

# On the number of generators of ideals in polynomial rings

By JEAN FASEL

## Abstract

For an ideal  $I$  in a noetherian ring  $R$ , let  $\mu(I)$  be the minimal number of generators of  $I$ . It is well known that there is a sequence of inequalities  $\mu(I/I^2) \leq \mu(I) \leq \mu(I/I^2) + 1$  that are strict in general. However, Murthy conjectured in 1975 that  $\mu(I/I^2) = \mu(I)$  for ideals in polynomial rings whose height equals  $\mu(I/I^2)$ . The purpose of this article is to prove a stronger form of the conjecture in case the base field is infinite of characteristic different from 2: Namely, the equality  $\mu(I/I^2) = \mu(I)$  holds for any ideal  $I$ , irrespective of its height.

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## Introduction

Let  $R$  be a ring,  $I \subset R$  a finitely generated ideal, and let  $\mu(I)$  be the minimal number of generators of  $I$  (as a  $R$ -module). In general, computing  $\mu(I)$  is an extremely difficult problem, even in the case of a polynomial ring  $k[T_1, \dots, T_d]$  over a field  $k$ . Indeed, Macaulay showed that given any integer  $r \geq 4$ , there exists a prime ideal  $\mathfrak{p} \subset \mathbb{C}[T_1, T_2, T_3]$  of height 2 such that  $\mu(\mathfrak{p}) \geq r$  [Abh73, Theorem]. The situation is much better when one assumes that the variety defined by  $\mathfrak{p} \subset k[T_1, \dots, T_d]$  is regular. In that case, Forster proved in [For64, §3, Satz 5] that any such ideal is generated by at most  $d+1$  elements, a result later improved to  $\mu(\mathfrak{p}) \leq d$  by Sathaye [Sat78, Introduction, Corollary] in case the base field is infinite and Mohan Kumar [MK78, Th. 4] in full generality.

Now let  $R$  be a noetherian ring, and let  $I \subset R$  be an ideal. It is easy to see that  $\mu(I) \geq \mu(I/I^2)$ , and an easy application of Nakayama's lemma shows

that there is an inequality  $\mu(I) \leq \mu(I/I^2) + 1$ . The example of a real maximal ideal in the real algebraic circle shows that this inequality is, in general, strict. In case  $R = k[T_1, \dots, T_d]$  is a polynomial ring over a field  $k$ , it was however conjectured in 1975 by M. P. Murthy in [Mur75, Question (b)] that the above inequality is actually an equality under some additional hypotheses.

**CONJECTURE 1 (Murthy).** *Let  $k$  be a field, and let  $R = k[T_1, \dots, T_d]$  for some  $d \in \mathbb{N}$ . Let  $n \in \mathbb{N}$ , and let  $I \subset R$  be any ideal of height  $n$  such that  $\mu(I/I^2) = n$ . Then  $\mu(I) = \mu(I/I^2) = n$ .*

This conjecture was resolved in the affirmative in 1978 by N. Mohan Kumar in case  $n := \mu(I/I^2) \geq \dim(R/I) + 2$  in [MK78, Th. 5], but the general case has remained open since then. The purpose of this article is to prove a stronger form of the conjecture in case  $k$  is infinite having characteristic different from 2.

**THEOREM 2.** *If  $k$  is an infinite field with  $\text{char}(k) \neq 2$  and  $R = k[T_1, \dots, T_d]$  for some  $d \in \mathbb{N}$ , then  $\mu(I) = \mu(I/I^2)$  holds for any ideal  $I \subset R$ .*

Observe that there are no assumptions on the height of  $I$  and that  $I/I^2$  can be generated by more elements than the height of  $I$ . Moreover, the result can easily be generalized to polynomial rings over regular local rings essentially of finite type over an infinite field of characteristic different from 2.

The method we use to establish our main theorem appears to be new. Indeed, we use naive homotopy theory and unstable  $K$ -theory of orthogonal groups, as we now explain. If  $I \subset R$  is an ideal and  $\omega_I : (R/I)^n \rightarrow I/I^2$  is a surjective homomorphism, then it is easy to see that there exist  $(a_1, \dots, a_n, s) \in I$  and  $b_1, \dots, b_n \in R$  such that  $I = \langle a_1, \dots, a_n, s \rangle$  and  $s(1-s) = \sum_{i=1}^n a_i b_i$  (see, e.g., [MK77, Lemma] or Lemma 2.0.1 below). Such an element  $(s, a_1, \dots, a_n, b_1, \dots, b_n)$  corresponds to a morphism of schemes  $\text{Spec } R \rightarrow Q_{2n}$ , where  $Q_{2n}$  is the smooth hypersurface in  $\mathbb{A}_{\mathbb{Z}}^{2n+1}$  given by the equation  $\sum x_i y_i = z(1-z)$ . However, this morphism depends on many choices and is not uniquely determined by the pair  $(I, \omega_I)$ . The situation improves if one considers morphisms up to naive homotopies.

Recall that two morphisms  $f_0, f_1 : \text{Spec } R \rightarrow Q_{2n}$  are said to be naively homotopic if there exists  $F : \text{Spec } R[T] \rightarrow Q_{2n}$  whose restrictions at  $T = 0$  and  $T = 1$  are, respectively,  $f_0$  and  $f_1$ . Considering the equivalence relation on  $\text{Hom}(\text{Spec } R, Q_{2n})$  generated by naive homotopies, we obtain a set  $\text{Hom}_{\mathbb{A}^1}(\text{Spec } R, Q_{2n})$ . We prove that the assignment sending a pair  $(I, \omega_I)$  to the class of  $(s, a_1, \dots, a_n, b_1, \dots, b_n)$  in the set  $\text{Hom}_{\mathbb{A}^1}(\text{Spec } R, Q_{2n})$  is well defined. Following Murthy ([Mur94, §5]), we write  $s(I, \omega_I)$  for this class and call it the *universal Segre class* of the pair  $(I, \omega_I)$ . This terminology can be justified as follows.

Let us consider contravariant pointed-set valued functors  $F$  associating with each smooth affine scheme  $X = \text{Spec } R$  a pointed set  $F(X)$  and with each pair  $(I, \omega_I)$  as above an element  $s_F(I, \omega_I) \in F(X)$ . Suppose moreover that  $F$  is homotopy invariant, i.e., that the map  $F(X) \rightarrow F(X \times \mathbb{A}^1)$  induced by the projection is a bijection, and that  $s_F(I, \omega_I) = \star$  (where  $\star$  is the base point of  $F(X)$ ) if there is a commutative diagram

$$\begin{array}{ccc} R^n & \xrightarrow{\Omega} & I \\ \downarrow & & \downarrow \\ (R/I)^n & \xrightarrow{\omega_I} & I/I^2, \end{array}$$

with  $\Omega$  surjective. As an example, one can consider the functor  $X \mapsto CH^n(X)$  and the Segre class considered by Murthy in [Mur94, §5] following Fulton’s construction ([Ful98, Ch. 4]). It is easy to check that the functor  $X \mapsto \text{Hom}_{\mathbb{A}^1}(X, Q_{2n})$  and the universal Segre class defined above is initial among pairs  $(F, s_F)$ ; i.e., any such pair  $(F, s_F)$  factors through the one we define.

In view of this property, one could state the following principle: the universal Segre class should be the precise obstruction to lifting  $\omega_I$  to a surjective homomorphism  $\Omega : R^n \rightarrow I$ . We are not able to establish this principle in complete generality, but we prove a strong enough form to resolve Murthy’s conjecture. More precisely, let  $R$  be a smooth  $k$ -algebra over an infinite field  $k$  having characteristic different from 2. It is easy to prove that  $Q_{2n}(R)$  is isomorphic to the set of elements  $v := (s, a_1, \dots, a_n, b_1, \dots, b_n)$  in  $R^{2n+1}$  such that  $q_{2n+1}(v) = 1$ , where  $q_{2n+1}$  is the quadratic form  $z^2 + \sum x_i y_i$ . If  $O_{2n+1}(R)$  denotes the orthogonal group of  $q_{2n+1}$  and  $EO_{2n+1}(R)$  its elementary subgroup (see Section 1), then by transport of structure, both groups act on  $Q_{2n}(R)$ . We can then consider the set of orbits  $Q_{2n}(R)/EO_{2n+1}(R)$  under this action and we prove that there is a natural bijection  $Q_{2n}(R)/EO_{2n+1}(R) \simeq \text{Hom}_{\mathbb{A}^1}(\text{Spec } R, Q_{2n})$  provided  $n \geq 2$ . The advantage of the left-hand term over the right-hand term is that  $EO_{2n+1}(R)$  is generated by elementary transformations that are easier to understand than abstract homotopies.

Given  $v = (s, a_1, \dots, a_n, b_1, \dots, b_n) \in Q_{2n}(R)$ , we say that  $v$  satisfies the *strong lifting property* provided there exists a sequence  $\mu_1, \dots, \mu_n \in R$  such that the ideal  $I = \langle s, a_1, \dots, a_n \rangle$  is actually generated by  $n$  explicit elements, namely,  $a_1 + \mu_1 s^2, \dots, a_n + \mu_n s^2$  for some  $\mu_1, \dots, \mu_n \in R$ . We prove next that the strong lifting property is preserved under the action of  $EO_{2n+1}(R)$  and therefore that  $v$  satisfies the strong lifting property if and only if it is in the orbit of  $v_0 := (0, \dots, 0) \in Q_{2n}(R)$ . In the final section, we put everything together by observing that the set  $Q_{2n}(k[T_1, \dots, T_m])/EO_{2n+1}(k[T_1, \dots, T_m])$  is reduced to a point.

Let us now spend a few lines on the ideas underlying the results of this paper. Let  $\mathcal{H}(k)$  be the  $\mathbb{A}^1$ -homotopy of schemes as developed by Morel and Voevodsky ([MV99]). One of the main ideas of this category is to give a refined notion of homotopy that allows us to import the full strength of homotopy theory into algebraic geometry. In particular, there exists for any smooth  $k$ -schemes  $X$  and  $Y$  a map  $\mathrm{Hom}_{\mathbb{A}^1}(X, Y) \rightarrow \mathrm{Hom}_{\mathcal{H}(k)}(X, Y)$ , which is in general neither injective nor surjective. However, it follows from [AHW15, Th. 4.2.2] that this map is a bijection in case  $X$  is affine and  $Y = Q_{2n}$ . In other words, the naive notion of homotopy and the refined one coincide in the framework of this paper. Moreover, the scheme  $Q_{2n}$  is isomorphic in  $\mathcal{H}(k)$  to a motivic sphere by [ADF14, Th. 2.2.5], and thus the obstruction set we consider can be thought of as a motivic cohomotopy set.

Besides the motivic motivations just sketched, which will be developed elsewhere, one can try to extend the results in different directions. First, it seems likely that the universal Segre class can actually be defined in the orbit set  $Q_{2n}(R)/EO_{2n+1}(R)$  for any (Noetherian) ring  $R$  and that it is the precise obstruction for  $\omega_I$  to lift to a surjection. Second, it would be nice to get rid of the assumptions that  $k$  is infinite and of characteristic different from 2 in Theorem 3.2.9.

*Notation.* Let  $R$  be a ring, and let  $a = (a_1, \dots, a_m) \in R^m$ . We denote by  $\langle a \rangle \subset R$  the ideal generated by  $a_1, \dots, a_m$ .

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## 1. Quadrics and naive homotopies

Let  $k$  be a field, and let  $\mathrm{Sch}_k$  be the category of separated schemes of finite type over  $k$ . Given objects  $X, Y \in \mathrm{Sch}_k$ , recall that two morphisms  $f_0, f_1 : X \rightarrow Y$  are said to be naively homotopic if there exists a morphism  $F : X \times \mathbb{A}^1 \rightarrow Y$  such that  $F(0) = f_0$  and  $F(1) = f_1$ . We can consider the

equivalence relation on  $\text{Hom}_{\text{Sch}}(X, Y)$  generated by the naive homotopies, and we write  $\text{Hom}_{\mathbb{A}^1}(X, Y)$  for the set of classes under this relation.

*Definition 1.0.1.* Write  $\pi_0(Y)$  for the presheaf (of sets)  $X \mapsto \text{Hom}_{\mathbb{A}^1}(X, Y)$ .

The following lemma can be found in [Swa72, Lemma 4.1].

**LEMMA 1.0.2.** *For any scheme  $X$ , the projection  $X \times \mathbb{A}^1 \rightarrow X$  yields a bijection*

$$\pi_0(Y)(X) \rightarrow \pi_0(Y)(X \times \mathbb{A}^1).$$

*Proof.* As  $\pi_0(Y)$  is a presheaf, we see that the map

$$\pi_0(Y)(X) \rightarrow \pi_0(Y)(X \times \mathbb{A}^1)$$

induced by the projection is split injective. Now, the multiplication morphism  $m : \mathbb{A}^1 \times \mathbb{A}^1 \rightarrow \mathbb{A}^1$  shows that the identity on  $\mathbb{A}^1$  is naively homotopic to the composite  $i_0 \circ p$ , where  $i_0 : \text{Spec } k \rightarrow \mathbb{A}^1$  is the inclusion at  $T = 0$ . If  $F \in \text{Hom}_{\text{Sch}}(X \times \mathbb{A}^1, Y)$ , then  $G := F \circ (1_X \times m) \in \text{Hom}_{\text{Sch}}(X \times \mathbb{A}^1 \times \mathbb{A}^1, Y)$  satisfies  $G(1) = F$  and  $G(0) = F \circ i_0 \circ p$ , thus showing that  $\pi_0(Y)(X) \rightarrow \pi_0(Y)(X \times \mathbb{A}^1)$  is also surjective. □

Suppose now that  $k$  is of characteristic different from 2. Let  $q_{2n}$  be the quadratic form on  $k^{2n}$  given by the equation  $\sum_{i=1}^n x_i y_i$  and  $q_{2n+1}$  be the quadratic form on  $k^{2n+1}$  given by the equation  $\sum_{i=1}^n x_i y_i + z^2$ . Let  $O_{2n}$  and  $O_{2n+1}$  be the algebraic groups of invertible matrices preserving respectively  $q_{2n}$  and  $q_{2n+1}$ , and let  $SO_{2n}$  and  $SO_{2n+1}$  be their subgroups of matrices of determinant 1. Embedding  $k^{2n}$  into  $k^{2n+1}$  as the first  $2n$  coordinates yields embeddings  $O_{2n} \rightarrow O_{2n+1}$  and  $SO_{2n} \rightarrow SO_{2n+1}$ ; we can thus consider the quotient presheaf defined on  $k$ -algebras by  $R \mapsto SO_{2n+1}(R)/SO_{2n}(R)$ . If  $Q'_{2n} \subset \mathbb{A}^{2n+1}$  is the smooth affine  $k$ -scheme defined by the equation  $q_{2n+1} = 1$ , then we see that  $v_0 := (0, \dots, 0, 1) \in Q'_{2n}(k)$ , and thus the assignment  $M \mapsto Mv_0$  defines a map  $SO_{2n+1}(R) \rightarrow Q'_{2n}(R)$  for any  $k$ -algebra  $R$  that is constant on  $SO_{2n}(R)$ . We therefore obtain a morphism of presheaves  $p_{2n+1} : SO_{2n+1}/SO_{2n} \rightarrow Q'_{2n}$ .

**LEMMA 1.0.3.** *Let  $R$  be a local  $k$ -algebra and  $n \geq 1$ . The map  $p_{2n+1}(R) : SO_{2n+1}(R)/SO_{2n}(R) \rightarrow Q'_{2n}(R)$  is a bijection.*

*Proof.* Let  $v \in Q'_{2n}(R)$ . Then the restrictions  $q_{|Rv}$  and  $q_{|Rv_0}$  of  $q$  to  $Rv$  and  $Rv_0$  are both nondegenerate. It follows from [Kne02, Ch. I, (4.4) and (4.5)] that there exists an orthogonal transformation  $M \in O_{2n+1}(R)$  such that  $Mv = v_0$ . Multiplying if necessary by a reflection fixing both  $v$  and  $v_0$ , we may suppose that  $M \in SO_{2n+1}(R)$  and  $p_{2n+1}$  is then surjective. To conclude, it suffices to check that the stabilizer of  $v$  is isomorphic to  $SO_{2n}(R)$ , which is obvious. □

As a consequence, we see that the  $SO_{2n}$ -torsor  $p_{2n+1} : SO_{2n+1} \rightarrow Q'_{2n}$  is Zariski locally trivial. This will allow us to give another description of the presheaf of sets  $\pi_0(Q'_{2n})$ , but we first need the definition of the subgroup  $EO_{2n+1}(R) \subset SO_{2n+1}(R)$ . It is the subgroup generated by the following elementary operations changing exactly two entries and leaving the other fixed (e.g. [CF16, §6.1, p. 117]):

(1) for any  $\lambda \in R$  and  $1 \leq i \leq n$ ,

$$(s, a_1, \dots, a_n, b_1, \dots, b_n) \mapsto (s + \lambda b_i, a_1, \dots, a_{i-1}, a_i - 2\lambda s - \lambda^2 b_i, a_{i+1}, \dots, b_n);$$

(2) for any  $\lambda \in R$  and  $1 \leq i \leq n$ ,

$$(s, a_1, \dots, a_n, b_1, \dots, b_n) \mapsto (s + \lambda a_i, a_1, \dots, b_{i-1}, b_i - 2\lambda s - \lambda^2 a_i, b_{i+1}, \dots, b_n);$$

(3) for any  $\lambda \in R$  and  $1 \leq i, j \leq n$  with  $i \neq j$ ,

$$(s, a_1, \dots, a_n, b_1, \dots, b_n)$$

$$\mapsto (s, a_1, \dots, a_{i-1}, a_i + \lambda a_j, a_{i+1}, \dots, b_{j-1}, b_j - \lambda b_i, b_{j+1}, \dots, b_n);$$

(4) for any  $\lambda \in R$  and  $1 \leq i < j \leq n$  with  $i \neq j$ ,

$$(s, a_1, \dots, a_n, b_1, \dots, b_n) \mapsto (s, a_1, \dots, a_{i-1}, a_i + \lambda b_j, \dots, a_j - \lambda b_i, a_{j+1}, \dots, b_n);$$

(5) for any  $\lambda \in R$  and  $1 \leq i < j \leq n$  with  $i \neq j$ ,

$$(s, a_1, \dots, a_n, b_1, \dots, b_n) \mapsto (s, a_1, \dots, b_{i-1}, b_i + \lambda a_j, \dots, b_j - \lambda a_i, b_{j+1}, \dots, b_n).$$

Let now  $SKO_1(2n+1)$  be the presheaf defined on  $k$ -algebras  $R$  by

$$SKO_1(2n+1)(R) := SO_{2n+1}(R)/EO_{2n+1}(R).$$

The following theorem gathers the results of several authors.

**THEOREM 1.0.4.** *Suppose that  $k$  is a field with  $\text{char}(k) \neq 2$  and let  $n \geq 2$ . Then the subgroup  $EO_{2n+1}(R) \subset SO_{2n+1}(R)$  is normal and the presheaf  $SKO_1(2n+1)$  is homotopy invariant for regular  $k$ -algebras essentially of finite type. In particular, we have  $SO_{2n+1}(k[X_1, \dots, X_m]) = SO_{2n+1}(k) \cdot EO_{2n+1}(k[X_1, \dots, X_m])$  for any  $m \geq 1$ .*

*Proof.* The statement on the normality of  $EO_{2n+1}(R)$  in  $SO_{2n+1}(R)$  can be found in [Tad86, Th. 0.3]; see also [VP07, Lemma 4] for additional references. The homotopy invariance of  $SKO_1(2n+1)$  can be found in [Sta14, Th. 1.3] in case the base field is perfect. However, as explained in the introduction of [Sta14, Th. 1.3], an argument of T. Vorst shows that in case  $R$  is essentially of finite type over a nonperfect field  $k$  one can reduce to the case  $k = \mathbb{F}_p$ , which is perfect ([Vor81, proof of Th. 3.3]).  $\square$

*Remark 1.0.5.* The restriction  $n \geq 2$  is necessary for both the normality of the elementary subgroup and the homotopy invariance in view of [Coh66, discussion after Prop. 7.3].

As a subgroup of  $SO_{2n+1}(R)$ , the group  $EO_{2n+1}(R)$  acts on  $Q'_{2n}(R)$ , and it is easy to check that its generators are naively homotopic to identity. Consequently, we obtain a surjective map  $\varphi_n(R) : Q'_{2n}(R)/EO_{2n+1}(R) \rightarrow \pi_0(Q'_{2n}(R))$  for any  $k$ -algebra  $R$ .

**THEOREM 1.0.6.** *Let  $k$  be an infinite field with  $\text{char}(k) \neq 2$ , and let  $R$  be an essentially smooth  $k$ -algebra. If  $n \geq 2$ , then the map*

$$\varphi_n(R) : Q'_{2n}(R)/EO_{2n+1}(R) \rightarrow \pi_0(Q'_{2n}(R))$$

*is a bijection.*

*Proof.* We follow the proof of [Fas11, Th. 2.1]. From Lemma 1.0.3, we know that the  $SO_{2n}$ -torsor

$$SO_{2n+1} \rightarrow Q'_{2n}$$

is Zariski locally trivial. This torsor corresponds to a universal orthogonal module  $(E, q)$  over  $Q'_{2n}$  of rank  $2n$  such that  $(E, q) \perp (\mathcal{O}_{Q'_{2n}}, q_0)$  (where  $q_0(x) = x^2$ ) is isometric to the trivial quadratic module  $(\mathcal{O}_{Q'_{2n}}^{2n+1}, q_{2n+1})$ . If  $\alpha \in Q'_{2n}(R[T])$ , then the pull-back of  $(E, q)$  along the map  $\alpha : \text{Spec } R[T] \rightarrow Q'_{2n}$  is a quadratic module  $(P, q)$  on  $R[T]$  that is Zariski-locally trivial. It follows from [AHW15, Th. 3.3.6] that  $P$  is extended, i.e., that  $(P, q) \simeq (P(0), q(0))$ . Now,  $(P(0), q(0))$  is the bundle obtained by pulling back along  $\alpha(0)$ , and it follows that there is an automorphism  $(P(0), q(0)) \perp (R, q_0) \simeq (R^{2n+1}, q_h)$  that we can extend to  $R[T]$ . The same argument as in [Fas11, Th. 2.1] shows that we have an automorphism between  $\alpha(0)$  and  $\alpha$  whose image at  $T = 0$  is the identity. We conclude from the above theorem that  $\alpha = \alpha(0) \cdot M$  for some  $M \in EO_{2n+1}(R[X])$ . It follows that  $\alpha(1) = \alpha(0)M(1)$ , and the result is proved.  $\square$

*Remark 1.0.7.* If  $n = 0$ , the statement of the theorem is still valid for trivial reasons but the theorem fails for  $n = 1$  because of Cohn’s example once again. To get the correct statement in that case, one would have to consider the subgroup of matrices in  $SO_3(R)$  that are homotopic to the identity.

## 2. The universal Segre class

Let  $Q_{2n}$  be the smooth quadric in  $\mathbb{A}_{\mathbb{Z}}^{2n+1}$  defined by the equation  $\sum_{i=1}^n x_i y_i = z(1 - z)$ . If  $R$  is a ring, then by definition an element  $v \in Q_{2n}(R)$  corresponds to a sequence of elements  $(x_1, \dots, x_n, y_1, \dots, y_n, z) \in R$  satisfying the above equation. Given  $v \in Q_{2n}(R)$ , we can consider the ideal  $I(v) := \langle x_1, \dots, x_n, z \rangle \subset R$ . If we write  $\bar{x}_i$  for the image of  $x_i$  under the map  $I(v) \rightarrow I(v)/I(v)^2$ , then the quotient  $I(v)/I(v)^2$  is generated by  $\{\bar{x}_1, \dots, \bar{x}_n\}$ , yielding a surjective homomorphism  $\omega_v : (R/I(v))^n \rightarrow I(v)/I(v)^2$ .

Conversely, any finitely generated ideal  $I$  in a ring  $R$  endowed with a surjective homomorphism  $\omega_I : (R/I)^n \rightarrow I/I^2$  yields an element of  $Q_{2n}(R)$  as shown by the following lemma (see [MK77, Lemma]).

LEMMA 2.0.1. *Let  $R$  be a commutative ring, and let  $I \subset R$  be a finitely generated ideal. Given elements  $a_1, \dots, a_n \in I$  such that  $I/I^2 = \langle \bar{a}_1, \dots, \bar{a}_n \rangle$ , there exist an element  $s \in I$  and elements  $b_1, \dots, b_n \in R$  such that  $I = \langle a_1, \dots, a_n, s \rangle$  and  $s(1-s) = \sum a_i b_i$ .*

*Proof.* By construction,  $C := I/\langle a_1, \dots, a_n \rangle$  is a finitely generated  $R$ -module such that  $C/IC = 0$ . It follows from Nakayama's lemma that there exists  $s \in I$  such that  $(1-s)C = 0$ . For any  $c \in I$ , we find, in particular, that  $c = \sum \lambda_i a_i + cs$  and therefore  $I = \langle a_1, \dots, a_n, s \rangle$ . Setting  $c = s$  gives the existence of the  $b_i$ .  $\square$

Given a pair  $(I, \omega_I)$ , we then obtain an element

$$s(I, \omega_I) := (a_1, \dots, a_n, b_1, \dots, b_n, s)$$

in  $Q_{2n}(R)$  by choosing lifts  $a_1, \dots, a_n$  of the generators of  $I/I^2$  given by  $\omega_I$  and applying the above lemma. There are many choices of such  $(2n+1)$ -tuples, and  $s(I, \omega_I)$  is therefore not well defined, but our aim is now to show that these choices do not matter if we consider elements of  $Q_{2n}(R)$  up to naive homotopy.

THEOREM 2.0.2. *Let  $R$  be a noetherian ring,  $n \in \mathbb{N}$  be an integer,  $I \subset R$  be an ideal, and  $\omega_I : (R/I)^n \rightarrow I/I^2$  be a surjective homomorphism. Then the class of  $s(I, \omega_I)$  in  $\pi_0(Q_{2n})(R)$  is independent of any choices.*

The proof of the theorem will consist of several lemmas.

LEMMA 2.0.3. *Let  $a = (a_1, \dots, a_n)$ ,  $b = (b_1, \dots, b_n)$  and  $b' := (b'_1, \dots, b'_n)$  be elements of  $R^n$ . Let  $s \in R$  be such that  $s(1-s) = ab^t = a(b')^t$ . Then the classes of  $(a, b, s)$  and  $(a, b', s)$  are the same in  $\pi_0(Q_{2n})(R)$ .*

*Proof.* Consider the morphism  $a : R^n \rightarrow R$  given by  $c \mapsto ac^t$ . Observe that  $d = b' - b$  belongs to the kernel of this map. Setting  $B := b + Td \in R[T]^n$ , we get  $B(0) = b$  and  $B(1) = b'$ . Moreover,  $aB^t = s(1-s)$ , and therefore  $(a, B, s)$  can be seen as an element of  $Q_{2n}(R[T])$ . As  $B(0) = b$  and  $B(1) = b'$ , the claim follows.  $\square$

LEMMA 2.0.4. *Let  $I \subset R$  be an ideal,  $a_1, \dots, a_n \in I$  and  $a = (a_1, \dots, a_n) \in R^n$ . Suppose that there exist  $s, s' \in I$  such that  $(1-s)I \subset \langle a_1, \dots, a_n \rangle$  and  $(1-s')I \subset \langle a_1, \dots, a_n \rangle$ . For any choices of  $b = (b_1, \dots, b_n)$  and  $b' = (b'_1, \dots, b'_n)$  such that  $(1-s)s = ab^t$  and  $(1-s')s' = a(b')^t$ , the classes of  $(a, b, s)$  and  $(a, b', s')$  are the same in  $\pi_0(Q_{2n})(R)$ .*

*Proof.* Let  $b = (b_1, \dots, b_n)$  and  $c = (c_1, \dots, c_n) \in R^n$  be such that

$$\begin{aligned} (1-s)s &= ab^t, \\ (1-s')s &= ac^t. \end{aligned}$$

As a consequence,  $ab^t + s^2 = ac^t + s's$ . Consider next the morphism  $(a, s) : R^{n+1} \rightarrow R$  defined by  $v \mapsto (a, s)v^t$ , and observe that  $(b - c, s - s')$  belongs to the kernel of this map. Let  $S(T) = s' + T(s - s')$ . As  $s'$  and  $(s - s')$  are in  $I$ , we see that  $S(T) \in I[T]$ . By construction, we have  $S(0) = s'$  and  $S(1) = s$  and we now check that  $(1 - S(T))I[T] \subset \langle a_1, \dots, a_n \rangle$ . Since  $I = \langle a_1, \dots, a_n, s \rangle$ , it suffices to check that  $(1 - S(T))s \in \langle a_1, \dots, a_n \rangle$ . As  $(a, s)(b - c, s - s')^t = 0$ , we have  $s(s - s') \in \langle a_1, \dots, a_n \rangle$ . Now

$$(1 - S(T))s = (1 - s' - T(s - s'))s = (1 - s')s - Ts(s - s').$$

Since  $(1 - s')s \in \langle a_1, \dots, a_n \rangle$ , we see that  $(1 - S(T))I[T] \subset \langle a_1, \dots, a_n \rangle$ . It follows that there exists  $B(T) = (B_1(T), \dots, B_n(T))$  such that  $(a, B(T), S(T)) \in Q_{2n}(R[T])$ . By definition, we get that the classes of  $(a, B(0), S(0)) = (a, B(0), s')$  and  $(a, B(1), S(1)) = (a, B(1), s)$  are the same in  $\pi_0(Q_{2n})(R)$ . The result now follows from [Lemma 2.0.3](#). □

**LEMMA 2.0.5.** *Let  $I \subset R$  be an ideal, and let  $a = (a_1, \dots, a_n)$  and  $a' = (a'_1, \dots, a'_n)$  be such that*

$$a_1, \dots, a_n, a'_1, \dots, a'_n \in I \text{ and } a_i - a'_i \in I^2$$

*for any  $i = 1, \dots, n$ . Suppose moreover that  $I/I^2 = \langle \bar{a}_1, \dots, \bar{a}_n \rangle$ . For any choice of  $s, s' \in I$ ,  $b = (b_1, \dots, b_n)$  and  $b' = (b'_1, \dots, b'_n)$  such that  $ab^t = s(1 - s)$  and  $a'(b')^t = s'(1 - s')$ , the classes of  $(a, b, s)$  and  $(a', b', s')$  are equal in  $\pi_0(Q_{2n})(R)$ .*

*Proof.* For any  $i = 1, \dots, n$ , let  $c_i = a'_i - a_i \in I^2$  and  $A(T) = (a_1 + Tc_1, \dots, a_n + Tc_n)$ . As  $I/I^2 = \langle \bar{a}_1, \dots, \bar{a}_n \rangle$ , we deduce that the classes of  $a_1 + Tc_1, \dots, a_n + Tc_n$  modulo  $I[T]^2$  generate  $I[T]/(I[T])^2$ . It follows that there exist  $S(T) \in I[T]$  and  $B[T] \in R[T]^n$  such that  $A(T)B(T)^t = S(T)(1 - S(T))$ . By definition, the classes of

$$(A(0), B(0), S(0)) = (a, B(0), S(0)) \text{ and } (A(1), B(1), S(1)) = (a', B(1), S(1))$$

coincide in  $\pi_0(Q_{2n})(R)$ . The result now follows from [Lemma 2.0.4](#). □

*Proof of Theorem 2.0.2.* Then let  $(I, \omega_I)$  be the pair of the statement. The procedure described in [Lemma 2.0.1](#) to associate  $s(I, \omega_I)$  to this pair depended *a priori* on the choice of lifts  $(a_1, \dots, a_n)$  of generators of  $I/I^2$ , then on the choice of an element  $s \in I$  such that  $(1 - s)I \subset \langle a_1, \dots, a_n \rangle$  and finally on  $b_1, \dots, b_n$  such that  $s(1 - s) = \sum a_i b_i$ . The different choices are addressed in [Lemma 2.0.5](#). □

**Definition 2.0.6.** Let  $I \subset R$  be an ideal and  $\omega_I : (R/I)^n \rightarrow I/I^2$  be a surjective homomorphism. We call *universal Segre class* of  $(I, \omega_I)$  the class of  $s(I, \omega_I)$  in the pointed set  $\pi_0(Q_{2n})(R)$ .

### 3. Main theorems

Our aim in this section is to prove that the universal Segre class of an ideal vanishes in  $\pi_0(Q_{2n})(R)$  if and only if the surjection  $\omega_I$  lifts (in a strong sense) to a surjection  $R^n \rightarrow I$ . We begin with a technical lemma (Lemma 3.1.2 below), whose proof is due to Satya Mandal.

#### 3.1. Quillen patching and a lifting lemma.

LEMMA 3.1.1. *Let  $R$  be a commutative ring, and let  $B$  be a (not necessarily commutative)  $R$ -algebra. Let  $f \in R$  and  $\theta \in (1 + B_f[T])^\times$ . There exists then an integer  $k \in \mathbb{N}$  such that for any  $g_1, g_2 \in R$  with  $g_1 - g_2 \in f^k R$ , there is a unit  $\psi \in (1 + TB[T])^\times$  such that  $\psi_f(T) = \theta(g_1 T)\theta(g_2 T)^{-1}$ . Moreover, if  $g_1 - g_2 \in f^{k+r} R$  for some  $r \geq 1$  then  $\psi \in (1 + f^r TB[T])^\times$  by construction.*

*Proof.* See [Qui76, Lemma 1]. □

LEMMA 3.1.2. *Let  $R$  be a regular  $k$ -algebra, and let  $s \in R$  and  $A_1, \dots, A_n, B_1, \dots, B_n \in R[T]$  such that  $V = (s, A_1, \dots, A_n, B_1, \dots, B_n) \in Q_{2n}(R[T])$ . Set  $I(V) = \langle s, A_1, \dots, A_n \rangle$ , and assume that  $I(V)(0) = \langle a_1, \dots, a_n \rangle$  where  $a_i = A_i(0)$ . Then  $I(V) = \langle C_1, \dots, C_n \rangle$  for some  $C_1, \dots, C_n \in R[T]$  such that  $A_i - C_i \in s^2 R[T]$ .*

*Proof.* As  $s(1 - s) = \sum A_i B_i$ , we see that  $I(V)_{1-s} = \langle A_1, \dots, A_n \rangle$ , and it follows that the map

$$f_1 : (R_{1-s}[T])^n \rightarrow I(V)_{1-s}$$

defined by  $e_i \mapsto A_i$  is surjective. On the other hand, we have  $I(V)_s = R_s[T]$  and, in particular,  $a_1, \dots, a_n \in I(V)_s$ . Since  $s \in I(V)(0) = \langle a_1, \dots, a_n \rangle$ , it follows that there exist  $\lambda_1, \dots, \lambda_n \in R$  such that  $\sum \lambda_i a_i = s$ , and therefore the map

$$f_2 : (R_s[T])^n \rightarrow I(V)_s = R_s[T]$$

defined by  $e_i \mapsto a_i$  is surjective. We obtain exact sequences of  $R_{s(1-s)}[T]$ -modules

$$\begin{aligned} 0 \longrightarrow P_1 \longrightarrow (R_{s(1-s)}[T])^n \xrightarrow{f_1} R_{s(1-s)}[T] \longrightarrow 0, \\ 0 \longrightarrow P_2 \longrightarrow (R_{s(1-s)}[T])^n \xrightarrow{f_2} R_{s(1-s)}[T] \longrightarrow 0, \end{aligned}$$

with  $P_1, P_2$ -projective  $R_{s(1-s)}[T]$  modules of rank  $n - 1$ . Since the  $k$ -algebra  $R_{s(1-s)}$  is regular, it follows from [Lin82, Theorem] (in case  $R$  is essentially of finite type, or [Pop89] for a more general statement) that both  $P_1$  and  $P_2$  are extended from  $R_{s(1-s)}$ . Now  $f_1(0) = f_2(0)$ , and therefore

$$P_1 \simeq P \otimes_{R_{s(1-s)}} R_{s(1-s)}[T] \simeq P_2$$

for some projective  $R_{s(1-s)}$ -module  $P$ . We have thus obtained an endomorphism  $\theta$  of  $(R_{s(1-s)}[T])^n$  such that the diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & P \otimes_{R_{s(1-s)}} R_{s(1-s)}[T] & \longrightarrow & (R_{s(1-s)}[T])^n & \xrightarrow{f_1} & R_{s(1-s)}[T] \longrightarrow 0 \\
 & & \parallel & & \downarrow \theta & & \parallel \\
 0 & \longrightarrow & P \otimes_{R_{s(1-s)}} R_{s(1-s)}[T] & \longrightarrow & (R_{s(1-s)}[T])^n & \xrightarrow{f_2} & R_{s(1-s)}[T] \longrightarrow 0
 \end{array}$$

commutes and  $\theta(0) = \text{Id}$ . In other words, we have  $\theta \in (1 + \text{End}(R^n)_{s(1-s)}[T])^\times$ , and we can use Quillen’s localization [Lemma 3.1.1](#) for both  $s$  and  $1 - s$ . It follows that there exists  $k \in \mathbb{N}$  such that

- (1) There exists  $\psi_1 \in (1 + T \text{End}(R^n)_s[T])^\times$  with the property that whenever  $g_1, g_2 \in (1 - s)^{k+2}R_s$ , we have  $(\psi_1(T))_{1-s} = \theta(g_1T)\theta(g_2T)^{-1}$ .
- (2) There exists  $\psi_2 \in (1 + s^2T \text{End}(R^n)_{1-s}[T])^\times$  such that we have  $(\psi_2(T))_s = \theta(g_1T)\theta(g_2T)^{-1}$  if  $g_1 - g_2 \in s^{k+2}R_{1-s}$ . Notice that, in particular,  $\psi_2 = \text{Id} \pmod{s^2}$ .

As  $s + (1 - s) = 1$ , it follows that there exist  $c, d \in R$  such that  $cs^{k+2} + d(1 - s)^{k+2} = 1$ . From the first property above and the fact that  $cs^{k+2} - 1 \in (1 - s)^{k+2}R$ , we derive that

$$(3.1) \quad (\psi_1(T))_{1-s} = \theta(T)\theta(cs^{k+2}T)^{-1}.$$

Now considering  $g_1 = cs^{k+2}$  and  $g_2 = 0$  and using the second property, we get

$$(3.2) \quad (\psi_2(T))_s = \theta(cs^{k+2}T)\theta(0)^{-1} = \theta(cs^{k+2}T).$$

Putting together (3.1) and (3.2), we get  $\theta = (\psi_1)_{1-s}(\psi_2)_s$ . Now let  $E$  be the patching of  $(R_{(1-s)}[T])^n$  and  $(R_s[T])^n$  along  $\theta$ . Patching  $f_1$  and  $f_2$ , we obtain a surjective homomorphism  $f : E \rightarrow I(V)$ . Now using the isomorphisms  $(\psi_2)^{-1} : (R_{(1-s)}[T])^n \rightarrow (R_{(1-s)}[T])^n$  and  $\psi_1 : (R_s[T])^n \rightarrow (R_s[T])^n$ , we obtain an isomorphism  $R[T]^n \rightarrow E$  and thus a surjective homomorphism  $R[T]^n \rightarrow I(V)$  corresponding to generators  $C_1, \dots, C_n$  of  $I(V)$ . To conclude, we have to check that  $A_i - C_i \in s^2R[T]$ , which follows easily from the fact that  $\psi_2$  is the identity modulo  $s^2$ . □

### 3.2. Lifting generators.

*Definition 3.2.1.* Let  $v = (s, a_1, \dots, a_n, b_1, \dots, b_n) \in Q_{2n}(R)$  and  $I(v) = \langle s, a_1, \dots, a_n \rangle$ . We say that the *strong lifting property* holds for  $v$  if there exist  $\mu_1, \dots, \mu_n \in R$  such that  $I(v) = \langle a_1 + \mu_1s^2, \dots, a_n + \mu_ns^2 \rangle$ .

We want to show that the strong lifting property is preserved by naive homotopies. If  $k$  is of characteristic different from 2, it is easy to see that the

morphisms  $\alpha_n : Q'_{2n} \rightarrow Q_{2n}$  given on sections by

$$\alpha_n(a_1, \dots, a_n, b_1, \dots, b_n, s) = \frac{1}{2}(a_1, \dots, a_n, b_1, \dots, b_n, 1 - s)$$

and  $\beta_n : Q_{2n} \rightarrow Q'_{2n}$  given by

$$\beta_n(a_1, \dots, a_n, b_1, \dots, b_n, s) = (2a_1, \dots, 2a_n, 2b_1, \dots, 2b_n, 1 - 2s)$$

are inverse to each other and thus yield isomorphisms  $Q_{2n} \rightarrow Q'_{2n}$ . It follows that the group  $EO_{2n+1}(R)$  acts on  $Q_{2n}(R)$ , and the generators of this group act as follows:

(1) for any  $\lambda \in R$  and  $1 \leq i \leq n$ ,

$$(s, a_1, \dots, a_n, b_1, \dots, b_n) \\ \mapsto (s - \lambda b_i, a_1, \dots, a_{i-1}, a_i - \lambda(1 - 2s) - \lambda^2 b_i, a_{i+1}, \dots, b_n);$$

(2) for any  $\lambda \in R$  and  $1 \leq i \leq n$ ,

$$(s, a_1, \dots, a_n, b_1, \dots, b_n) \\ \mapsto (s - \lambda a_i, a_1, \dots, b_{i-1}, b_i - \lambda(1 - 2s) - \lambda^2 a_i, b_{i+1}, \dots, b_n);$$

(3) for any  $\lambda \in R$  and  $1 \leq i, j \leq n$  with  $i \neq j$ ,

$$(s, a_1, \dots, a_n, b_1, \dots, b_n) \\ \mapsto (s, a_1, \dots, a_{i-1}, a_i + \lambda a_j, a_{i+1}, \dots, b_{j-1}, b_j - \lambda b_i, b_{j+1}, \dots, b_n);$$

(4) for any  $\lambda \in R$  and  $1 \leq i < j \leq n$  with  $i \neq j$ ,

$$(s, a_1, \dots, a_n, b_1, \dots, b_n) \\ \mapsto (s, a_1, \dots, a_{i-1}, a_i + \lambda b_j, a_{i+1}, \dots, a_{j-1}, a_j - \lambda b_i, a_{j+1}, \dots, b_n);$$

(5) for any  $\lambda \in R$  and  $1 \leq i < j \leq n$  with  $i \neq j$ ,

$$(s, a_1, \dots, a_n, b_1, \dots, b_n) \\ \mapsto (s, a_1, \dots, b_{i-1}, b_i + \lambda a_j, b_{i+1}, \dots, b_{j-1}, b_j - \lambda a_i, b_{j+1}, \dots, b_n).$$

As a corollary of [Theorem 1.0.6](#), we see that the set  $\pi_0(Q_{2n})(R)$  is isomorphic to the orbit set  $Q_{2n}(R)/EO_{2n+1}(R)$ . To prove that the strong lifting property is preserved by naive homotopies, it suffices to prove that this property is preserved by elementary operations of type (1)–(5) above. The typical statement of the following few lemmas will be of the form “for  $M$  an elementary operation of type (i), the strong lifting property holds for  $v$  if and only if it holds for  $vM$ .” We observe that since the inverse of an elementary operation of type (i) is an elementary operation of the same type, it suffices to establish the “only if” direction to prove the result.

**LEMMA 3.2.2.** *Let  $M$  be an elementary operation of type (3) or (5) above. Then the strong lifting property holds for  $v$  if and only if it holds for  $vM$ .*

*Proof.* In case  $M$  is of type (5), there is nothing to do since the operation does not change the generators of  $I(v)$ . If  $M$  is of type (3), then we see that

$$I(vM) = \langle a_1 + \mu_1 s^2, \dots, a_{i-1} + \mu_{i-1} s^2, a_i + \lambda a_j + (\mu_i + \lambda \mu_j) s^2, \\ a_{i+1} + \mu_{i+1} s^2, \dots, a_n + \mu_n s^2 \rangle$$

if  $I(v) = \langle a_1 + \mu_1 s^2, \dots, a_n + \mu_n s^2 \rangle$ . □

LEMMA 3.2.3. *Let  $M$  be an elementary operation of type (2) above. Then the strong lifting property holds for  $v$  if and only if it holds for  $vM$ .*

*Proof.* As  $I(v) = \langle a_1 + \mu_1 s^2, \dots, a_n + \mu_n s^2 \rangle$ , it follows that there exist  $\alpha_1, \dots, \alpha_n$  such that  $s = \sum_{j=1}^n \alpha_j (a_j + \mu_j s^2)$  and that there exist  $\beta_1, \dots, \beta_n$  such that  $a_i = \sum_{j=1}^n \beta_j (a_j + \mu_j s^2)$ . Therefore, we get

$$s - \lambda a_i = \sum_{j=1}^n (\alpha_j - \lambda \beta_j) (a_j + \mu_j s^2),$$

and it follows that  $I(vM) = \langle a_1 + \mu_1 s^2, \dots, a_n + \mu_n s^2 \rangle$  as well. □

PROPOSITION 3.2.4. *Let  $M$  be an elementary operation of type (4) above. Then the strong lifting property holds for  $v$  if and only if it holds for  $vM$ .*

*Proof.* Let

$$V = (s, a_1, \dots, a_{i-1}, a_i + T\lambda b_j, a_{i+1}, \dots, a_{j-1}, a_j - T\lambda b_i, a_{j+1}, \dots, a_n, b_1, \dots, b_n)$$

in  $Q_{2n}(R[T])$ , and let  $A_r = a_r + \mu_r s^2$  for  $1 \leq r \leq n$  such that  $r \neq i, j$ ,  $A_i := a_i + \mu_i s^2 + T\lambda b_j \in R[T]$  and  $A_j := a_j + \mu_j s^2 - T\lambda b_i \in R[T]$ . We have

$$\sum A_i b_i = s - (1 - \sum \mu_i b_i) s^2,$$

and it follows that  $\langle A, (1 - \sum \mu_i b_i) s \rangle = \langle A, s \rangle$  and that

$$W := (A_1, \dots, A_n, (1 - \sum \mu_i b_i) b_1, \dots, (1 - \sum \mu_i b_i) b_n, \\ (1 - \sum \mu_i b_i) s) \in Q_{2n}(R[T]).$$

As  $A_i(0) = a_i + \mu_i s^2$ , we have

$$I(W)(0) = I(v) = \langle A_1(0), \dots, A_n(0) \rangle = \langle A_1(0), \dots, A_n(0), (1 - \sum \mu_i b_i) s \rangle.$$

Applying Lemma 3.1.2, we see that  $I(W) = \langle C_1, \dots, C_n \rangle$  with  $A_i - C_i \in (1 - \sum \mu_i b_i)^2 s^2 R[T]$ . It follows that the strong lifting property also holds for  $I(vM)$ . □

COROLLARY 3.2.5. *Let  $v = (s, a_1, \dots, a_n, b_1, \dots, b_n)$  and  $1 \leq i, j \leq n$  such that  $i \neq j$ . Let  $v' \in Q_{2n}(R)$  be obtained by exchanging  $a_i$  and  $b_i$ , as well as  $a_j$  and  $b_j$ . Then the strong lifting property holds for  $v$  if and only if it holds for  $v'$ .*

*Proof.* In view of [Lemmas 3.2.2 and 3.2.3](#), as well as [Proposition 3.2.4](#) above, the result holds if we show that the permutation matrix  $M$  such that  $v' = vM$  is obtained using elementary operations of type (3)–(5) above. This can be obtained by performing, for instance, the following operations (for simplicity, we assume  $n = 2$ ,  $i = 1$  and  $j = 2$  but the argument is the same in general):

$$\begin{aligned}
 (s, a_1, a_2, b_1, b_2) &\stackrel{(4)}{\mapsto} (s, a_1 + b_2, a_2 - b_1, b_1, b_s) \\
 &\stackrel{(5)}{\mapsto} (s, a_1 + b_2, a_2 - b_1, b_1 + (a_2 - b_1), b_2 - (a_1 + b_2)) \\
 &= (s, a_1 + b_2, a_2 - b_1, a_2, -a_1) \\
 &\stackrel{(4)}{\mapsto} (s, a_1 + b_2 - a_1, a_2 - b_1 - a_2, a_2, -a_1) \\
 &= (s, b_2, -b_1, a_2, -a_1) \\
 &\stackrel{(3)}{\mapsto} (s, b_2, b_2 - b_1, a_2 + a_1, -a_1) \\
 &\stackrel{(3)}{\mapsto} (s, b_2 - (b_2 - b_1), b_2 - b_1, a_2 + a_1, -a_1 + (a_2 + a_1)) \\
 &= (s, b_1, b_2 - b_1, a_2 + a_1, a_2) \\
 &\stackrel{(3)}{\mapsto} (s, b_1, b_2 - b_1 + b_1, a_2 + a_1 - a_2, a_2) \\
 &= (s, b_1, b_2, a_1, a_2). \quad \square
 \end{aligned}$$

**COROLLARY 3.2.6.** *Let  $n \geq 2$ , and let  $M$  be an elementary operation of type (1) above. Then the strong lifting property holds for  $v$  if and only if it holds for  $vM$ .*

*Proof.* By [Corollary 3.2.5](#), we can replace  $a_i$  and  $a_j$  by  $b_i$  and  $b_j$  (for some  $i \neq j$ ) in  $v = (s, a_1, \dots, a_n, b_1, \dots, b_n)$ . The result now follows from [Lemma 3.2.3](#).  $\square$

We can finally state the main theorem of this section.

**THEOREM 3.2.7.** *Let  $k$  be an infinite field of characteristic different from 2, and let  $R$  be an essentially smooth  $k$ -algebra. Moreover, let  $n \geq 2$ ,  $v \in Q_{2n}(R)$ , and  $v_0 = (0, \dots, 0) \in Q_{2n}(R)$ . The strong lifting property holds for the row  $v$  if and only if  $v \in v_0EO_{2n+1}(R)$ .*

*Proof.* Suppose first that  $v \in v_0EO_{2n+1}(R)$ . In view of [Lemmas 3.2.2, 3.2.3, Proposition 3.2.4](#) and [Corollary 3.2.6](#), it suffices to prove that  $v_0$  satisfies the strong lifting property, which is obvious. Conversely, suppose that  $v$  satisfies the strong lifting property. There exist then  $\mu_1, \dots, \mu_n$  such that  $I(v) = \langle a_1 + \mu_1 s^2, \dots, a_n + \mu_n s^2 \rangle$ . Setting  $A_i := a_1 + T\mu_i s^2$ , we get

$$\sum_{i=1}^n A_i b_i = s - \left(1 - T \sum_{i=1}^n \mu_i b_i\right) s^2.$$

If  $S := s(1 - T \sum_{i=1}^n \mu_i b_i)$  and  $B_i = b_i(1 - T \sum_{i=1}^n \mu_i b_i)$ , we then get  $\langle A, s \rangle = \langle A, S \rangle$  and further  $V := (S, A_1, \dots, A_n, B_1, \dots, B_n) \in Q_{2n}(R[T])$ . Now  $v = V(0)$ , and  $v' := V(1) = (s', a'_1, \dots, a'_n, b'_1, \dots, b'_n)$  is such that  $I(v') = \langle a'_1, \dots, a'_n \rangle$ . Using elementary operations of type (2), we see that we may suppose  $s = 0$ . (Note that the  $b'_i$  might change.) For such a row,  $(0, Ta'_1, \dots, Ta'_n, Tb'_1, \dots, Tb'_n)$  is in  $Q_{2n}(R[T])$ , and we finally see that if the strong lifting property holds for  $v$ , then  $v$  is homotopic to  $v_0$ . The result now follows from the fact that  $\pi_0(Q_{2n})(R) = Q_{2n}(R)/EO_{2n+1}(R)$ .  $\square$

As an immediate corollary, we obtain our main theorems.

**THEOREM 3.2.8.** *Let  $k$  be an infinite field of characteristic different from 2, and let  $R$  be an essentially smooth  $k$ -algebra. Suppose that  $n \geq 2$ , and let  $I \subset R$  be an ideal equipped with a surjective map  $\omega_I : (R/I)^n \rightarrow I/I^2$ . If  $s(I, \omega_I) = v_0$  in  $\pi_0(Q_{2n})(R)$ , then there exists a surjective homomorphism  $R^n \rightarrow I$  lifting  $\omega_I$ .*

Recall now that if  $R$  is a ring and  $I \subset R$ , then the number  $\mu(I) \in \mathbb{N}$  is defined to be the minimal number of generators of  $I$ .

**THEOREM 3.2.9** (Murthy’s conjecture). *Let  $k$  be an infinite field of characteristic different from 2,  $m \in \mathbb{N}$ , and  $I \subset k[T_1, \dots, T_m]$  be an ideal. Then we have  $\mu(I) = \mu(I/I^2)$ .*

*Proof.* If  $n := \mu(I/I^2) = 1$ , then  $I$  is of height  $\leq 1$  and without loss of generality we can assume that it is of height 1. Since  $\mu(I/I^2) = 1$ , it follows that  $I$  is an invertible ideal and any such ideal in  $k[T_1, \dots, T_m]$  is principal. We may therefore suppose that  $n \geq 2$ . In view of the above theorem, it suffices to prove that  $\pi_0(Q_{2n})(k[T_1, \dots, T_m]) = v_0$ . By [Lemma 1.0.2](#), we have  $\pi_0(Q_{2n})(k[T_1, \dots, T_m]) = \pi_0(Q_{2n})(k)$  and we are reduced to show that  $EO_{2n+1}(k)$  acts transitively on  $Q_{2n}(k)$ . Using [Lemma 1.0.3](#), we see that  $SO_{2n+1}(k)$  acts transitively on  $Q_{2n}(k)$  and we have  $SO_{2n+1}(k) = EO_{2n+1}(k)$  by Gaussian elimination.  $\square$

*Remark 3.2.10.* The same proof shows that Murthy’s conjecture holds when the base  $R$  is a regular local  $k$ -algebra essentially of finite type.

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INSTITUT FOURIER, UNIVERSITÉ GRENOBLE ALPES, GRENOBLE, FRANCE

*E-mail*: [jean.fasel@gmail.com](mailto:jean.fasel@gmail.com)