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Indecomposable vector bundles and stable Higgs bundles over smooth projective curves

By Olivier Schiffmann

Abstract

We prove that the number of geometrically indecomposable vector bundles of fixed rank r and degree d over a smooth projective curve X defined over a finite field is given by a polynomial (depending only on r, d and the genus g of X) in the Weil numbers of X. We provide a closed formula — expressed in terms of generating series- for this polynomial. We also show that the same polynomial computes the number of points of the moduli space of stable Higgs bundles of rank r and degree d over X. This entails a closed formula for the Poincaré polynomial of the moduli spaces of stable Higgs bundles over a compact Riemann surface, and hence also for the Poincaré polynomials of the twisted character varieties for the groups $\mathrm{GL}(r)$.

Contents

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1. Introduction

Overview. The aim of this paper is to compute the Betti numbers of the moduli spaces of stable Higgs bundles of fixed rank and degree over compact Riemann surfaces of arbitrary genus. These moduli spaces (and the corresponding moduli stacks), introduced by Hitchin in the late 80's ([Hit87b]), have since played an important role in the study of moduli spaces of connections on Riemann surfaces, integrable systems, and more recently in number theory and in the theory of automorphic forms in connection with Ngo's proof of the Fundamental Lemma (see [Ngô06], [LN08]). We refer to Section 1.3 for the history of the problem of computing the Poincaré polynomial of these moduli spaces and, in particular, for the relation between this work and some conjectures of Hausel, Rodriguez-Villegas ([HRV08]) and Mozgovoy ([Moz12]). In order to determine the Betti numbers of these moduli spaces and following a strategy put forward by T. Hausel, we count the number of points of the same moduli spaces for curves defined over finite fields and then use the Weil conjectures. This point counting is done in two steps. First, by a geometric deformation argument inspired by the work of Crawley-Boevey and Van den Bergh in the context of quivers ([CBVdB04]) we relate the number of stable Higgs bundles of rank r and degree d on a curve X to the number $\mathcal{A}_{r,d}(X)$ of *geometrically indecomposable* vector bundles of the same rank r and degree d over X, counted up to isomorphism. The second step is to explicitly calculate the number of such indecomposable vector bundles. Counting isomorphism classes of vector bundles (more generally, coherent sheaves) of rank r and degree d amounts to computing the volume of the inertia stack of the stack $\mathbf{Coh}_{r,d}$ parametrizing such coherent sheaves on X. Unfortunately, this inertia stack is of infinite volume. To circumvent this difficulty, we introduce a suitable truncation $\mathbf{Coh}_{r,d}^{\geq 0}$ of the stack $\mathbf{Coh}_{r,d}$ based on the Harder-Narasimhan stratification, whose inertia stack *is* of finite volume. Using some now classical techniques developped by Harder and others, we reduce the computation of that volume to the computation of the integral of some Eisenstein series over the truncated stack $\mathbf{Coh}_{r,d}^{\geq 0}$. The actual explicit evaluation of that integral is then performed using the language and tools provided by the theory of Hall algebras, which seems the most convenient here. This yields an explicit formula for the number $\mathcal{A}_{r,d}^{\geq 0}(X)$ of isomorphism classes of geometrically indecomposable vector bundles of rank r and degree d in $\mathbf{Coh}_{r,d}^{\geq 0}$. Using the obvious relation $\mathcal{A}_{r,d}(X) = \mathcal{A}_{r,d+r}(X)$ (which comes from the action of the Picard group) and the fact that $\mathcal{A}_{r,d}^{\geq 0}(X) = \mathcal{A}_{r,d}(X)$ for $d \gg 0$, we obtain the desired formula for the number of geometrically indecomposable vector bundles in $\mathbf{Coh}_{r,d}$, and hence also a computation of the number of stable Higgs bundles of rank r and degree d over X.

We also have another motivation in mind for computing the number of isomorphism classes of geometrically indecomposable vector bundles, coming from the analogy between vector bundles on a (smooth) curve and representations of a quiver. Indeed, as shown by Kac and Stanley (see [Kac82], [Kac83, 3.15]), the number of geometrically indecomposable $\mathbf{F}_q$-representations of a given dimension $\mathbf{d}$ of a quiver Q is given by a polynomial $A_{Q,d}(q)$ in q, the so-called *Kac's A-polynomial*. This polynomial carries a lot of very interesting Lie-theoretic information related to the Kac-Moody algebra associated to Q (at least when Q has no edge loops) and its Yangian; see, for instance, [Hau10], [Oko13]. It is natural to expect similar Lie-theoretic interpretations for the polynomials counting indecomposable vector bundles on curves. (See Section 8.3 for some conjectures in that direction.)

In the remainder of this introduction we describe our results in more details and point the reader to the corresponding sections of this paper.

1.1. Let $g \geq 0$, and let X be a smooth projective geometrically connected curve of genus g defined over a finite field $\mathbf{F}_q$. Let l be a prime number not dividing q, and let $\sigma_1, \dots, \sigma_{2g}$ stand for the associated Weil numbers of X (i.e., the eigenvalues of the Frobenius acting on $H^1(\overline{X}, \overline{\mathbb{Q}_l})$, where $\overline{X} = X \times_{\mathrm{Spec}(\mathbf{F}_q)} \mathrm{Spec}(\overline{\mathbf{F}_q})$). Fixing an embedding $\iota : \overline{\mathbb{Q}_l} \to \mathbb{C}$ we may view the σ_i as complex numbers satisfying $\overline{\sigma_{2i-1}} = \sigma_{2i}$ and $\sigma_{2i-1}\sigma_{2i} = q$ for $i = 1, \dots, g$. Consider the torus

$$T_g = \{(\alpha_1, \dots, \alpha_{2g}) \in \mathbb{G}_m^{2g} \mid \alpha_{2i-1}\alpha_{2i} = \alpha_{2j-1}\alpha_{2j} \ \forall\, i, j\}.$$

The group $W_g = \mathfrak{S}_g \ltimes (\mathfrak{S}_2)^g$ naturally acts on T_g, and the collection $\{\sigma_1, \dots, \sigma_{2g}\}$ defines a canonical element σ_X in the quotient $T_g(\mathbb{C})/W_g$. We denote by the same letter q the size of the finite field $\mathbf{F}_q$ and the element $q = \alpha_{2i-1}\alpha_{2i} \in \mathbb{Q}[T_g]^{W_g}$, hoping that this will not create any confusion. Let K_g be the localization of $\mathbb{Q}[T_g]^{W_g}$ at the multiplicative set generated by $\{q^l - 1 \mid l \geq 1\}$.

1.2. For $r > 0$ and $d \in \mathbb{Z}$, let $\mathcal{A}_{r,d}(X)$ stand for the number of geometrically indecomposable vector bundles on X (i.e., vector bundles $\mathcal{V}$ over X such that $\mathcal{V} \otimes_{\mathbf{F}_q} \overline{\mathbf{F}_q}$ is indecomposable) of rank r and degree d. The finiteness of $\mathcal{A}_{r,d}(X)$ results from standard arguments based on the Harder-Narasimhan filtration; see, e.g., Section 2.1. Observe that $\mathcal{A}_{r,d}(X)$ only depends on the residue of d modulo r as the set of geometrically indecomposable bundles is stable under tensoring by any line bundle.

The first main result of this paper is the following:

THEOREM 1.1. *For any fixed genus g and any pair $(r, d) \in \mathbb{N} \times \mathbb{Z}$, there exists a unique element $A_{g,r,d} \in K_g$ such that for any smooth projective geometrically connected curve X of genus g defined over a finite field, we have*

$$\mathcal{A}_{r,d}(X) = A_{g,r,d}(\sigma_X).$$

Conjecturally, $A_{g,r,d}$ belongs to $\mathbb{Q}[T_g]^{W_g}$ and there is no need to consider the localization K_g (see Corollary 1.5 and Conjecture 1.7). The proof of Theorem 1.1 is effective; i.e., we can explicitly compute the polynomial $A_{g,r,d}$. We postpone giving the (rather involved) explicit formula until Section 1.4; see Theorem 1.6. Theorem 1.1 is proved in Section 4.

1.3. Let us now assume that r and d are relatively prime. Let $\mathrm{Higgs}^{\mathrm{st}}_{r,d}(X)$ stand for the moduli space of stable Higgs bundles over X (see Section 6.2). This is a (smooth, quasi-projective, cohomologically pure) variety defined over $\mathbf{F}_q$, and we may consider its set of $\mathbf{F}_q$-rational points $\mathrm{Higgs}^{\mathrm{st}}_{r,d}(X)(\mathbf{F}_q)$.

The second main result of this paper, whose proof is very much inspired by the work of Crawley-Boevey and Van den Bergh (see [CBVdB04]), is the following:

THEOREM 1.2. *Let (r,d) be relatively prime. There exists an (explicit) constant $C = C(r,d)$ such that for any smooth projective geometrically connected curve X of genus g defined over $\mathbf{F}_q$ with $char(\mathbf{F}_q) > C$, we have*

$$|\mathrm{Higgs}^{\mathrm{st}}_{r,d}(X)(\mathbf{F}_q)| = q^{1+(g-1)r^2}\mathcal{A}_{r,d}(X).$$

The proof which we provide for the above theorem is geometric and relies on the symplectic structure of $\mathrm{Higgs}^{\mathrm{st}}_{r,d}(X)$. In the companion paper [MS14] written in collaboration with S. Mozgovoy, we give a different proof of the above theorem based on Hall-theoretic methods which works in all characteristics, as well as a generalization to the case of the moduli spaces of twisted (or meromorphic) Higgs bundles. Combining Theorems 1.1 and 1.2, we see that the number of $\mathbf{F}_q$-rational points of $\mathrm{Higgs}^{\mathrm{st}}_{r,d}(X)$ is given by some explicit polynomial in K_g. Note that the *existence* of such a polynomial is already known from the work of Garcia-Prada, Heinloth and Schmitt; see [GPH13, Th. 1.], [GPHS14].

Theorems 1.1 and 1.2 have the following corollary:

COROLLARY 1.3. *Let (r,d) be relatively prime.*

(i) *For any smooth, geometrically connected projective curve X of genus g defined over a field $\mathbf{F}_q$ of characteristic $p > C(r,d)$, we have*

$$\sum_n (-1)^n \dim H^n_c(\mathrm{Higgs}^{\mathrm{st}}_{r,d}(\overline{X}), \overline{\mathbb{Q}_l})t^n = t^{2(1+(g-1)r^2)}A_{g,r,d}(t,t,\dots,t).$$

(ii) *Let $X_{\mathbb{C}}$ be a compact Riemann surface of genus g. Then*

$$\sum_n (-1)^n \dim H^n_c(\mathrm{Higgs}^{\mathrm{st}}_{r,d}(X_{\mathbb{C}}), \mathbb{Q})t^n = t^{2(1+(g-1)r^2)}A_{g,r,d}(t,t,\dots,t),$$

where H^n_c denotes singular cohomology with compact support.

The Poincaré polynomial of the moduli space of stable Higgs bundles on a compact Riemann surface $X_{\mathbb{C}}$ of genus g has been computed in rank 2 by Hitchin ([Hit87a, Th. 7.6]), in rank 3 by Gothen (see [Got94, Th. 1.2]) and in rank 4 by Garcia-Prada, Heinloth and Schmitt (see [GPHS14, Th. 2]). Hausel and Rodriguez-Villegas ([HRV08, Conj. 4.2.1]) derived in a conjectural formula for the mixed Hodge polynomial of the genus g *twisted character variety* for the group GL_r. The latter being homeomorphic to $\mathrm{Higgs}^{\mathrm{st}}_{r,d}(X_{\mathbb{C}})$ their formula yields, in particular, a conjectural formula for the Betti numbers $\dim H^n_c(\mathrm{Higgs}^{\mathrm{st}}_{r,d}(X_{\mathbb{C}}), \mathbb{Q})$. This conjecture was extended by Mozgovoy (cf. [Moz12, Conj. 2]) to a conjectural formula for the motive of $\mathrm{Higgs}^{\mathrm{st}}_{r,d}(X_{\overline{\mathbf{F}_q}})$, where $X_{\overline{\mathbf{F}_q}}$ is now a smooth projective curve of genus g defined over $\overline{\mathbf{F}_q}$. Our formula (see Theorem 1.6 below) bears a strong similarity to the formula conjectured by Hausel-Rodriguez-Villegas and to its extension by Mozgovoy: it essentially differs from theirs by the presence of the rational functions $H_\lambda(z)$ (although of course if the main conjectures of [HRV08] and [Moz12] are true, then these formulas and ours compute the same numbers).

Theorems 1.1 and 1.2 have the following two other corollaries. Let $\mu : \mathrm{Higgs}^{\mathrm{st}}_{r,d} \to \mathbb{A}$ be the Hitchin map (see, e.g., [Hit87a]), and let $\Lambda^{\mathrm{st}}_{r,d}$ denote the zero fiber of μ (the stable global nilpotent cone). It is known that $\Lambda^{\mathrm{st}}_{r,d}$ is a projective (in general, singular) lagrangian subvariety of $\mathrm{Higgs}^{\mathrm{st}}_{r,d}$. It is nevertheless cohomologically pure, and its number of points is (by [GPH13, Th. 1], [GPHS14]) again given by a certain polynomial in the Weil numbers of X.

Corollary 1.4. *Let r, d be relatively prime. Let X be a smooth projective and geometrically connected curve of genus g defined over the field k. The following hold*:

(i) $(k = \mathbf{F}_q)$. *We have*

$$|\Lambda^{\mathrm{st}}_{r,d}(\mathbf{F}_q)| = \overline{A}_{g,r,d}(\sigma_X),$$

where $\overline{A}_{g,r,d}(z_1, \ldots, z_{2g}) = q^{2(1+(g-1)r^2}A_{g,r,d}(z_1^{-1}, \ldots, z_{2g}^{-1})$ *is the Poincaré dual of* $A_{g,r,d}$.

(ii) $(k = \mathbf{F}_q, \mathbb{C})$. *We have*

$$\dim H^{1+(g-1)r^2}(\Lambda^{\mathrm{st}}_{r,d}, \mathbb{C}) = \dim H^{1+(g-1)r^2}(\mathrm{Higgs}^{\mathrm{st}}_{r,d}, \mathbb{C}) = A_{g,r,d}(0).$$

In other words, the number of irreducible components of $\Lambda^{\mathrm{st}}_{r,d}$ *is equal to* $A_{g,r,d}(0)$.

Corollary 1.5. *Let r, d be relatively prime. Then*

$$A_{g,r,d} \in \mathrm{Im}(\mathbb{N}[-z_1, \ldots, -z_{2g}]^{W_g} \to R_g).$$

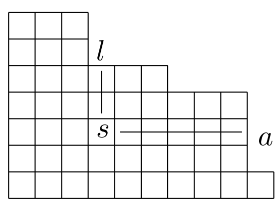
Moreover, $A_{g,r,d}$ *is unitary and of degree* $2(1+(g-1)r^2)$, *i.e.,* $A_{g,r,d} = q^{1+(g-1)r^2} + \cdots$, *where* $\cdots$ *stands for terms of degree strictly less than* $2(1+(g-1)r^2)$ *in the variables* $z_1, \ldots, z_{2g}$.

It would be a consequence of Conjecture 1.7 below that Corollary 1.5 holds without the coprimality assumption, thus yielding a global analog of Kac's positivity conjecture for A-polynomials of quivers. (The latter has recently been proved in [HLRV13].)

Theorem 1.2 and Corollaries 1.3, 1.4, and 1.5 are proved in Section 6.

1.4. Let us now give the precise expression for the polynomials $A_{g,r,d}$. This requires first introducing some notation.

Partitions. Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l > 0)$ be a partition. The Young diagram associated to λ is the set of boxes with integer coordinates (i,j) with $1 \leq i \leq \lambda_j$. For a box $s \in \lambda$, we denote by $a(s)$ (*armlength*), resp. $l(s)$ (*leglength*), the number of boxes in λ lying strictly to the right of (resp. strictly above) s. Here is an example for the partition $(10, 9^3, 6, 3^2)$:



$$a(s) = 5$$
$$l(s) = 2$$

Figure 1. Notations for partitions.

If λ, μ are partitions, then we set $\langle \lambda, \mu \rangle = \sum_i \lambda'_i \mu'_i$, where λ', μ' are the conjugate partitions of λ, μ.

Plethystic operators. Consider the space $K_g[[z,T]]$ of power series in the variables z, T. For $l \geq 1$, we define the l-th Adams operator ψ_l as the $\mathbb{Q}$-algebra map

$$\psi_l : K_g[[z,T]] \to K_g[[z,T]], \qquad \alpha_i \mapsto \alpha_i^l,\ z \mapsto z^l,\ T \mapsto T^l.$$

Set $K_g[[z,T]]^+ = zK_g[[z,T]] + TK_g[[z,T]]$. The plethystic exponential and logarithm functions are inverse maps

$$\mathrm{Exp} : K_g[[z,T]]^+ \longrightarrow 1 + K_g[[z,T]]^+, \qquad \mathrm{Log} : 1 + K_g[[z,T]]^+ \longrightarrow K_g[[z,T]]^+$$

respectively defined by

$$\mathrm{Exp}(f) = \exp\left(\sum_k \frac{1}{k}\psi_k(f)\right), \qquad \mathrm{Log}(f) = \sum_{k\geq 1} \frac{\mu(k)}{k}\psi_k\left(\log(f)\right),$$

where μ stands for the Mőbius function. These operators satisfy the usual properties, i.e., $\mathrm{Exp}(f+g) = \mathrm{Exp}(f)\mathrm{Exp}(g)$ and $\mathrm{Log}(fg) = \mathrm{Log}(f) + \mathrm{Log}(g)$. We refer, e.g., to [Moz06, §2] for more on these plethystic operators. Observe that $\mathrm{Exp}(z) = 1/(1-z)$.

Zeta function of the curve. We will need several versions of the zeta function of a curve. Recall that the zeta function is defined as

$$\zeta_X(z) = \exp\left(\sum_{k\geq 1} |X(\mathbb{F}_{q^k})| \frac{z^k}{k}\right) = \frac{\prod_i (1-\sigma_i z)}{(1-z)(1-qz)}.$$

This can be nicely expressed in terms of the plethystic exponential. Namely, set

$$\zeta(z) = \mathrm{Exp}\left((1 - \sum_i \alpha_i + q)z\right) = \frac{\prod_i (1-\alpha_i z)}{(1-z)(1-qz)}. \tag{1.1}$$

Then $\zeta_X(z) = \zeta(z)(\sigma_X)$. We will also need the following variants of $\zeta(z)$:

$$\widetilde{\zeta}(z) = z^{1-g}\zeta(z),$$

$$\zeta^*(q^{-u}z^v) = \begin{cases} \zeta(q^{-u}z^v) & \text{if } (u,v) \notin \{(1,0),(0,0)\}, \\ \prod_i (1-\alpha_i^{-1})/(1-q^{-1}) & \text{if } (u,v) = (1,0), \\ \prod_i (1-\alpha_i)/(1-q) & \text{if } (u,v) = (0,0). \end{cases} \tag{1.2}$$

Iterated residues. Let $f(z) \in K(z)$ be a rational function over some field K, and let $u \in K$. Write

$$f(z) = \sum_l a_l (1-u^{-1}z)^l$$

for the Laurent expansion of f around $z = u$, and set $\mathbb{R}\mathrm{es}_{z=u} f(z) = a_{-1} \in K$. This notation is a little bit nonstandard as $f(z)$ is a function rather than a differential form. We may apply this to a function $g(z_k,\ldots,z_1) \in \mathbb{C}(z_k,\ldots,z_1)$, viewed as an element of $\mathbb{C}(z_{k-1},\ldots,z_1)(z_k)$ and $u \in \mathbb{C}(z_{k-1},\ldots,z_1)$. In that case, $\mathbb{R}\mathrm{es}_{z_k=u} g(z_k,\ldots,z_1) \in \mathbb{C}(z_{k-1},\ldots,z_1)$. Now let $f(z_n,\ldots,z_1) \in \mathbb{C}(z_n,\ldots,z_1)$ be a rational function in variables $z_n,\ldots,z_1$, and let $(u_2,\ldots,u_n) \in \mathbb{C}^{n-1}$ be arbitrary complex numbers. We define the iterated residue of f along the collection of hyperplanes

$$\frac{z_n}{z_{n-1}} = u_n, \qquad \frac{z_{n-1}}{z_{n-2}} = u_{n-1}, \qquad \ldots, \frac{z_2}{z_1} = u_2$$

as follows:

$$\mathbb{R}\mathrm{es}_{\frac{z_n}{z_{n-1}}=u_n,\ldots,\frac{z_2}{1}=u_2} f(z_n,\ldots,z_1) = \mathbb{R}\mathrm{es}_{z_2=z_1u_2} \circ \cdots \circ \mathbb{R}\mathrm{es}_{z_n=z_{n-1}u_n} f(z_n,\ldots,z_1).$$

The result is an element of $\mathbb{C}(z_1)$. The result in general depends on the order in which the residues are taken. We will write $\mathrm{Res}_{z=u} f(z)dz$ for the (usual) residue of the differential form $f(z)dz$ at $z = u$.

We are now ready to introduce the ingredients entering in the explicit computation of $A_{g,r,d}$. Let $\lambda = (1^{r_1} 2^{r_2} \dots t^{r_t})$ be a partition. Let us set

$$J_\lambda(z) = \prod_{s \in \lambda} \zeta^*_X(q^{-1-l(s)} z^{a(s)}).$$

Next, write $n = l(\lambda) = \sum_i r_i$, and

$$r_{<i} = \sum_{k<i} r_k, \qquad r_{>i} = \sum_{k>i} r_k, \qquad r_{[i,j]} = \sum_{k=i}^{j} r_k,$$

and consider the multi-variable rational function

$$L(z_n, \dots, z_1) = \frac{1}{\prod_{i<j} \widetilde{\zeta}\left(\frac{z_i}{z_j}\right)} \sum_{\sigma \in \mathfrak{S}_n} \sigma \left\{ \prod_{i<j} \widetilde{\zeta}\left(\frac{z_i}{z_j}\right) \cdot \frac{1}{\prod_{i<n} \left(1 - q\frac{z_{i+1}}{z_i}\right)} \cdot \frac{1}{1 - z_1} \right\}.$$

Denote by $\mathbb{R}\mathrm{es}_\lambda$ the operator of successively taking the iterated residue along

$$\begin{array}{llll} \dfrac{z_n}{z_{n-1}} = q^{-1}, & \dfrac{z_{n-1}}{z_{n-2}} = q^{-1}, & \cdots, & \dfrac{z_{2+r_{<t}}}{z_{1+r_{<t}}} = q^{-1}, \\ \vdots & \vdots & \vdots & \\ \dfrac{z_{r_1}}{z_{r_1-1}} = q^{-1}, & \dfrac{z_{r_1-1}}{z_{r_1-2}} = q^{-1}, & \cdots, & \dfrac{z_2}{z_1} = q^{-1}. \end{array}$$

The result of such an operation is a function of $z_{1+r_{<t}}, \dots, z_{1+r_{<i}}, \dots, z_1$. Put

$$\widetilde{H}_\lambda(z_{1+r_{<t}}, \dots, z_{1+r_{<i}}, \dots, z_1) = \mathbb{R}\mathrm{es}_\lambda L(z_n, \dots, z_1)$$

and finally

$$H_\lambda(z) = \widetilde{H}_\lambda(z^t q^{-r_{<t}}, \dots, z^i q^{-r_{<i}}, \dots, z).$$

Note that if $r_i = 0$ for some i, then the function $\widetilde{H}_\lambda$ is independent of its i-th argument.

THEOREM 1.6. *Define rational functions $A_{g,r}(z) \in K_g(z)$ by the relation*

$$\sum_{r \geq 1} A_{g,r}(z) T^r = (q-1) \mathrm{Log} \left(\sum_\lambda q^{(g-1)\langle \lambda, \lambda \rangle} J_\lambda(z) H_\lambda(z) T^{|\lambda|} \right). \tag{1.3}$$

Then for any $d \in \mathbb{Z}$, we have

$$A_{g,r,d} = - \sum_{\xi \in \mu_r} \xi^{-d} \mathrm{Res}_{z=\xi} \left(A_{g,r}(z) \frac{dz}{z} \right),$$

where μ_r stands for the set of r-th roots of unity.

Examples. We list below the polynomials $A_{g,r,d}$ for $r \leq 2$:

$$A_{g,1,d} = \prod_{i=1}^{2g}(1-\alpha_i),$$
$$A_{g,2,d} = \prod_{i=1}^{2g}(1-\alpha_i) \cdot \left(\frac{\prod_i (1-q\alpha_i)}{(q-1)(q^2-1)} - \frac{\prod_i(1+\alpha_i)}{4(1+q)} \right.$$
$$\left. + \frac{\prod_i(1-\alpha_i)}{2(q-1)} \left[\frac{1}{2} - \frac{1}{q-1} - \sum_i \frac{1}{1-\alpha_i} \right] \right).$$

Conjecture 1.7. *The polynomial $A_{g,r,d}$ does not depend on d.*

In view of Theorem 1.6, this conjecture may be recast in purely combinatorial terms as follows:

Conjecture 1.8. *The rational function $A_{g,r}(z)$ is regular at nontrivial r-th roots of unity.*

It follows from the proof of Theorem 1.6 that $A_{g,r}(z)$ is regular outside of μ_r and has at most simple poles. Thus the above conjecture says that $(1-z)A_{g,r}(z)$ belongs to $K_g[z]$, in which case we would simply have

$$A_{g,r,d} = [(1-z)A_{g,r}(z)]_{|z=1} \qquad \forall\, d.$$

Conjecture 1.7 is, by Theorems 1.2 and 1.6, essentially equivalent to a conjecture by Hausel and Thaddeus (see [Hau05, Conj. 3.2]) claiming that the motive of $\mathrm{Higgs}^{\mathrm{st}}_{r,d}$ is independent of d. As supporting evidence, we prove Conjecture 1.7 by direct computation when r is prime; see Appendix C.

The constant term of $A_{g,r,d}$ can be described by a generating series formula similar to (1.3):

Corollary 1.9. *The values of $A_{g,r,d}(0)$ are computed from the following generating series. Set*

$$\sum_r A^0_{g,r}(z)T^r = -\mathrm{Log}\left(\sum_\lambda z^{(g-1)\langle\lambda,\lambda\rangle - l(\lambda)} K_\lambda(z) T^{|\lambda|} \right), \tag{1.4}$$

where for $\lambda = (1^{r_1}, 2^{r_2}, \ldots)$, we have set

$$K_\lambda(z) = \frac{1}{\prod_i \prod_{j=1}^{r_i}(1-z^{-j})}.$$

Then

$$A_{g,r,d}(0) = \sum_{\xi \in \mu_r} \xi^{-d} \mathrm{Res}_{z=\xi}\left(A^0_{g,r}(z)\frac{dz}{z} \right).$$

Examples. We list below the values $A_{g,r,d}(0)$ for $r \leq 4$:

$$A_{g,1,d}(0) = 1, \qquad A_{g,2,d}(0) = \binom{g}{1}, \qquad A_{g,3,d} = 4\binom{g}{2} + \binom{g}{1},$$
$$A_{g,4,d} = 32\binom{g}{3} + 20\binom{g}{2} + \binom{g}{1}.$$

It is easy to see that for any $r \geq 1$, $A_{g,r,d}(0)$ is a polynomial in g of degree $r-1$.

Remarks. (i) Just like $A_{g,r}(z)$, the rational function $A^0_{g,r}(z)$ has at most simple poles at r-th roots of unity and is conjectured to be regular outside of $z=1$.

(ii) Let Σ_g be the quiver with one vertex and g loops, and let $A_{\Sigma_g,r} \in \mathbb{N}[q]$ be Kac's A-polynomial counting geometrically indecomposable representation of Σ_g over $\mathbf{F}_q$ of dimension r (see [Kac82]). It is conjectured that $A_{g,r,d}(0) = A_{\Sigma_g,r}(1)$ — this, for instance, would follow from the main conjecture in [HRV08]. However, just as our formula (1.3) slightly differs from that conjectured in [HRV08], so does (1.4) slightly differ from Hua's formula computing $A_{\Sigma_g,r}(1)$; see [Hua00, Th. 4.9]. (In the latter case, the difference is only in the extra term $-l(\lambda)$ in (1.4) !)

Theorem 1.6 is proved in Section 5, as is Corollary 1.9.

1.5. In Sections 7 and 8, we point towards two types of possible extensions of the above results: counting indecomposable vector bundles equipped with quasi-parabolic structures along some (fixed) divisor D of X, and counting indecomposables with a prescribed Harder-Narasimhan polygon. We also propose an analog, in our context, of the famous conjecture of Kac (proved by Hausel — see [Hau10]) relating the constant terms of the Kac polynomials associated to a quiver Q with no edge loop to the dimensions of the root spaces of the corresponding Kac-Moody algebra.

2. Stacks of pairs and the Harder-Narasimhan truncation

2.1. We fix a smooth projective curve X over a finite field $\mathbf{F}_q$ as in 1.1. In this section we reduce the problem of counting geometrically indecomposable coherent sheaves on X to the computation of the volume of certain stacks, the *truncated stacks of pairs*. The truncation is defined in terms of the Harder-Narasimhan filtration. Let us first recall some standard notation. Denote by $\mathrm{Coh}(X)$ the category of coherent sheaves on X. By the class of a sheaf $\mathcal{F}$ we will mean the pair

$$[\mathcal{F}] = (\mathrm{rank}(\mathcal{F}), \deg(\mathcal{F})) \in (\mathbb{Z}^2)^+ = \{(r,d) \in \mathbb{N} \times \mathbb{Z} \mid d > 0 \; if \; r = 0\}.$$

If $\alpha = (r, d)$, then we put $\mathrm{rk}(\alpha) = r, \deg(\alpha) = d$. Let $K_0(\mathrm{Coh}(X))$ stand for the Grothendieck group of the category $\mathrm{Coh}(X)$, and let $\langle\ ,\ \rangle : K_0(X) \otimes_{\mathbb{Z}} K_0(X) \to \mathbb{Z}$ be the Euler form, defined by $\langle \mathcal{F}, \mathcal{G} \rangle = \dim \mathrm{Hom}(\mathcal{F}, \mathcal{G}) - \dim \mathrm{Ext}^1(\mathcal{F}, \mathcal{G})$. The form $\langle \mathcal{F}, \mathcal{G} \rangle$ only depends on the classes $[\mathcal{F}], [\mathcal{G}]$ and is given by

$$\langle (r, d), (r', d') \rangle = (1 - g)rr' + (rd' - r'd).$$

We let $(\ ,\ )$ stand for the symetrized Euler form, i.e., $(\alpha, \beta) = \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle$, so that

$$((r, d), (r', d')) = 2(1 - g)rr'.$$

For $\alpha \in (\mathbb{Z}^2)^+$, we denote by $\mathrm{Coh}_\alpha(X)$ the subcategory of coherent sheaves of class α. We consider the standard slope function $\mu(\mathcal{F}) = \deg(\mathcal{F})/\mathrm{rank}(\mathcal{F})$, and for any $\nu \in \mathbb{Q} \sqcup \{\infty\}$, we denote by $\mathrm{Coh}^{(\nu)}(X)$ the (abelian) subcategory of $\mathrm{Coh}(X)$ consisting of semistable sheaves of slope ν. From now on, unless there is some risk of confusion, we will drop the symbol X from the notation.

More generally, given a collection $\alpha_1 = (r_1, d_1), \ldots, \alpha_t = (r_t, d_t)$ of elements of $(\mathbb{Z}^2)^+$ satisfying $\mu(\alpha_1) > \cdots > \mu(\alpha_t)$, we denote by $\mathrm{Coh}^{(\alpha_1, \ldots, \alpha_t)}$ the full subcategory of $\mathrm{Coh}_{\alpha_1 + \cdots + \alpha_t}$ consisting of objects $\mathcal{F}$ with a Harder-Narasimhan filtration $\mathcal{F}_1 \subset \cdots \subset \mathcal{F}_t = \mathcal{F}$ satisfying $[\mathcal{F}_i/\mathcal{F}_{i-1}] = \alpha_i$ for all i. The Harder-Narasimhan filtration is stable under field extensions; see [HL10, Th. 1.3.7]. Observe that the number of isomorphism classes of coherent sheaves in $\mathrm{Coh}^{(\alpha_1, \ldots, \alpha_t)}$ is finite, since the number of semistable sheaves of any given class is finite, and $\dim \mathrm{Ext}^1(\mathcal{H}, \mathcal{G}) < \infty$ for any $(\mathcal{H}, \mathcal{G})$.

Let $\mathrm{Coh}^{\geq \nu}$ be the subcategory of sheaves $\mathcal{F}$ whose Harder-Narasimhan filtration

$$\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_t = \mathcal{F}$$

satisfies

$$\mu(\mathcal{F}_1) > \mu(\mathcal{F}_2/\mathcal{F}_1) > \cdots > \mu(\mathcal{F}/\mathcal{F}_{t-1}) \geq \nu,$$

i.e., sheaves $\mathcal{F}$ for which $\mu_{\min}(\mathcal{F}) \geq \nu$. We will write $\mathrm{Coh}_\alpha^{\geq \nu}$ for the full subcategory of $\mathrm{Coh}^{\geq \nu}$ consisting of objects belonging to Coh_α. The categories $\mathrm{Coh}^{\leq \nu}$ are defined in a similar fashion, using the condition $\mu_{\max}(\mathcal{F}) \leq \nu$. We will mostly be interested in the category $\mathrm{Coh}^{\geq 0}$. A sheaf belongs to $\mathrm{Coh}^{\geq 0}$ if and only if it belongs to some $\mathrm{Coh}^{(\alpha_1, \ldots, \alpha_t)}$ with $\mu(\alpha_1) > \cdots > \mu(\alpha_t) \geq 0$. For any given α, let us put

$$D(\alpha) = \{(\alpha_1, \ldots, \alpha_t) \mid \alpha = \sum \alpha_i,\ \mu(\alpha_1) > \cdots > \mu(\alpha_t) \geq 0\}.$$

The set $D(\alpha)$ is finite; therefore for any given α, the number of isomorphism classes of coherent sheaves in $\mathrm{Coh}^{\geq 0}$ of class α is finite.

The full subcategory $\mathrm{Coh}^{\geq 0}$ is stable under quotients and extensions. In particular, an object of $\mathrm{Coh}^{\geq 0}$ is isomorphic to a direct sum of indecomposable objects, all of which belong to $\mathrm{Coh}^{\geq 0}$. The category $\mathrm{Coh}^{\geq 0}$ is also preserved in a natural sense by extension of the base field. The relevance of $\mathrm{Coh}^{\geq 0}$

to our problem of computing the number of indecomposables stems from the following observation. Let us denote by $\mathcal{A}^{\geq 0}_{\alpha}(X)$ the number of geometrically indecomposable coherent sheaves in $\mathrm{Coh}^{\geq 0}$ of class α.

PROPOSITION 2.1. *Assume that $d > (g-1)r(r-1)$. Then any indecomposable vector bundle of rank r and degree d lies in* $\mathrm{Coh}^{\geq 0}$, *i.e.*,

$$\mathcal{A}^{\geq 0}_{r,d}(X) = \mathcal{A}_{r,d}(X).$$

Proof. Let $\mathcal{F}$ be any coherent sheaf of rank r and degree d, and let us denote by $\alpha_1 = (r_1, d_1), \ldots, \alpha_t = (r_t, d_t)$ its Harder-Narasimhan type, that is, $\mathcal{F} \in \mathrm{Coh}^{(\alpha_1,\ldots,\alpha_t)}$. By Serre duality, any sequence

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C} \longrightarrow 0$$

with $\mathcal{A} \in \mathrm{Coh}^{\geq \nu}$, $\mathcal{C} \in \mathrm{Coh}^{\leq \nu'}$ and $\nu - \nu' > 2g-2$ splits since then $\mathrm{Ext}^1(\mathcal{C}, \mathcal{A}) = \mathrm{Hom}(\mathcal{A}, \mathcal{C} \otimes \Omega_X) = 0$. In particular, if $\mu(\alpha_i) - \mu(\alpha_{i+1}) > 2g-2$ for some i, then $\mathcal{F}$ is decomposable. The proposition follows. □

2.2. Let us denote by $\mathrm{End}^{\mathrm{nil}}(\mathcal{F}) \subset \mathrm{End}(\mathcal{F})$ the set of nilpotent endomorphisms of a coherent sheaf $\mathcal{F}$. We consider the stacks

$$\mathbf{Nil}_{\alpha} = \langle (\mathcal{F}, \theta) \mid \mathcal{F} \in \mathrm{Coh}_{\alpha},\ \theta \in \mathrm{End}^{\mathrm{nil}}(\mathcal{F}) \rangle$$

where an isomorphism $j : (\mathcal{F}, \theta) \stackrel{\sim}{\to} (\mathcal{G}, \phi)$ is an isomorphism $j : \mathcal{F} \stackrel{\sim}{\to} \mathcal{G}$ such that $j\theta = \phi j$.

The stack $\mathbf{Nil}_{\alpha}$ is of infinite volume as soon as $\mathrm{rk}(\alpha) > 0$, i.e., the sum

$$\sum_{(\mathcal{F},\theta) \in \mathrm{Obj}(\mathbf{Nil}_{\alpha})/\sim} \frac{1}{|\mathrm{Aut}((\mathcal{F},\theta))|} = \sum_{\mathcal{F} \in \mathrm{Obj}(\mathrm{Coh}_{\alpha})/\sim} \frac{|\mathrm{End}^{\mathrm{nil}}(\mathcal{F})|}{|\mathrm{Aut}(\mathcal{F})|}$$

diverges. However, the full substack

$$\mathbf{Nil}^{\geq 0}_{\alpha} = \langle (\mathcal{F}, \theta) \in \mathbf{Nil}_{\alpha} \mid \mathcal{F} \in \mathrm{Coh}^{\geq 0}_{\alpha} \rangle$$

is of finite volume

$$\mathrm{vol}(\mathbf{Nil}^{\geq 0}_{\alpha}) = \sum_{\mathcal{F} \in \mathrm{Obj}(\mathrm{Coh}^{\geq 0}_{\alpha})/\sim} \frac{|\mathrm{End}^{\mathrm{nil}}(\mathcal{F})|}{|\mathrm{Aut}(\mathcal{F})|} < \infty$$

because there are only finitely many sheaves in $\mathrm{Coh}^{\geq 0}_{\alpha}$ up to isomorphism. Observe that $\mathbf{Nil}^{\geq 0}_{\alpha}$ is empty if $\deg(\alpha) < 0$.

The relation between the stack $\mathbf{Nil}^{\geq 0}_{\alpha}$ and the problem of counting indecomposable coherent sheaves is described by Propositions 2.1 and 2.2 below.

PROPOSITION 2.2. *The following relation holds in the ring* $\mathbb{Q}[[z^{(1,0)}, z^{(0,1)}]]$:

$$\sum_{\alpha} \mathrm{vol}(\mathbf{Nil}^{\geq 0}_{\alpha}) z^{\alpha} = \exp\left(\sum_{l \geq 1} \frac{1}{l} \sum_{\alpha} \frac{\mathcal{A}^{\geq 0}_{\alpha}(X \otimes \mathbf{F}_{q^l})}{q^l - 1} z^{l\alpha} \right).$$

Proof. We begin by collecting a few standard results on indecomposable coherent sheaves (see, e.g., [Ati56, Th. 2, Lemmata 6,7]).

LEMMA 2.3. *The following statements hold*:

(i) $\mathrm{Coh}(X)$ *is a Krull-Schmidt category*; *i.e.*, *every coherent sheaf* M *is isomorphic to a direct sum*

$$M = M_1^{\oplus n_1} \oplus \cdots \oplus M_s^{\oplus n_s},$$

where the M_i *are distinct indecomposables. The* (M_i, n_i) *are uniquely determined up to permutation.*

(ii) *Let* M *be indecomposable. Then* $k_M := \mathrm{End}(M)/\mathrm{rad}(\mathrm{End}(M))$ *is a field.*

(iii) *Let* M, M' *be distinct indecomposables. Then any composed map* $M \to M' \to M$ *lies in* $\mathrm{rad}(\mathrm{End}(M))$.

LEMMA 2.4. *Let* $M_1, \ldots, M_s$ *be distinct indecomposables and* $n_1, \ldots, n_s \in \mathbb{N}$. *Put* $M = \bigoplus_i M_i^{\oplus n_i}$. *Then*

$$\mathrm{rad}(\mathrm{End}(M)) = \bigoplus_{i \neq j} \mathrm{Hom}(M_i^{\oplus n_i}, M_j^{\oplus n_j}) \oplus \bigoplus_i \mathrm{rad}(\mathrm{End}(M_i))^{\oplus n_i}. \tag{2.1}$$

Proof. Let U denote the right-hand side of (2.1). It follows from Lemma 2.3(ii) that U is a nilpotent ideal in $\mathrm{End}(M)$ and $\mathrm{End}(M)/U \simeq \prod_i \mathfrak{gl}(n_i, k_{M_i})$ is a semisimple algebra. □

COROLLARY 2.5. *Let* M *be as in Lemma* 2.4. *Then we have*

$$\frac{|\mathrm{End}^{\,nil}(M)|}{|\mathrm{Aut}(M)|} = \prod_i \frac{|k_{M_i}|^{n_i(n_i-1)}}{|\mathrm{GL}(n_i, k_{M_i})|}. \tag{2.2}$$

Proof. Put $A = \mathrm{End}(M)$, and denote by $p : A \to A/\mathrm{rad}(A) \simeq \prod_i \mathfrak{gl}(n_i, k_{M_i})$ the natural projection. Then $\mathrm{End}^{\mathrm{nil}}(M) = p^{-1}(\prod_i \mathcal{N}_{n_i, k_{M_i}})$, where $\mathcal{N}_{n,k} \subset \mathfrak{gl}(n,k)$ denotes the nilpotent cone. Similarly, $\mathrm{Aut}(M) = p^{-1}(\prod_i \mathrm{GL}(n_i, k_{M_i}))$. The result now follows from Lemma 2.4 and the well-known formula $|\mathcal{N}_{n,k}| = |k|^{n(n-1)}$ (see [FH58, Th. 1]). □

We now start the proof of Proposition 2.2. We first introduce some useful notation. Let $\mathrm{Ind}^{\geq 0}$ stand for the set of isomorphism classes of indecomposables in $\mathrm{Coh}^{\geq 0}$. We choose a representative M_ι in each class ι and set $l_\iota = [k_{M_\iota} : \mathbf{F}_q]$. We have an obvious partition $\mathrm{Ind}^{\geq 0} = \bigsqcup_\alpha \mathrm{Ind}^{\geq 0}_\alpha$ according to the class $\alpha \in (\mathbb{Z}^2)^+$. Note that $\mathrm{Ind}^{\geq 0}_\alpha$ is empty if $\alpha \notin \mathbb{N}^2$. By Lemma 2.3(i) the set of isoclasses of objects in $\mathrm{Coh}^{\geq 0}$ is

$$\mathrm{Obj}(\mathrm{Coh}^{\geq 0})/\!\sim\; = \Big\{ \bigoplus_{\iota \in \mathrm{Ind}^{\geq 0}} M_\iota^{\oplus n_\iota} \mid n_\iota = 0 \text{ for almost all } \iota \Big\}.$$

Let $\Theta = \{(n_\iota) \in \mathbb{N}^{\mathrm{Ind}^{\geq 0}} \mid n_\iota = 0 \text{ for almost all } \iota\}$. Then, by Corollary 2.5,

$$\begin{aligned}\sum_\alpha \mathrm{vol}(\mathbf{Nil}_\alpha^{\geq 0}) z^\alpha &= \sum_{M \in \mathrm{Obj}(\mathrm{Coh}^{\geq 0})/\sim} \frac{|\mathrm{End}^{\mathrm{nil}}(M)|}{|\mathrm{Aut}(M)|} z^{[M]} \\ &= \sum_{(n_\iota) \in \Theta} \left\{ \prod_\iota \frac{q^{-l_\iota n_\iota}}{(1-q^{-l_\iota n_\iota}) \cdots (1-q^{-l_\iota})} z^{\sum_\iota n_\iota [M_\iota]} \right\} \\ &= \prod_{\iota \in \mathrm{Ind}^{\geq 0}} \left(\sum_{n \geq 0} \frac{q^{-l_\iota n}}{(1-q^{-l_\iota n}) \cdots (1-q^{-l_\iota})} z^{n[M_\iota]} \right).\end{aligned}$$

Note that the infinite product converges in the ring $\mathbb{Q}[[z^{(0,1)}, z^{(1,0)}]]$ because any element in $\mathbb{N}^2$ may be written in only finitely many different ways as a sum $\sum_\iota n_\iota [M_\iota]$. (Recall that each $\mathrm{Ind}_\alpha^{\geq 0}$ is of finite cardinality.)

Applying Heine's formula

$$\sum_{n \geq 0} \frac{u^n}{(1-v^n) \cdots (1-v)} = \exp\left(\sum_{l \geq 1} \frac{u^l}{l(1-v^l)} \right)$$

we get

$$\sum_\alpha \mathrm{vol}(\mathbf{Nil}_\alpha^{\geq 0}) z^\alpha = \exp\left(\sum_{l \geq 1} \sum_{\iota \in \mathrm{Ind}^{\geq 0}} \frac{z^{l[M_\iota]}}{l(q^{l l_\iota} - 1)} \right).$$

To prove Proposition 2.2 it only remains to show the next lemma:

Lemma 2.6. *The following relation holds*:

$$\sum_{l \geq 1} \sum_{\iota \in \mathrm{Ind}^{\geq 0}} \frac{z^{l[M_\iota]}}{l(q^{l l_\iota} - 1)} = \sum_{l \geq 1} \sum_\alpha \frac{\mathcal{A}_\alpha^{\geq 0}(X \otimes \mathbf{F}_{q^l})}{l(q^l - 1)} z^{l\alpha}. \tag{2.3}$$

Proof. Let us denote by $\mathrm{Ind}_{\alpha,l}^{\geq 0}$ the set of elements $\iota \in \mathrm{Ind}_\alpha^{\geq 0}$ satisfying $l_\iota = l$. If $\iota \in \mathrm{Ind}_{\alpha,l}^{\geq 0}$, then $M_\iota \otimes \overline{\mathbb{F}_q}$ splits as a direct sum of l geometrically indecomposable coherent sheaves $N_1, \ldots, N_l$ by Lemma 2.4.

The group $\mathbf{G} := \mathrm{Gal}(\overline{\mathbf{F}_q}/\mathbf{F}_q)$ acts naturally on the set of isoclasses of indecomposable coherent sheaves on $\overline{X} = X \otimes \overline{\mathbf{F}_q}$ of class α, preserving the subset of sheaves in $\mathrm{Coh}^{\geq 0}$. We denote by $(\sigma, L) \mapsto L^\sigma$ this action. For $\sigma \in \mathbf{G}$ and $M, M' \in \mathrm{Coh}(\overline{X})$, we write $M \simeq_\sigma M'$ if there exists a σ-semilinear isomorphism $M \xrightarrow{\sim} M'$. Let us fix an indecomposable coherent sheaf $N \in \mathrm{Coh}(\overline{X})$ and define an equivalence relation on $\mathbf{G}$ as follows: $\sigma \sim \tau$ if $N^\tau \simeq_{\sigma\tau^{-1}} N^\sigma$. The equivalence class $\mathbf{H}_N$ containing 1 is a subgroup of $\mathbf{G}$, and the other classes are the (right or left) $\mathbf{H}_N$-translates. Because N is coherent, the coset $\mathbf{G}/\mathbf{H}_N$ is finite, and we will call *Galois orbit* (of N) a collection of sheaves $\{N^{\sigma_1}, \ldots, N^{\sigma_r}\}$ with $\sigma_1, \ldots, \sigma_r$ a set of representatives of $\mathbf{G}/\mathbf{H}_N$. This construction yields a partition of the set of isomorphism classes of indecomposable coherent sheaves in $\mathrm{Coh}^{\geq 0}_{\alpha}(\overline{X})$ into Galois orbits. Let $\xi_{\alpha,d}$ stand for the set of Galois orbits of size d. We claim that

$$|\xi_{\alpha,d}| = |\mathrm{Ind}^{\geq 0}_{d\alpha,d}| \tag{2.4}$$

and

$$\mathcal{A}^{\geq 0}_{\alpha}(X \otimes \mathbf{F}_{q^l}) = \sum_{d|l} d|\xi_{\alpha,d}|. \tag{2.5}$$

Indeed, let $\{N_1, \ldots, N_d\}$ be a Galois orbit in $\xi_{\alpha,d}$, and set $M = \bigoplus_i N_i$. For any $\sigma \in \mathbf{G}$, there is a σ-semilinear isomorphism $f_\sigma : M \xrightarrow{\sim} M^\sigma$. The automorphism group $\mathrm{Aut}(M)$ is a connected algebraic group defined over $\overline{\mathbf{F}_q}$ (it is an open subset of the $\overline{\mathbf{F}_q}$-vector space $\mathrm{End}(M)$) which is solvable because M is a direct sum of nonisomorphic indecomposable coherent sheaves (see Lemma 2.3(ii)). By Steinberg's and Grothendieck's theorems (see [Ser94, III.2.4, Cor. 3] and [Spr66, Th. 3.5]), the Galois cohomology groups $H^i(\mathbf{G}, \mathrm{Aut}(M))$ are trivial (resp. neutral) for $i = 1, 2$. It follows that there exists a unique (up to isomorphism) effective descent data $(\tilde{f}_\sigma : M \xrightarrow{\sim} M^\sigma)_{\sigma\in\mathbf{G}}$ and thus a unique (up to isomorphism) object $M_0 \in \mathrm{Coh}(X)$ such that $M \simeq M_0 \otimes \overline{\mathbf{F}_q}$. The sheaf M_0 is indecomposable, as M does not contain any proper submodule $M' \subset M$ satisfying $M' \simeq_\sigma M'^\sigma$ for all $\sigma \in \mathbf{G}$. Therefore $M_0 \in \mathrm{Ind}^{\geq 0}_{d\alpha,d}$. The map

$$\xi_{\alpha,d} \to \mathrm{Ind}^{\geq 0}_{d\alpha,\alpha}, \qquad \{N_1, \ldots, N_d\} \mapsto M_0$$

thus constructed is a bijection, and (2.4) follows. This also implies that $\mathrm{Ind}^{\geq 0}_{\alpha,d}$ is empty unless d divides α.

For similar reasons, there is a bijection between isomorphism classes of geometrically indecomposable coherent sheaves of class α over $X \otimes \mathbf{F}_{q^l}$ and isomorphism classes of indecomposable coherent sheaves of class α over $\overline{X}$ satisfying $M \simeq_\sigma M^\sigma$ for all $\sigma \in \mathrm{Gal}(\overline{\mathbf{F}_q}/\mathbf{F}_{q^l})$. Equation (2.5) follows. Using (2.4) and (2.5), we compute

$$\begin{aligned}\sum_{l\geq 1}\sum_{\iota\in\mathrm{Ind}^{\geq 0}} \frac{z^{l[M_\iota]}}{l(q^{ll_\iota}-1)} &= \sum_{l\geq 1}\sum_{\beta}\sum_{d|\beta} \frac{|\mathrm{Ind}^{\geq 0}_{\beta,d}|}{l(q^{dl}-1)} z^{l\beta} = \sum_{l\geq 1}\sum_{\alpha}\sum_{d\geq 1} \frac{|\mathrm{Ind}^{\geq 0}_{d\alpha,d}|}{l(q^{dl}-1)} z^{dl\alpha}\\ &= \sum_{l\geq 1}\sum_{\alpha}\sum_{d\geq 1} \frac{|\xi_{\alpha,d}|}{l(q^{dl}-1)} z^{ld\alpha} = \sum_{l'\geq 1}\sum_{\alpha}\sum_{d|l'} \frac{d|\xi_{\alpha,d}|}{l'(q^{l'}-1)} z^{l'\alpha}\\ &= \sum_{l'\geq 1}\sum_{\alpha} \frac{\mathcal{A}^{\geq 0}_{\alpha}(X\otimes \mathbf{F}_{q^{l'}})}{l(q^{l'}-1)} z^{l'\alpha}\end{aligned}$$

as wanted. Lemma 2.6 and Proposition 2.2 are proved. □

Remark. In view of the definition of the plethystic exponential, Proposition 2.2 may heuristically be interpreted as the equality

$$\sum_\alpha \mathrm{vol}(\mathbf{Nil}_\alpha^{\geq 0}) z^\alpha = \mathrm{Exp}\left(\sum_\alpha \frac{\mathcal{A}_\alpha^{\geq 0}(X)}{q-1} z^\alpha\right).$$

Of course, this only makes sense *a posteriori*, once we know that $\mathcal{A}_\alpha^{\geq 0}(X)$ is a polynomial in the Weil numbers of X.

3. **Jordan stratification**

3.1. By Propositions 2.1 and 2.2, computing $\mathcal{A}_\alpha(X)$ (for all α and for all base field extensions of X) amounts to computing the volumes of the stacks $\mathbf{Nil}_\alpha^{\geq 0}$. We will achieve this by first stratifying $\mathbf{Nil}_\alpha$ according to Jordan types. The computation of the volume of each piece will be carried out in Section 5 Let $(\mathcal{F}, \theta) \in \mathbf{Nil}_\alpha$, and let s be such that $\theta^s = 0, \theta^{s-1} \neq 0$. There are two natural filtrations

$$\mathrm{Ker}(\theta) \subset \mathrm{Ker}(\theta^2) \subset \cdots \subset \mathrm{Ker}(\theta^s) = \mathcal{F},$$
$$\mathrm{Im}(\theta^{s-1}) \subset \mathrm{Im}(\theta^{s-2}) \subset \cdots \subset \mathrm{Im}(\theta^0) = \mathcal{F}$$

and a sequence of epimorphisms induced by θ

$$\mathcal{F}/\mathrm{Im}(\theta) \xrightarrow{d_1} \mathrm{Im}(\theta)/\mathrm{Im}(\theta^2) \xrightarrow{d_2} \cdots \xrightarrow{d_{s-1}} \mathrm{Im}(\theta^{s-1}). \tag{3.1}$$

We define the Jordan type of the pair $(\mathcal{F}, \theta)$ as follows:

$$J(\mathcal{F}, \theta) = (\alpha_1, \ldots, \alpha_s),$$

where $\alpha_i = [\mathrm{Ker}(d_i)]$. Note that we have

$$\begin{aligned} &[\mathrm{Im}(\theta^{i-1})] - [\mathrm{Im}(\theta^i)] = \alpha_i + \alpha_{i+1} + \cdots + \alpha_s, \qquad (i = 1, \ldots, s), \\ &\sum_i i\alpha_i = \alpha \end{aligned} \tag{3.2}$$

and that some of the α_i may be zero, but $\alpha_s \neq 0$.

The Jordan type of a pair $(\mathcal{F}, \theta)$ contains more information than the Jordan type (in the usual sense) of θ over the generic point of X, as it also keeps track of the degrees of the kernels of powers of θ. We put

$$J^{\mathrm{gen}}(\mathcal{F}, \theta) = (\mathrm{rk}(\alpha_1), \ldots, \mathrm{rk}(\alpha_t)),$$

where t is the largest index for which $\mathrm{rk}(\alpha_t) \neq 0$.

It may be helpful to visualize the pair $(\mathcal{F}, \theta)$ as a Young diagram. For instance, when $\theta^3 = 0$ we view $\mathcal{F}$ as shown in Figure 2, in which θ maps every box onto the box lying below it. Every region R which is saturated in the south and west directions corresponds to a canonical θ-stable subsheaf $\mathcal{F}_R$ of $\mathcal{F}$.

For instance, in Figure 2, the subsheaf $\mathrm{Ker}(\theta) + (\mathrm{Ker}(\theta^2) \cap \mathrm{Im}(\theta))$ corresponds to the region shown in Figure 3.

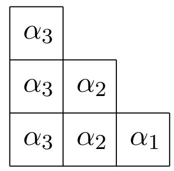


Figure 2. Jordan type of a nilpotent endomorphism.

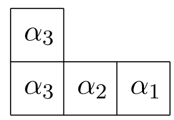


Figure 3. A canonical subsheaf.

For $\underline{\alpha} = (\alpha_1, \ldots, \alpha_r)$, we denote by $\mathbf{Nil}_{\underline{\alpha}}$ the stack consisting of pairs $(\mathcal{F}, \theta)$ with $J(\mathcal{F}, \theta) = \underline{\alpha}$. Hence we have a stratification

$$\mathbf{Nil}_\alpha = \bigsqcup_{|\underline{\alpha}|=\alpha} \mathbf{Nil}_{\underline{\alpha}},$$

where we have set $|\underline{\alpha}| = \sum_i i\alpha_i$.

We introduce several more stacks: $\mathbf{Coh}_\beta$ denotes the stack of coherent sheaves on X of class β; $\mathbf{Coh}_\beta^{\geq 0}$ is the full substack of $\mathbf{Coh}_\beta$ consisting of coherent sheaves which belong to $\mathrm{Coh}^{\geq 0}$; for $\underline{\beta} = (\beta_1, \ldots, \beta_s)$, we denote by $\widetilde{\mathbf{Coh}}_{\underline{\beta}}$ the stack whose objects are pairs $(\mathcal{H}, \mathcal{H}_\bullet)$, where $\mathcal{H}$ is a coherent sheaf on X of class $\beta_1 + \cdots + \beta_s$ and where $\mathcal{H}_\bullet$ is a filtration

$$\mathcal{H}_1 \subset \mathcal{H}_2 \subset \cdots \subset \mathcal{H}_s = \mathcal{H}$$

satisfying $[\mathcal{H}_i] = \beta_1 + \cdots + \beta_i$ for $i = 1, \ldots, s$; finally, we let $\widetilde{\mathbf{Coh}}_{\underline{\beta}}^{\geq 0}$ stand for the full substack of $\widetilde{\mathbf{Coh}}_{\underline{\beta}}$ consisting of pairs $(\mathcal{H}, \mathcal{H}_\bullet)$ with $\mathcal{H} \in \mathrm{Coh}^{\geq 0}$.

3.2. There is a natural functor

$$\pi_{\underline{\alpha}} : \mathbf{Nil}_{\underline{\alpha}} \to \mathbf{Coh}_{\alpha_1} \times \cdots \times \mathbf{Coh}_{\alpha_s}$$

sending a pair $(\mathcal{F}, \theta)$ to the tuple $(\mathcal{F}_1, \ldots, \mathcal{F}_s)$ where

$$\mathcal{F}_i = \mathrm{Ker}(d_i), \qquad i = 1, \ldots, s,$$

with $(d_1, \ldots, d_s)$ being defined as in (3.1).

The functor $\pi_{\underline{\alpha}}$ factors as the composition $\pi_{\underline{\alpha}} = \pi''_{\underline{\alpha}} \circ \pi'_{\underline{\alpha}}$ of the two functors $\pi'_{\underline{\alpha}} : \mathbf{Nil}_{\underline{\alpha}} \to \widetilde{\mathbf{Coh}}_{\underline{\alpha}}$ and $\pi''_{\underline{\alpha}} : \widetilde{\mathbf{Coh}}_{\underline{\alpha}} \to \mathbf{Coh}_{\alpha_1} \times \cdots \times \mathbf{Coh}_{\alpha_s}$ respectively defined by

$$\pi'_{\underline{\alpha}}(\mathcal{F}, \theta) = (\mathcal{H}, \mathcal{H}_\bullet),$$
$$\mathcal{H} = \mathcal{F}/\mathrm{Im}(\theta), \quad \mathcal{H}_1 = \mathrm{Ker}(d_1), \quad \mathcal{H}_2 = \mathrm{Ker}(d_2 \circ d_1), \ldots$$

and

$$\pi''_{\underline{\alpha}}(\mathcal{H},\mathcal{H}_\bullet)=(\mathcal{H}_1,\mathcal{H}_2/\mathcal{H}_1,\ldots,\mathcal{H}/\mathcal{H}_{s-1}).$$

Recall that $\langle\ ,\ \rangle$, resp. $(\ ,\ )$, stands for the Euler form, resp. symetrized Euler form (see Section 2.1). If $\phi:\mathcal{A}\to\mathcal{B}$ is a functor between groupoids and $\mathcal{B}'\subset\mathcal{B}$ is a full sub-groupoid, then $\phi^{-1}(\mathcal{B}')$ is by definition the full sub-groupoid of $\mathcal{A}$ whose objects satisfy the following condition: $\phi(A)\simeq B$ for some $B\in\mathcal{B}'$. This next proposition is crucial for us.

Proposition 3.1. *The following hold*:

(i) *for any* $(\mathcal{F}_1,\ldots,\mathcal{F}_s)\in\boldsymbol{Coh}_{\alpha_1}\times\cdots\times\boldsymbol{Coh}_{\alpha_s}$, *we have*

$$vol\Big(\pi_{\underline{\alpha}}^{-1}(\mathcal{F}_1,\ldots,\mathcal{F}_s)\Big)=q^{d(\underline{\alpha})},$$

where

$$d(\underline{\alpha})=-\Bigg\{\sum_i(i-1)\langle\alpha_i,\alpha_i\rangle+\sum_{i<j}i(\alpha_i,\alpha_j)\Bigg\};$$

(ii) *for any* $(\mathcal{F}_1,\ldots,\mathcal{F}_s)\in\boldsymbol{Coh}_{\alpha_1}\times\cdots\times\boldsymbol{Coh}_{\alpha_s}$, *we have*

$$vol\Big((\pi''_{\underline{\alpha}})^{-1}(\mathcal{F}_1,\ldots,\mathcal{F}_s)\Big)=q^{d''(\underline{\alpha})},$$

where

$$d''(\underline{\alpha})=-\sum_{i<j}\langle\alpha_j,\alpha_i\rangle;$$

(iii) *for any* $(\mathcal{H},\mathcal{H}_\bullet)\in\widetilde{\boldsymbol{Coh}}_{\underline{\alpha}}$, *we have*

$$vol\Big((\pi'_{\underline{\alpha}})^{-1}(\mathcal{H},\mathcal{H}^\bullet)\Big)=q^{d'(\underline{\alpha})},$$

where $d'(\underline{\alpha})=d(\underline{\alpha})-d''(\underline{\alpha})$;

(iv) *we have*

$$(\pi'_{\underline{\alpha}})^{-1}\Big(\widetilde{\boldsymbol{Coh}}^{\geq 0}_{\underline{\alpha}}\Big)=\boldsymbol{Nil}^{\geq 0}_{\underline{\alpha}}.$$

Proof. The proofs of statements (i)–(iii) are completely analogous to [GPHS14, Prop. 3.1, Cor. 3.2]. The fiber of $\pi_{\underline{\alpha}}$ over $(\mathcal{F}_1,\ldots,\mathcal{F}_s)$ classifies successive extensions between the sheaves $\mathcal{F}_i$, hence it is isomorphic to a suitable iteration of stack bundles of the form $\mathrm{RHom}(\mathcal{F}_i,\mathcal{F}_j)$. (Recall that $\mathrm{Coh}(X)$ is of global dimension one so the complex $\mathrm{RHom}(\mathcal{F}_i,\mathcal{F}_i)$ may only have cohomology in degrees 0 and 1.) One finds that the stack bundle $\mathrm{RHom}(\mathcal{F}_i,\mathcal{F}_j)$ occurs exactly j times if $j<i$, $i-1$ times if $j=i$ and i times if $j>i$. The stack bundle $\mathrm{RHom}(\mathcal{F}_i,\mathcal{F}_j)$ being of dimension $\langle\alpha_i,\alpha_j\rangle$, one obtains the given expression for the dimension $d(\underline{\alpha})$ of $\pi_{\underline{\alpha}}^{-1}(\mathcal{F}_1,\ldots,\mathcal{F}_s)$. Statements (ii) and (iii) are proved in the same way. We refer to [MS14, Prop. 5.1], where a more detailed argument is given.

We turn to (iv). Given $(\mathcal{F},\theta)\in\mathbf{Nil}_{\underline{\alpha}}$, we have to show that $\mathcal{F}\in\mathrm{Coh}^{\geq 0}$ if and only if $\mathcal{F}/\mathrm{Im}(\theta)\in\mathrm{Coh}^{\geq 0}$. As $\mathrm{Coh}^{\geq 0}$ is closed under quotients, $\mathcal{F}\in$

$\mathrm{Coh}^{\geq 0} \Rightarrow \mathcal{F}/\mathrm{Im}(\theta) \in \mathrm{Coh}^{\geq 0}$. To get the reverse implication, recall the sequence of surjective morphisms $\mathcal{F}/\mathrm{Im}(\theta) \twoheadrightarrow \mathrm{Im}(\theta)/\mathrm{Im}(\theta^2) \twoheadrightarrow \cdots \twoheadrightarrow \mathrm{Im}(\theta^{s-1})$. Hence if $\mathcal{F}/\mathrm{Im}(\theta) \in \mathrm{Coh}^{\geq 0}$, then so do $\mathrm{Im}(\theta^i)/\mathrm{Im}(\theta^{i+1})$ for $i = 1, \ldots, s-1$. But as $\mathrm{Coh}^{\geq 0}$ is also stable under extensions, this implies that $\mathcal{F} \in \mathrm{Coh}^{\geq 0}$. Proposition 3.1 is proved. □

COROLLARY 3.2. *We have*

$$\mathrm{vol}\big(\boldsymbol{Nil}^{\geq 0}_{\underline{\alpha}}\big) = q^{d'(\underline{\alpha})}\mathrm{vol}\big(\widetilde{\boldsymbol{Coh}}^{\geq 0}_{\underline{\alpha}}\big).$$

4. **Hall algebras of curves**

4.1. As we will show in Section 5, the volume of $\widetilde{\mathbf{Coh}}^{\geq 0}_{\underline{\alpha}}$ may be computed using some standard techniques in the theory of automorphic functions over function fields for the groups $\mathrm{GL}(n)$. We will use the language of Hall algebras, which seems the most convenient here and which we briefly recall in this section. We refer, e.g., to [Kap97], [KSV12] or [Sch12, Lect. 4] for details. This will also yield a proof of Theorem 1.1.

For any $\gamma \in (\mathbb{Z}^2)^+$, we set $\mathcal{I}_\gamma = \mathrm{Obj}(\mathbf{Coh}_\gamma)/\sim$, and we let $\mathcal{H}_\gamma = \mathrm{Fun}(\mathcal{I}_\gamma, \mathbb{C})$ be the $\mathbb{C}$-vector space of all functions $\mathcal{I}_\gamma \to \mathbb{C}$. There is a natural convolution diagram

$$\mathbf{Coh}_{\gamma_2} \times \mathbf{Coh}_{\gamma_1} \xleftarrow{p} \widetilde{\mathbf{Coh}}_{\gamma_2,\gamma_1} \xrightarrow{s} \mathbf{Coh}_{\gamma_1+\gamma_2},$$

where $p(\mathcal{H}, \mathcal{H}_1 \subset \mathcal{H}) = (\mathcal{H}/\mathcal{H}_1, \mathcal{H}_1)$ and $s(\mathcal{H}, \mathcal{H}_1 \subset \mathcal{H}) = \mathcal{H}$. This induces maps

$$\begin{aligned} m_{\gamma_2,\gamma_1} : \mathcal{H}_{\gamma_2} \otimes \mathcal{H}_{\gamma_1} &\to \mathcal{H}_{\gamma_1+\gamma_2} \\ f \otimes g &\mapsto q^{\frac{1}{2}\langle \gamma_2,\gamma_1\rangle} s_* p^*(f \boxtimes g) \end{aligned}$$

and

$$\begin{aligned} \Delta'_{\gamma_2,\gamma_1} : \mathcal{H}_{\gamma_1+\gamma_2} &\to \mathrm{Fun}(\mathcal{I}_{\gamma_2} \times \mathcal{I}_{\gamma_1}, \mathbb{C}) \\ h &\mapsto q^{\frac{1}{2}\langle \gamma_2,\gamma_1\rangle} p_* s^*(h). \end{aligned}$$

The exponent $\langle \gamma_2, \gamma_1 \rangle$ of $q^{1/2}$ occurring in these definitions is (up to a sign) the dimension of the smooth fibration p; it is an analogue of a cohomological Tate shift.

Note that $\mathrm{Fun}(\mathcal{I}_{\gamma_2} \times \mathcal{I}_{\gamma_1}, \mathbb{C})$ is a natural completion of $\mathcal{H}_{\gamma_2} \otimes \mathcal{H}_{\gamma_1}$. We will denote this completion by $\mathcal{H}_{\gamma_2} \hat{\otimes} \mathcal{H}_{\gamma_1}$. Taking the direct sum over all γ yields an algebra and a (topological) coalgebra structure on $\mathcal{H}' = \bigoplus_\gamma \mathcal{H}_\gamma$. As defined, this is not a bi-algebra in a strict sense (i.e., Δ' is not a morphism of algebras). Let $\mathbf{K} = \bigoplus_{\gamma \in \mathbb{Z}^2} \mathbb{C}\mathbf{k}_\gamma$ be the group algebra of $\mathbb{Z}^2$. The (extended) Hall algebra of X is the semidirect tensor product $\mathcal{H} = \mathcal{H}' \otimes \mathbf{K}$ with respect to the action

$$\mathbf{k}_\gamma f \mathbf{k}_{-\gamma} = q^{\frac{1}{2}(\gamma,\alpha)} f \qquad \text{for } f \in \mathcal{H}_\alpha.$$

It is equipped with a comultiplication satisfying

$$\Delta(\mathbf{k}_\gamma) = \mathbf{k}_\gamma \otimes \mathbf{k}_\gamma,$$
$$\Delta(f) = \sum_{\gamma_1+\gamma_2=\gamma} \Delta'_{\gamma_2,\gamma_1}(f) \cdot (\mathbf{k}_{\gamma_1} \otimes 1) \qquad \text{for } f \in \mathcal{H}_\gamma.$$

By a general theorem of Green (see [Gre95, Th. 1]), $\mathcal{H}$ is a (topological) bi-algebra. We will occasionally write $\Delta_{r,s}$ for the component of Δ of rank (r, s) (hence letting the degrees vary). Observe that $\mathbf{k}_{(0,1)}$ is central. When it bears no consequence, it is sometimes convenient to omit the degree in the notation (for instance, writing simply $\mathbf{k}_1$ for $\mathbf{k}_{1,d}$). We hope that that the reader will not find this slight abuse of notation too confusing.

Let $\mathcal{H}^{\text{fin}} \subset \mathcal{H}$ be the subalgebra of $\mathcal{H}$ consisting of functions with finite support. The algebra $\mathcal{H}^{\text{fin}}$ is equipped with a nondegenerate symmetric pairing defined by

$$(\mathbf{k}_\gamma \mid \mathbf{k}_\delta) = q^{\frac{1}{2}(\gamma,\delta)}, \qquad (1_\mathcal{F} \mid 1_\mathcal{G}) = \frac{\delta_{\mathcal{F},\mathcal{G}}}{|\text{Aut}(\mathcal{F})|},$$

which satisfies the Hopf property

$$(ab \mid c) = (a \otimes b \mid \Delta(c)) \qquad \forall\, a, b, c \in \mathcal{H}^{\text{fin}}.$$

4.2. We will use the following notation: for $\gamma \in (\mathbb{Z}^2)^+$, we denote by $1_\gamma, 1_\gamma^{\text{vec}}, 1_\gamma^{\geq 0}$ the characteristic functions of $\mathbf{Coh}_\gamma$ of the substack $\mathbf{Bun}_\gamma$ of $\mathbf{Coh}_\gamma$ parametrizing vector bundles and of $\mathbf{Coh}_\gamma^{\geq 0}$ respectively. Thus,

$$(1_\gamma \mid 1_\gamma) = \text{vol}(\mathbf{Coh}_\gamma), \;\; (1_\gamma^{\text{vec}} \mid 1_\gamma^{\text{vec}}) = \text{vol}(\mathbf{Bun}_\gamma), \;\; (1_\gamma^{\geq 0} \mid 1_\gamma^{\geq 0}) = \text{vol}(\mathbf{Coh}_\gamma^{\geq 0}).$$

Moreover, it is easy to see from the definitions that if $\gamma = (r, d)$ with $r \geq 1$, then

$$1_\gamma = \sum_{l \geq 0} q^{-\frac{1}{2}l} 1^{\text{vec}}_{\gamma-(0,l)} 1_{0,l}.$$

Unraveling the definitions we have that for any $\underline{\alpha} = (\alpha_1, \ldots, \alpha_s)$,

$$\text{vol}\left(\widetilde{\mathbf{Coh}}^{\geq 0}_{\underline{\alpha}}\right) = q^{-\frac{1}{2}\sum_{i>j}\langle \alpha_i, \alpha_j \rangle} \left(1_{\alpha_s} \cdots 1_{\alpha_1} \mid 1^{\geq 0}_{\sum \alpha_i}\right). \tag{4.1}$$

Theorem 4.1. *We have*

(i) *if* $\gamma = (r, d)$ *with* $r > 0$, *then*

$$(1_\gamma^{\text{vec}} \mid 1_\gamma^{\text{vec}}) = \frac{q^{(g-1)(r^2-1)}}{q-1} |Pic^0(X)| \zeta_X(q^{-2}) \cdots \zeta_X(q^{-r});$$

(ii)
$$\sum_{l \geq 0} (1_{0,l} \mid 1_{0,l}) s^l = \text{Exp}\left(\frac{|X(\mathbf{F}_q)|}{q-1} s\right) = \prod_{i=1}^{\infty} \zeta_X(q^{-i}s);$$

(iii) *if* $\gamma = (r,d)$ *with* $r > 0$, *then*

$$(1_\gamma \mid 1_\gamma) = \frac{q^{(g-1)(r^2-1)}}{q-1} |Pic^0(X)| \prod_{i=2}^{\infty} \zeta_X(q^{-i}).$$

Proof. The first statement is known as the Siegel formula; see [HN75, Prop. 2.3.4] and also [BD07, §6] for a motivic analog. The second statement is also known, but we indicate a proof in the appendix as we have not been able to locate a precise reference. The last statement is an easy consequence of (i) and (ii) together with the fact that

$$\mathrm{vol}(\mathbf{Coh}_\gamma) = \sum_{l \geq 0} q^{-rl} \mathrm{vol}(\mathbf{Bun}_{\gamma-(0,l)}) \mathrm{vol}(\mathbf{Coh}_{0,l}),$$

this last equality coming from the stratification of Coh_γ according to the length of the torsion part. We note that the cohomology (together with the action of the Frobenius) of the moduli stacks Coh_γ have been determined for all γ by J. Heinloth (see [Hei12, Th. 1]); the above formulas (ii) and (iii) may alternatively be deduced from loc. cit. using Behrend's trace formula. □

For any $r > 0$, we set

$$\mathrm{vol}_r = \mathrm{vol}(\mathbf{Bun}_{r,d}(X)).$$

This is independent of d, given explicitly in Theorem 4.1(i).

4.3. Our computation uses some well-known properties of Eisenstein series, which we recall in this paragraph. Several statements concern infinite series in several variables. If $f(z_1, \ldots, z_s) \in \mathbb{Q}(z_1, \ldots, z_s)$ is a rational function, then by its *expansion in the region* $z_1 \gg z_2 \gg \cdots \gg z_s$ we mean its expansion in the Laurent series ring

$$\mathbb{Q}[z_1^{\pm 1}, \ldots, z_s^{\pm 1}] \otimes_{\mathbb{Q}[z_1, \ldots, z_s]} \mathbb{Q}\left[\left[\frac{z_2}{z_1}, \ldots, \frac{z_s}{z_{s-1}}\right]\right]. \tag{4.2}$$

For any $r \geq 0$, let us consider the series

$$E_r(z) = \sum_{d \in \mathbb{Z}} 1_{r,d} z^d, \qquad E_r^{\mathrm{vec}}(z) = \sum_{d \in \mathbb{Z}} 1_{r,d}^{\mathrm{vec}} z^d,$$

which both belong to $\prod_{d \in \mathbb{Z}} (\mathcal{H}_{r,d}[z^{\pm 1}])$. The unicity of the torsion part of a coherent sheaf implies

$$E_r(z) = E_r^{\mathrm{vec}}(z) E_0(q^{-\frac{1}{2}r} z). \tag{4.3}$$

Note also that $E_0(z) = 1 + \sum_{d > 0} 1_{0,d} z^d$ is an invertible series. We consider the Eisenstein series

$$E_{r_s,\ldots,r_1}(z_s, \ldots, z_1) = E_{r_s}(z_s) \cdots E_{r_1}(z_1) \in \prod_{d \in \mathbb{Z}} (\mathcal{H}_{r,d}[[z_s^{\pm 1}, \ldots, z_1^{\pm 1}]]),$$

where $r=\sum_i r_i$, and we define $E^{\mathrm{vec}}_{r_s,\dots,r_1}(z_s,\dots,z_1)$ likewise. It was shown by Harder (see [Har74, Th. 1.6.6]) that for any coherent sheaf $\mathcal{F}$ of rank r, the value of $E^{\mathrm{vec}}_{r_s,\dots,r_1}(z_s,\dots,z_1)$ on $\mathcal{F}$ is the expansion in the region $z_1 \gg z_2 \gg \cdots \gg z_s$ of a rational function. The fact that this coefficient belongs to the space (4.2) is a consequence of the Harder-Narasimhan reduction theory: the slope of possible subsheaves of a given sheaf $\mathcal{F}$ is bounded above, and the number of subsheaves of a given class is finite. Whenever there is no risk of confusion, we will abbreviate $E_{r_s,\dots,r_1}(z_s,\dots,z_1)$ by $E_{\underline{r}}(\underline{z})$.

We summarize the properties of the Eisenstein series which we will use in the following theorem.

Theorem 4.2 (Harder). *The following hold*:

(i) $E_0(z)E_0(w)=E_0(w)E_0(z)$;

(ii) *for any* $r\geq 1$, *we have*

$$E_0(z)E^{\mathrm{vec}}_r(w)=\left(\prod_{i=0}^{r-1}\zeta\left(q^{-\frac{r}{2}+i}\frac{z}{w}\right)\right)\,E^{\mathrm{vec}}_r(w)E_0(z),$$

where the rational function $\prod_i \zeta_X\left(q^{-\frac{r}{2}+i}\frac{z}{w}\right)$ *is expanded in the region* $w\gg z$;

(iii) *for any* $r\geq 0$, *we have*

$$\Delta'(E_r(z))=\sum_{s+t=r} q^{\frac{1}{2}st(g-1)}E_s(q^{\frac{t}{2}}z)\otimes E_t(q^{-\frac{s}{2}}z),$$

$$\Delta'(E^{\mathrm{vec}}_r(z))=\sum_{s+t=r} q^{\frac{1}{2}st(g-1)}E^{\mathrm{vec}}_s(q^{\frac{t}{2}}z)E_0(q^{\frac{t-s}{2}}z)E_0^{-1}(q^{-\frac{t+s}{2}}z)\otimes E^{\mathrm{vec}}_t(q^{-\frac{s}{2}}z);$$

(iv) *for* $r\geq 1$, *we have*

$$E^{\mathrm{vec}}_r(q^{\frac{1}{2}(1-r)}z_1)=C\cdot \mathbb{R}\mathrm{es}_{\frac{z_r}{z_{r-1}}=\frac{z_{r-1}}{z_{r-2}}=\cdots=\frac{z_2}{z_1}=q^{-1}}\left(E^{\mathrm{vec}}_1(z_r)\cdots E^{\mathrm{vec}}_1(z_1)\right),$$

where

$$C=q^{-\frac{1}{4}(g-1)r(r-1)}\mathrm{vol}_1^{-r}\mathrm{vol}_r.$$

Proof. The first statement simply expresses the commutativity of Hecke operators, while the second expresses the fact that the constant functions are Hecke eigenfunctions; see, e.g., [SV11, Th. 6.3] for a proof in the language of Hall algebras. (The proof is given there when $g_X=1$, but the same proof works for an arbitrary curve.) The first statement (iii) is the formula for the constant term of the constant function; see, e.g., [SV11, Prop. 6.2]. The second statement of (iii) is a consequence of the first and the factorization (4.3). Finally, (iv) is the formula expressing the constant function on $\mathrm{Bun}_{\mathrm{GL}(r)}(X)$ as an iterated residue of an Eisenstein series; see [Har74, Th. 2.2.3] and Section 1.4 for our conventions concerning iterated residues. $\square$

Corollary 4.3. *For any tuple* $(r_1, \ldots, r_s)$ *and any coherent sheaf* $\mathcal{F}$ *of rank* $r_1 + \cdots + r_s$*, the coefficient* $E_{r_s,\ldots,r_1}(z_s, \ldots, z_1)(\mathcal{F})$ *is the expansion in the region* $z_1 \gg \cdots \gg z_s$ *of a rational function.*

Proof. Indeed, this follows from the analogous statement for the series $E^{\mathrm{vec}}_{r_s,\ldots,r_1}(z_s, \ldots, z_1)$ using the factorization (4.3) and the Hecke relations (i) and (ii) in Theorem 4.2. Observe that the coefficient of any torsion sheaf $\mathcal{T}$ in a rank zero Eisenstein series $E_{0,\ldots,0}(w_s, \ldots, w_1)$ is a polynomial in $w_1, \ldots, w_s$. □

4.4. We will need some appropriate truncations of the series $E_r(z)$ and $E^{\mathrm{vec}}_r(z)$. Put

$$1^{\geq 0}_{r,d} = 1_{\mathbf{Coh}^{\geq 0}_{r,d}}, \qquad 1^{\mathrm{vec},\geq 0}_{r,d} = 1_{\mathbf{Bun}^{\geq 0}_{r,d}}, \qquad 1^{<0}_{r,d} = 1_{\mathbf{Bun}^{<0}_{r,d}},$$

where $\mathbf{Bun}^{<0}_{r,d}$ is the full subgroupoid of $\mathbf{Bun}_{r,d}$ whose objects are vector bundles belonging to $\mathrm{Coh}^{<0}$. We also set

$$E^{\geq 0}_r(z) = \sum_{d\in\mathbb{Z}} 1^{\geq 0}_{r,d} z^d, \qquad E^{\mathrm{vec},\geq 0}_r(z) = \sum_{d\in\mathbb{Z}} 1^{\mathrm{vec},\geq 0}_{r,d} z^d, \qquad E^{<0}_r(z) = \sum_{d\in\mathbb{Z}} 1^{<0}_{r,d} z^d.$$

The unicity of the Harder-Narasimhan filtration yields the following relations:

$$\begin{aligned}
E_r(z) &= \sum_{\substack{s+t=r\\ s,t\geq 0}} q^{\frac{1}{2}(g-1)st} E^{<0}_s(q^{\frac{t}{2}} z) E^{\geq 0}_t(q^{-\frac{1}{2}s} z),\\
E^{\mathrm{vec}}_r(z) &= \sum_{\substack{s+t=r\\ s,t\geq 0}} q^{\frac{1}{2}(g-1)st} E^{<0}_s(q^{\frac{t}{2}} z) E^{\mathrm{vec},\geq 0}_t(q^{-\frac{1}{2}s} z),\\
E^{\geq 0}_r(z) &= E^{\mathrm{vec},\geq 0}_r(z) E_0(q^{-\frac{1}{2}r} z).
\end{aligned} \tag{4.4}$$

4.5. Let $\mathcal{H}^{\mathrm{sph}} \subset \mathcal{H}^{\mathrm{fin}}$ be the subalgebra generated by $\mathbf{K}$ and the characteristic functions $\mathbf{1}^{\mathrm{vec}}_{1,d}$ and $\mathbf{1}_{0,d}$ of the connected components of $\mathbf{Pic}(X)$ and $\mathbf{Coh}_0(X)$, the stack of torsion sheaves on X. This subalgebra is studied in [SV12] and [Sch11]. In particular, it is shown in [Sch11, Thm 3.1] that the characteristic function $1_{\mathrm{Coh}^{(\alpha_1,\ldots,\alpha_t)}}$ of any HN strata $\mathrm{Coh}^{(\alpha_1,\ldots,\alpha_t)}$ belongs to $\mathcal{H}^{\mathrm{sph}}$. One nice feature of $\mathcal{H}^{\mathrm{sph}}$ is that it possesses an *integral* (*or generic*) *form* in the following sense. Let us fix a genus $g \geq 0$, put $R_g = \mathbb{Q}[T_g]^{W_g}$ and recall that K_g is the localization of R_g at the set $\{q^l - 1 \mid l \geq 1\}$, where by definition $q(\sigma_1, \ldots, \sigma_{2g}) = \sigma_{2i-1}\sigma_{2i}$ for any $1 \leq i \leq g$. (see Section 1.1). For any choice of smooth projective curve X of genus g, there is a natural map $K_g \to \mathbb{C}, f \mapsto f(\sigma_X)$.

Theorem 4.4 ([SV12], [Sch11]). *There exists an* R_g*-Hopf algebra* ${}_R\mathcal{H}^{\mathrm{sph}}$ *equipped with a Hopf pairing*

$$(\ ) : {}_R\mathcal{H}^{\mathrm{sph}} \otimes {}_R\mathcal{H}^{\mathrm{sph}} \to K_g,$$

generated by elements ${}_R1_{0,l}, {}_R1^{\mathrm{vec}}_{1,d}, l \geq 1, d \in \mathbb{Z}$, *containing elements* ${}_R1_{\mathrm{Coh}^{\alpha_1,\ldots,\alpha_t}}$ *for any* HN *strata* $\mathrm{Coh}^{(\alpha_1,\ldots,\alpha_t)}$ *and having the following property*: *for any smooth connected projective curve* X *of genus* g *defined over a finite field* $\mathbf{F}_q$ *there exists a specialisation morphism of Hopf algebras*

$$\Psi_X : {}_R\mathcal{H}^{\mathrm{sph}} \otimes_{R_g} \mathbb{C} \twoheadrightarrow \mathcal{H}_X^{\mathrm{sph}}$$

such that

$$\Psi_X({}_R1_{\mathrm{Coh}^{(\alpha_1,\ldots,\alpha_t)}}) = 1_{\mathrm{Coh}^{(\alpha_1,\ldots,\alpha_t)}}$$

for any HN *strata* $\mathrm{Coh}^{(\alpha_1,\ldots,\alpha_t)}$.

Proof. The existence of ${}_R\mathcal{H}^{\mathrm{sph}}$ is shown in [SV12, 1.11]. The existence of the elements ${}_R1_{\mathrm{Coh}^{(\alpha_1,\ldots,\alpha_t)}}$ is proved in exactly the same fashion as in [Sch11, Th. 3.1]. □

Corollary 4.5. *For any tuple* $\underline{\alpha} = (\alpha_1, \ldots, \alpha_s)$, *there exists an element* $B^{\geq 0}_{g,\underline{\alpha}} \in K_g$ *such that*

$$vol\left(\widetilde{\boldsymbol{Coh}}^{\geq 0}_{\underline{\alpha}}(X)\right) = B^{\geq 0}_{g,\underline{\alpha}}(\sigma_X)$$

for any X.

Proof. By (4.1) and Theorem 4.4 it is enough to show that the pairing

$$(1_{\alpha_s} \cdots 1_{\alpha_1} \mid 1^{\geq 0}_{\sum \alpha_i}) \tag{4.5}$$

may be expressed as a pairing between certain explicit polynomials in elements $1_{\mathrm{Coh}^{(\beta_1,\ldots,\beta_t)}}$. On the one hand, we have

$$1^{\geq 0}_{\sum \alpha_i} = \sum_{\underline{\beta}} 1_{\mathrm{Coh}^{(\underline{\beta})}},$$

where $\underline{\beta}$ ranges among the (finite) set of all HN types $(\beta_1, \ldots, \beta_t)$ such that $\sum \beta_i = \sum \alpha_i$ and $\mu(\beta_1) \geq 0$. On the other hand, we have

$$1_{\alpha_s} \cdots 1_{\alpha_1} = \sum_{\underline{\beta}_s, \ldots, \underline{\beta}_1} 1_{\mathrm{Coh}^{(\underline{\beta}_s)}} \cdots 1_{\mathrm{Coh}^{(\underline{\beta}_1)}} \tag{4.6}$$

where the sum ranges over all tuples $(\underline{\beta}_s, \ldots, \underline{\beta}_1)$ of HN types of respective class $\alpha_s, \ldots, \alpha_1$. Write

$$\underline{\beta}_i = (\beta_1^{(i)}, \ldots, \beta_{t_i}^{(i)}), \qquad (1 \leq i \leq s).$$

We claim that the pairing $(1_{\mathrm{Coh}^{(\underline{\beta}_s)}} \cdots 1_{\mathrm{Coh}^{(\underline{\beta}_1)}} \mid 1^{\geq 0}_{\sum \alpha_i})$ may be nonzero only when

$$\mu(\alpha_s + \cdots + \alpha_{i+1} + \beta_1^{(i)} + \cdots + \beta_l^{(i)}) \geq 0 \tag{4.7}$$

for all possible choices of i and l. Indeed, if (4.7) does not hold, then there exists a coherent sheaf $\mathcal{F} \in \mathrm{Coh}^{\geq 0}$ satisfying $(1_{\mathrm{Coh}^{(\underline{\beta}_s)}} \cdots 1_{\mathrm{Coh}^{(\underline{\beta}_1)}} \mid \mathcal{F}) \neq 0$ having some quotient of negative slope. Observe that condition (4.7) reduces the number of summands in (4.6) contributing to (4.5) to a finite set. We are done. $\square$

From the above corollary one deduces that for any $\alpha \in (\mathbb{Z}^2)^+$, there exist an element $C^{\geq 0}_{g,\alpha} \in K_g$ such that

$$\mathrm{vol}\Big(\mathbf{Nil}^{\geq 0}_{\alpha}(X)\Big) = C^{\geq 0}_{g,\alpha}(\sigma_X)$$

for any X. Therefore using Proposition 2.2 we obtain the relation

$$\sum_{l\geq 1} \frac{1}{l} \sum_{\alpha} \frac{\mathcal{A}^{\geq 0}_{\alpha}(X \otimes \mathbf{F}_{q^l})}{q^l - 1} z^{l\alpha} = \log\left(\sum_{\alpha} C^{\geq 0}_{g,\alpha} z^{\alpha}\right). \tag{4.8}$$

This implies that for any α, there exists an element $A^{\geq 0}_{g,\alpha} \in K_g$ such that $\mathcal{A}^{\geq 0}_{\alpha}(X) = A^{\geq 0}_{g,\alpha}(\sigma_X)$ for any X. Indeed, this follows immediately from (4.8) for $\alpha = (r, d)$ with r and d relatively prime and from there by an easy induction on $\gcd(r, d)$ for an arbitrary pair α. Using Proposition 2.1 we therefore have

COROLLARY 4.6. *For any g and any α, there exists an element $A_{g,\alpha} \in K_g$ such that $\mathcal{A}_{\alpha}(X) = A_{g,\alpha}(\sigma_X)$ for any smooth projective curve X of genus g.*

4.6. To finish the proof of Theorem 1.1 it remains to prove the unicity of $A^{\geq 0}_{g,\alpha}$. For this, let us fix a prime number l, an embedding $\iota : \overline{\mathbb{Q}_l} \to \mathbb{C}$ and consider the collection $\mathcal{X}_g$ of all smooth projective geometrically connected curves X of genus g defined over some finite field $\mathbf{F}_q$ with l not dividing q. Setting

$$\mathcal{W} = \{\sigma_X \mid X \in \mathcal{X}_g\} \subset T_g/W_g,$$

we see that the unicity statement of Theorem 1.1 boils down to the following fact, whose proof is given in the appendix:

PROPOSITION 4.7. *The set $\mathcal{W}$ is Zariski dense in T_g/W_g.*

5. Volume of the stack of pairs

5.1. The aim of this section is to perform the computation of the pairing

$$\Big(1_{\alpha_s} \cdots 1_{\alpha_1} \mid 1^{\geq 0}_{\sum \alpha_i}\Big)$$

(and hence of the volume of $\widetilde{\mathbf{Coh}}^{\geq 0}_{\underline{\alpha}}$) and to prove Theorem 1.6. To this aim, let us introduce the following generating series:

$$G^{\geq 0}_{r_s,\ldots,r_1}(z_s, \ldots, z_1; w) := \left(E_{r_s,\ldots,r_1}(z_s, \ldots, z_1) \,\middle|\, E^{\geq 0}_{n}(w)\right),$$

where $n = \sum_i r_i$. Note that we allow some of the r_i to be zero. By Corollary 4.5, $G^{\geq 0}_{r_s,\ldots,r_1}(z_s, \ldots, z_1; w)$ belongs to the vector space $\mathbb{C}\left[\left[\frac{z_s}{z_{s-1}}, \frac{z_{s-1}}{z_{s-2}}, \ldots, \frac{z_2}{z_1}, z_1, w\right]\right]$. In addition, for any $l \geq 0$, the coefficient of w^l is the expansion in the region $z_1 \gg \cdots \gg z_s$ of some rational function in $z_1, \ldots, z_s$. Indeed, there are only finitely many sheaves $\mathcal{G}$ in $\mathrm{Coh}_{n,l}^{\geq 0}$ up to isomorphism, and by Harder's theorem, $E_{\underline{r}}(\underline{z})(\mathcal{G})$ is the expansion of a rational function (see Corollary 4.3).

Proposition 5.1. *For any* $r_s, \ldots, r_1$, *we have*

$$G^{\geq 0}_{r_s,\ldots,r_1}(z_s,\ldots,z_1;w) = X_{r_1,\ldots,r_s}(z_s,\ldots,z_1;w)\cdot Y^{\geq 0}_{r_s,\ldots,r_1}(z_r,\ldots,z_1;w),$$

where

$$Y^{\geq 0}_{r_s,\ldots,r_1}(z_s,\ldots,z_1;w) = \Big(E^{\mathrm{vec}}_{r_s,\ldots,r_1}(z_s,\ldots,z_1)\ \Big|\ E^{\geq 0}_{n}(w)\Big)$$

and

$$\begin{aligned} &X_{r_s,\ldots,r_1}(z_s,\ldots,z_1;w)\\ &\qquad = \mathrm{Exp}\left(\frac{|X(\mathbf{F}_q)|}{q-1}\left[\sum_i q^{-\frac{1}{2}(n+r_i)}z_i w + \sum_{i>j}\frac{z_i}{z_j}\big(q^{\frac{r_j}{2}}-q^{-\frac{r_j}{2}}\big)q^{-\frac{r_i}{2}}\right]\right). \end{aligned}$$

Proof. Let us abbreviate $\underline{r} = (r_s,\ldots r_1)$ and $\underline{z} = (z_s,\ldots,z_1)$. From the third relation in (4.4) and using (twice) the Hopf property of the pairing, we get

$$\begin{aligned} G^{\geq 0}_{\underline{r}}(\underline{z};w) &= \big(\Delta_{r_s,0}(E_{r_s}(z_s))\cdots\Delta_{r_1,0}(E_{r_1}(z_1)) \mid E^{\mathrm{vec},\geq 0}_n(w)\otimes E_0(q^{-\frac{n}{2}}w)\big)\\ &= \Big(\big(E_{r_s}(z_s)\mathbf{k}_0\otimes E_0(q^{-\frac{r_s}{2}}z_s)\big)\\ &\qquad \cdots\big(E_{r_1}(z_1)\mathbf{k}_0\otimes E_0(q^{-\frac{r_1}{2}}z_1)\big) \mid E^{\mathrm{vec},\geq 0}_n(w)\otimes E_0(q^{-\frac{n}{2}}w)\Big)\\ &= \Big(E_{r_s}(z_s)\cdots E_{r_1}(z_1)\mathbf{k}_0^s\otimes E_0(q^{-\frac{r_s}{2}}z_s)\\ &\qquad \cdots E_0(q^{-\frac{r_1}{2}}z_1) \mid E^{\mathrm{vec},\geq 0}_n(w)\otimes E_0(q^{-\frac{n}{2}}w)\Big)\\ &= \Big(E_{\underline{r}}(\underline{z}) \mid E^{\mathrm{vec},\geq 0}_n(w)\Big)\cdot\left(\prod_i E_0(q^{-\frac{r_i}{2}}z_i) \mid E_0(q^{-\frac{n}{2}}w)\right)\\ &= \Big(E_{\underline{r}}(\underline{z})) \mid E^{\mathrm{vec},\geq 0}_n(w)\Big)\cdot\prod_i\Big(E_0(q^{-\frac{r_i}{2}}z_i) \mid E_0(q^{-\frac{n}{2}}w)\Big)\\ &= \Big(E_{\underline{r}}(\underline{z}) \mid E^{\mathrm{vec},\geq 0}_n(w)\Big)\cdot\mathrm{Exp}\left(\frac{|X(\mathbf{F}_q)|}{q-1}\sum_i q^{-\frac{1}{2}(n+r_i)}z_i w\right). \end{aligned} \tag{5.1}$$

The last step of the above calculation uses Theorem 4.1(ii). Using (4.3) and the Hecke relations (see Theorem 4.2(i), (ii)) we get

(5.2)
$$\begin{aligned}
&\left(E_{\underline{r}}(\underline{z}) \mid E_n^{\mathrm{vec},\geq 0}(w)\right)\\
&\quad= \left(E_{r_s}^{\mathrm{vec}}(z_s)E_0(q^{-\frac{r_s}{2}}z_s)E_{r_{s-1}}^{\mathrm{vec}}(z_{s-1})\cdots E_{r_1}^{\mathrm{vec}}(z_1)E_0(q^{-\frac{r_1}{2}}z_1) \mid E_n^{\mathrm{vec},\geq 0}(w)\right)\\
&\quad= \prod_{i>j}\prod_{l=0}^{r_j-1}\zeta\left(q^{-\frac{r_j}{2}+l}q^{-\frac{r_i}{2}}\frac{z_i}{z_j}\right)\cdot\left(E_{\underline{r}}^{\mathrm{vec}}(\underline{z})\prod_{i=1}^{s}E_0(q^{-\frac{r_i}{2}}z_i) \mid E_n^{\mathrm{vec},\geq 0}(w)\right)\\
&\quad= \mathrm{Exp}\left(\frac{|X(\mathbf{F}_q)|}{q-1}\left[\sum_{i>j}\frac{z_i}{z_j}\big(q^{\frac{r_j}{2}}-q^{-\frac{r_j}{2}}\big)q^{-\frac{r_i}{2}}\right]\right)\cdot\left(E_{\underline{r}}^{\mathrm{vec}}(\underline{z}) \mid E_n^{\mathrm{vec},\geq 0}(w)\right).
\end{aligned}$$

Above we have made use of the fact that the vector bundle part of the product $E_{\underline{r}}^{\mathrm{vec}}(\underline{z})\prod_{i=1}^{s}E_0(q^{-\frac{r_i}{2}}z_i)$ is equal to $E_{\underline{r}}^{\mathrm{vec}}(\underline{z})$ together with the relation

$$\prod_{l=0}^{n-1}\zeta\left(q^{-\frac{n}{2}+l}u\right) = \mathrm{Exp}\left(\frac{|X(\mathbf{F}_q)|}{q-1}u\big(q^{\frac{n}{2}}-q^{-\frac{n}{2}}\big)\right).$$

Combining (5.1) and (5.2) yields the proposition. □

5.2. In order to compute the series $Y_{\underline{r}}^{\geq 0}(\underline{z};w)$ we introduce some further generating series

$$Y_{\underline{r}}^{*}(\underline{z};w) := \left(E_{\underline{r}}^{\mathrm{vec}}(\underline{z}) \,\middle|\, E_r^{*}(w)\right),$$

where $r=\sum r_i$ and where the symbol $*$ is either empty or belongs to the set $\{\geq 0, <0\}$. As before, these series belong to the vector space of formal sums $\mathbb{C}[[z_s^{\pm 1},\ldots,z_1^{\pm 1},w^{\pm 1}]]$. By construction, the coefficient $(1_{r_s,d_s}\cdots 1_{r_1,d_1} \mid 1_{r,d}^{*})$ of $z_s^{d_s}\cdots z_1^{d_1}w^d$ in $Y_{r_s,\ldots,r_1}^{*}(z_s,\ldots,z_1;w)$ is nonzero only if $d=\sum_i d_i$. We claim that

$$\begin{aligned}
&Y_{r_s,\ldots,r_1}^{\geq 0}(z_s,\ldots,z_1;w) \in \mathbb{C}\left[\left[\frac{z_s}{z_{s-1}},\frac{z_{s-1}}{z_{s-2}},\ldots,\frac{z_2}{z_1},z_1,w\right]\right],\\
&Y_{r_s,\ldots,r_1}^{<0}(z_s,\ldots,z_1;w) \in z_1^{-1}\mathbb{C}\left[\left[z_s^{-1},\frac{z_s}{z_{s-1}},\frac{z_{s-1}}{z_{s-2}},\ldots,\frac{z_2}{z_1},w^{-1}\right]\right].
\end{aligned} \tag{5.3}$$

Indeed, by definition, a coherent sheaf $\mathcal{G}$ in $\mathrm{Coh}_{r,d}^{\geq 0}$ is of positive degree and may only have positive degree quotient sheaves, hence any filtration

$$\mathcal{G}_1 \subset \mathcal{G}_2 \subset \cdots \subset \mathcal{G}_{s-1} \subset \mathcal{G}_s = \mathcal{G}$$

satisfies $\deg(\mathcal{G}/\mathcal{G}_{i-1}) = \sum_{l=s}^{i+1}\deg(\mathcal{G}_l/\mathcal{G}_{l-1}) \geq 0$. This yields the first inclusion in (5.3). The second one is proved in an analogous fashion. Note that the coefficients in w of $Y_{\underline{r}}^{\geq 0}(\underline{z};w)$ are expansions of some rational functions (because the same holds for $G_{\underline{r}}^{\geq 0}(\underline{z};w)$). This also applies to $Y_{\underline{r}}^{<0}(\underline{z};w)$.

To unburden the notation we will simply write $Y^*_{\underline{r}}(\underline{z};w)$ when the values of the r_i are understood and there is no risk of confusion. We will also write, as in Section 1.3,

$$r_{<i}=\sum_{k<i} r_k, \qquad r_{>i}=\sum_{k>i} r_k, \qquad r_{[i,j]}=\sum_{k=i}^{j} r_k, \qquad \text{etc.}$$

5.3. As (4.1) and Proposition 5.1 show, the volume of the moduli spaces $\widetilde{\mathbf{Coh}}^{\geq 0}_{\underline{\alpha}}(X)$ are essentially computed by the generating series $Y^{\geq 0}_{\underline{r}}(\underline{z};w)$ for suitable choices of $r_s,\ldots .r_1$. In order to determine these series, we will actually calculate all three types of series and use some induction process. We begin with the series $Y_{\underline{r}}(\underline{z};w)$, which is easy to compute.

LEMMA 5.2. *Assume that $r_i \geq 1$ for all i. Then*

$$Y_{\underline{r}}(\underline{z};w)=q^{\frac{1}{2}(g-1)\sum_{i>j} r_i r_j}\prod_i \mathrm{vol}_{r_i}\prod_i\left\{\sum_{l\in\mathbb{Z}} z_i^l w^l q^{\frac{1}{2}l(r_{<i}-r_{>i})}\right\}.$$

Proof. This is a direct consequence of Proposition 3.1(iii). Alternatively, we provide the details of a proof. We have

$$\begin{aligned}Y_{\underline{r}}(\underline{z};w)&=\left(E^{\mathrm{vec}}_{r_s}(z_s)\cdots E^{\mathrm{vec}}_{r_1}(z_1)\;\middle|\;E_r(w)\right)\\&=\left(E^{\mathrm{vec}}_{r_s}(z_s)\otimes\cdots\otimes E^{\mathrm{vec}}_{r_1}(z_1)\;\middle|\;\Delta'_{r_s,\ldots,r_1}(E_r(w))\right).\end{aligned}$$

By Theorem 4.2(iii),

$$\begin{aligned}\Delta'_{r_s,\ldots,r_1}(E_r(w))=q^{\frac{1}{2}(g-1)\sum_{i>j} r_i r_j}&E_{r_s}(q^{\frac{1}{2}r_{<s}}w)\\&\otimes\cdots\otimes E_{r_i}(q^{\frac{1}{2}(r_{<i}-r_{>i})}w)\otimes\cdots\otimes E_{r_1}(q^{-\frac{1}{2}r_{>1}}w).\end{aligned}$$

The lemma follows since by definition, $(1^{\mathrm{vec}}_{r,d}\mid 1_{r,d})=(1^{\mathrm{vec}}_{r,d}\mid 1^{\mathrm{vec}}_{r,d})=\mathrm{vol}_r$ for any $r\geq 1$ and any d. □

5.4. Our next task is to determine explicitly the series $Y^*_{1,\ldots,1}(z_s,\ldots,z_1;w)$, which we will simply abbreviate $Y^*_{\underline{1}}(\underline{z};w)$ when no confusion is likely. To begin, note that by Lemma 5.2,

$$\begin{aligned}Y_{\underline{1}}(\underline{z};w)=&\,q^{\frac{1}{2}(g-1)\frac{s(s-1)}{2}}\mathrm{vol}_1^s\\&\cdot\sum_{l_1,\ldots,l_s\in\mathbb{Z}}\left(z_1^{l_1}\cdots z_s^{l_s}w^{\sum l_i}q^{\sum\frac{1}{2}l_i(2i-s-1)}\right).\end{aligned}\tag{5.4}$$

PROPOSITION 5.3. *For any $s \geq 1$, we have*

$$Y_{\underline{1}}^{\geq 0}(\underline{z};w) = \frac{q^{\frac{1}{4}(g-1)s(s-1)}\mathrm{vol}_1^s}{\prod_{i<j}\widetilde{\zeta}\left(\frac{z_i}{z_j}\right)} \cdot \sum_{\sigma\in\mathfrak{S}_s}\sigma\left[\prod_{i<j}\widetilde{\zeta}\left(\frac{z_i}{z_j}\right)\cdot\frac{1}{\prod_{i<s}\left(1-q\frac{z_{i+1}}{z_i}\right)}\cdot\frac{1}{1-q^{\frac{1-s}{2}}z_1w}\right] \tag{5.5}$$

and

$$Y_{\underline{1}}^{<0}(\underline{z};w) = (-1)^s\frac{q^{\frac{1}{4}(g-1)s(s-1)}\mathrm{vol}_1^s}{\prod_{i<j}\widetilde{\zeta}\left(\frac{z_i}{z_j}\right)} \cdot \sum_{\sigma\in\mathfrak{S}_s}\sigma\left[\prod_{i<j}\widetilde{\zeta}\left(\frac{z_i}{z_j}\right)\cdot\frac{1}{\prod_{i<s}\left(1-q^{-1}\frac{z_i}{z_{i+1}}\right)}\cdot\frac{1}{1-q^{\frac{s-1}{2}}z_sw}\right], \tag{5.6}$$

where the rational functions are expanded in the regions $z_1 \gg z_2 \gg \cdots \gg z_s$, $w \ll 1$ and $z_1 \gg z_2 \gg \cdots \gg z_s, w \gg 1$ respectively, (i.e., in power series in the $\frac{z_{i+1}}{z_i}$ and w, resp. w^{-1}).

Proof. The proof proceeds by induction on s, using formulas (4.4) and (5.4). When $s=1$, we have $E_s^{\mathrm{vec},\geq 0}(w)=\sum_{d\geq 0}1_{1,d}^{\mathrm{vec}}, E_s^{<0}(w)=\sum_{d<0}1_{1,d}^{\mathrm{vec}}w^d$, and hence

$$Y_{\underline{1}}^{\geq 0}(z_1;w) = \sum_{d\geq 0}(1_{1,d}^{\mathrm{vec}} \mid 1_{1,d}^{\mathrm{vec}})(z_1w)^d = \frac{\mathrm{vol}_1}{1-z_1w},$$
$$Y_{\underline{1}}^{<0}(z_1;w) = \sum_{d<0}(1_{1,d}^{\mathrm{vec}} \mid 1_{1,d}^{\mathrm{vec}})(z_1w)^d = -\frac{\mathrm{vol}_1}{1-z_1w},$$

where we expand the rational functions in the regions $w \ll 1$ and $w \gg 1$ respectively. Next, fix $s > 1$ and assume that the proposition is proved for all $s' < s$. Using (4.4), we have

$$Y_{\underline{1}}(z_s,\ldots,z_1;w) = Y_{\underline{1}}^{\geq 0}(\underline{z};w) + Y_{\underline{1}}^{<0}(\underline{z};w) + \sum_{\substack{u+t=s\\u,t>0}} q^{\frac{1}{2}(g-1)ut}\Big(\Delta_{u,t}(E_{\underline{1}}^{\mathrm{vec}}(\underline{z})) \mid E_u^{<0}(q^{\frac{t}{2}}w)\otimes E_t^{\geq 0}(q^{-\frac{u}{2}}w)\Big). \tag{5.7}$$

Observe that in the above equation, the term $Y_{\underline{1}}^{\geq 0}(\underline{z};w)$ only contains positive powers of w while the term $Y_{\underline{1}}^{<0}(\underline{z};w)$ only contains strictly negative powers of w. This will make it possible to inductively extract simultaneously $Y_{\underline{1}}^{\geq 0}(\underline{z};w)$

and $Y_{\underline{1}}^{<0}(\underline{z};w)$ from (5.7). Now, from Theorem 4.2(iii),

(5.8)
$$\Delta(E_{\underline{1}}^{\mathrm{vec}}(z_s,\ldots,z_1)) = \Delta(E_1^{\mathrm{vec}}(z_s))\cdots\Delta(E_1^{\mathrm{vec}}(z_1)) = \prod_i^{\rightarrow}\Big(E_1^{\mathrm{vec}}(z_i)\otimes 1 + E_0(q^{\frac{1}{2}}z_i)E_0(q^{-\frac{1}{2}}z_i)^{-1}\mathbf{k}_1\otimes E_1^{\mathrm{vec}}(z_i)\Big).$$

Expanding (5.8) yields an expression of $\Delta(E_{\underline{1}}^{\mathrm{vec}}(\underline{z}))$ as a sum

$$\Delta(E_{\underline{1}}^{\mathrm{vec}}(\underline{z})) = \sum_\sigma X_\sigma$$

parametrized by maps $\sigma:\{1,\ldots,s\}\to\{1,2\}$, with

$$X_\sigma = \prod_i^{\rightarrow} C_{\sigma(i)}(z_i),$$

where

$$C_1(z) = E_1^{\mathrm{vec}}(z)\otimes 1, \qquad C_2(z) = E_0(q^{\frac{1}{2}}z)E_0(q^{-\frac{1}{2}}z)^{-1}\mathbf{k}_1\otimes E_1^{\mathrm{vec}}(z).$$

The component $\Delta_{u,t}(E_{\underline{1}}^{\mathrm{vec}}(\underline{z}))$ of $\Delta(E_{\underline{1}}^{\mathrm{vec}}(\underline{z}))$ is equal to the same sum, this time ranging over the set of maps $\sigma:\{1,\ldots,s\}\to\{1,2\}$ such that $|\sigma^{-1}(1)| = u, |\sigma^{-1}(2)| = t$. We will denote this set of maps $\mathrm{Sh}_{u,t}$, for (u,t)-*shuffles*. From Theorem 4.2 and the defining commutation relations involving $\mathbf{k}_1$ (see Section 4.1), we derive

$$X_\sigma = H_\sigma(\underline{z})\left(\prod_{i,\sigma(i)=1}^{\rightarrow} E_1^{\mathrm{vec}}(z_i)\prod_{j,\sigma(j)=2}^{\rightarrow} E_0(q^{\frac{1}{2}}z_j)E_0(q^{-\frac{1}{2}}z_j)^{-1}\mathbf{k}_1^t\right) \otimes \prod_{j,\sigma(j)=2} E_1^{\mathrm{vec}}(z_j),$$

where

$$H_\sigma(\underline{z}) = \prod_{\substack{(i,j),j>i,\\ \sigma(i)=1,\sigma(j)=2}} \frac{\widetilde{\zeta}\left(\frac{z_j}{z_i}\right)}{\widetilde{\zeta}\left(\frac{z_i}{z_j}\right)}.$$

Putting all the pieces together yields the following recursion formula:

$$Y_{\underline{1}}(z_s,\ldots,z_1;w) = Y_{\underline{1}}^{\geq 0}(z_s,\ldots,z_1;w) + Y_{\underline{1}}^{<0}(z_s,\ldots,z_1;w) + \sum_{\substack{u+t=s\\ u,t>0}}\sum_{\sigma\in\mathrm{Sh}_{u,t}} Y_\sigma(z_s,\ldots,z_1;w), \tag{5.9}$$

with

$$Y_\sigma(z_s,\ldots,z_1;w) = q^{\frac{1}{2}(g-1)ut}H_\sigma(\underline{z})Y_{\underline{1}}^{<0}(z_{i_u},\ldots,z_{i_1};q^{\frac{t}{2}}w)Y_{\underline{1}}^{\geq 0}(z_{j_t},\ldots,z_{j_1};q^{-\frac{u}{2}}w), \tag{5.10}$$

where $(i_u, \dots, i_1)$ (resp. $(j_t, \dots, j_1)$) are the reordering (in decreasing order) of the sets $\sigma^{-1}(1)$ (resp. $\sigma^{-1}(2)$). Note that the factor $\prod_j E_0(q^{\frac{1}{2}}z_j)E_0(q^{-\frac{1}{2}}z_j)^{-1}\mathbf{k}_1^t$ does not contribute as it does not change the vector bundle part, and hence does not change the scalar product with $E_u^{<0}(q^{\frac{t}{2}}w)$.

Equation (5.9) takes place in the vector space $\mathbb{C}[[z_s^{\pm 1}, \dots z_1^{\pm 1}, w^{\pm 1}]]$. Suppose that we have already determined the series $Y_{\underline{1}}^{<0}(z_u, \dots, z_1; w)$ and $Y_{\underline{1}}^{\geq 0}(z_t, \dots, z_1; w)$ for all $u, t < s$. Then from (5.9) and (5.4), we may derive $Y_{\underline{1}}^{\geq 0}(z_s, \dots, z_1; w)$ and $Y_{\underline{1}}^{<0}(z_s, \dots, z_1; w)$ — recall that $Y_{\underline{1}}^{\geq 0}(z_s, \dots, z_1; w)$ is a power series in w while $Y_{\underline{1}}^{<0}(z_s, \dots, z_1; w)$ is a power series in w^{-1}. In order to establish the statement of Proposition 5.3 for s, it therefore suffices to show that (5.9) holds with $Y_{\underline{1}}^{\geq 0}(z_s, \dots, z_1; w)$ and $Y_{\underline{1}}^{<0}(z_s, \dots, z_1; w)$ respectively given by (5.5) and (5.6). For this, let us consider the coefficients

$$Y_{\underline{1}}^{\geq 0}(\underline{z}; w) = \sum_{n \geq 0} y_n^{\geq 0}(\underline{z})w^n, \qquad Y_{\underline{1}}^{<0}(\underline{z}; w) = \sum_{n<0} y_n^{<0}(\underline{z})w^n,$$

$$Y_\sigma(\underline{z}; w) = \sum_n y_{\sigma,n}(\underline{z})w^n.$$

Note that $y_n^{\geq 0}(\underline{z})$ is zero for $n < 0$ while $y_n^{<0}(\underline{z})$ is zero when $n \geq 0$. Observe that, by construction of Y_σ (see (5.7)), $y_{\sigma,n}(\underline{z})$ belongs to the subspace of $\mathbb{C}[[z_s^{\pm 1}, \dots, z_1^{\pm 1}]]$ of formal series converging in the asymptotic region

$$U_\sigma := \{(z_s, \dots, z_1) \mid z_{i_1} \gg z_{i_2} \gg \cdots \gg z_{i_u} \gg z_{j_1} \gg \cdots \gg z_{j_t}\}.$$

Similarly, $y_n^{<0}(\underline{z})$ and $y_n^{\geq 0}(\underline{z})$ both belong to the subspace of $\mathbb{C}[[z_s^{\pm 1}, \dots, z_1^{\pm 1}]]$ of formal series converging in the asymptotic region

$$U_1 := \{(z_s, \dots, z_1) \mid z_1 \gg z_2 \gg \cdots \gg z_s\}.$$

The part of (5.9) in which w appears with the exponent n reads

(5.11)
$$q^{\frac{1}{2}(g-1)\frac{s(s-1)}{2}} \mathrm{vol}_1^s \sum_{\substack{l_1, \dots, l_s \in \mathbb{Z}, \\ \sum_i l_i = n}} z_1^{l_1} \cdots z_s^{l_s} q^{\sum \frac{1}{2} l_i(2i-s-1)} = y_n^{\geq 0}(\underline{z}) + y_n^{<0}(\underline{z}) + \sum_{u,\sigma} y_{\sigma,n}(\underline{z}).$$

Denote by $\mathbf{y}_n^{\geq 0}(\underline{z}), \mathbf{y}_n^{<0}(\underline{z})$ and $\mathbf{y}_{\sigma,n}(\underline{z})$ the rational functions of which $y_n^{\geq 0}(\underline{z}), y_n^{<0}(\underline{z})$ and $y_{\sigma,n}(\underline{z})$ are the expansion (each in its respective region). We would like to deduce from equation (5.11) a relation between these rational functions. First observe that the left-hand side of (5.11) may also be written as a sum of Laurent series, each of which is the expansion in a suitable asymptotic direction of some rational function. Indeed, setting $\delta(z) = \sum_{l \in \mathbb{Z}} z^l$, we have

$$\sum_{\substack{l_1, \dots, l_s \in \mathbb{Z}, \\ \sum_i l_i = n}} z_1^{l_1} \cdots z_s^{l_s} q^{\sum i l_i} = z_1^n \cdot \delta\left(q\frac{z_2}{z_1}\right) \cdots \delta\left(q\frac{z_s}{z_{s-1}}\right)$$

We can split each delta function $\delta(z)$ as $\delta(z) = \delta_+(z) + \delta_-(z)$ with $\delta_+(z)$ (resp. $\delta_-(z)$) converging in the region $z \ll 1$ (resp. $z \gg 1$) to the function $(1-z)^{-1}$ (resp. $-(1-z)^{-1}$). Any product

$$\delta_{\varepsilon_1}\left(q\frac{z_2}{z_1}\right)\cdots\delta_{\varepsilon_{s-1}}\left(q\frac{z_s}{z_{s-1}}\right)$$

with $\varepsilon_i \in \{1,-1\}$ converges to the rational function

$$f_{\varepsilon_1,\ldots,\varepsilon_{s-1}} = \prod_{i=1}^{s-1}\varepsilon_i\frac{1}{1-qz_{i+1}/z_i}$$

in the asymptotic region

$$U_\gamma = \{(z_s,\ldots,z_1) \mid z_{\gamma(1)} \gg z_{\gamma(2)} \gg \cdots \gg z_{\gamma(s)}\},$$

where $\gamma \in \mathfrak{S}_s$ is any permutation satisfying $\gamma^{-1}(i) < \gamma^{-1}(i+1)$ if $\varepsilon_i = 1$ and $\gamma^{-1}(i) > \gamma^{-1}(i+1)$ if $\varepsilon_i = -1$. Note that

$$\sum_{\underline{\varepsilon}} f_{\varepsilon_1,\ldots,\varepsilon_{s-1}} = 0. \tag{5.12}$$

We are in the situation of the following lemma:

Lemma 5.4. *Let $\mathbb{C}[z_s^{\pm1},\ldots,z_1^{\pm1}]_{\mathrm{loc}}$ be the localization of $\mathbb{C}[z_s^{\pm1},\ldots,z_1^{\pm1}]$ at the set of linear polynomials $z_i - cz_j$ for $c \in \mathbb{C}$. For any $\gamma \in \mathfrak{S}_s$, let*

$$\tau_\gamma : \mathbb{C}[z_s^{\pm1},\ldots,z_1^{\pm1}]_{\mathrm{loc}} \hookrightarrow \mathbb{C}[[z_s^{\pm1},\ldots,z_1^{\pm1}]]$$

be the expansion map in the region

$$U_\gamma = \{(z_s,\ldots,z_1) \mid z_{\gamma(1)} \gg z_{\gamma(2)} \gg \cdots \gg z_{\gamma(s)}\}.$$

Assume given elements $f_\gamma \in \mathbb{C}[z_s^{\pm1},\ldots,z_1^{\pm1}]_{\mathrm{loc}}$ satisfying $\sum_\gamma \tau_\gamma(f_\gamma) = 0$. Then $\sum_\gamma f_\gamma = 0$.

Proof. Write $f_\gamma = R_\gamma/Q_\gamma$ with $R_\gamma, Q_\gamma \in \mathbb{C}[z_s^{\pm1},\ldots,z_s^{\pm1}]$. Let $Q = \prod_\gamma Q_\gamma$. Then $0 = Q\sum_\gamma \tau_\gamma(f_\gamma) = \sum_\gamma \tau_\gamma(Qf_\gamma) = \sum_\gamma Qf_\gamma$, since τ_γ is a morphism of $\mathbb{C}[z_s^{\pm1},\ldots,z_1^{\pm1}]$-modules. Hence $\sum_\gamma f_\gamma = 0$. □

By Lemma 5.4 and (5.12), we have

$$\mathbf{y}_n^{\geq 0}(\underline{z}) = -\sum_\sigma \mathbf{y}_{\sigma,n}(\underline{z}), \qquad (n \geq 0),$$
$$\mathbf{y}_n^{<0}(\underline{z}) = -\sum_\sigma \mathbf{y}_{\sigma,n}(\underline{z}), \qquad (n < 0).$$

Using the induction hypothesis and the expansions

$$\frac{1}{1-q^{\frac{u-1}{2}}z_{i_u}q^{\frac{t}{2}}w} = -\sum_{n_-<0}(q^{\frac{s-1}{2}}z_{i_u}w)^{n_-},$$
$$\frac{1}{1-q^{\frac{1-t}{2}}z_{j_1}q^{-\frac{u}{2}}w} = \sum_{n_+\geq 0}(q^{\frac{1-s}{2}}z_{j_1}w)^{n_-},$$

we arrive at

$$\begin{aligned} &Y_\sigma(z_s,\ldots,z_1;w) = Z_\sigma(z_s,\ldots,z_1) \\ &\quad\cdot\sum_{\sigma_1\in\mathfrak{S}_u}\sigma_1\left[\prod_{l<h\leq u}\widetilde{\zeta}\Big(\frac{z_{i_l}}{z_{i_h}}\Big)\cdot\frac{1}{\prod_{l<u}(1-q^{-1}\frac{z_{i_l}}{z_{i_{l+1}}})}\sum_{n_-<0}\Big(q^{\frac{s-1}{2}}wz_{i_u}\Big)^{n_-}\right] \\ &\quad\cdot\sum_{\sigma_2\in\mathfrak{S}_t}\sigma_2\left[\prod_{k<m\leq t}\widetilde{\zeta}\Big(\frac{z_{j_k}}{z_{j_m}}\Big)\cdot\frac{1}{\prod_{k<t}(1-q\frac{z_{j_{k+1}}}{z_{j_k}})}\sum_{n_+\geq 0}\Big(q^{\frac{1-s}{2}}wz_{j_1}\Big)^{n_+}\right], \end{aligned} \tag{5.13}$$

where

$$\begin{aligned} Z_\sigma(z_s,\ldots,z_1) = (-1)^{u-1}q^{\frac{1}{4}(g-1)s(s-1)}\mathrm{vol}_1^s &\prod_{\substack{(i,j),i<j,\\ \sigma(i)=1,\sigma(j)=2}}\frac{\widetilde{\zeta}\big(\frac{z_j}{z_i}\big)}{\widetilde{\zeta}\big(\frac{z_i}{z_j}\big)} \\ &\cdot\prod_{l<h\leq u}\frac{1}{\widetilde{\zeta}\big(\frac{z_{i_l}}{z_{i_h}}\big)}\cdot\prod_{k<m\leq t}\frac{1}{\widetilde{\zeta}\big(\frac{z_{j_k}}{z_{j_m}}\big)} \end{aligned}$$

and all the denominators are to be expanded in the region U_σ.

Assume $n\geq 0$. Collecting terms in (5.13) with $n=n^++n_-$ yields

$$\begin{aligned} &\mathbf{y}_{\sigma,n}(z_s,\ldots,z_1) = -Z_\sigma(z_s,\ldots,z_1)q^{\frac{1-s}{2}n} \\ &\quad\cdot\sum_{\sigma_1,\sigma_2}\sigma_1\boxtimes\sigma_2\left[\frac{\prod_{l<h\leq u}\widetilde{\zeta}\big(\frac{z_{i_l}}{z_{i_h}}\big)\cdot\prod_{k<m\leq t}\widetilde{\zeta}\big(\frac{z_{j_k}}{z_{j_m}}\big)}{\prod_{l<u}(1-q^{-1}\frac{z_{i_l}}{z_{i_{l+1}}})\cdot\prod_{k<t}(1-q\frac{z_{j_{k+1}}}{z_{j_k}})}\cdot\frac{z_{j_1}^n}{1-q^{s-1}\frac{z_{i_u}}{z_{j_1}}}\right], \end{aligned}$$

where we used the following expansion in U_σ:

$$z_{j_1}^n q^{\frac{1-s}{2}n}\sum_{n_-<0}z_{i_u}^{n_-}z_{j_1}^{-n_-}q^{(s-1)n_-} = -\frac{z_{j_1}^n}{1-q^{s-1}\frac{z_{i_u}}{z_{j_1}}}.$$

Fixing (u,t) and letting σ vary we obtain a sum involving all permutations $\tau\in\mathfrak{S}_s$. Namely, there is a bijection

$$\mathrm{Sh}_{u,t}\times\mathfrak{S}_u\times\mathfrak{S}_t\mapsto\mathfrak{S}_s, \qquad (\sigma,\sigma_1,\sigma_2)\mapsto\tau,$$

where

$$\tau(1,2,\ldots,s) = (j_{\sigma_2(1)},j_{\sigma_2(2)},\ldots,i_{\sigma_1(1)},\ldots,i_{\sigma_1(u)}).$$

This yields (for a fixed (u,t))

$$\sum_{\sigma\in \mathrm{Sh}_{u,t}} \mathbf{y}_{\sigma,n}(\underline{z}) = a_{u,n} \sum_{\sigma,\sigma_1,\sigma_2} \prod_{\substack{\sigma(i)=1,\\ \sigma(j)=2}} \widetilde{\zeta}\Big(\frac{z_j}{z_i}\Big)$$

$$\cdot \prod_{i<j} \widetilde{\zeta}\Big(\frac{z_i}{z_j}\Big)^{-1} \sigma_1 \boxtimes \sigma_2 \left[\frac{\prod_{l<h\leq u} \widetilde{\zeta}\big(\frac{z_{i_l}}{z_{i_h}}\big) \cdot \prod_{k<m\leq t} \widetilde{\zeta}\big(\frac{z_{j_k}}{z_{j_m}}\big)}{\prod_{l<u}(1-q^{-1}\frac{z_{i_l}}{z_{i_{l+1}}}) \cdot \prod_{k<t}(1-q\frac{z_{j_{k+1}}}{z_{j_k}})} \cdot \frac{z_{j_1}^n}{1-q^{s-1}\frac{z_{i_u}}{z_{j_1}}}\right]$$

$$= a_{u,n} \prod_{i<j} \widetilde{\zeta}\Big(\frac{z_i}{z_j}\Big)^{-1} \sum_{\tau\in\mathfrak{S}_s} \tau \left[\frac{\prod_{i<j} \widetilde{\zeta}\big(\frac{z_i}{z_j}\big)}{\prod_{t<k<s}(1-q^{-1}\frac{z_k}{z_{k+1}}) \cdot \prod_{l<t}(1-q\frac{z_{l+1}}{z_l})} \cdot \frac{z_1^n}{1-q^{s-1}\frac{z_s}{z_1}}\right],$$

where

$$a_{u,n} = (-1)^u q^{\frac{1}{4}(g-1)s(s-1)+\frac{1-s}{2}n} \mathrm{vol}_1^s.$$

Then, summing over the set of pairs (u,t), we get

$$\mathbf{y}_n^{\geq 0}(\underline{z}) := -\sum_{u,\sigma} \mathbf{y}_{\sigma,n}(\underline{z})$$

$$= \mathrm{vol}_1^s \prod_{i<j} \frac{1}{\widetilde{\zeta}\big(\frac{z_i}{z_j}\big)} \cdot q^{\frac{1}{4}(g-1)s(s-1)+\frac{1-s}{2}n} \cdot \sum_{\tau\in\mathfrak{S}_s} \tau \left[\prod_{i<j} \widetilde{\zeta}\Big(\frac{z_i}{z_j}\Big) \cdot z_1^n \cdot \sum_{u=1}^{s-1} T_u(\underline{z}) \right],$$

where

$$T_u(\underline{z}) = (-1)^{u-1} \frac{1}{\prod_{l<t}(1-q\frac{z_{l+1}}{z_l}) \cdot \prod_{t<k<s}(1-q^{-1}\frac{z_k}{z_{k+1}}) \cdot (1-q^{s-1}\frac{z_s}{z_1})}.$$

Now,

$$\sum_{u=1}^{s-1} (-1)^u T_u(\underline{z}) = \frac{1}{\prod_{l<s}(1-q\frac{z_{l+1}}{z_l})(1-q^{s-1}\frac{z_s}{z_1})}$$

$$\cdot \Big\{ (1-q\frac{z_s}{z_{s-1}}) + q\frac{z_s}{z_{s-1}}(1-q\frac{z_{s-1}}{z_{s-2}}) + \cdots + q^{s-2}\frac{z_s}{z_2}(1-q\frac{z_2}{z_1}) \Big\}$$

$$= \frac{1}{\prod_{l<s}(1-q\frac{z_{l+1}}{z_l})(1-q^{s-1}\frac{z_s}{z_1})} \Big(1-q^{s-1}\frac{z_s}{z_1}\Big)$$

$$= \frac{1}{\prod_{l<s}(1-q\frac{z_{l+1}}{z_l})}.$$

Summing now over $n \geq 0$ we obtain

$$\sum_{n\geq 0} \mathbf{y}_n^{\geq 0}(\underline{z}) w^n$$

$$= \frac{q^{\frac{1}{4}(g-1)s(s-1)} \mathrm{vol}_1^s}{\prod_{i<j} \widetilde{\zeta}\big(\frac{z_i}{z_j}\big)} \sum_{\tau\in\mathfrak{S}_s} \tau \left[\prod_{i<j} \widetilde{\zeta}\Big(\frac{z_i}{z_j}\Big) \cdot \frac{1}{\prod_{i<s}\big(1-q\frac{z_{i+1}}{z_i}\big)} \cdot \frac{1}{1-q^{\frac{1-s}{2}} z_1 w} \right]$$

as wanted. This shows (5.5) for s. The computations of $\mathbf{y}_{\sigma,n}(\underline{z})$ and $\mathbf{y}_n^{<0}(\underline{z})$ for $n<0$ and hence the proof of (5.6) for s are entirely similar. Proposition 5.3 is proved. □

5.5. Proposition 5.3 allows us to compute the value of $Y_{\underline{r}}^{\geq 0}(\underline{z};w)$ (and hence also $G_{\underline{r}}^{\geq 0}(\underline{z};w)$) for an arbitrary sequence of nonnegative integers $\underline{r}=(r_i)$ by considering appropriate residues. Namely, by Theorem 4.2(iv), we have

(5.14)
$$\begin{aligned} &Y_{\underline{r}}^{\geq 0}(q^{\frac{1}{2}(1-r_t)}z_1^{(t)},\dots,q^{\frac{1}{2}(1-r_1)}z_1^{(1)};w)\\ &\quad = q^{a(\underline{r})}\mathrm{vol}_1^{-n}\prod_i \mathrm{vol}_{r_i}\cdot \mathbb{R}\mathrm{es}_{\underline{r}}\left[Y_{(1^n)}^{\geq 0}(z_{r_t}^{(t)},z_{r_t-1}^{(t)},\dots,z_1^{(t)},z_{r_{t-1}}^{(t-1)},\dots,z_1^{(1)};w)\right],\end{aligned}$$

where $\mathbb{R}\mathrm{es}_{\underline{r}}=\prod_{i=1}^t \mathbb{R}\mathrm{es}^{(i)}$, $\mathbb{R}\mathrm{es}^{(i)}$ being the operator of taking the iterated residue along
$$\frac{z_{r_i}^{(i)}}{z_{r_i-1}^{(i)}}=\frac{z_{r_i-1}^{(i)}}{z_{r_i-2}^{(i)}}=\cdots=\frac{z_2^{(i)}}{z_1^{(i)}}=q^{-1}$$
and where
$$a(\underline{r})=-\frac{1}{4}(g-1)\sum_i r_i(r_i-1).$$
In an effort to unburden the notation let us rename the variables $(z_{r_t}^{(t)},\dots,z_1^{(1)})$ as $(z_n,z_{n-1},\dots,z_1)$. In particular,
$$z_1^{(i)}=z_{1+r_{<i}}\qquad \forall\, i=1,\dots,t.$$
Using Proposition 5.3 we get
(5.15)
$$\begin{aligned} &Y_{\underline{r}}^{\geq 0}(q^{-\frac{1}{2}r_t}z_1^{(t)},\dots,q^{-\frac{1}{2}r_1}z_1^{(1)};w)=q^{b(\underline{r})}\prod_i \mathrm{vol}_{r_i}\\ &\quad\cdot \mathbb{R}\mathrm{es}_{\underline{r}}\left[\frac{1}{\prod_{i<j}\widetilde{\zeta}\left(\frac{z_i}{z_j}\right)}\sum_{\sigma\in\mathfrak{S}_n}\sigma\left\{\prod_{i<j}\widetilde{\zeta}\left(\frac{z_i}{z_j}\right)\cdot\frac{1}{\prod_{i<n}\left(1-q\frac{z_{i+1}}{z_i}\right)}\cdot\frac{1}{1-q^{-\frac{n}{2}}z_1w}\right\}\right],\end{aligned}$$
where
$$b(\underline{r})=\frac{1}{2}(g-1)\sum_{i<j}r_ir_j.$$

Of course taking appropriate residues in (5.6) yields similar formulas for $Y_{\underline{r}}^{<0}(q^{-\frac{1}{2}r_t}z_1^{(t)},\dots,q^{-\frac{1}{2}r_1}z_1^{(1)};w)$.

5.6. Fix some $r\geq 0$. By a *generic Jordan type* of weight r we will mean a finite (possibly empty) sequence $\underline{r}=(r_1,\dots,r_t)$ of nonnegative integers such that $\sum_i ir_i=r$ and $r_t\neq 0$. Observe that the assignment
$$(r_1,\dots,r_t)\mapsto(1^{r_1}2^{r_2}\dots t^{r_t})$$
sets up a bijection between the set $J_{\mathrm{gen}}(r)$ of generic Jordan types of weight r and the set of partitions of r. A *Jordan type* of weight (r,d) is a sequence

$\underline{\alpha} = (\alpha_1, \dots, \alpha_s)$ such that $\sum_i i\alpha_i = (r, d)$ and $\alpha_s \neq 0$. We denote by $J(r, d)$ the set of Jordan types of weight (r, d). There is a natural forgetful map

$$\pi : \bigsqcup_d J(r, d) \to J_{\mathrm{gen}}(r).$$

Let us fix a generic Jordan type $\underline{r} = (r_1, \dots, r_t)$ of weight $r \geq 0$. We will now compute the sum

$$\begin{aligned}\Xi_{\underline{r}}(z) &= \sum_{\underline{\alpha} \in \pi^{-1}(\underline{r})} \mathrm{vol}(\mathbf{Nil}^{\geq 0}_{\underline{\alpha}}) z^{\sum i\alpha_i} \\ &= \sum_{\underline{\alpha} \in \pi^{-1}(\underline{r})} q^{d'(\underline{\alpha}) - \frac{1}{2}\sum_{i>j}\langle \alpha_i, \alpha_j \rangle} \left(1_{\alpha_s} \cdots 1_{\alpha_1} \mid 1^{\geq 0}_{\sum \alpha_i} \right) z^{\sum i\alpha_i}.\end{aligned}$$

Let us fix some $s \geq t$ and let $\Xi^s_{\underline{r}}(z)$ be the restriction of the sum to the subset of Jordan types $\underline{\alpha} = (\alpha_1, \dots, \alpha_{s'})$ in $\pi^{-1}(\underline{r})$ for which $s' \leq s$. To unburden the notation, we set

$$T = z^{(1,0)}, \qquad z = z^{(0,1)}, \qquad n = \sum r_i.$$

We obtain

$$\begin{aligned}\Xi^s_{\underline{r}}(z) &= q^{e(\underline{r})} \sum_{d_1, \dots, d_s \in \mathbb{Z}} q^{\frac{1}{2}\sum_i d_i(r_{>i} - r_{<i})} \left(1_{r_s, d_s} z^{sd_s} \cdots 1_{r_1, d_1} z^{d_1} \mid 1^{\geq 0}_{n, \sum d_i} \right) T^r \\ &= q^{e(\underline{r})} G^{\geq 0}_{0^{s-t}, r_t, \dots, r_1}(x_s, \dots, x_1; 1) T^r,\end{aligned}$$

where

$$e(\underline{r}) = (g-1) \left[\sum_i (i-1) r_i^2 + \sum_{i<j} (2i - \frac{1}{2}) r_i r_j \right]$$

and $x_l = z^l q^{\frac{1}{2}(r_{>l} - r_{<l})}$ for $l = 1, \dots, s$. Using Proposition 5.1, we have

(5.16)

$$\begin{aligned}\Xi^s_{\underline{r}}(z) = q^{e(\underline{r})} \mathrm{Exp} \Bigg(\frac{|X(\mathbf{F}_q)|}{q-1} \Bigg[&\sum_{i=1}^s q^{-\frac{1}{2}(n + r_i)} x_i \\ &+ \sum_{i>j} \frac{x_i}{x_j} (q^{\frac{r_j}{2}} - q^{-\frac{r_j}{2}}) q^{-\frac{r_i}{2}} \Bigg] \Bigg) Y^{\geq 0}_{r_t, \dots, r_1}(x_t, \dots, x_1; 1) T^r\end{aligned}$$

since, clearly, $Y^{\geq 0}_{0^{s-t}, r_t, \dots, r_1}(x_s, \dots, x_1; 1) = Y^{\geq 0}_{r_t, \dots, r_1}(x_t, \dots, x_1; 1)$. The number of isomorphism classes of pairs $(\mathcal{F}, \theta) \in \mathbf{Nil}^{\geq 0}_{\alpha}$ being finite for any fixed $\alpha = (r, d)$, the stack $\mathbf{Nil}^{\geq 0}_{\underline{\alpha}}$ is empty for almost all $\underline{\alpha}$. In particular, the coefficient of $T^r z^d$ in $\Xi^s_{\underline{r}}$ stabilizes for any fixed d as s tends to infinity, since the possible length of the torsion part of a coherent sheaf in $\mathrm{Coh}^{\geq 0}_{\alpha}$ is bounded (by d). Taking the limit $s \to \infty$ in (5.16) of the sums inside the bracket, we obtain

$$\begin{aligned}\Xi_{\underline{r}}(z) = q^{e(\underline{r})}\mathrm{Exp}\Bigg(\frac{|X(\mathbf{F}_q)|}{q-1}\Bigg[\sum_{i,l\geq 1} q^{-r_{[i,i+l]}}(q^{r_{i+l}}-1)z^l \\ + \frac{z}{1-z}\Bigg]\Bigg) Y^{\geq 0}_{r_t,\dots,r_1}(x_t,\dots,x_1;1)T^r.\end{aligned} \tag{5.17}$$

Let $\lambda = (1^{r_1}2^{r_2}\cdots t^{r_t})$ be the partition associated to $\underline{r}$. Let λ° denote the the set of boxes $s \in \lambda$ satisfying $a(s) > 0$.

LEMMA 5.5. *We have*

$$\mathrm{Exp}\left(\frac{|X(\mathbf{F}_q)|}{q-1}\left[\sum_{i,l\geq 1} q^{-r_{[i,i+l]}}(q^{r_{i+l}}-1)z^l\right]\right) = \prod_{s\in\lambda^\circ}\zeta_X(q^{-1-l(s)}z^{a(s)}).$$

Proof. A direct verification using the formula

$$\mathrm{Exp}(|X(\mathbf{F}_q)|q^{-u}z^v) = \zeta_X(q^{-u}z^v).$$ □

Observing (see Theorem 4.1(i)) that

$$q^{-\sum_i r_i^2}\prod_i \mathrm{vol}_{r_i} = \prod_{s\in\lambda\setminus\lambda^\circ}\zeta^*_X(q^{-1-l(s)}z^{a(s)}),$$

and using (5.15), (5.17) and Lemma 5.5 we arrive at the following expression:

$$\Xi_{\underline{r}}(z) = q^{(g-1)\langle\lambda,\lambda\rangle}\cdot\prod_{s\in\lambda}\zeta^*_X(q^{-1-l(s)}z^{a(s)})\cdot H_{\underline{r}}(z)\cdot\mathrm{Exp}\left(\frac{|X(\mathbf{F}_q)|}{q-1}\cdot\frac{z}{1-z}\right)T^r, \tag{5.18}$$

where

$$\langle\lambda,\lambda\rangle = \sum_k(\lambda')_k^2 = \sum_i ir_i^2 + \sum_{i<j}2ir_ir_j,$$
$$H_{\underline{r}}(z) = \widetilde{H}_{\underline{r}}(z^tq^{-r_{<t}},\dots,z^iq^{-r_{<i}},\dots,z)$$

and

$$\begin{aligned}&\widetilde{H}_{\underline{r}}(z_{1+r_{<t}},\dots,z_{1+r_{<i}},\dots,z_1)\\ &\quad= \mathbb{R}\mathrm{es}_{\underline{r}}\left[\frac{1}{\prod_{i<j}\widetilde{\zeta}\left(\frac{z_i}{z_j}\right)}\sum_{\sigma\in\mathfrak{S}_n}\sigma\left\{\prod_{i<j}\widetilde{\zeta}\Big(\frac{z_i}{z_j}\Big)\cdot\frac{1}{\prod_{i<n}\left(1-q\frac{z_{i+1}}{z_i}\right)}\cdot\frac{1}{1-z_1}\right\}\right].\end{aligned}$$

5.7. Taking the sum over all generic Jordan types $\underline{r}$, using Proposition 2.2 and Corollary 4.6 and setting in accordance with Section 1.3,

$$J_\lambda(z) = \prod_{s\in\lambda}\zeta^*_X(q^{-1-l(s)}z^{a(s)}), \qquad H_\lambda(z) = H_{\underline{r}}(z),$$

when $\lambda = (1^{r_1}2^{r_2}\cdots)$, we get the following complicated but nevertheless explicit generating formula for the numbers $\mathcal{A}^{\geq 0}_{r,d}$:

$$\exp\left(\sum_{l\geq 1}\frac{1}{l}\sum_{r,d}\frac{\mathcal{A}^{\geq 0}_{r,d}(X\otimes_{\mathbf{F}_q}\mathbf{F}_{q^l})}{q^l-1}z^{ld}T^{lr}\right) = \sum_{\lambda}\left\{q^{(g-1)\langle\lambda,\lambda\rangle}J_\lambda(z)H_\lambda(z)T^{|\lambda|}\right\}\cdot \operatorname{Exp}\left(\frac{|X(\mathbf{F}_q)|}{q-1}\cdot\frac{z}{1-z}\right). \tag{5.19}$$

In the above, all the rational functions in z are expanded in the region $z \ll 1$, i.e., in $\mathbb{C}[[z]]$. Observe that $\mathcal{A}^{\geq 0}_{0,d}(X\otimes_{\mathbf{F}_q}\mathbf{F}_{q^l}) = |X(\mathbf{F}_{q^l})|$ since a geometrically indecomposable torsion sheaf on $X\otimes_{\mathbf{F}_q}\mathbf{F}_{q^l}$ is the indecomposable d-fold self extension of the structure sheaf of a rational point in $X(\mathbf{F}_{q^l})$. It follows that

$$\exp\left(\sum_{l\geq 1}\frac{1}{l}\sum_{d}\frac{\mathcal{A}^{\geq 0}_{0,d}(X\otimes_{\mathbf{F}_q}\mathbf{F}_{q^l})}{q^l-1}z^{ld}\right) = \operatorname{Exp}\left(\frac{|X(\mathbf{F}_q)|}{q-1}\cdot\frac{z}{1-z}\right),$$

and (5.19) simplifies to

$$\exp\left(\sum_{l\geq 1}\frac{1}{l}\sum_{r>0,d}\frac{\mathcal{A}^{\geq 0}_{r,d}(X\otimes_{\mathbf{F}_q}\mathbf{F}_{q^l})}{q^l-1}z^{ld}T^{lr}\right) = \sum_{\lambda}\left\{q^{(g-1)\langle\lambda,\lambda\rangle}J_\lambda(z)H_\lambda(z)T^{|\lambda|}\right\}. \tag{5.20}$$

Recall from Section 4.6 that the elements $A^{\geq 0}_{g,r,d} \in K_g$ defined by (4.8) are uniquely characterized by the property that $A^{\geq 0}_{g,r,d}(\sigma_X) = \mathcal{A}_{r,d}(X)$ for all smooth projective curves X. As a consequence we have the following equality in $K_g[[T,z]]$:

$$\operatorname{Exp}\left(\sum_{r>0,d}\frac{A^{\geq 0}_{g,r,d}}{q-1}z^dT^r\right) = \sum_{\lambda}\left\{q^{(g-1)\langle\lambda,\lambda\rangle}J_\lambda(z)H_\lambda(z)T^{|\lambda|}\right\}. \tag{5.21}$$

5.8. Tensoring by a line bundle of degree one induces a bijection between the set of geometrically indecomposable vector bundles on a curve X of rank r and degrees d and $d+r$ respectively. Therefore $A_{r,d}(X)$, and hence $A_{g,r,d}$ only depend on the class of d in $\mathbb{Z}/r\mathbb{Z}$. By Proposition 2.1 the integers $A^{\geq 0}_{g,r,d}(\sigma_X)$ are eventually periodic in d as $d\to\infty$, with period r. Thus so are the $A^{\geq 0}_{g,r,d}$. This means that if we consider the generating function

$$A^{\geq 0}_{g,r}(z) = \sum_{d\geq 0}A^{\geq 0}_{g,r,d}z^d,$$

then we have

$$A^{\geq 0}_{g,r}(z) = P_{g,r}(z) + \sum_{d=0}^{r-1}\frac{A_{g,r,d}z^d}{1-z^r} \tag{5.22}$$

for some polynomial $P_{g,r}(z) \in K_g[z]$. As a consequence of (5.22), the polynomials $A_{g,r,d}$ are expressed as

$$A_{g,r,d} = -\sum_{l\in\mathbb{Z}/r\mathbb{Z}} \mathrm{Res}_{z=\xi^l}\left(A_{g,r}(z)\frac{dz}{z}\right)\xi^{-ld}$$

for ξ a primitive r-th root of unity. This concludes the proof of Theorem 1.6.

5.9. To finish this section, we provide the proof of Corollary 1.9.

Proof of Corollary 1.9. We need to specialize (1.3) to $\alpha_1 = \cdots = \alpha_{2g} = 0$. To this end we rewrite the terms entering (1.3) as follows:

$$q^{(g-1)\langle\lambda,\lambda\rangle}J_\lambda(z) = \prod_{s\in\lambda^\diamond}\frac{\prod_i(\alpha_i q^{1+l(s)} - z^{a(s)})}{(q^{1+l(s)} - z^{a(s)})(q^{l(s)} - z^{a(s)})}\cdot\prod_{s\in\lambda\setminus\lambda^\diamond}\frac{\prod_i(\alpha_i - 1)}{q-1},$$

where $\lambda^\diamond$ denotes the set of $s \in \lambda$ satisfying $a(s) > 0$ or $l(s) > 0$. This expression is regular at the point $\alpha_1 = \cdots = \alpha_{2g} = 0$ and evaluates to

$$\begin{aligned} q^{(g-1)\langle\lambda,\lambda\rangle}J_\lambda(z)_{|\alpha_i=0} &= \prod_{s\in\lambda^\diamond} z^{2(g-1)a(s)}\cdot\prod_{\substack{s\in\lambda^\diamond\\ l(s)=0}}\frac{1}{1-z^{-a(s)}}\cdot(-1)^{|\lambda\setminus\lambda^\diamond|}\\ &= (-1)^{|\lambda\setminus\lambda^\diamond|}z^{(g-1)(\sum_i\lambda_i^2-\sum\lambda_i)}\prod_{\substack{s\in\lambda^\diamond\\ l(s)=0}}\frac{1}{1-z^{-a(s)}}. \end{aligned}$$

Next, we have

$$\begin{aligned} L(z_n,\ldots,z_1) = \sum_{\sigma\in\mathfrak{S}_n}\varepsilon(\sigma)\prod_{\substack{k<l\\ \sigma^{-1}(k)>\sigma^{-1}(l)}}\left(\frac{z_l}{z_k}\right)^{-g}\frac{\prod_i(1-\alpha_i\frac{z_l}{z_k})\cdot(\frac{z_l}{z_k}-q)}{\prod_i(\frac{z_l}{z_k}-\alpha_i)\cdot(1-q\frac{z_l}{z_k})}\\ \cdot\frac{1}{\prod_{j<n}(1-q\frac{z_{\sigma(j+1)}}{z_{\sigma(j)}})\cdot(1-z_{\sigma(1)})}, \end{aligned}$$

where $\varepsilon(\sigma)$ is the sign character. We see that for $\sigma \neq \mathrm{Id}$, the evaluation of the quantity

$$\mathbb{R}\mathrm{es}_\lambda\left[\prod_{\substack{k<l\\ \sigma^{-1}(k)>\sigma^{-1}(l)}}\left(\frac{z_l}{z_k}\right)^{-g}\frac{\prod_i(1-\alpha_i\frac{z_l}{z_k})\cdot(\frac{z_l}{z_k}-q)}{\prod_i(\frac{z_l}{z_k}-\alpha_i)\cdot(1-q\frac{z_l}{z_k})}\cdot\frac{1}{\prod_{j<n}(1-q\frac{z_{\sigma(j+1)}}{z_{\sigma(j)}})\cdot(1-z_{\sigma(1)})}\right]$$

at $z_{1+r_{<i}} = z^i q^{-r_{<i}}$ for $i = 1,\ldots,t$ is a rational function of $\alpha_1,\ldots,\alpha_{2g}$ with coefficients in $\mathbb{Q}(z)$ which is regular and vanishes at the point $\alpha_1 = \cdots = \alpha_{2g} = 0$. As a consequence, if we write $\lambda = (1^{r_1}, 2^{r_2},\ldots)$ and denote by

$i_1 < i_2 < \cdots < i_s$ the integers satisfying $r_{i_j} \neq 0$, then

$$\begin{aligned} H_\lambda(z)_{|\alpha_i=0} &= \frac{1}{(1-z^{i_1})(1-z^{i_2-i_1})\cdots(1-z^{i_s-i_{s-1}})} \\ &= (-1)^s \frac{z^{-i_s}}{(1-z^{-i_1})(1-z^{i_1-i_2})\cdots(1-z^{i_{s-1}-i_s})}. \end{aligned}$$

Observing that $s = |\lambda \backslash \lambda^\diamond|$, we get

$$q^{(g-1)\langle\lambda,\lambda\rangle} J_\lambda(z) H_{\lambda(z)} = z^{(g-1)(\langle\lambda',\lambda'\rangle - |\lambda'|) - l(\lambda')} K_{\lambda'}(z). \tag{5.23}$$

Finally, observe that since $A_{g,r}(z)$ has at most simple poles at r-th roots of unity, the same holds for $A_{g,r}(z)_{|\alpha_i=0}$, and hence the residue at r-th roots of unity is unchanged upon rescaling by a factor of z^{-r}. This allows us to remove the term $z^{-\sum_i \lambda_i} = z^{-r}$ in (5.23). We are done. □

6. **Relation to the number of points of Hitchin moduli spaces**

6.1. In this section, we relate the number of indecomposable vector bundles of a given class α to the number of stable Higgs bundles of the same class, under the assumption that the characteristic p of the field is large enough (with an explicit bound, depending on the genus g of X and the class α), thereby proving Theorem 1.2. Our method is directly inspired by that of Crawley-Boevey, Van den Bergh [CBVdB04] and Nakajima (appendix to loc. cit.) in the context of moduli spaces of representations of quivers, and it hinges on the construction of a smooth deformation $\mathcal{Y} \to \mathbb{A}^1$ of the moduli space of stable Higgs bundles

$$\begin{array}{ccccc} \mathrm{Higgs}^{\mathrm{st}}_{r,d} & \longrightarrow & \mathcal{Y} & \longleftarrow & \mathcal{Y}' \\ \downarrow & & \downarrow & & \downarrow \\ \{0\} & \longrightarrow & \mathbb{A}^1 & \longleftarrow & \mathbb{A}^1\backslash\{0\} \end{array}$$

preserving the number of $\mathbf{F}_q$-rational points and equipped with a projection map $p : \mathcal{Y} \to \mathrm{Bun}_{r,d}$ whose restriction to any fiber $\mathcal{Y}_t$ with $t \neq 0$ is a fibration over the constructible substack $\mathrm{Indec}_{r,d} \subset \mathrm{Bun}_{r,d}$ of indecomposable vector bundles. The construction of $\mathcal{Y}$ itself may appear slightly noncanonical as it involves an explicit local presentation of the stack $\mathbf{Higgs}^{\mathrm{st}}_{r,d}$ in terms of quot schemes. In doing so, we borrow some techniques developed in [ÁCK07].

6.2. Let us fix a smooth projective, geometrically connected curve X of genus g defined over $k = \mathbf{F}_q$, and let Ω_X be the canonical line bundle of X. A Higgs sheaf of rank r and degree d is a pair $(\mathcal{V}, \theta)$ with $\mathcal{V}$ a coherent sheaf of rank r and degree d and $\theta \in \mathrm{Hom}(\mathcal{V}, \mathcal{V} \otimes \Omega_X)$. A Higgs subsheaf of $(\mathcal{V}, \theta)$ is by definition a subsheaf $\mathcal{W} \subseteq \mathcal{V}$ such that $\theta(\mathcal{W}) \subseteq \mathcal{W} \otimes \Omega_X$. A Higgs sheaf $(\mathcal{V}, \theta)$ is called semistable (resp. stable) if for any proper Higgs subsheaf $\mathcal{W} \subset \mathcal{V}$, we have $\mu(\mathcal{W}) \leq \mu(\mathcal{V})$ (resp. $\mu(\mathcal{W}) < \mu(\mathcal{V})$). A Higg subsheaf $\mathcal{W} \subset \mathcal{V}$ satisfying $\mu(\mathcal{W}) > \mu(\mathcal{V})$ is called destabilizing. It is clear that as soon as $r > 0$, a semistable Higgs sheaf $(\mathcal{V}, \theta)$ is necessarily a Higgs bundle; i.e., $\mathcal{V}$ is a vector bundle.

Let $\mathbf{Higgs}_{r,d}(X)$ and $\mathbf{Coh}_{r,d}(X)$ respectively stand for the moduli stacks of Higgs sheaves and coherent sheaves over X of rank r and degree d. These are algebraic stacks defined over k, locally of finite type, of respective dimensions $2(g-1)r^2$ and $(g-1)r^2$. If (r,d) are coprime, we let $\mathbf{Higgs}^{\mathrm{st}}_{r,d}(X)$ be the open substack of $\mathbf{Higgs}_{r,d}(X)$ parametrizing stable Higgs bundles. The stack $\mathbf{Higgs}^{\mathrm{st}}_{r,d}(X)$ is a $\mathbb{G}_m$-gerbe over a smooth connected scheme over k, which we denote by $\mathrm{Higgs}^{\mathrm{st}}_{r,d}$.

Serre duality provides a canonical isomorphism

$$\mathrm{Ext}^1(\mathcal{V}, \mathcal{W})^* \simeq \mathrm{Hom}(\mathcal{W}, \mathcal{V} \otimes \Omega_X)$$

for any pair of coherent sheaves $(\mathcal{V}, \mathcal{W})$. Hence, the moduli stack $\mathbf{Higgs}_{r,d}(X)$ may alternatively be defined as the stack parametrizing pairs $(\mathcal{V}, \nu)$ with $\mathcal{V}$ a coherent sheaf over X of rank r and degree d and $\nu \in \mathrm{Ext}^1(\mathcal{V}, \mathcal{V})^*$. A Higgs subsheaf of such a pair $(\mathcal{V}, \nu)$ is a subsheaf $\mathcal{W} \subseteq \mathcal{V}$ satisfying the following condition:

$$a(\nu) \in b(\mathrm{Ext}^1(\mathcal{W}, \mathcal{W})^*), \tag{6.1}$$

where a, b are the canonical maps in the sequence

$$\mathrm{Ext}^1(\mathcal{V}, \mathcal{V})^* \xrightarrow{a} \mathrm{Ext}^1(\mathcal{V}, \mathcal{W})^* \xleftarrow{b} \mathrm{Ext}^1(\mathcal{W}, \mathcal{W})^* .$$

The stack $\mathbf{Higgs}^{\mathrm{st}}_{r,d}(X)$ thus parametrizes pairs $(\mathcal{V}, \nu)$ as above such that any proper Higgs subsheaf $\mathcal{W} \subset \mathcal{V}$ verifies $\mu(\mathcal{W}) < \mu(\mathcal{V}) = \frac{d}{r}$.

6.3. In this section we recall the definition and basic properties of quot schemes. These will be used in the next section to make explicit the construction of the stacks $\mathbf{Coh}_{r,d}(X)$ and $\mathbf{Higgs}_{r,d}(X)$.

We say that a vector bundle $\mathcal{F}$ is strongly generated by another vector bundle $\mathcal{G}$ if $\mathrm{Ext}^1(\mathcal{G}, \mathcal{F}) = 0$ and the canonical map $\mathcal{G} \otimes \mathrm{Hom}(\mathcal{G}, \mathcal{F}) \to \mathcal{F}$ is surjective. By definition, if $\mathcal{F}$ is strongly generated by $\mathcal{G}$, then $\dim(\mathrm{Hom}(\mathcal{G}, \mathcal{F})) = \langle \mathcal{G}, \mathcal{F} \rangle$. Observe that the notion of being 'strongly generated by' is transitive: if $\mathcal{F}_1$ is strongly generated by $\mathcal{F}_2$, which is itself strongly generated by $\mathcal{F}_3$, then $\mathcal{F}_1$ is strongly generated by $\mathcal{F}_3$.

Given a vector bundle $\mathcal{V}$ over X and a pair $\alpha = (r, d)$, the quot scheme $\mathrm{Quot}(\mathcal{V}, \alpha)$ is the k-scheme representing the functor $\mathrm{quot}_{\mathcal{V},\alpha} : (\mathrm{Aff}/k) \to \mathrm{Sets}$ which assigns to an affine k-scheme S the set of equivalence classes of epimorphisms

$$\phi_S : \mathcal{V} \boxtimes \mathcal{O}_S \twoheadrightarrow \mathcal{F},$$

where $\mathcal{F}$ is an S-flat coherent sheaf over $X \times S$ such that for any closed point $s \in S$, the sheaf $\mathcal{F}_{|s}$ over X is of rank r and degree d. Here, two epimorphisms ϕ_S, ϕ'_S are equivalent if $\mathrm{Ker}(\phi_S) = \mathrm{Ker}(\phi'_S)$. The quot scheme $\mathrm{Quot}(\mathcal{V}, \alpha)$ is a (generally singular) projective scheme. The tangent space to $\mathrm{Quot}(\mathcal{V}, \alpha)$ at a point $\phi : \mathcal{V} \twoheadrightarrow \mathcal{F}$ is equal to $\mathrm{Hom}(\mathrm{Ker}(\phi), \mathcal{F})$.

One constructs an explicit closed embedding in a projective variety as follows. There exists a line bundle $\mathcal{L}$ of sufficiently negative degree so that for any $\phi : \mathcal{V} \twoheadrightarrow \mathcal{F}$ with $\mathcal{F}$ of rank r and degree d, the sheaf $\mathrm{Ker}(\phi)$ is strongly generated by $\mathcal{L}$. Put

$$a = \dim(\mathrm{Hom}(\mathcal{L}, \mathcal{V})) = \langle \mathcal{L}, \mathcal{V} \rangle, \qquad b = \langle \mathcal{L}, \mathcal{V} - \alpha \rangle,$$

and let $\mathrm{Gr}(a, b)$ stand for the Grassmanian of b-dimensional subspaces of k^a. Fixing an identification $\mathrm{Hom}(\mathcal{L}, \mathcal{V}) \simeq k^a$ we obtain a map $j : \mathrm{Quot}(\mathcal{V}, \alpha) \to \mathrm{Gr}(a, b)$ by assigning to a point $\phi : \mathcal{V} \twoheadrightarrow \mathcal{F}$ the subspace $\mathrm{Hom}(\mathcal{L}, \mathrm{Ker}(\phi)) \subset \mathrm{Hom}(\mathcal{L}, \mathcal{V})$. This is a closed embedding (see, e.g., [LP97, Th. 4.4.5.]).

6.4. Let us fix a class $\alpha = (r, d)$ with $r > 0$, and r, d coprime. We will now give a construction of the stacks $\mathbf{Coh}_{r,d}$ and $\mathbf{Higgs}_{r,d}$, or at least of suitable open subset of these stacks. For reasons that will become clear later (see Section 6.7), we will use a variant of the standard construction, based on the choice of *two* line bundles instead of one, which we borrow from [ÁCK07].

LEMMA 6.1. *There exists a pair of line bundles* $(\mathcal{L}_1, \mathcal{L}_2) \in \mathrm{Pic}^{-d_1}(X) \times \mathrm{Pic}^{-d_2}(X)$ *such that the following hold*:

(a) *any semistable Higgs bundle* $(\mathcal{V}, \theta)$ *of class* α *is strongly generated by* $\mathcal{L}_1$;
(b) *any indecomposable vector bundle* $\mathcal{V}$ *of class* α *is strongly generated by* $\mathcal{L}_1$;
(c) *for any unstable Higgs sheaf* $(\mathcal{F}, \theta)$ *of class* α *there exists a destabilizing Higgs subsheaf* $\mathcal{G} \subset \mathcal{F}$ *which is strongly generated by* $\mathcal{L}_1$;
(d) *for any coherent sheaf* $\mathcal{V}$ *of class* α *and any epimorphism* $\phi : \mathcal{L}_1 \otimes V \twoheadrightarrow \mathcal{V}$, $\mathrm{Ker}(\phi)$ *is strongly generated by* $\mathcal{L}_2$;
(e) $\mathcal{L}_1$ *is strongly generated by* $\mathcal{L}_2$.

In particular, any sheaf strongly generated by $\mathcal{L}_1$ *is also strongly generated by* $\mathcal{L}_2$.

Proof. We first show the existence of a line bundle $\mathcal{L}_1$ satisfying (a), (b) and (c). The minimal slope $\mu_{\min}(\mathcal{F})$ of an indecomposable vector bundle $\mathcal{F}$ of class α is bounded below by some constant ν which only depends on α (see Proposition 2.1). An argument in all points similar shows that the minimal slope $\mu_{\min}(\mathcal{F})$ of the vector bundle underlying a semistable Higgs bundle of class α is likewise bounded below by a constant ν' which again only depends on α. Let $(\mathcal{F}, \theta)$ be an unstable Higgs sheaf of class α. By definition there exists a semistable Higgs subsheaf $\mathcal{G} \subset \mathcal{F}$ of slope $\mu(\mathcal{G}) > \mu(\alpha)$ and rank

$\mathrm{rk}(\mathcal{G}) \leq \mathrm{rk}(\alpha)$. Tensoring by a line bundle $\mathcal{O}(-nx)$ for some $x \in X$, if necessary we may assume that $\mu(\alpha) < \mu(\mathcal{G}) \leq \mu(\alpha) + 1$. Because $\mathrm{rank}(\mathcal{G}) \leq r$, there are only finitely many possibilities for the class $\overline{\mathcal{G}}$ of such a sheaf, and therefore by b) the family of all such semistable Higgs sheaves is also bounded. In particular, there exists a constant ν''' which only depends on α, such that any unstable Higgs sheaf of class α contains a destabilizing subsheaf $\mathcal{G}$ satisfying $\mu_{\min}(\mathcal{G}) \geq \nu'''$. For any $\nu \in \mathbb{Q}$, there exists $n \in \mathbb{Z}$ such that any semistable sheaf of slope $\sigma \geq \nu$ is strongly generated by any line bundle of degree $m \leq n$. It suffices to take n as above for $\nu = \min\{\nu', \nu'', \nu'''\}$. This proves the existence of a line bundle $\mathcal{L}_1$ satisfying (a), (b) and (c). Let us now fix such a line bundle. The set of HN types of sheaves $\mathcal{F}$ of class α which are generated by $\mathcal{L}_1$ is finite, as is the set of HN types of kernels of epimorphisms $\mathcal{L}_1 \otimes V \twoheadrightarrow \mathcal{F}$. Therefore there exists $\mathcal{L}_2$ such that any such kernel is strongly generated by $\mathcal{L}_2$. We may of course also assume that $\mathcal{L}_1$ is strongly generated by $\mathcal{L}_2$. We are done. $\square$

Set

$$l_1 = \langle \mathcal{L}_1, \alpha\rangle = (1-g+d_1)r + d, \quad l_2 = \langle \mathcal{L}_2, \alpha\rangle = (1-g+d_2)r + d, \quad V_i = k^{l_i}, \ i = 1, 2.$$

Consider the quot schemes

$$Q_{\mathcal{L}_1,\mathcal{L}_2} = \mathrm{Quot}((\mathcal{L}_1 \otimes V_1) \oplus (\mathcal{L}_2 \otimes V_2), \alpha), \qquad Q_{\mathcal{L}_1} = \mathrm{Quot}(\mathcal{L}_1 \otimes V_1, \alpha).$$

Points of $Q_{\mathcal{L}_1,\mathcal{L}_2}$ correspond to epimorphisms $\phi : (\mathcal{L}_1 \otimes V_1) \oplus (\mathcal{L}_2 \otimes V_2) \twoheadrightarrow \mathcal{F}$; we will usually write $\phi_i = \phi_{|\mathcal{L}_i \otimes V_i}$ for $i = 1, 2$. Denote by $Q^{\circ;\circ}_{\mathcal{L}_1,\mathcal{L}_2}$ the open subscheme of $Q_{\mathcal{L}_1,\mathcal{L}_2}$ parametrizing epimorphisms $\phi : (\mathcal{L}_1 \otimes V_1) \oplus (\mathcal{L}_2 \otimes V_2) \twoheadrightarrow \mathcal{F}$ for which the canonical maps

$$\phi_{i*} : V_i \to \mathrm{Hom}(\mathcal{L}_i, \mathcal{F}), \qquad i = 1, 2$$

are isomorphisms. (This implies, in particular, that $\mathcal{F}$ is strongly generated by $\mathcal{L}_1$ and hence by $\mathcal{L}_2$.) We define $Q^{\circ}_{\mathcal{L}_1} \subset Q_{\mathcal{L}_1}$ in the same fashion. The schemes $Q^{\circ;\circ}_{\mathcal{L}_1,\mathcal{L}_2}$ and $Q^{\circ}_{\mathcal{L}_1}$ are smooth. The group $G := \mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$ naturally acts on $Q_{\mathcal{L}_1,\mathcal{L}_2}$ and preserves $Q^{\circ;\circ}_{\mathcal{L}_1,\mathcal{L}_2}$. Similarly, the group $\mathrm{GL}(V_1)$ acts on $Q_{\mathcal{L}_1}$ and preserves $Q^{\circ}_{\mathcal{L}_1}$. The natural restriction map

$$[\phi : (\mathcal{L}_1 \otimes V_1) \oplus (\mathcal{L}_2 \otimes V_2) \twoheadrightarrow \mathcal{F}] \mapsto [\phi_1 : (\mathcal{L}_1 \otimes V_1) \twoheadrightarrow \mathcal{F}]$$

is a principal $\mathrm{GL}(V_2)$-bundle $Q^{\circ;\circ}_{\mathcal{L}_1,\mathcal{L}_2} \to Q^{\circ}_{\mathcal{L}_1}$. By Lemma 6.1(d), the stack quotient $[Q^{\circ;\circ}_{\mathcal{L}_1,\mathcal{L}_2}/G]$ (and hence $[Q^{\circ}_{\mathcal{L}_1}/\mathrm{GL}(V_1)]$) is isomorphic to the open substack $\mathbf{Coh}^{>\mathcal{L}_1}_{r,d}(X)$ of $\mathbf{Coh}_{r,d}(X)$ parametrizing coherent sheaves $\mathcal{V}$ of class α which are strongly generated by $\mathcal{L}_1$ (see, e.g., [LP97]).

For later purposes, we introduce the locally closed subscheme $Q^{\circ}_{\mathcal{L}_1,\mathcal{L}_2}$ of $Q_{\mathcal{L}_1,\mathcal{L}_2}$ which parametrizes epimorphisms $\phi : (\mathcal{L}_1 \otimes V_1) \oplus (\mathcal{L}_2 \otimes V_2) \twoheadrightarrow \mathcal{F}$ for which $\phi_{2*} : V_2 \to \mathrm{Hom}(\mathcal{L}_2, \mathcal{F})$ is an isomorphism and for which the restriction of ϕ to $\mathcal{L}_1 \otimes V_1$ is still an epimorphism. There is a natural map $Q^{\circ}_{\mathcal{L}_1,\mathcal{L}_2} \to Q_{\mathcal{L}_1}$ which is a principal $\mathrm{GL}(V_2)$-bundle.

The cotangent space $T^*_\phi Q_{\mathcal{L}_1,\mathcal{L}_2}$ to $Q_{\mathcal{L}_1,\mathcal{L}_2}$ at a point $\phi : (\mathcal{L}_1 \otimes V_1) \oplus (\mathcal{L}_2 \otimes V_2) \twoheadrightarrow \mathcal{F}$ is identified with $\mathrm{Hom}(\mathrm{Ker}(\phi), \mathcal{F})^*$. If $\phi \in Q_{\mathcal{L}_1,\mathcal{L}_2}$, then the restriction of the moment map

$$\mu : T^* Q^{\circ,\circ}_{\mathcal{L}_1,\mathcal{L}_2} \to \mathfrak{g}^* = \mathfrak{gl}(V_1)^* \times \mathfrak{gl}(V_2)^*$$

to $T^*_\phi Q_{\mathcal{L}_1,\mathcal{L}_2}$ is the composition $\mu_\phi = \nu_\phi \circ \kappa_\phi$ of the canonical restriction map

$$\mathrm{Hom}(\mathrm{Ker}(\phi), \mathcal{F})^* \to \mathrm{Hom}((\mathcal{L}_1 \otimes V_1) \oplus (\mathcal{L}_2 \otimes V_2), \mathcal{F})^*$$

arising from the long exact sequence

$$(6.2) \qquad \begin{aligned} &0 \longrightarrow \mathrm{Ext}^1(\mathcal{F},\mathcal{F})^* \longrightarrow \mathrm{Hom}(\mathrm{Ker}(\phi), \mathcal{F})^* \\ &\xrightarrow{\kappa_\phi} \mathrm{Hom}((\mathcal{L}_1 \otimes V_1) \oplus (\mathcal{L}_2 \otimes V_2), \mathcal{F})^* \xrightarrow{j} \mathrm{End}(\mathcal{F})^* \longrightarrow 0 \end{aligned}$$

with the map

$$\nu_\phi : \mathrm{Hom}((\mathcal{L}_1 \otimes V_1) \oplus (\mathcal{L}_2 \otimes V_2), \mathcal{F})^* = \bigoplus_i \mathrm{Hom}(V_i, \mathrm{Hom}(\mathcal{L}_i, \mathcal{F})) \to \bigoplus_i \mathrm{End}(V_i)^*$$

induced by composition with $\phi_i^* \in \mathrm{Hom}(V_i, \mathrm{Hom}(\mathcal{L}_i, \mathcal{F}))$.

The stack $[\mu^{-1}(0)/G]$ is isomorphic to the open substack $\mathbf{Higgs}^{>\mathcal{L}_1}_{r,d}(X)$ of $\mathbf{Higgs}_{r,d}(X)$ parametrizing Higgs bundles $(\mathcal{F}, \theta)$ with $\mathcal{F}$ of class α strongly generated by $\mathcal{L}_1$. In particular, by Lemma 6.1(b) the stack $[\mu^{-1}(0)/G]$ contains $\mathbf{Higgs}^{\mathrm{st}}_{r,d}(X)$ as an open substack.

6.5. We will now relate some appropriate fibers of the moment map $\mu : T^* Q^{\circ,\circ}_{\mathcal{L}_1,\mathcal{L}_2} \to \mathfrak{g}^*$ to indecomposable vector bundles. This explains why we considered the quot scheme construction with two line bundles instead of one. Recall that we have assumed r and d to be coprime. It easily follows that we may pick d_1, d_2 and $\mathcal{L}_1, \mathcal{L}_2$ verifying the hypothesis (a)–(e) of Lemma 6.1 in such a way that l_1 and l_2 are also coprime. Consider the element $\lambda \in \mathfrak{g}^* = \mathfrak{gl}(V_1)^* \times \mathfrak{gl}(V_2)^*$ defined by

$$\lambda(u_1, u_2) = l_2 \mathrm{Tr}(u_1) - l_1 \mathrm{Tr}(u_2).$$

From now on, *we will assume that* $p > l_1 l_2$. By construction we have

(i) $\lambda(\mathrm{Id}, \mathrm{Id}) = 0$;
(ii) $\lambda(e_1, e_2) \neq 0$ for any nontrivial pair of projectors $(e_1, e_2) \in \mathfrak{gl}(V_1) \times \mathfrak{gl}(V_2)$.

LEMMA 6.2. *Let $\phi : (\mathcal{L}_1 \otimes V_1) \oplus (\mathcal{L}_2 \otimes V_2) \twoheadrightarrow \mathcal{F}$ be a k-point of $Q^{\circ,\circ}_{\mathcal{L}_1,\mathcal{L}_2}$. We have $k\lambda \subset \mathrm{Im}(\mu_\phi)$ if and only if $\mathcal{F}$ is indecomposable.*

Proof. By (6.2) we have $k\lambda \in \mathrm{Im}(\mu_\phi)$ if and only if $j(\lambda) = 0$ if and only if $\lambda(f \circ \phi) = 0$ for all $f \in \mathrm{End}(\mathcal{F})$. Let us assume that $\mathcal{F}$ is decomposable, and let us fix a nontrivial decomposition $\mathcal{F} = \mathcal{G} \oplus \mathcal{H}$. As $\mathcal{F} \in \mathcal{C}_{>0}$, we have $\mathcal{G}, \mathcal{H} \in \mathcal{C}_{>0}$. In particular, $\mathrm{Ext}^1(\mathcal{L}_i, \mathcal{G}) = \mathrm{Ext}^1(\mathcal{L}_i, \mathcal{H}) = \{0\}$ and we have decompositions

$$\mathrm{Hom}(\mathcal{L}_i, \mathcal{F}) = \mathrm{Hom}(\mathcal{L}_i, \mathcal{G}) \oplus \mathrm{Hom}(\mathcal{L}_i, \mathcal{H}).$$

Let f be the projector onto $\mathcal{G}$ along $\mathcal{H}$. Thus $f \circ \phi$ is the projector onto $\mathrm{Hom}(\mathcal{L}_1, \mathcal{G}) \oplus \mathrm{Hom}(\mathcal{L}_2, \mathcal{G})$ along $\mathrm{Hom}(\mathcal{L}_1, \mathcal{H}) \oplus \mathrm{Hom}(\mathcal{L}_2, \mathcal{H})$. By (ii) above, $\lambda(f \circ \phi) \neq 0$ hence $\lambda \notin \mathrm{Im}(\mu_\phi)$.

Next let us assume that $\mathcal{F}$ is indecomposable (and thus also geometrically indecomposable as (r, d) are coprime). By Fitting's lemma, $\mathrm{End}(\mathcal{F})$ is a local k-algebra with $\mathrm{End}(\mathcal{F})/\mathrm{rad}(\mathrm{End}(\mathcal{F})) = k$, and therefore every endomorphism is of the form $f = c\mathrm{Id} + n$ for some nilpotent n. But then $f \circ \phi = c(\mathrm{Id}, \mathrm{Id}) + (n_1, n_2)$ for some nilpotent n_1, n_2. Using (i), we deduce that

$$\lambda(f \circ \phi) = c\lambda(\mathrm{Id}, \mathrm{Id}) + \lambda(n_1, n_2) = 0.$$

It follows that $\lambda \in \mathrm{Im}(\mu_\phi)$. □

6.6. Our next goal will be to construct and study the symplectic quotient of $Q^{\circ,\circ}_{\mathcal{L}_1,\mathcal{L}_2}$ by the group G. This will be done in Section 6.7. In the present section, following [ÁCK07] we embed $Q^{\circ,\circ}_{\mathcal{L}_1,\mathcal{L}_2}$ as a locally closed subvariety of the representation variety of an appropriate Kronecker quiver. Namely, put $h = \dim(\mathrm{Hom}(\mathcal{L}_2, \mathcal{L}_1)) = (1-g) + d_2 - d_1$ and let Kr stand for the quiver with vertex set $\{1, 2\}$ and h arrows from 1 to 2:

$$\underset{\bullet}{1} \xrightarrow{h} \underset{\bullet}{2}.$$

Set

$$\mathbb{V} = \mathrm{Hom}(\mathcal{L}_2, \mathcal{L}_1), \qquad E = \mathrm{Hom}(V_1 \otimes \mathbb{V}, V_2).$$

The group G acts on E by conjugation, and the quotient stack $[E/G]$ is the moduli stack of representations of Kr of dimension (l_1, l_2). There is a natural map $j : Q^{\circ,\circ}_{\mathcal{L}_1,\mathcal{L}_2} \to E$ sending the point $\phi : (\mathcal{L}_1 \otimes V_1) \oplus (\mathcal{L}_2 \oplus V_2) \twoheadrightarrow \mathcal{F}$ to the induced map

$$V_1 \otimes \mathbb{V} = \mathrm{Hom}(\mathcal{L}_2, \mathcal{L}_1 \otimes V_1) \xrightarrow{\phi_1} \mathrm{Hom}(\mathcal{L}_2, \mathcal{F}) \xrightarrow{\phi_{2*}^{-1}} V_2.$$

Lemma 6.1(d) guarantees that this defines an embedding of $Q^{\circ,\circ}_{\mathcal{L}_1,\mathcal{L}_2}$ in E as a smooth locally closed subvariety. Observe that the embedding j extends to an embedding $Q^{\circ}_{\mathcal{L}_1,\mathcal{L}_2} \to E$. In fact, set

$$E^\circ = \{u \in E \mid \mathrm{Im}(u) = V_2\}.$$

This is a principal $\mathrm{GL}(V_2)$-bundle over the Grassmanian $\mathrm{Gr}(hl_1, l_2)$. We have the following diagram:

$$\begin{array}{ccccc} Q^{\circ,\circ}_{\mathcal{L}_1,\mathcal{L}_2} & \longrightarrow & Q^{\circ}_{\mathcal{L}_1,\mathcal{L}_2} & \xrightarrow{j} & E^\circ \\ & & \downarrow & & \downarrow \\ & & Q_{\mathcal{L}_1} & \xrightarrow{j'} & \mathrm{Gr}(hl_1, l_2) \end{array} \tag{6.3}$$

in which the two vertical maps are $\mathrm{GL}(V_2)$-bundles and the horizontal maps are embeddings, with j' being the closed embedding described in Section 6.3.

Using the trace pairing, we may identify the cotangent space $T^*E = E \times E^*$ with the representation space of the double $\overline{\mathrm{Kr}}$ of Kr (that is, the quiver with vertex set $\{1, 2\}$, h arrows from 1 to 2 and h arrows from 2 to 1) of dimension (l_1, l_2):

$$\overset{1}{\bullet} \underset{h}{\overset{h}{\rightleftarrows}} \overset{2}{\bullet}$$

so that

$$T^*E \simeq \mathrm{Hom}(V_1 \otimes \mathbb{V}, V_2) \times \mathrm{Hom}(V_2 \otimes \mathbb{V}^*, V_1).$$

Fixing dual bases $\{v_1, \ldots, v_h\}$ and $\{v_1^*, \ldots, v_h^*\}$ of $\mathbb{V}, \mathbb{V}^*$, we may write an element of T^*E as a pair $(\underline{x}, \underline{y})$ with $\underline{x} = (x_1, \ldots, x_h)$, $\underline{y} = (y_1, \ldots, y_h)$ and $x_i \in \mathrm{Hom}(V_1, V_2), y_i \in \mathrm{Hom}(V_2, V_1)$. Using this identification, and identifying $\mathfrak{g}$ with $\mathfrak{g}^*$ via the usual trace pairing, the moment map for the action of G on T^*E reads

$$\mu : T^*E \to \mathfrak{g}^*, \qquad \mu(\underline{x}, \underline{y}) = \left(\sum_{i=1}^h y_i x_i, -\sum_{i=1}^h x_i y_i \right).$$

The Zariski closure $P = \overline{Q^{\circ,\circ}_{\mathcal{L}_1,\mathcal{L}_2}}$ of $Q^{\circ,\circ}_{\mathcal{L}_1,\mathcal{L}_2}$ in E is a (possibly singular) affine variety. Of course, since $Q^{\circ,\circ}_{\mathcal{L}_1,\mathcal{L}_2}$ is dense in $Q^{\circ}_{\mathcal{L}_1,\mathcal{L}_2}$, we have $P = \overline{Q^{\circ}_{\mathcal{L}_1,\mathcal{L}_2}}$. We will denote by $j : Q^{\circ,\circ}_{\mathcal{L}_1,\mathcal{L}_2} \to P$ and $i : P \to E$ the open, resp. closed, embeddings. There is a canonical projection $\pi : P \times E^* \to T^*P$ whose fibers are affine spaces. Namely, over a point $\underline{x} \in P$, the map π is the natural projection

$$E^* = T^*_x E \to T^*_x E/(T_x P)^\perp = T^*_x P.$$

The map π restricts to an affine fibration $\pi^\circ : Q^{\circ,\circ}_{\mathcal{L}_1,\mathcal{L}_2} \times E^* \to T^* Q^{\circ,\circ}_{\mathcal{L}_1,\mathcal{L}_2}$. The moment maps on $T^*Q^{\circ,\circ}_{\mathcal{L}_1,\mathcal{L}_2}, T^*P$ and T^*E fit in a commutative diagram

$$\begin{array}{ccccc}
Q^{\circ,\circ}_{\mathcal{L}_1,\mathcal{L}_2} \times E^* & \xrightarrow{j \times \mathrm{Id}} & P \times E^* & \xrightarrow{i \times \mathrm{Id}} & T^*E \\
\downarrow{\scriptstyle \pi^\circ} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \mu} \\
T^*Q^{\circ,\circ}_{\mathcal{L}_1,\mathcal{L}_2} & \xrightarrow{d^*j} & T^*P & & \\
& \searrow{\scriptstyle \mu} & & \searrow{\scriptstyle \mu} & \\
& & & & \mathfrak{g}^*,
\end{array}$$

where d^*j is the open embedding induced by j.

For the reader's convenience, we make explicit the map d^*j. We begin with the differential $dj_\phi : T_\phi Q^{\circ,\circ}_{\mathcal{L}_1,\mathcal{L}_2} \to T_{j(\phi)} P$ at a point $\phi : \bigoplus_i (\mathcal{L}_i \otimes V_i) \twoheadrightarrow \mathcal{F}$.

Recall that we have canonical identifications

$$\begin{aligned} T_\phi Q^{\circ,\circ}_{\mathcal{L}_1,\mathcal{L}_2} &= \mathrm{Hom}(\mathrm{Ker}(\phi),\mathcal{F}),\\ T_{j(\phi)}P\subset T_{j(\phi)}E &= E=\mathrm{Hom}(V_1\otimes\mathbb{V},V_2)\\ &\simeq \mathrm{Hom}(\mathrm{Hom}(\mathcal{L}_2,\mathcal{L}_1\otimes V_1),\mathrm{Hom}(\mathcal{L}_2,\mathcal{F})). \end{aligned}$$

Consider the exact sequences

$$0\longrightarrow \mathrm{Ker}(\phi_1)\longrightarrow \mathrm{Ker}(\phi)\xrightarrow{\rho_2}\mathcal{L}_2\otimes V_2\longrightarrow 0, \tag{6.4}$$

$$0\longrightarrow \mathrm{Ker}(\phi_2)\longrightarrow \mathrm{Ker}(\phi)\xrightarrow{\rho_1}\mathcal{L}_1\otimes V_1\longrightarrow 0. \tag{6.5}$$

The first exact sequence (6.4) is split as $\mathrm{Ext}^1(\mathcal{L}_2,\mathrm{Ker}(\phi_1))=0$ by Lemma 6.1(d). It follows that

$$\begin{aligned} \dim(\mathrm{Hom}(\mathcal{L}_2,\mathrm{Ker}(\phi))) &= \dim(V_2)+\dim(\mathrm{Hom}(\mathcal{L}_2,\mathrm{Ker}(\phi_2)))\\ &=\langle\mathcal{L}_2,\alpha\rangle+\langle\mathcal{L}_2,\mathcal{L}_1\otimes V_1-\alpha\rangle\\ &=\langle\mathcal{L}_2,\mathcal{L}_1\otimes V_1\rangle\\ &=\dim(\mathrm{Hom}(\mathcal{L}_2,\mathcal{L}_1\otimes V_1)). \end{aligned}\tag{6.6}$$

On the other hand, the exact sequence (6.5) gives rise to a sequence

$$0\longrightarrow \mathrm{Hom}(\mathcal{L}_2,\mathrm{Ker}(\phi_2))\longrightarrow \mathrm{Hom}(\mathcal{L}_2,\mathrm{Ker}(\phi))\xrightarrow{\rho_{1*}}\mathrm{Hom}(\mathcal{L}_2,\mathcal{L}_1\otimes V_1)\ ,$$

and since $\mathrm{Hom}(\mathcal{L}_2,\mathrm{Ker}(\phi_2))=0$, this yields by (6.6) a canonical isomorphism $\rho_{1*}:\mathrm{Hom}(\mathcal{L}_2,\mathrm{Ker}(\phi))\to\mathrm{Hom}(\mathcal{L}_2,\mathcal{L}_1\otimes V_1)$. The map dj_ϕ is equal to the composition

$$\begin{aligned} \mathrm{Hom}(\mathrm{Ker}(\phi),\mathcal{F}) &\xrightarrow{q}\mathrm{Hom}(\mathrm{Hom}(\mathcal{L}_2,\mathrm{Ker}(\phi)),\mathrm{Hom}(\mathcal{L}_2,\mathcal{F}))\\ &\xrightarrow{\rho_{1*}}\mathrm{Hom}(\mathrm{Hom}(\mathcal{L}_2,\mathcal{L}_1\otimes V_1),\mathrm{Hom}(\mathcal{L}_2,\mathcal{F})), \end{aligned}\tag{6.7}$$

with

$$q:\mathrm{Hom}(\mathrm{Ker}(\phi),\mathcal{F})\to\mathrm{Hom}(\mathrm{Hom}(\mathcal{L}_2,\mathrm{Ker}(\phi)),\mathrm{Hom}(\mathcal{L}_2,\mathcal{F}),\quad u\mapsto(a\mapsto u\circ a).$$

Because $j:Q^{\circ,\circ}_{\mathcal{L}_1,\mathcal{L}_2}\to P$ is an open embedding, $dj_\phi: T_\phi Q^{\circ,\circ}_{\mathcal{L}_1,\mathcal{L}_2}\to T_{j(\phi)}P$ is an isomorphism. The map d^*j_ϕ is the transpose isomorphism $T^*_\phi Q^{\circ,\circ}_{\mathcal{L}_1,\mathcal{L}_2}\to E^*/(T_{j(\phi)}P)^\perp$.

6.7. We may now consider GIT quotients of the various above spaces, following the method in [Kin94]. We will consider the stability condition associated to the character

$$\gamma:G\to k^*,(g_1,g_2)\mapsto\det(g_1)^{l_2}\det(g_2)^{-l_1}.$$

Let $(T^*E)^{\mathrm{ss}}, (T^*P)^{\mathrm{ss}}, (P\times E^*)^{\mathrm{ss}}$ denote the open sets of γ-semistable points, and let $T^*E/\!/G, (T^*P)/\!/G$ and $(P\times E^*)/\!/G$ denote the affine quotients. Because T^*E, T^*P and $P\times E^*$ are all affine varieties, there are proper maps

$$p:(T^*E)^{\mathrm{ss}}/\!/G \to (T^*E)/\!/G, \qquad p':(P\times E^*)^{\mathrm{ss}}/\!/G\to (P\times E^*)/\!/G,$$
$$p'':(T^*P)^{\mathrm{ss}}/\!/G\to (T^*P)/\!/G.$$

We have

$$(T^*E)^{\mathrm{ss}}/\!/G = \mathrm{Proj}\left(\bigoplus_{l\geq 0} k[T^*E]^{\gamma,l}\right), \qquad (T^*E)/\!/G = \mathrm{Spec}(k[T^*E]^G),$$

where

$$k[T^*E]^{\gamma,l} = \{f\in k[T^*E]\mid g\cdot f = \gamma(g)^l f \quad \forall\, g\in G\},$$

and there are similar descriptions in the cases of $P\times E^*$ and T^*P. Finally, the closed embedding $i\times \mathrm{Id}: P\times E^*\hookrightarrow T^*E$ and the surjective map $\pi: P\times E^*\to T^*P$ give rise to maps

$$(T^*P)/\!/G \xleftarrow{\overline{\pi}} (P\times E^*)/\!/G \xrightarrow{\overline{i\times \mathrm{Id}}} (T^*E)/\!/G\ .$$

Note that $\overline{\pi}$ is surjective while $\overline{i\times\mathrm{Id}}$ is a closed embedding.

Recall that a subrepresentation of a representation $(\underline{x},\underline{y})\in T^*E$ is a pair of subspaces $(W_1\subseteq V_1, W_2\subseteq W_2)$ such that $x_i(W_1)\subseteq W_2, y_i(W_2)\subseteq W_1$ for all i. Similarly, we will call subrepresentation of some $(\underline{x},\underline{y})\in T^*P$ a pair of subspaces $(W_1\subseteq V_1, W_2\subseteq W_2)$ such that $x_i(W_1)\subseteq W_2$ and $y_i\in \mathfrak{p}_W/(T_{\underline{x}}P)^{\perp}$, where $\mathfrak{p}_W=\{u\in \mathrm{Hom}(V_2,V_1)\mid u(W_2)\subseteq W_1\}$.

Lemma 6.3. *The following hold*:

(i) *a point* $(\underline{x},\underline{y})\in T^*E$ *is* γ*-semistable* (*resp.* γ*-stable*) *if and only if for any subrepresentation* $W=(W_1,W_2)$ *of* $(\underline{x},\underline{y})$, *we have* $l_1\mathrm{dim}(W_2)-l_2\mathrm{dim}(W_1)\geq 0$ (*resp.* >0),

(ii) *a point* $(\underline{x},\underline{y})\in T^*P$ *is* γ*-semistable* (*resp.* γ*-stable*) *if and only if for any subrepresentation* $W=(W_1,W_2)$ *of* $(\underline{x},\underline{y})$, *we have* $l_1\mathrm{dim}(W_2)-l_2\mathrm{dim}(W_1)\geq 0$ (*resp.* >0).

Proof. The first statement is well known and follows from the Hilbert-Mumford numerical criterion (see [Kin94, Prop. 3.1]). The second one can be proved along the same lines, or deduced from (i) together with the fact that P is closed in E. Note that the Hilbert-Mumford criterion is stated in [Kin94] for algebraically closed fields, but it holds over an arbitrary perfect field; see [Kem78, Cor. 4.3]. (Recall that the notion of semistability of representations of quivers (or of coherent sheaves) is stable under field extension; see [HL10, Th. 1.3.7].) □

Set $(T^*Q^{\circ,\circ}_{\mathcal{L}_1,\mathcal{L}_2})^{\mathrm{ss}} = T^*Q^{\circ,\circ}_{\mathcal{L}_1,\mathcal{L}_2}\cap (d^*j)^{-1}\left((T^*P)^{\mathrm{ss}}\right)$.

LEMMA 6.4. *We have* $(T^*P)^{\mathrm{ss}} = d^*j((T^*Q^{\circ,\circ}_{\mathcal{L}_1,\mathcal{L}_2})^{\mathrm{ss}})$, *i.e.*, $(T^*P)^{\mathrm{ss}} \subset d^*j(T^*Q^{\circ,\circ}_{\mathcal{L}_1,\mathcal{L}_2})$.

Proof. Let $\rho : T^*P \to P$ be the natural projection. Observe that $(T^*P)^{\mathrm{ss}} \subseteq \rho^{-1}(P \cap E^\circ)$. Indeed, if $(\underline{x},\underline{y}) \in T^*P$ with $\mathrm{Im}(\underline{x}) \subsetneq V_2$, then the subrepresentation $W = (V_1, \mathrm{Im}(\underline{x}))$ violates the semistability condition of Lemma 6.3(ii). Similarly, if $\underline{x} \in E$ satisfies $\bigcap_i \mathrm{Ker}(x_i) \neq \{0\}$, then $(\underline{x},\underline{y})$ is not semistable for any $\underline{y}$ since the subspace $W = (\bigcap_i \mathrm{Ker}(x_i), 0)$ violates the semistability condition. By diagram (6.3), $P \cap E^\circ = Q^\circ_{\mathcal{L}_1,\mathcal{L}_2}$. Moreover, by construction, if $P \cap E^\circ \ni \underline{x} = j(\phi : (\mathcal{L}_1 \otimes V_1) \oplus (\mathcal{L}_2 \otimes V_2) \twoheadrightarrow \mathcal{F})$ satisfies $\bigcap_i \mathrm{Ker}(x_i) = \{0\}$, then the map $\phi_{1*} : V_1 \to \mathrm{Hom}(\mathcal{L}_1, \mathcal{F})$ is injective, hence bijective as $\dim(V_1) = \dim(\mathrm{Hom}(\mathcal{L}_1, \mathcal{F}))$. This implies that $(T^*P)^{\mathrm{ss}} \subset \rho^{-1}(Q^{\circ,\circ}_{\mathcal{L}_1,\mathcal{L}_2})$. The lemma is proved. □

6.8. Put $A = k\lambda \subset \mathfrak{g}^*$ and

$$\mathcal{X} = \mu^{-1}(A) \subset T^*P, \qquad \mathcal{X}_t = \mu^{-1}(\{t\lambda\}), \qquad \mathcal{X}' = \mathcal{X} \backslash \mathcal{X}_0.$$

The idea is now to consider a GIT quotient $\mathcal{Y}$ of $\mathcal{X}$ and view the family of smooth varieties $\mathcal{Y} \to A$ as a deformation of the moduli space of stable Higgs bundles of rank r and degree d. Because the moment map $\mu : T^*P \to \mathfrak{g}^*$ is G-equivariant, we still have a map $\overline{\mu} : (T^*P)^{\mathrm{ss}}/\!/G \to A$. We set

$$\mathcal{Y} = \overline{\mu}^{-1}(A), \qquad \mathcal{Y}_t = \overline{\mu}^{-1}(\{t\lambda\}), \qquad \mathcal{Y}' = \mathcal{Y} \backslash \mathcal{Y}_0.$$

By construction, $\mathcal{Y} = \mathcal{X}^{\mathrm{ss}}/\!/G$, where $\mathcal{X}^{\mathrm{ss}} = \mathcal{X} \cap (T^*P)^{\mathrm{ss}}$. Observe that by Lemma 6.4 we have $\mathcal{X}^{\mathrm{ss}} \subset T^*Q^{\circ,\circ}_{\mathcal{L}_1,\mathcal{L}_2}$.

LEMMA 6.5. *The k-schemes $\mathcal{Y}, \mathcal{Y}'$ and $\mathcal{Y}_t$ for $t \in k$ are smooth. In addition, $\mathcal{X}' \subset \mathcal{X}^{\mathrm{ss}}$.*

Proof. Because l_1, l_2 are relatively prime, we have $l_2\dim(W_1) - l_1\dim(W_2) \neq 0$ for any proper pair of subspaces $W_1 \subset V_1, W_2 \subset V_2$. This implies that the notions of semistability and stability coincide in T^*P. The action of $PG := G/\mathbb{G}_m$ on $(T^*P)^{\mathrm{ss}}$ thus has finite stabilizers. On the other hand, the stabilizer for the action of G on any representation $(\underline{x},\underline{y}) \in T^*E$ is the automorphism group $\mathrm{Aut}((\underline{x},\underline{y}))$ which is open in $\mathrm{End}((\underline{x},\underline{y}))$ and hence connected. We deduce that the action of PG on T^*P has no finite stabilizers and, in particular, that the action of PG on $(T^*P)^{\mathrm{ss}}$ is free. It follows that $(T^*P)^{\mathrm{ss}}/\!/G = \big((T^*P)^{\mathrm{ss}} \cap T^*Q^{\circ,\circ}_{\mathcal{L}_1,\mathcal{L}_2}\big)/\!/G$ is smooth. The first statement will be proved once we show that the map $\overline{\mu} : (T^*P)^{\mathrm{ss}}/\!/G \to \mathfrak{g}^*$ is submersive. This is a consequence of [CBVdB04, Lemma 2.1.5]. (Note that the hypothesis that the field be algebraically closed is not used in the proof there.) We turn to the second statement. Let $u = (\underline{x},\underline{y}) \in \mathcal{X}'$, and let us assume that u is not semistable. Thus, by Lemma 6.3 there exists a subrepresentation (W_1, W_2) of u such that $l_2\dim(W_1) - l_1\dim(W_2) > 0$. There exists a lift $u' = (\underline{x}, \underline{y'}) \in P \times E^*$ of u for which (W_1, W_2) is also a subrepresentation. Moreover, we have $\mu(u') = \mu(u) = t\lambda$ with $t \neq 0$. But then $0 = \mathrm{Tr}(\mu(u')_{|W_1 \oplus W_2}) = t(l_2 \mathrm{dim}(W_1) - l_1 \mathrm{dim}(W_2))$ in contradiction with property (ii) of λ (see Section 6.5). ☐

Proposition 6.6. *There is a canonical isomorphism of schemes* $\mathcal{Y}_0 \simeq \mathrm{Higgs}^{\mathrm{st}}_{r,d}(X)$.

Proof. Let us fix a pair $(\phi, \theta) \in \mu^{-1}(0) \subset T^* Q^{\circ,\circ}_{\mathcal{L}_1,\mathcal{L}_2}$ with

$$\phi : (\mathcal{L}_1 \otimes V_1) \oplus (\mathcal{L}_2 \otimes V_2) \twoheadrightarrow \mathcal{V}.$$

We will say that (ϕ, θ) is μ-stable if $(\mathcal{V}, \theta)$ is a stable Higgs bundle (as in Section 6.2). We will say that (ϕ, θ) is γ-stable if $d^* j((\phi, \theta)) \in (T^*P)^{\mathrm{ss}}$ (i.e., is γ-semistable). Recall (see Section 6.4) that $\mathbf{Higgs}^{\mathrm{st}}_{r,d} \subset [\mu^{-1}(0)/G]$. The proof of Proposition 6.6 boils down to showing that $(\phi, \theta) \in \mu^{-1}(0)$ is μ-stable if and only if it is γ-stable.

Let us denote by $\mathcal{S}_X$ the (finite) set of subsheaves $\mathcal{W} \subset \mathcal{V}$ which are strongly generated by $\mathcal{L}_1$. We will also denote by $\mathcal{S}'_X$ the subset of $\mathcal{S}_X$ consisting of Higgs subsheaves. Likewise, let us denote by $\mathcal{S}_{\mathrm{Kr}}$ and $\mathcal{S}'_6$ the (finite) sets of submodules of $j(\phi)$ and $d^* j(\phi, \theta)$ respectively. There is a natural injective map

$$\psi : \mathcal{S}_X \to \mathcal{S}_{\mathrm{Kr}},$$
$$\mathcal{W} \mapsto (\mathrm{Hom}(\mathcal{L}_1, \mathcal{W}), \mathrm{Hom}(\mathcal{L}_2, \mathcal{W})) \subseteq (\mathrm{Hom}(\mathcal{L}_1, \mathcal{V}), \mathrm{Hom}(\mathcal{L}_2, \mathcal{V})) \simeq (V_1, V_2).$$

Lemma 6.7. *We have* $\mathcal{W} \in \mathcal{S}'_X$ *if and only if* $\psi(\mathcal{W}) \in \mathcal{S}'_{\mathrm{Kr}}$; *i.e.,* $\psi^{-1}(\mathcal{S}'_{\mathrm{Kr}}) = \mathcal{S}'_X$.

Proof. By definition a Higgs subsheaf of $(\mathcal{V}, \theta)$ is a subsheaf $\mathcal{W} \subset \mathcal{V}$ satisfying (6.1). For a subsheaf $\mathcal{W} \subset \mathcal{V}$ which is strongly generated by $\mathcal{L}_1$, we have a commutative diagram

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathrm{Ker}(\phi) & \longrightarrow & \bigoplus_i \mathcal{L}_i \otimes V_i & \xrightarrow{\sim} & \bigoplus_i \mathcal{L}_i \otimes \mathrm{Hom}(\mathcal{L}_i, \mathcal{V}) & \longrightarrow & \mathcal{V} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathrm{Ker}(\phi_{\mathcal{W}}) & \longrightarrow & \bigoplus_i \mathcal{L}_i \otimes W_i & \xrightarrow{\sim} & \bigoplus_i \mathcal{L}_i \otimes \mathrm{Hom}(\mathcal{L}_i, \mathcal{W}) & \longrightarrow & \mathcal{W} & \longrightarrow & 0,
\end{array}$$

where the upward arrows are canonical embeddings and where the subspaces $W_i \subset V_i$ are defined as $\phi_{i*}^{-1}(\mathrm{Hom}(\mathcal{L}_i, \mathcal{W}))$. This gives rise to a commutative diagram

$$\begin{array}{ccccc}
\mathrm{Ext}^1(\mathcal{V}, \mathcal{V})^* & \xrightarrow{i} & \mathrm{Hom}(\mathrm{Ker}(\phi_{\mathcal{V}}), \mathcal{V})^* & \xleftarrow{\pi} & \mathrm{Hom}(\mathrm{Hom}(\mathcal{L}_2, \mathrm{Ker}(\phi_{\mathcal{V}})), \mathrm{Hom}(\mathcal{L}_2, \mathcal{V}))^* \\
\downarrow a & & \downarrow a' & & \downarrow \\
\mathrm{Ext}^1(\mathcal{V}, \mathcal{W})^* & \xrightarrow{i'} & \mathrm{Hom}(\mathrm{Ker}(\phi_{\mathcal{V}}), \mathcal{W})^* & \xleftarrow{\pi'} & \mathrm{Hom}(\mathrm{Hom}(\mathcal{L}_2, \mathrm{Ker}(\phi_{\mathcal{V}})), \mathrm{Hom}(\mathcal{L}_2, \mathcal{W}))^* \\
\uparrow b & & \uparrow b' & & \uparrow \\
\mathrm{Ext}^1(\mathcal{W}, \mathcal{W})^* & \xrightarrow{i''} & \mathrm{Hom}(\mathrm{Ker}(\phi_{\mathcal{W}}), \mathcal{W})^* & \xleftarrow{\pi''} & \mathrm{Hom}(\mathrm{Hom}(\mathcal{L}_2, \mathrm{Ker}(\phi_{\mathcal{W}})), \mathrm{Hom}(\mathcal{L}_2, \mathcal{W}))^*.
\end{array}$$

The maps i, i', i'' are injective while the maps π, π', π'' are surjective. This diagram may be completed with an extra column of identifications

$$\begin{array}{ccc}
\mathrm{Hom}(\mathrm{Hom}(\mathcal{L}_2, \mathrm{Ker}(\phi_{\mathcal{V}})), \mathrm{Hom}(\mathcal{L}_2, \mathcal{V}))^* & \xrightarrow{\sim} & \mathrm{Hom}(V_1, V_2 \otimes \mathbb{V}^*)^* \\
\downarrow & & \downarrow a'' \\
\mathrm{Hom}(\mathrm{Hom}(\mathcal{L}_2, \mathrm{Ker}(\phi_{\mathcal{V}})), \mathrm{Hom}(\mathcal{L}_2, \mathcal{W}))^* & \xrightarrow{\sim} & \mathrm{Hom}(V_1, W_2 \otimes \mathbb{V}^*)^* \\
\uparrow & & \uparrow b'' \\
\mathrm{Hom}(\mathrm{Hom}(\mathcal{L}_2, \mathrm{Ker}(\phi_{\mathcal{W}})), \mathrm{Hom}(\mathcal{L}_2, \mathcal{W}))^* & \xrightarrow{\sim} & \mathrm{Hom}(W_1, W_2 \otimes \mathbb{V}^*)^*
\end{array}$$

in which the horizontal arrows are induced by the isomorphisms

$$\mathrm{Hom}(\mathcal{L}_2, \mathrm{Ker}(\phi_{\mathcal{V}})) \simeq \mathrm{Hom}(\mathcal{L}_2, \mathcal{L}_1 \otimes V_1), \quad \mathrm{Hom}(\mathcal{L}_2, \mathcal{V}) \simeq V_2$$

(see Section 6.5) and the similar isomorphisms with $\mathcal{W}$ instead of $\mathcal{V}$. Observe that π is identified with the projection

$$T^*_{j(\phi)}E \longrightarrow T^*_{j(\phi)}P \xrightarrow[d^*j^{-1}_{|\phi}]{\sim} T^*_{\phi}Q^{\circ;\circ}_{\mathcal{L}_1,\mathcal{L}_2}$$

(see (6.7)).

The subsheaf $\mathcal{W}$ is a Higgs subsheaf if and only if $a(\theta) \in b(\mathrm{Ext}^1(\mathcal{W}, \mathcal{W})^*)$. Now consider the morphism of exact sequences

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{Ext}^1(\mathcal{V}, \mathcal{W})^* & \xrightarrow{i'} & \mathrm{Hom}(\mathrm{Ker}(\phi_{\mathcal{V}}), \mathcal{W})^* & \xrightarrow{s'} & \mathrm{Hom}(\mathcal{L}_{\mathcal{V}}, \mathcal{W})^* \\
& \uparrow b & & \uparrow b' & & \uparrow c \\
0 \longrightarrow & \mathrm{Ext}^1(\mathcal{W}, \mathcal{W})^* & \xrightarrow{i''} & \mathrm{Hom}(\mathrm{Ker}(\phi_{\mathcal{W}}), \mathcal{W})^* & \xrightarrow{s''} & \mathrm{Hom}(\mathcal{L}_{\mathcal{W}}, \mathcal{W})^*
\end{array}$$

in which we have set for simplicity $\mathcal{L}_{\mathcal{V}} = \bigoplus_i \mathcal{L}_i \otimes V_i$ and $\mathcal{L}_{\mathcal{W}} = \bigoplus_i \mathcal{L}_i \otimes W_i$. Note that the map c is injective since $\mathcal{L}_{\mathcal{W}}$ is a direct summand of $\mathcal{L}_{\mathcal{V}}$. It follows that $a(\theta) \in b(\mathrm{Ext}^1(\mathcal{W}, \mathcal{W})^*)$ if and only if $i'(a(\theta)) \in b'(\mathrm{Hom}(\mathrm{Ker}(\phi_{\mathcal{W}}), \mathcal{W})^*)$, and this holds if and only $i(\theta)$ can be lifted to an element $\underline{y} \in \mathrm{Hom}(V_1, V_2 \otimes \mathbb{W})^*$ satisfying $a''(\underline{y}) \in b''(\mathrm{Hom}(W_1, W_2 \otimes \mathbb{V})^*)$. But this last condition is equivalent to the fact that (W_1, W_2) is a subrepresentation of $d^*j(\phi, \theta)$. The lemma is proved. □

Lemma 6.8. *Let $\mathcal{W} \in \mathcal{S}_X$ and $(W_1, W_2) = \psi(\mathcal{W})$. Then $\mu(\mathcal{W}) > \mu(\alpha)$ if and only if $l_1 \mathrm{dim}(W_2) < l_2 \mathrm{dim}(W_1)$.*

Proof. This is a straightforward computation (see, e.g., [ÁCK09, Lemma 3.2]). □

Let $(\phi, \theta) \in \mu^{-1}(0)$ be μ-unstable. Then by the defining property (d) of $(\mathcal{L}_1, \mathcal{L}_2)$ there exists a destabilizing subsheaf $\mathcal{W} \in \mathcal{S}'_X$ (see Section 6.4). Therefore $\psi(\mathcal{W})$ is a destabilizing subrepresentation of $d^*j(\phi, \theta)$. Conversely, assume that $d^*j(\phi,\theta)$ is γ-unstable. Following the authors of [ÁCK07] we will call tight a subrepresentation (W_1, W_2) of $d^*j(\phi,\theta)$ satisfying the following condition: if (W_1', W_2') is a subrepresentation of $d^*j(\phi,\theta)$ such that $W_1 \subseteq W_1'$ and $W_2 \supseteq W_2'$, then $W_1' = W_1$ and $W_2' = W_2$. Clearly, there exists a tight destabilizing subrepresentation (W_1, W_2) of $d^*j(\phi,\theta)$.[1] Observe that (W_1, W_2) is also tight as a submodule of the (nondoubled) Kronecker representation $j(\phi)$. Using [ÁCK07, Lemma 5.5.] we conclude that the submodule (W_1, W_2) is equal to $\psi(\mathcal{W})$ for some $\mathcal{W} \in \mathcal{S}_X$. By Lemmas 6.7 and 6.8 it follows that $\mathcal{W}$ is a destabilizing Higgs subsheaf of (ϕ,θ) and thus that (ϕ,θ) is μ-unstable. The proposition is proved. $\square$

6.9. Let us consider the action of $\mathbb{G}_m$ on T^*E given by

$$z \cdot (\underline{x}, \underline{y}) = (\underline{x}, z\underline{y}).$$

This action preserves $P \times E^*$ and descends to an action of $\mathbb{G}_m$ on T^*P, which in turn preserves $T^*Q^{\circ,\circ}_{\mathcal{L}_1,\mathcal{L}_2}$. Since the map $\mu : T^*Q^{\circ,\circ}_{\mathcal{L}_1,\mathcal{L}_2} \to \mathfrak{g}^*$ is equivariant (for the standard weight-one action of $\mathbb{G}_m$ on $\mathfrak{g}^*$), this action preserves $\mathcal{X}$ and thus induces a $\mathbb{G}_m$-action on $\mathcal{Y}$. Observe that the schemes $\mathcal{Y}_t$ with $t \neq 0$ are transformed into each other by the $\mathbb{G}_m$-action and, in particular, are all isomorphic.

Proposition 6.9. *The $\mathbb{G}_m$-action on $\mathcal{Y}$ is contracting, i.e., for any $y \in \mathcal{Y}$, the action map $\mathbb{G}_m \to \mathcal{Y}, z \mapsto z \cdot y$ extends to a map $\mathbb{A}^1 \to \mathcal{Y}$.*

Proof. It is enough to prove that the $\mathbb{G}_m$-action on 3

$$(T^*P)^{\mathrm{ss}}/\!/G = (T^*Q^{\circ,\circ}_{\mathcal{L}_1,\mathcal{L}_2})^{\mathrm{ss}}/\!/G$$

is contracting since $\mathcal{Y}$ is closed in $(T^*Q^{\circ,\circ}_{\mathcal{L}_1,\mathcal{L}_2})^{\mathrm{ss}}/\!/G$. Because the map $p'' : (T^*P)^{\mathrm{ss}}/\!/G \to (T^*P)/\!/G$ is proper, it is in turn enough to prove that the $\mathbb{G}_m$-action on $(T^*P)/\!/G$ is contracting. It is clear that the $\mathbb{G}_m$-action on $(T^*E)/\!/G$ is contracting. Since $(P \times E^*)/\!/G$ is a $\mathbb{G}_m$-invariant closed subvariety of $(T^*E)/\!/G$, the $\mathbb{G}_m$-action on $(P \times E^*)/\!/G$ is contracting as well. But there is a surjective $\mathbb{G}_m$-equivariant morphism $(P \times E^*)/\!/G \to (T^*P)/\!/G$, and hence the $\mathbb{G}_m$-action on $(T^*P)/\!/G$ is also contracting. The proposition is proved. $\square$

We may now apply the method of Nakajima in [Nak04]. For the reader's convenience, we repeat the argument here. Denote by $\mathcal{Z}$ the scheme of $\mathbb{G}_m$-fixed points in $\mathcal{Y}$, a smooth subscheme of $\mathcal{Y}_0$. An explicit description of $\mathcal{Z}$ is given and studied in [GPHS14] (the so-called moduli of chains on X). Let $\mathcal{Z} = \bigsqcup_i \mathcal{Z}_i$ denote the decomposition of $\mathcal{Z}$ into connected components. The tangent space to $\mathcal{Y}$ at a point $z \in \mathcal{Z}$ splits as a direct sum

$$T_z\mathcal{Y} = T_z\mathcal{Y}^+ \oplus T_z\mathcal{Z} \oplus T_z\mathcal{Y}^-,$$

[1] Such a representation may be thought of as a maximally destabilizing subrepresentation of $d^*j(\phi,\theta)$.

where $T_z\mathcal{Y}^+$, resp. $T_z\mathcal{Y}^-$, stands for the subspace over which the $\mathbb{G}_m$-action is of strictly positive (resp. strictly negative) weight. Let n_i be the dimension of $T_z\mathcal{Y}^+$ for $z \in \mathcal{Z}_i$. Replacing $\mathcal{Y}$ by $\mathcal{Y}_0$, one similarly defines integers n'_i. Observe that $n'_i = n_i - 1$ as ν is $\mathbb{G}_m$-equivariant and $\mathbb{G}_m$ acts on $L \simeq \mathbb{A}^1$ with weight one.

By Lemma 6.5(i) and Proposition 6.9, the Hesselink-Byaliniki-Birula decomposition for $\mathcal{Y}$ and $\mathcal{Y}_0$ provide locally closed partitions

$$\mathcal{Y} = \bigsqcup_i \mathcal{W}_i, \qquad \mathcal{Y}_0 = \bigsqcup_i \mathcal{W}'_i, \tag{6.8}$$

where $\mathcal{W}_i$ (resp. $\mathcal{W}'_i$) is an $\mathbb{A}^{n_i}$-fibration (resp. an $\mathbb{A}^{n'_i}$-fibration) over $\mathcal{Z}_i$; see [Hes81, Th. 5.7].

The decompositions (6.8) and the fibrations $\mathcal{W}_i \to \mathcal{Z}_i$, $\mathcal{W}'_i \to \mathcal{Z}_i$ are all defined over k. It follows on the one hand that

$$|\mathcal{Y}(k)| = |\mathcal{Y}_0(k)| + (q-1)|\mathcal{Y}_1(k)|$$

and on the other hand that

$$|\mathcal{Y}(k)| = \sum_i q^{n_i}|\mathcal{Z}_i(k)|, \qquad |\mathcal{Y}_0(k)| = \sum_i q^{n_i-1}|\mathcal{Z}_i(k)|.$$

We deduce that $|\mathcal{Y}_0(k)| = |\mathcal{Y}_1(k)|$. By Lemma 6.2,

$$|\mathcal{Y}_1(k)| = \sum_{\mathcal{F}\in I_{r,d}} q^{\dim(\mathrm{Ext}^1(\mathcal{F},\mathcal{F}))} |\{\phi \in Q \mid \phi : \mathcal{L}_1^{\oplus l_1} \oplus \mathcal{L}_2^{\oplus l_2} \twoheadrightarrow \mathcal{F}\}|/|PG(k)|,$$

where $I_{r,d}$ stands for the set of indecomposable (and hence geometrically indecomposable) vector bundles of rank r and degree d over X. For such a bundle, we have

$$|\{\phi \in Q \mid \phi : \mathcal{L}_1^{\oplus l_1} \oplus \mathcal{L}^{\oplus l_2} \twoheadrightarrow \mathcal{F}\}|/|PG(k)| = (q-1)/|\mathrm{Aut}(\mathcal{F})| = q/|\mathrm{End}(\mathcal{F})|,$$

from which we deduce that

$$|\mathcal{Y}_1(k)| = \sum_{\mathcal{F}\in I_{r,d}} q^{\dim(\mathrm{Ext}^1(\mathcal{F},\mathcal{F}))-\dim(\mathrm{End}(\mathcal{F}))+1} = q^{1-\langle\alpha,\alpha\rangle}|I_{r,d}| = q^{1+(g-1)r^2}|I_{r,d}|$$

as wanted. This finishes the proof of Theorem 1.2.

6.10. In this section we provide the (standard) proof of Corollary 1.3.

Proof of Corollary 1.3. We will first provide an independent proof of the following fact (due to [GPH13, Th. 1], [GPHS14]):

(a) *The Frobenius eigenvalues in* $H_c^n(\mathrm{Higgs}^{\mathrm{st}}_{r,d}(X \otimes \overline{\mathbf{F}_q}), \overline{\mathbb{Q}_l})$ *are all of the form*

$$\lambda = \prod_j \sigma_j^{n_j}, \qquad \sum n_i = n,$$

where $\sigma_X = (\sigma_1, \dots, \sigma_{2g})$.

So let $C_i = \{c_{i,j} \mid j \in K_i\}$ be the collection of Frobenius eigenvalues in $H^i_c(\mathrm{Higgs}^{\mathrm{st}}_{r,d}(\overline{X}), \overline{\mathbb{Q}_l})$, counted with multiplicity. It is known that $\mathrm{Higgs}^{\mathrm{st}}_{r,d}(\overline{X})$ is cohomologically pure. (See, e.g., [HRV13, Cor. 1.2.3] for the similar case of the mixed Hodge structure on the moduli space of stable Higgs bundles over a complex curve, or see Section 6.11 below.) Therefore $|c_{i,j}| = q^{i/2}$ for all $j \in K_i$. By Theorems 1.1 and 1.2 there exist a polynomial $B_{r,d} \in \mathbb{Q}[T_g]^{W_g}$ and a unitary polynomial $R(q) \in \mathbb{Z}[q]$ such that for any $l \geq 1$,

$$B_{r,d}(\sigma_1^l, \ldots, \sigma_{2g}^l) = \left(\sum_{i,j} (-1)^i c_{i,j}^l \right) R(q^l), \tag{6.9}$$

where $\sigma_X = (\sigma_1, \ldots, \sigma_{2g})$. Multiplying $B_{r,d}$ by some positive integer N if necessary and repeating each $c_{i,j}$ N times accordingly, we may assume that $B_{r,d} \in \mathbb{Z}[T_g]^{W_g}$. Expanding the product $\left(\sum_{i,j}(-1)^i c_{i,j}^l\right) R(q^l)$ and gathering together terms with the same sign, we may write (6.9) as an equality

$$\sum_{a \in A} u_a^l = \sum_{b \in B} v_b^l, \tag{6.10}$$

where u_a, v_b are either some monomials of the form $\sigma_1^{i_1} \cdots \sigma_{2g}^{i_{2g}}$ or of the form $q^n c_{i,j}$ for some i and $j \in K_i$. Because (6.10) holds for all l, we deduce that $\{u_a \mid a \in A\} = \{v_b \mid b \in B\}$. We may decompose the sets $\{u_a\}, \{v_b\}$ according to the complex norm, yielding for each n an equality

$$\{u_a \mid a \in A, |u_a| = n\} = \{v_b \mid b \in B, |v_b| = n\}.$$

Let d be the degree of $R(q)$, so that $R(q) = q^d + P(q)$ with $\deg(P) < d$. Set $l = \max\, \{l \mid K_l \neq \emptyset\}$. Depending on the parity of l, the monomials of the form $q^d c_{l,j}$ either all belong to $\{u_a \mid a \in A, |u_a| = q^{d+l/2}\}$ or all belong to $\{v_b \mid b \in B, |v_b| = q^{d+l/2}\}$. This implies that the $c_{l,j}, j \in K_l$ are all equal to monomials $\sigma_1^{i_1} \cdots \sigma_{2g}^{i_{2g}}$ with $\sum_k i_k = l$. Canceling from (6.10) all the terms arising in the products $c_{l,j} R(q)$ for $j \in K_l$ and arguing by induction, we deduce that the same holds for the $c_{i,j}$ with $j \in K_i$ and i arbitrary. This proves (a). In fact, the above argument shows the following. Write $A_{g,r,d} = \sum_{i_1,\ldots,i_{2g}} a_{i_1,\ldots,i_{2g}} (-z_1)^{i_1} \cdots (-z_{2g})^{i_{2g}}$. Then the multiplicity of the eigenvalue $\sigma_1^{i_1} \cdots \sigma_{2g}^{i_{2g}}$ is equal to $a_{i_1,\ldots,i_{2g}}$. In particular, this implies that $a_{i_1,\ldots,i_{2g}} \in \mathbb{N}$ for any $i_1, \ldots, i_{2g}$. Statement (i) of Corollary 1.3 easily follows.

Let us now turn to statement (ii). Let $X_{\mathbb{Q}}$ be a smooth projective curve of genus g defined over $\mathbb{Q}$, and let X_R be a spreading out of $X_{\mathbb{Q}}$ defined over some ring $R = \mathbb{Z}[\frac{1}{N}]$. Consider the R-scheme $\pi : \mathrm{Higgs}^{\mathrm{st}}_{r,d}(X_R) \to \mathrm{Spec}(R)$. The complex $R\pi_!(\overline{\mathbb{Q}_l})$ is locally constant over an open subset $U \subseteq \mathrm{Spec}(R)$. For any field k and any point $j_k : \mathrm{Spec}(k) \to \mathrm{Spec}(R)$, the proper base change theorem provides an isomorphism $j_k^* R\pi_!(\overline{\mathbb{Q}_l}) \simeq R\pi_{k,!}(\overline{\mathbb{Q}_l})$, where $\pi_k : \mathrm{Higgs}^{\mathrm{st}}_{r,d}(X_R \otimes k) \to$

Spec(k). If $j_{\mathbf{F}_q} \in U$, then $j^*_{\mathbf{F}_q} R\pi_!(\overline{\mathbb{Q}_l}) \simeq j^*_{\mathbb{Q}} R\pi_!(\overline{\mathbb{Q}_l}) \simeq$, where $j_{\mathbb{Q}} : \mathrm{Spec}(\mathbb{Q}) \to \mathrm{Spec}(R)$ is the generic point. As $j_{\mathbf{F}_q} \in U$ for $q \gg 0$, this yields an equality

$$\sum_n \dim(H^n_c(\mathrm{Higgs}^{\mathrm{st}}_{r,d}(X_R \otimes \mathbf{F}_q), \overline{\mathbb{Q}_l}))t^n = \sum_n \dim(H^n_c(\mathrm{Higgs}^{\mathrm{st}}_{r,d}(X_R \otimes \mathbb{Q}), \overline{\mathbb{Q}_l}))t^n.$$

Finally, by the Artin-Grothendieck comparison theorem,

$$\sum_n \dim(H^n_c(\mathrm{Higgs}^{\mathrm{st}}_{r,d}(X_R \otimes \mathbb{Q}), \overline{\mathbb{Q}_l}))t^n = \sum_n \dim(H^{n,sing}_c(\mathrm{Higgs}^{\mathrm{st}}_{r,d}(X_R \otimes \mathbb{C}), \mathbb{C}))t^n.$$

We conclude using the fact that the all the complex varieties $\mathrm{Higgs}^{\mathrm{st}}_{r,d}(X)$ as X runs through the set of Riemann surface of genus g are diffeomorphic (and all diffeomorphic to the genus g twisted character variety for the group $\mathrm{GL}(r)$; see [HRV08]). □

6.11. Finally, let us prove Corollaries 1.4 and 1.5.

Proof of Corollary 1.4. Assume that $k = \mathbf{F}_q$. We first recall the proof that the variety $\Lambda^{\mathrm{st}}_{r,d}$ is cohomologically pure and that the Frobenius eigenvalues in $H^i(\Lambda^{\mathrm{st}}_{r,d}, \overline{\mathbb{Q}_l})$ are all of the form $\prod_j \alpha_j^{l_j}$ with $\sum_i l_j = i$. Consider the $\mathbb{G}_m$-action on $\mathrm{Higgs}^{\mathrm{st}}_{r,d}$ defined by $\rho(z)(\mathcal{V}, \theta) = (\mathcal{V}, z\theta)$. Observe that the Hitchin map μ is naturally $\mathbb{G}_m$-equivariant for the weight-one action of $\mathbb{G}_m$ on the Hitchin base. Since μ is proper, it follows (as in Section 6.9) that this action is contracting. Let $Z = (\mathrm{Higgs}^{\mathrm{st}}_{r,d})^{\mathbb{G}_m}$ be the be the fixed point subvariety and $Z = \bigsqcup_i Z_i$ its decomposition into connected components. Each Z_i is a smooth subvariety of $\mathrm{Higgs}^{\mathrm{st}}_{r,d}$ which is included in $\Lambda^{\mathrm{st}}_{r,d}$ and hence is projective. The tangent space of $\mathrm{Higgs}^{\mathrm{st}}_{r,d}$ at a point $z_i \in Z_i$ decomposes according to the $\mathbb{G}_m$-character as

$$T_{z_i}\mathrm{Higgs}^{\mathrm{st}}_{r,d} = T^{>0}_{z_i} \oplus T_{z_i}Z_i \oplus T^{<0}_{z_i}.$$

We have Byalinicki-Birula-Hesselink decompositions (for ρ and ρ^{-1} respectively)

$$\mathrm{Higgs}^{\mathrm{st}}_{r,d} = \bigsqcup_i Y_i^+, \qquad \Lambda^{\mathrm{st}}_{r,d} = \bigsqcup_i Y_i^-,$$

where Y_i^+ is a locally trivial $\mathbb{A}^{n_i^+}$-fibration over Z_i and Y_i^- is a locally trivial $\mathbb{A}^{n_i^-}$-fibration over Z_i, where

$$n_i^+ = \dim T^{>0}_{z_i}, \qquad n_i^- = \dim T^{<0}_{z_i};$$

see [Hes81, Th. 5.7]. (This is independent of the choice of z_i.) Because $\Lambda^{\mathrm{st}}_{r,d}$ is lagrangian and Z_i is included in the smooth locus of $\Lambda^{\mathrm{st}}_{r,d}$, we have $n_i^+ = \frac{1}{2}\dim \mathrm{Higgs}^{\mathrm{st}}_{r,d} = 1 + (g-1)r^2$. The varieties Z_i being smooth and projective, they are pure, and hence so are the Y_i^+, Y_i^- (for the compactly supported cohomology). This implies that $\Lambda^{\mathrm{st}}_{r,d}$ and $\mathrm{Higgs}^{\mathrm{st}}_{r,d}$ are pure as well and that there is an equality in the Grothendieck group of $\mathrm{Gal}(\overline{k}/k)$-modules

$$H^n_c(\mathrm{Higgs}^{\mathrm{st}}_{r,d}, \overline{\mathbb{Q}_l}) \simeq \bigoplus_i H^{n-1-(g-1)r^2}_c(Z_i, \overline{\mathbb{Q}_l})((1-g)r^2 - 1), \tag{6.11}$$

where ( ) denotes a Tate twist. Similarly, there is an isomorphism

$$H_c^n(\Lambda^{\mathrm{st}}_{r,d}, \overline{\mathbb{Q}_l}) \simeq \bigoplus_i H_c^{n-n_i^{<0}}(Z_i, \overline{\mathbb{Q}_l})(-n_i^{<0}). \tag{6.12}$$

By Poincaré duality,

$$H_c^{2\dim Z_i - l}(Z_i, \overline{\mathbb{Q}_l})^*(-\dim Z_i) \simeq H_c^l(Z_i, \overline{\mathbb{Q}_l}). \tag{6.13}$$

Observe that $\dim Z_i = 1+(g-1)r^2 - n_i^{<0}$. Combining (6.11), (6.12) and (6.13) and taking the trace with respect to the Frobenius element yields statement (i). Statement (ii) for $k = \mathbf{F}_q$ follows by considering the appropriate cohomological degrees, and for $k = \mathbb{C}$ by the same type of arguments as in the proof of Corollary 1.5. □

Proof of Corollary 1.5. The first statement was shown in the course of the proof of Corollary 1.3. The second statement is a direct consequence of the fact that the moduli space $\mathrm{Higgs}^{\mathrm{st}}_{r,d}(X_{\mathbb{C}})$ is connected and of dimension $2(1+(g-1)r^2)$. □

7. **Extension to the parabolic case**

7.1. There is a result analogous to Theorem 1.1 for vector bundles with (quasi)-parabolic structure. As before, let X be a smooth projective curve defined over a finite field $\mathbf{F}_q$. Fix an effective divisor $D = \sum_{i=1}^N p_i x_i$ where for simplicity we assume that the x_i are $\mathbf{F}_q$-rational points of X. By definition a *quasi-parabolic vector bundle* $(\mathcal{V}, F^\bullet)$ *on* (X, D) is a vector bundle $\mathcal{V}$ on X equipped with a collection of filtrations

$$F_1^{(i)} \subseteq F_2^{(i)} \subseteq \cdots \subseteq F_{p_i}^{(i)} = \mathcal{V}_{|x_i}$$

for $i = 1, \ldots, N$. The sequence $(\dim(F_1^{(i)}), \dim(F_2^{(i)}), \ldots, \dim(F_{p_i}^{(i)}))$ is called the *dimension type* of $(\mathcal{V}, F^\bullet)$ at x_i.

Given $r > 0, d \in \mathbb{Z}$ and fixed dimension types $\mathbf{d}^{(i)} = d_1^{(i)} \leq \cdots \leq d_{p_i}^{(i)} = r$ for $i = 1, \ldots, N$, we let $\mathcal{A}_{r,d,\mathbf{d}^{(1)},\ldots,\mathbf{d}^{(N)}}(X)$ stand for the number of geometrically indecomposable quasi-parabolic bundles on (X, D) of rank r, degree d and dimension type $\mathbf{d}^{(i)}$ at x_i for all i. Again the finiteness of such number is a consequence of the existence of Harder-Narasimhan filtrations.

THEOREM 7.1. *For any* $g \geq 0$, *any positive integer* $N \geq 0$, *any collection of positive integers* $\mathbf{p} = (p_1, \ldots, p_N)$ *and any tuple* $\boldsymbol{\alpha} = (r, d, \mathbf{d}^{(1)}, \ldots, \mathbf{d}^{(N)})$ *satisfying*

$$(r, d) \in \mathbb{N} \times \mathbb{Z},$$

$$\mathbf{d}^{(i)} = (d_1^{(i)} \leq \cdots \leq d_{p_i}^{(i)} = r) \quad \forall i,$$

there exists a unique polynomial $A_{g,\mathbf{p},\boldsymbol{\alpha}} \in \mathbb{Q}[T_g]^{W_g}$ *such that for any smooth projective curve* X *of genus* g *defined over a finite field, for any divisor* $D = \sum_i p_i x_i$ *with* $x_i \in X(\mathbf{F}_q)$, *we have*

$$\mathcal{A}_{\boldsymbol{\alpha}}(X) = A_{g,\mathbf{p},\boldsymbol{\alpha}}(\sigma_X).$$

When $g = 0$, the above theorem settles Conjecture 9.2(ii) in [Sch04].

7.2. The proof of Theorem 7.1 is completely parallel to that of Theorem 1.1, using the spherical Hall algebra of the category of D-parabolic coherent sheaves over X in place of the spherical Hall algebra of X. Shuffle presentations for such Hall algebras are studied in [Lin15]. There is also an effective version of Theorem 7.1, whose proof is again similar to that of Theorem 1.6. This would then provide an answer to a question raised by Deligne in the context of the counting of the number of irreducible l-adic local systems on a curve defined over a finite field (see [DF13] or [Del15]). It is natural to expect that the results and methods of Section 6 extend to the parabolic setting as well. These extensions to the parabolic setting will be the subject of a companion paper.

8. **Refinements and conjectures**

To finish, we state a few refinements of the results of this paper and propose some conjectures, in particular, on the possible Lie-theoretic interpretations of the polynomials $A_{g,r,d}$.

8.1. Let $\nu \in \mathbb{Q}$. Denote by $\mathcal{A}^{\geq \nu}_{r,d}(X)$, resp. $\mathcal{A}^{\leq \nu}_{r,d}(X)$, the number of absolutely indecomposable vector bundles over X of class (r, d) lying in $\mathrm{Coh}^{\geq \nu}$, resp. $\mathrm{Coh}^{\leq \nu}$. The proof of Theorem 1.1 yields the following:

Corollary 8.1. *Fix* $g \geq 0$ *and* $\nu \in \mathbb{Q}$. *For any* $(r, d) \in (\mathbb{Z}^2)^+$, *there exist polynomials* $A^{\geq \nu}_{g,r,d}, A^{\leq \nu}_{g,r,d} \in K_g$ *such that for any smooth projective curve* X *of genus* g *defined over a finite field, we have*

$$\mathcal{A}^{\geq \nu}_{r,d}(X) = A^{\geq \nu}_{g,r,d}(\sigma_X), \qquad \mathcal{A}^{\leq \nu}_{r,d}(X) = A^{\leq \nu}_{g,r,d}(\sigma_X).$$

Remarks. (i) When $\mu((r, d)) = \nu$ we have $\mathrm{Coh}_{r,d}(X) \cap \mathrm{Coh}^{\geq \nu} = \mathrm{Coh}_{r,d}(X) \cap \mathrm{Coh}^{(\nu)}$. The above result thus implies that there exist polynomials counting the number of geometrically indecomposable semistable sheaves of any given slope ν.

(ii) We have $\mathrm{Coh}^{\geq 0} \supset \mathrm{Coh}^{\geq 1} \supset \cdots$. Thus for a given (r, d) with $r > 0$, $d \geq 0$, there is (for each curve X) a decreasing sequence of positive integers

$$\mathcal{A}^{\geq 0}_{r,d}(X) \geq \mathcal{A}^{\geq 1}_{r,d}(X) \geq \cdots \geq \mathcal{A}^{\geq \frac{d}{r}}_{r,d}(X).$$

Of course, taking $d \gg 0$, we have $\mathcal{A}^{\geq 0}_{r,d}(X) = \mathcal{A}_{r,d}(X)$. It is natural to hope that

$$A^{\geq i}_{g,r,d} - A^{\geq i+1}_{g,r,d} \in \mathrm{Im}(\mathbb{N}[-z_i]^{W_g}_i \to R_g), \qquad \text{for } i \geq 0.$$

When (r,d) are coprime it would be interesting to interpret the ensuing termwise decreasing sequence

$$A_{g,r,d}(t,\ldots,t) \geq \cdots \geq A^{\geq \frac{d}{r}}_{g,r,d}(t,\ldots,t)$$

of single variable polynomials as corresponding to some natural filtration in the cohomology of moduli spaces of Higgs bundles over complex curves (or of twisted character varieties).

(iii) The above remarks (i) and (ii) can be made also in the case of vector bundles equipped with quasi-parabolic structures (for any choice of slope function).

8.2. From Corollary 8.1, it seems natural to make the following conjecture. For $(\alpha_1,\ldots,\alpha_t)$ a Harder-Narasimhan type, let us denote by $\mathcal{A}_{\alpha_1,\ldots,\alpha_t}(X)$ the number of absolutely indecomposable vector bundles over X which belong to $\mathrm{Coh}^{(\alpha_1,\ldots,\alpha_t)}$.

Conjecture 8.2. *For any $g \geq 0$ and for any Harder-Narasimhan type $(\alpha_1,\ldots,\alpha_t)$, there exists a polynomial $A_{g,\alpha_1,\ldots,\alpha_t} \in \mathbb{Q}[T_g]^{W_g}$ such that for any smooth projective curve X of genus g defined over a finite field, we have*

$$\mathcal{A}_{\alpha_1,\ldots,\alpha_t}(X) = A_{g,\alpha_1,\ldots,\alpha_t}(\sigma_X).$$

Again, one may formulate an entirely similar conjecture in the case of vector bundles equipped with quasi-parabolic structures (for any choice of slope function). One may likewise formulate exactly the same conjecture in the context of representations of quivers.

8.3. In the context of quivers Kac conjectured (see [Kac83, Conj. 1]), and Hausel proved in general, that the constant term $A_{\mathbf{d}}(0)$ of the Kac polynomial attached to a quiver Q with no edge loop and a dimension vector $\mathbf{d}$ is equal to the multiplicity of the root $\sum_i d_i\alpha_i$ in the Kac-Moody Lie algebra $\mathfrak{g}_Q$ canonically associated to Q; see [Hau10, §3] for details.

In the context of a smooth projective curve one is therefore led to seek an analog of the Kac-Moody Lie algebra $\mathfrak{g}_Q$. Motivated by Ringel's theorem relating Hall algebras and quantum groups, we suggest the following construction. Let X be a smooth projective curve of genus g defined over an algebraically closed field, and let $\mathcal{H}^{\chi}_{\nu}$ be the space of all $\mathbb{C}$-valued constructible functions on the moduli stack $\mathbf{Coh}_{\nu}$. The space $\mathcal{H}^{\chi} := \bigoplus_{\nu} \mathcal{H}^{\chi}_{\nu}$ has the structure of a co-commutative Hopf algebra (see, e.g., [Lus91, §10.20] or [BTL12, Th. 4.3]) and is sometimes called the χ-Hall algebra of X. Let $\mathcal{H}^{\chi,\mathrm{sph}}$ stand for the sub Hopf algebra generated by the constant functions on $\mathbf{Coh}_{0,d}$ and $\mathbf{Bun}_{1,l}$ for $d \geq 0$ and $l \in \mathbb{Z}$. This Hopf algebra may be thought of as an $\alpha_i = 1$ limit of the spherical Hall algebra $\mathcal{H}^{\mathrm{sph}}$ of a curve of genus g defined over a finite field. We define the spherical Hall Lie algebra $\mathfrak{h}^{\mathrm{sph}}_X$ of X as the Lie algebra of primitive elements in $\mathcal{H}^{\chi,\mathrm{sph}}$. We conjecture that this Lie algebra is independent of the choice of X and has finite dimensional $\mathbb{Z}^2$ graded components. The analog of Kac's conjecture may now be formulated as follows:

CONJECTURE 8.3. *For any* $(r,d) \in (\mathbb{Z}^2)^+$, *we have* $A_{g,r,d}(0) = \dim \mathfrak{h}^{\mathrm{sph}}_{(r,d)}$.

In addition, one may formulate a version of Kac's conjecture directly in terms of the spherical Hall algebra $\mathcal{H}^{\mathrm{sph}}$ of the curve X itself (and its integral form ${}_R\mathcal{H}^{\mathrm{sph}}$), which are natural analogs in the context of curves of the quantum enveloping algebra $U_q(\mathfrak{n}_Q)$. These algebras are $(\mathbb{Z}^2)^+$-graded but with graded components of infinite dimension in general. In order to circumvent this difficulty, let us denote by $\mathcal{H}^{\mathrm{sph},\geq 0}_\nu$ the subspace of $\mathcal{H}^{\mathrm{sph}}_\nu$ consisting of those functions on $\mathbf{Coh}_\nu$ which are supported on the substack $\mathbf{Coh}^{\geq 0}_\nu$. It is easy to check that $\mathcal{H}^{\mathrm{sph},\geq 0} = \bigoplus_\nu \mathcal{H}^{\mathrm{sph},\geq 0}_\nu$ is an $(\mathbb{N}^2)$-graded algebra with finite dimensional graded components.

CONJECTURE 8.4. *The following equality holds in the ring of power series* $\mathbb{N}[[z^{(0,1)}, z^{(1,0)}]]$:

$$\sum_\nu \dim(\mathcal{H}^{\mathrm{sph},\geq 0}_\nu) z^\nu = \mathrm{Exp}\left(\sum_\nu A^{\geq 0}_{g,r,d}(0) z^\nu\right).$$

These conjectures may be directly checked for $g = 0, 1$ using the results in [Kap97], [BS12] respectively.

Remark. By [SV12, Th. 3.1], the spherical Hall algebra $\mathcal{H}^{\mathrm{sph}}$ of X is isomorphic to the spherical part of the K-theoretic Hall algebra

$$\mathbf{K}_g = \bigoplus_{r \geq 0} K^{\mathrm{GL}_r \times T_g}(\mathcal{C}_{g,r})$$

of the commuting variety $\mathcal{C}_g = \bigsqcup_r \mathcal{C}_{g,r}$, where

$$\mathcal{C}_{g,r} = \left\{(x_1, y_1 \ldots, x_g, y_g) \in \mathfrak{gl}_r(\mathbb{C})^{2g} \mid \sum_i [x_i, y_i] = 0\right\}.$$

We do not know how to geometrically describe the subalgebra $\mathbf{K}^{\geq 0}_g$ of $\mathbf{K}_g$ corresponding to $\mathcal{H}^{\mathrm{sph},\geq 0}$. However, one may expect the existence of a degeneration from $\mathbf{K}^{\geq 0}_g$ to the *cohomological* Hall algebra

$$\mathbf{C}_g = \bigoplus_{r \geq 0} H^\bullet_{\mathrm{GL}_r \times T_g}(\mathcal{C}_{g,r}).$$

(See [SV13, §7], where such a degeneration is performed (algebraically) in the case of $g = 1$.) In particular, one may consider an analog of Conjecture 8.4 in which $\mathcal{H}^{\mathrm{sph},\geq 0}$ is replaced by the spherical part of $\mathbf{C}_g$. (This suggests a relation between $A_{g,r,d}(0)$ and the Donaldson-Thomas invariants of the $2g$-loop quiver, with preprojective relations.)

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Appendix A. Volume of moduli stacks of torsion sheaves

Proof of Theorem 4.1(ii). Let $\mathbf{Coh}_{0,d}^{(x)}$ be the substack of $\mathbf{Coh}_{0,d}$ parametrizing torsion sheaves of degree d supported at a closed point x of X, and let $1_{0,d}^{(x)}$ stand for the characteristic function of $\mathbf{Coh}_{0,d}^{(x)}$. Observe that $1_{0,d}^{(x)} = 0$ unless $\deg(x) \,|\, d$ and

$$\begin{aligned}\mathrm{vol}(\mathbf{Coh}^{(x)}_{0,n\deg(x)}) = (1^{(x)}_{0,n\deg(x)} \mid 1^{(x)}_{0,n\deg(x)}) &= \frac{|\mathcal{N}_n(k_x)|}{|\mathrm{GL}_n(k_x)|} \\ &= \frac{q^{n(n-1)\deg(x)}}{(q^{n\deg(x)}-1)\cdots(q^{n\deg(x)}-q^{(n-1)\deg(x)})},\end{aligned}$$

where $k_x \simeq \mathbf{F}_{q^{\deg(x)}}$ is the residue field at x. Moreover,

$$\sum_{d\geq 0} 1_{0,d}s^d = \prod_{x\in X}\left(\sum_{n\geq 0} 1^{(x)}_{(0,n\deg(x))}s^{n\deg(x)}\right)$$

and

$$\sum_{l\geq 0}(1_{0,l} \mid 1_{0,l})s^l = \prod_{x\in X}\sum_{n\geq 0}(1^{(x)}_{0,n\deg(x)} \mid 1^{(x)}_{0,n\deg(x)})s^{n\deg(x)}.$$

Using Heine's formula, we obtain

$$\sum_{n\geq 0}(1^{(x)}_{0,n\deg(x)} \mid 1^{(x)}_{0,n\deg(x)})s^{n\deg(x)} = \exp\left(\sum_{l\geq 1}\frac{(sq^{-1})^{\deg(x)}}{l(1-q^{-l\deg(x)})}\right).$$

Using the relation

$$\sum_{d\,|\,l}\ \sum_{\substack{x\\ \deg(x)=d}} d = |X(\mathbf{F}_{q^l})|,$$

we finally obtain

$$\sum_{d\geq 0}(1_{0,d} \mid 1_{0,d})s^d = \exp\left(\sum_{l\geq 1}\frac{|X(\mathbf{F}_{q^l})|}{l(q^l-1)}s^l\right) = \mathrm{Exp}\left(\frac{|X(\mathbf{F}_q)|}{q-1}s\right)$$

as wanted.

Appendix B. Density of Weil numbers of smooth projective curves

Proof of Proposition 4.7. The set $\mathcal{W}$ is constructed as the collection of Weil numbers (in the l-adic cohomology) of smooth projective curves defined over finite fields, allowing both the curve and the finite field to vary (as long as the characteristic is different from l). Let us fix a finite field $\mathbf{F}_q$ with l not dividing q, a square root $q^{\frac{1}{2}}$ of q in $\overline{\mathbb{Q}_l}$ and let us denote by $\mathcal{W}_q$ the set of all (collections of) Weil numbers of smooth projective curves defined over $\mathbf{F}_q$. For any $\mathbf{F}_q$-scheme U, we write $\pi_1(U)$, resp. $\pi_1^{\mathrm{geom}}(U)$, for the fundamental group (resp. geometric fundamental group) of U.

In [KS99, §10.1, 10.2], Katz and Sarnak constructed a family $\rho : \mathfrak{X} \to U_g$ of smooth projective curves over $\mathbf{F}_q$ of genus g, satisfying the following property. Set $\mathcal{F} = R^1\rho_!(\overline{\mathbb{Q}_l})(1/2)$, a pure lisse sheaf of weight zero whose stalk at a point $\mathrm{Spec}(\mathbf{F}_{q^n}) \to U_g$ corresponding to a curve X defined over $\mathbf{F}_{q^n}$ is equal to $H^1(X \otimes \overline{\mathbf{F}_q}, \overline{\mathbb{Q}_l})(1/2)$. Let us also denote by $\rho : \pi_1(U) \to \mathrm{GL}(2g, \overline{\mathbb{Q}_l})$ the representation associated to $\mathcal{F}$ (well defined up to conjugation). Then the Zariski closure of $\rho(\pi_1^{\mathrm{geom}}(U_g))$ is equal to $\mathrm{Sp}(2g, \overline{\mathbb{Q}_l})$ (see [KS99, Ths. 10.1.16 and 10.2.2]). Moreover, $\rho(\pi_1(U_g)) \subset \mathrm{Sp}(2g, \overline{\mathbb{Q}_l})$.

To every point $x : \mathrm{Spec}(\mathbf{F}_{q^n}) \to U_g$, there corresponds a map $\pi_1(\mathrm{Spec}(\mathbf{F}_{q^n})) \to \pi_1(U_g)$, and hence a Frobenius element $\rho(\mathrm{Fr}_{x,n}) \in \mathrm{Sp}(2g, \overline{\mathbb{Q}_l})$ (well defined up to conjugation). Let $\rho(\mathrm{Fr}_{x,n})^{\mathrm{ss}}$ stand for the semi-simple part of $\rho(\mathrm{Fr}_{x,n})$. Using the embedding $\iota : \overline{\mathbb{Q}_l} \to \mathbb{C}$, we may view $\rho(\mathrm{Fr}_{x,n})^{\mathrm{ss}}$ as a semisimple conjugacy class in $\mathrm{Sp}(2g, \mathbb{C})$. Because $\mathcal{F}$ is pure of weight zero, the eigenvalues of $\rho(\mathrm{Fr}_{x,n})^{\mathrm{ss}}$ are all unitary; i.e., the conjugacy class of $\rho(\mathrm{Fr}_{x,n})^{\mathrm{ss}}$ intersects the maximal compact subgroup $K \subset \mathrm{Sp}(2g, \mathbb{C})$ in a K-conjugacy class which we denote by $C_{x,n}$. If $x : \mathrm{Spec}(\mathbf{F}_{q^n}) \to U_g$ corresponds to a curve X defined over $\mathbf{F}_{q^n}$, then $C_{x,n}$ is the conjugacy class whose eigenvalues are $(q^{-n/2}\sigma_1, \ldots, q^{-n/2}\sigma_{2g})$, where $(\sigma_1, \ldots, \sigma_{2g})$ are the Weil numbers of X. By Deligne's equidistribution theorem (see [Del74, 3.5.3] and [KS99, Th. 9.2.6]), the set of conjugacy classes $\mathcal{C}_{\leq n} := \{C_{x,m} \mid m \leq n, x \in U_g(\mathbf{F}_{q^m})\}$ becomes equidistributed for the Haar measure as n tends to infinity.

The maximal torus T of K is equal to

$$T = \{(z_1, \ldots, z_{2g}) \in (\mathbb{C}^*)^{2g} \mid |z_i| = 1, z_{2i-1}z_{2i} = 1 \; \forall\, i\} \simeq (S^1)^g.$$

Set $\mathcal{W}'_q = \bigcup_{n\geq 1} \mathcal{W}'_{q,n}$, where

$$\mathcal{W}'_{q,n} = \{q^{-n/2}\sigma_X = (q^{-n/2}\sigma_1, \ldots, q^{-n/2}\sigma_{2g}) \mid X \in U_g(\mathbf{F}_{q^n})\}.$$

Deligne's equidistribution theorem implies that $\mathcal{W}'_{q,n}$ is equidistributed in T/W_g as n tends to infinity. In particular, $\mathcal{W}'_q$ is dense in T (for the analytic topology). We claim[2] that this implies that $\mathcal{W}_q = \bigcup_{n\geq 1} \mathcal{W}_{q,n}$ is Zariski dense

[2] We thank Gaëtan Chenevier for providing us the argument.

in T_g/W_g. Indeed let $f \in \mathbb{C}[T_g]^{W_g}$ be a polynomial function vanishing on $\mathcal{W}_q W_g$. Consider the (real) algebraic map $r : T \times \mathbb{R}^* \to T_g, ((z_1, \dots, z_{2g}), t) \mapsto (tz_1, \dots, tz_{2g})$. The image of r contains $\mathcal{W}_q W_g$ and is Zariski dense in T_g. Assume that $f \neq 0$ so that $r^* f \neq 0$, and let us write $r^* f(\underline{z}, t) = \sum_i h_i(\underline{z}) t^i$. Rescaling by a power of t if necessary, we may assume that $h_0 \neq 0$ and $h_i = 0$ for $i > 0$. Let $\underline{z} \in T$ such that $h_0(\underline{z}) \neq 0$. Because each $\mathcal{W}'_{q,n}$ is finite and $\mathcal{W}'_q$ is dense in T, there exists a sequence $(\omega_i, n_i)_i$ with $\omega_i \in \mathcal{W}'_{q,n_i}$ and $n_i \mapsto \infty$ such that $\omega_i \mapsto \underline{z}$. The functions $h_i, i < 0$ being bounded on the compact set T, it follows that $r^* f(\omega_i, q^{n_i/2}) \mapsto h_0(\underline{z}) \neq 0$, in contradiction with our hypothesis on f. This proves that $\mathcal{W}_q W_g$ is dense in T_g and thus that $\mathcal{W}_q$ (and *a fortiori* $\mathcal{W}$) is dense in T_g/W_g. We are done. □

Appendix C. Proof of Conjecture 1.7 when r is prime

This is a straightforward computation. By the proof of Theorem 1.1, $A_{g,r}(z)$ may have poles only at r-th roots of unity, and these poles are of order at most one. If r is assumed to be prime, then all the nontrivial r-th roots of unity are primitive, and hence only occur as poles of terms in $A_{g,r}(z)$ containing a factor $(1-z^r)^{-1}$. Upon inspection, on easily sees that this factor arises (as a coefficient of T^r) on the right-hand side of (1.3) in only two terms, namely,

$$q^{\langle (r),(r)\rangle} J_{(r)}(z) H_{(r)}(z) = \frac{\prod_i (\alpha_i - 1)(\alpha_i - z) \cdots (\alpha_i - z^{r-1})}{(q-1)(q-z)\cdots(q-z^{r-1})} \cdot \frac{1}{(1-z)(1-z^2)\cdots(1-z^r)}$$

and

$$\frac{\mu(r)}{r} \psi_r \left(q^{\langle (1),(1)\rangle} J_{(1)}(z) H_{(1)}(z) \right) = -\frac{1}{r} \cdot \frac{\prod_i (\alpha_i^r - 1)}{(q^r - 1)(1 - z^r)}.$$

The result follows by a simple residue computation.

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Université de Paris-Sud Orsay, Orsay, France
E-mail: olivier.schiffmann@math.u-psud.fr