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# Perverse sheaves over real hyperplane arrangements

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## Abstract

Let $\mathcal{H}$ be an arrangement of real hyperplanes in $\mathbb{R}^n$. The complexification of $\mathcal{H}$ defines a natural stratification of $\mathbb{C}^n$. We denote by $\mathrm{Perv}(\mathbb{C}^n, \mathcal{H})$ the category of perverse sheaves on $\mathbb{C}^n$ smooth with respect to this stratification. We give a description of $\mathrm{Perv}(\mathbb{C}^n, \mathcal{H})$ as the category of representations of an explicit quiver with relations, whose vertices correspond to real faces of $\mathcal{H}$ (of all dimensions). The relations are of monomial nature: they identify some pairs of paths in the quiver. They can be formulated in terms of the oriented matroid associated to $\mathcal{H}$.

## Contents



## 0. Introduction

0.1. Let $X$ be a smooth complex algebraic variety and $\mathcal{X} = (X_\alpha)$ be a complex algebraic Whitney stratification of $X$. Fix a base field $\mathbf{k}$. One then has the abelian category $\mathrm{Perv}(X, \mathcal{X})$ of $\mathcal{X}$-smooth perverse sheaves of $\mathbf{k}$-vector spaces on $X$; see [2], [25]. Understanding this category is one of the central problems of topology of algebraic varieties. In many applications it is important to have an explicit description of $\mathrm{Perv}(X, \mathcal{X})$ as the category of $\mathbf{k}$-representations of some quiver with relations. General microlocal methods [1], [29], [18] provide a way to approach such a description in principle: an object $\mathcal{F} \in \mathrm{Perv}(X, \mathcal{X})$ gives rise, for each $\alpha$, to a "local system of Morse data" $\mathcal{L}_\alpha$ on some open part of the conormal bundle $T^*_{X_\alpha} X$, and the strategy is then to try to glue the categories formed by the $\mathcal{L}_\alpha$ into a single abelian category by an inductive procedure, adding one stratum at a time. However, there are only a very few higher-dimensional examples where a complete description has been obtained: normal crossings [15], Grassmannians with the Schubert stratification [7], rectangular matrices with the rank stratification [9].

0.2. In this paper we consider the case when $X = \mathbb{C}^n$ and $\mathcal{X}$ is given by an arrangement $\mathcal{H}_\mathbb{C}$ of linear hyperplanes in $X$ with real equations (so $\mathcal{H}_\mathbb{C}$ is the complexification of an arrangement $\mathcal{H}$ of hyperplanes in $\mathbb{R}^n$). We denote the corresponding category $\mathrm{Perv}(\mathbb{C}^n, \mathcal{H})$ and give a complete, combinatorial description of it in terms of the following data:

(1) The poset (partially ordered set) $(\mathcal{C}, \leq)$ of *faces*, i.e., convex locally closed subsets of all dimensions into which $\mathcal{H}$ stratifies $\mathbb{R}^n$. The order $\leq$ is given by inclusion of the closures of the faces; see Section 2.A.

(2) The concept of *collinearity*. We call an ordered triple of three faces $(A, B, C)$ collinear if there are points $a \in A, b \in B, c \in C$ such that $b$ lies in the straight line segment $[a, c]$.

Here the order is important. For example, a triple $(A, A, B)$ is always collinear, whereas $(A, B, A)$ is not if $B \neq A$.

Note that these data can be recovered from the *oriented matroid* associated to $\mathcal{H}$; see [5] and Proposition 7.4 below. See also (0.5) below.

Let us denote by $\mathrm{Rep}^{(2)}(\mathcal{C})$ the category formed by *double representations* of $\mathcal{C}$, i.e., of diagrams consisting of finite-dimensional $\mathbf{k}$-vector spaces $E_C, C \in \mathcal{C}$ and linear operators

$$\gamma_{C'C} : E_{C'} \longrightarrow E_C, \;\; \delta_{CC'} : E_C \longrightarrow E_{C'}, \;\; C' \leq C,$$

such that the $\gamma_{C'C}$ form a representation of $(\mathcal{C}, \leq)$, and the $\delta_{CC'}$ form a representation of the opposite poset $(\mathcal{C}, \geq)$, in $\mathbf{k}$-vector spaces. Our main result, Theorem 8.1, is that $\mathrm{Perv}(\mathbb{C}^n, \mathcal{H})$ is equivalent to the full subcategory in $\mathrm{Rep}^{(2)}(\mathcal{C})$ consisting of representations satisfying the following three conditions:

MONOTONICITY: *For any $C' \leq C$, we have $\gamma_{C'C}\delta_{CC'} = \mathrm{Id}_{E_C}$.*

This allows us to define, for any $A, B \in \mathcal{C}$, the transition map

$$\phi_{AB} = \gamma_{CB}\delta_{AC} : E_A \longrightarrow E_B,$$

where $C$ is any cell $\leq A, B$. For example, $\phi_{AA} = \mathrm{Id}_{E_A}$.

TRANSITIVITY: *If $(A, B, C)$ is a collinear triple of faces, then $\phi_{AC} = \phi_{BC}\phi_{AB}$.*

Note that this relation does *not* imply that $\phi_{AB}\phi_{BA} = \mathrm{Id}_{\mathrm{E_B}}$.

INVERTIBILITY: *If $C_1, C_2$ are faces of the same dimension $d$, lying in the same $d$-dimensional subspace, on the opposite sides of a $(d-1)$-dimensional face $D$, then $\phi_{C_1C_2}$ is an isomorphism.*

Let $L \subset X$ be *a flat*, i.e., an intersection of some hyperplanes from $\mathcal{H}_{\mathbb{C}}$, of dimension $d$. The last two conditions mean that the spaces $E_C$ and the maps $\phi_{CC'}$ where $C, C'$ run through all $d$-dimensional faces inside $L$, form a local system over $L^\circ = L \setminus \cup_{H\in\mathcal{H}_{\mathbb{C}}} H$. This follows from a description of the fundamental groupoid of $L^\circ$ given in Proposition 9.11.

0.3. The type of description of $\mathrm{Perv}(\mathbb{C}^n, \mathcal{H})$ that we obtain is quite different from those appearing in most of the earlier approaches. More precisely,

(a) It is of *monomial nature*: the conditions on the maps $\gamma_{C'C}, \delta_{CC'}$ do not appeal to the operations of addition or multiplication by scalars but only to composition of maps. For comparison, in the most classical case of $X = \mathbb{C}$ stratified by $\{0\}$ and $\mathbb{C} \setminus \{0\}$, the standard description [1], [15] is in terms of diagrams $\{\Phi \underset{u}{\overset{v}{\rightleftarrows}} \Psi\}$ such that $\mathrm{Id}_\Psi + vu$ is invertible, so it is not monomial. (See Section 9 for the comparison of the two descriptions in this case.) Note that with a monomial description one has the means to define what should be a "*perverse sheaf of sets*," or a "*perverse stack of categories*" on $\mathbb{C}^n$ smooth with respect to $\mathcal{H}_{\mathbb{C}}$; cf. [22].

(b) Since the strata $X_\alpha$ are, in our situation, generic parts of the complex flats $L_{\mathbb{C}}$ of $\mathcal{H}$; see Section 2.D, the local systems $\mathcal{L}_\alpha$ of Morse data in the standard approach are defined on some open parts of $T^*_{L_{\mathbb{C}}}\mathbb{C}^n$. Our linear algebra data provide maps ($\phi_{C_1,C_2}$ in the invertibility condition) that can be related, in the sense outlined in (d) below, to *half-monodromies* of appropriate $\mathcal{L}_\alpha$ corresponding to paths joining neighboring cells and going around the wall in the complex domain. This is different from a more straightforward approach when a local system is described by its monodromies corresponding to closed loops.

It is a known phenomenon in the theory of quantum groups that a typical monodromy matrix $M$ of the Knizhnik-Zamolodchikov equation has quite complicated matrix elements, whereas the two half-monodromies $M_+$ and $M_-$ of which it is composed via $M = M_+M_-$ are of much simpler (monomial) nature; cf. [36].

(c) From the purely topological point of view, our approach emphasizes not the fundamental groups but the *fundamental groupoids* of the complex strata, with as many base points as there are different faces in the same stratum. The simpler nature of relations is achieved therefore by introducing a certain redundancy in our description: to each complex stratum we associate not a single vector space (as in the standard approaches) but several isomorphic spaces $E_C$

corresponding to different cells $C$ in the stratum. Our approach can thus be seen as a natural development of the work of Salvetti [33], Gelfand-Rybnikov [16] and Björner-Ziegler [6] who studied the topology of complexified arrangements by real methods and formulated the results in terms of oriented matroids.

(d) The spaces $E_C$ appearing in our description are not obtained as the stalks of the local systems $\mathcal{L}_\alpha$ at appropriate points (as in most of the standard approaches). The indexing sets for the $E_C$ and the $\mathcal{L}_\alpha$ are already quite different. Instead, $E_C$ can be identified, noncanonically, with direct sums of several such stalks (involving several different $\mathcal{L}_\alpha$). This can be surmised already from the monotonicity condition that implies $\dim(E_{C'}) \geq \dim(E_C)$ for $C' \leq C$. In particular, the "biggest" space $E_0$ has the dimension equal to the sum of the ranks of all the local systems corresponding to all the strata. In our approach, $E_0$ can be identified with the *space of hyperfunction solutions* of the holonomic $\mathcal{D}$-module $\mathcal{M}$ corresponding to the perverse sheaf. More generally, for any face $C$, the space $E_C$ is the stalk, at any point $c \in C$, of the *sheaf of hyperfunction solutions* of $\mathcal{M}$, i.e., the space of such solutions defined in a small neighborhood of $c$ in $\mathbb{R}^n$.

The study of the spaces of hyperfunction solutions in 1 dimension (i.e., for $\mathbb{C}$ stratified by $\{0\}$ and $\mathbb{C} \setminus \{0\}$) goes back to the very origins of the theory of $\mathcal{D}$-modules as presented in Kashiwara's 1971 Master Thesis [24, Th. 4.2.7] and to the paper of Komatsu [27] from the same year. These works identify the dimension of the space $E_0$ in this case with what in the "standard" (much later) description would be denoted by $\dim(\Phi \oplus \Psi)$. Higher-dimensional generalizations were found in the papers of Takeuchi [39] and Schürmann [38] of which the first considers precisely the situation of the complexification of a real arrangement. Spaces that turn out to be identical with our $E_C$ have also appeared in the work of Bezrukavnikov, Finkelberg and one of the authors, [3], under the name "generalized vanishing cycles." In fact, it was conjectured in [3, p. 50] that a perverse sheaf $\mathcal{F}$ can be uniquely reconstructed from the linear algebra data equivalent to our $(E_C, \gamma_{C'C}, \delta_{CC'})$. From this point of view, we not only prove the conjecture of [3] but find an explicit characterization of the linear algebra data that can appear.

0.4. Our method is closest to that of Galligo, Granger and Maisonobe [15] (who attribute the original idea to Malgrange). That is, we construct a version of Cousin resolution of a perverse sheaf $\mathcal{F} \in \mathrm{Perv}(\mathbb{C}^n, \mathcal{H})$ using a stratification of $\mathbb{C}^n$ into "tube cells" $C + i\mathbb{R}^n$, $C \in \mathcal{C}$, so the terms of the resolution are direct sums of the sheaves of cohomology with supports in such tubes. It turns out that for each $C$, only the sheaf $\mathcal{E}_C = \underline{\mathbb{H}}^{\mathrm{codim}(C)}_{C+i\mathbb{R}^n}(\mathcal{F})$ is nonzero, and $E_C$ can be identified with the space of global sections of this sheaf. The main technical point of our study is that the entire Cousin complex $\mathcal{E}^\bullet$ formed by the $\mathcal{E}_C$ can be recovered from linear algebra data represented by the $E_C, \gamma_{C'C}$ and $\delta_{CC'}$ and that $\mathcal{H}_{\mathbb{C}}$-smoothness and perversity of $\mathcal{E}^\bullet \simeq \mathcal{F}$ are precisely equivalent to the three conditions above.

0.5. The concept of collinearity of a triple of faces contains, as a particular case, the familiar condition

$$l(w'w'') = l(w') + l(w''), \quad w', w'' \in W. \tag{$*$}$$

Here $W$ is the Weyl group of a root system $(\mathfrak{h}, \Delta)$. In this case we have the arrangement $\mathcal{H} = \{\alpha^\perp\}_{\alpha\in\Delta}$ of root hyperplanes in $\mathfrak{h}$. Chambers (open faces) of $\mathcal{H}$ form a $W$-torsor, so for any two chambers $A, B$, there is a unique $w_{AB} \in W$ such that $w_{AB} \cdot A = B$. A triple of chambers $(A, B, C)$ is collinear if and only if $w' = w_{BC}$ and $w'' = w_{AB}$ satisfy $(*)$. The transitivity property $\phi_{AC} = \phi_{BC}\phi_{AB}$ of our double representations is thus reminiscent of the classical Gindikin-Karpelevich factorization formula (cf. [20, Th.1]) and of the cocycle property of the principal series intertwiners [26], [37], [31] (which is another manifestation of that formula).

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## 1. **Generalities**

1.A. *Postnikov systems.* Let $\mathcal{D}$ be a triangulated category. A *left Postnikov system* in $\mathcal{D}$ is a diagram of exact triangles of the form

$$\begin{tikzcd}[column sep=small]
A = A_{01\cdots n} \arrow[dr, "\alpha_0"'] & & A_{1\cdots n} \arrow[ll] \arrow[dr, "\alpha_1"'] & & A_{2\cdots n} \arrow[ll] \arrow[dr, "\alpha_2"'] & & \cdots \arrow[ll] & & A_{n-1,n} \arrow[ll] \arrow[dr, "\alpha_{n-1}"'] \arrow[rr, leftrightarrow] & & A_n \arrow[dr, "\alpha_n"'] & & 0 \arrow[ll] \\
& A_0 \arrow[ur, "+1", "\beta_0"'] & & A_1 \arrow[ur, "+1", "\beta_1"'] & & \cdots & & \cdots \arrow[ur, "+1"] & & A_{n-1} \arrow[ur, "+1", "\beta_{n-1}"'] & & A_n. \arrow[ur] &
\end{tikzcd}$$

It gives rise to a sequence of objects and morphisms in $\mathcal{D}$

$$A_0 \xrightarrow{\delta_0} A_1[1] \xrightarrow{\delta_1} A_2[2] \xrightarrow{\delta_2} \cdots \xrightarrow{\delta_{n-1}} A_n[n], \quad \delta_i = \alpha_{i+1}\beta_i. \tag{1.1}$$

Similarly, a *right Postnikov system* in $\mathcal{D}$ is a diagram of exact triangles of the form

$$\begin{tikzcd}
A = A_{01\cdots n} \arrow[rr] & & A_{0\cdots n-1} \arrow[rr] \arrow[dl, "+1/\beta'_n"] & & A_{0\cdots n-2} \arrow[rr] \arrow[dl, "+1/\beta'_{n-1}"] & & \cdots \arrow[rr] & & A_{01} \arrow[rr] \arrow[dl, "+1"] & & A_0 \arrow[rr] \arrow[dl, "+1/\beta'_1"] & & 0 \arrow[dl] \\
& A_n \arrow[ul, "\alpha'_n"] & & A_{n-1} \arrow[ul, "\alpha'_{n-1}"] & & \cdots \arrow[ul, "\alpha'_{n-2}"] & & \cdots & & A_1 \arrow[ul, "\alpha'_1"] & & A_0. \arrow[ul, "\alpha'_0"] &
\end{tikzcd}$$

It gives rise to a sequence of objects and morphisms in $\mathcal{D}$

$$A_0 \xrightarrow{\delta'_0} A_1[1] \xrightarrow{\delta'_1} A_2[2] \xrightarrow{\delta'_2} \cdots \xrightarrow{\delta'_{n-1}} A_n[n], \quad \delta'_i = \beta'_{i+1}\alpha'_i. \tag{1.2}$$

PROPOSITION 1.3. *Suppose we have a left resp. right Postnikov system in $\mathcal{D}$. Then*

(a) *The sequence* (1.1) *resp.* (1.2) *is a complex in $\mathcal{D}$, i.e., $\delta_{i+1}\delta_i = 0$, resp. $\delta'_{i+1}\delta'_i = 0$.*

(b) *Suppose, in addition, that $\mathcal{D} = D^b(\mathcal{A})$ is the bounded derived category of an abelian category $\mathcal{A}$, and assume that each $A_i[i]$ is quasi-isomorphic to an object $B_i \in \mathcal{A}$, so $B_i = H^i(A_i)$ is the only cohomology object of $A_i$. Then the complex in $\mathcal{A}$*

$$B_0 \longrightarrow B_1 \longrightarrow \cdots \longrightarrow B_n$$

*induced from* (1.1) *resp.* (1.2) *by passing to $H^0$ is an object of $D^b(\mathcal{A}) = \mathcal{D}$, isomorphic to $A$.*

*Proof.* (a) follows because the composition of two consecutive morphisms in an exact triangle is equal to 0. Part (b) is proved by induction on the length of the Postnikov system. □

*Remark* 1.4. In general, the left or right Postnikov system as above exhibits $A$ as a total object of the complex (1.1) or (1.2); see [19, Ch. 4, §2].

1.B. *Filtered topological spaces.* By a *space* we mean a topological space homeomorphic to an open subset of a finite CW-complex. We fix a base field $\mathbf{k}$. For a space $X$, we denote by $\mathrm{Sh}_X$ the category of sheaves of $\mathbf{k}$-vector spaces on $X$ and by $D^b\,\mathrm{Sh}_X$ the corresponding bounded derived category. For any map $f : X \to Y$ of spaces, we have the standard functors $f^*, f^! : D^b\,\mathrm{Sh}_Y \to D^b\,\mathrm{Sh}_X$ and $f_*, f_! : D^b\,\mathrm{Sh}_X \to D^b\,\mathrm{Sh}_Y$. In particular, we reserve the notation $f_*$ for the derived direct image functor and denote the usual direct image functor on sheaves by $R^0 f_*$. If $j : Y \hookrightarrow X$ is an embedding of a locally closed subspace, then we denote by

$$\underline{R\Gamma}_Y(\mathcal{F}) = j_* j^! \mathcal{F}$$

the complex of "cohomology with support in $Y$."

If $V$ is a $\mathbf{k}$-vector space, then we denote by $\underline{V}_X$ the constant sheaf on $X$ with stalk $V$. If $i : Z \hookrightarrow X$ is the embedding of a closed subspace, then, by a slight abuse of notation, we consider $\underline{V}_Z$ as a sheaf on $X$ via the direct image functor $i_*$.

Let $X$ be a space and

$$\mathcal{X} = \left\{X_0 \subset X_1 \subset \cdots \subset X_n = X\right\}$$

be a filtration of $X$ by closed subspaces. We then have the locally closed subspaces $Y_d = X_d \setminus X_{d-1}$ and denote by

$$i_d : X_d \hookrightarrow X, \quad j_d : Y_d \hookrightarrow X$$

the embeddings.

Any complex of sheaves $\mathcal{F} \in D^b \operatorname{Sh}_X$ includes into two canonical Postnikov systems, which we call the *cohomological* (right) and *homological* (left) Postnikov systems of $\mathcal{F}$ relative to the filtration $\mathcal{X}$. The cohomological system has the form

$$\begin{tikzcd}[column sep=tiny]
\mathcal{F} = i_{n!}i_n^*\mathcal{F} \arrow[rr] & & (i_{n-1})_! i_{n-1}^*\mathcal{F} \arrow[rr] \arrow[dl, "+1"'] & & (i_{n-2})_! i_{n-2}^*\mathcal{F} \arrow[rr] \arrow[dl, "+1"'] & & \cdots \arrow[rr] & & i_{1!}i_1^*\mathcal{F} \arrow[rr] \arrow[dl, "+1"'] & & i_{0!}i_0^*\mathcal{F} \arrow[rr] \arrow[dl, "+1"'] & & 0 \arrow[dl] \\
& j_{n!}j_n^*\mathcal{F} \arrow[ul] & & (j_{n-1})_! j_{n-1}^*\mathcal{F} \arrow[ul] & & \cdots \arrow[ul] & & \cdots & & j_{1!}j_1^*\mathcal{F} \arrow[ul] & & j_{0!}j_0^*\mathcal{F} \arrow[ul] &
\end{tikzcd}$$

and gives rise to the complex in $D^b \operatorname{Sh}_X$

$$j_{0!}j_0^*\mathcal{F} \longrightarrow j_{1!}j_1^*\mathcal{F}[1] \longrightarrow \cdots \longrightarrow j_{n!}j_n^*\mathcal{F}[n]. \tag{1.5}$$

If all the $j_{\nu!}j_\nu^*\mathcal{F}[\nu]$ are quasi-isomorphic to single sheaves in degree 0, this complex has total object $\mathcal{F}$. The homological system has the form

$$\begin{tikzcd}[column sep=tiny]
\mathcal{F} = i_{n*}i_n^!\mathcal{F} \arrow[dr] & & (i_{n-1})_* i_{n-1}^!\mathcal{F} \arrow[ll] \arrow[dr] & & (i_{n-2})_* i_{n-2}^!\mathcal{F} \arrow[ll] \arrow[dr] & & \cdots \arrow[ll] & & i_{1*}i_1^!\mathcal{F} \arrow[ll] \arrow[dr] & & i_{0*}i_0^!\mathcal{F} \arrow[ll, leftrightarrow] \arrow[dr] & & 0 \arrow[ll] \\
& j_{n*}j_n^!\mathcal{F} \arrow[ur, "+1"] & & (j_{n-1})_* j_{n-1}^!\mathcal{F} \arrow[ur, "+1"] & & \cdots & & \cdots \arrow[ur, "+1"] & & j_{1*}j_1^!\mathcal{F} \arrow[ur, "+1"] & & j_{0*}j_0^!\mathcal{F} \arrow[ur] &
\end{tikzcd}$$

and gives rise to a complex in $D^b \operatorname{Sh}_X$

$$j_{n*}j_n^!\mathcal{F} \longrightarrow (j_{n-1})_*j_{n-1}^!\mathcal{F}[1] \longrightarrow \cdots \longrightarrow j_{0*}j_0^!\mathcal{F}[n]. \tag{1.6}$$

If all the $(j_{n-\nu})_*j_{n-\nu}^!\mathcal{F}[\nu]$ are quasi-isomorphic to single sheaves of degree 0, this complex has total object $\mathcal{F}$.

1.C. *Verdier duality.* For a space $X$, we denote by

$$\mathbb{D} : D^b \operatorname{Sh}_X \longrightarrow D^b \operatorname{Sh}_X$$

the Verdier duality functor; see [25, 43]. We recall that $\mathbb{D}$ interchanges the functors $f_*$ and $f_!$, as well as the functors $f^*$ and $f^!$ for any continuous map $f$ of spaces.

Let $X$ be a real analytic manifold of dimension $d$. We denote by $\mathrm{or}_X$ the orientation local system on $X$. This is the rank 1 local system of $\mathbf{k}$-vector spaces whose stalk at $x \in X$ is $H^d_c(U, \mathbf{k})$, where $U$ is any open neighborhood of $X$ homeomorphic to a $d$-ball. In this case

$$\mathbb{D}(\mathcal{F}) = \underline{R\operatorname{Hom}}(\mathcal{F}, \mathrm{or}_X[d]),$$

and we denote by

$$\mathcal{F}^\bigstar = \mathbb{D}(\mathcal{F})[-d] = \underline{R\operatorname{Hom}}(\mathcal{F}, \mathrm{or}_X) \tag{1.7}$$

the shifted Verdier duality that has the advantage of preserving local systems in degree 0.

We further denote by $D^b_{\mathrm{constr}} \operatorname{Sh}_X \subset D^b \operatorname{Sh}_X$ the subcategory formed by $\mathbb{R}$-constructible complexes, i.e., by complexes whose cohomology sheaves are $\mathbb{R}$-constructible [25]. The functor $\mathbb{D}$ preserves the subcategory $D^b_{\mathrm{constr}} \operatorname{Sh}_X$ and its restriction there is a perfect duality. The same is true for $\bigstar$.

If $\mathcal{X}$ is a filtration of $X$ by real analytic subsets, then $\mathbb{D}$ takes the cohomological Postnikov system for $\mathcal{F}$ to the homological Postnikov system for $\mathbb{D}(\mathcal{F})$ for any $\mathbb{R}$-constructible complex $\mathcal{F}$.

1.D. *Cellular sheaves and complexes.* For background on cellular spaces and sheaves, we refer the reader to [11], [32], [44], [45]. Here we present a (self-contained) synopsis of features needed in the rest of the paper.

By a *$d$-cell* we mean a topological space homeomorphic to an open $d$-ball. For a $d$-cell $\sigma$, we denote by

$$\mathrm{or}(\sigma) = H^d_c(\sigma, \mathbf{k})$$

its 1-dimensional orientation space. For constant sheaves on $\sigma$, the Verdier duality has the form

$$\mathbb{D}(\underline{V}_\sigma) = \underline{V^* \otimes_{\mathbf{k}} \mathrm{or}(\sigma)}_\sigma[d].$$

By a *cellular space* we will mean a space with a filtration $\mathcal{X}$ by closed subspaces such that each $X_d = X_d \setminus X_{d-1}$ is a disjoint union of finitely many $d$-cells. We denote by $\mathcal{C} = \mathcal{C}_\mathcal{X}$ the set of all cells of $X$ together with the partial order

$$\sigma' \leq \sigma \quad \Leftrightarrow \quad \sigma' \subset \overline{\sigma}.$$

By a slight abuse of language we will refer to the relation $\leq$ as *cell inclusion* although, set-theoretically, it is the closures of the cells that are included.

For a cell $\sigma$ of a cellular space $X$, we denote by $j_\sigma : \sigma \to X$ the corresponding embedding. A cellular space $X$ will be called *regular* if the closure of each $d$-cell is homeomorphic to a closed $d$-ball. A cellular space $X$ will be called *quasi-regular* if it can be represented as $X' - X''$, where $X'$ is a regular cellular space and $X''$ is a closed cellular subspace (union of some cells of $X'$) that is then also regular. In the sequel all cellular spaces will be assumed quasi-regular.

For any $\mathbf{k}$-vector space $V$, we have an identification in $D^b \operatorname{Sh}_X$:

$$\underline{V}_{\overline{\sigma}} = j_{\sigma *} \underline{V}_\sigma = R^0 j_{\sigma *} \underline{V}_\sigma.$$

A *cellular sheaf* on $X$ (with respect to $\mathcal{X}$) is, by definition, a sheaf $\mathcal{F}$ on $X$ such that each $j_\sigma^* \mathcal{F}$ is a constant sheaf on $\sigma$ with finite-dimensional stalks. We denote by $\operatorname{Sh}_{X,\mathcal{X}}$ the category of cellular sheaves on $X$ with respect to $\mathcal{X}$ and by $D^b_{\mathcal{X}}(\operatorname{Sh}_X) \subset D^b \operatorname{Sh}_X$ the full subcategory of complexes whose cohomology sheaves lie in $\operatorname{Sh}_{X,\mathcal{X}}$.

For a cellular sheaf $\mathcal{F}$ on $X$, we have the *linear algebra data* that consists of stalks $F_\sigma = H^0(\sigma, j_\sigma^* \mathcal{F})$ and *generalization maps* (terminology taken from [17])

$$\gamma_{\sigma'\sigma} : F_{\sigma'} \longrightarrow F_\sigma, \quad \sigma' \leq \sigma$$

defined as follows. Take a point $x' \in \sigma'$. Then $F_{\sigma'} = H^0(U', \mathcal{F})$, where $U'$ is a sufficiently small contractible neighborhood of $x'$ in $X$. Let $x \in U' \cap \sigma$. Then $F_\sigma = H^0(U, \mathcal{F})$, where $U$ is a sufficiently small contractible neighborhood of $x$ in $X$. Taking $U$ small enough, we can assume $U \subset U'$. Then $\gamma_{\sigma'\sigma}$ is given by the restriction map

$$F'_\sigma = H^0(U', \mathcal{F}) \xrightarrow{\text{Res}} H^0(U, \mathcal{F}) = F_\sigma.$$

The following is by now well known.

Proposition 1.8.

(a) *For $\mathcal{F} \in \operatorname{Sh}_{X,\mathcal{X}}$, the data $(F_\sigma, \gamma_{\sigma'\sigma})$ form a representation of the poset $(\mathcal{C}, \leq)$, i.e., a covariant functor $\mathcal{C} \to \operatorname{Vect}^{\text{fd}}_{\mathbf{k}}$.*
(b) *The construction in* (a) *defines an equivalence of* $\operatorname{Sh}_{X,\mathcal{X}}$ *with* $\operatorname{Rep}(\mathcal{C}) = \operatorname{Fun}(\mathcal{C}, \operatorname{Vect}^{\text{fd}}_{\mathbf{k}})$.
(c) *The natural functor*

$$\phi : D^b \operatorname{Rep}(\mathcal{C}) = D^b \operatorname{Sh}_{X,\mathcal{X}} \longrightarrow D^b_{\mathcal{X}}(\operatorname{Sh}_X)$$

*is an equivalence of triangulated categories.*

*Proof.* (a) and (b) are obvious; cf. the case of "simplicial complexes" considered in detail by Kashiwara [23]. In fact, the notion of "exit paths" invented by MacPherson allows one to formulate and prove analogs of these statements for arbitrary stratifications; see [41]. In our present case, the category of exit paths is equivalent to $(\mathcal{C}, \leq)$.

To see (c), note that the sheaf $j_{\sigma *}\underline{\mathbf{k}}$ corresponds via (b) to the injective object of $\mathrm{Rep}(\mathcal{C})$, i.e., to the covariant functor $R_\sigma : \tau \mapsto (\mathbf{k}\,\mathrm{Hom}_{\mathcal{C}}(\tau, \sigma))^*$ dual to the representable contravariant one. Therefore for $p \in \mathbb{Z}$,

$$\begin{aligned}
&\mathrm{Hom}_{D^b\,\mathrm{Sh}_{X,\mathcal{X}}}(j_{\sigma *}\underline{\mathbf{k}}, j_{\tau *}\underline{\mathbf{k}}[p]) = \mathrm{Ext}^p_{\mathrm{Sh}_{X,\mathcal{X}}}(j_{\sigma *}\underline{\mathbf{k}}, j_{\tau *}\underline{\mathbf{k}}) \\
&= \mathrm{Ext}^p_{\mathrm{Rep}(\mathcal{C})}(R_\sigma . R_\tau) = \begin{cases} \mathbf{k} & \text{if } p = 0,\ \sigma \geq \tau, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned} \tag{1.9}$$

On the other hand, adjointness between $j_\tau^*$ and $j_{\tau *}$ gives that

$$\mathrm{Hom}_{D^b\,\mathrm{Sh}_X}(j_{\sigma *}\underline{\mathbf{k}}, j_{\tau *}\underline{\mathbf{k}}[p]) = \mathrm{Hom}_{D^b\,\mathrm{Sh}_\tau}(j_\tau^* j_{\sigma *}\underline{\mathbf{k}}, \underline{\mathbf{k}}[p])$$

and so it is given by the right-hand side of (1.9) as well. This means that the morphism

$$\phi_{\mathcal{F},\mathcal{G}} : \mathrm{Hom}_{D^b\,\mathrm{Sh}_{X,\mathcal{X}}}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathrm{Hom}_{D^b\,\mathrm{Sh}_X}(\mathcal{F}, \mathcal{G})$$

is an isomorphism whenever both $\mathcal{F}$ and $\mathcal{G}$ are shifts of sheaves of the form $j_{\sigma *}\underline{\mathbf{k}}$, $\sigma \in \mathcal{C}$. Now, each object of $\mathrm{Sh}_{X,\mathcal{X}}$ has a finite resolution by direct sums of the injective objects $j_{\sigma *}\underline{\mathbf{k}}$. This means that $\phi_{\mathcal{F},\mathcal{G}}$ is an isomorphism for any $\mathcal{F}, \mathcal{G} \in D^b\,\mathrm{Sh}_{X,\mathcal{X}}$, i.e., that $\phi$ is fully faithful. To see that $\phi$ is essentially surjective, note that the cohomological Postnikov system of any object $\mathcal{F} \in D^b_{\mathcal{X}}(\mathrm{Sh}_X)$ shows that $\mathcal{F}$ lies in the smallest triangulated category containing all the $j_{\sigma *}\underline{\mathbf{k}}$. □

*Remark* 1.10. An object of $D^b\,\mathrm{Rep}(\mathcal{C})$ corresponding to $\mathcal{F} \in D^b_{\mathcal{X}}(\mathrm{Sh}_X)$ by (c) can be seen as "linear algebra data for $\mathcal{F}$ at the level of complexes," i.e., as a choice of actual complexes of vector spaces $F^\bullet_\sigma$ quasi-isomorphic to $R\Gamma(\sigma, j_\sigma^* \mathcal{F})$ and actual morphisms of complexes $\gamma_{\sigma'\sigma} : F^\bullet_{\sigma'} \to F^\bullet_\sigma$ forming a representation of $(\mathcal{C}, \leq)$ at the level of complexes.

PROPOSITION 1.11. *Let $X$ be a quasi-regular cellular space of dimension $n$ and $\mathcal{F}$ a cellular complex represented by $(F^\bullet_\sigma, \gamma_{\sigma'\sigma})$. Then the the complex $\mathbb{D}(\mathcal{F}[n])$ is quasi-isomorphic to the total complex of the double complex*

$$\bigoplus_{\mathrm{codim}(\sigma)=0} \underline{F^*_\sigma \otimes \mathrm{or}(\sigma)}_{\overline{\sigma}} \longrightarrow \bigoplus_{\mathrm{codim}(\sigma)=1} \underline{F^*_\sigma \otimes \mathrm{or}(\sigma)}_{\overline{\sigma}} \longrightarrow \cdots .$$

*Here each summand is a complex of constant sheaves on the closed cell.*

This is a combinatorial version of the Cousin complex.

*Proof.* First let $\mathcal{F}$ be a cellular sheaf (not a complex) on $X$, so the $F^\bullet_\sigma = F_\sigma$ are vector spaces. The cohomological Postnikov system for $\mathcal{F}$ relative to $\mathcal{X}$

gives a complex in $D^b \operatorname{Sh}_X$

$$\bigoplus_{\dim(\sigma)=0} j_{\sigma!}(\underline{F_\sigma}_\sigma) \longrightarrow \bigoplus_{\dim(\sigma)=1} j_{\sigma!}(\underline{F_\sigma}_\sigma)[1] \longrightarrow \cdots \tag{1.12}$$

whose total object is $\mathcal{F}$. Applying the exact functor $\mathbb{D}[-n]$ to (1.12), we get a complex in $D^b \operatorname{Sh}_X$ described in the proposition, with total object $\mathcal{F}$. By Proposition 1.3(b), this complex represents $\mathbb{D}(\mathcal{F}[n])$.

The general case follows by compatibility of $\mathbb{D}[-n]$ with forming total complexes of double complexes. □

*Remark* 1.13. If $\mathcal{F}$ is a cellular sheaf as above and $X$ is regular, then

$$R\Gamma(j_{\sigma!}\underline{F_\sigma}_\sigma) = R\Gamma_c(\sigma, F_\sigma) = F_\sigma \otimes \operatorname{or}(\sigma)[-\dim(\sigma)]$$

and so from (1.12), we get a complex in $D^b \operatorname{Vect}_{\mathbf{k}}^{\mathrm{fd}}$

$$\bigoplus_{\dim(\sigma)=0} F_\sigma \otimes \operatorname{or}(\sigma) \longrightarrow \bigoplus_{\dim(\sigma)=1} F_\sigma \otimes \operatorname{or}(\sigma) \longrightarrow \cdots$$

whose total object is $R\Gamma(X, \mathcal{F})$. By Proposition 1.3(b), this complex represents $R\Gamma(X, \mathcal{F})$. This is the "standard cellular cochain complex" of $\mathcal{F}$. Compare also with the well-known fact that the geometric realization of the simplicial nerve of $(\mathcal{C}, \leq)$ is, for a regular $X$, homeomorphic to $X$.

## 2. Background on real arrangements and the three stratifications

2.A. *Faces and sign vectors.* Let $V$ be a finite-dimensional real vector space and $\mathcal{H}$ be an arrangement of linear hyperplanes in $V$. Note that, in particular, $0 \in H$ for any hyperplane $H \in \mathcal{H}$. We choose, once and for all, a linear equation $f_H : V \to \mathbb{R}$ for each $H \in \mathcal{H}$. (The essential concepts that we define will not depend on the choice.)

By $\mathcal{L} = \mathcal{L}_\mathcal{H}$ we denote the poset of *flats* of $\mathcal{H}$, i.e., of linear subspaces of the form $\bigcap_{H \in \mathcal{I}} H$ for various subsets $\mathcal{I} \subset \mathcal{H}$. Note that $\mathcal{L}$ contains $V$ (for $\mathcal{I} = \emptyset$). We will assume that $\{0\} \in \mathcal{L}$. This can always be achieved by quotienting by the smallest flat of $\mathcal{H}$, without changing the combinatorial structure of the arrangement.

In the sequel we also assume $V = \mathbb{R}^n$ for simplicity. We denote by

$$\operatorname{sgn} : \mathbb{R} \longrightarrow \{+, -, 0\}$$

the standard sign function. By a *sign vector* we will mean a sequence $(s_H)_{H \in \mathcal{H}} \in \mathcal{H}^{\{+,-,0\}}$ assigning a "sign" to each element of $\mathcal{H}$. Each $x \in \mathbb{R}^n$ gives rise to a sign vector $(\operatorname{sgn} f_H(x))_{H \in \mathcal{H}}$. Level sets of this vector function subdivide $\mathbb{R}^n$ into locally closed subsets called *faces*. Thus $x$ and $x'$ lie in the same face, if and only if $\operatorname{sgn} f_H(x) = \operatorname{sgn} f_H(x')$ for each $H \in \mathcal{H}$. We will sometimes identify a face $C$ with the corresponding sign vector

$$C \leftrightarrow (C_H)_{H\in\mathcal{H}}, \quad C_H = \operatorname{sgn}(f_H|_C) \in \{+,-,0\}.$$

Faces are convex subsets of $\mathbb{R}^n$, each given by a system of linear equations and strict linear inequalities [3], [33], [35], [42]. Open faces will be called *chambers.* We denote by $\mathcal{C} = \mathcal{C}_{\mathcal{H}}$ the poset of faces, with the partial order given by $C' \leq C$, if $C' \subset \overline{C}$. Note that $\mathcal{C}$ does not depend on the choice of equations $f_H$, both as a poset and as a stratification. For $x \in \mathbb{R}^n$, we denote by $\sigma(x) = \sigma_{\mathcal{H}}(x) \in \mathcal{C}$ the face containing $x$. Similarly, if $X \subset \mathbb{R}^n$ is a subset contained in one face, we denote by $\sigma(X) = \sigma_{\mathcal{H}}(X)$ this face.

The faces form a quasi-regular cell decomposition of $\mathbb{R}^n$ that we also denote $\mathcal{C}$. (Taking the one-point compactification of $\mathbb{R}^n$ to a sphere, we embed $(\mathbb{R}^n, \mathcal{C})$ into a regular cellular space.)

We also write

$$C' <_1 C, \text{ if } C' \leq C \text{ and } \dim(C') = \dim(C) - 1$$

and call this relation *codimension* 1 *inclusion* of faces.

Alternatively, let us introduce a partial order $\leq$ on $\{+,-,0\}$ in which the only nontrivial inequalities are

$$0 \leq +, \quad 0 \leq - \tag{2.1}$$

(and $+$ and $-$ are incomparable). This is the order of inclusion of the subsets $\{x \geq 0\}$, $\{x \leq 0\}$ and $\{x = 0\}$ in $\mathbb{R}$. Further, introduce on $\mathcal{H}^{\{+,-,0\}}$ the Cartesian product partial order. Then

$$C \leq D \quad \Leftrightarrow \quad C_H \leq D_H \text{ for each } H \in \mathcal{H};$$

i.e., $\mathcal{C} \hookrightarrow \mathcal{H}^{\{+,-,0\}}$ is an embedding of posets.

2.B. *Functoriality of faces.* Any $L \in \mathcal{L}$ gives rise to two induced hyperplane arrangements:

- the arrangement $\mathcal{H} \cap L$ in $L$, formed by the $H \cap L$, $H \in \mathcal{H}$, $H \not\supset L$;
- the arrangement $\mathcal{H}/L$ in $V/L$, formed by the $H/L$, $H \in \mathcal{H}$, $H \supset L$.

Therefore we have the face stratifications $\mathcal{C}_{\mathcal{H}\cap L}$ and $\mathcal{C}_{\mathcal{H}/L}$ of these arrangements.

For any face $C \in \mathcal{C}$, we denote by $\mathbb{L}(C)$ the $\mathbb{R}$-linear subspace in $\mathbb{R}^n$ spanned by $C$ and denote by

$$\pi_C : \mathbb{R}^n \longrightarrow \mathbb{R}^n/\mathbb{L}(C)$$

the projection. Note that $\pi_C$ induces an isomorphism of posets

$$\overline{\pi}_C : \mathcal{C}^{\geq C} \xrightarrow{\simeq} \mathcal{C}_{\mathcal{H}/\mathbb{L}(C)}, \quad D \mapsto D/\mathbb{L}(C). \tag{2.2}$$

Here $\mathcal{C}^{\geq C}$ is the *star* of $\mathcal{C}$, i.e., the poset of $D \in \mathcal{C} = \mathcal{C}_{\mathcal{H}}$ such that $D \geq C$. For such a $D$, we denote $D/\mathbb{L}(C) = \pi_C(D)$ (which is a face of $\mathcal{H}/\mathbb{L}(C)$).

2.C. *Composition of faces.* Let $(\Sigma, \leq)$ be a poset. As usual, we write $a < b$ if $a \leq b$ and $a \neq b$. Following [6], introduce a binary operation $\star$ on $\Sigma$, called *composition*, by

$$a \star b = \begin{cases} b & \text{if } a < b, \\ a & \text{otherwise.} \end{cases}$$

This operation is associative but not commutative in general. If $\leq$ is a total order, then $a \circ b = \max(a, b)$ is commutative. Thus $\star$ can be seen as a generalization of the maximum to partially ordered sets.

We are interested in the case when $\Sigma = \{+, -, 0\}$ with the partial order (2.1). Define an associative operation $\circ$ on the set $\mathcal{H}^{\{+,-,0\}}$ of sign vectors by putting

$$(s \circ t)_H = s_H \star t_H, \quad H \in \mathcal{H}.$$

Proposition 2.3. *Let $C, D \in \mathcal{C}$ be two faces with corresponding sign vectors $(C_H)_{H\in\mathcal{H}}$ and $(D_H)_{H\in\mathcal{H}}$. Then*

(a) *The sign vector $(C_H \circ D_H)_{H\in\mathcal{H}}$ corresponds to a (necessarily unique) face $C \circ D$. We thus obtain an associative binary operation $\circ$ on $\mathcal{C}$.*
(b) *Explicitly, choose any $c \in C, d \in D$. Then*

$$C \circ D = \sigma\big((1-\epsilon)c + \epsilon d\big), \quad 0 < \epsilon \ll 1$$

*is the cell containing a small displacement of $c$ in the direction of $d$.*
(c) *Alternatively, $C \circ D$ is the minimal cell $K \in \mathcal{C}$ such that $K \geq C$ and $K + \mathbb{L}(C) \supset D$.*

*Remark* 2.4. The operation $\circ$ on $\mathcal{C}$ is the cornerstone of the "covector" axiomatization of oriented matroids; see [5, Axioms 4.1.1]. An arrangement of hyperplanes in $\mathbb{R}^n$ gives a *representable oriented matroid*; see loc. cit. §2.1. In this way, isomorphism classes of representable oriented matroids are in bijection with combinatorial equivalence classes of real hyperplane arrangements.

The characterization in (b) is also taken from loc. cit. §4.1.

The operation $\circ$ was introduced (in a slightly different context) by Tits under the name of "projection"; cf. [40, 3.19]. It plays a basic role in the studies of random walks on hyperplane arrangements; cf. [10] and references therein.

*Proof of Proposition 2.3.* Parts (a) and (b) follow from the next lemma, whose verification is left to the reader.

Lemma 2.5. *Let $x, y \in \mathbb{R}$. Then, for $0 < \epsilon \ll 1$, we have*

$$\operatorname{sgn}\big((1-\epsilon)x + \epsilon y\big) \;=\; \operatorname{sgn}(x) \star \operatorname{sgn}(y).$$

Part (c) follows from (b). Indeed, (b) implies that $(C \circ D) + \mathbb{L}(C) \supset D$. Conversely, suppose $K \geq C$ and $K + \mathbb{L}(C) \supset D$. We claim that $K \geq C \circ D$; that is, $K_H \geq C_H \star D_H$ for each $H \in \mathcal{H}$. We already know that $K_H \geq C_H$, since $K \geq C$. We need to prove that whenever $D_H > C_H$, we also have $K_H \geq D_H$. But $D_H > C_H$ means that $C_H = 0$ and $D_H \neq 0$; that is, $C \subset H$ but $D \not\subset H$. The statement that $K_H \geq D_H$ means therefore $K_H = D_H$; that is, $K$ and $D$ lie on the same side of $H$. But this is clear since $D \subset K + \mathbb{L}(C)$ and $\mathbb{L}(C) \subset H$ since $C \subset H$. □

For future reference, let us note yet another characterization of $C \circ D$. Consider the image $\pi_C(D) \subset \mathbb{R}^n/\mathbb{L}(C)$. This image lies in one face but may not itself be a face of the quotient arrangement $\mathcal{H}/\mathbb{L}(C)$. We denote by $\sigma_{\mathcal{H}/\mathbb{L}(C)}(\pi_C(D)) \in \mathcal{C}_{\mathcal{H}/\mathbb{L}(C)}$ the cell of $\mathcal{H}/\mathbb{L}(C)$ containing $\pi_C(D)$.

PROPOSITION 2.6. *$C \circ D = \overline{\pi}_C^{-1} \sigma_{\mathcal{H}/\mathbb{L}(C)}(\pi_C(D))$, where $\overline{\pi}_C$ is the isomorphism of posets from* (2.2).

*Proof.* This is a reformulation of Proposition 2.3(c). □

We further note the following monotonicity properties of the composition.

PROPOSITION 2.7.

(a) *If $D' \leq D$, then for any $C$, we have $C \circ D' \leq C \circ D$.*
(b) *For any $C$ and $D$, the flat $\mathbb{L}(C \circ D)$ is the minimal flat $L \in \mathcal{L}_{\mathcal{H}}$ containing both $\mathbb{L}(C)$ and $\mathbb{L}(D)$. In particular, if $C' \leq C$, then for any $D$, we have $\mathbb{L}(C' \circ D) \subset \mathbb{L}(C \circ D)$.*

*Proof.* (a) follows from the obvious fact that the operation $a \star b$ on a poset $\Sigma$ is monotone in the second argument. Part (b) follows because

$$\mathbb{L}(C \circ D) \;=\; \bigcap_{C_H \star D_H = 0} H,$$

and $C_H \star D_H = 0$ means that $C_H = D_H = 0$, i.e., $C, D \subset H$. □

*Remark* 2.8. Note that $\circ$ is not monotone in the first argument; that is, the condition $C' \leq C$ does not imply that $C' \circ D \leq C \circ D$. To obtain a counterexample, it suffices to take $C' = 0$. Then $C' \circ D = D$ for any $D$. So if $D \not\geq C$, then $0 \circ D \not\leq C \circ D$.

2.D. *Complexified arrangement and its stratifications.* Let $V_{\mathbb{C}} = \mathbb{C}^n$ be the complexification of $V = \mathbb{R}^n$. For $z \in \mathbb{C}^n$, we denote by $\Re(z), \Im(z) \in \mathbb{R}^n$ its real and imaginary parts. For $L \in \mathcal{L}$, we denote by $L_{\mathbb{C}} \subset \mathbb{C}^n$ its complexification. Accordingly, we write $\mathbb{L}_{\mathbb{C}}(X)$ for the $\mathbb{C}$-linear span of $X \subset \mathbb{C}^n$. Therefore we have the *complexified arrangement* $\mathcal{H}_{\mathbb{C}}$ of complex hyperplanes in $V_{\mathbb{C}} = \mathbb{C}^n$, formed by the $H_{\mathbb{C}}, H \in \mathcal{H}$. We will be interested in three natural stratifications of $\mathbb{C}^n$ induced by $\mathcal{H}$.

The *complex stratification* $\mathcal{S}^{(0)}$ consists of the open parts of the complexified flats

$$L^\circ_{\mathbb{C}} = L_{\mathbb{C}} \setminus \bigcup_{H \not\supset L} H_{\mathbb{C}}.$$

The $\mathbf{s}^{(2)}$*-stratification* $\mathcal{S}^{(2)} = \mathcal{C} + i\mathcal{C}$ consists of direct product cells $C + iD$, $C, D \in \mathcal{C}$. It makes $\mathbb{C}^n$ into a quasi-regular cellular space.

The $\mathbf{s}^{(1)}$*-stratification* $\mathcal{S}^{(1)}$ consists of cells $[C, D]$ defined for any *face interval*, by which we mean a pair $C, D \in \mathcal{C}$ such that $C \leq D$. By definition, $[C, D]$ consists of $z \in \mathbb{C}^n$ such that

(a) $\Im(z) \in C$,
(b) $\Re(z)$ lies in $\pi_C^{-1}(\pi_C(D))$.

For example, $[0, D] = D + i0$.

Note that $\dim[C, D] = \dim(C) + \dim(D)$. Notice also that for different pairs $(C, D) \neq (C', D')$, we have $[C, D] \cap [C', D'] = \emptyset$; i.e., the strata of $\mathcal{S}^{(1)}$ are precisely labelled by the pairs $(C \leq D)$. The stratification $\mathcal{S}^{(1)}$ also makes $\mathbb{C}^n$ into a quasi-regular cellular space.

The stratifications $\mathcal{S}^{(\nu)}$, $\nu = 1, 2$, were introduced and studied by Björner and Ziegler [6]. They use a slightly different definition, based on the following two complex generalizations of the sign function: the obvious one

$$\mathbf{s}^{(2)} : \mathbb{C} \longrightarrow \{+, -, 0\}^2, \quad \mathbf{s}^{(2)}(a + bi) = (\mathrm{sgn}(a), \mathrm{sgn}(b)),$$

and the less obvious one

$$\mathbf{s}^{(1)} : \mathbb{C} \longrightarrow \{i, j, +, -, 0\}, \quad \mathbf{s}^{(1)}(a + bi) = \begin{cases} i & \text{if } b > 0, \\ j & \text{if } b < 0, \\ + & \text{if } b = 0 \text{ and } a > 0, \\ - & \text{if } b = 0 \text{ and } a < 0, \\ 0 & \text{if } b = a = 0. \end{cases}$$

PROPOSITION 2.9. *Two vectors $z, w \in \mathbb{C}^n$ lie in the same stratum of $\mathcal{S}^{(\nu)}$, $\nu = 1, 2$, if and only if we have $\mathbf{s}^{(\nu)}(f_H(z)) = \mathbf{s}^{(\nu)}(f_H(w))$ for each $H \in \mathcal{H}$.*

*Proof.* Obvious for $\nu = 2$. For $\nu = 1$, this is the content of Theorem 5.1(ii) of [6]. □

We will view each $\mathcal{S}^{(\nu)}$, $\nu = 0, 1, 2$, as a poset, i.e., as the set of strata with the partial order $\leq$ defined by $S' \leq S$ if and only if $S' \subset \overline{S}$.

PROPOSITION 2.10.

(a) *Denoting by $\prec$ the relation of refinement of stratifications, we have*

$$\mathcal{S}^{(2)} \prec \mathcal{S}^{(1)} \prec \mathcal{S}^{(0)}.$$

*In particular, we have order preserving maps of posets $\mathcal{S}^{(2)} \to \mathcal{S}^{(1)} \to \mathcal{S}^{(0)}$ describing which stratum of $\mathcal{S}^{(\nu-1)}$ contains a given stratum of $\mathcal{S}^{(\nu)}$, $\nu = 1, 2$. These maps are as follows*:

(b) *The stratum of $\mathcal{S}^{(0)}$ containing a cell $[C, D] \in \mathcal{S}^{(1)}$, $C \leq D$, is $\mathbb{L}_{\mathbb{C}}(D)^\circ$.*

(c) *The stratum of $\mathcal{S}^{(1)}$ containing a cell $D + iC \in \mathcal{S}^{(2)}$, is $[C, C \circ D]$.*

(d) *For two cells $[C', D']$ and $[C, D]$ of $\mathcal{S}^{(1)}$, we have $[C', D'] \leq [C, D]$ if and only if $C' \leq C$ and $C \circ D' \leq D$.*

*Proof.* (a) Note that the stratification $\mathcal{S}^{(0)}$ can also be described in the style of Proposition 2.9, by using the sign-type function

$$\mathbf{s}^{(0)} : \mathbb{C} \longrightarrow \{0, *\}, \quad \mathbf{s}^{(0)}(z) = \begin{cases} 0 & \text{if } z = 0, \\ * & \text{if } z \neq 0. \end{cases}$$

The statement follows since the three stratifications of $\mathbb{C}$ induced by the three sign functions $\mathbf{s}^{(\nu)}$, $\nu = 0, 1, 2$, refine each other as claimed.

(b) The $\mathbb{R}$-linear span of $\pi_C^{-1}(\pi_C(D))$, the allowable range for $\Re(z)$, $z \in [C, D]$, is $\mathbb{L}(D)$. So $[C, D]$ cannot be contained in any $\mathbb{C}$-linear subspace strictly smaller than $\mathbb{L}_{\mathbb{C}}(D)$. Also, by construction it is indeed contained in $\mathbb{L}_{\mathbb{C}}(D)$, since the allowable range for $\Im(z)$ is $C \subset \overline{D}$.

(c) This follows by definition of $C \circ D$.

(d) This is Proposition 5.2 of [6]. □

## 3. Constructible complexes on arrangements

We keep the notation of the previous section, in particular, concerning the three stratifications $\mathcal{S}^{(\nu)}$ of $\mathbb{C}^n$, $\nu = 0, 1, 2$, induced by the arrangement $\mathcal{H}$.

3.A. *Constructible sheaves.* A sheaf $\mathcal{F} \in \mathrm{Sh}_{\mathbb{C}^n}$ will be called $\mathcal{S}^{(\nu)}$-*smooth* if it is locally constant on each stratum of $\mathcal{S}^{(\nu)}$. We will say that $\mathcal{F}$ is $\mathcal{S}^{(\nu)}$-*constructible* if it is $\mathcal{S}^{(\nu)}$-smooth with finite-dimensional stalks. A complex $\mathcal{F} \in D^b \mathrm{Sh}_{\mathbb{C}^n}$ will be called $\mathcal{S}^{(\nu)}$-smooth (resp. $\mathcal{S}^{(\nu)}$-constructible) if all the cohomology sheaves of $\mathcal{F}$ are $\mathcal{S}^{(\nu)}$-smooth (resp. $\mathcal{S}^{(\nu)}$-constructible). We denote by $D^b_{\mathcal{S}^{(\nu)}} \mathrm{Sh}_{\mathbb{C}^n} \subset D^b \mathrm{Sh}_{\mathbb{C}^n}$ the full subcategory of $\mathcal{S}^{(\nu)}$-constructible complexes.

In particular, for $\nu = 2, 1$, since $\mathcal{S}^{(\nu)}$ is cellular, an $\mathcal{S}^{(\nu)}$-smooth sheaf $\mathcal{F}$ is uniquely defined by its cellular stalks and generalization maps, which we denote, respectively, as follows:

$$\begin{gathered} \mathcal{F}\Big|_{D+iC}, \quad \gamma^{\mathcal{F}}_{D'+iC', D+iC} : \mathcal{F}\Big|_{D'+iC'} \longrightarrow \mathcal{F}\Big|_{D+iC}, \;\; D' \leq D, \; C' \leq C, \\ \mathcal{F}_{[C,D]} = \mathcal{F}\Big|_{[C,D]}, \quad \gamma^{\mathcal{F}}_{[C',D'],[C,D]} : \mathcal{F}_{[C',D']} \longrightarrow \mathcal{F}_{[C,D]}, \;\; [C', D'] \leq [C, D]. \end{gathered} \tag{3.1}$$

We use similar notation for the case when $\mathcal{F}$ is an $\mathcal{S}^{(\nu)}$-smooth complex. In this case the stalks are complexes of vector spaces and we can assume, by using

Proposition 1.8(c), that the generalization maps are morphisms of complexes forming a representation of the poset of cells.

3.B. *From $\mathcal{S}^{(1)}$-smoothness to $\mathcal{S}^{(0)}$-smoothness.* We will need to compare the conditions of smoothness with respect to different stratifications.

*Definition* 3.2. An inclusion of $\mathbf{s}^{(1)}$-cells $[C', D'] \leq [C, D]$ will be called *elementary* if one of the two following cases hold:

(1) $C' \leq C$ and $D' = D$, so we have a flag $C' \leq C \leq D$.
(2) $C = D$, $\mathbb{L}(D) = \mathbb{L}(D')$, $\dim(C') = \dim(D') - 1$, and $C' \leq D$. In other words, $C'$ is a codimension 1 "wall" separating $D$ and $D'$.

PROPOSITION 3.3. *Let $\mathcal{F}$ be an $\mathcal{S}^{(1)}$-smooth sheaf. The following are equivalent*:

(i) *$\mathcal{F}$ is $\mathcal{S}^{(0)}$-smooth*;
(ii) *the map $\gamma^{\mathcal{F}}_{[C',D'],[C,D]}$ is an isomorphism for each elementary inclusion $[C', D'] \leq [C, D]$.*

*Proof.* The tautological equivalent of (i) is

(iii) The map $\gamma^{\mathcal{F}}_{[C',D'],[C,D]}$ is an isomorphism for each inclusion $[C', D'] \leq [C, D]$ such that $\mathbb{L}(D') = \mathbb{L}(D)$.

Indeed, by Proposition 2.10(b), inclusions in (iii) are precisely all inclusions of $\mathbf{s}^{(1)}$-cells in the same $\mathbf{s}^{(0)}$-stratum. Clearly (iii) is stronger than (ii). So we need to prove that (ii) implies (iii).

To prove (iii), it is enough to fix $L \in \mathcal{L}$ and to concentrate on inclusions with $\mathbb{L}(D) = \mathbb{L}(D') = L$. For this, we do not need to consider any faces outside $L$, so we can and will assume that $L = \mathbb{R}^n$ and $D$ and $D'$ are $n$-dimensional, i.e., are chambers.

Let $\delta(D, D')$ be the *chamber distance* between $D$ and $D'$, i.e., the minimal length of a sequence

$$D = D_0, D_1, \cdots, D_l = D'$$

such that each $D_i$ is a chamber and each $D_p, D_{p+1}$ have a codimension 1 face in common. Thus $\delta(D, D') = 0$ means that $D' = D$.

Let us prove (iii) by induction on $\delta(D, D')$. If $\delta(D, D') = 0$, i.e., $D' = D$, then our inclusion is of type (1). Consider now an arbitrary inclusion

$$[C', D'] \leq [C, D], \;\; C' \leq C, \;\; C \circ D' = D,$$

with $D', D$ being chambers (open faces).

Take generic points $c \in C, d' \in D'$, and form the straight line interval $[c, d']$ oriented towards $d'$. By Proposition 2.3(b), this interval, after leaving $C$, first hits $D = C \circ D'$. Since $D'$ is a chamber, our interval will, after leaving $D$, hit some cell $C_1 \geq C'$ of positive codimension, then a chamber $D_1 \geq C_1$,

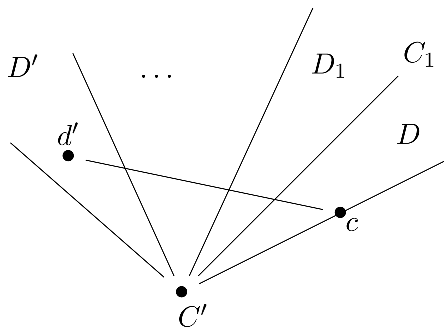


Figure 1. Reduction of an inclusion $[C', D'] \leq [C, D]$.

and so on, see Figure 1. Note, that by choosing $d'$ in a generic enough way, we can ensure that $C_1$ is of codimension 1, which we will assume. Note also that $\delta(D_1, D') < \delta(D, D')$.

We then have a commutative square in (the category corresponding to) the poset $(\mathcal{S}^{(1)}, \leq)$:

$$\begin{tikzcd} {[C', D']} \arrow[r, "p"] \arrow[d, "q"'] & {[C_1, D_1]} \arrow[d, "r"] \\ {[C, D]} \arrow[r, "s"'] & {[D, D].} \end{tikzcd}$$

In this square, $q$ is our inclusion in question, $p$ is an inclusion with smaller $\delta$, while $r$ is an inclusion of type (2), and $s$ is an inclusion of type (1). Now, the generalization maps for $\mathcal{F}$ can be seen as a functor $\gamma^{\mathcal{F}} : (\mathcal{S}^{(1)}, \leq) \to \mathrm{Vect}_{\mathbf{k}}$, so the above square gives a commutative square of vector spaces. By (ii) and our inductive assumption, $p, r, s$ are taken by $\gamma^{\mathcal{F}}$ into isomorphisms. It remains to deduce that $\gamma^{\mathcal{F}}$ takes $q$ to an isomorphism, invoking the following obvious

LEMMA 3.4. *If, in a commutative square of morphisms in any category, three out of four arrows are isomorphisms, then the fourth arrow is an isomorphism as well.*

Proposition 3.3 is proved. □

3.C. *Complexes of cohomology with support and their linear algebra data.* Let $\mathcal{F} \in D^b_{\mathcal{S}^{(0)}} \mathrm{Sh}_{\mathbb{C}^n}$. Define

$$\mathcal{R}^\bullet_{\mathcal{F}} = \underline{R\Gamma}_{\mathbb{R}^n}(\mathcal{F})[n] \ \in \ D^b_{\mathcal{C}}(\mathrm{Sh}_{\mathbb{R}^n}). \tag{3.5}$$

This is a cellular complex on $\mathbb{R}^n$ (with respect to $\mathcal{C}$, the stratification by faces of $\mathcal{H}$). Therefore it can be defined by the linear algebra data consisting of complexes and generalization maps

$$E^\bullet_C = E^\bullet_C(\mathcal{F}) = R\Gamma(C, \mathcal{R}^\bullet_{\mathcal{F}}), \quad \gamma_{C'C} : E^\bullet_{C'} \to E^\bullet_C, \quad C' \leq C \in \mathcal{C} \tag{3.6}$$

forming a representation of $(\mathcal{C}, \leq)$ in complexes of vector spaces.

More generally, let $C \in \mathcal{C}$ be a face of codimension $d$. Consider the "tube face"

$$j_C : \mathbb{R}^n + iC \hookrightarrow \mathbb{C}^n, \tag{3.7}$$

and form the complex

$$\mathcal{E}_C^\bullet = \mathcal{E}_C^\bullet(\mathcal{F}) = \underline{R\Gamma}_{\mathbb{R}^n+iC}(\mathcal{F})[d] = j_{C*}j_C^!(\mathcal{F})[d]. \tag{3.8}$$

Thus $\mathcal{E}_C^\bullet$ is an $\mathbb{R}$-constructible complex on $\mathbb{C}^n$, supported on the closure $\mathbb{R}^n + i\overline{C}$. We denote by $p_C, \overline{p}_C$ the composite projections

$$\begin{aligned} &\mathbb{R}^n + iC \xrightarrow{\Re} \mathbb{R}^n \longrightarrow \mathbb{R}^n/\mathbb{L}(C),\\ &\mathbb{R}^n + i\overline{C} \xrightarrow{\Re} \mathbb{R}^n \longrightarrow \mathbb{R}^n/\mathbb{L}(C). \end{aligned} \tag{3.9}$$

PROPOSITION 3.10. *The complex $\mathcal{E}_C^\bullet$, considered as a complex of sheaves on $\mathbb{R}^n + i\overline{C}$ has the form $\overline{p}_C^*\mathcal{E}_C^{\bullet,\mathrm{red}}$, where $\mathcal{E}_C^{\bullet,\mathrm{red}}$ is a complex on $\mathbb{R}^n/\mathbb{L}(C)$ cellular with respect to the stratification by faces of the quotient arrangement $\mathcal{H}/\mathbb{L}(C)$.*

*Proof.* It is enough to show that $j_C^!\mathcal{F}$ has the form $p_C^*\mathcal{E}_C^{\bullet,\mathrm{red}}$ for some $\mathcal{E}_C^{\bullet,\mathrm{red}}$ as above, since the derived direct image extension from $\mathbb{R}^n + iC$ to $\mathbb{R}^n + i\overline{C}$ will then proceed along the directions in which $p_C^*\mathcal{E}_C^{\bullet,\mathrm{red}}$ is constant.

In the remainder of the proof the word "manifold" will mean a $C^\infty$-manifold. For a complex of sheaves $\mathcal{G}$ on a manifold $X$, we denote by $SS(\mathcal{G}) \subset T^*X$ the micro-support of $\mathcal{G}$; see [25, Ch. V]. Our desired result about $j_C^!\mathcal{F}$ can be reformulated by saying that

$$\begin{aligned} SS(j_C^!\mathcal{F}) &\subset \bigcup_{\Lambda\in\mathcal{L}_{\mathcal{H}/\mathbb{L}(C)}} T^*_{p_C^{-1}(\Lambda)}(\mathbb{R}^n + iC)\\ &= \bigcup_{L\in\mathcal{L},\ L\supset C} T^*_{L+iC}(\mathbb{R}^n+iC) = \bigcup_{L\in\mathcal{L},\ L\supset C} (L+iC)\times L^\perp, \end{aligned} \tag{3.11}$$

where $L^\perp \subset \mathbb{R}^{n*}$ is the orthogonal to $L$. We recall the estimate for the micro-support of the direct image [25, Cor. 6.4.4] specialized to the case of a locally closed embedding.

For a locally closed submanifold $M$ of a manifold $X$ and a subset $S \subset X$, we denote by $K_M(S)$ the *normal cone* to $S$ along $M$; see [25, Def. 4.1.1]. It is a closed conic subset of $T_MX$, the normal bundle to $M$ in $X$. We note the following properties:

$$K_M(S_1 \cup S_2) = K_M(S_1) \cup K_M(S_2), \quad S_\nu \subset X,\ \nu = 1,2; \tag{3.12}$$

$$\begin{aligned} K_{M_1\times M_2}(S_1\times S_2) &= K_{M_1}(S_1)\times K_{M_2}(S_2) \subset T_{M_1}(X_1)\times T_{M_2}(X_2)\\ &= T_{M_1\times M_2}(X_1\times X_2), \quad M_\nu, S_\nu \subset X_\nu. \end{aligned} \tag{3.13}$$

Given a locally closed embedding of manifolds $f : Y \to X$, the conormal bundle $T^*_Y X$ is a Lagrangian submanifold in $T^*X$, and so its normal bundle there can be written as

$$T_{T^*_Y X}(T^*X) \;\simeq\; T^*(T^*_Y X).$$

In particular, the projection $T^*_Y X \to Y$ gives a closed embedding

$$T^*Y \;\subset\; T^*(T^*_Y X) \simeq T_{T^*_Y X}(T^*X).$$

For a complex of sheaves $\mathcal{F}$ on $X$, Corolarry 6.4.4 and Proposition 6.2.4 of [25] give

$$SS(f^!\mathcal{F}) \;\subset\; K_{T^*_Y X}(SS(\mathcal{F})) \cap T^*Y, \tag{3.14}$$

the intersection inside $T_{T^*_Y X}(T^*X)$.

We now specialize this to

$$f = j_C : Y = \mathbb{R}^n + iC \longrightarrow \mathbb{C}^n = X.$$

By our assumptions on $\mathcal{F}$,

$$SS(\mathcal{F}) \;\subset\; \bigcup_{L\in\mathcal{L}} T^*_{L_\mathbb{C}}\mathbb{C}^n = \bigcup_{L\in\mathcal{L}} L_\mathbb{C} \times L_\mathbb{C}^\perp \;\subset\; \mathbb{C}^n \times \mathbb{C}^{n*} \;= T^*\mathbb{C}^n = T^*X.$$

We further have

$$T^*_Y X = T^*_{\mathbb{R}^n + iC}\mathbb{C}^n = (\mathbb{R}^n + iC) \times i\mathbb{L}(C)^\perp \subset \mathbb{C}^n \times \mathbb{C}^{n*} \;= T^*X.$$

Therefore

$$T_{T^*_Y X}(T^*X) = \Big((\mathbb{R}^n + iC) \times i\mathbb{L}(C)^\perp\Big) \times \Big(i(\mathbb{R}^n/\mathbb{L}(C)) \times (\mathbb{R}^{n*} + i\mathbb{L}(C)^*)\Big),$$

in which $T^*Y = T^*(\mathbb{R}^n + iC)$ is embedded as the product of all the factors except $i\mathbb{L}(C)^\perp$ and $i(\mathbb{R}^n/\mathbb{L}(C))$, the coordinates corresponding to these factors being put to 0. Applying (3.12), we find

$$K_{T^*_Y X}(SS(\mathcal{F})) \;\subset\; \bigcup_{L\in\mathcal{L}} K_{(\mathbb{R}^n+iC)\times i\mathbb{L}(C)^\perp}(L_\mathbb{C} \times L_\mathbb{C}^\perp). \tag{3.15}$$

Lemma 3.16. *For $L \in \mathcal{L}$, we have one of two possibilities*:

(1) $L_\mathbb{C} \cap (\mathbb{R}^n + iC) = \emptyset$,

(2) $L \supset C$.

Assuming the lemma, we note that in case (1), $L_\mathbb{C} \times L_\mathbb{C}^\perp$ does not meet $(\mathbb{R}^n + iC) \times i\mathbb{L}(C)^\perp$ and so will not contribute to the union in (3.15). In case (2), we write, using (3.13):

$$\begin{aligned} K_{(\mathbb{R}^n+iC)\times i\mathbb{L}(C)^\perp}(L_\mathbb{C} \times L_\mathbb{C}^\perp) &= K_{(\mathbb{R}^n+iC)\times(0+i\mathbb{L}(C)^\perp)}\Big((L+iL)\times(L^\perp + iL^\perp)\Big) \\ &= \Big(K_{\mathbb{R}^n}(L) + iK_C(L)\Big) \times \Big(K_0(L^\perp) + iK_{\mathbb{L}(C)^\perp}(L^\perp)\Big) \\ &= \Big(L + i(C \times (L/\mathbb{L}(C)))\Big) \times \Big(iL^\perp + (L^\perp \times \{0\})\Big), \end{aligned}$$

since $L^\perp \subset \mathbb{L}(C)^\perp$.

Now, putting the coordinates from $i\mathbb{L}(C)^\perp$ and $i(\mathbb{R}^n/\mathbb{L}(C))$ to be 0, we get $(L+iC)\times L^\perp$, which is the contribution of $L\supset C$ into (3.11). This proves Proposition 3.10 modulo Lemma 3.16. □

*Proof of Lemma* 3.16. By taking intersections, the lemma reduces to the case when $L=H$ is a hyperplane from $\mathcal{H}$, which we now assume. Let $f_H:\mathbb{R}^n\to\mathbb{R}$ be a linear equation of $H$. If (1) does not hold for $H$, then there are $b\in\mathbb{R}^n, c\in C$ such that

$$0 \ = f_H(b+ic) = f_H(b)+if_H(c).$$

Since $f_H(b), f_H(c)\in\mathbb{R}$, this implies that $f_H(c)=0$ and so $f_H|_C=0$ and $H\supset C$. Lemma 3.16 and Proposition 3.10 are proved. □

Since $\mathcal{E}_C^{\bullet,\mathrm{red}}$ is a cellular complex, it is determined by linear algebra data, which we denote by

$$E_D^{C\bullet},\ \ D\in\mathcal{C}_{\mathcal{H}/\mathbb{L}(C)},\quad \gamma^C_{D'D}:E^{C\bullet}_{D'}\longrightarrow E^{C\bullet}_D,\ \ D'\le D. \tag{3.17}$$

PROPOSITION 3.18. *We have canonical identifications (quasi-isomorphisms of complexes of* $\mathbf{k}$*-vector spaces compatible with the maps* $\gamma$*)*

$$E_D^{C\bullet}\simeq E^\bullet_{\overline{\pi}_C^{-1}(D)}\otimes_{\mathbf{k}}\mathrm{or}(C),\qquad \gamma^C_{D'D}=\gamma_{\overline{\pi}_C^{-1}(D'),\overline{\pi}_C^{-1}(D)}\otimes\mathrm{Id}\,.$$

*Here* $\overline{\pi}_C$ *is the isomorphism of posets from* (2.2).

*Proof.* Let $K=\overline{\pi}_C^{-1}(D)$, so $K\ge C$. We start with recalling the definitions of $E_D^{C\bullet}$ and $E_K^\bullet$ side-by-side. First, we recall the notation $d=\mathrm{codim}_{\mathbb{R}^n}(C)$. Also let $\mathbb{R}^n_{\ge C}=\bigcup_{K'\ge C}K'$, which is an open subset of $\mathbb{R}^n$.

The natural projection $\pi:K\to D$ has contractible fibers. The cohomology sheaves of the complex $\underline{R\Gamma}_{\mathbb{R}^n+iC}(\mathcal{F})[d]$ are constant on $K+iC$, while the cohomology sheaves of $\underline{R\Gamma}_{\mathbb{R}^n}(\mathcal{F})[n]$ are constant on $K$. Set

$$\mathcal{G}'=\underline{R\Gamma}_{\mathbb{R}^n+i\mathbb{L}(C)}(\mathcal{F})[d],\quad \mathcal{G}=\mathcal{G}'|_{\mathbb{R}^n_{\ge C}+i\mathbb{L}(C)}.$$

Because $C$ is open in $\mathbb{L}(C)$ and $\mathbb{R}^n_{\ge C}$ in $\mathbb{R}^n$, we have

$$E_D^{C\bullet}=\mathcal{E}_C^{\bullet,\mathrm{red}}|_D=\underline{R\Gamma}_{\mathbb{R}^n+iC}(\mathcal{F})[d]\Big|_{K+iC}=\mathcal{G}\Big|_{K+iC}$$

(restriction of a cellular complex to a cell, considered as a complex of vector spaces). On the other hand, we have

$$E_K^\bullet=\mathcal{R}_{\mathcal{F}}|_K=\underline{R\Gamma}_{\mathbb{R}^n}(\mathcal{F})[n]\Big|_K=\underline{R\Gamma}_{\mathbb{R}^n}\mathcal{G}'[n-d]\Big|_K=\underline{R\Gamma}_{\mathbb{R}^n_{\ge C}}\mathcal{G}[n-d]\Big|_K.$$

To compare these, we note that $K=K+i0$ lies in $K+i\overline{C}$ but not in $K+iC$ proper. We use the following

LEMMA 3.19. *For any* $K'\ge C$*, the cohomology sheaves of* $\mathcal{G}$ *are constant on the whole of* $K'+i\mathbb{L}(C)$*. (Hence the restriction of* $\mathcal{G}$ *to* $K'+i\mathbb{L}(C)$ *is quasi-isomorphic to a constant complex of sheaves.)*

This is a consequence of the following analog of Lemma 3.16.

LEMMA 3.20. *Let $H$ be a hyperplane from $\mathcal{H}$ and $K' \geq C$. Then $H_{\mathbb{C}}$ meets $K' + i\mathbb{L}(C)$ if and only if $H_{\mathbb{C}}$ contains $K' + i\mathbb{L}(C)$.*

*Proof of Lemma* 3.20. If $f_H(k' + iy) = 0$ for some $k' \in K$, $y \in \mathbb{L}(C)$, then by taking the real part, we find $f_H(k') = 0$, and therefore $(f_H)|_{K'} = 0$. Now, since $C \leq K'$, $(f_H)|_C = 0$ as well and therefore $(f_H)|_{\mathbb{L}(C)} = 0$. Therefore $f_H(k' + iy) = f_H(k') + i f_H(y) = 0$ for each $k' \in K'$ and $y \in \mathbb{L}(C)$. □

Consider now the Cartesian square of embeddings (transverse intersection)

$$\begin{tikzcd}
K + i\mathbb{L}(C) \arrow[r, "v"] & \mathbb{R}^n_{\geq C} + i\mathbb{L}(C) \\
K = K + i0 \arrow[u, "u"] \arrow[r, "t"'] & \mathbb{R}^n_{\geq C} = \mathbb{R}^n_{\geq C} + i0. \arrow[u, "s"']
\end{tikzcd}$$

Lemma 3.19 implies that $s$ is noncharacteristic for $\mathcal{G}$, and therefore we have "local Poincaré duality" $s^!\mathcal{G} \simeq s^*\mathcal{G} \otimes \mathrm{or}(C)[d-n]$;see [25, Cor. 5.4.11]. Therefore we identify

$$\begin{aligned} E_K^\bullet &= \Gamma(K, t^* s^! \mathcal{G}[n-d]) \simeq \Gamma(K, t^* s^* \mathcal{G} \otimes \mathrm{or}(C)) \\ &= \Gamma(K, u^* v^* \mathcal{G} \otimes \mathrm{or}(C)) \overset{(3.19)}{=} \Gamma(K + i\mathbb{L}(C), v^*\mathcal{G} \otimes \mathrm{or}(C)) = E_D^{C\bullet} \otimes \mathrm{or}(C). \end{aligned}$$

Identification of the maps $\gamma^C_{D'D}$ is done similarly. Proposition 3.18 is proved. □

*Remark* 3.21. Let us indicate another, perhaps more geometric, view on Proposition 3.18. Let $x \in C$ and $T$ be a small transversal slice to $\mathbb{L}(C)$ at $x$, i.e., a subset of the form $S + x$ where $S$ is a small open ball in a linear subspace $M \subset \mathbb{R}^n$ such that $M \oplus \mathbb{L}(C) = \mathbb{R}^n$. Thus $T$ fits into a diagram of embeddings

$$T \xrightarrow{\epsilon} \mathbb{R}^n \xrightarrow{j_0} \mathbb{C}^n.$$

The composition of $\epsilon$ with the projection $q : \mathbb{R}^n \to \mathbb{R}^n/\mathbb{L}(C)$ is an embedding of a small ball into $\mathbb{R}^n/\mathbb{L}(C)$. Proposition 3.18 can be reformulated by saying that

$$(q\epsilon)^* \mathcal{E}_C^{\bullet,\mathrm{red}} = \epsilon^* \mathcal{R}_{\mathcal{F}} \otimes_{\mathbf{k}} \mathrm{or}(C).$$

Both sides of this proposed isomorphism can be identified as certain complexes of cohomology with support. First, by transversality and Poincaré duality,

$$\epsilon^* \mathcal{R}_{\mathcal{F}} = \epsilon^! \mathcal{R}_{\mathcal{F}} \otimes_{\mathbf{k}} \mathrm{or}(C)[n-d],$$

and so, since $\mathcal{R}_{\mathcal{F}} = j_0^! \mathcal{F}[n]$,

$$\epsilon^* \mathcal{R}_{\mathcal{F}} = (j_0\epsilon)^! \mathcal{F} \otimes_{\mathbf{k}} \mathrm{or}(C)[2n-d].$$

Second, take another point $y \in C$ and let $T' = T + iy \subset \mathbb{C}^n$ be the translation of $T$ by $iy$, fitting into the diagram of embeddings

$$T' \xrightarrow{\epsilon'} \mathbb{R}^n + iC \xrightarrow{j_C} \mathbb{C}^n.$$

Then $T'$ is transversal to the fibers of $p_C$, and therefore $(q\epsilon)^* \mathcal{E}_C^{\bullet,\mathrm{red}}$ is identified with $(\epsilon')^* \mathcal{E}_C^\bullet$, after identification of $T$ with $T'$ via the shift. Again, by transversality of $T'$ to the fibers of $p_C$ (and the fact that they are canonically oriented, being complex manifolds), we find that

$$(\epsilon')^* \mathcal{E}_C^\bullet = (\epsilon')^! \mathcal{E}_C^\bullet[2n-2d] = (j_C\epsilon')^! \mathcal{F}[2n-d].$$

Using the canonical identification $\mathrm{or}(C)^{\otimes 2} = \mathbf{k}$, the proposition thus reduces to the claim that the complexes of cohomology with support in $T$ and $T'$ for $\mathcal{F}$ are identified. Further, Lemma 3.20 can be seen as identifying the stratifications induced by $\mathcal{H}$ on $T$ and $T'$ and so allows one to prove the desired claim by a homotopy argument, deforming $y$ to 0.

Corollary 3.22.

(a) *The complex $\mathcal{E}_C^\bullet$ of sheaves on $\mathbb{C}^n$ is smooth with respect to the stratification $\mathcal{S}^{(1)}$ (and therefore to $\mathcal{S}^{(2)}$).*

(b) *The stalks and generalization maps of $\mathcal{E}_C^\bullet$ on cells of $\mathcal{S}^{(2)}$ are given by*

$$\mathcal{E}_C^\bullet\Big|_{iC_1+D} = \begin{cases} E^\bullet_{C\circ D} \otimes \mathrm{or}(C) & \textit{if } C_1 \le C, \\ 0 & \textit{otherwise}, \end{cases}$$

$$\gamma^{\mathcal{E}_C^\bullet}_{iC_1'+D', iC_1+D} = \gamma_{C\circ D', C\circ D} \otimes \mathrm{Id}, \quad C_1' \le C_1 \le C,\ D' \le D.$$

(c) *The stalks of $\mathcal{E}_C^\bullet$ on cells of $\mathcal{S}^{(1)}$ are given by*

$$\mathcal{E}_C^\bullet\Big|_{[C_1,D]} = \begin{cases} E^\bullet_{C\circ D} \otimes \mathrm{or}(C) & \textit{if } C_1 \le C, \\ 0 & \textit{otherwise}. \end{cases}$$

*Further, if $[C_1', D'] \le [C_1, D]$ is an inclusion of two $\mathcal{S}^{(1)}$-cells in the support of $\mathcal{E}_C^\bullet$, then $C \circ D' \le C \circ D$, and the corresponding generalization map for $\mathcal{E}_C^\bullet$ has the form*

$$\gamma^{\mathcal{E}_C^\bullet}_{[C_1',D'],[C_1,D]} = \gamma_{C\circ D', C\circ D} \otimes \mathrm{Id}\,.$$

In particular, putting $C_1 = D = \{0\}$ in part (b) of the corollary, we get quasi-isomorphisms

$$\begin{aligned} R\Gamma(\mathbb{C}^n, \mathcal{E}_C^\bullet) &\simeq R\Gamma(\mathbb{R}^n/\mathbb{L}(C), \mathcal{E}_C^{\bullet,\mathrm{red}}) \\ &\simeq E_0^{C\bullet} = E_C^\bullet \otimes \mathrm{or}(C). \end{aligned} \tag{3.23}$$

At the very right we have the complexes appearing in the description of the cellular complex $\mathcal{R}_{\mathcal{F}}^\bullet$.

*Proof of Corollary* 3.22. Part (a) follows from Proposition 3.18, because cells of $\mathcal{S}^{(1)}$ are the the lifts of cells of the quotient arrangements, by their definition in Section 2.D. Part (b) is a simple translation of Proposition 3.18 by using the interpretation of the $\circ$ operation in Proposition 2.6. The inclusion $C \circ D' \leq C \circ D$ in part (c) is proved as follows.

By Proposition 2.10(d), the condition $[C_1', D'] \leq [C_1, D]$ means $C_1' \leq C_1$ and $C_1 \circ D' \leq D$. The inclusion $C_1 \leq C$ implies that $C \circ C_1 = C$. Since the operation $\circ$ is associative and monotone in the second argument, we then have

$$C \circ D' = (C \circ C_1) \circ D' = C \circ (C_1 \circ D') \ \leq \ C \circ D.$$

Once this inclusion is established, part (c) becomes a reformulation of part (b) using the explicit relation between strata of $\mathcal{S}^{(2)}$ and $\mathcal{S}^{(1)}$ as given in Proposition 2.10(c). $\square$

## 4. **Perverse sheaves and double quivers**

4.A. *The Cousin complex.* We keep the notations of the previous section. Put $X = \mathbb{C}^n$, and consider the filtration of $X$ by closed subspaces

$$X_d = \bigcup_{\dim(C) \leq d} \mathbb{R}^n + iC, \tag{4.1}$$

so that

$$Y_d \ = X_d \setminus X_{d-1} \ = \bigsqcup_{\dim(C)=d} \mathbb{R}^n + iC.$$

Let $\mathcal{F} \in D^b_{\mathcal{S}^{(0)}} \mathrm{Sh}_{\mathbb{C}^n}$ be an $\mathcal{S}^{(0)}$-constructible complex. We consider the corresponding homological Postnikov system for $\mathcal{F}$. The associated complex in the derived category with total object $\mathcal{F}$ has the form

$$\mathcal{E}^{\bullet\bullet} = \mathcal{E}^{\bullet\bullet}(\mathcal{F}) = \left\{ \bigoplus_{\operatorname{codim}(C)=0} \mathcal{E}_C^\bullet \xrightarrow{\widetilde{\delta}} \bigoplus_{\operatorname{codim}(C)=1} \mathcal{E}_C^\bullet \xrightarrow{\widetilde{\delta}} \cdots \xrightarrow{\widetilde{\delta}} \mathcal{E}_0^\bullet \right\}. \tag{4.2}$$

Here $\mathcal{E}_C^\bullet$ has been defined in (3.8). In particular, by applying the functor $R\Gamma(\mathbb{C}^n, -)$, we get a Postnikov system in $D^b \operatorname{Vect}^{\mathrm{fd}}_{\mathbf{k}}$ whose associated complex with total object $R\Gamma(\mathbb{C}^n, \mathcal{F})$ has the form

$$\bigoplus_{\operatorname{codim}(C)=0} E_C^\bullet \otimes \mathrm{or}(C) \xrightarrow{\widetilde{\delta}} \bigoplus_{\operatorname{codim}(C)=1} E_C^\bullet \otimes \mathrm{or}(C) \xrightarrow{\widetilde{\delta}} \cdots \xrightarrow{\widetilde{\delta}} E_0^\bullet \otimes \mathrm{or}(0). \tag{4.3}$$

We will call (4.3) the *Cousin complex* of $\mathcal{F}$. In particular, the differential $\widetilde{\delta}$ of this complex splits into matrix elements

$$\widetilde{\delta}_{CC'} : E_C^\bullet \otimes \mathrm{or}(C) \longrightarrow E_{C'}^\bullet \otimes \mathrm{or}(C'), \quad C' <_1 C.$$

Note that for a codimension 1 face $C'$ of a convex polyhedron $C$, we have a canonical coorientation, i.e., a canonical trivialization of $\mathrm{or}(C)^* \otimes \mathrm{or}(C')$, which allows us to write the matrix elements as maps

$$\delta_{CC'} = \delta^{\mathcal{F}}_{CC'} : E^\bullet_C \longrightarrow E^\bullet_{C'}, \quad C' <_1 C. \tag{4.4}$$

PROPOSITION 4.5. *The maps of complexes $\delta_{CC'}$ commute, i.e., extend to a contravariant representation of the poset $(\mathcal{C}, \leq)$ in complexes of $\mathbf{k}$-vector spaces.*

*Proof.* Indeed, the condition $\widetilde{\delta}^2 = 0$ implies that the $\widetilde{\delta}_{CC'}$ anticommute, so the $\delta_{CC'}$ commute by the antisymmetry of the orientation isomorphisms. □

PROPOSITION 4.6. *We have canonical isomorphisms $E^\bullet_C(\mathcal{F}^\star) \simeq E^\bullet_C(\mathcal{F})^*$ (dual complexes and adjoint maps) in the derived category of vector spaces. Further, under these isomorphisms the morphisms of complexes $\delta^{\mathcal{F}^\star}_{CC'}$ (considered as morphisms in the derived category) are dual to $(\gamma^{\mathcal{F}}_{C'C})$.*

*Proof.* Since the stratification given by $\mathcal{H}_\mathbb{C}$ is invariant under the $\mathbb{C}^*$-action on $\mathbb{C}^n$, we can use cohomology with support in $i\mathbb{R}^n$ instead of $\mathbb{R}^n$ to define $\mathcal{R}_\mathcal{F}$ and $\mathcal{R}_{\mathcal{F}^\star}$. More precisely, for $t \in \mathbb{R}$, let $j_t : \mathbb{R}^n \to \mathbb{C}^n$ be the embedding given by the multiplication with $e^{it} \in \mathbb{C}^*$. Then the sheaves $\mathcal{R}_{\mathcal{F},t} = j_t^! \mathcal{F}[n]$ on $\mathbb{R}^n$ for $t \in [0, \pi/2]$ are canonically identified with each other because they unite into a sheaf on $[0, \pi/2] \times \mathbb{R}^n$ locally constant (and therefore constant) on each interval $[0, \pi/2] \times \{x\}$, $x \in \mathbb{R}^n$. So we denote by $j = j_{\pi/2}$ the embedding of $i\mathbb{R}^n$ and use $j^!$ to define $\mathcal{R}_{\mathcal{F}^\star}$.

Since Verdier duality interchanges $j^!$ and $j^*$, we have a canonical quasi-isomorphism

$$(\mathcal{R}_{\mathcal{F}^\star})^\star = \left(\underline{R\Gamma}_{i\mathbb{R}^n}(\mathcal{F}^\star)[n]\right)^\star \simeq \mathcal{F}|_{i\mathbb{R}^n}.$$

Now, $\mathcal{F}|_{i\mathbb{R}^n}$ can be understood by restricting to $i\mathbb{R}^n$ the homological Postnikov system corresponding to the filtration (4.1). This gives a complex in derived category with total object $\mathcal{F}|_{i\mathbb{R}^n}$ obtained by restricting $\mathcal{E}^{\bullet\bullet}(\mathcal{F})$ to $i\mathbb{R}^n$ which, after identifying $i\mathbb{R}^n$ back with $\mathbb{R}^n$, gives the following:

$$\bigoplus_{\mathrm{codim}(C)=0} \underline{E^\bullet_C \otimes \mathrm{or}(C)}_{\overline{C}} \xrightarrow{(\widetilde{\delta}_{CC'})} \bigoplus_{\mathrm{codim}(C)=1} \underline{E^\bullet_C \otimes \mathrm{or}(C)}_{\overline{C}} \xrightarrow{(\widetilde{\delta}_{CC'})} \cdots . \tag{4.7}$$

Indeed, for any $C$, we find from Corollary 3.22(b) that

$$\mathcal{E}^\bullet_C|_{i\mathbb{R}^n} = \underline{E^\bullet_C \otimes \mathrm{or}(C)}_{i\overline{C}}.$$

On the other hand, by its original definition (involving cohomology with support in $\mathbb{R}^n$), $\mathcal{R}_{\mathcal{F}^\star}$ is given by complexes of vector spaces $E^\bullet_C(\mathcal{F}^\star)$ and generalization maps $\gamma^{\mathcal{F}^\star}_{C'C}$. So by Proposition 1.11 the shifted Verdier dual $(\mathcal{R}_{\mathcal{F}^\star})^\star$

is the complex

$$\bigoplus_{\mathrm{codim}(C)=0} \underline{E^\bullet_C(\mathcal{F}^\bigstar)^* \otimes \mathrm{or}(C)}_{\overline{C}} \tag{4.8}$$
$$\xrightarrow{(\gamma^{\mathcal{F}^\bigstar}_{C'C"})^*} \bigoplus_{\mathrm{codim}(C)=1} \underline{E^\bullet_C(\mathcal{F}^\bigstar)^* \otimes \mathrm{or}(C)}_{\overline{C}} \xrightarrow{(\gamma^{\mathcal{F}^\bigstar}_{C'C"})^*} \cdots .$$

Comparing (4.7) and (4.8), we get our statement. □

4.B. *Perverse sheaves.* Let $\mathrm{Perv}(\mathbb{C}^n, \mathcal{H}) \subset D^b_{\mathcal{S}^{(0)}} \mathrm{Sh}_{\mathbb{C}^n}$ be the full subcategory of perverse sheaves. We choose the following normalization of the perversity conditions for a complex $\mathcal{F}$ (differing by a shift from that of [25, §10.3]):

- $(P^-)$ for each $p$, the sheaf $\underline{H}^p(\mathcal{F})$ is supported on a closed complex subspace of codimension $\geq p$;
- $(P^+)$ if $l : Z \to X$ is a locally closed embedding of a smooth analytic submanifold of codimension $p$, then the sheaf $\underline{H}^q(l^! \mathcal{F}) = \underline{\mathbb{H}}^q_Z(\mathcal{F})|_Z$ is zero for $q < p$.

With respect to this definition, a constant sheaf on $\mathbb{C}^n$ is perverse if put in degree 0. The normalized Verdier duality functor $\bigstar$ interchanges $(P^-)$ and $(P^+)$ and preserves $\mathrm{Perv}(\mathbb{C}^n, \mathcal{H})$.

PROPOSITION 4.9.

(a) *If $\mathcal{F}$ is perverse, then each $E^\bullet_C(\mathcal{F})$ is quasi-isomorphic to one vector space $E_C(\mathcal{F})$ in degree* 0.

(b) *For any $C \in \mathcal{C}$, the functor*

$$E_C : \mathrm{Perv}(\mathbb{C}^n, \mathcal{H}) \longrightarrow \mathrm{Vect}^{\mathrm{fd}}_{\mathbf{k}}, \quad \mathcal{F} \mapsto E_C(\mathcal{F})$$

*is an exact functor of abelian categories.*

*Proof.* The functor

$$E^\bullet_C : D^b_{\mathcal{H}_C} \mathrm{Sh}_{\mathbb{C}^n} \longrightarrow D^b \, \mathrm{Vect}^{\mathrm{fd}}_{\mathbf{k}}, \quad \mathcal{F} \mapsto E^\bullet_C(\mathcal{F})$$

is an exact functor of triangulated categories. So (b) will follow from (a), and we concentrate on (a).

Because of Proposition 4.6, it is enough to show that $H^i E^\bullet_C(\mathcal{F})$ vanishes for $i > 0$, since vanishing for $i < 0$ will then follow by duality. Since the $E^\bullet_C(\mathcal{F})$ are the linear algebra data describing the cellular complex $\mathcal{R}_\mathcal{F} = \underline{R\Gamma}_{\mathbb{R}^n}(\mathcal{F})[n]$, we need to show that

$$\underline{\mathbb{H}}^p_{\mathbb{R}^n}(\mathcal{F}) = 0, \quad p > n.$$

From the spectral sequence

$$\underline{H}^i_{\mathbb{R}^n}(\underline{H}^j(\mathcal{F})) \Rightarrow \underline{\mathbb{H}}^{i+j}_{\mathbb{R}^n}(\mathcal{F})$$

we see that it is enough to show that

$$\underline{H}^i_{\mathbb{R}^n}(\underline{H}^j(\mathcal{F})) = 0 \quad \text{for} \quad i > n - j.$$

But by $(P^+)$, the sheaf $\underline{H}^j(\mathcal{F})$ is supported on the union of complex flats $L_{\mathbb{C}}$ of $\mathcal{H}$ of codimension $\geq j$, i.e., of dimension $\leq n-j$. So our statement follows from the next lemma.

Lemma 4.10. *Let $\mathcal{G}$ be an $\mathcal{S}^{(0)}$-smooth sheaf on $\mathbb{C}^n$, with* $\dim_{\mathbb{C}}(\mathrm{Supp}(\mathcal{G})) \leq q$. *Then $\underline{H}^r_{\mathbb{R}^n}(\mathcal{G}) = 0$ for $r > q$.*

*Proof.* Since any $\mathcal{S}^{(0)}$-smooth sheaf is $\mathcal{S}^{(2)}$-smooth, $\mathcal{G}$ has a filtration with quotients of the form $j_!\mathcal{L}$, where $j$ is the embedding of a stratum of $\mathcal{S}^{(2)}$ and $\mathcal{L}$ is a local system on this stratum. So it is enough to show that for any such quotient, we have $\underline{H}^r_{\mathbb{R}^n}(j_!\mathcal{L}) = 0$ for $r > q$.

Now, a stratum of $\mathcal{S}^{(2)}$ is a product cell $A + iB$ where $A, B$ are cells of $\mathcal{H}$ whose dimensions we denote by $a, b$. We note the following:

(1) By our assumptions on $\mathcal{G}$, each product cell $A + iB$ for which $j_!\mathcal{L}$ can appear as a quotient of a filtration of $\mathcal{G}$ satisfies $a, b \leq q$. We therefore assume this.

(2) As $A + iB$ is a cell, a local system $\mathcal{L}$ on it must be trivial. So it is enough to assume that $\mathcal{L} = \underline{\mathbf{k}}_{A+iB}$, where $a, b \leq q$ by (1).

(3) The local cohomology sheaves $\underline{H}^r_{\mathbb{R}^n}(j_!\underline{\mathbf{k}}_{A+iB})$ vanish for $r \neq b$. Indeed, denote by $\varepsilon : \mathbb{R}^n \to \mathbb{C}^n$ the embedding. Then $\underline{R\Gamma}_{\mathbb{R}^n}(j_!\underline{\mathbf{k}}_{A+iB})$ can be written as $\varepsilon^! j_! \underline{\mathbf{k}}_{A+iB}$, which is Verdier dual to

$$\varepsilon^* j_* \mathbb{D}(\underline{\mathbf{k}}_{A+iB}) \simeq \varepsilon^* j_*(\underline{\mathbf{k}}_{A+iB}[a+b])$$

(isomorphism depending on a choice of orientation of $A + iB$ that we fix). But $j_*\underline{\mathbf{k}}_{A+iB} = \underline{\mathbf{k}}_{\overline{A}+i\overline{B}}$ is the constant sheaf on the closed cell, and therefore $\varepsilon^* j_* \underline{\mathbf{k}}_{A+iB} = \underline{\mathbf{k}}_{\overline{A}}$, the constant sheaf on the closed cell $\overline{A}$, considered as a sheaf on $\mathbb{R}^n$. Therefore (choosing a co-orientation of $A$ in $\mathbb{R}^n$), we identify

$$\varepsilon^! j_! \underline{\mathbf{k}}_{A+iB} = \mathbb{D}(\underline{\mathbf{k}}_{\overline{A}}[a+b]) \;\simeq\; j_{A!}\underline{\mathbf{k}}_A[-b], \quad j_A : A \hookrightarrow \mathbb{R}^n,$$

which has only one cohomology sheaf, namely $j_{A!}\underline{\mathbf{k}}_A$ in degree $b$. This proves the lemma and Proposition 4.9. □

Corollary 4.11.

(a) *If $\mathcal{F} \in \mathrm{Perv}(\mathbb{C}^n, \mathcal{H})$, then each complex $\mathcal{E}^\bullet_C(\mathcal{F})$ is quasi-isomorphic to a single sheaf $\mathcal{E}_C(\mathcal{F})$ in degree* 0, *and*

$$\mathcal{E}^\bullet(\mathcal{F}) = \left\{ \bigoplus_{\mathrm{codim}(C)=0} \mathcal{E}_C \xrightarrow{\widetilde{\delta}} \bigoplus_{\mathrm{codim}(C)=1} \mathcal{E}_C \xrightarrow{\widetilde{\delta}} \cdots \xrightarrow{\widetilde{\delta}} \mathcal{E}_0 \right\}$$

*is a complex of sheaves on $\mathbb{C}^n$ in the usual sense, quasi-isomorphic to $\mathcal{F}$.*

(b) *For any $C \in \mathcal{C}$, the functor*

$$\mathcal{E}_C : \mathrm{Perv}(\mathbb{C}^n, \mathcal{H}) \longrightarrow \mathrm{Sh}_{\mathbb{C}^n}, \quad \mathcal{F} \mapsto \mathcal{E}_C(\mathcal{F})$$

*is an exact functor of abelian categories.*

We will call $\mathcal{E}^\bullet(\mathcal{F})$ the *Cousin resolution* of $\mathcal{F}$.

*Proof.* (a) follows from Propositions 4.9(a) and 3.18. Part (b) follows, similarly to Proposition 4.9(b), from the exactness of $\mathcal{F} \mapsto \mathcal{E}_C^\bullet(\mathcal{F})$ on the derived category. □

*Definition* 4.12. By a *double representation* of the poset $(\mathcal{C}, \leq)$ we mean a datum $Q = (E_C, \gamma_{C'C}, \delta_{CC'})$ consisting of finite-dimensional $\mathbf{k}$-vector spaces $E_C, C \in C$ and linear maps

$$\gamma_{C'C} : E_{C'} \to E_C, \quad \delta_{CC'} : E_C \to E_{C'}, \quad C' \leq C,$$

so that $(\gamma_{C'C})$ is a covariant representation and $(\delta_{CC'})$ is a contravariant representation of $(\mathcal{C}, \leq)$.

Double representations of $\mathcal{C}$ form an abelian category, which we denote by $\mathrm{Rep}^{(2)}(\mathcal{C})$. This category has a perfect duality

$$Q \mapsto Q^* = (E_C^*, \delta_{CC'}^*, \gamma_{C'C}^*).$$

The results of this sections imply that we have an exact functor

$$\begin{aligned} &\mathcal{Q} : \mathrm{Perv}(\mathbb{C}^n, \mathcal{H}_\mathbb{C}) \longrightarrow \mathrm{Rep}^{(2)}(\mathcal{C}), \\ &\mathcal{F} \mapsto \mathcal{Q}(\mathcal{F}) = \big(E_C(\mathcal{F}), \gamma^{\mathcal{F}}_{C'C"}, \delta^{\mathcal{F}}_{CC'}\big), \end{aligned} \tag{4.13}$$

commuting with duality. We will call $\mathcal{Q}(\mathcal{F})$ the *double quiver* associated to the perverse sheaf $\mathcal{F}$.

Let us note the following converse to Proposition 4.9.

PROPOSITION 4.14. *Let $\mathcal{F} \in D^b_{\mathcal{S}^{(0)}} \mathrm{Sh}_{\mathbb{C}^n}$ be an $\mathcal{S}^{(0)}$-constructible complex such that each $E_C^\bullet(\mathcal{F})$, $C \in \mathcal{C}$, is quasi-isomorphic to a single vector space in degree* 0. *Then $\mathcal{F}$ is perverse.*

*Proof.* Our assumptions imply that we have a Cousin resolution $\mathcal{E}^\bullet(\mathcal{F})$ of $\mathcal{F}$ as in Corollary 4.11. So it is enough to show that $\mathcal{E}^\bullet(\mathcal{F})$ satisfies $(P^-)$ and $(P^+)$. By construction, $\mathcal{E}^p(\mathcal{F})$ is supported on the union of the $\mathbb{R}^n + i\overline{C}$ for $C \in \mathcal{C}$, $\mathrm{codim}(C) = p$. Therefore $\underline{H}^p(\mathcal{E}^\bullet(\mathcal{F}))$ is supported on the union of complex flats $L_\mathbb{C}$, $L \in \mathcal{L}$, that are contained in the above union of the $\mathbb{R}^n + i\overline{C}$. Each such $L$ must have codimension $\geq p$. So $\mathcal{E}^\bullet(\mathcal{F}) \sim \mathcal{F}$ satisfies $(P^-)$. Now, look at $\mathcal{F}^\bigstar$. The double quiver corresponding to $\mathcal{F}^\bigstar$, being, by Proposition 4.6, identified with $\mathcal{Q}(\mathcal{F})^*$, also consists of single vector spaces in degree 0. Therefore the above reasoning shows that $\mathcal{F}^\star$ satisfies $(P^-)$, and so $\mathcal{F}$ satisfies $(P^+)$ and so is perverse. □

4.C. *Relation to earlier works.*

4.C.1. Suppose that $\mathbf{k}$ (our coefficient field for perverse sheaves) is equal to $\mathbb{C}$. By the Riemann-Hilbert correspondence, any $\mathcal{F} \in \mathrm{Perv}(\mathbb{C}^n, \mathcal{H})$ can be represented as the solution sheaf of a holonomic $\mathcal{D}$-module $\mathcal{M}$ on $\mathbb{C}^n$:

$$\mathcal{F} = \underline{R\,\mathrm{Hom}}_{\mathcal{D}_{\mathbb{C}^n}}(\mathcal{M}, \mathcal{O}_{\mathbb{C}^n}).$$

In this case one can give another, analytic, proof of Proposition 4.9. Indeed, the complex $\mathcal{R}^\bullet_{\mathcal{F}}$ can be written as

$$\mathcal{R}^\bullet_{\mathcal{F}} = \underline{R\,\mathrm{Hom}}_{\mathcal{D}_{\mathbb{C}^n}}(\mathcal{M}, \underline{R\Gamma}_{\mathbb{R}^n}(\mathcal{O}_{\mathbb{C}^n}))[n].$$

By the classical result of Sato [34]

$$\underline{R\Gamma}_{\mathbb{R}^n}(\mathcal{O}_{\mathbb{C}^n})[n] \;\sim\; \underline{H}^n_{\mathbb{R}^n}(\mathcal{O}_{\mathbb{C}^n}) = j_*\mathcal{B}_{\mathbb{R}^n} \otimes_{\mathbb{C}} \mathrm{or}(\mathbb{R}^n), \quad j : \mathbb{R}^n \hookrightarrow \mathbb{C}^n$$

reduces to the sheaf of hyperfunctions $\mathcal{B}_{\mathbb{R}^n}$, so

$$\mathcal{R}^\bullet_{\mathcal{F}} = \underline{R\,\mathrm{Hom}}_{\mathcal{D}_{\mathbb{C}^n}}(\mathcal{M}, j_*\mathcal{B}_{\mathbb{R}^n}) \otimes_{\mathbb{C}} \mathrm{or}(\mathbb{R}^n)$$

is the complex of hyperfunction solutions of $\mathcal{M}$. The fact that this complex reduces to a single sheaf follows from the result of Lebeau [28] (see [21] for the proof of a more general statement) that implies that under our assumptions, $\underline{\mathrm{Ext}}^q_{\mathcal{D}_{\mathbb{C}^n}}(\mathcal{M}, j_*\mathcal{B}_{\mathbb{R}^n}) = 0$ for $q > 0$. So $\mathcal{R}^\bullet_{\mathcal{F}}$ is quasi-isomorphic to the sheaf of hyperfunction solutions of $\mathcal{M}$ in the nonderived sense.

In particular, $E_0(\mathcal{F}) = (\mathcal{R}_{\mathcal{F}})_0 = \Gamma(\mathbb{R}^n, \mathcal{R}_{\mathcal{F}})$ is the space of global hyperfunction solutions of $\mathcal{M}$, and its dimension was found by Takeuchi [39] to be the sum of multiplicities of $\mathcal{F}$ (or $\mathcal{M}$) along all the possible components of the characteristic variety:

$$\dim_{\mathbb{C}} E_0(\mathcal{F}) = \sum_{L \in \mathcal{L}} \mathrm{mult}_{T^*_{L_{\mathbb{C}}}\mathbb{C}^n}\, \mathcal{F}.$$

This generalizes the classical index formula of Kashiwara-Komatsu in dimension 1; see [24, Th. 4.2.7] [27]. See also [38, Th. 1.2] for a generalization to arrangements of nonlinear analytic subvarieties.

More generally, for any $C \in \mathcal{C}$, we deduce, by passing to the transverse slice to $\mathbb{L}(C)$, that

$$\dim E_C(\mathcal{F}) \;=\; \sum_{L \in \mathcal{L} \text{ s.t. } C \subset L} \mathrm{mult}_{T^*_{L_{\mathbb{C}}}\mathbb{C}^n}\, \mathcal{F}.$$

See Section 9 for a discussion of low-dimensional cases and identification of the $E_C(\mathcal{F})$ in these cases in terms of the standard functors of nearby and vanishing cycles.

4.C.2. The vector spaces $E_C(\mathcal{F})$ are the same as the spaces of "generalized vanishing cycles" introduced in [3, Part I, §3.3]. Our Proposition 4.9 corresponds to Theorem 3.9 of [3, Part I], which says that the complexes of generalized vanishing cycles reduce to single vector spaces, while the part of Proposition 4.6 pertaining to the spaces $E_C$ corresponds to Theorem 3.5 of [3, Part I]. Note that in our approach the more immediate maps among the $E_C$ are the $\gamma_{C'C}$, while in [3, Part I, §3.11] it is the $\delta_{CC'}$ (the "variation maps"). This is due to the fact that the definition of op. cit. is "Verdier dual" to ours: the spaces there are our $E_C(\mathcal{F}^*)^*$.

## 5. **Functorialities of the double quiver**

5.A. *Hyperbolic restriction.* Let $L \in \mathcal{L}$ be a flat of $\mathcal{H}$ of codimension $d$. Consider the embeddings

$$L_{\mathbb{C}} = L + iL \xrightarrow{k} \mathbb{R}^n + iL \xrightarrow{j} \mathbb{C}^n.$$

Recall that $L$ carries the induced arrangement $\mathcal{H} \cap L$ with poset of faces $\mathcal{C}_{\mathcal{H}\cap L} \simeq \mathcal{C}^{\leq L}$ naturally a subposet of $\mathcal{C} = \mathcal{C}_{\mathcal{H}}$. Therefore we have the restriction functor

$$\mathrm{Rep}^{(2)}(\mathcal{C}) \longrightarrow \mathrm{Rep}^{(2)}(\mathcal{C}^{\leq L}), \quad Q \mapsto Q^{\leq L},$$

which takes a double quiver $Q = (E_C, \gamma_{C'C}, \delta_{CC'})$ to its subdiagram involving only $C', C$ that are contained in $L$.

PROPOSITION 5.1. *Let $\mathcal{F} \in \mathrm{Perv}(\mathbb{C}^n, \mathcal{H})$ be an $\mathcal{S}^{(0)}$-smooth perverse sheaf with double quiver $Q$. Then $k^* j^! \mathcal{F}[d]$ is an object of $\mathrm{Perv}(L_{\mathbb{C}}, \mathcal{H} \cap L)$, and its associated double quiver is $Q^{\leq L}$.*

*Remark* 5.2. The statement about perversity of $k^* j^! \mathcal{F}[d]$ can be seen as a real analytic analog of the main result of Braden [8]. (See also [12] for a more in-depth treatment.) Extending the terminology of [8], we will call the perverse sheaf $k^* j^! \mathcal{F}[d]$ the *hyperbolic restriction* of $\mathcal{F}$ to $L_{\mathbb{C}}$.

*Proof of Proposition* 5.1. First, we notice that $k^* j^! \mathcal{F}$ is an $\mathcal{S}^{(0)}$-smooth constructible complex on $L_{\mathbb{C}}$. This follows at once from the estimate (3.14) for the singular support of the inverse image. Thus $k^* j^! \mathcal{F}[d]$ can be described by a double quiver of complexes of vector spaces, and we analyze this double quiver.

Recall that $\mathcal{F}$ is represented by its Cousin resolution $\mathcal{E}^\bullet = \mathcal{E}^\bullet(\mathcal{F})$ with $\mathcal{E}^p$ being the direct sum of cellular sheaves $\mathcal{E}_C$ for $C$ running over faces of $\mathcal{H}$ of codimension $p$.

If $C \subset L$, then $\mathbb{R}^n + i\overline{C}$, the support of $\mathcal{E}_C$, is contained in $\mathbb{R}^n + iL$, the source of $j$. Therefore $j^! \mathcal{E}_C = j^* \mathcal{E}_C$, and so

$$k^* j^! \mathcal{E}_C = k^* j^* \mathcal{E}_C = (\mathcal{E}_C)|_{L_{\mathbb{C}}}.$$

Recall that $\mathcal{E}_C = \overline{p}_C^* \mathcal{E}_C^{\mathrm{red}}$, where $\mathcal{E}_C^{\mathrm{red}}$ is a cellular sheaf on $\mathbb{R}^n/\mathbb{L}(C)$ given by the spaces $E_K, K \geq C$ from $Q$ and their $\gamma$-maps. So the restriction $(\mathcal{E}_C)|_{L_\mathbb{C}}$ is a similar pullback of the restriction of $\mathcal{E}_C^{\mathrm{red}}$ to $L/\mathbb{L}(C)$, which is given by the $E_K$ for $C \leq K \subset L$, so the proposition is true in the case $C \subset L$.

If $C \not\subset L$, then $C \cap L = \emptyset$, and so

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & \mathbb{R}^n + iC \arrow[d, "j_C"] \\ \mathbb{R}^n + iL \arrow[r, "j"'] & \mathbb{C}^n \end{tikzcd}$$

is a Cartesian square. Therefore, $j^! \mathcal{E}_C = j^! j_{C*}(j_C^! \mathcal{F}) = 0$ by base change [25, Prop. 3.1.9].

We conclude that $k^* j^! \mathcal{E}^\bullet[d]$ is an $\mathcal{S}^{(0)}$-constructible complex given by the double quiver $Q^{\leq L}$. Since $Q^{\leq L}$ consists of single vector spaces (not just complexes), Proposition 4.14 implies that $k^* j^! \mathcal{F}[d]$ is perverse. □

5.B. *Transversal slice.* Let $L \in \mathcal{L}$ be as before. The normal bundle $N_{L/\mathbb{R}^n}$ is canonically trivialized, with fiber $\mathbb{R}^n/L$. Recall that $\mathbb{R}^n/L$ carries the quotient arrangement $\mathcal{H}/L$. Choose a face $C \subset L$ open in $L$. Then the projection $\overline{\pi}_C : \mathcal{C}^{\geq C} \to \mathcal{C}_{\mathcal{H}/L}$ identifies $\mathcal{C}_{\mathcal{H}/L}$ with the subposet $\mathcal{C}^{\geq C} \subset \mathcal{C}$. This leads to another restriction functor

$$\mathrm{Rep}^{(2)}(\mathcal{C}) \longrightarrow \mathrm{Rep}^{(2)}(\mathcal{C}^{\geq C}), \quad Q \mapsto Q^{\geq C},$$

defined similarly to that in Section 5.A

As before, suppose we are given a perverse sheaf $\mathcal{F} \in \mathrm{Perv}(\mathbb{C}^n, \mathcal{H})$ with double quiver $Q$. We then have the *specialization* $\mathrm{Sp}_{L_\mathbb{C}}(\mathcal{F})$; see [25] for background. It is a perverse sheaf on (the total space of) the normal bundle $N_{L_\mathbb{C}/\mathbb{C}^n} = L_\mathbb{C} \times (\mathbb{C}^n/L_\mathbb{C})$. Choose some point $c \in C$ (this choice will be immaterial), and let $N_c \simeq \mathbb{C}^n/L_\mathbb{C}$ be the fiber of the above normal bundle over $c$. We can think of $N_c$ as a transversal slice to $L_\mathbb{C}$ at $c$. Note that $N_c$ is transverse to the characteristic variety of $\mathrm{Sp}_{L_\mathbb{C}}(\mathcal{F})$, and so we have the perverse sheaf

$$\mathcal{F}|_{N_c} := \mathrm{Sp}_{L_\mathbb{C}}(\mathcal{F})|_{N_c} \in \mathrm{Perv}(\mathbb{C}^n/L_\mathbb{C}, \mathcal{H}/L).$$

PROPOSITION 5.3. *The double quiver of $\mathcal{F}|_{N_c}$ is identified with $Q^{\geq C}$.*

*Proof.* Let $T \subset \mathbb{R}^n$ be an affine subspace forming a transversal slice to $L$ at $c$. We consider $T$ as an $\mathbb{R}$-vector space with origin $c$. Then the composition $T \hookrightarrow \mathbb{R}^n \to \mathbb{R}^n/L$ is an isomorphism of vector spaces. To understand the specialization $\mathrm{Sp}_{L_\mathbb{C}}(\mathcal{F})$ and the perverse sheaf $\mathcal{F}|_{N_c}$, we use the Cousin resolution $\mathcal{E}^\bullet = \mathcal{E}^\bullet(\mathcal{F})$ and analyze $\mathrm{Sp}_{L_\mathbb{C}}(\mathcal{E}_K)$ for each summand $\mathcal{E}_K$ of each term of $\mathcal{E}^\bullet$.

We claim that

$$\mathrm{Sp}_{L_{\mathbb{C}}}(\mathcal{E}_K) = \begin{cases} \mathcal{E}_{\overline{\pi}_C(K)}(\mathrm{Sp}_{L_{\mathbb{C}}}(\mathcal{F})) & \text{if } K \geq C, \\ 0 & \text{otherwise.} \end{cases} \tag{5.4}$$

Indeed, the small neighborhood of the origin in $T_{\mathbb{C}}$ meets only those closures of tube cells $\mathbb{R}^n + i\overline{K}$ for which $K \geq C$. If $K \geq C$, the statement follows from the fact that $T_{\mathbb{C}}$ is transversal to the characteristic varieties of all the sheaves involved and therefore taking cohomology with support commutes with restriction to $T_{\mathbb{C}}$.

Now, the statement about the double quiver of $\mathcal{F}|_{N_c}$ follows from (5.4) immediately. □

## 6. **The double quiver determines a perverse sheaf**

The goal of this section is to prove the following preliminary result.

THEOREM 6.1. *The functor* $\mathcal{Q} : \mathrm{Perv}(\mathbb{C}^n, \mathcal{H}) \to \mathrm{Rep}^{(2)}(\mathcal{C})$ *from* (4.13) *is fully faithful.*

6.A. *Orthogonality relations.* For an abelian category $\mathcal{A}$, let $C^b\mathcal{A}$ be the abelian category formed by bounded complexes over $\mathcal{A}$ and morphisms of complexes (not homotopy classes) in the usual sense. We start with

PROPOSITION 6.2. *The Cousin resolution functor*

$$\mathcal{E}^\bullet : \mathrm{Perv}(\mathbb{C}^n, \mathcal{H}) \longrightarrow C^b\,\mathrm{Sh}_{\mathbb{C}^n}, \quad \mathcal{F} \mapsto \mathcal{E}^\bullet(\mathcal{F})$$

*is fully faithful.*

*Proof.* Let $\mathcal{I}$ be the image of the functor $\mathcal{E}^\bullet$, i.e., the full subcategory in $C^b\,\mathrm{Sh}_{\mathbb{C}^n}$ consisting of complexes of sheaves of the form $\mathcal{E}^\bullet(\mathcal{F})$ for $\mathcal{F} \in \mathrm{Perv}(\mathbb{C}^n, \mathcal{H})$. Thus we need to show that $\mathcal{E}^\bullet : \mathrm{Perv}(\mathbb{C}^n, \mathcal{H}) \to \mathcal{I}$ is an equivalence of categories. Define the functor $\Xi : \mathcal{I} \to \mathrm{Perv}(\mathbb{C}^n, \mathcal{H})$ to fit into the commutative diagram of functors

$$\begin{tikzcd} \mathcal{I} \arrow[r, "\Xi"] \arrow[d, "\mathrm{emb}'"'] & \mathrm{Perv}(\mathbb{C}^n, \mathcal{H}) \arrow[d, "\mathrm{emb}''"] \\ C^b\,\mathrm{Sh}_{\mathbb{C}^n} \arrow[r, "\mathrm{can}"'] & D^b\,\mathrm{Sh}_{\mathbb{C}^n}\,. \end{tikzcd}$$

Here $\mathrm{emb}'$ is the embedding of $\mathcal{I}$ into the abelian category of complexes, $\mathrm{emb}''$ is the embedding of $\mathrm{Perv}(\mathbb{C}^n, \mathcal{H})$ into the derived category, and can is the canonical functor from the abelian category of complexes to the derived category. Thus $\Xi(\mathcal{G})$ for $\mathcal{G} \in \mathcal{I}$ is defined as the image of $\mathcal{G}$ in the derived category. We know this image to be a perverse sheaf, i.e., an object of $\mathrm{Perv}(\mathbb{C}^n, \mathcal{H})$. Indeed, since $\mathcal{G} \in \mathcal{I}$, we have that $\mathcal{G} = \mathcal{E}^\bullet(\mathcal{F})$ for some $\mathcal{F} \in \operatorname{Perv}(\mathbb{C}^n, \mathcal{H})$, and we know that $\mathcal{E}^\bullet(\mathcal{F})$ is quasi-isomorphic to $\mathcal{F}$ by Corollary 4.11.

We now prove that the functors $\operatorname{Perv}(\mathbb{C}^n, \mathcal{H}) \underset{\Xi}{\overset{\mathcal{E}^\bullet}{\rightleftarrows}} \mathcal{I}$ are quasi-inverse to each other. First, for $\mathcal{F} \in \operatorname{Perv}(\mathbb{C}^n, \mathcal{H})$, we notice that $\Xi(\mathcal{E}^\bullet(\mathcal{F}))$ is canonically identified with $\mathcal{F}$ by Corollary 4.11. Further, for $\mathcal{G} \in \mathcal{I}$ we prove that $\mathcal{E}^\bullet(\Xi(\mathcal{G}))$ is canonically isomorphic to $\mathcal{G}$ in the abelian category of complexes. This is an immediate consequence of the following "orthogonality relations."

LEMMA 6.3. *Let $\mathcal{F} \in \operatorname{Perv}(\mathbb{C}^n, \mathcal{H}_\mathbb{C})$. For any $C, C' \in \mathcal{C}$ we have*

$$\underline{R\Gamma}_{\mathbb{R}^n + iC'} \mathcal{E}_C(\mathcal{F}) = \begin{cases} \mathcal{E}_C(\mathcal{F}) & \textit{if } C' = C, \\ 0 & \textit{if } C' \neq C. \end{cases}$$

*Proof.* If $C' = C$, the statement of the lemma is obvious. If $C' \neq C$, then $C' \cap C = \emptyset$, and so

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & \mathbb{R}^n + iC \arrow[d, "j_C"] \\ \mathbb{R}^n + iC' \arrow[r, "j_{C'}"'] & \mathbb{C}^n \end{tikzcd}$$

is a Cartesian square. So base change [25, Prop. 3.1.9] implies that $j_{C'}^! j_{C*} \mathcal{G} = 0$ for any $\mathcal{G} \in D^b \operatorname{Sh}_{\mathbb{R}^n + iC}$. In particular,

$$\underline{R\Gamma}_{\mathbb{R}^n + iC'} \mathcal{E}_C(\mathcal{F}) = j_{C'*} j_{C'}^! j_{C*} j_C^! \mathcal{F} = 0.$$

Lemma 6.3 and Proposition 6.2 are proved. □

6.B. *Recovery of $\mathcal{E}^\bullet(\mathcal{F})$ from $\mathcal{Q}(\mathcal{F})$.* We prove Theorem 6.1 by the argument similar to that in the proof of Proposition 6.2. That is, as a first step, we explain how to "recover" the entire complex $\mathcal{E}^\bullet(\mathcal{F})$ from the double quiver $\mathcal{Q}(\mathcal{F})$. Next, the second step is to consider the image $\mathcal{J} \subset \operatorname{Rep}^{(2)}(\mathcal{C})$ of the functor $\mathcal{Q}$ and to interpret the "recovery" procedure by constructing a functor $\Theta : \mathcal{J} \to \operatorname{Perv}(\mathbb{C}^n, \mathcal{H})$ quasi-inverse to $\mathcal{Q} : \operatorname{Perv}(\mathbb{C}^n, \mathcal{H}) \to \mathcal{J}$.

The first step proceeds as follows. By Proposition 3.18, each sheaf $\mathcal{E}_C(\mathcal{F})$ is recovered from the data of $E_D(\mathcal{F})$, $D \geq C$ and of $\gamma_{D'D}$, $C \leq D' \leq D$. So the only remaining data are the matrix elements of the differentials. Eliminating the orientation torsors as in (4.4), we write these matrix elements as morphisms of sheaves

$$\underline{\delta}_{CC'} : \mathcal{E}_C(\mathcal{F}) \longrightarrow \mathcal{E}_{C'}(\mathcal{F}), \quad C' <_1 C. \tag{6.4}$$

Similarly to Proposition 4.5, we have the following fact.

PROPOSITION 6.5. *The morphisms of sheaves $\underline{\delta}_{CC'}$ commute, i.e., extend to a contravariant representation of $(\mathcal{C}, \leq)$ in $\operatorname{Sh}_{\mathbb{C}^n}$.*

In other words, we have a well-defined map $\underline{\delta}_{CC'}$ for any inclusion $C' < C$, not necessarily of codimension 1 obtained by composing the "elementary" maps corresponding to codimension 1 inclusions.

The data contained in $\underline{\delta}_{CC'}$ are precisely the induced morphisms on stalks of the sheaves $\mathcal{E}_C, \mathcal{E}_{C'}$ over all the cells $D + iC', D \in \mathcal{C}$, which are linear maps

$$\delta_{CC'|D} : E_{C\circ D} \longrightarrow E_{C'\circ D}. \tag{6.6}$$

So it is enough to express each $\delta_{CC'|D}$ through the $\gamma$ and $\delta$ maps of the double quiver $Q = \mathcal{Q}(\mathcal{E})$. We start with the following statement.

Proposition 6.7. *For each $C' \le C$, we have $\gamma_{C'C} \circ \delta_{CC'} = \mathrm{Id}_{E_C}$. In particular, each $\gamma_{C'C}$ is surjective, each $\delta_{CC'}$ is injective and* $\dim(E_{C'}) \ge \dim(E_C)$.

*Proof.* It is enough to prove the statement for $C' <_1 C$, which we assume. We use the fact that the maps of stalks induced by the morphism of sheaves $\underline{\delta}_{CC'}$ commute with the generalization maps from $C' + iC'$ to $C + iC'$. This translates into the commutativity of

$$\begin{tikzcd}
\mathcal{E}_C|_{C'+iC'} = E_C \arrow[r, "\delta_{CC'}=\delta_{CC'|C'}"] \arrow[d, "\mathrm{Id}"'] & E_{C'} = \mathcal{E}_{C'}|_{C'+iC'} \arrow[d, "\gamma_{C'C}"] \\
\mathcal{E}_C|_{C+iC'} = E_C \arrow[r, "\delta_{CC'|C}"'] & E_C = \mathcal{E}_{C'}|_{C+iC'}.
\end{tikzcd}$$

Here the vertical Id is the generalization map for $\mathcal{E}_C$ and $\gamma_{C'C}$ is the generalization map for $\mathcal{E}_{C'}$. Our statement follows therefore from the next lemma.

Lemma 6.8. *The map $\delta_{CC'|C}$ is equal to* $\mathrm{Id}_{E_C}$.

*Proof of the lemma.* Recall the identifications of stalks

$$\mathcal{E}_C|_{C+iC'} \xrightarrow{\alpha} E_C, \quad \mathcal{E}_{C'}|_{C+iC'} \xrightarrow{\alpha'} E_C$$

given in the proof of Proposition 3.18. Let $d = \mathrm{codim}(C)$. In the notation of 3.18, we take $K = C$ and form the complex $\mathcal{G} = \underline{R\Gamma}_{\mathbb{R}^n+i\mathbb{L}(C)}(\mathcal{F})[d]$, so we have

(1) the restriction of $\mathcal{G}$ on $C + i\mathbb{L}(C)$ reduces to a constant sheaf in degree 0, denote it $\mathcal{N}$;

(2) $\mathcal{E}_C|_{C+iC'} = \mathcal{G}|_{C+iC'}$, while $E_C = \underline{R\Gamma}_{\mathbb{R}^n+i0}(\mathcal{G})[n-d]|_{C+i0}$, and $\alpha$ is induced by the local Poincaré duality for $\mathcal{N}$.

In the same way, $\alpha'$ is induced by the local Poincaré duality for the constant sheaf $\mathcal{N}'$ obtained as the restriction to $C + i\mathbb{L}(C')$ of

$$\mathcal{G}' = \underline{R\Gamma}_{\mathbb{R}^n+i\mathbb{L}(C')}(\mathcal{F})[d+1] = \underline{R\Gamma}_{\mathbb{R}^n+i\mathbb{L}(C')}(\mathcal{G})[1].$$

The last equality above means that

$$\mathcal{N}' = \underline{H}^1_{C+i\mathbb{L}(C')}(\mathcal{N}),$$

which we interpret, by codimension 1 local Poincaré duality, as

$$\mathcal{N}' \;\simeq\; k^*\mathcal{N} \otimes \mathrm{or}(C'/C) \xrightarrow{\sim} k^*\mathcal{N}. \tag{6.9}$$

Here $k : C + i\mathbb{L}(C') \hookrightarrow C + i\mathbb{L}(C)$ is the embedding, and the last isomorphism comes from the identification of orientation torsors given by the codimension 1 inclusion $C' <_1 C$. Now, $\underline{\delta}_{CC'}$ is the coboundary map

$$\underline{\mathbb{H}}^d_{\mathbb{R}^n+iC}(\mathcal{F}) \longrightarrow \underline{\mathbb{H}}^{d+1}_{\mathbb{R}^n+iC'}(\mathcal{F}).$$

This means that, under our identifications, its stalk at $C + iC'$ becomes equal to the stalk, also at $C + iC'$, of the coboundary map for $\mathcal{N}$, which we write as

$$(j_*j^*\mathcal{N})|_{C+iC'} \xrightarrow{\delta} \underline{H}^1_{C+iC'}(\mathcal{N}) = (j')^*\mathcal{N}' \overset{(6.9)}{=} l^*\mathcal{N}.$$

Here

$$j : C+iC \hookrightarrow C+i\mathbb{L}(C), \quad j' : C+iC' \hookrightarrow C+i\mathbb{L}(C'), \quad l : C+iC' \hookrightarrow C+i\mathbb{L}(C)$$

are the embeddings. So our statement reduces to the following elementary fact. ("Codimension 1 Poincaré duality is given by the coboundary map.")

LEMMA 6.10. *Let $\epsilon : M_0 \to M$ be a closed codimension 1 embedding of $C^\infty$-manifolds and $J : M_+ \to M$ an open embedding such that the closure $\overline{J(M_+)}$ is a manifold with boundary $M_0$ (and so gives a trivialization of the orientation torsor $\mathrm{or}_{M_0/M}$). For any locally constant sheaf $\mathcal{K}$ on $M$, the coboundary map*

$$\epsilon^* J_* J^* \mathcal{K} \xrightarrow{\delta} \underline{H}^1_{M_0}(\mathcal{K})$$

*corresponds, after the identification $\epsilon^* J_* J^* \mathcal{K} = \epsilon^*\mathcal{K}$ and the Poincaré duality $\underline{H}^1_{M_0}(\mathcal{K}) \simeq \epsilon^*\mathcal{K}$, to the identity of $\epsilon^*\mathcal{K}$.*

This finishes the proof of Lemma 6.8 and Proposition 6.7. □

We now let $C, C', D$ be three arbitrary faces such that $C' \le C$. Put

$$K = C \circ D \ge C, \quad K' = C' \circ D \ge C'. \tag{6.11}$$

By the associativity of the operation $\circ$, we have $C \circ K' = K$ and $C' \circ K' = K'$. Note that because $\mathcal{E}_C$ and $\mathcal{E}_{C'}$ are pullbacks of sheaves on $\mathbb{R}^n/\mathbb{L}(C)$, resp. $\mathbb{R}^n/\mathbb{L}(C')$, we have

$$\delta_{CC'|D} = \delta_{CC'|K'}. \tag{6.12}$$

That is,

$$\delta_{CC'|D} : E_{C\circ D} = E_K \longrightarrow C_{C'\circ D} = E_{K'}$$

is equal to

$$\delta_{CC'|K'} : E_{C\circ K'} = E_K \longrightarrow E_{C'\circ K'} = E_{K'}.$$

To complete the recovery procedure of $\mathcal{F}$ from $\mathcal{Q}(\mathcal{F})$, we prove

PROPOSITION 6.13. *In the described situation, we have*

$$\delta_{CC'|D} = \gamma_{C'K'}\,\delta_{KC'} : E_K \longrightarrow E_{K'}.$$

*Proof.* We first use that the maps of stalks induced by the morphism of sheaves $\underline{\delta}_{CC'}$ commute with the generalization maps from $C' + iC'$ to $K' + iC'$. This gives the commutativity of

$$\begin{tikzcd} E_C \arrow[r, "\delta_{CC'}"] \arrow[d, "\gamma_{CK}"'] & E_{C'} \arrow[d, "\gamma_{C'K'}"] \\ E_K \arrow[r, "\delta_{CC'|K'}"'] & E_{K'}, \end{tikzcd}$$

i.e., the equality

$$\delta_{CC'|K'}\,\gamma_{CK} = \gamma_{C'K'}\,\delta_{CC'}.$$

We precompose this equality with $\delta_{KC}$:

$$\delta_{CC'|K'}\,\gamma_{CK}\,\delta_{KC} = \gamma_{C'K'}\,\delta_{CC'}\,\delta_{KC}.$$

Now, using Proposition 6.7 on the left and using the fact that the $\delta$ maps form a contravariant representation of $(\mathcal{C}, \leq)$ on the right, and also invoking (6.12), we get the desired statement. □

COROLLARY 6.14. *In the above situation, we also have*

$$\delta_{CC'|D} = \gamma_{0K'}\,\delta_{K0}.$$

6.C. *End of proof of Theorem* 6.1. We now perform the second step outlined at the beginning of Section 6.B by interpreting the above in a more categorical language. So we denote by $\mathcal{J} \subset \mathrm{Rep}^{(2)}(\mathcal{C})$ the image of the functor $\mathcal{Q}$ and construct a functor $\Theta$ as in the diagram

$$\mathrm{Perv}(\mathbb{C}^n, \mathcal{H}) \underset{\Theta}{\overset{\mathcal{Q}}{\rightleftarrows}} \mathcal{J}$$

so that the two functors are quasi-inverse. Explicitly, let $Q = (E_C, \gamma_{C'C}, \delta_{CC'})$ be a double quiver from $\mathcal{J}$. For each $C \in \mathcal{C}$, we define the sheaf $\mathcal{E}_C(Q)$ by postulating the formulas of Corollary 3.22(b), i.e., by setting

$$\mathcal{E}_C(Q)\Big|_{iC_1+D} = \begin{cases} E_{C\circ D} & \text{if } C_1 \leq C, \\ 0 & \text{otherwise,} \end{cases} \tag{6.15}$$

$$\gamma^{\mathcal{E}_C(Q)}_{iC_1'+D',\, iC_1+D} = \gamma_{C\circ D',\, C\circ D} \otimes \mathrm{Id}, \quad C_1' \leq C_1 \leq C,\ D' \leq D.$$

Next, for any $C' <_1 C$, we define a morphism of sheaves

$$\underline{\delta}_{CC'} : \mathcal{E}_C(Q) \longrightarrow \mathcal{E}_{C'}(Q)$$

by postulating the formulas of Proposition 6.13, i.e., by defining the action on the stalk over $D + iC'$, $D \in \mathcal{C}$, to be

$$\delta_{CC'|D} = \gamma_{C'K'}\,\delta_{KC'} : E_{C\circ D} = E_K \longrightarrow E_{K'} = E_{C'\circ D}, \tag{6.16}$$

where $K$ and $K'$ are defined by (6.11). Since we know that $Q \in \mathcal{J}$, i.e., $Q = \mathcal{Q}(\mathcal{F})$ for some $\mathcal{F} \in \operatorname{Perv}(\mathbb{C}^n, \mathcal{H})$, the results of n° B imply that these maps of stalks commute with generalization maps and so indeed define morphisms of sheaves $\underline{\delta}_{CC'}$. Further, for the same reason ($Q \in \mathcal{J}$), these morphisms commute and so assemble into a complex of sheaves

$$\Theta(Q) = \left\{ \bigoplus_{\operatorname{codim}(C)=0} \mathcal{E}_C(Q) \otimes \operatorname{or}(C) \xrightarrow{\delta} \bigoplus_{\operatorname{codim}(C)=1} \mathcal{E}_C(Q) \otimes \operatorname{or}(C) \xrightarrow{\delta} \cdots \xrightarrow{\delta} \mathcal{E}_0(Q) \right\}$$

which, moreover, lies in $\operatorname{Perv}(\mathbb{C}^n, \mathcal{H})$. This defines the functor $\Theta$, and the results of n° B mean that $\mathcal{Q}\Theta$ is naturally isomorphic to the identity functor of $\mathcal{J}$, while $\Theta\mathcal{Q}$ is naturally isomorphic to the identity functor of $\operatorname{Perv}(\mathbb{C}^n, \mathcal{H})$. $\square$

## 7. Algebraic relations in the double quiver

7.A. *The transitivity relations.* Let

$$Q = \Big( (E_C)_{C \in \mathcal{C}}, (\gamma_{C'C}, \delta_{CC'})_{C' \le C} \Big) \in \operatorname{Rep}^{(2)}(\mathcal{C}) \tag{7.1}$$

be a double representation of $\mathcal{C}$. In this section we find algebraic relations among the maps $\gamma_{C'C}, \delta_{CC'}$ that are necessary for $Q$ to have the form $Q = \mathcal{Q}(\mathcal{F})$ for some $\mathcal{F} \in \operatorname{Perv}(\mathbb{C}^n, \mathcal{H})$.

Call $Q$ *monotone* if $\gamma_{C'C}\,\delta_{CC'} = \operatorname{Id}$ for any $C' \le C$. By Proposition 6.7, any $\mathcal{Q}(\mathcal{F})$ is monotone.

Given a monotone $Q$, for any $A, B \in \mathcal{C}$, we define the *transition map*

$$\phi_{AB} = \gamma_{MB}\,\delta_{AM} : E_A \longrightarrow E_B, \quad M \le A, B. \tag{7.2}$$

By monotonicity, the choice of $M$ is immaterial; for example, one can take $M = 0$. Note that $\phi_{AB}$ is equal to $\gamma_{AB}$ if $A \le B$ and to $\delta_{AB}$ if $A \ge B$.

*Definition* 7.3. Three faces $A, B, C \in \mathcal{C}$ are called *collinear* if there are $a \in A, b \in B, c \in C$ such that $b \in [a, c]$; i.e., $b$ lies on the straight line segment between $a$ and $c$.

Collinearity is recovered from the oriented matroid corresponding to $\mathcal{H}$. More precisely,

Proposition 7.4. *Let us introduce a total order $\preceq$ on $\{+, -, 0\}$ that is induced by the standard order on $\mathbb{R}$, i.e., by $- \preceq 0 \preceq +$. Then the following are equivalent:*

(i) *the faces $A, B, C$ are collinear*;
(ii) *for each $H \in \mathcal{H}$, the sequence of sign vectors $(A_H, B_H, C_H)$ is monotone increasing or decreasing*: *either $A_H \preceq B_H \preceq C_H$, or $C_H \preceq B_H \preceq A_H$.*

*Proof.* (i)⇒(ii) is obvious. To see the converse, suppose that three cells $A, B, C$ are *not* collinear. Then there is a hyperplane $H \in \mathcal{H}$ such that $B$ lies on one side of $H$ and $A, C$ lie on the other side. This contradicts the monotonicity of $(A_H, B_H, C_H)$. □

THEOREM 7.5 (Transitivity relations). *Let $Q = \mathcal{Q}(\mathcal{F})$ for some $\mathcal{F} \in$* Perv$(\mathbb{C}^n, \mathcal{H})$. *Then for any collinear faces $A, B, C$, we have*

$$\phi_{AC} = \phi_{BC}\, \phi_{AB} : E_A \longrightarrow E_C.$$

The proof will be given after Example 7.9.

*Remark* 7.6 (Transitivity: long form). By marking all the faces meeting $[a, c]$, we get a face path (alternating sequence of inclusions)

$$A = B_1 \geq B_1' < B_2 < B_2' > \cdots < B_{m-1} > B_{m-1}' \leq B_m = C.$$

(There are four possibilities as to whether $A = B_1'$ or $B_{m-1}' = C$.) By iterating Theorem 7.5, we can reformulate in the equivalent form:

$$\phi_{AC} = \gamma_{B_{m-1}' B_m}\, \delta_{B_{m-1} B_{m-1}'} \cdots \gamma_{B_1' B_2}\, \delta_{B_1 B_1'}.$$

Note that it may be possible that $A$ and $C$ can be connected by a straight line segment in more than one inequivalent way, in which case $\phi_{AC}$ can be expressible through the $\gamma$ and $\delta$ maps in more than one way, producing additional algebraic relations in $\mathcal{Q}$; see Example 7.9.

*Example* 7.7 (Base change). It is sometimes convenient to view the poset $\mathcal{C}$ as a category, that is, to write an inclusion $C' \leq C$ as a morphism $u : C' \to C$. Then a double representation $Q$ can be viewed as a "bivariant theory" on $\mathcal{C}$: for a morphism $u$ as above, we write

$$u_* = \gamma_{C'C}, \; u^* = \delta_{CC'}.$$

In this language, a simplest instance of Theorem 7.5 can be viewed as a "base change property": we consider a (necessarily commutative) square of face inclusions

(7.8)
$$\begin{tikzcd} D \arrow[r, "v_1"] \arrow[d, "v_2"'] & A \arrow[d, "u_1"] \\ C \arrow[r, "u_2"'] & B. \end{tikzcd}$$

The condition that $A, B, C$ are collinear means that the square is coCartesian in the categorical sense: $B$ is the minimal face containing $A$ and $C$ in its closure. In this case, Theorem 7.5 says that

$$v_{2*}v_1^* = u_2^* u_{1*} : E_A \longrightarrow E_C.$$

Indeed, the left-hand side is $\phi_{AC}$, while $u_{1*} = \gamma_{AB} = \phi_{AB}$ and $u_2^* = \delta_{BC} = \phi_{BC}$.

*Example* 7.9 (Zifferblatt relations). Another extreme instance of Theorem 7.5 corresponds to the case when $C = -A$ is the opposite cell to $A$. Suppose $\dim(A) \geq 2$, and let $L$ be a 2-dimensional subspace such that $\dim(L \cap A) = 2$. The arrangement $\mathcal{H} \cap L$ then cuts $L \setminus \{0\}$ into some even number $2m$ of 2-dimensional open cones, which we number cyclically $\overline{B}_1, \ldots, \overline{B}_{2m}$ and the same number of 1-dimensional open rays $\overline{B}'_1, \ldots, \overline{B}'_{2m}$. Let $B_\nu, B'_\nu$ be the faces of $\mathcal{C}$ that intersect $L$ in $\overline{B}_\nu, \overline{B}'_\nu$; see Figure 2.

Note that there are two inequivalent ways to join a point of $A$ with a point of $C$ by a straight line segment inside $L$ not passing through 0, represented by the segments $[a, c]$ and $[a', c']$ in Figure 2. So the long form of the transitivity relations (Remark 7.6) gives the *Zifferblatt relation* (we borrow the term from [30]):

$$\begin{aligned}\gamma_{B'_m,C}\,\delta_{B_m,B'_m}\,\gamma_{B'_{m-1},B_m}\cdots\delta_{B_2,B'_2}\,\gamma_{B'_1,B_2}\,\delta_{A,B'_1}\\ = \gamma_{B'_{m+1},C}\,\delta_{B_{m+2},B'_{m+1}}\,\gamma_{B'_{m+2},B_{m+2}}\cdots\delta_{B_{2m},B'_{2m-1}}\,\gamma_{B'_{2m},B_{2m}}\,\delta_{A,B'_{2m}}.\end{aligned}$$

(Both sides of this equality are equal to $\phi_{A,C} = \gamma_{0,C} \circ \delta_{A,0}$.)

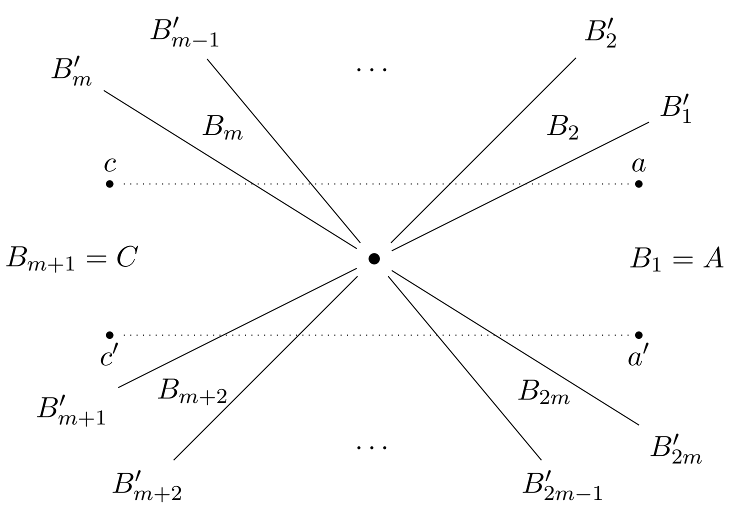


Figure 2. The Zifferblatt.

7.B. *Proof of Theorem* 7.5.

*Step* 1: *Base change.* We first consider the situation of a coCartesian square from Example 7.7, as it will serve as an inductive step in treating more general cases. The assumption that $A, B, C$ are collinear implies that $C \circ A = B$.

LEMMA 7.10. *In the situation of Example 7.7, the map*

$$\delta_{AD|C} : E_{A \circ C} = E_B \longrightarrow E_C = E_{D \circ C}$$

*is equal to* $\delta_{BC}$.

*Proof of the lemma.* We consider the commutative square of the $\mathcal{E}$-sheaves corresponding to (7.8) and the corresponding commutative square of stalks over $C + iD$. These squares have the form

$$\begin{tikzcd}
\mathcal{E}_D & \mathcal{E}_A \arrow[l, "\underline{\delta}_{AD}"'] \\
\mathcal{E}_C \arrow[u, "\underline{\delta}_{CD}"] & \mathcal{E}_B, \arrow[l, "\underline{\delta}_{BC}"] \arrow[u, "\underline{\delta}_{BA}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
E_C & E_B \arrow[l, "\delta_{AD|C}"'] \\
E_C \arrow[u, "\mathrm{Id}"] & E_B, \arrow[l, "\delta_{BC}"] \arrow[u, "\mathrm{Id}"']
\end{tikzcd}$$

whence the lemma. □

To deduce our particular case of Theorem 7.5 from the lemma, we spell out the condition that the maps of stalks induced by $\underline{\delta}_{AD}$ commute with generalization from $D + iD$ to $C + iD$. This gives a commutative square of vector spaces

$$\begin{tikzcd}
E_B \arrow[r, "\delta_{AD|C} = \delta_{BC}"] & E_C \\
E_A \arrow[u, "\gamma_{AB}"] \arrow[r, "\delta_{AD|D} = \delta_{AD}"'] & E_D \arrow[u, "\gamma_{DC}"']
\end{tikzcd}$$

in which the path through $E_D$ gives, as the composite map, $\phi_{AC}$ while the path through $E_B$ consists of $\delta_{BC} = \phi_{BC}$ and $\gamma_{AB} = \phi_{AB}$.

*Step* 2: *Case when* $C \neq -A$. In this case the segment $[a, c]$ does not pass through 0 and so its $\mathbb{R}$-linear span is a 2-dimensional subspace $L \subset \mathbb{R}^n$. As in Example 7.9, we then have an induced arrangement $\mathcal{H} \cap L$ of lines in $L$, and there are various possibilities as to whether the faces

$$\overline{A} = A \cap L, \ \overline{B} = B \cap L, \ \overline{C} = C \cap L$$

have dimension 1 or 2. We first consider the case $\dim(\overline{A}) = 2$, depicted in Figure 3. Denote by $\overline{A}'_1, \overline{A}'_2$ the two rays bordering $\overline{A}$ so that $\overline{A}'_2$ meets $[a, c]$. Further, denote by $\overline{D}$ the next 2-dimensional face in the direction from $a$ to $c$.

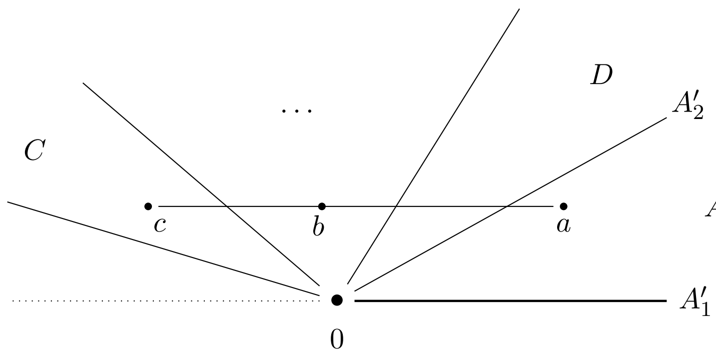


Figure 3. Case $C \neq -A$.

Let $D, A'_1, A'_2 \in \mathcal{C}$ be the faces whose intersections with $L$ are $\overline{D}, \overline{A}'_1, \overline{A}'_2$. Our assumption that $C \neq -A$ implies that the intersection with $L$ of all the faces meeting $[a, c]$ lie on the same side of the line $\mathbb{L}(\overline{A}'_1)$.

Consider the square of inclusions $0 \leq A'_1, A'_2 \leq A$. Let us write the corresponding commutative squares of $\mathcal{E}$-sheaves and of stalks of these sheaves at $C = C + i0 = [0, C]$. Note that $A \circ C = A$, since the interval $[a, c]$ spends, after leaving $a$, nonzero time inside $A$. Therefore the square in question has the form

$$\begin{array}{ccc}
\mathcal{E}_A\Big|_C = E_A & \xrightarrow{\mathrm{Id}} & E_A = E_{A'_1 \circ C} = \mathcal{E}_{A'_1}\Big|_C \\
{\scriptstyle \phi_{AD}}\Big\downarrow & & \Big\downarrow{\scriptstyle \phi_{AC}} \\
\mathcal{E}_{A'_2}\Big|_C = E_{A'_2 \circ C} = E_D & \xrightarrow[\phi_{DC}]{} & \mathcal{E}_0\Big|_C = E_C,
\end{array}$$

so $\phi_{AC} = \phi_{DC}\phi_{AD}$. Similarly, by considering the commutative square of stalks over $B + i0$, we find that $\phi_{AB} = \phi_{DB}\phi_{AD}$. Therefore we reduce to proving that $\phi_{DC} = \phi_{BC}\phi_{DB}$, which can be further reduced in a similar way. So proceeding by induction, we reduce the situation to the case $B = A$ or $B = A'_2$. If $B = A$, there is nothing to prove. If $B = A'_2$, then

$$\phi_{AC} = \gamma_{0C}\delta_{A0} = \gamma_{0C}\delta_{B0}\delta_{AB} = \phi_{BC}\delta_{AB} = \phi_{BC}\phi_{AB}.$$

This concludes the treatment of the case $\dim(\overline{A}) = 2$.

The case when $\dim(\overline{A}) = 1$ is analyzed inductively in a similar way, reducing it to the situation of Example 7.7, which has been analyzed in Step 1. This concludes the analysis of Step 2.

*Step* 3: *Case* $C = -A$. In this case we use the notation of Example 7.9. We consider the segment $[a,c]$. Note that by Step 2, we have

$$\phi_{B_1' B_m'} := \gamma_{0 B_m'} \circ \delta_{B_1' 0} = \delta_{B_m B_m'} \circ \gamma_{B_{m-1}' B_m} \circ \cdots \circ \delta_{B_2 B_2'} \circ \gamma_{B_1' B_2}.$$

Therefore, using the fact that the $\gamma$ and $\delta$ maps commute with compositions of inclusions, we have

$$\begin{aligned}\phi_{A,C} := \gamma_{0,C}\, \delta_{A,0} &= \gamma_{B_m',C}\, \gamma_{0,B_m'}\, \delta_{B_1',0}\, \delta_{A,B_1'} \\ &= \gamma_{B_m',C} \left( \delta_{B_m,B_m'}\, \gamma_{B_{m-1}',B_m} \cdots \delta_{B_2,B_2'}\, \gamma_{B_1',B_2} \right) \delta_{A,B_1'},\end{aligned}$$

which is the long form of the transitivity relation. The case of the segment $[a',c']$ is treated in the same way.

This concludes the proof of Theorem 7.5. □

## 8. **Equivalence of categories**

8.A. *The main result.* Let $\mathcal{H}$ be an arrangement of hyperplanes in $\mathbb{R}^n$, with the poset of faces $\mathcal{C}$, and $\mathrm{Rep}^{(2)}(\mathcal{C})$ be the corresponding category of double representations $Q = (E_C, \gamma_{C'C}, \delta_{CC'})$. Let $\mathcal{A} = \mathcal{A}_{\mathcal{H}} \subset \mathrm{Rep}^{(2)}(\mathcal{C})$ be the full subcategory of double representations satisfying the following three conditions:

(Mon) *Monotonicity*: $\gamma_{C'C}\, \delta_{CC'} = \mathrm{Id}_{E_C}$, $C' \le C$. This allows us to define transition maps $\phi_{AB} = \gamma_{CB}\, \delta_{AC} : E_A \to E_B$, where $C$ is an arbitrary face $\le A, B$.

(Tran) *Transitivity*: $\phi_{AC} = \phi_{BC}\, \phi_{AB}$ for any three collinear faces $A, B, C$.

(Inv) *Invertibility*: Let $C_1, C_2$ be two faces of the same dimension $d$ with the same linear span $\mathbb{L}(C_1) = \mathbb{L}(C_2)$, which lie on opposite sides of a face $D_1$ of dimension $d-1$, so $C_1 >_1 D_1 <_1 C_2$. Then $\phi_{C_1C_2} = \gamma_{D_1C_2}\, \delta_{C_1D_1} : E_{C_1} \to E_{C_2}$ is an isomorphism.

The following is the main result of this paper.

THEOREM 8.1. *The functor* $\mathcal{F} \mapsto \mathcal{Q}(\mathcal{F})$ *defines an equivalence of categories* $\mathrm{Perv}(\mathbb{C}^n, \mathcal{H}) \to \mathcal{A}$.

In view of Theorem 6.1, it suffices to prove the following statement.

*Reformulation* 8.2. In order for $\mathcal{Q} \in \mathrm{Rep}^{(2)}(\mathcal{C})$ to have the form $\mathcal{Q}(\mathcal{F})$ for some $\mathcal{F} \in \mathrm{Perv}(\mathbb{C}^n, \mathcal{H})$, it is necessary and sufficient that $\mathcal{Q}$ satisfies (Mon), (Tran) and (Inv).

The proof will occupy the remainder of this section.

8.B. *Necessity.* The necessity of (Mon) and (Tran) has already been proved in Proposition 6.7 and Theorem 7.5. Let us prove the necessity of (Inv). Because of Proposition 5.1, it is enough to consider the case when $C_1$ and $C_2$ are faces open in $\mathbb{R}^n$, because the general case will then follow by considering the hyperbolic restriction to $L_{\mathbb{C}}$, where $L = \mathbb{L}(C_1) = \mathbb{L}(C_2)$.

Assuming the $C_1$ and $C_2$ are open, consider the Cousin resolution $\mathcal{E}^\bullet$ of $\mathcal{F}$. Note that $\mathcal{F}$ is $\mathcal{S}^{(0)}$-smooth, while the summands $\mathcal{E}_C$ of $\mathcal{E}^\bullet$ are only $\mathcal{S}^{(1)}$-smooth. Look at the $\mathcal{S}^{(0)}$-smooth sheaf

$$\underline{H}^0(\mathcal{F}) \ = \mathrm{Ker}\left\{\bigoplus_{C \text{ open}} \mathcal{E}_C \xrightarrow{\tilde{\delta}} \bigoplus_{\mathrm{codim}(D)=1} \mathcal{E}_D\right\}.$$

The $\mathcal{S}^{(1)}$-cells $[D_1, C_2] \leq [C_2, C_2]$ lie in the same stratum $(\mathbb{C}^n)^\circ$ of $\mathcal{S}^{(0)}$. Therefore the generalization map for $\underline{H}^0(\mathcal{F})$ from $[D_1, C_2]$ to $[C_2, C_2]$ must be an isomorphism. But, applying Corollary 3.22(c), we find that up to tensoring with $\mathrm{or}(\mathbb{R}^n)$, we have

$$\begin{aligned}\underline{H}^0(\mathcal{F})|_{[D_1,C_2]} &= \mathrm{Ker}\left\{E_{C_1} \oplus E_{C_2} \xrightarrow{\phi_{C_1C_2} - \mathrm{Id}} E_{C_2}\right\},\\ \underline{H}^0(\mathcal{F})_{[C_2,C_2]} &= E_{C_2},\end{aligned}$$

and the generalization map is induced by the projection $E_{C_1} \oplus E_{C_2} \to E_{C_2}$. In order for this projection to restrict to an isomorphism $\mathrm{Ker}(\phi_{C_1C_2} - \mathrm{Id}) \to E_{C_2}$, the map $\phi_{C_1C_2}$ must be an isomorphism.

8.C. *Sufficiency*: *construction of the complex* $\mathcal{E}^\bullet$. To prove the sufficiency of the three conditions in Reformulation 8.2, we start with a double representation $Q$ satisfying them, construct a complex $\mathcal{E}^\bullet = \mathcal{E}^\bullet(Q)$ and then prove that it is a perverse sheaf with double quiver $Q$.

In this procedure, the *construction* of the complex $\mathcal{E}^\bullet(Q)$ will rely only on the properties (Mon) and (Tran), while (Inv) will be needed to ensure that this complex is an object of $\mathrm{Perv}(\mathbb{C}^n, \mathcal{H})$.

The construction will be done by the same procedure as in Section 6.C. That is, we define the $\mathcal{S}^{(2)}$-smooth sheaves $\mathcal{E}_C = \mathcal{E}_C(Q)$ by the formulas (6.15). In fact, it follows from the definition that each $\mathcal{E}_C(Q)$ is $\mathcal{S}^{(1)}$-smooth. We next define, for any faces $C' \leq C$ and $D$, the map of the stalks

$$\delta_{CC'|D} = \delta^Q_{CC'|D} : E_{C\circ D} \longrightarrow E_{C'\circ D}$$

by the formulas (6.16), that is,

$$\delta_{CC'|D} = \gamma_{C'K'}\, \delta_{KC'} : E_{C\circ D} = E_K \longrightarrow E_{K'} = E_{C'\circ D},$$

where $K = C \circ D$ and $K' = C' \circ D$. Note that

$$\delta_{CC'|D} = \phi_{C\circ D, C'\circ D}. \tag{8.3}$$

PROPOSITION 8.4. *The $\delta_{CC'|D}$ commute with the generalization maps and so define morphisms of sheaves*

$$\underline{\delta}_{CC'} : \mathcal{E}_C \longrightarrow \mathcal{E}_{C'}, \quad C' \leq C.$$

*Proof.* For any faces $C' \leq C$ and $D' \leq D$, we need to prove the commutativity of the square

$$
\begin{tikzcd}
E_{C'\circ D'} \arrow[d, "\gamma_{C'\circ D',C'\circ D}"'] & E_{C\circ D'} \arrow[l, "\delta_{CC'|D'}"'] \arrow[d, "\gamma_{C\circ D',C\circ D}"] \\
E_{C'\circ D} & E_{C\circ D}. \arrow[l, "\delta_{CC'|D}"]
\end{tikzcd} \tag{8.5}
$$

In order to do this, we use the bivariant notation of Example 7.7 and consider the diagram of inclusions depicted by arrows

$$
\begin{tikzcd}
C'\circ D' \arrow[dd, "v'"'] & & & & C\circ D' \arrow[dd, "v"] \\
& C' \arrow[ul, "w'"'] \arrow[dl, "u'"] \arrow[r, "s"] & & C \arrow[ur, "w"] \arrow[dr, "u"'] & \\
C'\circ D & & & & C\circ D.
\end{tikzcd}
$$

Then

$$
\begin{aligned}
\gamma_{C'\circ D',C'\circ D}\,\delta_{CC'|D'} = v'_*(w'_* s^* w^*) = u'_* s^* w^* = \gamma_{C',C'\circ D}\,\delta_{C\circ D',C'} = \phi_{C\circ D',C'\circ D},\\
\delta_{CC'|D}\,\gamma_{C\circ D',C\circ D} = (u'_* s^* u^*) v_* = \phi_{C\circ D,C'\circ D}\,\phi_{C\circ D',C\circ D}.
\end{aligned}
$$

In the last identification we used that $v_* = \gamma_{C\circ D',C\circ D}$ is the same as $\phi_{C\circ D',C\circ D}$. So the commutativity of (8.5) would follow from (Tran) if we establish the next lemma.

LEMMA 8.6. *For any faces $C' \leq C$ and $D' \leq D$, the cells $C\circ D', C\circ D, C'\circ D$ form a collinear triple.*

*Proof of the lemma.* Choose points $c' \in C', c \in C, d' \in D', d \in D$. Assume that they are in general position; i.e., $T = \operatorname{Conv}\{c', c, d', d\}$ is a tetrahedron in an affine 3-space (the case $\dim(T) \leq 2$ is analyzed easily). We need to find points $x \in C' \circ D$, $y \in C \circ D$, $z \in C \circ D'$ such that $y \in [x, z]$.

To choose a possible $x$, we can draw the interval from $c'$ to any point $d_1 \in (d', d]$ (any such $d_1$ satisfies $d_1 \in D$) and take any point on this interval sufficiently close to $c'$. The supply of $x$ thus obtained contains a neighborhood of the vertex $c'$ in the triangle $\triangle(c'd'd)$, with the edge $[c', d']$ removed. See Figure 4.

To choose a possible $y$, we can draw the interval from any $c_2 \in (c', c]$ to any $d_2 \in (d', d]$ (as any such $c_2, d_2$ satisfy $c_2 \in C, d_2 \in D$) and take any point on this interval sufficiently close to $c_2$. The supply of $y$ thus obtained contains a neighborhood of the edge $(c', c]$ in $T$, with the faces $\triangle(cc'd')$ and $\triangle(dc'd')$ removed.

To choose a possible $z$, we can similarly draw the interval from any $c_3 \in (c', c]$ to $d'$ and take any point on this interval sufficiently close to $c_3$.

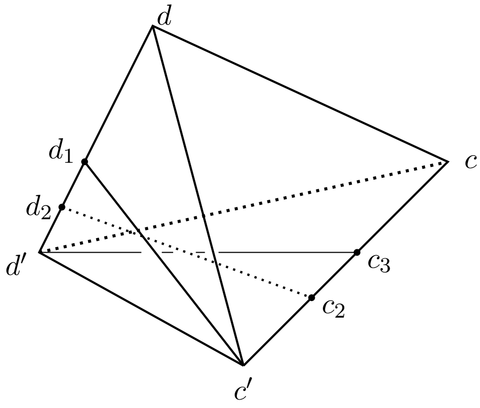


Figure 4. Collinearity of $C \circ D'$, $C \circ D$ and $C' \circ D$.

The supply of $z$ thus obtained covers a neighborhood of the edge $(c'c]$ in the triangle $\triangle(d'c'c)$.

From this description it is clear that one can start from any admissible $x \in [c', d_1]$ and take $y \in [c_2, d_2]$ sufficiently close to $c_2$, very near the face $\triangle(d'c'c)$. Then the interval $[x, y]$, continued after $y$, will hit $\triangle(d'c'c)$ in a point $z$ very close to the edge $[c', c] \ni c_2$, so such $z$ will be obtained by the above construction, i.e., will lie in $C \circ D'$. This proves Lemma 8.6 and Proposition 8.4. □

Proposition 8.7. *The morphisms of sheaves $\underline{\delta}_{CC'}$, $C' \leq C$, commute with each other, i.e., give rise to a contravariant representation of* $(\mathcal{C}, \leq)$ *in* $\mathrm{Sh}_{\mathbb{C}^n}$.

*Proof.* We have to prove the identity

$$\phi_{K_1', K''}\, \phi_{K, K_1'} = \phi_{K_2', K''}\, \phi_{K, K_2'} : E_K \longrightarrow E_{K''} \tag{8.8}$$

for any four faces $K, K_1', K_2', K''$ with the following property: there exists a square of codimension 1 inclusions

$$C >_1 C_1', C_2' >_1 C''$$

and a face $D \geq C''$ such that

$$\begin{aligned} K &= C \circ D, \quad K'' = C'' \circ D, \\ K_1' &= C_1' \circ D, \quad K_2' = C_2' \circ D. \end{aligned} \tag{8.9}$$

This would correspond to the commutativity of the stalks of the square

$$\begin{tikzcd} \mathcal{E}_C \arrow[r, "\underline{\delta}_{CC_1'}"] \arrow[d, "\underline{\delta}_{CC_2'}"'] & \mathcal{E}_{C_1'} \arrow[d, "\underline{\delta}_{C_1'C''}"] \\ \mathcal{E}_{C_2'} \arrow[r, "\underline{\delta}_{C_2'C''}"'] & \mathcal{E}_{C''} \end{tikzcd}$$

over the face $[C'', D]$. Note that the $[C'', D]$ for $D \geq C''$ form a cell decomposition of the tube cell $\mathbb{R}^n + iC''$, and the sheaves in question are direct image extensions of some sheaves from tube cells to their closures, so checking the commutativity of the above square over

$$\mathbb{R}^n + iC'' \ \subset \ \operatorname{supp}(\mathcal{E}_{C''}) = \mathbb{R}^n + i\overline{C}''$$

is enough.

To prove (8.8), we first remark that for any faces $A, B$ the triple $A, A \circ B, B$ is collinear, since $A \circ B$ is defined in terms of points on the interval $[a, b]$ for $a \in A, b \in B$. We also note that in (8.9) we have $K'' = D$. Consider the diagram of inclusions

$$\begin{tikzcd}
K'' = D & K'_1 & K \\
C'' \arrow[u] \arrow[r] & C'_1 \arrow[u] \arrow[r] & C. \arrow[u]
\end{tikzcd}$$

In this diagram, $C'_1, K'_1, K''$ form a collinear triple, since $K'_1 = C'_1 \circ D = C'_1 \circ K''$. Therefore by (Tran),

$$\phi_{C'_1, K''} = \phi_{K'_1, K''}\, \phi_{C'_1, K'_1} = \phi_{K'_1, K''}\, \gamma_{C'_1, K'_1}.$$

Now, the left-hand side of the putative equality (8.8) is transformed as follows:

$$\begin{aligned}
\phi_{K'_1, K''}\, \phi_{K, K'_1} &= \phi_{K'_1, K''}\, \gamma_{C'_1, K'_1}\, \delta_{K, C'_1} \\
&= \phi_{C'_1, K''}\, \delta_{K, C'_1} = \gamma_{C'', K''}\, \delta_{C'_1, C''}\, \delta_{K, C'_1} \\
&= \gamma_{C'', K''}\, \delta_{K, C''} = \phi_{K, K''}.
\end{aligned}$$

Considering a similar diagram but with $C'_2, K'_2$ instead of $C'_1, K'_1$, we find that the right-hand side of (8.8) is also equal to $\phi_{K, K''}$. □

8.D. *The complex $\mathcal{E}^\bullet(Q)$ is $\mathcal{S}^{(0)}$-smooth: inclusions of type* (1). Proposition 8.7 implies that we have a complex of sheaves $\mathcal{E}^\bullet = \mathcal{E}^\bullet(Q)$ with

$$\mathcal{E}^p(Q) = \bigoplus_{\operatorname{codim}(C) = p} \mathcal{E}_C \otimes \operatorname{or}(C)$$

and the differential $\widetilde{\underline{\delta}}$ given by the $\underline{\delta}_{CC'}$ and satisfying $\widetilde{\underline{\delta}}^2 = 0$. By construction, each summand of $\mathcal{E}^\bullet$, and thus $\mathcal{E}^\bullet$ itself, is $\mathcal{S}^{(1)}$-smooth.

PROPOSITION 8.10. *The complex $\mathcal{E}^\bullet$ is $\mathcal{S}^{(0)}$-smooth.*

*Proof.* By Proposition 3.3, it is enough to show that for any elementary inclusion $[C_1, D_1] \leq [C_2, D_2]$ of $\mathcal{S}^{(1)}$-cells, the generalization map

$$\gamma_{[C_1, D_1], [C_2, D_2]} : \mathcal{E}^\bullet|_{[C_1, D_1]} \longrightarrow \mathcal{E}^\bullet_{[C_2, D_2]}$$

is a quasi-isomorphism of complexes of vector spaces. We first consider an inclusion of type (1)

$$[C_1, D] \leq [C_2, D], \quad C_1 < C_2 \leq D.$$

Note that every such inclusion is a composition of inclusions with $C_1 <_1 C_2$. (If this condition does not hold, it suffices to choose a maximal chain of faces $C_1 = C'_0 <_1 C'_1 <_1 \cdots <_1 C'_p = C_2$ between $C_1$ and $C_2$ and consider the inclusions $[C'_i, D] < [C'_{i+1}, D]$.) So we assume $C_1 <_1 C_2$.

Lemma 8.11. *$\gamma_{[C_1,D],[C_2,D]}$ is a surjective morphism of complexes of vector spaces, with kernel*

$$K^\bullet = \Bigg\{ \bigoplus_{\substack{\operatorname{codim}(C)=0 \\ C \geq C_1, C \not\geq C_2}} E_{C\circ D} \otimes \operatorname{or}(C) \xrightarrow{\widetilde{\underline{\delta}}} \bigoplus_{\substack{\operatorname{codim}(C)=1 \\ C \geq C_1, C \not\geq C_2}} E_{C\circ D} \otimes \operatorname{or}(C) \xrightarrow{\widetilde{\underline{\delta}}} \cdots \Bigg\},$$

*where $\widetilde{\underline{\delta}}$ has, as matrix elements, the $\delta_{CC'}$ for relevant $C, C'$ tensored with the coorientations of the adjacent faces.*

*Proof.* Recall that we defined the sheaf $\mathcal{E}_C$ by formulas (6.15). Therefore the stalk $\mathcal{E}_C|_{[C_1,D]}$ is equal to $E_{C\circ D}$ if $C \geq C_1$ and to $0$ if $C \not\geq C_1$, and similarly for $\mathcal{E}_C|_{[C_2,D]}$. The generalization maps were also defined by formulas of Corollary 3.22(c). This implies that the matrix elements of $\gamma_{[C_1,D],[C_2,D]}$ are either Id or 0, whence our statement. □

Therefore we need to prove that $K^\bullet$ is an exact complex.

Lemma 8.12. *Consider the increasing filtration $F$ of $K^\bullet$ by graded subspaces defined by*

$$F_d K^p = \bigoplus_{\substack{\operatorname{codim}(C)=p \\ C \geq C_1, C \not\geq C_2 \\ \dim(C\circ D) \leq d}} E_{C\circ D} \otimes \operatorname{or}(C).$$

*This filtration is compatible with the differential $\widetilde{\underline{\delta}}$.*

*Proof.* The matrix element $\widetilde{\underline{\delta}}_{CC'}$ is nonzero only if $C' \leq C$. In this case $\mathbb{L}(C'\circ D) \subset \mathbb{L}(C\circ D)$ by Proposition 2.7(b), and so $\dim(C'\circ D) \leq \dim(C\circ D)$. □

We are therefore reduced to proving that each $\operatorname{gr}^F_d K^\bullet$ is exact.

Lemma 8.13. *Fix $d \geq 0$, and consider all $d$-dimensional flats $M \in \mathcal{L}$ containing $C_1$. Let*

$$(\operatorname{gr}^F_d K)^p_M = \bigoplus_{\substack{\operatorname{codim}(C)=p \\ C \geq C_1, C \not\geq C_2 \\ \mathbb{L}(C\circ D)=M}} E_{C\circ D} \otimes \operatorname{or}(C).$$

*Then* $(\mathrm{gr}_d^F K)_M^\bullet$ *is a subcomplex in* $\mathrm{gr}_d^F K^\bullet$, *and we have a decomposition into a direct sum of complexes*

$$\mathrm{gr}_d^F K^\bullet = \bigoplus_{M \supset C_1} (\mathrm{gr}_d^F K)_M^\bullet.$$

*Proof.* Look at a summand $E_{C \circ D} \otimes \mathrm{or}(C)$ with

$$\mathrm{codim}(C) = p,\ C \geq C_1,\ C \not\geq C_2,\ \dim(C \circ D) = d.$$

The differential in $\mathrm{gr}_d^F K^\bullet$ can take this summand only to summands of the form $E_{C' \circ D}$, where

$$C' <_1 C,\ \mathrm{codim}(C) = p+1,\ C' \geq C_1,\ C' \not\geq C_2,\ \dim(C' \circ D) = d.$$

Since $\mathbb{L}(C' \circ D) \subset \mathbb{L}(C \circ D)$ by Proposition 2.7(b) and since $\dim\mathbb{L}(C' \circ D) = \dim\mathbb{L}(C \circ D)$, we conclude that $\mathbb{L}(C' \circ D) = \mathbb{L}(C \circ D)$. So the complex $\mathrm{gr}_D^F K^\bullet$ splits into a direct sum of subcomplexes corresponding to all possible values of $M = \mathbb{L}(C \circ D)$. □

Denote $G_M^\bullet = (\mathrm{gr}_d^F K)_M^\bullet$. We are therefore reduced to proving that $G_M^\bullet$ is exact for each $M$. So we fix $M$ and note that by the above,

$$G_M^p = \bigoplus_{\substack{\mathrm{codim}(C)=p \\ M \supset C \geq C_1,\ C \not\geq C_2 \\ C \circ D \text{ open in } M}} E_{C \circ D} \otimes \mathrm{or}(C).$$

Note that we can represent $G_M^\bullet$ as the quotient $G_{1,M}^\bullet / G_{2,M}^\bullet$, where $G_{1,M}^p$ is the direct sum as above but without the restriction $C \not\geq C_2$, and $G_{2,M}^p$ is a similar direct sum but with the additional restriction $C \geq C_2$. So we are reduced to proving that the embedding $G_{2,M}^\bullet \hookrightarrow G_{1,M}^\bullet$ is a quasi-isomorphism.

In order for the summand corresponding to a face $C$ to be present in $G_{1,M}^p$, it is necessary that not only $C \subset M$, but also that $D \subset M$, since $M = \mathbb{L}(C \circ D)$ is the minimal flat of $\mathcal{H}$ containing $\mathbb{L}(C)$ and $\mathbb{L}(D)$. This means that $G_{1,M}^\bullet$ (as well as $G_{2,M}^\bullet$) is entirely described in terms of faces of $\mathcal{H}$ contained in $M$, i.e., in terms of the restricted double quiver $\mathcal{Q}^{\leq M}$. So by passing to the restricted configuration $\mathcal{H} \cap M$, if necessary, we can and will assume that $M = \mathbb{R}^n$, and we write $G_\nu^\bullet = G_{\nu, \mathbb{R}^n}^\bullet$, $\nu = 1, 2$. Further, since $C$ runs over faces containing $C_1$, the $G_\nu^\bullet$ are entirely described in terms of the restricted double quiver $\mathcal{Q}^{\geq C_1}$. So by passing to the quotient configuration if necessary, we can and will assume that $C_1 = \{0\}$, and therefore $\dim(C_2) = 1$.

Under all these assumptions, let us give a geometric interpretation of the $G_\nu^\bullet$. Let $\mathcal{C}_D \subset \mathcal{C}$ be the set of faces $C$ such that $C \circ D$ is open in $\mathbb{R}^n$, and let $|\mathcal{C}_D|$ be the union of cells from $\mathcal{C}_D$.

Lemma 8.14. *Let* $C, D \in \mathcal{C}$. *Then the following are equivalent:*

(i) $C \circ D$ *is open in* $\mathbb{R}^n$,

(ii) $C$ *does not lie in any hyperplane* $H \in \mathcal{H}$ *containing* $D$.

COROLLARY 8.15.
$$|\mathcal{C}_D| \ = \mathbb{R}^n \setminus \bigcup_{\substack{H\in\mathcal{H}\\ H\supset D}} H$$
*is the union of convex, hence contractible components labelled by the (open) chambers of the quotient arrangement $\mathcal{H}/\mathbb{L}(D)$.*

*Proof of Lemma 8.14.* We recall the interpretation of $C \circ D$ in terms of the projection $\pi_C : \mathbb{R}^n \to \mathbb{R}^n/\mathbb{L}(C)$ given in Proposition 2.6. That is, $C \circ D$ is the unique face from $\mathcal{C}^{\geq C}$ that projects onto $\sigma(\pi_C(D))$, the cell of $\mathcal{H}/\mathbb{L}(C)$ containing $\pi_C(D)$. So for $C \circ D$ to be open in $\mathbb{R}^n$, it is necessary and sufficient that $\sigma(\pi_C(D))$ be open in $\mathbb{R}^n/\mathbb{L}(C)$. This means that $\pi_C(D)$ does not lie in any hyperplane of $\mathcal{H}/\mathbb{L}(C)$; i.e., $D$ does not lie in any hyperplane of $\mathcal{H}$ containing $C$. In other words, the sets of hyperplanes of $\mathcal{H}$ containing $C$ and $D$ must be disjoint. $\square$

Let us look at an arbitrary matrix element of the differential in $G_1^\bullet$,
$$\underline{\widetilde{\delta}}_{CC'} : E_{C\circ D} \otimes \mathrm{or}(C) \longrightarrow E_{C'\circ D} \otimes \mathrm{or}(C').$$
It can be nonzero only if $C_1 \leq C' <_1 C \subset M$ and both $C \circ D$ and $C' \circ D$ are open in $M$. If these conditions are satisfied, then (8.3)
$$\underline{\widetilde{\delta}}_{CC'} = \phi_{C\circ D, C'\circ D} \otimes \epsilon_{CC'},$$
where $\epsilon_{CC'}$ is the identification of orientation torsors induced by $C' <_1 C$. We note that in this case $\phi_{C\circ D,C'\circ D}$ is an isomorphism. Indeed, since both $C \circ D$ and $C' \circ D$ are open in $M$, we can choose generic points $x \in C \circ D$, $x' \in C' \circ D$ so that the interval $[x, x'] \subset M$ intersects only flats of $\mathcal{H} \cap M$ that have codimension 1 in $M$. Therefore we can write all the faces intersecting $[x, x']$, in the order from $x$ to $x'$, as
$$F_0 = C \circ D >_1 F'_0 <_1 F_1 >_1 F'_1 <_1 \cdots >_1 F'_{p-1} <_1 F_p = C \circ D',$$
where the $F_j$ are open in $M$, while the $F'_j$ have codimension 1 in $M$. Therefore each $\phi_{F_j,F_{j+1}}$ is an isomorphism by (Inv), while
$$\phi_{C\circ D,C'\circ D} = \phi_{F_0,F_p} = \phi_{F_{p-1},F_p} \cdots \phi_{F_0,F_1}$$
by (Tran) (long form of the transitivity relations).

We now note that the isomorphisms $\phi_{C\circ D,C'\circ D}$ define a cellular locally constant (and therefore, by Corollary 8.15, constant) sheaf $\mathcal{G}$ on $|\mathcal{C}_D|$, and the above argument shows that
$$\mathcal{G}_1^\bullet = C^{\mathrm{cell}}_{n-\bullet}(|\mathcal{C}_D|, \mathcal{G})$$
is the standard cellular chain complex of $\mathcal{G}$ shifted to so as to start in degree 0 (and to have differential of degree $-1$). Similarly, let $\mathcal{C}_D^{\geq C_2}$ be the subposet in

$\mathcal{C}_D$ formed by $C$ satisfying $C \geq C_2$ and $|\mathcal{C}_D^{\geq C_2}|$ be the union of faces from this subposet. Then

$$\mathcal{G}_2^\bullet = C_{n-\bullet}^{\mathrm{cell}}(|\mathcal{C}_D^{\geq C_2}|, \mathcal{G}).$$

Therefore, the acyclicity of $G_1^\bullet / G_2^\bullet$ will follow if we prove that the embedding of spaces $|\mathcal{C}_D^{\geq C_2}| \hookrightarrow |\mathcal{C}_D|$ is a homotopy equivalence.

Recall that $C_2 \leq D$, $\dim(C_2) = 1$ is, by our assumption, a half-line. Therefore for any connected component $U \subset |\mathcal{C}_D|$, the intersection $U \cap |\mathcal{C}_D^{\geq C_2}|$ is a nonempty convex set so it is contractible as well. This proves that $|\mathcal{C}_D^{\geq C_2}| \hookrightarrow |\mathcal{C}_D|$ is a homotopy equivalence and so $\gamma_{[C_1,D],[C_2,D]}$ is a quasi-isomorphism.

8.E. *Inclusions of type* (2). To finish the proof of Proposition 8.10, we need to consider elementary inclusions of type (2). We write such an inclusion as $[C_1, D_1] \leq [D_2, D_2]$, where $D_1, D_2$ are two faces of the same dimension $m$, lying in the same $m$-dimensional flat $L = \mathbb{L}(D_1) = \mathbb{L}(D_2)$ on the opposite side of an $(m-1)$-dimensional cell $C_1 \leq D_1, D_2$.

The argument runs very similarly to the case of an inclusion of type (1) considered in Section 8.D. We indicate the changes, using the same notation as in 8.D for the intermediate complexes and treating the grading in these complexes as implicit.

LEMMA 8.16. *The morphism of complexes $\gamma_{[C_1,D_1],[D_2,D_2]}$ is surjective with kernel*

$$K^\bullet = \bigoplus_{\substack{C \geq C_1 \\ C \not\geq D_2}} E_{C \circ D_1} \otimes \mathrm{or}(C),$$

*graded by* $\mathrm{codim}(C)$.

*Proof.* By Corollary 3.22(c),

$$\mathcal{E}^\bullet_{[C_1,D_1]} = \bigoplus_{C \geq C_1} E_{C \circ D_1} \otimes \mathrm{or}(C), \quad \mathcal{E}^\bullet_{[D_2,D_2]} = \bigoplus_{C \geq D_2} E_{C \circ D_2} \otimes \mathrm{or}(C).$$

The set of admissible $C$ in the sum for $\mathcal{E}^\bullet_{[D_2,D_2]}$ is clearly a subset in the set of admissible $C$ in the sum for $\mathcal{E}^\bullet_{[C_1,D_1]}$, because $C_1 < D_2$. Further, for $C \geq D_2$, the corresponding summand in $\mathcal{E}^\bullet_{[C_1,D_1]}$ is equal to the corresponding summand in $\mathcal{E}^\bullet_{[D_2,D_2]}$. Indeed, since $D_1$ and $D_2$ are adjacent (have the same dimension $m$ and lie on the opposite sides of $C_1$ in an $m$-flat), we have $D_2 \circ D_1 = D_2$. We also have $C \circ D_2 = C$ since $C \geq D_2$, so by the associativity of $\circ$, we find

$$C \circ D_1 = (C \circ D_2) \circ D_1 = C \circ (D_2 \circ D_1) = C \circ D_2.$$

Further, for $C \geq D_2$, Corollary 3.22(c) shows that the matrix element of $\gamma^{\mathcal{E}^\bullet}_{[C_1,D_1],[D_2,D_2]}$ on the summand corresponding to $C$ is the identity. Recalling that the matrix elements of $\gamma^{\mathcal{E}^\bullet}_{[C_1,D_1],[D_2,D_2]}$ can act only between summands labelled by the same $C$, we conclude that it is the projection onto the direct sum of summands labelled by $C \geq D_2$, as claimed. □

Now, as before, we have the increasing filtration $F$ in $K^\bullet$ by $\dim(C \circ D_1)$ with quotients that split into direct sum over $M \in \mathcal{L}$ of complexes

$$G_M^\bullet = \bigoplus_{\substack{C \geq C_1,\, C \not\geq D_2 \\ \mathbb{L}(C \circ D_1) = M}} E_{C \circ D_1} \otimes \mathrm{or}(C).$$

So we need to prove that each $G_M^\bullet$ is exact. As in Section 8.D, we have $G_M^\bullet = G_1^\bullet / G_2^\bullet$, where $G_1^\bullet$ is the direct sum over $C \geq C_1$ such that $L(C \circ D_1) = M$ (so it is exactly the same complex as in 8.D) and $G_2^\bullet$ is the subcomplex formed by $E_{C \circ D_1} \otimes \mathrm{or}(C)$ for $C \geq D_2$. As before, we can and will assume that $C_1 = 0$ and $M = \mathbb{R}^n$. Thus $D_1, D_2$ are two opposite half-lines of the same line: $D_2 = -D_1$. So $G_1^\bullet$ is the cellular chain complex of a local system $\mathcal{G}$ on $|\mathcal{C}_{D_1}|$ and $G_2^\bullet$ is the chain complex of $\mathcal{G}$ on $|\mathcal{C}_{D_1}^{\geq D_2}|$. Since $D_2 = -D_1$, the subspace $|\mathcal{C}_{D_1}^{\geq D_2}|$ is again a union of convex connected components, one inside each convex connected component of $|\mathcal{C}_{D_1}|$, so the embedding $G_2^\bullet \hookrightarrow G_1^\bullet$ is a quasi-isomorphism, and Proposition 8.10 is proved.

8.F. *End of the proof of Reformulation* 8.2. Given $Q \in \mathrm{Rep}^{(2)}(\mathcal{C})$ satisfying (Mon), (Tran) and (Inv), we have associated to it an $\mathcal{S}^{(0)}$-constructible complex $\mathcal{E}^\bullet(Q)$ on $\mathbb{C}^n$. Note that the orthogonality relations of Lemma 6.3 apply to the $\mathcal{E}_C(Q)$ and imply that the linear data of $\mathcal{E}^\bullet(Q)$ are given by $Q$. Since $Q$ consists of single vector spaces (not just complexes), Proposition 4.14 implies that $\mathcal{E}^\bullet(Q)$ is a perverse sheaf, with double quiver $Q$. This proves Reformulation 8.2 and thus Theorem 8.1.

## 9. **Examples and complements**

9.A. *The case of dimension* 1. Suppose $n = 1$. The real vector space $\mathbb{R}$ has the unique hyperplane $\{0\}$. The arrangement consisting of this hyperplane will be denoted simply by 0. It has three faces: $\mathbb{R}_+ = \{x > 0\}$, $\mathbb{R}_- = \{x < 0\}$ and $\{0\}$. The category $\mathrm{Perv}(\mathbb{C}, 0)$ consists of perverse sheaves on $\mathbb{C}$ smooth with respect to the stratification consisting of $\{0\}$ and $\mathbb{C} \setminus \{0\}$. The classical description [15] identifies $\mathrm{Perv}(\mathbb{C}, 0)$ with the category $\mathcal{P}$ of diagrams of finite-dimensional vector spaces

$$\Phi \underset{u}{\overset{v}{\rightleftarrows}} \Psi\,, \quad vu + \mathrm{Id}_\Psi \text{ invertible}.$$

The spaces $\Psi$ and $\Phi$ associated to $\mathcal{F} \in \mathrm{Perv}(\mathbb{C}, 0)$ are canonically identified with the spaces of nearby and vanishing cycles of $\mathcal{F}$ with respect to the standard coordinate function on $\mathbb{C}$, while $vu + \mathrm{Id}_\Psi$ is the monodromy on the space of nearby cycles; see [1].

On the other hand, our description from Theorem 8.1 identifies $\mathrm{Perv}(\mathbb{C},0)$ with the category $\mathcal{A}$ formed by double representations

$$\mathcal{Q}=\left\{\ E_- \underset{\delta_-}{\overset{\gamma_-}{\leftrightarrows}} E_0 \underset{\delta_+}{\overset{\gamma_+}{\rightleftarrows}} E_+\ \right\}\quad \text{such that}$$

$$\gamma_-\delta_-=\mathrm{Id}_{E_-},\ \ \gamma_+\delta_+=\mathrm{Id}_{E_+},$$
$$\gamma_-\delta_+: E_+\longrightarrow E_-,\ \ \gamma_+\delta_-: E_-\to E_+\quad \text{are invertible.}$$

Let us construct an equivalence between $\mathcal{P}$ and $\mathcal{A}$ directly. In fact, it is convenient to reformulate the definition of $\mathcal{A}$ slightly. Given $\mathcal{Q}\in\mathcal{A}$, consider endomorphisms

$$P_+=\delta_+\gamma_+,\ \ P_-=\delta_-\gamma_-\ \ \in\ \ \mathrm{End}(E_0).$$

These endomorphisms are idempotent:

$$P_+^2=\delta_+\gamma_+\delta_+\gamma_+=\delta_+\,\mathrm{Id}\,\gamma_+=\delta_+\gamma_+=P_+,$$

and similarly for $P_-$. The spaces $E_\pm$ are identified with $\mathrm{Im}(P_\pm)$ via $\delta_\pm$. The conditions of invertibility of $\gamma_\pm\delta_\mp$ is expressed by

$$\text{(9.1)}\qquad \begin{cases} P_-:\mathrm{Im}(P_+)\longrightarrow \mathrm{Im}(P_-),\\ P_+:\mathrm{Im}(P_-)\longrightarrow \mathrm{Im}(P_+)\end{cases}\qquad \text{are isomorphisms.}$$

This establishes the following.

LEMMA 9.2. *The category $\mathcal{A}$ is equivalent to the category $\mathcal{B}$ formed by data $(E_0,P_+,P_-)$ consisting of a finite-dimensional $\mathbf{k}$-vector space $E_0$ and two idempotents $P_+,P_-:E_0\to E_0$ satisfying* (9.1).

So we will construct an equivalence $\mathcal{B}\simeq\mathcal{P}$. Given $(E_0,P_+,P_-)\in\mathcal{B}$, we put

$$\Phi:=\mathrm{Ker}(P_-)\ \underset{u=P_--\mathrm{Id}}{\overset{v=P_+}{\rightleftarrows}}\ \Psi:=\mathrm{Im}(P_+).$$

Then $vu=P_+(P_--\mathrm{Id})$ is, as an endomorphism of $\mathrm{Im}(P_+)$, equal to $P_+P_--P_+$. Now, on $\mathrm{Im}(P_+)$ we have $P_+=\mathrm{Id}$. So $vu+\mathrm{Id}=P_+P_-$ as an endomorphism of $\mathrm{Im}(P_+)$, and so it is the composition of two invertible maps

$$\mathrm{Im}(P_+)\xrightarrow{P_-}\mathrm{Im}(P_-)\xrightarrow{P_+}\mathrm{Im}(P_+)$$

and hence invertible. This defines a functor $F:\mathcal{B}\to\mathcal{P}$. Note that the two maps above have the meaning of half-monodromies from the upper to the lower half plane, so $vu+\mathrm{Id}$ is the full monodromy.

Let us also define the functor $G : \mathcal{P} \to \mathcal{B}$ as follows. Given an object $\{ \Phi \underset{u}{\overset{v}{\rightleftarrows}} \Psi \} \in \mathcal{P}$, we put

$$E_0 = \Phi \oplus \Psi, \quad P_+ = \begin{pmatrix} 0 & 0 \\ v & 1 \end{pmatrix}, \quad P_- = \begin{pmatrix} 0 & u \\ 0 & 1 \end{pmatrix}. \tag{9.3}$$

Then $P_\pm$ are idempotents.

PROPOSITION 9.4. *The functors $F$ and $G$ are quasi-inverse to each other.*

*Proof.* Let us find $FG$. For $P_\pm$ defined above, we have

$$\mathrm{Im}(P_+) = \Psi, \quad \mathrm{Ker}(P_+) = \left\{ \begin{pmatrix} \phi \\ \psi \end{pmatrix} \middle| \, v\phi + \psi = 0 \right\} = \text{Graph of } (-v) : \Phi \to \Psi,$$
$$\mathrm{Im}(P_-) = \text{Graph of } u : \Psi \to \Phi, \quad \mathrm{Ker}(P_-) = \Phi.$$

The map $P_+ : \mathrm{Ker}(P_-) \to \mathrm{Im}(P_+)$ coincides with $v$. Further, $P_- - \mathrm{Id}$ restricted to $\Psi \subset \Phi \oplus \Psi$ gives $u : \Psi \to \Phi$. Therefore $FG$ is isomorphic to the identity functor of $\mathcal{P}$.

Conversely, suppose $(E_0, P_+, P_-) \in \mathcal{B}$. We then have two direct sum decompositions of $E_0$:

$$E_0 \;\; = \Phi \oplus \mathrm{Im}(P_-) = \mathrm{Ker}(P_+) \oplus \Psi.$$

The condition that $P_- : \mathrm{Im}(P_+) \to \mathrm{Im}(P_-)$ is an isomorphism implies that $\mathrm{Im}(P_+) \cap \mathrm{Ker}(P_-) = 0$ which, by the dimension count, implies that we have a direct sum decomposition

$$E_0 = \mathrm{Im}(P_+) \oplus \mathrm{Ker}(P_-) = \Psi \oplus \Phi.$$

With respect to this decomposition, we find that $P_\pm$ are given by the matrices in (9.3). So $GF$ is isomorphic to the identity functor of $\mathcal{B}$. □

*Remark* 9.5. Note that the composite equivalence $\mathcal{A} \to \mathcal{B} \xrightarrow{F} \mathcal{P}$ is compatible with (and so can be considered as induced by) the identifications of $\mathcal{A}$ and $\mathcal{P}$ with $\mathrm{Perv}(\mathbb{C}, 0)$, constructed in [15] and in this paper, respectively. Indeed, let $\mathcal{F} \in \mathrm{Perv}(\mathbb{C}, 0)$. Then, in the original construction of [15], the object of $\mathcal{A}$ corresponding to $\mathcal{F}$ is given by:

$$\Phi(\mathcal{F}) = \mathbb{H}^1_{\mathbb{R}_{\geq 0}}(\mathbb{C}, \mathcal{F}) = \Gamma(\mathbb{R}_{\geq 0}, \underline{\mathbb{H}}^1_{\mathbb{R}_{\geq 0}}(\mathcal{F})), \quad \Psi(\mathcal{F}) = \Gamma(\mathbb{R}_{>0}, \underline{\mathbb{H}}^1_{\mathbb{R}_{\geq 0}}(\mathcal{F})), \tag{9.6}$$

and the map $v$ is the generalization map for the $\mathbb{R}$-constructible sheaf $\underline{\mathbb{H}}^1_{\mathbb{R}_{\geq 0}}(\mathcal{F})$ on $\mathbb{R}_{\geq 0}$. On the other hand, the object $Q(\mathcal{F}) \in \mathcal{P}$ corresponding to $\mathcal{F}$ by (4.13) has

$$E_0(\mathcal{F}) = \mathbb{H}^1_{\mathbb{R}}(\mathbb{C}, \mathcal{F}) = \Gamma(\mathbb{R}, \underline{\mathbb{H}}^1_{\mathbb{R}}(\mathcal{F})), \quad E_\pm(\mathcal{F}) = \Gamma(\mathbb{R}_{\gtrless 0}, \underline{\mathbb{H}}^1_{\mathbb{R}}(\mathcal{F})), \tag{9.7}$$

and $\gamma_\pm$ is the generalization map for the $\mathbb{R}$-constructible sheaf $\underline{\mathbb{H}}^1_{\mathbb{R}}(\mathcal{F})$ on $\mathbb{R}$. The functor $F$ sends $Q = Q(\mathcal{F})$ into an object of $\mathcal{A}$ with

$$\Phi = \Phi_Q := \mathrm{Ker}(P_-) = \mathrm{Ker}(\gamma_-), \quad \Psi = \Psi_Q := \mathrm{Im}(P_+) = \mathrm{Im}(\delta_+).$$

Now, the identification $\mathrm{Ker}(\gamma_-) \to \Phi(\mathcal{F})$ is obtained from the long exact sequence relating (hyper)cohomology with supports in $\mathbb{R}$, $\mathbb{R}_{\geq 0}$ and $\mathbb{R}_{<0}$. The identification $\mathrm{Im}(\delta_+) \to \Psi(\mathcal{F})$ is obtained by comparing (9.6) and (9.7) and noting that $\underline{\mathbb{H}}^1_{\mathbb{R}_{\geq 0}}(\mathcal{F})$ and $\underline{\mathbb{H}}^1_{\mathbb{R}}(\mathcal{F})$ are identified on $\mathbb{R}_{>0}$. We leave to the reader the identification of the arrows between the vector spaces thus identified.

*Remark* 9.8. In addition to the above equivalence, which we denote here as $F_+ : \mathcal{B} \xrightarrow{\sim} \mathcal{P}$, there exists another one, $F_- : \mathcal{B} \xrightarrow{\sim} \mathcal{P}$, where for $x = (E_0, P_+, P_-) \in \mathcal{B}$, we set

$$F_-(x) = \left\{ \Phi := \mathrm{Ker}(P_+) \underset{u=P_+-\mathrm{Id}}{\overset{v=P_-}{\rightleftarrows}} \Psi := \mathrm{Im}(P_-) \right\}.$$

Furthermore, there are two invertible natural transformations ("half-monodromies") $t_\pm : F_\pm \xrightarrow{\sim} F_\mp$ given by

$$t_+(x) = (P_+ - \mathrm{Id}, P_-) : \; F_+(x) \xrightarrow{\sim} F_-(x)$$

and

$$t_-(x) = (P_- - \mathrm{Id}, P_+) : \; F_-(x) \xrightarrow{\sim} F_+(x).$$

These data define a $\mathbb{C}^*$-local system of equivalences $\mathcal{B} \to \mathcal{P}$; this is a particular case of the *microlocalization*, similar to [14].

The category $\mathrm{Perv}(\mathbb{C}, 0)$ admits an involution $\mathfrak{F} : \mathrm{Perv}(\mathbb{C}, 0) \to \mathrm{Perv}(\mathbb{C}, 0)$, which in the classical description looks as follows:

$$\mathfrak{F}\left\{ \Phi \underset{u}{\overset{v}{\rightleftarrows}} \Psi \right\} \simeq \left\{ \Psi \underset{v}{\overset{u}{\rightleftarrows}} \Phi \right\}.$$

(This involution is close to the geometric Fourier transform but does not coincide with it; cf. [4, Prop. 4.5].)

We leave to the reader the verification of the following.

Proposition 9.9. *In terms of the identification* $\mathrm{Perv}(\mathbb{C}, 0) \simeq \mathcal{B}$,

$$\mathfrak{F}(E_0, P_1, P_2) \simeq (E_0, 1 - P_1, 1 - P_2),$$

*and in terms of the identification* $\mathrm{Perv}(\mathbb{C}, 0) \simeq \mathcal{A}$,

$$\mathfrak{F}\left\{ E_- \underset{\delta_-}{\overset{\gamma_-}{\leftrightarrows}} E_0 \underset{\delta_+}{\overset{\gamma_+}{\rightleftarrows}} E_+ \right\} \simeq \left\{ \mathrm{Ker}(\gamma_-) \underset{\text{embedding}}{\overset{\mathrm{Id}-\delta_-\gamma_-}{\leftrightarrows}} E_0 \underset{\text{embedding}}{\overset{\mathrm{Id}-\delta_+\gamma_+}{\rightleftarrows}} \mathrm{Ker}(\gamma_+) \right\}.$$

9.B. *Real affine arrangements.* Let $\mathcal{H}$ be an arrangement of *affine* hyperplanes in $\mathbb{R}^n$ and $\mathcal{H}_\mathbb{C}$ be the complexified arrangement of affine hyperplanes in $\mathbb{C}^n$. We have then the category $\mathrm{Perv}(\mathbb{C}^n, \mathcal{H})$ of $\mathcal{H}$-smooth perverse sheaves on $\mathbb{C}^n$, similarly to the case of linear arrangements.

Such categories are important for the geometric description of tensor structures on the categories of quantum group representations [3].

Theorem 8.1 is extended to this case as follows. We have the poset of faces $(\mathcal{C}, \leq)$, defined similarly to the linear case. A triple of faces $(A, B, C)$ is called *collinear* if

(C1) there exists a face $D \leq A, B, C$;
(C2) there exists points $a \in A, b \in B, c \in C$ such that $b \in [a, c]$.

Condition (C1) holds automatically in the linear case (take $D = 0$). As with any poset, we have the category $\mathrm{Rep}^{(2)}(\mathcal{C})$ of double representations $Q = (E_C, \gamma_{C'C}, \delta_{CC'})$ of $\mathcal{C}$.

The condition of *monotonicity* on $Q$ is defined just as in the linear case: $\gamma_{C'C}\delta_{CC'} = \mathrm{Id}$ for any $C' \leq C$. This allows us to define the transition maps $\phi_{AB} = \gamma_{CB}\delta_{AC} : E_A \to E_B$ for any two faces $A, B$ such that there is a face $C \leq A, B$.

The condition of *transitivity* is defined by requiring that $\phi_{AC} = \phi_{BC}\phi_{AB}$ for any triple of cases $(A, B, C)$ collinear in the new sense above.

Finally, the condition of *invertibility* of $Q$ is defined completely similarly to the linear case.

We denote by $\mathcal{A} = \mathcal{A}_\mathcal{H}$ the full subcategory in $\mathrm{Rep}^{(2)}(\mathcal{C})$ formed by $Q$ that are monotone, transitive and invertible.

THEOREM 9.10. *The category* $\mathrm{Perv}(\mathbb{C}^n, \mathcal{H}_\mathbb{C})$ *is equivalent to* $\mathcal{A}_\mathcal{H}$.

*Proof.* Locally (in a neighborhood of any point of $\mathbb{R}^n$), an affine arrangement looks like a linear one. Therefore, applying known properties of the linear case, we establish the following statements:

(1) For $\mathcal{F} \in \mathrm{Perv}(\mathbb{C}^n, \mathcal{H})$, the complex $\mathcal{R}_\mathcal{F} = \underline{R\Gamma}_{\mathbb{R}^n}(\mathcal{F})[n]$ is reduced to one sheaf in degree 0, constructible with respect to the (cellular) stratification $\mathcal{C}$.
(2) Denoting by $E_C(\mathcal{F})$ the stalk of $\mathcal{R}_\mathcal{F}$ at $C \in \mathcal{C}$, we have a canonical identification $\mathcal{E}_C(\mathcal{F}^\bigstar) = E_C(\mathcal{F})^*$.
(3) Denoting by $\gamma^\mathcal{F}_{C'C} : E_{C'}(\mathcal{F}) \to E_C(\mathcal{F})$, $C' \leq C$, the generalization maps for $\mathcal{R}_F$, and putting $\delta^\mathcal{F}_{CC'} = (\gamma^{\mathcal{F}^\bigstar}_{C'C})^*$, we get an object
$$\mathcal{Q}(\mathcal{F}) = \big(E_C(\mathcal{F}),\, \gamma^\mathcal{F}_{C'C},\, \delta^\mathcal{F}_{CC'}\big) \,\in\, \mathrm{Rep}^{(2)}(\mathcal{C}).$$
This object lies in $\mathcal{A}$.

Therefore we get a functor $\mathcal{Q} : \mathrm{Perv}(\mathbb{C}^n, \mathcal{H}_\mathbb{C}) \to \mathcal{A}$ taking $\mathcal{F}$ to $\mathcal{Q}(\mathcal{F})$. To prove that it is an equivalence, we use the fact that perverse sheaves form a stack. That is, we upgrade $\mathcal{Q}$ to a morphism of stacks of abelian categories on $\mathbb{R}^n$,

$$\underline{\mathcal{Q}} : \underline{\mathrm{Perv}}_{(\mathbb{C}^n,\mathcal{H})} \longrightarrow \underline{\mathcal{A}},$$

defined as follows. For an open $U \subset \mathbb{R}^n$, the category $\underline{\mathrm{Perv}}_{(\mathbb{C}^n,\mathcal{H})}(U)$ consists of perverse sheaves on $U + i\mathbb{R}^n$ smooth with respect to the stratification cut out on $U + i\mathbb{R}^n$ by $\mathcal{H}$. We further denote by $\mathcal{C}_U \subset \mathcal{C}$ the subset of faces meeting $U$ and extend the concept of collinearity to $\mathcal{C}_U$ be requiring in (C1) that $D \in \mathcal{C}_U$. Then we define $\underline{\mathcal{A}}(U) \subset \mathrm{Rep}^{(2)}(\mathcal{C}_U)$ to be the full subcategory specified by the conditions of monotonicity, transitivity and invertibility. The same arguments as before define a functor $\underline{\mathcal{Q}}(U) : \underline{\mathrm{Perv}}_{(\mathbb{C}^n,\mathcal{H})}(U) \to \underline{\mathcal{A}}(U)$, and these functors unite into a morphism of stacks $\underline{\mathcal{Q}}$. Note that $\mathcal{Q}$ is obtained from $\underline{\mathcal{Q}}$ by passing to the categories of global sections. So it is enough to show that $\underline{\mathcal{Q}}$ is an equivalence of stacks, a statement that can be checked locally, at the level of stalks. But the stalk functor of $\underline{\mathcal{Q}}$ over any point $x \in \mathbb{R}^n$ is a similar functor $\mathcal{Q}$ for the (essentially) linear configuration formed by hyperplanes from $\mathcal{H}$ containing $x$. So it is an equivalence by Theorem 8.1. □

The method of using the stacky nature of perverse sheaves to obtain descriptions in new situations was applied by Dupont to the case of smooth toric varieties [13]. Another approach to proving Theorem 9.10 would be to add a variable to make an affine arrangement into a linear one.

9.C. *The fundamental groupoid of the open stratum.* Let us write $V = \mathbb{R}^n$, so $V_\mathbb{C} = \mathbb{C}^n$, and let $V_\mathbb{C}^\circ \subset V_\mathbb{C}$ be the open stratum (the complement of all the hyperplanes $L_\mathbb{C}$, for $L \in \mathcal{H}$). Let $\mathcal{C}_0$ be the set of open faces (chambers) of $\mathcal{H}$, so each $A \in \mathcal{C}_0$ is a contractible set contained in $V_\mathbb{C}^\circ$. Denote $\pi_1(V_\mathbb{C}^\circ, \mathcal{C}_0)$ the fundamental groupoid of $V_\mathbb{C}^\circ$ with respect to a set of base points consisting of one point in each $A \in \mathcal{C}_0$. The construction of this paper leads to a new description of this groupoid.

PROPOSITION 9.11. *$\pi_1(V_\mathbb{C}^\circ, \mathcal{C}_0)$ is isomorphic to the groupoid $\mathfrak{G}$ defined by generators and relations as follows*:

(0) *Objects $x_A$, $A \in \mathcal{C}_0$.*
(1) *Generating morphisms $\varphi_{AB} : x_A \to x_B$ for each ordered pair $(A, B)$ of chambers. We assume that $\varphi_{AA} = \mathrm{Id}_{x_A}$.*
(2) *Relations $\varphi_{AC} = \varphi_{BC}\varphi_{AB}$ for any collinear triple $(A, B, C)$ of chambers.*

A more familiar description of $\pi_1(V_\mathbb{C}^\circ, \mathcal{C}_0)$ is the one from the work of Salvetti [33], which we now recall.

PROPOSITION 9.12. *$\pi_1(V_\mathbb{C}^\circ, \mathcal{C}_0)$ is isomorphic to the groupoid $\mathfrak{S}$ defined by generators and relations as follows*:

(0) *Objects $x_A$, $A \in \mathcal{C}_0$.*

(1) *Generating morphisms* $\psi_{AB} : x_A \to x_B$ *for each ordered pair* $(A, B)$ *of chambers that are adjacent, i.e., lie on opposite sides of a codimension* 1 *face* $C$.

(2) *The Zifferblatt relations*

$$\psi_{B_m,C}\,\psi_{B_{m-1},B_m}\cdots\psi_{A,B_2} = \psi_{B_{m+2},C}\,\psi_{B_{m+3},B_{m+2}}\cdots\psi_{B_{2m},B_{2m-1}}\,\psi_{A,B_{2m}}$$

*for any codimension* 2 *face* $F$ *and any chamber* $A > F$. *Here we number all the chambers* $> F$ *around the circle as* $A = B_1, B_2, \ldots, B_{m+1} = C, B_{m+1}, \ldots, B_{2m}$, *as in Figure 2.*

Proposition 9.12 can be obtained by noticing that $\mathcal{S}^{(1)}$-cells $[C, A]$ for $A$ being a chamber form a cell decomposition of $V_{\mathbb{C}}^{\circ}$. Among them the cells $[A, A]$ are precisely the open ones. Therefore one can form the dual CW-complex. denote it $\mathcal{S}al$, homotopy equivalent to $V_{\mathbb{C}}^{\circ}$, in which each cell $[A, A]$ will give rise to a vertex, each cell $[C, A]$ with $\operatorname{codim}(C) = 1$ will give rise to an edge, and each cell $[F, A]$ with $\operatorname{codim}(C) = 2$ will give rise to a $2m$-gon, with $2m = \#\{B \in \mathcal{C}_0 |\, B > F\}$ and so on. The groupoid $\mathfrak{S}$ is the groupoid whose presentation is obtained in the standard way, from the 2-skeleton of $\mathcal{S}al$; see [6], [33] for more details.

We now prove Proposition 9.11. Define a functor $F : \mathfrak{S} \to \mathfrak{G}$ to be identity on the objects and to send $\psi_{AB}$ (with $A, B$ adjacent) to $\varphi_{AB}$. Note that the relations of $\mathfrak{S}$ are satisfied in $\mathfrak{G}$ (see Example 7.9) so $F$ is well defined. Let us prove that $F$ is an isomorphism of groupoids.

For this, we define a functor $G : \mathfrak{G} \to \mathfrak{S}$, also identical on objects, as follows. Let $A, B$ be two chambers. Choose any two generic points $a \in A, b \in B$ so that the interval $[a, b] \subset \mathbb{R}^n$ does not intersect any faces of codimension $\geq 2$. Denote the chambers intersecting $[a, b]$, if written in the direction from $a$ to $b$, by $C_1 = A, C_2, \ldots, C_r = B$. Then each $(C_i, C_{i+1})$ form an adjacent pair, so the generator $\psi_{C_i,C_{i+1}}$ of $\mathfrak{S}$ is defined, and we put

$$G(\varphi_{AB}) = \psi_{C_{r-1},B}\,\psi_{C_{r-2},C_{r-1}}\cdots\psi_{A,C_2}. \tag{9.13}$$

LEMMA 9.14. *The right-hand side of* (9.13), *considered as an element of* $\operatorname{Hom}_{\mathfrak{S}}(x_A, x_B)$, *is independent of the choice of generic points* $a \in A, b \in B$.

*Proof.* It is enough to prove that if we keep $a$ and replace $b$ by another generic point $b' \in B$, or if we keep $b$ and replace $a$ by another generic point $a' \in A$, then the right-hand side of (9.13) will give the same morphism. Let us consider the first situation; the second one is treated similarly.

Given $a, b, b'$, consider the plane triangle $\Delta = \operatorname{Conv}\{a, b, b'\}$ inside the affine 2-plane $P$ spanned by $a, b, b'$. The side $[b, b']$ of $\Delta$ lies inside $B$. Note that $\mathcal{H}$ induces an affine arrangement of lines inside $P$, and by our assumption, flats of $\mathcal{H}$ of codimension $\geq 3$ do not meet $P$. Each codimension 2 face $L$ of $\mathcal{H}$ intersecting $\Delta$ does so at an interior point of $\Delta$. Now, around each such point $L \cap \Delta$, we have a Zifferblatt situation. That is, we can deform the path $\xi_0 = [a, b'] \cup [b', b]$ into $\xi_1 = [a, b]$ in a family of paths $(\xi_t)_{t \in [0,1]}$, keeping the endpoints $a, b$ fixed so that at every moment $t$ we cross at most one of the points $L \cap \Delta$. Associating to each intermediate path $\xi_t$ the product of generators $\psi$ similar to (9.13), we see that after crossing each $L \cap \Delta$, the product remains unchanged in virtue of the relations of $\mathfrak{S}$. □

With the lemma established, we see that $G$ preserves the relations of $\mathfrak{G}$ by its very definition: for collinear chambers $A, B, C$ with points $a, b, c$ such that $c \in [a, b]$, we use the intervals $[a, b], [b, c], [a, c]$ to define the values of $G$ on $\varphi_{AB}, \varphi_{BC}, \varphi_{AC}$. So $G$ is indeed a functor, and we see that it is inverse to $F$ by looking at the action of $F$ and $G$ on the generators. Proposition 9.11 is proved. □

## References

[1] A. A. Beĭlinson, How to glue perverse sheaves, in *K-Theory, Arithmetic and Geometry* (Moscow, 1984–1986), *Lecture Notes in Math.* **1289**, Springer-Verlag, New York, 1987, pp. 42–51. MR 0923134. Zbl 0651.14009. http://dx.doi.org/10.1007/BFb0078366.

[2] A. A. Beĭlinson, J. Bernstein, and P. Deligne, Faisceaux pervers, in *Analysis and Topology on Singular Spaces, I* (Luminy, 1981), *Astérisque* **100**, Soc. Math. France, Paris, 1982, pp. 5–171. MR 0751966. Zbl 0536.14011.

[3] R. Bezrukavnikov, M. Finkelberg, and V. Schechtman, *Factorizable Sheaves and Quantum Groups*, *Lecture Notes in Math.* **1691**, Springer-Verlag, New York, 1998. MR 1641131. Zbl 0938.17016. http://dx.doi.org/10.1007/BFb0092343.

[4] R. Bezrukavnikov and M. Kapranov, Microlocal sheaves and quiver varieties, 2015. arXiv 1506.07050.

[5] A. Björner, M. Las Vergnas, B. Sturmfels, N. White, and G. M. Ziegler, *Oriented Matroids*, second ed., *Encyclopedia Math. Appl.* **46**, Cambridge Univ. Press, Cambridge, 1999. MR 1744046. Zbl 0944.52006. http://dx.doi.org/10.1017/CBO9780511586507.

[6] A. Björner and G. M. Ziegler, Combinatorial stratification of complex arrangements, *J. Amer. Math. Soc.* **5** (1992), 105–149. MR 1119198. Zbl 0754.52003. http://dx.doi.org/10.2307/2152753.

[7] T. Braden, Perverse sheaves on Grassmannians, *Canad. J. Math.* **54** (2002), 493–532. MR 1900761. Zbl 1009.32019. http://dx.doi.org/10.4153/CJM-2002-017-6.

[8] T. Braden, Hyperbolic localization of intersection cohomology, *Transform. Groups* **8** (2003), 209–216. MR 1996415. Zbl 1026.14005. http://dx.doi.org/10.1007/s00031-003-0606-4.

[9] T. Braden and M. Grinberg, Perverse sheaves on rank stratifications, *Duke Math. J.* **96** (1999), 317–362. MR 1666554. Zbl 0958.14009. http://dx.doi.org/10.1215/S0012-7094-99-09609-6.

[10] K. S. Brown and P. Diaconis, Random walks and hyperplane arrangements, *Ann. Probab.* **26** (1998), 1813–1854. MR 1675083. Zbl 0938.60064. http://dx.doi.org/10.1214/aop/1022855884.

[11] J. M. Curry, *Sheaves, Cosheaves and Applications*, ProQuest LLC, Ann Arbor, MI, 2014, Ph.D. Thesis, University of Pennsylvania. MR 3259939. Available at http://gateway.proquest.com/openurl?url_ver=Z39.88-2004&rft_val_fmt=info:ofi/fmt:kev:mtx:dissertation&res_dat=xri:pqm&rft_dat=xri:pqdiss:3623819.

[12] V. Drinfeld and D. Gaitsgory, On a theorem of Braden, *Transform. Groups* **19** (2014), 313–358. MR 3200429. Zbl 1314.14039. http://dx.doi.org/10.1007/s00031-014-9267-8.

[13] D. Dupont, Faisceaux pervers sur les variétés toriques lisses, *C. R. Math. Acad. Sci. Paris* **348** (2010), 853–856. MR 2677979. Zbl 1197.14013. http://dx.doi.org/10.1016/j.crma.2010.06.011.

[14] M. Finkelberg and V. Schechtman, Microlocal approach to Lusztig's symmetries, 2014. arXiv 1401.5885.

[15] A. Galligo, M. Granger, and P. Maisonobe, $D$-modules et faisceaux pervers dont le support singulier est un croisement normal, *Ann. Inst. Fourier* (*Grenoble*) **35** (1985), 1–48. MR 0781776. Zbl 0572.32012. http://dx.doi.org/10.5802/aif.996.

[16] I. M. Gel′fand and G. L. Rybnikov, Algebraic and topological invariants of oriented matroids, *Dokl. Akad. Nauk SSSR* **307** (1989), 791–795. MR 1020668. Zbl 0717.33009.

[17] S. Gelfand and R. MacPherson, Verma modules and Schubert cells: a dictionary, in *Paul Dubreil and Marie-Paule Malliavin Algebra Seminar*, 34*th Year* (Paris, 1981), *Lecture Notes in Math.* **924**, Springer-Verlag, New York, 1982, pp. 1–50. MR 0662251. Zbl 0512.22009. http://dx.doi.org/10.1007/BFb0092924.

[18] S. Gelfand, R. MacPherson, and K. Vilonen, Microlocal perverse sheaves, 2005. arXiv math/0509440.

[19] S. Gelfand and Y. Manin, *Methods of Homological Algebra*, second ed., *Springer Monogr. Math.*, Springer-Verlag, New York, 2003. MR 1950475. Zbl 1006.18001. http://dx.doi.org/10.1007/978-3-662-12492-5.

[20] S. G. Gindikin and F. I. Karpelevič, An integral associated with Riemannian symmetric spaces of non-positive curvature, *Izv. Akad. Nauk SSSR Ser. Mat.* **30** (1966), 1147–1156. MR 0222219. Zbl 0165.55705. Available at http://mi.mathnet.ru/eng/izv/v30/i5/p1147.

[21] N. Honda and P. Schapira, A vanishing theorem for holonomic modules with positive characteristic varieties, *Publ. Res. Inst. Math. Sci.* **26** (1990), 529–534. MR 1068865. Zbl 0728.58036. http://dx.doi.org/10.2977/prims/1195170961.

[22] M. Kapranov and V. Schechtman, Perverse Schobers, 2014. arXiv 1411.2772.

[23] M. Kashiwara, The Riemann-Hilbert problem for holonomic systems, *Publ. Res. Inst. Math. Sci.* **20** (1984), 319–365. MR 0743382. Zbl 0566.32023. http://dx.doi.org/10.2977/prims/1195181610.

[24] M. Kashiwara, *Algebraic Study of Systems of Partial Differential Equations*, *Mém. Soc. Math. France* **63**, Soc. Math. France, Paris, 1995. MR 1384226. Zbl 0877.35003. Available at http://www.numdam.org/item?id=MSMF_1995_2_63__R1_0.

[25] M. Kashiwara and P. Schapira, *Sheaves on Manifolds*, *Grundl. math. Wissen.* **292**, Springer-Verlag, New York, 1990. MR 1074006. Zbl 0709.18001. http://dx.doi.org/10.1007/978-3-662-02661-8.

[26] A. W. Knapp, The Gindikin-Karpelevič formula and intertwining operators, in *Lie Groups and Symmetric Spaces*, *Amer. Math. Soc. Transl. Ser.* 2 **210**, Amer. Math. Soc., Providence, RI, 2003, pp. 145–159. MR 2018359. Zbl 1041.22007.

[27] H. Komatsu, On the index of ordinary differential operators, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **18** (1971), 379–398. MR 0303568. Zbl 0232.34026.

[28] G. Lebeau, Annulation de la cohomologie hyperfonction de certains modules holonomes, *C. R. Acad. Sci. Paris Sér. A-B* **290** (1980), A313–A316. MR 0567756. Zbl 0423.58017.

[29] R. MacPherson and K. Vilonen, Elementary construction of perverse sheaves, *Invent. Math.* **84** (1986), 403–435. MR 0833195. Zbl 0597.18005. http://dx.doi.org/10.1007/BF01388812.

[30] Y. I. Manin and V. V. Schechtman, Arrangements of real hyperplanes and Zamolodchikov equations, in *Group Theoretical Methods in Physics, Vol. I* (Yurmala, 1985), VNU Sci. Press, Utrecht, 1986, pp. 151–165. MR 0919748. Zbl 0699.58069.

[31] C. Mœglin and J.-L. Waldspurger, *Décomposition Spectrale et Séries d'Eisenstein*, Une paraphrase de l'Écriture. [A paraphrase of Scripture], *Progress in Mathematics* **113**, Birkhäuser, Basel, 1994. MR 1261867. Zbl 0794.11022.

[32] A. Polishchuk, Perverse sheaves on a triangulated space, *Math. Res. Lett.* **4** (1997), 191–199. MR 1453053. Zbl 0905.18007. http://dx.doi.org/10.4310/MRL.1997.v4.n2.a2.

[33] M. Salvetti, Topology of the complement of real hyperplanes in $\mathbf{C}^N$, *Invent. Math.* **88** (1987), 603–618. MR 0884802. Zbl 0594.57009. http://dx.doi.org/10.1007/BF01391833.

[34] M. Sato, T. Kawai, and M. Kashiwara, Microfunctions and pseudo-differential equations, in *Hyperfunctions and Pseudo-Differential Equations* (Proc. Conf., Katata, 1971; dedicated to the memory of André Martineau), *Lecture Notes in Math.* **287**, Springer-Verlag, New York, 1973, pp. 265–529. MR 0420735. Zbl 0277.46039.

[35] V. V. Schechtman and A. N. Varchenko, Arrangements of hyperplanes and Lie algebra homology, *Invent. Math.* **106** (1991), 139–194. MR 1123378. Zbl 0754.17024. http://dx.doi.org/10.1007/BF01243909.

[36] V. V. Schechtman and A. N. Varchenko, Quantum groups and homology of local systems, in *Algebraic Geometry and Analytic Geometry* (Tokyo,

1990), *ICM-90 Satell. Conf. Proc.*, Springer-Verlag, New York, 1991, pp. 182–197. MR 1260946. Zbl 0760.17014.

[37] G. Schiffmann, Intégrales d'entrelacement et fonctions de Whittaker, *Bull. Soc. Math. France* **99** (1971), 3–72. MR 0311838. Zbl 0223.22017. Available at http://www.numdam.org/item?id=BSMF_1971__99__3_0.

[38] J. Schürmann, On the dimension formula for the hyperfunction solutions of some holonomic $D$-modules, *Publ. Res. Inst. Math. Sci.* **42** (2006), 1–8. MR 2215432. Zbl 1106.32008. http://dx.doi.org/10.2977/prims/1166642055.

[39] K. Takeuchi, Dimension formulas for the hyperfunction solutions to holonomic $D$-modules, *Adv. Math.* **180** (2003), 134–145. MR 2019219. Zbl 1066.32014. http://dx.doi.org/10.1016/S0001-8708(02)00099-3.

[40] J. Tits, *Buildings of Spherical Type and Finite BN-Pairs*, *Lecture Notes in Math.* **386**, Springer-Verlag, New York, 1974. MR 0470099. Zbl 0295.20047. http://dx.doi.org/10.1007/978-3-540-38349-9.

[41] D. Treumann, Exit paths and constructible stacks, *Compos. Math.* **145** (2009), 1504–1532. MR 2575092. Zbl 1185.32022. http://dx.doi.org/10.1112/S0010437X09004229.

[42] A. Varchenko, *Multidimensional Hypergeometric Functions and Representation Theory of Lie Algebras and Quantum Groups*, *Adv. Ser. Math. Phys.* **21**, World Scientific Publishing Co., River Edge, NJ, 1995. MR 1384760. Zbl 0951.33001. http://dx.doi.org/10.1142/2467.

[43] J. L. Verdier, *Dualité dans la Cohomologie des espaces Localement Compacts*, *Sém. Bourbaki* 1965–1966 **300**, 1966. Zbl 0268.55006.

[44] M. Vybornov, Mixed algebras and quivers related to cell complexes, *Comm. Algebra* **25** (1997), 3985–3992. MR 1481582. Zbl 0919.18007. http://dx.doi.org/10.1080/00927879708826102.

[45] M. Vybornov, Constructible sheaves on simplicial complexes and Koszul duality, *Math. Res. Lett.* **5** (1998), 675–683. MR 1666864. Zbl 0935.18011. http://dx.doi.org/10.4310/MRL.1998.v5.n5.a10.

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