Annals of Mathematics **182** (2015), 1067–1091
http://dx.doi.org/10.4007/annals.2015.182.3.4

A complete complex hypersurface in the ball of $\mathbb{C}^N$

By Josip Globevnik

Abstract

In 1977, P. Yang asked whether there exist complete immersed complex submanifolds $\varphi\colon M^k \to \mathbb{C}^N$ with bounded image. A positive answer is known for holomorphic curves ($k = 1$) and partial answers are known for the case when $k > 1$. The principal result of the present paper is a construction of a holomorphic function on the open unit ball $\mathbb{B}_N$ of $\mathbb{C}^N$ whose real part is unbounded on every path in $\mathbb{B}_N$ of finite length that ends on $b\mathbb{B}_N$. A consequence is the existence of a complete, closed complex hypersurface in $\mathbb{B}_N$. This gives a positive answer to Yang's question in all dimensions k, N, $1 \leq k < N$, by providing properly embedded complete complex manifolds.

1. Introduction and the main result

Denote by Δ the open unit disc in $\mathbb{C}$ and by $\mathbb{B}_N$ the open unit ball in $\mathbb{C}^N$, $N \geq 2$.

In 1977 P. Yang asked whether there exist complete immersed complex submanifolds $\varphi\colon M^k \to \mathbb{C}^N$ with bounded image [Yan77b], [Yan77a]. The first answer was obtained by P. Jones [Jon79] who constructed a bounded complete immersion $\varphi\colon \Delta \to \mathbb{C}^2$ and a complete proper holomorphic embedding $\varphi\colon \Delta \to \mathbb{B}_4$. Since then there has been a series of results on bounded complete holomorphic curves ($k = 1$) *immersed* in $\mathbb{C}^2$ [MUY09], [AL13], [AF13] the most recent being that every bordered Riemann surface admits a complete proper holomorphic immersion to $\mathbb{B}_2$ and a complete proper holomorphic embedding to $\mathbb{B}_3$ [AF13]. The more difficult complete *embedding* problem for $k = 1$ and $N = 2$ has been solved only very recently by A. Alarcón and F. J. López [AL] who proved that every convex domain in $\mathbb{C}^2$ contains a complete, properly embedded complex curve.

In the present paper we are interested primarily in the higher dimensional case ($k > 1$) where there are partial answers that are easy consequences of the results for complete curves. For instance, it is known that for any $k \in \mathbb{N}$, there are complete bounded embedded complex k-dimensional submanifolds of

$\mathbb{C}^{2k}$, and it is an open question whether, in this case, $N = 2k$ is the minimal possible dimension [AL]. In the present paper we consider the case where φ is a proper holomorphic embedding. In this case $\varphi(M^k)$ is a closed submanifold. We restate the definition of completeness for this case:

Definition 1.1. A closed complex submanifold M of $\mathbb{B}_N$ is complete if every path $p\colon [0,1) \to M$ such that $|p(t)| \to 1$ as $t \to 1$ has infinite length.

Note that this coincides with the standard definition of completeness since the paths $p\colon [0,1) \to M$ such that $|p(t)| \to 1$ as $t \to 1$ are precisely the paths that leave every compact subset of M as $t \to 1$.

Here is our main result.

THEOREM 1.1. *Let $N \geq 2$. There is a holomorphic function f on $\mathbb{B}_N$ such that $\Re f$ is unbounded on every path of finite length that ends on $b\mathbb{B}_N$.*

So our function f has the property that if $p\colon [0,1] \to \overline{\mathbb{B}_N}$ is a path of finite length such that $|p(t)| < 1$ $(0 \leq t < 1)$ and $|p(1)| = 1$, then $t \to \Re\big(f(p(t)\big)$ is unbounded on $[0,1)$.

The following corollary answers the question of Yang in all dimensions k and N by providing properly embedded complete complex manifolds.

COROLLARY 1.2. *For each k, N, $1 \leq k < N$, there is a complete, closed, k-dimensional complex submanifold of $\mathbb{B}_N$.*

Proof. We first prove the corollary for $k = N-1$; that is, we first prove the existence of the hypersurface mentioned in the title. Let f be the function given by Theorem 1.1. By Sard's theorem one can choose $c \in \mathbb{C}$ such that the level set $M = \{z \in \mathbb{B}_N\colon f(z) = c\}$ is a closed submanifold of $\mathbb{B}_N$. Let $p\colon [0,1) \to M$ be a path such that $p(t) \to b\mathbb{B}_N$ as $t \to 1$. Assume that p has finite length. Then there is a point w on $b\mathbb{B}_N$ such that $\lim_{t\to 1} p(t) = w$. By the properties of f, $\Re f$ is unbounded on $p([0,1))$. On the other hand, $f((p(t)) = c$ $(0 \leq t < 1)$, a contradiction. So p must have infinite length. This proves that M is complete and so completes the proof of the corollary for $k = N-1$. Assume now that $1 \leq k \leq N-2$. By the first part of the proof there is a complete, closed, k-dimensional complex submanifold M of $\mathbb{B}_{k+1} \subset \mathbb{B}_N$. Clearly M is a complete, closed k-dimensional manifold of $\mathbb{B}_N$. This completes the proof. $\square$

Remark. If we want to have a connected, complete closed complex submanifold of $\mathbb{B}_N$, then we simply take a connected component of M as above. Note also that the same function f gives many complete closed complex manifolds of $\mathbb{B}_N$ since, by Sard's theorem, one can use the same reasoning for almost every c in the range of f.

2. Outline of the proof of Theorem 1.1

Let $M \in \mathbb{N}$. For $x \in \mathbb{R}^M \setminus \{0\}$ and $\alpha \in \mathbb{R}$, write

$$H(x,\alpha) = \{y \in \mathbb{R}^M : \langle y|x\rangle = \alpha\}, \quad K(x,\alpha) = \{y \in \mathbb{R}^M : \langle y|x\rangle \le \alpha\}.$$

Assume that $x_i \in \mathbb{R}^M \setminus \{0\}$ $(1 \le i \le n)$ and that

$$P = \bigcap_{i=1}^{n} K(x_i, 1) \tag{2.1}$$

is a bounded set. Then P is a *convex polytope*, that is, the convex hull of a finite set. So P is a compact convex set that contains the origin in its interior. A convex subset F of P is called a *face* of P if any closed segment with endpoints in P whose relative interior meets F is contained in F. A *k-face* is a face F with $\dim F = k$; that is, the affine hull of F is k-dimensional. A face of dimension $M-1$ is called a *facet* of P. Let P be a convex polytope such that the representation (2.1) is irreducible; that is,

$$P \neq \bigcap_{i=1, i\neq k}^{n} K(x_i, 1) \quad \text{for each} \quad k, \ 1 \le k \le n.$$

Then

$$bP = \bigcup_{i=1}^{n} H(x_i, 1) \cap P$$

and the sets $F_i = H(x_i, 1) \cap P$, $1 \le i \le n$, are precisely the facets of P. See [Brø83] for the details.

Given a convex set G, denote by $\mathrm{ri}(G)$ the relative interior of G in the affine hull of G. What remains of the boundary of a convex polytope P after we have removed relative interiors of all facets F_i, $1 \le i \le n$, we call the *skeleton* of P (or more precisely, the $(M-2)$-skeleton of P, the union of all $(M-2)$-dimensional faces of P) and denote by $\mathrm{skel}(P)$. Thus

$$\mathrm{skel}(P) = \bigcup_{i=1}^{n} \big[F_i \setminus \mathrm{ri}(F_i)\big].$$

To prove Theorem 1.1 we first prove

THEOREM 2.1. *Let $\mathbb{B}$ be the open unit ball of $\mathbb{R}^M$, $M \ge 3$. There is a sequence of convex polytopes P_n, $n \in \mathbb{N}$, such that*

$$P_1 \subset \mathrm{Int} P_2 \subset P_2 \subset \mathrm{Int} P_3 \subset \cdots \subset \mathbb{B}, \qquad \bigcup_{j=1}^{\infty} P_j = \mathbb{B},$$

such that if $w_j \in \mathrm{skel}(P_j)$ $(j \in \mathbb{N})$, then

$$\sum_{j=1}^{\infty} |w_{j+1} - w_j| = \infty; \tag{2.2}$$

that is, the series in (2.2) diverges.

In the proof of Theorem 1.1 we shall use the following

Corollary 2.2. *Let P_n, $n \in \mathbb{N}$ be the sequence of convex polytopes from Theorem 2.1. Let θ_n be a decreasing sequence of positive numbers such that $\sum_{n=1}^{\infty} \theta_n < \infty$. For each $n \in \mathbb{N}$, let $\mathcal{U}_n \subset bP_n$ be the θ_n-neighborhood of $\operatorname{skel}(P_n)$ in bP_n; that is, $\mathcal{U}_n = \{w \in bP_n \colon \operatorname{dist}(w, \operatorname{skel}(P_n)) < \theta_n\}$. Let $p\colon [0,1) \to \mathbb{B}$ be a path such that $|p(t)| \to 1$ as $t \to 1$ and such that for all sufficiently large $n \in \mathbb{N}$, $p([0,1))$ meets bP_n only at $\mathcal{U}_n$. Then p has infinite length.*

Once we have proved Corollary 2.2 we prove Theorem 1.1 as follows. Let $\mathbb{B}_N$ be the open unit ball of $\mathbb{C}^N$, $N \geq 2$. Let P_n, $n \in \mathbb{N}$, be a sequence of convex polytopes as in Theorem 2.1 with $M = 2N$, and let $\mathcal{U}_n$, $n \in \mathbb{N}$, be as in Corollary 2.2. Given $\varepsilon_n > 0$ and $L_n < \infty$ we use an idea from [GS82] to construct a function f_n, holomorphic on $\mathbb{B}_N$, such that $|f_n| < \varepsilon_n$ on P_{n-1} and such that $\Re f_n > L_n$ on $bP_n \setminus \mathcal{U}_n$. By choosing L_n and ε_n inductively in the right way, we then see that $f = \sum_{n=1}^{\infty} f_n$ has all the required properties.

3. Beginning of the proof of Theorem 2.1

Let w_n be a sequence in $\mathbb{B}$ such that $|w_n| \to 1$ as $n \to \infty$. If w_n does not converge, then (2.2) holds, and so to prove Theorem 2.1 it is enough to consider only the convergent sequences w_n.

First, we try to explain the idea of the most important part of the proof. Suppose for a moment that we have a sequence P_n of convex polytopes with the desired properties and that there is an increasing sequence R_n of positive numbers converging to 1 such that

$$bP_n \subset R_n\overline{\mathbb{B}} \setminus R_{n-1}\overline{\mathbb{B}} \quad (n \in \mathbb{N}).$$

Let $W = U \times (1-\nu, 1+\nu)$ be a small open neighborhood of $z = (0, 0, \ldots, 0, 1)$ in $\mathbb{R}^M$, where U is a small open ball in $\mathbb{R}^{M-1}$ centered at the origin and $\nu > 0$ is small. Assume that $U \times \{1-\nu\} \subset R_1\mathbb{B}$.

Let π be the orthogonal projection onto $\mathbb{R}^{M-1}$, so

$$\pi(x_1, \ldots, x_M) = (x_1, \ldots, x_{M-1}).$$

For each n, consider C_n, the part of $bP_n \cap W$ consisting of the facets of P_n contained in W. The projection π is one-to-one on C_n, and for each of these facets, its image under π is a convex polytope in U that is a cell of a partition of $\pi(C_n)$ into convex polytopes. Call this partition $\mathcal{L}_n$, and notice that as $n \to \infty$, $\pi(C_n)$ tends to U. If we remove from each cell of $\mathcal{L}_n$ its relative interior, then we get what we call the skeleton of $\mathcal{L}_n$, denoted by $\operatorname{skel}(\mathcal{L}_n)$. Clearly $\pi(\operatorname{skel}(P_n) \cap C_n) = \operatorname{skel}(\mathcal{L}_n)$. Since, by our assumption at the moment, every sequence w_n contained in W that meets $\operatorname{skel}(P_n)$ for all sufficiently large n must satisfy (2.2), looking at $z_n = \pi(w_n)$ we conclude that every sequence $z_n \in U$ such that

$z_n \in \operatorname{skel}(\mathcal{L}_n)$ for all sufficiently large n must satisfy $\sum_{n=1}^{\infty} |z_{n+1} - z_n| = \infty$. The idea now is to reverse the direction of reasoning. Let R_0 be so close to 1 that $U \times \{1-\nu\} \subset R_0\mathbb{B}$. In a typical induction step of constructing our polytopes the data will be a partition $\mathcal{L}$ of $\mathbb{R}^{M-1}$ into convex polytopes and ρ and r, $R_0 < \rho < r < 1$. Denote by $\mathcal{C}$ the union of those cells of the partition $\mathcal{L}$ that are contained in U, and let $\mathcal{V}$ be the set of their vertices. We will "lift" $\mathcal{V}$ to $b(r\mathbb{B})$ by putting $V = (\pi|W \cap b(r\mathbb{B}))^{-1}(\mathcal{V})$. We want V to be the set of vertices of a convex polyhedral surface C such that $\pi(C) = \mathcal{C}$ and such that π maps the facets of C precisely onto the cells of $\mathcal{C}$. We will do this in such a way that C stays out of $\rho\overline{\mathbb{B}}$ — for this, the cells of C, and consequently the cells of $\mathcal{C}$ will have to be sufficiently small, of size proportional to $\sqrt{r-\rho}$. Then we will construct a convex polytope P such that C will be a part of its boundary bP and such that $\rho\overline{\mathbb{B}} \subset \operatorname{Int} P \subset P \subset r\overline{\mathbb{B}}$.

There is a potential problem already at the first step. Namely, the points of V need not be the vertices of a *convex* surface C. For this to happen we will need two things: $\mathcal{L}$ will have to be a true Delaunay partition of $\mathbb{R}^{M-1}$, and the ball U in the definition of W will have to be sufficiently small so that the part of $b(r\mathbb{B})$ contained in W will be sufficiently flat.

4. A Delaunay tessellation of $\mathbb{R}^{M-1}$

Perturb the canonical orthonormal basis in $\mathbb{R}^{M-1}$ a little to get an $(M-1)$-tuple of vectors $e_1, e_2, \ldots, e_{M-1}$ in general position so that the lattice

$$\Lambda = \Big\{ \sum_{i=1}^{M-1} n_i e_i \ : n_i \in Z,\ 1 \le i \le M-1 \Big\} \tag{4.1}$$

will be generic and, in particular, no more than M points of Λ will lie on the same sphere.

For each point $x \in \Lambda$, there is the *Voronei cell* $V(x)$ consisting of those points of $\mathbb{R}^{M-1}$ that are at least as close to x as to any other $y \in \Lambda$, so

$$V(x) = \{y \in \mathbb{R}^{M-1} \colon \operatorname{dist}(y,x) \le \operatorname{dist}(y,z) \text{ for all } z \in \Lambda\}.$$

In our case it is easy to see how to get $V(0)$. Consider the finite set $E = \{\sum_{j=1}^{M-1} n_i e_i : -1 \le n_i \le 1, 1 \le i \le M-1\}$, and for each $x \in E \setminus \{0\}$, look at $K(x, |x|^2/2)$, that is, at the halfspace that contains the origin and is bounded by the hyperplane passing through $x/2$ that is perpendicular to x. Then

$$V(0) = \bigcap_{x \in E\setminus\{0\}} K(x, |x|^2/2).$$

This is a convex polytope. It is known that the Voronei cells form a tessellation of R^{M-1} and in our case they are all congruent, of the form $V(0) + x,\ x \in \Lambda$ [CS88].

There is a *Delaunay cell* for each point that is a vertex of a Voronei cell. It is the convex polytope that is the convex hull of the points in Λ closest to that point — these points are all on a sphere centered at this point. In our case, when there are no more than M points of Λ on a sphere, Delaunay cells are $(M-1)$-simplices. Delaunay cells form a tessellation of R^{M-1} [CS88]. It is a *true* Delaunay tessellation; that is, for each cell, the circumsphere of each cell S contains no other points of Λ than the vertices of S. We shall denote by $\mathcal{D}(\Lambda)$ the family of all simplices — cells of the Delaunay tessellation for the lattice Λ.

By periodicity there are only finitely many simplices $S_1, \dots, S_\ell$ such that every other simplex of $\mathcal{D}(\Lambda)$ is of the form $S_i + w$ where $w \in \Lambda$ and $1 \le i \le \ell$. It is then clear by periodicity that there is an $\eta > 0$ such that for every simplex $S \in \mathcal{D}(\Lambda)$ in η-neighborhood of the closed ball bounded by the circumsphere of S, there are no other points of Λ than the vertices of S.

We shall typically replace the lattice Λ by the lattice $\Lambda + q = \{x + q \colon x \in \Lambda\}$ where $q \in \mathbb{R}^{M-1}$ or, more generally, by the lattice $\sigma(\Lambda+q)$ where $\sigma > 0$ is small. Again, we shall denote by $\mathcal{D}(\sigma(\Lambda+q))$ the family of all simplices - cells of the Delaunay tessellation for $\sigma(\Lambda+q)$. These are the simplices of the form $\sigma(S+q)$ where $S \in \mathcal{D}(\Lambda)$. Passing from Λ to $\sigma(\Lambda+q)$ everything in the reasoning will change proportionally. In particular, for every simplex $S \in \mathcal{D}(\sigma(\Lambda+q))$ in $(\sigma\eta)$-neighborhood of the closed ball bounded by the circumsphere of S, there will be no other points of $\sigma(\Lambda+q)$ than the vertices of S. We shall also need the notion of the *skeleton* of the Delaunay tessellation for $\sigma(\Lambda+q)$. This is what remains after we remove the interiors of all $S \in \mathcal{D}(\sigma(\Lambda+q))$, hence

$$\operatorname{skel}\big(\mathcal{D}(\sigma(\Lambda+q))\big) = \bigcup_{S \in \mathcal{D}(\sigma(\Lambda+q))} \big[S \setminus \operatorname{Int} S\big] = \mathbb{R}^{M-1} \setminus \Big[\bigcup_{S \in \mathcal{D}(\sigma(\Lambda+q))} \operatorname{Int} S\Big].$$

The author is grateful to John M. Sullivan who suggested the use of a generic lattice for our purpose here.

5. Lifting the lattice from $\mathbb{R}^{M-1}$ to the sphere

Let z, $W = U \times (1-\nu, 1+\nu)$ and π be as in Section 3. Let $\Lambda \subset \mathbb{R}^{M-1}$ be as in (4.1).

Fix R_0, $0 < R_0 < 1$, so large that $U \times \{1-\nu\} \subset R_0\mathbb{B}$, and assume that $R_0 < \rho < r < 1$. The part of the sphere $b(r\mathbb{B})$ in W can now be written as a graph of a real analytic function, call it ψ_r, so

$$b(r\mathbb{B}) \cap W = \{(x, \psi_r(x)) \colon x \in U\},$$

where

$$\psi_r(x) = \psi_r(x_1, \dots, x_{M-1}) = \Big(r^2 - \sum_{j=1}^{M-1} x_j^2\Big)^{1/2}. \tag{5.1}$$

Note that $(\operatorname{grad} \psi_r)(0) = 0$, $R_0 < r < 1$.

The map π maps $W \cap b(r\mathbb{B})$ in a one-to-one way onto U. We shall "lift" $(\sigma\Lambda)$ from U to $b(r\mathbb{B}) \cap W$ by the inverse of this map, that is, by the map $x \mapsto (x, \psi_r(x))$. We want to get a convex polyhedral surface C with vertices $w = (v, \psi_r(v))$, where v are the vertices of those cells of the Delaunay tessellation for $\sigma\Lambda$ that are contained in U, and we want that π maps the facets of the surface C precisely onto the Delaunay cells of $\sigma\Lambda$ contained in U. Let us describe the conditions for this to happen. Let S be a simplex of the Delaunay tessellation for $\sigma\Lambda$. Let $v_1, \ldots, v_M$ be the vertices of S. We want that the simplex with vertices $w_j = (v_j, \psi(v_j))$, $1 \le j \le M$, is a facet of a convex poyhedral surface. For this to happen, all other points $w = (v, \psi_r(v))$, $v \in \sigma\Lambda \cap U$, $v \neq v_1, \ldots, v_M$, must lie in the open halfspace bounded by the hyperplane Π through w_j, $1 \le j \le M$, which contains the origin; that is, they must lie on $b(r\mathbb{B})$ outside the "small" sphere $\Gamma = \Pi \cap b(r\mathbb{B})$. Since $\pi | W \cap b(r\mathbb{B})$ is one-to-one, this happens if and only if the points $v \in \sigma\Lambda$ that are the vertices of the Delaunay cells of $\sigma\Lambda$ contained in U and are different from $v_1, \ldots, v_M$, are outside the projection $\pi(\Gamma)$, an ellipsoid in $\mathbb{R}^{M-1}$.

As we shall see, this will happen for all such simplices S if the ball $U \subset \mathbb{R}^{M-1}$ centered at the origin is small enough so that the the gradient of ψ_r and thus the Lipschitz constant of ψ_r is small enough on U. The choice of U will depend only on η from Section 4, and the same reasoning will work for any $\sigma > 0$.

LEMMA 5.1. *Let $\pi\colon \mathbb{R}^M \to \mathbb{R}^{M-1}$ be the standard projection*

$$\pi(x_1, \ldots, x_M) = (x_1, \ldots, x_{M-1}).$$

Let Λ be the lattice in $\mathbb{R}^{M-1}$ as in (4.1), and let $\eta > 0$. There is a constant $\omega > 0$ such that for every $\sigma > 0$, the following holds. Let $S \subset \mathbb{R}^{M-1}$ be a simplex belonging to $\mathcal{D}(\sigma\Lambda)$. Suppose that ψ is a Lipschitz function in a neighborhood of S with Lipschitz constant $\le \omega$. Let $v_1, \ldots, v_M$ be the vertices of S, and let $w_1, \ldots, w_M$ be the points in $\mathbb{R}^M$ given by $w_j = (v_j, \psi(v_j))$, $1 \le j \le M$. Let Π be the hyperplane in $\mathbb{R}^M$ containing the points $w_1, \ldots, w_M$ and let Γ be the sphere in Π containing these points; that is, let Γ be the circumsphere of the $(M-1)$-simplex in Π with vertices $w_1, \ldots, w_M$. Then $\pi(\Gamma)$ is contained in the $(\sigma\eta)$-neighborhood of the circumsphere of the simplex S.

6. **Proof of Lemma 5.1**

Let $S \in \mathcal{D}(\Lambda)$, and let $\eta > 0$. If we replace ψ with $\psi + c$, where c is a constant, Π will change to $\Pi + (0, c)$, Γ to $\Gamma + (0, c)$, and consequently $\pi(\Gamma)$ will not change. Thus, $\pi(\Gamma)$ remains unchanged if we subtract $\psi(v_M)$ from each $\psi(v_j)$, $1 \le j \le M$. Thus, $\pi(\Gamma)$ will be determined precisely once we know $\beta_1 = \psi(v_1) - \psi(v_M), \ldots, \beta_{M-1} = \psi(v_{M-1}) - \psi(v_M)$. We shall show that $\pi(\Gamma)$ changes continuously with $(\beta_1, \cdots, \beta_{M-1})$ near $(0, 0, \ldots, 0)$ if $w_1 =$

$(v_1,\beta_1),\dots,w_{M-1}=(v_{M-1},\beta_{M-1})$ and $w_M=(v_M,0)$. Note that when $\beta_1=\cdots=\beta_{M-1}=0$, then $\Gamma=\pi(\Gamma)$ is the circumsphere of S in $\mathbb{R}^{M-1}$. Let $w_0=(w_{01},\dots,w_{0,M-1},1)$ be the vector in $\mathbb{R}^M$ perpendicular to Π whose last component equals 1. So w_0 must be perpendicular to w_j-w_M, $1\le j\le M-1$, so $\langle w_j-w_M|w_0\rangle=0$ $(1\le j\le M-1)$ which, if $v_j=(v_{j1},\dots,v_{j,M-1}),1\le j\le M-1$, is the system of linear equations

$$(v_{j1}-v_{M1})w_{01}+\cdots+(v_{j,M-1}-v_{M,M-1})w_{0,M-1}=-\beta_j\ \ (1\le j\le M-1).$$

This is a system of $M-1$ linear equations for $M-1$ unknowns $w_{01},\dots,w_{0,M-1}$ whose matrix is nonsingular since, S being a $(M-1)$-simplex, the vectors v_j-v_M, $1\le j\le M-1$, are linearly independent. Its solution depends linearly on $(\beta_1,\dots,\beta_{M-1})$. When $\beta_1=\cdots=\beta_{M-1}=0$ the solution is the zero vector. In this case $w_0=(0,\dots,0,1)$. Let $z=(z_1,\dots,z_M)$ be the center of the sphere in Π that contains $w_1,\dots,w_M$. Then z is in Π, and so

$$\langle z-w_M|w_0\rangle=0. \tag{6.1}$$

Further, for each i, $1\le i\le M-1$, z is at equal distance from w_i and w_M, so z is contained in the hyperplane in $\mathbb{R}^M$ that passes through the midpoint of the segment joining w_i and w_M, and it is perpendicular to this segment, so z must satisfy

$$\langle[z-(w_i+w_M)/2]|[w_i-w_M]\rangle=0.$$

Thus,

$$\langle z|[w_i-w_M]\rangle=(1/2)\langle[w_i+w_M]|[w_i-w_M]\rangle\ \ (1\le i\le M-1).$$

Together with (6.1) this becomes the following system of linear equations for $z_1,\dots,z_M$:

$$\begin{aligned}
&z_1(v_{i1}-v_{M1})+\cdots+z_{M-1}(v_{i,M-1}-v_{M,M-1})+z_M\beta_i\\
&\qquad\qquad\qquad\qquad=(|w_i|^2-|w_M|^2)/2\ \ (1\le i\le M-1),\\
&z_1w_{01}+\cdots+z_{M-1}w_{0,M-1}+z_M=v_{M1}w_{01}+\cdots+v_{M,M-1}w_{0,M-1}.
\end{aligned}$$

Its matrix

$$\begin{bmatrix}
v_{11}-v_{M1} & \dots & v_{1,M-1}-v_{M,M-1} & \beta_1\\
 & \dots & & \\
v_{M-1,1}-v_{M1} & \dots & v_{M-1,M-1}-v_{M,M-1} & \beta_{M-1}\\
w_{01} & \dots & w_{0,M-1} & 1
\end{bmatrix}$$

is nonsingular for $\beta_1=\cdots=\beta_{M-1}=0$ when $w_{01}=\cdots=w_{0,M-1}=0$. The matrix depends continuously on $(\beta_1,\dots,\beta_{M-1})$ and so do the right sides $(1/2)(|v_i|^2-|v_M|^2+\beta_i^2)$, $1\le i,\le M-1$, and, since w_0 depends continuously on $(\beta_1,\dots,\beta_{M-1})$, also $v_{M1}w_{01}+\cdots+v_{M,M-1}w_{0,M-1}$ depends continuously on $(\beta_1,\dots,\beta_{M-1})$. So the solution $z=(z_1,\dots,z_M)$, the center of the sphere Γ, depends continuously on $(\beta_1,\dots,\beta_{M-1})$ near $(0,0,\dots,0)$ and so does its radius $|z - w_M| = \big((z_1 - v_1)^2 + \cdots + (z_{M-1} - v_{M-1})^2 + z_M^2\big)^{1/2}$. Recall that Π passes through $w_M = (v_M, 0)$ and its perpendicular direction w_0 changes continuously with $(\beta_1, \ldots, \beta_{M-1})$ so Π changes continuously with $(\beta_1, \ldots, \beta_{M-1})$. We have seen that the center z of the sphere Γ in Π and its radius also change continuously with $(\beta_1, \ldots, \beta_{M-1})$ near $(0, 0, \ldots, 0)$. Thus, $\pi(\Gamma)$ changes continuously with $(\beta_1, \ldots, \beta_{M-1})$ near the origin where $\pi(\Gamma) = \Gamma$ is the circumsphere of S when $\beta_1 = \beta_2 = \cdots = \beta_{M-1} = 0$. Thus, $\pi(\Gamma)$ is contained in the η-neighborhood of the circumsphere of the simplex S in $\mathbb{R}^{M-1}$ provided that $\psi(v_1) - \psi(v_M), \ldots, \psi(v_{M-1}) - \psi(v_M)$ are small enough. If ψ is a Lipschitz function with the Lipschitz constant ω, then $|\psi(v_i) - \psi(v_M)| \le \omega|v_i - v_M|,\ 1 \le i \le M-1$, so there is an ω such that if ψ is a Lipschitz function with the Lipschitz constant not exceeding ω, then $\pi(\Gamma)$ is contained in the η-neighborhood of the circumsphere of the simplex S. Recall that every simplex in $\mathcal{D}(\Lambda)$ is of the form $S_i + x,\ 1 \le i \le \ell,\ x \in \Lambda$. Repeating the reasoning above for each $S_i,\ 1 \le i \le \ell$, we get the Lipschitz constant that works for every simplex S in $\mathcal{D}(\Lambda)$. This completes the proof for $\sigma = 1$.

Now, let $\sigma > 0$ be arbitrary and let $S \subset \mathbb{R}^{M-1}$ be a simplex in $\mathcal{D}(\sigma\Lambda)$. Let ψ be a Lipschitz function with Lipschitz constant not exceeding ω in a neighborhood of S, so its graph is given by $x_M = \psi(x_1, \ldots, x_{M-1})$. Introduce new coordinates $X_1, \ldots, X_M$ in $\mathbb{R}^M$ by $x_j = \sigma X_j,\ 1 \le j \le M$. In new coordinates we have $\sigma X_M = \psi(\sigma X_1, \ldots, \sigma X_{M-1})$, so $X_M = \Psi(X_1, \ldots, X_{M-1}) = (1/\sigma)\psi(\sigma X_1, \ldots, \sigma X_{M-1})$. Both ψ and Ψ are Lipschitz functions with the same Lipschitz constants, so in new coordinates Ψ is a Lipschitz function in a neighborhood of S which, in new coordinates, belongs to $\mathcal{D}(\Lambda)$. Thus, applying the first part of the proof we see that in new coordinates $\pi(\Gamma)$ is contained in the η-neighborhood of the circumsphere of S. In follows that in old coordinates $\pi(\Gamma)$ is contained in the $(\sigma\eta)$-neighborhood of the circumsphere of S. This completes the proof.

7. Polyhedral convex surface contained in a spherical shell

Let Λ be as in (4.1), let $\eta > 0$ be as in Section 4, and let ω be the one given by Lemma 5.1. Again let $W = U \times (1 - \nu, 1 + \nu)$, where $\nu > 0$ is small and U is a small open ball centered at the origin in $\mathbb{R}^{M-1}$, and let $R_0 < 1$ be so large that $U \times \{1 - \nu\} \subset R_0\mathbb{B}$. For every $r,\ R_0 < r < 1$, $W \cap b(r\mathbb{B}) = \{(x, \psi_r(x)) \colon x \in U\}$, where the function ψ_r is as in (5.1). We have $(\mathrm{grad}(\psi_r))(x) = -(r^2 - |x|^2)^{-1/2}x \ \ (x \in U)$ so we may, passing to a smaller U if necessary, assume that $|(\mathrm{grad}\psi_r)(x)| \le \omega\ (x \in U,\ R_0 < r < 1)$ so that for each $r,\ R_0 < r < 1,\ \psi_r$ is a Lipschitz function on U with Lipschitz constant not exceeding ω.

Let $\sigma > 0$ be small, and let $R_0 < r < 1$. Let ψ_r be as in (5.1). Then $x \mapsto \Psi_r = (x, \psi_r(x))$ is a one-to-one map from U onto $W \cap b(r\mathbb{B})$. We now look at the points $\Psi_r(x), x \in (\sigma\Lambda) \cap U$ and want to see them as vertices of a convex polyhedral hypersurface in $\mathbb{R}^M$.

Consider a simplex $S \in \mathcal{D}(\sigma\Lambda)$ that is contained in U. Let $v_1, \dots, v_M$ be its vertices. We can extend the restriction of the function ψ_r to this set of vertices to a function φ_r on all S by putting

$$\varphi_r\Big(\sum_{j=1}^M \alpha_j v_j\Big) = \sum_{j=1}^M \alpha_j \psi_r(v_j) \ \ \Big(0 \le \alpha_j \le 1,\ 1 \le j \le M,\ \sum_{j=1}^M \alpha_j = 1\Big)$$

to get an affine function φ_r on S so that $x \mapsto \Phi_r(x) = (x, \varphi_r(x))$ is an affine map mapping S to $\Phi_r(S)$, the simplex with vertices $\Psi_r(v_1), \dots, \Psi_r(v_M)$. We do this for every simplex $S \in \mathcal{D}(\sigma\Lambda)$ that is contained in U. Thus, we get a piecewise linear function φ_r on the union of the simplices $S \in \mathcal{D}(\sigma\Lambda)$ contained in U and so the union $C_r(\sigma)$ of all these $\Phi_r(S)$, the graph of the function φ_r, is then a polyhedral surface in $\mathbb{R}^M$. We shall show that the function φ_r is convex so that $C_r(\sigma)$ is a convex polyhedral surface. Later we shall show that the part of $C_r(\sigma)$ contained in $W_0 = U_0 \cap (1-\nu, 1+\nu)$ with U_0 being a ball in R^{M-1} centered at the origin, strictly smaller than U, is a part of the boundary bP of a suitable convex polytope P.

Given $S \in \mathcal{D}(\sigma\Lambda),\ S \subset U$, let Π be the hyperplane in $\mathbb{R}^M$ that contains $\Phi_r(S)$. Then $\Pi \cap b(r\mathbb{B})$ is the sphere in Π that is the circumsphere of $\Phi_r(S)$, which was denoted by Γ in Section 5. By Lemma 5.1, $\pi(\Gamma)$ is contained in the $(\sigma\eta)$-neighborhood of the circumsphere of S in $\mathbb{R}^{M-1}$. We know that the $\sigma\eta$-neighborhood of the closed ball in $\mathbb{R}^{M-1}$ bounded by the circumsphere of S contains no other points of $\sigma\Lambda$ than the vertices of S, which implies that all points of $\Psi_r(U \cap (\sigma\Lambda))$ other than the vertices of $\Phi_r(S)$ lie outside of the small "spherical cap" that Π cuts out of $b(r\mathbb{B})$, that is, outside of the "small" part of $b(r\mathbb{B})$ bounded by Γ. This shows that all other vertices of the simplices in $C_r(\sigma)$ that are not the vertices of $\Phi_r(S)$ are contained in the *open* halfspace of $\mathbb{R}^M$ bounded by Π that contains the origin. Thus, $\Phi_r(S)$ is a facet of $C_r(\sigma)$. Since this holds for every $S \in \mathcal{D}(\sigma\Lambda), S \subset U$, it follows that the surface $C_r(\sigma)$ is convex.

The simplices $\Phi_r(S)$ where $S \in \mathcal{D}(\sigma\Lambda),\ S \subset U$, have all their vertices on $b(r\mathbb{B})$. We want to estimate how far into $r\mathbb{B}$ they reach. To do this, we need the following

PROPOSITION 7.1. *Let $0 < r < 1$, let $a \in b(r\mathbb{B})$, and let $A \subset b(r\mathbb{B})$ be a set such that $|x - a| \le \gamma$ for all $x \in A$, where $\gamma < r$. Then the convex hull of A misses $\rho\overline{\mathbb{B}}$ where $\rho = r - \frac{\gamma^2}{2r}$.*

Proof. A is contained in $\{x \in b(r\mathbb{B})\colon |x - a| \le \gamma\}$. With no loss of generality assume that $a = (r, 0, \dots, 0)$. Then

$$\begin{aligned}
A &\subset \{x \in b(r\mathbb{B})\colon (x_1 - r)^2 + x_2^2 + \cdots + x_M^2 \le \gamma^2\}\\
&\subset \{x \in b(r\mathbb{B})\colon r^2 - 2x_1 r + r^2 \le \gamma^2\}\\
&= \{x \in b(r\mathbb{B})\colon 2r^2 - 2x_1 r < \gamma^2\}\\
&= \left\{x \in b(r\mathbb{B})\colon x_1 > r - \frac{\gamma^2}{2r}\right\}\\
&\subset \left\{x \in r\overline{\mathbb{B}}\colon x_1 > r - \frac{\gamma^2}{2r}\right\}.
\end{aligned}$$

The last set is a convex set that contains A and misses $\rho\overline{\mathbb{B}}$, which completes the proof. □

Denote by d the length of the longest edge of simplices in $\mathcal{D}(\Lambda)$ so that σd is the length of the longest edge of the simplices in $\mathcal{D}(\sigma\Lambda)$. Since ψ_r is a Lipschitz function with the Lipschitz constant not exceeding ω, the length of the longest edge of the simplices $\Phi_r(S)$ where $S \in \mathcal{D}(\sigma\Lambda)$, $S \subset U$, does not exceed $\sqrt{1+\omega^2}\sigma d$. Now, we use Proposition 7.1. If $R_0 < r < 1$, then $r - \frac{\gamma^2}{2r} > r - \frac{\gamma^2}{2R_0}$. Thus, putting

$$\lambda = \frac{(1+\omega^2)d^2}{2R_0},$$

we get the following

Proposition 7.2. *If $R_0 < r < 1$, then the simplices $\Phi_r(S)$, where $S \subset \mathcal{D}(\sigma\Lambda)$, $S \subset U$, miss $\rho\overline{\mathbb{B}}$ where $\rho = r - \sigma^2\lambda$.*

8. **A convex polytope with a prescribed part of the boundary**

We keep the meaning of R_0, U, d and λ. Recall that U is an open ball in $\mathbb{R}^{M-1}$ centered at the origin. Let μ be its radius. Let $0 < \mu_0 < \mu_1 < \mu_2 < \mu_3 < \mu$, and let $U_i = \{x \in \mathbb{R}^{M-1}\colon |x| < \mu_i\}$, $W_i = U_i \times (1-\nu, 1+\nu)$, $0 \le i \le 3$.

Choose $\sigma_0 > 0$ so small that

$$\sigma_0 d < \min\{\mu - \mu_3, \mu_3 - \mu_2, \mu_2 - \mu_1, \mu_1 - \mu_0\}. \tag{8.1}$$

Then, since the maximal edge length of simplices in $\mathcal{D}(\sigma\Lambda)$ equals σd, it follows that if $0 < \sigma < \sigma_0$, then

- the simplices $S \in \mathcal{D}(\sigma\Lambda)$ that meet U_0 are contained in U_1,
- the simplices $S \in \mathcal{D}(\sigma\Lambda)$ that are contained in U cover U_3.

Proposition 8.1. *There is a $\kappa > 0$ such that whenever $R_0 \le R \le 1$ and $R < R' < R + \kappa$, then each hyperplane in $\mathbb{R}^M$ that meets $W_2 \cap \left(R'\overline{\mathbb{B}} \setminus R\overline{\mathbb{B}}\right)$ and misses $W_3 \cap R\overline{\mathbb{B}}$ misses $R\overline{\mathbb{B}}$.*

Proof. Suppose that there is no such $\kappa > 0$. Then there are a sequence R_n, $R_0 \le R_n \le 1$ $(n \in \mathbb{N})$, and a sequence $x_n \in W_2$, such that $|x_n| > R_n$ $(n \in \mathbb{N})$ and such that $|x_n| - R_n \to 0$ as $n \to \infty$, and for each n, a hyperplane H_n through x_n that misses $W_3 \cap R_n\overline{\mathbb{B}}$ and meets $R_n\overline{\mathbb{B}} \setminus W_3$. Since $|x_n| - R_n \to 0$ as $n \to \infty$ we may, passing to subsequences if necessary, with no loss of generality assume that R_n converges to an R and x_n converges to $x \in b(R\mathbb{B}) \cap \overline{W_2}$. Since for each n, H_n misses $W_3 \cap R_n\overline{\mathbb{B}}$, it follows that H_n converges to H, the hyperplane through x tangent to $b(R\mathbb{B})$ at x. In particular, $H \cap (R\overline{\mathbb{B}} \setminus W_3)$ is empty, so for sufficiently large n, $H_n \cap (R_n\overline{\mathbb{B}} \setminus W_3)$ must be empty, a contradiction. This completes the proof. □

With no loss of generality, passing to a smaller σ_0 if necessary, we may assume that $\sigma_0^2\lambda < \kappa$. Suppose now that $0 < \sigma < \sigma_0$, and let $R_0 \le \rho < r < 1$ where $\rho = r - \sigma^2\lambda$.

We know that the union $C_r(\sigma)$ of the simplices $\Phi_r(S)$, where $S \in \mathcal{D}(\sigma\Lambda)$, $S \subset U$, is a convex polyhedral surface that, by Proposition 7.2, is contained in $r\overline{\mathbb{B}} \setminus \rho\overline{\mathbb{B}}$. Each of these simplices $\Phi_r(S)$ is contained in a hyperplane H. We want that these hyperplanes miss $\rho\overline{\mathbb{B}}$. Note that by (8.1) the simplices in $\mathcal{D}(\sigma\Lambda)$, contained in U, cover U_3. So the function φ_r is well defined on U_3 and its graph $C_r(\sigma) \cap W_3$ is contained in $W_3 \cap (r\overline{\mathbb{B}} \setminus \rho\overline{\mathbb{B}})$. The function φ_r is piecewise linear and convex. Thus, if $S \in \mathcal{D}(\sigma\Lambda)$ meets U_2 then, by (8.1), $S \subset U_3$ and by the convexity of φ_r, the graph of $\varphi_r|U_3$ lies on one side of the hyperplane H that contains $\Phi_r(S)$ which, in particular, implies that H misses $W_3 \cap \rho\overline{\mathbb{B}}$ and thus, by Proposition 8.1, H misses $\rho\overline{\mathbb{B}}$. This shows that the part of $C_r(\sigma)$ contained in W_2 can be described in terms of the hyperplanes that miss $\rho\overline{\mathbb{B}}$. So we find $x_1, \ldots, x_n \in b\mathbb{B}$ and $\alpha_1, \ldots, \alpha_n$, $\rho < \alpha_i \le r$ $(1 \le i \le n)$, such that

$$G_1 = \{x \in \overline{\mathbb{B}} \colon \langle x|x_i\rangle \le \alpha_i, 1 \le i \le n\}$$

is a convex set containing $\rho\overline{\mathbb{B}}$ in its interior, and is such that $W_2 \cap bG_1 = W_2 \cap C_r(\sigma)$.

PROPOSITION 8.2. *Let $R_0 < r < 1$, let $0 < \sigma < \sigma_0$, and let $\rho = r - \sigma^2\lambda > R_0$. There is a convex polytope P which contains $\rho\overline{\mathbb{B}}$ in its interior, such that $bP \subset r\overline{\mathbb{B}} \setminus \rho\overline{\mathbb{B}}$, and such that every $\Phi_r(S)$ where $S \in \mathcal{D}(\sigma\Lambda)$, $S \subset U_1$, is a facet of P.*

Proposition 8.2 implies, in particular, that

$$W_0 \cap \operatorname{skel}(P) = \Phi_r\big(U_0 \cap \operatorname{skel}(\mathcal{D}(\sigma\Lambda))\big)$$

so that

$$\pi\big(W_0 \cap \operatorname{skel}(P)\big) = U_0 \cap \operatorname{skel}\big(\mathcal{D}(\sigma\Lambda)\big).$$

Proof. To prove Proposition 8.2 we will find another convex set G_2 whose boundary outside W_2 will be a polyhedral convex surface approximating $b(r\mathbb{B})$ and such that $W_1 \cap bG_2 = W_1 \cap r\overline{\mathbb{B}}$ and then put $P = G_1 \cap G_2$. To do this we first choose $\rho_1 < r$ so close to r that if H is a hyperplane in $\mathbb{R}^M$ passing through a point $x \in b(\rho_1\mathbb{B}) \setminus W_2$ tangent to $b(\rho_1\mathbb{B})$, then $H \cap W_1 \cap r\overline{B} = \emptyset$. We will now use a finite number of these hyperplanes to modify the part of $b(r\mathbb{B})$ outside W_1 to get a convex polyhedral hypersurface contained in $r\overline{\mathbb{B}} \setminus \rho_1\mathbb{B}$ that will be a part of bG_2. To do this, we need

PROPOSITION 8.3. *Let $x, y \in b\mathbb{B}$. Suppose that ry is in the halfspace $\{z \in \mathbb{R}^M : \langle z|x\rangle \leq \rho_1\}$, that is, in the halfspace bounded by the hyperplane through $\rho_1 x$, tangent to $b(\rho_1\mathbb{B})$ that contains the origin. Then $|x - y| \geq \sqrt{2(1 - \rho_1/r)}$.*

Proof. Our assumption implies that $\langle ry|x\rangle \leq \rho_1$ so $\langle x|y\rangle \leq \rho_1/r$, and so $|y - x|^2 = 2 - 2\langle x|y\rangle \geq 2 - 2\rho_1/r = 2(1 - \rho_1/r)$, which completes the proof. □

Note that if $z \in b\mathbb{B}$, then $\{y : \langle y|z\rangle \leq \rho_1\}$ is the halfspace bounded by the hyperplane through $\rho_1 z$ tangent to $b(\rho_1\mathbb{B})$ that contains the origin.

PROPOSITION 8.4. *Let $\mathcal{S}$ be a subset of $b\mathbb{B}$. Let $0 < \rho_1 < r$, and let $0 < \delta < \sqrt{2(1 - \rho_1/r)}$. Assume that $z_1, \ldots, z_m \in \mathcal{S}$ are such that*

$$\mathcal{S} \subset \cup_{j=1}^m (z_j + \delta\mathbb{B}). \tag{8.2}$$

Then the convex polyhedron

$$Q = \bigcap_{j=1}^m \{y : \langle y|z_j\rangle \leq \rho_1\}$$

does not meet $r\mathcal{S}$.

Proof. Suppose that $y \in \mathcal{S}$ is such that $ry \in Q$; that is, $\langle ry|z_j\rangle \leq \rho_1$ for all $j,\ 1 \leq j \leq m$. By Proposition 8.3 it follows that $|y - z_j| \geq \sqrt{2(1 - \rho_1/r)} > \delta$ for all $j, 1 \leq j \leq m$, which contradicts (8.2). This completes the proof. □

We now proceed to finish the proof of Proposition 8.2. Let $\mathcal{T} = b(r\mathbb{B}) \setminus W_2$. Choose $\delta,\ 0 < \delta < \sqrt{2(1 - \rho_1/r)}$, and then choose $z_1, \ldots, z_m \in b\mathbb{B}$ such that

$$\frac{1}{r}\mathcal{T} \subset \cup_{j=1}^m (z_j + \delta\mathbb{B}).$$

Set

$$G_2 = \{y \in r\overline{\mathbb{B}} : \langle y|z_j\rangle \leq \rho_1\ (1 \leq j \leq m)\},$$

and let $P = G_1 \cap G_2$, so

$$P = \{x \in \overline{\mathbb{B}} : \langle x|x_i\rangle \leq \alpha_i, 1 \leq i \leq n,\ \langle x|z_j\rangle \leq \rho_1,\ 1 \leq j \leq m\}.$$

By construction, P contains $\rho\overline{\mathbb{B}}$ in its interior. Moreover, it is easy to see that

$$P = \{x \in \mathbb{R}^M : \langle x|x_i\rangle \leq \alpha_i, 1 \leq i \leq n,\ \langle x|z_j\rangle \leq \rho_1,\ 1 \leq j \leq m\},$$

so P is a convex polytope contained in $r\overline{\mathbb{B}}$ and, by construction, is such that every $\Phi_r(S)$ where $S \in \mathcal{D}(\sigma\Lambda)$, $S \subset U_1$, is a facet of P. Proposition 8.2 is proved.

It is clear that all we have done so far will work in the same way for any lattice $\sigma(\Lambda + q)$. Summing up what we have proved so far we get our main Lemma 8.5. Recall that $\pi(z_1, \ldots, z_M) = (z_1, \ldots, z_{M-1})$.

LEMMA 8.5. *There are* R_0, $0 < R_0 < 1$, $\nu > 0$, $\sigma_0 > 0$, $\lambda > 0$, *and a small open ball* $U_0 \subset \mathbb{R}^{M-1}$ *centered at the origin, such that* $U_0 \times \{1-\nu\} \subset R_0\mathbb{B}$ *and such that if* $W_0 = U_0 \times (1-\nu, 1+\nu)$, *then the following holds*: *For each* σ, $0 < \sigma < \sigma_0$, *for each* r *such that*

$$R_0 < r - \lambda\sigma^2 < r < 1,$$

and for each $q \in \mathbb{R}^{M-1}$, *there is a convex polytope* P *contained in* $r\overline{\mathbb{B}}$ *and containing* $(r - \lambda\sigma^2)\overline{\mathbb{B}}$ *in its interior and such that* π *maps* $W_0 \cap \operatorname{skel}(P)$ *onto* $U_0 \cap \operatorname{skel}(\mathcal{D}(\sigma(\Lambda + q)))$.

9. **Small blocks of convex polytopes**

Let Λ be as in (4.1), and let $E(\Lambda)$ be the fundamental parallelotope for Λ; that is,

$$E(\Lambda) = \{\theta_1 e_1 + \cdots + \theta_{M-1}e_{M-1} \colon 0 \le \theta_i < 1,\ 1 \le i \le M-1\}.$$

Given $q \in \mathbb{R}^{M-1}$, define $\mathcal{S}(q) = \operatorname{skel}(\mathcal{D}(\Lambda + q))$. Clearly $\mathcal{S}(q) = \mathcal{S}(0) + q$. Recall that all our tessellations are periodic so

$$\mathcal{S}(q) + \sum_{j=1}^{M-1} n_j e_j = \mathcal{S}(q)$$

for every $q \in \mathbb{R}^{M-1}$ and every $n_j \in Z$, $1 \le j \le M-1$. Thus, if $w \in \mathcal{S}(q_1) \cap \mathcal{S}(q_2)$, there are n_j, $1 \le j \le M-1$ such that if $w_0 = w - \sum_{j=1}^{M-1} n_j e_j \in E(\Lambda)$, then $w_0 \in E(\Lambda) \cap \mathcal{S}(q_1) \cap \mathcal{S}(q_2)$. Thus, if $\mathcal{S}(0) \cap \mathcal{S}(q_1) \cap \cdots \cap \mathcal{S}(q_{M-1}) \cap E(\Lambda) = \emptyset$, then $\mathcal{S}(0) \cap \mathcal{S}(q_1) \cap \cdots \cap \mathcal{S}(q_{M-1}) = \emptyset$.

PROPOSITION 9.1. *Given* $\varepsilon > 0$, *there are* $q_1, \ldots, q_{M-1}$, $|q_i| < \varepsilon$, $1 \le i \le M-1$, *such that* $\mathcal{S}(0) \cap \mathcal{S}(q_1) \cap \cdots \cap \mathcal{S}(q_{M-1}) = \emptyset$.

We need the following

PROPOSITION 9.2. *Let* H *be a hyperplane in* $\mathbb{R}^{M-1}$. *Let* $\tilde{H}$ *be the hyperplane in* $\mathbb{R}^{M-1}$ *parallel to* H *that passes through the origin, and assume that* $q \in \mathbb{R}^{M-1}$, $q \notin \tilde{H}$. *Let* L *be a* k*-plane in* $\mathbb{R}^{M-1}$ *where* $1 \le k \le M-2$. *Then either* $L \subset H + tq$ *for some* $t \in \mathbb{R}$ *or else* L *intersects* $H + tq$ *transversely for every* $t \in \mathbb{R}$.

Proof. Obvious. $\square$

We shall say that a k-plane L is transverse to a hyperplane G if it is not contained in G. In this case either L misses G or else L intersects G transversely (and $L \cap G$ is a $(k-1)$-plane). So the proposition says that L is transverse to the hyperplane $H + tq$ for each t except for perhaps one value of t.

Proof of Proposition 9.1. Take a large ball B centered at the origin, and consider the family of all those hyperplanes that contain a facet of a simplex $S \in \mathcal{D}(\Lambda)$ contained in B. There are finitely many of these hyperplanes. Denote them by $L_1, \ldots, L_p$ and their union by $\mathcal{L}$. For each j, $1 \leq j \leq p$, let $\tilde{L}_j$ be the hyperplane parallel to L_j passing through the origin. Choose $q \in \mathbb{R}^{M-1}$ so that q belongs to no $\tilde{L}_j$, $1 \leq j \leq p$. Let $\varepsilon > 0$. By the dicussion at the beginning of this section the proposition will be proved once we have proved that there are t_j, $\varepsilon > t_1 > \cdots > t_{M-1} > 0$ such that

$$\mathcal{L} \cap (\mathcal{L} + t_1 q) \cap \cdots \cap (\mathcal{L} + t_{M-1} q) = \emptyset,$$

and then we put $q_j = t_j q$, $1 \leq j \leq M-1$.

By Proposition 9.2, for each j, $1 \leq j \leq p$, and for each t, $0 < t < \varepsilon$, except perhaps finitely many, $L_j + tq$ is transverse to each L_k, $1 \leq k \leq p$. So there is a t_1, $0 < t_1 < \varepsilon$, that works for all L_j, $1 \leq j \leq p$, so that $\mathcal{L} \cap (\mathcal{L} + t_1 q)$ is a union of finitely many $(M-3)$-planes. Suppose that $1 \leq \ell \leq M-3$, and suppose that we have found $t_1, \ldots, t_\ell$, $\varepsilon > t_1 > t_2 > \cdots > t_\ell > 0$, such that $\mathcal{L} \cap (\mathcal{L} + t_1 q) \cap \cdots \cap (\mathcal{L} + t_\ell q)$ is a finite union of $(M-2-\ell)$-planes. Applying Proposition 9.2 we find $t_{\ell+1}$, $0 < t_{\ell+1} < t_\ell$, such that $\mathcal{L} \cap (\mathcal{L} + t_1 q) \cap \cdots \cap (\mathcal{L} + t_{\ell+1} q)$ is a finite union of $(M-3-\ell)$-planes. Thus, step-by-step we arrive at the point where $\mathcal{L} \cap (\mathcal{L} + t_1 q) \cap \cdots \cap (\mathcal{L} + t_{M-2} q)$ is a finite set of points whose intersection with $\mathcal{L} + t_{M-1} q$ with a suitable chosen t_{M-1}, $0 < t_{M-1} < t_{M-2}$ is empty. This completes the proof. □

LEMMA 9.3. *Let $q_0 = 0$, and let $q_1, \ldots, q_{M-1}$ be as in Proposition 9.1. Let*

$$\mathcal{S}_i = \operatorname{skel}\big(\mathcal{D}(\Lambda + q_i)\big) \quad (0 \leq i \leq M-1).$$

There is a $\mu > 0$ such that whenever $x_i \in \mathcal{S}_i$, $0 \leq i \leq M-1$, we have

$$|x_1 - x_0| + |x_2 - x_1| + \cdots + |x_{M-1} - x_{M-2}| \geq \mu. \tag{9.1}$$

Proof. Assume that there is no $\mu > 0$ such that (9.1) holds whenever $x_i \in \mathcal{S}_i$, $0 \leq i \leq M-1$. Then there are sequences $x_{i,n} \in \mathcal{S}_i$, $0 \leq i \leq M-1$, $n \in \mathbb{N}$ such that

$$|x_{1n} - x_{0,n}| + |x_{2n} - x_{1n}| + \cdots + |x_{M-1,n} - x_{M-2,n}| \tag{9.2}$$

tends to zero as $n \to \infty$. Notice that $\mathcal{S}_i$ are periodic, that is,

$$\mathcal{S}_i = \mathcal{S}_i + \sum_{k=1}^{M-1} m_k e_k \quad (0 \le i \le M-1)$$

whenever $m_k \in Z$, $1 \le k \le M-1$. Thus, adding for each n a suitable $\sum_{k=1}^{M-1} m_{k,n} e_k$ to all $x_{0n}, x_{1n}, \dots, x_{M-1,n}$ where $m_{k,n} \in Z, 1 \le k \le M-1$ (note that doing this, the sum (9.2) remains unchanged), we may, with no loss of generality, assume that $x_{0n} \in E(\Lambda)$ for all n. Therefore, by compactness, we may, after passing to a subsequence if necessary, assume that x_{0n} converges to some x_0. Since $\mathcal{S}_0$ is closed, $x_0 \in \mathcal{S}_0$. Since (9.2) tends to zero as $n \to \infty$, it follows that for each j, $0 \le j \le M-1$, the sequence $x_{jn} \in \mathcal{S}_j$ converges to the same limit x_0 that must be in $\mathcal{S}_j$ since $\mathcal{S}_j$ is closed. Thus, x_0 is contained in the intersection $\mathcal{S}_0 \cap \dots \cap \mathcal{S}_{M-1}$, contradicting the fact that this intersection is empty. This completes the proof. $\square$

Let q_i, $0 \le i \le M-1$ be as in Lemma 9.3. For each $\sigma > 0$, we have

$$\mathrm{skel}\big(\mathcal{D}(\sigma(\Lambda + q))\big) = \sigma \mathrm{skel}\big(\mathcal{D}(\Lambda + q)\big),$$

so by Lemma 9.3 it follows that if $\sigma > 0$ and if $x_i \in \mathrm{skel}\big(\mathcal{D}(\sigma(\Lambda + q_i))\big)$, $0 \le i \le M-1$, then

$$|x_1 - x_0| + |x_2 - x_1| + \dots + |x_{M-1} - x_{M-2}| \ge \sigma\mu.$$

LEMMA 9.4. *Let $0 < \sigma < \sigma_0$, and suppose that*

$$R_0 < r - M\sigma^2\lambda < r < 1.$$

There are convex polytopes Q_j, $0 \le j \le M-1$, such that

$$((r - M\sigma^2\lambda)\overline{\mathbb{B}} \subset \mathrm{Int}Q_0 \subset \mathrm{Int}Q_1 \subset \dots \subset Q_{M-1} \subset r\overline{\mathbb{B}}$$

such that for each j, $0 \le j \le M-1$,

$$\pi(W_0 \cap \mathrm{skel}(Q_j)) = U_0 \cap \mathrm{skel}(\mathcal{D}(\sigma(\Lambda + q_j)).$$

Thus,

$$\begin{cases} \text{if } x_j \in W_0 \cap \mathrm{skel}(Q_j) \quad (0 \le j \le M-1), \text{ then} \\ |x_1 - x_0| + \dots + |x_{M-1} - x_{M-2}| \ge \sigma\mu. \end{cases} \tag{9.3}$$

Proof. Let $0 \le j \le M-1$. By Lemma 8.5 there is a convex polytope Q_j containing $\big(r - (M-j)\sigma^2\lambda\big)\overline{\mathbb{B}}$ in its interior and contained in

$$\big(r - (M - (j+1))\sigma^2\lambda\big)\overline{\mathbb{B}}$$

such that π maps $W_0 \cap \mathrm{skel}(Q_j)$ onto $U_0 \cap \mathrm{skel}\big(\mathcal{D}(\sigma(\Lambda + q_j))\big)$. Thus, if x_j, $0 \le j \le M-1$, are as in (9.3), then $\pi(x_j) \in \mathrm{skel}\big(\mathcal{D}(\sigma(\Lambda + q_j))\big)$ $(0 \le j \le M-1)$, and hence by the discussion preceding Lemma 9.4, we have

$$|\pi(x_1) - \pi(x_0)| + \cdots + |\pi(x_{M-1} - \pi(x_{M-2})| \geq \sigma\mu$$

so

$$|x_1 - x_0| + \cdots + |x_{M-1} - x_{M-2}| \geq \sigma\mu.$$

This completes the proof. ☐

We shall call the family $\{Q_0, Q_1, \ldots, Q_{M-1}\}$ as above a *small block of convex polytopes* with boundaries contained in $r\overline{\mathbb{B}} \setminus (r - M\sigma^2\lambda)\overline{\mathbb{B}}$. More generally, if $A\colon \mathbb{R}^M \to \mathbb{R}^M$ is a rotation, that is, $A \in SO(M)$, then we will call the family $\{A(Q_0), A(Q_1), \ldots, A(Q_{M-1})\}$ also a small block of convex polytopes.

10. **Large blocks of convex polytopes**

In previous section we constructed a small block of convex polytopes; that is, given ρ, $R_0 < \rho - M\sigma^2\lambda < \rho < 1$, we constructed convex polytopes Q_j, $0 \leq j \leq M-1$, such that

$$(\rho - M\sigma^2\lambda)\overline{\mathbb{B}} \subset \operatorname{Int} Q_0 \subset Q_0 \subset \cdots \subset \operatorname{Int} Q_{M-1} \subset Q_{M-1} \subset \rho\overline{\mathbb{B}}$$

and such that (9.3) holds. An analogous statement holds if we apply a rotation A to all polytopes Q_j, $1 \leq j \leq M-1$, to get a new small block of convex polytopes $R_j = A(Q_j)$, $0 \leq j \leq M-1$, that have the property that if $x_j \in A(W_0) \cap \operatorname{skel}(R_j)$ $(0 \leq j \leq M-1)$, then

$$|x_1 - x_0| + \cdots + |x_{M-1} - x_{M-2}| \geq \sigma\mu.$$

It is perhaps appropriate to mention that different convex polytopes Q' and Q'' in the family of convex polytopes that we are constructing always have their boundaries in disjoint spherical shells so that if $Q' \subset \operatorname{Int} Q''$ and if A is a rotation, then $A(Q') \subset \operatorname{Int} Q''$.

We now choose rotations $A_1 = \mathrm{Id}, A_2, \ldots, A_L$ so that the open sets

$$W_{0j} = A_j(W_0),\ 1 \leq j \leq L,\ \text{cover } b\mathbb{B};\ \text{that is, } b\mathbb{B} \subset \bigcup_{j=1}^{L} W_{0j}. \tag{10.1}$$

We now construct what we call a *large block of convex polytopes* that will have a property analogous to (9.3) for a sequence x_j, $0 \leq j \leq M-1$ contained in any of the sets W_{0j}, $1 \leq j \leq L$. Roughly speaking, we shall take $\rho_0 < \rho_1 < \cdots < \rho_L$, and for each spherical shell $\mathcal{S}_k = \rho_k\overline{\mathbb{B}} \setminus \rho_{k-1}\overline{\mathbb{B}}$, $1 \leq k \leq L$, we shall construct a small block $\mathcal{B}_k$ of convex polytopes with boundaries contained in $\mathcal{S}_k$ that has the property (9.3) for $Q_j \in \mathcal{B}_k$, $0 \leq j \leq M-1$. Then we will rotate each $\mathcal{B}_k$ by A_k to form an L-tuple of smal blocks $A_1(\mathcal{B}_1), A_2(\mathcal{B}_2), \ldots, A_L(\mathcal{B}_L)$ and then arrange all the convex polytopes of these $A_j(\mathcal{B}_j)$ into a single sequence. Here is the exact formulation.

Lemma 10.1. *Given σ, $0 < \sigma < \sigma_0$, and r such that*

$$R_0 < r - ML\sigma^2\lambda < r < 1,$$

there is a family of convex polytopes C_j, $0 \le j \le ML-1$, such that

$$(r - ML\sigma^2\lambda)\overline{\mathbb{B}} \subset \mathrm{Int}C_0 \subset C_0 \subset \mathrm{Int}C_1 \subset \cdots \subset \mathrm{Int}C_{ML-1} \subset C_{ML-1} \subset r\overline{\mathbb{B}},$$

which has the property that if $1 \le k \le L$ and if $x_j \in W_{0k} \cap \mathrm{skel}C_j$, $0 \le j \le ML-1$, then

$$|x_1 - x_0| + |x_2 - x_1| + \cdots + |x_{ML-1} - x_{ML-2}| \ge \sigma\mu.$$

We shall call the family $\mathcal{C} = \{C_0, C_1, \ldots, C_{ML-1}\}$ as above *a large block of convex polytopes* with boundaries contained in $r\overline{\mathbb{B}} \setminus (r - ML\sigma^2\lambda)\overline{\mathbb{B}}$.

Proof. Let

$$\rho_j = r - M(L-j)\sigma^2\lambda \ \ (0 \le j \le L).$$

For each j, $1 \le j \le L$, there is a small block $\mathcal{B}_j$ of convex polytopes with boundaries contained in $\rho_j\overline{\mathbb{B}} \setminus \rho_{j-1}\overline{\mathbb{B}}$ such that (9.3) holds.

Let A_j, $1 \le j \le L$, be rotations of $\mathbb{R}^M$ satisfying (10.1). For each j, $1 \le j \le L$, form a new small block

$$\mathcal{A}_j = \{A_j(P)\colon P \in \mathcal{B}_j\} = \{C_{j0}, C_{j1}, \ldots, C_{j,M-1}\},$$

where

$$\rho_{j-1}\overline{\mathbb{B}} \subset \mathrm{Int}(C_{j0}) \subset \mathrm{Int}(C_{j1}) \subset \cdots \subset C_{j,M-1} \subset \rho_j\overline{\mathbb{B}}$$

such that if

$$x_i \in W_{0k} \cap \mathrm{skel}(C_{ki})\ , \quad 0 \le i \le M-1,$$

then

$$|x_1 - x_0| + \cdots + |x_{M-1} - x_{M-2}| \ge \sigma\mu.$$

Now, write all C_{ji} into a single sequence

$$C_{10}, C_{11}, \ldots, C_{1,M-1}, C_{20}, \ldots, C_{2,M-1}, \ldots, C_{L0}, C_{L1}, \ldots, C_{L,M-1};$$

in other words,

$$C_{(j-1)M+i} = C_{ji} \ \ (1 \le j \le L,\ 0 \le i \le M-1).$$

It is easy to see that the convex polytopes C_0, $C_1, \cdots, C_{LM-1}$ have all the required properties. This completes the proof. $\square$

11. Completion of the proof of Theorem 2.1 and the proof of Corollary 2.2

We keep the meaning of R_0 and σ_0. Recall that by (10.1) the open sets $W_{0j} = A_j(W_0)$, $1 \leq j \leq L$, cover $b\mathbb{B}$. Thus

$$\begin{cases} \text{if } x_n \in \mathbb{B} \text{ converges to } x \in b\mathbb{B}, \text{ then there are } n_0 \\ \text{and } j,\ 1 \leq j \leq L, \text{ such that } x_n \in W_{0j}\ (n \geq n_0). \end{cases} \tag{11.1}$$

To complete the proof of Theorem 2.1 we shall construct a sequence r_j, $R_0 < r_1 < \cdots < r_j < \cdots < 1$, converging to 1, and for each $j \in \mathbb{N}$, we shall construct a large block $\mathcal{C}_j = \{C_{j0}, C_{j1}, \ldots, C_{j,LM-1}\}$ of convex polytopes such that

$$r_j\overline{\mathbb{B}} \subset \operatorname{Int} C_{j0} \subset C_{j0} \subset \operatorname{Int} C_{j1} \subset \cdots \subset \operatorname{Int} C_{j,LM-1} \subset C_{j,LM-1} \subset r_{j+1}\overline{\mathbb{B}} \tag{11.2}$$

so that writing all polytopes of all large blocks into a single sequence, i.e.,

$$P_{(j-1)LM+k} = C_{jk} \quad (0 \leq k \leq LM-1, j \in \mathbb{N}), \tag{11.3}$$

we get our sequence P_n of convex polytopes with the desired properties. To do this, choose r_1, $R_0 < r_1 < 1$, and a decreasing sequence of positive numbers σ_j, $\sigma_1 < \sigma_0$, such that

$$\sum_{j=1}^{\infty} \sigma_j^2 = \frac{1-r_1}{ML\lambda} \quad \text{and such that} \quad \sum_{j=1}^{\infty} \sigma_j \quad \text{diverges,} \tag{11.4}$$

and then let $r_{j+1} = r_j + ML\sigma_j^2\lambda$ $(j \in \mathbb{N})$. Note that the equality in (11.4) means that the sequence r_j converges to 1 as $j \to \infty$.

Use Lemma 10.1 to show that for each $j \in \mathbb{N}$, there is a large block

$$\mathcal{C}_j = \{C_{j0}, C_{j1}, \ldots, C_{j,LM-1}\}$$

of convex polytopes satisfying (11.2) and having the property that

$$\begin{cases} \text{if for some } k,\ 1 \leq k \leq L,\ x_\ell \in W_{0k} \cap \operatorname{skel}(C_{j\ell}) \text{ for each } \ell,\ 0 \leq \ell \leq LM-1, \\ \text{then } |x_1 - x_0| + |x_2 - x_1| + \cdots + |x_{LM-1} - x_{LM-2}| \geq \sigma_j \mu. \end{cases} \tag{11.5}$$

Define the sequence P_n of convex polytopes by writing all polytopes C_{jk} into a single sequence as in (11.3). Obviously

$$P_0 \subset \operatorname{Int} P_1 \subset P_1 \subset \cdots \subset \mathbb{B}, \bigcup_{j=0}^{\infty} P_j = \mathbb{B}.$$

Now, let $w_n \in \operatorname{skel}(P_n)$ $(n \in \mathbb{N})$. To complete the proof of Theorem 2.1 we must show (2.2). We know that it is enough to show this for sequences w_n that converge. So assume that w_n converges. The properties of P_n imply that the limit of the sequence w_n is contained in $b\mathbb{B}$. By (11.1) there are $k, 1 \leq k \leq L$, and n_0 such that $w_n \in W_{0k}$ $(n \geq n_0)$. Let j_0 be so large that $j_0 ML \geq n_0$. By

(11.5), for each $j \geq j_0$, the large block of polytopes $\mathcal{C}_j$ adds at least $\sigma_j \mu$ to the sum of the absolute values of differences of consequtive w_j-s; that is, for each $j \geq j_0$, we have

$$|w_{(j-1)ML+1} - w_{(j-1)ML}| + \cdots + |w_{jML-1} - w_{jML-2}| \geq \sigma_j \mu.$$

It follows that for each $j \geq j_0$, there is a $N(j) < \infty$ such that

$$\sum_{i=1}^{N(j)} |w_i - w_{i-1}| \geq \sum_{k=j_0}^{j} \sigma_k \mu.$$

The fact that the series $\sum_{i=1}^{\infty} \sigma_j$ diverges implies (2.2). The proof of Theorem 2.1 is complete.

Proof of Corollary 2.2. Let $p\colon [0,1) \to \mathbb{B}$ be a path such that $|p(t)| \to 1$ as $t \to 1$ and such that for all sufficiently large $n \in \mathbb{N}$, $p([0,1))$ meets bP_n only at $\mathcal{U}_n$. Since $|p(t)| \to 1$ as $t \to 1$, it follows that $p(t)$ has to leave each P_n so there are an n_0 and a sequence t_j,

$$t_{n_0} < t_{n_0+1} < \cdots < 1, \lim_{n\to\infty} t_n = 1,$$

such that $p(t_n) \in bP_n$ for each $n \geq n_0$. Thus, by our assumption, passing to a larger n_0 if necessary, we may assume that $p(t_n) \in \mathcal{U}_n$ for each $n \geq n_0$. Thus, for each $n \geq n_0$, there is an $x_n \in \operatorname{skel}(P_n)$ such that $|x_n - p(t_n)| < \theta_n$. For $n \geq n_0$, we have $|p(t_{n+1}) - p(t_n)| \geq |x_{n+1} - x_n| - |p(t_{n+1} - x_{n+1}| - |p(t_n) - x_n| \geq |x_{n+1} - x_n| - \theta_{n+1} - \theta_n$. It follows that

$$\sum_{n=n_0}^{\infty} |p(t_{n+1}) - p(t_n)| \ \geq \sum_{n=n_0}^{\infty} |x_{n+1} - x_n| - 2\sum_{n=n_0}^{\infty} \theta_n.$$

Since, by Theorem 2.1, the series $\sum_{n=n_0}^{\infty} |x_{n+1} - x_n|$ diverges and since the series $\sum_{n=n_0}^{\infty} \theta_n$ converges, it follows that the series

$$\sum_{n=n_0}^{\infty} |p(t_{n+1}) - p(t_n)| \tag{11.6}$$

diverges. Since the sequence t_m increases, it follows that the length of $p([t_{n_0}, 1))$ is bounded from below by the sum of the series (11.6). Since this series diverges, it follows that p has infinite length. This completes the proof of Corollary 2.2. □

12. Proof of Theorem 1.1

As we know, every convex polytope $P \subset \mathbb{R}^M$ that contains the origin in its interior can be written as

$$P = \bigcap_{i=1}^{n} K(x_i, 1) = \bigcap_{i=1}^{n} \{y \in \mathbb{R}^M : \langle y|x_i\rangle \leq 1\}, \tag{12.1}$$

with $x_i \in \mathbb{R}^M \setminus \{0\}$, $1 \le i \le n$. We assume that the representation (12.1) is irreducible, so

$$bP = \bigcup_{i=1}^{n} H(x_i, 1) \cap P = \bigcup_{i=1}^{n} \{y \in \mathbb{R}^M : \langle y|x_i\rangle = 1\} \cap P,$$

and the sets $F_j = H(x_j, 1) \cap P$, $1 \le j \le n$, are precisely the facets of P. Recall that $\mathrm{skel}(P) = \bigcup_{i=1}^{n} [F_i \setminus \mathrm{ri}(F_i)]$.

PROPOSITION 12.1. *Let P be as above. Let $\theta > 0$. There is an $\eta > 0$ such that for each i, $1 \le i \le n$, the set*

$$bP \cap \{y \in \mathbb{R}^M : 1 - \eta < \langle y|x_i\rangle < 1\}$$

is contained in the θ-neighborhood of $\mathrm{skel}(P)$ *in bP.*

Proof. Assume that Proposition 12.1 does not hold so that there are i, $1 \le i \le n$, and $\theta > 0$ such that for each $\eta > 0$, there is some $y \in bP$ such that $1 - \eta < \langle y|x_i\rangle < 1$ and $\mathrm{dist}(y, \mathrm{skel}(P)) \ge \theta$. So there is a sequence $y_n \in bP$ such that $\langle y_n|x_i\rangle < 1$ $(n \in \mathbb{N})$, $\langle y_n|x_i\rangle \to 1$ as $n \to \infty$ and such that $\mathrm{dist}(y_n, \mathrm{skel}(P) \ge \theta$ for all n. By compactness we may, after passing to a subsequence if necessary, assume that y_n converges to $y_0 \in bP$. Clearly $y_0 \in H(x_i, 1)$. Since $y_0 \in bP$, it follows that y_0 belongs to the facet $F_i = P \cap H(x_i, 1)$. Since $\mathrm{dist}(y_0, \mathrm{skel}(P)) \ge \theta$, it follows that $y_0 \in \mathrm{ri}(F_i)$. On the other hand, since $y_n \in bP \setminus F_i$, it follows that $y_n \in \cup_{j=1, j\ne i} F_j$. Passing to a subsequence if necessary we may assume that there is a $j \ne i$, such that $y_n \in F_j$ for all n. Since F_j is closed, it follows that $y_0 \in F_j$. Thus y_0, a relative interior point of the facet F_i, belongs to a different facet F_j, which is impossible. This completes the proof. □

Remark. Note that if $\mathcal{U}$ is the θ-neighborhood of $\mathrm{skel}(P)$ and if η is as above, then for each j, $1 \le j \le n$, the set $\{y \in \mathbb{R}^M : \langle y|x_j\rangle \le 1 - \eta\}$ contains $\cup_{i=1, i\ne j}^{n} [F_i \setminus \mathcal{U}]$.

We now move to $\mathbb{C}^N = \mathbb{R}^{2N}$ and denote by $\langle\,|\,\rangle$ the Hermitian inner product in $\mathbb{C}^N$. Note that $\Re(\langle\,|\,\rangle)$ is then the standard inner product in $\mathbb{R}^{2N}$.

LEMMA 12.2. *Let P be a convex polytope in $\mathbb{C}^N$, and let $K \subset \mathrm{Int}(P)$ be a compact set. Let $\theta > 0$, and let $\mathcal{U} \subset bP$ be the θ-neighborhood of* $\mathrm{skel}(P)$ *in bP. Given $\varepsilon > 0$ and $L < \infty$, there is a polynomial $f : \mathbb{C}^N \to \mathbb{C}$ such that*

$$\Re(f(z)) \ge L \quad (z \in bP \setminus \mathcal{U}) \quad \textit{and} \quad |f(z)| < \varepsilon \quad (z \in K).$$

Proof. With no loss of generality assume that the origin is an interior point of P. There are $n \in \mathbb{N}$ and $w_1, w_2, \ldots, w_n \in \mathbb{C}^N \setminus \{0\}$ such that

$$P = \bigcap_{i=1}^{n} \{z \in \mathbb{C}^N : \Re(\langle z|w_i\rangle) \le 1\}, \tag{12.2}$$

where we may assume that the representation (12.2) is irreducible so that $bP = \bigcup_{i=1}^n F_i$, where $F_i = \{z \in \mathbb{C}^N : \Re(\langle z|w_i\rangle) = 1\} \cap P \quad (1 \le i \le n)$ are the facets of P.

Since P is compact, there is an $R < \infty$ such that

$$|\langle z|w_i\rangle| \le R \quad (z \in P, 1 \le i \le n). \tag{12.3}$$

By Proposition 12.1 there is an $\eta > 0$ such that for each j, $1 \le j \le n$,

$$bP \cap \{z \in \mathbb{C}^N : 1 - \eta < \Re(\langle z|w_j\rangle) < 1\} \subset \mathcal{U}.$$

Passing to a smaller η if necessary we may assume that

$$K \subset \{z \in \mathbb{C}^N : \Re(\langle z|w_j\rangle) \le 1 - \eta\} \ \text{ for each } \ j,\ 1 \le j \le n. \tag{12.4}$$

By the remark following Proposition 12.1, for each j, $1 \le j \le n$, we have

$$\bigcup_{i=1, i\ne j}^{n} [F_i \setminus \mathcal{U}] \subset \{z \in \mathbb{C}^N : \Re(\langle z|w_j\rangle) \le 1 - \eta\}. \tag{12.5}$$

Let $\varepsilon > 0$ and $L < \infty$. By the Runge theorem there is a polynomial $\Phi \colon \mathbb{C} \to \mathbb{C}$ such that

$$|\Phi(\zeta) - (L + \varepsilon)| < \varepsilon/n \quad (\zeta \in R\overline{\Delta},\ \Re(\zeta) \ge 1), \tag{12.6}$$

$$|\Phi(\zeta)| < \varepsilon/n \quad (\zeta \in R\overline{\Delta},\ \Re(\zeta) \le 1 - \eta). \tag{12.7}$$

For each j, $1 \le j \le n$, consider the polynomial $f_j(z) = \Phi(\langle z|w_j\rangle)$. By (12.4),

$$|f_j(z)| < \varepsilon/n \quad (z \in K), \tag{12.8}$$

and by (12.5) and (12.7),

$$|f_j(z)| < \varepsilon/n \quad \Big(z \in \bigcup_{i=1, i\ne j}^{n} F_i \setminus \mathcal{U}\Big). \tag{12.9}$$

Further, if $z \in F_j$, then $\Re(\langle z|w_j\rangle) = 1$, and so by (12.6),

$$|f_j(z) - (L + \varepsilon)| < \varepsilon/n \quad (z \in F_j). \tag{12.10}$$

Now, let $f = \sum_{j=1}^n f_j$. If $1 \le j \le n$ and if $z \in F_j \setminus \mathcal{U}$, then by (12.9) and (12.10),

$$|f(z) - (L + \varepsilon| \le |f_j(z) - (L + \varepsilon)| + \Big|\sum_{i=1, i\ne j}^{n} f_i(z)\Big| \le \varepsilon/n + (n-1)\varepsilon/n = \varepsilon,$$

which implies that

$$\Re(f(z) \ge L \ (z \in F_j \setminus \mathcal{U}, 1 \le j \le n)$$

so $\Re(f(z)) \ge L \quad (z \in bP \setminus \mathcal{U})$. Finally, by (12.8), $|f(z)| < \varepsilon \quad (z \in K)$. This completes the proof. $\square$

Proof of Theorem 1.1. Let P_n be the sequence of convex polytopes from Theorem 2.1, and let θ_n be a decreasing sequence of positive numbers such that $\sum_{n=1}^{\infty}\theta_n < \infty$. For each n, let $\mathcal{U}_n \subset bP_n$ be the θ_n-neighborhood of skel(P_n) in bP_n. The theorem will be proved once we have constructed a holomorphic function f on $\mathbb{B}_N$ such that

$$\Re(f(z)) \geq n \quad (z \in bP_n \setminus \mathcal{U}_n,\ n \in \mathbb{N}). \tag{12.11}$$

To see this, let f satisfy (12.11) and suppose that $p\colon [0,1) \to \mathbb{B}_N$ is a path such that $\lim_{t\to 1}|p(t)| = 1$. Suppose that f is bounded on $p([0,1))$. By (12.11) there is some n_0 such that for each $n \geq n_0$, $p([0,1))$ meets bP_n only at $\mathcal{U}_n$. By Corollary 2.2 it follows that p has infinite length.

We shall construct a sequence f_n of polynomials from $\mathbb{C}^N$ to $\mathbb{C}$ such that for each $n \in \mathbb{N}$,

(i) $\Re(f_n(z)) \geq n+1$ on $bP_n \setminus \mathcal{U}_n$,
(ii) $|f_{n+1}(z) - f_n(z)| \leq 1/2^{n+1}$ on P_n.

Suppose that we have done this. By (ii) the sequence converges uniformly on compacta in $\mathbb{B}_N$ so the limit f is holomorphic on $\mathbb{B}_N$. If $z \in bP_n \setminus \mathcal{U}_n$, then we have

$$f(z) = f_n(z) + \sum_{j=n}^{\infty}[f_{j+1}(z) - f_j(z)].$$

So by (ii), $|f(z)-f_n(z)|<1$ on $bP_n\setminus\mathcal{U}_n$, and therefore $\Re(f(z))\geq\Re(f_n(z))-1\geq n$ on $bP_n \setminus \mathcal{U}_n$ so that f satisfies (12.11).

We construct f_n by induction. Clearly there is a polynomial f_1 that satisfies (i) for $n = 1$. Suppose that for some $m \in \mathbb{N}$ we have constructed a polynomial f_m that satisfies

$$\Re(f_m(z)) \geq m+1 \quad \text{on} \quad bP_m \setminus \mathcal{U}_m.$$

Choose $T < \infty$ so large that

$$\Re(f_m(z)) + T \geq m+2 \quad \text{on} \quad bP_{m+1}. \tag{12.12}$$

By Lemma 12.2 there is a polynomial g such that

$$\Re(g(z)) \geq T \quad \text{on} \quad bP_{m+1} \setminus \mathcal{U}_{m+1} \tag{12.13}$$

and

$$|g(z)| \leq (1/2)^{m+1} \quad \text{on} \quad P_m. \tag{12.14}$$

Put $f_{m+1} = f_m + g$. By (12.13), we have

$$\Re(f_{m+1}) = \Re(f_m+g) = \Re(f_m)+\Re(g) \geq \Re(f_m)+T \geq m+2 \text{ on } bP_{m+1}\setminus\mathcal{U}_{m+1},$$

and by (12.14), we have $|f_{m+1}-f_m|<(1/2)^{m+1}$ on P_m. Theorem 1.1 is proved. $\square$

13. Concluding remarks

We have proved Theorem 2.1 in $\mathbb{R}^M$ with $M \geq 3$. Theorem 2.1 holds also in $\mathbb{R}^2$ where the proof is much simpler. One can use a sequence of pairs of regular polygons.

Having in mind the length of the proof of Theorem 2.1 one could say that the principal result of the present paper is Theorem 2.1. It belongs to convex geometry and is not related to complex analysis. In its complex analysis consequence, Theorem 1.1, the *real part* of the holomorphic function f is unbounded on every path of finite length in $\mathbb{B}_N$ that ends on $b\mathbb{B}_N$. Notice that by the maximum principle the zero sets of (real) pluriharmonic functions on $\mathbb{B}_N, N \geq 2$, have no compact components. Applying Sard's theorem to the real part of the function f obtained in Theorem 1.1 we get

THEOREM 13.1. *Given $N \geq 2$, there is a complete, closed, real hypersurface of $\mathbb{B}_N$ that is the zero set of a (real) pluriharmonic function on $\mathbb{B}_N$.*

In the special case when $k = 1$ and $N = 2$, our Corollary 1.2 provides the existence of a complete, properly embedded complex curve in $\mathbb{B}_2$. The existence of such a curve also follows from a recent paper of Alarcón and López [AL]. Their proof is completely different from the one presented here. However, neither of the proofs provides any information about the topology of the curve so the following question remains open:

Question 13.1. *Does there exist a complete proper holomorphic embedding $f\colon \Delta \to \mathbb{B}_2$?*

Knowing now that for each $N \geq 2$ there are complete closed complex hypersurfaces in $\mathbb{B}^N$, one may also ask

Question 13.2. *Given $N \geq 2$, does there exist a complete proper holomorphic embedding $f\colon \mathbb{B}_N \to \mathbb{B}_{N+1}$?*

Acknowledgements. The author is grateful to David Eppstein and John M. Sullivan for helpful suggestions. He is also grateful to Tomaž Pisanski for his interest. This work was supported by the Research Program P1-0291 from ARRS, Republic of Slovenia.

References

[AF13] A. ALARCÓN and F. FORSTNERIČ, Every bordered Riemann surface is a complete proper curve in a ball, *Math. Ann.* **357** (2013), 1049–1070. MR 3118624. Zbl 1288.32014. http://dx.doi.org/10.1007/s00208-013-0931-4.

[AL13] A. Alarcón and F. J. López, Null curves in $\mathbb{C}^3$ and Calabi-Yau conjectures, *Math. Ann.* **355** (2013), 429–455. MR 3010135. Zbl 1269.53061. http://dx.doi.org/10.1007/s00208-012-0790-4.

[AL] A. Alarcón and F. J. López, Complete bounded complex curves in $\mathbb{C}^2$, to appear in *J. Europ. Math. Soc.* arXiv 1305.2118v2.

[Brø83] A. Brøndsted, *An Introduction to Convex Polytopes*, *Grad. Texts in Math.* **90**, Springer-Verlag, New York, 1983. MR 0683612. Zbl 0509.52001.

[CS88] J. H. Conway and N. A. Sloane, *Sphere Packings, Lattices and Groups*, *Grundl. Math. Wissen.* **290**, Springer-Verlag, New York, 1988. MR 0920369. Zbl 0634.52002. http://dx.doi.org/10.1007/978-1-4757-2016-7.

[GS82] J. Globevnik and E. L. Stout, Holomorphic functions with highly noncontinuable boundary behavior, *J. Analyse Math.* **41** (1982), 211–216. MR 0687952. Zbl 0564.32009. http://dx.doi.org/10.1007/BF02803401.

[Jon79] P. W. Jones, A complete bounded complex submanifold of $\mathbf{C}^3$, *Proc. Amer. Math. Soc.* **76** (1979), 305–306. MR 0537094. Zbl 0418.32006. http://dx.doi.org/10.2307/2043009.

[MUY09] F. Martin, M. Umehara, and K. Yamada, Complete bounded holomorphic curves immersed in $\mathbb{C}^2$ with arbitrary genus, *Proc. Amer. Math. Soc.* **137** (2009), 3437–3450. MR 2515413. Zbl 1177.53056. http://dx.doi.org/10.1090/S0002-9939-09-09953-5.

[Yan77a] P. Yang, Curvature of complex submanifolds of C^n, in *Several Complex Variables* (Proc. Sympos. Pure Math., Vol. XXX, Part 2, Williams Coll., Williamstown, Mass., 1975), Amer. Math. Soc., Providence, R.I., 1977, pp. 135–137. MR 0450606. Zbl 0409.53043.

[Yan77b] P. Yang, Curvatures of complex submanifolds of $\mathbf{C}^n$, *J. Differential Geom.* **12** (1977), 499–511 (1978). MR 0512921. Zbl 0355.53035. Available at http://projecteuclid.org/euclid.jdg/1214434221.

(Received: January 9, 2014)
(Revised: May 12, 2015)

Institute of Mathematics, Physics and Mechanics, Ljubljana, Slovenia
E-mail: josip.globevnik@fmf.uni-lj.si