies on the fact that the degree of  $\chi$  is divisible by a sufficiently high power of the even number  $q-1$ . It thus also works for  $q \equiv 3 \pmod{4}$  and does even yield the second assertion.  $\square$

**PROPOSITION 7.4.** *Let  $G = \mathbf{G}^F$  be as above. Then all unipotent characters of  $G$  of odd degree lie in the principal series of  $G$ .*

*Proof.* We distinguish two cases. If  $q \equiv 1 \pmod{4}$ , then our claim is contained in [Mal07, Cor. 6.6]. So for the rest of the proof we may suppose that  $q \equiv 3 \pmod{4}$ .

Assume if possible that  $\chi$  is a unipotent character of  $G$  of odd degree and not lying in the principal series. So  $\chi$  lies above a cuspidal unipotent character  $\lambda \neq 1_L$  of a Levi subgroup  $L \leq G$ . Let  $f_\chi, f_\lambda \in \mathbb{Q}[X]$  denote the degree polynomials of  $\chi, \lambda$  respectively. As  $\chi(1)$  is odd and  $4|(q+1)$ , we have that  $\chi$  must lie in the principal 2-series of  $G$  by [Mal07, Cor. 6.6] applied with  $d=2$ . Thus  $f_\chi$  is prime to  $X+1$  according to [BMM93, Prop. 2.4] (an analogue of our Lemma 7.1).

Now by what we recalled before Lemma 7.2 there exists a rational function  $g \in \mathbb{Q}(X)$  with numerator and denominator products of cyclotomic polynomials times an integer, both prime to  $X-1$ , such that  $f_\chi = g \cdot f_\lambda$ , and such that  $g(1)$  is the degree of an irreducible character of the relative Weyl group  $W(\lambda)$  of  $(L, \lambda)$  in  $G$ . Write  $f_\lambda = (X+1)^k f_1$  with a nonnegative integer  $k$  such that



$f_1 \in \mathbb{Q}[X]$  is prime to  $X + 1$ . Then by our observations on  $f_\chi$  and  $f_\lambda$  there exists a rational function  $g_1$  such that  $g = g_1/(X + 1)^k$  and both numerator and denominator of  $g_1$  are prime to  $X^2 - 1$ . Then  $f_\chi = g_1 \cdot f_1$ .

Now let  $\Phi_i$  be a cyclotomic polynomial dividing  $g_1$ . Then  $\Phi_i(q)$  is odd unless  $i = 2^{j+1}$  for some  $j \geq 1$ , in which case we have  $\Phi_i(q)_2 = (q^{2^j} + 1)_2 = 2 = \Phi_i(1)_2$ . Thus,  $g_1(q)$  is divisible by the same 2-power as  $g_1(1)$ , which is an integer. Since  $f_1(q)$  is even by Proposition 7.3, we conclude that

$$\chi(1) = g(q) \cdot f_\lambda(q) = g_1(q) \cdot f_1(q) \equiv g_1(1) \cdot f_1(q) \pmod{2}$$

is even as well, a contradiction. □

7.C. Characters of odd degree and the principal series.

LEMMA 7.5. Let  $\mathbf{H}$  be simple of adjoint type  $B_l$  ( $l \geq 1$ ),  $C_l$  ( $l \geq 2$ ),  $D_{2l}$  ( $l \geq 2$ ) or  $E_7$ , and  $F : \mathbf{H} \rightarrow \mathbf{H}$  a Steinberg endomorphism. Let  $s \in \mathbf{H}^F$  be semisimple centralising a Sylow 2-subgroup of  $\mathbf{H}^F$ . Then  $s^2 = 1$ .

This was observed in [Mal08a, Lemma 4.1] for type  $B_l$ ; the proof given there carries over word by word, since in all listed cases the longest element of the Weyl group acts by inversion on a maximal torus.

Recall that an element of a connected reductive algebraic group  $\mathbf{H}$  is called *quasi-isolated* if its centraliser is not contained in any proper Levi subgroup of  $\mathbf{H}$ .

LEMMA 7.6. Let  $\mathbf{H}$  be simple of adjoint type  $B_l$  ( $l \geq 2$ ),  $C_l$  ( $l \geq 3$ ) or  $D_l$  ( $l \geq 4$ ), and  $F : \mathbf{H} \rightarrow \mathbf{H}$  a Frobenius endomorphism defining an  $\mathbb{F}_q$ -rational structure such that  $q \equiv 3 \pmod{4}$ . Let  $s \in \mathbf{H}$  be semisimple quasi-isolated with disconnected centraliser  $\mathbf{C} = \mathbf{C}_{\mathbf{H}}(s)$  such that  $\mathbf{C}^F$  contains a Sylow 2-subgroup of  $\mathbf{H}^F$ . Then  $\mathbf{C}^F$  is as in Table 1.

$\mathbf{H}^F$	$\mathbf{C}^F$	conditions
$B_l(q)$	$B_{l-2k}(q) \cdot D_{2k}(q).2$	$1 \leq k \leq l/2$
$B_{2l+1}(q)$	$B_{2k}(q) \cdot {}^2D_{2(l-k)+1}(q).2$	$0 \leq k \leq l$
$C_{2l}(q)$	$(C_l(q) \cdot C_l(q)).2$	
$D_l(q)$	$(D_k(q) \cdot D_{l-k}(q)).2$	$1 \leq k < l/2$
$D_{4l}(q)$	$(D_{2l}(q) \cdot D_{2l}(q)).4$	
${}^2D_l(q)$	$(D_k(q) \cdot {}^2D_{l-k}(q)).2$	$2 \leq k \leq l - 1, k \neq l/2$

Table 1. Disconnected centralisers of 2-central elements

Here  $D_1(q)$ ,  ${}^2D_1(q)$  are to be interpreted as tori of order  $q - 1$ ,  $q + 1$  respectively.

*Proof.* The conjugacy classes of quasi-isolated elements  $s$  in classical groups of adjoint type were classified in [Bon05, Tab. 2]. From that list, we may exclude those  $s$  with connected centraliser. It then remains to determine the

various rational forms of  $\mathbf{H}$  and  $\mathbf{C}$  and to decide when  $s$  is 2-central. We treat the cases individually. For  $\mathbf{H}$  of adjoint type  $\mathbf{B}_l$ , the table contains all examples from loc. cit. For  $\mathbf{H}$  of type  $\mathbf{C}_l$ , an easy calculation shows that only the listed case occurs. Similarly, it can be checked in type  $\mathbf{D}_l$  from the order formulas that only the listed types of disconnected centralisers can possibly contain a Sylow 2-subgroup.  $\square$

We thus obtain the following classification of characters of odd degree:

**THEOREM 7.7.** *Let  $\mathbf{G}$  be simple, of simply connected type, not of type  $\mathbf{A}_l$ , with  $F$ ,  $\mathbf{G}^*$  as introduced in Section 2.B. Let  $\chi \in \text{Irr}_2(G)$ . Then either  $\chi$  lies in the principal series of  $G$ , or  $q \equiv 3 \pmod{4}$ ,  $G = \text{Sp}_{2l}(q)$  with  $l \geq 1$  odd,  $\chi \in \mathcal{E}(G, s)$  with  $\mathbf{C}_{G^*}(s) = \mathbf{B}_{2k}(q) \cdot {}^2\mathbf{D}_{l-2k}(q)$  where  $0 \leq k \leq (l-3)/2$ , and  $\chi$  lies in the Harish-Chandra series of a cuspidal character of degree  $\frac{1}{2}(q-1)$  of a Levi subgroup  $\text{Sp}_2(q) \times (q-1)^{l-1}$ .*

*Proof.* We follow the line of arguments in [Mal07, §7]. Let  $\chi \in \text{Irr}(G)$  be a character of odd degree and not lying in the principal series of  $G$ . Then the degree polynomial of  $\chi$  is divisible by  $X-1$ , by Lemma 7.1. Let  $s \in G^*$  be semisimple such that  $\chi \in \mathcal{E}(G, s)$ , and set  $\mathbf{C} := \mathbf{C}_{G^*}(s)$ ,  $C := \mathbf{C}^F$  and  $C^\circ := \mathbf{C}^{\circ F}$ . Let  $\Psi(\chi) \in \mathcal{E}(C, 1)$  denote the unipotent Jordan correspondent of  $\chi$ . Then by Lusztig’s Jordan decomposition  $\chi(1) = |G^* : C|_{p'} \Psi(\chi)(1)$  (see (7.1)), both  $|G^* : C|$  and  $\Psi(\chi)(1)$  have to be odd. Thus  $\Psi(\chi)$  lies above a unipotent character of  $\mathbf{C}^{\circ F}$  of odd degree, and hence in the principal series of  $C$  by Proposition 7.4. So its degree polynomial is prime to  $X-1$  by Lemma 7.2 and (7.2). Hence the order polynomial of  $|G^* : C|$  must be divisible by  $X-1$  by our assumption. On the other hand, as  $|G^* : C|$  is odd,  $C$  contains a Sylow 2-subgroup of  $G^*$ .

If  $q \equiv 1 \pmod{4}$ , then we may argue as follows. By [Mal07, Thm. 5.9],  $\mathbf{C}$  must contain a Sylow 1-torus of  $\mathbf{G}^*$ . But then the order polynomial of  $|G^* : C|$  cannot be divisible by  $X-1$ , a contradiction.

So now assume that  $q \equiv 3 \pmod{4}$ . Then again by [Mal07, Thm. 5.9],  $\mathbf{C}$  must contain a Sylow 2-torus of  $\mathbf{G}^*$ . The order  $|C|$  is given by a polynomial in  $q$  of the form  $cf(q)$ , where  $c = |C : C^\circ|$  and  $f \in \mathbb{Z}[X]$  is monic. Note that  $\mathbf{C}/\mathbf{C}^\circ$  is isomorphic to a subgroup of the fundamental group of  $\mathbf{G}$ , hence of the center of  $\mathbf{G}$  (see [MT11, Prop. 14.20]). In particular, as  $\mathbf{G}$  is simple and not of type  $\mathbf{A}_l$ , we have  $|C : C^\circ|_2 \leq 4$ , and in fact  $|C : C^\circ|_2 \leq 2$  unless  $\mathbf{G}$  is of type  $\mathbf{D}_l$ . As  $X-1$  divides the order polynomial of  $G^*$  divided by  $f$ , we are done if either  $G$  has odd order center, or if  $\mathbf{C}$  is connected.

So  $\mathbf{G}$  is of type  $\mathbf{B}_l, \mathbf{C}_l, \mathbf{D}_l$  or  $\mathbf{E}_7$ . For  $\mathbf{G}$  not of type  $\mathbf{D}_l$  with  $l$  odd, we know by Lemma 7.5 applied to  $\mathbf{H} := \mathbf{G}^*$  that  $s$  must be an involution. For  $\mathbf{G}$  of type  $\mathbf{E}_7$ , the 2-central involutions of  $G^*$  have centraliser of type  $\mathbf{D}_6(q)\mathbf{A}_1(q)$ , whose order polynomial is divisible by the full power  $(X-1)^7$  of  $X-1$  occurring

in the polynomial order of  $G^*$ , contrary to what we showed. For the groups of classical types  $B_l$ ,  $C_l$ ,  $D_l$  and  ${}^2D_l$ , let us first observe that  $\mathbf{C}$  cannot be contained inside a proper  $F$ -stable Levi subgroup  $\mathbf{L}$  of  $\mathbf{G}^*$ , because  $L = \mathbf{L}^F$  has even index in  $G^*$ . Indeed, a Sylow 2-subgroup of  $G^*$ , or  $L$ , is contained in the normaliser of a Sylow 2-torus of  $\mathbf{G}^*$  (see [MT11, Cor. 25.17]), respectively of  $\mathbf{L}$ . But this normaliser is an extension of that Sylow 2-torus by the Weyl group of  $G^*$ ,  $L$  respectively. The claim then follows since any proper parabolic subgroup of a Weyl group  $W$  of type  $B_l$  or  $D_l$  has even index in  $W$ .

Thus,  $s$  is quasi-isolated in  $\mathbf{G}^*$ , and hence occurs in Table 1. For  $\mathbf{G}$  of type  $B_l$ , the dual group is of adjoint type  $C_l$ , and there the listed centraliser does contain a Sylow 1-torus. For  $\mathbf{G}$  of type  $C_l$  and so  $\mathbf{G}^*$  of type  $B_l$ , the listed centralisers either contain a Sylow 1-torus or are given in the statement with  $l$  odd. As the order polynomial of  $\mathbf{C}^\circ$  is divisible by  $(X - 1)^{l-1}$  in these cases, the degree polynomial of  $\chi$  is divisible by  $X - 1$  just once, so by Lemma 7.1,  $\chi$  lies in the Harish-Chandra series of a cuspidal character of a Levi subgroup  $L$  of  $G$  of rank 1, hence a Levi subgroup of type  $A_1$ . Now  $G$  has two conjugacy classes of such Levi subgroups, one with connected center lying in the stabiliser  $\mathrm{GL}_l(q)$  of a maximally isotropic subspace, the other isomorphic to  $\mathrm{Sp}_2(q) \times (q - 1)^{l-1}$ . The degrees of cuspidal characters of these two types of subgroups are  $q - 1$ , and also  $\frac{1}{2}(q - 1)$  for the second type. The latter ones are thus the only ones of odd degree. An easy variation of the proof of Proposition 7.4, using that  $X - 1$  is prime to  $X + 1$  now shows that if  $\chi$  has odd degree, it must lie above the cuspidal characters of  $\mathrm{Sp}_2(q) \times (q - 1)^{l-1}$  of degree  $\frac{1}{2}(q - 1)$ .

So finally assume that  $\mathbf{G}$  is of type  $D_l$ . Recall that  $\mathbf{C}^\circ$  cannot contain a Sylow 1-torus of  $\mathbf{G}^*$ . The only centralisers in Table 1 not containing a Sylow 1-torus of the ambient group are  $D_k(q) \cdot {}^2D_{l-k}(q) \cdot 2$  with  $l$  even and  $k$  odd inside  ${}^2D_l(q)$ . But these do not contain a Sylow 2-torus, contrary to what we know has to happen. So we get no example in type  $D_l$ . □

*Remark 7.8.* The precise conditions on  $k$  for  $B_{2k}(q) \cdot {}^2D_{l-2k}(q)$  in Theorem 7.7 to contain a Sylow 2-subgroup of  $G^* = \mathrm{SO}_{2l+1}(q)$  are worked out in [Mal08a, Prop. 4.2]. In fact, in [Mal08a, Thms. 4.10 and 4.11] it is shown that  $G = \mathrm{Sp}_{2l}(q)$  satisfies the inductive McKay condition for the prime 2 if  $q$  is an odd power of  $p$ .

The following consequence will be used in the proof of Proposition 7.10:

LEMMA 7.9. *Let  $\mathbf{G}$  be simple, of simply connected type, not of type  $A_l$  or  $C_l$ . Let  $\chi \in \mathrm{Irr}_2(G)$ . Then  $\chi = R_T^G(\lambda)_\eta$ , where  $T$  is a maximally split torus of  $G$ ,  $\lambda \in \mathrm{Irr}(T)$  is such that  $2 \nmid |W : W(\lambda)|$  and  $\eta \in \mathrm{Irr}(W(\lambda))$  is of odd degree.*

*Proof.* By Theorem 7.7 every  $\chi \in \mathrm{Irr}_2(G)$  occurs in the principal series; that is, it lies in the Harish-Chandra series of a linear character  $\lambda \in \mathrm{Irr}(T)$ . According to our remarks before Lemma 7.2 we have that  $\chi(1) = |G :$

$T|_{p'} D_\chi(q) \lambda(1)$ , where  $\lambda(1) = 1$ . Write  $f'_\chi \in \mathbb{Q}[X]$  for the degree polynomial of  $\chi$ . Since  $\Phi_i(q)_2 \geq \Phi_i(1)_2$  for all  $i \geq 2$ , it follows that  $f'_\chi(1)$  is odd. Now replacing  $|G : T|_{p'}$  by the order polynomial and then specialising at  $q = 1$  we obtain that  $|W| D_\chi(1) = |W| \eta(1)/|W(\lambda)|$  is odd, for  $\eta \in \text{Irr}(W(\lambda))$  the label of  $\chi$ , whence the two integers  $\eta(1)$  and  $|W : W(\lambda)|$  are odd.  $\square$

This completes the proof of [Theorem 2](#). Indeed if  $d = 1$ , then [Theorem 2](#) follows from [Theorem 6.4\(a\)](#) and [Lemma 6.6](#). Suppose that  $\ell|(q-1)$ , but  $d \neq 1$ . Then  $\ell = d = 2$ , and one may assume by [Proposition 6.1](#) that  $\mathbf{G}^F$  is not of type  $\text{Sp}_{2n}(q)$  with  $4|(q+1)$ . In this case, [Theorem 2](#) follows by combining [Lemma 6.6](#), [Corollary 5.3](#), [Theorem 6.4\(c\)](#) and [Theorem 7.7](#).

7.D. *Proof of Theorem 1.* We now combine the above results on Harish-Chandra induction and on characters of odd degree to complete the proof that every simple group satisfies the inductive McKay condition for  $\ell = 2$ , and thus [Theorem 1](#) holds.

We have seen in [Theorem 6.4](#) that our results are sufficient to prove that the inductive McKay condition holds for the prime 2, whenever  $4|(q-1)$  and  $\Phi$  is of type  $B_l, C_l$  or  $E_7$ . The result even applies to the simple groups that are the quotients of  ${}^2D_l(q)$  or  ${}^2E_6(q)$ . Taking into account earlier results summarised in [Proposition 6.1](#) the only cases that are left to consider when  $4|(q-1)$  are the simple groups associated with  $D_{l,\text{sc}}(q)$  and  $E_{6,\text{sc}}(q)$ .

The following specific considerations are tailored to the case where  $\ell = 2$ :

**PROPOSITION 7.10.** *Let  $G = \mathbf{G}^F$  be as above. Let  $\chi \in \text{Irr}_{2'}(G)$ . Then  $\chi$  extends to its inertia group in  $GD$ . Thus the assumption [2.1\(ii.2\)](#) holds.*

*Proof.* The statement is trivial whenever  $D_\chi$  is cyclic. If  $G \not\cong D_{4,\text{sc}}(q)$ , the Sylow  $r$ -subgroups of  $D$  are cyclic for any odd prime  $r$ . Note that  $\det \chi$  is trivial since  $G$  is perfect. Then [[Isa76](#), Thm. 6.25] shows that  $\chi$  extends to its inertia group in  $GD_2$ , where  $D_2$  is a Sylow 2-subgroup of  $D$ . According to [[Isa76](#), (11.31)] this implies that  $\chi$  extends to  $GD_\chi$ .

It remains to consider the case where  $G \cong D_{4,\text{sc}}(q)$ . Following the considerations above we have to show that  $\chi$  extends to its inertia group in  $GD_3$  for any Sylow 3-subgroup  $D_3$  of  $D$ . Assume that  $D_\chi$  has a noncyclic Sylow 3-subgroup.

According to [Theorem 7.7](#) there exists a character  $\lambda \in \text{Irr}(T)$ , where  $T$  is a maximally split maximal torus of  $G$ , such that  $\chi$  is a constituent of  $R_T^G(\lambda)$ . According to [Lemma 7.9](#),  $\chi$  corresponds to some character  $\eta \in \text{Irr}(W(\lambda))$  of odd degree such that  $\chi$  has multiplicity  $\eta(1)$  in  $R_T^G(\lambda)$ . Moreover,  $2 \nmid |W : W(\lambda)|$ .

Let  $\gamma \in D, F' \in \langle F_0 \rangle$  be such that  $\langle \gamma, F' \rangle$  is the Sylow 3-subgroup of  $D_\chi$ . Direct computations show that any Sylow 2-subgroup of  $W$  is self-normalising

in  $W$  and can be chosen to be  $\gamma$ -stable. Let  $P$  be such a  $\gamma$ -stable Sylow 2-subgroup of  $W$ . Then after some  $N$ -conjugation of  $\lambda$  we can assume that  $W(\lambda)$  contains  $P$ .

Then there exist elements  $n, n' \in N$  such that  $n\gamma$  and  $n'F'$  stabilise  $\lambda$  and  $P$ . Since  $P$  is  $\langle \gamma, F' \rangle$ -invariant, the elements  $\pi(n), \pi(n')$  are contained in  $N_W(P) = P$ . As linear character,  $\lambda$  has an extension  $\tilde{\lambda}$  to  $\langle T, F', \gamma \rangle$ . Since  $\gamma$  and  $F'$  stabilise the unipotent radical  $U$  of the Borel subgroup  $B$  of  $G$ ,  $\tilde{\lambda}$  lifts to a character  $\hat{\lambda} \in \text{Irr}(\langle B, F', \gamma \rangle)$ . The induced character  $\Gamma = \hat{\lambda}^{\langle G, F', \gamma \rangle}$  is then an extension of the character  $R_T^G(\lambda)$ . Hence  $\Gamma \upharpoonright_G$  has  $\chi$  as constituent with odd multiplicity  $\eta(1)$ .

Since  $|W : W(\lambda)|$  is odd and  $P$  has index 3 in  $W$ , we either have  $W(\lambda) = W$  or  $W(\lambda) = P$ . In the first case,  $\lambda = 1$  since  $W$  acts fixed point freely on  $\text{Irr}(T)$  and so  $\chi$  is unipotent, in which case the statement is an easy consequence of [Mal08a, Thm. 2.4]. Else, the character  $\eta$  of  $W(\lambda)$  of odd degree is linear since  $W(\lambda)$  is a 2-group, and so  $\chi$  has multiplicity one in  $R_T^G(\lambda)$ . Hence  $\Gamma$  has a unique constituent  $\tilde{\chi}$  that is an extension of  $\chi$ . This completes the proof.  $\square$

*Proof of Theorem 1.* By [IMN07, Thm. B] it is sufficient to show that all non-abelian simple groups  $S$  satisfy the inductive McKay condition. For the simple groups not of Lie type, this is known by [Mal08b, Thm. 1.1]. So now assume that  $S$  is of Lie type, and not as in Proposition 6.1. Observe that  $\ell = 2$  implies that  $d_\ell(q) \in \{1, 2\}$ . By Theorem 2 it suffices to verify the assumptions in Theorem 6.4(b) and (d). Condition 2.1(ii.1) holds since all characters of odd degree only lie in very specific Harish-Chandra series thanks to Theorem 7.7 and characters in those series, more precisely the structure of their stabilisers, have been studied in Corollary 5.3. The requirement 2.1(ii.2) is satisfied for characters in  $\text{Irr}_{2'}(G)$  thanks to Proposition 7.10.  $\square$

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