Regularity of area minimizing currents III: blow-up

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Abstract

This is the last of a series of three papers in which we give a new, shorter proof of a slightly improved version of Almgren’s partial regularity of area minimizing currents in Riemannian manifolds. Here we perform a blow-up analysis deducing the regularity of area minimizing currents from that of Dir-minimizing multiple valued functions.

0. Introduction

In this paper we complete the proof of a slightly improved version of the celebrated Almgren’s partial regularity result for area minimizing currents in a Riemannian manifold (see [1]), namely, Theorem 0.3 below.

Assumption 0.1. Let \( \varepsilon_0 \in ]0,1[ \), \( m, \bar{n} \in \mathbb{N} \setminus \{0\} \) and \( l \in \mathbb{N} \). We denote by

(M) \( \Sigma \subset \mathbb{R}^{m+n} = \mathbb{R}^{m+\bar{n}+l} \) an embedded \((m+\bar{n})\)-dimensional submanifold of class \( C^{3,\varepsilon_0} \);

(C) \( T \) an integral current of dimension \( m \) with compact support \( \text{spt}(T) \subset \Sigma \), area minimizing in \( \Sigma \).

In this paper we follow the notation of [6] concerning balls, cylinders and disks. In particular, \( B_r(x) \subset \mathbb{R}^{m+n} \) will denote the Euclidean ball of radius \( r \) and center \( x \).

Definition 0.2. For \( T \) and \( \Sigma \) as in Assumption 0.1, we define

\[
\text{Reg}(T) := \left\{ x \in \text{spt}(T) : \text{spt}(T) \cap B_r(x) \text{ is a } C^{3,\varepsilon_0} \text{ submanifold for some } r > 0 \right\},
\]

(0.1)

\[
\text{Sing}(T) := \text{spt}(T) \setminus \left( \text{spt}(\partial T) \cup \text{Reg}(T) \right).
\]

(0.2)

The partial regularity result proven first by Almgren [1] under the more restrictive hypothesis \( \Sigma \in C^5 \) gives an estimate on the Hausdorff dimension \( \dim_H(\text{Sing}(T)) \) of \( \text{Sing}(T) \).
Theorem 0.3. $\dim_H(\text{Sing}(T)) \leq m - 2$ for any $m, \bar{n}, l, T$ and $\Sigma$ as in Assumption 0.1.

In this note we complete the proof of Theorem 0.3, based on our previous works [3], [4], [5], [6], thus providing a new, and much shorter, account of one of the most fundamental regularity result in geometric measure theory; we refer to [4] for an extended general introduction to all these works. The proof is carried by contradiction. In the sequel we will always assume the following.

Assumption 0.4 (Contradiction). There exist $m \geq 2, \bar{n}, l, \Sigma$ and $T$ as in Assumption 0.1 such that $\mathcal{H}^{m-2+\alpha}(\text{Sing}(T)) > 0$ for some $\alpha > 0$.

The hypothesis $m \geq 2$ in Assumption 0.4 is justified by the well-known fact that $\text{Sing}(T) = \emptyset$ when $m = 1$. (In this case $\text{spt}(T) \setminus \text{spt}(\partial T)$ is locally the union of finitely many nonintersecting geodesic segments.) Starting from Assumption 0.4, we make a careful blow-up analysis, split in the following steps.

0.1. Flat tangent planes. We first reduce to flat blow-ups around a given point, which in the sequel is assumed to be the origin. These blow-ups will also be chosen so that the size of the singular set satisfies a uniform estimate from below (cf. Section 1).

0.2. Intervals of flattening. For appropriate rescalings of the current around the origin, we take advantage of the center manifold constructed in [6], which gives a good approximation of the average of the sheets of the current at some given scale. However, since it might fail to do so at different scales, in Section 2 we introduce a stopping condition for the center manifolds and define appropriate intervals of flattening $I_j = [s_j, t_j]$. For each $j$, we construct a different center manifold $M_j$ and approximate the (rescaled) current with a suitable multi-valued map on the normal bundle of $M_j$.

0.3. Finite order of contact. A major difficulty in the analysis is to prove that the minimizing current has finite order of contact with the center manifold. To this aim, in analogy with the case of harmonic multiple valued functions (cf. [3, §3.4]), we introduce a variant of the frequency function and prove its almost monotonicity and boundedness. This analysis, carried in Sections 3, 4 and 5, relies on the variational formulas for images of multiple valued maps as computed in [5] and on the careful estimates of [6]. Our frequency function differs from that of Almgren and allows for simpler estimates.

0.4. Convergence to Dir-minimizer and contradiction. Based on the previous steps, we can blow-up the Lipschitz approximations from the center manifold $M_j$ in order to get a limiting Dir-minimizing function on a flat $m$-dimensional domain. We then show that the singularities of the rescaled currents converge to singularities of that limiting Dir-minimizer, contradicting the partial regularity of [3, §3.6] and, hence, proving Theorem 0.3.
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1. Flat tangent cones

Definition 1.1 (Q-points). For Q ∈ N, we denote by D_Q(T) the points of density Q of the current T, and we set

\[ \text{Reg}_Q(T) := \text{Reg}(T) \cap D_Q(T) \quad \text{and} \quad \text{Sing}_Q(T) := \text{Sing}(T) \cap D_Q(T). \]

Definition 1.2 (Tangent cones). For any \( r > 0 \) and \( x \in \mathbb{R}^{m+n} \), \( \iota_{x,r} : \mathbb{R}^{m+n} \to \mathbb{R}^{m+n} \) is the map \( y \mapsto \frac{y-x}{r} \) and \( T_{x,r} := (\iota_{x,r})_* T \). The classical monotonicity formula (see [10] and [4, Lemma A.1]) implies that, for every density \( Q \), \( \lim_{r \to 0} \iota_{x,r}^{-1}(N) = 0 \) and \( \iota_{x,r}^{-1}(\mathbb{R}^{m+n} - \text{spt}(T)) \to \partial T \) a.e. \( x \in \text{spt}(T) \setminus \text{spt}(\partial T) \), there is a subsequence (not relabeled) for which \( T_{x,r_k} \) converges to an integral cycle \( S \) which is a cone (i.e., \( S_{0,r} = S \) for all \( r > 0 \) and \( \partial S = 0 \)) and is (locally) area minimizing in \( \mathbb{R}^{m+n} \). Such a cone will be called, as usual, a tangent cone to \( T \) at \( x \).

Fix \( \alpha > 0 \). By Almgren’s stratification theorem (see [10, Th. 35.3]), for \( \mathcal{H}^{m-2+\alpha} \)-a.e. \( x \in \text{spt}(T) \setminus \text{spt}(\partial T) \), there exists a subsequence of radii \( r_k \downarrow 0 \) such that \( T_{x,r_k} \) converge to an integer multiplicity flat plane. Similarly, for measure-theoretic reasons, if \( T \) is as in Assumption 0.4, then for \( \mathcal{H}^{m-2+\alpha} \)-a.e. \( x \in \text{spt}(T) \setminus \text{spt}(\partial T) \), there is a subsequence \( s_k \downarrow 0 \) such that \( \liminf_{k\to \infty} \mathcal{H}_\infty^{m-2+\alpha}(D_Q(T_{x,s_k}) \cap B_1) > 0 \) (see again [10]). Obviously there would then be \( Q \in N \) and \( x \in \text{Sing}_Q(T) \) where both subsequences exist. The two subsequences might, however, differ. In the next proposition we show the existence of one point and a single subsequence along which both conclusions hold. For the relevant notation (concerning, for instance, excess and height of currents), we refer to [4], [6].

Proposition 1.3 (Contradiction sequence). Under Assumption 0.4, there are \( m,n,Q \geq 2, \Sigma \) and \( T \) as in Assumption 0.1, reals \( \alpha, \eta > 0 \), and a sequence \( r_k \downarrow 0 \) such that \( 0 \in D_Q(T) \) and the following holds:

\[
\lim_{k \to +\infty} E(T_{0,r_k}, B_{6\sqrt{m}}) = 0, \\
\lim_{k \to +\infty} H_\infty^{m-2+\alpha}(D_Q(T_{0,r_k}) \cap B_1) > \eta, \\
H^m((B_1 \cap \text{spt}(T_{0,r_k})) \setminus D_Q(T_{0,r_k})) > 0 \quad \forall k \in \mathbb{N}.
\]

The proof is based on the following lemma.

Lemma 1.4. Let \( S \) be an \( m \)-dimensional area minimizing integral cone in \( \mathbb{R}^{m+n} \) such that \( \partial S = 0 \), \( Q = \Theta(S,0) \in \mathbb{N} \), \( H^m(D_Q(S)) > 0 \) and \( H^{m-1}(\text{Sing}_Q(S)) = 0 \). Then, \( S \) is an \( m \)-dimensional plane with multiplicity \( Q \).
Thus, we conclude that some regular submanifold $\Gamma$ and set

$$U := \bigcup_{x \in \text{Reg}_Q(S)} B_{r_x}(x).$$

Obviously, $\text{Reg}_Q(S) \subset U$; hence, by assumption, it is not empty. Fix $x \in \text{spt}(S) \cap \partial U$. Next let $(x_k)_{k \in \mathbb{N}} \subset \text{Reg}_Q(S)$ be such that $\text{dist}(x, B_{r_{x_k}}(x_k)) \to 0$. We necessarily have that $r_{x_k} \to 0$; otherwise we would have $x \in B_{2r_{x_k}}(x_k)$ for some $k$, which would imply $x \in \text{Reg}_Q(S) \subset U$, i.e., a contradiction. Therefore, $x_k \to x$ and, by [10, Th. 35.1],

$$Q = \limsup_{k \to +\infty} \Theta(S, x_k) \leq \Theta(S, x) = \lim_{\lambda \downarrow 0} \Theta(S, \lambda x) \leq \Theta(S, 0) = Q.$$

This implies $x \in D_Q(S)$. Since $x \in \partial U$, we must then have $x \in \text{Sing}_Q(S)$. Thus, we conclude that $H^{m-1}(\text{spt}(S) \cap \partial U) = 0$. It follows from the standard theory of rectifiable currents (cf. Lemma A.2) that $S' := S \setminus U$ has 0 boundary in $\mathbb{R}^{m+n}$. Moreover, since $S$ is an area minimizing cone, $S'$ is also an cone. By definition of $U$ we have $\Theta(S', x) = Q$ for $\|S'\|$-a.e. $x$ and, by semicontinuity,

$$Q \leq \Theta(S', 0) \leq \Theta(S, 0) = Q.$$

We apply Allard’s theorem and deduce that $S'$ is regular; i.e., $S'$ is an $m$-plane with multiplicity $Q$. Finally, from $\Theta(S', 0) = \Theta(S, 0)$, we infer $M(S \setminus B_1) = M(S' \setminus B_1)$ and then $S' = S$. 

Proof of Proposition 1.3. Let $m > 1$ be the smallest integer for which Theorem 0.3 fails. By Theorem A.3 there must be an integer rectifiable area minimizing current $R$ of dimension $m$ and a positive integer $Q$ such that the Hausdorff dimension of $\text{Sing}_Q(R)$ is larger than $m-2$. (Note that Theorem A.3 is just a corollary of a well-known stratification theorem by Almgren; cf. [1], [10], [11].) We fix the smallest $Q$ for which such a current $R$ exists. Recall that, by the upper semicontinuity of the density and a straightforward application of Allard’s regularity theorem (see Theorem A.1), $\text{Sing}_1(R) = \emptyset$; i.e., $Q > 1$.

Let $\alpha \in [0, 1]$ be such that $H^{m-2+\alpha}(\text{Sing}_Q(R)) > 0$. By [10, Th. 3.6] there exists a point $x \in \text{Sing}_Q(R)$ such that $\text{Sing}_Q(R)$ has positive $H^{m-2+\alpha}$-upper density; i.e., assuming without loss of generality $x = 0$ and $\partial R \setminus B_1 = 0$, there exists $r_k \downarrow 0$ such that

$$\lim_{k \to +\infty} H_{\infty}^{m-2+\alpha}(\text{Sing}_Q(R_{0, r_k}) \cap B_1) = \lim_{k \to +\infty} \frac{H_{\infty}^{m-2+\alpha}(\text{Sing}_Q(R) \cap B_{r_k})}{r_k^{m-2+\alpha}} > 0.$$ 

Up to a subsequence (not relabeled) we can assume that $R_{0, r_k} \to S$, with $S$ a tangent cone. If $S$ is a multiplicity $Q$ flat plane, then we set $T := R$ and we are done. Indeed, (1.3) is satisfied by Theorem A.1, because $0 \in \text{Sing}(R)$ and $\|R\| \geq H^m \text{spt}(R)$.
Therefore assume that \( S \) is not an \( m \)-dimensional plane with multiplicity \( Q \). Taking into account the convergence of the total variations for minimizing currents [10, Th. 34.5] and the upper semicontinuity of \( \mathcal{H}_\infty^{-2+\alpha} \) under the Hausdorff convergence of compact sets, we get
\[
(1.4) \quad \mathcal{H}_\infty^{-2+\alpha}(D_Q(S) \cap \tilde{B}_1) \geq \liminf_{k \to +\infty} \mathcal{H}_\infty^{-2+\alpha}(D_Q(R_{0,r_k}) \cap \tilde{B}_1) > 0.
\]
We claim that (1.4) implies
\[
(1.5) \quad \mathcal{H}_\infty^{-2+\alpha}(\text{Sing}_Q(S)) > 0.
\]
Indeed, if all points of \( D_Q(S) \) are singular, then this follows from (1.4) directly. Otherwise, \( \text{Reg}_Q(S) \) is not empty and, hence, \( \mathcal{H}^m(D_Q(S) \cap \tilde{B}_1) > 0 \). In this case we can apply Lemma 1.4 and infer that, since \( S \) is not regular, then \( \mathcal{H}^{m-1}(\text{Sing}_Q(S)) > 0 \) and (1.5) holds.

We can, hence, find \( x \in \text{Sing}_Q(S) \setminus \{0\} \) and \( r_k \downarrow 0 \) such that
\[
\lim_{k \to +\infty} \mathcal{H}_\infty^{-2+\alpha}(\text{Sing}_Q(S_{x,r_k}) \cap \tilde{B}_1) = \lim_{k \to +\infty} \frac{\mathcal{H}_\infty^{-2+\alpha}(\text{Sing}_Q(S) \cap B_{r_k}(x))}{r_k^{m-2+\alpha}} > 0.
\]
Up to a subsequence (not relabeled), we can assume that \( S_{x,r_k} \) converges to \( S_1 \). Since \( S_1 \) is a tangent cone to the cone \( S \) at \( x \neq 0 \), \( S_1 \) splits off a line, i.e., \( S_1 = S_2 \times [\mathbb{R} v] \), for some area minimizing cone \( S_2 \) in \( \mathbb{R}^{m-1+n} \) and some \( v \in \mathbb{R}^{m+n} \) (cf. the arguments in [10, Lemma 35.5]). Since \( m \) is, by assumption, the smallest integer for which Theorem 0.3 fails, \( \mathcal{H}^{m-3+\alpha}(\text{Sing}(S_2)) = 0 \) and, hence, \( \mathcal{H}_\infty^{-2+\alpha}(\text{Sing}_Q(S_1)) = 0 \). On the other hand, arguing as we did for (1.4), we have
\[
\mathcal{H}_\infty^{-2+\alpha}(D_Q(S_1) \cap \tilde{B}_1) \geq \limsup_{k \to +\infty} \mathcal{H}_\infty^{-2+\alpha}(D_Q(S_{x,r_k}) \cap \tilde{B}_1) > 0.
\]
Thus \( \text{Reg}_Q(S_1) \neq \emptyset \) and, hence, \( \mathcal{H}^m(D_Q(S_1)) > 0 \). We can apply Lemma 1.4 again and conclude that \( S_1 \) is an \( m \)-dimensional plane with multiplicity \( Q \).

Therefore, the proposition follows taking \( T := \tau \bar{s} S \), with \( \tau \) the translation map \( y \mapsto y - x \) and \( \Sigma \) the tangent plane at 0 to the original Riemannian manifold. \( \square \)

2. Intervals of flattening

For the sequel, we fix the constant \( c_s := \frac{-1}{64\sqrt{m}} \) and notice that \( 2^{-N_0} < c_s \), where \( N_0 \) is the parameter introduced in [6, Assumption 1.8]. It is always understood that the parameters \( \beta_2, \delta_2, \gamma_2, \kappa, C_{\epsilon}, C_h, M_0, N_0 \) in [6] are fixed in such a way that all the theorems and propositions therein are applicable; cf. [6, §1.2]. In particular, all constants which will depend upon these parameters will be called geometric and denoted by \( C_0 \). On the contrary, we will highlight the dependence of the constants upon the parameters introduced in this paper \( p_1, p_2, \ldots \) by writing \( C = C(p_1, p_2, \ldots) \).
We also recall the notation introduced in [6, Assumption 1.3]. If $\Sigma \cap B_{7\sqrt{m}}$ has no boundary in $B_{\sqrt{m}}$ and for any $p \in \Sigma \cap B_{\sqrt{m}}$ there is a map $\Psi_p : T_p \Sigma \supset \Omega \to (T_p \Sigma)^- \,$ parametrizing it, then $c(\Sigma \cap B_{7\sqrt{m}}) := \sup_{p \in \Sigma \cap B_{7\sqrt{m}}} \|D\Psi_p\|_{C^{2,\alpha}}$. Obviously, these assumptions might fail for a general $\Sigma$. (In fact, $c(\Sigma \cap B_{7\sqrt{m}})$ need not be well defined.) However, having fixed a point $q \in \Sigma$, given its $C^{3,\alpha}$ regularity, $c(t_{q,r}(\Sigma) \cap B_{7\sqrt{m}})$ is well defined whenever $r$ is sufficiently small and converges to 0 as $r \downarrow 0$. In particular, by Proposition 1.3 and simple rescaling arguments, we assume in the sequel the following.

**Assumption 2.1.** Let $\varepsilon_3 \in ]0,\varepsilon_2[$. Under Assumption 0.4, there exist $m, n, Q \geq 2$, $\alpha, \eta > 0$, $T$ and $\Sigma$ for which

(a) there is a sequence of radii $r_k \downarrow 0$ as in Proposition 1.3;

(b) the following holds:

\[
\begin{align*}
(2.1) \quad & T_0 \Sigma = \mathbb{R}^{m+n} \times \{0\}, \quad \text{spt}(\partial T) \cap B_{6\sqrt{m}} = \emptyset, \quad 0 \in D(Q(T), \\
(2.2) \quad & \|T\|(B_{6\sqrt{m}}) \leq r^m \left( Q \omega_m (6\sqrt{m})^m + \varepsilon_3^2 \right) \quad \text{for all } r \in (0,1), \\
(2.3) \quad & c(\Sigma \cap B_{7\sqrt{m}}) \leq \varepsilon_3.
\end{align*}
\]

2.1. **Defining procedure.** We set

\[
(2.4) \quad \mathcal{R} := \left\{ r \in ]0,1] : E(T, B_{6\sqrt{m}}) \leq \varepsilon_3^2 \right\}.
\]

Observe that, if $\{s_k\} \subset \mathcal{R}$ and $s_k \uparrow s$, then $s \in \mathcal{R}$. We cover $\mathcal{R}$ with a collection $\mathcal{F} = \{I_j\}_j$ of intervals $I_j = ]s_j, t_j]$ defined as follows: $t_0 := \max\{t : t \in \mathcal{R}\}$. Next assume, by induction, that $t_j$ is defined (and hence also $t_0 > s_0 \geq t_1 > s_1 \geq \ldots > s_{j-1} \geq t_j$), and consider the following objects:

- $T_j := ((t_0, t_j) \times \Sigma_j) \cap T \subseteq B_{6\sqrt{m}} \cap B_{7\sqrt{m}}$; moreover, consider for each $j$ an orthonormal system of coordinates so that, if we denote by $\pi_0$ the $m$-plane $\mathbb{R}^m \times \{0\}$, then $E(T_j, B_{6\sqrt{m}}, \pi_0) = E(T_j, B_{6\sqrt{m}})$ (alternatively we can keep the system of coordinates fixed and rotate the currents $T_j$).

- Let $\mathcal{M}_j$ be the corresponding center manifold constructed in [6, Th. 1.17] applied to $T_j$ and $\Sigma_j$ with respect to the $m$-plane $\pi_0$; the manifold $\mathcal{M}_j$ is then the graph of a map $\varphi_j : \pi_0 \supset [-4,4]^m \to \pi_0^1$, and we set $\Phi_j(x) := (x, \varphi_j(x)) \in \pi_0 \times \pi_0^1$.

Then, we consider the Whitney decomposition $\mathcal{W}^{(j)}$ of $[-4,4]^m \subset \pi_0$ as in [6, Def. 1.10 and Prop. 1.11] (applied to $T_j$), and we define

\[
(2.5) \quad s_j := t_j \max \left( \{e_\varepsilon^{-1} \ell(L) : L \in \mathcal{W}^{(j)} \text{ and } e_\varepsilon^{-1} \ell(L) \geq \text{dist}(0,L) \} \cup \{0\} \right).
\]

We will prove below that $s_j/t_j < 2^{-5}$. In particular, this ensures that $[s_j, t_j]$ is a (nontrivial) interval. Next, if $s_j = 0$, we stop the induction. Otherwise we let $t_{j+1}$ be the largest element in $\mathcal{R} \cap ]0, s_j]$ and proceed as above. Moreover, note the following simple consequence of (2.5):
such that this implies that the inductive procedure either never stops, leading to position have side-length at most 2 \( - 2 \).

\( \ell(L) = c_s \bar{r} \) and \( L \cap \bar{B}_r(0, \pi_0) \neq \emptyset. \)

In particular, the latter inequality is true for every \( \rho \in [0, 3] \) if \( s_j = 0 \).

2.2. First consequences. The following is a list of easy consequences of the definition. Given two sets \( A \) and \( B \), we define their separation as the number \( \text{sep}(A, B) \) := \( \inf \{|x - y| : x \in A, y \in B\} \).

**Proposition 2.2.** Assuming \( \varepsilon_3 \) sufficiently small, the following holds:

(i) \( s_j < \frac{t_j}{2} \) and the family \( \mathcal{F} \) is either countable and \( t_j \downarrow 0 \), or finite and \( I_j \) \( = [0, t_j] \) for the largest \( j \);

(ii) the union of the intervals of \( \mathcal{F} \) cover \( \mathcal{R} \), and for \( k \) large enough, the radii \( r_k \) in Assumption 2.1 belong to \( \mathcal{R} \);

(iii) if \( r \in ]\frac{s_j}{3}, 3[ \) and \( J \in \mathcal{W}_n^{(j)} \) intersects \( B := p_{\pi_0}(\mathcal{B}(p_j)) \), with \( p_j := \Phi_j(0) \), then \( J \) is in the domain of influence \( \mathcal{W}_n^{(j)}(H) \) (see [6, Def. 3.3]) of a cube \( H \in \mathcal{W}_e^{(j)} \) with

\[ \ell(H) \leq 3 c_s r \quad \text{and} \quad \max \{\text{sep}(H, B), \text{sep}(H, J)\} \leq 3 \sqrt{m} \ell(H) \leq \frac{3r}{16}; \]

(iv) \( \mathbf{E}(T_j, \mathcal{B}_r) \leq C_0 \varepsilon^2 \pi^{-2 \beta_2} \) for every \( r \in ]\frac{s_j}{3}, 3[ \);

(v) \( \sup \{\text{dist}(x, \mathcal{M}_j) : x \in \text{spt}(T_j) \cap p_j^{-1}(\mathcal{B}(p_j))\} \leq C_0 (m_j^3) \frac{1}{\sqrt{m}} r^{-1 + \beta_2} \) for every \( r \in ]\frac{s_j}{3}, 3[ \), where \( m_j^3 := \max \{c(\Sigma_j)^2, \mathbf{E}(T_j, \mathcal{B}_{6\sqrt{m}})\} \).

**Proof.** We start by noticing that \( s_j \leq \frac{t_j}{\pi} \) follows from the inequality \( 2^{-N_0} < c_s \) (cf. [6, Assumption 1.8]) because all cubes in the Whitney decomposition have side-length at most \( 2^{-N_0 - 6} \) (cf. [6, Prop. 1.11]). In particular, this implies that the inductive procedure either never stops, leading to \( t_j \downarrow 0 \), or it stops because \( s_j = 0 \) and \( [0, t_j] \subset \mathcal{R} \), thus proving (i). The first part of (ii) follows straightforwardly from the choice of \( t_{j+1} \), and the last assertion holds from \( \mathbf{E}(T_j, \mathcal{B}_{6\sqrt{m} r_k}) \to 0 \).

Regarding (iii), let \( H \in \mathcal{W}_e^{(j)} \) be as in [6, Def. 3.3] and choose \( k \in \mathbb{N} \setminus \{0\} \) such that \( \ell(H) = 2^k \ell(J) \). Observe that \( \|D \Phi_j\|_{C^{2, \alpha}} \leq C_0 \varepsilon_3 \) by [6, Th. 1.17]. If \( \varepsilon_3 \) is sufficiently small, we can assume

\[ B_{r/2}(0, \pi_0) \subset B \subset B_r(0, \pi_0). \]
Now, by [6, Cor. 3.2], $\text{sep}(H,J) \leq 2\sqrt{m}\ell(H)$ and
\[
\text{sep}(B,H) \leq \text{sep}(H,J) + 2\sqrt{m}\ell(J) \leq 3\sqrt{m}\ell(H).
\]
Both the inequalities claimed in (iii) are then trivial when $r > \frac{1}{4}$, because $\ell(H) \leq 2^{-N_0-6} \leq 2^{-5}c_s \leq 2^{-9}/\sqrt{m}$. Therefore assume $r \leq \frac{1}{4}$, and note that $H$ intersects $B_{2r+3\sqrt{m}\ell(H)}$. Let $\rho := 2r + 3\sqrt{m}\ell(H)$. Observe that $2r < \rho < 1$.

By the definition of $s_j$, we have that
\[
\ell(H) < c_s \left(2r + 3\sqrt{m}\ell(H)\right) = 2c_s r + \frac{3\ell(H)}{16}.
\]
Therefore, we conclude that $\ell(H) \leq 3c_s r$ and $\text{sep}(H,B) \leq 9\sqrt{m}c_s r < 3r/16$.

We now turn to (iv). If $r \geq 2^{-N_0}$, then obviously
\[
\epsilon(T_j, B_r) \leq (4\sqrt{m} 2^{N_0})^{m+2-2\delta_2} r^{2-2\delta_2} \epsilon(T_j, B_{4\sqrt{m}}) \\
\leq (4\sqrt{m} 2^{N_0})^{m+2-2\delta_2} r^{2-2\delta_2} c_3^2.
\]
Otherwise, let $k \geq N_0$ be the smallest natural number such that $2^{-k+1} > r$, and let $L \in \mathcal{W}^{(j),k} \cup \mathcal{W}^{(j),k}$ be a cube so that $0 \in L$ (cf. [6, Def. 1.10], $\ell(H) = 2^{-k}$). By [6, Prop. 4.2(v)], $|p_L| \leq (\sqrt{m} + C_0 (m_j)_{1/2m}) \leq 2\sqrt{m}\ell(H)$ and so it follows easily that $B_r \subset B_L$. From condition (Go) we have $L \not\in \mathcal{W}^{(j)}$. Thus, by [6, Prop. 1.11], we get
\[
\epsilon(T_j, B_r) \leq C_0 \epsilon(T_j, B_L) \leq C_0 c_3^2 r^{2-2\delta_2}.
\]
Finally, (v) follows from [6, Cor. 2.2 (ii)], because by (Go), for every $r \in [s_j, 3[,$ every cube $L \in \mathcal{W}^{(j)}$ which intersects $B_r(0, \pi_0)$ satisfies $\ell(L) < c_s r$. \hfill \Box

3. Frequency function and first variations

Consider the following Lipschitz (piecewise linear) function $\phi : [0 + \infty] \rightarrow [0, 1]$ given by
\[
\phi(r) := \begin{cases} 
1 & \text{for } r \in [0, \frac{1}{2}], \\
2 - 2r & \text{for } r \in \left[\frac{1}{2}, 1\right], \\
0 & \text{for } r \in \left[1, +\infty\right].
\end{cases}
\]
For every interval of flattening $I_j = [s_j, t_j]$, let $N_j$ be the normal approximation of $T_j$ on $\mathcal{M}_j$ in [6, Th. 2.4].

**Definition 3.1 (Frequency functions).** For every $r \in [0, 3]$, we define
\[
D_j(r) := \int_{\mathcal{M}_j} \phi\left(\frac{d_j(p)}{r}\right) |DN_j|^2(p) \, dp
\]
and
\[
H_j(r) := -\int_{\mathcal{M}_j} \phi'\left(\frac{d_j(p)}{r}\right) \frac{|N_j|^2(p)}{d(p)} \, dp,
\]
where $d_j(p)$ is the geodesic distance on $\mathcal{M}_j$ between $p$ and $\Phi_j(0)$. If $H_j(r) > 0$, we define the frequency function $I_j(r) := \frac{r D_j(r)}{H_j(r)}$.

The following is the main analytical estimate of the paper, which allows us to exclude infinite order of contact among the different sheets of a minimizing current.

**THEOREM 3.2 (Main frequency estimate).** If $\varepsilon_3$ is sufficiently small, then there exists a geometric constant $C_0$ such that, for every $[a, b] \subset [\frac{s_j}{l_j}, 3]$ with $H_j|_{[a, b]} > 0$, we have

\[
I_j(a) \leq C_0(1 + I_j(b)).
\]

To simplify the notation, in this section we drop the index $j$ and omit the measure $H_j$ in the integrals over regions of $\mathcal{M}$. The proof exploits four identities collected in Proposition 3.5, which will be proved in the next sections.

**Definition 3.3.** We let $\partial_\hat{r}$ denote the derivative with respect to arclength along geodesics starting at $\Phi(0)$. We set

\[
E(r) := -\int_{\mathcal{M}} \phi' \left(\frac{d(p)}{r}\right) \sum_{i=1}^{Q} \langle N_i(p), \partial_\hat{r}N_i(p) \rangle \, dp,
\]

\[
G(r) := -\int_{\mathcal{M}} \phi' \left(\frac{d(p)}{r}\right) d(p) |\partial_\hat{r}N(p)|^2 \, dp \quad \text{and}
\]

\[
\Sigma(r) := \int_{\mathcal{M}} \phi \left(\frac{d(p)}{r}\right) |N|^2(p) \, dp.
\]

**Remark 3.4.** Observe that all these functions of $r$ are absolutely continuous and, therefore, classically differentiable at almost every $r$. Moreover, the following rough estimate easily follows from [6, Th. 2.4] and the condition (Go):

\[
D(r) \leq \int_{B_r(\Phi(0))} |DN|^2 \leq C_0 m_0 r^{m+2-2\delta_2} \quad \text{for every} \quad r \in \left[\frac{s_j}{l_j}, 3\right].
\]

Indeed, since $N$ vanishes identically on the set $K$ of [6, Th. 2.4], it suffices to sum the estimate of [6, Th. 2.4, (2.3)] over all the different cubes $L$ (of the corresponding Whitney decomposition) for which $\Phi(L)$ intersects the geodesic ball $B_r$.

**Proposition 3.5 (First variation estimates).** For every $\gamma_3$ sufficiently small, there is a constant $C = C(\gamma_3) > 0$ such that, if $\varepsilon_3$ is sufficiently small, $[a, b] \subset \left[\frac{s_j}{l_j}, 3\right]$ and $I \geq 1$ on $[a, b]$, then the following inequalities hold for almost every $r \in [a, b]$,

\[
\left|H'(r) - \frac{m-1}{r}H(r) - \frac{2}{r}E(r)\right| \leq CH(r),
\]

\[
\left|D(r) - r^{-1}E(r)\right| \leq CD(r)^{1+\gamma_3} + C\varepsilon_3^2 \Sigma(r),
\]
but to simplify the notation we keep denoting them by ε. Thus treat the corresponding constants in the inequalities as geometric ones.

\[ (3.12) \]

We assume for the moment the proposition and prove the theorem.

**Proof of Theorem 3.2.** Set \( \Omega(r) := \log \left( \max \{ I(r), 1 \} \right) \). Fix a \( \gamma_3 > 0 \) and an \( \varepsilon_3 \) sufficiently small so that the conclusion of Proposition 3.5 holds. We can thus treat the corresponding constants in the inequalities as geometric ones, but to simplify the notation we keep denoting them by \( C \).

To prove (3.1) it is enough to show that there is nothing to prove. If \( \Omega(a) > 0 \), let \( b' \in ]a, b[ \) be the supremum of \( t \) such that \( \Omega > 0 \) on \( ]a, t[ \). If \( b' < b \), then \( \Omega(b') = 0 \leq \Omega(b) \). Therefore, by possibly substituting \( ]a, b[ \) with \( ]a, b'[ \), we can assume that \( \Omega > 0 \), i.e., \( I > 1 \), on \( ]a, b[ \).

By Proposition 3.5, if \( \varepsilon_3 \) is sufficiently small, then

\[ (3.9) \]

\[
\frac{D(r)}{2} \leq \frac{E(r)}{r} \leq 2D(r),
\]

from which we conclude that \( E > 0 \) over the interval \( ]a, b'[ \). For simplicity, set \( F(r) := D(r)^{-1} - rE(r)^{-1} \), and compute

\[ -\Omega'(r) = \frac{H'(r)}{H(r)} - \frac{D'(r)}{D(r)} - \frac{1}{r} \leq \frac{H'(r)}{H(r)} - \frac{rD'(r)}{E(r)} - D'(r)F(r) - \frac{1}{r}. \]

Again by Proposition 3.5,

\[ (3.10) \]

\[
\frac{H'(r)}{H(r)} \leq \frac{m - 1}{r} + C + \frac{2}{r} \frac{E(r)}{H(r)},
\]

\[ (3.11) \]

\[
|F(r)| \leq C \frac{r(D(r)^{1+\gamma_3} + \Sigma(r))}{D(r)} \leq C \frac{\Sigma(r)}{D(r)},
\]

\[ (3.12) \]

\[
-\frac{rD'(r)}{E(r) \leq \left( C - \frac{m - 2}{r} \right) \frac{rD(r)}{E(r)} - \frac{2}{r} \frac{G(r)}{E(r)} + C \frac{rD(r)^{\gamma_3}D'(r) + D(r)^{1+\gamma_3}}{E(r)} \leq C - \frac{m - 2}{r} + \frac{C}{r} D(r)|F(r)| - \frac{2}{r} \frac{G(r)}{E(r)} + CD(r)^{\gamma_3 - 1}D'(r) + C \frac{D(r)^{\gamma_3}}{r} \leq C - \frac{m - 2}{r} + \frac{2}{r} \frac{G(r)}{E(r)} + CD(r)^{\gamma_3 - 1}D'(r) + C r^{\gamma_3 m - 1}. \]
By Cauchy-Schwartz, we have

\begin{equation}
\frac{\mathbf{E}(r)}{r \mathbf{H}(r)} \leq \frac{\mathbf{G}(r)}{r \mathbf{E}(r)}.
\end{equation}

Thus, by (3.4), (3.10), (3.12) and (3.13), we conclude

\begin{equation}
-\Omega'(r) \leq C + Cr^{\gamma_3 m - 1} + Cr \mathbf{D}(r)^{\gamma_3 - 1} \mathbf{D}'(r) - \mathbf{D}'(r) \mathbf{F}(r)
\end{equation}

\begin{equation}
\Omega(a) - \Omega(b) \leq C + C (\mathbf{D}(b)^{\gamma_3} - \mathbf{D}(a)^{\gamma_3})
\end{equation}

Integrating (3.14), we conclude that

\begin{equation}
\int \frac{\phi'(|z|)}{|z|} |N|^2(\exp(rz)) J \exp(rz) \, dx.
\end{equation}

Therefore, differentiating under the integral sign, we easily get (3.5):

\begin{equation}
\mathbf{H}'(r) = -\frac{\phi'(|z|)}{|z|} |N|^2(\exp(rz)) J \exp(rz) \, dz
\end{equation}

The rest of the section is devoted to the proof of Proposition 3.5.

3.1. Estimates on \( \mathbf{H}' \): Proof of (3.5). Set \( q := \Phi(0) \). Let \( \exp : B_3 \subset T_q \mathcal{M} \rightarrow \mathcal{M} \) be the exponential map and \( J \exp \) its Jacobian. Note that \( d(\exp(y), q) = |y| \) for every \( y \in B_3 \). By the area formula, setting \( y = rz \), we can write \( \mathbf{H} \) in the following way:

\[ \mathbf{H}(r) = -r^{m-1} \int_{T_q \mathcal{M}} \frac{\phi'(|z|)}{|z|} |N|^2(\exp(rz)) J \exp(rz) \, dx. \]

Therefore, differentiating under the integral sign, we easily get (3.5):

\begin{equation}
\mathbf{H}'(r) = -\frac{\phi'(|z|)}{|z|} |N|^2(\exp(rz)) J \exp(rz) \, dz
\end{equation}

3.2. \( \mathbf{O} \) and \( \mathbf{O}' \): Proof of (3.8). We show the following more precise estimates.

**Lemma 3.6.** There exists a dimensional constant \( C_0 > 0 \) such that

\begin{equation}
\mathbf{O}(r) \leq C_0 r^2 \mathbf{D}(r) + C_0 r \mathbf{H}(r)
\end{equation}

\begin{equation}
\mathbf{O}'(r) \leq C_0 \mathbf{H}(r)
\end{equation}

\begin{equation}
\int_{B_r(q)} |N|^2 \leq C_0 \mathbf{O}(r) + C_0 r \mathbf{H}(r),
\end{equation}

where we used that \( \frac{d}{dr} J \exp(rz) = O(1) \), because \( \mathcal{M} \) is a \( C^{3,\kappa} \) submanifold and hence \( \exp \) is a \( C^{2,\kappa} \) map (see Proposition A.4).
\[
\int_{B_r(q)} |DN|^2 \leq C_0 D(r) + C_0 r D'(r).
\]

In particular, if \( I \geq 1 \), then (3.8) holds and
\[
\int_{B_r(q)} |N|^2 \leq C_0 r^2 D(r).
\]

**Proof.** To simplify the notation we drop the subscript 0 from the geometric constants. Observe that \( \psi(p) := \phi\left(\frac{d(p)}{r}\right) N(p) \) is a Lipschitz function with compact support in \( B_r(q) \). We therefore use the Poincaré inequality: \( \Sigma(r) = \int_M \psi \leq C r \int_M |D\psi| \). (The constant \( C \) depends on the smoothness of \( M \).) We compute
\[
\Sigma(r) \leq -C \int_M \phi'(r^{-1} d(p)) |N|^2(p) + C r \int_M \phi(r^{-1} d(p)) |N||DN|
\]
\[
\leq C r H(r) + C \Sigma(r)^{1/2} (r^2 D(r))^{1/2} \leq C r H(r) + \frac{1}{2} \Sigma(r) + C r^2 D(r),
\]
which gives the first part of (3.15). The remaining inequality is straightforward:
\[
\Sigma'(r) = - \int_{M} \frac{d(p)}{r^2} \phi' \left( \frac{d(p)}{r} \right) |N|^2(p) \leq CH(r).
\]

Since \( \phi' = 0 \) on \([0, \frac{1}{2}]\) and \( \phi' = -2 \) on \([\frac{1}{2}, 1]\), we easily deduce
\[
\int_{B_r(q) \setminus B_{r/2}(q)} |N|^2 \leq r H(r),
\]
\[
r D'(r) = - \int \frac{d(p)}{r} \phi' \left( \frac{d(p)}{r} \right) |DN|^2 \geq \int_{B_r(q) \setminus B_{r/2}(q)} |DN|^2.
\]
On the other hand, since \( \phi = 1 \) on \([0, \frac{1}{2}]\), (3.16) and (3.17) readily follow. Therefore, in the hypothesis \( I \geq 1 \), i.e., \( H \leq r D \), we conclude (3.8) from (3.15).

3.3. **First variations.** To prove the remaining estimates in Proposition 3.5 we exploit the first variation of \( T \) along some vector fields \( X \). The variations are denoted by \( \delta T(X) \). We fix a neighborhood \( U \) of \( M \) and the normal projection \( p : U \to M \) as in [6, Assumption 2.1]. Observe that \( p \in C^{2,\kappa} \) and [5, Assumption 3.1] holds. We will consider

- The outer variations, where \( X(p) = X_o(p) := \phi\left(\frac{d(p)}{r}\right) (p - p(p)) \).
- The inner variations, where \( X(p) = X_i(p) := Y(p(p)) \) with
\[
Y(p) := \frac{d(p)}{r} \phi \left( \frac{d(p)}{r} \right) \frac{\partial}{\partial r} \quad \forall p \in M.
\]

\( (\frac{\partial}{\partial r} \) is the unit vector field tangent to the geodesics emanating from \( \Phi(0) \) and pointing outwards.)
Note that $X_i$ is the infinitesimal generator of a one-parameter family of bilipschitz homeomorphisms $\Phi_\varepsilon$ defined as $\Phi_\varepsilon(p) := \Psi_\varepsilon(p) + p - p(p)$, where $\Psi_\varepsilon$ is the one-parameter family of bilipschitz homeomorphisms of $\mathcal{M}$ generated by $Y$.

Consider now the map $F(p) := \sum_i [p + N_i(p)]$ and the current $T_F$ associated to its image; cf. [5] for the notation. Observe that $X_i$ and $X_o$ are supported in $p^{-1}(B_r(q))$ but none of them is compactly supported. However, recalling Proposition 2.2(v) and the minimizing property of $T$ in $\Sigma$, we deduce that $\delta T(X) = \delta T(X^T) + \delta T(X^\perp) = \delta T(X^\perp)$, where $X = X^T + X^\perp$ is the decomposition of $X$ in the tangent and normal components to $T\Sigma$. Then, we have

\begin{equation}
(3.19) \quad |\delta T_F(X)| \leq |\delta T_F(X) - \delta T(X)| + |\delta T(X^\perp)|
\end{equation}

\begin{equation*}
\leq \int_{\text{spt}(T) \setminus \text{Im}(F)} |\text{div}_F X| \, d\|T\| + \int_{\text{Im}(F) \setminus \text{spt}(T)} |\text{div}_T F X| \, d\|T_F\|
\end{equation*}

\begin{equation*}
+ \left| \int_{\text{Im}(F) \setminus \text{spt}(T)} \text{div}_T F X \, d\|T\| \right|.
\end{equation*}

For simplicity, now set $\varphi_r(p) := \phi \left( \frac{\partial p}{\partial r} \right)$. We wish to apply [5, Th. 4.2] to conclude

\begin{equation}
(3.20) \quad \delta T_F(X_o) = \int_{\mathcal{M}} \left( \varphi_r |DN|^2 + \sum_{i=1}^Q N_i \otimes \nabla \varphi_r : DN_i \right) + \sum_{j=1}^3 \text{Err}_j^o,
\end{equation}

where the errors $\text{Err}_j^o$ correspond to the terms $\text{Err}_j$ of [5, Th. 4.2]. This would imply

\begin{equation}
(3.21) \quad \text{Err}_1^o = -Q \int_{\mathcal{M}} \varphi_r(H_M, \eta \circ N),
\end{equation}

\begin{equation}
(3.22) \quad |\text{Err}_2^o| \leq C_0 \int_{\mathcal{M}} |\varphi_r| |A|^2 |N|^2,
\end{equation}

\begin{equation}
(3.23) \quad |\text{Err}_3^o| \leq C_0 \int_{\mathcal{M}} \left( |N| |A| + |DN|^2 \right) \left( |\varphi_r||DN|^2 + |D\varphi_r||DN| |N| \right),
\end{equation}

where $H_M$ is the mean curvature vector of $\mathcal{M}$. Note that [5, Th. 4.2] requires the $C^1$ regularity of $\varphi_r$. We overcome this technical obstruction applying [5, Th. 4.2] to a standard smoothing of $\phi$ and then passing into the limit. (The obvious details are left to the reader.) Plugging (3.20) into (3.19), we then conclude

\begin{equation}
(3.24) \quad \left| \mathbf{D}(r) - r^{-1} \mathbf{E}(r) \right| \leq \sum_{j=1}^5 |\text{Err}_j^o|,
\end{equation}
where $\text{Err}_4$ and $\text{Err}_5$ correspond respectively to $\text{Err}_4$ and $\text{Err}_5$ of (3.19) when $X = X_o$. With the same argument, but this time applying [5, Th. 4.3] to $X = X_i$, we get

\[(3.25)\]

$$\delta T_F(X_i) = \frac{1}{2} \int_M \left( |DN|^2 \text{div}_Y - 2 \sum_{i=1}^{Q} \langle DN_i : (DN_i \cdot D_M Y) \rangle \right) + \sum_{j=1}^{3} \text{Err}_j,$$

where this time the errors $\text{Err}_j$ correspond to the error terms $\text{Err}_j$ of [5, Th. 4.3]; i.e.,

\[(3.26)\]

$$\text{Err}_1^i = -Q \int_M \left( \langle H_M, \eta \circ N \rangle \text{div}_M Y + \langle D_Y H_M, \eta \circ N \rangle \right),$$

\[(3.27)\]

$$|\text{Err}_2^i| \leq C_0 \int_M |A|^2 \left( |DY||N|^2 + |Y||N| |DN| \right),$$

\[(3.28)\]

$$|\text{Err}_3^i| \leq C_0 \int_M \left( |Y||A||DN|^2 (|N| + |DN|) + |DY| (|A||N|^2|DN| + |DN|^4) \right).$$

Straightforward computations (again appealing to Proposition A.4) lead to

\[(3.29)\]

$$D_M Y(p) = \phi' \left( \frac{d(p)}{r} \right) \frac{d(p)}{r^2} \frac{\partial}{\partial \hat{r}} \otimes \frac{\partial}{\partial \hat{r}} + \phi \left( \frac{d(p)}{r} \right) \left( \frac{1}{r^2} + O(1) \right),$$

\[(3.30)\]

$$\text{div}_M Y(p) = \phi' \left( \frac{d(p)}{r} \right) \frac{d(p)}{r^2} + \phi \left( \frac{d(p)}{r} \right) \left( \frac{m}{r} + O(1) \right).$$

Plugging (3.29) and (3.30) into (3.25) and using (3.19), we then conclude

\[(3.31)\]

$$\left| D'(r) - (m - 2)r^{-1} D(r) - 2r^{-2} G(r) \right| \leq C_0 D(r) + \sum_{j=1}^{5} |\text{Err}_j^i|.$$ 

Proposition 3.5 is then proved by the estimates of the errors terms done in the next section.

4. Error estimates

We start with some preliminary considerations, keeping the notation and convention of the previous section (and dropping the subscript when dealing with the maps of Theorem 3.2 and Proposition 3.5).

4.1. Families of subregions. Set $q := \Phi(0)$. We select a family of subregions of $B_r(p) \subset M$. Denote by $B$ and $\partial B$ respectively $p_{\pi}(B_r(q))$ and $p_{\pi}(\partial B_r(q))$, where $\pi$ is the reference $m$-dimensional plane of the construction of the center manifold $M$. Since $\|\Phi\|_{C^{3,\alpha}} \leq C \varepsilon^{1/m}$ (cf. [6, Th. 1.17]), by Proposition A.4 we can assume that $B$ is a $C^2$ convex set which at any boundary
point \( p \) contains an interior sphere of radius \( r/2 \) passing through \( p \). Thus,

\[
\forall z \in \partial B \text{ there is a ball } B_{r/2}(y, \pi) \subset B \text{ whose closure touches } \partial B \text{ at } z.
\]

**(4.1)** \( \forall z \in \partial B \) there is a ball \( B_{r/2}(y, \pi) \subset B \) whose closure touches \( \partial B \) at \( z \).

**Definition 4.1** (Family of cubes). We first define a family \( \mathcal{T} \) of cubes in the Whitney decomposition \( \mathcal{W} \) as follows:

(i) \( \mathcal{T} \) includes all \( L \in \mathcal{W}_h \cup \mathcal{W}_e \) which intersect \( B \);

(ii) if \( L' \in \mathcal{W}_e \) intersects \( B \) and belongs to the domain of influence \( \mathcal{W}_n(L) \) of the cube \( L \in \mathcal{W}_e \) as in [6, Cor. 3.2], then \( L \in \mathcal{T} \).

**Definition 4.2** (Associated balls \( B^L \)). By Proposition 2.2(iii), \( \ell(L) \leq 3c_dr \leq r \) and \( \operatorname{sep}(L, B) \leq 3\sqrt{m}\ell(L) \) for each \( L \in \mathcal{T} \). Let \( x_L \) be the center of \( L \) and

(a) if \( x_L \in \overline{B} \), we then set \( s(L) := \ell(L) \) and \( B^L := B_{s(L)}(x_L, \pi) \);

(b) otherwise, we consider the ball \( B_{r(L)}(x_L, \pi) \subset \pi \) whose closure touches \( \overline{B} \) at exactly one point \( p(L) \), we set \( s(L) := r(L) + \ell(L) \) and define \( B^L := B_{s(L)}(x_L, \pi) \).

Observe that, when \( L \in \mathcal{T} \cap \mathcal{W}_h \), then \( s(L) \) is at most \((\sqrt{m} + 1)\ell(L) \). We proceed to select a countable family \( \mathcal{F} \) of pairwise disjoint balls \( \{B^L\} \). We let \( S := \sup_{L \in \mathcal{T}} s(L) \) and start selecting a maximal subcollection \( \mathcal{F}_1 \) of pairwise disjoint balls with radii larger than \( S/2 \). Clearly, \( \mathcal{F}_1 \) is finite. In general, at the stage \( k \), we select a maximal subcollection \( \mathcal{F}_k \) of pairwise disjoint balls which do not intersect any of the previously selected balls in \( \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_{k-1} \) and which have radii \( r \in [2^{-k}S, 2^{1-k}S] \). Finally, we set \( \mathcal{F} := \bigcup_k \mathcal{F}_k \).

**Definition 4.3** (Family of cube-ball pairs \( (L, B(L)) \in \mathcal{Z} \)). Recalling (4.1) and \( \ell(L) \leq r \), it easy to see that there exist balls \( B_{\ell(L)/4}(q_L, \pi) \subset B^L \cap B \) which lie at distance at least \( \ell(L)/4 \) from \( \partial B \). We denote by \( B(L) \) one such ball and by \( \mathcal{Z} \) the collection of pairs \( (L, B(L)) \) with \( B^L \in \mathcal{F} \).

Next, we partition the cubes of \( \mathcal{W} \) which intersect \( B \) into disjoint families \( \mathcal{W}'(L) \) labeled by \( (L, B(L)) \in \mathcal{Z} \) in the following way. (Observe that \( \mathcal{W}'(L) \) and \( \mathcal{W}_n(L) \) are different families and should not be confused!) Let \( H \in \mathcal{W} \) have nonempty intersection with \( B \). If \( H \) is itself in \( \mathcal{F} \), we then select \( L \in \mathcal{F} \) with \( B^L \cap B^H \neq \emptyset \) and assign \( H \in \mathcal{W}'(L) \). Otherwise \( H \) is in the domain of influence of some \( J \in \mathcal{W}_e \). By Proposition 2.2, the separation between \( J \) and \( H \) is at most \( 3\sqrt{m}\ell(J) \) and, hence, \( H \subset B_{4\sqrt{m}\ell(J)}(x_J) \). By construction there is a \( B^L \in \mathcal{F} \) with \( B^J \cap B^L \neq \emptyset \) and radius \( s(L) \geq \frac{s(J)}{2} \). We then prescribe \( H \in \mathcal{W}'(L) \). Observe that \( s(L) \leq 4\sqrt{m}\ell(L) \) and \( s(J) \leq \ell(J) \). Therefore, \( \ell(J) \leq 8\sqrt{m}\ell(L) \) and \( |x_J - x_L| \leq 5s(L) \leq 20\sqrt{m}\ell(L) \). This implies that

\[
H \subset B_{4\sqrt{m}\ell(J)}(x_J) \subset B_{4\sqrt{m}\ell(J) + 20\sqrt{m}\ell(L)}(x_L) \subset B_{30\sqrt{m}\ell(L)}(x_L).
\]
The inclusion $H \subset B_{30\sqrt{m(L)}}(x_L)$ also holds in case $H \in \mathcal{T}$, as can be easily seen by simply setting $J = H$ and using the same computations. For later reference, we collect the main properties of the above construction.

**Lemma 4.4.** The following holds:

(i) if $(L, B(L)) \in \mathcal{Z}$, then $L \in \mathcal{W} \cup \mathcal{W}_h$, the radius of $B(L)$ is $\ell(L) = \ell(L) B(L) \subset B^L \cap B$ and $\text{sep}(B(L), \partial B) \geq \frac{\ell(L)}{4}$;

(ii) if the pairs $(L, B(L)), (L', B(L')) \in \mathcal{Z}$ are distinct, then $L$ and $L'$ are distinct and $B(L) \cap B(L') = \emptyset$;

(iii) the cubes $\mathcal{W}$ which intersect $B$ are partitioned into disjoint families $\mathcal{W}(L)$ labeled by $(L, B(L)) \in \mathcal{Z}$ such that, if $H \in \mathcal{W}(L)$, then $H \subset B_{30 \sqrt{m(L)}}(x_L)$.

4.2. Basic estimates in the subregions. For notational convenience, we order the family $\mathcal{Z} = \{(L_i, B(L_i))\}_{i \in \mathbb{N}}$ and set

$$B^i := \Phi(B(L_i)), \quad U_i = \bigcup_{H \in \mathcal{W}(L_i)} \Phi(H) \cap B_r(q).$$

(Recall that $q = \Phi(0)$.) Observe that the separation between $B^i$ and $\partial B_r(q)$ is larger than that between $B(L_i)$ and $\partial B = p_0(\partial B_r(q))$. Thus, by Lemma 4.4(i),

$$\varphi_r(p) = \phi\left(\frac{d(p)}{r}\right) \text{satisfies}$$

$$\inf_{p \in B^i} \varphi_r(p) \geq (4r)^{-1} \ell_i,$$

where $\ell_i := \ell(L_i)$. From this and Lemma 4.4(iii), we also obtain

$$\sup_{p \in U_i} \varphi_r(p) - \inf_{p \in U_i} \varphi_r(p) \leq C \text{Lip}(\varphi_r) \ell_i \leq C \frac{\ell_i}{r} \leq C \inf_{p \in B^i} \varphi_r(p),$$

which translates into

$$\sup_{p \in U_i} \varphi_r(p) \leq C \inf_{p \in B^i} \varphi_r(p).$$

Moreover, set $\mathcal{V}_i := U_i \cap ((\text{spt}(T_F) \setminus \text{spt}(T)) \cup (\text{spt}(T) \setminus \text{spt}(T_F)))$ and observe that $\mathcal{V}_i \subset U_i \setminus K$, where $K$ is the coincidence set of [6, Th. 2.4]. From [6, Th. 2.4], we derive the following estimates:

$$\int_{U_i} |\eta \circ N| \leq C_0 \mathbf{m}_0 \ell_i^{2+\gamma_2} + C_0 \int_{U_i} |N|^{2+\gamma_2},$$

$$\int_{U_i} |DN|^2 \leq C_0 \mathbf{m}_0 \ell_i^{m+2-2\delta_2},$$

$$\|N\|_{C^0(U_i)} + \sup_{p \in \text{spt}(T) \cap p^{-1}(U_i)} |p - p(p)| \leq C_0 \mathbf{m}_0^{\frac{1}{2m}} \ell_i^{1+\beta_2},$$

$$\text{Lip}(N|_{U_i}) \leq C_0 \mathbf{m}_0^{\gamma_2} \ell_i^{\gamma_2},$$

$$\mathbf{M}(T \cap p^{-1}(\mathcal{V}_i)) + \mathbf{M}(T_F \cap p^{-1}(\mathcal{V}_i)) \leq C_0 \mathbf{m}_0^{1+\gamma_2} \ell_i^{m+2-2+\gamma_2}.$$
This in turn implies that \( \sum_{H \in \mathcal{H}(L_i)} \ell(H)^{m+\varepsilon} \leq C_0 \ell^{m+\varepsilon}_i \), because \( \ell(H) \leq \ell_i \) for any \( H \in \mathcal{H}(L) \). Thus

- (4.4) follows summing the estimate of [6, Th. 2.4 (2.4)] applied with \( a = 1 \) to \( \Phi(H) \) with \( H \in \mathcal{H}(L_i) \);
- (4.5) follows from summing the estimate of [6, Th. 2.4 (2.3)] applied to \( \Phi(H) \) with \( H \in \mathcal{H}(L_i) \);
- (4.6) follows from [6, Th. 2.4 (2.1)] and [6, Cor. 2.2 (ii)];
- (4.7) follows from [6, Th. 2.4 (2.1)];
- (4.8) follows summing [6, Th. 2.4 (2.2)] applied to \( \Phi(H) \) with \( H \in \mathcal{H}(L_i) \).

The last ingredient for the completion of the proof of Proposition 3.5 are the following three key estimates, which are derived from the analysis of the construction of the center manifold in [6].

**Lemma 4.5.** Under the assumptions of Proposition 3.5, it holds that

\[
(4.9) \quad \sum_i \left( \inf_{B_i} \phi_r \right) m_0 \ell_i^{m+2+2\gamma/4} \leq C_0 D(r),
\]

\[
(4.10) \quad \sum_i m_0 \ell_i^{m+2+2\gamma/4} \leq C_0 (D(r) + rD'(r))
\]

for some geometric constant \( C_0 \). Moreover, for every \( t > 0 \), there exist \( C_0 > 0 \) and \( a > 0 \) such that, for \( C(t) = C^t \) and \( \gamma(t) = at \), we have

\[
(4.11) \quad \sup_t m_0 \left[ \ell_i^t + \left( \inf_{B_i} \phi_r \right)^{t/2} \ell_i^{t/2} \right] \leq C(t)D(r)^\gamma(t).
\]

**Proof.** Recall that, from [6, Props. 3.1 and 3.4] and (4.2) we have, for some geometric positive constant \( c_0 \),

\[
(4.12) \quad \int_{B_i} \phi_r |N|^2 \geq c_0 m_0^{1/2} \inf_{B_i} \phi_r \ell_i^{m+2+2\beta_2} \geq c_0 m_0^{1/2} \left[ \ell_i^t + \left( \inf_{B_i} \phi_r \right) \ell_i^{t/2} \right]^{1/(2a)} \quad \text{if } L_i \in \mathcal{H},
\]

\[
(4.13) \quad \int_{B_i} \phi_r |DN|^2 \geq c_0 m_0 \inf_{B_i} \phi_r \ell_i^{m+2-2\delta_2} \geq c_0 m_0^{1/2} \left[ \ell_i^t + \left( \inf_{B_i} \phi_r \right) \ell_i^{t/2} \right]^{1/(2a)} \quad \text{if } L_i \in \mathcal{E},
\]

where we just need \( a \leq \min\{1/(2(m+2+2\beta_2)), 1/(2(m+2-2\delta_2))\} \). (Note that (4.12) follows from [6, Prop. 3.1 (S3)] because \( s(L_i) \leq (\sqrt{m} + 1)\ell(L_i) \) for \( L_i \in \mathcal{H}_r \).) Therefore, by Lemma 3.6, (4.2), (4.12) and (4.13), it follows easily that

\[
2^{-t} c_0 a t m_0^{1/2} \left[ \ell_i^t + \left( \inf_{B_i} \phi_r \right)^{t/2} \ell_i^{t/2} \right] \leq \left( \int_{B_i} \phi_r |DN|^2 \right)^{at} + \left( \int_{B_i} \phi_r |N|^2 \right)^{at} \leq 2^t \left( \int_{B_i} \phi_r (|DN|^2 + |N|^2) \right)^{at} \leq C_0^2 D(r)^{at}.
\]
Finally, arguing as above, we conclude that

\[ \sum_i \left( \inf_{B_i} \varphi_r \right) \ell_i^{m+2+\gamma_2/4} \leq C \sum_i \int_{B_i} \varphi_r (|D N|^2 + |N|^2)^{3.18} \leq C \mathcal{D}(r). \]

Finally, arguing as above, we conclude that

\[ \sum_i \ell_i^{m+2+\gamma_2/4} \leq C \int_{B_r(q)} (|D N|^2 + |N|^2)^{3.18} \leq C (\mathcal{D}(r) + r \mathcal{D}'(r)) \]

and, hence, (4.10) follows from Lemma 3.6.

4.3. Proof of Proposition 3.5: (3.6) and (3.7). We can now pass to estimate the errors terms in (3.6) and (3.7) in order to conclude the proof of Proposition 3.5. Unless otherwise specified, the constants denoted by \( C \) will be assumed to be geometric (i.e., to depend only upon the parameters introduced in [6]).

Errors of type 1. By [6, Th. 1.12], the map \( \varphi \) defining the center manifold satisfies \( \|D \varphi\|_{C^{2, \kappa}} \leq C \ell_0^{1/2} \), which in turn implies \( \|H_M\|_{L^\infty} + \|D H_M\|_{L^\infty} \leq C \ell_0^{1/2} \). (Recall that \( H_M \) denotes the mean curvature of \( M \).) Therefore, by (4.3), (4.4), (4.9) and (4.11), we get

\[
|\text{Err}_1| \leq C_0 \int_M \varphi_r |H_M| \eta \circ N \\
\leq C_0 \ell_0^{1/2} \sum_j \left( \sup_{U_j} \varphi_r \right) \ell_j^{2+\gamma_2} + C_0 \int_{U_j} \varphi_r |N|^{2+\gamma_2} \\
\leq C \mathcal{D}(r)^{1+\gamma_3} + C \sum_j \ell_0^{1/2} \ell_j^{2+\gamma_2} \int_{U_j} \varphi_r |N|^2 \leq C(\gamma_3) \mathcal{D}(r)^{1+\gamma_3},
\]

provided \( \gamma_3 > 0 \) is sufficiently small depending only upon \( m, \beta_2, \delta_2 \) and \( \gamma_2 \). Analogously,

\[
|\text{Err}_1| \leq C r^{-1} \int_M \left( |H_M| + |D Y H_M| \right) \eta \circ N \\
\leq C r^{-1} \ell_0^{1/2} \sum_j \left( \ell_0 \ell_j^{2+\gamma_2} + C \int_{U_j} |N|^{2+\gamma_2} \right) \\
\leq C(\gamma_3) r^{-1} \mathcal{D}(r)^{\gamma_3} (\mathcal{D}(r) + r \mathcal{D}'(r)).
\]

Errors of type 2. From \( \|A\|_{C^0} \leq C \|D \varphi\|_{C^2} \leq C \ell_0^{1/2} \leq C \varepsilon_3 \), it follows that \( \text{Err}_2 \leq C \varepsilon_3^3 \Sigma(r) \). Moreover, since \( |D X_i| \leq Cr^{-1} \), Lemma 3.6 gives

\[
|\text{Err}_2| \leq C r^{-1} \int_{B_r(p_0)} |N|^2 + C \int \varphi_r |N||D N| \leq C \mathcal{D}(r).
\]
Errors of type 3. Clearly, we have

\[ |Err_3^0| \leq \int_{I_1} \varphi_r \left( |DN|^2 |N| + |DN|^3 \right) \]
\[ + Cr^{-1} \int_{B_r(q)} |DN|^3 |N| + C r^{-1} \int_{B_r(q)} |DN||N|^2. \]

We separately estimate the three terms (recall that \( \gamma_2 > 4 \delta_2 \)):

\[ I_1 \leq \int_{B_r(p_0)} \varphi_r (|N|^2 |DN| + |DN|^3) \]
\[ \leq I_3 + C \sum_j \sup_{U_j} \varphi_r m_0^{1+\gamma_2} \ell_j^{m+2+\gamma_2/2} \]
\[ \leq I_3 + C(\gamma_3)D(r)^{1+\gamma_3}, \]

\[ I_2 \leq Cr^{-1} \sum_j m_0^{1+1/2m+\gamma_2} \ell_0^{m+2+\beta_2+\gamma_2/2} \]
\[ \leq C \sum_j m_0^{1+1/2m+\gamma_2} \ell_j^{m+2+\beta_2+\gamma_2/2} \inf_{B_j} \varphi_r \]
\[ \leq C(\gamma_3)D(r)^{1+\gamma_3}, \]

\[ I_3 \leq Cr^{-1} \sum_j m_0^{2} \ell_j^{\gamma_2} \int_{U_j} |N|^2 \]
\[ \leq C_{\gamma_3} r^{-1} D(r)^{\gamma_3} \int_{B_r(q)} |N|^2 \]
\[ \leq C(\gamma_3)D(r)^{1+\gamma_3}, \]

provided \( \gamma_3 > 0 \) is sufficiently small. For what concerns the inner variations, we have

\[ |Err_3^i| \leq C \int_{B_r(q)} \left( r^{-1}|DN|^3 + r^{-1}|DN|^2 |N| + r^{-1}|DN||N|^2 \right). \]

The last integrand corresponds to \( I_3 \), while the remaining part can be estimated as follows:

\[ \int_{B_r(q)} r^{-1}(|DN|^3 + |DN|^2 |N|) \leq C \sum_j r^{-1}(m_0^{\gamma_2} \ell_j^{\gamma_2} + m_0^{1/2m} \ell_j^{1+\beta_2}) \int_{U_j} |DN|^2 \]
\[ \leq C(\gamma_3) r^{-1} D(r)^{\gamma_3} \int_{B_r(q)} |DN|^2 \]
\[ \leq C(\gamma_3) D(r)^{\gamma_3} \left( D'(r) + r^{-1} D(r) \right). \]
Errors of type 4. Explicitly, we compute
\[ |DX_o(p)| \leq 2|p - p(p)||Dd(p(p), q)| \]
\[ + \varphi_r(p)|D(p - p(p))| \leq C \left( \frac{|p - p(p)|}{r} + \varphi_r(p) \right). \]

It follows readily from (3.19), (4.6) and (4.8) that
\[ |\text{Err}_4^2| \leq \sum_i C \left( r^{-1} m_0^{1/2} \ell_i^{1+\beta_2} + \sup_{\mathcal{U}_i} \varphi_r \right) m_0^{1+\gamma_2} \ell_i^{m+2+\gamma_2} \]
\[ \leq C \sum_i \left[ m_0^{\gamma_2} \ell_i^{\gamma_2/4} \right] \inf_{\mathcal{B}_i} \varphi_r m_0 \ell_i^{2+\gamma_2/4} \]
\[ \leq C(\gamma_3)D(r)^{1+\gamma_3}. \]

Similarly, since |DX_i| \leq Cr^{-1}, we get
\[ \text{Err}_4^i \leq Cr^{-1} \sum_j \left( m_0^{\gamma_2} \ell_j^{\gamma_2/4} \right) m_0 \ell_j^{m+2+\gamma_2/2} \]
\[ \leq C(\gamma_3)D(r)^{\gamma_3} (D'(r) + r^{-1}D(r)). \]

Errors of type 5. Integrating by part Err_5, we get
\[ \text{Err}_5 = \left| \int \langle X^\perp, h(\bar{T}(p)) \rangle d\|T\| \right| \leq \left| \int_{I_2} \langle X^\perp, h(\bar{T}(p)) \rangle d\|T\| \right| \]
\[ + \int_{\text{spt}(T) \setminus \text{Im}(F)} |X^\perp| |h(\bar{T}(p))| d\|T\| + \int_{\text{Im}(F) \setminus \text{spt}(T)} |X^\perp| |h(\bar{T}(p))| d\|T\|, \]

where \( h(\bar{\lambda}) \) is the trace of \( A_\Sigma \) on the \( m \)-vector \( \bar{\lambda} \), i.e., \( h(\bar{\lambda}) := \sum_{k=1}^m A_\Sigma(v_k, v_k) \) with \( v_1, \ldots, v_m \) orthonormal vectors such that \( v_1 \wedge \ldots \wedge v_m = \bar{\lambda} \).

Since |X| \leq C, \( I_1 \) can be easily estimated as Err_4:
\[ I_2 \leq C \sum_j (\sup_{\mathcal{U}_i} \varphi_r) m_0^{1+\gamma_2} \ell_j^{m+2+\gamma_2} \leq C(\gamma_3)D^{1+\gamma_3}(r). \]

For what concerns \( I_2 \), we argue differently for the outer and the inner variations. For \( \text{Err}_5^o \), observe that \( |X^{\alpha_\perp}(p)| = \varphi_r(p)|p|p_{\Sigma^{\alpha_\perp}}(p - p(p))| \). On the other hand, we also have
\[ |p_{\Sigma^{\alpha_\perp}}(p - p(p))| \leq C\epsilon(\Sigma)|p - p(p)|^2 \leq C m_0^{1/2} |p - p(p)|^2 \quad \forall \ p \in \Sigma. \]
Therefore, we can estimate
\[ I_2^o \leq C m_0 \int \varphi_r |N|^2 \leq C \epsilon^2(\Sigma)(r). \]
For the inner variations, denote by \( \nu_1, \ldots, \nu_l \) an orthonormal frame for \( T_p \Sigma_\perp \) of class \( C^{2, \alpha_0} \) (cf. [5, App. A]) and set\( h^j_j := -\sum_{k=1}^m \langle D_v \nu_j(p), v_k \rangle \) whenever \( v_1 \wedge \ldots \wedge v_m = \tilde{\nu} \) is an \( m \)-vector of \( T_p \Sigma \) (with \( v_1, \ldots, v_m \) orthonormal). For the sake of simplicity, we write

\[
h^j_p := h^j_j(T_F(p)) \quad \text{and} \quad h_p = \sum_{j=1}^l h^j_j \nu_j(p),
\]

\[
h^j_p(p) := h^j_j(M(p(p))) \quad \text{and} \quad h_p(p) = \sum_{j=1}^l h^j_j \nu_j(p(p)).
\]

Consider the exponential map \( \exp_p : T_p \Sigma \to \Sigma \) and its inverse \( \exp_p^{-1} \). Recall that

- the geodesic distance \( d_\Sigma(p,q) \) is comparable to \( |p - q| \) up to a constant factor;
- \( \nu_j \) is \( C^{2, \alpha_0} \) and \( \|Dv_j\|_{C^{1, \alpha_0}} \leq C m_0^{1/2} \);
- \( \exp_p \) and \( \exp_p^{-1} \) are both \( C^{2, \alpha_0} \) and \( \|d \exp_p\|_{C^{1, \alpha_0}} + \|d \exp_p^{-1}\|_{C^{1, \alpha_0}} \leq m_0^{1/2} \);
- \( |h^j_j| \leq C \|A\Sigma\|_{C^0} \leq C m_0^{1/2} \),

where all the constants involved are just geometric. We then conclude that

\[
\begin{align*}
|X_\perp(p) - h_p(p)| &= |h_p - h_p(p)| \\
&= \sum_j \nu_j(p) (h^j_p - h^j_p(p)) + \sum_j (\nu_j(p) - \nu_j(p(p))) h^j_j(p) \\
&= \sum_j \nu_j(p) (h^j_p - h^j_p(p)) \\
&\quad + \sum_j Dv_j(p) \cdot \exp_p^{-1}(p) h^j_j(p(p)) + O(|p - p(p)|^2).
\end{align*}
\]

On the other hand, \( X_i(p) = Y(p(p)) \) is tangent to \( \mathcal{M} \) in \( p(p) \) and hence orthogonal to \( h_p(p) \). Thus

\[
\begin{align*}
\langle X_i(p), h_p(p) \rangle &= \langle X_i(p), (h_p - h_p(p)) \rangle \\
&= \sum_j \langle X_i(p), Dv_j(p) \rangle \cdot \exp_p^{-1}(p) h^j_j(p(p)) \\
&\quad + \sum_j (\nu_j(p, X_i(p)) (h^j_p - h^j_p(p)) + O(|p - p(p)|^2) \\
&= \sum_j \langle X_i(p), Dv_j(p) \rangle \cdot \exp_p^{-1}(p) h^j_j(p(p)) \\
&\quad + O \left( |T_F(p) - \tilde{\mathcal{M}}(p)|, |p - p(p)|, |p - p(p)|^2 \right),
\end{align*}
\]

where we used elementary calculus to infer that \( |\langle X_i(p), \nu_j(p) \rangle| \leq C |p - p(p)| \) and

\[
|h^j_p - h^j_p(p)| \leq C \left( |T_F(p) - \tilde{\mathcal{M}}(p)| + |p - p(p)| \right).
\]
We only need that the constants $C$ appearing in the above inequalities are bounded by a geometric factor. In fact, they enjoy explicit bounds in terms of $m_0^{1/2}$ which are at least linear, but such degree of precision is not needed. Finally, recalling that $p \in \text{spt}(T_F)$, we can bound $|p - p(p)| \leq |N(p)|$ and $|\bar{T}_F(p) - \bar{M}(p(p))| \leq C|DN(p(p))|$. We therefore conclude the estimate
\[
\langle X_i(p), h_p \rangle = \sum_j \langle X_i(p(p)), D\nu_j(p(p)) \cdot \text{ex}^{-1}_{p(p)}(p) \rangle h^j_p + O(|N|^2(p) + |DN|^2(p)).
\]
We combine it with the expansion of the area functional in [5, Th. 3.2] to conclude the estimate on $I_2$. Recalling that $p(F_i(x)) = x$, we get
\[
I_2 = \left| \int \langle X_i, h_p \rangle d\|T_F\| \right| = \left| \sum_{i=1}^{Q} \int_M \langle Y, h_{F_i} \rangle J F_i \right|
\leq \int_M \sum_{j=1}^{Q} \sum_{i=1}^{Q} \langle Y(x), D\nu_j(x) \cdot \text{ex}^{-1}_x(F_i(x)) \rangle h^j_x dH^m(x)
+ C \int_M \varphi_r(|N|^2 + |DN|^2).
\]
Using the Taylor expansion for $\text{ex}^{-1}_x$ at $x$ (and recalling that $F_i(x) - x = N_i(x)$), we conclude
\[
\left| \sum_{i=1}^{Q} \text{ex}^{-1}_x(F_i(x)) \right| \leq |d \text{ex}^{-1}_x(\eta \circ N(x))| + O(|N|^2) \leq C|\eta \circ N(x)| + C|N|^2.
\]
Next consider that $|\langle Y, D\nu_j \cdot v \rangle| \leq C\varphi_r\|A_{\Sigma}\|_{C^0}|v| \leq C\varphi_r m^{1/2}_0 |v|$ for every tangent vector $v$ and $|h^j_x| \leq C\|A_{\Sigma}\|_{C^0} \leq m^{1/2}_0$. We thus conclude with the estimate
\[
I_2 \leq C m_0 \int_M \varphi_r |\eta \circ N| + C \int_M \varphi_r (|N|^2 + |DN|^2) =: J_1 + J_2.
\]
Clearly $J_1$ can be estimated as $\text{Err}_1^2$ and $J_2$ as $\text{Err}_2^2$, thus concluding the proof.

5. Boundedness of the frequency

In this section we prove that the frequency function $I_j$ remains bounded along the different center manifolds corresponding to the intervals of flattening. To simplify the notation, we set $p_j := \Phi_j(0)$ and simply write $B_\rho$ in place of $B_\rho(p_j)$.

**THEOREM 5.1** (Boundedness of the frequency functions). Let $T$ be as in **Assumption 2.1**. If the intervals of flattening are $j_0 < \infty$, then there is $\rho > 0$ such that
\[
H_{j_0} > 0 \text{ on } ]0, \rho[ \quad \text{and} \quad \limsup_{r \to 0} I_{j_0}(r) < \infty.
\]
If the intervals of flattening are infinitely many, then there is a number \( j_0 \in \mathbb{N} \) and a geometric constant \( j_1 \in \mathbb{N} \) such that

\[
(5.2) \quad H_j > 0 \text{ on } [\frac{s_j}{t_j}, 2^{-j_1}3[ \text{ for all } j \geq j_0, \sup_{j \geq j_0} \sup_{r \in [\frac{s_j}{t_j}, 2^{-j_1}3[} I_j(r) < \infty,
\]

\[
(5.3) \quad \sup \left\{ I_j(r), \frac{r^2 J_{B_r} |DN_j|^2}{J_{B_r} |N_j|^2} \right\} : j \geq j_0 \text{ and } \max \left\{ \frac{s_j}{t_j}, \frac{3}{2^{j_1}} \right\} \leq r < 3 \right\} < \infty.
\]

(In the latter inequality we understand \( I_j(r) = \infty \) when \( H_j(r) = 0 \).)

**Proof.** Consider the first alternative. We claim that for every \( r > 0 \), there is a radius \( 0 < \rho < r \) such that \( H(\rho) = H_{j_0}(\rho) > 0 \). Otherwise, \( N_{j_0} \) vanishes identically on some \( B_r \). By [6, Props. 3.1 and 3.4] and Proposition 2.2(iii) this is possible only if no cube of the Whitney decomposition \( \mathcal{W}(j_0) \) intersects the projection of \( B_r \) onto the plane \( \pi \) (the reference plane for the construction of the center manifold). But then \( T_{j_0} \) would coincide with \( Q[\mathcal{M}] \) in \( B_{3r/4} \) and 0 would be a regular point of \( T_{j_0} \) and, therefore, of \( T \).

Next we claim that \( H(r) > 0 \) for every \( r \leq \rho \). If not, let \( r_0 \) be the largest zero of \( H \) which is smaller than \( \rho \). By Theorem 3.2, there is a constant \( C \) such that \( I(r) \leq C(1 + I(\rho)) \) for every \( r \in [0, \rho[ \). By letting \( r \downarrow r_0 \), we then conclude

\[
r_0 D(r_0) \leq C(1 + I(\rho)) H(r_0) = 0,
\]

that is, \( N_j |_{B_{r_0}} \equiv 0 \), which we have already excluded. Therefore, since \( H > 0 \) on \( ]0, \rho[ \), we can now apply Theorem 3.2 to conclude (5.1).

In the second case, we partition the extrema \( t_j \) of the intervals of flattening into two different classes: the class \( (A) \) when \( t_j = s_{j-1} \) and the class \( (B) \) when \( t_j < s_{j-1} \). If \( t_j \) belongs to \( (A) \), set \( r := \frac{s_{j-1}}{t_{j-1}} \). Let \( L \in \mathcal{W}(j-1) \) be a cube of the Whitney decomposition such that \( c_s \leq \ell(L) \) and \( L \cap B_r(0, \pi) \neq \emptyset \). We are in the position to apply [6, Prop. 3.7] for the comparison of two center manifolds. There exists a constant \( \bar{c}_s > 0 \) such that

\[
\int_{B_2 \cap M_j} |N_j|^2 \geq \bar{c}_s m_0^j := \bar{c}_s \max \left\{ E(T_j, B_{6\sqrt{m}}, c(\Sigma_j)^2) \right\},
\]

which obviously gives \( \int_{B_3} |N_j|^2 \geq cm_0^j \). By [6, (2.7)] (or alternatively by (3.4)), we then conclude

\[
(5.4) \quad \int_{B_3} |N_j|^2 \geq \bar{c} \int_{B_3} |DN_j|^2,
\]

where \( \bar{c} \) is a positive geometric constant. By the Hölder inequality and Sobolev embedding (cf. [3, Prop. 2.11]), there are geometric constants \( C_0 \) and \( \bar{a} = \dots \)
\( m(1 - \frac{2}{q}) > 0 \) such that

\[
\int_{B_{\frac{3}{2}}} |N_j|^2 \leq \left( H^m \left( B_{\frac{3}{2}} \right) \right)^{1-\frac{2}{q}} \left( \int_{B_{\frac{3}{2}}} |N_j|^q \right)^{\frac{2}{q}} \\
\leq C_0 2^{-J\alpha} \int_{B_3} |N_j|^2 + C_0 2^{-J\alpha} \int_{B_3} |DN_j|^2 \\
\leq C_0 2^{-J\alpha - \frac{1}{2}} \int_{B_3} |N_j|^2 \quad \text{for any} \ J \in \mathbb{N}.
\]

(In the above we can set \( q = 2^* \) when \( m \geq 3 \) and choose any \( q < \infty \) larger than 2 for \( m = 2 \); note also that since the curvature of the manifold \( M_j \) is bounded by \( m_0 \), we can assume that \( H^m(B_\rho) \) is comparable to the \( m \)-dimensional volume of the corresponding euclidean ball for every \( \rho < 3 \).) If we choose \( J = j_1 \) for a large enough \( j_1 \) (depending only upon \( \bar{c}, \alpha \) and \( C_0 \)), we achieve

\[
\int_{B_3 \setminus B_{\frac{3}{2}}} |N_j|^2 \geq \frac{1}{2} \int_{B_3} |N_j|^2 \geq \frac{\bar{c}}{2} \int_{B_3} |DN_j|^2.
\]

In turn we conclude the existence of one annulus \( A(k(j)) := B_{3/(2^{k(j)}) \setminus B_{3/(2^{k(j)}+1)}} \) with

\[
\int_{A(k(j))} |N_j|^2 \geq \frac{\bar{c}}{2j_1} \int_{B_3} |DN_j|^2 \quad \text{and} \quad k(j) \leq j_1.
\]

\( H_{N_j}(k(j)) \) is bounded from below by the integral on the left-hand side of (5.7), whereas the right-hand side bounds \( D_{N_j}(2^{-k(j)}3) \) from above. Thus \( I_{N_j}(2^{-k(j)}3) \) is smaller than a constant which depends upon \( \bar{c}, \alpha \) and \( C_0 \). Arguing as in the first alternative, we can apply Theorem 3.2 to conclude the positivity of \( H_{N_j} \) and to gain a uniform upper bound for \( I_{N_j} \) on the interval \( \left[ 2^{-k(j)}3, 3 \right] \). Since the latter contains \( \left[ \frac{5}{4}, 2^{j_1}3 \right] \), we conclude the validity of (5.2). (If one or both the intervals are trivial, namely \( \left( \frac{5}{4}, 2^{j_1}3 \right] \) is larger than the right endpoint, then there is nothing to prove.) On the other hand, for every \( r \in [2^{-k(j)}3, 3[ \), by (5.7) we certainly have

\[
\int_{B_r} |N_j|^2 \geq \frac{\bar{c}}{2j_1} \int_{B_3} |DN_j|^2,
\]

from which (5.3) readily follows.

In the case \( t_j \) belongs to the class \( (B) \) then, by construction there is \( \eta_j \in ]0, 1[ \) such that \( E((t_0,t_j)^tT, B_{6\sqrt{m}(1+\eta_j)}) > \varepsilon_3^2 \). Up to extraction of a subsequence, we can assume that \( (t_0,t_j)^tT \) converges to a cone \( S \). The convergence is strong enough to conclude that the excess of the cone is the limit of the excesses of the sequence. Moreover (since \( S \) is a cone), the excess \( E(S, B_r) \) is
independent of $r$. We then conclude

$$
\varepsilon_3^2 \leq \liminf_{j \to \infty, j \in \{B\}} \mathbf{E}(T_j, \mathcal{B}_3).
$$

Thus, by Lemma 5.2 below, we conclude $\liminf_{j \to \infty, j \in \{B\}} \mathbf{H}_{N_j}(3) > 0$. Since $D_{N_j}(3) \leq C m^2 \leq C \varepsilon_3^2$, we achieve that $\limsup_{j \to \infty, j \in \{B\}} I_{N_j}(3) < +\infty$ and conclude as before.

**Lemma 5.2.** Assume the intervals of flattening are infinitely many and $r_j \in [\frac{r}{4}, 3]$ is a subsequence (not relabeled) with $\lim_j \| N_j \|_{L^2(B_{r_j} \setminus B_{r_j/2})} = 0$. If $\varepsilon_3$ is sufficiently small, then, $\mathbf{E}(T_j, \mathcal{B}_{r_j}) \to 0$.

**Proof.** Note that, if $r_j \to 0$, then necessarily $\mathbf{E}(T_j, \mathcal{B}_{r_j}) \to 0$ by Proposition 2.2(iv). Therefore, up to a subsequence, we can assume the existence of $c > 0$ such that

$$
r_j \geq c \text{ and } \mathbf{E}(T_j, \mathcal{B}_{6m/3}) \geq c.
$$

After the extraction of a further subsequence, we can assume the existence of $r$ such that

$$
\int_{B_r \setminus \mathcal{B}_{\frac{3}{4}r}} |N_j|^2 \to 0
$$

and the existence of an area minimizing cone $S$ such that $(\iota_0 f_j)_{\#} T \to S$. Note that, by (5.8), $S$ is not a multiplicity $Q$ flat $m$-plane. Consider the orthogonal projection $q_j : \mathbb{R}^{m+n} \to \pi_j$, where $\pi_j$ is the $m$-dimensional plane of the construction of the center manifold $\mathcal{M}_j$. Assuming $\varepsilon_3$ is sufficiently small, we have $U_j := B_{13/16}(\pi) \setminus B_{13/16}(\pi) \subset q_j(\mathcal{B}_r \setminus \mathcal{B}_{3/4r})$. Consider the Whitney decomposition $\mathcal{W}^{(j)}$ leading to the construction of $\mathcal{M}_j$. If no cube of the decomposition intersects $U^j$, then $N_j$ vanishes identically on it. Otherwise, set

$$
d_j := \max \{ \ell(J) : J \in \mathcal{W}^{(j)} \text{ and } J \cap U_j \neq \emptyset \}.
$$

Let $J_j \in \mathcal{W}^{(j)}$ be such that $U_j \cap J_j \neq \emptyset$ and $d_j = \ell(J_j)$. If the stopping condition for $J_j$ is either (HT) or (EX), recalling that $\ell(J_j) \leq c \varepsilon r$, we choose a ball $B^j \subset U_j$ of radius $\frac{d_j}{2}$ and at distance at most $\sqrt{md_j}$ from $J_j$. If the stopping condition for $J_j$ is (NN), $J_j$ is in the domain of influence of $K_j \in \mathcal{W}_e^{(j)}$. By Proposition 2.2 we can then choose a ball $B^j \subset U_j$ of radius $\ell(K_j) \frac{3}{8}$ at distance at most $3\sqrt{m\ell(K_j)}$ from $K_j$. If the stopping condition is (HT), by [6, Prop. 3.1] we then have

$$
\int_{\mathcal{B}_{r} \setminus \mathcal{B}_{\frac{3}{4}r}} |N_j|^2 \geq \int_{\mathcal{B}_{r}(B^j)} |N_j|^2 \geq c(m_0^2)^{\frac{1}{m}} d_j^{m+2+2\beta_2}.
$$
If the stopping condition is either (NN) or (EX), by [6, Prop. 3.1] and [6, Prop. 3.4] we have

\[(5.10) \quad \int_{B_{s} \setminus B_{\frac{s}{4}}} |N_j|^2 \geq \int_{\Phi_j(\mathcal{B})} |N_j|^2 \geq \frac{c d_j^{2}}{J_{j}^{(0)}} \int_{\Phi_j(\mathcal{B})} |DN_j|^2 \geq c m_{j}^{2} d_j^{m_{j}+4-2\delta_{j}}.\]

In both cases we conclude that \(d_j \to 0.\)

By [6, Cor. 2.2], \(\text{spt}(T_j) \cap \Phi_j(U_j)\) is contained in a \(d_j\)-tubular neighborhood of \(\mathcal{M}_j\), which we denote by \(\hat{U}_j\). Moreover, again assuming that \(\varepsilon_{3}\) is sufficiently small, we can assume \(B_t \setminus B_{s} \cap \mathcal{M}_j \subset \Phi_j(U_j)\) for some appropriate choice of \(s < t\), independent of \(j\). Finally, by [6, Th. 1.17] we can assume that (up to subsequences) \(\mathcal{M}_j\) converges to \(\mathcal{M}\) in \(C^3\). We thus conclude that \(\mathcal{S}(B_t \setminus B_{s})\) is supported in \(\mathcal{M} \cap (B_t \setminus B_{t/2})\) and, hence, by the constancy theorem, \(\mathcal{S}(B_t \setminus B_{s}) = Q_0 \left[\mathcal{M} \cap (B_t \setminus B_{t/2})\right]\) for some integer \(Q_0\). Observe also that, if \(p_j : \hat{U}_j \to \mathcal{M}_j\) is the least distance projection onto \(\mathcal{M}_j\), then by [6, Th. 2.4] we also have \((p_j)_{\sharp}(T_j \mathcal{L}(B_t \setminus B_{t/2})) = Q \left[\mathcal{M} \cap (B_t \setminus B_{t/2})\right]\). We therefore conclude that \(Q_0 = Q\). Since \(S\) is a cone without boundary, \(\partial(S \mathcal{L} B_t) = Q \left[\mathcal{M} \cap \partial B_t\right]\), i.e., \(S \mathcal{L} B_t = Q [0] \times [\mathcal{M} \cap \partial B_t]\). By Allard’s regularity theorem (which can be applied because \(\Theta(S, 0) = \lim_j \Theta(T_j, 0) = Q\), \(S\) is regular in a neighborhood of 0 and, therefore, it is an \(m\)-plane with multiplicity \(Q\), which gives the desired contradiction.

\[\Box\]

A corollary of Theorem 5.1 is the following.

**Corollary 5.3 (Reverse Sobolev).** Let \(T\) be as in Assumption 2.1. Then, there exists a constant \(C > 0\) which depends on \(T\) but not on \(j\) such that, for every \(j\) and for every \(r \in \left[\frac{r_j}{T_j}, 1\right]\), there is \(s \in \left[\frac{3}{2} r, 3 r\right]\) such that

\[(5.11) \quad \int_{B_{s}(\Phi_j(0))} |DN_j|^2 \leq \frac{C}{r^2} \int_{B_{s}(\Phi_j(0))} |N_j|^2.\]

**Proof.** If the second alternative in Theorem 5.1 holds, if \(r \geq 2^{-j} 3\) and if \(I_j(3r)\) is larger than the ratio

\[\frac{(3r)^2 \int_{B_{sr}(\Phi_j(0))} |DN_j|^2}{\int_{B_{sr}(\Phi_j(0))} |N_j|^2},\]

then the claim follows from (5.3). Therefore, without loss of generality, we can assume that \(I_j(3r)\) is bounded by a constant \(C^*\), which depends on \(T\) but not on \(j\).

We start observing that, by the Coarea Formula,

\[H_j(3r) = \int_{B_{3r}(\Phi_j(0)) \setminus B_{3r/2}(\Phi_j(0))} |N_j|^2 \frac{2}{d(p)} = 2 \int_{3r/2}^{3r} t \int_{\partial B_{s}(\Phi_j(0))} |N_j|^2 dt,\]
whereas, using Fubini,

$$\int_{\frac{3r}{2}}^{3r} \int_{B_{r}(\Phi_j(0))} |DN_j|^2 \, dt = \int_{\mathcal{M}_j} |DN_j|^2(x) \int_{3r/2}^{3r} 1_{|x|,\infty}(t) \, dt \, d\mathcal{H}^m(x) = \frac{3}{2} rD_j(3r).$$

Since we are assuming that $I_j(3r) \leq C^*$,

$$\int_{\frac{3r}{2}}^{3r} dt \int_{B_{r}(\Phi_j(0))} |DN_j|^2 = \frac{3}{2} rD_j(3r) \leq C^* H_j(3r) = C^* \int_{\frac{3}{2}r}^{3r} dt \frac{1}{t} \int_{\partial B_{r}(\Phi_j(0))} |N_j|^2.
$$

Therefore, there must be $s \in [\frac{3}{2}r, 3r]$ such that

$$\int_{B_{s}(\Phi_j(0))} |DN_j|^2 \leq \frac{C^*}{s} \int_{\partial B_{s}(\Phi_j(0))} |N_j|^2. \tag{5.12}$$

Now fix any $\sigma \in [s/2, s]$ and any point $x \in \partial B_{s}(\Phi_j(0))$. Consider the geodesic line $\gamma$ passing through $x$ and $\Phi_j(0)$, and let $\hat{\gamma}$ be the arc on $\gamma$ having one endpoint $\bar{x}$ in $\partial B_{\sigma}(\Phi_j(0))$ and one endpoint equal to $x$. Using [3, Prop. 2.1(b)] and the fundamental theorem of calculus, we easily conclude

$$|N_j(x)|^2 \leq |N_j(\bar{x})|^2 + 2 \int_{\hat{\gamma}} |DN_j||N_j|.
$$

Integrating this inequality in $x$ and recalling that $\sigma > s/2$, we then easily conclude

$$\int_{\partial B_{s}(\Phi_j(0))} |N_j|^2 \leq C \int_{\partial B_{\sigma}(\Phi_j(0))} |N_j|^2 + C \int_{B_{s}(\Phi_j(0)) \setminus B_{s/2}(\Phi_j(0))} |N_j||DN_j|,$$

where the constant $C$ depends only on the curvature of $\mathcal{M}_j$, which is bounded independently of $j$. We further integrate in $\sigma$ between $s/2$ and $s$ to achieve

$$\frac{s}{2} \int_{\partial B_{s}(\Phi_j(0))} |N_j|^2 \leq C \int_{B_{s}(\Phi_j(0)) \setminus B_{s/2}(\Phi_j(0))} (|N_j|^2 + s |N_j||DN_j|)
$$

$$\leq \frac{s^2}{4C^*} \int_{B_{s}(\Phi_j(0))} |DN_j|^2 + \tilde{C} \int_{B_{s}(\Phi_j(0))} |N_j|^2, \tag{5.13}$$

where $C^*$ is the constant in (5.12) and the constant $\tilde{C}$ depends on the curvature of $\mathcal{M}_j$ and on $C^*$. Combining (5.13) with (5.12), we easily conclude (5.11). \hfill \Box

6. Final blow-up sequence and capacitary argument

6.1. Blow-up maps. Let $T$ be a current as in the Assumption 2.1. By Proposition 2.2 we can assume that for each radius $r_k$, there is an interval of flattening $I_{j(k)} = [s_{j(k)}, t_{j(k)}]$ containing $r_k$. We next define the sequence of “blow-up maps” which will lead to the proof of Almgren’s partial regularity
result Theorem 0.3. To this aim, for \( k \) large enough, we define \( \tilde{s}_k \) so that the radius \( \frac{\tilde{s}_k}{r_{j(k)}} \in \left[ \frac{3}{2}, \frac{3r_k}{r_{j(k)}} \right] \) is the radius provided in Corollary 5.3 applied to \( r = \frac{r_k}{r_{j(k)}} \). We then set \( \tilde{r}_k := \frac{\tilde{s}_k}{r_{j(k)}} \) and rescale and translate currents and maps accordingly:

(BU1) \( \tilde{T}_k = (\iota_{0,\tilde{r}_k})_* T_{j(k)} = ((\iota_{0,\tilde{r}_k} t_{j(k)})_* T) \cup B_{6\sqrt{m}/\tilde{r}_k}, \quad \tilde{\Sigma}_k = \iota_{0,\tilde{r}_k}(\Sigma_{j(k)}) \) and \( \tilde{\mathcal{M}}_k := \iota_{0,\tilde{r}_k}(\mathcal{M}_{j(k)}) \);

(BU2) \( \tilde{N}_k : \tilde{\mathcal{M}}_k \to \mathbb{R}^{m+n} \) are the rescaled \( \tilde{\mathcal{M}}_k \)-normal approximations given by

\[
\tilde{N}_k(p) = \frac{1}{\tilde{r}_k} N_{j(k)}(\tilde{r}_k p).
\]

Since by assumption \( T_0 \Sigma = \mathbb{R}^{m+n} \times \{0\} \), the ambient manifolds \( \tilde{\Sigma}_k \) converge to \( \mathbb{R}^{m+n} \times \{0\} \) locally in \( C^{3,\text{iso}} \) (more precisely, to a "large portion" of \( \mathbb{R}^{m+n} \times \{0\} \), because \( B_{6\sqrt{m}/\tilde{r}_k} \subset B_{6\sqrt{m}/\tilde{r}_k} \)). Moreover, since \( \frac{1}{2} < \frac{r_k}{r_{j(k)}} < 1 \), it follows from Proposition 1.3 that

\[
E(\tilde{T}_k, B_\frac{1}{2}) \leq CE(T, B_{r_k}) \to 0.
\]

By the standard regularity theory of area minimizing currents and Assumption 2.1, this implies that \( \tilde{T}_k \) locally converge (and supports converge locally in the Hausdorff sense) to (a large portion of) a minimizing tangent cone which is an \( m \)-plane with multiplicity \( Q \) contained in \( \mathbb{R}^{m+n} \times \{0\} \). Without loss of generality, we can assume that \( \tilde{T}_k \) locally converge to \( Q \oplus \pi_0 \). Moreover, from Proposition 1.3 it follows that

\[
\mathcal{H}^{m-2+\alpha}_\infty(D_Q(\tilde{T}_k) \cap B_1) \geq C_0 r_k^{-(m-2+\alpha)} \mathcal{H}^{m-2+\alpha}_\infty(D_Q(T) \cap B_{r_k}) \geq \eta > 0,
\]

where \( C_0 \) is a geometric constant.

In the next lemma, we show that the rescaled center manifolds \( \tilde{\mathcal{M}}_k \) converge locally to the flat \( m \)-plane \( \pi_0 \), thus leading to the following natural definition for the blow-up maps \( N^k : B_3 \subset \mathbb{R}^m \to A_Q(\mathbb{R}^{m+n}) \):

\[
N^k(x) := h_k^{-1} N_k(e_k(x)),
\]

where \( h_k := \| \tilde{N}_k \|_{L^2(B_\frac{1}{2})} \) and \( e_k : B_3 \subset \mathbb{R}^m \simeq T_{\tilde{p}_k} \tilde{\mathcal{M}}_k \to \tilde{\mathcal{M}}_k \) denotes the exponential map at \( \tilde{p}_k = \Phi_{j(k)}(0)/\tilde{r}_k \). (Here and in what follows we assume, w.l.o.g., to have applied a suitable rotation to each \( \tilde{T}_k \) so that the tangent plane \( T_{\tilde{p}_k} \tilde{\mathcal{M}}_k \) coincides with \( \mathbb{R}^m \times \{0\} \)).

**Lemma 6.1 (Vanishing lemma).** Under the Assumption 2.1, the following hold:

(i) we can assume, without loss of generality, \( \tilde{r}_k m^j_{0(k)} \to 0 \);

(ii) the rescaled center manifolds \( \tilde{\mathcal{M}}_k \) converge (up to subsequences) to \( \mathbb{R}^m \times \{0\} \) in \( C^{3,\kappa/2}(B_4) \) and the maps \( e_k \) converge in \( C^{2,\kappa/2} \) to the identity map \( \text{id} : B_3 \to B_3 \).
(iii) there exists a constant $C > 0$, depending only $T$, such that, for every $k$,

\begin{equation}
\int_{B_2} |DN_k|^2 \leq C.
\end{equation}  

Proof. To show (i), note that, if $\liminf_k \bar{r}_k > 0$, we can extract a further subsequence and assume that $\limsup_k \bar{r}_k > 0$. Observe that then $\bar{r} := \limsup_k \frac{t_{j(k)}}{r_k} < \infty$. Since $r_k \downarrow 0$, we necessarily conclude that $t_{j(k)} \downarrow 0$ and hence $c(\Sigma_{j(k)}) \rightarrow 0$. Moreover, $E(T, B_6 \sqrt{m_{j(k)}}) \leq C(\bar{r}) E(\bar{T}_k, B_6 \sqrt{m_{\bar{r}_k}^{-1}}) \rightarrow 0$ because $\bar{T}_k$ converges to $Q[\pi_0]$. We conclude $\bar{r}_k m_{j(k)} \rightarrow 0$. On the other hand, if $\lim_k \bar{r}_k = 0$, then (i) follows trivially from the fact that $m_{j(k)}$ is a bounded sequence.

Next, using $\bar{r}_k m_{j(k)} \rightarrow 0$ and the estimate of [6, Th. 1.17], it easily follows that $\bar{M}_k - \bar{p}_k$ converge (up to subsequences) to a plane in $C^{3,\kappa/2}(B_4)$. By Proposition 2.2(v) we easily deduce that such a plane is in fact $\pi_0$. Since $0 \in \text{supp} \ T_{j(k)}$, we conclude for the same reason that $\bar{M}_k$ is converging to $\pi_0$ as well. Therefore, by Proposition A.4 the maps $e_k$ converge to the identity in $C^{2,\kappa/2}$. (Indeed, by standard arguments they must converge to the exponential map on the — totally geodesic! — submanifold $\mathbb{R}^m \times \{0\}$.) Finally, (iii) is a simple consequence of Corollary 5.3. \hfill $\Box$

The main result about the blow-up maps $N_k^b$ is the following.

**Theorem 6.2 (Final blow-up).** Up to subsequences, the maps $N_k^b$ converge strongly in $L^2(B_2^2)$ to a function $\bar{N}_\infty^b : B_2^2 \rightarrow \mathcal{A}_Q(\{0\} \times \mathbb{R}^n \times \{0\})$ which is Dir-minimizing in $B_t$ for every $t \in \frac{5}{3}, \frac{3}{2}$ and satisfies $\|\bar{N}_\infty^b\|_{L^2(B_2^2)} = 1$ and $\eta \circ \bar{N}_\infty^b \equiv 0$.

We postpone the proof of Theorem 6.2 to the next section and now show Theorem 0.3.

**6.2. Proof of Theorem 0.3: Capacitary argument.** Let $N_\infty^b$ be as in Theorem 6.2 and

$$
\bar{\mathcal{Y}} := \{x \in B_1 : N_\infty^b(x) = Q[\{0\}]\}.
$$

Since $\eta \circ N_\infty^b \equiv 0$ and $\|N_\infty^b\|_{L^2(B_{3/2})} = 1$, from the regularity of Dir-minimizing $Q$-valued functions (cf. [3, Prop. 3.22]), we know that $\mathcal{H}^{m-2+\alpha}_\infty(\bar{\mathcal{Y}}) = 0$. We show in the following three steps that this contradicts Assumption 0.4.

**Step 1.** We cover $\bar{\mathcal{Y}}$ by balls $\{B_{\sigma_i}(x_i)\}$ in such a way that

$$
\sum_i \omega_{m-2+\alpha}(4\sigma_i)^{m-2+\alpha} \leq \frac{\eta}{2},
$$

where

$$
\mathcal{H}_E(\mathcal{Y}) \leq \int_{\mathcal{Y}} |E|^2 \leq C \int_{\mathcal{Y}} H^{m-2+\alpha}.
$$

Since $\mathcal{H}^{m-2+\alpha}_\infty(\bar{\mathcal{Y}}) = 0$, we conclude that $\|\mathcal{H}^{m-2+\alpha}_\infty\|_{L^2(B_{3/2})} = 0$. This contradicts Assumption 0.4.

**Step 2.** Using the capacitary property of $\mathcal{Y}$, we can extract a further sequence $\bar{N}_k$ which converges strongly in $L^2(B_2^2)$ to a function $\bar{N}_\infty^b : B_2^2 \rightarrow \mathcal{A}_Q(\{0\} \times \mathbb{R}^n \times \{0\})$. Since $\mathcal{H}^{m-2+\alpha}_\infty(\bar{\mathcal{Y}}) = 0$, we conclude that $\|\bar{N}_\infty^b\|_{L^2(B_2^2)} = 1$ and $\eta \circ \bar{N}_\infty^b \equiv 0$.

**Step 3.** Using the capacitary property of $\bar{\mathcal{Y}}$, we can extract a further sequence $\bar{N}_k$ which converges strongly in $L^2(B_2^2)$ to a function $\bar{N}_\infty^b : B_2^2 \rightarrow \mathcal{A}_Q(\{0\} \times \mathbb{R}^n \times \{0\})$. Since $\mathcal{H}^{m-2+\alpha}_\infty(\bar{\mathcal{Y}}) = 0$, we conclude that $\|\bar{N}_\infty^b\|_{L^2(B_2^2)} = 1$ and $\eta \circ \bar{N}_\infty^b \equiv 0$.

Finally, we show that $\bar{N}_\infty^b$ is Dir-minimizing in $B_t$ for every $t$. For this, we use the capacitary property of $\mathcal{Y}$ and the fact that $\mathcal{H}^{m-2+\alpha}_\infty(\bar{\mathcal{Y}}) = 0$. This contradicts Assumption 0.4.

**Theorem 6.3.** Up to subsequences, the maps $N_k^b$ converge strongly in $L^2(B_2^2)$ to a function $\bar{N}_\infty^b : B_2^2 \rightarrow \mathcal{A}_Q(\{0\} \times \mathbb{R}^n \times \{0\})$ which is Dir-minimizing in $B_t$ for every $t$ and satisfies $\|\bar{N}_\infty^b\|_{L^2(B_2^2)} = 1$ and $\eta \circ \bar{N}_\infty^b \equiv 0$. \hfill $\Box$
where $\eta$ is the constant in (6.2). By the compactness of $\Upsilon$, such a covering can be chosen finite. We can therefore choose a $\bar{\sigma} > 0$ so that the $5\bar{\sigma}$-neighborhood of $\Upsilon$ is covered by $\{B_{\sigma_i}(x_i)\}$. Denote by $\Lambda_k$ the set of multiplicity $Q$ points of $\tilde{T}_k$ far away from the singular set $\Upsilon$:

$$\Lambda_k := \{p \in D_Q(\tilde{T}_k) \cap B_1 : \text{dist}(p, \Upsilon) > 4\bar{\sigma}\}.$$ 

Clearly, $H^{m-2+\alpha}_\infty(\Lambda_k) \geq \frac{\eta}{2}$. Let $V$ denote the neighborhood of $\Upsilon$ of size $2\bar{\sigma}$. By the H"{o}lder continuity of Dir-minimizing functions (cf. [3, Th. 2.9]) there is a positive constant $1 > \vartheta > 0$ such that $|N_\infty^b(x)|^2 \geq 2\vartheta$ for every $x \not\in V$.

We next introduce a parameter $\sigma > 0$ whose choice will be specified only at the very end; throughout the rest of the proof it will only be required to be sufficiently small. In particular, $\sigma < \bar{\sigma}$ will surely imply that

$$\int_{B_{2\vartheta}(x)} |N_\infty^b|^2 \geq 2\vartheta \quad \forall x \in B_{\frac{\bar{\sigma}}{4}} \text{ with } \text{dist}(x, \Upsilon) \geq 4\bar{\sigma}.$$ 

Therefore, from Theorem 6.2 we infer that, for sufficiently large $k$’s,

$$\int_{B_{2\vartheta}(x)} G(\bar{N}_k, Q \circ \bar{N}_k)^2 \geq \vartheta h_k^2 \quad \forall x \in \Gamma_k := p_M(\Lambda_k).$$

**Step 2.** For every $p \in \Lambda_k$, consider $\bar{x}_k(p) = p_{\pi_k}(p)$ (where $\pi_k$ is the reference plane for the center manifold related to $T_{j(k)}$) and

$$\bar{x}_k(p) := (\bar{z}_k(p), \bar{r}_k^{-1} \varphi_{j(k)}(\bar{r}_k z_k(p))).$$

Observe that $\bar{x}_k(p) \in \bar{M}_k$. We next claim the existence of a suitably chosen geometric constant $1 > c_0 > 0$ (in particular, independent of $\sigma$) such that, when $k$ is large enough, for each $p \in \Lambda_k$ there is a radius $\varrho_p \leq 2\sigma$ with the following properties:

$$\frac{c_0 \vartheta}{\sigma^\alpha} h_k^2 \leq \frac{1}{\varrho_p^{m-2+\alpha}} \int_{B_{\varrho_p}(\bar{x}_k(p))} |D\bar{N}_k|^2,$$

$$B_{\varrho_p}(\bar{x}_k(p)) \subset B_{4\varrho_p}(p).$$

In order to show this claim, fix such a point $p$, consider the points $q_k := \bar{r}_k p$, $z_k := \bar{r}_k \bar{z}_k(p)$ and $x_k = \bar{r}_k \bar{x}_k(p) = (z_k, \varphi_{j(k)}(z_k))$. Observe that $q_k \in D_Q(T_{j(k)})$.

By [6, Prop. 3.1], $z_k$ cannot belong to some $L \in \mathcal{W}_{\rho}^{(j(k))}$ (otherwise $B_{16r_L}(p_L)$ would contain a multiplicity $Q$ point of $T_{j(k)}$, contradicting statement (S1) in [6, Prop. 3.1]). We thus distinguish two possibilities:

(Exc) either $z_k$ belongs to some $L_k \in \mathcal{W}_{\rho}^{(j(k))} \cup \mathcal{W}_n^{(j(k))}$;

(Con) or it belongs to the set $\Gamma_{j(k)}$. 
Case (Exc). Observe that if $L_k \in \mathcal{W}_n^{(j(k))}$, by Proposition 2.2(iii), there exists a cube $H_k \in \mathcal{W}_e^{(j(k))}$ such that $L_k$ belongs to the domain of influence of $H_k$ and $\text{sep}(B_{\frac{3r_k}{16}}, H_k) \leq 3r_k/16$. Thus $H_k$ intersects $B_{\frac{19r_k}{16}}(0, \pi)$.

We wish now to apply [6, Prop. 3.5], with the $s$ in there equal to $\tilde{r}_k$ and the $T$ in there equal to $T_{j(k)}$. The aim is to infer

$$\ell_k := \tilde{r}_k^{-1} \sup \{\ell(L) : L \in \mathcal{W}_e^{(j(k))} \text{ and } L \cap B_{\frac{19r_k}{16}}(0, \pi) \neq \emptyset\} = o(1).$$

First observe that, taking into account the inequality $1 \leq \tilde{r}_k t_{j(k)}/r_k \leq 2$, a simple scaling argument gives

$$\mathcal{H}^{m-2+\alpha}(DQ(T_{j(k)}), B_{t_{j(k)}}) \geq \left(\frac{r_k}{t_{j(k)}}\right)^{m-2+\alpha} \mathcal{H}^{m-2+\alpha}(DQ(T_{0,r_k}) \cap B_{t_{j(k)}/r_k}) \geq \left(\frac{r_k}{t_{j(k)}}\right)^{m-2+\alpha} \mathcal{H}^{m-2+\alpha}(DQ(T_{0,r_k}) \cap B_1) \geq (1.2) \eta \left(\frac{\tilde{r}_k}{2}\right)^{m-2+\alpha},$$

which verifies [6, (3.4)]. We next need to verify [6, (3.3)] and consider therefore $L \in \mathcal{W}^{(j)}$ which intersects $B_{3r_k}(0, \pi)$. Since $\tilde{r}_k > s_{j(k)}/t_{j(k)}$, by (Go) we have $\ell(L) < 3cs\tilde{r}_k \leq \tilde{r}_k$. Now, for any fixed $\hat{\alpha} > 0$, we can apply [6, Prop. 3.5] provided $\min\{\tilde{r}_k, m_j^{0(k)}\}$ is small enough, which is the case for $k$ large enough by Lemma 6.1(i). Thus [6, Prop. 3.5] implies $\limsup_k \hat{\ell}_k \leq \hat{\alpha}$, and the arbitrariness of the latter parameter implies (6.8).

For $k$ large enough, we can then apply [6, Prop. 3.6] with $\eta_2 = \frac{\vartheta}{\pi}$. (In particular, this condition on how large $k$ must be is independent of the point $p$.)

The proposition will be applied to $L_k$, if $L_k \in \mathcal{W}_n^{(j(k))}$, or to $H_k$ above, if $L_k \in \mathcal{W}_n^{(j(k))}$. We thus set

$$J_k = \begin{cases} H_k & \text{if } L_k \in \mathcal{W}_n^{(j(k))}, \\ L_k & \text{if } L_k \in \mathcal{W}_e^{(j(k))} \end{cases}$$

and conclude the existence of a constant $\tilde{s} < 1$ such that

$$\int_{B_{\vartheta t_{j(k)}(x_k)}} \mathcal{G}(\eta \circ N_{j(k)}, Q[\eta \circ N_{j(k)}])^2 \leq \frac{\vartheta}{4w_\omega t_{j(k)}^{m-2}} \int_{B_{\vartheta t_{j(k)}(x_k)}} |D\eta \circ N_{j(k)}|^2.$$

By (6.8), provided $k$ is large enough, we have $t(p) := \frac{\ell(L)}{r_k} \leq \tilde{\ell}_k \leq \sigma$. Therefore, rescaling to $\tilde{M}_k$, there exists $t(p) \leq \tilde{\ell}_k$ such that

$$\int_{B_{\vartheta t(p)(x_k(p))}} \mathcal{G}(\bar{\eta}_k \circ N_k, Q[\bar{\eta} \circ N_k])^2 \leq \frac{\vartheta}{4w_\omega t(p)^{m-2}} \int_{B_{\vartheta t(p)(x_k(p))}} |D\bar{\eta}_k|^2.$$

Moreover, from Proposition 2.2(v) and Lemma 6.1, for $k$ large enough, we get

$$|p - \bar{x}_k(p)| \leq C(m_0^{j(k)} \frac{t^{\beta_2}}{t^2} t(p) < \tilde{s} t(p).$$
The previous lines we have repeatedly used $\phi$ such that (6.6) holds. Indeed, if this were not the case, for simplicity we set $j = 0$. 

We now argue by contradiction to infer the existence of $\varrho = \varrho_{\rho}$. (In the case (Con), we have already shown it; see (6.11).)

Indeed, we can use (6.9) in the case (Exc). (In the case (Con), we have already shown it; see (6.11).)

We now argue by contradiction to infer the existence of $\varrho = \varrho_{\rho}$. (In the case (Exc), we have already shown it; see (6.11).)

Having chosen $t(p)$ in both cases, we next show the existence of $\varrho = \varrho_{\rho}$ such that (6.6) holds. Observe that (6.7) will be an obvious consequence of (6.10). Notice that if

\begin{equation}
\int_{\mathcal{B}_{r_0}((\bar{x}_k(p)))} |D\bar{N}_k|^2 \geq h_k^2,
\end{equation}

then (6.6) follows with $\varrho_{\rho} = t(p)$. If (6.12) does not hold, then

\begin{equation}
\int_{\mathcal{B}_{r_0}((\bar{x}_k(p)))} |D\bar{N}_k|^2 \leq \frac{\varrho}{4} h_k^2.
\end{equation}

Indeed, we can use (6.9) in the case (Exc). (In the case (Con), we have already shown it; see (6.11).)

In the previous lines we have repeatedly used $|Df| \leq |D\bar{N}_k|$, the classical Poincaré inequality and the following simple Morrey-type estimate (which is...
also a consequence of the Poincaré inequality)

\[(\bar{f}_t - \bar{f}_1)^2 \leq \frac{C_0}{t^{m-2}} \int_{B_{2t}(\bar{x}_k(p))} |Df|^2.\]

Note that such a constant \(C_0\) (and the constant for the Poincaré inequality) depends only upon the regularity of the underlying manifold \(\bar{M}_k\) and, hence, can be assumed independent of \(k\). Summarizing, if (6.6) were to fail for every radius in the interval \([\bar{s}t(p), 2\sigma]\), from (6.14) we would conclude

\[\int_{B_{2\sigma}(\bar{x}_k(p))} f^2 \leq h^2_k \vartheta \left( \frac{1}{\sqrt{2}} + C_0 \frac{1}{\sigma^{n/2}} \sum_{i=0}^{j-1} (2^{1-j} \sigma)^{n/2} \right)^2 \]

\[\leq h^2_k \vartheta \left( \frac{1}{\sqrt{2}} + c_0 C(\alpha) \right)^2.\]

Since \(C(\alpha)\) depends on \(\alpha, m\) and \(Q\), but does not depend on \(k\), for \(c_0\) chosen sufficiently small, the latter inequality would contradict (6.5). Note that (6.7) follows by a simple triangular inequality.

**Step 3.** Finally, we show that (6.6) and (6.7) lead to a contradiction. Consider a covering of \(\Lambda_k\) with balls \(B_i := B_{2^0 \varrho p_i}(p_i)\) with the property that the corresponding balls \(B_{4^0 \varrho_p(\bar{x}_k(p_i))}\) are disjoint. We then can estimate

\[\eta^2 \leq C_0 \sum_i \varrho_{p_i}^{m-2+\alpha} \leq C_0 \frac{\sigma^\alpha}{c_0} \int_{B_{2\sigma}(\bar{x}_k(p))} |D\bar{N}_k|^2 \]

\[\leq C_0 \frac{\sigma^\alpha}{h_k^2} \int_{B_2} |D\bar{N}_k|^2 \leq C \frac{\sigma^\alpha}{\vartheta},\]

where \(C_0 > 0\) is a dimensional constant. In the last line we have used that, thanks to (6.7), the balls \(B_{v_{p_i}}(\bar{M}_k(p_i))\) are pairwise disjoint and that, provided \(\sigma\) is smaller than \(\frac{1}{32}\) and \(k\) large enough, they are all contained in \(B_{3^2}\). Since \(\vartheta\) and \(c_0\) are independent of \(\sigma\), the above inequality reaches the desired contradiction as soon as \(\sigma\) is fixed sufficiently small. This will only require a sufficiently small \(\bar{\ell}_k\), which by (6.8) is ensured for \(k\) sufficiently large.

### 7. Harmonicity of the limit

In this section we prove Theorem 6.2 and conclude our argument. We continue to follow the notation of the previous section; in particular, recall the maps defined in (BU1) and (BU2) of Section 6.1

**7.1. First estimates.** Without loss of generality, we might translate the manifolds \(\bar{M}_k\) so that the rescaled points \(\bar{p}_k = \bar{r}^{-1}_k \Phi_{j_k}(0)\) all coincide with the origin. Let \(\bar{F}_k : B_2 \subset \bar{M}_k \rightarrow \mathcal{A}_Q(\mathbb{R}^{m+n})\) be the multiple valued map given
by \( F_k(x) := \sum_i \left[ x + (N_k)_i(x) \right] \) and, to simplify the notation, set \( p_k := p_{\mathcal{M}_k} \). We start by showing the existence of a suitable exponent \( \gamma > 0 \) such that

\[
\text{Lip}(N_k|_{B_{3/2}}) \leq Ch_k^{\gamma} \quad \text{and} \quad \|N_k\|_{C^0(B_{3/2})} \leq C(m_0^{j(k)} r_k)^{\gamma},
\]

(7.1)

\[
\mathcal{M}((T_{F_k} - T_k) \circ p_k^{-1}(B_{3/2})) \leq Ch_k^{2+2\gamma},
\]

(7.2)

\[
\int_{B_{3/2}} |\eta \circ \tilde{N}_k| \leq Ch_k^2.
\]

(7.3)

Indeed, set \( p_{j(k)} = \Phi_{j(k)}(0) \). Using the domain decomposition of Section 4.1 (note that \( \frac{3}{2} \tilde{r}_k \in [\frac{s_{j(k)}}{r_{j(k)}}, 3] \)) and arguing in an analogous way, we infer that

\[
\|N_{j(k)}\|_{C^0(B_{3/2}(p_{j(k)}))} \leq C(m_0^{j(k)})^{\frac{1}{2m}} r_k^{1+\beta_2}
\]

and

\[
\text{Lip}(N_{j(k)}|_{B_{3/2}(p_{j(k)})}) \leq C(m_0^{j(k)})^{\gamma_2} \max_i \ell_i^{\gamma_2}
\]

\[
\mathcal{M}((T_{F_{j(k)}} - T_{j(k)}) \circ p_k^{-1}(B_{3/2}(p_{j(k)}))) \leq \sum_i (m_0^{j(k)})^{1+\gamma_2} \ell_i^{m+2+\gamma_2},
\]

\[
\int_{B_{3/2}(p_{j(k)})} |\eta \circ N_{j(k)}| \leq C m_0^{j(k)} r_k \sum_i \ell_i^{2+m+\gamma_2} + C \tilde{r}_k \int_{B_{3/2}(p_{j(k)})} |N_{j(k)}|^2,
\]

where this time for the latter inequality, we have used [6, Th. 2.4 (2.4)] with \( a = \tilde{r}_k \). On the other hand, again by the arguments of Section 4.1 (see, for instance, (4.12), (4.13) and (4.14)) and Corollary 5.3, we see that

\[
\sum_i m_0^{j(k)} \ell_i^{m+2+\gamma_2} \leq C_0 \int_{B_{3/2}(p_{j(k)})} |DN_{j(k)}|^2 + |N_{j(k)}|^2
\]

(7.4)

\[
\leq C \tilde{r}_k^{-2} \int_{B_{\tilde{r}_k}(p_{j(k)})} |N_{j(k)}|^2,
\]

from which (7.1)-(7.3) follow by a simple rescaling. (The constant \( C \) on the right-hand side of (7.4) depends on \( T \) but not on \( k \).)

It is then clear that the strong \( L^2 \) convergence of \( N_k^b \) is a consequence of these bounds and of the Sobolev embedding (cf. [3, Prop. 2.11]) whereas, by (7.3),

\[
\int_{B_{3/2}} |\eta \circ N_{\infty}^b| = \lim_{k \to +\infty} \int_{B_{3/2}} |\eta \circ N_k^b| \leq C \lim_{k \to +\infty} h_k = 0.
\]

Finally, note that \( N_{\infty}^b \) must take its values in \( \{0\} \times \mathbb{R}^d \times \{0\} \). Indeed, considering the tangential part of \( N_k \) given by \( \tilde{N}_k^T(x) := \sum_i \left[ p_{T^k \Sigma_k} (N_k(x))_i \right] \), it is simple
to verify that \( G(\overline{N}_k, \overline{N}_k^T) \leq C_0|\overline{N}_k|^2 \), which leads to
\[
\int_{B_{\delta/2}} G(N_k^b, h_k^{-1}\overline{N}_k^T \circ e_k)^2 \leq C_0h_k^{-2} \int_{B_{\delta/2}} |\overline{N}_k|^4
\]
and, by the convergence of \( \overline{\Sigma}_k \) to \( \mathbb{R}^{m+n} \times \{0\} \), gives the claim.

7.2. A suitable trivialization of the normal bundle. By Lemma 6.1, we can consider for every \( \mathcal{M}_k \) an orthonormal frame of \( (TM_k)^\perp, \nu^k_1, \ldots, \nu^k_n, \omega^k_1, \ldots, \omega^k_n \) with the property that \( \nu^j_k(x) \in T_x\Sigma_k, \omega^j_k(x) \perp T_x\Sigma_k \) and (cf. [5, Lemma A.1])
\[
\nu^j_k \to e_{m+j} \quad \text{and} \quad \omega^j_k \to e_{m+n+j} \quad \text{in} \quad C^{2,\kappa/2}(\mathcal{M}_k) \quad \text{as} \quad k \uparrow \infty.
\]
(For every \( j \), here \( e_1, \ldots, e_{m+n+l} \) is the standard basis of \( \mathbb{R}^{m+n+l} = \mathbb{R}^{m+n} \).)

We next claim the existence of maps \( \psi_k : \mathcal{M}_k \times \mathbb{R}^n \to \mathbb{R}^l \) converging to 0 in \( C^{2,\kappa/2} \) (uniformly bounded in \( C^{2,\kappa} \)) and of \( \delta > 0 \) (independent of \( k \)) such that, for every \( v \in T_p\mathcal{M}_k \) with \( |v| \leq \delta \),
\[
p + v \in \overline{\Sigma}_k \iff v^+ = \psi_k(p, v^T),
\]
with \( v^T = (\langle v, \nu^k_1 \rangle, \ldots, \langle v, \nu^k_n \rangle) \in \mathbb{R}^n \) and \( v^+ = (\langle v, \omega^k_1 \rangle, \ldots, \langle v, \omega^k_n \rangle) \in \mathbb{R}^l \). To see this, consider the map
\[
\Phi_k : \mathcal{M}_k \times \mathbb{R}^n \times \mathbb{R}^l \ni (p, z, w) \mapsto p + z^j \nu^k_j + w^j \omega^k_j \in \mathbb{R}^{m+n},
\]
where we use the Einstein convention of summation over repeated indices. It is simple to show that the frame can be chosen so that \( D\Phi_k(0,0) = \text{Id} \) and, hence, \( \Phi_k^{-1}(\Sigma_k) \) can be written locally as a graph of a function \( \psi_k \) satisfying the claim above.

Note that, by construction, we also have that \( \psi_k(p, 0) = |D_w\psi_k(p, 0)| = 0 \) for every \( p \in \mathcal{M}_k \), which in turn implies
\[
|D_x\psi_k(x, w)| \leq C|w|^{1+\kappa}, \quad |D_w\psi_k(x, w)| \leq C|w| \quad \text{and} \quad |\psi_k(x, w)| \leq C|w|^2.
\]
Now given any \( Q \)-valued map \( u = \sum_i [u_i] : \mathcal{M}_k \to A_Q(\{0\} \times \mathbb{R}^n \times \{0\}) \) with \( \|u\|_{L^\infty} \leq \delta \), we can consider the map \( \mathbf{u}_k := \psi_k(x, u) \) defined by
\[
x \mapsto \sum_i \left[ (u_i)^j \nu^k_j(x) + \psi^j_k(x, u_i(x)) \omega^k_j(x) \right],
\]
where we set \( (u_i)^j := \langle u_i(x), e_{m+j} \rangle \), \( \psi^j_k(x, u_i(x)) := \langle \psi_k(x, u_i(x)), e_{m+n+j} \rangle \).
(Again we use Einstein’s summation convention.) Then, the differential map \( D\mathbf{u}_k := \sum_i [D(u_i)]_i \) is given by
\[
D(\mathbf{u}_k)_i = D(u_i)^j \nu^k_j + \left[ D_x\psi^j_k(x, u_i) + D_w\psi^j_k(x, u_i)Du_i \right] \omega^k_j
+ (u_i)^j D\nu^k_j + \psi^j_k(x, u_i)D\omega^k_j.
\]
Taking into account that \( \|Du^k\|_{C^0} + \|D\psi^k\|_{C^0} \to 0 \) as \( k \to +\infty \), by (7.5) we deduce that
\[
\int \left( |Du_k|^2 - |Du|^2 \right) \leq C \int \left( |Du|^2|u| + |Du||u|^{1+\kappa} + |u|^{2+2\kappa} \right) + o(1) \int \left( |u|^2 + |Du|^2 \right).
\]
(7.6)

Now we clearly have \( \tilde{N}_k(x) = \psi_k(x, \bar{u}_k) \) for some Lipschitz \( \bar{u}_k : \tilde{\mathcal{M}}_k \to \mathcal{A}_Q(\mathbb{R}^n) \) with \( \|\bar{u}_k\|_{L^\infty} = o(1) \) by (7.1). Setting \( u_k^b := \bar{u}_k \circ e_k \), we conclude from (5.11), (7.1) and (7.6) that
\[
\lim_{k \to +\infty} \int_{B_{\frac{3}{2}}} \left( |DN_k^b|^2 - h_k^{-2}|Du_k^b|^2 \right) = 0,
\]
and \( N_{\infty}^b \) is the limit of \( h_k^{-1}u_k^b \).

7.3. Competitor function. We now show the Dir-minimizing property of \( N_{\infty}^b \). Clearly, there is nothing to prove if its Dirichlet energy vanishes. We can therefore assume that there exists \( c_0 > 0 \) such that
\[
c_0 h_k^2 \leq \int_{B_{\frac{3}{2}}} |D\tilde{N}_k|^2.
\]
(7.8)

Assume there is a radius \( t \in \left[ \frac{5}{4}, \frac{3}{2} \right] \) and a function \( f : B_{\frac{3}{2}} \to \mathcal{A}_Q(\mathbb{R}^n) \) such that
\[
f|_{B_{\frac{3}{2}} \setminus B_t} = N_{\infty}^b|_{B_{\frac{3}{2}} \setminus B_t} \quad \text{and} \quad \text{Dir}(f, B_t) \leq \text{Dir}(N_{\infty}^b, B_t) - 2\delta
\]
for some \( \delta > 0 \). We can apply [4, Prop. 3.5] to the functions \( h_k^{-1}u_k^b \) and find \( r \in [t, 2] \) and competitors \( v_k^b \) such that, for \( k \) large enough,
\[
v_k^b|_{\partial B_r} = u_k^b|_{\partial B_r}, \quad \text{Lip}(v_k^b) \leq Ch_k^\gamma, \quad |v_k^b| \leq C(m_k^r \rho_k)^\gamma,
\]
\[
\int_{B_{\frac{3}{2}}} |\eta \circ v_k^b| \leq Ch_k^2 \quad \text{and} \quad \int_{B_{\frac{3}{2}}} |Dv_k^b|^2 \leq \int |Du_k^b|^2 - \delta h_k^2,
\]
where \( C > 0 \) is a constant independent of \( k \) and \( \gamma \) the exponent of (7.1)-(7.3).

Clearly, by Lemma 6.1 and (7.5), the maps \( \tilde{N}_k = \psi_k(x, v_k^b \circ e_k^{-1}) \) satisfy
\[
\tilde{N}_k \equiv \tilde{N}_k \quad \text{in} \quad B_{\frac{3}{2}} \setminus B_t, \quad \text{Lip}(\tilde{N}_k) \leq Ch_k^\gamma, \quad |\tilde{N}_k|^1 \leq C(m_k^r \rho_k)^\gamma,
\]
\[
\int_{B_{\frac{3}{2}}} |\eta \circ \tilde{N}_k| \leq Ch_k^2 \quad \text{and} \quad \int_{B_{\frac{3}{2}}} |D\tilde{N}_k|^2 \leq \int_{B_{\frac{3}{2}}} |DN_k|^2 - \delta h_k^2.
\]

7.4. Competitor current. Consider finally the map \( \tilde{F}_k(x) = \sum_i [x + \tilde{N}_i(x)] \).

The current \( T_{\tilde{F}_k} \) coincides with \( T_{\tilde{F}_k} \circ p_k^{-1}(B_{\frac{3}{2}} \setminus B_t) \). Define the function \( \varphi_k(p) = \text{dist}_{\tilde{\mathcal{M}}_k}(0, p_k(p)) \), and for each \( s \in \left[ t, \frac{3}{2} \right] \), consider the slices \( \langle T_{\tilde{F}_k} - T_k, \varphi_k, s \rangle \). By (7.2) we have
\[
\int_t^{\frac{3}{2}} M(\langle T_{\tilde{F}_k} - T_k, \varphi_k, s \rangle) \leq Ch_k^{2+\gamma}.
\]
Thus for each $k$, we can find a radius $\sigma_k \in [t, \frac{3}{2}t]$ on which $M(\langle T_{\tilde{F}_k} - T_k, \varphi_k, \sigma_k \rangle) \leq C k^{2+\gamma}$. By the isoperimetric inequality (see [4, Rem. 4.3]) there is a current $S_k$ such that
\[
\partial S_k = \langle T_{\tilde{F}_k} - T_k, \varphi_k, \sigma_k \rangle, \quad M(S_k) \leq C k^{(2+\gamma)/(m-1)} \quad \text{and spt}(S_k) \subset \Sigma_k.
\]
Our competitor current is, then, given by
\[
\tilde{T}_k := T_k \mathbb{L}(p_k^{-1}(\mathcal{M}_k \setminus B_{\sigma_k})) + S_k + T_{\tilde{F}_k} \mathbb{L}(p_k^{-1}(B_{\sigma_k})).
\]
Note that $\tilde{T}_k$ is supported in $\Sigma_k$ and is an admissible competitor for $T_k$. On the other hand, by (7.2) and the bound on $M(S_k)$, we have
\[
(7.9) \quad M(\tilde{T}_k) - M(T_k) \leq M(\tilde{F}_k) - M(F_k) + C k^{2+2\gamma}.
\]
Denote by $A_k$ and by $H_k$ respectively the second fundamental forms and mean curvatures of the manifolds $\mathcal{M}_k$. Using the Taylor expansion of [5, Th. 3.2], we achieve
\[
(7.10) \quad M(\tilde{T}_k) - M(T_k) \leq \frac{1}{2} \int_{B_{\rho}} \left( |D\tilde{N}_k|^2 - |D\tilde{N}_k|^2 \right) \\
+ C \|H_k\| C^\infty \int \left( |\eta \circ \tilde{N}_k| + |\eta \circ \tilde{N}_k| \right) \\
+ \|A_k\| C^\infty \int \left( |\tilde{N}_k|^2 + |\tilde{N}_k|^2 \right) + o(h_k^2) \leq -\frac{\delta}{2} h_k^2 + o(h_k^2),
\]
where in the last inequality we have taken into account Lemma 6.1. Clearly, (7.10) contradicts the minimizing property of $\tilde{T}_k$ for $k$ large enough and concludes the proof.

**Appendix A. Some technical lemmas**

The following is a special case of Allard’s $\varepsilon$-regularity theory (see [10, Chap. 5]).

**Theorem A.1.** Assume that $T$ is area minimizing, $x \in D_Q(T)$ and that $\|T\|((\text{spt}(T) \cap U) \setminus D_Q) = 0$ in some neighborhood $U$ of $x$. Then, $x \in \text{Reg}(T)$. In particular, $D_1(T) \subset \text{Reg}(T)$.

**Proof.** By simple considerations on the density, the tangent cones at $x$ must necessarily be all $m$-dimensional planes with multiplicity $Q$. This allows us to apply Allard’s theorem and conclude that, in a neighborhood of $x$, spt$(T)$ is necessarily the graph of a $C^{1,\kappa_0}$ function for some $\kappa_0 > 0$. Let $u : \mathbb{R}^m \to \mathbb{R}^{n+l}$ be the corresponding function and $\Psi : \mathbb{R}^{m+n} \to \mathbb{R}^l$ a $C^{3,\delta_0}$ function whose graph describes $\Sigma$. Let $\tilde{u}$ consist of the first $\tilde{n}$ coordinates functions of $u$. We then have that $\tilde{u}$ minimizes an elliptic functional of the form $\int \Phi(x, \tilde{u}(x), D\tilde{u}(x)) \, dx$ where $(x, v, p) \mapsto \Phi(x, v, p)$ and $(x, v, p) \mapsto D_p \Phi(x, v, p)$.
are of class $C^{2,\varepsilon_0}$. We can then apply the classical regularity theory to conclude that $\bar{u} \in C^{3,\varepsilon_0}$ (see, for instance, [9, Th. 9.2]), thereby concluding that $x$ belongs to $\text{Reg}(T)$ according to Definition 0.2. Next fix any $x \in D_1(T)$. By the upper semicontinuity of the density $\Theta$ (cf. [10]), $\Theta < \frac{3}{2}$ in a neighborhood $U$ of $x$, which implies $\|T\|(\text{spt}(T) \cap U \setminus D_1) = 0$. □

Next, we prove the following technical lemma.

**Lemma A.2.** Let $T$ be an integer rectifiable current of dimension $m$ in $\mathbb{R}^{m+n}$ with locally finite mass and $U$ an open set such that $H^{m-1}(\partial U \cap \text{spt}(T)) = 0$ and $(\partial T) \cap U = 0$. Then $\partial(T \cap U) = 0$.

**Proof.** Consider $V \subset \subset \mathbb{R}^{m+n}$. By the Slicing Theorem [7, 4.2.1] applied to $\text{dist}(\cdot, \partial U)$ we conclude that $S_r := T(V \cap U \cap \{\text{dist}(x, \partial U) > r}\}$ is a normal current in $N^m(V)$ for almost every $r$. Since $M(T(V \cap U) - S_r) \to 0$ as $r \downarrow 0$, we conclude that $T(V \cap U)$ is in the closure of $N^m(V)$. Thus, by [7, 4.1.17], $T \cap U$ is a flat chain in $\mathbb{R}^{m+n}$. By [7, 4.1.12], $\partial(T \cap U)$ is also a flat chain. It is easy to check that $\text{spt}(\partial(T \cap U)) \subset \partial U \cap \text{spt}(T)$. Thus we can apply [7, Th. 4.1.20] to conclude that $\partial(T \cap U) = 0$. □

Recall the following theorem. (For the proof, see [10, Th. 35.3].)

**Theorem A.3.** If $T$ is an integer rectifiable area minimizing current in $\Sigma$, then

$$H^{m-3+\alpha}(\text{spt}(T) \setminus \left(\text{spt}(\partial T) \cup \bigcup_{Q \in \mathbb{N}} D_Q(T)\right)) = 0 \quad \forall \alpha > 0.$$  

We finally prove the following result (first proved by Allard in an unpublished note and hence reported in [1]).

**Proposition A.4.** Set $\pi := \mathbb{R}^m \times \{0\} \subset \mathbb{R}^{m+n}$, and let $\mathcal{M}$ be the graph of a $C^{3,\kappa}$ function $\varphi : \pi \supset B_3(0) \to \mathbb{R}^m$, with $\varphi(0) = 0$. Then the exponential map $\exp : B_3(0) \to \mathcal{M}$ belongs to the class $C^{2,\kappa}$. Moreover, if $\|\varphi\|_{C^{3,\kappa}}$ is sufficiently small, then the set $p_\pi(\exp(B_r(0))) \subset \pi$ is (for all $r < 3$) a convex set and the maximal curvature of its boundary is less than $\frac{2}{r}$.

**Proof.** Consider any $C^{3,\kappa}$ chart $x : \mathcal{M} \to \Omega$, for instance, the one induced by the graphical structure. It is then obvious that the components $g_{ij}$ of the Riemannian metric (induced by the Euclidean ambient space on the submanifold $\mathcal{M}$) are $C^{2,\kappa}$. We let $\nabla$ be the Levi-Civita connection on $\mathcal{M}$ for which $g$ is parallel and consider the corresponding Christoffel symbols $\Gamma^i_{jk}$ (in the fixed coordinate patch). Using the standard formula which expresses the Christoffel symbols $\Gamma^i_{jk}$ in terms of the metric $g_{sr}$ (see, for instance, [8, Prop. 2.54]), it is easy to see that the former are $C^{1,\kappa}$. The careful reader will notice that these objects are usually defined in standard textbooks assuming that the metric is $C^\infty$, but in order to have a unique Levi-Civita connection it is enough that the
metric is $C^1$; see the proof of [8, Th. 2.51]. (In fact, the Levi-Civita connection on $M$ can also be recovered by differentiating in the Euclidean ambient space and projecting the result onto the tangent space to $M$.) Similarly, for $C^2$ metrics, we can use the intrinsic definition of the Riemann curvature tensor as in [8, Def. 3.3]. From the standard formula in [8, 3.16] we easily conclude that the components of this tensor are $C^0,\kappa$. However, by [3, Lemma A.1] we can choose a $C^2,\kappa$ orthonormal frame $\nu_1, \ldots, \nu_n : \Omega \to \mathbb{R}^{m+n}$ for the normal bundle of $M$ and the curvature tensor can be computed via the Gauss’ equations as in [8, 5.8b]). We thus conclude that the components of the Riemann tensor are in fact $C^1,\kappa$. Again, although the references above carry on all computations in the $C^\infty$ setting, it can be easily checked that these work in a straightforward way under our regularity assumptions.

Next let $\Phi(t, v) := \exp(v t)$. (The fact that the exponential map is well defined will be justified in few lines.) Fix a $C^3,\kappa$ coordinate patch on $M$ where 0 is the origin, using the graphical structure of $M$ over $T_0M$. Set $t \mapsto \gamma(t) = \Phi(t, v)$, and use the notation $\gamma_i'(t)$ for the components of $\gamma'$ in the fixed chart $x : \Omega \to M$. (So, $\gamma_i'(t) = \sum_j \gamma_j(t) \frac{\partial}{\partial x_j}$.) $\gamma$ satisfies the system of differential equations

$$\gamma''_i(t) + \sum_{jk} \Gamma^j_{ik}(\gamma(t)) \gamma'_j(t) \gamma'_k(t) = 0,$$

with the initial conditions $\gamma(0) = 0$ and $\gamma'(0) = v$; cf. [8, Def. 2.77]. Thus it follows that the maps $\Phi$ and $\partial_t \Phi$ are $C^1,\kappa$; incidentally, this shows that the exponential map is well defined. (In fact, standard textbooks on ODEs only provide $C^1$ regularity. However, the usual proof of $C^1$ regularity via the Gronwall Lemma on the linearized ODEs for the derivative $\partial_v \Phi$ can be easily modified to prove $\partial_w \Phi \in C^{0,\alpha}$; cf. [2, §9].)

Now fix a tangent vector $e$ at 0, a point $p = \exp(v) \in M$ and perform a parallel transport of $e$ along the (“radial”) geodesic segment $[0, 1] \ni t \mapsto \exp(tv)$ to define $e(p)$. We claim that the corresponding vector field is $C^1,\kappa$. Indeed, fix any orthonormal tangent frame $f_1, \ldots, f_m$ which is $C^2,\kappa$. Let

$$e(\exp(tv)) = \sum_i \alpha_{v,i}(t) f_i(\Phi(t, v)) = \sum_i \alpha_{v,i}(t) \sum_k \varphi_{ik}(\Phi(t, v)) \frac{\partial}{\partial x_k},$$

where the functions $\varphi_{ik}$ are $C^2,\kappa$. Recall that the a vector field $X(t) = \sum_j X_j(t) \frac{\partial}{\partial x_j}$ along a $C^1$ curve $c$ with tangent $c'(t) = \sum_i c'_i(t) \frac{\partial}{\partial x_i}$ is parallel if and only if

$$X'_i(t) = -\sum_{j,k} \Gamma^i_{jk}(c(t)) c'_j(t) X_k(t);$$

cf. [8, Th. 2.68 and eq. (2.69)]. We therefore conclude that the coefficients $\alpha_{v,i}(t)$ must satisfy a system of ODEs of the form

$$\alpha'_{v,i}(t) = -\sum_j \alpha_{v,j}(t) F_{ij}(\Phi(t, v), \partial_t \Phi(t, v)).$$
where \((t,v) \mapsto F_{ij}(\Phi(t,v), \partial_t \Phi(t,v))\) are \(C^{1,\kappa}\) maps. Thus the existence of \(e\) and the claimed regularity of \((t,v) \mapsto \alpha_{v,i}(t)\) follow from the standard theory of ODEs.

Recall also that the parallel transport keeps the angle between vectors constant; cf. [8, Prop. 2.74]. We conclude that there exists an orthonormal frame \(e_1, \ldots, e_m\) of class \(C^{1,\kappa}\) which is parallel along geodesic rays emanating from the origin. Next, consider the map \((w,v,t) \mapsto \partial_w \Phi(t,v)\) where \(w\) varies in \(\mathbb{R}^m\). Fix \(w\) and \(v\), and consider again the curve \(\gamma(t)\) above and the vector \(\eta_{v,w}(t) = \partial_w \Phi(t,v)\). We claim that \(\eta\) satisfies the Jacobi equation along the geodesic \(\gamma\), with initial data \(\eta_{v,w}(0) = 0\) and \(\eta'_{v,w}(0) = w\). More precisely, if we write the vector field in the frame \(e_i\) as \(\eta(t) = \sum_i \eta_i(t) e_i(\gamma(t))\), the Jacobi equation is

\[
\eta''_{v,w,i}(t) = - \sum_j R_{\gamma(t)}(e_j(\gamma(t)), \gamma'(t), \gamma'(t), e_i(\gamma(t))) \eta_{v,w,j}(t),
\]

where \(R\) depends on the Riemann tensor (cf. [8, Th. 3.43]). Note that we do not have the usual smoothness assumptions under which (A.1) is derived in standard textbooks. We can however proceed by regularizing our manifold \(\mathcal{M}\) via convolution of the function of which the manifold is a graph. We fix the corresponding graphical charts for the regularized manifolds and observe that the exponential maps in these coordinates have uniform \(C^{1,\kappa}\) bounds from the corresponding ODEs and thus will converge to \(\Phi\) in \(C^1\). Similarly, one concludes the obvious convergence statements for the Riemann tensor and thus the right-hand side of (A.1) for the corresponding objects converge uniformly. This justifies, in the limit, that \(\eta_{v,w,i}\) is twice differentiable (in time) and that (A.1) holds.

Taking into account that \(\gamma(t) = \Phi(t,v)\) and \(\gamma'(t) = \partial_t \Phi(t,v)\), we conclude that \(\eta_{v,w,i}\) satisfies an ODE of the type \(\eta''_{v,w,i}(t) = \Lambda(t,v, \eta_{v,w,i}(t))\) where the function \(\Lambda\) is \(C^{1,\kappa}\) in all its entries. We thus conclude that the map \((v,w,t) \mapsto \eta_{v,w}(t) = \partial_w \Phi(t,v)\) is a \(C^{1,\kappa}\) map. Since \(d\exp(v)(w) = \partial_w \Phi(v,1)\), this implies that the exponential map is \(C^{2,\kappa}\).

As for the last assertion, for \(\|\varphi\|_{C^{3,\kappa}}\) sufficiently small, we conclude from the discussion above that \(p_{\pi} \circ \exp\) is \(C^2\) close to the identity, which implies the desired statement. \(\square\)

References


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