

On Schmidt and Summerer parametric geometry of numbers

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Abstract

Recently, W. M. Schmidt and L. Summerer introduced a new theory that allowed them to recover the main known inequalities relating the usual exponents of Diophantine approximation to a point in \mathbb{R}^n and to discover new ones. They first note that these exponents can be computed in terms of the successive minima of a parametric family of convex bodies attached to the given point. Then they prove that the n -tuple of these successive minima can in turn be approximated up to bounded difference by a function from a certain class. In this paper, we show that the same is true within a smaller and simpler class of functions, which we call rigid systems. We also show that conversely, given a rigid system, there exists a point in \mathbb{R}^n whose associated family of convex bodies has successive minima that approximate that rigid system up to bounded difference. As a consequence, the problem of describing the joint spectrum of a family of exponents of Diophantine approximation is reduced to combinatorial analysis.

1. Introduction

In two recent outstanding papers [17] and [18], W. M. Schmidt and L. Summerer study the joint behavior of the n successive minima of certain one parameter families of convex bodies in \mathbb{R}^n , as a function of the parameter. Then, they show how their results can be used to recover important inequalities relating standard exponents of Diophantine approximation attached to points in \mathbb{R}^n , and they find new ones. This general direction of research was already suggested by Schmidt in his 1982 survey of open problems in Diophantine approximation [15], and a conjecture that he made in this context was recently solved by N. G. Moshchevitin [11]. The goal of the present paper is to simplify and to complete some aspects of the new theory of Schmidt and Summerer. It also aims at promoting their wonderful idea which, the author is convinced, will have a major impact in Diophantine approximation as it provides a new simple way of thinking about problems of simultaneous approximation.

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Let $n \geq 2$ be an integer, let $\mathbf{x} \cdot \mathbf{y}$ denote the standard scalar product of vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, and let $\|\mathbf{x}\| = (\mathbf{x} \cdot \mathbf{x})^{1/2}$ denote the Euclidean norm of \mathbf{x} . Up to re-scaling, the families of convex bodies considered by Schmidt and Summerer are dual to

$$\mathcal{C}_{\mathbf{u}}(e^q) := \{\mathbf{x} \in \mathbb{R}^n; \|\mathbf{x}\| \leq 1, |\mathbf{x} \cdot \mathbf{u}| \leq e^{-q}\} \quad (q \geq 0),$$

where \mathbf{u} is a given unit vector in \mathbb{R}^n . However, working with these alternative families brings little difference and, for our purpose, is more convenient. Thus, all the results that we present here can easily be translated to the setting of [18]. For each $j = 1, \dots, n$ and each $q \geq 0$, we denote by $L_{\mathbf{u},j}(q)$ the smallest real number $L \geq 0$ such that $e^L \mathcal{C}_{\mathbf{u}}(e^q)$ contains at least j linearly independent points of \mathbb{Z}^n . Then, we group these successive minima into a single map $\mathbf{L}_{\mathbf{u}}: [0, \infty) \rightarrow \mathbb{R}^n$ by setting

$$\mathbf{L}_{\mathbf{u}}(q) = (L_{\mathbf{u},1}(q), \dots, L_{\mathbf{u},n}(q)) \quad (q \geq 0).$$

Schmidt and Summerer establish many properties of this map. We will recall them in the next section. Here we simply mention that each component $L_{\mathbf{u},j}: [0, \infty) \rightarrow \mathbb{R}$ is continuous and piecewise linear with slopes 0 and 1, which means that its right derivative is 0 or 1 at each point $q \geq 0$, and the same for its left derivative at each point $q > 0$. Equivalently, the graph of $L_{\mathbf{u},j}$ is a connected polygon composed of line segments of slopes 0 and 1. We also have $0 \leq L_{\mathbf{u},1}(q) \leq \dots \leq L_{\mathbf{u},n}(q)$ for each $q \geq 0$, which means that $\mathbf{L}_{\mathbf{u}}$ takes values in the set

$$\Delta_n = \{(x_1, \dots, x_n) \in \mathbb{R}^n; x_1 \leq \dots \leq x_n\}$$

of monotone increasing n -tuples of real numbers.

Schmidt and Summerer show that each function $\mathbf{L}_{\mathbf{u}}$ can be approximated up to bounded difference by functions from a certain class (see the next section). One of our main results is that the same property holds within a simpler class. To describe it, we follow [18, §3] and define the *combined graph* of a set of real valued functions on an interval I to be the union of their graphs in $I \times \mathbb{R}$. For a function $\mathbf{P} = (P_1, \dots, P_n): [c, \infty) \rightarrow \Delta_n$, and a subinterval I of $[c, \infty)$, we define the *combined graph of \mathbf{P} above I* to be the combined graph of its components P_1, \dots, P_n restricted to I . If \mathbf{P} is continuous and if the real numbers $q \geq c$ at which $P_1(q), \dots, P_n(q)$ are not all distinct form a discrete subset of $[c, \infty)$, then the map \mathbf{P} is uniquely determined by its combined graph over the full interval $[c, \infty)$. We also denote by $\Phi_n: \mathbb{R}^n \rightarrow \Delta_n$ the continuous map that lists the coordinates of a point in monotone increasing order. We can now introduce our basic combinatorial object.

Definition 1.1. Let $\delta \in (0, \infty)$, and let $s \in \mathbb{N}^* \cup \{\infty\} = \{\infty, 1, 2, 3, \dots\}$. A *canvas with mesh δ and cardinality s* in \mathbb{R}^n is a triple consisting of a sequence of

points $(\mathbf{a}^{(i)})_{0 \leq i < s}$ in Δ_n together with two sequences of integers $(k_i)_{0 \leq i < s}$ and $(\ell_i)_{0 \leq i < s}$ of the same cardinality s such that, for each index i with $0 \leq i < s$,

- (C1) the coordinates $(a_1^{(i)}, \dots, a_n^{(i)})$ of $\mathbf{a}^{(i)}$ form a strictly increasing sequence of positive integer multiples of δ ;
- (C2) we have $1 \leq k_0 \leq \ell_0 = n$ and $1 \leq k_i < \ell_i \leq n$ if $i \geq 1$;
- (C3) if $i + 1 < s$, then $k_i \leq \ell_{i+1}$, $a_{\ell_{i+1}}^{(i)} + \delta \leq a_{\ell_{i+1}}^{(i+1)}$ and

$$(a_1^{(i)}, \dots, \widehat{a_{k_i}^{(i)}}, \dots, a_n^{(i)}) = (a_1^{(i+1)}, \dots, \widehat{a_{\ell_{i+1}}^{(i+1)}}, \dots, a_n^{(i+1)}),$$

where the hat on a coordinate means that it is omitted.

Thus, in such a sequence $(\mathbf{a}^{(i)})_{0 \leq i < s}$, each point $\mathbf{a}^{(i+1)}$ with $i + 1 < s$ is obtained from the preceding point $\mathbf{a}^{(i)}$ by replacing one of its coordinates by a larger multiple of δ , different from all other coordinates of $\mathbf{a}^{(i)}$, and then by re-ordering the new n -tuple. In particular, this sequence uniquely determines the sequence $(k_i)_{0 \leq i < s}$ up to its last term k_{s-1} if $s < \infty$, and the full sequence $(\ell_i)_{0 \leq i < s}$ since $\ell_0 = n$. When s is finite, we also define $\ell_s = n$ and

$$\mathbf{a}^{(s)} = (a_1^{(s)}, \dots, a_{n-1}^{(s)}, \infty) \in \mathbb{R}^{n-1} \times \{\infty\}$$

so that (C3) holds for $i = s - 1$.

Definition 1.2. To each canvas of mesh $\delta > 0$ as in Definition 1.1, we associate the function $\mathbf{P}: [q_0, \infty) \rightarrow \Delta_n$ given by

$$\mathbf{P}(q) = \Phi_n(a_1^{(i)}, \dots, \widehat{a_{k_i}^{(i)}}, \dots, a_n^{(i)}, a_{k_i}^{(i)} + q - q_i) \quad (0 \leq i < s, q_i \leq q < q_{i+1}),$$

where $q_i = a_1^{(i)} + \dots + a_n^{(i)}$ ($0 \leq i < s$) and $q_s = \infty$ if $s < \infty$. We say that such a function is a *rigid n -system with mesh δ* and that $(q_i)_{0 \leq i < s}$ is its sequence of *switch numbers*.

Since $a_{k_i}^{(i)} + q_{i+1} - q_i = a_{\ell_{i+1}}^{(i+1)}$ when $i + 1 < s$, such a map \mathbf{P} is continuous. Its combined graph over an interval $[q_i, q_{i+1})$ with $0 \leq i < s$ consists of $n - 1$ horizontal half-open line segments and one half-open line segment of slope 1. Their left end-points are the points $(q_i, a_j^{(i)})$ ($1 \leq j \leq n$) and, if $i + 1 < s$, their right end-points are $(q_{i+1}, a_j^{(i+1)})$ ($1 \leq j \leq n$). In this context, the condition (C2), imposed on the canvas, translates into the fact that, for each index i with $1 \leq i < s$, the straight line extending the line segment of slope 1 over $[q_i, q_{i+1})$ lies to the right of the straight line that extends the line segment of slope 1 over the preceding interval $[q_{i-1}, q_i)$. Figure 1 illustrates this by showing the combined graph of a rigid 5-system with mesh 1 attached to the canvas $\{(1, 2, 4, 5, 8), (1, 2, 4, 7, 8), (1, 4, 5, 7, 8)\}$ of cardinality $s = 3$ with $k_2 = 1$.

The main goal of this paper is to prove the following result.

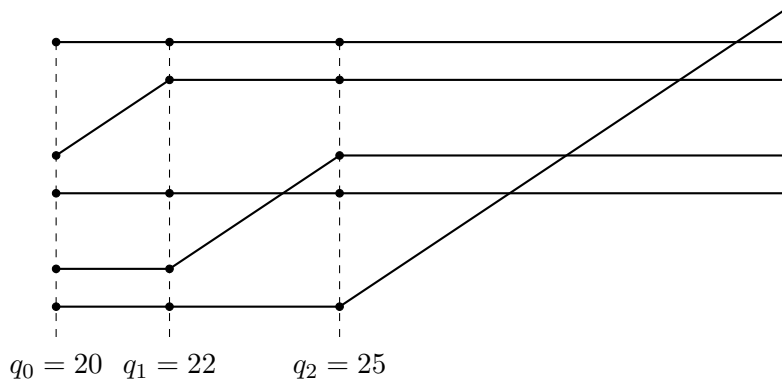


Figure 1. The combined graph of a rigid 5-system.

THEOREM 1.3. *Let $n \geq 2$ be an integer, and let $\delta \in (0, \infty)$. For each unit vector \mathbf{u} of \mathbb{R}^n , there exists a rigid system $\mathbf{P}: [q_0, \infty) \rightarrow \Delta_n$ with mesh δ such that $\mathbf{L}_{\mathbf{u}} - \mathbf{P}$ is bounded on $[q_0, \infty)$. Conversely, for each rigid system $\mathbf{P}: [q_0, \infty) \rightarrow \Delta_n$ with mesh δ , there exists a unit vector \mathbf{u} in \mathbb{R}^n such that $\mathbf{L}_{\mathbf{u}} - \mathbf{P}$ is bounded on $[q_0, \infty)$.*

To mention but one application, recall that, to each unit vector \mathbf{u} in \mathbb{R}^n , one attaches several exponents of Diophantine approximation that measure how well \mathbf{u} can be approximated by rational vector subspaces of \mathbb{R}^n of a given dimension d (see [1], [8], [16]). For $d = n - 1$, one uses $\tau(\mathbf{u})$, respectively $\hat{\tau}(\mathbf{u})$, defined as the supremum of all real numbers $\tau > 0$ for which the system of inequalities

$$\|\mathbf{x}\| \leq X \quad \text{and} \quad |\mathbf{x} \cdot \mathbf{u}| \leq X^{-\tau}$$

admits a nonzero solution \mathbf{x} in \mathbb{Z}^n for arbitrarily large values of X , respectively for all sufficiently large values of X . For dimension $d = 1$, one uses the dual exponents $\lambda(\mathbf{u})$, respectively $\hat{\lambda}(\mathbf{u})$, defined as the supremum of all $\lambda > 0$ such that

$$\|\mathbf{x}\| \leq X \quad \text{and} \quad \|\mathbf{x} \wedge \mathbf{u}\| \leq X^{-\lambda}$$

admits a nonzero solution \mathbf{x} in \mathbb{Z}^n for arbitrarily large values of X , respectively for all sufficiently large values of X , where $\|\mathbf{x} \wedge \mathbf{u}\|$ represents the norm of $\mathbf{x} \wedge \mathbf{u}$ for the Euclidean structure of $\wedge^2 \mathbb{R}^n$ inherited from \mathbb{R}^n (see the next section). With this notation, Theorem 1.3 admits the following consequence.

COROLLARY 1.4. *Let $n \in \mathbb{N}$ with $n \geq 2$, and let $\delta > 0$. The map $\theta: \mathbb{R}^4 \rightarrow \mathbb{R}^4$ given by*

$$\theta(\tau, \hat{\tau}, \hat{\lambda}, \lambda) = \left(\frac{1}{\tau + 1}, \frac{1}{\hat{\tau} + 1}, \frac{\hat{\lambda}}{\hat{\lambda} + 1}, \frac{\lambda}{\lambda + 1} \right)$$

establishes a bijection between the set of quadruples $(\tau(\mathbf{u}), \hat{\tau}(\mathbf{u}), \hat{\lambda}(\mathbf{u}), \lambda(\mathbf{u}))$ where \mathbf{u} runs through all unit vectors of \mathbb{R}^n with \mathbb{Q} -linearly independent coordinates, and the set of quadruples

$$(1.1) \quad \left(\liminf_{q \rightarrow \infty} \frac{P_1(q)}{q}, \limsup_{q \rightarrow \infty} \frac{P_1(q)}{q}, \liminf_{q \rightarrow \infty} \frac{P_n(q)}{q}, \limsup_{q \rightarrow \infty} \frac{P_n(q)}{q} \right),$$

where $\mathbf{P} = (P_1, \dots, P_n)$ runs through all rigid n -systems with mesh δ for which P_1 is unbounded.

The proof is clear based on the general philosophy developed by Schmidt and Summerer in their above-mentioned papers. Namely, if a unit vector \mathbf{u} and a rigid n -system $\mathbf{P} = (P_1, \dots, P_n)$ with mesh δ are such that the difference $\mathbf{L}_{\mathbf{u}} - \mathbf{P}$ is bounded, then

$$\liminf_{q \rightarrow \infty} \frac{P_1(q)}{q} = \liminf_{q \rightarrow \infty} \frac{L_{\mathbf{u},1}(q)}{q} = \frac{1}{\tau(\mathbf{u}) + 1},$$

with similar equalities for the three other components of the quadruple (1.1). Moreover, in that case, \mathbf{u} has \mathbb{Q} -linearly independent coordinates if and only if P_1 is unbounded.

The results of Schmidt and Summerer in [18] imply that θ maps injectively the quadruples $(\tau(\mathbf{u}), \hat{\tau}(\mathbf{u}), \hat{\lambda}(\mathbf{u}), \lambda(\mathbf{u}))$ to those of the form (1.1) where \mathbf{P} runs through the larger set of what they call *proper (n, γ) -systems* (see Section 2.5 below) for an appropriate $\gamma > 0$. Based on this, they recover the celebrated Khintchine’s and Jarník’s transference principles [4], [5], [6] as well as more recent results of Bugeaud, Laurent and Moshchevitin from [1], [7], [8], [12]. They also prove new results [17], [18], [19]. Recently, Laurent gave in [7] a complete description of the joint spectrum of $(\tau, \hat{\tau}, \hat{\lambda}, \lambda)$ in dimension $n = 3$. For larger dimension $n \geq 4$, the problem is open and the above corollary reduces it to combinatorial analysis.

In a similar way, Corollary 1.4 could be extended to deal with all the intermediate Diophantine exponents introduced by Laurent in [8] on the basis of earlier work of Schmidt [16]. This is done and used in [14] to compute the joint spectrum of these exponents, thereby answering a question of Laurent from [8]. Theorem 1.3 also solves a conjecture of Schmidt and Summerer in [18, §4] about another related family of Diophantine exponents. Note however that the present study does not apply to the more general exponents introduced by German in [2] (see also [16]).

In the next section, we recast in our setting the result of Schmidt and Summerer that approximates the maps $\mathbf{L}_{\mathbf{u}}$ by (n, γ) -systems. We also present there the intermediate results of geometry of number that are involved in the proof. In Section 3, we use the same results to analyze the situation where a rigid system with large mesh is, in comparison, well approximated by the

map $\mathbf{L}_{\mathbf{u}}$ for some unit vector \mathbf{u} . The results that we obtain there complement Theorem 1.3. They also motivate the constructions of Section 5 by which we prove the second assertion of Theorem 1.3 for rigid systems with sufficiently large mesh; cf. [13] for a special case of this construction. The last piece of the puzzle is provided by Sections 6 and 7, which construct an approximation to an arbitrary (n, γ) -system by a rigid system with given sufficiently large mesh. The process is first to transform the (n, γ) -system into what we call a *reduced system* (Section 6), then to approximate the resulting system by a step function and finally to construct a canvas out of this data (Section 7). The proof of our main Theorem 1.3 follows in Section 8, in quantitative form. The reader may go there directly to get a precise idea on the role of all intermediate results.

2. The theory of Schmidt and Summerer

We start by recalling the basic notions and results from geometry of numbers that we will need throughout this paper. In few places, we provide a short argument in order to be able to specify the constants involved. We also present the central result of Schmidt and Summerer theory and its proof, which we adapt to our slightly different but equivalent (dual) context. We hope that this will help the reader firstly because this makes our account relatively self contained, and secondly because, in the next section, we use the same notions and intermediate results to gather information about the inverse problem raised by this theory. All results stated below are thus either classical or due to Schmidt and Summerer.

Let V be a real Euclidean vector space of finite dimension $N \geq 1$. We use the following standard terminology (see [3]). By a *convex body* of V , we mean a compact convex neighborhood of 0, stable under multiplication by -1 . By a *lattice* Λ of V , we mean a discrete subgroup of V of rank N . Its *co-volume* is the volume of the parallelepiped spanned by a basis of Λ or, equivalently, the volume of V/Λ .

Suppose that \mathcal{C} is a convex body of V and Λ is a lattice of V . For each $j = 1, \dots, N$, we define the *j -th minimum of \mathcal{C} with respect to Λ* , denoted $\lambda_j(\mathcal{C})$, to be the smallest real number $\lambda > 0$ such that $\lambda\mathcal{C}$ contains at least j elements of Λ that are linearly independent over \mathbb{R} . Although this notation does not refer to the lattice Λ , this should not cause any ambiguity since, in all situations that we consider, the underlying lattice will be clear from the context. For each $\mathbf{x} \in V$, we further define $\lambda_{\mathbf{x}}(\mathcal{C})$, also denoted $\lambda(\mathbf{x}, \mathcal{C})$, to be the smallest real number $\lambda \geq 0$ such that $\mathbf{x} \in \lambda\mathcal{C}$. Then, there exist elements $\mathbf{x}_1, \dots, \mathbf{x}_N$ of Λ that are linearly independent over \mathbb{R} and satisfy $\lambda(\mathbf{x}_j, \mathcal{C}) = \lambda_j(\mathcal{C})$ for $j = 1, \dots, N$. The function from V to \mathbb{R} mapping a point $\mathbf{x} \in V$ to $\lambda_{\mathbf{x}}(\mathcal{C})$ is called the *distance function* of \mathcal{C} . The notation $\lambda_{\mathbf{x}}(\mathcal{C})$ with the point \mathbf{x} in index

stresses the fact that, in the theory of Schmidt and Summerer, the convex body \mathcal{C} is varying.

2.1. *A general family of convex bodies.* Let V be as above, and let Λ be a lattice of V with co-volume 1. We choose a decomposition of V into an orthogonal sum $V = U \perp W$ of two vector subspaces U and W with $W \neq 0$, and we put $K = \dim_{\mathbb{R}}(W)$. Motivated by [18], we consider the family of convex bodies of V given by

$$\mathcal{C}(Q) := \{\mathbf{x} \in V; \|\mathbf{x}\| \leq 1 \text{ and } \|\text{proj}_W(\mathbf{x})\| \leq Q^{-1}\} \quad (Q \geq 1),$$

where proj_W stands for the orthogonal projection on W . For each $j = 1, \dots, N$, we define a function $L_j: [0, \infty) \rightarrow \mathbb{R}$ by

$$L_j(q) = \log \lambda_j(\mathcal{C}(e^q)) \quad (q \geq 0).$$

Clearly, we have $L_1(q) \leq \dots \leq L_N(q)$ for each $q \geq 0$, and so we get a map $\mathbf{L}: [0, \infty) \rightarrow \Delta_N$ by putting

$$\mathbf{L}(q) = (L_1(q), \dots, L_N(q)) \quad (q \geq 0).$$

Then, Minkowski's second convex body theorem yields the following result.

LEMMA 2.1. *We have $|L_1(q) + \dots + L_N(q) - Kq| \leq N \log(N)$ for each $q \geq 0$.*

Proof. Let $Q \geq 1$. According to Minkowski's second convex body theorem, we have

$$2^N/N! \leq \lambda_1(\mathcal{C}(Q)) \cdots \lambda_N(\mathcal{C}(Q)) \text{vol}(\mathcal{C}(Q)) \leq 2^N.$$

To estimate the volume of $\mathcal{C}(Q)$, we choose an orthonormal basis $(\mathbf{e}_1, \dots, \mathbf{e}_K)$ of W , extend it to an orthonormal basis $(\mathbf{e}_1, \dots, \mathbf{e}_N)$ of V , and form the parallelepiped P given by

$$P = \left\{ \mathbf{x} \in V; \max_{1 \leq j \leq K} |\mathbf{x} \cdot \mathbf{e}_j| \leq Q^{-1} \text{ and } \max_{K < j \leq N} |\mathbf{x} \cdot \mathbf{e}_j| \leq 1 \right\}.$$

Since $N^{-1}P \subset \mathcal{C}(Q) \subset P$, we obtain $(2/N)^N Q^{-K} \leq \text{vol}(\mathcal{C}(Q)) \leq 2^N Q^{-K}$. Thus

$$(N!)^{-1} \leq \lambda_1(\mathcal{C}(Q)) \cdots \lambda_N(\mathcal{C}(Q)) Q^{-K} \leq N^N,$$

and the conclusion follows by taking logarithms. □

2.2. *Trajectories of points and combined graphs.* Let the notation be as in Section 2.1. For each $\mathbf{x} \in V$ and each $Q \geq 1$, we find

$$\lambda_{\mathbf{x}}(\mathcal{C}(Q)) = \lambda(\mathbf{x}, \mathcal{C}(Q)) = \max\{\|\mathbf{x}\|, Q\|\text{proj}_W(\mathbf{x})\|\}.$$

When $\mathbf{x} \neq 0$, this number is positive, and so we obtain a function $L_{\mathbf{x}}: [0, \infty) \rightarrow \mathbb{R}$ by putting

$$(2.1) \quad L_{\mathbf{x}}(q) := \log \lambda_{\mathbf{x}}(\mathcal{C}(e^q)) = \max\{\log \|\mathbf{x}\|, q + \log \|\text{proj}_W(\mathbf{x})\|\} \quad (q \geq 0).$$

It is continuous and piecewise linear. If $\text{proj}_W(\mathbf{x}) = 0$, it is constant equal to $\log \|\mathbf{x}\|$. Otherwise, it has slope 0 and then 1. We also note that, if \mathbf{x} and \mathbf{y} are linearly dependent nonzero elements of V , then $L_{\mathbf{x}}$ and $L_{\mathbf{y}}$ differ by a constant. In particular, they have the same derivative $L'_{\mathbf{x}}(q) = L'_{\mathbf{y}}(q)$ at each point $q > 0$ at which they are differentiable.

For a fixed nonzero \mathbf{x} in V , the function $L_{\mathbf{x}}$ describes the position of \mathbf{x} with respect to the varying family of convex bodies $\mathcal{C}(e^q)$. For this reason, we call its graph the *trajectory* of \mathbf{x} . Explicitly, this is the set $\{(q, L_{\mathbf{x}}(q)); 0 \leq q\}$. The inclusion

$$\{(q, L_j(q)); 0 \leq q, 1 \leq j \leq N\} \subseteq \{(q, L_{\mathbf{x}}(q)); 0 \leq q, \mathbf{x} \in \Lambda \setminus \{0\}\}$$

may thus be expressed by saying that the *combined graph* of L_1, \dots, L_N is contained in the combined graph of the functions $L_{\mathbf{x}}$ with $\mathbf{x} \in \Lambda \setminus \{0\}$, namely, the union of the trajectories of these points. The goal is, in a sense, to compare these two sets. Clearly, we have

$$L_1(q) = \inf\{L_{\mathbf{x}}(q); \mathbf{x} \in \Lambda \setminus \{0\}\} \quad (q \geq 0).$$

The next crucial result, due to Schmidt and Summerer, goes a step further.

LEMMA 2.2. *The functions L_1, \dots, L_N are continuous and piecewise linear with slopes 0 and 1. At each point $q > 0$ where L_1 changes slope from 1 to 0, we have $L_1(q) = L_2(q)$.*

Proof. Since $\mathcal{C}(e^{q_1}) \supseteq \mathcal{C}(e^{q_2}) \supseteq e^{q_1 - q_2} \mathcal{C}(e^{q_1})$ for each choice of $q_2 \geq q_1 \geq 0$, we have

$$L_j(q_1) \leq L_j(q_2) \leq L_j(q_1) + (q_2 - q_1) \quad (1 \leq j \leq N, 0 \leq q_1 \leq q_2).$$

Thus L_1, \dots, L_N are continuous. The inequality $\log \|\mathbf{x}\| \leq \lambda_{\mathbf{x}}(q)$, valid for $q \geq 0$ and $\mathbf{x} \in V \setminus \{0\}$, shows that any bounded region of $[0, \infty) \times \mathbb{R}$ meets only finitely trajectories of points $\mathbf{x} \in \Lambda \setminus \{0\}$. As the latter cover the graphs of L_1, \dots, L_N and consist of at most two line segments of slope 0 and 1, we conclude that L_1, \dots, L_N are piecewise linear with slopes 0 and 1. Finally, suppose that L_1 changes slope from 1 to 0 at a point $q > 0$. Then, there exist $\varepsilon > 0$ and two nonzero points \mathbf{x} and \mathbf{y} in Λ such that

$$L_1(t) = \begin{cases} L_{\mathbf{x}}(t) & \text{for } t \in [q - \varepsilon, q], \\ L_{\mathbf{y}}(t) & \text{for } t \in [q, q + \varepsilon]. \end{cases}$$

This implies that $L'_{\mathbf{x}}(q) = 1$ and $L'_{\mathbf{y}}(q) = 0$. So, the points \mathbf{x} and \mathbf{y} are linearly independent. As they both belong to $\exp(L_1(q))\mathcal{C}(e^q)$, we conclude that $L_2(q) = L_1(q)$. □

2.3. *The main family of convex bodies.* Fix an integer $n \geq 2$ and a unit vector \mathbf{u} of \mathbb{R}^n . We apply the preceding considerations to the decomposition

$$\mathbb{R}^n = U \perp W, \quad \text{where } W = \langle \mathbf{u} \rangle_{\mathbb{R}} \quad \text{and} \quad U := W^\perp = \{ \mathbf{x} \in \mathbb{R}^n; \mathbf{x} \cdot \mathbf{u} = 0 \},$$

using the standard integer lattice $\Lambda = \mathbb{Z}^n$. Since $\|\text{proj}_W(\mathbf{x})\| = |\mathbf{x} \cdot \mathbf{u}|$ for each $\mathbf{x} \in \mathbb{R}^n$, this gives rise to the family of convex bodies

$$\mathcal{C}_{\mathbf{u}}(Q) := \{ \mathbf{x} \in \mathbb{R}^n; \|\mathbf{x}\| \leq 1 \text{ and } |\mathbf{x} \cdot \mathbf{u}| \leq Q^{-1} \} \quad (Q \geq 1),$$

and its associated map $\mathbf{L}_{\mathbf{u}} = (L_{\mathbf{u},1}, \dots, L_{\mathbf{u},n}): [0, \infty) \rightarrow \Delta_n$ where $L_{\mathbf{u},j}(q) = \log \lambda_j(\mathcal{C}_{\mathbf{u}}(e^q))$. By Lemmas 2.1 and 2.2, the functions $L_{\mathbf{u},1}, \dots, L_{\mathbf{u},n}$ are continuous, piecewise linear with slopes 0 and 1, and they satisfy

$$(2.2) \quad |L_{\mathbf{u},1}(q) + \dots + L_{\mathbf{u},n}(q) - q| \leq n \log(n) \quad (q \geq 0).$$

In particular, these functions are monotone increasing. By (2.1), the trajectory of a nonzero point $\mathbf{x} \in \mathbb{Z}^n$, with respect to the family of convex bodies $\mathcal{C}_{\mathbf{u}}$, is the graph of the function $L_{\mathbf{x}}: [0, \infty) \rightarrow \mathbb{R}$ given by

$$(2.3) \quad L_{\mathbf{x}}(q) = L(\mathbf{x}, q) := \max\{\log \|\mathbf{x}\|, q + \log |\mathbf{x} \cdot \mathbf{u}|\} \quad (q \geq 0).$$

Equivalently, we note that $\lambda(\mathbf{x}, \mathcal{C}_{\mathbf{u}}(Q)) = \max\{\|\mathbf{x}\|, Q|\mathbf{x} \cdot \mathbf{u}|\} \quad (Q \geq 1)$.

2.4. *Families of pseudo-compound convex bodies.* Let \mathbf{u} and U be as in Section 2.3. We fix an integer $k \in \{1, \dots, n\}$ and endow the vector space $V = \wedge^k \mathbb{R}^n$ with the unique structure of Euclidean space such that, for any orthonormal basis $(\mathbf{e}_1, \dots, \mathbf{e}_n)$ of \mathbb{R}^n , the products $\mathbf{e}_{j_1} \wedge \dots \wedge \mathbf{e}_{j_k}$ with indices $1 \leq j_1 < \dots < j_k \leq n$ form an orthonormal basis of $\wedge^k \mathbb{R}^n$. We also define $\Lambda = \wedge^k \mathbb{Z}^n$ to be the lattice of co-volume 1 spanned by all products $\mathbf{x}_1 \wedge \dots \wedge \mathbf{x}_k$ with $\mathbf{x}_1, \dots, \mathbf{x}_k \in \mathbb{Z}^n$. We have the orthogonal sum decomposition

$$\wedge^k \mathbb{R}^n = U^{(k)} \perp W^{(k)}, \quad \text{where } U^{(k)} = \wedge^k U \quad \text{and} \quad W^{(k)} = \wedge^{k-1} U \wedge \langle \mathbf{u} \rangle_{\mathbb{R}}.$$

In accordance with the general construction of Section 2.1, we set

$$N := \dim_{\mathbb{R}}(\wedge^k \mathbb{R}^n) = \binom{n}{k} \quad \text{and} \quad K := \dim_{\mathbb{R}}(W^{(k)}) = \binom{n-1}{k-1}$$

and, for each $Q \geq 1$, we define

$$\mathcal{C}_{\mathbf{u}}^{(k)}(Q) = \{ \omega \in \wedge^k \mathbb{R}^n; \|\omega\| \leq 1 \text{ and } \|\text{proj}_{W^{(k)}}(\omega)\| \leq Q^{-1} \}.$$

We also form the associated map $\mathbf{L}_{\mathbf{u}}^{(k)} = (L_{\mathbf{u},1}^{(k)}, \dots, L_{\mathbf{u},N}^{(k)}): [0, \infty) \rightarrow \Delta_N$ given by

$$L_{\mathbf{u},j}^{(k)}(q) = \log \lambda_j(\mathcal{C}_{\mathbf{u}}^{(k)}(e^q)) \quad (q \geq 0, 1 \leq j \leq N).$$

By Lemmas 2.1 and 2.2, its components are continuous piecewise linear functions with slopes 0 and 1, and they satisfy

$$(2.4) \quad |L_{\mathbf{u},1}^{(k)}(q) + \dots + L_{\mathbf{u},N}^{(k)}(q) - Kq| \leq N \log(N) \quad (q \geq 0).$$

Moreover, we have the equality $L_{\mathbf{u},2}^{(k)}(q) = L_{\mathbf{u},1}^{(k)}(q)$ at each point $q > 0$ where $L_{\mathbf{u},1}^{(k)}$ changes slope from 1 to 0.

When $k = 1$, we have $\wedge^1 \mathbb{R}^n = \mathbb{R}^n$, $W^{(1)} = \langle \mathbf{u} \rangle_{\mathbb{R}}$, $\mathcal{C}_{\mathbf{u}}^{(1)}(Q) = \mathcal{C}_{\mathbf{u}}(Q)$ ($Q \geq 1$), and $\mathbf{L}_{\mathbf{u}}^{(1)} = \mathbf{L}_{\mathbf{u}}$. In general, we use $\mathcal{C}_{\mathbf{u}}^{(k)}(Q)$ as an approximation of the k -th compound convex body of $\mathcal{C}_{\mathbf{u}}(Q)$, namely, the convex hull of the exterior products of k elements of $\mathcal{C}_{\mathbf{u}}(Q)$ (see [1, Lemma 3]). The next lemma shows that this compound body is contained in $k\mathcal{C}_{\mathbf{u}}^{(k)}(Q)$.

LEMMA 2.3. *Let $Q \geq 1$, and let $\mathbf{x}_1, \dots, \mathbf{x}_k \in \mathcal{C}_{\mathbf{u}}(Q)$. Then, we have $\mathbf{x}_1 \wedge \dots \wedge \mathbf{x}_k \in k\mathcal{C}_{\mathbf{u}}^{(k)}(Q)$.*

Proof. For $j = 1, \dots, k$, define $\mathbf{y}_j = \text{proj}_U(\mathbf{x}_j)$ so that $\mathbf{x}_j = \mathbf{y}_j + (\mathbf{x}_j \cdot \mathbf{u})\mathbf{u}$. Upon writing $\omega = \mathbf{x}_1 \wedge \dots \wedge \mathbf{x}_k$, we find

$$\omega = \mathbf{y}_1 \wedge \dots \wedge \mathbf{y}_k + \sum_{j=1}^k (-1)^{j+k} (\mathbf{x}_j \cdot \mathbf{u}) \mathbf{y}_1 \wedge \dots \wedge \widehat{\mathbf{y}}_j \wedge \dots \wedge \mathbf{y}_k \wedge \mathbf{u}.$$

Thus the sum on the right is $\text{proj}_{W^{(k)}}(\omega)$ and therefore

$$\|\text{proj}_{W^{(k)}}(\omega)\| \leq \sum_{j=1}^k |\mathbf{x}_j \cdot \mathbf{u}| \|\mathbf{y}_1\| \cdots \|\widehat{\mathbf{y}}_j\| \cdots \|\mathbf{y}_k\| \leq kQ^{-1}.$$

As $\|\omega\| \leq \|\mathbf{x}_1\| \cdots \|\mathbf{x}_k\| \leq 1 \leq k$, we conclude that $\omega \in k\mathcal{C}_{\mathbf{u}}^{(k)}(Q)$. □

According to (2.1), the trajectory of a nonzero point $\omega \in \wedge^k \mathbb{Z}^n$ with respect to the family $\mathcal{C}_{\mathbf{u}}^{(k)}$ is the graph of the function $L_{\omega}: [0, \infty) \rightarrow \mathbb{R}$ given by

$$(2.5) \quad L_{\omega}(q) = L(\omega, q) := \max\{\log \|\omega\|, q + \log \|\text{proj}_{W^{(k)}}(\omega)\|\} \quad (q \geq 0).$$

With this notation, the previous lemma generalizes as follows.

LEMMA 2.4. *Let $\mathbf{y}_1, \dots, \mathbf{y}_k$ be linearly independent elements of \mathbb{Z}^n . We have*

$$L(\mathbf{y}_1 \wedge \dots \wedge \mathbf{y}_k, q) \leq L(\mathbf{y}_1, q) + \dots + L(\mathbf{y}_k, q) + \log(k) \quad (q \geq 0).$$

Proof. Fix a choice of $q \geq 0$, and set $\omega = \mathbf{y}_1 \wedge \dots \wedge \mathbf{y}_k$. For each $j = 1, \dots, k$, we have $\lambda_j^{-1} \mathbf{y}_j \in \mathcal{C}_{\mathbf{u}}(e^q)$, where $\lambda_j := \exp(L(\mathbf{y}_j, q))$. Thus the product $(\lambda_1 \dots \lambda_k)^{-1} \omega$ belongs to $k\mathcal{C}_{\mathbf{u}}^{(k)}(e^q)$, and so $L(\omega, q) \leq \log(k\lambda_1 \dots \lambda_k)$. □

In the case where $k = n$, we have $U^{(n)} = 0$. Then, our convex bodies are balls

$$\mathcal{C}_{\mathbf{u}}^{(n)}(Q) = \{\omega \in \wedge^n \mathbb{R}^n; \|\omega\| \leq Q^{-1}\} \quad (Q \geq 1).$$

So we find $L(\omega, q) = q + \log \|\omega\|$ for any nonzero $\omega \in \wedge^n \mathbb{R}^n$ and any $q \geq 0$. In particular, this gives $L_{\mathbf{u},1}^{(n)}(q) = q$ ($q \geq 0$), and the preceding lemma admits the following consequence.

LEMMA 2.5. *Let $\mathbf{y}_1, \dots, \mathbf{y}_n$ be linearly independent elements of \mathbb{Z}^n . We have*

$$\log \|\mathbf{y}_1 \wedge \dots \wedge \mathbf{y}_n\| \leq L(\mathbf{y}_1, q) + \dots + L(\mathbf{y}_n, q) - q + \log(n) \quad (q \geq 0).$$

The next estimate essentially goes back to Mahler [9], [10] (see also [3, §15.2]).

LEMMA 2.6. *For each $q \geq 0$, let $(S_{\mathbf{u},1}^{(k)}(q), \dots, S_{\mathbf{u},N}^{(k)}(q))$ denote the sequence of all sums $L_{\mathbf{u},j_1}(q) + \dots + L_{\mathbf{u},j_k}(q)$ with $1 \leq j_1 < \dots < j_k \leq n$ written in monotone increasing order. Then, we have*

$$-\log(n) \leq S_{\mathbf{u},j}^{(k)}(q) - L_{\mathbf{u},j}^{(k)}(q) \leq 2^n n \log(n) \quad (q \geq 0, 1 \leq j \leq N).$$

Proof. Fix a choice of $q \geq 0$, and choose linearly independent points $\mathbf{y}_1, \dots, \mathbf{y}_n$ of \mathbb{Z}^n that realize the successive minima of $\mathcal{C}_{\mathbf{u}}(e^q)$ in the sense that $L(\mathbf{y}_j, q) = L_{\mathbf{u},j}(q)$ for $j = 1, \dots, n$. Then, denote by (μ_1, \dots, μ_N) the set of numbers $L(\mathbf{y}_{j_1} \wedge \dots \wedge \mathbf{y}_{j_k}, q)$ arranged in monotone increasing order (with the convention that $1 \leq j_1 < \dots < j_k \leq n$). Since the products $\mathbf{y}_{j_1} \wedge \dots \wedge \mathbf{y}_{j_k}$ are linearly independent elements of $\wedge^k \mathbb{Z}^n$, we have $L_{\mathbf{u},j}^{(k)}(q) \leq \mu_j$ for $j = 1, \dots, N$. By Lemma 2.4, we also have

$$L(\mathbf{y}_{j_1} \wedge \dots \wedge \mathbf{y}_{j_k}, q) \leq L_{\mathbf{u},j_1}(q) + \dots + L_{\mathbf{u},j_k}(q) + \log k \quad (1 \leq j_1 < \dots < j_k \leq n),$$

and so $\mu_j \leq S_{\mathbf{u},j}^{(k)}(q) + \log(k)$ for $j = 1, \dots, N$. Combining these two observations gives

$$L_{\mathbf{u},j}^{(k)}(q) \leq S_{\mathbf{u},j}^{(k)}(q) + \log(k) \quad (1 \leq j \leq N).$$

On the other hand, the estimates (2.2) and (2.4) yield

$$\sum_{j=1}^N (S_{\mathbf{u},j}^{(k)}(q) + \log(k) - L_{\mathbf{u},j}^{(k)}(q)) = K \sum_{j=1}^n L_{\mathbf{u},j}(q) + N \log(k) - \sum_{j=1}^N L_{\mathbf{u},j}^{(k)}(q) \leq c,$$

where $c = Kn \log(n) + N \log(k) + N \log(N) = Kn \log(n) + N \log(kN)$. Since $nK = kN \leq n^k$, we have $c \leq 2Kn \log(n) \leq 2^n n \log(n)$, and the conclusion follows. □

We simply need the following consequence of the previous lemma.

LEMMA 2.7. *Put $c_1 = 2^n n \log(n)$. For each $q \geq 0$, we have*

- (i) $|L_{\mathbf{u},1}^{(k)}(q) - L_{\mathbf{u},1}(q) - \dots - L_{\mathbf{u},k}(q)| \leq c_1$;
- (ii) $|L_{\mathbf{u},2}^{(k)}(q) - L_{\mathbf{u},1}(q) - \dots - L_{\mathbf{u},k-1}(q) - L_{\mathbf{u},k+1}(q)| \leq c_1$ if $k < n$;
- (iii) $|L_{\mathbf{u},j}^{(n-1)}(q) + L_{\mathbf{u},n+1-j}(q) - q| \leq c_1 + n \log(n)$ for $j = 1, \dots, n$.

Proof. The first two estimates follow immediately from Lemma 2.6 because the smallest sum $L_{\mathbf{u},j_1}(q) + \dots + L_{\mathbf{u},j_k}(q)$ is obtained for the choice of indices $j_1 = 1, \dots, j_k = k$, while the next one, when $k < n$, is obtained by choosing $j_1 = 1, \dots, j_{k-1} = k - 1$ and $j_k = k + 1$. If $k = n - 1$, then the j -th sum is $S_{\mathbf{u},j}^{(n-1)}(q) = L_{\mathbf{u},1}(q) + \dots + L_{\mathbf{u},n}(q) - L_{\mathbf{u},n+1-j}(q)$, and so (iii) follows from (2.2). \square

2.5. *The approximation theorem of Schmidt and Summerer.* Fix an integer $n \geq 2$. The following definition is extracted from [18, §2] and adapted to our context.

Definition 2.8. Let $\gamma, q_0 \geq 0$. An (n, γ) -system on the half-line $[q_0, \infty)$ is a function $\mathbf{P} = (P_1, \dots, P_n): [q_0, \infty) \rightarrow \mathbb{R}^n$ that satisfies the following conditions:

- (S1) $-\gamma \leq P_j(q) \leq P_{j+1}(q) + \gamma \quad (1 \leq j < n, q_0 \leq q)$;
- (S2) $P_j(q_1) \leq P_j(q_2) + \gamma \quad (1 \leq j \leq n, q_0 \leq q_1 \leq q_2)$;
- (S3) for $j = 1, \dots, n$, the function $M_j := P_1 + \dots + P_j: [q_0, \infty) \rightarrow \mathbb{R}$ is continuous and piecewise linear with slopes 0 and 1;
- (S4) $M_n(q) = q \quad (q_0 \leq q)$;
- (S5) if, for some $j \in \{1, \dots, n - 1\}$, the function M_j changes slope from 1 to 0 at a point $q > q_0$, then $P_{j+1}(q) \leq P_j(q) + \gamma$.

Up to the value of γ , the next result comes from [18, §2]. It shows the fundamental importance of the notion of (n, γ) -systems. For the sake of completeness and because our context is slightly different, we also recall its proof below.

THEOREM 2.9 (Schmidt-Summerer, 2013). *Let $\gamma = 6n2^n \log(n)$. For each unit vector \mathbf{u} of \mathbb{R}^n , there exists an (n, γ) -system $\mathbf{P}: [0, \infty) \rightarrow \mathbb{R}^n$ such that*

$$\sup_{q \geq 0} \|\mathbf{P}(q) - \mathbf{L}_{\mathbf{u}}(q)\|_{\infty} \leq \gamma.$$

Proof. Set $M_0 := 0$ and $M_k := L_{\mathbf{u},1}^{(k)}$ for $k = 1, \dots, n$. By Lemma 2.7(i), we have

$$|M_k(q) - L_{\mathbf{u},1}(q) - \dots - L_{\mathbf{u},k}(q)| \leq c_1 \quad (1 \leq k \leq n, 0 \leq q).$$

Then, upon defining $P_k = M_k - M_{k-1}$ for $k = 1, \dots, n$, we obtain

$$(2.6) \quad |P_k(q) - L_{\mathbf{u},k}(q)| \leq 2c_1 \quad (1 \leq k \leq n, 0 \leq q).$$

So, in order to complete the proof of the theorem, it suffices to show that $\mathbf{P} := (P_1, \dots, P_n)$ is an $(n, 6c_1)$ -system on $[0, \infty)$. The conditions (S1) and (S2) derive immediately from (2.6) because for each $q \geq 0$, we have

$$\begin{aligned} P_k(q) &\geq L_{\mathbf{u},k}(q) - 2c_1 \geq -2c_1, \\ P_k(q) &\leq L_{\mathbf{u},k}(q) + 2c_1 \leq L_{\mathbf{u},k+1}(q) + 2c_1 \leq P_{k+1}(q) + 4c_1 \quad (1 \leq k < n), \end{aligned}$$

while for $0 \leq q_1 \leq q_2$, we find

$$P_k(q_1) \leq L_{\mathbf{u},k}(q_1) + 2c_1 \leq L_{\mathbf{u},k}(q_2) + 2c_1 \leq P_k(q_2) + 4c_1 \quad (1 \leq k \leq n).$$

For $k = 1, \dots, n$, we also have $P_1 + \dots + P_k = M_k - M_0 = M_k$, and we know that $M_k = L_{\mathbf{u},1}^{(k)}$ is continuous and piecewise linear with slopes 0 and 1. Moreover, as noticed in Section 2.4, we have $M_n(q) = L_{\mathbf{u},1}^{(n)}(q) = q$ for each $q \geq 0$. Thus (S3) and (S4) are automatically satisfied. Finally, (S5) also holds because if, for some $k \in \{1, \dots, n - 1\}$, the function $M_k = L_{\mathbf{u},1}^{(k)}$ changes slope from 1 to 0 at a point $q > 0$, then $L_{\mathbf{u},2}^{(k)}(q) = L_{\mathbf{u},1}^{(k)}(q)$ and so the estimates (i) and (ii) of Lemma 2.7 yield $L_{\mathbf{u},k+1}(q) - L_{\mathbf{u},k}(q) \leq 2c_1$, which in turn implies that $P_{k+1}(q) \leq P_k(q) + 6c_1$. \square

Note that the properties of an (n, γ) -system are simpler when $\gamma = 0$. For example, the condition (S3) implies that the components of an (n, γ) -system are continuous piecewise linear functions with slopes $-1, 0$ or 1 . However, when $\gamma = 0$, it follows from (S2) that these components are monotone increasing and so, possess only the slopes 0 and 1 . Moreover, (S1) shows that an $(n, 0)$ -system takes values in Δ_n while this may also fail for a general (n, γ) -system. A general description of $(n, 0)$ -systems is given in [18, §3]. We conclude this section with the following observation, which describes the rigid n -systems as a subset of the set of $(n, 0)$ -systems, leaving its proof to the reader.

LEMMA 2.10. *Let $\delta \in (0, \infty)$. The rigid n -systems of mesh δ are the $(n, 0)$ -systems $(P_1, \dots, P_n): [q_0, \infty) \rightarrow \mathbb{R}^n$ with the property that, for $q = q_0$ and for each $q > q_0$ at which at least one of the functions $P_1 + \dots + P_j$ ($1 \leq j < n$) changes slope from 0 to 1, the numbers $P_1(q), \dots, P_n(q)$ are n distinct multiples of δ .*

In particular, a rigid system $(P_1, \dots, P_n): [q_0, \infty) \rightarrow \mathbb{R}^n$ satisfies

$$(2.7) \quad P_1(q) + \dots + P_n(q) = q \quad (q \geq q_0).$$

This is an important property that we will use repeatedly.

3. A special case

In this section, we fix an integer $n \geq 2$ and assume that a rigid n -system \mathbf{P} of large mesh is, in comparison, very closely approximated by the map $\mathbf{L}_{\mathbf{u}}$ for some unit vector \mathbf{u} of \mathbb{R}^n . We derive from this the existence of a sequence of n -tuples of integer points with strong properties. This partly explains and motivates the constructions of Section 5 where, given an arbitrary rigid system \mathbf{P} of sufficiently large mesh, we construct a unit vector \mathbf{u} such that $\mathbf{P} - \mathbf{L}_{\mathbf{u}}$ is bounded. Our goal is thus to prove the following complement to Theorem 1.3.

THEOREM 3.1. *Let δ, ε be positive real numbers with $\delta \geq 6(n\varepsilon + c_1)$, where c_1 is as in Lemma 2.7, and let $\mathbf{P} = (P_1, \dots, P_n): [q_0, \infty) \rightarrow \mathbb{R}^n$ be a rigid system of mesh δ . Suppose that there exists a unit vector $\mathbf{u} \in \mathbb{R}^n$ such that $\|\mathbf{P}(q) - \mathbf{L}_{\mathbf{u}}(q)\|_\infty \leq \varepsilon$ for any $q \geq q_0$. Consider the sequences $(q_i)_{0 \leq i < s}$, $(k_i)_{0 \leq i < s}$ and $(\ell_i)_{0 \leq i < s}$ attached to \mathbf{P} as in Definition 1.2, and set $q_s = \infty$ if $s < \infty$. Then, for each integer i with $0 \leq i < s$, there exists an n -tuple of linearly independent integer points $(\mathbf{x}_1^{(i)}, \dots, \mathbf{x}_n^{(i)})$ with the following properties:*

- (1) $\|\mathbf{P}(q) - \Phi_n(L(\mathbf{x}_1^{(i)}, q), \dots, L(\mathbf{x}_n^{(i)}, q))\|_\infty \leq \varepsilon \quad (q_i \leq q < q_{i+1});$
- (2) $(\mathbf{x}_1^{(i)}, \dots, \widehat{\mathbf{x}_{k_i}^{(i)}}, \dots, \mathbf{x}_n^{(i)}) = (\mathbf{x}_1^{(i+1)}, \dots, \widehat{\mathbf{x}_{\ell_{i+1}}^{(i+1)}}, \dots, \mathbf{x}_n^{(i+1)})$ if $i + 1 < s$;
- (3) $|\log \|\mathbf{x}_j^{(i)}\| - P_j(q_i)| \leq \varepsilon \quad (j = 1, \dots, n)$ if $i \geq 1$;
- (4) $\mathbf{x}_{\ell_{i+1}}^{(i+1)} \in \langle \mathbf{x}_1^{(i)}, \dots, \mathbf{x}_{\ell_{i+1}}^{(i)} \rangle_{\mathbb{R}}$ if $i + 1 < s$;
- (5) $0 \leq \log |\det(\mathbf{x}_1^{(i)}, \dots, \mathbf{x}_n^{(i)})| \leq n\varepsilon + \log(n)$;
- (6) $|\log \|\mathbf{x}_1^{(i)} \wedge \dots \wedge \widehat{\mathbf{x}_{k_i}^{(i)}} \wedge \dots \wedge \mathbf{x}_n^{(i)}\| - \sum_{j \neq k_i} \log \|\mathbf{x}_j^{(i)}\|| \leq n\varepsilon + 2c_1$ if $i \geq 1$.

Property (1) means that, over each interval $[q_i, q_{i+1})$, the combined graph of \mathbf{P} is contained in an ε -neighborhood of the union of the trajectories of $\mathbf{x}_1^{(i)}, \dots, \mathbf{x}_n^{(i)}$. Together with the hypothesis, it implies that, for each q in that interval, the latter points realize the logarithms of the successive minima of $\mathcal{C}_{\mathbf{u}}(e^q)$ within 2ε . Because of the specific form of the combined graph of \mathbf{P} , we also infer from (1) that exactly one of the points $\mathbf{x}_j^{(i)}$ must have a trajectory of slope 1 over $[q_i + 2\varepsilon, \infty)$ while all other points have trajectories of slope 0 over $[0, q_{i+1} - 2\varepsilon)$ (see Lemma 3.3). If $i + 1 < s$, it follows from (2) that this exceptional point must be $\mathbf{x}_{k_i}^{(i)}$. Over the next interval $[q_{i+1}, q_{i+2})$, its trajectory is replaced by that of a new point $\mathbf{x}_{\ell_{i+1}}^{(i+1)}$ while those of the other points are kept. Figure 2 illustrates this in an example with $n = 5$. In this picture the solid lines represent the combined graph of \mathbf{P} and the dotted lines the trajectories of the points $\mathbf{x}_j^{(i)}$.

Properties (3)–(6) are useful additional information about the points $\mathbf{x}_j^{(i)}$. For example, (5) tells us that each n -tuple $(\mathbf{x}_1^{(i)}, \dots, \mathbf{x}_n^{(i)})$ spans a subgroup of \mathbb{Z}^n with uniformly bounded index, while (6) implies that the angles between any two of the points $\mathbf{x}_1^{(i)}, \dots, \widehat{\mathbf{x}_{k_i}^{(i)}}, \dots, \mathbf{x}_n^{(i)}$ are bounded away from 0. In Section 5, dealing with the inverse problem, we construct n -tuples of integer points with stronger properties. In particular, we request that each n -tuple $(\mathbf{x}_1^{(i)}, \dots, \mathbf{x}_n^{(i)})$ is a basis of \mathbb{Z}^n and that the points $\mathbf{x}_1^{(i)}, \dots, \widehat{\mathbf{x}_{k_i}^{(i)}}, \dots, \mathbf{x}_n^{(i)}$ are almost orthogonal in a sense that is defined in Section 4.

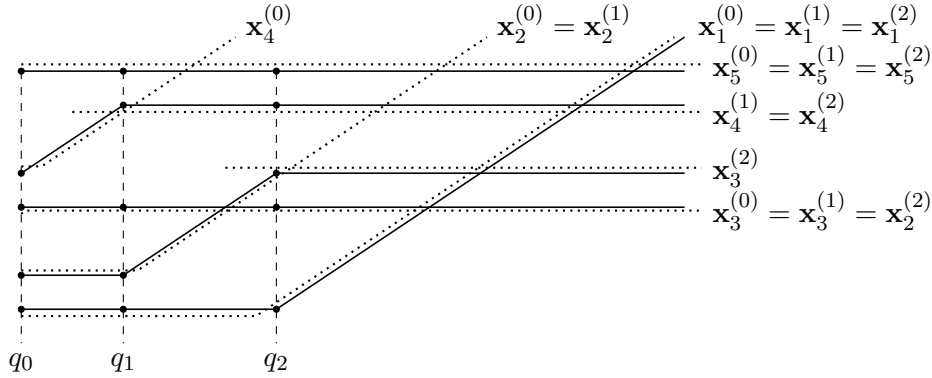


Figure 2. The combined graph of a rigid 5-system and the trajectories of integer points that approximate it within ε .

Preliminary observations towards the proof of Theorem 3.1. Suppose that the hypotheses of Theorem 3.1 are fulfilled. For each $\mathbf{x} \in \mathbb{R}^n \setminus \{0\}$ and each $j \in \{1, \dots, n\}$, we set

$$L(\mathbf{x}, \infty) = \lim_{q \rightarrow \infty} L(\mathbf{x}, q), \quad L_{\mathbf{u},j}(\infty) = \lim_{q \rightarrow \infty} L_{\mathbf{u},j}(q), \quad P_j(\infty) = \lim_{q \rightarrow \infty} P_j(q).$$

We also define

$$V_j(q) = \langle \mathbf{x} \in \mathbb{Z}^n \setminus \{0\}; L(\mathbf{x}, q) \leq L_{\mathbf{u},j}(q) \rangle_{\mathbb{R}} \quad (1 \leq j \leq n, 0 \leq q \leq \infty),$$

$$V_0(q) = 0 \quad (0 \leq q \leq \infty).$$

The vector space $V_j(\infty)$ is interesting only when $L_{\mathbf{u},j}(\infty) < \infty$ or, equivalently, when $P_j(\infty) < \infty$. In that case, $V_j(\infty)$ is orthogonal to \mathbf{u} and we have $L(\mathbf{x}, q) = \log \|\mathbf{x}\|$ ($q \geq 0$) for any $\mathbf{x} \in V_j(\infty) \setminus \{0\}$. Otherwise, we have $V_j(\infty) = \mathbb{R}^n$. We start with three observations.

LEMMA 3.2. *Let $j \in \{1, \dots, n - 1\}$. The family of vector spaces $V_j(q)$ is constant, with dimension j , on any subinterval of $[q_0, \infty)$ on which we have $P_{j+1}(q) > P_j(q) + 2\varepsilon$.*

Proof. Since we have $\|\mathbf{P}(q) - \mathbf{L}_{\mathbf{u}}(q)\|_{\infty} \leq \varepsilon$ for each $q \geq q_0$, the condition $P_{j+1}(q) > P_j(q) + 2\varepsilon$ implies that $L_{\mathbf{u},j+1}(q) > L_{\mathbf{u},j}(q)$ and thus $\dim_{\mathbb{R}} V_j(q) = j$. For any given $q \geq q_0$ with this property, there exists a connected neighborhood I of q in $[q_0, \infty)$ and a real number r such that $L_{\mathbf{u},j}(t) < r < L_{\mathbf{u},j+1}(t)$ for each $t \in I$. Then we have

$$V_j(t) = \langle \mathbf{x} \in \mathbb{Z}^n \setminus \{0\}; L(\mathbf{x}, t) < r \rangle_{\mathbb{R}} \quad (t \in I),$$

and so the family $V_j(t)$ is monotone decreasing on I . As it has constant dimension j , it is therefore constant. Thus the family $V_j(q)$ is locally constant and therefore constant on any subinterval of $[q_0, \infty)$ on which $P_{j+1} > P_j + 2\varepsilon$. \square

LEMMA 3.3. *Let i be an integer with $0 \leq i < s$. Put $k = k_i$. Then, there exists a point $\mathbf{x} \in \mathbb{Z}^n$ with $\mathbf{x} \notin V_{k-1}(q_i)$ such that*

$$(3.1) \quad \left| L(\mathbf{x}, t) - (P_k(q_i) + \max\{0, t - q_i\}) \right| \leq \varepsilon \quad (t \geq 0).$$

In particular, the trajectory of \mathbf{x} has slope 0 on $[0, q_i - 2\varepsilon]$ if $i \geq 1$ and slope 1 on $[q_i + 2\varepsilon, \infty)$.

Proof. Let $I = \{t \geq q_0; |t - q_i| < \delta - 2\varepsilon\}$. If $k \geq 2$, then we have $P_k(q_i) \geq P_{k-1}(q_i) + \delta$ and so $P_k(t) > P_{k-1}(t) + 2\varepsilon$ for any $t \in I$. By Lemma 3.2, this implies that $V_{k-1}(t) = V_{k-1}(q_i)$ has dimension $k - 1$ for each $t \in I$. This is also true if $k = 1$ because then $V_{k-1}(t) = 0$ for each $t \geq 0$.

Since $\dim_{\mathbb{R}} V_{k-1}(q_i) = k - 1$, there exists $\mathbf{x} \in \mathbb{Z}^n$ with $\mathbf{x} \notin V_{k-1}(q_i)$ such that $L(\mathbf{x}, q_i) = L_{\mathbf{u},k}(q_i)$. We claim that this point satisfies condition (3.1). For each $t \in I$, we have $\mathbf{x} \notin V_{k-1}(t)$, and therefore

$$L(\mathbf{x}, t) \geq L_{\mathbf{u},k}(t) \geq P_k(t) - \varepsilon \quad (t \in I).$$

As $L(\mathbf{x}, t)$ has slope 0 and 1, we also have

$$L(\mathbf{x}, t) \leq L(\mathbf{x}, q_i) + \max\{0, t - q_i\} \quad (t \geq 0).$$

By construction, the function P_k is constant on $[q_i - \delta, q_i]$ if $i \geq 1$, and it has slope 1 on $[q_i, q_i + \delta]$. Since $L(\mathbf{x}, q_i) = L_{\mathbf{u},k}(q_i) \leq P_k(q_i) + \varepsilon$, the previous inequality yields

$$L(\mathbf{x}, t) \leq P_k(q_i) + \max\{0, t - q_i\} + \varepsilon = P_k(t) + \varepsilon \quad (t \in I).$$

Thus, (3.1) holds for each $t \in I$. Since $\delta > 4\varepsilon$, this implies that $L(\mathbf{x}, t)$ is constant on $[0, q_i - 2\varepsilon]$ if $i \geq 1$, and that it has slope 1 on $[q_i + 2\varepsilon, \infty)$. Thus the validity of (3.1) extends from I to the whole interval $[0, \infty)$. \square

LEMMA 3.4. *Suppose that $P_j(\infty) < \infty$ for some index j with $1 \leq j < n$. Then $V_j(\infty)$ has dimension j and there exists a real number $q \geq 0$ such that $V_j(t) = V_j(\infty)$ for each $t \geq q$. Moreover, there exists a point $\mathbf{x} \in \mathbb{Z}^n \cap V_j(\infty)$ with $\mathbf{x} \notin V_{j-1}(\infty)$ such that $|L(\mathbf{x}, t) - P_j(\infty)| \leq \varepsilon$ for each $t \geq 0$.*

Proof. Given a real number $q \geq 0$, there are finitely many nonzero points $\mathbf{x} \in \mathbb{Z}^n$ such that $L(\mathbf{x}, t) \leq L_{\mathbf{u},j}(t)$ for some $t \geq q$ because this implies that $\log \|\mathbf{x}\| \leq P_j(t) + \varepsilon \leq P_j(\infty) + \varepsilon$. Moreover, the set of these points \mathbf{x} is monotone decreasing as a function of q . Choose q so that it has minimal cardinality. Then, all points \mathbf{x} with the above property have $L(\mathbf{x}, t)$ constant, equal to $\log \|\mathbf{x}\|$, for $t \geq q$. Thus, $L_{\mathbf{u},j}$ is also constant on $[q, \infty)$ and we find

$$V_j(t) = \left\langle \mathbf{x} \in \mathbb{Z}^n \setminus \{0\}; \mathbf{x} \cdot \mathbf{u} = 0, \log \|\mathbf{x}\| \leq L_{\mathbf{u},j}(\infty) \right\rangle_{\mathbb{R}} = V_j(\infty)$$

for each $t \geq q$. If q is sufficiently large, we also have $P_{j+1}(q) > P_j(q) + 2\varepsilon$ because $P_{j+1}(\infty) > P_j(\infty) + \delta$. Then, $V_j(\infty) = V_j(q)$ has dimension j according to Lemma 3.2.

By the above, we also have $\dim_{\mathbb{R}} V_{j-1}(\infty) = j - 1$ if $j \geq 2$ because then $P_{j-1}(\infty) < \infty$. This remains true if $j = 1$ since $V_0(\infty) = 0$. Thus $V_j(\infty) \neq V_{j-1}(\infty)$. Then, assuming q sufficiently large so that $V_{j-1}(q) = V_{j-1}(\infty)$, we have $V_j(q) \neq V_{j-1}(q)$ and consequently there exists $\mathbf{x} \in \mathbb{Z}^n \cap V_j(q)$ with $\mathbf{x} \notin V_{j-1}(q)$ such that $L(\mathbf{x}, q) = L_{\mathbf{u},j}(q)$. This point has the requested property because $L(\mathbf{x}, t) = L_{\mathbf{u},j}(\infty)$ for $t \geq 0$, and $|L_{\mathbf{u},j}(\infty) - P_j(\infty)| \leq \varepsilon$. \square

Thanks to Lemmas 3.2 and 3.4, we note that $\dim_{\mathbb{R}} V_j(q_i) = j$ for each pair of integers $i (\neq \infty)$ and j with $0 \leq i \leq s$ and $0 \leq j \leq n$.

Construction of the points $\mathbf{x}_j^{(i)}$ and proof of Properties (1)–(3). We denote by \mathcal{S} the set of all maximal horizontal line segments of positive length contained in the combined graph of \mathbf{P} , together with the point $(q_0, P_{k_0}(q_0))$ (which is thus the only line segment of length 0 in \mathcal{S}).

To each $S \in \mathcal{S}$ we associate a point $\mathbf{x}_S \in \mathbb{Z}^n$ in the following way. If S is bounded, its right end-point is $(q_i, P_{k_i}(q_i))$ for some integer i with $0 \leq i < s$. Then, we choose for \mathbf{x}_S any point \mathbf{x} with the property stated in Lemma 3.3. Otherwise, S is contained in $[0, \infty) \times \{P_j(\infty)\}$ for some index j with $1 \leq j < n$ such that $P_j(\infty) < \infty$. Then, we choose for \mathbf{x}_S any point \mathbf{x} with the property stated in Lemma 3.4.

Finally, for each pair of integers i and j with $0 \leq i < s$ and $1 \leq j \leq n$, there exists a unique line segment S in \mathcal{S} containing the point $(q_i, P_j(q_i))$, and we define $\mathbf{x}_j^{(i)} = \mathbf{x}_S$. In the degenerate case where $s < \infty$, we have $P_j(\infty) < \infty$ for each $j = 1, \dots, n - 1$. Then, for those j , we define $\mathbf{x}_j^{(s)} = \mathbf{x}_S$ where S is the unique unbounded line segment of \mathcal{S} contained in $[0, \infty) \times \{P_j(\infty)\}$. By construction, these points $\mathbf{x}_1^{(s)}, \dots, \mathbf{x}_{n-1}^{(s)}$ are linearly independent.

With these definitions, property (2) in Theorem 3.1 is automatically satisfied. More precisely, we have

$$(3.2) \quad (\mathbf{x}_1^{(i)}, \dots, \widehat{\mathbf{x}_{k_i}^{(i)}}, \dots, \mathbf{x}_n^{(i)}) = (\mathbf{x}_1^{(i+1)}, \dots, \widehat{\mathbf{x}_{\ell_{i+1}}^{(i+1)}}, \dots, \mathbf{x}_n^{(i+1)}) \quad (0 \leq i < s)$$

upon defining $\ell_s = n$ if $s < \infty$. Property (1) is also clear in view of the description of the trajectories of the points \mathbf{x} given by Lemmas 3.3 and 3.4, as illustrated on Figure 2 for $n = 5$. Indeed, for any fixed index i with $0 \leq i < s$, we have

$$(3.3) \quad \sup_{q_i \leq t < q_{i+1}} \left| L(\mathbf{x}_{k_i}^{(i)}, t) - P_{k_i}(q_i) - (t - q_i) \right| \leq \varepsilon,$$

$$(3.4) \quad \sup_{q_i \leq t < q_{i+1}} \left| L(\mathbf{x}_j^{(i)}, t) - P_j(q_i) \right| \leq \varepsilon \quad (1 \leq j \leq n, j \neq k_i).$$

Since $\mathbf{P}(t) = \Phi_n(P_1(q_i), \dots, \widehat{P_{k_i}(q_i)}, \dots, P_n(q_i), P_{k_i}(q_i) + t - q_i)$ for each t in $[q_i, q_{i+1})$, and since $\|\Phi(\mathbf{p}) - \Phi(\mathbf{p}')\|_{\infty} \leq \|\mathbf{p} - \mathbf{p}'\|_{\infty}$ for any $\mathbf{p}, \mathbf{p}' \in \mathbb{R}^n$, this yields

(1). Moreover, if $i \geq 1$, we have $|L(\mathbf{x}_j^{(i)}, t) - P_j(q_i)| \leq \varepsilon$ for each $t \in [0, q_i]$ and each $j = 1, \dots, n$. Upon setting $t = 0$, this yields property (3).

Linear independence of the n -tuples $(\mathbf{x}_1^{(i)}, \dots, \mathbf{x}_n^{(i)})$. This is the most delicate part of the argument. It will come out of the last of the following four lemmas.

LEMMA 3.5. *Let i, j be integers with $0 \leq i < s$ and $0 \leq j < n$. If P_{j+1} is constant on $[q_i, \infty]$, then $V_j(q_i) = V_j(\infty)$. If P_{j+1} is constant on $[q_i, q_r]$ for some integer r with $i \leq r < s$, then $V_j(q_i) = V_j(q_r)$.*

Proof. If $j = 0$, this is clear since $V_0(q) = 0$ for any $q \in [0, \infty]$. Suppose that $j \geq 1$. In the first case, we choose $q \geq q_i$ sufficiently large so that, by Lemma 3.4, we have $V_j(q) = V_j(\infty)$. In the second case, we set $q = q_r$. In both cases, we have $P_{j+1}(t) \geq P_j(t) + \delta$ for each $t \in [q_i, q]$ and so, by Lemma 3.2, we conclude that $V_j(q_i) = V_j(q)$. \square

LEMMA 3.6. *Let i, j be integers with $0 \leq i \leq s$ and $1 \leq j \leq n$, and let ω be a generator of the one-dimensional vector space $\wedge^j V_j(q_i)$.*

- (1) *If $0 < i \leq s$ and $j < \ell_i$, then $L(\omega, t)$ is constant on $[0, q_i - \delta/3]$ (understood as $[0, \infty]$ if $i = s$, since in that case $q_s = \infty$).*
- (2) *If $0 \leq i < s$ and $j \geq k_i$, then $L(\omega, t)$ has slope 1 on $[q_i + \delta/3, \infty)$.*

Proof. Since $V_j(q_i)$ has dimension j , its j -th exterior power has dimension 1. For any other generator ω' of the latter vector space, the difference $L(\omega, t) - L(\omega', t)$ is constant. So, we may assume that $\omega = \mathbf{y}_1 \wedge \dots \wedge \mathbf{y}_j$ where $\mathbf{y}_1, \dots, \mathbf{y}_j$ are linearly independent points of \mathbb{Z}^n such that $L(\mathbf{y}_m, q_i) = L_{\mathbf{u}, m}(q_i)$ for $m = 1, \dots, j$.

If $i = s$ and $j < \ell_s = n$, then $\mathbf{y}_1, \dots, \mathbf{y}_j$ belong to $U = \mathbf{u}^\perp$ (because $V_j(\infty) \subseteq U$) and so $\omega \in \wedge^j U$. By virtue of formula (2.5) with k replaced by j , this implies that $L(\omega, t) = \log \|\omega\|$ is constant for all $t \geq 0$.

From now on, we may therefore assume that $i < s$, and so $q_i < \infty$. As ω is a nonzero element of $\wedge^j \mathbb{Z}^n$, we have

$$L(\omega, t) \geq L_{\mathbf{u}, 1}^{(j)}(t) \quad (t \geq 0).$$

By Lemma 2.7(i), we also have

$$L_{\mathbf{u}, 1}^{(j)}(t) \geq L_{\mathbf{u}, 1}(t) + \dots + L_{\mathbf{u}, j}(t) - c_1 \geq M_j(t) - j\varepsilon - c_1 \quad (t \geq q_0),$$

where $M_j = P_1 + \dots + P_j$. On the other hand, since $L(\omega, t)$ has slopes 0 and 1, it satisfies

$$L(\omega, t) \leq L(\omega, q_i) + \max\{0, t - q_i\} \quad (t \geq 0),$$

while Lemma 2.4 gives

$$\begin{aligned} L(\omega, q_i) &\leq L(\mathbf{y}_1, q_i) + \cdots + L(\mathbf{y}_j, q_i) + \log n \\ &= L_{\mathbf{u},1}(q_i) + \cdots + L_{\mathbf{u},j}(q_i) + \log n \leq M_j(q_i) + j\varepsilon + c_1. \end{aligned}$$

Combining these estimates, we conclude that

$$|L(\omega, t) - M_j(t)| \leq |M_j(q_i) + \max\{0, t - q_i\} - M_j(t)| + (\delta/6) \quad (t \geq q_0)$$

since $j\varepsilon + c_1 \leq \delta/6$. If $0 < i < s$ and $j < \ell_i$, then M_j is constant on $[q_i - \delta, q_i]$ and the latter estimate gives $|L(\omega, t) - M_j(q_i)| \leq \delta/6$ for each $t \in [q_i - \delta, q_i]$. So, $L(\omega, t)$ is constant on $[0, q_i - \delta/3]$. If $0 \leq i < s$ and $j \geq k_i$, then M_j has slope 1 on $[q_i, q_i + \delta]$ and the same estimate yields $|L(\omega, t) - M_j(t)| \leq \delta/6$ for each $t \in [q_i, q_i + \delta]$. Then, $L(\omega, t)$ has slope 1 on $[q_i + \delta/3, \infty)$. \square

LEMMA 3.7. *Let $i, j \in \mathbb{N}$ with $0 \leq i < s$ and $k_i \leq j < \ell_{i+1}$. Then, $V_j(q_i) \neq V_j(q_{i+1})$.*

Proof. Suppose on the contrary that $V_j(q_i) = V_j(q_{i+1})$, and let ω be a generator of the j -th exterior power of this vector space. By Lemma 3.6 applied successively to the points q_i and q_{i+1} , the function $L(\omega, t)$ has slope 1 on $[q_i + \delta/3, \infty)$ and is constant on $[0, q_{i+1} - \delta/3]$. As $q_{i+1} \geq q_i + \delta$, this is impossible. \square

LEMMA 3.8. *For each pair of integers i, j with $0 \leq i < s$ and $0 \leq j \leq n$, we have $V_j(q_i) = \langle \mathbf{x}_1^{(i)}, \dots, \mathbf{x}_j^{(i)} \rangle_{\mathbb{R}}$. In particular, $\mathbf{x}_1^{(i)}, \dots, \mathbf{x}_n^{(i)}$ are linearly independent for each index i with $0 \leq i < s$.*

Proof. The second assertion follows from the first because $V_n(q_i) = \mathbb{R}^n$ for each i . We prove the first assertion by induction on j . For $j = 0$, it is clear since $V_0(q_i) = 0$ for each i . Suppose that it holds for some integer j with $0 \leq j < n$. Fix an arbitrary index i with $0 \leq i < s$. Since $V_{j+1}(q_i)$ has dimension $j + 1$ and contains $V_j(q_i)$ as a vector subspace of dimension j , we are reduced to showing that $\mathbf{x}_{j+1}^{(i)} \in V_{j+1}(q_i) \setminus V_j(q_i)$, in order to complete the induction step. If $j < n - 1$, we find that

$$L(\mathbf{x}_{j+1}^{(i)}, q_i) \leq P_{j+1}(q_i) + \varepsilon < P_{j+2}(q_i) - \varepsilon \leq L_{\mathbf{u},j+2}(q_i),$$

where the first inequality comes from (3.3) and (3.4). Thus, $\mathbf{x}_{j+1}^{(i)}$ belongs to $V_{j+1}^{(i)}$ if $j < n - 1$. If $j = n - 1$, the conclusion is the same because $V_n(q_i) = \mathbb{R}^n$. So, it remains to show that $\mathbf{x}_{j+1}^{(i)} \notin V_j(q_i)$.

Let S be the line segment in \mathcal{S} that contains the point $(q_i, P_{j+1}(q_i))$ so that $\mathbf{x}_{j+1}^{(i)} = \mathbf{x}_S$. If P_{j+1} is constant on $[q_i, \infty)$, then Lemma 3.5 gives $V_j(q_i) = V_j(\infty)$ and, by construction, we have $\mathbf{x}_S \notin V_j(\infty)$. We may therefore assume the existence of a largest integer r with $i \leq r < s$ such that P_{j+1} is constant on

$[q_i, q_r]$. Then the same lemma shows that $V_j(q_i) = V_j(q_r)$. By construction, we also have $\mathbf{x}_{j+1}^{(i)} = \mathbf{x}_{j+1}^{(r)} = \mathbf{x}_S$. Thus, upon replacing i by r if necessary, we may assume that P_{j+1} is nonconstant on $[q_i, q_{i+1}]$. This means that $k_i \leq j+1 \leq \ell_{i+1}$. If $j+1 = k_i$, then $(q_i, P_{j+1}(q_i))$ is the right end-point of S and, by construction, we have $\mathbf{x}_S \notin V_j(q_i)$. So, we may further assume that $k_i \leq j < \ell_{i+1}$. By Lemma 3.7, this implies that $V_j(q_i) \neq V_j(q_{i+1})$. However, by the induction hypothesis, we have

$$V_j(q_i) = \langle \mathbf{x}_1^{(i)}, \dots, \mathbf{x}_j^{(i)} \rangle_{\mathbb{R}},$$

$$V_j(q_{i+1}) = \langle \mathbf{x}_1^{(i+1)}, \dots, \mathbf{x}_j^{(i+1)} \rangle_{\mathbb{R}} = \langle \mathbf{x}_1^{(i)}, \dots, \widehat{\mathbf{x}_{k_i}^{(i)}}, \dots, \mathbf{x}_{j+1}^{(i)} \rangle_{\mathbb{R}},$$

where the second expression for $V_j(q_{i+1})$ comes from (3.2). (If $i + 1 = s$, the second formula is not part of the induction hypothesis but follows from the definition of the points $\mathbf{x}_1^{(s)}, \dots, \mathbf{x}_{n-1}^{(s)}$.) So, we conclude that $\mathbf{x}_{j+1}^{(i)} \notin V_j(q_i)$. \square

Proof of property (4). Let $i \in \mathbb{N}$ with $i + 1 < s$, and put $\ell = \ell_{i+1}$. By Lemma 3.8, we have $\mathbf{x}_\ell^{(i+1)} \in V_\ell(q_{i+1})$, and we simply need to show that $\mathbf{x}_\ell^{(i+1)} \in V_\ell(q_i)$. If $\ell = n$, this is clear since $V_n(q_i) = \mathbb{R}^n$. Otherwise, the function $P_{\ell+1}$ is constant on $[q_i, q_{i+1}]$. So, Lemma 3.5 gives $V_\ell(q_i) = V_\ell(q_{i+1})$ and the result follows.

Proof of property (5). By Lemma 3.8, the integer $|\det(\mathbf{x}_1^{(i)}, \dots, \mathbf{x}_n^{(i)})| = \|\mathbf{x}_1^{(i)} \wedge \dots \wedge \mathbf{x}_n^{(i)}\|$ is positive for each i . So, its logarithm is bounded below by 0. By Lemma 2.5 together with the estimates (3.3)–(3.4) and formula (2.7), it is also bounded above by

$$\log(n) - q_i + \sum_{j=1}^n L(\mathbf{x}_j^{(i)}, q_i) \leq \log(n) - q_i + \sum_{j=1}^n (P_j(q_i) + \varepsilon) = n\varepsilon + \log(n).$$

Proof of property (6). Fix an index i with $1 \leq i < s$. We set $k = k_i$ and define

$$\mathbf{y}_j = \mathbf{x}_1^{(i)} \wedge \dots \wedge \widehat{\mathbf{x}_j^{(i)}} \wedge \dots \wedge \mathbf{x}_n^{(i)} \quad (1 \leq j \leq n).$$

Since $\mathbf{y}_k, \dots, \mathbf{y}_n$ are $n - k + 1$ linearly independent elements of $\bigwedge^{n-1} \mathbb{Z}^n$, we have

$$L_{\mathbf{u}, n-k+1}^{(n-1)}(t) \leq \max\{L(\mathbf{y}_k, t), \dots, L(\mathbf{y}_n, t)\} \quad (t \geq 0).$$

We use once again the fact that the functions P_j with $j \neq k$ are constant on the interval $[q_i, q_i + \delta]$, while $P_k(t) = P_k(q_i) + t - q_i$ for each $t \in [q_i, q_i + \delta]$. In view of (3.3) and (3.4), this implies that

$$|L(\mathbf{x}_j^{(i)}, t) - P_j(t)| \leq \varepsilon \quad (1 \leq j \leq n, q_i \leq t \leq q_i + \delta).$$

From now on, we restrict to values of t in the smaller interval $[q_i, q_i + \delta/2]$ as this ensures that $P_j(t) \geq P_k(t) + \delta/2$ for $j = k+1, \dots, n$. For each $j = 1, \dots, n$,

Lemma 2.4 gives

$$\begin{aligned} L(\mathbf{y}_j, t) &\leq \log(n) + L(\mathbf{x}_1^{(i)}, t) + \cdots + \widehat{L(\mathbf{x}_j^{(i)}, t)} + \cdots + L(\mathbf{x}_n^{(i)}, t) \\ &\leq c_1 + (n - 1)\varepsilon + P_1(t) + \cdots + \widehat{P_j(t)} + \cdots + P_n(t) \\ &= c_1 + (n - 1)\varepsilon + t - P_j(t) \quad (\text{by (2.7)}), \end{aligned}$$

while Lemma 2.7(iii) yields

$$L_{\mathbf{u}, n-k+1}^{(n-1)}(t) \geq t - L_{\mathbf{u}, k}(t) - 2c_1 \geq t - P_k(t) - 2c_1 - \varepsilon.$$

As $n\varepsilon + 3c_1 < \delta/2$, this implies that $L_{\mathbf{u}, n-k+1}^{(n-1)}(t) > L(\mathbf{y}_j, t)$ for $j = k + 1, \dots, n$.

Therefore, we must have $L_{\mathbf{u}, n-k+1}^{(n-1)}(t) \leq L(\mathbf{y}_k, t)$, which in turn, in view of the above estimates, implies that

$$t - P_k(t) - 2c_1 - \varepsilon \leq L(\mathbf{y}_k, t) \leq c_1 + (n - 1)\varepsilon + t - P_k(t) \quad (q_i \leq t \leq q_i + \delta/2).$$

As the interval $[q_i, q_i + \delta/2]$ has length greater than $3c_1 + n\varepsilon$ and as the difference $t - P_k(t)$ is constant on this interval, the function $L(\mathbf{y}_k, t)$, having slope 0 then 1, must be constant on $[0, q_i]$ and so we obtain

$$\log \|\mathbf{y}_k\| = L(\mathbf{y}_k, q_i) \geq q_i - P_k(q_i) - 2c_1 - \varepsilon.$$

On the other hand, we have

$$\log \|\mathbf{y}_k\| \leq \sum_{j \neq k} \log \|\mathbf{x}_j^{(i)}\| \leq \sum_{j \neq k} (P_j(q_i) + \varepsilon) = (n - 1)\varepsilon + q_i - P_k(q_i)$$

using property (3) and formula (2.7). Then (6) follows.

4. Distance and height

Here we gather definitions and preliminary results that we will need in the next section to deal with the inverse problem to Schmidt’s and Summerer’s theory.

Let $n \geq 2$ be an integer. We say that a vector subspace V of \mathbb{R}^n is *defined over* \mathbb{Q} if it spanned by elements of \mathbb{Q}^n . If $V \neq 0$, this is equivalent to asking that $V \cap \mathbb{Z}^n$ is a lattice in V . Then, following Schmidt in [16], we define the *height* of V , denoted $H(V)$, to be the co-volume of this lattice in V . It is given by the formula

$$H(V) = \|\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_m\|,$$

where $(\mathbf{x}_1, \dots, \mathbf{x}_m)$ is any basis of $V \cap \mathbb{Z}^n$, using the natural Euclidean norm on $\wedge^m \mathbb{R}^n$ (see Section 2.4). In particular, we note that $H(\mathbb{R}^n) = 1$. We also set $H(0) = 1$. The next result is well known.

LEMMA 4.1. *Let $(\mathbf{x}_1, \dots, \mathbf{x}_n)$ be a basis of \mathbb{Z}^n , and let \mathbf{u} be a unit vector of \mathbb{R}^n perpendicular to $V := \langle \mathbf{x}_1, \dots, \mathbf{x}_{n-1} \rangle_{\mathbb{R}}$. Then, we have $H(V) = |\mathbf{x}_n \cdot \mathbf{u}|^{-1}$.*

Proof. We find

$$1 = \|\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_n\| = \|\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_{n-1}\| |\mathbf{x}_n \cdot \mathbf{u}| = H(V) |\mathbf{x}_n \cdot \mathbf{u}|. \quad \square$$

We define the (projective) *distance* between two nonzero points \mathbf{x} and \mathbf{y} in \mathbb{R}^n by

$$\text{dist}(\mathbf{x}, \mathbf{y}) := \frac{\|\mathbf{x} \wedge \mathbf{y}\|}{\|\mathbf{x}\| \|\mathbf{y}\|}.$$

It represents the sine of the acute angle between the lines spanned by \mathbf{x} and \mathbf{y} in \mathbb{R}^n . As a function on $(\mathbb{R}^n \setminus \{0\})^2$, it is continuous and symmetric. It also satisfies the triangle inequality:

$$\text{dist}(\mathbf{x}, \mathbf{z}) \leq \text{dist}(\mathbf{x}, \mathbf{y}) + \text{dist}(\mathbf{y}, \mathbf{z}) \quad (\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n \setminus \{0\}).$$

For any nonzero point $\mathbf{x} \in \mathbb{R}^n$ and any nonzero subspace V of \mathbb{R}^n , we define the distance from \mathbf{x} to V by

$$\text{dist}(\mathbf{x}, V) := \inf\{\text{dist}(\mathbf{x}, \mathbf{y}) ; \mathbf{y} \in V \setminus \{0\}\} = \inf\{\text{dist}(\mathbf{x}, \mathbf{y}) ; \mathbf{y} \in S^n \cap V\},$$

where S^n denotes the unit sphere of \mathbb{R}^n . Since $S^n \cap V$ is compact and since $\text{dist}(\mathbf{x}, \mathbf{y})$ is a continuous function of $\mathbf{y} \in \mathbb{R}^n \setminus \{0\}$, this infimum is in fact a minimum, achieved by at least one point \mathbf{y} in $S^n \cap V$. We also define $\text{dist}(\mathbf{x}, 0) = 1$ to be consistent with the following result.

LEMMA 4.2. *Let $\mathbf{x} \in \mathbb{R}^n \setminus \{0\}$, and let V be any subspace of \mathbb{R}^n . Then, we have*

$$\text{dist}(\mathbf{x}, V) = \frac{\|\text{proj}_{V^\perp}(\mathbf{x})\|}{\|\mathbf{x}\|}.$$

Moreover, we have $\text{dist}(\mathbf{x}, V) \geq \text{dist}(\mathbf{x}, U)$ for any subspace U of \mathbb{R}^n containing V .

Proof. As the formula gives $\text{dist}(\mathbf{x}, 0) = 1$, we may assume that $V \neq 0$. Since $\mathbb{R}^n = V \perp V^\perp$, we have $\wedge^2 \mathbb{R}^n = \wedge^2 V \perp (V^\perp \wedge V) \perp \wedge^2 V^\perp$. In particular, $\wedge^2 V$ and $V^\perp \wedge V$ are orthogonal subspaces of $\wedge^2 \mathbb{R}^n$. Write $\mathbf{x} = \mathbf{v} + \mathbf{w}$ with $\mathbf{v} \in V$ and $\mathbf{w} \in V^\perp$ so that $\mathbf{w} = \text{proj}_{V^\perp}(\mathbf{x})$. For any $\mathbf{y} \in V \setminus \{0\}$, we find

$$\text{dist}(\mathbf{x}, \mathbf{y}) = \frac{\|(\mathbf{v} \wedge \mathbf{y}) + (\mathbf{w} \wedge \mathbf{y})\|}{\|\mathbf{x}\| \|\mathbf{y}\|} \geq \frac{\|\mathbf{w} \wedge \mathbf{y}\|}{\|\mathbf{x}\| \|\mathbf{y}\|} = \frac{\|\mathbf{w}\|}{\|\mathbf{x}\|}$$

with equality if and only if $\mathbf{v} \wedge \mathbf{y} = 0$. As there exists $\mathbf{y} \in V \setminus \{0\}$ satisfying the latter condition, this proves the first assertion. The second one is clear. \square

Finally, given nonzero subspaces V_1 and V_2 of \mathbb{R}^n , we define the distance from V_1 to V_2 by

$$\text{dist}(V_1, V_2) := \sup\{\text{dist}(\mathbf{x}, V_2) ; \mathbf{x} \in V_1 \setminus \{0\}\} = \sup\{\text{dist}(\mathbf{x}, V_2) ; \mathbf{x} \in S^n \cap V_1\}.$$

Since $S^n \cap V_1$ is compact and since the above lemma shows that $\text{dist}(\mathbf{x}, V_2)$ is a continuous function of \mathbf{x} on $\mathbb{R}^n \setminus \{0\}$, this supremum is also achieved by

some point $\mathbf{x} \in S^n \cap V_1$. Note that this distance is not, in general, a symmetric function of V_1 and V_2 . However, it satisfies the following property.

LEMMA 4.3. *Let $\mathbf{x} \in \mathbb{R}^n \setminus \{0\}$, and let V_1, V_2 be nonzero subspaces of \mathbb{R}^n . Then, we have*

$$\text{dist}(\mathbf{x}, V_2) \leq \text{dist}(\mathbf{x}, V_1) + \text{dist}(V_1, V_2).$$

Moreover, if V is also a nonzero subspace of \mathbb{R}^n , then

$$\text{dist}(V, V_2) \leq \text{dist}(V, V_1) + \text{dist}(V_1, V_2).$$

Proof. Choose $\mathbf{y} \in S^n \cap V_1$ such that $\text{dist}(\mathbf{x}, V_1) = \text{dist}(\mathbf{x}, \mathbf{y})$, and choose $\mathbf{z} \in S^n \cap V_2$ such that $\text{dist}(\mathbf{y}, V_2) = \text{dist}(\mathbf{y}, \mathbf{z})$. Then

$$\begin{aligned} \text{dist}(\mathbf{x}, V_2) &\leq \text{dist}(\mathbf{x}, \mathbf{z}) \leq \text{dist}(\mathbf{x}, \mathbf{y}) + \text{dist}(\mathbf{y}, \mathbf{z}) \\ &= \text{dist}(\mathbf{x}, V_1) + \text{dist}(\mathbf{y}, V_2) \leq \text{dist}(\mathbf{x}, V_1) + \text{dist}(V_1, V_2). \end{aligned}$$

The second inequality follows from this by choosing a point $\mathbf{x} \in V$ that satisfies $\text{dist}(V, V_2) = \text{dist}(\mathbf{x}, V_2)$ and then using $\text{dist}(\mathbf{x}, V_1) \leq \text{dist}(V, V_1)$. \square

The next lemma shows a case where the distance between two vector subspaces is a symmetric function of the subspaces.

LEMMA 4.4. *Let V_1, V_2 be nonzero subspaces of co-dimension 1 inside a subspace U of \mathbb{R}^n . For $i = 1, 2$, choose a unit vector \mathbf{u}_i in $U \cap V_i^\perp$. Then, we have*

$$\text{dist}(V_1, V_2) = \text{dist}(\mathbf{u}_1, \mathbf{u}_2).$$

Proof. For the first assertion, we may assume that $V_1 \neq V_2$ because otherwise $\text{dist}(V_1, V_2) = 0 = \text{dist}(\mathbf{u}_1, \mathbf{u}_2)$. Under this hypothesis, we define $W := V_1 \cap V_2$ and, for $j = 1, 2$, we choose a unit vector \mathbf{v}_j in $V_j \cap W^\perp$. Then $(\mathbf{u}_1, \mathbf{v}_1)$ and $(\mathbf{u}_2, \mathbf{v}_2)$ are two orthonormal bases of $U \cap W^\perp$, and so

$$\text{dist}(\mathbf{u}_1, \mathbf{u}_2) = \|\mathbf{u}_1 \wedge \mathbf{u}_2\| = |\mathbf{v}_1 \cdot \mathbf{u}_2|.$$

Let \mathbf{x} be a nonzero element of V_1 . Upon writing $\mathbf{x} = \mathbf{w} + a\mathbf{v}_1$ with $\mathbf{w} \in W$ and $a \in \mathbb{R}$, we find, using Lemma 4.2,

$$\text{dist}(\mathbf{x}, V_2) = \frac{\|\text{proj}_{V_2^\perp}(\mathbf{w} + a\mathbf{v}_1)\|}{\|\mathbf{w} + a\mathbf{v}_1\|} = \frac{|a| |\mathbf{v}_1 \cdot \mathbf{u}_2|}{\|\mathbf{w} + a\mathbf{v}_1\|} \leq |\mathbf{v}_1 \cdot \mathbf{u}_2|$$

with equality if and only if $\mathbf{w} = 0$. Thus, $\text{dist}(V_1, V_2) = |\mathbf{v}_1 \cdot \mathbf{u}_2| = \text{dist}(\mathbf{u}_1, \mathbf{u}_2)$. \square

Definition 4.5. We say that a nonempty sequence $(\mathbf{x}_1, \dots, \mathbf{x}_m)$ of vectors of \mathbb{R}^n is *almost orthogonal* if it is linearly independent and satisfies

$$\text{dist}(\mathbf{x}_j, \langle \mathbf{x}_1, \dots, \mathbf{x}_{j-1} \rangle_{\mathbb{R}}) \geq 1 - \frac{1}{2^{j-1}} \quad (2 \leq j \leq m).$$

Thus, any sequence (\mathbf{x}) consisting of just one nonzero vector $\mathbf{x} \in \mathbb{R}^n$ is almost orthogonal. By Lemma 4.2, it follows that any nonempty subsequence of an almost orthogonal sequence is almost orthogonal. We conclude this section with two more facts linked with this notion.

LEMMA 4.6. *Let $(\mathbf{x}_1, \dots, \mathbf{x}_m)$ be an almost orthogonal primitive m -tuple of points of \mathbb{Z}^n , and let $U := \langle \mathbf{x}_1, \dots, \mathbf{x}_m \rangle_{\mathbb{R}}$. Then, we have*

$$e^{-2} \|\mathbf{x}_1\| \cdots \|\mathbf{x}_m\| \leq H(U) \leq \|\mathbf{x}_1\| \cdots \|\mathbf{x}_m\|.$$

Proof. Since $H(U) = \|\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_m\|$, the upper bound is clear. For the lower bound, we use induction to show that

$$(4.1) \quad \|\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_k\| \geq \|\mathbf{x}_1\| \cdots \|\mathbf{x}_k\| \prod_{j=2}^k (1 - 2^{1-j}) \quad (1 \leq k \leq m).$$

Since $\prod_{j=2}^m (1 - 2^{1-j}) \geq \prod_{j=2}^{\infty} \exp(-2^{2-j}) = e^{-2}$, this will complete the proof. For $k = 1$, the inequality (4.1) is clear. Suppose that it holds for some integer k with $1 \leq k < m$. Set $V = \langle \mathbf{x}_1, \dots, \mathbf{x}_k \rangle_{\mathbb{R}}$. Using Lemma 4.2, we find

$$\begin{aligned} \|\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_{k+1}\| &= \|\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_k\| \|\text{proj}_{V^\perp}(\mathbf{x}_{k+1})\| \\ &= \|\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_k\| \|\mathbf{x}_{k+1}\| \text{dist}(\mathbf{x}_{k+1}, V) \\ &\geq \|\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_k\| \|\mathbf{x}_{k+1}\| (1 - 2^{-k}). \end{aligned}$$

So, (4.1) also holds with k replaced by $k + 1$. □

LEMMA 4.7. *Let $k, \ell, m \in \{1, \dots, n\}$ with $k < \ell \leq m$, let U be a subspace of \mathbb{R}^n of dimension m defined over \mathbb{Q} , and let $(\mathbf{y}_1, \dots, \mathbf{y}_m)$ be a basis of $U \cap \mathbb{Z}^n$. Define*

$$V_1 = \langle \mathbf{y}_1, \dots, \widehat{\mathbf{y}}_\ell, \dots, \mathbf{y}_m \rangle_{\mathbb{R}} \quad \text{and} \quad V_2 = \langle \mathbf{y}_1, \dots, \widehat{\mathbf{y}}_k, \dots, \mathbf{y}_m \rangle_{\mathbb{R}}.$$

Then, we have

$$(4.2) \quad \text{dist}(V_1, V_2) = \frac{H(V_1 \cap V_2)H(U)}{H(V_1)H(V_2)}.$$

Moreover, if $(\mathbf{y}_1, \dots, \widehat{\mathbf{y}}_\ell, \dots, \mathbf{y}_m)$ and $(\mathbf{y}_1, \dots, \widehat{\mathbf{y}}_k, \dots, \mathbf{y}_m)$ are almost orthogonal, then

$$(4.3) \quad \text{dist}(V_1, V_2) \leq \frac{e^4 H(U)}{\|\mathbf{y}_1\| \cdots \|\mathbf{y}_m\|}.$$

Proof. Set $W = V_1 \cap V_2 = \langle \mathbf{y}_1, \dots, \widehat{\mathbf{y}}_k, \dots, \widehat{\mathbf{y}}_\ell, \dots, \mathbf{y}_m \rangle_{\mathbb{R}}$. For $j = 1, 2$, we choose unit vectors \mathbf{u}_j in $U \cap V_j^\perp$ and \mathbf{v}_j in $V_j \cap W^\perp$. Then $(\mathbf{u}_1, \mathbf{v}_1)$ and $(\mathbf{u}_2, \mathbf{v}_2)$ are orthonormal bases of $U \cap W^\perp$. So, $\mathbf{v}_1 \wedge \mathbf{v}_2 = \pm \mathbf{u}_1 \wedge \mathbf{u}_2$ and, by Lemma 4.4, we obtain $\|\mathbf{v}_1 \wedge \mathbf{v}_2\| = \text{dist}(V_1, V_2)$.

Define $\omega = \mathbf{y}_1 \wedge \cdots \wedge \widehat{\mathbf{y}}_k \wedge \cdots \wedge \widehat{\mathbf{y}}_\ell \wedge \cdots \wedge \mathbf{y}_m$, so that $H(W) = \|\omega\|$. Upon writing $\mathbf{y}_k = \mathbf{w}_1 + a_1\mathbf{v}_1$ and $\mathbf{y}_\ell = \mathbf{w}_2 + a_2\mathbf{v}_2$ with $\mathbf{w}_1, \mathbf{w}_2 \in W$ and $a_1, a_2 \in \mathbb{R}$, we find

$$\begin{aligned} H(V_j) &= \|\omega \wedge (\mathbf{w}_j + a_j\mathbf{v}_j)\| = \|\omega\| |a_j| \quad (j = 1, 2), \\ H(U) &= \|\omega \wedge (\mathbf{w}_1 + a_1\mathbf{v}_1) \wedge (\mathbf{w}_2 + a_2\mathbf{v}_2)\| \\ &= \|\omega\| \|a_1a_2\mathbf{v}_1 \wedge \mathbf{v}_2\| = \|\omega\| |a_1a_2| \operatorname{dist}(V_1, V_2), \end{aligned}$$

and (4.2) follows because $a_1a_2 \neq 0$.

If $(\mathbf{y}_1, \dots, \widehat{\mathbf{y}}_\ell, \dots, \mathbf{y}_m)$ and $(\mathbf{y}_1, \dots, \widehat{\mathbf{y}}_k, \dots, \mathbf{y}_m)$ are almost orthogonal, Lemma 4.6 gives

$$H(V_1) \geq \frac{\|\mathbf{y}_1\| \cdots \|\mathbf{y}_m\|}{e^2 \|\mathbf{y}_\ell\|} \quad \text{and} \quad H(V_2) \geq \frac{\|\mathbf{y}_1\| \cdots \|\mathbf{y}_m\|}{e^2 \|\mathbf{y}_k\|}.$$

Then (4.3) follows from (4.2) using

$$H(W) = \|\omega\| \leq (\|\mathbf{y}_1\| \cdots \|\mathbf{y}_m\|) / (\|\mathbf{y}_k\| \|\mathbf{y}_\ell\|). \quad \square$$

5. The inverse problem for rigid systems

The goal of this section is to prove a partial converse to Schmidt’s and Summerer’s Theorem 2.9 for rigid systems with large mesh. Motivated by Theorem 3.1, we start by constructing recursively a sequence of bases of \mathbb{Z}^n with several strong properties. The key to the recurrence is provided by the following lemma.

LEMMA 5.1. *Let $h, k, \ell \in \{1, \dots, n\}$ with $h \leq \ell$ and $k < \ell$, let $(\mathbf{x}_1, \dots, \mathbf{x}_n)$ be a basis of \mathbb{Z}^n , and let A be a real number with $A \geq 2^\ell(\|\mathbf{x}_1\| + \cdots + \|\mathbf{x}_\ell\|)$. Then, there exists a basis $(\mathbf{y}_1, \dots, \mathbf{y}_n)$ of \mathbb{Z}^n satisfying*

- (1) $(\mathbf{y}_1, \dots, \widehat{\mathbf{y}}_\ell, \dots, \mathbf{y}_n) = (\mathbf{x}_1, \dots, \widehat{\mathbf{x}}_h, \dots, \mathbf{x}_n)$;
- (2) $\mathbf{y}_\ell \in \mathbf{x}_h + \langle \mathbf{x}_1, \dots, \widehat{\mathbf{x}}_h, \dots, \mathbf{x}_\ell \rangle_{\mathbb{Z}}$;
- (3) $A \leq \|\mathbf{y}_\ell\| \leq 2A$;
- (4) $\operatorname{dist}(\mathbf{y}_\ell, \langle \mathbf{y}_1, \dots, \widehat{\mathbf{y}}_k, \dots, \mathbf{y}_{\ell-1} \rangle_{\mathbb{R}}) \geq 1 - \frac{1}{2^{\ell-1}}$.

Moreover, if $(\mathbf{x}_1, \dots, \widehat{\mathbf{x}}_h, \dots, \mathbf{x}_\ell)$ is almost orthogonal, then $(\mathbf{y}_1, \dots, \widehat{\mathbf{y}}_k, \dots, \mathbf{y}_\ell)$ is almost orthogonal.

Note that condition (4) is stronger than needed in order to establish that the sequence $(\mathbf{y}_1, \dots, \widehat{\mathbf{y}}_k, \dots, \mathbf{y}_\ell)$ is almost orthogonal. In view of the definition, a lower bound of $1 - 1/2^{\ell-2}$ would suffice instead of $1 - 1/2^{\ell-1}$. However, this stronger requirement will show its importance later. Note also that (4) trivially holds when $\ell = 2$.

Proof. We use (1) as a definition of the vectors $\mathbf{y}_1, \dots, \widehat{\mathbf{y}}_\ell, \dots, \mathbf{y}_n$. Then, $(\mathbf{y}_1, \dots, \mathbf{y}_n)$ is a basis of \mathbb{Z}^n for any choice of \mathbf{y}_ℓ in $\mathbf{x}_h + \langle \mathbf{x}_1, \dots, \widehat{\mathbf{x}}_h, \dots, \mathbf{x}_\ell \rangle_{\mathbb{Z}}$.

A fortiori, it is a basis of \mathbb{Z}^n for any choice of \mathbf{y}_ℓ satisfying (2). To prove the existence of a point \mathbf{y}_ℓ satisfying (2), (3) and (4), we first observe that the hypothesis $h \leq \ell$ yields

$$(\mathbf{y}_1, \dots, \mathbf{y}_{\ell-1}) = (\mathbf{x}_1, \dots, \widehat{\mathbf{x}}_h, \dots, \mathbf{x}_\ell).$$

Consider the chain of subspaces $W \subset V \subset U$ of relative codimension 1 given by

$$\begin{aligned} W &= \langle \mathbf{y}_1, \dots, \widehat{\mathbf{y}}_k, \dots, \mathbf{y}_{\ell-1} \rangle_{\mathbb{R}}, \\ V &= \langle \mathbf{y}_1, \dots, \mathbf{y}_{\ell-1} \rangle_{\mathbb{R}} = \langle \mathbf{x}_1, \dots, \widehat{\mathbf{x}}_h, \dots, \mathbf{x}_\ell \rangle_{\mathbb{R}}, \\ U &= \langle \mathbf{x}_1, \dots, \mathbf{x}_\ell \rangle_{\mathbb{R}}. \end{aligned}$$

Choose unit vectors $\mathbf{u} \in U \cap V^\perp$ and $\mathbf{v} \in V \cap W^\perp$. Then $\{\mathbf{u}, \mathbf{v}\}$ is an orthonormal basis of $U \cap W^\perp$ and we have $\text{proj}_{V^\perp}(\mathbf{x}_h) = c\mathbf{u}$, where $c = \mathbf{x}_h \cdot \mathbf{u}$. Put $B = (3/2)A$. Since the vector $c\mathbf{u} + B\mathbf{v} - \mathbf{x}_h$ belongs to V , we can write

$$c\mathbf{u} + B\mathbf{v} = \mathbf{x}_h + \sum_{j=1}^{\ell-1} c_j \mathbf{y}_j$$

for some $c_1, \dots, c_{\ell-1} \in \mathbb{R}$. We choose

$$\mathbf{y}_\ell := \mathbf{x}_h + \sum_{j=1}^{\ell-1} [c_j] \mathbf{y}_j = c\mathbf{u} + B\mathbf{v} + \sum_{j=1}^{\ell-1} \varepsilon_j \mathbf{y}_j,$$

where $\varepsilon_j = [c_j] - c_j \in [0, 1)$ for $j = 1, \dots, \ell - 1$. Then, condition (2) is fulfilled and

$$\left| \|\mathbf{y}_\ell\| - B \right| \leq \|\mathbf{y}_\ell - B\mathbf{v}\| \leq |c| + \sum_{j=1}^{\ell-1} \|\mathbf{y}_j\| \leq \sum_{j=1}^{\ell} \|\mathbf{x}_j\| \leq \frac{B}{2^\ell},$$

so that $(1 - 2^{-\ell})B \leq \|\mathbf{y}_\ell\| \leq (1 + 2^{-\ell})B$. Since $\ell \geq 2$, this shows that (3) holds. Moreover, we find that

$$\|\text{proj}_{W^\perp}(\mathbf{y}_\ell)\| \geq |\mathbf{y}_\ell \cdot \mathbf{v}| = |B + \varepsilon_k \mathbf{y}_k \cdot \mathbf{v}| \geq B - \|\mathbf{y}_k\| \geq (1 - 2^{-\ell})B$$

and, therefore,

$$\text{dist}(\mathbf{y}_\ell, W) = \frac{\|\text{proj}_{W^\perp}(\mathbf{y}_\ell)\|}{\|\mathbf{y}_\ell\|} \geq \frac{1 - 2^{-\ell}}{1 + 2^{-\ell}} \geq 1 - \frac{1}{2^{\ell-1}}.$$

Thus \mathbf{y}_ℓ also satisfies (4). Finally, if $(\mathbf{x}_1, \dots, \widehat{\mathbf{x}}_h, \dots, \mathbf{x}_\ell) = (\mathbf{y}_1, \dots, \mathbf{y}_{\ell-1})$ is almost orthogonal, then its subsequence $(\mathbf{y}_1, \dots, \widehat{\mathbf{y}}_k, \dots, \mathbf{y}_{\ell-1})$ is almost orthogonal, and by (4) we conclude that $(\mathbf{y}_1, \dots, \widehat{\mathbf{y}}_k, \dots, \mathbf{y}_\ell)$ is almost orthogonal. \square

The next lemma provides us with an initial basis of \mathbb{Z}^n to start the recurrence.

LEMMA 5.2. *Let k be an integer with $1 \leq k \leq n$, and let A_1, \dots, A_n be real numbers satisfying $A_1 \geq 8$ and $A_j \geq 2^{j+3}A_{j-1}$ for $j = 2, \dots, n$. Then there exists a basis $(\mathbf{y}_1, \dots, \mathbf{y}_n)$ of \mathbb{Z}^n that satisfies the following conditions:*

- (i) $(\mathbf{y}_1, \dots, \mathbf{y}_{n-1})$ and $(\mathbf{y}_1, \dots, \widehat{\mathbf{y}}_k, \dots, \mathbf{y}_n)$ are almost orthogonal,
- (ii) $A_j \leq \|\mathbf{y}_j\| \leq 2A_j$ for $j = 1, \dots, n$,
- (iii) $\text{dist}(\mathbf{y}_n, \langle \mathbf{y}_1, \dots, \widehat{\mathbf{y}}_k, \dots, \mathbf{y}_{n-1} \rangle_{\mathbb{R}}) \geq 1 - 1/2^{n-1}$ if $k < n$.

Note that, like in the preceding lemma, the lower bound $1 - 1/2^{n-1}$ imposed in condition (iii) is larger than the lower bound $1 - 1/2^{n-2}$ that follows from condition (i) if $k < n$.

Proof. Let $(\mathbf{e}_1, \dots, \mathbf{e}_n)$ denote the canonical basis of \mathbb{Z}^n . Lemma 5.1 allows us to construct recursively a sequence of integer points $\mathbf{x}_1, \dots, \mathbf{x}_n \in \mathbb{Z}^n$ starting with $\mathbf{x}_1 = \mathbf{e}_1$ that, for each $j = 2, \dots, n$, satisfy the following properties:

- (1) $(\mathbf{x}_1, \dots, \mathbf{x}_{j-1}, \mathbf{e}_j, \dots, \mathbf{e}_n)$ is a basis of \mathbb{Z}^n ;
- (2) $\mathbf{x}_j \in \mathbf{e}_j + \langle \mathbf{x}_1, \dots, \mathbf{x}_{j-1} \rangle_{\mathbb{Z}}$;
- (3) $A_{j-1} \leq \|\mathbf{x}_j\| \leq 2A_{j-1}$;
- (4) $\text{dist}(\mathbf{x}_j, \langle \mathbf{x}_2, \dots, \mathbf{x}_{j-1} \rangle_{\mathbb{R}}) \geq 1 - 1/2^{j-1}$ if $j \geq 3$.

Indeed, condition (1) holds for $j = 2$ because $\mathbf{x}_1 = \mathbf{e}_1$. Suppose that we have constructed $\mathbf{x}_1, \dots, \mathbf{x}_{\ell-1}$ for some ℓ with $2 \leq \ell \leq n$ so that (1) holds when $2 \leq j \leq \ell$ while (2), (3) and (4) hold when $2 \leq j \leq \ell - 1$. We apply Lemma 5.1 to the basis $(\mathbf{x}_1, \dots, \mathbf{x}_{\ell-1}, \mathbf{e}_{\ell}, \dots, \mathbf{e}_n)$ of \mathbb{Z}^n with the choice of $h = \ell$, $k = 1$ and $A = A_{\ell-1}$. Since

$$2^{\ell}(\|\mathbf{x}_1\| + \dots + \|\mathbf{x}_{\ell-1}\| + \|\mathbf{e}_{\ell}\|) \leq 2^{\ell+1} \left(1 + \sum_{2 \leq j \leq \ell-1} A_{j-1}\right) \leq A_{\ell-1},$$

it provides a new point \mathbf{x}_{ℓ} of \mathbb{Z}^n that satisfies condition (1) for $j = \ell + 1$ as well as conditions (2), (3) and (4) for $j = \ell$. This proves our claim and, in the case $\ell = n$, shows that $(\mathbf{x}_1, \dots, \mathbf{x}_n)$ is a basis of \mathbb{Z}^n . Moreover, since (4) holds for $j = 3, \dots, n$, we also note that $(\mathbf{x}_2, \dots, \mathbf{x}_n)$ is almost orthogonal.

If $k < n$, once again we apply Lemma 5.1, this time to the basis $(\mathbf{x}_1, \dots, \mathbf{x}_n)$ with the choice of $h = 1$, $\ell = n$ and the given value of k . Since

$$2^n(\|\mathbf{x}_1\| + \dots + \|\mathbf{x}_n\|) \leq 2^{n+1} \left(1 + \sum_{2 \leq j \leq n} A_{j-1}\right) \leq A_n,$$

it provides a new basis of \mathbb{Z}^n of the form $(\mathbf{y}_1, \dots, \mathbf{y}_n) = (\mathbf{x}_2, \dots, \mathbf{x}_n, \mathbf{y}_n)$ with all the requested properties. Otherwise, it suffices to choose $(\mathbf{y}_1, \dots, \mathbf{y}_n) = (\mathbf{x}_2, \dots, \mathbf{x}_n, \mathbf{x}_1 + B\mathbf{x}_2)$ with $B = \lceil A_n/\|\mathbf{x}_2\| \rceil + 1$. \square

From now on, we fix a choice of $s \in \mathbb{N}^* \cup \{\infty\}$ and put

$$C = 2^{n+3}e^4.$$

We suppose that, for each integer i with $0 \leq i < s$, we are given a point $\mathbf{A}^{(i)} = (A_1^{(i)}, \dots, A_n^{(i)})$ in \mathbb{R}^n and integers k_i and ℓ_i satisfying the conditions

$$(5.1) \quad 1 \leq k_0 \leq \ell_0 = n \quad \text{and} \quad 1 \leq k_i < \ell_i \leq n \quad \text{if} \quad i \geq 1,$$

$$(5.2) \quad A_1^{(i)} \geq C, \quad A_j^{(i)} \geq A_{j-1}^{(i)} C \quad \text{for} \quad j = 2, \dots, n,$$

$$(5.3) \quad k_{i-1} \leq \ell_i \quad \text{and} \quad A_{\ell_i}^{(i)} \geq A_{\ell_i}^{(i-1)} C \quad \text{if} \quad i \geq 1,$$

$$(5.4) \quad (A_1^{(i)}, \dots, \widehat{A_{\ell_i}^{(i)}}, \dots, A_n^{(i)}) = (A_1^{(i-1)}, \dots, \widehat{A_{k_{i-1}}^{(i-1)}}, \dots, A_n^{(i-1)}) \quad \text{if} \quad i \geq 1.$$

We now combine the previous lemmas to establish the following result.

PROPOSITION 5.3. *For each integer i with $0 \leq i < s$, there exists a basis $(\mathbf{x}_1^{(i)}, \dots, \mathbf{x}_n^{(i)})$ of \mathbb{Z}^n that satisfies the following properties:*

- (1) $(\mathbf{x}_1^{(0)}, \dots, \mathbf{x}_{n-1}^{(0)})$ is almost orthogonal;
- (2) $(\mathbf{x}_1^{(i)}, \dots, \widehat{\mathbf{x}_{k_i}^{(i)}}, \dots, \mathbf{x}_n^{(i)})$ is almost orthogonal;
- (3) $A_j^{(i)} \leq \|\mathbf{x}_j^{(i)}\| \leq 2A_j^{(i)}$ for $j = 1, \dots, n$;
- (4) $\text{dist}\left(\mathbf{x}_{\ell_i}^{(i)}, \langle \mathbf{x}_1^{(i)}, \dots, \widehat{\mathbf{x}_{k_i}^{(i)}}, \dots, \mathbf{x}_{\ell_i-1}^{(i)} \rangle_{\mathbb{R}}\right) \geq 1 - \frac{1}{2^{\ell_i-1}}$ if $k_i < \ell_i$;
- (5) $\mathbf{x}_{\ell_i}^{(i)} \in \mathbf{x}_{k_{i-1}}^{(i-1)} + \langle \mathbf{x}_1^{(i-1)}, \dots, \widehat{\mathbf{x}_{k_{i-1}}^{(i-1)}}, \dots, \mathbf{x}_{\ell_i}^{(i-1)} \rangle_{\mathbb{Z}}$ if $i \geq 1$;
- (6) $(\mathbf{x}_1^{(i)}, \dots, \widehat{\mathbf{x}_{\ell_i}^{(i)}}, \dots, \mathbf{x}_n^{(i)}) = (\mathbf{x}_1^{(i-1)}, \dots, \widehat{\mathbf{x}_{k_{i-1}}^{(i-1)}}, \dots, \mathbf{x}_n^{(i-1)})$ if $i \geq 1$.

Moreover, if \mathbf{u}_i denotes a unit vector orthogonal to $\langle \mathbf{x}_1^{(i)}, \dots, \widehat{\mathbf{x}_{k_i}^{(i)}}, \dots, \mathbf{x}_n^{(i)} \rangle_{\mathbb{R}}$ then, for each pair of integers i and j with $0 \leq i < j < s$, we have

$$\text{dist}(\mathbf{u}_i, \mathbf{u}_j) \leq \frac{2e^4}{\|\mathbf{x}_1^{(i+1)}\| \dots \|\mathbf{x}_n^{(i+1)}\|}.$$

Proof. We prove the first assertion by induction on i . To construct the initial basis $(\mathbf{x}_1^{(0)}, \dots, \mathbf{x}_n^{(0)})$, we apply Lemma 5.2 with the choice of $k = k_0$ and $A_j = A_j^{(0)}$ for $j = 1, \dots, n$. Thanks to (5.2), the hypotheses of the lemma are satisfied and, since $\ell_0 = n$, the basis that we obtain satisfies the conditions (1) to (4) of the proposition for $i = 0$. Conditions (5) and (6) are vacuous.

Now, let t be an integer with $1 \leq t < s$. Suppose that, for $i = 0, \dots, t - 1$, we have constructed a basis $(\mathbf{x}_1^{(i)}, \dots, \mathbf{x}_n^{(i)})$ of \mathbb{Z}^n that satisfies all conditions (1)–(6). To construct the next basis, we apply Lemma 5.1 to $(\mathbf{x}_1^{(t-1)}, \dots, \mathbf{x}_n^{(t-1)})$ for the choice of $h = k_{t-1}$, $k = k_t$, $\ell = \ell_t$ and $A = A_{\ell_t}^{(t)}$. Since (3) holds for $i = t - 1$, we find with the help of (5.2) and (5.3) that

$$2^{\ell_t} (\|\mathbf{x}_1^{(t-1)}\| + \dots + \|\mathbf{x}_{\ell_t}^{(t-1)}\|) \leq 2^{\ell_t+1} (A_1^{(t-1)} + \dots + A_{\ell_t}^{(t-1)}) \leq 2^{\ell_t+2} A_{\ell_t}^{(t-1)} \leq A_{\ell_t}^{(t)}.$$

So, the lemma produces a new basis $(\mathbf{x}_1^{(t)}, \dots, \mathbf{x}_n^{(t)})$ of \mathbb{Z}^n that satisfies the conditions (4)–(6) for $i = t$, as well as $A_{\ell_t}^{(t)} \leq \|\mathbf{x}_{\ell_t}^{(t)}\| \leq 2A_{\ell_t}^{(t)}$. Then, combining

our hypothesis that (3) holds for $i = t - 1$ with the equalities (6) and (5.4) for $i = t$, we conclude that (3) holds for $i = t$.

To complete the inductive step, it only remains to show that the sequence $(\mathbf{x}_1^{(t)}, \dots, \widehat{\mathbf{x}_{k_t}^{(t)}}, \dots, \mathbf{x}_n^{(t)})$ is almost orthogonal. This is the most delicate part of the argument. To prove this, we show by induction on m that its subsequence $(\mathbf{x}_1^{(t)}, \dots, \widehat{\mathbf{x}_{k_t}^{(t)}}, \dots, \mathbf{x}_m^{(t)})$ is almost orthogonal for $m = \ell_t, \dots, n$. When $m = \ell_t$, this follows from our construction, thanks to the last assertion of Lemma 5.1, because $(\mathbf{x}_1^{(t-1)}, \dots, \widehat{\mathbf{x}_{k_{t-1}}^{(t-1)}}, \dots, \mathbf{x}_{\ell_t}^{(t-1)})$ is almost orthogonal. If $\ell_t = n$, we are done. Otherwise, fix an integer m with $\ell_t \leq m < n$ and assume that our claim is true for that value of m . Since $\ell_0 = n > m$, there exists an index r with $0 \leq r < t$ such that

$$(5.5) \quad \ell_r > m \quad \text{and} \quad \ell_{r+1}, \dots, \ell_t \leq m.$$

In particular, we have $k_r, \dots, k_t \leq m$. Thus $(\mathbf{x}_1^{(i)}, \dots, \widehat{\mathbf{x}_{k_i}^{(i)}}, \dots, \mathbf{x}_m^{(i)})$ is almost orthogonal for $i = r, \dots, t$, while

$$(\mathbf{x}_1^{(i)}, \dots, \widehat{\mathbf{x}_{\ell_i}^{(i)}}, \dots, \mathbf{x}_m^{(i)}) = (\mathbf{x}_1^{(i-1)}, \dots, \widehat{\mathbf{x}_{k_{i-1}}^{(i-1)}}, \dots, \mathbf{x}_m^{(i-1)})$$

is almost orthogonal for $i = r + 1, \dots, t$. Define

$$U^{(i)} = \langle \mathbf{x}_1^{(i)}, \dots, \mathbf{x}_m^{(i)} \rangle_{\mathbb{R}} \quad \text{and} \quad V^{(i)} = \langle \mathbf{x}_1^{(i)}, \dots, \widehat{\mathbf{x}_{k_i}^{(i)}}, \dots, \mathbf{x}_m^{(i)} \rangle_{\mathbb{R}} \quad (r \leq i \leq t).$$

As $V^{(i-1)} = \langle \mathbf{x}_1^{(i)}, \dots, \widehat{\mathbf{x}_{\ell_i}^{(i)}}, \dots, \mathbf{x}_m^{(i)} \rangle_{\mathbb{R}}$ for $i = r + 1, \dots, t$, Lemma 4.7 gives

$$\text{dist}(V^{(i-1)}, V^{(i)}) \leq e^4 \frac{H(U^{(i)})}{\|\mathbf{x}_1^{(i)}\| \cdots \|\mathbf{x}_m^{(i)}\|} \quad (r + 1 \leq i \leq t).$$

By (5), we also have $U^{(i-1)} = V^{(i-1)} + \langle \mathbf{x}_{k_{i-1}}^{(i-1)} \rangle_{\mathbb{R}} = V^{(i-1)} + \langle \mathbf{x}_{\ell_i}^{(i)} \rangle_{\mathbb{R}} = U^{(i)}$ for $i = r + 1, \dots, t$. Thus $U^{(r)} = U^{(r+1)} = \dots = U^{(t)}$ and, therefore, using Lemma 4.3, we obtain

$$\text{dist}(V^{(r)}, V^{(t)}) \leq \sum_{i=r+1}^t \text{dist}(V^{(i-1)}, V^{(i)}) \leq e^4 \sum_{i=r+1}^t \frac{H(U^{(r)})}{\|\mathbf{x}_1^{(i)}\| \cdots \|\mathbf{x}_m^{(i)}\|}.$$

With the help of (6), (3), (5.2) and (5.3), we also note that

$$(5.6) \quad \frac{\|\mathbf{x}_1^{(i-1)}\| \cdots \|\mathbf{x}_m^{(i-1)}\|}{\|\mathbf{x}_1^{(i)}\| \cdots \|\mathbf{x}_m^{(i)}\|} = \frac{\|\mathbf{x}_{k_{i-1}}^{(i-1)}\|}{\|\mathbf{x}_{\ell_i}^{(i)}\|} \leq \frac{2A_{k_{i-1}}^{(i-1)}}{A_{\ell_i}^{(i)}} \leq \frac{2A_{\ell_i}^{(i-1)}}{A_{\ell_i}^{(i)}} \leq \frac{2}{C}$$

for $i = r + 1, \dots, t$. This yields

$$\begin{aligned} \text{dist}(V^{(r)}, V^{(t)}) &\leq 2e^4 \frac{H(U^{(r)})}{\|\mathbf{x}_1^{(r+1)}\| \cdots \|\mathbf{x}_m^{(r+1)}\|} \\ &\leq 2e^4 \frac{\|\mathbf{x}_1^{(r)}\| \cdots \|\mathbf{x}_m^{(r)}\|}{\|\mathbf{x}_1^{(r+1)}\| \cdots \|\mathbf{x}_m^{(r+1)}\|} \leq \frac{4e^4}{C} \leq \frac{1}{2^n}. \end{aligned}$$

According to (6) and (5.5), we also have $\mathbf{x}_{m+1}^{(r)} = \mathbf{x}_{m+1}^{(r+1)} = \cdots = \mathbf{x}_{m+1}^{(t)}$. Therefore, we conclude, by Lemma 4.3, that

$$\text{dist}(\mathbf{x}_{m+1}^{(t)}, V^{(t)}) \geq \text{dist}(\mathbf{x}_{m+1}^{(t)}, V^{(r)}) - \text{dist}(V^{(r)}, V^{(t)}) \geq \text{dist}(\mathbf{x}_{m+1}^{(r)}, V^{(r)}) - \frac{1}{2^n}.$$

We recall at this point that $\ell_r > m$. If $r > 0$ and $\ell_r > m + 1$, then $(\mathbf{x}_1^{(r)}, \dots, \mathbf{x}_{m+1}^{(r)})$ is almost orthogonal because it is a subsequence of

$$(\mathbf{x}_1^{(r)}, \dots, \widehat{\mathbf{x}_{\ell_r}^{(r)}}, \dots, \mathbf{x}_n^{(r)}) = (\mathbf{x}_1^{(r-1)}, \dots, \widehat{\mathbf{x}_{k_{r-1}}^{(r-1)}}, \dots, \mathbf{x}_n^{(r-1)}),$$

which is almost orthogonal. The same is true if $r = 0$ and $n = \ell_0 > m + 1$ because, in that case, it is a subsequence of $(\mathbf{x}_1^{(0)}, \dots, \mathbf{x}_{n-1}^{(0)})$ which, by condition (1), is almost orthogonal. So, if $\ell_r > m + 1$, we obtain

$$\text{dist}(\mathbf{x}_{m+1}^{(r)}, V^{(r)}) \geq \text{dist}(\mathbf{x}_{m+1}^{(r)}, U^{(r)}) = \text{dist}(\mathbf{x}_{m+1}^{(r)}, \langle \mathbf{x}_1^{(r)}, \dots, \mathbf{x}_m^{(r)} \rangle_{\mathbb{R}}) \geq 1 - \frac{1}{2^m}.$$

If $\ell_r = m + 1$, the inequality $\text{dist}(\mathbf{x}_{m+1}^{(r)}, V^{(r)}) \geq 1 - 1/2^m$ also holds as a direct consequence of (4) for $i = r$ because $k_r \leq m < \ell_r$. (In fact, this is precisely the reason why we need this condition (4).) So, in all cases, we conclude that

$$\text{dist}(\mathbf{x}_{m+1}^{(t)}, V^{(t)}) \geq \text{dist}(\mathbf{x}_{m+1}^{(r)}, V^{(r)}) - \frac{1}{2^n} \geq 1 - \frac{1}{2^m} - \frac{1}{2^n} \geq 1 - \frac{1}{2^{m-1}}.$$

Thus $(\mathbf{x}_1^{(t)}, \dots, \widehat{\mathbf{x}_{k_t}^{(t)}}, \dots, \mathbf{x}_{m+1}^{(t)})$ is almost orthogonal, and our claim follows by the induction principle.

Having proved the first part of the proposition, we know that the sequence $(\mathbf{x}_1^{(i)}, \dots, \widehat{\mathbf{x}_{k_i}^{(i)}}, \dots, \mathbf{x}_n^{(i)})$ is almost orthogonal for each i with $0 \leq i < s$. Define

$$V^{(i)} = \langle \mathbf{x}_1^{(i)}, \dots, \widehat{\mathbf{x}_{k_i}^{(i)}}, \dots, \mathbf{x}_n^{(i)} \rangle_{\mathbb{R}} \quad (0 \leq i < s).$$

When $i \geq 1$, condition (6) gives $V^{(i-1)} = \langle \mathbf{x}_1^{(i)}, \dots, \widehat{\mathbf{x}_{\ell_i}^{(i)}}, \dots, \mathbf{x}_n^{(i)} \rangle_{\mathbb{R}}$ and so $V^{(i-1)} + V^{(i)} = \langle \mathbf{x}_1^{(i)}, \dots, \mathbf{x}_n^{(i)} \rangle_{\mathbb{R}} = \mathbb{R}^n$. Since $H(\mathbb{R}^n) = 1$, we deduce from Lemmas 4.4 and 4.7 that

$$\text{dist}(\mathbf{u}_{i-1}, \mathbf{u}_i) = \text{dist}(V^{(i-1)}, V^{(i)}) \leq \frac{e^4}{\|\mathbf{x}_1^{(i)}\| \cdots \|\mathbf{x}_n^{(i)}\|}.$$

We also note that (5.6) holds with $m = n$ for any integer i with $1 \leq i < s$. So, for integers r, t with $0 \leq r < t < s$, we conclude that

$$\begin{aligned} \text{dist}(\mathbf{u}_r, \mathbf{u}_t) &\leq \sum_{i=r+1}^t \text{dist}(\mathbf{u}_{i-1}, \mathbf{u}_i) \\ &\leq \sum_{i=r+1}^t \frac{e^4}{\|\mathbf{x}_1^{(i)}\| \cdots \|\mathbf{x}_n^{(i)}\|} \leq \frac{2e^4}{\|\mathbf{x}_1^{(r+1)}\| \cdots \|\mathbf{x}_n^{(r+1)}\|}. \quad \square \end{aligned}$$

The next result completes the above proposition by constructing a unit vector \mathbf{u} and by estimating the distance function of the points $\mathbf{x}_1^{(i)}, \dots, \mathbf{x}_n^{(i)}$ with respect to the convex body $\mathcal{C}_{\mathbf{u}}(Q)$ in appropriate ranges for Q .

PROPOSITION 5.4. *Let the notation be as in the previous proposition. Define*

$$Q_i = A_1^{(i)} \cdots A_n^{(i)} \quad (0 \leq i < s),$$

and set $Q_s = \infty$ if $s \neq \infty$. Then there exists a unit vector $\mathbf{u} \in \mathbb{R}^n$ with the property that, for each i with $0 \leq i < s$ and each $Q \in [Q_i, Q_{i+1})$, we have

- (1) $A_j^{(i)} \leq \lambda(\mathbf{x}_j^{(i)}, \mathcal{C}_{\mathbf{u}}(Q)) \leq 8e^4 A_j^{(i)}$ for any $j \in \{1, \dots, n\}$ with $j \neq k_i$;
- (2) $\frac{A_{k_i}^{(i)} Q}{2^n Q_i} \leq \lambda(\mathbf{x}_{k_i}^{(i)}, \mathcal{C}_{\mathbf{u}}(Q)) \leq \frac{8A_{k_i}^{(i)} Q}{Q_i}$.

Proof. We will use freely the set of inequalities $A_j^{(i)} \leq \|\mathbf{x}_j^{(i)}\| \leq 2A_j^{(i)}$ ($1 \leq j \leq n, 0 \leq i < s$) coming from condition (3) of Proposition 5.3, as well as the inequalities (5.2)–(5.4) involving the numbers $A_j^{(i)}$. The construction of the point \mathbf{u} is based on the second assertion of Proposition 5.3. When $s = \infty$, this statement implies that the image of the sequence $(\mathbf{u}_i)_{i \geq 1}$ in $\mathbb{P}^{n-1}(\mathbb{R})$ converges to the class of a unit vector $\mathbf{u} \in \mathbb{R}^n$ such that

$$\text{dist}(\mathbf{u}_i, \mathbf{u}) \leq \frac{2e^4}{\|\mathbf{x}_1^{(i+1)}\| \cdots \|\mathbf{x}_n^{(i+1)}\|} \quad (0 \leq i < s).$$

When $s \neq \infty$, these inequalities remain true for the choice of $\mathbf{u} = \mathbf{u}_{s-1}$, provided that we interpret the right-hand side as 0 when $i = s - 1$. We now replace each \mathbf{u}_i by $\pm \mathbf{u}_i$ so that $\mathbf{u}_i \cdot \mathbf{u} \geq 0$. Then, we obtain

$$\|\mathbf{u}_i - \mathbf{u}\| \leq 2\text{dist}(\mathbf{u}_i, \mathbf{u}) \leq \frac{4e^4}{Q_{i+1}} \quad (0 \leq i < s).$$

For the rest of the argument, we fix an integer i with $0 \leq i < s$ and a number $Q \in [Q_i, Q_{i+1})$. Our goal is to estimate

$$(5.7) \quad \lambda(\mathbf{x}_j^{(i)}, \mathcal{C}_{\mathbf{u}}(Q)) = \max\{\|\mathbf{x}_j^{(i)}\|, |\mathbf{x}_j^{(i)} \cdot \mathbf{u}|Q\} \quad (1 \leq j \leq n).$$

For each $j \in \{1, \dots, n\}$, we have

$$(5.8) \quad |\mathbf{x}_j^{(i)} \cdot (\mathbf{u} - \mathbf{u}_i)| \leq \|\mathbf{x}_j^{(i)}\| \|\mathbf{u} - \mathbf{u}_i\| \leq \frac{8e^4 A_j^{(i)}}{Q_{i+1}}.$$

When $j \neq k_i$, the vector \mathbf{u}_i is orthogonal to $\mathbf{x}_j^{(i)}$. Then, since $Q < Q_{i+1}$, we obtain

$$|\mathbf{x}_j^{(i)} \cdot \mathbf{u}| Q \leq 8e^4 A_j^{(i)},$$

and the inequalities (1) follow from (5.7).

Since $(\mathbf{x}_1^{(i)}, \dots, \mathbf{x}_n^{(i)})$ is a basis of \mathbb{Z}^n and since its subsequence obtained by deleting $\mathbf{x}_{k_i}^{(i)}$ is almost orthogonal (and orthogonal to \mathbf{u}_i), Lemmas 4.1 and 4.6 give

$$\frac{\|\mathbf{x}_{k_i}^{(i)}\|}{\|\mathbf{x}_1^{(i)}\| \cdots \|\mathbf{x}_n^{(i)}\|} \leq |\mathbf{x}_{k_i}^{(i)} \cdot \mathbf{u}_i| = \frac{1}{H(\langle \mathbf{x}_1^{(i)}, \dots, \widehat{\mathbf{x}_{k_i}^{(i)}}, \dots, \mathbf{x}_n^{(i)} \rangle_{\mathbb{R}})} \leq \frac{e^2 \|\mathbf{x}_{k_i}^{(i)}\|}{\|\mathbf{x}_1^{(i)}\| \cdots \|\mathbf{x}_n^{(i)}\|},$$

and therefore

$$(5.9) \quad \frac{A_{k_i}^{(i)}}{2^{n-1} Q_i} \leq |\mathbf{x}_{k_i}^{(i)} \cdot \mathbf{u}_i| \leq \frac{e^2 A_{k_i}^{(i)}}{Q_i}.$$

If $i + 1 < s$, we also note that

$$\frac{Q_{i+1}}{Q_i} = \frac{A_{\ell_{i+1}}^{(i+1)}}{A_{k_i}^{(i)}} \geq \frac{A_{\ell_{i+1}}^{(i+1)}}{A_{\ell_{i+1}}^{(i)}} \geq C = 2^{n+3} e^4,$$

and so (5.8) yields

$$|\mathbf{x}_{k_i}^{(i)} \cdot \mathbf{u} - \mathbf{x}_{k_i}^{(i)} \cdot \mathbf{u}_i| = |\mathbf{x}_{k_i}^{(i)} \cdot (\mathbf{u} - \mathbf{u}_i)| \leq \frac{A_{k_i}^{(i)}}{2^n Q_i}.$$

This inequality also holds if $i + 1 = s$ because in that case $\mathbf{u} = \mathbf{u}_i$. So, using (5.9), we deduce that

$$\frac{A_{k_i}^{(i)}}{2^n Q_i} \leq |\mathbf{x}_{k_i}^{(i)} \cdot \mathbf{u}| \leq \frac{8A_{k_i}^{(i)}}{Q_i}.$$

The estimates (2) then follow from (5.7) with $j = k_i$ because $Q \geq Q_i$. □

We conclude this section with the following result, which in quantitative form establishes Theorem 1.3 for rigid n -systems with sufficiently large mesh.

THEOREM 5.5. *Let $\delta \geq 4 + (n + 3) \log 2$, and let $\mathbf{P}: [q_0, \infty) \rightarrow \mathbb{R}^n$ be a rigid n -system with mesh δ . Then, there exists a unit vector $\mathbf{u} \in \mathbb{R}^n$ such that*

$$\sup_{q \geq q_0} \|\mathbf{P}(q) - \mathbf{L}_{\mathbf{u}}(q)\|_{\infty} \leq n \log(8e^4 n).$$

Proof. Consider the sequence of points $(\mathbf{a}^{(i)})_{0 \leq i < s}$ and the sequences of integers $(k_i)_{0 \leq i < s}$ and $(\ell_i)_{0 \leq i < s}$ that form the canvas attached to \mathbf{P} as in Definitions 1.1 and 1.2. For each integer i with $0 \leq i < s$, we write $\mathbf{a}^{(i)} = (a_1^{(i)}, \dots, a_n^{(i)})$ and set

$$A_j^{(i)} := \exp(a_j^{(i)}) \quad (1 \leq j \leq n).$$

Then, conditions (C1)–(C3) of Definition 1.1 yield (5.1)–(5.4), and thus Propositions 5.3 and 5.4 apply. Consider the bases $(\mathbf{x}_1^{(i)}, \dots, \mathbf{x}_n^{(i)})$ of \mathbb{Z}^n provided by the first proposition for $0 \leq i < s$ and the unit vector \mathbf{u} of \mathbb{R}^n provided by the second one. We claim that this unit vector has the required property. To show this, we first note that, in agreement with Definition 1.2 and Proposition 5.4, we have

$$q_i = a_1^{(i)} + \dots + a_n^{(i)} = \log Q_i \quad (0 \leq i < s)$$

and $q_s = \log Q_s = \infty$ if $s \neq \infty$. Then, we fix an arbitrary integer i with $0 \leq i < s$ and a real number $q \in [q_i, q_{i+1})$. We also set $Q = e^q$, so that $Q \in [Q_i, Q_{i+1})$. Since $(\mathbf{x}_1^{(i)}, \dots, \mathbf{x}_n^{(i)})$ is a basis of \mathbb{Z}^n , the successive minima of $\mathcal{C}_{\mathbf{u}}(Q)$ are bounded above by a permutation of the numbers $\lambda(\mathbf{x}_j^{(i)}, \mathcal{C}_{\mathbf{u}}(Q))$ ($1 \leq j \leq n$), and so, in view of Proposition 5.4, they are bounded above by a permutation of the n numbers $8A_{k_i}^{(i)}Q/Q_i$ and $8e^4A_j^{(i)}$ ($1 \leq j \leq n, j \neq k_i$). Taking logarithms, we deduce that the numbers $L_{\mathbf{u},j}(q) := \log \lambda_j(\mathcal{C}_{\mathbf{u}}(e^q))$ with $1 \leq j \leq n$ are bounded above by a permutation of $a_{k_i}^{(i)} + q - q_i + c$ and $a_j^{(i)} + c$ ($1 \leq j \leq n, j \neq k_i$), where $c = \log(8e^4)$. In view of the formula for $\mathbf{P}(q)$ in Definition 1.2, this means that

$$L_{\mathbf{u},j}(q) \leq P_j(q) + c \quad (1 \leq j \leq n),$$

where P_j denotes the j -th component of \mathbf{P} . By (2.2) and (2.7), we also have

$$q - n \log(n) \leq \sum_{j=1}^n L_{\mathbf{u},j}(q) \quad \text{and} \quad \sum_{j=1}^n (P_j(q) + c) = q + nc.$$

We conclude that $0 \leq (P_j(q) + c) - L_{\mathbf{u},j}(q) \leq n(\log(n) + c)$ for each $j = 1, \dots, n$, and so

$$\|\mathbf{L}_{\mathbf{u}}(q) - \mathbf{P}(q)\|_{\infty} \leq n(\log(n) + c) = n \log(8e^4n). \quad \square$$

6. Reduced systems

In this section and the next one, we turn to the problem of approximating a given (n, γ) -system on $[0, \infty)$ by rigid n -systems of sufficiently large mesh with $n \geq 2$. In the current section, we simply narrow the class of (n, γ) -systems that we need to consider.

Definition 6.1. Let $\gamma, q_0 \geq 0$. An (n, γ) -reduced system on $[q_0, \infty)$ is an (n, γ) -system $\mathbf{P} = (P_1, \dots, P_n): [q_0, \infty) \rightarrow \mathbb{R}^n$ with the property that, for any $j = 1, \dots, n - 1$, any $a \geq q_0$ and any $b \geq a + n\gamma$ such that $P_1 + \dots + P_j$ is constant on $[a, b]$, the functions P_1, \dots, P_j are constant on $[a, b - n\gamma]$.

For example, any $(n, 0)$ -system is already an $(n, 0)$ -reduced system. The goal of this section is to prove the following result.

PROPOSITION 6.2. *Let $\mathbf{P}: [0, \infty) \rightarrow \mathbb{R}^n$ be an (n, γ) -system on $[0, \infty)$ for some $\gamma \geq 0$. There exists an $(n, 2n\gamma)$ -reduced system $\tilde{\mathbf{P}}: [0, \infty) \rightarrow \mathbb{R}^n$ such that $\|\mathbf{P} - \tilde{\mathbf{P}}\|_\infty \leq n\gamma$.*

The proof goes through several steps, based on the following observation, whose proof is left to the reader.

LEMMA 6.3. *Let $a, b \in \mathbb{R}$ with $a < b$, and let $M: [a, b] \rightarrow \mathbb{R}$ be a continuous piecewise linear function with slopes 0 and 1. Then the map $\tilde{M}: [a, b] \rightarrow \mathbb{R}$ given by*

$$\tilde{M}(q) = \max\{M(a), M(b) + q - b\} \quad \text{for any } q \in [a, b]$$

is continuous and piecewise linear with slopes 0 and 1. Moreover, it satisfies

$$(6.1) \quad M(a) = \tilde{M}(a) \leq \tilde{M}(q) \leq M(q) \leq M(b) = \tilde{M}(b)$$

for each $q \in [a, b]$.

Write $\mathbf{P} = (P_1, \dots, P_n)$, and set $M_j = P_1 + \dots + P_j$ for each index $j = 0, \dots, n$. We define recursively a sequence of continuous piecewise linear functions $\tilde{M}_n, \dots, \tilde{M}_1$ from $[0, \infty)$ to \mathbb{R} with slopes 0 and 1 in the following way. We first set $\tilde{M}_n = M_n$ and $\tilde{M}_{n-1} = M_{n-1}$. Then, assuming that \tilde{M}_j has been constructed for some index j with $2 \leq j \leq n - 1$, we form the set \mathcal{E}_j of all maximal subintervals of $[0, \infty)$ with nonempty interior on which \tilde{M}_j is constant. The end-points of these intervals are the points of $(0, \infty)$ where \tilde{M}_j is not differentiable, and possibly the point 0. So, they form a discrete subset of $[0, \infty)$. We define $\tilde{M}_{j-1}: [0, \infty) \rightarrow \mathbb{R}$ by

$$\tilde{M}_{j-1}(q) = \begin{cases} \max\{M_{j-1}(a), M_{j-1}(b) + q - b\} & \text{if } q \in [a, b] \in \mathcal{E}_j, \\ M_{j-1}(a) & \text{if } q \in [a, \infty) \in \mathcal{E}_j, \\ M_{j-1}(q) & \text{if } q \notin I \text{ for any } I \in \mathcal{E}_j. \end{cases}$$

Lemma 6.3 applied to the restriction of M_{j-1} to any bounded interval $[a, b]$ in \mathcal{E}_j shows that \tilde{M}_{j-1} and M_{j-1} agree at the end-points of such an interval. Moreover, \tilde{M}_{j-1} and M_{j-1} agree at the point a if $[a, \infty) \in \mathcal{E}_j$. Therefore, \tilde{M}_{j-1} is continuous. Clearly it is piecewise linear with slopes 0 and 1. Once $\tilde{M}_n, \dots, \tilde{M}_1$ have been constructed, we set $\tilde{M}_0 = 0$.

LEMMA 6.4. *Let $j \in \{1, \dots, n - 1\}$.*

- (i) *For any $q \geq 0$, we have $\tilde{M}_j(q) \leq M_j(q) \leq \tilde{M}_j(q) + (n - j - 1)\gamma$.*
- (ii) *Suppose $j \geq 2$. Then, for any compact subinterval $[c, d]$ of $[0, \infty)$ on which \tilde{M}_j is constant, we have $0 \leq M_{j-1}(d) - \tilde{M}_{j-1}(c) \leq (n - j)\gamma$.*

Proof. We proceed by descending induction on j . For $j = n - 1$, assertion (i) is clear because $\tilde{M}_{n-1} = M_{n-1}$. If $n = 2$, there is nothing more to prove. Suppose that $n \geq 3$ and that (i) holds for some integer $j \in \{2, \dots, n - 1\}$. Let $[c, d]$ be a compact subinterval of $[0, \infty)$ on which \tilde{M}_j is constant. If $[c, d]$ is not contained in any interval of \mathcal{E}_j , we must have $c = d$ and $\tilde{M}_{j-1}(c) = M_{j-1}(c)$, so (ii) holds. Otherwise $[c, d]$ is contained in an interval I of \mathcal{E}_j of the form $[a, b]$ or $[a, \infty)$. In the first case, Lemma 6.3 shows that $\tilde{M}_{j-1}(a) = M_{j-1}(a)$ and $\tilde{M}_{j-1}(c) \leq M_{j-1}(c)$. This is also true in the second case. Then, combining this with the fact that M_{j-1} and \tilde{M}_{j-1} are monotone increasing, we deduce that

$$M_{j-1}(d) - \tilde{M}_{j-1}(c) \geq M_{j-1}(c) - \tilde{M}_{j-1}(c) \geq 0$$

and

$$\begin{aligned} M_{j-1}(d) - \tilde{M}_{j-1}(c) &\leq M_{j-1}(d) - \tilde{M}_{j-1}(a) \\ &= M_{j-1}(d) - M_{j-1}(a) \\ &= (M_j(d) - M_j(a)) + (P_j(a) - P_j(d)). \end{aligned}$$

Since \mathbf{P} is an (n, γ) -system and since $a \leq d$, we also have $P_j(a) - P_j(d) \leq \gamma$. Moreover, the induction hypothesis gives $M_j(d) \leq \tilde{M}_j(d) + (n - j - 1)\gamma$. As \tilde{M}_j is constant on $I \supseteq [a, d]$, we further have $\tilde{M}_j(d) = \tilde{M}_j(a) = M_j(a)$. This means that $M_j(d) - M_j(a) \leq (n - j - 1)\gamma$. So, we conclude that

$$0 \leq M_{j-1}(d) - \tilde{M}_{j-1}(c) \leq (n - j)\gamma,$$

and therefore (ii) holds for that value of j . Applying this result with $c = d = q$ for an arbitrary $q \geq 0$, we obtain $\tilde{M}_{j-1}(q) \leq M_{j-1}(q) \leq \tilde{M}_{j-1}(q) + (n - j)\gamma$. Thus (i) holds with j replaced by $j - 1$. This completes the induction step and proves the lemma. □

LEMMA 6.5. *Let $j \in \{1, \dots, n - 1\}$. Suppose that \tilde{M}_j is constant on some compact subinterval $[c, d]$ of $[0, \infty)$ with $d \geq c + n\gamma$. Then \tilde{M}_{j-1} is constant on $[c, d - n\gamma]$.*

Proof. If $j = 1$, this is true because $\tilde{M}_0 = 0$. Suppose that $j \geq 2$. Then, $[c, d]$ is contained in some interval of \mathcal{E}_j . If this interval is of the form $[a, \infty)$, then \tilde{M}_{j-1} is constant on $[a, \infty)$ and so it is constant on $[c, d]$. Otherwise this interval is of the form $[a, b]$ and, on this interval, \tilde{M}_{j-1} has slope 0 and then slope 1. If \tilde{M}_{j-1} is not constant on $[c, d - n\gamma]$, this means that it has slope 1

on $[d - n\gamma, b]$, and so we obtain

$$n\gamma = \tilde{M}_{j-1}(d) - \tilde{M}_{j-1}(d - n\gamma) \leq \tilde{M}_{j-1}(d) - \tilde{M}_{j-1}(c).$$

This is impossible because Lemma 6.4 gives

$$\tilde{M}_{j-1}(d) - \tilde{M}_{j-1}(c) \leq M_{j-1}(d) - \tilde{M}_{j-1}(c) \leq (n - j)\gamma. \quad \square$$

For each $j = 1, \dots, n$, we define $\tilde{P}_j = \tilde{M}_j - \tilde{M}_{j-1}$ so that

$$\tilde{M}_j = \tilde{P}_1 + \dots + \tilde{P}_j \quad \text{for } j = 1, \dots, n.$$

We claim that the resulting map $\tilde{\mathbf{P}} = (\tilde{P}_1, \dots, \tilde{P}_n): [0, \infty) \rightarrow \mathbb{R}^n$ satisfies all the conditions of Proposition 6.2. It clearly satisfies conditions (S3) and (S4) of Definition 2.8. The next two lemmas show that it satisfies all the other requirements.

LEMMA 6.6. *We have*

- (i) $|P_j(q) - \tilde{P}_j(q)| \leq (n - 1)\gamma \quad (1 \leq j \leq n, 0 \leq q);$
- (ii) $-n\gamma \leq \tilde{P}_j(q) \leq \tilde{P}_{j+1}(q) + 2n\gamma \quad (1 \leq j < n, 0 \leq q);$
- (iii) $\tilde{P}_j(q_1) \leq \tilde{P}_j(q_2) + 2n\gamma \quad (1 \leq j \leq n, 0 \leq q_1 \leq q_2).$

Moreover, if, for some $j \in \{1, \dots, n - 1\}$, the function \tilde{M}_j is constant on some compact subinterval $[c, d]$ of $[0, \infty)$ with $d \geq c + n^2\gamma$, then $\tilde{P}_1, \dots, \tilde{P}_j$ are constant on $[c, d - n^2\gamma]$.

Proof. Let $q \geq 0$. Lemma 6.4(i) shows that $0 \leq M_j(q) - \tilde{M}_j(q) \leq (n - 1)\gamma$ for $j = 1, \dots, n - 1$. This is also true for $j = 0$ and $j = n$ because in those cases we have $M_j = \tilde{M}_j$. Then part (i) follows from the equalities

$$P_j(q) - \tilde{P}_j(q) = (M_j(q) - \tilde{M}_j(q)) - (M_{j-1}(q) - \tilde{M}_{j-1}(q)) \quad (1 \leq j \leq n).$$

Let $j \in \{1, \dots, n\}$. If $j < n$, the inequality $-\gamma \leq P_j(q) \leq P_{j+1}(q) + \gamma$ combined with the estimates of part (i) yields

$$\begin{aligned} \tilde{P}_j(q) &\geq P_j(q) - (n - 1)\gamma \geq -n\gamma, \\ \tilde{P}_j(q) &\leq P_j(q) + (n - 1)\gamma \leq P_{j+1}(q) + n\gamma \leq \tilde{P}_{j+1}(q) + (2n - 1)\gamma. \end{aligned}$$

Similarly, if $0 \leq q_1 \leq q_2$, the inequality $P_j(q_1) \leq P_j(q_2) + \gamma$ yields

$$\tilde{P}_j(q_1) \leq P_j(q_1) + (n - 1)\gamma \leq P_j(q_2) + n\gamma \leq \tilde{P}_j(q_2) + (2n - 1)\gamma.$$

This proves (ii) and (iii). Finally, if \tilde{M}_j is constant on $[c, d]$ for some index $j \in \{1, \dots, n - 1\}$ and some compact subinterval $[c, d]$ of $[0, \infty)$ of length at least $n^2\gamma$, then Lemma 6.5 shows that $\tilde{M}_j, \dots, \tilde{M}_1$ are constant on $[c, d - nj\gamma]$ and so $\tilde{P}_1, \dots, \tilde{P}_j$ are also constant on the latter interval. \square

LEMMA 6.7. *Let $j \in \{2, \dots, n\}$. Suppose that \tilde{M}_{j-1} changes slope from 1 to 0 at a point $q > 0$. Then we have $\tilde{P}_j(q) \leq \tilde{P}_{j-1}(q) + 2n\gamma$.*

Proof. Suppose first that \tilde{M}_{j-1} coincides with M_{j-1} on some open neighborhood of q . Then, as M_{j-1} changes slope from 1 to 0 at q , we have $P_j(q) \leq P_{j-1}(q) + \gamma$, and the conclusion follows from Lemma 6.6(i).

Suppose now that such a neighborhood does not exist. Then q belongs to an interval I of \mathcal{E}_j and, since \tilde{M}_{j-1} changes slope at most once on I , going from slope 0 to slope 1, the point q lies on the boundary of I . We claim that there is an interior point p of I at which M_{j-1} changes slope from 1 to 0. If $I = [a, b]$ is a compact interval, this is clear because otherwise M_{j-1} would agree with \tilde{M}_{j-1} on the whole interval $[a, b]$ and therefore would agree on an open interval containing $[a, b]$. If instead $I = [a, \infty)$ is unbounded, then $q = a$ and, since \tilde{M}_{j-1} is constant on $[a, \infty)$, the function M_{j-1} , by differing locally from \tilde{M}_{j-1} , should have slope 1 in a right neighborhood $[a, a + \varepsilon)$ of a for some $\varepsilon > 0$. However, M_{j-1} cannot have constant slope 1 on $[a, \infty)$ because, by Lemma 6.4(i), the difference $M_{j-1} - \tilde{M}_{j-1}$ is bounded. Thus the claim holds in that case as well. By definition of an (n, γ) -system, we have

$$P_j(p) \leq P_{j-1}(p) + \gamma.$$

Put $c = \min\{p, q\}$ and $d = \max\{p, q\}$. Since \tilde{M}_j is constant on $I \supseteq [c, d]$, we find, with the help of Lemma 6.4, that

$$\begin{aligned} \tilde{P}_j(q) - P_j(p) &= (\tilde{M}_j(q) - M_j(p)) + (M_{j-1}(p) - \tilde{M}_{j-1}(q)) \\ &\leq (\tilde{M}_j(p) - M_j(p)) + (M_{j-1}(d) - \tilde{M}_{j-1}(c)) \leq (n - j)\gamma. \end{aligned}$$

Since $p \leq d$, we also have $P_{j-1}(p) \leq P_{j-1}(d) + \gamma$, and thus

$$\begin{aligned} P_{j-1}(p) - \tilde{P}_{j-1}(q) &\leq P_{j-1}(d) - \tilde{P}_{j-1}(q) + \gamma \\ &= (M_{j-1}(d) - \tilde{M}_{j-1}(q)) + (\tilde{M}_{j-2}(q) - M_{j-2}(d)) + \gamma \\ &\leq (M_{j-1}(d) - \tilde{M}_{j-1}(c)) + (\tilde{M}_{j-2}(d) - M_{j-2}(d)) + \gamma \\ &\leq (n - j + 1)\gamma. \end{aligned}$$

Combining the three displayed estimates, we get

$$\tilde{P}_j(q) - \tilde{P}_{j-1}(q) \leq 2(n - j + 1)\gamma \leq 2n\gamma. \quad \square$$

7. Approximation by rigid n -systems

We shall now prove the following result.

PROPOSITION 7.1. *Let $\gamma, \delta \in \mathbb{R}$ with $0 \leq \gamma < \delta/(2n^2)$, and let $\mathbf{P}: [0, \infty) \rightarrow \mathbb{R}^n$ be an (n, γ) -reduced system. Put $q_0 = n(n + 1)\delta/2$. Then there exists a rigid n -system $\mathbf{R}: [q_0, \infty) \rightarrow \mathbb{R}^n$ of mesh δ such that $\|\mathbf{P}(q) - \mathbf{R}(q)\|_\infty \leq 3n^2\delta$ for each $q \geq q_0$.*

We first note that a change of variables reduces the proof to the case where $\delta = 1$. Indeed, suppose for the moment that the proposition holds in that case,

and let us use the more suggestive terminology *rigid integral n-system* to denote a rigid n -system of mesh 1. Under the hypotheses of the proposition, we form the map $\tilde{\mathbf{P}}: [0, \infty) \rightarrow \mathbb{R}^n$ given by $\tilde{\mathbf{P}}(q) = \delta^{-1}\mathbf{P}(q\delta)$ for each $q \geq 0$. Then $\tilde{\mathbf{P}}$ is an $(n, \gamma/\delta)$ -reduced system on $[0, \infty)$. As $\gamma/\delta < 1/(2n^2)$, there exists a rigid integral n -system $\tilde{\mathbf{R}}: [q_0, \infty) \rightarrow \mathbb{R}^n$ with $q_0 = n(n+1)/2$ such that $\|\tilde{\mathbf{P}}(q) - \tilde{\mathbf{R}}(q)\|_\infty \leq 3n^2$ for each $q \geq q_0$. The map $\mathbf{R}: [q_0\delta, \infty) \rightarrow \mathbb{R}^n$ given by $\mathbf{R}(q) = \delta\tilde{\mathbf{R}}(q/\delta)$ for each $q \geq q_0\delta$ is then a rigid n -system of mesh δ that satisfies $\|\mathbf{P}(q) - \mathbf{R}(q)\|_\infty \leq 3n^2\delta$ for each $q \geq q_0\delta$ as requested.

So, from now on, we assume that $\delta = 1$. We fix a real number γ with $0 \leq \gamma < 1/(2n^2)$ and an (n, γ) -reduced system $\mathbf{P} = (P_1, \dots, P_n)$ on $[0, \infty)$. For each $j \in \{1, \dots, n\}$, we define a new function $\bar{P}_j: [0, \infty) \rightarrow \mathbb{R}$ by putting

$$\bar{P}_j(q) = \gamma + \sup\{P_j(t); 0 \leq t \leq q\} \quad (q \geq 0).$$

Then $\bar{P}_1, \dots, \bar{P}_n$ are continuous piecewise linear functions with slopes 0 and 1. Conditions (S1) and (S2) of Definition 2.8 respectively imply that, for each $q \geq 0$, they satisfy

$$(7.1) \quad 0 \leq \bar{P}_j(q) \leq \bar{P}_{j+1}(q) + \gamma \quad (1 \leq j < n),$$

$$(7.2) \quad P_j(q) + \gamma \leq \bar{P}_j(q) \leq P_j(q) + 2\gamma \quad (1 \leq j \leq n).$$

Moreover, we note that $\bar{P}_1 = \gamma + P_1$ and $\bar{P}_n = \gamma + P_n$ since P_1 and P_n do not take negative slope.

We also define recursively a sequence of functions E_1, \dots, E_n from $[0, \infty)$ to \mathbb{N}^* by putting, for each $q \geq 0$,

$$E_1(q) = \lfloor \bar{P}_1(q) \rfloor + 1,$$

$$E_j(q) = \max \{E_{j-1}(q) + 1, \lfloor \bar{P}_j(q) - 2(j-1)\gamma \rfloor + 1\} \quad \text{for } j = 2, \dots, n,$$

where $\lfloor x \rfloor$ stands for the integral part of a real number x . Then each E_j is monotone increasing and right continuous. Moreover, the set of points of discontinuity of E_j is discrete and, at such a point q , we have $E_j(q) = E_j(q^-) + 1$, where $E_j(q^-)$ is a shorthand for $\lim_{t \rightarrow q^-} E_j(t)$. These functions make up a map

$$\mathbf{E} = (E_1, \dots, E_n): [0, \infty) \rightarrow (\mathbb{N}^*)^n$$

whose values form strictly increasing sequences of positive integers

$$0 < E_1(q) < \dots < E_n(q).$$

Condition (S4) of Definition 2.8 together with (7.2) and the fact that each \bar{P}_j has slopes 0 and 1 implies that

$$\bar{P}_j(q) \leq q + \bar{P}_j(0) \leq q + \sum_{k=1}^n (P_k(0) + 2\gamma) = q + 2n\gamma < 1 \quad (1 \leq j \leq n, 0 \leq q \leq 1/2),$$

and so we obtain

$$(7.3) \quad \mathbf{E}(q) = (1, 2, \dots, n) \quad (0 \leq q \leq 1/2).$$

The next lemma compares the functions E_j and \bar{P}_j .

LEMMA 7.2. *For each $q \geq 0$, we have*

- (i) $\bar{P}_j(q) - 2(j - 1)\gamma < E_j(q) \leq \bar{P}_j(q) + j(1 + \gamma)$ for $j = 1, \dots, n$;
- (ii) $\|\mathbf{E}(q) - \mathbf{P}(q)\|_\infty \leq n + 1$;
- (iii) $|(E_1(q) + \dots + E_n(q)) - q| \leq n(n + 1)$.

Proof. Fix $q \geq 0$. For $j = 1$, the inequality (i) is clear because $E_1(q)$ is the smallest integer that is greater than $\bar{P}_1(q)$. Suppose that this inequality holds for some integer j with $1 \leq j < n$. We find

$$\bar{P}_{j+1}(q) - 2j\gamma < \lfloor \bar{P}_{j+1}(q) - 2j\gamma \rfloor + 1 \leq E_{j+1}(q)$$

and, using the induction hypothesis together with (7.1), we obtain

$$\begin{aligned} E_{j+1}(q) &\leq \max\{E_j(q) + 1, \bar{P}_{j+1}(q) - 2j\gamma + 1\} \\ &\leq \max\{\bar{P}_j(q) + j(1 + \gamma) + 1, \bar{P}_{j+1}(q) + 1\} \\ &\leq \bar{P}_{j+1}(q) + (j + 1)(1 + \gamma). \end{aligned}$$

So, the inequality (i) is also satisfied with j replaced by $j + 1$. Consequently, it holds for $j = 1, \dots, n$. Using (7.2) and the fact that $\gamma \leq 1/(2n^2)$, this implies that

$$|E_j(q) - P_j(q)| \leq \max\{2(j - 1)\gamma, j(1 + \gamma) + 2\gamma\} \leq n + 1 \quad (1 \leq j \leq n),$$

which proves (ii). Then (iii) follows because $P_1(q) + \dots + P_n(q) = q$. □

LEMMA 7.3. *Let (a, b) be an open subinterval of $[0, \infty)$ of length $b - a \leq 1$. Each of the functions E_1, \dots, E_n admits at most one point of discontinuity on (a, b) .*

Proof. Suppose on the contrary that, for some $j \in \{1, \dots, n\}$, the function E_j admits at least two points of discontinuity $q_1 < q_2$ on (a, b) . Let j be the minimal index with this property, and choose q_1, q_2 so that E_j is constant on (q_1, q_2) . Then, we have

$$E_j(q_1^-) = H - 1, \quad E_j(q_1) = E_j(q_2^-) = H \quad \text{and} \quad E_j(q_2) = H + 1$$

for some $H \in \mathbb{Z}$. Since \bar{P}_j is continuous, we deduce from Lemma 7.2 that

$$\bar{P}_j(q_1) \leq E_j(q_1^-) + 2(j - 1)\gamma = H - 1 + 2(j - 1)\gamma.$$

As \bar{P}_j has slope at most 1 and as $q_2 - q_1 < b - a \leq 1$, this gives

$$\bar{P}_j(q_2) < \bar{P}_j(q_1) + 1 \leq H + 2(j - 1)\gamma,$$

and thus $\lfloor \bar{P}_j(q_2) - 2(j - 1)\gamma \rfloor + 1 \leq H$. Since $E_j(q_2) = H + 1$, this implies that $j \geq 2$ and that E_{j-1} is discontinuous at the point q_2 , with $E_{j-1}(q_2) = H$. By virtue of the choice of j , this point q_2 must be the only point of discontinuity of

E_{j-1} on (a, b) . So, E_{j-1} is constant on (a, q_2) , equal to $E_{j-1}(q_2^-) = H - 1$. This means, in particular, that $E_{j-1}(q_1^-) = H - 1 = E_j(q_1^-)$, a contradiction. \square

Let Σ denote the set of all points of discontinuity of \mathbf{E} in $[0, \infty)$. Since, by (7.3), \mathbf{E} is constant on $[0, 1/2]$, the above lemma shows that Σ is a discrete subset of $[1/2, \infty)$. It is also infinite because, by Lemma 7.2(iii), the sum $E_1 + \dots + E_n$ is unbounded while it increases by at most n at each point of discontinuity of \mathbf{E} . We now study

$$\bar{\Sigma} := \Sigma + [-n\gamma, n\gamma] = \{t \in \mathbb{R}; |t - q| \leq n\gamma \text{ for some } q \in \Sigma\} \subset [0, \infty).$$

LEMMA 7.4. *The set $\bar{\Sigma}$ has infinitely many connected components. Any such component I is a closed interval of length at least $2n\gamma$ and at most $2n^2\gamma < 1$, on which each of the functions E_1, \dots, E_n admits at most one point of discontinuity.*

Proof. Since Σ is a discrete subset of \mathbb{R} , the set $\bar{\Sigma}$ is a union of disjoint closed intervals of length at least $2n\gamma$. Let I be one of these, and let $[a, b]$ be a compact subinterval of I . The intersection of Σ with $[a - n\gamma, b + n\gamma]$ consists of finitely many points $q_1 < \dots < q_s$. For these points, we have

$$(7.4) \quad [a, b] \subseteq [q_1 - n\gamma, q_1 + n\gamma] \cup \dots \cup [q_s - n\gamma, q_s + n\gamma]$$

and $q_{i+1} \leq q_i + 2n\gamma$ for $i = 1, \dots, s - 1$. If $s \geq n + 1$, then one of the functions E_1, \dots, E_n admits at least two points of discontinuity among q_1, \dots, q_{n+1} . By Lemma 7.3, this is impossible because $[q_1, q_{n+1}]$ has length at most $2n^2\gamma < 1$. Thus, we must have $s \leq n$, and (7.4) yields $b - a \leq 2ns\gamma \leq 2n^2\gamma$. This shows that I is bounded of length at most $2n^2\gamma < 1$ and so, again by Lemma 7.3, each of the functions E_1, \dots, E_n has at most one point of discontinuity in I . Finally, $\bar{\Sigma}$ consists of infinitely many disjoint such intervals because it contains Σ which is infinite and discrete. \square

In view of the above lemma, we can write

$$\bar{\Sigma} = [c_0, d_0] \cup [c_1, d_1] \cup \dots$$

for an infinite sequence of real numbers $0 \leq c_0 < d_0 < c_1 < d_1 < \dots$ with

$$2n\gamma \leq d_i - c_i < 1 \quad (i \geq 0).$$

For each $i \geq 0$, we denote by $\Omega^{(i)}$ the nonempty set of indices $j \in \{1, \dots, n\}$ for which E_j is not constant on $[c_i, d_i]$, and we define

$$(7.5) \quad \begin{aligned} k^{(i)} &:= \max\{j \in \Omega^{(i)}; j = 1 \text{ or } E_{j-1}(d_i) < E_j(d_i) - 1\}, \\ \ell^{(i)} &:= \min\{j \in \Omega^{(i)}; j = n \text{ or } E_{j+1}(c_i) > E_j(c_i) + 1\}. \end{aligned}$$

Our next goal is to show that $\ell^{(i)} \geq k^{(i+1)}$ for each $i \geq 0$. This is the most delicate point. Once it is settled, the construction of the requested rigid integral n -system goes quickly as the reader could see by going directly to Lemma 7.8.

To prove the above inequality, we use freely the estimates (7.1) and (7.2). For each $j \in \{1, \dots, n\}$, we set, as usual, $M_j = P_1 + \dots + P_j$. We also use the fact that, if $j < n$ and if $P_j + \gamma < P_{j+1}$ on a subinterval $[a, b]$ of $[0, \infty)$, then, by Condition (S5) of Definition 2.8, the function M_j is concave up on $[a, b]$: either its slope is constant on $[a, b]$, or it is 0 on $[a, c]$ and 1 on $[c, b]$ for some $c \in (a, b)$. We write $f'(q^-)$ to denote the left derivative of a function f at a point q and, for each $q \in \Sigma$, we denote by $\Omega(q)$ the nonempty set of indices $j \in \{1, \dots, n\}$ such that E_j is discontinuous at q . We first establish two lemmas whose proofs are illustrated in Figure 3.

LEMMA 7.5. *Let $q \in \Sigma$ and $k \in \Omega(q)$. Suppose that $k > 1$ and that $E_{k-1}(q) < E_k(q) - 1$. Then, we have $P'_k(q^-) = 1$ and $P_k(q) = E_k(q^-) + (2k - 3)\gamma$. Moreover, the function M_{k-1} is constant on $[q - 1, q]$.*

Proof. Set $H = E_k(q^-)$ so that $E_k(q) = H + 1$ and $E_{k-1}(q) \leq H - 1$. Since $E_{k-1}(q) + 1 < E_k(q)$, we must have

$$[\bar{P}_k(t) - 2(k - 1)\gamma] + 1 = E_k(t)$$

for each t in some open neighborhood of q . As \bar{P}_k is continuous and piecewise linear with slopes 0 and 1, while E_k has a discontinuity at q , this means that $\bar{P}'_k(q^-) = 1$ and that $\bar{P}_k(q) = H + 2(k - 1)\gamma$. In view of the definition of \bar{P}_k , this in turn implies that $P'_k(q^-) = 1$ and that $P_k(q) = \bar{P}_k(q) - \gamma = H + (2k - 3)\gamma$, which is the first assertion of the lemma.

Since $E_{k-1}(q) \leq H - 1$, Lemma 7.2 shows that, for any $t \in [0, q]$, we have

$$P_{k-1}(t) \leq \bar{P}_{k-1}(q) < E_{k-1}(q) + 2(k - 2)\gamma \leq H - 1 + 2(k - 2)\gamma.$$

As P_k is continuous and piecewise linear with slope at most 1, we conclude that, for any $t \in [q - 1, q]$, we have

$$P_k(t) \geq P_k(q) - 1 = H - 1 + (2k - 3)\gamma > P_{k-1}(t) + \gamma.$$

Thus M_{k-1} is concave up on $[q - 1, q]$. However, the inequalities

$$0 \leq M'_{k-1}(q^-) = M'_k(q^-) - P'_k(q^-) \leq 1 - P'_k(q^-) = 0$$

show that $M'_{k-1}(q^-) = 0$, and therefore M_{k-1} must be constant on $[q - 1, q]$. \square

LEMMA 7.6. *Let $q \in \Sigma$ and $\ell \in \Omega(q)$. Suppose that $\ell < n$ and that $E_{\ell+1}(q^-) > E_\ell(q^-) + 1$. Choose $r > q$ such that E_ℓ is constant on $[q, r)$. Then, M_ℓ is concave up on the interval $[q - 1, r]$.*

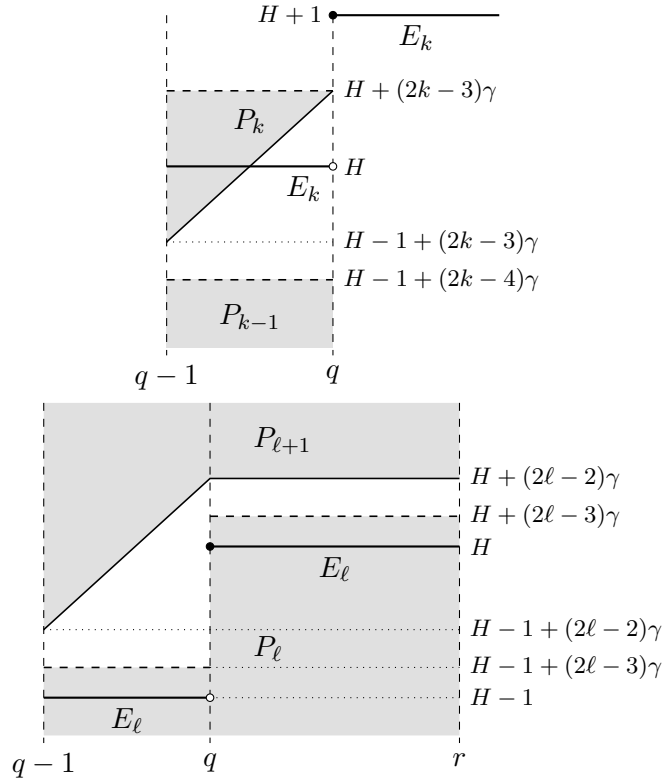


Figure 3. Illustrations for the proofs of Lemma 7.5 on top and of Lemma 7.6 on bottom. The shaded regions contain the graphs of P_{k-1} and P_k on top and those of P_ℓ and $P_{\ell+1}$ on bottom, showing the gap between them.

Proof. Set $H = E_\ell(q)$ so that $E_\ell(q^-) = H - 1$ and $E_{\ell+1}(q^-) \geq H + 1$. Choose $\varepsilon > 0$ such that E_ℓ and $E_{\ell+1}$ are constant on $[q - \varepsilon, q)$. For each $t \in [q - \varepsilon, q)$, we have $E_\ell(t) + 1 = H < E_{\ell+1}(t)$, and therefore

$$H + 1 \leq E_{\ell+1}(t) = \lfloor \bar{P}_{\ell+1}(t) - 2l\gamma \rfloor + 1 \leq \bar{P}_{\ell+1}(t) - 2l\gamma + 1.$$

Letting t tend to q and using the fact that $\bar{P}_{\ell+1}$ is continuous, we deduce that

$$(7.6) \quad \bar{P}_{\ell+1}(q) \geq H + 2l\gamma.$$

By Lemma 7.3, the function E_ℓ has no point of discontinuity on $(q - 1, q)$. Since it is right continuous, it is therefore constant equal to $H - 1$ on $[q - 1, q)$. Then, for any $t \in [q - 1, q)$, Lemma 7.2 yields

$$P_\ell(t) \leq \bar{P}_\ell(t) - \gamma < H - 1 + (2\ell - 3)\gamma.$$

Since $P_{\ell+1}$ has slope at most 1, we deduce from (7.6) and the preceding estimate that, for $t \in [q - 1, q)$, we have

$$P_{\ell+1}(t) \geq P_{\ell+1}(q) - 1 \geq \bar{P}_{\ell+1}(q) - 2\gamma - 1 \geq H - 1 + (2\ell - 2)\gamma > P_{\ell}(t) + \gamma.$$

On the other hand, since E_{ℓ} is constant equal to H on $[q, r)$, Lemma 7.2 combined with (7.6) shows that, for any $t \in [q, r)$, we have

$$P_{\ell}(t) + \gamma \leq \bar{P}_{\ell}(t) < H + 2(\ell - 1)\gamma \leq \bar{P}_{\ell+1}(q) - 2\gamma \leq \bar{P}_{\ell+1}(t) - 2\gamma \leq P_{\ell+1}(t).$$

Thus $P_{\ell+1}(t) > P_{\ell}(t) + \gamma$ for any $t \in [q - 1, r)$, and consequently, M_{ℓ} is concave up on $[q - 1, r]$. □

Our next lemma uses in a crucial way the hypothesis that \mathbf{P} is reduced.

LEMMA 7.7. *We have $\ell^{(i)} \geq k^{(i+1)}$ for each $i \geq 0$.*

Proof. Suppose on the contrary that $\ell^{(i)} < k^{(i+1)}$ for some $i \geq 0$. For simplicity, we write $\ell := \ell^{(i)}$ and $k' := k^{(i+1)}$. Let $q \in [c_i + n\gamma, d_i - n\gamma]$ be the single point of discontinuity of E_{ℓ} on $[c_i, d_i]$, and let $q' \in [c_{i+1} + n\gamma, d_{i+1} - n\gamma]$ be the one of $E_{k'}$ on $[c_{i+1}, d_{i+1}]$. Since $\ell < k'$, we have $k' > 1$ and $\ell < n$. The definition of $k' = k^{(i+1)}$ in (7.5) yields

$$E_{k'-1}(q') \leq E_{k'-1}(d_{i+1}) < E_{k'}(d_{i+1}) - 1 = E_{k'}(q') - 1.$$

Thus Lemma 7.5 applies and shows that $M_{k'-1}$ is constant on $[q' - 1, q']$. As \mathbf{P} is (n, γ) -reduced, and as $q' - n\gamma \geq c_{i+1}$, we deduce that $P_1, \dots, P_{k'-1}$ are constant on $[q' - 1, c_{i+1}]$. Note that this is an interval of positive length because $c_{i+1} > d_{i+1} - 1 \geq q' - 1$. Since $\ell < k'$, we conclude that $M_{\ell} = P_1 + \dots + P_{\ell}$ is also constant on $[q' - 1, c_{i+1}]$.

Similarly, we find that

$$E_{\ell+1}(q^-) \geq E_{\ell+1}(c_i) > E_{\ell}(c_i) + 1 = E_{\ell}(q^-) + 1.$$

Moreover, the function E_{ℓ} is constant on $[q, c_{i+1}]$ (because \mathbf{E} is constant on $[d_i, c_{i+1}]$). Then, Lemma 7.6 shows that M_{ℓ} is concave up on $[q - 1, c_{i+1}]$. As it is constant on $[q' - 1, c_{i+1}]$, it must therefore be constant on the whole interval $[q - 1, c_{i+1}]$. Using again the fact that \mathbf{P} is (n, γ) -reduced, this implies that P_1, \dots, P_{ℓ} are constant on $[q - 1, c_{i+1} - n\gamma]$, and thus E_1, \dots, E_{ℓ} are also constant on the same interval. This is a contradiction because q is an interior point of this interval (since $q \leq d_i - n\gamma < c_{i+1} - n\gamma$) and, by hypothesis, E_{ℓ} is discontinuous at the point q . □

The next two lemmas complete the proof of Proposition 7.1 when $\delta = 1$.

LEMMA 7.8. *Define $t_i := E_1(c_i) + \dots + E_n(c_i)$ for each $i \geq 0$. Then, we have $t_0 = n(n+1)/2$ and there exists a rigid integer n -system $\mathbf{R}: [t_0, \infty) \rightarrow \Delta_n$ such that $\mathbf{R}(t_i) = \mathbf{E}(c_i)$ for each $i \geq 0$.*

Proof. Since \mathbf{E} is constant on $[0, c_0]$, we have $\mathbf{E}(c_0) = \mathbf{E}(0) = (1, \dots, n)$ by (7.3), and thus $t_0 = n(n+1)/2$. For each $i \geq 0$, we partition $\Omega^{(i)}$ into maximal subsets of consecutive integers $\{k, \dots, \ell\}$ such that $E_k(c_i), \dots, E_\ell(c_i)$ are also consecutive integers, and we order these subsets in a sequence $\Omega_1^{(i)}, \dots, \Omega_{r_i}^{(i)}$ so that $\min \Omega_r^{(i)} > \max \Omega_{r+1}^{(i)}$ if $1 \leq r < r_i$. Then we group all of these into a single sequence

$$(\Omega_i)_{i \geq 0} = (\Omega_1^{(0)}, \dots, \Omega_{r_0}^{(0)}, \Omega_1^{(1)}, \dots, \Omega_{r_1}^{(1)}, \dots),$$

and we define

$$k_i = \min \Omega_i, \quad \ell_{i+1} = \max \Omega_i \quad (i \geq 0).$$

Clearly, we have $1 \leq k_i \leq \ell_{i+1} \leq n$ for each $i \geq 0$. By definition of $k^{(i)}$ and $\ell^{(i)}$ in (7.5), we also note that

$$k^{(i)} = \min \Omega_1^{(i)} \quad \text{and} \quad \ell^{(i)} = \max \Omega_{r_i}^{(i)} \quad (i \geq 0).$$

Thus the inequality $\ell^{(i)} \geq k^{(i+1)}$ of Lemma 7.7 translates into the relation $\max \Omega_{r_i}^{(i)} \geq \min \Omega_1^{(i+1)}$ for each $i \geq 0$. Since we also have $\max \Omega_r^{(i)} > \min \Omega_{r+1}^{(i)}$ when $1 \leq r < r_i$, we conclude that $\max \Omega_i \geq \min \Omega_{i+1}$ for each $i \geq 0$, and so $\ell_i \geq k_i$ for each $i \geq 1$. Upon setting $\ell_0 = n$, the latter inequality extends to all $i \geq 0$.

For each $i \geq 0$ and each $r = 1, \dots, r_i$, we define an integer point

$$\mathbf{a}_r^{(i)} = (a_{r,1}^{(i)}, \dots, a_{r,n}^{(i)})$$

by

$$a_{r,j}^{(i)} = \begin{cases} E_j(c_i) & \text{if } j \in \Omega_r^{(i)} \cup \dots \cup \Omega_{r_i}^{(i)}, \\ E_j(c_{i+1}) & \text{otherwise.} \end{cases}$$

We also set $\mathbf{a}_{r_{i+1}}^{(i)} = \mathbf{a}_1^{(i+1)}$. Then we form the sequence

$$(\mathbf{a}^{(i)})_{i \geq 0} = (\mathbf{a}_1^{(0)}, \dots, \mathbf{a}_{r_0}^{(0)}, \mathbf{a}_1^{(1)}, \dots, \mathbf{a}_{r_1}^{(1)}, \dots).$$

For fixed $i \geq 0$, we note that

$$E_j(c_{i+1}) = \begin{cases} E_j(c_i) + 1 & \text{if } j \in \Omega^{(i)}, \\ E_j(c_i) & \text{otherwise} \end{cases} \quad (1 \leq j \leq n),$$

because the map \mathbf{E} is constant on $[d_i, c_{i+1}]$ and, by Lemma 7.4, each of its component E_j with $j \in \Omega^{(i)}$ admits exactly one point of discontinuity in $[c_i, d_i]$ while the other components are constant on $[c_i, d_i]$. This means that

$$\mathbf{a}_1^{(i)} = \mathbf{E}(c_i) \quad (i \geq 0)$$

and that, for each $r = 1, \dots, r_i$, the point $\mathbf{a}_{r+1}^{(i)}$ is obtained from $\mathbf{a}_r^{(i)}$ by adding 1 to each of its coordinates with index in $\Omega_r^{(i)}$. Since these coordinates are consecutive integers and since $\mathbf{a}_{r_{i+1}}^{(i)} = \mathbf{a}_1^{(i+1)}$, we conclude that, in general, for

each $i \geq 0$, the coordinates of $\mathbf{a}^{(i)}$ with index in Ω_i are consecutive integers and that $\mathbf{a}^{(i+1)}$ is obtained from $\mathbf{a}^{(i)}$ by adding 1 to each of them. As $\Omega_i = \{k_i, \dots, \ell_{i+1}\}$, this yields

$$(a_1^{(i)}, \dots, \widehat{a_{k_i}^{(i)}}, \dots, a_n^{(i)}) = (a_1^{(i+1)}, \dots, \widehat{a_{\ell_{i+1}}^{(i+1)}}, \dots, a_n^{(i+1)}) \text{ and } a_{\ell_{i+1}}^{(i+1)} = a_{\ell_{i+1}}^{(i)} + 1$$

for each $i \geq 0$.

The triple consisting of $(\mathbf{a}^{(i)})_{i \geq 0}$, $(k_i)_{i \geq 0}$ and $(\ell_i)_{i \geq 0}$ is almost a canvas. In view of the above, it satisfies conditions (C1) and (C3) of Definition 1.1 and the slightly weaker condition $1 \leq k_i \leq \ell_i \leq n$ ($i \geq 0$) instead of (C2). Nevertheless, we can associate to it a map $\mathbf{R}: [t_0, \infty) \rightarrow \Delta_n$ as in Definition 1.2. By construction this map satisfies $\mathbf{R}(t_i) = \mathbf{E}(c_i)$ for each $i \geq 0$ because each point $\mathbf{E}(c_i)$ belongs to the sequence $(\mathbf{a}^{(i)})_{i \geq 0}$. Finally, let $(i_m)_{0 \leq m < s}$ denote the sequence of integers $i \geq 0$ with $i = 0$ or $k_i < \ell_i$, listed in increasing order. We leave it to the reader to check that the triple $(\mathbf{a}^{(i_m)})_{0 \leq m < s}$, $(k_{i_m})_{0 \leq m < s}$, $(\ell_{i_m})_{0 \leq m < s}$ is a canvas and that \mathbf{R} is the rigid system attached to it. \square

LEMMA 7.9. *Let the notation be as in the preceding lemma. We have $\|\mathbf{R}(t) - \mathbf{P}(t)\|_\infty \leq 3n^2$ for each $t \in [t_0, \infty)$.*

Proof. Fix a choice of $t \geq t_0$, and let $i \geq 0$ denote the index for which $t \in [t_i, t_{i+1})$. Using Lemma 7.2(iii), we find

$$\begin{aligned} |c_i - t| &\leq |c_i - t_i| + t_{i+1} - t_i \\ &= \left| c_i - \sum_{j=1}^n E_j(c_i) \right| + \sum_{j=1}^n (E_j(c_{i+1}) - E_j(c_i)) \leq n(n+1) + n. \end{aligned}$$

As the components of \mathbf{P} are continuous and piecewise linear with slopes 0 and 1, we deduce that

$$\|\mathbf{P}(c_i) - \mathbf{P}(t)\|_\infty \leq |c_i - t| \leq n(n+2).$$

By Lemma 7.2(ii), we also have

$$\|\mathbf{R}(t_i) - \mathbf{P}(c_i)\|_\infty = \|\mathbf{E}(c_i) - \mathbf{P}(c_i)\|_\infty \leq n+1.$$

Finally, since the components of \mathbf{R} are monotone increasing on $[t_i, t_{i+1}]$, we find

$$\|\mathbf{R}(t) - \mathbf{R}(t_i)\|_\infty \leq \|\mathbf{R}(t_{i+1}) - \mathbf{R}(t_i)\|_\infty = \|\mathbf{E}(c_{i+1}) - \mathbf{E}(c_i)\|_\infty = 1.$$

Combining the three preceding displayed inequalities, we conclude that

$$\|\mathbf{R}(t) - \mathbf{P}(t)\|_\infty \leq n(n+2) + (n+1) + 1 \leq 3n^2. \quad \square$$

8. Proof of Theorem 1.3

Since any rigid n -system is an $(n, 0)$ -system, the first result below proves the second assertion of our main Theorem 1.3 in a quantitative form.

THEOREM 8.1. *Let $q_0 \geq 0$, and let $\mathbf{P}: [q_0, \infty) \rightarrow \mathbb{R}^n$ be an $(n, 0)$ -system. Then there exists a unit vector \mathbf{u} of \mathbb{R}^n such that $\|\mathbf{P}(q) - \mathbf{L}_{\mathbf{u}}(q)\|_{\infty} \leq 3n^2(n+9)$ for each $q \geq q_0$.*

Proof. We first note that \mathbf{P} can be extended to an $(n, 0)$ -system on $[0, \infty)$ in the following way. Put $t_0 = 0$ and $t_i = P_1(q_0) + \dots + P_i(q_0)$ for $i = 1, \dots, n$. Then we have $t_n = q_0$, and we define

$$\mathbf{P}(q) = \Phi_n(0, \dots, 0, P_1(q_0), \dots, P_{i-1}(q_0), q - t_{i-1}) \quad (t_{i-1} \leq q \leq t_i, 1 \leq i \leq n),$$

so that, for $i = 1, \dots, n$, the combined graph of \mathbf{P} over $[t_{i-1}, t_i]$ consists of $n - 1$ horizontal line segments, not necessarily distinct, with ordinates $0, \dots, 0, P_1(q_0), \dots, P_{i-1}(q_0)$ and one line segment of slope 1 joining the points $(t_{i-1}, 0)$ and $(t_i, P_i(q_0))$.

In view of the above observation, we may assume that $q_0 = 0$. Put $\delta = n + 7$ and $\tilde{q}_0 = n(n + 1)\delta/2$. Since all $(n, 0)$ -systems are reduced, Proposition 7.1 shows the existence of a rigid n -system $\mathbf{R}: [\tilde{q}_0, \infty) \rightarrow \mathbb{R}^n$ of mesh δ such that $\|\mathbf{P}(q) - \mathbf{R}(q)\|_{\infty} \leq 3n^2\delta$ for each $q \geq \tilde{q}_0$. For this rigid system, Theorem 5.5 shows in turn the existence of a unit vector \mathbf{u} in \mathbb{R}^n such that $\|\mathbf{R}(q) - \mathbf{L}_{\mathbf{u}}(q)\|_{\infty} \leq n \log(8e^4n)$ for each $q \geq \tilde{q}_0$. Then, we have

$$(8.1) \quad \|\mathbf{P}(q) - \mathbf{L}_{\mathbf{u}}(q)\|_{\infty} \leq 3n^2\delta + n \log(8e^4n) \quad (q \geq \tilde{q}_0).$$

However, for $q \in [0, \tilde{q}_0]$, the coordinates of $\mathbf{P}(q)$ are nonnegative and bounded above by $P_1(q) + \dots + P_n(q) = q \leq \tilde{q}_0$ while those of $\mathbf{L}_{\mathbf{u}}(q)$ are also nonnegative and bounded above by $L_{\mathbf{u},1}(q) + \dots + L_{\mathbf{u},n}(q) \leq \tilde{q}_0 + n \log(n)$ thanks to (2.2). Thus the estimate (8.1) extends to all $q \geq 0$ and the conclusion follows because $3n^2\delta + n \log(8e^4n) \leq 3n^2(n + 9)$. \square

Since any rigid n -system of a given mesh $\delta > 0$ is also a rigid system of mesh δ/N for each integer $N \geq 1$, our last result below implies the first assertion of Theorem 1.3, thereby completing the proof of that theorem.

THEOREM 8.2. *Let $\delta > 24n^4 2^n \log(n)$, and let \mathbf{u} be a unit vector of \mathbb{R}^n . Put $q_0 = n(n + 1)\delta/2$. Then there exists a rigid n -system $\mathbf{R}: [q_0, \infty) \rightarrow \mathbb{R}^n$ of mesh δ such that $\|\mathbf{L}_{\mathbf{u}}(q) - \mathbf{R}(q)\|_{\infty} \leq 4n^2\delta$ for each $q \geq q_0$.*

Proof. Put $\gamma = 6n2^n \log(n)$. By Theorem 2.9, there exists an (n, γ) -system $\mathbf{P}: [0, \infty) \rightarrow \mathbb{R}^n$ such that $\|\mathbf{L}_{\mathbf{u}}(q) - \mathbf{P}(q)\|_{\infty} \leq \gamma$ for each $q \geq 0$. Then, by Proposition 6.2, there exists an $(n, 2n\gamma)$ -reduced system $\tilde{\mathbf{P}}: [0, \infty) \rightarrow \mathbb{R}^n$ for which $\|\mathbf{P}(q) - \tilde{\mathbf{P}}(q)\|_{\infty} \leq n\gamma$ for each $q \geq 0$. Finally, since $\delta > 4n^3\gamma$,

Proposition 7.1 provides a rigid n -system $\mathbf{R}: [q_0, \infty) \rightarrow \mathbb{R}^n$ with mesh δ satisfying $\|\tilde{\mathbf{P}}(q) - \mathbf{R}(q)\|_\infty \leq 3n^2\delta$ for each $q \geq q_0$. Thus, for $q \geq q_0$, we get $\|\mathbf{L}_u(q) - \mathbf{R}(q)\|_\infty \leq (n+1)\gamma + 3n^2\delta \leq 4n^2\delta$. \square

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