Errata of “Isoparametric hypersurfaces with \((g, m) = (6, 2)\)”

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Abstract

We give a correction of the proof of the homogeneity of isoparametric hypersurfaces with \((g, m) = (6, 2)\).

1. Introduction

In [2], [3], and [4], we discuss the homogeneity of isoparametric hypersurfaces \(M\) with six principal curvatures by investigating the kernel of the shape operators of the focal submanifolds. In fact, \(M\) is homogeneous if and only if the kernel is independent of the normal direction [1], [3, §15]. Using this, we reprove the homogeneity for multiplicity \(m = 1\) in [2], [4] and try to prove it for \(m = 2\) in [3]. However, in Sections 8 and 13.3 of [3], there are some inappropriate arguments.

The purpose of this paper is to correct Section 8 and Proposition 13.6 in [3], where the argument to exclude the case \(\dim K = 1, 2\) or \(\dim E(c) = 4\) fails. The correction is now achieved. In Section 3 we rewrite the entire Section 8 [3]. We exclude the case \(\dim E = 4\) in Section 5 and the case \(\dim E = 5\) in Section 6. Then in Sections 7 and 8, we settle the case \(\dim E(c)\) or \(\dim E = 6\).

Thus we obtain

Theorem 1.1. Isoparametric hypersurfaces with \((g, m) = (6, 2)\) are homogeneous.

Remark 1.2. In addition to the revision of Sections 8 and 13.3 of [3], we need some minor changes as follows: There are typos: In (i) on page 81, \(Y_1^V\) and \(Y_2^V\) should be \(\bar{Y}_1^V\) and \(\bar{Y}_2^V\). In (94) on page 84, \(\frac{1}{\sqrt{3}}\) in \(\hat{e}_2\) and \(\hat{e}_4\) should be \(\sqrt{3}\). The notation \(v_i\) in the fourth to ninth lines of page 95 might be confusing, and we had better replace it by, say, \(\omega_i\). All other parts of [3] are correct as they are.


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2. A brief summary of Sections 1–7 in [3]

Let \( M \) be an isoparametric hypersurface in \( S^{13} \) with \((g, m) = (6, 2)\), where \( g \) is the number of distinct principal curvatures and \( m \) is the multiplicity which is common among different principal curvatures when \( g = 6 \). The ambient sphere \( S^{13} \) is singularly foliated by parallel hypersurfaces of \( M \) and two focal submanifolds \( M_{\pm} \). Choosing a unit normal vector field \( \xi \) of \( M \), we denote the principal curvatures by \( \lambda_1 > \cdots > \lambda_6 \) and their curvature distributions by \( D_i, i = 1, \ldots, 6 \). We take an orthonormal frame \( e_i, e_i \) of each \( D_i \). We write \( \xi \) for \( i \) and \( \bar{i} \). Consider the focal submanifold \( M_+ \) at which each leaf \( L_6(p) \) of \( D_6 \) collapses into a point \( \bar{p} = \cos \theta p + \sin \theta \xi_p \) where \( p \in M \) and \( \theta = \cot^{-1} \lambda_6 \).

Then \( T_{\bar{p}}M_+ = \bigoplus_{i=1}^{5} D_i(q) \) and \( T_{\bar{p}}^{L_6}M_+ = \mathbb{R} \eta_q \oplus D_6(q) \) hold for all \( q \in D_6(p) \) and \( \eta_q = -\sin \theta q + \cos \theta \xi_q \). Another focal submanifold \( M_- \) is obtained by replacing \( D_6 \) by \( D_1 \) and \( \theta \) by \( \bar{\theta} = \cot^{-1} \lambda_1 \). Note that \( T_{\bar{p}}M_- = \bigoplus_{i=2}^{6} D_i(q) \) and \( T_{\bar{p}}^{L_6}M_- = \mathbb{R} \bar{\eta}_q \oplus D_1(q) \) for all \( q \in D_1(p) \) and \( \bar{\eta}_q = -\sin \bar{\theta} q + \cos \bar{\theta} \xi_q \). By the argument in Sections 1–7 and 15 of [3], we know the following:

**Fact.** (1) The shape operators \( B_n \) of \( M_+ \) with respect to a unit normal \( n \in T^1 M_+ \) are isospectral with eigenvalues \( \mu_1 = \sqrt{3} = -\mu_5, \mu_2 = 1/\sqrt{3} = -\mu_4 \) and \( \mu_3 = 0 \). The eigenspace of \( \mu_i \) of \( B_{\eta_p} \) is given by \( D_i(p) \).

(2) At the focal point \( \bar{p} \), the unit sphere \( S^2 \) in \( T_{\bar{p}}^{L_6}M_+ \) is identified with the leaf \( L_6(p) \) of \( D_6 \). Take \( \zeta = \epsilon_6(p) \) in \( T_{\bar{p}}^{L_6}M_+ \). The geodesic \( c = \{ p(t) \} \) of \( S^2 = L_6(p) \) through \( p \) in the direction \( \zeta \) corresponds to a one parameter family of normal vectors \( \cos t \eta + \sin t \zeta \) of \( M_+ \). Then the shape operator \( L(t) = \cos tB_\eta + \sin tB_\zeta \) of \( M_+ \) has \( \ker L(t) = D_3(p(t)) \).

(3) \( M \) is homogeneous if and only if \( \ker L(t) \) is independent of \( t \) and \( \zeta \), namely, if and only if \( D_3 \) is invariant on each \( L_6 \).

All these hold if we replace \( M_+ \) by \( M_- \) and index \( i \) by \( i + 1 \) modulo 6.

Now, for a geodesic \( c \) of \( L_6(p) \), put

(1) \( E(c) = \text{span}_t \ker L(t) \).

Then Theorem 1.1 is proved if we show \( \dim E(c) = 2 \) for any \( c \) of any \( L_6 \) (see [3, 15]). Recall [3, (42)]

\[
E(c) = \text{span}\{ e_3(q), \nabla_{e_6}^k e_3(q), k = 1, 2, \ldots \}
\]

(2) \( W(c) = \text{span}\{ \tilde{\nabla}_{e_3} e_6(q), \nabla_{e_6}^k \tilde{\nabla}_{e_2} e_6(q), k = 1, 2, \ldots \} \),

which do not depend on the choice of \( q \in c \). Note that
Lemma 2.1 ([3, Lemmas 5.3, 5.4, and (46) of Lemma 6.1]). \(W(c) \subset E(c)^\perp\). Moreover, \(L(t)\) maps \(E(c)\) onto \(W(c)\) for any \(t\), and \(\dim W(c) = \dim E(c) - 2\) holds.

For a fixed \(L_6(p)\), we put
\[
E = \text{span}\{E(c) \mid c : \text{a geodesic of } L_6(p)\}.
\]

3. Dimension of \(E(c)\)

To investigate the dimension of \(E(c)\) or \(E\) under the supposition \(\dim E(c) > 2\), we need a special frame of \(D_3(t)\) along a geodesic \(c = \{p(t)\}\) of \(L_6(p)\), parametrized by \(t\) so that \(p(0) = p(2\pi)\). For a vector field \(v(t)\) along \(c\), we call \(v(t)\) even when \(v(t + \pi) = v(t)\), and odd when \(v(t + \pi) = -v(t)\). We sometimes denote \(p(t) = c(t)\).

Lemma 3.1. If \(e_3(t)\) is an even (odd, resp.) vector along \(c\), then \(e_3(t), \nabla_{e_6} e_3(t), \nabla^2_{e_6} e_3(t), \ldots\) are all even (odd, resp.) vectors. On the other hand, \(\nabla_{e_3} e_6(t), \nabla_{e_6} \nabla_{e_3} e_6(t), \nabla^2_{e_6} \nabla_{e_3} e_6(t), \ldots\) are all odd (even, resp.) vectors.

Proof. The former is clear from \(\nabla^k_{e_6} e_3(t + \pi) = \nabla^k_{e_6} e_3(t)\). The latter follows from \(L(t + \pi) = -L(t)\) and \(L(t)(\nabla_{e_6} e_3(t)) = c_1 \nabla_{e_3} e_6(t)\) (see [3, Lemma 5.1, (36)]). Then its derivatives in the direction \(e_6(t)\) are all odd. The case when \(e_3(t)\) is odd is similar.

Lemma 3.2. \(\dim E(c)\) must be even.

Proof. There are no odd dimensional subspace of \(TM_+\) parallel along \(c\) and consisting of odd vectors, because of the continuity of the determinant of a moving frame. By [3, Lemma 7.7], we can choose \(e_3(t), e_3(t)\) so that \(E(c)\) consists of all even or all odd vectors. By Lemma 3.1, evenness and oddness of the vectors in \(E(c)\) and in \(W(c)\) are opposite. Since both \(E(c)\) and \(W(c)\) are parallel and \(\dim W(c) = \dim E(c) - 2\) (Lemma 2.1), \(\dim E(c)\) must be even.

Lemma 3.3. If a differentiable field \(e_3(t)\) spans a 2-dimensional space \(K = \text{span}\{e_3(t)\}\), then \(e_3(t)\) is an odd vector.

Remark 3.4. A typical case is when \(e_3(t) = \cos t u + \sin t v\) for orthonormal vectors \(u\) and \(v\). Usually, the coefficient functions are general odd functions and \(u\) and \(v\) are not necessarily orthonormal.

Proof. Assume \(\dim K = 2\); then it follows \(\nabla_{e_6} e_3(p) \not\equiv 0\) modulo \(D_3(p)\) ([3, Rem. 5.2]). Using \(q = p(\pi/2)\), we can express \(K = \text{span}\{e_3(p), e_3(q)\}\). Thus we have
\[
e_3(t) = a(t)e_3(p) + b(t)e_3(q) \in K.
\]
Recall [3, (37)]

\[ B_\xi(e_3(p)) = -\nabla_{e_3}e_6(p). \]

Because \( e_3(q) \in \text{Ker} \pi/2 = \ker B_\xi \), exchanging \( p \) and \( q \), we have

\[ B_\eta(e_3(q)) = \nabla_{e_3}e_6(q), \]

since \( B_\eta = -\pi/2 + \pi/2 \) and \( B_\xi = \pi/2 \). Therefore, denoting \( c(t) = \cos t \) and \( s(t) = \sin t \), by (4) we have

\[ 0 = L(t)e_3(t) = (c(t)B_\eta + s(t)B_\xi)(a(t)e_3(p) + b(t)e_3(q)) \]
\[ = b(t)c(t)B_\eta(e_3(q)) + a(t)s(t)B_\xi(e_3(p)) \]
\[ = b(t)c(t)\nabla_{e_3}e_6(q) - a(t)s(t)\nabla_{e_3}e_6(p) \]

for all \( t \). From this it follows

\[ \nabla_{e_3}e_6(q) = u\nabla_{e_3}e_6(p) \]

for some nonzero \( u \). Thus \( W = L(t)K \) is a 1-dimensional space consisting of \( \nabla_{e_3}e_6(t) \) which is a nonzero and hence a positive scalar multiple of \( \nabla_{e_3}e_6(p) \) (see [3, Rem. 5.2]). Then \( \nabla_{e_3}e_6(t) \) is an even vector, and so \( e_3(t) \) is an odd vector.

**Lemma 3.4.** If there exists a constant \( e_3 \) along two geodesics \( c \) and \( \bar{c} \) of \( L_0(p) \), then \( e_3 \) is constant all over \( L_0(p) \).

**Proof.** Recall that if \( e_3 \) coincides at two nonantipodal points on a geodesic \( c \), then \( e_3 \) is constant along \( c \) ([3, Lemma 7.1]). Thus if \( e_3 \) is constant along \( c \cup \bar{c} \), \( e_3 \) is constant along any geodesic joining a point on \( c \) and a point on \( \bar{c} \), and hence by the continuity, constant all over \( L_0(p) \).

Let \( e_3(t), e_3(t) \) be an orthonormal frame of \( D_3(t) \) along a geodesic \( c(t) \). For each \( t \), put \( W(t) = \text{span} \{ \nabla_{e_3}e_6(t), \nabla_{e_3'}e_6(t) \} \subset W(c) \).

**Lemma 3.5.** \( \dim W(t) \) is independent of \( t \) and takes values 0, 1 or 2.

**Proof.** If \( \nabla_{e_3}e_6(t_0) \) and \( \nabla_{e_3'}e_6(t_0) \) are dependent at some \( t_0 \), then there exists \( e_3'(t_0) = ae_3(t_0) + be_3(t_0) \) such that \( \nabla_{e_3'}e_6 = 0 \), and hence \( e_3' \) is constant along \( c \) (see [3, Lemma 7.1]). Thus \( \dim W(t) = 1 \) unless \( e_3'(t) \), which is orthogonal to \( e_3'(t_0) \), is also constant, in which case \( \dim W(t) = 0 \). Therefore, we have \( \dim W(t) = 0, 1 \) or 2 independent of \( t \).

Let \( \Gamma \) be the space of oriented geodesics of \( L_0(p) \) for each \( p \), which is diffeomorphic to \( S^2 \). Then \( d : \Gamma \ni c \mapsto d(c) = \dim W(t) \in \{ 0, 1, 2 \} \) is well defined by this lemma and is lower-semicontinuous. Thus \( U = \{ c \in \Gamma \mid d(c) = \max_{\Gamma} d \} \) is an open subset of \( \Gamma \). When \( \max_{\Gamma} d = 0, D_3 = D_3(p) \) is constant along \( L_3(p) \). Consider the following cases:

(i) \( \max_{\Gamma} d = 1 \),
(ii) \( \max_{\Gamma} d = 2 \).
LEMMA 3.6. When (i) is the case, there exists $e_3$ which is constant all over $L_6(p)$.

Proof. Since $U$ is open, we may assume that a family of geodesics $e^s$ through $p$ in the direction $e_6^s(p) = \cos se_6(p) + \sin se_6(p)$ belongs to $U$. Then for each $s$, some $e_3^s(p) \in D_3(p)$ is constant along $e^s$. If $e_3^0(p) = e_3^1(p)$ holds for some $0 < s < \pi$, then $e_3 = e_3^0(p)$ is constant all over $L_6(p)$ by Lemma 3.4.

When $e_3^0(p)$ and $e_3^1(p)$ are independent in $D_3(p)$ for all $s \neq 0$ modulo $\pi$, $e_3^s(p)$ lies in $D_3(p) \cap D_3(p_s)$ for each $p_s \in c^s \cap \gamma$, where $\gamma$ is any fixed geodesic transversal to $c^s$. Hence $e_3^s(p) \in E(\gamma)$ spans the 2-dimensional space $K = D_3(p)$ along $\gamma$, where $K$ is as in Lemma 3.3. Also, without loss of generality, we may consider that there exists a constant $e_3$ along $\gamma$, and so $E(\gamma) \subset D_3(p) + \{e_3\}$. However since $\dim E(\gamma)$ is even (Lemma 3.2), this implies $E(\gamma) = D_3(p)$. Because $\gamma$ is any geodesic transversal to $c^s$, $E = D_3(p)$ follows from [3, Lemma 7.3], which is not the case.

PROPOSITION 3.7. If there exists some geodesic $c$ of $L_6(p)$ such that $\dim E(c) > 2$, then (i) never occurs on $M_±$.

Proof. Note that $\dim F(\gamma) > 2$ also holds by [3, Lemma 7.6]. We may consider $d(\gamma)$ defined for a geodesic $\gamma$ of $L_1(p)$, where (i) or (ii) occurs similarly. Assume (i) is the case for $M_−$. Choose any $p_1 \in L_6(p)$, and let $p_3$ be as in [3, Fig. 1]. Then on $L_1(p_3)$, there exists $e_4(p_3)$ which is constant all over $L_1(p_3)$ by the previous lemma, and so is $e_6(p_1)$ all over $L_3(p_1)$. This means $0 = \nabla e_1^se_4(p_3) = \nabla e_3^se_6(p_1)$, and hence along the geodesic $c$ of $L_6(p_1)$ in the direction $e_6$, $D_3$ is constant ([3, Rem. 5.2]). Since $p_1 \in L_6(p)$ is arbitrarily, this means that at each point of $L_6(p)$, there exists a geodesic along which $D_3$ is constant. Thus by [3, Lemma 7.3], $\dim E = 2$ follows, a contradiction. Thus (i) cannot occur on $M_-$, and neither on $M_+$.

LEMMA 3.8. When (ii) is the case, the subset $U_1 = \{c \in \Gamma \mid d(c) \leq 1\}$ has no interior points.

Proof. Lemma 3.6 and the proof of Proposition 3.7 are valid on $U_1$ if it has interior points.

We call $c \in U$ “generic.” Up to here, we do not assume a specific value of $\dim E(c)$.

4. $\dim E(c) = 4$

When $\dim E(c) > 2$ for some geodesic $c$ of $L_6(p)$, we only need to consider the case (ii) by Proposition 3.7.

LEMMA 4.1. When $\dim E(c) = 4$ for $c \in U$, we can take $e_3(t)$ so that $\nabla e_3 e_6(t)$ is parallel to $\nabla e_3 e_6(p)$, and $K = \text{span}_t\{e_3(t)\}$ is of dimension 2. We
can express \( e_3(t) = a(t)e_3(p) + b(t)\nabla_{e_3}e_3(p) \), or \( \tilde{a}(t)e_3(p) + \tilde{b}(t)e_3(q) \), where \( a(t), b(t), \tilde{a}(t), \tilde{b}(t) \) are odd functions, and \( q \in c \) is not antipodal to \( p \).

Proof. Since (ii) is the case, \( \dim W(t) = 2 \) for each \( t \). Since \( W(t) \) and \( \nabla_{e_3}e_6(p) \) are contained in \( W(c) \) which is of dimension 2 (Lemma 2.1), we can find \( \tilde{e}_3(t) \) so that \( \nabla_{\tilde{e}_3}e_6(t) \) is parallel to \( \nabla_{e_3}e_6(p) \). We rewrite \( \tilde{e}_3(t) \) by \( e_3(t) \), and put \( K = \mathrm{span}_c\{e_3(t)\} \). From \( \dim L(t)K = 1, \dim K = 2 \) or 3 follows. If \( \dim K = 3, \ker L(t) \subset K \) for any \( t \), which contradicts that \( e_3(p) \) is not contained in \( K \), since \( \nabla_{e_3}e_6(p) \) is independent of \( \nabla_{e_3}e_6(p) \) (see Lemma 7.1 [3]). The remaining part is as in the proof of Lemma 3.3. \( \square \)

**Remark 4.2.** Replacing \( e_3(t) \) by \( e_3(t) \), we may consider that \( e_3(t) \) also spans a 2-dimensional subspace \( K_2(c) \) of \( E(c) \). Thus we have \( E(c) = K_1(c) + K_2(c) \), which is not necessarily an orthogonal decomposition, where

\[
K_1(c) = \mathrm{span}\{e_3(p), \nabla_{e_3}e_3(p)\}, \quad K_2(c) = \mathrm{span}\{e_3(p), \nabla_{e_6}e_3(p)\}.
\]

5. \( \dim E = 4 \)

In this section, we exclude the case \( \dim E = 4 \) where \( E = \mathrm{span}_c E(c) \).

Suppose \( \dim E = 4 \), and let \( S_E^3 \) be the unit sphere of \( E \cong \mathbb{R}^4 \). For each \( x \in L_6(p) \), consider the unit circle \( S_1^x \subset D_3(x) \subset E \), where \( D_3(x) = \ker B_{n_3} \).

When there is no constant \( e_3 \) along any geodesic of \( L_6(p) \), \( S_1^x \) does not intersect \( S_1^y \) for \( x, y \) belonging to an open hemisphere \( U \) of \( L_6(p) \), since \( e_3(x) = e_3(y) \) implies that \( e_3 \) is constant along the geodesic joining \( x \) and \( y \); see [3, Lemma 7.1]. Thus if \( y \) moves in an open neighborhood \( U' \subset U \) of \( x \), namely, in 2-parameters \( (s, t) \), \( S_1^y \) moves in 2-parameters in \( S_3^E \) without intersection continuously and hence generates an open neighborhood \( \Omega \cong U' \times S^1 \) of \( e_3(x) \) in \( S_3^E \).

**Lemma 5.1.** When \( \dim E = 4 \), let \( S = \bigcup_{x \in L_6(p)} S_1^x \subset S_3^E \). If along any geodesic of \( L_6(p) \) there is no constant \( e_3 \), then \( S = S_3^E \).

Proof. Obviously, \( S \) is a nonempty closed subset of \( S_3^E \). On the other hand, for \( e_3(x) \in S \) at \( x \in L_6(p) \), the above \( \Omega \) is an open neighborhood of \( e_3(x) \) contained in \( S \). Hence \( S \) is open. Since \( S_3^E \) is connected, the lemma follows. \( \square \)

**Lemma 5.2.** When \( \dim E = 4 \), there exists a constant \( e_3 \) along some geodesic \( c \).

Proof. We have a rank 2 vector bundle over \( L_6(p) \) with fiber \( D_3(x) \) at \( x \in L_6(p) \). Suppose that along any geodesic of \( D_6(p) \), there is no constant \( e_3 \). Then for any \( v \in S_3^E \), there exists \( x \in L_6(p) \) such that \( e_3(x) = v \) by Lemma 5.1. Here, for any antipodal pair \( x, -x \) of \( L_6(p) \), \( D_3(x) = D_3(-x) \) and so \( S_1^x = S_1^{-x} \) holds. On the other hand, under our assumption, \( D_3(y) \cap D_3(x) = \{0\} \) if \( y \neq -x \) and so \( S_1^x \cap S_1^y = \emptyset \).
Thus we can define $\pi : S^3_E \to L_6(p)/\mathbb{Z}_2$ with the local triviality $\pi^{-1}(U') \cong U' \times S^1$ where $U'$ is as above, and obtain an $S^1$ fibration $\pi : S^3_E \to L_6(p)/\mathbb{Z}_2 \cong S^2/\mathbb{Z}_2 = \mathbb{R}P^2$. However, this is impossible by the Thom-Gysin sequence. Namely, if there exists an $S^1$ bundle $(S^3_E, \mathbb{R}P^2, S^1)$, in the exact sequence for the $\mathbb{Z}_2$ homology of this bundle,

$$
\to H_q(S^3_E) \to H_q(\mathbb{R}P^2) \to H_{q-2}(\mathbb{R}P^2) \to H_{q-1}(S^3_E) \to \to
$$

putting $q = 3$, we have a contradiction. \hfill $\square$

Let $c$ be a geodesic appearing in the lemma on which $e_3(t)$ is constant, or equally, $\nabla_{e_6} e_3(t) = 0$ holds. Let $p \in c$ and $c$ be in the direction $e_6$. Along a generic geodesic $c^s$ ($s \neq 0, \pi$) in the direction $e^s_6 = \cos s e_6 + \sin s e_6$ at $p$, take $e^3_3(t)$ spanning the 2-dimensional space $K^c_1 = \{e_3(p), \nabla e^s_6 e_3(p)\}$, which is possible by Proposition 3.7. Here, $K^c_1$ is independent of $s(\neq 0, \pi)$, because

$$
\nabla e^s_6 e_3(p) = \cos s \nabla e_6 e_3(p) + \sin s \nabla e_6 e_3(p) = \sin s \nabla e_6 e_3(p).
$$

Thus for any $s, s' (\neq 0, \pi)$ and $q \in c^s$, there exists $x \in c^{s'}$ such that $e^s_3(q) = e^{s'}_3(x)$ (see Lemma 4.1).

Now, take $q \in L_6(p) \setminus c$ first, and let $c^s$ be the geodesic through $p, q$. Then above argument implies that for any $s' (\neq 0, \pi, s)$, there exists $x \in c^{s'}$ such that $e_3(q) = e_3(x)$. Hence $e_3$ is constant along the geodesic $\gamma$ joining $q$ and $x$ by [3, Lemma 7.1]. As $q$ is arbitrary, this implies the case (i), which contradicts Proposition 3.7. Thus we obtain

**Proposition 5.3.** Neither $\dim E = 4$ nor $\dim F = 4$ can occur.

6. $\dim E(c) = 4$ and $\dim E > 4$

Next, when $\dim E(c) = 4$, we show $\dim E = 6$. Along generic geodesics $c$ and $\hat{c}$ through $p$, put

$$(6) \quad \hat{E} = E(c) + E(\hat{c}) = D_3(p) + \text{span}\{\nabla e_6 e_3(p), \nabla e_6 e_3(p), \nabla e_6 e_3(p), \nabla e_6 e_3(p)\}.$$

**Lemma 6.1.** $\hat{E} = E$ and $\dim E = 6$.

**Proof.** Let $c^s$ be the geodesic through $p$ in the direction $e^s_6 = \cos s e_6 + \sin s e_6$. By Proposition 3.7 and Lemma 4.1, it is easy to see $E(c^s) \subset \hat{E}$. For any geodesic $\gamma$ transversal to $c^s$, take $p^s \in c^s \cap \gamma$. Then from $D_3(p^s) \subset E(c^s) \subset \hat{E}$ for every $s$, we know $E(\gamma) \subset \hat{E}$. Since $\gamma$ is arbitrary, we conclude $\hat{E} = E = \text{span}_\gamma E(\gamma)$, which is parallel along $L_6(p)$. By Lemma 4.1 again, vectors spanning $\hat{E}$ in (6) are odd. Thus we obtain $\dim E = 6$ by Proposition 5.3. \hfill $\square$

Now, put

$$
W = \text{span}_{s,t}\{\nabla e_2 e_6(t)\} = \text{span}\{\nabla e_2 e_6(p), \nabla e_3 e_6(p), \nabla e_3 e_6(p), \nabla e_3 e_6(p)\}.
$$
PROPOSITION 6.2. When \( \dim E(c) = 4 \), \( W \) is orthogonal to \( E \), and all the shape operators \( L(s, t) = \cos s \cos t B_\eta + \cos s \sin t B_\xi + \sin s B_\zeta \) map \( E \) onto \( W \), where \( \zeta = e_6 \) and \( \bar{\zeta} = e_6 \).

Proof. From [3, (43)], at any point of \( L_6 \),

\[
\langle \nabla_{e_6} e_2, \nabla_{e_2} e_6 \rangle = 0 \tag{7}
\]

holds if two \( e_6 \) are both \( e_6 \), or both \( e_6 \), or by the global symmetry (at \( p_3 \) for \( M_- \)), if two \( e_2 \) are both \( e_3 \), or both \( e_3 \). Hence we need to show

\[
\langle \nabla_{e_6} e_3, \nabla_{e_3} e_6 \rangle = 0, \tag{8}
\]

\[
\langle \nabla_{e_6} e_3, \nabla_{e_3} e_6 \rangle = 0. \tag{9}
\]

Since \( 0 = \langle \nabla_{e_6+e_6} e_3, \nabla_{e_3} (e_6 + e_6) \rangle = \langle \nabla_{e_6} e_3, \nabla_{e_3} e_6 \rangle + \langle \nabla_{e_6} e_3, \nabla_{e_3} e_6 \rangle \), it is sufficient to show either one of (8) or (9). Recall that \( e_3(t) \) is chosen as in Lemma 4.1 along \( c \), and we extend \( e_3(t) \), which is orthogonal to \( e_3(t) \), to \( e_3(s, t) \) as in Lemma 4.1 along each geodesic \( c'(s) \) through \( c(t) \) in the direction \( e_6(t) \). Then at \( p_3^t \in c \cap c' \), we have

\[
\langle \nabla_{e_6} e_3(p_3^t), \nabla_{e_3} e_6(p_3^t) \rangle = -\langle \nabla_{e_6} e_3(p_3^t), \nabla_{e_3} e_6(p_3^t) \rangle
\]

since \( \nabla_{e_6} e_3(t) \) is odd and \( \nabla_{e_3} e_6 \) is even. Thus we have \( p_0 \in c \) at which

\[
\langle \nabla_{e_6} e_3(p_0), \nabla_{e_3} e_6(p_0) \rangle = 0,
\]

namely, (8), and hence (9) hold. Thus \( W \) is orthogonal to \( E \) (by (2) and the statement after it). Since \( E \) is parallel and of dimension 6, \( W = E^\perp \) is parallel, and \( B_\eta(E) = W \).

We know already that \( L(s, 0) = \cos s B_\eta + \sin s B_\zeta \) maps \( E(c) \) onto \( W(c) \subset W \) ([3, Lemma 5.4]). Thus we need to show that \( B_\zeta \) maps \( \nabla_{e_6} e_2 \) into \( W \). Using [3, (36)], this follows from

\[
B_\zeta(\nabla_{e_6} e_2) = c_0 \nabla_{e_6} \left( B_\eta(\nabla_{e_6} e_2) \right) - c_0 B_\eta(\nabla_{e_6} \nabla_{e_6} e_2)
\]

\[
= c_0 c_1 \nabla_{e_6} \nabla_{e_6} e_6 - c_0 B_\eta(\nabla_{e_6} \nabla_{e_6} e_2).
\]

In fact, all the second derivatives such as \( \nabla_{e_6} \nabla_{e_6} e_2 \) are contained in \( E \) since \( E \) is parallel, and \( \nabla_{e_6} \nabla_{e_6} e_6 \in W \) since \( W = E^\perp \) is parallel. Hence \( B_\zeta \) maps \( E \) onto \( W \). Similarly, \( B_\zeta \) maps \( E \) onto \( W \). \( \square \)

By this proposition, even when \( \dim E(c) = 4 \), we can express

\[
L(t) = \cos t B_\eta + \sin t B_\zeta = \begin{pmatrix} 0 & R \\ t R & S \end{pmatrix}, \quad T = t RR,
\]

with respect to the decomposition \( E^6 \oplus W^4 \) for any \( \zeta \in D_6(p) \). In particular, we can apply the argument [3, §§9–13.2] to this case replacing \( E(c) \) by \( E \), and putting \( Y = 0 \) in [3, (106)]. All the results hold as in the case \( \dim E(c) = 6 \). Among the most important are Proposition 12.2 and Corollary 12.3, where
under the assumption \(ab \neq 0\), \(\sigma\) and \(\tau\) become constant along \(c\). The arguments in [3, \S 13] are true except for the proof of Proposition 13.6 and Lemma 13.9.

7. Eigenvalues of \(T\)

Recall [3, Props. 10.1 and 10.3]. Then in both cases (A) \(\dim E(c) = 6\), and (B) \(\dim E(c) = 4\) with \(\dim E = 6\), \(B_\eta\) is given by one of the following with respect to \(E(c) \oplus W(c)\), and \(E \oplus W\), respectively:

\[(0) \text{ } ab \neq 0, T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}, T_1 = \begin{pmatrix} \sigma & 0 \\ 0 & 1/\sigma \end{pmatrix}, T_2 = \begin{pmatrix} \tau & 0 \\ 0 & 1/\tau \end{pmatrix},
\]
\[
S = \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}, S_1 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad S_2 = \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix},
\]
\[
\sigma + 1/\tau + a^2 = 10/3, \quad \tau + 1/\sigma + b^2 = 10/3.
\]

(I) \(a = b = 0\) and \(T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}, T = \begin{pmatrix} \tau & 0 \\ 0 & 1/\tau \end{pmatrix}, S = 0.
\]

(II) \(a \neq 0, b = 0\) and \(T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}, S = \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}.
\]

In fact, if \(ab \equiv 0\) holds in an open neighborhood of the space of geodesics of \(L_6(p)\), either (I) or (II) occurs since \(\sigma + 1/\sigma + a^2 = 10/3\) and a similar formula holds for \(\tau\) \(b\) [3, (72)]. Note that \(a \neq 0\) is equivalent with \(a \beta \neq 0\), as the latter implies \(\tau \neq 1/3, 3\). Similarly, \(b \neq 0\) corresponds to \(\gamma \delta \neq 0\) (the last line of [3, Prop. 11.1]). Therefore, Case (0) occurs only when \(ab \neq 0\) which is the case \(a \beta, \gamma \delta \neq 0\).

The argument in [3, \S\S 12, 13.1, 13.2], treating the case \(ab \neq 0\) are quite important, and Corollary 12.3 is most notable. Based on these results, we show

**Proposition 7.1.** \(\text{When } ab \neq 0, \sigma = \tau \in (1/3, 3)\) holds.

**Proof.** In the following, we use the notation in [3, \S 12] and the orthonormal basis \(X_1, Z_1\) given by [3, (91), (92)].

Because \(\sigma, \tau\) are constant along the geodesic \(c\) by [3, Cor. 12.3], differentiating \(L(t)X_1(t) = \nu_1Z_1(t)\) by \(t\) where \(\nu_1 = \sqrt{\sigma}, \nu_2 = 1/\sqrt{\sigma}, \nu_1 = \sqrt{\tau}, \nu_2 = 1/\sqrt{\tau}\), we obtain
\[
L_t(t)X_1(t) + L(t)X_1(t) = \nu_1Z_1(t).
\]

Note that \(X_1(t) = H(t)X_1(t), Z_1(t) = H(t)Z_1(t)\) by [3, (27)], where we use again that \(\nu_1\)'s are constant. Hence putting \(t = 0\), and denoting \(X_1(0) = X_1, Z_1(0) = Z_1\) etc., we have

\[
(11) \quad B_\zeta X_1 = -B_\eta H(0)X_1 + \nu_1H(0)Z_1.
\]

Since \(ab \neq 0\), using [3, (116)], we may put \(H(0) = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix}\), where
\[
J_1 = \begin{pmatrix} H_0 & X & Y \\ -tX & H_1 & Z \\ -tY & -tZ & H_2 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & x & y & z \\ -x & 0 & u & v \\ -y & -u & 0 & w \\ -z & -v & -w & 0 \end{pmatrix}.
\]
Then (11) is expressed as
\[
\begin{pmatrix}
0 & M \\
1' & N
\end{pmatrix}
\begin{pmatrix}X_1 \\
0
\end{pmatrix} = -\begin{pmatrix}
0 & A \\
1' & D
\end{pmatrix}
\begin{pmatrix}X_1 \\
0
\end{pmatrix} + \nu_i \begin{pmatrix}J_1 & 0 \\
0 & J_2
\end{pmatrix}
\begin{pmatrix}0 \\
Z_i
\end{pmatrix},
\]
and hence we obtain
\[
(12) \quad B_0 X_1 = 1'M X_1 = -1'A J_1 X_1 + \nu_i J_2 Z_i.
\]
Here and there, we abuse \((0 V) = V) or \((V') = V)

Since we can express
\[
(13) \quad A = \begin{pmatrix}0_{2,4} \\
A
\end{pmatrix}, \quad \bar{A} = \text{diag}(\sqrt{\sigma} \quad 1/\sqrt{\tau} \quad \sqrt{\tau} \quad 1/\sqrt{\tau}),
\]
where \(0_{i,j}\) denote the \(i \times j\) zero matrix, from \(X_1 \perp D_3\) we have
\[
t^nA J_1 X_1 = \begin{pmatrix}0_{4,2} & t^nA \\
H_0 & X \\
H_1 & Y \\
-t^nY & -t^nZ & H_2
\end{pmatrix}
\begin{pmatrix}0_{2,1} \\
X_1
\end{pmatrix}
\]
\[
= t^n\bar{A} \begin{pmatrix}H_1 & Z \\
-t^nZ & H_2
\end{pmatrix} X_1 = \begin{pmatrix}0 & x/\sqrt{\sigma} & y/\sqrt{\sigma} & z/\sqrt{\sigma} \\
-x/\sqrt{\sigma} & 0 & u/\sqrt{\tau} & v/\sqrt{\tau} \\
-y/\sqrt{\tau} & -u/\sqrt{\tau} & 0 & w/\sqrt{\tau} \\
-z/\sqrt{\tau} & -v/\sqrt{\tau} & -w/\sqrt{\tau} & 0
\end{pmatrix} X_1.
\]

Now, suppose \(\sigma \neq \tau\), namely, \(a^2 \neq b^2\). Then by \([3, \text{Prop. } 13.3], [3, (138)]\) follows, and hence differentiating \(U_2\) at \(t = 0\), we have
\[
J_2Z_i = \begin{pmatrix}0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} Z_i.
\]
Substituting these into (12), we obtain
\[
\begin{align*}
t^nM_1 &= -\sqrt{\sigma} Z_1 + x/\sqrt{\sigma} Z_2 + y/\sqrt{\tau} Z_1 + z/\sqrt{\tau} Z_2, \\
t^nM_2 &= -x/\sqrt{\sigma} Z_1 + u/\sqrt{\tau} Z_1 + v/\sqrt{\tau} Z_2, \\
t^nM_1 &= \sqrt{\tau} Z_1 - y/\sqrt{\sigma} Z_1 - u/\sqrt{\tau} Z_2 + w/\sqrt{\tau} Z_2, \\
t^nM_2 &= -z/\sqrt{\sigma} Z_1 - v/\sqrt{\tau} Z_2 - w/\sqrt{\tau} Z_1.
\end{align*}
\]
Therefore, putting \(t^nM = (l_1 \quad l_2 \quad l_3 \quad l_4 \quad l_5 \quad l_6)\), by (12) we have
\[
(14) \quad (l_3 \quad l_4 \quad l_5 \quad l_6) = \begin{pmatrix}0 & -x/\sqrt{\sigma} & \sqrt{\tau} - y/\sqrt{\sigma} & -z/\sqrt{\tau} \\
x/\sqrt{\sigma} & 0 & -u/\sqrt{\tau} & -v/\sqrt{\tau} \\
-x/\sqrt{\sigma} + y/\sqrt{\tau} & u/\sqrt{\tau} & 0 & -w/\sqrt{\tau} \\
z/\sqrt{\tau} & v/\sqrt{\tau} & w/\sqrt{\tau} & 0
\end{pmatrix}.
\]
From this and (13), it follows
\[
\begin{pmatrix}
0 & -x & \tau - y\sqrt{\sigma\tau} & -z\sqrt{\sigma/\tau} \\
x & 0 & -u\sqrt{\tau/\sigma} & -v/\sqrt{\sigma\tau} \\
-\sigma + y\sqrt{\sigma\tau} & u\sqrt{\tau/\sigma} & 0 & -w \\
z\sqrt{\sigma/\tau} & v/\sqrt{\sigma\tau} & w & 0
\end{pmatrix}.
\]
Therefore, we obtain
\[
^tAM + ^tMA = \begin{pmatrix}
0 & 0 & \tau - \sigma & 0 \\
0 & 0 & 0 & 0 \\
\tau - \sigma & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]
On the other hand, we know
\[
^tAA = \text{diag}(\sigma, 1/\sigma, \tau, 1/\tau),
\]
and so
\[
^tMM = U_2^tAA^tU_2 = \begin{pmatrix}
(s + \tau)/2 & 0 & (s - \tau)/2 & 0 \\
0 & 1/\sigma & 0 & 0 \\
(s - \tau)/2 & 0 & (s + \tau)/2 & 0 \\
0 & 0 & 0 & 1/\tau
\end{pmatrix}
\]
follows, where \(U_2\) is given by [3, (138)]. Thus in \(^t(cA + sM)(cA + sM) = c^2(^tAA) + s^2(^tMM) + cs(^tAM + ^tMA)\), where \(c = \cos t, s = \sin t\), the second and the fourth columns and rows make \(\begin{pmatrix}1/\sigma & 0 \\ 0 & 1/\tau\end{pmatrix}\). On the other hand, the first and the third columns and rows yield
\[
\begin{pmatrix}
c^2\sigma + s^2(\sigma + \tau)/2 & s^2(\sigma - \tau)/2 + cs(\tau - \sigma) \\
s^2(\sigma - \tau)/2 + cs(\tau - \sigma) & c^2\tau + s^2(\sigma + \tau)/2
\end{pmatrix},
\]
which has eigenvalues \(\sigma\) and \(\tau\) for all \(c, s\). Then as its determinant
\[
(c^2\sigma + s^2(\sigma + \tau)/2)(c^2\tau + s^2(\sigma + \tau)/2) - \{s^2(\sigma - \tau)/2 + cs(\tau - \sigma)\}^2
\]
should be identically \(\sigma\tau\), noting the coefficient of \(cs^3\), we obtain \(\sigma = \tau\), a contradiction. Thus when \(ab \neq 0, \sigma = \tau \neq 3, 1/3\), occurs. \(\Box\)

8. Proof of Proposition 13.6 of [3]

In the proof of Proposition 13.6 in [3], the exclusion of \(\dim K = 4\) or \(\dim K = 2\) fails in Lemma 13.9, where we use an incorrect result in [3, §8]. In both cases (A) \(\dim E(c) = 6\) and (B) \(\dim E(c) = 4\) and \(\dim E = 6\), we give a correct proof here.

First, we remark that Case (II) is excluded in [3, Prop. 14.1] independent of the other argument, and the proof is also applicable to \(E\) when (B) occurs. Therefore, we may consider only the cases (0) and (I).
We emphasize \( \alpha \beta \neq 0 \) in Case (0). In this case, \( W(c) \) (Case (A)), or \( W(\text{Case (B)}) \) is contained in the space spanned by vectors given by \([3, (92)]\), where \( \sigma = \tau, \alpha = \gamma, \beta = \delta \) by Proposition 7.1:

\[
Z_1 = \frac{1}{\sqrt{3}}(3\alpha(e_1 - e_5) + \sqrt{3}(e_2 - e_4)), \quad Z_2 = \beta(e_1 + e_5) - \alpha(e_2 + e_4),
\]
\[
Z_1 = \frac{1}{\sqrt{3}}(3\alpha(e_1 - e_5) + \sqrt{3}(e_2 - e_4)), \quad Z_2 = \beta(e_1 + e_5) - \alpha(e_2 + e_4).
\]

Here \( Z_2, Z_2 \) are parallel along \( c \) ([3, Prop. 13.4]).


**Proposition 8.1.** When Case (0) occurs, Case (A) is impossible.

**Proof.** Suppose Case (0) and Case (A) occur. We restate the argument in the beginning of §13.3 [3]. Since \( \dim W(c) = 4 \), denoting by \( Z_2 \) the orthogonal complement of \( Z_2 \) in \( W(c) \), we know \( \dim (Z_2 \cap W(t)) = 3 + 2 - 4 \geq 1 \). Thus we can choose \( e_3(t) \) so that \( \nabla e_3 e_6(t) \in Z_2^\perp \) for all \( t \). Then \( K = \text{span}\{e_3(t)\} \) is mapped into \( Z_2^\perp \) by \( L(t) \), and so \( \dim K \leq 5 \). As we know \( \dim K \neq 3, 5 \) by the first part of Lemma 13.9, and by Lemma 13.10 of [3], which are correct, we may consider the case \( \dim K = 4 \) or 2.

When \( \dim K = 4 \), \( L(t)K = \text{span}\{Z_1(t), Z_1(t), Z_2\} \) for each \( t \). Thus \( K \) contains \( e_3(t), X_1(t), X_1(t), X_2(t), \) which implies that

\[
K = \text{span}\{e_3(t), X_1(t), X_1(t), X_2(t)\}
\]

for each \( t \). Then the orthogonal complement of \( K \) in \( E(c) \) is given by \( K^\perp = \text{span}\{e_3(t), X_3(t)\} \) for each \( t \), which is parallel along \( c \). Thus using a frame at \( p \), we may express \( K = \text{span}\{e_3, X_1, X_2, X_1\} \) and \( K^\perp = \text{span}\{e_3(t)\} = \text{span}\{e_3, X_2\} \).

Since \( Z_2 \) and \( Z_2 \) are constant along \( c \), \( Z_2 = \cos s Z_2 + \sin s Z_2 \) is constant along \( c \). Apply the above argument to \( Z_s \) for \( s \neq \pi/2 \) modulo \( \pi \). Namely, if we take \( e_3(t) \) along \( c \) so that \( \nabla e_3 e_6(t) \) is orthogonal to \( Z_s \), the space \( K^s = \text{span}\{e_3^s(t)\} \) is of dimension 4 or 2. If \( \dim K^s = 4 \), then \( e_3^s(t) \) which is orthogonal to \( e_3^s(t) \) spans the 2-dimensional space \( (K^s)^\perp = \{e_3^s, X_2^s\} \), where \( X_2^s = \cos s X_2 + \sin s X_2 \). Since \( e_3(t) \) and \( e_3^s(t) \) are independent because so are \( \nabla e_3 e_6(t) \) and \( \nabla e_3 e_6(t) \), we obtain

\[
D_3(t) = \text{span}\{e_3(t), e_3^s(t)\} \subset \{e_3, e_3^s, X_2, X_2^s\},
\]

which implies \( \dim E(c) = 4 \) because of (1), a contradiction. Thus \( \dim K^s = 2 \), but again in this case, \( e_3(t) \) and \( e_3^s(t) \) are independent, and we have

\[
D_3(t) = \text{span}\{e_3(t), e_3^s(t)\} \subset \{e_3, e_3^s, X_2, X_2^s\},
\]

where \( X_2^s = -\sin s X_2 + \cos s X_2 \), which contradicts \( \dim E(c) = 6 \). The case \( \dim K = 2 \) is similarly excluded. \( \square \)
8.2. Case (B).

Proposition 8.2. When (B) occurs, Case (0) is impossible. Hence Case (0) never occurs.

Proof. When (B) is the case, Lemma 6.1 implies that $E = E(c) + E(\bar{c})$ is of dimension 6 and $W = W(c) + W(\bar{c})$ is of dimension 4, where $\bar{c}$ is a geodesic orthogonal to $c$ at $p$. In fact, this is true for generic $\bar{c}$ transversal to $c$.

By [3, Prop. 13.4] applied to $W$, $Z_2$, $Z_2$ are constant. Also by Lemma 4.1, we may consider that $K = \text{span}\{e_3(t)\}$ and $\bar{K} = \text{span}\{e_3(t)\}$ are 2-dimensional, and $Z_2 = \nabla_{e_3} e_6(t)/\nabla_{e_3} e_6(t)$, $Z_2 = \nabla_{e_3} e_6(t)/\nabla_{e_3} e_6(t)$ hold. Thus we obtain

$$W(c) = \text{span}\{Z_2, Z_2\}. \quad (20)$$

As we assume Case (0) for generic geodesic $c^s$ in the direction $e_6^s = \cos s e_6 + \sin s e_6$, there exist $Z_2^s, Z_2^s$ constant along $c^s$ and $W(c^s) = \text{span}\{Z_2^s, Z_2^s\}$. Note that these $Z_2^s, Z_2^s$ are different from those in the last subsection (which was along $c$). Since $W(c^s) \subset W = \{Z_1, Z_2, Z_1, Z_2\}$, we may express

$$Z_2^s = \beta^s(e_1^s + e_5^s) - \alpha^s(e_2^s + e_4^s) = x^s Z_1 + y^s Z_2 + z^s Z_1 + w^s Z_2, \quad (21)$$

$$Z_2^s = \beta^s(e_1^s + e_5^s) - \alpha^s(e_2^s + e_4^s) = \bar{x}^s Z_1 + \bar{y}^s Z_2 + \bar{z}^s Z_1 + \bar{w}^s Z_2$$

for some $e_i^s \in D_i(p)$ and $\alpha^s, \beta^s$. As their $D_1$ component and $D_5$ component have the same length, we obtain

$$\left( x^s \frac{\sqrt{3}\alpha}{\sqrt{\sigma}} + y^s \beta \right)^2 + \left( z^s \frac{\sqrt{3}\alpha}{\sqrt{\sigma}} + w^s \beta \right)^2 = \left( -x^s \frac{\sqrt{3}\alpha}{\sqrt{\sigma}} + y^s \beta \right)^2 + \left( -z^s \frac{\sqrt{3}\alpha}{\sqrt{\sigma}} + w^s \beta \right)^2$$

for each $s$, and a similar formula holds for $\bar{x}^s$ etc. Here, $\sigma = 2\left(3\alpha^2 + \beta^2/3\right)$ as in [3, (99)]. From this and $\alpha\beta \neq 0$, it follows

$$x^s y^s + z^s w^s = 0, \quad \bar{x}^s \bar{y}^s + \bar{z}^s \bar{w}^s = 0.$$  

Rotating $Z_2^s, Z_2^s$ in $W(c^s)$, we may assume $\bar{y}^s \equiv 0$ for each $s$. Moreover, since $e_6^s = \cos s e_6 + \sin s e_6$ is odd in $s$, $y^s = \langle \nabla_{e_3} e_6^s, Z_2 \rangle$ is odd in $s$. Hence there exists some $s_0$ such that $\bar{y}^s = 0$, and we have

$$z^{s_0} w^{s_0} = 0 \quad \text{and} \quad \bar{z}^{s_0} \bar{w}^{s_0} = 0. \quad (22)$$

Lemma 8.3. Under the above assumption, $W(c^{s_0}) = \text{span}\{Z_1, Z_1\}$ holds.

Proof. For the moment, we omit $s_0$ in (22). We have four cases. The case $z = \bar{z} = 0$ causes $W(c^{s_0}) = \text{span}\{Z_1, Z_2\}$, which is impossible in view of (21) (see also (19)). Next, when $w = \bar{w} = 0$ holds, the conclusion follows. When
Observe that $\bar{q}$ is parallel along $D$. Thus putting $H = \{Z_2, Z_3\}$, we obtain $W(c_i) = H$ for any $q \in c$.

Now, let $c_1 = e^{q_i}$ and $c_2 = e^{q_i}$ for any $q \in c$, $p \neq \pm q$. Note that $W(c_1) = H = W(c_2)$. For $x \in c_1 \cap c_2$, we can express $E(c_i) = D_3(x) \oplus J_i$ for some 2-dimensional $J_i$ perpendicular to $D_3(x)$, $i = 1, 2$, which are mapped by $B_{n_0}$ onto $H$. Hence, $J_1 = J_2$, and so $E(c_1) = E(c_2)$ holds. Next, for any geodesic $\gamma$ transversal to $c_1$ and $c_2$, take $x_i \in \gamma \cap c_i$. Then $\dim E(\gamma) = 4$ implies $E(\gamma) = D_3(x_1) + D_3(x_2) \subset E(c_1) + E(c_2) = E(c_1)$. Thus we obtain $E(\gamma) = E(c_1)$. Since any point $y \in L_6(p)$ lies on some geodesic transversal to $c_1$ and $c_2$ unless $y$ lies on $c_1$ or $c_2$, $D_3(y) \subset E(c_1)$ always holds. Hence $E = E(c_1)$ and $\dim E = 4$ follows, which contradicts Proposition 5.3. 

By this proposition and by the remark in the beginning of this section, only Case (I) is possible on both $M_{\pm}$, which is excluded in [3, Prop. 14.4]. Note that the argument is available to both cases (A) and (B). Thus we obtain

**Theorem 8.4.** The focal submanifolds of an isoparametric hypersurface with $(g, m) = (6, 2)$ have the shape operators $B_n$ whose kernel does not depend on $n$.

This proves Theorem 1.1 by the argument in Section 15 of [3].

**References**


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