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Birational boundedness for holomorphic symplectic varieties, Zarhin's trick for $K3$ surfaces, and the Tate conjecture

By François Charles

Abstract

We investigate boundedness results for families of holomorphic symplectic varieties up to birational equivalence. We prove the analogue of Zarhin's trick for $K3$ surfaces by constructing big line bundles of low degree on certain moduli spaces of stable sheaves, and proving birational versions of Matsusaka's big theorem for holomorphic symplectic varieties.

As a consequence of these results, we give a new geometric proof of the Tate conjecture for $K3$ surfaces over finite fields of characteristic at least 5, and a simple proof of the Tate conjecture for $K3$ surfaces with Picard number at least 2 over arbitrary finite fields — including fields of characteristic 2.

1. Introduction

1.1. *Main results.* The main goal of this paper is to investigate geometric and arithmetic finiteness results for $K3$ surfaces and related objects. The initial inspiration for the results of this text is the paper [Zar74]. The main insight of Zarhin — "Zarhin's trick" — is the fact that if A is an abelian variety over an arbitrary field k, then $(A \times \widehat{A})^4$ admits a principal polarization. In particular, while the set of isomorphism classes of polarized abelian varieties of fixed dimension g does not form a limited family if $g > 1$, it does map naturally to the moduli space of principally polarized abelian varieties of dimension $8g$. As proved by Tate in [Tat66], this implies the Tate conjecture for abelian varieties over finite fields.

It is well known that the Tate conjecture for divisors in general is related to finiteness results for certain classes of algebraic varieties over finite fields or number fields. The aforementioned argument of Zarhin shows that, in the case of abelian varieties over finite fields, these are consequences of boundedness results that hold over arbitrary fields.

The goal of this paper is to discuss an analogue of this circle of ideas for $K3$ surfaces, and explain applications to new proofs of the Tate conjecture for divisors on these surfaces. While some of our results are not new, one of the goal of this paper is to emphasize the role of certain geometric objects — moduli spaces of twisted and untwisted sheaves — regarding the existence of divisors on surfaces. It seems that our results are the first occurrence of new versions of Zarhin's trick since Zarhin's original paper.

We start by investigating Zarhin's trick for $K3$ surfaces and proceed in two steps. The first step is to construct big line bundles on moduli spaces of sheaves on $K3$ surfaces. A simplified version of our result, stated in detail in Theorem 2.10, is the following.

THEOREM 1.1. *Let k be a field, and let d be a positive integer. Then there exists a positive integer r such that for infinitely many positive integers m, if (X, H) is a polarized $K3$ surface of degree $2md$ over k, then there exists a smooth, 4-dimensional, projective moduli space $\mathcal{M}$ of stable sheaves on X and a line bundle L on $\mathcal{M}$ satisfying $c_1(L)^4 = r$ and $q(L) > 0$, where q is the Beauville-Bogomolov form on $\mathcal{M}$.*

We actually give a congruence condition on m that ensures it satisfies the property above.

The space $\mathcal{M}$ is a natural analogue of the product $(A \times \widehat{A})^4$ appearing in Zarhin's trick. Indeed, at least over $\mathbb{C}$, it is an irreducible holomorphic symplectic variety — that is, it is simply connected and its space of holomorphic 2-forms is spanned by a single symplectic form. Furthermore, it is deformation-equivalent to the Hilbert scheme $X^{[2]}$ of 2 points on X.

By an important theorem of Huybrechts [Huy99, 3.10], either L or its dual is big. This means that there exists a power of L that induces a birational map from $\mathcal{M}$ onto a subvariety of projective space. Another major theme — and the second step — of this paper is investigating the extent to which a birational version of Matsusaka's big theorem holds in this setting. We formulate an optimistic possible result as a question.

Question. Let $\mathcal{M}$ be either a complex projective holomorphic symplectic variety or — in positive characteristic — a smooth projective moduli space of stable sheaves of dimension $2n$ on a $K3$ surface, and let L be a big line bundle on $\mathcal{M}$ with $c_1(L)^{2n} = r$ and $q(L) > 0$, where q is the Beauville-Bogomolov form. Do there exist integers N, d and a depending only on r and n such that the complete linear system $|aL|$ induces a birational map from $\mathcal{M}$ onto a subvariety of degree at most d of $\mathbb{P}^m$ with $m \leq N$?

One could even ask whether the integer a can be chosen independently of r. If one could control the singularities of a general member of $|L|$, this would follow in characteristic zero from Theorem 1.3 in [HMX14].

We are not able to answer the question — in positive characteristic, even the case where L is assumed to be ample is not known — but give partial results in that direction. For $K3$ surfaces, geometric considerations following [SD74] allow us to answer the question in any characteristic, up to replacing L by a different line bundle L' with self-intersection bounded in terms of r. This is Proposition 3.1.

In higher dimension, we do not give a complete answer, as we do not understand the geometry of linear systems well enough in that case; see however related considerations in [O'G05]. Over the field of complex numbers, we can use the period map and the global Torelli theorem of [Ver13] to answer the question in Theorem 3.3, again possibly changing the bundle L. This has the following consequence.

THEOREM 1.2. *Let n and r be two positive integers. Then there exist a scheme S of finite type over $\mathbb{C}$ and a projective morphism $\mathcal{X} \to S$ such that if X is a complex irreducible holomorphic symplectic variety of dimension $2n$ and L is a line bundle on X with $c_1(L)^{2n} = r$ and $q(L) > 0$, where q is the Beauville-Bogomolov form, then there exists a complex point s of S such that $\mathcal{X}_s$ is birational to X.*

In other words, the holomorphic symplectic varieties as above form a birationally bounded family.

We deal with finite fields using a similar strategy — in that case, the period map is replaced by the Kuga-Satake construction. For technical reasons that could be circumvented using heavier machinery as in [KMP15], we assume that the characteristic is at least 5. The finiteness result we obtain is Proposition 3.16.

The main application of our results is to the Tate conjecture for $K3$ surfaces over finite fields. In odd characteristic, it has been proved in [Mau14, Cha13], and independently in [MP15]. The first of these proofs relies on results of Borcherds on the Picard group of Shimura varieties, while the second one uses construction of canonical models of certain Shimura varieties. We follow a different approach that first appeared in spirit in [ASD73] and was discussed in [LMS14]. In the latter paper, it is proved that the Tate conjecture for $K3$ surfaces over a finite field k is equivalent to the finiteness of the set of isomorphism classes of $K3$ surfaces over k. By refining the arguments of [LMS14], we are able to use a version of this criterion, together with both our version of Zarhin's trick above and our birational boundedness results, and give a new proof of the following theorem.

THEOREM 1.3. *Let X be a $K3$ surface over a finite field of characteristic at least 5. Then X satisfies the Tate conjecture.*

It was the hope of the author that the techniques of this paper would be able to give a proof of the Tate conjecture for $K3$ surfaces over finite fields that does not rely on Kuga-Satake varieties. However, our proof of birational boundedness results for higher-dimensional holomorphic symplectic varieties in positive characteristic turned out to require this construction. The reason why it appears is that since the birational geometry of holomorphic symplectic varieties might be hard to control, it is very helpful to translate the problem in terms of abelian varieties where the birational geometry is trivial and Matsusaka's big theorem is known even in positive characteristic. It seems possible that further understanding of the underlying geometry might answer the question above along the lines of Proposition 3.1, at least in characteristic zero.

Our last result, which is new only in characteristic 2 but whose proof is in any case significantly simpler than all the other proofs of the Tate conjecture, should be seen as a modern rephrasing of the main result of [ASD73] that dealt with elliptic $K3$ surfaces. It realizes the hope described in the previous paragraph, as it does not use the Kuga-Satake construction nor p-adic methods, making use instead of general geometric results. As opposed to the proofs of the Tate conjecture for various classes of $K3$ surfaces that appeared after [ASD73], it does not rely on the geometry of moduli spaces of $K3$ surfaces, but rather on the geometry of the surfaces themselves.

THEOREM 1.4. *Let X be a $K3$ surface over a finite field of arbitrary characteristic. If the Picard number of X is at least 2, then X satisfies the Tate conjecture.*

It is perhaps interesting to notice that, after a finite extension of the base field, the hypothesis of the theorem above is satisfied as soon as X satisfies the Tate conjecture. If this holds, the Picard number of $X_{\overline{k}}$ should be even; see, for instance, [dJK00]. It would be very interesting to find a direct proof of this fact.

The paper is split in three parts, which are independent in some respect. In Section 2, we prove a version of Zarhin's trick for $K3$ surfaces over arbitrary fields. This relies on the study of moduli spaces of stable sheaves on $K3$ surfaces and their cohomology, as initiated by Mukai.

Section 3 is devoted to birational versions of Matsusaka's big theorem for holomorphic symplectic varieties, over $\mathbb{C}$ and finite fields. For $K3$ surfaces, we explain how to use results of Saint-Donat to prove the desired results, while in the other cases we need a finer analysis of some moduli spaces via period maps. This leads to technical complications in positive characteristic, which arise, in particular, due to the fact that there does not seem to be a satisfying definition of holomorphic symplectic varieties over arbitrary fields.

In Section 4, we apply the aforementioned results to the Tate conjecture for $K3$ surfaces over finite fields. We follow the strategy of [LMS14] for the most part by using moduli spaces of twisted sheaves. We also discuss a simple proof of Theorem 1.4.

While the last section is arithmetic in nature, we hope that the first two might be of some interest even for complex geometers.

1.2. *A preliminary result.* We will need the following lifting result. It is certainly well known to experts and follows easily from [LM14]; see also [LO15, Prop. A.1].

Proposition 1.5. *Let X be a $K3$ surface over an algebraically closed field k of positive characteristic, and let $L_1, \ldots, L_r$ be line bundles on X. Assume that L_1 is ample, and let W be the ring of Witt vectors of k. If $r \leq 10$, there exists a finite flat morphism $S \to \operatorname{Spec} W$, where S is the spectrum of a discrete valuation ring, and a smooth projective relative $K3$ surface $\mathcal{X} \to S$ such that*

(i) *The special fiber of $\mathcal{X} \to S$ is isomorphic to X;*
(ii) *The image of the specialization map*
$$\operatorname{Pic}(\mathcal{X}) \to \operatorname{Pic}(X)$$
contains the classes of $L_1, \ldots, L_r$.

Proof. If X has finite height, the result follows from [LM14, Cor. 4.2]. In general, in the deformation space of (X, L_1) over k, which has dimension 19, the complement of the locus of surfaces of finite height has dimension 9 by [Art74], and the deformation space of $(X, L_1, \ldots, L_r)$ has codimension $r - 1$. As a consequence, $(X, L_1, \ldots, L_r)$ is a specialization of a $K3$ surface with finite height, which allows us to conclude. □

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2. A variant of Zarhin's trick for $K3$ surfaces

2.1. *Moduli spaces of stable sheaves on $K3$ surfaces.* The goal of this section is to describe the geometry of moduli spaces of stable sheaves on $K3$ surfaces. Over the field of complex numbers, these results are well known due to the work of Mukai, O'Grady and Yoshioka. We explain below how to extend them to arbitrary fields.

If X is a $K3$ surface over a field k, we denote by $\mathrm{NS}(X)$ the group of line bundles on X modulo numerical equivalence. Since the cohomology groups of X have no torsion, the first Chern class map induces an injective homomorphism

$$c_1 : \mathrm{NS}(X) \to H^2(X_{\overline{k}}, \mathbb{Z}_\ell(1))$$

for any prime number ℓ different from the characteristic of k, where $\overline{k}$ denotes an algebraic closure of k. In particular, $\mathrm{NS}(X)$ identifies with a subgroup of $\mathrm{NS}(X_{\overline{k}})$.

Mukai lattices were first defined in [Muk87b] for complex $K3$ surfaces. Recently, they have been defined and studied in great generality in the paper [LO15]. For our purposes, we only need very basic definitions, which recall now.

Definition 2.1. Let k be a field with algebraic closure $\overline{k}$, and let X be a $K3$ surface over k. Let ℓ be a prime number that is invertible in k.

(i) The *ℓ-adic Mukai lattice of X* is the free $\mathbb{Z}_\ell$-module

$$\widetilde{H}(X_{\overline{k}}, \mathbb{Z}_\ell) := H^0(X_{\overline{k}}, \mathbb{Z}_\ell) \oplus H^2(X_{\overline{k}}, \mathbb{Z}_\ell(1)) \oplus H^4(X_{\overline{k}}, \mathbb{Z}_\ell(2))$$

endowed with the *Mukai pairing*

$$\langle (a,b,c), (a',b',c') \rangle = bb' - ac' - a'c.$$

(ii) Let ω be the numerical equivalence class of a closed point in $X_{\overline{k}}$. A *Mukai vector* on X is an element v of

$$N(X) := \mathbb{Z} \oplus \mathrm{NS}(X) \oplus \mathbb{Z}\omega.$$

We denote by $\mathrm{rk}(v)$ the *rank* of v, that is, its first component, and by $c_1(v)$ its component in $\mathrm{NS}(X)$. We identify a Mukai vector and its image in the ℓ-adic Mukai lattice under the natural injection

$$N(X) \to \widetilde{H}(X_{\overline{k}}, \mathbb{Z}_\ell).$$

(iii) Let $\mathcal{F}$ be a coherent sheaf on X. The *Mukai vector of $\mathcal{F}$* is

$$v(\mathcal{F}) := ch(\mathcal{F})\sqrt{td_X} = \mathrm{rk}(\mathcal{F}) + c_1(\mathcal{F}) + (\chi(\mathcal{F}) - \mathrm{rk}(\mathcal{F}))\omega.$$

If $\mathcal{F}$ and $\mathcal{G}$ are two coherent sheaves on X, let $\chi(\mathcal{F},\mathcal{G}) = \sum_i (-1)^i \mathrm{Ext}^i(\mathcal{F},\mathcal{G})$. If $\mathcal{F}$ is locally free, then $\chi(\mathcal{F},\mathcal{G}) = \chi(\mathcal{F}^\vee \otimes \mathcal{G})$. By the Riemann-Roch theorem (see [Muk87b, Prop. 2.2]) we have the following.

Proposition 2.2. *Let $\mathcal{F}$ and $\mathcal{G}$ be two coherent sheaves on X. Then*

$$\chi(\mathcal{F},\mathcal{G}) = -v(\mathcal{F}).v(\mathcal{G}).$$

Given a Mukai vector v and a polarization H on X — that is, H is an isomorphism class of ample line bundles on X — we denote by $\mathcal{M}_H(X,v)$ the moduli space of Gieseker-Maruyama H-stable sheaves $\mathcal{F}$ on X such that $v(\mathcal{F}) = v$. This moduli space is well defined as a quasi-projective scheme over

k in arbitrary characteristic by work of Langer [Lan04, Th. 0.2]. When X is fixed, we will denote this moduli space by $\mathcal{M}_H(v)$.

Over the field of complex numbers, it is customary to require the polarization H to be generic with respect to v; see [O'G97, §1]. Over an arbitrary field, we will state our results under a stronger assumption on v.

Definition 2.3. Let X be a $K3$ surface over a field k, and let H be a polarization on X. We say that a Mukai vector v on X satisfies (C) if

(i) the vector v is primitive, $\mathrm{rk}(v) > 0$ and $v^2 > 0$;
(ii) writing $v = \mathrm{rk}(v) + c_1(v) + \lambda\omega$, then

$$\gcd(\mathrm{rk}(v), H.c_1(v), \lambda) = 1.$$

The following theorem describes the geometry of the moduli space in arbitrary characteristic. We will refine some of these results below under additional assumptions.

THEOREM 2.4. *Let k be a field with algebraic closure $\overline{k}$, and let X be a $K3$ surface over k. Let v be a Mukai vector on X satisfying condition* (C).

(i) *The space $\mathcal{M}_H(v)$ is a smooth, projective, geometrically irreducible variety of dimension v^2+2 over k. It is deformation-equivalent to the Hilbert scheme $X^{[n]}$ parametrizing subschemes of dimension 0 and length $n = \frac{v^2+2}{2}$ in X. It is endowed with a natural symplectic structure up to homothety; i.e., the space of global sections of the sheaf $\Omega^2_{X/k}$ is 1-dimensional and is spanned by a form that is everywhere nondegenerate.*
(ii) *If k is the field $\mathbb{C}$ of complex numbers, then $\mathcal{M}_H(v)$ is an irreducible holomorphic symplectic variety.*
(iii) *If ℓ is a prime number that is invertible in k, then the ℓ-adic cohomology group $H^2(\mathcal{M}_H(v)_{\overline{k}}, \mathbb{Z}_\ell(1))$ is endowed with a canonical quadratic form q satisfying the formula*

$$\forall\alpha \in H^2(\mathcal{M}_H(v)_{\overline{k}}, \mathbb{Z}_\ell(1)), (2n)!q(\alpha)^n = (n!)2^n\alpha^{2n}, \tag{2.1}$$

where $2n$ is the dimension of $\mathcal{M}_H(v)$.
(iv) *There exists a canonical quadratic form on $\mathrm{NS}(\mathcal{M}_H(v)_{\overline{k}})$. If $p = \mathrm{char}(k) > 0$, then this quadratic form has values in $\mathbb{Z}[1/p]$. If $p = 0$, then it has values in $\mathbb{Z}$. For any $\ell \neq p$, the first Chern class map*

$$c_1 : \mathrm{NS}(\mathcal{M}_H(v)_{\overline{k}}) \otimes \mathbb{Z}_\ell \to H^2(\mathcal{M}_H(v)_{\overline{k}}, \mathbb{Z}_\ell(1))$$

is an isometry.
(v) *Let ℓ be a prime number that is invertible in k. Let $v^\perp$ be the orthogonal complement of v in the ℓ-adic Mukai lattice of X — by a slight abuse of notation, we do not make explicit the dependence in ℓ. Then there exists*

a canonical, $\mathrm{Gal}(\overline{k}/k)$*-equivariant, isomorphism*

$$\theta_{v,\ell} : v^{\perp} \to H^2(\mathcal{M}_H(v)_{\overline{k}}, \mathbb{Z}_\ell(1))$$

that is an isometry.

(vi) *Assume that k is either algebraically closed or finite. Let $v^{\perp} \cap N(X)$ be the orthogonal complement of v in the lattice $N(X)$ of Mukai vectors on X. There exists an injective isometry*

$$\theta_v : v^{\perp} \cap N(X) \to \mathrm{NS}(\mathcal{M}_H(v))$$

such that the following diagram commutes:

$$\begin{tikzcd}
v^{\perp} \cap N(X) \arrow[r, "\theta_v"] \arrow[d, "c_1"] & \mathrm{NS}(\mathcal{M}_H(v)) \arrow[d, "c_1"] \\
v^{\perp} \arrow[r, "\theta_{v,\ell}"] & H^2(\mathcal{M}_H(v)_{\overline{k}}, \mathbb{Z}_\ell(1))
\end{tikzcd}$$

for any prime number ℓ as above.

(vii) *Assume that k is either algebraically closed or finite. Then the cokernel of θ_v is a p-primary torsion group, where p is the characteristic of k.*

Proof. Let $\mathcal{F}$ be a semistable torsion-free sheaf of Mukai vector v. Write $v = \mathrm{rk}(v) + c_1(v) + \lambda\omega$. By Proposition 2.2 we have, for any integer d,

$$\chi(\mathcal{F}(d)) = \frac{\mathrm{rk}(v)}{2} d^2 H^2 + (H.c_1(v))d + \lambda + \mathrm{rk}(v).$$

Since $H.c_1(v), \mathrm{rk}(v)$ and λ are relatively prime, this implies — since $\mathcal{F}$ is semistable — that if $\mathcal{G}$ is any coherent subsheaf of $\mathcal{F}$, then

$$\frac{\chi(\mathcal{F}(d))}{\mathrm{rk}(\mathcal{F})} > \frac{\chi(\mathcal{G}(d))}{\mathrm{rk}(\mathcal{G})}$$

for any large enough integer d. Equivalently, $\mathcal{F}$ is stable. As a consequence, Theorem 0.2 of [Lan04] shows that $\mathcal{M}_H(v)$ is projective. By [Muk84, Cor. 0.2], it is smooth, endowed with a natural symplectic structure and, if nonempty, it is pure of dimension $v^2 + 2$.

Let $n = \frac{v^2+2}{2}$. If $k = \mathbb{C}$, [O'G97, Main Theorem] (see also [Yos01, Th. 8.1]) shows that the moduli space $\mathcal{M}_H(v)$ is an irreducible holomorphic symplectic variety birational to $X^{[n]}$. In particular, $\mathcal{M}_H(v)$ is not empty. By the main theorem of [Huy99], $\mathcal{M}_H(v)$ is deformation-equivalent to $X^{[n]}$. The Lefschetz principle shows that these statements hold over any algebraically closed field of characteristic zero.

Now assume k is an arbitrary field. We want to show that $\mathcal{M}_H(v)$ is deformation-equivalent to $X^{[n]}$. For this, we can assume that k is algebraically closed and, by the discussion above, that k has positive characteristic. By Proposition 1.5, we can find a finite flat morphism $S \to \operatorname{Spec} W$, where W is the ring of Witt vectors of k, and a lifting $\mathcal{X} \to S$ of X over S such that both H and $c_1(v)$ lift to $\mathcal{X}$. As a consequence, the Mukai vector v also lifts to $\mathcal{X}$.

Consider the relative moduli space $\mathcal{M}_H(\mathcal{X}, v)$, which exists by [Lan04, Th. 0.2]. By [Muk84, Th. 1.17], it is smooth over S. Since its generic fiber is deformation-equivalent to $\mathcal{X}_\eta^{[n]}$, $\mathcal{M}_H(X, v)$ is deformation-equivalent to $X^{[n]}$. This shows (i) and (ii).

We now prove items (iii)–(v). For these, we can assume that k is algebraically closed. First assume that $k = \mathbb{C}$. Then the Beauville-Bogomolov quadratic form on $H^2(\mathcal{M}_H(v), \mathbb{Z}_\ell(1))$, as defined in [Bea83, Rem. 3 after Th. 5], satisfies equation (2.1) by [O'G05, 4.14]. By the comparison theorem between singular and ℓ-adic cohomology, this shows (iii) for $k = \mathbb{C}$, hence for k algebraically closed of characteristic zero. For general k, lifting as before by Proposition 1.5, the smooth base change theorem gives (iii).

Let ℓ be a prime number invertible in k. The cycle class map gives an injection

$$c_1 : \mathrm{NS}(\mathcal{M}_H(v)) \to H^2(\mathcal{M}_H(v), \mathbb{Z}_\ell(1)).$$

As a consequence, the quadratic form q on $H^2(\mathcal{M}_H(v), \mathbb{Z}_\ell(1))$ induces a quadratic form on $\mathrm{NS}(\mathcal{M}_H(v))$ with values in $\mathbb{Z}_\ell$, which we denote by q as well. We show that it actually takes values in $\mathbb{Q}$ and is independent of ℓ. This will imply (iv). If $k = \mathbb{C}$, this holds because the Beauville-Bogomolov form is actually defined on singular cohomology with integer coefficients. This shows that the result holds if k has characteristic zero. Assume that k has positive characteristic, and choose a lifting $\mathcal{X} \to S$ of X, H and v to characteristic zero as above.

Let $H_{\mathcal{M}} \in \mathrm{NS}(\mathcal{M}_H(v))$ be an ample line bundle on $\mathcal{M}_H(X, v)$ that lifts to $\mathcal{M}_H(\mathcal{X}, v)$. Since $H_{\mathcal{M}}$ lifts to characteristic zero, the argument above shows that $q(H_{\mathcal{M}})$ is an integer independent of ℓ.

Let $\overline{\eta}$ be a generic geometric point of S. By [Bea83, Th. 5 and end of p. 775], there exists a rational number λ, independent of ℓ, such that for any $\alpha \in H^2(\mathcal{M}_H(\mathcal{X}_{\overline{\eta}}, v), \mathbb{Z}_\ell(1))$ such that $\alpha \cup H_{\mathcal{M}}^{2n-1} = 0$, we have

$$q(\alpha) = \lambda\alpha^2 \cup H_{\mathcal{M}}^{2n-2}. \tag{2.2}$$

Indeed, this is true over $\mathbb{C}$ by the result of Beauville quoted above, and thus holds over any algebraically closed field of characteristic zero. Furthermore, the same formula holds for $\mathcal{M}_H(v)$ by the smooth base change theorem. This readily implies that the quadratic form q on $\mathrm{NS}(\mathcal{M}_H(v))$ takes values in $\mathbb{Q}$ and is independent of ℓ. This proves (iv).

Over the field of complex numbers, the map $\theta_{v,\ell}$ is defined on the level of singular cohomology with coefficients in $\mathbb{Z}$ in [Muk87a, 5.14], and (v) holds by the main theorem of [O'G97]. By the same arguments as above, it holds over an arbitrary algebraically closed field. Since the morphism is canonical, it is Galois-equivariant.

We now prove (vi). The map $\theta_{v,\ell}$ is induced by an algebraic correspondence with coefficients in $\mathbb{Q}$; see again [Muk87a, 5.14]. As a consequence, it induces a map

$$\theta_v : (v^\perp \cap N(X)) \otimes \mathbb{Q} \to \mathrm{NS}(\mathcal{M}_H(v)) \otimes \mathbb{Q}.$$

This map is clearly compatible with $\theta_{v,\ell}$ via the cycle class map. By the definition of the quadratic forms involved, this implies that θ_v is an injective isometry.

We claim that $\theta_{v,\overline{k}}$ is defined over $\mathbb{Z}$, that is, that it sends $v^\perp \cap N(X_{\overline{k}})$ to $\mathrm{NS}(\mathcal{M}_H(v)_{\overline{k}})$. If k has characteristic zero, this is due to the fact that $\theta_{v,\ell}$ is defined over $\mathbb{Z}_\ell$ for any prime number ℓ and that the cokernel of the cycle class map

$$c_1 : \mathrm{NS}(\mathcal{M}_H(v)_{\overline{k}}) \to H^2(\mathcal{M}_H(v)_{\overline{k}}, \mathbb{Z}_\ell(1))$$

has no torsion.

To show that $\theta_{v,\overline{k}}$ is defined over $\mathbb{Z}$ for arbitrary k, we lift once again to characteristic zero. Given $\alpha \in N(X_{\overline{k}})$, we can lift X, H, v and α to characteristic zero by Theorem 1.5 and apply the claim to the generic fiber. This shows (vi) if k is algebraically closed.

Assume now that k is a finite field. Since $H^1(\mathcal{M}_H(v)_{\overline{k}}, \mathbb{Z}_\ell) = 0$ (as follows from (ii) if k is the field of complex numbers and from a lifting argument in general), the Hochschild-Serre spectral sequence shows that the map

$$H^2(\mathcal{M}_H(v), \mathbb{Z}_\ell(1)) \to H^2(\mathcal{M}_H(v)_{\overline{k}}, \mathbb{Z}_\ell(1))^{\mathrm{Gal}(\overline{k}/k)}$$

is an isomorphism, where the left-hand side denotes continuous étale cohomology. Furthermore, a classical argument involving the Kummer exact sequence shows that the cycle class map

$$\mathrm{Pic}(\mathcal{M}_H(v)) \to H^2(\mathcal{M}_H(v), \mathbb{Z}_\ell(1))$$

has a torsion-free cokernel. As a consequence, the cokernel of the map

$$\mathrm{Pic}(M_H(v)) \to H^2(M_H(v)_{\overline{k}}, \mathbb{Z}_l(1))$$

is torsion-free.

Now let $\alpha \in N(X) \cap v^\perp$. By the argument above, some multiple of $\theta_{v,\ell}(\alpha)$ is the Chern class of an element of $\mathrm{Pic}(\mathcal{M}_H(v))$. This shows that $\theta_{v,\ell}(\alpha)$ belongs to $\mathrm{NS}(\mathcal{M}_H(v))$.

Let k be as in (vii). To show the result, we need to show that if ℓ is any prime number invertible in k, then

$$\theta_v \otimes \mathbb{Z}_\ell : (v^\perp \cap N(X)) \otimes \mathbb{Z}_\ell \to \mathrm{NS}(\mathcal{M}_H(v)) \otimes \mathbb{Z}_\ell$$

is an isomorphism. We already know that $\theta_v \otimes \mathbb{Z}_\ell$ is injective. Furthermore, in the commutative diagram

$$\begin{tikzcd}
(v^\perp \cap N(X)) \otimes \mathbb{Z}_\ell \arrow[r, "\theta_v \otimes \mathbb{Z}_\ell"] \arrow[d, "c_1"] & \mathrm{NS}(\mathcal{M}_H(v)) \otimes \mathbb{Z}_\ell \arrow[d, "c_1"] \\
v^\perp \arrow[r, "\theta_{v,\ell}"] & H^2(\mathcal{M}_H(v)_{\overline{k}}, \mathbb{Z}_\ell(1))
\end{tikzcd}$$

the lower horizontal map is an isomorphism and the cokernel of the two vertical maps are torsion-free as shown above. This implies that the cokernel of $\theta_v \otimes \mathbb{Z}_\ell$ is torsion-free. To show that $\theta_v \otimes \mathbb{Z}_\ell$ is an isomorphism, we now have to show that

$$\theta_v \otimes \mathbb{Q}_\ell : (v^\perp \cap N(X)) \otimes \mathbb{Q}_\ell \to \mathrm{NS}(\mathcal{M}_H(v)) \otimes \mathbb{Q}_\ell$$

is an isomorphism. Let $H_{\mathcal{M}}$ be the homological equivalence class of an ample divisor on $\mathcal{M}_H(v)$, and consider the composition ϕ_ℓ

$$\begin{tikzcd}
(v^\perp \cap N(X)) \otimes \mathbb{Q}_\ell \arrow[r, "\theta_v"] & \mathrm{NS}(\mathcal{M}_H(v)) \otimes \mathbb{Q}_\ell \arrow[r, "c_1"] & H^2(\mathcal{M}_H(v)_{\overline{k}}, \mathbb{Q}_\ell(1)) \arrow[d, "\cup H_{\mathcal{M}}^{2n-2}"] \\
v^\perp \otimes \mathbb{Q}_\ell & & H^{4n-2}(\mathcal{M}_H(v)_{\overline{k}}, \mathbb{Q}_\ell(1)), \arrow[ll, "\theta_{v,\ell}^\vee"']
\end{tikzcd}$$

where

$$\theta_{v,\ell}^\vee : H^{2n-2}(\mathcal{M}_H(v), \mathbb{Z}_\ell(1)) \to v^\perp$$

is the Poincaré dual of $\theta_{v,\ell}$. Then ϕ_ℓ is injective by the hard Lefschetz theorem, (v) and (vi). Furthermore, it sends $(v^\perp \cap N(X)) \otimes \mathbb{Q}_\ell$ into itself since it is induced by an algebraic correspondence and is Galois-equivariant. As a consequence, it induces an automorphism of $(v^\perp \cap N(X)) \otimes \mathbb{Q}_\ell$. Furthermore, by the same argument, the composition

$$\begin{aligned}
\mathrm{NS}(\mathcal{M}_H(v)) \otimes \mathbb{Q}_\ell &\longrightarrow H^2(\mathcal{M}_H(v)_{\overline{k}}, \mathbb{Q}_\ell(1)) \\
&\longrightarrow H^{4n-2}(\mathcal{M}_H(v)_{\overline{k}}, \mathbb{Q}_\ell(1)) \longrightarrow v^{\perp_\ell} \otimes \mathbb{Q}_\ell
\end{aligned}$$

is injective and maps into $(v^\perp \cap N(X)) \otimes \mathbb{Q}_\ell$. This implies that $\theta_v \otimes \mathbb{Q}_\ell$ is surjective and concludes the proof. $\square$

Definition 2.5. With the notation of Theorem 2.4, the quadratic forms on $\mathrm{NS}(\mathcal{M}_H(v)_{\overline{k}})$ and $H^2(\mathcal{M}_H(v)_{\overline{k}}, \mathbb{Z}_\ell(1))$ defined in (iii) and (iv) are called the *Beauville-Bogomolov form.*

Under suitable assumptions on the characteristic of the base field and the $K3$ surface X, we can also both describe the de Rham cohomology groups of $\mathcal{M}_H(v)$ and extend the description of the Néron-Severi group of $\mathcal{M}_H(v)$ to some nonalgebraically closed fields.

Proposition 2.6. *Let k be an algebraically closed field of characteristic $p > 0$, and let (X, H) be a polarized $K3$ surface over k. Let v be a Mukai vector on X satisfying condition* (C). *Let W be the ring of Witt vectors of k.*

Assume that the triple (X, v, H) lifts to a projective $K3$ surface over W. Then the Hodge to de Rham spectral sequence of $\mathcal{M}_H(v)$ degenerates at E_1 if $p > v^2+2$. In general, the Hodge numbers $h^{p,q} = H^q(\mathcal{M}_H(v), \Omega^p_{X/k})$ of $\mathcal{M}_H(v)$ satisfy the equalities

(1) $h^{1,0} = h^{0,1} = 0$ *if* $p > 2$;

(2) $h^{2,0} = h^{0,2} = 1$ *and* $h^{1,1} = 21$ *if* $p > 3$.

Proof. As above, a projective lift of (X, v, H) to W induces a projective lift $\mathcal{M}_H(\mathcal{X}, v)$ of $\mathcal{M}_H(v)$ to W. By the main result of [DI87], and since $p > v^2+2 = \dim \mathcal{M}_H(v)$, this implies that the Hodge to de Rham spectral sequence of $\mathcal{M}_H(v)$ degenerates at E_1.

In characteristic at least 3, the result of [DI87] still shows that $E_1^{p,q} = E_\infty^{p,q}$ if $p + q = 1$. In characteristic at least 5, the equality also holds if $p + q = 2$.

The statement regarding the Hodge numbers can be rephrased as saying that the Hodge numbers of $\mathcal{M}_H(v)$ are the same as those of $\mathcal{M}_H(\mathcal{X}_\eta, v)$, where $\mathcal{X}_\eta$ is the generic fiber of $\mathcal{X}$ over W. By the universal coefficient theorem for crystalline cohomology, it is enough to show that the second and third crystalline cohomology groups of $\mathcal{M}_H(v)$ are torsion-free. By the integral comparison theorem of Fontaine-Messing [FM87], this is the case as soon as the corresponding p-adic cohomology groups of $\mathcal{M}_H(\mathcal{X}_\eta, v)$ are torsion-free. Since $\mathcal{M}_H(\mathcal{X}_\eta, v)$ is deformation-equivalent to a Hilbert scheme of points on a $K3$ surface in characteristic zero, it suffices to show that if S is any projective complex $K3$ surface, then $H^2(S^{[n]}, \mathbb{Z})$ and $H^3(S^{[n]}, \mathbb{Z})$ are both torsion-free.

The fact that $H^2(S^{[n]}, \mathbb{Z})$ is torsion-free is proved in [Bea83, Rem. after Prop. 6]. We now show that $H^3(S^{[n]}, \mathbb{Z}) = 0$.

Following [Bea83], let $S^{(n)}$ be the n-fold symmetric product of S, and let $\varepsilon : S^{[n]} \to S^{(n)}$ be the Hilbert-Chow morphism. Let $\pi : S^n \to S^{(n)}$ be the natural map. Let D be the diagonal in $S^{(n)}$, that is, the locus of elements $x_1 + \cdots + x_n$ such that $x_i = x_j$ for some $i \neq j$, and let D_* be the open subset of D consisting of zero-cycles of the form $2x_1 + \cdots + x_{n-1}$ where the x_i are all distinct. We define $S^{(n)}_* = S^{(n)} \setminus (D \setminus D_*)$, $S^{[n]}_* = \varepsilon^{-1}(S^{(n)}_*)$ and $S^n_* = \pi^{-1}(S^{(n)}_*)$. Then by [Bea83, §6], $S^{[n]} \setminus S^{[n]}_*$ has codimension 2 in $S^{[n]}$ and $S^{[n]}$ is the quotient by the symmetric group of the blow-up of S^n_* along the diagonal $\Delta_* = \pi^{-1}(D_*)$.

From the description above, it is straightforward to check that $H^3(S^{[n]}_*, \mathbb{Z}) = 0$. Since $S^{[n]} \setminus S^{[n]}_*$ has codimension 2 in $S^{[n]}$, the restriction morphism

$$H^3(S^{[n]}, \mathbb{Z}) \to H^3(S^{[n]}_*, \mathbb{Z})$$

is injective, which shows the result. $\square$

Corollary 2.7. *Let k be a field of characteristic $p > 2$ that is either finite or algebraically closed, and let (X, H) be a polarized $K3$ surface over k. Let v be a Mukai vector on X satisfying condition* (C). *Let W be the ring of Witt vectors of k.*

Assume that the triple (X, v, H) lifts to a projective $K3$ surface $\mathcal{X}$ over W. Let η be the generic point of $\operatorname{Spec} W$.

Then there exists an isometry

$$\theta_{v,\eta} : v^{\perp} \cap N(\mathcal{X}_\eta) \to \mathrm{NS}(\mathcal{M}_H(\mathcal{X}_\eta, v))$$

such that the following diagram commutes

$$\begin{tikzcd} v^{\perp} \cap N(\mathcal{X}_\eta) \arrow[r, "\theta_{v,\eta}"] \arrow[d] & \mathrm{NS}(\mathcal{M}_H(\mathcal{X}_\eta, v)) \arrow[d] \\ v^{\perp} \cap N(X) \arrow[r, "\theta_v"] & \mathrm{NS}(\mathcal{M}_H(X, v)), \end{tikzcd}$$

where the vertical maps are the specialization maps.

Proof. By the proof of (vi) in Theorem 2.4, we know that there exists an isometry

$$\theta_{v,\eta} : (v^{\perp} \cap N(\mathcal{X}_\eta)) \otimes \mathbb{Q} \to \mathrm{NS}(\mathcal{M}_H(\mathcal{X}_\eta, v)) \otimes \mathbb{Q}$$

making the analog of the diagram above commute. We need to show that $\theta_{v,\eta}$ sends $v^{\perp} \cap N(\mathcal{X}_\eta)$ to $\mathrm{NS}(\mathcal{M}_H(\mathcal{X}_\eta, v))$.

By [EJ11] — which is stated over $\mathbb{Z}_p$ but whose proof extends verbatim to our setting over W — the equality $H^1(\mathcal{M}_H(v), \mathcal{O}_{\mathcal{M}_H(v)}) = 0$ proved in Proposition 2.6 implies that the cokernel of the specialization map

$$\mathrm{NS}(\mathcal{M}_H(\mathcal{X}_\eta, v)) \to \mathrm{NS}(\mathcal{M}_H(X, v))$$

is torsion-free, which shows the result. □

Corollary 2.8. *Let k be an algebraically closed field of characteristic p, and let X be a $K3$ surface over k. Let H be a polarization on X, and let v be a Mukai vector on X that satisfies condition* (C) *of Definition* 2.3. *If $p > 0$, we can find nonnegative integers λ and t such that $0 < \lambda \le v^2$ and*

$$|\mathrm{disc}(\mathrm{NS}(X))| = p^t \lambda |\mathrm{disc}(\mathrm{NS}(\mathcal{M}_H(v)))|.$$

If $p = 0$, then

$$|\mathrm{disc}(\mathrm{NS}(X))| \le v^2 |\mathrm{disc}(\mathrm{NS}(\mathcal{M}_H(v)))|.$$

Proof. We treat the case where $p > 0$. By Theorem 2.4, (vi) and (vii), we have an injective isometry

$$\theta_v : v^{\perp} \cap N(X) \to \mathrm{NS}(\mathcal{M}_H(v)),$$

where $v^\perp$ is the orthogonal of v in $N(X) = \mathbb{Z} \oplus \mathrm{NS}(X) \oplus \mathbb{Z}\omega$. The cokernel of the map above is a p-primary torsion group. By [LMS14, Lemma 2.1.1], we have

$$|\mathrm{disc}(v^\perp \cap N(X))| = p^r |\mathrm{disc}(\mathrm{NS}(\mathcal{M}_H(v)))|$$

for some nonnegative integer r. Furthermore, we have a natural injection of lattices with torsion cokernel

$$\mathbb{Z}v \oplus (v^\perp \cap N(X)) \hookrightarrow N(X).$$

Since the discriminant of $\mathbb{Z}v \oplus v^\perp$ is $v^2 \mathrm{disc}(v^\perp \cap N(X))$, this implies by the same argument that

$$|\mathrm{disc}(N(X))| \leq v^2 |\mathrm{disc}(v^\perp \cap N(X))|.$$

Finally, since as a lattice, $N(X) \simeq \mathrm{NS}(X) \oplus U$, where U is the hyperbolic plane, we get the result. ☐

2.2. *Low-degree line bundles on moduli spaces of stable sheaves on $K3$ surfaces.* If n is an integer, let Λ_{2n} denote the lattice

$$\Lambda_{2n} = \langle 2n \rangle \oplus U,$$

where U is the hyperbolic plane.

Proposition 2.9. *Let d be a positive integer, and let Λ be a rank 2 positive definite sublattice of Λ_{2d}. There exists a positive integer N and nonzero integers a, b such that if m is any positive integer satisfying*

(i) $m = 1[N]$;
(ii) *m is prime to a and b, and both a and b are quadratic residues modulo m;*

then there exists a primitive embedding of Λ into Λ_{2md}.

Proof. We use a result of Nikulin that describes primitive embeddings of even lattices. We describe here its content in our case.

Let Λ be any even positive-definite lattice of rank 2. Fix a positive integer n. Let $A_\Lambda = \Lambda^*/\Lambda$ be the discriminant group of Λ. It is endowed with a natural quadratic form q_Λ with values in $\mathbb{Q}/2\mathbb{Z}$. Similarly, let $A_{2n} = \mathbb{Z}/2n\mathbb{Z}$ be the discriminant group of the even lattice Λ_{2n}, and let q_{2n} be the natural quadratic form on A_{2n}. Then $q_{2n}(1) = \frac{1}{2n} \in \mathbb{Q}/2\mathbb{Z}$.

It is proved in [Nik79, Prop. 1.15.1] that primitive embeddings of Λ into Λ_{2n} are in one-to-one correspondence with the tuples (V, W, γ, t), where $V \subset A_\Lambda$ and $W \subset A_{2n}$ are subgroups, $\gamma : V \to W$ is an isomorphism respecting the restrictions of q_Λ and q_{2n} to V and W respectively and t is a positive integer such that the quadratic form

$$(q_\Lambda \oplus (-q_{2n}))_{|\Gamma_\gamma^\perp/\Gamma_\gamma}$$

is isomorphic to the quadratic form on $\mathbb{Z}/2t\mathbb{Z}$ that sends 1 to $\frac{1}{2t}$, where

$$\Gamma_\gamma = \{(a,b) \in V \oplus W | \gamma(a) = b\}.$$

Note that Γ_γ is a cyclic group as it can be identified to $W \subset \mathbb{Z}/2n\mathbb{Z}$.

In the setting of the proposition, the primitive embedding of Λ into Λ_{2d} corresponds to a tuple (V, W, γ, t). Let m be a positive integer such that $m = 1[4dt]$ and $m = 1[|A_\Lambda|]$. Assume also that m is prime to $(2n)!$. Note that multiplication by m is the identity in A_Λ. We will make further assumptions on m later on.

The map

$$A_\Lambda \oplus A_{2d} \xrightarrow{\mathrm{Id} \oplus m\mathrm{Id}} A_\Lambda \oplus A_{2md}$$

is injective and respects the quadratic forms $q_\Lambda \oplus (-q_{2d})$ on the left and $q_\Lambda \oplus (-q_{2md})$ on the right. Indeed, since $m = 1[4d]$, we have

$$q_{2md}(m) = \frac{m^2}{2md} = \frac{m}{2d} = \frac{1}{2d} \in \mathbb{Q}/2\mathbb{Z}.$$

Let W' be the image of W in A_{2nd}, and let

$$\gamma' : V \to W'$$

be the isometry induced by γ.

By assumption, the group $\Gamma_\gamma^\perp/\Gamma_\gamma$ is isomorphic to $\mathbb{Z}/2t\mathbb{Z}$. We can also find an element (x_0, y_0) of $\Gamma_\gamma^\perp \subset A_\Lambda \oplus A_{2d}$ that maps to a generator of $\Gamma_\gamma^\perp/\Gamma_\gamma$ and such that

$$q_\lambda(x_0) - \frac{y_0^2}{2d} = \frac{1}{2t} \in \mathbb{Q}/2\mathbb{Z}.$$

We see y_0 as an integer between 1 and $2d$. Then we can consider (x_0, y_0) as an element of $A_\Lambda \oplus A_{2md}$.

Let (α, β) be a generator of Γ_γ, where $\alpha \in A_\Lambda$ and $\beta \in A_{2d}$ is considered as an integer. Then $(\alpha, m\beta)$ is a generator of $\Gamma_{\gamma'}$. Let $b_\lambda, b_{2d}, b_{2md}$ be the bilinear forms with values in $\mathbb{Q}/\mathbb{Z}$ associated with q_Λ, q_{2d} and q_{2md} respectively. We have

$$b_\Lambda(x_0, \alpha) - b_{2d}(y_0, \beta) = b_\Lambda(x_0, \alpha) - \frac{y_0\beta}{2d} = 0 \in \mathbb{Q}/\mathbb{Z}$$

since $(x_0, y_0) \in \Gamma_\gamma^\perp$. This implies that

$$b_\Lambda(x_0, \alpha) - b_{2md}(y_0, m\beta) = b_\Lambda(x_0, \alpha) - \frac{my_0\beta}{2md} = 0 \in \mathbb{Q}/\mathbb{Z},$$

which shows that (x_0, y_0) belongs to $\Gamma_{\gamma'}^\perp$ in $A_\Lambda \oplus A_{2md}$.

By construction and since m is prime to y_0, the order of (x_0, y_0), seen as an element of $\Gamma_{\gamma'}^\perp$, in the group $\Gamma_{\gamma'}^\perp/(\Gamma_{\gamma'}^\perp \cap \mathrm{Im}(A_\Lambda \oplus A_{2d}))$, is exactly m. Furthermore, $m(x_0, y_0)$ is the image of the element $(mx_0, y_0) = (x_0, y_0) \in A_\Lambda \oplus A_{2d}$, which maps to a generator of $\Gamma_\gamma^\perp/\Gamma_\gamma$ by assumption.

Since Γ_γ and $\Gamma_{\gamma'}$ are canonically isomorphic, the discussion above shows that the group $\Gamma_{\gamma'}^\perp/\Gamma_{\gamma'}$ is cyclic of order $2tm$ and has a generator v such that

$$q(v) = q_\lambda(x_0) - \frac{y_0^2}{2md} = \frac{1}{2t} + \frac{y_0^2}{2d} - \frac{y_0^2}{2md} = \frac{1}{2t} + \frac{y_0^2(m-1)}{2md} \in \mathbb{Q}/2\mathbb{Z},$$

where q is the natural quadratic form on $\Gamma_{\gamma'}^\perp/\Gamma_{\gamma'}$.

To show the result — after adding conditions on m according to the statement of the proposition — we need to find a generator v' of $\Gamma_{\gamma'}^\perp/\Gamma_{\gamma'}$ such that $q(v') = \frac{1}{2mt} \in \mathbb{Q}/2\mathbb{Z}$. Writing $v' = \lambda v$, we need to find an integer λ such that

(i) λ is prime to $2mt$,

(ii) $\lambda^2(\frac{1}{2t} + \frac{y_0^2(m-1)}{2md}) - \frac{1}{2mt} \in 2\mathbb{Z}$.

The second condition can be rephrased as the congruence

$$\lambda^2(md + ty_0^2(m-1)) - d = 0[4mdt].$$

From now on, we only consider integers λ such that $\lambda = 1[4dt]$. Since by assumption $m = 1[4dt]$, this implies that the condition above is always satisfied modulo $4dt$. As a consequence, we only have to consider the condition modulo m, which becomes

$$\lambda^2 ty_0^2 + d = 0[m].$$

Note that, as above, y_0 is prime to m. Choosing m such that both $-d$ and t are both quadratic residues modulo m, this shows that we can find a suitable λ, which concludes the proof. □

THEOREM 2.10. *Let k be a field, and let d be a positive integer. Then there exist a positive integer r, a positive integer N and nonzero integers a, b such that if (X, H) is a polarized $K3$ surface of degree $2md$ over k, where m is any positive integer satisfying*

(i) $m = 1[N]$;

(ii) *m is prime to a and b, and both a and b are quadratic residues modulo m;*

then there exists a Mukai vector v on X satisfying condition (C) *such that*

(i) *$c_1(v)$ is proportional to $c_1(H)$;*

(ii) *the moduli space $\mathcal{M}_H(v)$ has dimension* 4*;*

(iii) *there exists a line bundle L on $\mathcal{M}_H(v)$ satisfying $c_1(L)^4 = r$ and $q(L) > 0$.*

If n is any integer not divisible by 2 *or* 3*, we can assume that $q(L)$ is an integer prime to n and that there exists an ample line bundle A on $\mathcal{M}_H(v)$ such that $q(A)$ is an integer prime to n. Furthermore, if k is algebraically closed or finite, the same result holds even when n is divisible by* 3*.*

Finally, assume that k has characteristic $p > 2$, and let W be the ring of Witt vectors of an algebraic closure $\overline{k}$ of k. If the pair $(X_{\overline{k}}, H)$ lifts to W, then we can assume that the triple $(\mathcal{M}_H(v)_{\overline{k}}, L, A)$ lifts to W.

Proof. In the lattice Λ_{2d}, consider a positive-definite rank-2 sublattice Λ containing elements v and w with $v^2 = 2$ and $v.w = 1$. Let l be an element of Λ such that $l.v = 0$, and let $r = \frac{(4)!}{(2!)2^2}(l^2)^2 = 3(l^2)^2$. Note that we can indeed choose l so that l^2 is prime to n if n is odd.

By Proposition 2.9, we can find integers N, a and b as above such that if m is any positive integer satisfying the conditions of the theorem, then Λ_{2md} contains Λ as a primitive sublattice.

Let X be any $K3$ surface over k with an ample line bundle H of self-intersection $2md$, with m as above. Then the lattice $N(X)$ of Mukai vectors on X contains the sublattice $\mathbb{Z} \oplus \mathbb{Z}H \oplus \mathbb{Z}\omega \simeq \Lambda_{2md}$. As a consequence, there exists an injection

$$\Lambda \hookrightarrow \Lambda_{2md} \hookrightarrow N(X).$$

Seeing $v \in \Lambda$ as an element of $N(X)$, write $v = \mathrm{rk}(v) + c_1(v) + \lambda\omega$. By assumption there exists $w \in \Lambda \subset \mathbb{Z} \oplus \mathbb{Z}H \oplus \mathbb{Z}\omega$ such that $v.w = 1$. In particular, the vector $v \in N(X)$ is primitive. Furthermore, we have

$$\gcd(\mathrm{rk}(v), c_1(v).H, \lambda) = 1.$$

In particular, since 2 divides $c_1(v).H$, $\mathrm{rk}(v)$ and λ cannot both vanish. After composing the embedding of Λ into Λ_{2md} by a suitable automorphism of Λ_{2md}, we can assume that $\mathrm{rk}(v) > 0$. Since $v^2 > 0$, this shows that v satisfies condition (C). As a consequence of Theorem 2.4(i), the moduli space $\mathcal{M}_H(v)$ has dimension $v^2 + 2 = 4$.

We first assume that k is algebraically closed or finite. Let n be an odd integer. Then the vector $l \in \Lambda \subset N(X)$ defined above lies in $v^\perp \cap N(X)$ by assumption. By Theorem 2.4(vi), we have an injection of lattices

$$v^\perp \cap N(X) \hookrightarrow \mathrm{NS}(\mathcal{M}_H(v)).$$

By (iii) and (iv) of the same theorem, the image of l in $\mathrm{NS}(\mathcal{M}_H(v))$ is the class of a line bundle L on $\mathcal{M}_H(v)$ such that

$$c_1(L)^4 = 3(l^2)^2 = r.$$

The integer $q(L)$ is prime to n.

Let A_0 be an ample divisor on $\mathcal{M}_H(v)$. We can assume that $q(A_0)$ is an integer after raising A_0 to a sufficiently large p-th power. If λ is large enough and n divides λ, then $A = A_0^{\otimes\lambda} \otimes L$ is ample and $q(A)$ is congruent to $q(L)$ modulo n, which implies that $q(A)$ is prime to n.

If k is an arbitrary field, the construction above provides a line bundle L on $\mathcal{M}_H(v)_{\overline{k}}$ with Galois-invariant first Chern class. This implies that L itself is Galois-invariant.

Consider the exact sequence

$$\mathrm{Pic}(\mathcal{M}_H(v)) \to \mathrm{Pic}(\mathcal{M}_H(v)_{\overline{k}})^{\mathrm{Gal}(\overline{k}/k)} \to \mathrm{Br}(k) \to \mathrm{Br}(\mathcal{M}_H(v)).$$

Since $\mathcal{M}_H(v)$ is deformation-equivalent to $X^{[2]}$, $c_4(\mathcal{M}_H(v))$ is a zero-cycle of degree $324 = 2^2 \times 3^4$; see [EGL01, Rem. 5.5]. Such a zero-cycle induces a map $\mathrm{Br}(X) \to \mathrm{Br}(k)$ such that the composition $\mathrm{Br}(k) \to \mathrm{Br}(X) \to \mathrm{Br}(k)$ is multiplication by 324. In particular, the cokernel of the map $\mathrm{Pic}(\mathcal{M}_H(v)) \to \mathrm{Pic}(\mathcal{M}_H(v)_{\overline{k}})^{\mathrm{Gal}(\overline{k}/k)}$ is killed by multiplication by 324. This shows that $L^{\otimes 324}$ satisfies the conclusion of the theorem.

Finally, assume that k has characteristic $p > 2$, and assume that the pair $(X_{\overline{k}}, H)$ lifts to a projective $K3$ surface $(\mathcal{X}, H)$ over W. Then since $c_1(v)$ is a multiple of $c_1(H)$, v lifts to $\mathcal{X}$ as well. The result then follows directly from Corollary 2.7. □

Remark 2.11. Even over an arbitrary field, it is possible to ensure that $q(L)$ and $q(A)$ are prime to 3 by considering a 6-dimensional moduli space of sheaves: these have top Chern class of degree $3200 = 2^7 \times 5^2$.

3. Finiteness results for holomorphic symplectic varieties

Theorem 2.10 was devoted to constructing irreducible holomorphic symplectic varieties of dimension 4 — or, in positive characteristic, reduction of such varieties — together with a line bundle L of low positive self-intersection. By a theorem of Huybrechts [Huy99, Cor. 3.10], either L or its dual is big. We now investigate finiteness results for families of such varieties.

In characteristic zero, we prove that given positive integers n and r, the family of irreducible holomorphic varieties X such that there exists a line bundle L on X with $c_1(L)^{2n} = r$ is birationally bounded. Unfortunately, our proof does not make explicit any of the natural constants involved. It relies on the global period map and the local Torelli theorem.

Over finite fields of characteristic $p > 3$, we show a finiteness result for Néron-Severi groups of such varieties. The proof relies on the Kuga-Satake construction as a replacement for the period map.

3.1. *The case of $K3$ surfaces.* Before dealing with higher-dimensional varieties below, we treat the much easier case of $K3$ surfaces. The following result is certainly well known to experts. The proof is very close to arguments in [SD74], but since this paper assumes that the characteristic is odd, we make sure that the statement is correct in arbitrary characteristic.

Proposition 3.1. *Let r be a positive integer. Then there exist positive integers N and d such that if X is a $K3$ surface over an algebraically closed field k with a line bundle L such that $L^2 = r$, then there exists a line bundle L' on X with $h^0(X, L') \leq N$ such that the complete linear system $|L'|$ induces a birational map from X to a subvariety of $\mathbb{P}|L'|$ of degree at most d.*

Proof. In this proof, we will identify L and $c_1(L)$. We can assume that L is big and nef. Indeed, by [Ogu79, 1.10 and p.371] (see also [Huy14, Ch. 8, par. 2]), there exists a line bundle L' with the same self-intersection as L that is big and nef. In particular, we have $h^1(L) = 0$ by [Huy14, Prop. 3.1].

By the Riemann-Roch theorem, we have

$$h^0(L) + h^0(L^\vee) = \frac{r}{2} + 2 \geq 3.$$

Since L is nef, this shows that L is effective and $h^0(L) \geq 3$.

We now rephrase the argument of [SD74, Prop. 8.1]. Let F be the fixed part of the linear system $|L|$. Assume that $F \neq 0$. Let $M = L \otimes \mathcal{O}(-F)$. Then $M^2 \geq 0$. By [SD74, 2.6 and 2.7.4], which does not make use of any hypothesis on the characteristic, either $M^2 > 0$ or we can find an irreducible curve E on X with arithmetic genus 1 such that $M = \mathcal{O}(mE)$ with $m = h^0(L) - 1 \geq 2$. Furthermore, we can then find an irreducible rational curve Γ in F such that $F.E = 1$.

Let $L' = M$ if $M^2 > 0$ and $L' = mE + \Gamma$ in the other case. Write $L = L' + \Delta$. Assume that $\Delta \neq 0$. Then by the Hodge index theorem (see [Huy14, Ch. 1, Rem. 2.4(iii)]), $L'.\Delta > 0$. Now $h^0(L') = h^0(L) = \frac{L^2}{2} + 2$ and $h^1(L') = 0$ by [SD74, Lemma 2.2]. This shows that $L'.L' = L.L$. In other words, we have

$$2L'.\Delta + \Delta^2 = 0.$$

However, since $L'.\Delta > 0$, this implies that

$$L.\Delta = L'.\Delta + \Delta^2 < 0,$$

which contradicts the fact that L is nef. This shows that $\Delta = 0$, i.e., $L = L'$.

Now up to replacing L by $2L$, it is readily seen that we can assume that L has no fixed part. By [SD74, Prop. 2.6] (see also [Huy14, Ch. 2, Rem. 3.7(ii)]), we can write $L = \mathcal{O}(C)$, where C is an irreducible curve on X. Furthermore, we have $h^1(L) = 0$ by [Huy14, Prop. 3.1] again.

The discussion above readily implies that the image of the rational map ϕ_L from X to $\mathbb{P}^{\frac{r}{2}+1}$ induced by the complete linear system $|L|$ has dimension 2. Furthermore, ϕ_L is either birational or has generic degree 2 onto its image. Indeed, the degree of the image of ϕ_L in projective space is at least $\frac{r}{2}$.

It is stated in the literature that ϕ_{2L} is birational onto its image. If the characteristic is odd, this follows from the existence of a smooth, irreducible global section of L as in [SD74]. In general, this is stated as "well known" in [ASD73, after Lemma 5.17]. We briefly give an argument that shows that ϕ_{4L} is birational onto its image.

The line bundle L is big and nef, and the fixed part of L vanishes. By [Huy14, Chap. 2, Cor. 3.14(i)], L is base point free. We assume that ϕ_L is of generic degree 2 onto its image. Let C be an irreducible curve belonging to $|L|$.

Let C_{red} be the reduced corresponding curve. Then as cycles on X, we have $C = aC_{\mathrm{red}}$ where $a = 1$ or $a = 2$; in characteristic 2, it might happen that a is necessarily 2. By the adjunction formula, dualizing sheaf of C over k is $L_{|C}$. It has degree r. The arithmetic genus of C is $1 + \frac{r}{2}$. In the following, we assume that $a = 2$, as the generically reduced case is easier.

The exact sequence

$$0 \to (i-1)L \to iL \to (iL)_{|C} \to 0$$

together with the vanishing of $H^1(X, (i-1)L)$ shows that for any $i > 0$, we have a surjection $H^0(X, iL) \to H^0(C, iL)$.

The integral curve C_{red} is generically smooth. Let p be a closed point of C_{red} lying in the smooth locus. Then there exists a line bundle $D(p)$ of degree 2 on C that restricts to $\mathcal{O}(p)$ on C_{red}. By Riemann-Roch, for any p and q as above, the morphism

$$H^0(C, 3L_{|C}) \to 3L_{|C}/(3L_{|C} - D(p) - D(q))$$

is surjective. Indeed, the degree of $3L_{|C} - D(p) - D(q)$ is $3r - 4 > r = \deg(\omega_{C/k})$; note that r is even, so that $r > 1$. As a consequence, we obtain the surjectivity of the morphism

$$H^0(C_{\mathrm{red}}, 3L) \to 3L_{|C_{\mathrm{red}}}/3L_{|C_{\mathrm{red}}}(-p-q).$$

This shows that the restriction of $3L$ to C induces an immersion on the smooth locus of C_{red}, hence an application of degree 1 onto its image. This shows that ϕ_{4L} is birational onto its image.

Since $h^1(X, 4L) = 0$ and $4L$ is base point free, this shows that ϕ_{4L} induces a birational map from X to a subvariety of degree r of $\mathbb{P}^N$ with $N = \frac{r}{2} + 1$. $\square$

Since $K3$ surfaces are minimal, this shows that the surfaces X as in the proposition above form a bounded family. In particular, we get the following result.

COROLLARY 3.2. *Let k be a finite field with algebraic closure $\overline{k}$, and let r be a positive integer. Then there exist finitely many $\overline{k}$-isomorphism classes of $K3$ surfaces X over k such that there exists a line bundle L on $X_{\overline{k}}$ with $L^2 = r$.*

If the characteristic of k is odd, then there exist only finitely many isomorphism classes over k of such $K3$ surfaces.

Proof. After replacing k by a finite extension K of fixed degree, we can assume that any line bundle on $X_{\overline{k}}$ is defined over X. Let N, d and L' be as in Proposition 3.1. Then L' is defined over K, and so is the image of X under the rational map defined by the complete linear system associated to L'.

The theory of Chow forms shows that there are only finitely many subvarieties of $\mathbb{P}^N_K$ of degree at most d. As a consequence, the number of K-birational classes of surfaces X as in the statement is finite. Since $K3$ surfaces are minimal, this shows the first result. The second statement is a consequence of the first and of [LMS14, Prop. 2.4.1]. □

3.2. *A birational version of Matsusaka's big theorem for holomorphic symplectic varieties.* The goal of this section is to prove the following result. It should be seen as a — weak — birational version of Matsusaka's big theorem. It will not be used in the proof of the Tate conjecture.

THEOREM 3.3. *Let n and r be two positive integers. Then we can find constants a, N and d such that if X is a complex irreducible holomorphic symplectic variety of dimension $2n$ and L is a line bundle on X with $c_1(L)^{2n} = r$ and $q(L) > 0$, where q is the Beauville-Bogomolov form, then there exists a line bundle L' with $c_1(L')^{2n} = r$ such that $h^0(X, aL') \leq N$ and such that the complete linear system $|aL'|$ induces a birational map from X to a subvariety of $\mathbb{P}|aL'|$ of degree at most d.*

In particular, there exists a scheme S of finite type over $\mathbb{C}$, and a projective morphism $\mathcal{X} \to S$ such that if (X, L) is any pair as above, there exists a complex point s of S such that $\mathcal{X}_s$ is birational to X.

Remark 3.4. Our proof relies crucially on the existence of the global period map. Recent work of Amerik-Verbitsky [AV14] on the cone conjecture for holomorphic symplectic varieties should make it possible to give a geometric proof of Theorem 3.3 along the lines of Proposition 3.1.

We start with two lemmas.

LEMMA 3.5. *There exist finitely many pairs $(X_1, L_1), \ldots, (X_s, L_s)$ where the X_i are complex irreducible holomorphic symplectic manifolds of dimension $2n$ and L_i is an ample line bundle on X_i with $c_1(L_i)^{2n} = r$ such that if (X, L) is a pair as in Theorem 3.3, then either (X, L) or $(X, L^{\otimes -1})$ is deformation-equivalent to (X_i, L_i) for some i.*

In the statement above and in the proof below, we are considering deformations of complex varieties over bases that are complex manifolds that are not necessarily projective. Finiteness of deformation types for holomorphic symplectic varieties with given topological invariants has been proved in [Huy03b].

Proof. Let (X, L) be a pair as in Theorem 3.3. By [Huy99, Th. 3.11] and [Huy03a], X is projective. By the local Torelli theorem for X [Bea83, Th. 5], we can find a small deformation (X', L') of the pair (X, L) such that $\mathrm{Pic}(X')$ has rank 1. By the aforementioned theorem of Huybrechts, X' is projective, which implies, up to replacing L by its dual, that L' is ample.

Consider pairs (X', L') where X' is smooth projective of dimension $2n$, $K_{X'} = 0$ and L' is an ample line bundle with $c_1(L)^{2n} = r$. By Kollár-Matsusaka's refinement of Matsusaka's big theorem [KM83], the family of such pairs (X, L) is bounded. As a consequence, we can find finitely many pairs $(X_1, L_1), \dots, (X_s, L_s)$ where the X_i are complex irreducible holomorphic symplectic manifolds of dimension $2n$ and L_i is an ample line bundle on X_i with $c_1(L_i)^{2n} = r$ such that any pair (X', L') as in the paragraph above is deformation-equivalent to one of the (X_i, L_i). □

LEMMA 3.6. *Let S be a noetherian scheme over $\mathbb{C}$, and let $\mathcal{X} \to S$ be a projective morphism. Let L be a line bundle over $\mathcal{X}$ such that for every complex point s of S, the restriction L_s of L to $\mathcal{X}_s$ is big. Then there exist integers a, N and d such that for any complex point s of S, $h^0(\mathcal{X}_s, aL_s) \leq N$ and the complete linear system $|aL_s|$ induces a birational map from $\mathcal{X}_s$ to a subvariety of $\mathbb{P}|aL_s|$ of degree at most d.*

Proof. We use noetherian induction on S. It suffices to show that if S is nonempty, there exists a nonempty open subset U of S and constants a, N, d such that the conclusion of the lemma holds on U.

Since L_s is big for any complex point s of S, Baire's theorem shows that if η is any generic point of S, then L_η is big. This readily shows the result. □

Proof of Theorem 3.3. By Lemma 3.5, we can restrict our attention to the pairs (X, L) that are deformation-equivalent to a given (X_0, L_0) where L_0 is ample. Note that this implies that L is big.

We denote by $l \in \Lambda$ the element $c_1(L)$ and by Λ the lattice $H^2(X_0, \mathbb{Z})$ endowed with its Beauville-Bogomolov form. Note that $l^2 > 0$. Let Λ_{prim} be the orthogonal complement of l in Λ, and let D be the period domain associated to Λ_{prim}, that is,

$$D = \{x \in \mathbb{P}(\Lambda_{\mathrm{prim}} \otimes \mathbb{C}) | x^2 = 0, x.\overline{x} > 0\}.$$

Let $\widetilde{O}(\Lambda_{\mathrm{prim}})$ be the group

$$\widetilde{O}(\Lambda_{\mathrm{prim}}) := \{g_{|\Lambda_{\mathrm{prim}}} | g \in O(\Lambda), g(l) = l\}.$$

We will freely identify $\widetilde{O}(\Lambda_{\mathrm{prim}})$ with a subgroup of $O(\Lambda)$ when needed.

Let M be the monodromy group of (X_0, L_0); see, for instance, [Mar11, Def. 1.1(5)]. Then M can be identified with a subgroup of $\widetilde{O}(\Lambda_{\mathrm{prim}})$. By a result of Sullivan [Sul77], M has finite index in $\widetilde{O}(\Lambda_{\mathrm{prim}})$; the result of Sullivan deals with the unpolarized case (see the discussion in [Ver13, Th. 3.5]), but the polarized case follows from [Mar11, Prop. 1.9].

Let Γ be a subgroup of finite index in both M and a torsion-free arithmetic subgroup of $\widetilde{O}(\Lambda_{\mathrm{prim}})$. By the theorem of Baily-Borel [BB66], the quotient $\Gamma\backslash D$ is a normal quasi-projective variety.

To any triple (X, L, ϕ) where X is an irreducible holomorphic symplectic variety, L is a line bundle on L such that the pair (X, L) is deformation-equivalent to (X_0, L_0) and ϕ is an isomorphism $\phi : H^2(X, \mathbb{Z}) \to \Lambda$ sending $c_1(L)$

to l, we can associate its *period point* $\mathcal{P}(X, L, \phi)$. The element $\phi(H^{2,0}(X)) \subset \Lambda \otimes \mathbb{C}$ belongs to D, and we define $\mathcal{P}(X, L, \phi)$ to be the image of $\phi(H^{2,0}(X))$ in $\Gamma\backslash D$. If γ is any element of Γ, then $\mathcal{P}(X, L, \phi) = \mathcal{P}(X, L, \gamma \circ \phi)$.

Let S be a smooth quasi-projective complex scheme, and let $\mathcal{X} \to S$ be a smooth projective morphism whose fibers are irreducible holomorphic symplectic varieties. Let L be a line bundle on $\mathcal{X}$ such that the pairs $(\mathcal{X}_s, L_s)$ are deformation-equivalent to (X_0, L_0) for any complex point s of S. Assume for simplicity that S is connected and fix s and such a deformation. Then, using parallel transport, we can identify Λ and $H^2(\mathcal{X}_s, \mathbb{Z})$. By definition of the monodromy group M, the monodromy representation $\rho : \pi_1(S, s) \to O(\Lambda)$ factors through M. If ρ factors through the finite index subgroup $\Gamma \subset M$, then the construction above induces a period map

$$\mathcal{P} : S \to \Gamma\backslash D.$$

By a result of Borel [Bor72, Th. 3.10], $\mathcal{P}$ is algebraic.

Let (X, L, ϕ) and (X', L', ψ) be two triple as above. Assume that ϕ (resp. ψ) is induced by parallel transport along a deformation of (X, L) (resp. (X', L')) to (X_0, L_0). Then the global Torelli theorem of Verbitsky [Ver13] shows that if $\mathcal{P}(X, L, \phi) = \mathcal{P}(X', L', \psi)$, then X and X' are birational. In case L and L' are ample, this is the statement of [Mar11, Th. 1.10], and the general case can be deduced either by using a small deformation to the ample case or by using [Mar11, Prop. 1.9] to reduce to the general global Torelli theorem.

We claim that there exists $S, \mathcal{X}$ and L as above such that the monodromy representation of each connected component of S factors through Γ and such that the image of $\mathcal{P}$ is $\Gamma\backslash D$. By noetherian induction, we can find $S, \mathcal{X}$ and L, as well as a Zariski open subset U of $\Gamma\backslash D$, such that the image of the period map $\mathcal{P} : S \to \Gamma\backslash D$ contains U and such that U is maximal with respect to this property. We assume by contradiction that U is strictly contained in $\Gamma\backslash D$.

Let Z be an irreducible component of the complement of U in $\Gamma\backslash D$, and let z be a very general complex point of Z. Using the surjectivity of the period map [Huy99, Th. 8.1], we can find a triple (X_z, L_z, ϕ), where (X_z, L_z) is deformation-equivalent to (X_0, L_0) and $\phi : H^2(X_z, L_z) \to \Lambda$ is induced by parallel transport, such that $\mathcal{P}(X_z, L_z) = z$.

By the aforementioned theorem of Huybrechts, X_z is projective. Let H_z be an ample line bundle on X_z. By the local Torelli theorem [Bea83, Th. 5], the pair (X_z, L_z) can be deformed over a small open subset of $Z(\mathbb{C})$ — for the usual topology. Since z is a very general point of Z, the whole Néron-Severi group of X_z deforms above this open subset, hence so does H_z. This shows that this deformation can be algebraized.

As a consequence, resolving singularities of the base and passing to a finite cover, we can find a smooth projective morphism $\mathcal{X}_T \to T$, where T is a smooth complex quasi-projective variety whose fibers are irreducible holomorphic symplectic varieties, and L a line bundle on $\mathcal{X}_T$ where the pairs $(\mathcal{X}_t, L_t)$ are deformation-equivalent to (X_0, L_0) for any complex point t of T, such that the modnodromy representation on T factors through Γ and the image of the period map

$$\mathcal{P} : T \to \Gamma\backslash D$$

is a Zariski-open subset V of Z. Since Z is an irreducible component of $(\Gamma\backslash D) \setminus U$, we can shrink V so that V is open in $(\Gamma\backslash D) \setminus U$.

Now taking the disjoint union of the families $\mathcal{X} \to S$ and $\mathcal{X}_T \to T$, we get a family as above such that the image of the period map

$$\mathcal{P} : S \sqcup T \to \Gamma\backslash D$$

contains $U \cup V$, which is open in $\Gamma\backslash D$ and strictly contains U. This is the desired contradiction.

Now let $S, \mathcal{X}$ and L be as above such that the image of the period map is $\Gamma\backslash D$. By Lemma 3.6, we can find integers k, N and d such that for any complex point s of S, $h^0(\mathcal{X}_s, kL_s) \leq N$ and the complete linear system $|kL_s|$ induces a birational map from $\mathcal{X}_s$ to a subvariety of $\mathbb{P}|kL_s|$ of degree at most d.

Let (X, L) be any pair as in the theorem that is deformation-equivalent to (X_0, L_0). Let $\phi : H^2(X, \mathbb{Z}) \to \Lambda$ be induced by parallel transport. By construction of S, we can find a complex point s of S such that $\mathcal{P}(X, L, \phi) = \mathcal{P}(s)$. As noted above, this implies by the global Torelli theorem that X_s is birational to X. Since X and X_s have trivial canonical bundle, such a birational map is an isomorphism outside a closed subscheme of codimension at least 2. In particular, it induces an isomorphism between the Picard groups of X and X_s. Let L' be the image of L_s in the Picard group of X_s. Then (X, L') satisfies the condition of the theorem. □

Remark 3.7. While using it simplifies slightly the phrasing of the proof, the global Torelli theorem of Verbitsky — as well as the surjectivity of the period map — could be replaced by the local Torelli theorem.

3.3. *A variant of the Kuga-Satake construction and birational boundedness in positive characteristic.* The goal of this section is to extend part of the boundedness result above to positive characteristic. To facilitate the exposition, we will prove a weaker result. The proof is very similar to that of Theorem 3.3, but we replaced the complex period map with the Kuga-Satake construction.

It is very likely that the construction by Pera in [MP16] of integral models of Shimura varieties of orthogonal type provides a period map that is sufficient to translate with only minor changes the proof of Theorem 3.3 to a positive characteristic setting — which would also take care of the case of characteristic 3. However, since one of the goals of this paper is to investigate the extent to which one can refrain from using too much of the theory of integral models of Shimura varieties, we decided to provide a slightly more elementary — though certainly related — proof. To simplify certain arguments, we will work in characteristic at least 5.

The Kuga-Satake construction associates an abelian variety to a polarized Hodge structure of weight 2 with $h^{2,0} = 1$. As shown by Deligne in [Del72], when applied to the primitive second cohomology group of a $K3$ surface, it is given by an absolute Hodge cycle.

At least over the field of complex numbers, the Hodge-theoretic definition of the Kuga-Satake construction makes it possible to apply it to any irreducible holomorphic symplectic variety endowed with a line bundle such that $q(L) > 0$, q being the Beauville-Bogomolov form; in that case, the orthogonal of $c_1(L)$ in $H^2(X, \mathbb{Z})$ is indeed a polarized Hodge structure of weight 2 with $h^{2,0} = 1$. Most of the usual results on the arithmetic of the Kuga-Satake construction extend to this setting without any change in the proofs, as we explain in this section.

The following is the situation we will be considering.

Setup. Let k be a perfect field of characteristic $p > 3$, and let W be the ring of Witt vectors of k. Let K be the fraction field of W. Fix an embedding of K into the field $\mathbb{C}$ of complex numbers. Let T be a smooth, irreducible W-scheme, and let $\pi : \mathcal{M} \to T$ be a smooth projective morphism. Let L and H be two line bundles on $\mathcal{M}$. We assume that H is relatively ample. We fix a k-point 0 of T.

We assume that $\mathcal{M}_{\mathbb{C}} \to T_{\mathbb{C}}$ is a family of irreducible holomorphic symplectic manifolds and that for any complex point t of T, the restriction L_t of L to $\mathcal{M}_t$ satisfies $q(L_t) > 0$, where q is the Beauville-Bogomolov form. We assume that there is no torsion in the second and third singular cohomology groups of the fibers of $\mathcal{M}_{\mathbb{C}} \to T_{\mathbb{C}}$.

Let Λ be a lattice isomorphic to $H^2(\mathcal{M}_t, \mathbb{Z})$ for any complex point t of T. Let l and h be elements of Λ that are mapped to $c_1(L_t)$ and $c_1(H_t)$ under such an isomorphism. Let Λ_l, Λ_h and $\Lambda_{l,h}$ be the orthogonal complement of l, h and $\mathbb{Z}l + \mathbb{Z}h$ in Λ respectively. We assume that the reduction modulo p of the restriction of q to Λ_l is nondegenerate.

For the sake of later reference, we will turn the preceding situation into a definition.

Definition 3.8. In the setup above, we say that the triple $(\mathcal{M}_0, H_0, L_0)$ is *admissible* and that the lattice Λ_l is a *primitive lattice* for (M_0, L_0). The quadratic form of the lattice is called the *Beauville-Bogomolov quadratic form.* It induces a quadratic form on the Néron-Severi group of $\mathcal{M}_0$.

We say that $(\mathcal{M}_0, H_0, L_0)$ is *strongly admissible* if in the setup above, we can ensure the following condition: let $\overline{\eta_k}$ be a geometric generic point of the special fiber T_k of T above W. Then $\mathcal{M}_{\overline{\eta_k}}$ is ordinary in degree 2 — that is, its second crystalline cohomology group has no torsion, and its Newton and Hodge polygons coincide — and $\mathrm{NS}(\mathcal{M}_{\overline{\eta_k}}) \otimes \mathbb{Q}$ is generated over $\mathbb{Q}$ by $c_1(L)$ and $c_1(H)$.

Remark 3.9. By [DI87], the Hodge to de Rham spectral sequence of $\mathcal{M}_0$ satisfies $E_1^{p,q} = E_\infty^{p,q}$ if $p+q=1$ or $p+q=2$. Furthermore, as in Proposition 2.6, the hypotheses ensure that $h^{2,0}(\mathcal{M}_0) = h^{0,2}(\mathcal{M}_0) = 1$.

The following result shows that moduli spaces of sheaves on $K3$ surfaces tend to be strongly admissible. Let k be a field with algebraic closure $\overline{k}$, and let X be a $K3$ surface over k. Recall from [Ogu79, Ex. 1.10] that X is said to be superspecial if the Hodge and conjugate filtration of X on $H^2_{dR}(X_{\overline{k}}/\overline{k})$ coincide. Superspecial $K3$ surfaces define isolated points in the moduli space of polarized $K3$ surfaces and correspond to the singular locus of this moduli space; see [Ogu79, Prop. 2.2].

Proposition 3.10. *In the situation of Theorem 2.10, assume that k has characteristic at least 5 and that (X, H) is a polarized, nonsuperspecial $K3$ surface over k. Then we can find a polarization A on $\mathcal{M}_H(v)$ such that $(\mathcal{M}_H(v), A, L)$ is strongly admissible.*

Proof. Dividing H by an integer if necessary, we can assume that H is primitive. Let $\widehat{X} \to \widehat{T}$ be the formal universal deformation of the pair (X, H). Since X is not superspecial, $\widehat{T}$ is formally smooth of dimension 19, i.e., $\widehat{T}$ is isomorphic to $\mathrm{Spf}\, W[[t_1, \dots, t_{19}]]$. In Theorem 2.10, $c_1(v)$ is proportional to H, so v lifts to $\widehat{X}$. Consider the relative moduli space $\mathcal{M}_H(\widehat{X}, v)$ over $\widehat{T}$. It is smooth and projective. As a consequence, we can find an ample line bundle A on $\mathcal{M}_H(v)$ that lifts to $\mathcal{M}_H(\widehat{X}, v)$. Since $\mathcal{M}_H(\widehat{X}, v) \to \widehat{T}$ is algebraizable, and by Proposition 2.6, the only thing that remains to be proved to show that $(\mathcal{M}_H(v), A, L)$ is strongly admissible is that L lifts to $\mathcal{M}_H(\widehat{X}, v)$.

Let P be the W-point of $\widehat{T}$ that corresponds to $t_1 = \cdots = t_{19} = 0$. By Theorem 2.10, we know that L lifts to $\mathcal{M}_H(\widehat{X}, v)_P$. We prove by induction on n that L lifts to the n-th infinitesimal neighborhood of P in $\widehat{T}$. Note that such liftings are unique since $H^1(\mathcal{M}_H(v), \mathcal{O}_{\mathcal{M}_H(v)}) = 0$. Furthermore, Proposition 2.6 also shows that the formation of $R^2\pi_*\mathcal{O}_{\mathcal{M}_H(\widehat{X},v)}$ is compatible with base change.

We just showed that the result is true for $n = 0$. Assume that L lifts to the n-th infinitesimal neighborhood P_n of P in $\widehat{T}$. The obstruction to lifting L to the $n+1$st infinitesimal neighborhood of P belongs to

$$H^2\left(\mathcal{M}_H(\widehat{X}, v)_{P_n}, \mathcal{O}_{\mathcal{M}_H(\widehat{X},v)_P}\right).$$

By Proposition 2.6 again, this group is a free W-module of rank 1. However, Theorem 2.4(vi) shows that some power of L actually lifts to $\widehat{T}$, so this obstruction is torsion. This shows that the obstruction vanishes and concludes the proof. $\square$

We now investigate the Kuga-Satake construction in the setup above. If ℓ is a prime number different from p, we write $R^2\pi_*\mathbb{Z}_{\ell,\,\mathrm{prim}}$ for the orthogonal of $c_1(L)$ in $R^2\pi_*\mathbb{Z}_\ell$. Let $n \geq 3$ be an integer prime to p. Up to replacing k by a finite extension whose degree only depends on n and the pair (Λ, l), we can assume that the family $\mathcal{M} \to T$ is endowed with a spin structure of level n with respect to $R^2\pi_*\mathbb{Z}_{\ell,\,\mathrm{prim}}$. We refer to [Cha13, 3.2] and to [And96], [Riz10], [Mau14] for definitions and details.

Let $\mathcal{S}_{n,l,h}$, $\mathcal{S}_{n,h}$ and $\mathcal{S}_{n,l}$ be the orthogonal Shimura varieties with spin level n associated to $\Lambda_{l,h}$, Λ_h and Λ_l respectively; see [Cha13, 3.6]. Then these three varieties are all defined over $\mathbb{Q}$ and we have closed embeddings of $\mathcal{S}_{n,l,h}$ into $\mathcal{S}_{n,h}$ and $\mathcal{S}_{n,l}$. These are both defined over $\mathbb{Q}$.

The period map $\mathcal{P}$, as defined for instance in the previous section, gives a morphism

$$\mathcal{P} : T_{\mathbb{C}} \to \mathcal{S}_{n,l,h}.$$

By the argument of [Cha13, Prop. 16], which is essentially contained in [And96, App. 1], the composition

$$T_{\mathbb{C}} \xrightarrow{\mathcal{P}} \mathcal{S}_{n,l,h} \longrightarrow \mathcal{S}_{n,h}$$

is defined over K. Since the second map is a closed immersion defined over $\mathbb{Q}$, this shows that

$$\mathcal{P} : T_{\mathbb{C}} \to \mathcal{S}_{n,l,h}$$

is defined over K. Let

$$\mathcal{P}_l : T_{\mathbb{C}} \to \mathcal{S}_{n,l}$$

be the composition with $\mathcal{S}_{n,l,h} \to \mathcal{S}_{n,l}$.

The Kuga-Satake construction induces a morphism

$$KS : \mathcal{S}_{n,l} \to \mathcal{A}_{g,d',n,\mathbb{Q}}, \tag{3.1}$$

where $\mathcal{A}_{g,d',n,\mathbb{Q}}$ is the moduli space over $\mathbb{Q}$ of abelian varieties of dimension g with a polarization of degree d'^2 and level n structure, for some integers g and d', where d' is prime to p; see [And96]. Let

$$\kappa_K : T_K \to \mathcal{A}_{g,d',n,K}$$

be the composition of KS with $\mathcal{P}$.

Let $\psi : \mathcal{A}_K \to T_K$ be the abelian scheme induced by κ_K, and let $C = C(\Lambda_l)$ be the Clifford algebra associated to Λ_l. Then there is a canonical injection of C into the ring of endomorphisms of the scheme $\mathcal{A}_K$ and, as shown in [Del72, 6.5], we have an isomorphism of ℓ-adic sheaves of algebras on T_K

$$C(R^2\pi_*\mathbb{Z}_\ell(1)_{\text{prim}}) \simeq \mathrm{End}_C(R^1\psi_*\mathbb{Z}_\ell); \tag{3.2}$$

here we are using the Kuga-Satake construction with respect to the full Clifford algebra.

By [Riz10, 6.1.2], the data above is sufficient to show that the Kuga-Satake morphism κ_K can be extended in a unique way to a morphism over W

$$\kappa : T \to \mathcal{A}_{g,d',n}$$

where $\mathcal{A}_{g,d',n}$ denotes the moduli space over W.

Definition 3.11. The morphism

$$\kappa : T \to \mathcal{A}_{g,d',n}$$

is the Kuga-Satake mapping.

We now recall some properties of the Kuga-Satake construction. Assume that k is algebraically closed. Recall that 0 is a k-point of T, and let $\widehat{T}$ be the formal neighborhood of 0 in T. As in [Mau14, §6], and by the argument of [Cha13, Prop. 13], we have the following canonical primitive strict embedding of filtered Frobenius crystals:

$$R^2\pi_*\Omega^\bullet_{\widehat{\mathcal{M}}/\widehat{T}}(1)_{\text{prim}} \hookrightarrow \mathrm{End}_C(R^1\psi_*\Omega^\bullet_{\widehat{\mathcal{A}}/\widehat{T}}). \tag{3.3}$$

It is compatible with (3.2) and the Beauville-Bogomolov form via the comparison theorems. In particular, we get a primitive isometry

$$H^2_{\text{cris}}(\mathcal{M}_0/W) \hookrightarrow \mathrm{End}(H^1_{\text{cris}}(\mathcal{A}_0/W)). \tag{3.4}$$

LEMMA 3.12. *Let Λ_l be a lattice, and let A_0 be an abelian variety of dimension g over k, together with a level n structure and a polarization of degree d'. Then there are only finitely many subspaces $V \subset \mathrm{End}(H^1_{\text{cris}}(\mathcal{A}_0/W))$ that arise as the image of some $H^2_{\text{cris}}(\mathcal{M}_0/W)$ for some admissible triple $(\mathcal{M}_0, H_0, L_0)$ with primitive lattice Λ_l.*

Proof. By the main construction and result of [Kis10], and since l^2 is not divisible by p, the Shimura variety $\mathcal{S}_{n,l}$ admits a smooth canonical integral model $\overline{\mathcal{S}_{n,l}}$ over W, and the Kuga-Satake morphism $KS : \mathcal{S}_{n,l} \to \mathcal{A}_{g,d',n,\mathbb{Q}}$ extends to a finite, unramified morphism

$$\overline{KS} : \overline{\mathcal{S}_{n,l}} \to \mathcal{A}_{g,d',n}. \tag{3.5}$$

Let P be the canonical locally $\mathcal{O}_{\mathcal{S}_{n,l}}$-module endowed with a connection ∇ and a Hodge filtration of weight 0 on $\mathcal{S}_{n,l}$ over K. Denote again by $\psi : \mathcal{A} \to \overline{\mathcal{S}_{n,l}}$

the abelian scheme induced by $\overline{KS}$. Then by definition of the Kuga-Satake construction, we have a morphism over $\mathcal{S}_{n,l}$,

$$P \hookrightarrow \mathrm{End}(R^1\psi_*\Omega^\bullet_{\mathcal{A}/\mathcal{S}_{n,l}}), \tag{3.6}$$

analogous to (3.3). It is compatible with the Hodge filtrations and the connexions on both sides.

Now let $\mathcal{A}_0$ be as in the lemma. It suffices to show that there are a finite number of subspaces $V' \subset \mathrm{End}(H^1_{\mathrm{cris}}(\mathcal{A}_0/W)\otimes K)$ that arise as stated. Such a V' is obtained by first picking a preimage of the point of $\mathcal{A}_{g,d',n}$ corresponding to $\mathcal{A}_0$, then lifting this preimage to a W-point of $\overline{\mathcal{S}_{n,l}}$. If A_K is the corresponding abelian variety over K, then the relation (3.6) induces a subspace of $\mathrm{End}(H^1_{dR}(\mathcal{A}_K/K))$. Via the comparison theorem between the de Rham cohomology of $\mathcal{A}_K$ and the crystalline cohomology of $\mathcal{A}_0$, this induces the subspace V' of $\mathrm{End}(H^1_{\mathrm{cris}}(\mathcal{A}_0/W)\otimes K)$.

Since (3.6) is compatible with the Gauss-Manin connection, it is readily seen that the construction above only depends on the choice of a preimage of the point corresponding to $\mathcal{A}_0$ in $\overline{\mathcal{S}_{n,l}}$ under $\overline{KS}$. Since $\overline{KS}$ is finite, this shows the result. □

Remark 3.13. The proof of the lemma above is the only appearance in the text of canonical integral models of Shimura varieties. While they make the proof more natural, we do not really make full use of their properties, and they could be replaced by any model that allows us to extend the Kuga-Satake morphism over W.

Remark 3.14. The idea of the proof could be extended to show that there are only finitely many birational equivalence classes of varieties $\mathcal{M}_0$ as above that have a given Kuga-Satake variety. We will prove a weaker result instead.

The following lemma appears in the proof of [Cha13, Prop. 22]. For $K3$ surfaces, it is stated and proved in [MP15, Prop. 4.17 (4)]; see also [Ben15, Prop. 2.3].

LEMMA 3.15. *Assume that k is algebraically closed. Let $(\mathcal{M}_0, H_0, L_0)$ be a strongly admissible triple over k, and let A_0 be the abelian variety associated to $\mathcal{M}_0$ under the Kuga-Satake mapping. Let $i : H^2_{\mathrm{cris}}(\mathcal{M}_0/W) \hookrightarrow \mathrm{End}(H^1_{\mathrm{cris}}(\mathcal{A}_0/W))$ be the morphism* (3.4).

Then we have an isomorphism of lattices

$$\{\alpha \in \mathrm{NS}(\mathcal{M}_0) | q(\alpha, c_1(L_0)) = 0\} \simeq \mathrm{End}(A_0) \cap \mathrm{Im}(i).$$

Proof. The arguments of the proof of [Cha13, Prop. 22] — which are essentially the same as those of [MP15, Prop. 4.17 (4)] — apply without any change. For the sake of completeness, we briefly recall the argument. We work in the setup above.

Let α be a class in the Néron-Severi group of $\mathcal{M}_0$ that is orthogonal to $c_1(L_0)$. Since $h^{0,2}(\mathcal{M}_0) = 1$, the assumption on the geometric generic fiber of T ensures that α lifts to characteristic zero. In particular, it lifts to a Hodge class. Via the Kuga-Satake correspondence and the Hodge conjecture for endomorphisms of abelian varieties, it induces an endomorphism of a lift of A_0, hence of A_0 itself. This endomorphism belongs to the image of i by construction.

Now let β be a class in $\mathrm{End}(A_0) \cap \mathrm{Im}(i)$. The argument of [Cha13, end of Prop. 22], which is a rephrasing of [Ogu79, Th. 2.9], shows that there exists a lift of $\mathcal{M}_0$ parametrized by T such that β lifts to an endomorphism of the induced Kuga-Satake abelian variety. In particular, it lifts to a Hodge class, which by the Hodge conjecture for divisors allows us to conclude as before that β was induced by a line bundle on $\mathcal{M}_0$, orthogonal to $c_1(L)$. □

The two results above directly imply the following finiteness result, which is a weak version of Theorem 3.3.

Proposition 3.16. *Let k be a finite field of characteristic at least* 5. *Let r and n be two positive integers. Then there exist finitely many lattices $\Lambda_1, \ldots, \Lambda_r$ such that if $(\mathcal{M}_0, H_0, L_0)$ is a strongly admissible triple over k with $\dim(X) = 2n$ and $c_1(L_0)^{2n} = r$, then*

$$\mathrm{NS}(\mathcal{M}_{0,\overline{k}}) \simeq \Lambda_i$$

for some integer i, where the left-hand side is endowed with the Beauville-Bogomolov form.

Proof. By assumption $(\mathcal{M}_0, H_0, L_0)$ lifts to $\mathbb{C}$. We can apply Lemma 3.5 to show that there exist finitely many lattices that can appear as a primitive lattice for $(\mathcal{M}_0, L_0)$ and that $q(L_0)$ is uniformly bounded. As a consequence, we can restrict our attention to those strongly admissible triple with primitive lattice Λ_l for some fixed Λ_l.

Fix some integer $n \leq 3$. After replacing k by a finite extension whose degree only depends on n and Λ_l so that spin structures are defined on suitable deformations of $(\mathcal{M}_0, H_0, L_0)$ as before, we can construct the Kuga-Satake abelian variety A_0 together with the canonical morphism

$$i : H^2_{\text{cris}}(\mathcal{M}_0/W) \hookrightarrow \mathrm{End}(H^1_{\text{cris}}(\mathcal{A}_0/W)).$$

If d' is as in Definition 3.11, then A_0 is a polarized abelian variety over k of degree d', together with a level n structure. Since k is finite, there are only finitely many such A_0. Furthermore, given A_0, Lemma 3.12 shows that there are only finitely many subspaces $V \subset \mathrm{End}(H^1_{\text{cris}}(A_0/W))$ that arise as the image of a morphism i as above.

By Lemma 3.15, this discussion shows that the lattice

$$c_1(L_0)^{\perp} := \{\alpha \in \mathrm{NS}(\mathcal{M}_{0,\overline{k}}) | q(\alpha, c_1(L_0)) = 0\}$$

can take only finitely many values as the strongly admissible triple varies. The inequality

$$|\mathrm{disc}(\mathrm{NS}(\mathcal{M}_{0,\overline{k}}))| \leq q(L_0)|\mathrm{disc}(c_1(L_0)^{\perp})|$$

shows that the discriminant of $\mathrm{NS}(\mathcal{M}_{0,\overline{k}})$ is bounded. Since the set of isomorphism classes of lattices with bounded rank and discriminant is finite by [Cas78, Ch. 9, Th. 1.1], this shows the result. □

3.4. *Finiteness results for $K3$ surfaces over finite fields.* The following weak finiteness result for $K3$ surfaces over finite fields will be the key to the proof of the Tate conjecture for $K3$ surfaces.

PROPOSITION 3.17. *Let k be a finite field of characteristic at least 5. Let $\overline{k}$ be an algebraic closure of k, and let W be the ring of Witt vectors of $\overline{k}$.*

Let d and t_0 be positive integers. Then there exist a positive integer N and nonzero integers a, b such that there exist only finitely many polarized nonsuperspecial $K3$ surfaces (X, H) of degree $2md$ over k, where m is a positive integer satisfying

(i) $m = 1[N]$;
(ii) *m is prime to a and b, and both a and b are quadratic residues modulo m;*
(iii) *the p-adic valuation of the discriminant of $\mathrm{NS}(X_{\overline{k}})$ is at most t_0.*

Proof. We fix integers r, n, N, a, b as in Theorem 2.10. As a consequence, if (X, H) is a polarized $K3$ surface over k as above, we can find a Mukai vector v on X satisfying condition (C) of Definition 2.3 such that the moduli space $\mathcal{M}_H(v)$ has dimension 4 and there exists a line bundle L on $\mathcal{M}_H(v)$ satisfying $c_1(L)^4 = r$ and $q(L) > 0$. Proposition 3.10, there exists an ample line bundle A on $\mathcal{M}_H(v)$ such that $(\mathcal{M}_H(v), A, L)$ is strongly admissible.

As a consequence of Proposition 3.16, we can find finitely many lattices $\Lambda_1, \ldots, \Lambda_s$, depending only on d, N, a, b, such that if X, H and v are as above, then

$$\mathrm{NS}(\mathcal{M}_H(v)_{\overline{k}}) \simeq \Lambda_i.$$

Let p be the characteristic of k. By Corollary 2.8, we can write

$$|\mathrm{disc}(\mathrm{NS}(X_{\overline{k}}))| = p^t \lambda |\mathrm{disc}(\Lambda_i)|$$

for some $\lambda \leq v^2 = n - 2$ and some nonnegative integer t. Since the p-adic valuation of $\mathrm{disc}(\mathrm{NS}(X_{\overline{k}}))$ is bounded by assumption, this shows that the discriminant of $\mathrm{NS}(X_{\overline{k}})$ is bounded independently of X. Since the set of lattices with bounded rank and discriminant is finite by [Cas78, Ch. 9, Th. 1.1], Corollary 3.2 shows that the set of isomorphism classes of $K3$ surfaces X as in the statement is finite. □

4. The Tate conjecture for $K3$ surfaces over finite fields

Let us first recall the Tate conjecture for varieties over finite fields.

Conjecture 4.1 (Tate conjecture [Tat95]). *Let X be a smooth projective variety over a finite field k with algebraic closure $\overline{k}$. Let G be the absolute Galois group of k, and let ℓ be a prime number invertible in k. Then the cycle class map*

$$\mathrm{NS}(X) \otimes \mathbb{Z}_\ell \to H^2(X_{\overline{k}}, \mathbb{Z}_\ell(1))^G$$

is an isomorphism.

As shown in [Tat95, Th. 5.2] and [Mil75, Th. 6.1] (see the addendum at http://www.jmilne.org/math/articles/add/1975a.pdf for the case of characteristic 2) the validity of this conjecture is independent of ℓ, and is equivalent to the finiteness of the Brauer group of X.

This section is devoted to the application of moduli spaces of twisted sheaves to the Tate conjecture for $K3$ surfaces. This circle of ideas originates in [LMS14], though it is in the line of [ASD73]. As explained [LMS14], the failure of the Tate conjecture for a given $K3$ surface X over a finite field — equivalently, the failure for the ℓ-primary part of the Brauer group of X to be finite — can be rephrased as the existence of an infinite family of so-called twisted Fourier-Mukai partners of X over k. Finiteness results as we proved above allow us to control such families, thus proving Theorems 1.3 and 1.4.

We will need a variation on the techniques of [LMS14] to adapt their results to a slightly more flexible setting and make it work in arbitrary characteristic. After recalling basic facts on moduli spaces of twisted sheaves, we give a very short proof of Theorem 1.4 and prove Theorem 1.3.

4.1. *Moduli spaces on twisted sheaves on $K3$ surfaces.* We briefly recall the theory of moduli spaces of twisted sheaves on a $K3$ surface. We refer to the discussion in [LMS14, 3.1–3.4] for details.

Let X be a $K3$ surface over a field k. Let ℓ be a prime number invertible in k. An *ℓ-adic B-field* on X is an element

$$B = \alpha/\ell^n \in H^2(X, \mathbb{Q}_\ell(1)),$$

where $\alpha \in H^2(X, \mathbb{Z}_\ell(1))$ is primitive. The Brauer class associated to α is the image of α under the composition

$$H^2(X, \mathbb{Z}_\ell(1)) \to H^2(X, \mu_{\ell^n}) \to \mathrm{Br}(X)[\ell^n].$$

It is denoted by $[\alpha_n]$.

Let $B = \alpha/\ell^n$ be an ℓ-adic B-field on X, and write $r = \ell^n$. Following [Yos06, (3.4)], we define

$$T_{-\alpha/r} : \widetilde{H}(X_{\overline{k}}, \mathbb{Z}_\ell) \to \widetilde{H}(X_{\overline{k}}, \mathbb{Q}_\ell), x \mapsto x \cup e^{-\alpha/r}.$$

Let $N^{\alpha/r}(X)$ be the preimage of $N(X)$ by $T_{-\alpha/r}$. This is the group denoted by $CH^{\alpha/r}(X,\mathbb{Z})$ in [LMS14]. By [LMS14, Lemma 3.3.3], we have

$$N^{\alpha/r}(X) = \{(ar, D + a\alpha, c\omega)|(a, c \in \mathbb{Z}, D \in \mathrm{NS}(X)\} \subset \widetilde{H}((X_{\overline{k}}, \mathbb{Z}_\ell). \tag{4.1}$$

Elements of $N^{\alpha/r}(X)$ are called *twisted Mukai vectors* on X.

Let $\mathcal{X} \to X$ be a μ_r-gerbe representing the class $[\alpha_n]$. Given an $\mathcal{X}$-twisted sheaf on X, we can define its Mukai vector as an element of $N^{\alpha/r}(X)$. Let v be a primitive element in $N^{\alpha/r}(X)$. Assume that $\mathrm{rk}(v) = r$ and $v^2 = 0$. Then by [LMS14, Prop. 3.4.1], the stack of simple $\mathcal{X}$-twisted sheaves on X with Mukai vector v is a μ_r-gerbe over a $K3$ surface $\mathcal{M}(v)$, denoted by $M_{\mathcal{X}}(v)$ in [LMS14].

The discriminant of $N^{\alpha/r}(X_{\overline{k}})$ is easy to compute.

Lemma 4.2. *With the notations above, we have*

$$|\mathrm{disc}(N^{\alpha/r}(X_{\overline{k}}))| = r^2|\mathrm{disc}(\mathrm{NS}(X_{\overline{k}}))|.$$

Proof. Let $D_1, \dots, D_s$ be a basis of the free $\mathbb{Z}$-module $\mathrm{NS}(X_{\overline{k}})$. Then by (4.1), a basis of $N^{\alpha/r}(X_{\overline{k}}))$ is given by

$$(r, \alpha, 0), (0, 0, 1), (0, D_1, 0), \dots, (0, D_s, 0).$$

The result follows immediately. $\square$

We now relate the Néron-Severi group of a 2-dimensional moduli space of twisted sheaves on a $K3$ surface with that of the Néron-Severi group of the $K3$ surface. The following discussion parallels Theorem 2.4. We do not repeat the arguments allowing us to deduce results over arbitrary fields from the results over the field of complex numbers.

Let ℓ be a prime number that is invertible in k. Theorem 3.19(ii) in [Yos06] shows that there exists a canonical bijective isometry

$$v^{\perp,\ell}/\mathbb{Z}_\ell v \to H^2(\mathcal{M}(v)_{\overline{k}}, \mathbb{Z}_\ell(1))$$

induced by an algebraic correspondence, where $v^{\perp,\ell}$ is the orthogonal of v in the ℓ-adic Mukai lattice of X. Note that v is isotropic by assumption, so $\mathbb{Z}_\ell v \subset v^{\perp,\ell}$.

The exact same argument as in the proof of 2.4, (vi) and (vii) shows that, if k is algebraically closed or finite, there exists an injective isometry

$$\theta_v : v^\perp/\mathbb{Z}v \to \mathrm{NS}(\mathcal{M}(v)), \tag{4.2}$$

where $v^\perp$ is the orthogonal of v in the lattice of twisted Mukai vectors on $X_{\overline{k}}$, $N^{\alpha/r}(X)$. Furthermore, the cokernel of θ_v is a p-primary torsion group, where p is the characteristic of k.

Proposition 4.3. *With the notations above, let n_v be the positive integer defined by*

$$v.N^{\alpha/r}(X_{\overline{k}}) = n_v\mathbb{Z}.$$

If k has positive characteristic p, then there exists a nonnegative integer t such that

$$(n_v)^2 p^t \mathrm{disc}(\mathrm{NS}(\mathcal{M}(v))_{\overline{k}}) = r^2 \mathrm{disc}(\mathrm{NS}(X_{\overline{k}})).$$

If k has characteristic zero, then

$$(n_v)^2 \mathrm{disc}(\mathrm{NS}(\mathcal{M}(v))_{\overline{k}}) = r^2 \mathrm{disc}(\mathrm{NS}(X_{\overline{k}})).$$

Proof. By (4.2), and as in Corollary 2.8, we can find a nonnegative integer t such that

$$|\mathrm{disc}(v^{\perp}/\mathbb{Z}v)| = p^t |\mathrm{disc}(\mathrm{NS}(\mathcal{M}(v)_{\overline{k}}))|.$$

We now relate the discriminant of $v^{\perp}/\mathbb{Z}v$ to that of $N^{\alpha/r}(X_{\overline{k}})$. Let $e_1, \ldots, e_t$ be elements of $v^{\perp}$ that form a basis of $v^{\perp}/\mathbb{Z}v$. Then $v, e_1, \ldots, e_t$ is a basis of $v^{\perp}$. Let w be an element of $N^{\alpha/r}(X_{\overline{k}})$ such that $v.w = n_v$. Then $w, v, e_1, \ldots, e_t$ is a basis of $N^{\alpha/r}(X_{\overline{k}})$. Computing the discriminant of $N^{\alpha/r}(X_{\overline{k}})$ in this basis, we get

$$|\mathrm{disc}(N^{\alpha/r}(X_{\overline{k}}))| = (n_v)^2 |\mathrm{disc}(v^{\perp}/\mathbb{Z}v)|.$$

Using Lemma 4.2, we finally get

$$r^2 |\mathrm{disc}(\mathrm{NS}(X_{\overline{k}})| = (n_v)^2 p^t |\mathrm{disc}(\mathrm{NS}(\mathcal{M}(v)_{\overline{k}}))|. \qquad \square$$

4.2. *Finiteness statements and the Tate conjecture for $K3$ surfaces.* The goal of this section is to prove Theorem 1.3. In [LMS14], the authors prove the following statement.

THEOREM 4.4. *Let k be a finite field of characteristic at least 5. Assume that there are only finitely many $K3$ surfaces defined over each finite extension of k. Then the Tate conjecture holds for all $K3$ surfaces over $\overline{k}$.*

We will not be able to use the theorem above directly, and we will rely on a simplified version of the argument of [LMS14], which holds in arbitrary characteristic.

From now on, let X be a $K3$ surface over a finite field k of characteristic p. Up to replacing k by a finite extension, we can and will assume that $\mathrm{NS}(X_{\overline{k}}) = \mathrm{NS}(X)$.

If ℓ is a prime number different from p, we denote by $T(X, \mathbb{Z}_\ell)$ the orthogonal complement of $\mathrm{NS}(X) \otimes \mathbb{Z}_\ell$ in $H^2(X, \mathbb{Z}_\ell(1))$. Note that by the Hodge index theorem for surfaces, the intersection form on $\mathrm{NS}(X)$ is nondegenerate.

The following is a slightly modified version of Lemma 3.5.1 of [LMS14] that gets rid of the hypothesis on the characteristic of k and does not make use of [ASD73].

LEMMA 4.5. *Assume that $\mathrm{Br}(X)$ is infinite. There exist prime numbers $p_1, \ldots, p_r$ such that if ℓ is big enough and p_i is a square modulo ℓ for all i, then there exists $\alpha \in T(X, \mathbb{Z}_\ell)$ such that $\alpha^2 = 1$.*

Proof. Let $\overline{k}$ be an algebraic closure of k, and let G be the absolute Galois group of k. Since k is finite, the group $H^2(G, H^0(X_{\overline{k}}, \mathbb{Z}_\ell(1)))$ vanishes. Using the Hochschild-Serre spectral sequence, this shows that the natural map

$$H^2(X, \mathbb{Z}_\ell(1)) \to H^2(X_{\overline{k}}, \mathbb{Z}_\ell(1))^G$$

is an isomorphism. By Proposition 2.1.2 of [LMS14] applied to the primitive inclusion of $H^2(X_{\overline{k}}, \mathbb{Z}_\ell(1))^G$ into the unimodular lattice $H^2(X_{\overline{k}}, \mathbb{Z}_\ell(1))$, we can assume by choosing ℓ large enough that the discriminant of $H^2(X, \mathbb{Z}_\ell(1))$ has ℓ-adic valuation zero. As a consequence, again if ℓ is big enough, the discriminant of $T(X, \mathbb{Z}_\ell)$ has ℓ-adic valuation zero. Furthermore, there are only finitely many prime numbers p_i such that the discriminant of $T(X, \mathbb{Z}_\ell)$ can have nonzero p_i-adic valuation as ℓ-varies again by [LMS14, Prop. 2.1.2].

By Hensel's lemma, we can prove the result after tensoring $T(X, \mathbb{Z}_\ell)$ with $\mathbb{F}_\ell$. Again by [Tat95] and [Mil75], the hypothesis on $\mathrm{Br}(X)$ implies that the rank of $T(X, \mathbb{Z}_\ell)$ is nonzero for any $\ell \neq \mathrm{char}(k)$.

If the rank of $T(X, \mathbb{Z}_\ell)$ is at least 2, then $T(X, \mathbb{Z}_\ell) \otimes \mathbb{F}_\ell$ represents 1 by general results. If the rank is 1, the result holds since its discriminant is a square by assumption. □

For the sake of reference, we also state the following easy lemma.

Lemma 4.6. *Let $x_1, \dots, x_r$ be finitely many integers. Then there exist infinitely many prime numbers ℓ such that all the x_i are quadratic residues modulo ℓ.*

Proof. We can assume that $x_1 = -1, x_2 = 2$ and the remaining x_i are distinct odd prime numbers. Then choosing ℓ to be congruent to 1 modulo 8 ensures that x_1 and x_2 are quadratic residues modulo ℓ. Using the quadratic reciprocity law and Dirichlet's theorem on primes in arithmetic progressions allows us to conclude. □

Now let D be a line bundle of degree $2d$ on X, and let t_0 be the p-adic valuation of $\mathrm{disc}(\mathrm{NS}(X))$. Let N, a and b be as in Proposition 3.17. Let ℓ be a big enough prime number, different from p, such that $2, a, b, -2d$ are quadratic residues modulo ℓ.

Assume that X does not satisfy the Tate conjecture, so that $\mathrm{Br}(X)$ is infinite. In addition to the above, choose a prime ℓ that satisfies the condition of Lemma 4.5. This is possible by Lemma 4.6. As in [LMS14, Prop. 3.5.4], the assumptions on ℓ and Lemma 4.5 show that we can find $\gamma \in T(X, \mathbb{Z}_\ell)$ such that $\gamma^2 = -2d$.

If n is a positive integer, let

$$v_n = (\ell^n, \gamma + D, 0) \in N^{\gamma/\ell^n}(X).$$

As in [LMS14, Prop. 3.5.4], we have $v_n^2 = 0$. As in Section 4.1, the twisted moduli space $X_n := \mathcal{M}_{H_n}(v_n)$ is a $K3$ surface over k.

By Proposition 4.3,

$$\lambda_n^2 p^t \mathrm{disc}(\mathrm{NS}((X_n)_{\overline{k}})) = \ell^{2n}\mathrm{disc}(\mathrm{NS}(X_{\overline{k}})).$$

Here λ_n is an integer such that

$$v_n.N^{\gamma/\ell^n} = \lambda_n\mathbb{Z}.$$

Since $v_n.(0, D, 0) = -2d$, we have $\lambda_n^2 \leq 4d^2$, so that the ℓ-adic valuation of $\mathrm{disc}(\mathrm{NS}((X_n)_{\overline{k}}))$ goes to infinity as n goes to infinity.

We now use the surfaces X_n to prove Theorems 1.3 and 1.4, i.e., that X satisfies the Tate conjecture if the characteristic is at least 5 or if the Picard number is at least 2. The second one does not rely on Theorem 2.10 and only uses Proposition 3.1. It should be seen as a modern rephrasing of [ASD73].

Proof of Theorem 1.4. Assume that X has Picard number at least 2. Then we can assume that d is negative and find a divisor B on X, orthogonal to D, such that $B^2 = 2e > 0$.

Let $b_n = (0, B, 0) \in N^{\gamma/\ell^n}(X)$. Then $b_n.v_n = 0$ and $b_n^2 = 2e$. By equation (4.1), this shows that there exists a divisor B_n on X with $B_n^2 = 2e > 0$. Corollary 3.2 implies that the surfaces $X_{n,\overline{k}}$ fall into finitely many isomorphism classes, which is in contradiction with the fact that the ℓ-adic valuation of $\mathrm{disc}(\mathrm{NS}((X_n)_{\overline{k}}))$ goes to infinity as n goes to infinity. □

Proof of Theorem 1.3. Assume that the characteristic of k is at least 5. We can assume that X is not superspecial. Indeed, these admit supersingular deformations that are not superspecial by [Ogu79, Rem. 2.7], so the Tate conjecture for these follow from the result for nonsuperspecial surfaces and [Art74, Th. 1.1].

We assume that D is ample. By equation (4.1), we have

$$h_n := (\ell^{2n}, \ell^n\gamma, -2d) \in N^{\gamma/\ell^n}(X_{\overline{k}}).$$

Furthermore, we have

$$h_n.v_n = \ell^n\gamma^2 + 2d\ell^n = 0$$

and

$$h_n^2 = \ell^{2n}\gamma^2 + 4d\ell^{2n} = 2d\ell^{2n}.$$

By equation (4.2), this shows that there exists a line bundle H_n on X_n with $c_1(B)^2 = 2d\ell^{2n}$. It is easy to show that H_n is ample. Indeed, if $\mathcal{X} \to S$ is a deformation of (X, H) over a discrete valuation ring with generic fiber of Picard rank 1, then $\mathcal{M}(v)$ lifts to a projective scheme over S with generic fiber of Picard rank 1, generated by a lift of H_n.

Using again the equality

$$\lambda_n^2 p^t \mathrm{disc}(\mathrm{NS}((X_n)_{\overline{k}})) = r^2\mathrm{disc}(\mathrm{NS}(X_{\overline{k}})),$$

with $\lambda_n^2 \leq 4d^2$, we get that the p-adic valuation of $\mathrm{disc}(\mathrm{NS}((X_n)_{\overline{k}}))$ is bounded independently of n.

We now restrict to the positive integers n such that $\ell^{2n} = 1[N]$. By Proposition 3.17, there exist only finitely many $K3$ surfaces over k satisfying the condition above, which is in contradiction with the fact that the ℓ-adic valuation of $\mathrm{disc}(\mathrm{NS}((X_n)_{\overline{k}}))$ goes to infinity as n goes to infinity and proves that X satisfies the Tate conjecture. □

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Laboratoire de Mathématiques d'Orsay, Université Paris-Sud, Orsay, France
E-mail: francois.charles@math.u-psud.fr