

Functoriality, Smith theory, and the Brauer homomorphism

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Abstract

If σ is an automorphism of order p of the semisimple group \mathbf{G} , there is a natural correspondence between mod p cohomological automorphic forms on \mathbf{G} and \mathbf{G}^σ . We describe this correspondence in the global and local settings.

1. Introduction

1.1. Let \mathbf{G} be a semisimple group over a number field F , with Langlands-dual ${}^L\hat{\mathbf{G}}$, and let k be an algebraically closed field of positive characteristic p . By “mod p automorphic forms” for \mathbf{G} we shall mean Hecke eigenclasses in the cohomology of congruence subgroups with k -coefficients. We *make no assumption that these cohomology classes lift to characteristic zero*; i.e., there may be no automorphic form in the classical sense associated to this eigenclass. Indeed, the primary novelty of our techniques is exactly in this case.

Now let σ be an order p automorphism of \mathbf{G} , defined over F , with a connected fixed point subgroup \mathbf{G}^σ . In this paper we show that there is a close relationship between mod p automorphic forms on \mathbf{G} and mod p automorphic forms on \mathbf{G}^σ : we construct a homomorphism (Section 4.3)

$$\begin{array}{ccc} \text{Hecke algebra for } \mathbf{G} \text{ at } v, & \xrightarrow{\psi_v} & \text{Hecke algebra for } \mathbf{G}^\sigma \text{ at } v, \\ \text{with } k \text{ coefficients} & & \text{with } k \text{ coefficients,} \end{array}$$

which is a slight variant of the “Brauer homomorphism” of modular representation theory. We prove (see Theorem 5.8)

MAIN THEOREM. *If a mod p automorphic form for \mathbf{G}^σ has Satake parameters $\{a_v\}$, then there exists a mod p automorphic form for \mathbf{G} with Satake parameters $\{\psi_v^*(a_v)\}$.*

What is the relationship between the parameters of these forms at ramified places? This is answered by Theorem 6.5, based on the notion of “linkage” (Definition 6.2) of local representations. Roughly speaking, these results suggest that local functoriality should be realized by Tate cohomology.

Finally, we compute ψ_v in terms of dual groups in Section 8; roughly, ψ_v can be computed in terms of a “norm map” on dual tori. While technical, this result is indispensable: we have no other way of verifying that the lift of the theorem is a “functorial lift” in the sense of Langlands, i.e., arises from a map of dual groups. Theorem 8.10 allows us to do this quickly in any given case. We give several examples in the next section of its application, and we briefly discuss the general situation in Section 1.3.

1.2. *Examples and relationship to other techniques for functoriality.* As mentioned, the main novelty of our result is precisely that it gives functoriality for mod p classes that do not lift to characteristic zero, i.e., torsion classes. For many groups \mathbf{H} , such situations are generic, in a sense that we will recall below, and we know of no cases where our Main Theorem, when applied to genuine torsion classes, can be deduced from existing methods.

On the other hand, when we begin with characteristic zero cohomology classes for \mathbf{H} , our results about functoriality are generally (not always — see Sections 1.2.2 or 1.2.3) weaker than what can be obtained by classical methods, that is to say, comparison of trace formulae. In those cases, there is still an interesting new result embedded in the method itself giving a “physical” constraint on the lift — see 1.2.5 below.

As far as torsion classes go, we remark

- (a) If $\mathbf{G}(F \otimes \mathbf{R})$ does not admit discrete series — for example, whenever F is not totally real, or $\mathbf{G} = \mathrm{GL}_n$ and $n \geq 3$ — it is expected that such torsion classes in fact occur far more frequently than their characteristic zero counterparts — see [3], [31], [27], [19] for evidence in that direction.
- (b) For a complete theory, therefore, the notion of “functoriality” must be enlarged to apply to torsion forms. However, the existing methods for functoriality in characteristic zero are all of analytic character (e.g., trace formula, θ -correspondence, converse theorems) and do not seem to apply to such classes.

Note that, when discussing functoriality for mod p classes, it is natural to regard the dual groups and L -groups to be algebraic groups over k , rather than over \mathbf{C} . We will follow this convention throughout the paper; see Section 2.5.

To illustrate these points, we discuss some examples in more detail. In Section 9 we explain how to deduce these from the Main Theorem and the results of Section 8.

1.2.1. *Cyclic base change.* Let E/F be a Galois field extension whose Galois group is a p -group and \mathbf{G} be a reductive group over F . Then our results give a base change map

$$\text{mod } p \text{ forms on } \mathbf{G}/F \longrightarrow \text{mod } p \text{ forms on } \mathbf{G}/E;$$

i.e., the Satake parameters of \mathbf{G}/E and \mathbf{G}/F are related in the usual fashion for base change. See Section 9.4 for the precise statement.

We do not know how to approach this result for torsion classes in any other way. We also note that to our knowledge the characteristic zero version of this statement is not proven for exceptional groups.

1.2.2. *Exotic transfer from Sp_{2n} to GL_{2n} .* Let α be a (characteristic zero) cohomological form for Sp_{2n} over the number field F . Then there is an exotic lift that produces from α a Hecke eigenform $\tilde{\alpha}$ in the mod 2 cohomology of SL_{2n} , whose Satake parameters are related to those of α by means of the embedding $\iota : \mathrm{SO}_{2n+1} \rightarrow \mathrm{GL}_{2n}$ of dual groups in characteristic 2.

In most cases, we do not expect the class $\tilde{\alpha}$ to lift to characteristic zero, because ι does not lift to characteristic zero. In particular, we do not know how to construct $\tilde{\alpha}$ by classical methods.

1.2.3. *mod 2 Eisenstein series.* Let $n_1, \dots, n_r \geq 1$, and put $N = \sum n_i$. Our result implies the existence of “mod 2 Eisenstein series”

$$\text{mod 2 forms on } \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_d} \longrightarrow \text{mod 2 forms on } \mathrm{GL}_N$$

and similar transfers when we replace GL by other classical groups (whether or not they are split, so the term “Eisenstein series” is less appropriate).

More precisely, we mean that the Satake parameters on GL_N are related to those for $\mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_d}$ by means of the “direct sum” inclusion $\mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_d} \hookrightarrow \mathrm{GL}_N$ of dual groups.

We draw attention to this simple-seeming example because the analog in any characteristic $p > 2$ is presumably *false*: the Galois representation attached to the form on GL_N does not, in general, satisfy the “oddness” property of Ash (see [2, Conj. 1]).¹ We do not know the relationship of these classes to boundary cohomology; we have called them “Eisenstein” because of their parameters, but it may be that they are more closely related to CAP representations. We thank the referee for this observation.

1.2.4. *Exceptional groups.* Our Main Theorem provides a large class of new examples of functoriality for exceptional groups. Here is the simplest one: let U_3 be a unitary group associated to the quadratic extension $F(\sqrt{-3})/F$; so if $\sqrt{-3} \in F$, then this is simply GL_3 . Our result gives a lift

$$\text{mod 3 forms on } \mathrm{SU}_3 \rightarrow \text{mod 3 forms on } \mathrm{G}_2$$

associated, at the level of L -groups, to the inclusion of

$$(1.2.1) \quad \iota : \mathrm{PGL}_3 \rtimes \langle w \rangle \hookrightarrow \mathrm{G}_2,$$

¹Even if the GL_{n_i} -forms all lift to characteristic zero and we can construct a corresponding real-analytic Eisenstein series, the failure of “oddness” presumably forces the corresponding Eisenstein series to be noncohomological, no matter how the lifting is carried out.

where w acts by the pinned outer automorphism, and the PGL_3 arises from the short roots of G_2 .² This embedding exists only in characteristic 3, and the proof is significantly less formal than the other examples: one must compute the “eigenvalues” of $\iota(w)$ and compare it to the difference between sums of positive roots for \mathbf{H} and \mathbf{G} .

There is a long list of similar examples involving exceptional groups; see also Section 1.3.

1.2.5. *Cyclic base change in a characteristic zero setting.* We now return to the example of Section 1.2.1, but in a “characteristic zero” setting; as mentioned, our results will be weaker than those obtainable directly via the twisted trace formula, but we will obtain the additional information.

Take \mathbf{G} to be the group of a definite quaternion algebra D over a totally real field F and $E \supset F$ to be a totally real Galois extension, with Galois group a p -group. Then mod p automorphic forms for \mathbf{G} can be lifted to characteristic zero. In this case, under modest local restrictions, our mod p base change can be “boot-strapped” to characteristic zero base change, by using Galois representations and modularity lifting theorems. Thus we prove usual base change for forms in characteristic zero.

On the other hand, this proof can hardly be said to be easier than the proof via the twisted trace formula since, e.g., to attach Galois representations in this context already requires a version of the trace formula. And the twisted trace formula gives more, such as a precise characterization of the image. However, our method also yields some further interesting results.

- (a) For a mod p Hecke eigenform f on F , our method shows that there is a mod p Hecke form f_E on E that is a generalized eigenvector of the Hecke algebra with the base-changed eigenvalues, and *such that f is obtained from f_E by restriction.* (In this setting, the domain of f is naturally a subset of the domain of f_E .) If the quaternion algebra D is not definite, the situation is more complicated: the analog of f and f_E occur in different cohomological degrees. However, one may lift f_E to an $\langle \sigma \rangle$ -equivariant cohomology class \tilde{f}_E in such a way that the restriction $\mathrm{Res}(\tilde{f}_E)$ is related to f by means of the action of $H^*(\langle \sigma \rangle)$ on $\langle \sigma \rangle$ -equivariant cohomology.

This was observed in this context by Clozel [13]. The entire discussion of the foregoing paragraph also applies in the general context of our main theorem; there is nothing special about cyclic base change. However, we do not discuss these results further in this paper.

²For the *long root* embedding $\mathrm{SL}_3 \hookrightarrow G_2$ and totally real F , and characteristic zero forms, there is an approach via θ -correspondence (see [16]); but we see no way of passing between these results.

- (b) At ramified places, the relationship between f_E, f is much more complicated; it is given by the rather mysterious operation of base-change for local representations. However, our results suggest and partly prove that, when the representations are reduced mod p , it degenerates to a much simpler operation, namely, taking Tate cohomology; see Section 6.

1.3. *σ -dual homomorphisms.* From the above results, it is natural to expect that the “lift” furnished by the Main Theorem is a “functorial lift” in the sense of Langlands: that ψ_v is induced by a homomorphism of L -groups

$${}^L\hat{\psi} : {}^L\hat{H} \rightarrow {}^L\hat{G}.$$

In the setting of our paper, it is natural to construct the dual groups as algebraic groups over k , rather than over \mathbf{C} . We call such an ${}^L\hat{\psi}$ a *σ -dual homomorphism*; see Definition 9.1 for a precise discussion.

Using Theorem 8.10 and case by case arguments, we have verified that

A σ -dual homomorphism exists whenever \mathbf{G} is simply connected³ and \mathbf{H} is semisimple — with three possible exceptions⁴ when \mathbf{G} has type E_6 .

In other words, the correspondence of the Main Theorem is indeed a “functorial correspondence” in the sense of the Langlands program — a fact that is not at all apparent from the definition of ψ_v . In many instances, such as Section 1.2.2, the existence of a σ -dual homomorphism is related to delicate properties of the relevant groups in characteristic p , with the Galois component of the L -group entering in an interesting way when $p > 2$.

Details will appear elsewhere; roughly speaking, we can treat many inner cases at once by the general methods of Section 9.7 and many outer cases at once by methods generalizing that of Section 9.6.

1.4. *Further discussion.* In the local setting, this correspondence is related to the “Brauer correspondence” of modular representation theory. In the global setting, the correspondence can be viewed as a kind of “mod p Eisenstein series”; like Eisenstein series, many phenomena related to the Langlands program simplify but yet do not become trivial.

Indeed, Eisenstein series are, in a sense, dual to the operation of restricting an automorphic form to the boundary. Here we observe that the symmetric space for \mathbf{G}^σ , embedded in the symmetric space for \mathbf{G} , behaves *with respect to characteristic p homology* like a kind of “interior boundary.” A better-known

³If we relax this assumption, an ${}^L\hat{\psi}$ need not exist. The difficulties arising here are related to the ambiguity of square roots in the Satake transform and can possibly be fixed by replacing the L -group with a variant of the “ C -group” discussed in [9]

⁴It is likely that these cases can be treated similarly, but at the time of this writing we have not completed the verification.

example of this phenomenon is that “restriction to supersingular points” gives, for classical modular forms, a geometric construction of the Jacquet–Langlands correspondence modulo p ; see [33].

The main technical tool to prove the “interior boundary” property is Smith theory, or \mathbf{Z}/p -equivariant localization. It is related to the prior paper [36] of the first-named author. If we call Y (resp. X) the locally symmetric space for \mathbf{G}^σ (resp. \mathbf{G}) and put $\Gamma = \langle \sigma \rangle$, then the inclusion $Y \hookrightarrow X$ induces (almost) an isomorphism on equivariant cohomology $H_\Gamma^*(X) \rightarrow H_\Gamma^*(Y)$, and what remains is “just” to pass from equivariant cohomology to usual cohomology and to understand Hecke actions.

Given a sufficiently good chain-level understanding of the σ -action on the cohomology of X , for instance, a compatible triangulation of X , our method gives an explicit recipe for lifting automorphic forms on Y to X . The recipe can be presented as a spectral sequence (see proof of Theorem 4.4). We do not expect it to degenerate and indeed the differentials seem to carry interesting information. It will be interesting to study this further.

One cannot be too optimistic about “lifting” the method to characteristic zero in any direct way. The proof uses special properties of the Frobenius at various points; in fact the homomorphism of dual groups mentioned above need not lift to characteristic zero, as in the example above, or the inner examples of [36].

We note some related work. One inspiration for this paper was trying to understand the ideas behind the Glauberman correspondence [17]. In [1], Ash has used Smith theory to produce Hecke eigenclasses in the homology of GL_n over certain fields. Another closely related paper is the recent work of Clozel [13], whose methods are similar to ours, specialized to the case of \mathbf{G} a definite quaternion algebra over a totally real number field. That paper, moreover, makes intriguing use of this idea in the context of an infinite p -adic tower (see Section 1.5). The paper [36] of the first author, already mentioned, studies a similar story in the setting of the local geometric Satake correspondence, when σ is inner. L. Clozel has pointed out that the arguments of Section 8 resemble some of the constructions in the theory of twisted endoscopy, as in [24], but in our case these constructions are on the dual side and in characteristic p . The paper of Kionke [23] applies the Smith inequalities to ℓ -adic analytic towers of locally symmetric spaces for \mathbf{G} .

1.5. *Open questions.* We mention four interesting open questions.

- (i) *The ramified correspondence:* We formulate a conjecture in Section 6 relating local functoriality to Tate cohomology. This is a problem solely in the representation theory of p -adic groups. N. Ronchetti has obtained some evidence for this, in the setting of cyclic base change for GL_n and supercuspidal representations.

- (ii) *Behavior in a p -adic tower:* In the case of a definite quaternion algebra, Clozel [13], [12] formulates a theory of “automorphic forms over $\mathbf{Q}(\zeta_{p^\infty})$.” Can one make a similar theory for cohomological forms on an arbitrary group, using the ideas of the current paper?
- (iii) *Spectral sequences:* Study more carefully the higher differentials in the spectral sequence of Theorem 4.4. In our context, these higher differentials cannot always be zero, and it would be interesting to understand their arithmetic importance.
- (iv) In general, the locally symmetric space Y for \mathbf{G}^σ is only part, a union of connected components, of the full space of σ -fixed points of the locally symmetric space X of \mathbf{G} . The other components appear to be locally symmetric spaces for different F -forms of \mathbf{G}^σ . Smith theory realizes their cohomology as a subquotient of the cohomology of X , but we have not investigated the compatibility with Hecke actions.

1.6. *Plan of the paper.* Section 2 summarizes some of our notation and Section 3 some basic facts about Tate cohomology for cyclic groups. Sections 4 and 5 describe the Brauer homomorphism and describe the proof of the first Main Theorem. Section 6 describes the situation at ramified places. The results of this section are also used in the later parts of the paper. Section 7 consists of “folklore results” on the Satake transform. We have stated and proved them here because we do not know of a reference with characteristic p coefficients. Section 8 computes the unramified Brauer homomorphism in terms of Satake parameters. Rather than compute directly, we deduce the result by applying the results of Section 6 to unramified representations. In Section 9 we discuss the examples of Section 1.2 in detail.

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2. Notation

2.1. *Notation used throughout.* Let p be a prime number, and let k be an algebraic closure of the field with p elements. Let Λ denote the ring of Witt vectors of k (thus, p is a uniformizer for Λ), and let $\Lambda[p^{-1}]$ denote the fraction field of Λ . For $x \in \Lambda$, let $\bar{x} \in k$ denote its reduction mod p .

The symbol σ denotes a generator for a group of order p . We write $\langle \sigma \rangle$ for this group.

Let F be a number field, Γ_F its absolute Galois group, \mathcal{O}_F its ring of integers, \mathbf{A} its adèle ring, \mathbf{A}_f its ring of finite adèles, and set $F_\infty = F \otimes_{\mathbf{Q}} \mathbf{R}$. If v is a place of F , then F_v denotes the completion of F at v . By $\text{cyclo} : \Gamma_F \rightarrow \mathbf{F}_p^* \subset k^*$ we mean the cyclotomic character, i.e., the action of Γ_F on p th roots of unity in F .

If W is a k -vector space, we denote by $W^{(p)}$ the Frobenius-twist of W , i.e., the space with the same underlying vectors but scalar multiplication modified: if \cdot is the scalar multiplication in W , then the scalar multiplication $*$ in $W^{(p)}$ is given by $\lambda * w = \lambda^{1/p} \cdot w$; equivalently, $W^{(p)} = W \otimes_{(k, \text{Frob})} k$, so that $w\lambda \otimes \mu = w \otimes \lambda^p \mu$ for $\lambda \in k$.

When we write homology H_* or cohomology H^* of a topological space, we will always understand the coefficients to be taken in k , unless otherwise specified.

If $X = \text{Spec}(R)$ is an affine algebraic variety over k and G an algebraic group acting on X , we denote by $X//G$ the geometric quotient, i.e., the spectrum of R^G .

2.2. Nonabelian cohomology. If σ is an order p automorphism of a group M , we let $H^1(\sigma; M)$ denote the nonabelian cohomology of $\langle \sigma \rangle$ with coefficients in M , i.e., cocycles $j : \langle \sigma \rangle \rightarrow M$ modulo coboundaries. Elements of $H^1(\sigma; M)$ may be equivalently regarded as elements $j(\sigma) \cdot \sigma \in M \cdot \sigma \subset M \rtimes \langle \sigma \rangle$ modulo M -conjugacy, or elements $j(\sigma) \in M$ up to twisted conjugacy.

If $N \subset M$ is a σ -stable subgroup, then we have the “long exact sequence” of nonabelian cohomology [32, §§I.5.4–I.5.5]

$$N^\sigma \hookrightarrow M^\sigma \rightarrow (M/N)^\sigma \rightarrow H^1(\sigma; N) \rightarrow H^1(\sigma; M).$$

If N is normal in M and nilpotent of order prime to p , then $H^1(\sigma; M) \rightarrow H^1(\sigma; M/N)$ is a bijection.

2.3. Algebraic groups and level structures. Let \mathbf{G} be a connected reductive algebraic group over F . If v is a place of F , then $\mathbf{G}(F_v)$ is a locally compact topological group that we denote by G_v . If v is a finite place, K_v will denote an open compact subgroup of G_v . If v is an archimedean place, K_v denotes a maximal compact subgroup of G_v , and finally K_∞ denotes a maximal compact subgroup of $G_\infty = \mathbf{G}(F_\infty)$.

By a level structure for \mathbf{G} , we mean an open compact subgroup $K \subset \mathbf{G}(\mathbf{A}_f)$ of the form $\prod_v K_v$. Note that for such a level structure, K_v is a “standard” maximal compact for almost all v (i.e., is obtained by taking \mathcal{O}_v -points of an integral model of \mathbf{G} over \mathcal{O}_F).

When a level K is fixed, then for V a set of places of F , we denote by G_V the restricted product $\prod'_{v \in V} G_v = \{(g_v)_{v \in V} \mid g_v \in K_v \text{ for all but finitely many } v\}$.

When V is a set of finite places, we denote by K_V the product $\prod_{w \in V} K_w$ and by $K^{(V)}$ the complementary subgroup

$$(2.3.1) \quad K^{(V)} := K_\infty \prod_{w \notin V} K_w,$$

where the product is taken over finite places w not belonging to V .

2.4. Canonical torus. If F is any field, and \mathbf{G} is connected and reductive over F , then there is a canonical algebraic torus defined over F attached to \mathbf{G} . We denote it by $\mathbf{T}_G^{\text{can}}$. We will follow [14, §1.1].

If \mathbf{G} is quasisplit, we may describe $\mathbf{T}_G^{\text{can}}$ as the limit $\varinjlim \mathbf{B}/\mathbf{R}_u(\mathbf{B})$ over F -rational Borel subgroups of \mathbf{G} , of the quotient torus of the Borels. In general, we pass to an extension over which \mathbf{G} is quasisplit and then descend the torus thus constructed; then there may be no inclusion of $\mathbf{T}_G^{\text{can}}$ into \mathbf{G} defined over F .

Let X^* denote the character lattice of $\mathbf{T}_G^{\text{can}} \times_F \bar{F}$ and X_* the dual lattice. Then (X^*, X_*) supports a canonical based root datum $\Psi(\mathbf{G})$.

Automorphisms of \bar{F}/F induce an action

$$\theta_G : \Gamma_F \rightarrow \text{Aut}(\Psi(\mathbf{G})) \subset \text{Aut}(X^*).$$

The permutation representation of Γ_F on simple roots determines an étale algebra over F , and if \mathbf{G} is semisimple, the F -rational points of $\mathbf{T}_G^{\text{can}}$ are naturally identified with the units in this algebra.

2.5. Dual groups and L-groups. By default, we will regard all dual groups as reductive algebraic groups over k , i.e., in characteristic p .

The dual root datum to $\Psi(\mathbf{G})$ determines a pinned reductive algebraic group over k , which we denote by \hat{G} . Recall that a “pinning” is data $(\hat{T}, \hat{B}, \{X_i\})$ where $\hat{T} \subset \hat{B} \subset \hat{G}$ are a maximal torus and Borel subgroup of \hat{G} , and each X_i is a nonzero vector in a simple root space of $\text{Lie}(\hat{B})$. A pinning determines a splitting $\text{Out}(\hat{G}) \rightarrow \text{Aut}(\hat{G})$ and an identification $\text{Out}(\hat{G}) \simeq \text{Aut}(\Psi(\mathbf{G}))$.

In fact, the dual group and its pinning can be constructed over the prime field \mathbf{F}_p . It follows that $\hat{G}, \hat{B}, \hat{T}$ can be equipped with Frobenius endomorphisms that are defined over k , which we will denote by Frob .

The construction $\Psi(\mathbf{G}) \rightarrow \Psi(\mathbf{G})^\vee \rightarrow (\hat{G}, \hat{B}, \{X_i\})$ is functorial, and one obtains a Γ_F action $\Gamma_F \rightarrow \text{Out}(\hat{G}) \subset \text{Aut}(\hat{G})$. We let ${}^L\hat{G}$ denote the semidirect product $\hat{G} \rtimes \Gamma_F$. We regard \hat{G} and ${}^L\hat{G}$ as algebraic groups over k (the latter with an infinite component group), and we denote their groups of k -points by $\hat{G}(k)$ and ${}^L\hat{G}(k)$.

If α is a root in $\Psi(\mathbf{G})$, it determines a coroot for \hat{G} . We will use the notation α_* for this coroot (although a couple of times we will abuse notation and drop the subscript). Similarly, if α^\vee is a coroot for $\Psi(\mathbf{G})$, then we use α_*^\vee for the associated root in \hat{G} .

Suppose \mathbf{H} is another algebraic group with L -group ${}^L\hat{H}$. As k is algebraically closed and \hat{H}, \hat{G} are scheme-theoretically reduced, any algebraic morphism ${}^L\hat{H} \rightarrow {}^L\hat{G}$ is determined by its induced morphism on k -points ${}^L\hat{H}(k) \rightarrow {}^L\hat{G}(k)$. For our purposes (Langlands functoriality), we may therefore usually ignore the difference between them. (Note, however, that as k has positive characteristic, the map ${}^L\hat{H} \rightarrow {}^L\hat{G}$ can have nontrivial fibers as a map of schemes and yet induce an injection, or even an isomorphism, ${}^L\hat{H}(k) \rightarrow {}^L\hat{G}(k)$.)

2.6. *Local L -groups.* In the construction of Section 2.5, we may replace F by F_v for any finite place v , producing a group $\hat{G} \rtimes \Gamma_{F_v}$. When \mathbf{G} splits over an unramified extension of F_v , we work with the smaller group ${}^L\hat{G}_v := \hat{G} \rtimes \langle \text{Frob}_v \rangle$. Here $\langle \text{Frob}_v \rangle$ denotes the discrete infinite cyclic group that topologically generates the unramified quotient of Γ_{F_v} .

2.7. *Parabolics and Levis in \mathbf{G} and \hat{G} .* A $\mathbf{G}(\overline{F})$ -conjugacy class of parabolic subgroups $\mathbf{P} \subset \mathbf{G} \times_F \overline{F}$ distinguishes a subset Δ_P of the simple roots of $\Psi(\mathbf{G})$: the set of α for which $-\alpha$ is a root of \mathbf{P} (for a conjugation action of \mathbf{T}^{can} induced by an arbitrary Borel subgroup $\mathbf{B} \subset \mathbf{P}$ and splitting of $\mathbf{B} \rightarrow \mathbf{T}^{\text{can}}$, all defined over \overline{F}).

We define a corresponding Levi subgroup \hat{L} of \hat{G} — the subgroup generated by \hat{T} and the coroot homomorphisms $\text{SL}_2 \rightarrow \hat{G}$ corresponding to $\alpha \in \Delta_P$. Then the abelianization \mathbf{L}^{ab} of the quotient Levi \mathbf{L} of \mathbf{P} is dual, as a torus, to the center of \hat{L} — with notation as above, the character group of \mathbf{L}^{ab} is identified with those elements of $X^*(\mathbf{T}^{\text{can}})$ trivial on $\langle \alpha^\vee : \alpha \in \Delta_P \rangle$, whereas the co-character group of $Z(\hat{L})$ is identified with the orthogonal complement of $\{\alpha_*^\vee : \alpha \in \Delta_P\}$ in $X_*(\hat{T})$.

2.8. *Frobenius maps.* We will use the notation “Frob” for the Frobenius endomorphism of any k -group scheme equipped with a descent to \mathbf{F}_p . This applies, in particular, to any group of the form \hat{G} ; the fixed point subgroup of Frob is exactly the discrete set of \mathbf{F}_p -points of \hat{G} .

2.9. *Class field theory.* We will recall part of the Langlands correspondence for tori [26]. Let \mathbf{T} be an algebraic torus over a number field F . We say \mathbf{T} is unramified at v if $T_v := \mathbf{T}(F_v)$ splits over an unramified extension of F_v . A homomorphism $T_v \rightarrow k^*$ is unramified if it is trivial on the maximal compact subgroup of T_v . An idele class character $\mathbf{T}(F) \backslash \mathbf{T}(\mathbf{A}_f) \rightarrow k^*$ is said to be unramified at v if its restriction to T_v is unramified.

Set ${}^L\hat{T} = \hat{T} \rtimes \Gamma_F$ as in Section 2.5. When \mathbf{T} is unramified at v , set ${}^L\hat{T}_v = \hat{T} \rtimes \langle \text{Frob}_v \rangle$ as in Section 2.6. Let A_v denote the F_v -points of the maximal split subtorus $\mathbf{A}_v \subset \mathbf{T}$. (We leave in the subscript v because this depends on the place v). Then when \mathbf{T} is unramified at v , restriction gives an

isomorphism (see [6, §9.5]):

$$(2.9.1) \quad \{\text{unramified characters of } T_v\} \xrightarrow{\sim} \{\text{unramified characters of } A_v\}.$$

There is also a natural surjective homomorphism $\hat{T} \rightarrow \hat{A}_v$ that identifies \hat{A}_v with the Frob_v -coinvariants of \hat{T} . Therefore, to an unramified character of T_v is associated an element of the coinvariants \hat{T}_{Frob_v} ; put another way, this gives a natural bijection between unramified characters of T_v and conjugacy classes of splittings $\langle \text{Frob}_v \rangle \rightarrow {}^L\hat{T}_v$. Both the element of \hat{T}_{Frob_v} associated to an unramified character χ , and the splitting of $\langle \text{Frob}_v \rangle \rightarrow {}^L\hat{T}_v$ associated to χ , will be called the “Langlands parameter” of χ .

We can describe this more directly: Since $X^*(\mathbf{T})^{\text{Frob}_v} \rightarrow X^*(\mathbf{A})$ is an injection with finite cokernel, and k^* is divisible, we obtain a surjection

$$X^*(\mathbf{T})^{\text{Frob}_v} \otimes k^* \twoheadrightarrow X^*(\mathbf{A}) \otimes k^*.$$

Thus each element of $X^*(\mathbf{T})^{\text{Frob}_v} \otimes k^*$ gives an unramified character χ , and every unramified character χ arises thus, although possibly not uniquely. Explicitly, if $\alpha \in X^*(\mathbf{T})^{\text{Frob}_v}$ and $\lambda \in k^*$, then the unramified character associated to $\alpha \otimes \lambda$ is given by $t \in T_v \mapsto \lambda^{v(\alpha(t))}$.

One obtains the parameter of χ via the maps

$$(2.9.2) \quad X^*(\mathbf{T}) \otimes k^* = X_*(\hat{T}) \otimes k^* = \hat{T}(k).$$

(That is, the character χ is parametrized by the splitting $\text{Frob}_v \mapsto t_\chi \text{Frob}_v$, where $t_\chi \in \hat{T}(k)$ is the element thus produced.)

2.10. Hecke algebras. Let G be a locally compact, totally disconnected group. If S is a discrete set with a continuous left G -action and compact stabilizers, let $\text{Fun}_G(S \times S)$ (pronounced “funguses”) denote the set of k -valued functions on $S \times S$ that are invariant for the diagonal action of G , and whose support is a union of finitely many G -orbits. $\text{Fun}_G(S \times S)$ has an algebra structure with multiplication given by

$$(2.10.1) \quad (h_1 * h_2)(x, z) = \sum_{y \in S} h_1(x, y)h_2(y, z).$$

If $k[S]$ denotes the vector space spanned by S , then there is a left action of $\text{Fun}_G(S \times S)$ on $k[S]$ given by

$$h * s = \sum_{t \in S} h(s, t)t.$$

If S has finitely many G -orbits, then $\text{Fun}_G(S \times S)$ has a two-sided unit and the action on $k[S]$ identifies $\text{Fun}_G(S \times S)$ with the ring of G -endomorphisms of $k[S]$.

The standard example is when $K \subset G$ is an open compact subgroup and $S = G/K$. In that case, $\text{Fun}_G(G/K \times G/K)$ can be identified with

finitely supported functions on the double coset space $K \backslash G / K$, via $h(K, gK) = h(KgK)$. We abbreviate this case by $\mathcal{H}(G, K)$. We will also use the notation $\mathcal{H}(G, K; \mathbf{F}_q)$ for that subalgebra of $\mathcal{H}(G, K)$ consisting of functions valued in $\mathbf{F}_q \subset k$.

The theories of left and right $\mathcal{H}(G, K)$ -modules are equivalent via the anti-involution $KgK \leftrightarrow Kg^{-1}K$; nevertheless, we wish to record some explicit formulae for these actions with some attention paid to the difference between left and right:

The identification $V^K = \text{Hom}_G(k[G/K], V)$ gives the K -invariants of a left G -module the structure of a right $\mathcal{H}(G, K)$ -module. When V is a left G -module, an explicit formula for this action is

$$(2.10.2) \quad v * h = \sum_{gK \in G/K} g^{-1}vh(K, gK).$$

When X is a set with a right G -action, the k -vector space $k[X]$ spanned by X carries a left G -module structure extending $g \cdot x = xg^{-1}$ linearly. Then (2.10.2) specializes to the following right $\mathcal{H}(G, K)$ -action on $k[X/K] \simeq k[X]^K$:

$$(2.10.3) \quad xK * h = \sum_{gK \in G/K} xgK h(K, gK) \quad \text{for } xK \in X/K.$$

2.11. *Hecke actions on homology and cohomology.* Suppose that

- (i) G , as in Section 2.10, is a locally compact, totally disconnected group, and $K \subset G$ an open compact subgroup;
- (ii) X is a locally compact Hausdorff topological space with continuous right G -action, such that the restriction of this action to K is free and proper.

Note that the assumptions force K to be profinite. Let $K_i \subset K$ be a collection of open normal subgroups with $\bigcap_i K_i = \{e\}$; then the natural map

$$(2.11.1) \quad \pi : X \rightarrow \varprojlim X/K_n$$

is a homeomorphism. In fact, it is easily verified to be a continuous bijection. Now, we need to check that the image of any closed set $Z \subset X$ is also closed in $\varprojlim X/K_n$. Choose $y \notin \pi(Z)$. We want an open set containing y and disjoint from $\pi(Z)$. We may find a compact neighbourhood A of y in X/K with $\pi^{-1}A$ also compact. Then $\pi(Z) \cap A = \pi(Z \cap \pi^{-1}(A))$ is compact inside $\varprojlim X/K_n$. Thus there is an open set $N \ni y$ that is disjoint from $\pi(Z) \cap A$, and then $A^\circ \cap N$ is the required open set.

Then there is a right action of $\mathcal{H}(G, K)$ on the k -homology of X/K and a left action on the k -cohomology of X/K . We give an explicit construction of it: The set of singular m -simplices $\text{Hom}(\Delta^m, X)$ has a right G -action. Since the K -action is free on X , it is free also on $\text{Hom}(\Delta^m, X)$ and the quotient is naturally identified with the set of singular m -simplices in X/K . Each map $X/K_n \rightarrow$

X/K is a covering space and, by (2.11.1), each m -simplex in X/K lifts to X uniquely up to $\varprojlim K/K_n = K$. Thus, we get a right action of $\mathcal{H}(G, K)$ on

$$C_m(X/K) = k[\mathrm{Hom}(\Delta^m, X)/K] = k[\mathrm{Hom}(\Delta^m, X)]^K$$

for each m by applying the discussion around (2.10.3) with $S = \mathrm{Hom}(\Delta^m, X)$. As the face maps are maps of $\mathcal{H}(G, K)$ -modules, the action descends to a right action of $\mathcal{H}(G, K)$ on $H_m(X/K; k)$. The identification of $H^m(X/K)$ with the dual vector space to $H_m(X/K)$ gives it a left $\mathcal{H}(G, K)$ -action.

3. Tate cohomology

3.1. *Definition of Tate cohomology.* Let M be an abelian group with an action of the cyclic group $\langle \sigma \rangle$ of order p generated by σ . Set

$$T^0(M) := \ker(1 - \sigma)/\mathrm{Im}(N) \text{ and } T^1(M) := \ker(N)/\mathrm{Im}(1 - \sigma),$$

where $N = 1 + \sigma + \cdots + \sigma^{p-1}$ is the “norm.” In other words, $T^i(M)$ is the cohomology of the 2-periodic chain complex whose differentials alternate between $1 - \sigma$ and N . Because of this each short exact sequence of σ -modules induces a long exact sequence

$$T^0(M') \rightarrow T^0(M) \rightarrow T^0(M'') \rightarrow T^1(M') \rightarrow T^1(M) \rightarrow T^1(M'') \rightarrow T^0(M').$$

3.2. *Tate on smooth functions on ℓ -spaces.* If X is a Hausdorff, locally compact, totally disconnected space (an “ ℓ -space,” in the terminology of [4]), write $C_c^\infty(X; \Lambda)$ or $C_c^\infty(X; k)$ for the space of Λ - or k -valued functions on X that are locally constant (“smooth”) and compactly supported. If σ acts continuously on X , then we may form $T^i(C_c^\infty(X; ?))$. These groups can be computed in terms of X^σ , as follows:

(1) for $i = 0$ or 1 , restricting to fixed points descends to an isomorphism

$$T^i(C_c^\infty(X; k)) \xrightarrow{\sim} C_c^\infty(X^\sigma; k);$$

(2) the map

$$T^0(C_c^\infty(X; \Lambda)) \rightarrow C_c^\infty(X^\sigma; k)$$

given by restricting to fixed points and reducing mod p (in either order) is an isomorphism, while $T^1(C_c^\infty(X; \Lambda)) = 0$.

We are going to prove a more general result in the proposition below.

3.3. *Tate on sheaves on ℓ -spaces.* Let X be as in Section 3.2, and let \mathcal{F} be a sheaf of k - or Λ -modules on X . Write $\Gamma_c(X; \mathcal{F})$ for the space of compactly supported sections of \mathcal{F} . For instance, if \mathcal{F} is the constant sheaf with stalk k or Λ , then $\Gamma_c(X; \mathcal{F}) = C_c^\infty(X; k)$ or $C_c^\infty(X; \Lambda)$. The assignment $\mathcal{F} \mapsto \Gamma_c(X, \mathcal{F})$ is a covariant exact functor [5, §1.3].

If σ acts on X and \mathcal{F} is σ -equivariant, then σ can be regarded as a map of sheaves $\mathcal{F}|_{X^\sigma} \xrightarrow{\sigma} \mathcal{F}|_{X^\sigma}$ and we may define

$$\begin{aligned} \mathrm{T}^0(\mathcal{F}|_{X^\sigma}) &= \ker(1 - \sigma)/\mathrm{Im}(N) \quad (\text{a sheaf on } X^\sigma), \\ \mathrm{T}^1(\mathcal{F}|_{X^\sigma}) &= \ker(N)/\mathrm{Im}(1 - \sigma) \quad (\text{a sheaf on } X^\sigma). \end{aligned}$$

A compactly supported section of \mathcal{F} can be restricted to a compactly supported section of $\mathcal{F}|_{X^\sigma}$. This map preserves the σ -actions inducing a map

$$(3.3.1) \quad \mathrm{T}^i(\Gamma_c(X; \mathcal{F})) \rightarrow \Gamma_c(X^\sigma; \mathrm{T}^i(\mathcal{F})).$$

PROPOSITION. *The map (3.3.1) is an isomorphism.*

In the proof we will use “existence of fundamental domains”: if σ acts freely on a compact ℓ -space X , there is a fundamental domain, i.e., a closed and open subset $F \subset X$ so that X is the disjoint union of $\sigma^i F$ for $0 \leq i \leq p-1$. Indeed, take a cover of X by finitely many closed-and-open sets U_i so that $\sigma(U_i) \cap U_i = \emptyset$, and take the algebra of sets generated (under intersection and complement) by the U_i and their images under $\langle \sigma \rangle$. The minimal nonempty elements of that algebra give a finite compatible partition of X , each block of the partition being disjoint from its σ -image. The desired domain now follows by taking representatives for the σ -orbits on the blocks.

Proof. There is a short exact sequence (cf. [4, 1.16])

$$0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$$

of σ -equivariant sheaves where \mathcal{F}'' is supported on X^σ and the stalks of \mathcal{F}' vanishes along X^σ . It suffices to treat the cases $\mathcal{F} = \mathcal{F}''$ and $\mathcal{F} = \mathcal{F}'$. For \mathcal{F}'' , the proposition follows from the exactness of Γ_c applied to the 2-periodic chain complex defining T^* . So it is enough to check vanishing of $\mathrm{T}^i(\Gamma_c(U; \mathcal{F}'))$ when U^σ is empty.

We check for T^0 , the other case being similar. If $f \in \Gamma_c(U, \mathcal{F}')$ is σ -invariant, choose a σ -invariant compact $U' \subset U$ containing the support of f , take a fundamental domain $F' \subset U'$ for the σ -action, and note $f = \mathrm{N}(f')$, where f' is a section that agrees with f on F' and is zero off F' . \square

3.4. *Tate on rings.* There is an algebraic relative of Section 3.2: If A is any commutative unital k -algebra with a σ -action, set $\bar{A} = \mathrm{T}^0 A = A^\sigma / NA$. It has a *ring* structure because $N \cdot A$ is an ideal in A^σ . Then we have a bijection

$$(3.4.1) \quad \mathrm{Hom}(A, k)^\sigma \xrightarrow{\sim} \mathrm{Hom}(\bar{A}, k).$$

defined by restriction of characters.

Proof. For short, let us say an “extension” of a character $\psi : \bar{A} \rightarrow k$ is a character $\chi : A \rightarrow k$ such that χ , when restricted to A^σ , factors as $A^\sigma \rightarrow \bar{A} \xrightarrow{\psi}$

k . There is a ring homomorphism $A \rightarrow \bar{A}$ given by $a \mapsto (aa^\sigma \cdots a^{\sigma^{p-1}})$. When we restrict this to A^σ , it gives the p th power of the tautological map $A^\sigma \rightarrow \bar{A}$.

Thus given a character χ on \bar{A} , the formula

$$\tilde{\chi}(a) = \chi(aa^\sigma \cdots a^{\sigma^{p-1}})^{1/p}$$

defines an extension to A . This is the unique extension of χ to A : it is clearly the only possible σ -fixed extension and, in fact, any extension of χ to A must be σ -fixed. To see this, note that as χ is trivial on NA , we must have $\sum_{i=0}^{p-1} \tilde{\chi}^{\sigma^i} = 0$ for any extension $\tilde{\chi}$. By linear independence of characters [8, Ch. V, §6.1, Th. 1], it follows that $\tilde{\chi}^{\sigma^i} = \tilde{\chi}$, i.e., $\tilde{\chi}_1 \in \text{Hom}(A, k)^\sigma$. \square

The proof shows, more generally, that given any commutative integral domain B , any homomorphism $\chi : \bar{A} \rightarrow B$ has the property that χ^p extends uniquely to A (same argument, replacing B by its quotient field to invoke the linear independence of characters).

4. The Brauer homomorphism

4.1. *σ -actions, σ -plain subgroups.* Let G be a locally compact, totally disconnected group and $K \subset G$ an open compact subgroup. Suppose σ acts on G with $\sigma(K) = K$ and $\sigma^p = 1$. Write G^σ and K^σ for the fixed subgroups.

If X is a right G -space on which K -acts freely, then $\mathcal{H}(G, K)$ acts on the chains and cochains of X/K and $\mathcal{H}(G^\sigma, K^\sigma)$ acts on the chains and cochains of X^σ/K^σ . Note the difference between $(X/K)^\sigma$ and its subspace X^σ/K^σ — the former usually does not carry a $\mathcal{H}(G^\sigma, K^\sigma)$ -action.

We wish to relate the Hecke modules $H^*(X/K)$ and $H^*(X^\sigma/K^\sigma)$. The relationship becomes much simpler under a technical hypothesis on K . We say that $K \subset G$ is *σ -plain* if both of the following conditions hold:

(a) The inclusion

$$G^\sigma/K^\sigma \hookrightarrow (G/K)^\sigma : gK^\sigma \mapsto gK$$

is a bijection, or equivalently if G^σ acts on $(G/K)^\sigma$ with a single orbit, or equivalently if $H^1(\sigma, K) \rightarrow H^1(\sigma, G)$ has trivial fiber above the trivial class.

(b) K is virtually prime-to- p , i.e., there is a finite index subgroup $K' \subset K$ that is a projective limit of prime-to- p finite groups. In particular, $H^1(\sigma, K)$ is finite (Section 2.2).

4.2. *The Brauer homomorphism.* Suppose σ acts on G and K as in Section 4.1. The algebra $\mathcal{H}(G, K)$ has an action of σ (i.e., $h^\sigma(x^\sigma) = h(x)$ for $x \in G/K \times G/K$). Write $\mathcal{H}(G, K)^\sigma$ for the σ -invariant part of $\mathcal{H}(G, K)$. If K is σ -plain, then the *Brauer homomorphism* is the map

$$\text{Br} : \mathcal{H}(G, K)^\sigma \rightarrow \mathcal{H}(G^\sigma, K^\sigma)$$

just given by restricting h from $G/K \times G/K$ to

$$((G/K)^\sigma \times (G/K)^\sigma) = (G^\sigma/K^\sigma) \times G^\sigma/K^\sigma.$$

Since the summands of (2.10.1) are invariant under the action of σ on G/K , and k has characteristic p , the Brauer map is an algebra homomorphism. A similar construction is called the “Brauer homomorphism” in the modular representation theory of finite groups, and we call it by the same name here.

Set $N = 1 + \sigma + \sigma^2 + \cdots + \sigma^{p-1}$; i.e., N is the “norm element” in the group ring $k[\sigma]$. Then $N \cdot (1 - \sigma) = (1 - \sigma) \cdot N = 0$. If σ acts on a set S , and we thereby regard $k[S]$ as a $k[\sigma]$ -module, there are canonical identifications

$$(4.2.1) \quad \ker(1 - \sigma)/\text{Im}(N) = k[S^\sigma], \quad \ker(N)/\text{Im}(1 - \sigma) = k[S^\sigma].$$

Note that these are the groups T^i from Section 3.1; indeed, (4.2.1) is Proposition 3.3 in the special case of $\mathcal{F} = \text{constant sheaf}$. The identification on the left sends $s \in S^\sigma$ to $s + \text{Im}(N)$, and on the right it sends $s \in S^\sigma$ to $s + \text{Im}(1 - \sigma)$. The Brauer homomorphism is compatible with these identifications in a sense we now describe. (See also (6.2.1) for a more general statement.)

Suppose that \tilde{S} is a right G -set with compatible σ -action such that K acts freely, $S = \tilde{S}/K$, and $h \in \mathcal{H}(G, K)^\sigma$. Then the map $i : \tilde{S}^\sigma/K^\sigma \rightarrow S^\sigma$ is injective as the K action is free. Moreover, $\mathcal{H}(G, K)^\sigma$ acts on $k[S^\sigma] \simeq T^0 k[S]$ by means of (4.2.1), through the quotient $\mathcal{H}(G, K)^\sigma \rightarrow T^0(\mathcal{H}(G, K))$.

We claim that, in fact, $k[\tilde{S}^\sigma/K^\sigma]$ is a $\mathcal{H}(G, K)^\sigma$ -direct summand of $k[S^\sigma]$, and

$$(4.2.2) \quad \begin{aligned} &\text{The action of } T^0 h \text{ for } h \in \mathcal{H}(G, K)^\sigma \text{ on } k[\tilde{S}^\sigma/K^\sigma] \\ &\text{coincides with } \text{Br}(h) \in \mathcal{H}(G^\sigma, K^\sigma). \end{aligned}$$

Moreover, the identical statement holds also for T^1 .

It is enough to show the same statement with “direct summand” replaced by “submodule”; then one notes that the natural k -valued bilinear pairing on $k[S^\sigma]$ — given by $\langle \sum a_s s, \sum b_s s \rangle = \sum a_s b_s$ — has the property that

$$\langle a * T^0 h, b \rangle = \langle a, b * T^0 h' \rangle,$$

where $h \mapsto h'$ is the antiinvolution of the Hecke algebra sending KgK to $Kg^{-1}K$. Moreover, this bilinear pairing is nondegenerate on $k[\tilde{S}^\sigma/K^\sigma]$. That shows that $k[\tilde{S}^\sigma/K^\sigma]$ is actually a summand.

Now to check (4.2.2): For $\tilde{s} \in \tilde{S}^\sigma$, we have

$$\tilde{s}K^\sigma * \text{Br}(h) = \sum_{g \in G^\sigma/K^\sigma} \tilde{s}gK^\sigma h(K, gK)$$

and

$$\tilde{s}K * h = \sum_{g \in G/K} \tilde{s}gKh(K, gK).$$

Considered in $k[S]^\sigma$, these elements differ by

$$\sum_{g \in G/K - (G^\sigma/K^\sigma)} \tilde{g}K h(K, gK).$$

Our assumption that K is σ -plain means that σ acts freely on $G/K - (G^\sigma/K^\sigma)$; in particular, the element above belongs to the image of N from (4.2.1).

4.3. *Normalized Brauer homomorphism.* Suppose that both $\mathcal{H}(G, K)$ and $\mathcal{H}(G^\sigma, K^\sigma)$ are commutative integral domains. Then according to Section 3.4, the p th power of Br extends uniquely to a homomorphism

$$\widetilde{\text{Br}} : \mathcal{H}(G, K) \rightarrow \mathcal{H}(G^\sigma, K^\sigma).$$

This map is not k -linear but rather Frobenius semilinear. However, we may twist it to be linear: $\mathcal{H}(G, K)$ has an \mathbf{F}_p -structure, i.e.

$$\mathcal{H}(G, K) = \text{Fun}_G(G/K \times G/K; \mathbf{F}_p) \otimes_{\mathbf{F}_p} k.$$

The *normalized Brauer homomorphism*, which we denote with a lower case “b,” is the unique k -linear homomorphism

$$\text{br} : \mathcal{H}(G, K) \longrightarrow \mathcal{H}(G^\sigma, K^\sigma)$$

that agrees with $\widetilde{\text{Br}}$ on $\text{Fun}_G(G/K \times G/K; \mathbf{F}_p)$. An explicit formula for br is given by

$$(4.3.1) \quad \text{br}(h)(K^\sigma, gK^\sigma) = \left((h * \cdot^p \cdot * h)(K, gK) \right)^{\frac{1}{p}}.$$

4.4. THEOREM. *Let G, K, X be as in Section 2.11. Suppose that σ acts compatibly on G, K, X , so that $G^\sigma, K^\sigma, X^\sigma$ also satisfy the conditions of Section 2.11. Suppose in addition that X/K has finite cohomological dimension and that K is σ -plain in the sense of Section 4.1. In this situation, as described above,*

$$(4.4.1) \quad H^*(X/K) \text{ is a left } \mathcal{H}(G, K)\text{-module,}$$

$$(4.4.2) \quad H^*(X^\sigma/K^\sigma) \text{ is a left } \mathcal{H}(G^\sigma, K^\sigma)\text{-module.}$$

Then we have

- (a) *If we regard these as $\mathcal{H}(G, K)^\sigma$ -modules (via restriction for (4.4.1) and via Br for (4.4.2)), then every composition factor of (4.4.2) is also a composition factor of (4.4.1).*
- (b) *Suppose that we are in the setting of Section 4.3 — i.e., suppose that $\mathcal{H}(G, K)$ and $\mathcal{H}(G^\sigma, K^\sigma)$ are both commutative integral domains — so the normalized Brauer homomorphism br is defined. Suppose that $\chi : \mathcal{H}(G^\sigma, K^\sigma) \rightarrow k$ is a character that appears as an eigenvalue of (4.4.2); i.e., there exists an element of $H^*(X^\sigma/K^\sigma)$ annihilated by all $h - \chi(h)$ for $h \in \mathcal{H}(G^\sigma, K^\sigma)$.*

Then also $\chi \circ \text{br}$ appears as an eigenvalue of (4.4.1).

Proof. Note that (a) implies (b). Indeed, suppose that χ is as in (b). By (a), we have

$$(4.4.3)$$

$\chi \circ \text{Br}$ appears as an eigenvalue of (4.4.1) when restricted to $\mathcal{H}(G, K)^\sigma$.

We have seen in Section 3.4 that $(\chi \circ \widetilde{\text{Br}})^{1/p}$ is the unique extension of $\chi \circ \text{Br}$ from $\mathcal{H}(G, K)^\sigma$ to $\mathcal{H}(G, K)$. So, supposing (4.4.3), we see that $(\chi \circ \widetilde{\text{Br}})^{1/p}$ appears as an eigenvalue of (4.4.1). But this implies that $\chi \circ \text{br}$ appears as an eigenvalue of (4.4.1): the isomorphisms

$H^*(X/K) = H^*(X/K; \mathbf{F}_p) \otimes_{\mathbf{F}_p} k$, $\mathcal{H}(G, K) = \text{Fun}_G(G/K \times G/K, \mathbf{F}_p) \otimes_{\mathbf{F}_p} k$ yield semilinear actions of $\text{Aut}(k)$ on $\mathcal{H}(G, K)$ and $H^*(X/K)$. We have also

$$\alpha^\tau(h^\tau) = (\alpha(h))^\tau$$

for $\alpha \in \mathcal{H}$, $h \in H^*(X/K)$, $\tau \in \text{Aut}(k)$. So if $h \in H^*(X/K)$ corresponds to the eigenvalue $(\chi \circ \widetilde{\text{Br}})^{1/p}$, then h^τ corresponds to the eigenvalue $\chi \circ \text{br}$, where τ is the Frobenius automorphism.

The proof of (a) is an application of “fixed point localization” methods of Smith, Borel, and Quillen. We give a treatment here that is well adapted to keeping track of the Hecke action. The statement of the theorem for homology implies the statement for cohomology — let us prove the homology version.

Consider the “Smith double complex”

$$C_*^{\text{Smith}} := \left[\dots \xleftarrow{N} C_*(X/K) \xleftarrow{1-\sigma} C_*(X/K) \xleftarrow{N} C_*(X/K) \xleftarrow{1-\sigma} \dots \right].$$

The map σ is not a map of $\mathcal{H}(G, K)$ -modules but it is a map of $\mathcal{H}(G, K)^\sigma$ -modules, so C_*^{Smith} is a double complex of $\mathcal{H}(G, K)^\sigma$ -modules. It leads to two spectral sequences of $\mathcal{H}(G, K)^\sigma$ -modules:

- The spectral sequence ${}^{hv}E$, in which the differential on the zeroth page is the horizontal differential and on the first page is the vertical differential.
- The spectral sequence ${}^{vh}E$, in which the differential on the zeroth page is the vertical differential and on the first page is the horizontal differential.

If C_* is bounded, then both ${}^{hv}E$ and ${}^{vh}E$ converge to the homology of the total complex of C_*^{Smith} . Let us abbreviate the horizontal differential (which alternates between $1 - \sigma$ or N) by d^h and the vertical differential (which is the standard singular differential on $C_*(X/K)$) by d^v . We can compute higher differentials in these spectral sequences by the following standard device, see, e.g., [25, Lemma 2.1]. If $x \in {}^{hv}E_{ij}^0$ is an element that survives to ${}^{hv}E_{ij}^r$, and $(x_1, \dots, x_r) \in {}^{hv}E_{ij}^0 \times {}^{hv}E_{i+1, j-1}^0 \times \dots \times {}^{hv}E_{i+r-1, j-r+1}^0$ is a sequence of elements with $x = x_1$ and $d^v(x_s) = d^h(x_{s+1})$ for $s < r$, then $d^v(x_r)$ is a representative for $d^r(x)$.

We complete the proof in three steps.

(1) *Degeneration of ${}^{hv}E$.* The ${}^{hv}E$ spectral sequence is analyzed as follows: The differential on the zeroth page is d^h , so by equation (4.2.1), the first page ${}^{hv}E^1$ is naturally identified with $C_*((X/K)^\sigma)$; i.e.,

$${}^{hv}E_{ij}^1 = C_j((X/K)^\sigma) \quad d^1 : C_j \rightarrow C_{j-1} \text{ is the singular differential.}$$

It follows that ${}^{hv}E_{ij}^2 = H_j((X/K)^\sigma)$. Now, if $\zeta \in {}^{hv}E_{ij}^0 = C_j(X/K)$ has $d^h(\zeta) = 0$ then, by (4.2.1),

$$\zeta = \zeta' + d^h \varepsilon$$

for some ζ' belonging to ${}^{hv}E_{ij}^1 = C_j((X/K)^\sigma)$ and $\varepsilon \in {}^{hv}E_{i+1,j}^1 = C_j((X/K)^\sigma)$. If ζ survives to ${}^{hv}E^2$, we must have, in addition, $d^v \zeta \in \text{im}(d^h)$, or equivalently $d^v \zeta' \in \text{im}(d^h)$. But $d^v \zeta' \in C_{j-1}((X/K)^\sigma)$; by another application of (4.2.1), $d^v \zeta'$ is identically zero. In other words, every element of ${}^{hv}E^2$ is represented by a *cycle* $\zeta' \in C_j((X/K)^\sigma)$. Then $(\zeta', d^v(\zeta') = 0, 0, 0, \dots, 0)$ is a sequence we may use to compute $d^r(\zeta') = 0$ for all $r \geq 2$. Thus, ${}^{hv}E^2 = {}^{hv}E^\infty$.

(2) *Compatibility with the Brauer homomorphism.* In other words, (1) shows that the homology of the total complex of C_*^{Smith} has a filtration (by $\mathcal{H}(G, K)^\sigma$ -submodules) whose associated graded is $H_*((X/K)^\sigma)$. We claim that our assumptions imply that X^σ/K^σ is a union of connected components of $(X/K)^\sigma$.

To prove the claim, let $Y \subset X$ be the inverse image of $(X/K)^\sigma$. As K acts freely, for each $y \in Y$, there is a unique $\kappa(y) \in K$ such that $y\kappa(y) = \sigma(y)$. The map κ is K -equivariant for the σ -twisted conjugation action on K . The graph of κ is the set of all $(y, k) \in Y \times K$ with $\sigma(y) = yk$; in particular, it is a closed set. The projection of this graph to Y is a homeomorphism. So κ is a continuous function $Y \rightarrow K$. It descends to a continuous function

$$Y/K = (X/K)^\sigma \rightarrow \sigma\text{-twisted conjugacy classes for } K,$$

where we give the right-hand side the quotient topology. But the space of σ -twisted conjugacy classes for K is a finite set, because we assumed that K is σ -plain, and because each σ -twisted conjugacy class is closed in K , the topology on this finite set is the discrete topology. It follows that X^σ/K^σ is a union of connected components of $(X/K)^\sigma$.

Thus, on the first page, ${}^{hv}E_{ij}^1 = C_i((X/K)^\sigma)$ has $C_i(X^\sigma/K^\sigma)$ as a vector space summand. By equation (4.2.2) — applied with \tilde{S} equal to the free K -set of singular i -simplices in X — the action of $\mathcal{H}(G, K)^\sigma$ on this summand factors through Br and it is actually a $\mathcal{H}(G, K)^\sigma$ -submodule. Passing to homology, we conclude that $H_*(X^\sigma/K^\sigma)$ is a $\mathcal{H}(G, K)^\sigma$ -submodule of ${}^{hv}E_{ij}^2$.

(3) *Convergence of ${}^{vh}E^r$.* In this last step, observe that ${}^{vh}E_{ij}^1 = H_j(X/K)$ and that since $H_*(X/K)$ vanishes in large degrees, we have a convergent spectral sequence

$$(4.4.4) \quad {}^{vh}E_{ij}^1 = H_j(X/K) \implies H_*(\text{Tot}(C^{\text{Smith}}))$$

of $\mathcal{H}(G, K)^\sigma$ -modules. Therefore, by (2) we obtain the desired statement. We have exhibited $H_*(X^\sigma/K^\sigma)$, as a composition factor of $H_*(X/K)$, where both are regarded as modules under $\mathcal{H}(G, K)^\sigma$. Indeed, even better, we can identify ${}^{vh}E_{ij}^2$ with the Tate cohomology $T^i H_j(X/K)$; and we have thus actually exhibited $H_*(X^\sigma/K^\sigma)$ as a subquotient of $T^* H_*(X/K)$. \square

5. Cyclic group actions on locally symmetric spaces

5.1. *Definition.* Let \mathbf{G}, K_∞ and level structure $K \subset \mathbf{G}(\mathbf{A}_f)$ be as in Section 2.3. Let $[G]_K$ denote the double coset space

$$[G]_K := \mathbf{G}(F) \backslash \mathbf{G}(\mathbf{A}) / (K_\infty \times K).$$

If $K \cap \mathbf{G}(F)$ is torsion-free, then the homology and cohomology of $[G]_K$ carry the action of the Hecke algebra $\mathcal{H}(\mathbf{G}(\mathbf{A}_f), K)$ described in Section 2.11. For general K , one should regard $[G]_K$ as an orbifold and take homology and cohomology in this sense, in which case a more careful discussion defines an action of \mathcal{H} as well, but we will restrict our attention to the torsion-free case.

The Hecke algebra $\mathcal{H}(\mathbf{G}(\mathbf{A}_f), K)$ is a restricted tensor product over finite places

$$\mathcal{H}(\mathbf{G}(\mathbf{A}_f), K) = \bigotimes'_v \mathcal{H}(G_v, K_v),$$

where the restricted product is taken with respect to the identity element in $\mathcal{H}(G_v, K_v)$. When V is a set of finite places, we write $\mathcal{H}(G_V, K_V) := \bigotimes'_{v \in V} \mathcal{H}(G_v, K_v)$; we sometimes abbreviate this to simply \mathcal{H}_V .

5.2. *Good places.* We call a place v *good* with respect to the algebraic group \mathbf{G} , level structure K , and prime p if

- (i) The residue characteristic of \mathcal{O}_v is not equal to p
- (ii) $\mathbf{G} \times_F F_v$ is quasisplit over F_v and split over an unramified extension of F_v .
- (iii) K_v is a hyperspecial subgroup of G_v . In other words, K_v is a maximal compact subgroup of the form $\mathfrak{G}(\mathcal{O}_v)$, where \mathfrak{G} is a reductive smooth model for $\mathbf{G} \times_F F_v$ over \mathcal{O}_v .

For any K , all but finitely many places are good. At a good place, $\mathcal{H}(G_v, K_v)$ is a commutative integral domain and its characters are understood via the Satake isomorphism; for this, see Section 7.2.

5.3. *Characters of the Hecke algebra appearing in cohomology.* Let V be a set of finite good places, and suppose that χ is a character $\chi : \mathcal{H}(G_V, K_V) \rightarrow k$. We say “ χ appears in the cohomology of $[G]_K$ ” if there is $h \in H^*([G]_K)$ such that h transforms under $\mathcal{H}(G_V, K_V)$ by χ .

The following result shows that it is enough to consider “sufficiently small” level structures. In particular, as long as V excludes at least one finite place, one may always assume that the relevant locally symmetric spaces are manifolds and not merely orbifolds.

5.4. PROPOSITION. *Suppose that $K = \prod K_v$ and $K' = \prod K'_v$, where $K'_v \subset K_v$ for all v with equality $K'_v = K_v$ for $v \in V$. If χ appears in the cohomology of $[G]_K$, then it also appears in the cohomology of $[G]_{K'}$.*

Proof. The finite group K/K' acts on the cohomology of $[G]_{K'}$. For all $v \in V$, we have $K_v = K'_v$, and the actions of \mathcal{H}_v and K/K' on $H^*([G]_{K'})$ commute. The spectral sequence

$$E_2^{ij} = H^i(K/K'; H^j([G]_{K'})) \implies H^{i+j}([G]_K)$$

is a spectral sequence of \mathcal{H}_V -modules. Thus, a character of \mathcal{H}_V that occurs in the cohomology of $[G]_K$ also occurs in $H^*(K/K'; H^*([G]_{K'}))$. The bar model for the K/K' -cohomology of $H^*([G]_{K'})$ shows that the character must appear in $H^*([G]_{K'})$ itself. \square

5.5. σ -action. Let \mathbf{G} be, as in Section 2.3, a reductive algebraic group over F ; suppose that the automorphism σ acts on \mathbf{G} with order p , and set $\mathbf{H} = \mathbf{G}^\sigma$. We will always assume that \mathbf{H} is connected.

We may treat either \mathbf{G} or \mathbf{H} as a special case of the setup of Section 5.1, and we make the following parallel notation and assumptions:

- (a) Fix level structures K for \mathbf{G} and U for \mathbf{H} , and suppose that K is σ -stable with fixed points $U = K^\sigma$.
- (b) Fix a maximal compact $K_\infty \subset \mathbf{G}(F_\infty)$ in such a way that K_∞ is σ -invariant and K_∞ intersects $\mathbf{H}(F_\infty)$ in a maximal compact subgroup U_∞ .

This is always possible: inside the disconnected group $\mathbf{G}(F_\infty) \rtimes \langle \sigma \rangle$, we may find a maximal compact subgroup that contains $U_\infty \times \langle \sigma \rangle$, and then we just take its intersection with $\mathbf{G}(F_\infty)$.

- (c) Write $[G]_K = \mathbf{G}(F) \backslash \mathbf{G}(\mathbf{A}_F) / K_\infty K$ and $[H]_U = \mathbf{H}(F) \backslash \mathbf{H}(\mathbf{A}_F) / U_\infty U$.
- (d) K is “sufficiently small,” in that $\mathbf{G}(F) \cap K_\infty K$ is trivial; so also $\mathbf{H}(F) \cap U_\infty U$ is trivial. By Proposition 5.4, this will entail no real loss of generality.

5.6. PROPOSITION. *Say that a finite place v is σ -good with respect to $\mathbf{G}, K, \mathbf{H}, U$ if*

- (a) v is good with respect to K and U in the sense of Section 5.2, and
- (b) $K_v \subset G_v$ is a σ -plain subgroup in the sense of Section 4.1.

If \mathbf{H} is connected, then all but finitely many places of F are σ -good.

We remark that Brian Conrad and Gopal Prasad explained to us how to obtain a much sharper result by reducing to a corresponding assertion for tori.

Proof. Let \mathfrak{G} be a model of \mathbf{G} over \mathcal{O}_F . We must check that the map $H^1(\sigma, \mathfrak{G}(\mathcal{O}_v)) \rightarrow H^1(\sigma, G_v)$ has trivial fiber above the trivial class, for almost all v . (It is easy to check the remaining conditions are valid for almost all v .)

Consider the morphism of \mathcal{O}_F -schemes $g \mapsto g^{-1}\sigma(g)$ from \mathfrak{G} to itself. Its image I is constructible, i.e., a finite union of locally closed sets. On the other hand, it intersects the generic fiber of \mathfrak{G} in a *closed* set J : in characteristic zero, the conjugacy class of σ is closed. ([22, Cor. 5.8]: σ is automatically semisimple, being of finite order.) Let \bar{J} be the closure of J inside \mathfrak{G} . The symmetric difference $(\bar{J} \setminus I) \cup (I \setminus \bar{J})$, considered as a subset of \mathfrak{G} , is a constructible set that does not intersect the generic fiber. The projection of this symmetric difference to $\text{Spec}(\mathcal{O}_F)$ is (being constructible and disjoint from the generic point) a finite set of closed points.

Let T be the corresponding set of places, together with all places at which \mathfrak{G} or \mathfrak{G}^σ are not smooth and all places of residue characteristic dividing p . In what follows, replace \mathfrak{G} by its restriction to $\mathcal{O}[\frac{1}{T}]$. Then, by choice of T , the image of $g \mapsto g^{-1}\sigma(g)$ is a closed subset J' of \mathfrak{G} .

We will prove the proposition for $v \notin T$. If $y \in \mathfrak{G}(\mathcal{O}_v)$ represents an element of $H^1(\sigma, \mathfrak{G}(\mathcal{O}_v))$ that becomes trivial in $H^1(\sigma, G_v)$, then $y = g^{-1}\sigma(g)$ for some $g \in G_v$, and we wish to show that in this case we can find $x \in \mathfrak{G}(\mathcal{O}_v)$ with $y = x^{-1}\sigma(x)$. Indeed, suppose given $g \in G_v$ with the property that $g^{-1}\sigma(g) \in \mathfrak{G}(\mathcal{O}_v)$. We will verify that the \mathcal{O}_v -scheme defined by

$$\mathfrak{X} = \{x \in \mathfrak{G} : x^{-1}\sigma(x) = g^{-1}\sigma(g)\}$$

has an \mathcal{O}_v -point.

Now \mathfrak{X} has a $\overline{\mathbf{F}_v}$ -point: By assumption, $g^{-1}\sigma(g)$ yields a map $\text{Spec}(\mathcal{O}_v) \rightarrow \mathfrak{G}$ sending the generic point of $\text{Spec}(\mathcal{O}_v)$ to an element of J' . Because J' is closed, the special point of $\text{Spec}(\mathcal{O}_v)$ is also sent to an element of J' ; i.e., there exists $y \in \mathfrak{G}(\overline{\mathbf{F}_v})$ with $y^{-1}\sigma(y) = g^{-1}\sigma(g)$ modulo v , as desired.

Therefore, \mathfrak{X} also has a point over \mathbf{F}_v , because $\mathfrak{X}(\overline{\mathbf{F}_v})$ is a torsor under $\mathbf{H}(\overline{\mathbf{F}_v})$ and Steinberg's theorem [34, Th. 1.9] says that the Galois cohomology of the connected algebraic group \mathbf{H} is trivial over the finite field \mathbf{F}_v .

In other words, there exists $x \in \mathfrak{G}(\mathcal{O}_v)$ such that

$$(xg)^{-1}\sigma(xg) \in \Delta_v := \ker(\mathfrak{G}(\mathcal{O}_v) \rightarrow \mathfrak{G}(\mathbf{F}_v));$$

i.e., it defines a class in $H^1(\sigma, \Delta_v)$. But Δ_v has pro-order that is relatively prime to p , so that class must vanish; i.e., there exists $\delta \in \Delta_v$ such that $(xg)^{-1}\sigma(xg) = \delta^{-1}\sigma(\delta)$. In other words, the class of $g^{-1}\sigma(g) = y^{-1}\sigma(y)$ where $y = \delta x^{-1} \in \mathfrak{G}(\mathcal{O}_v)$, as desired. \square

5.7. *Analysis of connected components.* Let V be any nonempty finite set of σ -good places. Write $K^{(V)} = K_\infty \prod_{w \notin V} K_w$. We are going to apply the discussion of Section 4.4, with

$$(5.7.1) \quad X = \mathbf{G}(F) \backslash \mathbf{G}(\mathbf{A}_F) / K^{(V)}$$

and the acting groups (“ G, K ” from Section 4.4)

$$G_V = G \text{ from Section 4.4} = \prod_{w \in V} \mathbf{G}(F_w),$$

$$K_V = K \text{ from Section 4.4} = \prod_{w \in V} K_w.$$

Since we assumed that $\mathbf{G}(F) \cap K_\infty K$ is trivial, the group K_V acts freely on X . The main issue is to precisely analyze how the fixed locus X^σ is related to \mathbf{H} .

PROPOSITION. *The natural map $[H]_U \rightarrow X^\sigma / K_V^\sigma$ maps $[H]_U$ homeomorphically onto a union of components of X^σ / K_V^σ .*

Proof. We use the description of the fixed point set by means of nonabelian cohomology, as carried out in the papers of Rohlfs and Speh (see [29, §0.4]). Namely, there is a map

$$(5.7.2) \quad \epsilon : X^\sigma / K_V^\sigma \rightarrow H^1(\sigma, \mathbf{G}(F)) \times H^1(\sigma, K^{(V)})$$

whose construction we recall: For $g \in \mathbf{G}(\mathbf{A}_F)$, the double coset $\mathbf{G}(F)gK^{(V)} \in X$ is σ -fixed if and only if one can find $\gamma \in \mathbf{G}(F)$ and $\kappa \in K^{(V)}$ such that $\sigma(g) = \gamma g \kappa$ inside $\mathbf{G}(\mathbf{A}_F)$. Consider this equality at a place $w \in V$. It shows that actually $\gamma = \sigma(g_w)g_w^{-1}$; in particular, it satisfies $\sigma^{p-1}(\gamma) \cdots \sigma(\gamma)\gamma = e$. Then computing $\sigma^p(g)$ we see that also $\kappa \sigma(\kappa) \cdots \sigma^{p-1}(\kappa) = e$. In other words, $\sigma \mapsto \gamma^{-1}$ and $\sigma \mapsto \kappa$ define cocycles in $H^1(\sigma, \mathbf{G}(F))$ and $H^1(\sigma, K^{(V)})$; these classes depend only on the double coset.

The map ϵ of (5.7.2) is locally constant. In fact choose $x \in X^\sigma$ and a representative $g \in \mathbf{G}(\mathbf{A}_F)$. Let U_∞ be a σ -fixed open neighbourhood of K_∞ inside $\mathbf{G}(F_\infty)$, and let $U = U_\infty \cdot K$. Suppose g is, as above, so that the double coset $\mathbf{G}(F)gK^{(V)}$ is σ -fixed, and γ, κ are as above. Suppose that gu also defines a σ -fixed element of X , i.e.,

$$\sigma(g)\sigma(u) = \gamma' g u \kappa' \implies \gamma g \kappa \sigma(u) = \gamma' g u \kappa'$$

and, in particular,

$$\gamma g K \cap \gamma' g K \cdot U_\infty \cdot U_\infty^{-1} \neq \emptyset.$$

Because the action of $\mathbf{G}(F)$ on $\mathbf{G}(\mathbf{A}_F) / K_\infty K$ is properly discontinuous and free by assumption (d) of Section 5.5, this implies that $\gamma' = \gamma$ if U_∞ is chosen sufficiently small. (Recall that K_∞ is chosen σ -invariant, and so one may choose U_∞ to be an arbitrarily small open neighbourhood of it.)

We also then have $\kappa = u \kappa' (\sigma(u))^{-1}$, and thus the corresponding classes in $H^1(\sigma, K^{(V)})$ are also equal. Indeed, this is now clear for the projection to the

latter component of $K^{(V)} \simeq K_\infty \times \prod_{w \notin V} K_w$. To handle the K_∞ component we observe that $H^1(\sigma, K_\infty)$ is finite, and for each class in $H^1(\sigma, K_\infty)$, the set of representing cocycles is closed; thus the induced topology on $H^1(\sigma, K_\infty)$ is the discrete one. Thus, if we take U_∞ sufficiently small, the classes $H^1(\sigma, K_\infty)$ corresponding to κ_∞ and κ'_∞ are then forced to be equal.

The natural $[H]_U \rightarrow X^\sigma/K_V^\sigma$ is injective. If the double cosets of $h, h' \in \mathbf{H}(\mathbf{A}_F)$ map to the same point, we have $h = \gamma h' k$ with $\gamma \in \mathbf{G}(F), k \in K^{(V)} K_V^\sigma$. Considering components at a place $w \in V$, we see that $h_w = \gamma h'_w k_w$ and, in particular, γ is σ -invariant. Then k too is σ -invariant and belongs to $(K_\infty K)^\sigma = U_\infty U$.

Finally, the image of $[H]_U \rightarrow X^\sigma/K_V^\sigma$ is, by definition, precisely the fiber of ϵ above the trivial class, i.e., a union of connected components. This map from $[H]_U$ to its image is now a proper continuous bijection (the properness is a statement of Borel–Prasad (cf. Lemma 2.7 of [1]); it can also be deduced as in Lemma 4.15 of [28]) and so is a homeomorphism. \square

At this point we are ready to prove the first theorem from the introduction.

5.8. THEOREM. *Let $\mathbf{G}, \mathbf{H}, U, K, \sigma$ be as in Section 5.5, and let V be a nonempty set of σ -good places. Then $\mathcal{H}(G_V, K_V)$ and $\mathcal{H}(H_V, U_V)$ are both commutative integral domains and, in particular, the normalized Brauer homomorphism $\text{br} : \mathcal{H}(G_V, K_V) \rightarrow \mathcal{H}(H_V, U_V)$ of Section 4.3 is defined. If $\chi : \mathcal{H}(H_V, U_V) \rightarrow k$ is an eigenvalue occurring in the cohomology of $[H]_U$, then the character $\tilde{\chi} = \chi \circ \text{br} : \mathcal{H}(G_V, K_V) \rightarrow k$ occurs in the cohomology of $[G]_K$.*

Proof. The fact that $\mathcal{H}(G_V, K_V)$ and $\mathcal{H}(H_V, U_V)$ are commutative integral domains is well known (at least in the context where the coefficient ring is \mathbf{C} rather than k , but the same proof works); we summarize the proof in Theorem 7.2.

It suffices to prove the theorem when V is finite. Suppose that V is infinite and the theorem is false, i.e., $\tilde{\chi}$ does not occur in the cohomology of $[G]_K$. Since that cohomology is finite-dimensional, there is certainly a finite subset $V' \subset V$ such that the restriction of $\tilde{\chi}$ to $\mathcal{H}(G_{V'}, K_{V'})$ does not occur in the cohomology of $[G]_K$.

Now suppose V is finite. We saw above that, with X as in (5.7.1), $H^*([H]_U)$ is a direct summand of $H^*(X^\sigma/K_V^\sigma)$. Indeed, as is clear by inspection, it is even a $\mathcal{H}(H_V, U_V)$ -submodule. Now apply Theorem 4.4. \square

6. Representation theory

Let $\mathbf{G}, K, \mathbf{H}, U$ be as in Section 5.5. For any finite place v , the Hecke algebras $\mathcal{H}(G_v, K_v)$ and $\mathcal{H}(H_v, U_v)$ describe portions of the categories of G_v -modules and of H_v -modules. In this section we make precise a sense in which the Brauer homomorphism of Section 4.2 “lifts” to a functor between

categories of representations. This is relevant both to understand the situation at ramified places and for the proof of the theorem of Section 8.

6.1. *Linkage and the Brauer homomorphism.* Let $\mathbf{G}, K, \mathbf{H}, U$ be as in Section 5.5. Fix a finite place v of F , of residue characteristic $\neq p$. We do not require that K_v is maximal compact. In particular, $\mathcal{H}(G_v, K_v)$ is not necessarily commutative.

We consider irreducible k -linear representations Π of G_v . These are always understood to be continuous; i.e., every vector in Π has open stabilizer (often called “smooth”). We will only consider *admissible* representations: $\Pi^{K'}$ is finite-dimensional over k for every compact open subgroup $K' \subset G$. Say that such a representation is σ -fixed if it is isomorphic to $\Pi \circ \sigma$.

PROPOSITION. *If Π is σ -fixed, then there is a unique action of σ on Π compatible with the σ -action on G_v .*

Note that this is a variation on a standard result for representations of characteristic zero — for example, compare to [17, Th. 1] for the case of finite groups.

Proof. If A is a k -linear isomorphism from Π to itself that intertwines Π with $\Pi \circ \sigma$, then we claim A^p must be a scalar. If that scalar is λ , then $\sigma = \lambda^{-1/p} A$ is a σ -action on Π compatible with the σ -action on G_v .

To prove the claim, choose a prime-to- p open subgroup $K_v^0 \subset K_v$. Then $\Pi^{K_v^0}$ is a finite-dimensional irreducible representation of the Hecke algebra $\mathcal{H}(G_v, K_v)$, so by Schur’s lemma A^p acts as a scalar on $\Pi^{K_v^0}$. Since the image of the action map $G_v/K_v^0 \times \Pi^{K_v^0} \rightarrow \Pi$ generates Π , A^p must act by the same scalar on the entirety of Π . \square

For a σ -fixed Π with its action of σ , we may then consider the Tate cohomology $T^i \Pi$ for $i \in \{0, 1\}$. It carries an action of H_v .

6.2. *Definition.* We say that an irreducible representation π of H_v is *linked with Π* if the Frobenius-twist $\pi^{(p)}$ (see Section 2.1) occurs as a Jordan-Holder constituent of $T^0(\Pi)$ or $T^1(\Pi)$.

As a motivating example, which may explain the role of the Frobenius-twist, take $\mathbf{G} = \mathbf{H}^p$ and σ to act by cyclic permutation. Then the irreducible representation π_v of H_v is linked with the irreducible representation $\pi_v^{\otimes p}$ of $G_v \simeq H_v^p$.

The notion of linkage is a representation theoretic version of the Brauer homomorphism. Let Π be a σ -fixed representation of G_v . We may apply T^* to the $\mathcal{H}(G_v, K_v)$ -module Π^{K_v} . The σ -equivariant inclusion map $\Pi^{K_v} \rightarrow \Pi$ induces $T^*(\Pi^{K_v}) \rightarrow T^*(\Pi)$, which in fact takes values in the $\mathcal{H}(H_v, U_v)$ -module $T^*(\Pi)^{U_v}$.

We now suppose that $H_v/U_v = (G_v/K_v)^\sigma$, as in Section 4.1; i.e., K_v is σ -plain in the notation of that section. Then the (unnormalized) Brauer homomorphism $\text{Br} : \mathcal{H}(G_v, K_v)^\sigma \rightarrow \mathcal{H}(H_v, U_v)$ is compatible with linkage in that the diagram

$$(6.2.1) \quad \begin{array}{ccc} \mathbf{T}^*(\Pi^{K_v}) & \longrightarrow & \mathbf{T}^*(\Pi)^{U_v} \\ \mathbf{T}^*(h) \downarrow & & \downarrow \text{Br}(h) \\ \mathbf{T}^*(\Pi^{K_v}) & \longrightarrow & \mathbf{T}^*(\Pi)^{U_v} \end{array}$$

commutes for any $h \in \mathcal{H}(G_v, K_v)^\sigma$.

Proof. We give the proof for \mathbf{T}^0 . If $x \in \Pi^{K_v}$ is σ -fixed, then the image of $x + N(\Pi^{K_v})$ in $\mathbf{T}^0(\Pi)^{U_v}$ is $x + N(\Pi)$, and to verify (6.2.1) we have to show that the equation

$$\text{Br}(h) * (x + N(\Pi)) = (h * x) + N(\Pi)$$

holds. The left-hand side is

$$\sum_{gU_v \in H_v/U_v} \text{Br}(h)(U_v, gU_v)g(x + N(\Pi)) = \sum_{gU_v \in H_v/U_v} h(K_v, gK_v)(gx + N(\Pi))$$

and the right-hand side is

$$\left(\sum_{gK_v \in G_v/K_v} h(K_v, gK_v)gx \right) + N(\Pi),$$

so (6.2.1) reduces to checking

$$\sum_{gK_v \in G_v/K_v - H_v/U_v} h(K_v, gK_v)gx \in N(\Pi).$$

Since we have assumed $(G_v/K_v)^\sigma = H_v/U_v$, σ acts freely on the set indexing the sum, which therefore does belong to $N(\Pi)$. A similar computation shows (6.2.1) holds for $x + (1 - \sigma)(\Pi^{K_v}) \in \mathbf{T}^1(\Pi^{K_v})$. \square

6.3. *Conjectures.* It seems very reasonable to believe the following.

Let Π be a σ -fixed irreducible admissible representation of G_v . Then \mathbf{T}^Π is of finite length as an H_v -representation.*

The conjecture is motivated by the analogy with Eisenstein series formulated in the introduction. Viewing the functor $\mathbf{T}^i\Pi$ as an analog of the Jacquet functor, the conjecture is a counterpart to the fact that the Jacquet functor carries admissibles to admissibles [11, Th. 3.3.1]. The analogy, together with computations we have carried out in the case of depth zero base change for GL_n , suggests another conjecture, which we will leave in a slightly less precise form:

Linkage is compatible with the Langlands functorial transfer associated to a σ -dual homomorphism ${}^L\hat{\psi} : {}^L\hat{H} \rightarrow {}^L\hat{G}$ (Definition 9.1).

In particular, if the σ -fixed representation Π of G_v is linked with the representation π of H_v , we should expect ${}^L\hat{\psi}$ to carry the Langlands parameter of π to the Langlands parameter of Π . In other words, just as the Jacquet functor realizes functoriality between an L -group and a Levi subgroup, we expect that the Tate cohomology functor should realize functoriality for the σ -dual homomorphism of Definition 9.1.

6.4. *Ramified places.* Fix a finite set V of places of F and a level structure $K \subset \mathbf{G}$, where each place $v \in V$ is good (Section 5.2) with respect to K .

Let S be a finite set of finite places, disjoint from V , and put $G_S = \prod_{w \in S} G_w$. Consider all level structures K' that agree with K away from the set S , that is to say,

$$(6.4.1) \quad K' = \prod_{v \in S} K'_v \cdot \prod_{v \notin S} K_v.$$

The V -Hecke algebra $\mathcal{H}(G_V, K_V)$ is a commutative integral domain acts on the cohomology of $[G]_{K'}$. Let $\chi : \mathcal{H}(G_V, K_V) \rightarrow k$ be a homomorphism. We may form the G_S -module

$$\pi(\chi) := \chi\text{-component of } \varinjlim_{K'} H^*([G]_{K'}),$$

where by χ -component we mean in fact the localization at the maximal ideal defined by χ , i.e., the generalized eigenspace corresponding to χ . Strictly speaking, as we have defined it, this depends on both χ and V , but we have suppressed the dependence on V in the notation.

The precise determination of $\pi(\chi)$ is an interesting and difficult question; it is the subject of the mod p Langlands correspondence [15]. In any case, $\pi(\chi)$ and all of its irreducible subquotients are admissible. If we take K' small enough that $K'_S := \prod_{v \in S} K'_v$ has pro-order that is prime-to- p , then $\pi(\chi)^{K'_S}$ is identified with the χ -component of cohomology of $H^*([\mathbf{G}]_{K'})$. If we shrink K' further, we may ensure that $[\mathbf{G}]_{K'}$ is a manifold and has finite-dimensional cohomology.

We are ready to formulate the exact relationship between linkage and the functoriality associated to a σ -dual homomorphism.

6.5. THEOREM. *Let $\mathbf{G}, \mathbf{H}, K, U, \sigma$ be as in Section 5.5, and suppose \mathbf{G} is semisimple and \mathbf{H} is connected. Let V be a finite set of σ -good places (see Proposition 5.6) and S a finite set of finite places disjoint from V and all primes above p .*

Let $\chi : \mathcal{H}(H_V, U_V) \rightarrow k$ be a character, $\psi = \chi \circ \text{br}$. Let $\pi = \pi_\chi$ and $\Pi = \Pi(\psi)$ be the representations of (respectively) H_S and G_S attached to χ as in Section 6.4. Then any irreducible subquotient of the H_S -module $\pi(\chi)$ is linked with an irreducible subquotient of the G_S -module $\Pi(\psi)$.

Note a minor weakness compared to Theorem 5.8: the set V above is required to be finite. It is likely this can be relaxed, and it seems harmless in practice. More seriously one could ask for a more precise statement — for example, a complete determination of one space in terms of the other — but we do not pursue this here.

Proof. Let $K' \leq G_S$ be an open compact subgroup as above (see (6.4.1) and prior discussion), now assumed σ -stable. Let $U' = (K')^\sigma \leq H_S$.

We proceed just as in Section 5.7 and Theorem 5.8 but in cohomology rather than homology. That furnishes an embedding of

$$H^*([H]_{U'}) \text{ as a subquotient of } T^*H^*([G]_{K'})$$

equivariantly for the action of $\mathcal{H}(H_V, U_V)$. In fact, we can do this compatibly at all levels at once, thus embedding $\varinjlim H^*([H]_{U'})$ as a subquotient of $T^*(\varinjlim H^*([G]_{K'}))$ in an $H_S \times \mathcal{H}(H_V, U_V)$ -equivariant fashion. We explicate this a little.

Proceed as in the proof of Theorem 5.8, but form the associated “Smith double complex” from the direct limit of cochain complexes for the $[G]_{K'}$. The sequences are convergent because the cohomological dimension of $[G]_{K'}$ is bounded independent of $[K']$. Our reasoning as before shows that the h -complex converges to $\varinjlim H^*([G]_{K'}^\sigma)$ whereas the E^2 term of the vh -complex is $\varinjlim T^*H^*([G]_{K'})$. Moreover, $\varinjlim H^*([H]_{U'})$ is an $H_S \times \mathcal{H}(H_V, K_V)$ -summand of $\varinjlim H^*([G]_{K'}^\sigma)$.

Now, localizing at a character of $\mathcal{H}(H_V, K_V)$, we see that any irreducible constituent τ of $\pi(\chi)$ (as an H_S -module) is a composition factor of $T^*\Pi(\psi')$, where $\psi' = \chi \circ \text{Br}$ is considered as a character of $\mathcal{H}(G_V, K_V)^\sigma$. By an argument with Frobenius acting on the coefficients, similar to that given earlier, we see that $\tau^{(p)}$ is a composition factor of $T^*\Pi((\psi')^p)$. Finally, because $\chi \circ \text{br}$ is the unique extension of $(\psi')^p$ to $\mathcal{H}(G_V, K_V)$, we see that $\Pi((\psi')^p) = \Pi(\psi)$. \square

7. Satake parameters

In this background section, we recall the Satake isomorphism and the notion of modularity. Then we reformulate the Brauer map in terms of Satake parameters. We require Satake parameters and a Satake correspondence at places v whose residue characteristic is different from the characteristic of k , which is quite similar to the case of complex coefficients. We give some details because references for the (limited) amount we need are not always available.

7.1. Restricted Weyl group. Let v be a good place for \mathbf{G} , and let $A_v \subset B_v$ be a maximally split torus and Borel subgroup of G_v . Let $\mathbf{A}_v \subset \mathbf{B}$ be the corresponding algebraic groups, and \mathbf{T} the quotient torus of \mathbf{B} . The “restricted Weyl group” of \mathbf{G} at v is the quotient $N_{G_v}(A_v)/Z_{G_v}(A_v)$, i.e., the normalizer of A_v divided by the centralizer of A_v . We denote it by $W_{0,v}$. The correspondence

between unramified characters of T_v and splittings of $\hat{T}(k) \rtimes \text{Frob}_v \rightarrow \langle \text{Frob}_v \rangle$ from Section 2.9 is compatible with the natural action of $W_{0,v}$ on each side.

The restricted Weyl group also acts on the dual torus \hat{A}_v to A_v . We will need the following assertion.

PROPOSITION. *The action of any $w \in W_{0,v}$ on \hat{A}_v is induced by an element $n \in \hat{G}(k)$ normalizing \hat{T} and fixed by Frob_v .*

This result is proven in [6, Lemma 6.2] over the complex numbers.

Proof. As in [6, §6.1] we must show that each Frob_v -fixed class in the Weyl group of (\hat{G}, \hat{T}) has a Frob_v -fixed representative within $\hat{G}(k)$. It is proven by Steinberg [34, p. 173] that σ -fixed points of the Weyl group \hat{W} are generated by *basic reflections* W_D indexed by orbits D of Frob on simple roots on (\hat{G}, \hat{T}) . It suffices to find a Frob_v -fixed representative in $\hat{G}(k)$ for these generators W_D .

The basic reflection W_D is characterized (item (3) of *loc. cit.*) as the unique element of $\langle w_\alpha \rangle_{\alpha \in D} \subset \hat{W}$ with the property that W_D sends D set-wise into $-D$. Equivalently, W_D is the long element of the Weyl of the Levi subgroup \hat{M}_D obtained when we adjoin each of the $\alpha \in D$ to \hat{T} .

This reduces us to a special case of the proposition, where $\hat{G} = \hat{M}_D$, $w = W_D$ is the long element, and the pinned automorphism Frob_v acts transitively on the simple roots of \hat{G} . It suffices to produce a Frob_v -fixed representative inside the derived group $[\hat{G}, \hat{G}](k)$, and then inside $\hat{G}'(k)$ where \hat{G}' is the simply-connected cover of $[\hat{G}, \hat{G}]$, so we may furthermore assume that \hat{G} is simply connected.

A simply-connected \hat{G} is a product of its almost simple factors. As Frob_v acts transitively on the simple roots, these factors are all isomorphic, say $\hat{G} \cong \hat{G}_0^{\times r}$. Then Frob_v^r still acts transitively on the simple roots of the almost simple group \hat{G}_0 . To finally reduce to the case $r = 1$ and $\hat{G} = \hat{G}_0$ is almost simple, note that if $n_0 \in \hat{G}_0(k)$ is a Frob_v^r -fixed representative for the long element of the Weyl group of \hat{G}_0 , then

$$(n_0, \text{Frob}_v(n_0), \text{Frob}_v^2(n_0), \dots, \text{Frob}_v^{r-1}(n_0)) \in \hat{G}_0^{\times r}$$

gives the desired representative for the long element of the Weyl group of \hat{G} .

From the classification, A_1 and A_2 are the only Dynkin diagrams that admit an automorphism acting transitively on the vertices. Thus when $r = 1$ and $\hat{G} = \hat{G}_0$, we must have either $\hat{G} = \text{SL}_2$ with the trivial Frob_v -action or $\hat{G} = \text{SL}_3$ with the Frob_v action given by

$$g \mapsto \omega(g^t)^{-1}\omega \quad \omega = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

In the first case it is vacuous that there is a Frob_v -fixed representative for the long element. In the second case we can take for this representative the matrix ω , completing the proof. \square

7.2. *The Satake isomorphism.* We now describe the Satake isomorphism. We begin with a statement of the main ingredients, but presented “over k ” and with no choices of square roots made. For this statement, we will require the following twisted action of the Weyl group $W_{0,v}$ on \hat{A}_v :

$$(7.2.1) \quad w * a = wa \cdot \sqrt{\frac{\Sigma_G^*}{w\Sigma_G^*}}(q_v) \quad \text{for } w \in W_0 \text{ and } a \in \hat{A}_v(k)$$

and Σ_G^* is the co-character of \hat{T} given by the sum of all positive coroots. Note that $\Sigma^*/w\Sigma_G^*$ is divisible by 2 in that cocharacter lattice; thus $\sqrt{\frac{\Sigma_G^*}{w\Sigma_G^*}}(q_v)$ makes sense, and we can then project to \hat{A}_v via $\hat{T} \rightarrow \hat{A}_v$.

THEOREM. *Let $G_v = \mathbf{G}(F_v)$ be a reductive v -adic group, and let $K_v \subset G_v$ be a maximal compact subgroup satisfying the conditions of Section 5.2, which is “in good position” with respect to A_v ; i.e., $A_v \cap K_v$ is a maximal compact subgroup of A_v . Let \hat{A}_v be as in Sections 2.9 and 7.1. The following hold:*

- (i) *There is a natural isomorphism (see (7.2.6) below)*

$$(7.2.2) \quad \mathcal{H}(G_v, K_v) \xrightarrow{\sim} (W_{0,v}, *)\text{-invariant regular functions on } \hat{A}_v.$$

In particular, $\mathcal{H}(G_v, K_v)$ is a commutative integral domain.

- (ii) *There is a natural identification*

$$(7.2.3) \quad \begin{array}{ccc} (W_{0,v}, \cdot)\text{-invariant regular} & \xleftarrow{\sim} & \text{regular functions on} \\ \text{functions on } \hat{A}_v & & \hat{G} \rtimes \text{Frob}_v // \hat{G}, \end{array}$$

where the $W_{0,v}$ -action on \hat{A}_v is now the usual one.

Note that integrality in the Satake transform was first considered by Satake himself [30, §5, remark after Th. 1] and much stronger results than (i) are given in Henniart–Vignéras [21, §7]. We now specify the isomorphism of (i) and give references for the proof; we prove (ii) in Section 7.3.

The usual Satake isomorphism (see, e.g., [18]) is defined thus: Let B_v be the F_v -points of a Borel containing A_v and N_v the points of its unipotent radical. Let $\delta_{\mathbf{R}_{>0}}$ be the modular character of the Borel, i.e., the composite

$$(7.2.4) \quad B_v \rightarrow F_v^* \rightarrow q_v^{\mathbf{Z}} \subset \mathbf{R}_{>0},$$

where the first map is the sum Σ_G of all positive roots, evaluated on F_v -points. Let $\delta_{\mathbf{R}_{>0}}^{1/2}$ denote the positive square root of $\delta_{\mathbf{R}_{>0}}$. The usual Satake transform

$f \mapsto \mathcal{S}(f)$ produces from $f \in \mathcal{H}(G_v, K_v)$ a function $\mathcal{S}(f)$ on A_v given by

$$\mathcal{S}(f)(t) = \delta_{\mathbf{R}_{>0}}^{1/2}(t) \cdot \int_{N_v} f(K_v, tnK_v) dn,$$

where the measure dn is normalized so that $N_v \cap K_v$ has mass 1. Since $\mathcal{S}(f)$ is compactly supported, and constant on $(A_v \cap K_v)$ -cosets, we may regard it as an element of the group ring $\mathbf{C}[X_*(\mathbf{A}_v)] = \mathbf{C}[X^*(\hat{A}_v)]$. Then \mathcal{S} defines an algebra isomorphism

$$\mathcal{H}(G_v, K_v) \text{ with } \mathbf{Z}[q_v^{\pm 1/2}]\text{-coefficients} \xrightarrow{\sim} \mathbf{Z}[q_v^{\pm 1/2}], [X^*(A_v)]^{W_{0,v}}$$

where the $W_{0,v}$ -action on $X^*(\hat{A}_v)$ is the “untwisted” one from Section 7.1. (To check this is an isomorphism is an “upper triangular” argument; see [20, Lemma 10.2.1].) To avoid choice of square roots, define a modified Satake transform \mathcal{S}^* by

$$(7.2.5) \quad \mathcal{S}^*(f)(t) = \int f(K_v, tnK_v) dn,$$

i.e., $\mathcal{S}^* := \delta^{-1/2}\mathcal{S}$. Then \mathcal{S}^* gives (see [21, §7.11]) an isomorphism from the complex Hecke algebra onto the $(W_{0,v}, *)$ -invariant subring of $\mathbf{C}[X^*(\hat{A}_v)]$. One deduces that \mathcal{S}^* defines an isomorphism of the corresponding algebras with $\mathbf{Z}[q_v^{-1}]$ coefficients. By tensoring with k , we get the desired isomorphism

$$(7.2.6) \quad \mathcal{S}^* : \mathcal{H}(G_v, K_v) \rightarrow k[X^*(\hat{A}_v)]^{(W_{0,v},*)}.$$

7.3. Invariant theory lemma. The proof of part (ii) of the theorem of Section 7.2 depends on the following lemma.

LEMMA. *Every W_0 -invariant function on \hat{A} arises from a \hat{G} -invariant function on $\hat{G} \rtimes \text{Frob}_v$.*

Proof. We will actually show that the ring of W_0 -invariant regular functions on \hat{A} is spanned by traces of representations of $\hat{G} \rtimes \text{Frob}_v$. (It will follow that functions on $\hat{G} \rtimes \langle \text{Frob}_v \rangle // \hat{G}$ are spanned by traces of representations of $\hat{G} \rtimes \langle \text{Frob}_v \rangle$.)

The ring of regular functions on \hat{A} is naturally identified with the group ring $k[X^*(\hat{T})_v^{\text{Frob}_v}]$. The $W_{0,v}$ -invariant regular functions have a basis parametrized by $W_{0,v}$ -orbits on $X^*(\hat{T})^{\text{Frob}_v}$.

Every $W_{0,v}$ -orbit on $X^*(\hat{T})^{\text{Frob}_v} = X_*(\mathbf{T})^{\text{Frob}_v}$ contains a dominant element: if we look on the dual side, the relative Weyl group for \mathbf{A} (i.e., the Weyl group of the relative root system that might not be reduced) has a relatively dominant element in its orbit, which implies it is dominant considered as a cocharacter of \mathbf{T} — although it may lie on a wall. Let us denote the basis element corresponding to $W_{0,v}\nu$, where ν is dominant, by ω_ν .

For each dominant weight ν , let V_ν denote the corresponding Weyl module for \hat{G} , i.e., by Borel-Weil

$$V_\nu = H^0(\hat{G}/\hat{B}; \mathcal{O}(\nu)).$$

Note that V_ν need not be irreducible, since we are in characteristic p , but it does not matter for us. If ν is Frob_v -invariant, then (as Frob_v leaves \hat{B} stable) the line bundle $\mathcal{O}(\nu)$ acquires a $\hat{G} \rtimes \langle \text{Frob}_v \rangle$ -equivariant structure, and V_ν is canonically a $\hat{G} \rtimes \langle \text{Frob}_v \rangle$ -module.

Let W be the full Weyl group for \hat{T} , i.e., the quotient $N(\hat{T})/\hat{T}$. Let $|W\nu| \subset X^*(\hat{T})$ denote the convex hull of the W -orbit of ν . For $\lambda \in |W\nu|$, let $V_\nu(\lambda)$ denote the corresponding weight space of $V(\nu)$. For $t \in \hat{T}$, we compute

$$(7.3.1) \quad \chi_\nu(t \rtimes \text{Frob}_v) = \sum_{\lambda \in |W\nu|^{\text{Frob}_v}} \lambda(t) \text{Tr}(\text{Frob}_v|_{V_\nu(\lambda)})$$

(where $|W\nu|^{\text{Frob}_v}$ is the Frob_v -fixed elements of $|W\nu|$). We will show that

$$(7.3.2) \quad \sum_{\lambda \in (W\nu)^{\text{Frob}_v}} \lambda(t) \text{Tr}(\text{Frob}_v|_{V_\nu(\lambda)}) = \text{Tr}(\text{Frob}_v|_{V_\nu(\nu)}) \omega_\nu.$$

The left-hand side is the dominant term of the right-hand side of (7.3.1), so that (7.3.2) implies

$$\chi_\nu(t \rtimes \text{Frob}_v) = \text{Tr}(\text{Frob}_v|_{V_\nu(\nu)}) \cdot \omega_\nu + \sum_{\nu'} a_{\nu'} \omega_{\nu'},$$

where every $\nu' \in X^*(\hat{T})^{\text{Frob}_v}$ has the property that $|\nu'| < |\nu|$. After observing that $\text{Tr}(\text{Frob}_v|_{V_\nu(\nu)})$ is nonzero and $V_\nu(\nu)$ is one-dimensional, it follows by induction that ω_ν can be written as a linear combination of characters of $V_{\nu'}$, where $\nu' \in X^*(\hat{T})^{\text{Frob}_v}$ and $\nu' \leq \nu$.

Let us prove (7.3.2). We may write $W\nu$ as W/W_ν , where W_ν is the Frob_v -stable parabolic subgroup fixing ν . Recall that each coset of W_ν has a minimum element in the Bruhat ordering on W . As Frob_v preserves the Bruhat ordering on W , it follows that each Frob_v -fixed coset is represented by a Frob_v -fixed element of W , i.e., by an element of W_0 . As Frob_v and W_0 commute, the trace $\text{Tr}(\text{Frob}_v|_{V_\nu(\lambda)})$ is therefore constant on the unique W_0 -orbit on $(W\nu)^{\text{Frob}_v}$. This completes the proof. \square

Proof of part (ii). We are now ready for the proof of part (ii) of the theorem of Section 7.2. Any \hat{G} -invariant regular function on $\hat{G} \rtimes \text{Frob}_v$ gives by restriction a regular function on $\hat{T} \rtimes \text{Frob}_v$. As $t\text{Frob}_v$ and $t(t')^{\text{Frob}_v}(t')^{-1}\text{Frob}_v$ are conjugate by t' , this restricted function descends to a regular function on the Frob_v -coinvariants on \hat{T}_v ; that is to say, it descends to \hat{A}_v .

The proposition of Section 7.1 implies that this function is $W_{0,v}$ -invariant as well. Indeed, if $a \in \hat{A}_v$ is represented by $t \in \hat{T}$, and $w \in W_{0,v}$, then according to the proposition, wa is represented by $ntn^{-1} \in \hat{T}$ for some Frob_v -fixed

$n \in \hat{G}$. A \hat{G} -invariant function on $\hat{G} \rtimes \text{Frob}_v$ therefore induces a $W_{0,v}$ -invariant function on \hat{A}_v .

After the lemma, it only remains to check that any \hat{G} -invariant regular function on $\hat{G} \rtimes \text{Frob}_v$ that induces the zero function on \hat{A}_v (and thus on \hat{T}), is zero.

The image of the action map $\alpha : \hat{G} \times (\hat{T} \rtimes \text{Frob}_v) \rightarrow \hat{G} \rtimes \text{Frob}_v$ is of dimension $\dim(\hat{G}) + \dim(\hat{T}) - \dim(M)$, where M is the set-wise stabilizer of $\hat{T} \rtimes \text{Frob}_v$ in $\hat{G} \rtimes \text{Frob}_v$. A Lie algebra computation shows that the identity component of M is \hat{T} ; consequently, α is dominant. This completes the proof. \square

7.4. *Local pseudoroots.* In the usual Satake isomorphism, one chooses the positive square root of the modular character. When working with k coefficients, there is no preferred square root. We will now discuss allowable choice of square roots of the modular character, called “pseudoroots.” Once a pseudoroot is chosen, we obtain in (7.4.5) the “usual” statement of the Satake isomorphism. A more intrinsic approach can be given with the “ C -group,” but the approach below is much closer to the literature.

With notation as in the theorem, a *pseudoroot at v* is either of the following pieces of data, which correspond to one another under the bijection of Section 2.9:

$$(7.4.1) \quad \hat{A}_v \simeq \text{unramified characters of } T_v.$$

- (a) A pseudoroot is a choice of element $\alpha_0 \in \hat{A}_v$ with the following properties:
 - (i) $\alpha_0^2 = \Sigma_G^*(q_v)$;
 - (ii) α_0 is invariant under the $*$ -action of $W_{0,v}$ (see (7.2.1)).
- (b) A pseudoroot is an unramified character $\delta^{1/2}$ of T_v (equivalently, by pull-back, an unramified character of B_v) with the following properties: Let v be a good place. By proceeding as in (7.2.4) but using the natural map $q_v^{\mathbf{Z}} \rightarrow k^*$ instead of the inclusion $q_v^{\mathbf{Z}} \subset \mathbf{R}$, we get the “ k -valued modular character”

$$(7.4.2) \quad \delta : B_v \rightarrow F_v^* \xrightarrow{|\cdot|_v} q_v^{\mathbf{Z}} \rightarrow k^*$$

or $\delta = |\Sigma_G|_v$ for short. Then we should have

- (i) $(\delta^{1/2})^2 = \delta$, and
- (ii) $\frac{w\delta^{1/2}}{\delta^{1/2}} = \left| \sqrt{\frac{w\Sigma_G}{\Sigma_G}} \right|_v$ for $w \in W_{0,v}$.

Note that $w\Sigma_G/\Sigma_G$ is a square.

Let us note that pseudoroots always exist. If we choose a square root $\sqrt{q_v} \in k^*$, we could take $\alpha_0 = \Sigma_G^*(\sqrt{q_v})$. In some cases, there are particularly

natural choices, e.g.

$$(7.4.3) \quad \alpha_0 = \begin{cases} 1, & k \text{ has characteristic } 2, \\ \sqrt{\Sigma_G^*(q_v)}, & \Sigma_G^* \text{ is divisible by } 2 \text{ in } X_*(\hat{T}). \end{cases}$$

Once we have fixed a pseudoroot, we obtain a Satake isomorphism over k in the usual form. The rule $a \mapsto a\alpha_0$ defines an isomorphism

$$(7.4.4) \quad \hat{A}_v/(W_{0,v}, \text{ usual action}) \longrightarrow \hat{A}_v/(W_{0,v}, \text{ twisted action}).$$

Thus, by composing (7.2.2), pullback under (7.4.4), and (7.2.3), we arrive at an identification

$$(7.4.5) \quad \text{characters of } \mathcal{H}(G_v, K_v) \simeq \hat{G} \rtimes \text{Frob}_v // \hat{G},$$

which we refer to, in short, as the Satake isomorphism. Given a character of $\mathcal{H}(G_v, K_v)$, the associated class in $\hat{G} \rtimes \text{Frob}_v // \hat{G}$ will be called its Satake parameter.

In particular, having fixed a pseudoroot, each element of $\hat{G}(k)$ gives a character $\mathcal{H}(G_v, K_v) \rightarrow k$, and two elements g_1, g_2 give the same character precisely when $g_1 \cdot \text{Frob}, g_2 \cdot \text{Frob}$ have the same projection to ${}^L\hat{G} // \hat{G}$.

7.5. Parabolic induction, ρ^- , and Satake parameters. As in Section 7.4, fix a pseudoroot $\alpha_0 \in \hat{A}_v$, with associated unramified character χ_0 of T_v .

As in (7.4.5), this gives rise to an identification

$$(7.5.1) \quad \text{characters of } \mathcal{H}(G_v, K_v) \simeq \hat{G} \rtimes \text{Frob}_v // \hat{G},$$

and we will explicitly determine how this map interacts with parabolic induction.

Let θ be an unramified character of T_v . We can form the unnormalized parabolic induction $J_B^G(\theta)$. This is the submodule of $C_c^\infty(G_v; k)$ given by those $s : G_v \rightarrow k$ that obey $s(bg) = \theta(b)s(g)$ for $b \in B_v$. The K_v -invariant subspace is a one-dimensional k -vector space (because $G_v = B_v K_v$), generated by the vector v^0 whose restriction to K_v is identically 1. For $h \in \mathcal{H}(G_v, K_v)$, we have⁵

$$(7.5.2) \quad hv^0 = \langle \mathcal{S}^* h, \theta \rangle \cdot v^0,$$

with \mathcal{S}^* as in (7.2.5).

In other words, the Hecke algebra acts on the K_v -fixed vector by the character obtained by pulling back θ via \mathcal{S}^* . Let $\chi_\theta : \mathcal{H}(G_v, K_v) \rightarrow k$ be this

⁵Indeed, we directly compute $hv^0(e) = \int_{n \in N_v, a \in A_v, k \in K_v} h(an)\theta(a)dn_v da_v$; note that the measure on G_v , normalized so that the measure of K_v is 1, also decomposes [10, §4.1] via $g = ank$ as $da \cdot dn \cdot dk$, where the measures on A_v, N_v are normalized so that the measures of $A_v \cap K_v, N_v \cap K_v$ is 1.

character of the Hecke algebra. Tracing through the definitions, we arrive at the following formula for the Satake parameter of χ_θ :

Under the identification (7.4.5), the character χ_θ is sent to

$$(7.5.3) \quad t_\theta \cdot \rho_G^-(\text{Frob}_v) \in \hat{G}(k) \rtimes \text{Frob}_v // \hat{G},$$

where ρ_G^- is the Langlands parameter (Section 2.9) of χ_0^{-1} and $t_\theta \in \hat{T}$ is such that $\text{Frob}_v \mapsto t_\theta \text{Frob}_v$ is the Langlands parameter of θ .

Explicitly, ρ_G^- (or simply ρ^- when the group is understood) is a Langlands parameter for T_v , i.e., a splitting of ${}^L\hat{T}_v \rightarrow \langle \text{Frob}_v \rangle$, and it satisfies

$$(7.5.4) \quad \rho_G^-(\text{Frob}_v) = \widetilde{\alpha_0^{-1}} \text{Frob}_v,$$

where $\widetilde{\alpha_0^{-1}}$ is a lift of α_0^{-1} under $\hat{T}_v \rightarrow \hat{A}_v$.

The parameter ρ_G^- of the inverse pseudoroot will play a very important role for us in what follows. We will also use ρ_G to denote the Langlands parameter of χ_0 itself, but ρ_G^- will occur much more. Note that either ρ_G or ρ_G^- determine the pseudoroot. Later on, when we deal with multiple groups G, H , we will denote the corresponding data as $\rho_G, \rho_H, \rho_G^-, \rho_H^-$.

For later use, let us examine the situation when we induce from a parabolic that is not minimal. Suppose P_v is a parabolic subgroup and θ an unramified character of \mathbf{L}^{ab} , the abelianized Levi subgroup for the parabolic P_v , and its Langlands parameter therefore is a twisted conjugacy class in the dual torus $Z(\hat{L})$ (cf. Section 2.7). Let $l_\theta \in Z(\hat{L})$ be a representative. Then the Satake parameter of the character of $\mathcal{H}(G_v, K_v)$ on $J_P^G(\theta)$ is

$$(7.5.5) \quad l_\theta \cdot \rho_G^-(\text{Frob}_v).$$

Indeed, to verify this we just choose a Borel subgroup $B_v \subset P_v$ and note that $J_P^G(\theta) \subset J_B^G(\theta)$, and use the previous formula.

8. The Satake parameters of the Brauer homomorphism

8.1. *Computing the Brauer homomorphism.* Let \mathbf{G} and $\mathbf{H} = \mathbf{G}^\sigma$ be as in Section 5.5. Suppose that \mathbf{H} is connected. Let v be a σ -good place (Section 5.6), let $\gamma \in \Gamma_F$ be a Frobenius element at v , and choose a local pseudoroot at v (Section 7.4) for both \mathbf{G} and \mathbf{H} .

By Theorem 7.2 and (7.4.5), the normalized Brauer homomorphism gives a map

$$(8.1.1) \quad \text{Spec}(\text{br}) : \hat{H} \rtimes \gamma // \hat{H} \rightarrow \hat{G} \rtimes \gamma // \hat{G}.$$

We may write the domain and codomain of this map as quotients of the maximal tori $\hat{T}_H \subset \hat{H}$ and $\hat{T}_G \subset \hat{G}$, respectively. In this section, we prove the following

THEOREM. *There exists a homomorphism $\hat{N} : \hat{T}_H \rightarrow \hat{T}_G$ with the following property. For every k -point x of $\hat{H} \rtimes \gamma // \hat{H}$, there is a $t \in \hat{T}_H(k)$ such that $t\rho_H^-(\gamma) \in {}^L\hat{T}_H$ is a representative for x and $\hat{N}(t)\rho_G^-(\gamma)$ is a representative for the image of x under the map (8.1.1).*

Here ρ_H^-, ρ_G^- are the splittings attached to the local pseudoroots at v for \mathbf{H} and \mathbf{G} , as in (7.5.4). Note the theorem does *not* assert that (8.1.1) can be extended to a commutative square of the form

$$(8.1.2) \quad \begin{array}{ccc} \hat{T}_H & \dashrightarrow & \hat{T}_G \\ \downarrow & & \downarrow \\ \hat{H} \rtimes \gamma // \hat{H} & \longrightarrow & \hat{G} \rtimes \gamma // \hat{G}. \end{array}$$

We need a more precise version, given as Theorem 8.10, after setting up some notation. It gives a finite list of allowable choices for \hat{N} , indexed by what we call “ γ -admissible Borel classes,” whose theory we give over Sections 8.3–8.9.

Roughly speaking, the proof of the theorem goes like this. We compute the effect of Tate cohomology on suitable spherical representations and deduce the computation of the Brauer homomorphism from (6.2.1). In turn, spherical representations are realized in the spaces of sections of suitable line bundles over flag varieties; the main technical step is extending a Borel subgroup of H_v to a σ -stable parabolic subgroup of G_v , to produce “compatible” flag varieties for G_v and H_v . It is at this step that we need to make choices — the “ γ -admissible Borel classes” mentioned above.

The arguments in this section resemble the arguments used to prove Theorem 3.3.A of [24], which produces a map similar to (8.1.1), but on the dual side and in characteristic zero. We are grateful to Laurent Clozel for bringing this to our attention; it would be interesting to investigate further.

8.2. Outline of this section. In Sections 8.3–8.9, we will work with \mathbf{G} and \mathbf{H} over \bar{F} . The Galois group makes its mark through its image in $\text{Out}(\mathbf{H})$ and $\text{Out}(\mathbf{G})$. For a fixed element $\gamma \in \text{Out}(\mathbf{H})$, we define certain subgroups \mathbf{T}_H^γ , $\mathbf{L}_G(\gamma)$, and a restricted class of Borels \mathbf{B}_G and parabolics \mathbf{P}_G (the former called “ γ -admissible Borels”). Until Section 8.11, *all these groups are defined over \bar{F}* . In Section 8.11 we return to rationality issues. If γ is a Frobenius element at v , some of these groups (but not \mathbf{B}_G) are defined over F_v .

8.3. PROPOSITION. *Let \mathbf{T}_H be a maximal torus in \mathbf{H} . Then the centralizer of \mathbf{T}_H is a maximal torus in \mathbf{G} .*

(We repeat: in Sections 8.3–8.9, all subgroups are to be taken as defined over \bar{F} .)

Proof. Let $x \in \mathbf{T}_H$ be a regular semisimple element of \mathbf{H} . By [35, §8.9], we may find a σ -stable maximal torus and Borel of \mathbf{G} containing x . Let $\mathbf{T}_G \subset \mathbf{G}$

be such a maximal torus. It must contain \mathbf{T}_H ; indeed, \mathbf{T}_H is the identity component of $\mathbf{T}_G^\sigma = \mathbf{T}_G \cap \mathbf{H}$. (In fact, since \mathbf{H} is connected and \mathbf{T}_H is a maximal torus in \mathbf{H} , we actually have $\mathbf{T}_H = \mathbf{T}_G^\sigma$.) The roots of \mathbf{T}_G on the centralizer of \mathbf{T}_H are those $\beta \in \Phi(\mathbf{T}_G, \mathbf{G})$ that are trivial on \mathbf{T}_H or, equivalently, that vanish on the Lie algebra of \mathbf{T}_H . We will show that there are no such β , and therefore the centralizer is equal to \mathbf{T}_G .

The map $\mathrm{Lie}(\mathbf{T}_G)^* \rightarrow \mathrm{Lie}(\mathbf{T}_H)^*$ identifies the codomain with the σ -coinvariants of the domain. Thus any β that vanishes on $\mathrm{Lie}(\mathbf{T}_H)$ belongs to the image of $1 - \sigma$ or, equivalently, to the kernel of $1 + \sigma + \cdots + \sigma^{p-1}$. If β is a positive (resp. negative) root, then each $\sigma^i(\beta)$ is also positive (resp. negative) and, in particular, $\beta + \sigma(\beta) + \cdots + \sigma^{p-1}(\beta) \neq 0$. This completes the proof. \square

In what follows, once we have fixed a choice of \mathbf{T}_H , we will use the notation \mathbf{T}_G to denote the centralizer of \mathbf{T}_H inside \mathbf{G} .

8.4. *The torus $(\mathbf{T}_H^\gamma)^\circ$.* We continue with the notation of the previous subsection. Let $\mathbf{B}_H \subset \mathbf{H}$ be a Borel containing \mathbf{T}_H . Let γ be an outer automorphism of \mathbf{H} (for instance, the image of an element of $\Gamma_F \rightarrow \mathrm{Out}(\mathbf{H})$ induced by the F -rational structure of \mathbf{H}). Then there exists a unique representative for γ in $\mathrm{Aut}(\mathbf{H})$ that preserves \mathbf{T}_H and \mathbf{B}_H ; we denote this representative also by γ . Let \mathbf{T}_H^γ denote the group of γ -fixed points and $(\mathbf{T}_H^\gamma)^\circ$ the identity component of \mathbf{T}_H^γ .

PROPOSITION.

- (1) *The centralizer of $(\mathbf{T}_H^\gamma)^\circ$ in \mathbf{H} is \mathbf{T}_H .*
- (2) *The cone of coweights in $(\mathbf{T}_H^\gamma)^\circ$ that are positive on $\Phi(\mathbf{T}_H, \mathbf{B}_H)$ is “open”; i.e., it does not lie in any hyperplane in $X_*((\mathbf{T}_H^\gamma)^\circ)$.*

Proof. The proof of Proposition 8.3, with σ replaced by γ and \mathbf{G} replaced by \mathbf{H} , establishes (1). Let us prove (2). Let m denote the order of γ in $\mathrm{Out}(\mathbf{H})$, and consider the operator ν on $X_*(\mathbf{T}_H)$ carrying χ to $\chi + \gamma \circ \chi + \cdots + \gamma^{m-1} \circ \chi$. After tensoring with \mathbf{Q} , the image of ν coincides with the kernel of $1 - \gamma$, in particular, the image of ν is not contained in any hyperplane of $X_*(\mathbf{T}_H)$. Part (2) now follows from the fact that ν preserves the property of being positive on \mathbf{B}_H . \square

8.5. *The Levi $\mathbf{L}_G(\gamma)$ and its derived group.* We let $\mathbf{L}_G(\gamma)$ denote the centralizer in \mathbf{G} of the torus $(\mathbf{T}_H^\gamma)^\circ$. It is a Levi subgroup [7, Th. 4.15], and $\Phi(\mathbf{T}_G, \mathbf{L}_G(\gamma)) \subset \Phi(\mathbf{T}_G, \mathbf{G})$ is given by those $\beta : \mathbf{T}_G \rightarrow \mathbb{G}_m$ that are trivial on $(\mathbf{T}_H^\gamma)^\circ$.

Let us call a cocharacter $\chi : \mathbb{G}_m \rightarrow (\mathbf{T}_H^\gamma)^\circ$ *generic* if its centralizer in \mathbf{G} is $\mathbf{L}_G(\gamma)$. Each $\delta \in \Phi(\mathbf{T}_G, \mathbf{G}) - \Phi(\mathbf{T}_G, \mathbf{L}_G(\gamma))$ defines an orthogonal hyperplane $H_\delta \subset X_*((\mathbf{T}_H^\gamma)^\circ)$, and genericity is equivalent to $\chi \notin \bigcup_\delta H_\delta$.

Note that $\mathbf{L}_G(\gamma)$ is σ -stable, as is $[\mathbf{L}_G(\gamma), \mathbf{L}_G(\gamma)]$. We have the following fixed-point computations.

PROPOSITION. *The following hold:*

- (1) $\mathbf{L}_G(\gamma)^\sigma = \mathbf{T}_H$.
- (2) $[\mathbf{L}_G(\gamma), \mathbf{L}_G(\gamma)]^\sigma$ is a maximal torus in $[\mathbf{L}_G(\gamma), \mathbf{L}_G(\gamma)]$.

Proof. Part (1) follows immediately from part (1) of the proposition of Section 8.4. We claim that σ induces an inner automorphism of $[\mathbf{L}_G, \mathbf{L}_G]$. Since the fixed points of an inner automorphism of finite order contain a maximal torus, and $[\mathbf{L}_G, \mathbf{L}_G]^\sigma$ is contained in a torus by part (1), we can conclude (2).

Let us prove the claim. In fact we will prove that if \mathfrak{g} is a semisimple Lie algebra over \bar{F} and a an automorphism of \mathfrak{g} of prime order p , then if a is not inner, we cannot have \mathfrak{g}^a contained in a Cartan subalgebra. Indeed we may find a pinning $(\mathfrak{t}, \mathfrak{b}, \{e_\alpha\}_{\alpha \in I})$ of \mathfrak{g} such that $a = \theta \text{ad}_x$, where θ is a pinned automorphism and $x \in \mathfrak{t}$. If θ is nontrivial, then it also has order p , and there is a simple root α with $\alpha, \theta(\alpha), \theta^2(\alpha), \dots, \theta^{p-1}(\alpha)$ all distinct.

From the classification of semisimple Lie algebras by Dynkin diagrams, either we may choose α so that the $\theta^i(\alpha)$ are all orthogonal, or else $p = 2$ and \mathfrak{g} has a factor of the form \mathfrak{sl}_3 on which θ acts by transpose-inverse. In the second case, take α to be one of the simple roots of the \mathfrak{sl}_3 -factor. In either case one computes that the elements $e_\alpha + a(e_\alpha) + \dots + a^{p-1}(e_\alpha)$ and $e_{-\alpha} + a(e_{-\alpha}) + \dots + a^{p-1}(e_{-\alpha})$ do not commute. \square

8.6. *Admissible Borels.* As in Sections 8.3–8.4, let $\mathbf{T}_H \subset \mathbf{B}_H \subset \mathbf{H}$ be torus and Borel subgroups inside \mathbf{H} . As before, \mathbf{T}_G denotes the centralizer of \mathbf{T}_H in \mathbf{G} . We will say that a Borel \mathbf{B}_G is γ -admissible with respect to $\sigma, \mathbf{T}_H, \mathbf{B}_H$ if it contains \mathbf{T}_G and there exists a cocharacter $\chi : \mathbb{G}_m \rightarrow (\mathbf{T}_H^\gamma)^\circ$ with the following properties:

- (i) χ is positive for \mathbf{B}_H : all nontrivial roots $\beta \in \Phi(\mathbf{T}_H, \mathbf{B}_H)$ satisfy $\langle \beta, \chi \rangle > 0$.
- (ii) χ is nonnegative for \mathbf{B}_G : all nontrivial roots $\delta \in \Phi(\mathbf{T}_G, \mathbf{B}_G)$ satisfy $\langle \delta, \chi \rangle \geq 0$.
- (iii) χ is generic in the sense of Section 8.5.

A tuple $(\mathbf{T}_H, \mathbf{B}_H, \mathbf{T}_G, \mathbf{B}_G)$, where $\mathbf{T}_H \subset \mathbf{B}_H$, $\mathbf{T}_G = Z_{\mathbf{G}}(\mathbf{T}_H)$, and \mathbf{B}_G is an admissible Borel, will be called a “ γ -admissible Borel tuple.” The group $\mathbf{H}(\bar{F})$ acts on γ -admissible Borel tuples by conjugation — an orbit of this action is called a “ γ -admissible Borel class.” When \mathbf{H} has a fixed F -rational structure and $\gamma \in \Gamma$, and γ' is the image of γ in $\text{Out}(\mathbf{H})$, we will abuse notation and say “ γ -admissible Borel tuple” in place of “ γ' -admissible Borel tuple.” Given a homomorphism $\Gamma \rightarrow \text{Out}(\mathbf{H})$, we will say that a tuple $(\mathbf{T}_H, \mathbf{B}_H, \mathbf{T}_G, \mathbf{B}_G)$ is “ Γ -admissible” if it is γ -admissible for every γ in the image.

The significance of admissibility is the following lemma, which realizes \mathbf{B}_H as the σ -fixed points of a parabolic in \mathbf{G} (cf. discussion after (8.1.2)).

8.7. LEMMA. *If \mathbf{B}_G is a γ -admissible Borel, then $\mathbf{L}_G(\gamma)$ and \mathbf{B}_G generate a σ -stable parabolic subgroup \mathbf{P}_G whose σ -fixed points are $\mathbf{P}_G^\sigma = \mathbf{B}_H$.*

Proof. We will construct $\mathbf{P} = \mathbf{P}_G$ by different means and then show that its σ -fixed points are \mathbf{B}_H and that it is generated by $\mathbf{L}_G(\gamma)$ and \mathbf{B}_G . Let \mathfrak{g} and \mathfrak{h} denote the \overline{F} -linear Lie algebras of \mathbf{G} and \mathbf{H} . Similarly, let \mathfrak{t}_H and \mathfrak{t}_G denote the Lie algebras of \mathbf{T}_H and \mathbf{T}_G . We have root space decompositions

$$\mathfrak{g} = \mathfrak{t}_G \oplus \bigoplus_{\beta \in \Phi(\mathbf{T}_G, \mathbf{G})} \mathfrak{g}_\beta, \quad \mathfrak{h} = \mathfrak{t}_H \oplus \bigoplus_{\delta \in \Phi(\mathbf{T}_H, \mathbf{H})} \mathfrak{h}_\delta.$$

Suppose χ witnesses the γ -admissibility of \mathbf{B}_G , i.e., χ obeys (i), (ii), and (iii) of Section 8.6.

Let $\mathbf{P} \subset \mathbf{G}$ denote the parabolic subgroup containing \mathbf{B}_G whose Lie algebra is the sum of \mathfrak{t}_G and those root spaces \mathfrak{g}_β with $\langle \beta, \chi \rangle \geq 0$. Then \mathbf{P} contains \mathbf{B}_G . It is clear that \mathbf{P} is σ -stable. To see that $\mathbf{P}^\sigma = \mathbf{B}_H$, note that by assumption (i) for χ , the Lie algebra of $\mathbf{P} \cap \mathbf{H}$ is the Lie algebra of \mathbf{B}_H , i.e., $(\mathbf{P} \cap \mathbf{H})^\circ = \mathbf{B}_H$, and \mathbf{B}_H is its own normalizer in \mathbf{H} .

It remains to show that \mathbf{P}_G is generated by $\mathbf{L}_G(\gamma)$ and \mathbf{B}_G . Since both $\mathbf{L}_G(\gamma)$ and \mathbf{B}_G are connected, the subgroup they generate is connected as well, so this can be checked on Lie algebras, i.e., it is enough to see $\mathfrak{p} = \mathfrak{l}_G(\gamma) + \mathfrak{b}_G$. We already have $\mathfrak{p} \supset \mathfrak{b}_G$, and $\mathfrak{p} \supset \mathfrak{l}_G(\gamma)$ follows from

$$\mathfrak{l}_G(\gamma) = \mathfrak{t}_G \oplus \bigoplus_{\beta | \langle \beta, \chi \rangle = 0} \mathfrak{g}_\beta.$$

As all three spaces contain \mathfrak{t}_G , to show that $\mathfrak{p} \subset \mathfrak{l}_G(\gamma) + \mathfrak{b}_G$, it is enough to prove that a root of \mathbf{T}_G on \mathfrak{p} is either a root of $\mathfrak{l}_G(\gamma)$ or a root of \mathfrak{b}_G . Suppose $\mathfrak{g}_\beta \subset \mathfrak{p}$ but $\mathfrak{g}_\beta \not\subset \mathfrak{l}_G(\gamma)$, then $\langle \beta, \chi \rangle > 0$. By assumption (ii) for χ , it follows that $\mathfrak{g}_{-\beta}$ is not a root for \mathbf{B}_G . But \mathbf{B}_G is a Borel subgroup. We have $\Phi(\mathbf{T}_G, \mathbf{G}) = \Phi(\mathbf{T}_G, \mathbf{B}_G) \amalg (-\Phi(\mathbf{T}_G, \mathbf{B}_G))$, so $\mathfrak{g}_{-\beta} \not\subset \mathfrak{b}_G$ implies $\mathfrak{g}_\beta \subset \mathfrak{b}_G$. This completes the proof. \square

Note that the proof has shown that $\mathbf{L}_G(\gamma)$ is the standard Levi factor of \mathbf{P}_G , generated by \mathbf{T} , the root subgroups for simple roots α_i of \mathbf{B}_G with $\langle \alpha_i, \chi \rangle = 0$, and the roots subgroups for $-\alpha_i$. γ -admissible Borels always exist, in fact,

8.8. LEMMA. *Fix a Borel subgroup and maximal torus $\mathbf{B}_H \supset \mathbf{T}_H$ in \mathbf{H} .*

- (1) *For any $\gamma \in \text{Out}(\mathbf{H})$, there is a Borel subgroup $\mathbf{B}_G \subset \mathbf{G}$ that is γ -admissible with respect to $\mathbf{B}_H, \mathbf{T}_H$.*
- (2) *For any group homomorphism $\Gamma \rightarrow \text{Out}(\mathbf{H})$ whose image is cyclic of prime order, there is a Borel subgroup $\mathbf{B}_G \subset \mathbf{G}$ that is Γ -admissible with respect to $\mathbf{B}_H, \mathbf{T}_H$.*

The second assertion of the lemma is not used in the proof of Theorem 8.1, but it is useful in applying it.

Proof. By part (2) of the proposition of Section 8.4, we may find a cocharacter of \mathbf{T}_H^γ that is positive on \mathbf{B}_H , i.e., that obeys (i) and (iii) of the conditions for admissibility. Fix such a χ . By “perturbing χ in $X_*(\mathbf{T}_H)$,” we may find a Borel \mathbf{B}_G obeying (ii). More specifically, let $\epsilon : \mathbb{G}_m \rightarrow \mathbf{T}_H$ be any cocharacter that does not vanish on any roots of $\Phi(\mathbf{T}_G, \mathbf{G})$. For $N \in \mathbf{Z}$ sufficiently large, the cocharacter $N\chi + \epsilon$ also does not vanish on any root of \mathbf{G} and therefore determines a positive system in $\Phi(\mathbf{T}_G, \mathbf{G})$. Let \mathbf{B}_G be the corresponding Borel, i.e., with $\delta \in \Phi(\mathbf{T}_G, \mathbf{B}_G)$ if and only if $\langle \delta, N\chi + \epsilon \rangle \geq 0$. By taking N sufficiently large, we have $\frac{1}{N}\langle \delta, \epsilon \rangle > -1$ and therefore $\langle \delta, \chi \rangle \geq 0$ for all $\delta \in \Phi(\mathbf{T}_G, \mathbf{B}_G)$.

To prove the second assertion it suffices to show that \mathbf{B}_G can be chosen simultaneously γ -admissible and 1-admissible. When $\gamma = 1$, we have $(\mathbf{T}_H^\gamma)^\circ = \mathbf{T}_H$, so the cocharacter $N\chi + \epsilon$ also witnesses the 1-admissibility of \mathbf{B}_G . \square

8.9. *The norm and dual norm homomorphisms.* With \mathbf{T}_H a maximal torus of \mathbf{H} and \mathbf{T}_G its centralizer in \mathbf{G} , we define the norm homomorphism $N : \mathbf{T}_G \rightarrow \mathbf{T}_H$ by

$$N(t) = t \cdot t^\sigma \cdot \dots \cdot t^{\sigma^{p-1}}.$$

If we choose $\mathbf{B}_H \supset \mathbf{T}_H$ and $\mathbf{B}_G \supset \mathbf{T}_G$, we get an induced map

$$X^*(\mathbf{B}_H) \simeq X^*(\mathbf{T}_H) \xrightarrow{N^*} X^*(\mathbf{T}_G) \simeq X^*(\mathbf{B}_G),$$

which in turn induces a map $\hat{T}_H \rightarrow \hat{T}_G$, which we call the *dual norm*. Note that (as there is no direct identification of \mathbf{T} with \mathbf{T}^{can} — Section 2.4) the dual norm depends on \mathbf{B}_H and \mathbf{B}_G .

When \mathbf{B}_G is γ -admissible with respect to $\mathbf{T}_H, \mathbf{B}_H$, the parabolic \mathbf{P}_G of Lemma 8.7 determines (Section 2.7) a Levi subgroup $\hat{L}_\gamma \subset \hat{G}$ containing \hat{T}_G . The natural projection $\mathbf{T}_G \rightarrow \mathbf{L}_G(\gamma)^{\text{ab}}$ dualizes to an inclusion

$$(8.9.1) \quad Z(\hat{L}_\gamma) \hookrightarrow \hat{T}_G.$$

Now we state a refined version of the theorem of Section 8.1, whose proof will occupy the rest of this section.

8.10. THEOREM. *Let \mathbf{G} and $\mathbf{H} = \mathbf{G}^\sigma$ be as in Section 5.5, with \mathbf{H} connected. Let v be a σ -good place (Section 5.6), let $\gamma \in \Gamma_F$ be a Frobenius element at v , and choose a local pseudoroot at v (Section 7.4) for both \mathbf{G} and \mathbf{H} .*

Let $\hat{N} : \hat{T}_H \rightarrow \hat{T}_G$ be the dual norm homomorphism attached, as described in Section 8.9, to a γ -admissible tuple $(\mathbf{T}_H, \mathbf{B}_H, \mathbf{T}_G, \mathbf{B}_G)$. Let \hat{L}_γ be the associated dual Levi, i.e., the Levi of \hat{G} associated to the parabolic $\mathbf{P}_G = (\mathbf{L}_G(\gamma), \mathbf{B}_G)$ described in Lemma 8.7.

- (1) For every k -point x of $\hat{H} \rtimes \gamma // \hat{H}$, there is a $t \in \hat{T}_H(k)$ such that $t\rho_H^-(\gamma) \in {}^L\hat{T}_H$ is a representative for x , and $\hat{N}(t)\rho_G^-(\gamma)$ is a representative for the image of x under the map (8.1.1).
- (2) Moreover, t can be chosen such that $\hat{N}(t)$ lies in the center of \hat{L}_γ .

8.11. *Rationality.* Now fix a place v of F at which \mathbf{H} is quasisplit. Then we may choose \mathbf{T}_H and \mathbf{B}_H to be F_v -rational. The group $\mathbf{T}_G := Z_G(\mathbf{T}_H)$ is also F_v -rational.

If $\gamma \in \text{Out}(\mathbf{H})$ is the outer automorphism corresponding to the Frobenius, then $(\mathbf{T}_H^\gamma)^\circ$ is the maximal split subtorus of \mathbf{T}_H , and its centralizer $\mathbf{L}_G(\gamma)$ is F_v -rational. When v is implicit, we write $\mathbf{L}_G := \mathbf{L}_G(\gamma)$ for short.

If \mathbf{B}_G is any γ -admissible Borel (relative to $\mathbf{T}_H, \mathbf{B}_H$), then the corresponding parabolic \mathbf{P}_G of Lemma 8.7 is F_v -rational, because any character into the split torus $(\mathbf{T}_H^\gamma)^\circ$ is automatically F -rational and \mathbf{P}_G can be defined via the nonnegative weight spaces for such a character. Note that we cannot necessarily arrange for \mathbf{B}_G itself to be F_v -rational (nor will we need it), even if \mathbf{G} is quasisplit at v . As before we write \hat{L}_γ for the standard Levi subgroup of \hat{G} associated to the parabolic \mathbf{P}_G .

8.12. LEMMA. *Using notation as above, $\mathbf{T}_G \cap [\mathbf{L}_G, \mathbf{L}_G]$ is an anisotropic maximal torus in $[\mathbf{L}_G, \mathbf{L}_G]$.*

Proof. We appeal to the following basic structural properties of Levi subgroups: $\mathbf{T}_G \cap [\mathbf{L}_G, \mathbf{L}_G]$ is a maximal torus in $[\mathbf{L}_G, \mathbf{L}_G]$, and $[\mathbf{L}_G, \mathbf{L}_G] \cap Z(\mathbf{L}_G)$ is finite. As $Z(\mathbf{L}_G)$ contains $(\mathbf{T}_H^\gamma)^\circ$, which is the maximal split subtorus of \mathbf{T}_H , we can prove that $\mathbf{T}_G \cap [\mathbf{L}_G, \mathbf{L}_G]$ is anisotropic by proving

$$\mathbf{T}_H \cap [\mathbf{L}_G, \mathbf{L}_G] = \mathbf{T}_G \cap [\mathbf{L}_G, \mathbf{L}_G].$$

It is obvious that the left-hand side is contained in the right-hand side. Since $\mathbf{T}_G^\sigma = \mathbf{T}_H$, to show equality it is enough to show that any element of the right-hand group is σ -fixed. This follows from part (2) of the proposition of Section 8.5. \square

8.13. LEMMA. *Let \mathbf{T}_1 and \mathbf{T}_2 be algebraic tori over F_v , and suppose that $f : \mathbf{T}_1 \rightarrow \mathbf{T}_2$ has anisotropic kernel. Then precomposition with f induces a surjection*

$$\begin{aligned} & \{\text{unramified } k^*\text{-valued characters of } \mathbf{T}_2\} \\ & \rightarrow \{\text{unramified } k^*\text{-valued characters of } \mathbf{T}_1\}. \end{aligned}$$

Proof. Indeed, let $T_{v,i}^\circ$ be the maximal compact subgroup of $\mathbf{T}_i(F_v)$. Then f induces a map $T_{1,v}/T_{1,v}^\circ \rightarrow T_{2,v}/T_{2,v}^\circ$ that is *injective*: Its kernel is a compact subgroup of a free abelian group and thus trivial. Since k^* is injective as an abelian group, the result follows. \square

8.14. *Extension of characters.* Let us say that a homomorphism $\mathbf{B}_H(F_v) \rightarrow k^*$ or $\mathbf{P}_G(F_v) \rightarrow k^*$ is unramified if it factors through an unramified character of $\mathbf{B}_H^{\text{ab}}(F_v) = \mathbf{T}_H(F_v)$ or $\mathbf{P}_G^{\text{ab}}(F_v) = \mathbf{L}_G^{\text{ab}}(F_v)$.

PROPOSITION. *With notation as in Section 8.11, let χ be an unramified character of $\mathbf{B}_H(F_v)$.*

- (1) χ extends to a σ -invariant unramified character χ^* of $\mathbf{P}_G(F_v)$.
- (2) χ^* may be chosen in such a way that we may choose representatives $t_\chi \in \hat{T}_H$ and $t_{\chi^*} \in Z(\hat{L}_\gamma) \subset \hat{T}_G$ for the Langlands parameters of χ and χ^* , respectively, with

$$(8.14.1) \quad \hat{N}(t_\chi) = (t_{\chi^*})^p,$$

where \hat{N} is the dual norm map associated to the admissible Borel \mathbf{B}_G (Section 8.9).

Proof. Let \mathbf{K} be the kernel of the natural projection $\mathbf{T}_G \hookrightarrow \mathbf{L}_G \rightarrow \mathbf{L}_G^{\text{ab}}$. Let $\mathbf{N}(\mathbf{K}) \subset \mathbf{T}_H$ be the image of \mathbf{K} under the norm map of Section 8.9. By Lemma 8.12, \mathbf{K} and $\mathbf{N}(\mathbf{K})$ are anisotropic tori.

The composite $\mathbf{T}_G \xrightarrow{N} \mathbf{T}_H \rightarrow \mathbf{T}_H/\mathbf{N}(\mathbf{K})$ is trivial on \mathbf{K} , so it factors through \mathbf{L}_G^{ab} . Consider the commutative squares of F_v -algebraic tori and of associated dual tori

$$\begin{array}{ccc} \mathbf{T}_G & \xrightarrow{g} & \mathbf{L}_G^{\text{ab}} \\ \mathbf{N} \downarrow & & \downarrow f \\ \mathbf{T}_H & \xrightarrow{\pi} & \mathbf{T}_H/\mathbf{N}(\mathbf{K}) \end{array} \qquad \begin{array}{ccc} \widehat{\mathbf{T}_H/\mathbf{N}(\mathbf{K})} & \xrightarrow{\hat{\pi}} & \hat{T}_H \\ \hat{f} \downarrow & & \downarrow \hat{N} \\ Z(\hat{L}_\gamma) & \longrightarrow & \hat{T}_G, \end{array}$$

where we use the γ -admissible Borel class to identify the duals of $\mathbf{T}_G, \mathbf{T}_H$, and \mathbf{L}_G^{ab} with \hat{T}_G, \hat{T}_H , and $Z(\hat{L}_\gamma)$ as in Section 8.9.

Since $\mathbf{N}(\mathbf{K})$ is anisotropic, there is (by Lemma 8.13) an unramified character of $\mathbf{T}_H/\mathbf{N}(\mathbf{K})$, call it $\bar{\chi}$, with $\pi^*\bar{\chi} = \chi$. Set $\psi = f^*\bar{\chi}$, an unramified character of $\mathbf{L}_G^{\text{ab}}(F_v)$. Then for $t \in \mathbf{T}_H(F_v) \subset \mathbf{T}_G(F_v)$, we have

$$\psi(g(t)) = \bar{\chi}(\pi \circ N(t)) = \chi(t)^p.$$

In other words, $\chi^* := \psi^{1/p}$ extends χ .

If $t_{\bar{\chi}}$ is a representative for the Langlands parameter of $\bar{\chi}$, then its image $t_\chi = \hat{\pi}(t_{\bar{\chi}}) \in \hat{T}_H$ is a representative for the Langlands parameter for χ , $t_\psi = \hat{f}(t_{\bar{\chi}}) \in Z(\hat{L}_\gamma)$ is a representative for the Langlands parameter of ψ , and $t_\psi^{1/p}$ is a representative for the Langlands parameter of $\chi^* = \psi^{1/p}$. (These facts are all readily deduced from (2.9.2).) So (8.14.1) is a consequence of the commutativity of the right-hand square. \square

8.15. *Proof of Theorem 8.10.* As in the hypotheses of the theorem, let v be a place of F , let $\gamma \in \Gamma_F$ be a Frobenius element at v , and let $\mathbf{T}_H, \mathbf{B}_H, \mathbf{T}_G, \mathbf{B}_G$ be a γ -admissible Borel tuple. Let \mathbf{P}_G be the corresponding parabolic, as described in (Lemma 8.7, and let \hat{N} be the corresponding dual norm (Section 8.9).

Fix $\theta \in \hat{H} \rtimes \gamma // \hat{H}$. We have to find $t \in \hat{T}_H$ such that the image of $t\rho_{\hat{H}}^-(\gamma)$ in $\hat{H} \rtimes \gamma // \hat{H}$ coincides with θ and is, moreover, carried by the Brauer homomorphism to the image of $\hat{N}(t)\rho_G^-(\gamma)$ in $\hat{G} \rtimes \gamma // \hat{G}$.

We may regard (via Satake (7.4.5)) θ as a character $\mathcal{H}(H_v, U_v) \rightarrow k$. By Section 7.5, there is an unramified homomorphism $\chi : \mathbf{B}(F_v) \rightarrow k^*$ such that θ is the character by which $\mathcal{H}(H_v, U_v)$ acts on the U_v -invariants of the unnormalized induction $J_{\mathbf{B}_H}^H(\chi)$.

We will show that we may take $t = t_\chi$, the element of $\hat{T}_H(k)$ corresponding to χ . We have already seen in Section 7.5 that $t\rho_{\hat{H}}^-(\gamma) \in \hat{H} \rtimes \gamma // \hat{H}$ is the Satake parameter for $\theta : \mathcal{H}(G_v, U_v) \rightarrow k$.

Now by Section 8.14, χ extends to a σ -invariant character χ^* of $\mathbf{P}_G(F_v)$. The parabolic induction $J_{\mathbf{P}_G}^G(\chi^*)$ can be regarded as the global sections of a certain sheaf of k -vector spaces (call it \mathcal{F}) over the space $X = (\mathbf{G}/\mathbf{P}_G)(F_v)$; similarly, the parabolic induction $J_{\mathbf{B}_H}^G(\chi)$ can be regarded as the sections of $\mathcal{F}|_Y$ on $Y = (\mathbf{H}/\mathbf{B}_H)(F_v)$. In fact, Y is open and closed inside X^σ , since the algebraic flag variety \mathbf{H}/\mathbf{B}_H is a connected component of $(\mathbf{G}/\mathbf{P}_G)^\sigma$. Note also that the σ -action on the sheaf $\mathcal{F}|_Y$ is trivial.

By the discussion of (3.3.1), restriction from X to Y gives a surjection

$$(8.15.1) \quad \mathrm{T}^0(J_{\mathbf{P}_G}^G(\chi^*)) \rightarrow J_{\mathbf{B}_H}^H(\chi)$$

that carries a nonzero K_v -invariant vector on the left (the function that is identically 1 on K_v) to a nonzero U_v -invariant vector on the right (the function that is identically 1 on U_v). Let Θ be the character by which $\mathcal{H}(G_v, K_v)$ acts on $J_{\mathbf{P}_G}^G(\chi^*)^{K_v}$. As \mathbf{P}_G and χ^* are σ -fixed, we may apply (6.2.1) and conclude

$$\Theta|_{\mathcal{H}(G_v, K_v)^\sigma} = \theta \circ \mathrm{Br}.$$

Thus by (7.5.2), $(\theta \circ \mathrm{Br})^p$ is the character by which $\mathcal{H}(G_v, K_v; \mathbf{F}_p)^\sigma$ acts on $J_{\mathbf{P}_G}^G((\chi^*)^p)^{K_v}$. Recall from Section 4.3 that br is the linear extension of Br^p from the \mathbf{F}_p -valued Hecke algebra. Since $\theta \circ \mathrm{br}$ extends the σ -invariant character $(\theta \circ \mathrm{Br})^p : \mathcal{H}(G_v, K_v; \mathbf{F}_p)^\sigma \rightarrow k$ and such an extension is unique (Section 3.4), it follows that $\theta \circ \mathrm{br}$ is the character by which $\mathcal{H}(G_v, K_v)$ acts on $J_{\mathbf{P}_G}^G((\chi^*)^p)^{K_v}$. In other words, by Section 7.5 again, the Satake parameter of $\theta \circ \mathrm{br}$ in $\hat{G} \rtimes \gamma // \hat{G}$ has a representative of the form $t_{\chi^*}^p \rho_G^-(\gamma)$, where $t_{\chi^*} \in Z(\hat{L})$ is a Langlands parameter for χ^* .

The conclusion of the theorem now follows from (8.14.1).

9. Functoriality and examples

Let us now make precise when the lift from the Main Theorem can be considered as a “functorial lift.” After this discussion (which takes up Sections 9.1–9.3), the remainder of the section is devoted to explaining why all the examples of Section 1.2 can be considered as functorial lifts and explicating the maps of L -groups involved. (As mentioned in Section 1.3, we have in fact carried out such computations in some generality, but here we will give only examples.)

Continue with the notation of Section 5.5. As in Section 2.5, let \hat{G} and \hat{H} be the dual groups to \mathbf{G}, \mathbf{H} , and let ${}^L\hat{G}$ and ${}^L\hat{H}$ be the L -groups to \mathbf{G}, \mathbf{H} . We define a *global* pseudoroot to be a k^* -valued idele class character of $\mathbf{T}_G^{\text{can}}$ (the canonical F -torus of \mathbf{G} — Section 2.4) which restricts to a local pseudoroot at almost every good place v ; that is to say, the associated character of a Borel subgroup $B_v \subset \mathbf{G}(F_v)$, via $B_v \rightarrow \mathbf{T}_G^{\text{can}}(F_v)$, is a pseudoroot in the sense of the discussion around (7.4.2).

9.1. *Definition.* Fix a global pseudoroot for \mathbf{H} and a global pseudoroot for \mathbf{G} . A *sigma-dual homomorphism*, relative to these choices of global pseudoroots, is a map

$$(9.1.1) \quad {}^L\hat{\psi} : {}^L\hat{H} \rightarrow {}^L\hat{G}$$

such that, for almost all places v , the induced maps on local L -groups $\psi_v : {}^L\hat{H}_v \rightarrow {}^L\hat{G}_v$ fits into a commutative diagram:

$$(9.1.2) \quad \begin{array}{ccc} \hat{H} \rtimes \text{Frob}_v // \hat{H} & \xrightarrow{{}^L\hat{\psi}} & \hat{G} \rtimes \text{Frob}_v // \hat{G} \\ (7.4.5) \downarrow & & \downarrow (7.4.5) \\ \text{Hom}(\mathcal{H}(H_v, U_v), k) & \xrightarrow{\text{br}} & \text{Hom}(\mathcal{H}(G_v, K_v), k). \end{array}$$

Here br is the map on characters induced by the normalized Brauer homomorphism of Section 4.3, and the identification of (7.4.5) uses the local pseudoroots for H_v, G_v associated to the fixed global pseudoroots.

In the presence of a σ -dual homomorphism we may reformulate Theorem 5.8 as a functorial lift. Say that $\rho : \Gamma_F \rightarrow {}^L\hat{G}(k)$ is *modular*, with respect to a fixed choice of global pseudoroot for \mathbf{G} , if there is a level structure K and a class $h \in H^*([G]_K)$ such that

For all but finitely many good places v , the class h is an eigenvector for the \mathcal{H}_v -action whose eigenvalue $\chi : \mathcal{H}_v \rightarrow k$ coincides with $\rho(\text{Frob}_v)$ under Satake.

THEOREM. *Fix global pseudoroots for both \mathbf{H} and \mathbf{G} . Suppose there is a σ -dual homomorphism ${}^L\hat{\psi} : {}^L\hat{H} \rightarrow {}^L\hat{G}$. If $\rho : \Gamma_F \rightarrow {}^L\hat{H}$ is modular for \mathbf{H} , then ${}^L\hat{\psi} \circ \rho$ is modular for \mathbf{G} .*

9.2. *Remark.* The dependence of this discussion on pseudoroots is, at first, a little disturbing. However, we note that if \mathbf{G} is simply connected, one can always choose canonical pseudoroots for both \mathbf{H} and \mathbf{G} in a sense to be described momentarily (Section 9.3) and we have verified (see discussion of Section 1.3) that there indeed exists a σ -dual homomorphism with respect to that canonical choice.

If \mathbf{G} is not simply connected, there can be no σ -dual homomorphism at all, no matter how pseudoroots are chosen. A simple example is provided by $\mathbf{G} = \mathrm{PGL}_2$ over $F = \mathbf{Q}$ and σ an inner automorphism of order 3, with fixed points isomorphic to $\mathbf{H} = \mathrm{PSO}(x^2 + 3y^2)$. In that case, there is no homomorphism of L -groups

$$\mathbb{G}_m \rtimes \Gamma_{\mathbf{Q}} \rightarrow \mathrm{SL}_2 \times \Gamma_{\mathbf{Q}},$$

because the image of complex conjugation (considered in $\Gamma_{\mathbf{Q}}$, on the left-hand side) when projected to SL_2 must be an order 2 element that normalizes but does not centralize a nontrivial torus, and none such exists.

These issues can be likely circumvented by using a variant of the C -group introduced in [9], but we do not discuss this here. Our present ad hoc formulation is closer to the existing literature.

9.3. *Canonical pseudoroots.* We mentioned in (7.4.3) some situations where one can distinguish a local pseudoroot. There is also a canonical choice of global pseudoroot in those situations:

- (a) *When the characteristic of k is two:* In this case we refer to the trivial character as the canonical pseudoroot.
- (b) *When the half-sum of positive roots $\frac{\Sigma_G}{2} : \mathbf{T}_G^{\mathrm{can}} \rightarrow \mathbb{G}_m$ exists in the character lattice for $\mathbf{T}_G^{\mathrm{can}}$:* In this case, we obtain the canonical pseudoroot by pulling back the “cyclotomic” idele class character of $\mathbb{G}_m(\mathbf{A}) \rightarrow k^*$ via $\frac{\Sigma_G}{2}$. (The “cyclotomic” idele class character corresponds to the Hecke character that sends a prime-to- p ideal to its norm in k^* .)

When both these apply, these canonical choices agree: the half-sum of positive roots determines the trivial character. We shall simply say “there is a canonical pseudoroot” in these cases. In both cases, for almost all v , the associated pseudoroot gives rise, as in (7.5.3), to a splitting $\rho_G^- : \langle \mathrm{Frob}_v \rangle \rightarrow {}^L \hat{T}_v$. In case (a) we have $\rho_G^-(\gamma) = \gamma$; in case (b) it is given by

$$(9.3.1) \quad \rho_G^-(\gamma) = \begin{cases} \gamma, & \text{case (a),} \\ \frac{\Sigma_G^*}{2}(\mathrm{cyclo}(\gamma)) \rtimes \gamma, & \text{case (b).} \end{cases}$$

All the examples that follow, except Section 9.4, have canonical pseudoroots, and we understand these to be chosen in what follows.

9.4. *Cyclic base change.* We examine the situation of Section 1.2.1. It suffices, by using a composition series for $\text{Gal}(E/F)$, to examine the case when E/F is Galois of order p .

Thus let E/F be a cyclic extension, let \mathbf{H} be an F -group for which the sum of positive roots is divisible by 2, and let $\mathbf{G} = \text{Res}_{E/F}(\mathbf{H} \otimes_F E)$. Then a generator σ for $\text{Aut}(E/F)$ induces an automorphism of \mathbf{G} fixing \mathbf{H} .

We claim that the canonical “diagonal-restriction” map on L -groups ${}^L\hat{\psi} : {}^L\hat{H} \rightarrow {}^L\hat{G}$ is a σ -dual homomorphism. (For generalities on the L -group of a restriction of scalars, we refer to [6, §5]). In other words, the lift furnished by the main theorem is “base change.”

Over \bar{F} , we can identify \mathbf{G} with $\mathbf{H}^{\text{Hom}_F(E, \bar{F})}$, and the action of Γ_F by outer automorphisms is that “induced” from the action of Γ_E by outer automorphisms on \mathbf{H} . So if $\mathbf{T}_H \subset \mathbf{B}_H$ is a torus and Borel subgroup of \mathbf{H} over \bar{F} , then we may form the corresponding Borel \mathbf{B}_G , i.e., the Borel given by $\mathbf{B}_H^{\text{Hom}_F(E, \bar{F})}$ in the above identification; and \mathbf{B}_G is γ -admissible with respect to $(\mathbf{T}_H, \mathbf{B}_H)$. Indeed, we can directly verify the defining properties from Section 8.6. For any $\gamma \in \Gamma_F$, there exists a γ -fixed cocharacter $\chi : \mathbb{G}_m \rightarrow \mathbf{T}_H$ that is positive for \mathbf{B}_H , by an averaging argument, and this cocharacter is also visibly positive for \mathbf{B}_G . The corresponding norm

$$\mathbf{T}_G = \mathbf{T}_H^{\text{Hom}_F(E, \bar{F})} \rightarrow \mathbf{T}_H$$

is simply given by the product map, and the dual norm is just the restriction of ${}^L\hat{\psi}$ to \hat{T}_H .

The claim now follows from Theorem 8.10 so long as the pseudoroots are chosen “compatibly,” in that

$$(9.4.1) \quad \rho_G^- = {}^L\hat{\psi} \circ \rho_H^-$$

9.5. *Mod 2 Eisenstein series.* We examine the situation of Section 1.2.3. It is enough to examine the case of $\mathbf{H} = \text{GL}_a \times \text{GL}_b \hookrightarrow \mathbf{G} = \text{GL}_n$ (the general case factoring as a sequence of such inclusions).

Now \mathbf{H} is the fixed points of an inner involution, namely conjugation by

$$\sigma = \text{diag}(\underbrace{1, 1, \dots, 1}_a, \underbrace{-1, -1, \dots, -1}_b).$$

As both groups are split, ${}^L\hat{H} = \text{GL}_a \times \text{GL}_b \times \Gamma_F$ and ${}^L\hat{G} = \text{GL}_n \times \Gamma_F$, defined over the algebraically closed field k of characteristic 2. Let $\iota : \text{GL}_a \times \text{GL}_b \hookrightarrow \text{GL}_n$ be the standard inclusion. The homomorphism $\Gamma \rightarrow \text{Out}(\mathbf{H})$ is trivial, and if we take $\mathbf{T}_H \subset \mathbf{B}_H$ to be (respectively) the diagonal torus and upper triangular Borel in \mathbf{H} and $\mathbf{T}_G \subset \mathbf{B}_G$ to be the diagonal torus and upper triangular Borel of \mathbf{G} , we verify that $(\mathbf{T}_H, \mathbf{B}_H, \mathbf{T}_G, \mathbf{B}_G)$ is Γ -admissible. The norm map $\mathbf{T}_G \rightarrow \mathbf{T}_H$ is simply squaring (note that $\mathbf{T}_H = \mathbf{T}_G$), and the associated dual norm map $\text{GL}_a \times \text{GL}_b \rightarrow \text{GL}_n$ is given on diagonal maximal tori

by $\text{Frob} \circ \iota$. Then Theorem 8.10 implies that

$${}^L\hat{\psi} : (g, \gamma) \in {}^L\hat{H} \mapsto (\text{Frob} \circ \iota(g), \gamma) \in {}^L\hat{G}$$

is a σ -dual homomorphism with respect to canonical pseudoroots. Twisting by the inverse of Frobenius, the desired result follows.

9.6. *Exotic transfer from Sp_{2n} .* Let \mathbf{G} be the split form of SL_{2n} over F . Let $J \in \text{SL}_{2n}(\mathbf{Z})$ be the standard skew-symmetric matrix representing a symplectic form on \mathbf{Z}^{2n} , and let σ be the automorphism of order 2 given by $\sigma(g) = J(g^T)^{-1}J^{-1}$. Then the fixed group $\mathbf{H} \cong \text{Sp}_{2n, F}$. As both groups are split, we have ${}^L\hat{H} = \text{SO}_{2n+1} \times \Gamma_F$ and ${}^L\hat{G} = \text{PGL}_{2n} \times \Gamma_F$, defined over the algebraically closed field k of characteristic 2.

More precisely, SO_{2n+1} is the group of automorphisms, of determinant one, of the quadratic form

$$x_{2n+1}^2 + \sum_{i=1}^n x_i x_{2n+1-i},$$

with its pinning by diagonal matrices \hat{T}_H and upper-triangular matrices \hat{B}_H . As an SO_{2n+1} -module, k^{2n+1} is an extension of a $2n$ -dimensional representation by the trivial module (generated by x_{2n+1}). We let $\iota : \text{SO}_{2n+1} \rightarrow \text{PGL}_{2n}$ denote the projection into PGL_{2n} of this $2n$ -dimensional quotient of k^{2n+1} .

We claim that

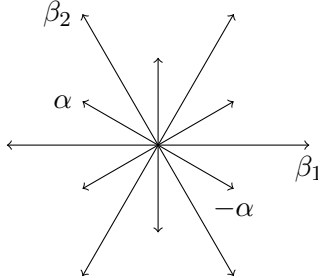
$${}^L\hat{\psi}(h, \gamma) := (\iota(h), \gamma)$$

defines a σ -dual homomorphism ${}^L\hat{H} \rightarrow {}^L\hat{G}$ (with respect to the canonical pseudoroots). Note that ι carries \hat{T}_H to \hat{T}_G . By Theorem 8.10, it suffices to show that one may choose an admissible Borel tuple (Section 8.6) whose dual norm \hat{N} has $\hat{N}(a_v) = \iota(a_v)$ for $a_v \in \hat{T}_H$. But if \mathbf{B}_G and \mathbf{B}_H are the standard Borels of \mathbf{G} and \mathbf{H} , consisting of upper-triangular matrices, and $\mathbf{T}_H, \mathbf{T}_G$ the diagonal maximal tori of \mathbf{H} and \mathbf{G} , then $(\mathbf{T}_H, \mathbf{B}_H, \mathbf{T}_G, \mathbf{B}_G)$ is Γ -admissible. The norm $\mathbf{T}_G \rightarrow \mathbf{T}_H$ carries the diagonal matrix (t_1, \dots, t_{2n}) to $(t_1 t_{2n}, t_2 t_{2n-1}, \dots, t_{2n} t_1)$ and the dual map \hat{N} coincides with ι as desired.

9.7. *Transfer from SL_3 to G_2 .* Let \mathbf{G} be a form of G_2 over F , and let σ be an F -automorphism of \mathbf{G} of order 3 such that $\mathbf{H} := \mathbf{G}^\sigma$ is a form of SL_3 . This example will be significantly more complicated. However, it is possible to set up the method in a uniform way to treat many instances of inner automorphisms.

Over \bar{F} , the automorphism σ is inner — it can be written as ad_s where s generates the center of $\mathbf{H}(\bar{F})$. In particular, if \mathbf{T} is a maximal torus of $\mathbf{H} \times_F \bar{F}$ (necessarily containing s), then it is also a maximal torus of \mathbf{G} . We fix such a \mathbf{T} and a Borel subgroup $\mathbf{B}_H \subset \mathbf{H} \times_F \bar{F}$ containing \mathbf{T} . Let us furthermore suppose that \mathbf{T} is F -rational.

Having fixed \mathbf{B}_H , let o be an outer automorphism $\mathrm{SL}_3 \times_F \overline{F}$ fixing \mathbf{B}_H and switching the simple roots β_1 and β_2 for \mathbf{T} on \mathbf{B}_H . There are precisely two roots of \mathbf{T} on $\mathbf{G} \times_F \overline{F}$ that are orthogonal to $(\beta_1 + \beta_2)^\vee$. They are short, antipodal to each other and exchanged by o . Let us call them α and $o(\alpha) = -\alpha$ in such a way that α makes an obtuse angle with β_1 and $-\alpha$ makes an obtuse angle with β_2 .



Corresponding to $\pm\alpha$ are the two Borels of $\mathbf{G} \times_F \overline{F}$ containing \mathbf{B}_H , one with simple roots $\{\alpha, \beta_1\}$ and one with simple roots $\{-\alpha, \beta_2\}$. Let us call the first \mathbf{B}_G and the second $\mathrm{ad}_{w(o)}\mathbf{B}_G$. Here $w(o)$ denotes the reflection in the Weyl group W_G of \mathbf{G}, \mathbf{T} that carries α to $-\alpha$ and is the unique element of W_G that exchanges these Borels; in other words, the action of $w(o)$ on \mathbf{T} coincides with the action of o . Note that \mathbf{B}_G and $\mathrm{ad}_{w(o)}\mathbf{B}_G$ are Γ -admissible (in the sense of Section 8.6) with respect to \mathbf{B}_H, \mathbf{T} . Indeed up to scaling there is a unique covector positive on β_1 and β_2 and vanishing on $\pm\alpha$ witnessing the o -admissibility of these Borels, in the sense of obeying conditions (i)–(iii) of 8.6. Since these Borels are clearly 1-admissible, they are Γ -admissible.

In Sections 8.5 and 8.9 we defined Levi subgroups $\mathbf{L}_G(o)$ and \hat{L}_o in \mathbf{G} and \hat{G} . In the present case, \hat{L}_o is obtained from \hat{G} adjoining the root groups associated to α_*^\vee and $-\alpha_*^\vee$ to \hat{T}_G and is isomorphic to GL_2 . We now choose a lift $\dot{w}(o) \in N_{\hat{G}}(\hat{T}_G)$ of $w(o) \in W_G \simeq W_{\hat{G}}$, as follows. Let $\iota : \mathrm{SL}_2 \rightarrow \hat{G}$ be the coroot homomorphism associated to α_* , and let $J \in \mathrm{SL}_2$ be an order 4 element normalizing the diagonal subgroup of SL_2 .⁶ Let $t_0 \in Z(\hat{L}_o)$ be such that $\iota(J)t_0$ has order 2, and put $\dot{w}(o) = \iota(J)t_0 = t_0\iota(J)$.

The triple $\mathbf{B}_H \supset \mathbf{T} \subset \mathbf{B}_G$ determines an identification of canonical tori $\mathbf{T}_H^{\mathrm{can}} \simeq \mathbf{T} \simeq \mathbf{T}_G^{\mathrm{can}}$ and, therefore, of dual tori $\hat{T}_H \simeq \hat{T}_G$. Let us denote this identification by ψ_1 — it carries the coroot $\beta_{1,*}$ to $\beta_{1,*}$ and $\beta_{2,*}$ to $\beta_{1,*} + 3\alpha_*$. Note that ψ_1 differs from the dual norm map. In fact,

PROPOSITION. *The following hold:*

- (1) *the identification ψ_1 extends to an inclusion of dual groups $\psi'_1 : \hat{H} \hookrightarrow \hat{G}$;*
- (2) *the composite $\psi'_1 \circ \mathrm{Frob} : \hat{H} \rightarrow \hat{H} \hookrightarrow \hat{G}$ extends the dual norm map $\hat{T}_H \rightarrow \hat{T}_G$ of Section 8.9.*

⁶For instance, take $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Proof. As we are in characteristic 3, there is a subgroup of \hat{G} isomorphic to \hat{H} , namely the subgroup \hat{H}_1 generated by unipotent root groups $\mathbb{G}_a \subset \hat{G}$ associated to the six short roots of \hat{G} .⁷ As the roots of \mathbf{T} on $\mathbf{H} \times_F \overline{F} \simeq \mathrm{SL}_3$ are precisely the long roots of \mathbf{T} on $\mathbf{G} \times_F \overline{F} \simeq \mathrm{G}_2$, the map ψ_1 carries the roots of \hat{T}_H on \hat{H} bijectively to the short roots of \hat{T}_G on \hat{G} . Thus in characteristic 3, ψ_1 extends, proving (1). As σ acts trivially on \mathbf{T} with order 3, the induced norm map on $X_*(\mathbf{T})$ is given by multiplication by 3. This proves (2). \square

Pulling back by $\psi_1 : \hat{T}_H \rightarrow \hat{T}_G$ carries the simple roots $\beta_{1,*}^\vee$ to $\beta_{1,*}^\vee$ and $(\beta_{1,*} + 3\alpha_*)^\vee = \beta_{1,*}^\vee + \alpha_*^\vee$ to $\beta_{2,*}^\vee$. A choice of nonzero vector in the $\beta_{1,*}^\vee$ - and $(\beta_{1,*}^\vee + \alpha_*^\vee)$ -root spaces determines a pinning of \hat{H}_1 , and the extension ψ'_1 is uniquely determined by the requirement that it preserves pinnings.

As $\dot{w}(o)$ has order 2 and exchanges the simple root spaces associated to $\beta_{1,*}^\vee$ and $\beta_{1,*}^\vee + \alpha_*^\vee$, it preserves a pinning of \hat{H}_1 . Let us choose one and take ψ'_1 to be the extension that matches this pinning with the pinning of \hat{H} .

Now for each $\gamma \in \Gamma_F$, put $\varpi_\gamma = 1$ or $\dot{w}(o)$ according to whether the action of γ on the root datum of \mathbf{H} is trivial or not or, equivalently, according to whether the cyclotomic character of γ is 1 or $-1 \pmod{3}$. Define

$$(9.7.1) \quad {}^L\hat{\psi}(h \rtimes \gamma) = \psi_1(\mathrm{Frob}(h))\varpi_\gamma \rtimes \gamma.$$

Since \mathbf{G} is an inner form, γ acts trivially on \hat{G} and, furthermore, since ψ_1 and $\dot{w}(o)$ preserve pinnings, it follows that ${}^L\hat{\psi}$ defines a homomorphism. Let us show it is a σ -dual homomorphism (again, with respect to canonical pseudo-roots).

Let x be an element of $\hat{H} \rtimes \gamma // \hat{H}$, and let y be its image under pull-back by the normalized Brauer homomorphism. By Theorem 8.10 and the Γ -admissibility of \mathbf{B}_G , we may find a representative $t\rho_{\overline{H}}(\gamma)$ for x such that $\hat{N}(t)\rho_{\overline{G}}(\gamma)$ is a representative for y . Moreover, $\hat{N}(t)$ can be assumed to be in the center of \hat{L}_o when γ acts nontrivially on \mathbf{H} .

We must verify that ${}^L\hat{\psi}(t\rho_{\overline{H}}(\gamma))$ and $\hat{N}(t)\rho_{\overline{G}}(\gamma)$ project to the same element of $\hat{G} \rtimes \gamma // \hat{G}$. As ${}^L\hat{\psi}$ is a homomorphism extending the dual norm, this reduces to showing that

$$(9.7.2) \quad \hat{N}(t){}^L\hat{\psi}(\rho_{\overline{H}}(\gamma)) \sim \hat{N}(t)\rho_{\overline{G}}(\gamma),$$

where \sim denotes that the two sides have the same image in $\hat{G} \rtimes \gamma // \hat{G}$. In fact we will show that the two sides of (9.7.2) are conjugate. This is clear when γ projects to the identity in $\mathrm{Out}(\mathbf{H})$ because $\rho_{\overline{G}}(\gamma) = \gamma$ (see (9.3.1)). Let us

⁷One may see this concretely as follows, using the description of G_2 as the automorphisms of a 3-form on k^7 . The trace on $\mathfrak{gl}_3(k)$ descends to a PGL_3 -equivariant homomorphism $\mathfrak{pgl}_3(k) \rightarrow k$, whose kernel is 7-dimensional. For the 3-form, take $\langle [X, Y], Z \rangle$, where $[,]$ is the Lie bracket and \langle, \rangle is the Killing form.

suppose now that γ projects to $o \in \text{Out}(\mathbf{H})$. In that case $\text{cyclo}(\gamma) = -1$, and using (9.3.1) we compute

$$\begin{aligned} \rho_H^-(\gamma) &= \alpha_*(-1)^3 \beta_{1,*}(-1)^2 \rtimes \gamma & \rho_G^-(\gamma) &= \alpha_*(-1)^5 \beta_{1,*}(-1)^3 \rtimes \gamma \\ &= \alpha_*(-1) \rtimes \gamma, & &= \alpha_*(-1) \beta_{1,*}(-1) \rtimes \gamma. \end{aligned}$$

(Note the slight abuse in notation in the formula for $\rho_H^-(\gamma)$: we regard α_* as a coweight of \hat{T}_H via the identification ψ_1 .) So (9.7.2) reduces (using (9.7.1), that $\alpha_*(-1) \in \hat{T}_H$ and that $\text{Frob}(-1) = -1$) to showing that

$$\hat{N}(t)\alpha_*(-1)\dot{w}(o) \rtimes \gamma \sim \hat{N}(t)\alpha_*(-1)\beta_{1,*}(-1) \rtimes \gamma.$$

As γ acts trivially on \hat{G} it is enough to show that

$$\hat{N}(t)\alpha_*(-1)\dot{w}(o) \sim \hat{N}(t)\alpha_*(-1)\beta_{1,*}(-1).$$

As $w(o)$ exchanges $\pm\alpha$, and $\alpha_*(-1) = \alpha_*(-1)^{-1}$, the expression $\hat{N}(t)\alpha_*(-1)$ belongs to the center of \hat{L}_o . Thus it is enough to show that $\dot{w}(o)$ and $\beta_{1,*}(-1)$ are conjugate inside \hat{L}_o . As $\hat{L}_o \simeq \text{GL}_2$, this follows from the fact that they both have order 2, and neither is central.

References

- [1] A. ASH, Smith theory and Hecke operators, *J. Algebra* **259** (2003), 43–58. MR 1953707. Zbl 1045.11034. [http://dx.doi.org/10.1016/S0021-8693\(02\)00547-1](http://dx.doi.org/10.1016/S0021-8693(02)00547-1).
- [2] A. ASH, Direct sums of mod p characters of $\text{GAL}(\overline{\mathbb{Q}}/\mathbb{Q})$ and the homology of $\text{GL}(n, \mathbb{Z})$, *Comm. Algebra* **41** (2013), 1751–1775. MR 3062822. Zbl 06194716. <http://dx.doi.org/10.1080/00927872.2011.649508>.
- [3] N. BERGERON and A. VENKATESH, The asymptotic growth of torsion homology for arithmetic groups, *J. Inst. Math. Jussieu* **12** (2013), 391–447. MR 3028790. Zbl 1266.22013. <http://dx.doi.org/10.1017/S1474748012000667>.
- [4] I. N. BERNSTEIN and A. V. ZELEVINSKY, Induced representations of reductive p -adic groups. I, *Ann. Sci. École Norm. Sup.* **10** (1977), 441–472. MR 0579172. Zbl 0412.22015. Available at http://www.numdam.org/item?id=ASENS_1977_4_10_4_441_0.
- [5] J. BERNSTEIN, Course notes of a course on p -adic groups.
- [6] A. BOREL, Automorphic L -functions, in *Automorphic Forms, Representations and L-Functions* (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977, Part 2), *Proc. Sympos. Pure Math.* **33**, Amer. Math. Soc., Providence, R.I., 1979, pp. 27–61. MR 0546608. Zbl 0412.10017.
- [7] A. BOREL and J. TITS, Groupes réductifs, *Publ. Math. Inst. Hautes Études Sci.* **27** (1965), 55–150. MR 0207712. Zbl 0145.17402. <http://dx.doi.org/10.1007/BF02684375>.
- [8] N. BOURBAKI, *Algebra II. Chapters 4–7, Elements of Mathematics (Berlin)*, Springer-Verlag, New York, 2003, translated from the 1981

- French edition by P. M. Cohn and J. Howie, Reprint of the 1990 English edition [Springer, Berlin; MR1080964. MR 1994218. Zbl 1017.12001. <http://dx.doi.org/10.1007/978-3-642-61698-3>.
- [9] K. BUZZARD and T. GEE, The conjectural connections between automorphic representations and Galois representations, 2010, preprint. arXiv 1009.0785.
- [10] P. CARTIER, Representations of p -adic groups: a survey, in *Automorphic Forms, Representations and L-Functions* (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977, Part 1), *Proc. Sympos. Pure Math.* **33**, Amer. Math. Soc., Providence, R.I., 1979, pp. 111–155. MR 0546593. Zbl 0421.22010.
- [11] W. CASSELMAN, Introduction to the theory of admissible representations.
- [12] L. CLOZEL, Formes modulaires mod p , changement de base et théorie d’Iwasawa, *Rend. Mat. Appl.* (7) **35** (2014), 35–46. MR 3241362.
- [13] L. CLOZEL, Formes modulaires sur la \mathbb{Z}_p -extension cyclotomique de \mathbb{Q} , *Pacific J. Math.* **268** (2014), 259–274. MR 3227435. Zbl 06335993. <http://dx.doi.org/10.2140/pjm.2014.268.259>.
- [14] P. DELIGNE and G. LUSZTIG, Representations of reductive groups over finite fields, *Ann. of Math.* **103** (1976), 103–161. MR 0393266. Zbl 0336.20029. <http://dx.doi.org/10.2307/1971021>.
- [15] M. EMERTON, A local-global compatibility conjecture in the p -adic Langlands programme for GL_2/\mathbb{Q} , *Pure Appl. Math. Q.* **2** (2006), 279–393. MR 2251474. Zbl 1254.11106. <http://dx.doi.org/10.4310/PAMQ.2006.v2.n2.a1>.
- [16] W. T. GAN and G. SAVIN, Real and global lifts from PGL_3 to G_2 , *Int. Math. Res. Not.* **2003** (2003), 2699–2724. MR 2017248. Zbl 1037.22033. <http://dx.doi.org/10.1155/S1073792803130607>.
- [17] G. GLAUBERMAN, Correspondences of characters for relatively prime operator groups., *Canad. J. Math.* **20** (1968), 1465–1488. MR 0232866. Zbl 0167.02602. <http://dx.doi.org/10.4153/CJM-1968-148-x>.
- [18] B. H. GROSS, On the Satake isomorphism, in *Galois Representations in Arithmetic Algebraic Geometry* (Durham, 1996), *London Math. Soc. Lecture Note Ser.* **254**, Cambridge Univ. Press, Cambridge, 1998, pp. 223–237. MR 1696481. Zbl 0996.11038. <http://dx.doi.org/10.1017/CBO9780511662010.006>.
- [19] P. GUNNELLS and D. YASAKI, On the growth of torsion in the cohomology of arithmetic groups, in preparation.
- [20] T. J. HAINES and S. ROSTAMI, The Satake isomorphism for special maximal parahoric Hecke algebras, *Represent. Theory* **14** (2010), 264–284. MR 2602034. Zbl 1251.22013. <http://dx.doi.org/10.1090/S1088-4165-10-00370-5>.
- [21] G. HENNIART and M.-F. VIGNÉRAS, A Satake isomorphism for representations modulo p of reductive groups over local fields, *J. reine angew. Math.* **701** (2015), 33–75. MR 3331726. Zbl 06424795.
- [22] D. JOYNER, On finite dimensional representations of non-connected reductive groups, *J. Lie Theory* **10** (2000), 269–284. MR 1774860. Zbl 0986.20043.
- [23] S. KIONKE, On lower bounds for cohomology growth in p -adic analytic towers, *Math. Z.* **277** (2014), 709–723. MR 3229961. Zbl 06323331. <http://dx.doi.org/10.1007/s00209-013-1273-3>.

- [24] R. E. KOTTWITZ and D. SHELSTAD, *Foundations of Twisted Endoscopy*, *Astérisque* **255**, 1999. MR 1687096. Zbl 0958.22013.
- [25] D. KRAINES and C. SCHOCHET, Differentials in the Eilenberg-Moore spectral sequence, *J. Pure Appl. Algebra* **2** (1972), 131–148. MR 0300285. Zbl 0237.55016. [http://dx.doi.org/10.1016/0022-4049\(72\)90018-7](http://dx.doi.org/10.1016/0022-4049(72)90018-7).
- [26] R. P. LANGLANDS, Representations of abelian algebraic groups, *Pacific J. Math.* **181** (1997), 231–250, Olga Taussky-Todd: in memoriam. MR 1610871. <http://dx.doi.org/10.2140/pjm.1997.181.231>.
- [27] W. MÜLLER and J. PFAFF, On the growth of torsion in the cohomology of arithmetic groups, *Math. Ann.* **359** (2014), 537–555. MR 3201905. Zbl 06312297. <http://dx.doi.org/10.1007/s00208-014-1014-x>.
- [28] V. PLATONOV and A. RAPINCHUK, *Algebraic Groups and Number Theory*, *Pure Appl. Math.* **139**, Academic Press, Boston, 1994, translated from the 1991 Russian original by Rachel Rowen. MR 1278263. Zbl 0841.20046.
- [29] J. ROHLFS and B. SPEH, Lefschetz numbers and twisted stabilized orbital integrals, *Math. Ann.* **296** (1993), 191–214. MR 1219899. Zbl 0808.11039. <http://dx.doi.org/10.1007/BF01445102>.
- [30] I. SATAKE, Theory of spherical functions on reductive algebraic groups over p -adic fields, *Publ. Math. Inst. Hautes Études Sci.* **18** (1963), 5–69. MR 0195863. Zbl 0122.28501. <http://dx.doi.org/10.1007/BF02684781>.
- [31] M. H. ŞENGÜN, On the integral cohomology of Bianchi groups, *Exp. Math.* **20** (2011), 487–505. MR 2859903. Zbl 1269.22007. <http://dx.doi.org/10.1080/10586458.2011.594671>.
- [32] J-P. SERRE, *Cohomologie Galoisienne*, *Lecture Notes in Math.* **5**, Springer-Verlag, New York, 1973. MR 0404227. Zbl 0812.12002.
- [33] J-P. SERRE, Two letters on quaternions and modular forms (mod p), *Israel J. Math.* **95** (1996), 281–299, with introduction, appendix and references by R. Livné. MR 1418297. Zbl 0870.11030. <http://dx.doi.org/10.1007/BF02761043>.
- [34] R. STEINBERG, Regular elements of semisimple algebraic groups, *Publ. Math. Inst. Hautes Études Sci.* **25** (1965), 49–80. MR 0180554. Zbl 0136.30002. <http://dx.doi.org/10.1007/BF02684397>.
- [35] R. STEINBERG, *Endomorphisms of Linear Algebraic Groups*, *Mem. Amer. Math. Soc.* **80**, Amer. Math. Soc., Providence, RI, 1968. MR 0230728. Zbl 0164.02902.
- [36] D. TREUMANN, Smith theory and geometric Hecke algebras, 2011, preprint. arXiv 1107.3798.

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