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Positivity for cluster algebras

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To the memory of Andrei Zelevinsky

Abstract

We prove the positivity conjecture for all skew-symmetric cluster algebras.

1. Introduction

Cluster algebras have been introduced by Fomin and Zelevinsky in [13] in the context of total positivity and canonical bases in Lie theory. Since then cluster algebras have been shown to be related to various fields in mathematics including representation theory of finite dimensional algebras, Teichmüller theory, Poisson geometry, combinatorics, Lie theory, tropical geometry and mathematical physics.

A cluster algebra is a subalgebra of a field of rational functions in N variables $x_1, x_2, \ldots, x_N$, given by specifying a set of generators, the so-called *cluster variables*. These generators are constructed in a recursive way, starting from the initial variables $x_1, x_2, \ldots, x_N$, by a procedure called *mutation*, which is determined by the choice of a skew-symmetric $N \times N$ integer matrix B or, equivalently, by a quiver Q. Although each mutation is an elementary operation, it is very difficult to compute cluster variables in general because of the recursive character of the construction.

Finding explicit computable direct formulas for the cluster variables is one of the main open problems in the theory of cluster algebras and has been studied by many mathematicians. In 2002, Fomin and Zelevinsky showed that every cluster variable is a Laurent polynomial in the initial variables $x_1, x_2, \ldots, x_N$, and they conjectured that this Laurent polynomial has positive coefficients [13].

This *positivity conjecture* has been proved in the following special cases:

- *Acyclic cluster algebras.* These are cluster algebras given by a quiver that is mutation equivalent to a quiver without oriented cycles. In this case,

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positivity has been shown in [17], building on [4], [16], [25], [26], using monoidal categorifications of quantum cluster algebras and perverse sheaves over graded quiver varieties. The bipartite case has been shown first in [25]. In the special case where the initial seed is acyclic, a different proof has been given later in [9] using Donaldson-Thomas theory. Very recently, after our proof of positivity was available, this approach has also been used to prove positivity for all rank 4 cluster algebras in [7].

- *Cluster algebras from surfaces.* In this case, positivity has been shown in [24] building on [28], [30], [29], using the fact that each cluster variable in such a cluster algebra corresponds to a curve in an oriented Riemann surface, and the Laurent expansion of the cluster variable is determined by the crossing pattern of the curve with a fixed triangulation of the surface [11], [12]. The construction and the proof of the positivity conjecture have been generalized to non skew-symmetric cluster algebras from orbifolds in [10].

Our approach in this paper is different. We prove positivity almost exclusively by elementary algebraic computation. The advantage of this approach is that we do not need to restrict to a special type of cluster algebras. Our main result is the following.

THEOREM 1.1. *The positivity conjecture holds in every skew-symmetric cluster algebra.*

Our argument provides a method for the computation of the Laurent expansions of cluster variables, and some examples of explicit computations were given in our earlier work [23]. We point out that direct formulas for the Laurent polynomials have been obtained earlier in several special cases. The most general results are the following:

- A formula involving the Euler-Poincaré characteristic of quiver Grassmannians obtained in [15], [8] using categorification and generalizing results in [5], [6]. While this formula shows a very interesting connection between cluster algebras and geometry, it is of limited computational use, since the Euler-Poincaré characteristics of quiver Grassmannians are hard to compute. In particular, this formula does not show positivity. On the other hand, the positivity result in this paper proves the positivity of the Euler-Poincaré characteristics of the quiver Grassmannians involved; see Section 6.
- An elementary combinatorial formula for cluster algebras from surfaces given in [24].
- A formula for cluster variables corresponding to string modules as a product of 2×2 matrices obtained in [1], generalizing a result in [2].

The main tools of the proof of Theorem 1.1 are modified versions of two formulas for the rank 2 case, one obtained by the first author in [18] and the other obtained by both authors in [22]. These formulas allow for the computation of the Laurent expansions of a given cluster variable with respect to any seed close enough to the variable, in the sense that there is a sequence of mutations $\mu_d, \mu_e, \mu_d, \mu_e, \dots$ using only two directions d and e that links seed and variable. The general result then follows by inductive reasoning.

We actually show the stronger result, Theorem 4.1, that for every cluster variable u and for every cluster $\mathbf{x}$, there exists a connected rank 2 subtree $\mathbb{T}_2$ of the exchange tree containing $\mathbf{x}$ such that u can be expressed as a sum of four positive Laurent polynomials in the variables of four clusters closest to $\mathbf{x}$ in $\mathbb{T}_2$, and such that the variables that are not contained in all four clusters appear only with positive powers. Because of rank 2 positivity, this implies that u is a positive Laurent polynomial in every cluster in $\mathbb{T}_2$.

The proof of Theorem 4.1 is by induction on the number of rank 2 mutation subsequences of the mutation sequence from u to $\mathbf{x}$. It uses two different rank 2 formulas, to express cluster variables that are two rank 2 sequences away from $\mathbf{x}$ as Laurent polynomials in four clusters including $\mathbf{x}$. First we compute the Laurent expansion $\mathcal{L}_1$ of the cluster variable after one rank 2 mutation sequence as a sum of two Laurent polynomials in two adjacent clusters such that the variables that are not contained in both clusters appear only with positive powers. Then we compute the Laurent expansions in four clusters including $\mathbf{x}$ of all cluster variables appearing in $\mathcal{L}_1$ using the second rank 2 sequence and then substitute these in $\mathcal{L}_1$. We then show that the variables that are not contained in all four clusters appear only with positive powers.

If the cluster algebra is not skew-symmetric, it is shown in [27], [19] that (an adaptation of) the second rank 2 formula still holds. We therefore expect that our argument can be generalized to prove the positivity conjecture for non skew-symmetric cluster algebras. A noncommutative version of the formula has been given in [21], [27].

The article is organized as follows. We start by recalling some definitions and results from the theory of cluster algebras in Section 2. In Section 3, we study mutation sequences of rank 3 quivers, and we recall the definition of compatible pairs as well as results from our previous work [23] in Section 3. The positivity conjecture is proved, using Theorem 4.1, in Section 4, and Theorem 4.1 is proved in Section 5 following the outline above. As an application, we show that certain quiver Grassmannians have positive Euler-Poincaré characteristic in Section 6.

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2. Cluster algebras

In this section, we review some notions from the theory of cluster algebras introduced by Fomin and Zelevinsky in [13]. Our definition follows the exposition in [14].

To define a cluster algebra $\mathcal{A}$ we must first fix its ground ring. Let $(\mathbb{P}, \oplus, \cdot)$ be a *semifield*, i.e., an abelian multiplicative group endowed with a binary operation of (*auxiliary*) *addition* $\oplus$ that is commutative, associative, and distributive with respect to the multiplication in $\mathbb{P}$. The group ring $\mathbb{Z}\mathbb{P}$ will be used as a *ground ring* for $\mathcal{A}$.

One important choice for $\mathbb{P}$ is the tropical semifield; in this case we say that the corresponding cluster algebra is of *geometric type*. Let $\mathrm{Trop}(u_1, \ldots, u_m)$ be an abelian group (written multiplicatively) freely generated by the u_j. We define $\oplus$ in $\mathrm{Trop}(u_1, \ldots, u_m)$ by

$$\prod_j u_j^{a_j} \oplus \prod_j u_j^{b_j} = \prod_j u_j^{\min(a_j,b_j)}, \tag{1}$$

and we call $(\mathrm{Trop}(u_1, \ldots, u_m), \oplus, \cdot)$ a *tropical semifield*. Note that the group ring of $\mathrm{Trop}(u_1, \ldots, u_m)$ is the ring of Laurent polynomials in the variables u_j.

As an *ambient field* for $\mathcal{A}$, we take a field $\mathcal{F}$ isomorphic to the field of rational functions in N independent variables (here N is the *rank* of $\mathcal{A}$), with coefficients in $\mathbb{Q}\mathbb{P}$. Note that the definition of $\mathcal{F}$ does not involve the auxiliary addition in $\mathbb{P}$.

Definition 2.1. A *labeled seed* in $\mathcal{F}$ is a triple $(\mathbf{x}, \mathbf{y}, B)$, where

- $\mathbf{x} = (x_1, \ldots, x_N)$ is an N-tuple from $\mathcal{F}$ forming a *free generating set* over $\mathbb{Q}\mathbb{P}$;
- $\mathbf{y} = (y_1, \ldots, y_N)$ is an N-tuple from $\mathbb{P}$; and
- $B = (b_{ij})$ is an $N \times N$ integer matrix that is *skew-symmetrizable*.

That is, $x_1, \ldots, x_N$ are algebraically independent over $\mathbb{Q}\mathbb{P}$, and

$$\mathcal{F} = \mathbb{Q}\mathbb{P}(x_1, \ldots, x_N).$$

We refer to $\mathbf{x}$ as the (labeled) *cluster* of a labeled seed $(\mathbf{x}, \mathbf{y}, B)$, to the tuple $\mathbf{y}$ as the *coefficient tuple*, and to the matrix B as the *exchange matrix*.

We use the notation $[x]_+ = \max(x, 0)$, $[1, N] = \{1, \ldots, N\}$, and

$$\mathrm{sgn}(x) = \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

Definition 2.2. Let $(\mathbf{x}, \mathbf{y}, B)$ be a labeled seed in $\mathcal{F}$, and let $k \in [1, N]$. The *seed mutation* μ_k in direction k transforms $(\mathbf{x}, \mathbf{y}, B)$ into the labeled seed $\mu_k(\mathbf{x}, \mathbf{y}, B) = (\mathbf{x}', \mathbf{y}', B')$ defined as follows:

- The entries of $B' = (b'_{ij})$ are given by

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k, \\ b_{ij} + \operatorname{sgn}(b_{ik})\ [b_{ik}b_{kj}]_+ & \text{otherwise.} \end{cases} \tag{2}$$

- The coefficient tuple $\mathbf{y}' = (y'_1, \ldots, y'_N)$ is given by

$$y'_j = \begin{cases} y_k^{-1} & \text{if } j = k, \\ y_j y_k^{[b_{kj}]_+}(y_k \oplus 1)^{-b_{kj}} & \text{if } j \neq k. \end{cases} \tag{3}$$

- The cluster $\mathbf{x}' = (x'_1, \ldots, x'_N)$ is given by $x'_j = x_j$ for $j \neq k$, whereas $x'_k \in \mathcal{F}$ is determined by the *exchange relation*

$$x'_k = \frac{y_k \ \prod x_i^{[b_{ik}]_+} + \prod x_i^{[-b_{ik}]_+}}{(y_k \oplus 1)x_k} \,. \tag{4}$$

We say that two exchange matrices B and B' are *mutation-equivalent* if one can get from B to B' by a sequence of mutations. A sequence of mutations $\mu_d, \mu_e, \mu_d, \mu_e, \ldots$ using only mutations in two directions d and e is called a rank 2 mutation sequence.

Definition 2.3. Consider the *N-regular tree* $\mathbb{T}_N$ whose edges are labeled by the numbers $1, \ldots, N$, so that the N edges emanating from each vertex receive different labels. A *cluster pattern* is an assignment of a labeled seed $\Sigma_t = (\mathbf{x}_t, \mathbf{y}_t, B_t)$ to every vertex $t \in \mathbb{T}_N$, such that the seeds assigned to the endpoints of any edge $t \overset{k}{-\!-} t'$ are obtained from each other by the seed mutation in direction k. The components of Σ_t are written as

$$\mathbf{x}_t = (x_{1;t}\,, \ldots, x_{N;t})\,, \quad \mathbf{y}_t = (y_{1;t}\,, \ldots, y_{N;t})\,, \quad B_t = (b^t_{ij})\,. \tag{5}$$

Clearly, a cluster pattern is uniquely determined by an arbitrary seed.

Definition 2.4. Given a cluster pattern, we denote

$$\mathcal{X} = \bigcup_{t \in \mathbb{T}_N} \mathbf{x}_t = \{x_{i;t} \,:\, t \in \mathbb{T}_N\,,\ 1 \le i \le N\}, \tag{6}$$

the union of clusters from all seeds in the pattern. The elements $x_{i;t} \in \mathcal{X}$ are called *cluster variables*. The *cluster algebra* $\mathcal{A}$ associated with a given pattern is the $\mathbb{ZP}$-subalgebra of the ambient field $\mathcal{F}$ generated by all cluster variables: $\mathcal{A} = \mathbb{ZP}[\mathcal{X}]$. We denote $\mathcal{A} = \mathcal{A}(\mathbf{x}, \mathbf{y}, B)$, where $(\mathbf{x}, \mathbf{y}, B)$ is any seed in the underlying cluster pattern.

The cluster algebra is called *skew-symmetric* if the matrix B is skew-symmetric. In this case, it is often convenient to represent the $N \times N$ matrix B by a quiver Q_B with vertices $1, 2, \ldots, N$ and $[b_{ij}]_+$ arrows from vertex i to vertex j.

In [13], Fomin and Zelevinsky proved the remarkable *Laurent phenomenon* and posed the following *positivity conjecture*.

THEOREM 2.5 (Laurent Phenomenon). *For any cluster algebra $\mathcal{A}$ and any seed Σ_t, each cluster variable x is a Laurent polynomial over $\mathbb{ZP}$ in the cluster variables from $\mathbf{x}_t = (x_{1;t}, \ldots, x_{N;t})$.*

CONJECTURE 2.6 (Positivity Conjecture). *For any cluster algebra $\mathcal{A}$, any seed Σ_t, and any cluster variable x, the Laurent polynomial expansion of x in the cluster $\mathbf{x}_t$ has coefficients that are nonnegative integer linear combinations of elements in $\mathbb{P}$.*

Our main result is the proof of this conjecture for skew-symmetric cluster algebras.

3. Expansion formulas

In this section, we recall from [23] how to compute the Laurent expansions of those cluster variables that are obtained from the initial cluster by a mutation sequence involving only two vertices. The main tools are the rank 2 formula from [22] (in the parametrization of [19]) and the rank 2 formula from [18].

Let r be a positive integer, and let $\{c_n^{[r]}\}_{n\in\mathbb{Z}}$ be the sequence defined by the recurrence relation

$$c_n^{[r]} = rc_{n-1}^{[r]} - c_{n-2}^{[r]},$$

with the initial condition $c_1^{[r]} = 0$, $c_2^{[r]} = 1$. For example, if $r = 2$, then $c_n^{[r]} = n - 1$; if $r = 3$, the sequence $c_n^{[r]}$ takes the following values:

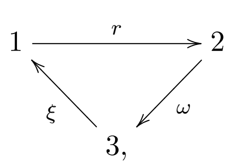
$$\ldots, -3, -1, 0, 1, 3, 8, 21, 55, 144, \ldots .$$

LEMMA 3.1 ([23, Lemma 3.1]). *Let $n \geq 3$. We have*

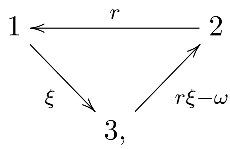
$$c_{n-1}^{[r]} c_{n+k-3}^{[r]} - c_{n+k-2}^{[r]} c_{n-2}^{[r]} = c_k^{[r]}$$

for $k \in \mathbb{Z}$. In particular, we have $(c_{n-1}^{[r]})^2 - c_n^{[r]} c_{n-2}^{[r]} = 1$.

3.1. *Nonacyclic mutation classes of rank* 3. We start by collecting some basic results on quivers of rank 3. First let us recall how mutations act on a rank 3 quiver. Given a quiver

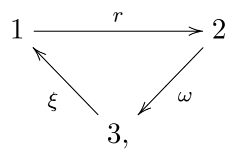


where $r, \xi, \omega \geq 0$ denote the number of arrows, then its mutation in 1 is the quiver



where we agree that if $r\xi - \omega < 0$, then there are $|r\xi - \omega|$ arrows from 2 to 3.

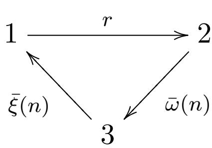
Lemma 3.2. *Let Q be the quiver*



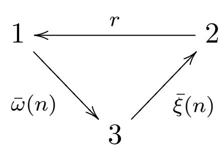
where $r \geq 0$ and $\xi, \omega \in \mathbb{Z}$ denote the number of arrows, and suppose that one of ξ, ω is nonnegative. Consider the mutation sequence

$$Q = Q_0 \overset{1}{\text{———}} Q_1 \overset{2}{\text{———}} Q_2 \overset{1}{\text{———}} Q_3 \overset{2}{\text{———}} \cdots .$$

Then Q_n is



if n is even,



if n is odd, where

$$\begin{cases} \bar{\xi}(n) = c^{[r]}_{n+2}\xi - c^{[r]}_{n+1}\omega \ \text{ and } \ \bar{\omega}(n) = c^{[r]}_{n+1}\xi - c^{[r]}_{n}\omega & \text{if } c^{[r]}_{\ell+1}\xi - c^{[r]}_{\ell}\omega > 0 \\ & \text{for } 1 \leq \ell \leq n, \\ \bar{\xi}(n) = c^{[r]}_{n-1}\omega - c^{[r]}_{n}\xi \ \text{ and } \ \bar{\omega}(n) = c^{[r]}_{n+1}\xi - c^{[r]}_{n}\omega & \text{if } c^{[r]}_{n+1}\xi - c^{[r]}_{n}\omega \leq 0 \\ & \text{and } c^{[r]}_{\ell}\xi - c^{[r]}_{\ell-1}\omega > 0 \\ & \text{for } 1 \leq \ell \leq n, \\ \bar{\xi}(n) = c^{[r]}_{n-1}\omega - c^{[r]}_{n}\xi \ \text{ and } \ \bar{\omega}(n) = c^{[r]}_{n-2}\omega - c^{[r]}_{n-1}\xi & \text{if } c^{[r]}_{n}\xi - c^{[r]}_{n-1}\omega \leq 0 \\ & \text{and } n \geq 2. \end{cases}$$

If both $\xi < 0$ and $\omega < 0$, then

$$\bar{\xi}(n) = -c^{[r]}_{n+1}\omega - c^{[r]}_{n}\xi \quad \text{and} \quad \bar{\omega}(n) = -c^{[r]}_{n}\omega - c^{[r]}_{n-1}\xi.$$

Remark 3.3. One may rephrase in terms of the following notation: let

$$\bar{s}(r, \xi, \omega, n) = \bar{s}(n) := c^{[r]}_{n+1}\xi - c^{[r]}_{n}\omega$$

for $n \geq 0$. Then we have

$$\begin{cases} \bar{\xi}(n) = \bar{s}(n+1),\ \bar{\omega}(n) = \bar{s}(n) & \text{if } \bar{s}(1), \dots, \bar{s}(n) > 0, \\ \bar{\xi}(n) = -\bar{s}(n-1),\ \bar{\omega}(n) = \bar{s}(n) & \text{if } \bar{s}(n) \leq 0 \\ & \text{and } \bar{s}(1), \dots, \bar{s}(n-1) > 0, \\ \bar{\xi}(n) = -\bar{s}(n-1),\ \bar{\omega}(n) = -\bar{s}(n-2) & \text{if } \bar{s}(n-1) \leq 0 \text{ and } n \geq 2. \end{cases}$$

Remark 3.4. (1) The three cases in the lemma, when one of ξ, ω is negative, arise from the different possibilities for the orientation of the arrows at vertex 3. In the first case, all quivers $Q_0, \dots, Q_{n-1}$ are cyclically oriented; in the second case, the quivers $Q_0, \dots, Q_{n-2}$ are cyclic and the quivers Q_{n-1}, Q_n, Q_{n+1} are acyclic. In the third case, if $r > 1$, there exists $m < n$ such that Q_{m-1}, Q_m, Q_{m+1} are acyclic, and the quivers Q_p, with $p > m+1$, are cyclic.

(2) If $r > 1$ the three cases exhaust all the possibilities. Indeed, if $c_{n+1}^{[r]}\xi - c_n^{[r]}\omega \leq 0$ and $c_n^{[r]}\xi - c_{n-1}^{[r]}\omega > 0$, then

$$\frac{c_{n+1}^{[r]}}{c_n^{[r]}} \leq \frac{\omega}{\xi} < \frac{c_n^{[r]}}{c_{n-1}^{[r]}} < \frac{c_{n-1}^{[r]}}{c_{n-2}^{[r]}} < \cdots \qquad \text{if } \xi > 0,$$

$$\frac{c_n^{[r]}}{c_{n+1}^{[r]}} \geq \frac{\xi}{\omega} > \frac{c_{n-1}^{[r]}}{c_n^{[r]}} > \frac{c_{n-2}^{[r]}}{c_{n-1}^{[r]}} > \cdots \qquad \text{if } \omega > 0,$$

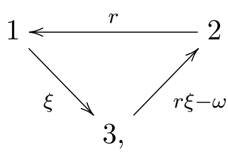
and thus $c_\ell^{[r]}\xi - c_{\ell-1}^{[r]}\omega > 0$ for $1 \leq \ell \leq n$.

If $r = 0, 1$, then $c_n^{[r]}$ is periodic. More precisely,

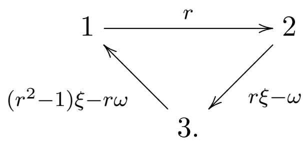
$$c_n^{[r]} = \begin{cases} \dots 0, 1, 1, 0, -1, -1, 0, 1, 1, 0, -1, -1, \dots & \text{for } r = 1, \\ \dots 0, 1, 0, -1, 0, 1, 0, -1, \dots & \text{for } r = 0. \end{cases}$$

(3) The case where both $\xi < 0$ and $\omega < 0$ is listed here for the sake of completeness, but this case is not used in the rest of the paper, because the quiver obtained after one mutation belongs to the other case. Note that if $\ell = 1$, the condition $c_{\ell+1}^{[r]}\xi - c_\ell^{[r]}\omega > 0$ becomes $\xi > 0$, and the condition $c_{\ell-1}^{[r]}\omega - c_\ell^{[r]}\xi < 0$ becomes $\omega > 0$.

Proof. This is generalization of a result in [3], where the first case is considered. If $c_{\ell+1}^{[r]}\xi - c_\ell^{[r]}\omega > 0$, for $1 \leq \ell \leq n$, we proceed by induction on n. For $n = 1$, the quiver Q_1, obtained from Q by mutation in 1, is the following:

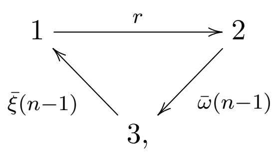


and for $n = 2$, the quiver Q_2, obtained from Q by mutation in 1 and 2, is the following:



In both cases, the result follows from $c_1^{[r]} = 0, c_2^{[r]} = 1, c_3^{[r]} = r, c_4^{[r]} = r^2 - 1$.

Suppose that $n > 2$. If n is odd, then by induction we know that the quiver Q_n is obtained by mutating the following quiver in vertex 1:

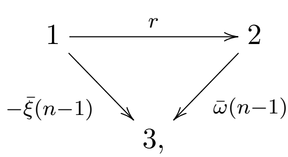


and the result follows from $\bar{\omega}(n) = \bar{\xi}(n-1)$ and

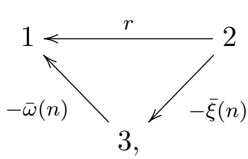
$$r\bar{\xi}(n-1)-\bar{\omega}(n-1) = rc_{n+1}^{[r]}\xi - rc_n^{[r]}\omega - c_n^{[r]}\xi + c_{n-1}^{[r]}\omega = c_{n+2}^{[r]}\xi - c_{n+1}^{[r]}\omega = \bar{\xi}(n). \tag{7}$$

The proof is similar in the case where n is even.

If $c_{n+1}^{[r]}\xi - c_n^{[r]}\omega \leq 0$ and $c_{\ell-1}^{[r]}\omega - c_\ell^{[r]}\xi < 0$, for $1 \leq \ell \leq n$, and if n is odd, then the quiver Q_n is obtained by mutating the following quiver in vertex 1:

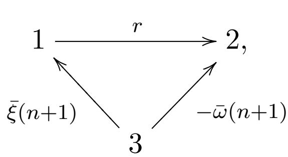


and the result follows from $\bar{\omega}(n) = \bar{\xi}(n-1)$ and $\bar{\xi}(n) = -\bar{\omega}(n-1)$. So the resulting quiver Q_n is



and $c_n^{[r]}\omega - c_{n+1}^{[r]}\xi \geq 0$. This shows the statement of the lemma in the second case.

Now mutating in vertex 2 results in Q_{n+1},



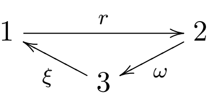
where $\bar{\omega}(n+1) = \bar{\xi}(n) = c_{n-1}^{[r]}\omega - c_n^{[r]}\xi$ and $\bar{\xi}(n+1) = -\bar{\omega}(n) = c_n^{[r]}\omega - c_{n+1}^{[r]}\xi$. This proves the third case of the lemma.

If n is even, then the proof is similar.

For the case where both $\xi < 0$ and $\omega < 0$, it is easy to check the claimed identity for $n = 1, 2$, and Q_n is nonacyclic for $n \geq 2$, thus the same proof as above applies. □

LEMMA 3.5. *In the situation of Lemma 3.2, if $r \geq 2$ and $\xi \geq \omega > 0$, then Q_n is cyclically oriented for all $n \geq 0$.*

Proof. An easy induction, using $c_{\ell+1}^{[r]} = rc_\ell^{[r]} - c_{\ell-1}^{[r]}$, shows that we never quit the first case of Lemma 3.2. □

Definition 3.6. Let Q_0 be the quiver  with $r \geq 0$ and $\omega, \xi \in \mathbb{Z}$, and let

$$Q_0 \overset{1}{\text{——}} Q_1 \overset{2}{\text{——}} Q_2 \overset{1}{\text{——}} Q_3 \cdots\cdots Q_m$$

be a sequence of mutations in directions 1 and 2. The sequence $(Q_0, \ldots, Q_m)$ of quivers is said to be of *almost cyclic type* if one of the following holds:

(1) $r \geq 2$ and $c_n^{[r]}\xi - c_{n-1}^{[r]}\omega > 0$ for $1 \leq n \leq m$;
(2) $r \geq 2$ and $c_n^{[r]}\xi - c_{n-1}^{[r]}\omega \leq 0$ for $1 \leq n \leq m$;
(3) $r = 1$, $m \leq 2$ and $c_n^{[r]}\xi - c_{n-1}^{[r]}\omega > 0$ for $1 \leq n \leq m$;
(4) $r = 1$, $m \leq 2$ and $c_n^{[r]}\xi - c_{n-1}^{[r]}\omega \leq 0$ for $1 \leq n \leq m$;
(5) $r = 0$.

The sequence $(Q_0, \ldots, Q_m)$ of quivers is said to be of *acyclic type* if one of the following holds:

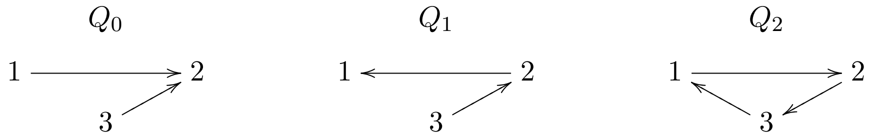
(6) $r \geq 2$, $m \geq 2$, and $c_{n+1}^{[r]}\xi - c_n^{[r]}\omega \leq 0$ and $c_{n-1}^{[r]}\omega - c_n^{[r]}\xi < 0$ for some $1 \leq n \leq m-1$;
(7) $r = 1$, $m = 2$, $\xi \leq 0$ and $\omega > 0$.

Remark 3.7. Conditions (3) and (4) are equivalent to conditions (3′) and (4′) below, respectively:

(3′) $r = 1, m = 1, \omega > 0$ or $r = 1, m = 2, \omega > 0, \xi > 0$;
(4′) $r = 1, m = 1, \omega \leq 0$ or $r = 1, m = 2, \omega \leq 0, \xi \leq 0$.

Remark 3.8. The quantities $c_n^{[r]}\xi - c_{n-1}^{[r]}\omega$ are the number of arrows in the quivers $Q_0, \ldots, Q_m$; see Lemma 3.2. If each of the quivers $Q_1, \ldots, Q_{m-1}$ has an oriented cycle, then the sequence $(Q_0, \ldots, Q_m)$ is of almost cyclic type. Thus being of almost cyclic type does not depend on the cyclicity of first and the last quiver. Condition (1) of the definition means that quivers $Q_0, \dots, Q_{m-2}$ are cyclic, and condition (2) means that quivers $Q_2, \dots, Q_m$ are cyclic.

Observe that it is possible that certain quivers in an almost cyclic sequence are acyclic. For example, the sequence



satisfies condition (4) and is therefore almost cyclic.

Remark 3.9. If $r \geq 2$, then conditions (1), (2) and (6) exhaust all possibilities. Thus in this case the sequence $(Q_0, \dots, Q_m)$ is either of almost cyclic type or of acyclic type.

Remark 3.10. In the case where $r = 1$, we only consider sequences with $m \leq 2$. The reason for this is that in this case the quiver Q_5 is the same as the quiver Q_0 with the vertices 1 and 2 switched. Thus the quivers obtained from Q_0 by a sequence of length m with $3 \leq m \leq 5$ are the same as the quivers obtained from Q_5 by a sequence of length $5 - m$ with $0 \leq 5 - m \leq 2$.

Remark 3.11. If the sequence is of acyclic type satisfying condition (6) with some $n \leq m-1$, then all quivers $Q_{n+1}, \dots, Q_m$ satisfy the third condition of Lemma 3.2.

3.2. *Compatible pairs.* Let (a_1, a_2) be a pair of nonnegative integers. A *Dyck path* of type $a_1 \times a_2$ is a lattice path from $(0,0)$ to (a_1, a_2) that never goes above the main diagonal joining $(0,0)$ and (a_1, a_2). Among the Dyck paths of a given type $a_1 \times a_2$, there is a (unique) *maximal* one denoted by $\mathcal{D} = \mathcal{D}^{a_1 \times a_2}$. It is defined by the property that any lattice point strictly above $\mathcal{D}$ is also strictly above the main diagonal.

It will be convenient to extend this definition to negative integers a_1, a_2. If $a_1 < 0$, then the notation $\mathcal{D}^{a_1 \times a_2}$ means $\mathcal{D}^{0 \times a_2}$ and, similarly, if $a_2 < 0$, then the notation $\mathcal{D}^{a_1 \times a_2}$ means $\mathcal{D}^{a_1 \times 0}$. If both $a_1, a_2 < 0$, then $\mathcal{D}^{a_1 \times a_2}$ means $\mathcal{D}^{0 \times 0}$.

Let $\mathcal{D} = \mathcal{D}^{a_1 \times a_2}$. Let $\mathcal{D}_1 = \mathcal{D}_1^{a_1 \times a_2} = \{u_1, \dots, u_{a_1}\}$ be the set of horizontal edges of $\mathcal{D}$ indexed from left to right, and let $\mathcal{D}_2 = \mathcal{D}_2^{a_1 \times a_2} = \{v_1, \dots, v_{a_2}\}$ be the set of vertical edges of $\mathcal{D}$ indexed from bottom to top. Given any points A and B on $\mathcal{D}$, let AB be the subpath starting from A, and going in the Northeast direction until it reaches B (If we reach (a_1, a_2) first, we continue from $(0,0)$.) By convention, if $A = B$, then AA is the subpath that starts from A, then passes (a_1, a_2) and ends at A. If we represent a subpath of $\mathcal{D}$ by its set of edges, then for $A = (i,j)$ and $B = (i', j')$, we have

$$AB = \begin{cases} \{u_k, v_\ell : i < k \leq i', j < \ell \leq j'\} & \text{if } B \text{ is to the Northeast of } A, \\ \mathcal{D} - \{u_k, v_\ell : i' < k \leq i, j' < \ell \leq j\} & \text{otherwise.} \end{cases}$$

We denote by $(AB)_1$ the set of horizontal edges in AB, and by $(AB)_2$ the set of vertical edges in AB. Also let AB° denote the set of lattice points on the subpath AB excluding the endpoints A and B. (Here $(0,0)$ and (a_1, a_2) are regarded as the same point.)

Here is an example for $(a_1, a_2) = (6, 4)$:

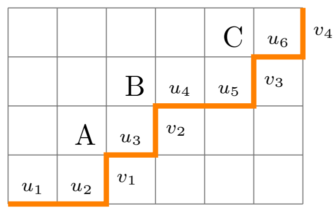


Figure 1. A maximal Dyck path.

Let $A = (2, 1)$, $B = (3, 2)$ and $C = (5, 3)$. Then

$$(AB)_1 = \{u_3\}, \qquad (AB)_2 = \{v_2\},$$
$$(BA)_1 = \{u_4, u_5, u_6, u_1, u_2\}, \quad (BA)_2 = \{v_3, v_4, v_1\}.$$

The point C is in BA° but not in AB°. The subpath AA has length 10 (not 0).

Definition 3.12. Let r be a positive integer. For $S_1 \subseteq \mathcal{D}_1$, $S_2 \subseteq \mathcal{D}_2$, we say that the pair (S_1, S_2) is *r-compatible* if for every $u \in S_1$ and $v \in S_2$, denoting by E the left endpoint of u and F the upper endpoint of v, there exists a lattice point $A \in EF^\circ$ such that

$$|(AF)_1| = r|(AF)_2 \cap S_2| \text{ or } |(EA)_2| = r|(EA)_1 \cap S_1|. \tag{8}$$

Remark 3.13. We often say *compatible* instead of *r-compatible* if r is clear from the context.

For $r = 3$, the pair $(\{u_1, u_2\}, \{v_3, v_4\})$ is not compatible in $\mathcal{D}^{6\times 4}$ (Figure 2), but it is compatible in $\mathcal{D}^{7\times 4}$ (Figure 3).

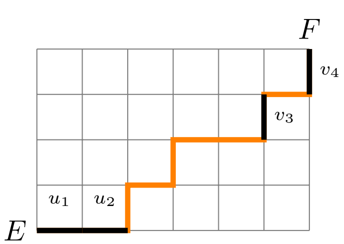


Figure 2

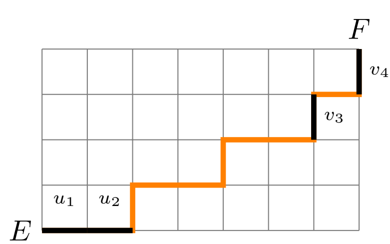


Figure 3

3.3. *Expansion formulas.* The formulas in this subsection are proved in [23]. Let $\mathcal{A}$ be a skew-symmetric cluster algebra of geometric type of arbitrary rank N. Let $\mathbf{x}_t = \{x_{1;t}, \ldots, x_{N;t}\}$ and $\mathbf{x}_{t'} = \{x_{1;t'}, \ldots, x_{N;t'}\}$ be two clusters such that there exists a sequence μ of mutations in directions 1 and 2, transforming $\mathbf{x}_{t'}$ into $\mathbf{x}_t$. Suppose that the last mutation in μ is in direction 1. Thus μ is of one of the following two forms:

$$\mu = t' \xrightarrow{1} \cdot \xrightarrow{2} \cdot \xrightarrow{1} \cdot \cdots\cdots \cdot \xrightarrow{2} \cdot \xrightarrow{1} t$$

or

$$\mu = t' \xrightarrow{2} \cdot \xrightarrow{1} \cdot \xrightarrow{2} \cdot \cdots\cdots \cdot \xrightarrow{2} \cdot \xrightarrow{1} t$$

Observe that $x_{f,t'} = x_{f,t}$ for all $f = 3, 4, \ldots, N$. Denote by n the number of seeds in the sequence μ including t' and t. Let Q_t be the quiver of the seed at t, let r be the number of arrows $1 \to 2$, where we suppose without loss of generality that $r \geq 0$, and let $\xi_f \in \mathbb{Z}$ be the number of arrows $f \to 1$ and $\omega_f \in \mathbb{Z}$ the number of arrows $2 \to f$ in Q_t. Given $p, q \geq 0$, define

$$A_i = A_i(p,q) = \begin{cases} pc_{i+1}^{[r]} + qc_i^{[r]} & \text{if } \mu = t' \xrightarrow{1} \cdot \xrightarrow{2} \cdot \xrightarrow{1} \cdot \cdots\cdots \cdot \xrightarrow{2} \cdot \xrightarrow{1} t, \\ qc_{i+1}^{[r]} + pc_i^{[r]} & \text{if } \mu = t' \xrightarrow{2} \cdot \xrightarrow{1} \cdot \xrightarrow{2} \cdot \cdots\cdots \cdot \xrightarrow{2} \cdot \xrightarrow{1} t, \end{cases}$$

and

$$\alpha = \begin{cases} q & \text{if } \mu = t' \xrightarrow{1} \cdot \xrightarrow{2} \cdot \xrightarrow{1} \cdot \cdots\cdots \cdot \xrightarrow{2} \cdot \xrightarrow{1} t, \\ p & \text{if } \mu = t' \xrightarrow{2} \cdot \xrightarrow{1} \cdot \xrightarrow{2} \cdot \cdots\cdots \cdot \xrightarrow{2} \cdot \xrightarrow{1} t. \end{cases}$$

The following lemma is a straightforward consequence of Lemma 3.1.

LEMMA 3.14 ([23, Lemma 3.12]). *For any i, we have*

(a) $A_i = rA_{i-1} - A_{i-2}$,
(b) $A_i^2 - A_{i+1}A_{i-1} = p^2 + q^2 + rpq$. □

We are now ready to state our first expansion formula.

THEOREM 3.15 ([23, Th. 3.13]). *For all $p, q \geq 0$, we have*

$$x_{1;t'}^p x_{2;t'}^q = \sum_{(S_1,S_2)} x_{1;t}^{r|S_2| - A_{n-1}} x_{2;t}^{r|S_1| - A_{n-2}} \prod_{f=3}^{N} x_{f;t}^{\xi_f(A_{n-1} - |S_1|) - \omega_f |S_2| - M_f},$$

where $M_f = \min_{(S_1,S_2)} \{\xi_f(A_{n-1} - |S_1|) - \omega_f |S_2|\}$, and the sum is over all $(S_1 = \cup_{i=1}^{p+q} S_1^i, S_2 = \cup_{i=1}^{p+q} S_2^i)$ such that

$$(S_1^i, S_2^i) \text{ is a compatible pair in } \begin{cases} \mathcal{D}^{c_{n-1}^{[r]} \times c_{n-2}^{[r]}} & \text{if } 1 \leq i \leq \alpha, \\ \mathcal{D}^{c_n^{[r]} \times c_{n-1}^{[r]}} & \text{if } \alpha + 1 \leq i \leq p + q. \end{cases}$$

Remark 3.16. It can be shown that the summation on the right-hand side in Theorem 3.15 can be taken over all compatible pairs in $\mathcal{D}^{A_{n-1}\times A_{n-2}}$ instead, without changing the sum; see [19, Th. 1.11].

Remark 3.17. The term M_f in the exponent of $x_{f;t}$ in Theorem 3.15 comes from Fomin-Zelevinsky's separation of addition formula [14, Th. 3.7].

We shall need a precise value for M_f. As a first step, we determine which pair (S_1, S_2) can realize the minimum M_f. Let

$$a_{1,i} = \begin{cases} c_{n-1}^{[r]} & \text{if } 1 \le i \le \alpha, \\ c_n^{[r]} & \text{if } \alpha+1 \le i \le p+q \end{cases}$$

and

$$a_{2,i} = \begin{cases} c_{n-2}^{[r]} & \text{if } 1 \le i \le \alpha, \\ c_{n-1}^{[r]} & \text{if } \alpha+1 \le i \le p+q. \end{cases}$$

LEMMA 3.18 ([23, Lemma 3.8]). *In the setting of Theorem 3.15, consider the values* $\xi_f(A_{n-1} - |S_1|) - \omega_f |S_2|$ *obtained from the following three cases:*

- $S_1^i = \mathcal{D}_1^{a_{1,i}\times a_{2,i}}$ *and* $S_2^i = \emptyset$ *for all* $1 \le i \le p+q$;
- $S_1^i = \emptyset$ *and* $S_2^i = \emptyset$ *for all* $1 \le i \le p+q$;
- $S_1^i = \emptyset$ *and* $S_2^i = \mathcal{D}_2^{a_{1,i}\times a_{2,i}}$ *for all* $1 \le i \le p+q$.

Then one of the (possibly nondistinct) three values is equal to M_f.

Proof. This follows from the proof of Lemma 3.8 in [23] replacing $c_n^{[r]}$ by A_n. □

Before stating our second formula, we need to introduce some notation. For arbitrary (possibly negative) integers A, B, we define the modified binomial coefficient as follows:

$$\left[\begin{array}{c} A \\ B \end{array}\right] := \begin{cases} \prod_{i=0}^{A-B-1} \frac{A-i}{A-B-i} & \text{if } A > B, \\ 1 & \text{if } A = B, \\ 0 & \text{if } A < B. \end{cases}$$

If $A \ge 0$, then $\left[\begin{smallmatrix} A \\ B \end{smallmatrix}\right] = \left[\begin{smallmatrix} A \\ A-B \end{smallmatrix}\right]$ is just the usual binomial coefficient; in particular, $\left[\begin{smallmatrix} A \\ B \end{smallmatrix}\right] = 0$ if $A \ge 0$ and $B < 0$.

For a sequence of integers (τ_j) (respectively (τ'_j)), we define a sequence of weighted partial sums (s_i) (respectively (s'_i)) as follows:

$$s_0 = 0, \qquad s_i = \sum_{j=0}^{i-1} c_{i-j+1}^{[r]} \tau_j = c_{i+1}^{[r]}\tau_0 + c_i^{[r]}\tau_1 + \cdots + c_2^{[r]}\tau_{i-1},$$

$$s'_0 = 0, \qquad s'_i = \sum_{j=0}^{i-1} c_{i-j+1}^{[r]} \tau'_j = c_{i+1}^{[r]}\tau'_0 + c_i^{[r]}\tau'_1 + \cdots + c_2^{[r]}\tau'_{i-1}.$$

For example, $s_1 = c_2^{[r]}\tau_0 = \tau_0$, $s_2 = c_3^{[r]}\tau_0 + c_2^{[r]}\tau_1 = r\tau_0 + \tau_1$.

LEMMA 3.19 ([23, Lemma 3.15]). $s_n = rs_{n-1} - s_{n-2} + \tau_{n-1}$.

Definition 3.20. Let $\mathcal{L}(\tau_0, \tau_1, \ldots, \tau_{n-3})$ denote the set of all elements $(\tau'_0, \tau'_1, \ldots, \tau'_{n-3}) \in \mathbb{Z}^{n-2}$ satisfying the conditions

(1) $0 \leq \tau'_i \leq \tau_i$ for $0 \leq i \leq n-4$,
(2) $s'_{n-3} = kc^{[r]}_{n-2}$ and $s'_{n-2} = kc^{[r]}_{n-1}$ for some integer $0 \leq k \leq p$.

We define a partial order on $\mathcal{L}(\tau_0, \tau_1, \ldots, \tau_{n-3})$ by

$$(\tau'_0, \tau'_1, \ldots, \tau'_{n-3}) \leq_{\mathcal{L}} (\tau''_0, \tau''_1, \ldots, \tau''_{n-3}) \text{ if and only if } \tau'_i \leq \tau''_i \text{ for } 0 \leq i \leq n-4.$$

Then we define $\mathcal{L}_{\max}(\tau_0, \tau_1, \ldots, \tau_{n-3})$ to be the set of the maximal elements of of $\mathcal{L}(\tau_0, \tau_1, \ldots, \tau_{n-3})$ with respect to $\leq_{\mathcal{L}}$.

Our second expansion formula is the following.

THEOREM 3.21 ([23, Th. 3.17]). *Let $\widetilde{x_{1;t}}$ be the cluster variable obtained by mutating $\mathbf{x}_t$ in direction 1. Let $\omega'_f = \xi_f$ and ξ'_f be the number of arrows from 1 to f, or f to 2 respectively, in the quiver obtained from Q_t by mutating in the vertex 1. Then*

$$x^p_{1;t'} x^q_{2;t'} = \sum_{\tau_0, \tau_1, \ldots, \tau_{n-3}} \left(\prod_{i=0}^{n-3} \begin{bmatrix} A_{i+1} - rs_i \\ \tau_i \end{bmatrix} \right) x_{2;t}^{rs_{n-3} - A_{n-2}} \widetilde{x_{1;t}}^{A_{n-1} - rs_{n-2}} \times \prod_{f=3}^{N} x_{f;t}^{\xi'_f s_{n-2} - \omega'_f s_{n-3} - M'_f}, \tag{9}$$

where the summation runs over all integers $\tau_0, \ldots, \tau_{n-3}$ satisfying

$$\begin{cases} 0 \leq \tau_i \leq A_{i+1} - rs_i \ (0 \leq i \leq n-4), \ \tau_{n-3} \leq A_{n-2} - rs_{n-3}, \\ (s_{n-2} - s'_{n-2}) A_{n-3} \geq (s_{n-3} - s'_{n-3}) A_{n-2} \end{cases} \tag{10}$$

for any $(\tau'_0, \ldots, \tau'_{n-3}) \in \mathcal{L}_{\max}(\tau_0, \ldots, \tau_{n-3})$, and $M'_f = 0$ if the sequence of full subquivers on vertices 1, 2, f in μ from t' to t is of almost cyclic type, and $M'_f = \xi'_f A_{n-2} - \omega'_f A_{n-3}$ if the sequence is of acyclic type from t' up to $\mu_1(t)$. In particular, the exponent of $x_{f;t}$ is nonnegative.

Proof. The proof is exactly the same as the proof of [23, Th. 3.17] except for the M'_f in the exponent of $x_{f,t}$. The precise value for M'_f follows by comparing terms with the formula in Theorem 3.15 and using Lemma 3.18. The statement about nonnegative exponents also follows from Theorem 3.15. □

Combining the two formulas of Theorems 3.15 and 3.21, we get the following mixed formula, which has the advantage that the exponents of $\widetilde{x_{1;t}}, x_{1;t}$ and $x_{f;t}$ are nonnegative.

THEOREM 3.22 ([23, Th. 3.21]).

(11)

$$x^p_{1;t'}x^q_{2;t'} = \sum_{\substack{\tau_0,\tau_1,\dots,\tau_{n-3}\\ A_{n-1}-rs_{n-2}\geq 0}} \left(\prod_{i=0}^{n-3}\binom{A_{i+1}-rs_i}{\tau_i}\right)\widetilde{x_{1;t}}^{A_{n-1}-rs_{n-2}}x_{2;t}^{rs_{n-3}-A_{n-2}}$$

$$\times\prod_{f=3}^{N}x_{f;t}^{\xi'_f s_{n-2}-\omega'_f s_{n-3}-M'_f},$$

$$+\sum_{\substack{(S_1,S_2)\\ r|S_2|-A_{n-1}>0}} x_{1;t}^{r|S_2|-A_{n-1}}x_{2;t}^{r|S_1|-A_{n-2}}\prod_{f=3}^{N}x_{f;t}^{\xi_f(A_{n-1}-|S_1|)-\omega_f|S_2|-M_f},$$

where (S_1, S_2) *are as in Theorem 3.15, and where* $M'_f = 0$ *if the sequence of full subquivers on vertices* 1, 2, f *in* μ *is of almost cyclic type, and* $M_f = \xi_f A_{n-1} - \omega_f A_{n-2}$ *if the sequence of full subquivers on vertices* 1, 2, f *in* μ *is of acyclic type. In particular, the exponents of* $x_{f;t}$ *are nonnegative.*

COROLLARY 3.23. *For any cluster monomial* u *in the variables of* $\mathbf{x}'$, *there exist two polynomials*

$$f_1 \in \mathbb{Z}_{\geq 0}\mathbb{P}[\widetilde{x_{1;t}}, x_{2;t}^{\pm 1}, x_{3;t}, \dots, x_{N;t}] \quad \textit{and} \quad f_2 \in \mathbb{Z}_{\geq 0}\mathbb{P}[x_{1;t}, x_{2;t}^{\pm 1}, x_{3;t}, \dots, x_{N;t}]$$

such that

$$u = f_1 + f_2.$$

We end this subsection with the following rank 2 result, which we will need later.

THEOREM 3.24 ([23, Th. 3.26]). *Let* $a \geq \frac{A_n}{r}$ *be an integer. Then the sum*

$$\sum_{\substack{\tau_0,\tau_1,\dots,\tau_{n-2}\\ s_{n-1}=a}} \prod_{i=0}^{n-2}\left[\begin{array}{c}A_{i+1}-rs_i\\ \tau_i\end{array}\right] x_1^{rs_{n-2}-A_{n-1}}x_2^{r(A_{n-1}-a)-A_{n-2}},$$

where the summation runs over all integers $\tau_0, \dots, \tau_{n-2}$ *satisfying* (10) *with* n *replaced by* $n+1$, *is divisible by* $(1 + x_1^r)^{ra-A_n}$, *and the resulting quotient has nonnegative coefficients.*

4. Main result

In this section we present our main results. The positivity conjecture (Theorem 4.2) follows from the following result.

THEOREM 4.1. *Let* $\mathcal{A}$ *be a skew-symmetric cluster algebra of geometric type, and let* $\mathbf{x}_t = \{x_{1;t}, x_{2;t}, \dots, x_{N;t}\}$ *be a cluster in* $\mathcal{A}$. *Let* u *be a cluster variable, and let* $\mathbf{x}_{t_0}$ *be a cluster containing* u *such that the distance between*

t_0 and t in the exchange tree of labeled seeds is minimal. Let μ be the unique sequence of mutations relating the seeds t_0 and t in the exchange tree of labeled seeds. Denote by d', d the directions of the last two mutations in the sequence μ, thus

$$\mu = \; t_0 \cdots\cdots \cdot \overline{} \; t'' \xrightarrow{d'} t' \xrightarrow{d} t,$$

and let $e \in \{1, 2, \ldots, n\}, e \neq d, d'$. Let $\widetilde{x_{d;t}}, \widetilde{x_{e;t}}$ be the cluster variables obtained from $\mathbf{x}_t$ by mutation in direction d, e, respectively, and let $\widetilde{\widetilde{x_{e;t}}}$ be the cluster variable obtained from $\mathbf{x}_t$ by the two step mutation first in d and then in e. Then there exist polynomials

$$A_t \in \mathbb{Z}_{\geq 0}\mathbb{P}[\widetilde{x_{d;t}}, \widetilde{\widetilde{x_{e;t}}}; (x_{f;t}^{\pm 1})_{f \neq d,e}], \qquad B_t \in \mathbb{Z}_{\geq 0}\mathbb{P}[\widetilde{x_{d;t}}, x_{e;t}; (x_{f;t}^{\pm 1})_{f \neq d,e}],$$

$$C_t \in \mathbb{Z}_{\geq 0}\mathbb{P}[x_{d;t}, x_{e;t}; (x_{f;t}^{\pm 1})_{f \neq d,e}], \qquad D_t \in \mathbb{Z}_{\geq 0}\mathbb{P}[x_{d;t}, \widetilde{x_{e;t}}; (x_{f;t}^{\pm 1})_{f \neq d,e}]$$

such that

$$u = A_t + B_t + C_t + D_t.$$

Moreover, the polynomials A_t, B_t, C_t, D_t are unique up to intersection of polynomial rings. In particular,

$$u \in \mathbb{Z}_{\geq 0}\mathbb{P}[x_{d;t}, x_{e;t}, \widetilde{x_{d;t}}, \widetilde{x_{e;t}}, \widetilde{\widetilde{x_{e;t}}}; (x_{f;t}^{\pm 1})_{f \neq d,e}].$$

The proof of this theorem is given in Section 5. The positivity conjecture follows easily.

Theorem 4.2 (Positivity Conjecture). *Let $\mathcal{A}(Q)$ be a skew-symmetric cluster algebra, let $\mathbf{x}_t$ be any cluster, and let u be any cluster variable. Then the Laurent expansion of u with respect to the cluster $\mathbf{x}_t$ is a Laurent polynomial in $\mathbf{x}_t$ whose coefficients are nonnegative integer linear combinations of elements of $\mathbb{P}$.*

Proof. Because of Fomin-Zelevinsky's separation of addition formula [14, Th. 3.7], it suffices to prove the result in the case where $\mathcal{A}(Q)$ is of geometric type. Let $\mathbf{x}_{t_0}$ be an arbitrary cluster, and let $u \in \mathbf{x}_{t_0}$ be a cluster variable. Let μ be the unique sequence of mutations relating the seed t_0 to the seed t in the exchange tree of labeled seeds. Let d, e be the last two directions in the sequence μ. Consider the maximal rank 2 mutation subsequence in directions e, d at the end of μ. This subsequence connects t to a seed t_1', and we denote by t_1 the seed one step away from t_1' on this subsequence. Thus we have

$$\mu = \; t_0 \cdots\cdots \cdot \xrightarrow{d'} t_1' \xrightarrow{e} t_1 \xrightarrow{d} \cdot \xrightarrow{e} \cdot \xrightarrow{d} \cdot \cdots\cdots \cdot \xrightarrow{d} \cdot \xrightarrow{e} t$$

or

$$\mu = \; t_0 \cdots\cdots \cdot \xrightarrow{d'} t_1' \xrightarrow{d} t_1 \xrightarrow{e} \cdot \xrightarrow{d} \cdot \xrightarrow{e} \cdot \cdots\cdots \cdot \xrightarrow{d} \cdot \xrightarrow{e} t,$$

with $d' \neq d, e$. Then applying Theorem 4.1 at the seed t_1 with respect to the directions d and e, we get that

$$u \in \mathbb{Z}_{\geq 0}\mathbb{P}[x_{d;t_1}, x_{e;t_1}, \widetilde{x_{d;t_1}}, \widetilde{x_{e;t_1}}, \widetilde{\widetilde{x_{d;t_1}}}; (x_{f;t_1}^{\pm 1})_{f \neq d,e}]$$

or

$$u \in \mathbb{Z}_{\geq 0}\mathbb{P}[x_{d;t_1}, x_{e;t_1}, \widetilde{x_{d;t_1}}, \widetilde{x_{e;t_1}}, \widetilde{\widetilde{x_{e;t_1}}}; (x_{f;t_1}^{\pm 1})_{f \neq d,e}].$$

Moreover, if $f \neq d, e$, then $x_{f;t'} = x_{f;t}$ is a cluster variable in $\mathbf{x}_t$. On the other hand, each of the variables $x_{d;t_1}, x_{e;t_1}, \widetilde{x_{d;t_1}}, \widetilde{x_{e;t_1}}, \widetilde{\widetilde{x_{d;t_1}}}$, and $\widetilde{\widetilde{x_{e;t_1}}}$ is obtained from the cluster $\mathbf{x}_t$ by a mutation sequence using only the two directions e and d, and therefore Theorem 3.15 implies that these variables are Laurent polynomials in $\mathbf{x}_t$ with coefficients in $\mathbb{Z}_{\geq 0}\mathbb{P}$. It follows that, after substitution into u, we get an expansion for u as a Laurent polynomial with nonnegative coefficients in the initial cluster $\mathbf{x}_t$. □

Remark 4.3. For $N > 2$, one can prove Theorem 4.2 directly from Theorem 4.1 without using induction. We prefer the proof above, since it illustrates the inductive nature of Theorem 4.1.

5. Proof of Theorem 4.1

We use induction on the length ℓ of the sequence of mutations μ.

If $\ell = 0$, then $u = x_{i;t}$ for some i, which is of the form C_t.

If $\ell = 1$, then μ consists of a single mutation in direction d and $u = \widetilde{x_{d;t}}$ is of the form B_t.

If $\ell = 2$, then μ is a sequence of two mutations $t_0 \overset{d'}{\text{——}} t' \overset{d}{\text{——}} t$, with $d' \neq d$ and $e \neq d, d'$. Then

$$u = (\text{binomial in } \mathbf{x}_{t'})\, x_{d';t}^{-1} = (\text{binomial in } (\mathbf{x}_t \setminus \{x_{d;t}\}) \cup \{\widetilde{x_{d;t}}\})\, x_{d',t}^{-1}$$

and this is of the form B_t, since $d' \neq d, e$ and the binomial has coefficients in $\mathbb{Z}_{\geq 0}\mathbb{P}$.

Now let $\ell \geq 3$. Then μ is a sequence of the form $t_0 \cdots\cdots \cdot \text{——} t'' \overset{d'}{\text{——}} t' \overset{d}{\text{——}} t$, with $d' \neq d$. Consider the maximal rank 2 mutation subsequence in directions d, d' at the end of μ. This subsequence connects t to a seed t^{**}, and we denote by t^* the seed one step away from t^{**} on this subsequence. Thus we have

$$\mu = t_0 \cdots\cdots \cdot \overset{d''}{\text{——}} t^{**} \overset{d}{\text{——}} t^* \overset{d'}{\text{——}} \cdot \overset{d}{\text{——}} \cdot \overset{d'}{\text{——}} \cdot \cdots\cdots \cdot \overset{d'}{\text{——}} \cdot \overset{d}{\text{——}} t$$

or

$$\mu = t_0 \cdots\cdots \cdot \overset{d''}{\text{——}} t^{**} \overset{d'}{\text{——}} t^* \overset{d}{\text{——}} \cdot \overset{d'}{\text{——}} \cdot \overset{d}{\text{——}} \cdot \cdots\cdots \cdot \overset{d'}{\text{——}} \cdot \overset{d}{\text{——}} t,$$

with $d'' \neq d, d'$.

Consider the subsequence of mutations μ^* connecting t_0 to t^*. Since this sequence is shorter than the sequence μ, we can conclude by induction that the statement holds in the seed t^* with directions d, d'. Thus, if μ is as in the first case,

$$u = A_{t^*} + B_{t^*} + C_{t^*} + D_{t^*}, \tag{12}$$

where

$$A_{t^*} \in \mathbb{Z}_{\geq 0}\mathbb{P}[\widetilde{\widetilde{x_{d';t^*}}}, \widetilde{x_{d;t^*}}; (x_{f;t^*}^{\pm 1})_{f\neq d,d'}], \quad B_{t^*} \in \mathbb{Z}_{\geq 0}\mathbb{P}[x_{d';t^*}, \widetilde{x_{d;t^*}}; (x_{f;t^*}^{\pm 1})_{f\neq d,d'}],$$
$$C_{t^*} \in \mathbb{Z}_{\geq 0}\mathbb{P}[x_{d';t^*}, x_{d;t^*}; (x_{f;t^*}^{\pm 1})_{f\neq d,d'}], \quad D_{t^*} \in \mathbb{Z}_{\geq 0}\mathbb{P}[\widetilde{x_{d';t^*}}, x_{d;t^*}; (x_{f;t^*}^{\pm 1})_{f\neq d,d'}].$$

If the sequence μ is as in the second case, the roles of d and d' are interchanged. Without loss of generality, we assume we are in the first case.

Consider the variables appearing in these expressions one by one. If $f \neq d, d'$ then $x_{f;t^*} = x_{f;t}$ is in $\mathbf{x}_t$ and may have a negative exponent in the desired expression for u. The variables $\widetilde{\widetilde{x_{d';t^*}}}, \widetilde{x_{d;t^*}}, x_{d';t^*}, x_{d;t^*}$ and $\widetilde{x_{d';t^*}}$ lie on a rank 2 mutation sequence from t in the directions d and d', and Corollary 3.23 implies that all the cluster monomials involving such variables (up to the $x_f^{\pm}$) have expansions of the form $f_1 + f_2$ with

$$f_1 \in \mathbb{Z}_{\geq 0}\mathbb{P}[\widetilde{x_{d;t}}, x_{d';t}^{\pm 1}; x_{f;t} : f \neq d, d'] \text{ and } f_2 \in \mathbb{Z}_{\geq 0}\mathbb{P}[x_{d;t}, x_{d';t}^{\pm 1}; x_{f;t} : f \neq d, d'].$$

Substituting these expansions into (12) shows that

$$u = B'_t + C'_t, \tag{13}$$

with

$$B'_t \in \mathbb{Z}_{\geq 0}\mathbb{P}[\widetilde{x_{d;t}}, x_{d';t}^{\pm 1}; (x_{f;t^*}^{\pm 1})_{f\neq d,d'}], \qquad C'_t \in \mathbb{Z}_{\geq 0}\mathbb{P}[x_{d;t}, x_{d';t}^{\pm 1}; (x_{f;t^*}^{\pm 1})_{f\neq d,d'}],$$

We now have to study the exponents of $x_{e;t}$ in B'_t and C'_t. If these exponents are nonnegative, then u is of the form $B_t + C_t$ and we are done. But if $x_{e;t}$ appears with negative exponents in B'_t, then we have to rewrite B'_t in the form $A_t + B_t$, and if $x_{e;t}$ appears with negative exponents in C'_t, then we have to rewrite C'_t in the form $C_t + D_t$, with A_t, B_t, C_t, D_t as in the statement of Theorem 4.1. We prove that this is always possible in Proposition 5.5. To do so, we have to go back in the mutation sequence μ up to the last mutations in direction e.

More precisely, consider the last maximal rank 2 subsequence ν of μ containing e. Let e' be the other direction occurring in ν. If

$$\nu = \; t_{w_B} \overset{e'}{\text{-----}} t_{w_C} \overset{e}{\text{-----}} t_{w_D} \overset{e'}{\text{-----}} \; \overset{e}{\text{-----}} \; \cdots\cdots \; \overset{e'}{\text{-----}} t_{w_Z},$$

then let t_{w_A} be the seed obtained from t_{w_B} by mutating in direction e. If

$$\nu = \; t_{w_B} \overset{e}{\text{-----}} t_{w_C} \overset{e'}{\text{-----}} t_{w_D} \overset{e}{\text{-----}} \; \overset{e'}{\text{-----}} \; \cdots\cdots \; \overset{e'}{\text{-----}} t_{w_Z},$$

then let t_{w_A} be the seed obtained from t_{w_B} by mutating in direction e'. We do not label any of the seeds between t_{w_D} and t_{w_E}.

Without loss of generality, we may assume that $t_{w_Z} = t^*$ and that $e' = d$. Indeed, otherwise we can change the sequence μ by inserting two consecutive mutations in direction e as follows

$$\mu' = \ t_0 \cdots\cdots \cdot \overset{d''}{\text{——}} t^{**} \overset{e}{\text{——}} t_{w_B} \overset{e}{\text{——}} t^{**} \overset{d}{\text{——}} t^* \overset{d'}{\text{——}} \cdot \cdots\cdots \cdot \overset{d'}{\text{——}} \cdot \overset{d}{\text{——}} t,$$

letting ν be the mutation sequence $t_{w_B} \overset{e}{\text{——}} t^{**} \overset{d}{\text{——}} t^*$.

Thus, without loss of generality, we assume the sequence μ is as follows:
(14)

$$\mu = t_0 \cdots\cdots t_{w_B} \text{——} t_{w_C} \cdots\cdots \cdot \overset{d}{\text{——}} \cdot \overset{e}{\text{——}} t^{**} \overset{d}{\text{——}} t^* \overset{d'}{\text{——}} \cdot \cdots\cdots \cdot \overset{d'}{\text{——}} \cdot \overset{d}{\text{——}} t.$$

PROPOSITION 5.1. *The variable $x_{e;t}$ may have negative exponents in one of the expressions B'_t or C'_t in equation (13), but not in both.*

Proof. This is proved in Section 5.1. □

Let $A_{i;1} = A_i$ (respectively $\tau_{i;1} = \tau_i, s_{i;1} = s_1$) be the sequence of integers defined in Section 3.3 with respect to r the number of arrows between e and d at the seed t_{w_E}, where $E = A, B, C$, or D.

PROPOSITION 5.2. *Suppose that at least one monomial of B'_t has a negative exponent in $x_{e;t}$. Then each cluster monomial of the form $x^p_{e;t_{w_E}} x^q_{d;t_{w_E}}$, where $E = A, B, C$, or D, has the following form:*

$$\begin{aligned}
&\sum_{v\geq 0} (\textit{Laurent polynomial in cluster variables of } \mathbf{x}_t \cup \mathbf{x}_{t'} \setminus \{x_{e;t}\})\, x^v_{e;t} \\
&\quad + \sum_{\theta>0} x^{-\theta}_{e;t'} \sum_{\varsigma\geq 0} \sum_{\mathfrak{r}} \lambda_{\mathfrak{r}} \mathfrak{r} \\
&\quad \times \sum_{\substack{\tau_{0;1},\tau_{1;1},\dots,\tau_{n_1-2;1} \\ s_{n_1-1;1}=A_{n_1-1;1}-\varsigma}} \ \sum_{j=0}^{\sum_{w=1}^{n_2-3}\tau_{w;2}} d_j \binom{\left\lfloor (A_{n_1-1;1}-\varsigma)\frac{A_{n_1-1;1}}{A_{n_1;1}} \right\rfloor - s_{n_1-2;1}}{j} \qquad (15) \\
&\quad \times \left(\prod_{w=0}^{n_1-2} \begin{bmatrix} A_{w+1;1} - r_1 s_{w;1} \\ \tau_{w;1} \end{bmatrix} \right) \\
&\quad \times \left(\frac{\prod_i x_{i;t'}^{[b^{t'}_{i,e}]_+}}{\prod_i x_{i;t'}^{[-b^{t'}_{i,e}]_+}} \right)^{\mathrm{sgn}(2b^{t'}_{d,e}+1)\left(\left\lfloor (A_{n_1-1;1}-\varsigma)\frac{A_{n_1-1;1}}{A_{n_1;1}} \right\rfloor - s_{n_1-2;1}\right)},
\end{aligned}$$

where $\mathfrak{r} \in \mathbb{ZP}_{\geq 0}[\{x_{d;t'}\} \cup (\mathbf{x}^{\pm 1}_{t'} \setminus \{x^{\pm 1}_{d;t'}, x^{\pm 1}_{e;t'}\})]$, $\lambda_{\mathfrak{r}} \in \{0,1\}$ and d_j are nonnegative integers depending on the summation indices, and the integers $\tau_{i,\ell}$ satisfy condition (10) with $n = n_1 + 1$ if $\ell = 1$; and $n = n_2$ if $\ell = 2$ for some integers n_1, n_2. The second subindex $\ell = 1, 2$ in $s_{i,;\ell}, A_{i;\ell}, \tau_{i;\ell}$ refers to the fact that these integers are defined in terms of n_ℓ.

Proof. This is proved in Section 5.2. □

Remark 5.3. The condition that at least one monomial of B'_t has a negative exponent in $x_{e;t}$ does not depend on the variable u but rather on the orientation of the quivers in the mutation sequence ν.

Proposition 5.4. *Suppose that at least one monomial of C'_t has a negative exponent in $x_{e;t}$. Then each cluster monomial of the form $x^p_{e;t_{w_E}} x^q_{f;t_{w_E}}$, where $E = A, B, C$, or D, has the following form:*

$$
\begin{aligned}
&\sum_{v\geq 0}(\textit{Laurent polynomial in cluster variables of } \mathbf{x}_t \cup \mathbf{x}_{t'} \setminus \{x_{e;t}\})\, x^v_{e;t} \\
&\quad + \sum_{\theta>0} x^{-\theta}_{e;t'} \sum_{\varsigma\geq 0}\sum_{\mathfrak{r}} \lambda_{\mathfrak{r}}\mathfrak{r} \sum_{\substack{\tau_{0;1},\tau_{1;1},\ldots,\tau_{n_1-2;1}\\ s_{n_1-1;1}=A_{n_1-1;1}-\varsigma}} \\
&\quad \times \sum_{j=0}^{A_{n_1-2;1}-r_1\varsigma-\theta} d_j \binom{\left\lfloor (A_{n_1-1;1}-\varsigma)\frac{A_{n_1-1;1}}{A_{n_1;1}}\right\rfloor - s_{n_1-2;1}}{j} \qquad (16)\\
&\quad \times \left(\prod_{w=0}^{n_1-2}\begin{bmatrix} A_{w+1;1}-r_1 s_{w;1} \\ \tau_{w;1}\end{bmatrix}\right) \\
&\quad \times \left(\frac{\prod_i x^{[b^t_{i,e}]_+}_{i;t}}{\prod_i x^{[-b^t_{i,e}]_+}_{i;t}}\right)^{\operatorname{sgn}(2b^t_{d,e}+1)\left(\left\lfloor (A_{n_1-1;1}-\varsigma)\frac{A_{n_1-1;1}}{A_{n_1;1}}\right\rfloor - s_{n_1-2;1}\right)},
\end{aligned}
$$

where $\mathfrak{r} \in \mathbb{ZP}_{\geq 0}[\{x_{d;t}\} \cup (\mathbf{x}^{\pm 1}_t \setminus \{x^{\pm 1}_{d;t}, x^{\pm 1}_{f;t}\})]$, $\lambda_{\mathfrak{r}} \in \{0,1\}$ and d_j are nonnegative integers depending on the summation indices, and the integers $\tau_{i,\ell}$ satisfy condition (10) with $n = n_1 + 1$ if $\ell = 1$ for some integer n_1.

Proof. The proof of Proposition 5.4 is similar to the proof of Proposition 5.2. □

Proposition 5.5. *With the notation in equation (13), we have*

(1) *B'_t is of the form $A_t + B_t$,*
(2) *C'_t is of the form $C_t + D_t$,*

with A_t, B_t, C_t, D_t as in the statement of Theorem 4.1.

Proof. This is proved in Section 5.3. □

This completes the proof of Theorem 4.1 modulo Propositions 5.1, 5.2 and 5.5.

5.1. *Proof of Proposition 5.1.* By induction, we can assume that u can be written as a Laurent polynomial in the clusters $t_{w_A}, t_{w_B}, t_{w_C}$, and t_{w_D} in such a way that the variables $x_{d;-}$ and $x_{e;-}$ appear only with nonnegative exponents.

Thus, in order to prove Proposition 5.1, we must compute the $\mathbf{x}_t$-expansions of cluster monomials $x^p_{d;t_{w_E}} x^q_{e;t_{w_E}}$ with $p, q \geq 0$.

Recall from (14) that $t^* = t_{w_Z}$ and our mutation sequence is of the following form:

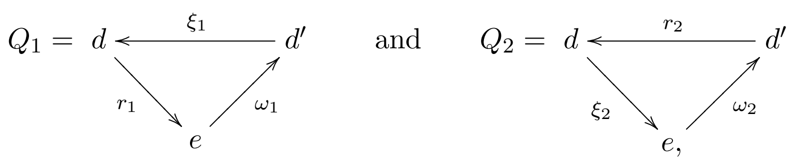
$$t_{w_E} \cdots\cdots \cdot \overset{d}{\text{———}} \cdot \overset{e}{\text{———}} t^{**} \overset{d}{\text{———}} t^* \overset{d'}{\text{———}} \cdot \cdots\cdots \cdot \overset{d'}{\text{———}} t' \overset{d}{\text{———}} t. \tag{17}$$

Let Q_0, Q_1 and Q_2, respectively, be the full subquivers on vertices d, d', e of the quiver at the seeds t_{w_E}, t^{**} and t', respectively. Note that the notation here is not the same as in Lemma 3.2. Denote by n_1 the number of seeds between t_{w_E} and t^* inclusively and by n_2 the number of seeds between t^{**} and t inclusively. Note that n_2 is even. We define $s_{i,;\ell}, A_{i;\ell}$ and $\tau_{i;\ell}$ with $\ell = 1, 2$ as in Section 3.3, but replacing n with $n_\ell - 2$.

We shall often use Lemma 3.2 to compute the relations between the number of arrows in the quivers Q_0, Q_1 and Q_2. The integer n in the statement of Lemma 3.2 denotes the number of mutations between two quivers; thus we have

$$n = \begin{cases} n_1 - 2 & \text{between } Q_0 \text{ and } Q_1, \\ n_2 - 2 & \text{between } Q_1 \text{ and } Q_2. \end{cases}$$

Let



where r_1 (respectively ω_1 and ξ_1) is the number of arrows from d to e (respectively from e to d' and from d' to d) in Q_1, and r_2 (respectively ω_2 and ξ_2) is the number of arrows from d' to d (respectively from e to d', and from d to e) in Q_2. Without loss of generality, we assume that $r_1 \geq 0$ and $r_2 \geq 0$. Thus we also have $\xi_1 \geq 0$, since $r_2 = \xi_1$. Note however, that ξ_2, ω_1 and ω_2 may be negative. For the rest of this proof, we set $x_{f;t_{w_E}} = 1$ for all $f \neq d, d', e$.

Recall that the sequences of quivers of *almost cyclic type* and of *acyclic type* were defined in Definition 3.6. We need to consider four cases:

(a) Both the sequence of quivers from t_{w_E} to $t_{w_Z} = t^*$ and the sequence of quivers from $\mu_d(t_{w_Z}) = t^{**}$ to t are of almost cyclic type.
(b) The sequence of quivers from t_{w_E} to $t_{w_Z} = t^*$ is of almost cyclic type, and the sequence of quivers from $\mu_d(t_{w_Z}) = t^{**}$ to t is of acyclic type.
(c) The sequence of quivers from t_{w_E} to $t_{w_Z} = t^*$ is of acyclic type, and the sequence of quivers from $\mu_d(t_{w_Z}) = t^{**}$ to t is of almost cyclic type.
(d) Both the sequence of quivers from t_{w_E} to $t_{w_Z} = t^*$ and the sequence of quivers from $\mu_d(t_{w_Z}) = t^{**}$ to t are of acyclic type.

Let us suppose first that we are in case (a) or (b); that is, the sequence of quivers from t_{w_B} to $t_{w_Z} = t^*$ is of almost cyclic type. Using Theorem 3.22, we see that $x^p_{d;t_{\omega_E}} x^q_{e;t_{\omega_E}}$ is equal to

(18)
$$\sum_{\substack{\tau_{0;1},\cdots,\tau_{n_1-3;1}\\ A_{n_1-1;1}-r_1 s_{n_1-2;1}\geq 0}} \left(\prod_{w=0}^{n_1-3}\left(\begin{array}{c} A_{w+1;1}-r_1 s_{w;1}\\ \tau_{w;1}\end{array}\right)\right)$$
$$\times \quad \widetilde{x_{d;t_{\omega_Z}}}^{A_{n_1-1;1}-r_1 s_{n_1-2;1}} x_{e;t_{\omega_Z}}^{r_1 s_{n_1-3;1}-A_{n_1-2;1}} x_{d';t_{\omega_Z}}^{\omega_1 s_{n_1-2;1}-\xi_1 s_{n_1-3;1}}$$

(19)
$$+\ x_{d;t_{\omega_Z}}^{-A_{n_1-1;1}} x_{e;t_{\omega_Z}}^{-A_{n_1-2;1}} \sum_{\substack{(S_1,S_2)\\ -A_{n_1-1;1}+r_1|S_2|>0}} x_{d;t_{\omega_Z}}^{r_1|S_2|} x_{e;t_{\omega_Z}}^{r_1|S_1|} x_{d';t_{\omega_Z}}^{\xi_1(A_{n_1-1;1}-|S_1|)-(\xi_1 r_1-\omega_1)|S_2|-M_{d'}}.$$

From now on, we restrict ourselves to the most difficult case where the full rank 3 subquivers with vertices d, d', e at both seeds t_{w_E} and t are nonacyclic; in other words, $\omega_1 c^{[r_1]}_{n_1} - \xi_1 c^{[r_1]}_{n_1-1} > 0$ and $c^{[\xi_1]}_{n_2} r_1 - c^{[\xi_1]}_{n_2-1}\omega_1 > 0$. In particular, $\xi_2 \geq 0$. But the same argument can be adapted to the case where the full rank 3 subquiver at either t_{w_E} or t is acyclic. We focus on the former case, because the positivity conjecture for acyclic cluster algebras was already proved in [17]. Thus assume that $(\omega_1 c^{[r_1]}_{n_1} - \xi_1 c^{[r_1]}_{n_1-1}) > 0$. In this case, $M_{d'} = 0$.

Let $p_2 = A_{n_1-1;1} - r_1 s_{n_1-2;1}$ and $q_2 = \omega_1 s_{n_1-2;1} - \xi_1 s_{n_1-3;1}$ be the exponents of $\widetilde{x_{d;t_{\omega_Z}}}$ and $x_{d';t_{\omega_Z}}$ in (18), respectively. Define $A_{i;2} = p_2 c^{[r_2]}_{i+1} + q_2 c^{[r_2]}_i$.

In case (a), applying Theorem 3.22 to $\widetilde{x_{d;t_{\omega_Z}}}^{p_2} x^{q_2}_{d';t_{\omega_Z}}$ in (18), we have that the part of (18) that contributes to C'_t is equal to

(20)
$$\sum_{\substack{\tau_{0;1},\cdots,\tau_{n_1-3;1}\\ A_{n_1-1;1}-r_1 s_{n_1-2;1}\geq 0}} \left(\prod_{w=0}^{n_1-3}\left[\begin{array}{c} A_{w+1;1}-r_1 s_{w;1}\\ \tau_{w;1}\end{array}\right]\right) x_{e;t_{\omega_Z}}^{r_1 s_{n_1-3;1}-A_{n_1-2;1}}$$
$$\times \sum_{\substack{(S_1,S_2)\\ r_2|S_2|-A_{n_2-1;2}>0}} x_{d;t}^{r_2|S_2|-A_{n_2-1;2}} x_{d';t}^{r_2|S_1|-A_{n_2-2;2}} x_{e;t}^{\xi_2(A_{n_2-1;2}-|S_1|)-(\xi_2 r_2-\omega_2)|S_2|}$$
$$= \sum_{\substack{\tau_{0;1},\cdots,\tau_{n_1-3;1}\\ A_{n_1-1;1}-r_1 s_{n_1-2;1}\geq 0}} \left(\prod_{w=0}^{n_1-3}\left[\begin{array}{c} A_{w+1;1}-r_1 s_{w;1}\\ \tau_{w;1}\end{array}\right]\right)$$
$$\times \sum_{\substack{(S'_1,S'_2)\\ r_2|S'_2|-A_{n_2-1;2}>0}} x_{d;t}^{r_2|S'_2|-A_{n_2-1;2}} x_{d';t}^{r_2|S'_1|-A_{n_2-2;2}} x_{e;t}^{\xi_2(A_{n_2-1;2}-|S'_1|)-(\xi_2 r_2-\omega_2)|S'_2|+r_1 s_{n_1-3;1}-A_{n_1-2;1}},$$

where (S_1', S_2') is a family of compatible pairs satisfying the condition in Theorem 3.22.

Since $r_2|S_2'| - A_{n_2-1;2} > 0$ in this last expression, then $A_{n_2-1;2}/r_2|S_2'| < 1$ and thus

(21)
$$\frac{r_1 A_{n_2-1;2}}{c^{[r_2]}_{n_2-1}\, r_2} = \frac{r_1|S_2'|}{c^{[r_2]}_{n_2-1}} \frac{A_{n_2-1;2}}{r_2|S_2'|} < \frac{r_1|S_2'|}{c^{[r_2]}_{n_2-1}} \le \xi_2(A_{n_2-1;2} - |S_1'|) - (\xi_2 r_2 - \omega_2)|S_2'|,$$

where the last inequality follows from Lemma 5.7 below. Using $r_2 = \xi_1$ and the definition of $A_{n_2-1;2}$, we get

$$\begin{aligned}
& r_1 s_{n_1-3;1} - A_{n_1-2;1} + \xi_2(A_{n_2-1;2} - |S_1'|) - (\xi_2 r_2 - \omega_2)|S_2'| \\
& > r_1 s_{n_1-3;1} - A_{n_1-2;1} \\
& + \frac{r_1\left(c^{[\xi_1]}_{n_2}(A_{n_1-1;1} - r_1 s_{n_1-2;1}) + c^{[\xi_1]}_{n_2-1}(\omega_1 s_{n_1-2;1} - \xi_1 s_{n_1-3;1})\right)}{c^{[\xi_1]}_{n_2-1}\xi_1} \\
& = -A_{n_1-2;1} + \frac{r_1\left(c^{[\xi_1]}_{n_2}(A_{n_1-1;1} - r_1 s_{n_1-2;1}) + c^{[\xi_1]}_{n_2-1}\omega_1 s_{n_1-2;1}\right)}{c^{[\xi_1]}_{n_2-1}\xi_1} \\
& = -A_{n_1-2;1} + \frac{r_1\left(c^{[\xi_1]}_{n_2} A_{n_1-1;1} - (c^{[\xi_1]}_{n_2} r_1 - c^{[\xi_1]}_{n_2-1}\omega_1) s_{n_1-2;1}\right)}{c^{[\xi_1]}_{n_2-1}\xi_1} \\
& \underset{(s_{n_1-2;1} \le A_{n_1-1;1}/r_1)}{\ge} -A_{n_1-2;1} + \frac{r_1\left(c^{[\xi_1]}_{n_2} A_{n_1-1;1} - (c^{[\xi_1]}_{n_2} r_1 - c^{[\xi_1]}_{n_2-1}\omega_1)\frac{A_{n_1-1;1}}{r_1}\right)}{c^{[\xi_1]}_{n_2-1}\xi_1}
\end{aligned}$$

(22)
$$\begin{aligned}
& = \frac{1}{\xi_1}(\omega_1 A_{n_1-1;1} - \xi_1 A_{n_1-2;1}) \\
& = \frac{1}{\xi_1}(\omega_1(p c^{[r_1]}_{n_1} + q c^{[r_1]}_{n_1-1}) - \xi_1(p c^{[r_1]}_{n_1-1} + q c^{[r_1]}_{n_1-2})) \\
& = \frac{1}{\xi_1}(p(\omega_1 c^{[r_1]}_{n_1} - \xi_1 c^{[r_1]}_{n_1-1}) + q(\omega_1 c^{[r_1]}_{n_1-1} - \xi_1 c^{[r_1]}_{n_1-2})) \\
& > 0,
\end{aligned}$$

where the last inequality holds, because from the first case of Lemma 3.2, we see that

$$\begin{aligned}
& \{\omega_1 c^{[r_1]}_{n_1} - \xi_1 c^{[r_1]}_{n_1-1}, \omega_1 c^{[r_1]}_{n_1-1} - \xi_1 c^{[r_1]}_{n_1-2}\} \\
& \quad = \{(\text{the number of arrows between } d' \text{ and } d \text{ in the seed } t_{w_E}), \\
& \qquad (\text{the number of arrows between } d' \text{ and } e \text{ in the seed } t_{w_E})\}.
\end{aligned}$$

Thus in case (a), the exponent of $x_{e;t}$ in the expansion of (20) is positive. Thus the terms with negative exponents on $x_{e;t}$ in (18) are all in B_t'. The proof that the terms with negative exponents on $x_{e;t}$ in (19) are also all in B'_t uses a similar argument. For the sake of completeness, we include the details.

Let $p_3 = r_1|S_2| - A_{n_1-1;1}$ and $q_3 = \xi_1(A_{n_1-1;1} - |S_1|) - (\xi_1 r_1 - \omega_1)|S_2|$, and let $A_{i;3} = p_3 c_{i+1}^{[r_2]} + q_3 c_i^{[r_2]}$. Applying Theorem 3.22 to $x_{d;t_{\omega_Z}}^{p_3} x_{d';t_{\omega_Z}}^{q_3}$ in (19), the exponents of $x_{e;t}$ in the part of (19) that contributes to C'_t are of the form

$$\xi_2(A_{n_2-2;3} - |S'_1|) - (\xi_2 r_2 - \omega_2)|S'_2| + r_1|S_1| - A_{n_1-2;1}. \tag{23}$$

We will derive the same conclusion as above, namely

$$(23) > \frac{1}{\xi_1}(\omega_1 A_{n_1-1;1} - \xi_1 A_{n_1-2;1}).$$

Using $r_2 = \xi_1$, the definition of $A_{n_2-2;3}$ and inequality (21) with $n_2 - 1$ replaced by $n_2 - 2$, we get

$$\begin{aligned}
(23) &> r_1|S_1| - A_{n_1-2;1} \\
&\quad + \frac{r_1}{\xi_1 c_{n_2-2}^{[\xi_1]}}\Big(c_{n_2-1}^{[\xi_1]}(r_1|S_2| - A_{n_1-1;1}) \\
&\qquad\qquad + c_{n_2-2}^{[\xi_1]}(\xi_1(A_{n_1-1;1} - |S_1|) - (\xi_1 r_1 - \omega_1)|S_2|)\Big) \\
&= -A_{n_1-2;1} + \frac{r_1}{\xi_1 c_{n_2-2}^{[\xi_1]}}\Big(c_{n_2-1}^{[\xi_1]}(r_1|S_2| - A_{n_1-1;1}) \\
&\qquad\qquad + c_{n_2-2}^{[\xi_1]}(\xi_1 A_{n_1-1;1} - (\xi_1 r_1 - \omega_1)|S_2|)\Big) \\
&= -A_{n_1-2;1} + \frac{r_1}{\xi_1 c_{n_2-2}^{[\xi_1]}}\Big(c_{n_2-1}^{[\xi_1]} r_1|S_2| + c_{n_2-3}^{[\xi_1]} A_{n_1-1;1} - c_{n_2-2}^{[\xi_1]}(\xi_1 r_1 - \omega_1)|S_2|\Big) \\
&= \frac{1}{\xi_1 c_{n_2-2}^{[\xi_1]}}\Big(r_1 c_{n_2-3}^{[\xi_1]} A_{n_1-1;1} - \xi_1 c_{n_2-2}^{[\xi_1]} A_{n_1-2;1} \\
&\qquad\qquad + r_1|S_2|(r_1 c_{n_2-1}^{[\xi_1]} - (\xi_1 r_1 - \omega_1)c_{n_2-2}^{[\xi_1]})\Big) \\
&\underset{(r_1|S_2| - A_{n_1-1;1} > 0)}{>} \frac{1}{\xi_1 c_{n_2-2}^{[\xi_1]}}\Big(r_1 c_{n_2-3}^{[\xi_1]} A_{n_1-1;1} - \xi_1 c_{n_2-2}^{[\xi_1]} A_{n_1-2;1} \\
&\qquad\qquad + A_{n_1-1;1}(r_1 c_{n_2-1}^{[\xi_1]} - (\xi_1 r_1 - \omega_1)c_{n_2-2}^{[\xi_1]})\Big) \\
&= \frac{1}{\xi_1 c_{n_2-2}^{[\xi_1]}}\Big(r_1 c_{n_2-3}^{[\xi_1]} A_{n_1-1;1} - \xi_1 c_{n_2-2}^{[\xi_1]} A_{n_1-2;1} + A_{n_1-1;1}(\omega_1 c_{n_2-2}^{[\xi_1]} - r_1 c_{n_2-3}^{[\xi_1]})\Big) \\
&= \frac{1}{\xi_1 c_{n_2-2}^{[\xi_1]}}\Big(A_{n_1-1;1}\omega_1 c_{n_2-2}^{[\xi_1]} - \xi_1 c_{n_2-2}^{[\xi_1]} A_{n_1-2;1}\Big) \\
&= \frac{1}{\xi_1}(\omega_1 A_{n_1-1;1} - \xi_1 A_{n_1-2;1}).
\end{aligned}$$

In case (b), applying Theorem 3.22 to $\widetilde{x_{d;t_{\omega_Z}}}^{p_2} x_{d';t_{\omega_Z}}^{q_2}$ in (18), we see that the part of (18) that contributes to B'_t is equal to

(24)

$$\sum_{\substack{\tau_{0;1},\tau_{1;1},\dots,\tau_{n_1-3;1}\\ A_{n_1-1;1}-r_1s_{n_1-2;1}\geq 0}} \left(\prod_{w=0}^{n_1-3}\left[\begin{array}{c} A_{w+1;1}-r_1s_{w;1} \\ \tau_{w;1}\end{array}\right]\right) x_{e;t}^{r_1s_{n_1-3;1}-A_{n_1-2;1}}$$

$$\times \sum_{\substack{\tau_{0;2},\tau_{1;2},\dots,\tau_{n_2-3;2}\\ A_{n_2-1;2}-r_2s_{n_2-2;2}>0}} \left(\prod_{w=0}^{n_2-3}\left[\begin{array}{c} A_{w+1;2}-r_2s_{w;2} \\ \tau_{w;2}\end{array}\right]\right)$$

$$\times \widetilde{x_{d;t}}^{A_{n_2-1;2}-r_2s_{n_2-2;2}} x_{d';t}^{r_2s_{n_2-3;2}-A_{n_2-2;2}} x_{e;t}^{\omega_2 s_{n_2-2;2}-\xi_2 s_{n_2-3;2}-M_e}$$

$$= \sum_{\substack{\tau_{0;1},\tau_{1;1},\dots,\tau_{n_1-3;1}\\ A_{n_1-1;1}-r_1s_{n_1-2;1}\geq 0}} \left(\prod_{w=0}^{n_1-3}\left[\begin{array}{c} A_{w+1;1}-r_1s_{w;1} \\ \tau_{w;1}\end{array}\right]\right)$$

$$\times \sum_{\substack{\tau_{0;2},\tau_{1;2},\dots,\tau_{n_2-3;2}\\ A_{n_2-1;2}-r_2s_{n_2-2;2}>0}} \left(\prod_{w=0}^{n_2-3}\left[\begin{array}{c} A_{w+1;2}-r_2s_{w;2} \\ \tau_{w;2}\end{array}\right]\right)$$

$$\times \widetilde{x_{d;t}}^{A_{n_2-1;2}-r_2s_{n_2-2;2}} x_{d';t}^{r_2s_{n_2-3;2}-A_{n_2-2;2}} x_{e;t}^{\omega_2 s_{n_2-2;2}-\xi_2 s_{n_2-3;2}+r_1s_{n_1-3;1}-A_{n_1-2;1}-M_e},$$

where $M_e=\omega_2A_{n_2-2;2}-\xi_2A_{n_2-3;2}$, and $s_{i;2}$ is as defined before Lemma 3.19 but in terms of p_2, q_2, and r_2, thus $s_{i;2}=\sum_{j=0}^{i-1} c_{i-j+1}^{[r_2]}\tau_{j;2}$.

In this last expression, we have

$$\frac{A_{n_2-1;2}}{r_2} > s_{n_2-2;2}. \tag{25}$$

We want to show that the exponent of $x_{e;t}$ is positive; that is,

$$\omega_2 s_{n_2-2;2}-\xi_2 s_{n_2-3;2}+r_1s_{n_1-3;1}-A_{n_1-2;1}-(\omega_2A_{n_2-2;2}-\xi_2A_{n_2-3;2})>0. \tag{26}$$

Thanks to Lemma 5.6 below, we have

$$s_{n_2-2;2}A_{n_2-2;2} > s_{n_2-3;2}A_{n_2-1;2},$$

and thus it suffices to show that

(27)

$$\left(\omega_2-\xi_2\frac{A_{n_2-2;2}}{A_{n_2-1;2}}\right) s_{n_2-2;2}+r_1s_{n_1-3;1}-A_{n_1-2;1}-(\omega_2A_{n_2-2;2}-\xi_2A_{n_2-3;2})>0$$

$$\iff r_1s_{n_1-3;1}-A_{n_1-2;1}-(\omega_2A_{n_2-2;2}-\xi_2A_{n_2-3;2})$$

$$> \left(\xi_2\frac{A_{n_2-2;2}}{A_{n_2-1;2}}-\omega_2\right) s_{n_2-2;2},$$

and by (25) it suffices to show

(28)
$$
\begin{aligned}
& r_1 s_{n_1-3;1} - A_{n_1-2;1} - (\omega_2 A_{n_2-2;2} - \xi_2 A_{n_2-3;2}) > \left(\xi_2 \frac{A_{n_2-2;2}}{A_{n_2-1;2}} - \omega_2\right) \frac{A_{n_2-1;2}}{r_2} \\
\iff \quad & (r_1 s_{n_1-3;1} - A_{n_1-2;1} + (\xi_2 A_{n_2-3;2} - \omega_2 A_{n_2-2;2}))\, r_2 \\
& > \xi_2 A_{n_2-2;2} - \omega_2 A_{n_2-1;2} \\
\overset{\text{Lemma 3.14}}{\iff} \quad & \xi_2 A_{n_2-4;2} - \omega_2 A_{n_2-3;2} > r_2 \left(A_{n_1-2;1} - r_1 s_{n_1-3;1}\right).
\end{aligned}
$$

Since

$$A_{i;2} = p_2 c_{i+1}^{[r_2]} + q_2 c_i^{[r_2]} = (A_{n_1-1;1} - r_1 s_{n_1-2;1}) c_{i+1}^{[r_2]} + (\omega_1 s_{n_1-2;1} - \xi_1 s_{n_1-3;1}) c_i^{[r_2]},$$

inequality (28) is equivalent to

(29)
$$
\begin{aligned}
& (A_{n_1-1;1} - r_1 s_{n_1-2;1})(\xi_2 c_{n_2-3}^{[r_2]} - \omega_2 c_{n_2-2}^{[r_2]}) \\
& \quad + (\omega_1 s_{n_1-2;1} - \xi_1 s_{n_1-3;1})(\xi_2 c_{n_2-4}^{[r_2]} - \omega_2 c_{n_2-3}^{[r_2]}) \\
& \quad > r_2 \left(A_{n_1-2;1} - r_1 s_{n_1-3;1}\right) \\
\overset{\text{Lemma 3.2}}{\iff} \quad & (A_{n_1-1;1} - r_1 s_{n_1-2;1})\omega_1 + (\omega_1 s_{n_1-2;1} - \xi_1 s_{n_1-3;1}) r_1 \\
& \quad > \xi_1 \left(A_{n_1-2;1} - r_1 s_{n_1-3;1}\right) \\
\iff \quad & A_{n_1-1;1}\omega_1 > \xi_1 A_{n_1-2;1}.
\end{aligned}
$$

Note that cases (a) and (b) agree on the first mutation sequence, and so equation (22) is valid in both cases, and it implies $A_{n_1-1;1}\omega_1 > \xi_1 A_{n_1-2;1}$. The proof that the exponents of $x_{e;t}$ in the part of (19) that contributes to B'_t are nonnegative uses a similar argument. We give an outline as follows.

Applying Theorem 3.22 to $x_{d;t_{\omega_Z}}^{p_3} x_{d';t_{\omega_Z}}^{q_3}$ in (19), the exponents of $x_{e;t}$ in the part of (19) that contributes to B'_t are of the form

(30) $$\omega_2 s_{n_2-3;2} - \xi_2 s_{n_2-4;2} + r_1|S_1| - A_{n_1-2;1} - (\omega_2 A_{n_2-3;2} - \xi_2 A_{n_2-4;2}).$$

Using the same process as above, we get the following analogue of (28):

(31) $$\xi_2 A_{n_2-5;2} - \omega_2 A_{n_2-4;2} > r_2(A_{n_1-2;1} - r_1|S_1|),$$

which is obtained by replacing n_2 with $n_2 - 1$ and $s_{n_1-3;1}$ with $|S_1|$. Since $A_{i;2} = q_3 c_{i+1}^{[r_2]} + p_3 c_i^{[r_2]}$, inequality (31) is equivalent to

$$
\begin{aligned}
& p_3(\xi_2 c_{n_2-5}^{[r_2]} - \omega_2 c_{n_2-4}^{[r_2]}) + q_3(\xi_2 c_{n_2-4}^{[r_2]} - \omega_2 c_{n_2-3}^{[r_2]}) > r_2(A_{n_1-2;1} - r_1|S_1|) \\
\overset{\text{Lemma 3.2}}{\iff} \quad & (r_1|S_2| - A_{n_1-1;1})(\xi_1 r_1 - \omega_1) + (\xi_1(A_{n_1-1;1} - |S_1|) \\
& \quad - (\xi_1 r_1 - \omega_1)|S_2|) r_1 > \xi_1(A_{n_1-2;1} - r_1|S_1|) \\
\iff & \omega_1 A_{n_1-1;1} > \xi_1 A_{n_1-2;1},
\end{aligned}
$$

which is exactly the same as (29). Thus the exponents of $x_{e;t}$ in B'_t are nonnegative.

This completes the proof of Proposition 5.1 in cases (a) and (b), modulo the following two lemmas.

LEMMA 5.6. $s_{n_2-2;2}A_{n_2-2;2} > s_{n_2-3;2}A_{n_2-1;2}$.

Proof. It follows from Definition 3.20 that $s'_{n_2-2}c^{[r_2]}_{n_2-2} = s'_{n_2-3}c^{[r_2]}_{n_2-1}$, and using $(c^{[r_2]}_{n_2-1})^2 > c^{[r_2]}_{n_2}c^{[r_2]}_{n_2-2}$ from Lemma 3.1, this implies

$$s'_{n_2-2}c^{[r_2]}_{n_2-1} > s'_{n_2-3}c^{[r_2]}_{n_2}. \tag{32}$$

On the other hand, the second line of (10) in Theorem 3.21 together with $A_{n_2-2}A_{n_2-2} > A_{n_2-1}A_{n_2-3}$ from Lemma 3.14 implies

$$(s_{n_2-2} - s'_{n_2-2})A_{n_2-2} > (s_{n_2-3} - s'_{n_2-3})A_{n_2-1}$$

and, using the definition of A_{n_2-i}, we get

$$(s_{n_2-2} - s'_{n_2-2})(p_2c^{[r_2]}_{n_2-1} + q_2c^{[r_2]}_{n_2-2}) > (s_{n_2-3} - s'_{n_2-3})(p_2c^{[r_2]}_{n_2} + q_2c^{[r_2]}_{n_2-1}).$$

Now (32) implies the statement. □

LEMMA 5.7. *Let* $(S_1 = \cup_{i=1}^{p+q} S_1^i, S_2 = \cup_{i=1}^{p+q} S_2^i)$ *such that*

$$(S_1^i, S_2^i) \text{ is a compatible pair in } \begin{cases} \mathcal{D}^{c^{[r_2]}_{n_2} \times c^{[r_2]}_{n_2-1}} & \text{if } 1 \le i \le p_2, \\ \mathcal{D}^{c^{[r_2]}_{n_2-1} \times c^{[r_2]}_{n_2-2}} & \text{if } p_2+1 \le i \le p_2+q_2, \end{cases}$$

where p_2, q_2 *are arbitrary nonnegative integers. Then*

$$\frac{r_1|S_2|}{c^{[r_2]}_{n_2-1}} \le \xi_2(A_{n_2-1;2} - |S_1|) - (\xi_2 r_2 - \omega_2)|S_2|.$$

Proof. This is proved in [20, Lemma 4.10] using colored subpaths with $|\beta|_2 = A_{n_2-1;2} - |S_1|$ and $|\beta|_1 = |S_2|$. This is also proved in [19, Prop. 4.1, Case 6] with $a_2 = A_{n_2-1;2}, b = c = r_2$. □

Now suppose we are in the case (c). Using Theorem 3.22, we see that $x^p_{d;t_{\omega_E}} x^q_{e;t_{\omega_E}}$ is equal to the following expression, which is almost the same as the expression in (18) and (19) up to the exponent $M'_{d'}$:

(33)

$$\sum_{\substack{\tau_{0;1},\cdots,\tau_{n_1-3;1} \\ A_{n_1-1;1}-r_1s_{n_1-2;1}\ge 0}} \left(\prod_{w=0}^{n_1-3} \left[\begin{array}{c} A_{w+1;1} - r_1s_{w;1} \\ \tau_{w;1} \end{array} \right] \right)$$
$$\times \quad \widetilde{x_{d;t_{\omega_Z}}}^{A_{n_1-1;1}-r_1s_{n_1-2;1}} x_{e;t_{\omega_Z}}^{r_1s_{n_1-3;1}-A_{n_1-2;1}} x_{d';t_{\omega_Z}}^{\omega_1s_{n_1-2;1}-\xi_1s_{n_1-3;1}-M'_{d'}}$$

$$+\ x_{d;t_{\omega_Z}}^{-A_{n_1-1;1}} x_{e;t_{\omega_Z}}^{-A_{n_1-2;1}} \sum_{\substack{(S_1,S_2)\\ -A_{n_1-1;1}+r_1|S_2|>0}} x_{d;t_{\omega_Z}}^{r_1|S_2|} x_{e;t_{\omega_Z}}^{r_1|S_1|} x_{d';t_{\omega_Z}}^{\xi_1(A_{n_1-1;1}-|S_1|)-(\xi_1 r_1-\omega_1)|S_2|-M_{d'}}. \tag{34}$$

Let $p_2 = r_1|S_2| - A_{n_1-1;1}$ and $q_2 = \xi_1(A_{n_1-1;1} - |S_1|) - (\xi_1 r_1 - \omega_1)|S_2| - M_{d'}$ be the exponents of $x_{d;t_{\omega_Z}}$ and $x_{d';t_{\omega_Z}}$ in (34), respectively. Applying Theorem 3.22 to $x_{d;t_{\omega_Z}}^{p_2} x_{d';t_{\omega_Z}}^{q_2}$, we see that the part of (34) that contributes to C'_t is equal to

(35)
$$\begin{aligned}
&\sum_{\substack{(S_1,S_2)\\ -A_{n_1-1;1}+r_1|S_2|>0}} x_{e;t_{\omega_Z}}^{r_1|S_1|-A_{n_1-2;1}} \\
&\quad\times \sum_{\substack{(S'_1,S'_2)\\ A_{n_2-2;2}<r_2|S'_2|}} x_{d;t}^{r_2|S'_2|-A_{n_2-2;2}} x_{d';t}^{r_2|S'_1|-A_{n_2-3;2}} x_{e;t}^{\xi_2(A_{n_2-2;2}-|S'_1|)-(\xi_2 r_2-\omega_2)|S'_2|} \\
&= \sum_{\substack{(S_1,S_2)\\ -A_{n_1-1;1}+r_1|S_2|>0}} \sum_{\substack{(S'_1,S'_2)\\ A_{n_2-2;2}<r_2|S'_2|}} x_{d;t}^{r_2|S'_2|-A_{n_2-2;2}} x_{d';t}^{r_2|S'_1|-A_{n_2-3;2}} \\
&\quad\times x_{e;t}^{\xi_2(A_{n_2-2;2}-|S'_1|)-(\xi_2 r_2-\omega_2)|S'_2|+r_1|S_1|-A_{n_1-2;1}}.
\end{aligned}$$

Since $A_{n_2-2;2} < r_2|S'_2|$ in this expression, we have $A_{n_2-2;2}/r_2|S'_2| < 1$ and thus

(36)
$$\frac{r_1 A_{n_2-2;2}}{c_{n_2-2}^{[r_2]} r_2} = \frac{r_1|S'_2|}{c_{n_2-2}^{[r_2]}} \frac{A_{n_2-2;2}}{r_2|S'_2|} < \frac{r_1|S'_2|}{c_{n_2-2}^{[r_2]}} \le \xi_2(A_{n_2-2;2} - |S'_1|) - (\xi_2 r_2 - \omega_2)|S'_2|,$$

where the last inequality is proved in Lemma 5.7. On the other hand,

$$\begin{aligned}
A_{n_2-2;2} &= p_2 c_{n_2-1}^{[r_2]} + q_2 c_{n_2-2}^{[r_2]} \\
&= (r_1|S_2| - A_{n_1-1;1}) c_{n_2-1}^{[r_2]} \\
&\quad + \xi_1(A_{n_1-1;1} - |S_1|) - (\xi_1 r_1 - \omega_1)|S_2| - M_{d'}) c_{n_2-2}^{[r_2]}.
\end{aligned} \tag{37}$$

The exponent of $x_{e;t}$ is

$$\begin{aligned}
&\xi_2(A_{n_2-2;2} - |S'_1|) - (\xi_2 r_2 - \omega_2)|S'_2| + r_1|S_1| - A_{n_1-2;1} \\
&\overset{(36)}{>} \frac{r_1 A_{n_2-2;2}}{c_{n_2-2}^{[r_2]} r_2} + r_1|S_1| - A_{n_1-2;1} \\
&\overset{(37)}{=} \frac{r_1}{c_{n_2-2}^{[r_2]} r_2} \Big((r_1|S_2| - A_{n_1-1;1}) c_{n_2-1}^{[r_2]} \\
&\qquad + (\xi_1(A_{n_1-1;1} - |S_1|) - (\xi_1 r_1 - \omega_1)|S_2| - M_{d'}) c_{n_2-2}^{[r_2]} \Big) \\
&\quad + r_1|S_1| - A_{n_1-2;1}
\end{aligned}$$

$$\overset{\text{Lemma 3.14}}{=} \frac{r_1}{c^{[r_2]}_{n_2-2}\, r_2} \Big((r_1|S_2| - A_{n_1-1;1}) c^{[r_2]}_{n_2-1} - ((\xi_1 r_1 - \omega_1)|S_2| - M_{d'}) c^{[r_2]}_{n_2-2} \Big) + A_{n_1;1}$$

$$\overset{\text{Lemma 3.1}}{=} \frac{r_1}{c^{[r_2]}_{n_2-2}\, r_2} \Big((\omega_1 c^{[r_2]}_{n_2-2} - r_1 c^{[r_2]}_{n_2-3})|S_2| - A_{n_1-1;1} c^{[r_2]}_{n_2-1} - M_{d'} c^{[r_2]}_{n_2-2} \Big) + A_{n_1;1}$$

$$\overset{(*)}{\geq} \frac{r_1}{c^{[r_2]}_{n_2-2}\, r_2} \Big((\omega_1 c^{[r_2]}_{n_2-2} - r_1 c^{[r_2]}_{n_2-3}) \frac{A_{n_1-1;1}}{r_1} - A_{n_1-1;1} c^{[r_2]}_{n_2-1} - M_{d'} c^{[r_2]}_{n_2-2} \Big) + A_{n_1;1}$$

$$= \frac{r_1}{c^{[r_2]}_{n_2-2}\, r_2} \Big((\omega_1 c^{[r_2]}_{n_2-2} - r_1 c^{[r_2]}_{n_2-3}) \frac{A_{n_1-1;1}}{r_1} - A_{n_1-1;1} c^{[r_2]}_{n_2-1} - (\xi_1 A_{n_1-1;1} - (\xi_1 r_1 - \omega_1) A_{n_1-2;1}) c^{[r_2]}_{n_2-2} \Big) + A_{n_1;1},$$

where inequality $(*)$ holds because $p_2 = r_1|S_2| - A_{n_1-1;1} \geq 0$. We shall show below that the quiver Q_2 is cyclic, and therefore ξ_2 and r_2 have the same sign, so $r_2 > 0$, since $\xi_2 > 0$. This implies that the above expression has the same sign as

$$r_1 \Big((\omega_1 c^{[r_2]}_{n_2-2} - r_1 c^{[r_2]}_{n_2-3}) \frac{A_{n_1-1;1}}{r_1} - A_{n_1-1;1} c^{[r_2]}_{n_2-1} - (\xi_1 A_{n_1-1;1} - (\xi_1 r_1 - \omega_1) A_{n_1-2;1}) c^{[r_2]}_{n_2-2} \Big) + A_{n_1;1} c^{[r_2]}_{n_2-2}\, \xi_1$$

$$= (\omega_1 c^{[r_2]}_{n_2-2} - r_1 c^{[r_2]}_{n_2-3}) A_{n_1-1;1} + (r_1(\xi_1 r_1 - \omega_1) A_{n_1-2;1} - r_1 \xi_1 A_{n_1-1;1} + A_{n_1;1}\, \xi_1) c^{[r_2]}_{n_2-2} - r_1 A_{n_1-1;1} c^{[r_2]}_{n_2-1}$$

$$= (\omega_1 c^{[r_2]}_{n_2-2} - r_1 c^{[r_2]}_{n_2-3}) A_{n_1-1;1} + (\xi_1(r_1^2 - 1) - \omega_1\, r_1) A_{n_1-2;1} c^{[r_2]}_{n_2-2} - r_1 A_{n_1-1;1} c^{[r_2]}_{n_2-1}$$

$$= (\omega_1 c^{[r_2]}_{n_2-2} - r_1 \xi_1 c^{[r_2]}_{n_2-2}) A_{n_1-1;1} + (\xi_1(r_1^2 - 1) - \omega_1\, r_1) A_{n_1-2;1} c^{[r_2]}_{n_2-2},$$

and this expression has the same sign as

$$(\omega_1 - r_1\xi_1) A_{n_1-1;1} + (\xi_1(r_1^2 - 1) - \omega_1\, r_1) A_{n_1-2;1} = \xi_1 A_{n_1-4;1} - \omega_1 A_{n_1-3;1}. \tag{38}$$

This expression is positive because of Lemma 3.2.

Let us now show that the quiver Q_2 is cyclic in this case. Since we are in case (c), the mutation sequence from t^{**} to t is of almost cyclic type. Moreover, this sequence is of length at least 3 and the quiver Q_2 is the quiver at the seed t' one step before reaching the seed t in this sequence. We need to consider conditions (1) and (2) of Definition 3.6. If condition (2) holds, then all the quivers after the second mutation in this sequence are cyclic so, in particular, Q_2 is cyclic. Suppose now that condition (1) holds. In our situation this condition says $c_n^{[r_2]}\xi_2 - c_{n-1}^{[r_2]}\omega_2 > 0$ for $1 \le n \le m$, where m is the length of the mutation sequence. Using $n = 1$ and 2, this implies that $\omega_2 > 0$ and $\xi_2 > 0$, and thus Q_1 is cyclic. Because we are in case (c), the mutation sequence from t_{w_E} to t^* is of acyclic type, and it follows that there is an acyclic quiver in one of the seeds preceding Q_1 in that sequence. The facts that $r_1 > 1$ and that the last mutation to get to Q_1 is in direction e then imply that $\xi_1 > \omega_1$, because ξ_1 is the number of arrows opposite to the vertex e. But then the first mutation in the sequence from Q_1 to Q_2 is in direction d and afterwards the sequence alternates between directions d' and d, and therefore the number of arrows in the quivers of this sequence grows. In particular, all quivers in that sequence are cyclic.

The proof that the exponents of $x_{e;t}$ in the part of (33) that contributes to C'_t are nonnegative uses a similar argument, which is given below. Let $p_3 = A_{n_1-1;1} - r_1 s_{n_1-2;1}$ and $q_3 = \omega_1 s_{n_1-2;1} - \xi_1 s_{n_1-3;1} - M'_{d'}$, and let $A_{i;3} = p_3 c_{i+1}^{[r_2]} + q_3 c_i^{[r_2]}$. Applying Theorem 3.22 to $\widetilde{x_{d;t_{\omega_Z}}}^{p_3} x_{d';t_{\omega_Z}}^{q_3}$ in (33), the exponents of $x_{e;t}$ in the part of (33) that contributes to C'_t are of the form

$$\xi_2(A_{n_2-2;3} - |S'_1|) - (\xi_2 r_2 - \omega_2)|S'_2| + r_1 s_{n_1-3;1} - A_{n_1-2;1}. \tag{39}$$

Using (36) with $n_2 - 2$ replaced by $n_2 - 1$, we have

$$\begin{aligned}
(39) &> \frac{r_1 A_{n_2-1;3}}{c_{n_2-1}^{[r_2]} r_2} + r_1 s_{n_1-3;1} - A_{n_1-2;1} \\
&= \frac{r_1}{c_{n_2-1}^{[r_2]}\xi_1}\left(p_3 c_{n_2}^{[r_2]} + q_3 c_{n_2-1}^{[r_2]}\right) + r_1 s_{n_1-3;1} - A_{n_1-2;1} \\
&= \frac{r_1}{c_{n_2-1}^{[r_2]}\xi_1}\left(p_3 c_{n_2}^{[r_2]} + (\omega_1 s_{n_1-2;1} - \xi_1 s_{n_1-3;1} - M'_{d'}) c_{n_2-1}^{[r_2]}\right) \\
&\quad + r_1 s_{n_1-3;1} - A_{n_1-2;1} \\
&= \frac{r_1}{c_{n_2-1}^{[r_2]}\xi_1}\left(p_3 c_{n_2}^{[r_2]} + (\omega_1 s_{n_1-2;1} - M'_{d'}) c_{n_2-1}^{[r_2]}\right) - A_{n_1-2;1} \\
&= \frac{r_1}{c_{n_2-1}^{[r_2]}\xi_1}\left((A_{n_1-1;1} - r_1 s_{n_1-2;1}) c_{n_2}^{[r_2]} + (\omega_1 s_{n_1-2;1} - M'_{d'}) c_{n_2-1}^{[r_2]}\right) - A_{n_1-2;1} \\
&= \frac{r_1}{c_{n_2-1}^{[r_2]}\xi_1}\left(A_{n_1-1;1} c_{n_2}^{[r_2]} - s_{n_1-2;1}(r_1 c_{n_2}^{[r_2]} - \omega_1 c_{n_2-1}^{[r_2]}) - M'_{d'} c_{n_2-1}^{[r_2]}\right) - A_{n_1-2;1} \\
&\underset{(A_{n_1-1;1} - r_1 s_{n_1-2;1} \ge 0)}{>} \frac{r_1}{c_{n_2-1}^{[r_2]}\xi_1}\Big(A_{n_1-1;1} c_{n_2}^{[r_2]} - \frac{A_{n_1-1;1}}{r_1}(r_1 c_{n_2}^{[r_2]} - \omega_1 c_{n_2-1}^{[r_2]}) \\
&\qquad\qquad - M'_{d'} c_{n_2-1}^{[r_2]}\Big) - A_{n_1-2;1}
\end{aligned}$$

$$= \frac{r_1}{c^{[r_2]}_{n_2-1}\xi_1}\left(\frac{A_{n_1-1;1}}{r_1}\omega_1 c^{[r_2]}_{n_2-1} - M'_{d'}c^{[r_2]}_{n_2-1}\right) - A_{n_1-2;1}$$
$$= \frac{r_1}{c^{[r_2]}_{n_2-1}\xi_1}\left(\frac{A_{n_1-1;1}}{r_1}\omega_1 c^{[r_2]}_{n_2-1} - (\omega_1 A_{n_1-2;1} - \xi_1 A_{n_1-3;1})c^{[r_2]}_{n_2-1}\right) - A_{n_1-2;1}$$
$$= \frac{\omega_1 A_{n_1-1;1}}{\xi_1} - \frac{r_1}{\xi_1}(\omega_1 A_{n_1-2;1} - \xi_1 A_{n_1-3;1}) - A_{n_1-2;1}$$
$$= -\frac{\omega_1 A_{n_1-3;1}}{\xi_1} + A_{n_1-4;1},$$

which has the same sign as

$$\xi_1 A_{n_1-4;1} - \omega_1 A_{n_1-3;1},$$

which is equal to (38).

Thus the exponents of $x_{e;t}$ in C'_t are nonnegative, and this shows Proposition 5.1 in the case (c).

To complete the proof of the proposition, we analyze case (d). Suppose that the sequence from t_{w_B} to t^* is of acyclic type and consider the quiver Q_1 at the seed $t^{**} = \mu_d(t^*)$. Our first goal is to show that $\mu_e Q_1$ is acyclic. Suppose the contrary. Thus condition (7) in Definition 3.6 does not hold. Hence condition (6) in Definition 3.6 implies that the number of arrows r_1 from d to e in Q_1 is at least 2. In this case, the acyclic quivers in the sequence from t_{w_B} to t^* form a connected subsequence, and thus $\mu_e Q_1$ being cyclic implies that Q_1 and $\mu_d Q_1$ are cyclic too. The third case of Lemma 3.2 with $\omega = \xi_1$ and $r = \omega_1$ implies that

$$\xi_1 \geq \frac{c^{[r_1]}_n}{c^{[r_1]}_{n-1}}\omega_1 > \omega_1,$$

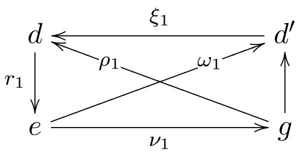
where the last inequality holds since $c^{[r_1]}_n > c^{[r_1]}_{n-1}$. Moreover, using the third case of Lemma 3.2 with $\omega = \omega_1$ and $r = r_1$ together with the fact that the mutation sequence from Q_1 to Q_2 is of acyclic type, we also see that $\omega_1 \geq r_1$. Thus $\xi_1 > \omega_1 \geq r_1 \geq 2$, and therefore Lemma 3.5 implies that all quivers in the sequence from t^* to t are cyclic, a contradiction to the assumption that we are in case (d).

Thus $\mu_e Q_1$ is acyclic. By the same reasoning, we get that Q_1 is acyclic, and then by symmetry, we also have that the quivers in the seeds t^* and $\mu_{d'}(t^*)$ are acyclic.

Thus we have shown that the case (d) occurs if and only if the four quivers in the consecutive seeds $\mu_e(t^{**})$, t^{**}, t^* and $\mu_{d'}(t^*)$ are acyclic. In this case, the sequence from t^* to t is of almost cyclic type and the result can be shown by the same argument as in the case (c). This completes the proof of Proposition 5.1.

5.2. *Proof of Proposition* 5.2. Proposition 5.2 applies in cases (a) and (c). We assume that we are in case (a). Case (c) is similar.

Let g be an arbitrary vertex different from d, d', e and let

$Q_1 =$ 

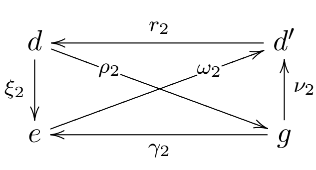
be the full subquiver with vertices $d, d', e'g$ of the quiver at the seed $\mu_d(t_{w_Z}) = t^{**}$, where r_1 (respectively ω_1, ξ_1, ν_1, ρ_1) is the number of arrows from d to e (respectively from e to d', from d' to d, from e to g, from g to d). Recall that $r_1 \geq 0$. Although the mutations in the sequence μ are in directions d, d', e only, we need to study how the variable $x_{g,t}$ behaves in the expansion formulas in order to show certain divisibility properties. On the other hand, it suffices to consider only one of the $x_{g;t}$ with $g \neq d, d', e$.

We present the case where the subquiver with vertices d, e, g in some seed between $\mu_d(t_{w_E})$ and $\mu_d(t_{w_Z})$ is acyclic. The case where all these subquivers are nonacyclic is easier; in fact, only the exponent of $x_{g,t_{w_Z}}$ would change. Thus for the rest of this proof, we set $x_{f;t} = 1$ for all $f \neq d, d', e'g$.

Since the mutation sequence relating the seeds t_{w_E} and $t_{w_Z} = t^*$ consists in mutations in directions d and e, applying Theorem 3.21 to $x^p_{d;t_{w_E}} x^q_{e;t_{w_E}}$ yields[1]

(40)
$$\sum_{\tau_{0;1},\tau_{1;1},\ldots,\tau_{n_1-3;1}} \left(\prod_{w=0}^{n_1-3} \begin{bmatrix} A_{w+1;1} - r_1 s_{w;1} \\ \tau_{w;1} \end{bmatrix} \right) \widetilde{x}_{d;t_{w_Z}}^{A_{n_1-1;1} - r_1 s_{n_1-2;1}} x_{e;t_{w_Z}}^{r_1 s_{n_1-3;1} - A_{n_1-2;1}}$$
$$\times\, x_{d';t_{w_Z}}^{\omega_1 s_{n_1-2;1} - \xi_1 s_{n_1-3;1}} x_{g;t_{w_Z}}^{\nu_1 s_{n_1-2;1} - \rho_1 s_{n_1-3;1} - (\nu_1 A_{n_1-2;1} - \rho_1 A_{n_1-3;1})}.$$

As before, let n_2 be the number of seeds between $\mu_d(t_{w_Z}) = t^{**}$ and t inclusive. Recall that n_2 is even. Let

$Q_2 =$ 

[1]The term $(\nu_1 A_{n_1-2;1} - \rho_1 A_{n_1-3;1})$ in the exponent of $x_{g;t_{w_Z}}$ is equal to the M_g in Theorem 3.21, and it is nonzero because of the assumption that the subquiver with vertices d, e, g in some seed between $\mu_d(t_{w_E})$ and $\mu_d(t_{w_Z})$ is acyclic. On the other hand, the term $M_{d'}$ in the exponent of $x_{d';t_{w_Z}}$ is zero, because we are in case (a).

be the quiver at the seed $\mu_d(t) = t'$, where r_2 (respectively ω_2, ξ_2, ν_2, ρ_2, γ_2) is the number of arrows from d' to d (respectively from e to d', from d to e, from g to d', from d to g, from g to e). Recall that $r_2 \geq 0$.

Next we want to use Lemma 3.2 to compute the number of arrows in Q_2 in terms of the number of arrows in Q_1. The mutation sequence relating the quivers Q_1 and Q_2 is a rank 2 sequence in directions d and d', starting with d and ending with d'. In particular, the number of mutations in this sequence (which is denoted n in Lemma 3.2) is even and we have $n = n_2 - 2$. First we apply Lemma 3.2 to the subquiver on vertices d, d', e. Since we are in case (a), the condition $c^{[r_2]}_{n_2-1} r_1 - c^{[r_2]}_{n_2-2}\omega_1 > 0$ holds. Therefore we see from Lemma 3.2 that

$$
\begin{aligned}
r_2 &= \xi_1, \\
\omega_2 &= c^{[r_2]}_{n_2-1} r_1 - c^{[r_2]}_{n_2-2}\omega_1, \\
\xi_2 &= c^{[r_2]}_{n_2} r_1 - c^{[r_2]}_{n_2-1}\omega_1.
\end{aligned}
\tag{41}
$$

Next, we use Lemma 3.2 on the subquiver on vertices d, d' and g. Since we assumed that the mutation sequence on this subquiver is of acyclic type and because of Remark 3.11, we use the third case of Lemma 3.2 with $\bar{\omega}(n) = -\rho_1$, $\xi = r_2$, $\omega = \nu_2$, and $r = \rho_2$ to obtain

$$\rho_1 = c^{[r_2]}_{n_2-3}\nu_2 - c^{[r_2]}_{n_2-4}\rho_2.$$

Finally, we show by induction on n_2 that

$$\gamma_2 = (c^{[r_2]}_{n_2-2}\rho_2 - c^{[r_2]}_{n_2-1}\nu_2) r_1 - (c^{[r_2]}_{n_2-3}\rho_2 - c^{[r_2]}_{n_2-2}\nu_2)\omega_1 - \nu_1. \tag{42}$$

If $n_2 = 4$, then the above formula becomes $\gamma_2 = (\rho_2 - r_2\nu_2)r_1 - (-\nu_2)\omega_1 - \nu_1$. This is easily checked, since Q_1 and Q_2 are related by sequence of two mutations. Suppose $n_2 > 4$. We want to show that if equation (42) computes the number of arrows from e to g in the quiver Q_2, then it also computes the number of arrows from e to g in the quiver $\mu_{d'}\mu_d Q_2$, if n_2 is replaced by $n_2 + 2$, and ρ_2 (respectively ν_2) is replaced by the number of arrows in $\mu_{d'}\mu_d Q_2$ from g to d (respectively d' to g).

Denoting by ρ_2' (respectively ν_2') the number of arrows from g to d (respectively d' to g) in $\mu_{d'}\mu_d Q_2$, we observe that $\rho_2' = (r_2(r_2\rho_2 - \nu_2) - \rho_2)$, $\nu_2' = (r_2\rho_2 - \nu_2)$, and the number of arrows from e to g is still γ_2.

Now using the definition $c^{[r_2]}_{n-2} = r_2 c^{[r_2]}_{n-1} - c^{[r_2]}_{n}$, we see that

$$
\begin{aligned}
c^{[r_2]}_{n_2-2}\rho_2 - c^{[r_2]}_{n_2-1}\nu_2 &= (r_2 c^{[r_2]}_{n-1} - c^{[r_2]}_{n})\rho_2 - (r_2 c^{[r_2]}_{n} - c^{[r_2]}_{n+1})\nu_2 \\
&= (r_2(r_2 c^{[r_2]}_{n} - c^{[r_2]}_{n+1}) - c^{[r_2]}_{n})\rho_2 - (r_2 c^{[r_2]}_{n} - c^{[r_2]}_{n+1})\nu_2 \\
&= c^{[r_2]}_{n}(r_2(r_2\rho_2 - \nu_2) - \rho_2) - c^{[r_2]}_{n+1}(r_2\rho_2 - \nu_2) \\
&= c^{[r_2]}_{n}\rho_2' - c^{[r_2]}_{n+1}\nu_2'.
\end{aligned}
$$

Similarly,

$$c^{[r_2]}_{n_2-3}\rho_2 - c^{[r_2]}_{n_2-2}\nu_2 = c^{[r_2]}_{n_2-1}\rho'_2 - c^{[r_2]}_{n_2}\nu'_2.$$

This proves formula (42).

Now let $p_2 = A_{n_1-1;1} - r_1 s_{n_1-2;1}$, $q_2 = \omega_1 s_{n_1-2;1} - \xi_1 s_{n_1-3;1}$ be the exponents of $\widetilde{x_{d;t_{w_Z}}}$ and $x_{d';t_{w_Z}}$ in (40) respectively. Applying Theorem 3.21 to $\widetilde{x_{d;t_{w_Z}}}^{p_2} x^{q_2}_{d';t_{w_Z}}$, we see that (40) is equal to

(43)

$$\sum_{\tau_{0;1},\tau_{1;1},\dots,\tau_{n_1-3;1}} \left(\prod_{w=0}^{n_1-3} \begin{bmatrix} A_{w+1;1} - r_1 s_{w;1} \\ \tau_{w;1} \end{bmatrix}\right) x_{e;t}^{r_1 s_{n_1-3;1} - A_{n_1-2;1}}$$
$$\times\, x_{g;t}^{\nu_1 s_{n_1-2;1} - \rho_1 s_{n_1-3;1} - (\nu_1 A_{n_1-2;1} - \rho_1 A_{n_1-3;1})}$$
$$\times \sum_{\tau_{0;2},\tau_{1;2},\dots,\tau_{n_2-3;2}} \left(\prod_{w=0}^{n_2-3} \begin{bmatrix} A_{w+1;2} - r_2 s_{w;2} \\ \tau_{w;2} \end{bmatrix}\right)$$
$$\times\, \widetilde{x_{d;t}}^{A_{n_2-1;2} - r_2 s_{n_2-2;2}} x_{d';t}^{r_2 s_{n_2-3;2} - A_{n_2-2;2}} x_{e;t}^{\omega_2 s_{n_2-2;2} - \xi_2 s_{n_2-3;2}}$$
$$\times\, x_{g;t}^{\nu_2 s_{n_2-2;2} - \rho_2 s_{n_2-3;2} - (\nu_2 A_{n_2-2;2} - \rho_2 A_{n_2-3;2})}$$

$$= \sum_{\tau_{0;1},\tau_{1;1},\cdots,\tau_{n_1-3;1}} \left(\prod_{w=0}^{n_1-3} \begin{bmatrix} A_{w+1;1} - r_1 s_{w;1} \\ \tau_{w;1} \end{bmatrix}\right)$$
$$\times \sum_{\tau_{0;2},\tau_{1;2},\cdots,\tau_{n_2-3;2}} \left(\prod_{w=0}^{n_2-3} \begin{bmatrix} A_{w+1;2} - r_2 s_{w;2} \\ \tau_{w;2} \end{bmatrix}\right)$$
$$\times\, \widetilde{x_{d;t}}^{A_{n_2-1;2} - r_2 s_{n_2-2;2}} x_{d';t}^{r_2 s_{n_2-3;2} - A_{n_2-2;2}} x_{e;t}^{\omega_2 s_{n_2-2;2} - \xi_2 s_{n_2-3;2} + r_1 s_{n_1-3;1} - A_{n_1-2;1}}$$
$$\times\, x_{g;t}^{\nu_2 s_{n_2-2;2} - \rho_2 s_{n_2-3;2} - (\nu_2 A_{n_2-2;2} - \rho_2 A_{n_2-3;2}) + \nu_1 s_{n_1-2;1} - \rho_1 s_{n_1-3;1}}$$
$$\times\, x_{g;t}^{-(\nu_1 A_{n_1-2;1} - \rho_1 A_{n_1-3;1})},$$

where $A_{i;2}$ and $s_{i;2}$ are as defined before Lemmas 3.14 and 3.19 but in terms of p_2, q_2, and r_2, thus $A_{i;2} = p_2 c^{[r_2]}_{i+1} + q_2 c^{[r_2]}_i$ and $s_{i;2} = \sum_{j=0}^{i-1} c^{[r_2]}_{i-j+1}\tau_{j;2}$.

Let θ be a positive integer, and let P_θ be the sum of all the terms in the sum above for which the exponent of $x_{e;t}$ is equal to $-\theta$. Thus $-\theta$ is equal to

$$\omega_2 s_{n_2-2;2} - \xi_2 s_{n_2-3;2} + r_1 s_{n_1-3;1} - A_{n_1-2;1}.$$

5.2.1. *Computation of exponents.* In this subsubsection, we compute the exponents in the expression in (43). The main computation continues in 5.2.2. It is convenient to introduce ς such that $\tau_{0;2} = \varsigma - s_{n_1-3;1}$. Then

$$s_{n_2-2;2} = c^{[r_2]}_{n_2-1}(\varsigma - s_{n_1-3;1}) + \sum_{j=1}^{n_2-3} c^{[r_2]}_{n_2-1-j}\tau_{j;2}$$

and

$$s_{n_2-3;2} = c^{[r_2]}_{n_2-2}(\varsigma - s_{n_1-3;1}) + \sum_{j=1}^{n_2-4} c^{[r_2]}_{n_2-2-j}\tau_{j;2}.$$

Using equation (41), the expressions for $s_{n_2-2;2}$ and $s_{n_2-3;2}$ and the fact that $c^{[\xi]}_1 = 0$, we have

$$\begin{aligned}
&\omega_2 s_{n_2-2;2} - \xi_2 s_{n_2-3;2}\\
&= (c^{[\xi_1]}_{n_2-1} r_1 - c^{[\xi_1]}_{n_2-2}\omega_1)\left[c^{[\xi_1]}_{n_2-1}(\varsigma - s_{n_1-3;1}) + \sum_{j=1}^{n_2-3} c^{[\xi_1]}_{n_2-1-j}\tau_{j;2}\right]\\
&\quad - (c^{[\xi_1]}_{n_2} r_1 - c^{[\xi_1]}_{n_2-1}\omega_1)\left[c^{[\xi_1]}_{n_2-2}(\varsigma - s_{n_1-3;1}) + \left(\sum_{j=1}^{n_2-3} c^{[\xi_1]}_{n_2-2-j}\tau_{j;2}\right) - c^{[\xi_1]}_1 \tau_{n_2-3;2}\right]\\
&= (\varsigma - s_{n_1-3;1})\, r_1 \left((c^{[\xi_1]}_{n_2-1})^2 - c^{[\xi_1]}_{n_2} c^{[\xi_1]}_{n_2-2}\right)\\
&\quad + \sum_{j=1}^{n_2-3} \tau_{j;2}\left[r_1\left(c^{[\xi_1]}_{n_2-1} c^{[\xi_1]}_{n_2-1-j} - c^{[\xi_1]}_{n_2} c^{[\xi_1]}_{n_2-2-j}\right)\right.\\
&\qquad\qquad \left. + \omega_1\left(-c^{[\xi_1]}_{n_2-2} c^{[\xi_1]}_{n_2-1-j} + c^{[\xi_1]}_{n_2-1} c^{[\xi_1]}_{n_2-2-j}\right)\right]\\
&= (\varsigma - s_{n_1-3;1})\, r_1 + \sum_{j=1}^{n_2-3} \tau_{j;2}\left[r_1\left(-c^{[\xi_1]}_{-j}\right) + \omega_1 c^{[\xi_1]}_{1-j}\right] \qquad \text{(by Lemma 3.1)}.
\end{aligned}$$

And since $-c^{[\xi_1]}_{-j} = c^{[\xi_1]}_{j+2}$, we get

$$\begin{aligned}
-\theta &= r_1(\varsigma - s_{n_1-3;1}) + \sum_{j=1}^{n_2-3} \tau_{j;2}(c^{[\xi_1]}_{j+2} r_1 - c^{[\xi_1]}_{j+1}\omega_1) + r_1 s_{n_1-3;1} - A_{n_1-2;1}\\
&= -A_{n_1-2;1} + r_1\varsigma + \sum_{j=1}^{n_2-3} \tau_{j;2}(c^{[\xi_1]}_{j+2} r_1 - c^{[\xi_1]}_{j+1}\omega_1).
\end{aligned} \tag{44}$$

Also, the exponents of $\widetilde{x_{d;t}}$ and $x_{d';t}$ in (43) can be expressed as follows:

$$\begin{aligned}
A_{n_2-1;2} - r_2 s_{n_2-2;2} &= c^{[\xi_1]}_{n_2} p_2 + c^{[\xi_1]}_{n_2-1} q_2\\
&\quad - \xi_1\left(c^{[\xi_1]}_{n_2-1}(\varsigma - s_{n_1-3;1}) + \sum_{j=1}^{n_2-3} c^{[\xi_1]}_{n_2-1-j}\tau_{j;2}\right)\\
&= c^{[\xi_1]}_{n_2}(A_{n_1-1;1} - r_1 s_{n_1-2;1}) + c^{[\xi_1]}_{n_2-1}(\omega_1 s_{n_1-2;1} - \xi_1 s_{n_1-3;1})\\
&\quad - \xi_1\left(c^{[\xi_1]}_{n_2-1}(\varsigma - s_{n_1-3;1}) + \sum_{j=1}^{n_2-3} c^{[\xi_1]}_{n_2-1-j}\tau_{j;2}\right)
\end{aligned}$$

$$= c^{[\xi_1]}_{n_2}(A_{n_1-1;1} - r_1 s_{n_1-2;1}) + c^{[\xi_1]}_{n_2-1}\omega_1 s_{n_1-2;1}$$
$$- \xi_1 \left(c^{[\xi_1]}_{n_2-1}\varsigma + \sum_{j=1}^{n_2-3} c^{[\xi_1]}_{n_2-1-j}\tau_{j;2} \right),$$

and similarly

$$\xi_1 s_{n_2-3;2} - A_{n_2-2;2} = \xi_1 \left(c^{[\xi_1]}_{n_2-2}\varsigma + \sum_{j=1}^{n_2-4} c^{[\xi_1]}_{n_2-2-j}\tau_{j;2} \right)$$
$$- \left(c^{[\xi_1]}_{n_2-1}(A_{n_1-1;1} - r_1 s_{n_1-2;1}) + c^{[\xi_1]}_{n_2-2}\omega_1 s_{n_1-2;1} \right).$$

Recall that $\rho_1 = c^{[r_2]}_{n_2-3}\nu_2 - c^{[r_2]}_{n_2-4}\rho_2$. The exponent of $x_{g;t}$ in (43) is equal to

$$\begin{aligned}
&\nu_2 s_{n_2-2;2} - \rho_2 s_{n_2-3;2} - (\nu_2 A_{n_2-2;2} - \rho_2 A_{n_2-3;2}) \\
&\quad + \nu_1 s_{n_1-2;1} - \rho_1 s_{n_1-3;1} - (\nu_1 A_{n_1-2;1} - \rho_1 A_{n_1-3;1}) \\
&= \nu_2 \left(c^{[\xi_1]}_{n_2-1}(\varsigma - s_{n_1-3;1}) + \sum_{j=1}^{n_2-3} c^{[\xi_1]}_{n_2-1-j}\tau_{j;2} \right) \\
&\quad - \rho_2 \left(c^{[\xi_1]}_{n_2-2}(\varsigma - s_{n_1-3;1}) + \sum_{j=1}^{n_2-4} c^{[\xi_1]}_{n_2-2-j}\tau_{j;2} \right) \\
&\quad - \nu_2 \left(c^{[\xi_1]}_{n_2-1}(A_{n_1-1;1} - r_1 s_{n_1-2;1}) + c^{[\xi_1]}_{n_2-2}(\omega_1 s_{n_1-2;1} - \xi_1 s_{n_1-3;1}) \right) \\
&\quad + \rho_2 \left(c^{[\xi_1]}_{n_2-2}(A_{n_1-1;1} - r_1 s_{n_1-2;1}) + c^{[\xi_1]}_{n_2-3}(\omega_1 s_{n_1-2;1} - \xi_1 s_{n_1-3;1}) \right) \\
&\quad + \nu_1 s_{n_1-2;1} - \rho_1 s_{n_1-3;1} \\
&\quad - (\nu_1 A_{n_1-2;1} - \rho_1 A_{n_1-3;1}) \\
&= -\gamma_2 s_{n_1-2;1} + (\nu_2 c^{[\xi_1]}_{n_2-1} - \rho_2 c^{[\xi_1]}_{n_2-2})(\varsigma - A_{n_1-1;1}) \\
&\quad + \nu_2 \sum_{j=1}^{n_2-3} c^{[\xi_1]}_{n_2-1-j}\tau_{j;2} - \rho_2 \sum_{j=1}^{n_2-4} c^{[\xi_1]}_{n_2-2-j}\tau_{j;2} - (\nu_1 A_{n_1-2;1} - \rho_1 A_{n_1-3;1}).
\end{aligned}$$

5.2.2. *Back to main computation.* Using the computations from 5.2.1 and fixing $\varsigma, \tau_{1;2}, \ldots, \tau_{n_2-3;2}$ in (43), we obtain

$$\begin{aligned}
(43) = &\sum_{\tau_{0;1},\tau_{1;1},\cdots,\tau_{n_1-3;1}} \left(\prod_{w=0}^{n_1-3} \begin{bmatrix} A_{w+1;1} - r_1 s_{w;1} \\ \tau_{w;1} \end{bmatrix} \right) \left(\prod_{w=0}^{n_2-3} \begin{bmatrix} A_{w+1;2} - r_2 s_{w;2} \\ \tau_{w;2} \end{bmatrix} \right) \\
&\times \widetilde{x_{d;t}}^{\,c^{[\xi_1]}_{n_2}(A_{n_1-1;1} - r_1 s_{n_1-2;1}) + c^{[\xi_1]}_{n_2-1}\omega_1 s_{n_1-2;1} - \xi_1 \left(c^{[\xi_1]}_{n_2-1}\varsigma + \sum_{j=1}^{n_2-3} c^{[\xi_1]}_{n_2-1-j}\tau_{j;2} \right)} \\
&\times x_{d';t}^{\,\xi_1 \left(c^{[\xi_1]}_{n_2-2}\varsigma + \sum_{j=1}^{n_2-4} c^{[\xi_1]}_{n_2-2-j}\tau_{j;2} \right) - \left(c^{[\xi_1]}_{n_2-1}(A_{n_1-1;1} - r_1 s_{n_1-2;1}) + c^{[\xi_1]}_{n_2-2}\omega_1 s_{n_1-2;1} \right)} \\
&\times x_{e;t}^{\,-A_{n_1-2;1} + r_1\varsigma + \sum_{j=1}^{n_2-3} (c^{[\xi_1]}_{j+2} r_1 - c^{[\xi_1]}_{j+1}\omega_1)\tau_{j;2}}
\end{aligned}$$

$$\times\, x_{g;t}^{-\gamma_2 s_{n_1-2;1}+(\nu_2 c^{[\xi_1]}_{n_2-1}-\rho_2 c^{[\xi_1]}_{n_2-2})(\varsigma-A_{n_1-1;1})+\nu_2\sum_{j=1}^{n_2-3}c^{[\xi_1]}_{n_2-1-j}\tau_{j;2}}$$
$$\times\, x_{g;t}^{-\rho_2\sum_{j=1}^{n_2-4}c^{[\xi_1]}_{n_2-2-j}\tau_{j;2}-(\nu_1 A_{n_1-2;1}-\rho_1 A_{n_1-3;1})}.$$

Now we collect all powers involving $s_{n_1-2;1}$ and write (43) as a product $\phi\varphi$, where ϕ is a Laurent monomial in $\widetilde{x_{d;t}}, x_{d';t}, x_{e;t}, x_{g;t}$ that in the expression of Proposition 5.2 is absorbed either in the first summation, if the exponent of $x_{e;t}$ is positive, or in the second summation inside $x_{e;t'}^{-\theta}\mathfrak{r}$ if the exponent of $x_{e;t}$ is negative. On the other hand, φ is equal to

$$\sum_{\tau_{0;1},\tau_{1;1},\dots,\tau_{n_1-3;1}}\left(\prod_{w=0}^{n_1-3}\left[\begin{array}{c}A_{w+1;1}-r_1 s_{w;1}\\ \tau_{w;1}\end{array}\right]\right)\left(\prod_{w=0}^{n_2-3}\left[\begin{array}{c}A_{w+1;2}-r_2 s_{w;2}\\ \tau_{w;2}\end{array}\right]\right)$$
$$\times\left(\frac{\widetilde{x_{d;t}}^{\,c^{[\xi_1]}_{n_2}r_1-c^{[\xi_1]}_{n_2-1}\omega_1}x_{g;t}^{\gamma_2}}{x_{d';t}^{\,c^{[\xi_1]}_{n_2-1}r_1-c^{[\xi_1]}_{n_2-2}\omega_1}}\right)^{\left\lfloor(A_{n_1-1;1}-\varsigma)\frac{A_{n_1-1;1}}{A_{n_1;1}}\right\rfloor-s_{n_1-2;1}}.$$

Note that the 0-th term of the second product can be identified with an (n_1-2)-nd term in the first product, since $A_{1;2}=A_{n_1-1;1}-r_1 s_{n_1-2;1}$ by the definition of $A_{w;2}$ right after equation (19) and $s_{0,2}=0$. The above expression is therefore equal to

$$\sum_{\tau_{0;1},\tau_{1;1},\dots,\tau_{n_1-3;1}}\left(\prod_{w=0}^{n_1-2}\left[\begin{array}{c}A_{w+1;1}-r_1 s_{w;1}\\ \tau_{w;1}\end{array}\right]\right)\left(\prod_{w=1}^{n_2-3}\binom{A_{w+1;2}-r_2 s_{w;2}}{\tau_{w;2}}\right)$$
$$\times\left(\frac{\widetilde{x_{d;t}}^{\,c^{[\xi_1]}_{n_2}r_1-c^{[\xi_1]}_{n_2-1}\omega_1}x_{g;t}^{\gamma_2}}{x_{d';t}^{\,c^{[\xi_1]}_{n_2-1}r_1-c^{[\xi_1]}_{n_2-2}\omega_1}}\right)^{\left\lfloor(A_{n_1-1;1}-\varsigma)\frac{A_{n_1-1;1}}{A_{n_1;1}}\right\rfloor-s_{n_1-2;1}}, \tag{45}$$

where $\tau_{n_1-2;1}=A_{n_1-1;1}-r_1 s_{n_1-2;1}-\tau_{0;2}=A_{n_1-1;1}-r_1 s_{n_1-2;1}-\varsigma+s_{n_1-3;1}$. The exponent $\left\lfloor(A_{n_1-1;1}-\varsigma)\frac{A_{n_1-1;1}}{A_{n_1;1}}\right\rfloor-s_{n_1-2;1}$ is nonnegative by Lemma 5.13 below.

Now we show that the Laurent monomial ϕ has nonnegative degree on $\widetilde{x_{d;t}}$. We want to show that

$$c^{[\xi_1]}_{n_2}A_{n_1-1;1}-\xi_1\left(c^{[\xi_1]}_{n_2-1}\varsigma+\sum_{j=1}^{n_2-3}c^{[\xi_1]}_{n_2-1-j}\tau_{j;2}\right)$$
$$-(c^{[\xi_1]}_{n_2}r_1-c^{[\xi_1]}_{n_2-1}\omega_1)(A_{n_1-1;1}-\varsigma)\frac{A_{n_1-1;1}}{A_{n_1;1}}\geq 0. \tag{46}$$

Inequality (50) yields

$$(c_{j+2}^{[\xi_1]}r_1 - c_{j+1}^{[\xi_1]}\omega_1)\frac{A_{n_1-1;1}}{A_{n_1-2;1}}\left(c_{n_2}^{[\xi_1]} - (c_{n_2}^{[\xi_1]}r_1 - c_{n_2-1}^{[\xi_1]}\omega_1)\frac{A_{n_1-1;1}}{A_{n_1;1}}\right)\tau_{j;2} > \xi_1 c_{n_2-1-j}^{[\xi_1]}\tau_{j;2},$$

so

$$\sum_{j=1}^{n_2-3}(c_{j+2}^{[\xi_1]}r_1 - c_{j+1}^{[\xi_1]}\omega_1)\frac{A_{n_1-1;1}}{A_{n_1-2;1}}\left(c_{n_2}^{[\xi_1]} - (c_{n_2}^{[\xi_1]}r_1 - c_{n_2-1}^{[\xi_1]}\omega_1)\frac{A_{n_1-1;1}}{A_{n_1;1}}\right)\tau_{j;2} > \sum_{j=1}^{n_2-3}\xi_1 c_{n_2-1-j}^{[\xi_1]}\tau_{j;2}.$$

Thanks to $\theta > 0$ and equation (44), this implies that

(47)
$$(A_{n_1-2;1} - r_1\varsigma)\frac{A_{n_1-1;1}}{A_{n_1-2;1}}\left(c_{n_2}^{[\xi_1]} - (c_{n_2}^{[\xi_1]}r_1 - c_{n_2-1}^{[\xi_1]}\omega_1)\frac{A_{n_1-1;1}}{A_{n_1;1}}\right) > \sum_j \xi_1 c_{n_2-1-j}^{[\xi_1]}\tau_{j;2}.$$

Then to prove (46), it is enough to show that

$$c_{n_2}^{[\xi_1]}A_{n_1-1;1} - \xi_1 c_{n_2-1}^{[\xi_1]}\varsigma - (c_{n_2}^{[\xi_1]}r_1 - c_{n_2-1}^{[\xi_1]}\omega_1)(A_{n_1-1;1} - \varsigma)\frac{A_{n_1-1;1}}{A_{n_1;1}}$$

is greater than the left-hand side of (47). The terms without ς are all canceled. Hence it remains to show that

$$\begin{aligned}
& r_1\frac{A_{n_1-1;1}}{A_{n_1-2;1}}\left(c_{n_2}^{[\xi_1]} - (c_{n_2}^{[\xi_1]}r_1 - c_{n_2-1}^{[\xi_1]}\omega_1)\frac{A_{n_1-1;1}}{A_{n_1;1}}\right) \\
&\qquad > \xi_1 c_{n_2-1}^{[\xi_1]} - (c_{n_2}^{[\xi_1]}r_1 - c_{n_2-1}^{[\xi_1]}\omega_1)\frac{A_{n_1-1;1}}{A_{n_1;1}} \\
\Longleftrightarrow\ & r_1 c_{n_2}^{[\xi_1]}\frac{A_{n_1-1;1}}{A_{n_1-2;1}} - \xi_1 c_{n_2-1}^{[\xi_1]} > (c_{n_2}^{[\xi_1]}r_1 - c_{n_2-1}^{[\xi_1]}\omega_1)\left(r_1\frac{A_{n_1-1;1}}{A_{n_1-2;1}} - 1\right)\frac{A_{n_1-1;1}}{A_{n_1;1}} \\
\Longleftrightarrow\ & r_1 c_{n_2}^{[\xi_1]}\frac{A_{n_1-1;1}}{A_{n_1-2;1}} - \xi_1 c_{n_2-1}^{[\xi_1]} \\
&\qquad > (c_{n_2}^{[\xi_1]}r_1 - c_{n_2-1}^{[\xi_1]}\omega_1)\left(\frac{r_1 A_{n_1-1;1} - A_{n_1-2;1}}{A_{n_1-2;1}}\right)\frac{A_{n_1-1;1}}{A_{n_1;1}} \\
\overset{\text{Lemma 3.14}}{\Longleftrightarrow}\ & r_1 c_{n_2}^{[\xi_1]}\frac{A_{n_1-1;1}}{A_{n_1-2;1}} - \xi_1 c_{n_2-1}^{[\xi_1]} > (c_{n_2}^{[\xi_1]}r_1 - c_{n_2-1}^{[\xi_1]}\omega_1)\frac{A_{n_1-1;1}}{A_{n_1-2;1}} \\
\Longleftrightarrow\ & \omega_1\frac{A_{n_1-1;1}}{A_{n_1-2;1}} - \xi_1 > 0.
\end{aligned}$$

The last inequality follows from the first case of Lemma 3.2. This shows (46), and thus the exponent of $\widetilde{x_{d;t}}$ in ϕ is nonnegative.

Now Proposition 5.2 follows from the following lemma.

LEMMA 5.8.

$$\prod_{w=1}^{n_2-3}\binom{A_{w+1;2}-r_2s_{w;2}}{\tau_{w;2}}=\sum_{i=0}^{\sum_{w=1}^{n_2-3}\tau_{w;2}} d_i\binom{\left\lfloor (A_{n_1-1;1}-\varsigma)\frac{A_{n_1-1;1}}{A_{n_1;1}}\right\rfloor - s_{n_1-2;1}}{i}$$

for some $d_i\in\mathbb{N}$, which are independent of $s_{n_1-2;1}$.

Proof. First suppose that one of ξ_1, r_1, ω_1 is at most one. The equation in Lemma 5.8 without the requirement that $d_i\in\mathbb{N}$ is always true in any case (a), (b), (c), (d). The divisibility in Lemma 5.14 without the positivity statement is always true, so the argument in Section 5.3 shows that Proposition 5.5 without the positivity statement is always true. Therefore $x^p_{d;t_{\omega_E}}x^q_{e;t_{\omega_E}}$ is a linear combination of the following rank 2 cluster monomials:

$$\widetilde{x_{d;t}}^{p'}\widetilde{x_{e;t}}^{q'},\ \ \widetilde{x_{d;t}}^{p'}x_{e;t}{}^{q'},\ \ x_{d;t}{}^{p'}x_{e;t}{}^{q'},\ \ x_{d;t}{}^{p'}\widetilde{x_{e;t}}^{q'}.$$

Now if one of ξ_1, r_1, ω_1 is at most one, then some quiver that is mutation equivalent to the subquiver with vertices d, d', e is acyclic [23], and it follows from [31, eq. (5.10)] that the coefficients of the linear combination are nonnegative. This implies that $d_i\in\mathbb{N}$.

In what follows, we assume that $\xi_1, r_1, \omega_1\geq 2$.

Once we know that there are nonnegative integers a and b such that

$$A_{w+1;2}-r_2s_{w;2}=a\left(\left\lfloor (A_{n_1-1;1}-\varsigma)\frac{A_{n_1-1;1}}{A_{n_1;1}}\right\rfloor - s_{n_1-2;1}\right)+b,$$

then it is clear, by Lemma 5.11 below, that

$$\binom{A_{w+1;2}-r_2s_{w;2}}{\tau_{w;2}}=\sum_{i=0}^{\tau_{w;2}} d'_i\binom{\left\lfloor (A_{n_1-1;1}-\varsigma)\frac{A_{n_1-1;1}}{A_{n_1;1}}\right\rfloor - s_{n_1-2;1}}{i}$$

for some $d'_i\in\mathbb{N}$, and by Lemma 5.12 below, for any nonnegative integers j and k,

$$\binom{\left\lfloor (A_{n_1-1;1}-\varsigma)\frac{A_{n_1-1;1}}{A_{n_1;1}}\right\rfloor - s_{n_1-2;1}}{j}\binom{\left\lfloor (A_{n_1-1;1}-\varsigma)\frac{A_{n_1-1;1}}{A_{n_1;1}}\right\rfloor - s_{n_1-2;1}}{k}$$
$$=\sum_{i=0}^{j+k} d''_i\binom{\left\lfloor (A_{n_1-1;1}-\varsigma)\frac{A_{n_1-1;1}}{A_{n_1;1}}\right\rfloor - s_{n_1-2;1}}{i}$$

for some $d''_i\in\mathbb{N}$. Then it follows that

$$\prod_{w=1}^{n_2-3}\binom{A_{w+1;2}-r_2s_{w;2}}{\tau_{w;2}}=\sum_{i=0}^{\sum_{w=1}^{n_2-3}\tau_{w;2}} d_i\binom{\left\lfloor (A_{n_1-1;1}-\varsigma)\frac{A_{n_1-1;1}}{A_{n_1;1}}\right\rfloor - s_{n_1-2;1}}{i}.$$

Thus we need to show the existence of the nonnegative integers a and b. Using the definitions of $A_{w+1;2}$ and ς as well as the fact that $r_2 = \xi_1$, we get

$$A_{w+1;2} - r_2 s_{w;2} = c^{[\xi_1]}_{w+2}(A_{n_1-1;1} - r_1 s_{n_1-2;1}) + c^{[\xi_1]}_{w+1}\omega_1 s_{n_1-2;1} - \xi_1 \left(c^{[\xi_1]}_{w+1}\varsigma + \sum_{j=1}^{w-1} c^{[\xi_1]}_{w+1-j}\tau_{j;2} \right),$$

which can be written as

$$A_{w+1;2} - r_2 s_{w;2} = (c^{[\xi_1]}_{w+2} r_1 - c^{[\xi_1]}_{w+1}\omega_1)\left(\left\lfloor (A_{n_1-1;1} - \varsigma)\frac{A_{n_1-1;1}}{A_{n_1;1}} \right\rfloor - s_{n_1-2;1} \right) + C(w), \tag{48}$$

where $C(w)$ is some function of w, which is independent of $s_{n_1-2;1}$ and which we give explicitly below. Note that

$$c^{[\xi_1]}_{w+2} r_1 - c^{[\xi_1]}_{w+1}\omega_1 > 0,$$

because, by Lemma 3.2, this is the number of arrows between some pair of vertices in some seed between t_{ω_Z} and t. Thus is suffices to show that $C(w)$ is nonnegative:

$$C(w) = (c^{[\xi_1]}_{w+2} r_1 - c^{[\xi_1]}_{w+1}\omega_1)\Bigg((A_{n_1-1;1} - \varsigma)\frac{A_{n_1-1;1}}{A_{n_1;1}} - \left\lfloor (A_{n_1-1;1} - \varsigma)\frac{A_{n_1-1;1}}{A_{n_1;1}} \right\rfloor \Bigg) + \tilde{C}(w)\theta(w),$$

where

$$\tilde{C}(w) = c^{[\xi_1]}_{w+2} - (c^{[\xi_1]}_{w+2} r_1 - c^{[\xi_1]}_{w+1}\omega_1)\frac{A_{n_1-1;1}}{A_{n_1;1}}$$

and

$$\theta(w) = A_{n_1-1;1} - \frac{\xi_1 c^{[\xi_1]}_{w+1} - (c^{[\xi_1]}_{w+2} r_1 - c^{[\xi_1]}_{w+1}\omega_1)\frac{A_{n_1-1;1}}{A_{n_1;1}}}{c^{[\xi_1]}_{w+2} - (c^{[\xi_1]}_{w+2} r_1 - c^{[\xi_1]}_{w+1}\omega_1)\frac{A_{n_1-1;1}}{A_{n_1;1}}}\varsigma - \sum_{j=1}^{w-1} \frac{\xi_1 c^{[\xi_1]}_{w+1-j}}{c^{[\xi_1]}_{w+2} - (c^{[\xi_1]}_{w+2} r_1 - c^{[\xi_1]}_{w+1}\omega_1)\frac{A_{n_1-1;1}}{A_{n_1;1}}}\tau_{j;2}.$$

We want to show that $C(w)$ is nonnegative for $1 \leq w \leq n_2 - 3$, for which it suffices to show that $\tilde{C}(w)$ and $\theta(w)$ are nonnegative.

First we show that $\tilde{C}(w)$ are nonnegative for $w \geq 1$. Note that $\tilde{C}(w) = \xi_1\tilde{C}(w-1) - \tilde{C}(w-2)$. Then if we show $\tilde{C}(1) > 0 \geq \tilde{C}(0)$, then the induction on w will show that $\tilde{C}(w)$ is increasing with w. By Lemma 3.14, we have

$\tilde{C}(0) = 1 - r_1 \frac{A_{n_1-1;1}}{A_{n_1;1}} \leq 0$. On the other hand,

$$\tilde{C}(1) = \xi_1 - (\xi_1 r_1 - \omega_1)\frac{A_{n_1-1;1}}{A_{n_1;1}} = \xi_1 \left(\frac{A_{n_1;1} - r_1 A_{n_1-1;1}}{A_{n_1;1}}\right) + \omega_1 \frac{A_{n_1-1;1}}{A_{n_1;1}}$$
$$= \xi_1 \left(\frac{-A_{n_1-2;1}}{A_{n_1;1}}\right) + \omega_1 \frac{A_{n_1-1;1}}{A_{n_1;1}},$$

which is positive because of equation (22).

Next we show that $\theta(w)$ are nonnegative for all w such that $1 \leq w \leq n_2-3$. Recall from (44) that

$$\theta = A_{n_1-2;1} - r_1\varsigma - \sum_{j=1}^{n_2-3} (c_{j+2}^{[\xi_1]} r_1 - c_{j+1}^{[\xi_1]}\omega_1)\tau_{j;2} > 0.$$

Multiplying with $\frac{A_{n_1-1;1}}{A_{n_1-2;1}}$ yields

$$A_{n_1-1;1} - \frac{r_1 A_{n_1-1;1}}{A_{n_1-2;1}}\varsigma - \sum_{j=1}^{w-1} \frac{(c_{j+2}^{[\xi_1]} r_1 - c_{j+1}^{[\xi_1]}\omega_1)A_{n_1-1;1}}{A_{n_1-2;1}}\tau_{j;2} > 0.$$

So it is enough to show that if $\xi_1, r_1, \omega_1 \geq 2$, then

(49) $$\frac{r_1 A_{n_1-1;1}}{A_{n_1-2;1}} > \frac{\xi_1 c_{w+1}^{[\xi_1]} - (c_{w+2}^{[\xi_1]} r_1 - c_{w+1}^{[\xi_1]}\omega_1)\frac{A_{n_1-1;1}}{A_{n_1;1}}}{c_{w+2}^{[\xi_1]} - (c_{w+2}^{[\xi_1]} r_1 - c_{w+1}^{[\xi_1]}\omega_1)\frac{A_{n_1-1;1}}{A_{n_1;1}}}$$

and

(50) $$\frac{(c_{j+2}^{[\xi_1]} r_1 - c_{j+1}^{[\xi_1]}\omega_1)A_{n_1-1;1}}{A_{n_1-2;1}} > \frac{\xi_1 c_{w+1-j}^{[\xi_1]}}{c_{w+2}^{[\xi_1]} - (c_{w+2}^{[\xi_1]} r_1 - c_{w+1}^{[\xi_1]}\omega_1)\frac{A_{n_1-1;1}}{A_{n_1;1}}}.$$

This ends the proof of Lemma 5.8, modulo inequalities (49) and (50), which are proved in the following subsection. □

5.2.3. *Proof of* (49) *and* (50). It follows from Lemma 3.14 that the left-hand side of (49) is equal to $(A_{n_1-2;1} + A_{n_1;1})/A_{n_1-2;1} = 1 + A_{n_1;1}/A_{n_1-2;1}$. Thus (49) is equivalent to

(51)

$$1 + \frac{A_{n_1;1}}{A_{n_1-2;1}} > \frac{\xi_1 c_{w+1}^{[\xi_1]} A_{n_1;1} - (c_{w+2}^{[\xi_1]} r_1 - c_{w+1}^{[\xi_1]}\omega_1)A_{n_1-1;1}}{c_{w+2}^{[\xi_1]} A_{n_1;1} - (c_{w+2}^{[\xi_1]} r_1 - c_{w+1}^{[\xi_1]}\omega_1)A_{n_1-1;1}}$$
$$\iff 1 + \frac{A_{n_1;1}}{A_{n_1-2;1}} > 1 + \frac{(\xi_1 c_{w+1}^{[\xi_1]} - c_{w+2}^{[\xi_1]})A_{n_1;1}}{c_{w+2}^{[\xi_1]} A_{n_1;1} - (c_{w+2}^{[\xi_1]} r_1 - c_{w+1}^{[\xi_1]}\omega_1)A_{n_1-1;1}}$$
$$\iff \frac{1}{A_{n_1-2;1}} > \frac{\xi_1 c_{w+1}^{[\xi_1]} - c_{w+2}^{[\xi_1]}}{c_{w+2}^{[\xi_1]} A_{n_1;1} - (c_{w+2}^{[\xi_1]} r_1 - c_{w+1}^{[\xi_1]}\omega_1)A_{n_1-1;1}}$$

$$\overset{\text{by recursive definition of } c_w^{[\xi_1]}}{\Longleftrightarrow} \quad \frac{1}{A_{n_1-2;1}} > \frac{c_w^{[\xi_1]}}{c_{w+2}^{[\xi_1]}A_{n_1;1} - (c_{w+2}^{[\xi_1]}r_1 - c_{w+1}^{[\xi_1]}\omega_1)A_{n_1-1;1}}$$

$$\overset{\text{Lemma 3.14}}{\Longleftrightarrow} \quad \frac{1}{A_{n_1-2;1}} > \frac{c_w^{[\xi_1]}}{c_{w+1}^{[\xi_1]}\omega_1 A_{n_1-1;1} - c_{w+2}^{[\xi_1]}A_{n_1-2;1}}$$

$$\Longleftrightarrow \quad c_{w+1}^{[\xi_1]}\omega_1 A_{n_1-1;1} - c_{w+2}^{[\xi_1]}A_{n_1-2;1} > c_w^{[\xi_1]}A_{n_1-2;1}$$

$$\Longleftrightarrow \quad c_{w+1}^{[\xi_1]}\omega_1 A_{n_1-1;1} - A_{n_1-2;1}(c_w^{[\xi_1]} + c_{w+2}^{[\xi_1]}) > 0$$

$$\overset{\text{by recursive definition of } c_w^{[\xi_1]}}{\Longleftrightarrow} \quad c_{w+1}^{[\xi_1]}(\omega_1 A_{n_1-1;1} - \xi_1 A_{n_1-2;1}) > 0.$$

This last inequality holds by equation (22), and thus this completes the proof of (49).

To prove inequality (50) we start with inequality (51) above. Suppose first that

$$c_w^{[\xi_1]} \geq \frac{\xi_1 c_{w+1-j}^{[\xi_1]}A_{n_1;1}}{(c_{j+2}^{[\xi_1]}r_1 - c_{j+1}^{[\xi_1]}\omega_1)A_{n_1-1;1}}.$$

Then (51) implies

$$\frac{(c_{j+2}^{[\xi_1]}r_1 - c_{j+1}^{[\xi_1]}\omega_1)A_{n_1-1;1}}{A_{n_1-2;1}} > \frac{\xi_1 c_{w+1-j}^{[\xi_1]}A_{n_1;1}}{c_{w+1}^{[\xi_1]}\omega_1 A_{n_1-1;1} - c_{w+2}^{[\xi_1]}A_{n_1-2;1}},$$

which is equivalent to inequality (50), because $A_{n_1-2;1} = rA_{n_1-1;1} - A_{n_1;1}$, by Lemma 3.14.

Suppose to the contrary that

$$c_w^{[\xi_1]} < \frac{\xi_1 c_{w+1-j}^{[\xi_1]}A_{n_1;1}}{(c_{j+2}^{[\xi_1]}r_1 - c_{j+1}^{[\xi_1]}\omega_1)A_{n_1-1;1}}.$$

Then

$$c_{j+2}^{[\xi_1]}r_1 - c_{j+1}^{[\xi_1]}\omega_1 < \frac{\xi_1 c_{w+1-j}^{[\xi_1]}A_{n_1;1}}{c_w^{[\xi_1]}A_{n_1-1;1}},$$

and dividing by $c_{j+1}^{[\xi_1]}$ and using $A_{n_1;1} = r_1 A_{n_1-1;1} - A_{n_1-2;1}$ yields

$$\frac{c_{j+2}^{[\xi_1]}}{c_{j+1}^{[\xi_1]}}r_1 - \omega_1 < \frac{\xi_1 c_{w+1-j}^{[\xi_1]}(r_1 A_{n_1-1;1} - A_{n_1-2;1})}{c_{j+1}^{[\xi_1]}c_w^{[\xi_1]}A_{n_1-1;1}} \leq \frac{\xi_1 c_{w+1-j}^{[\xi_1]}r_1}{c_{j+1}^{[\xi_1]}c_w^{[\xi_1]}}. \tag{52}$$

Case 1. Suppose that $j \geq 2$. Recall that $c_1^{[\xi_1]} = 0, c_2^{[\xi_1]} = 1, c_3^{[\xi_1]} = \xi_1$, and then (52) implies

$$\frac{c_{j+2}^{[\xi_1]}}{c_{j+1}^{[\xi_1]}}r_1 - \omega_1 < \frac{c_{w-1}^{[\xi_1]}r_1}{c_w^{[\xi_1]}},$$

so

$$\omega_1 > \left(\frac{c_{j+2}^{[\xi_1]}}{c_{j+1}^{[\xi_1]}} - \frac{c_{w-1}^{[\xi_1]}}{c_w^{[\xi_1]}} \right) r_1 > \left(\lim_{j\to\infty} \frac{c_{j+2}^{[\xi_1]}}{c_{j+1}^{[\xi_1]}} - \lim_{w\to\infty} \frac{c_{w-1}^{[\xi_1]}}{c_w^{[\xi_1]}} \right) r_1,$$

where the last inequality holds since $\frac{c_{j+2}^{[\xi_1]}}{c_{j+1}^{[\xi_1]}} < \frac{c_{j+3}^{[\xi_1]}}{c_{j+2}^{[\xi_1]}}$, for all $j \geq 1$. Computing the limits, we obtain

$$\omega_1 > \left(\frac{\xi_1 + \sqrt{\xi_1^2 - 4}}{2} - \frac{\xi_1 - \sqrt{\xi_1^2 - 4}}{2} \right) r_1 = r_1\sqrt{\xi_1^2 - 4}. \tag{53}$$

- If $\xi_1 \geq 3$, then (53) implies $\omega_1 > r_1(\xi_1 - 1)$, thus $\omega_1 \geq 5$ and $\omega_1 - \xi_1 > \xi_1 r_1 - r_1 - \xi_1 \geq 1$.
- If $\xi_1 = 2$ and $\omega_1 \geq 4$ then, since $0 < A_{n_1-2;1}/A_{n_1-1;1} < 1$, we have

$$\omega_1 - \xi_1 \frac{A_{n_1-2;1}}{A_{n_1-1;1}} \geq 2.$$

- If $\xi_1 = 2$, $\omega_1 = 3$ and $r_1 \geq 3$, then we still have

$$\omega_1 - \xi_1 \frac{A_{n_1-2;1}}{A_{n_1-1;1}} \geq 2.$$

- If $\xi_1 = 2$, $\omega_1 = 3$ and $r_1 = 2$, then the subquiver obtained from t^* by mutating at d' is acyclic, so we do not consider this case.
- If $\xi_1 = \omega_1 = 2$ and $r_1 \geq 3$, then $c_{j+2}^{[\xi_1]} r_1 - c_{j+1}^{[\xi_1]}\omega_1 \geq 2$ and $\omega_1 - \xi_1 \frac{A_{n_1-2;1}}{A_{n_1-1;1}} \geq 1$.

In any of the above cases, we have

$$\begin{aligned}
& (c_{j+2}^{[\xi_1]} r_1 - c_{j+1}^{[\xi_1]}\omega_1)(\omega_1 - \xi_1 \frac{A_{n_1-2;1}}{A_{n_1-1;1}}) \geq 2 \\
\Longrightarrow \quad & (c_{j+2}^{[\xi_1]} r_1 - c_{j+1}^{[\xi_1]}\omega_1)(\omega_1 - \xi_1 \frac{A_{n_1-2;1}}{A_{n_1-1;1}}) \geq \frac{\xi_1 c_{w+1-j}^{[\xi_1]}}{c_{w+1}^{[\xi_1]}} \\
\Longrightarrow \quad & (c_{j+2}^{[\xi_1]} r_1 - c_{j+1}^{[\xi_1]}\omega_1)(\omega_1 A_{n_1-1;1} - \xi_1 A_{n_1-2;1}) \geq \frac{\xi_1 c_{w+1-j}^{[\xi_1]} A_{n_1-1;1}}{c_{w+1}^{[\xi_1]}}.
\end{aligned} \tag{54}$$

Since $c_w^{[\xi_1]} = \xi_1 c_{w-1}^{[\xi_1]} - c_{w-2}^{[\xi_1]}$, and thus $\xi_1 c_{w-1}^{[\xi_1]} > c_{w-2}^{[\xi_1]}$, it follows that

$$\begin{aligned}
& (c_{j+2}^{[\xi_1]} r_1 - c_{j+1}^{[\xi_1]}\omega_1)(c_{w+1}^{[\xi_1]}\omega_1 A_{n_1-1;1} - c_{w+2}^{[\xi_1]} A_{n_1-2;1}) \geq \xi_1 c_{w+1-j}^{[\xi_1]} A_{n_1-1;1} \\
\Longleftrightarrow \quad & (c_{j+2}^{[\xi_1]} r_1 - c_{j+1}^{[\xi_1]}\omega_1)(c_{w+1}^{[\xi_1]}\omega_1 A_{n_1-1;1} - c_{w+2}^{[\xi_1]} A_{n_1-2;1}) A_{n_1-1;1} \\
& \quad \geq \xi_1 c_{w+1-j}^{[\xi_1]} A_{n_1-1;1}^2 \\
\overset{\text{Lemma 3.14}}{\Longrightarrow} \quad & (c_{j+2}^{[\xi_1]} r_1 - c_{j+1}^{[\xi_1]}\omega_1)(c_{w+1}^{[\xi_1]}\omega_1 A_{n_1-1;1} - c_{w+2}^{[\xi_1]} A_{n_1-2;1}) A_{n_1-1;1} \\
& \quad > \xi_1 c_{w+1-j}^{[\xi_1]} A_{n_1;1} A_{n_1-2;1}
\end{aligned}$$

$$\iff \quad \frac{(c^{[\xi_1]}_{j+2} r_1 - c^{[\xi_1]}_{j+1}\omega_1) A_{n_1-1;1}}{A_{n_1-2;1}} > \frac{\xi_1 c^{[\xi_1]}_{w+1-j} A_{n_1;1}}{c^{[\xi_1]}_{w+1}\omega_1 A_{n_1-1;1} - c^{[\xi_1]}_{w+2} A_{n_1-2;1}}$$

$$\overset{\text{Lemma 3.14}}{\iff} \quad \frac{(c^{[\xi_1]}_{j+2} r_1 - c^{[\xi_1]}_{j+1}\omega_1) A_{n_1-1;1}}{A_{n_1-2;1}}$$

$$> \frac{\xi_1 c^{[\xi_1]}_{w+1-j} A_{n_1;1}}{c^{[\xi_1]}_{w+2} A_{n_1;1} - (c^{[\xi_1]}_{w+2} r_1 - c^{[\xi_1]}_{w+1}\omega_1) A_{n_1-1;1}},$$

which proves inequality (50) in these cases.

Next suppose that $\xi_1 = r_1 = \omega_1 = 2$. In this case, $c^{[\xi_1]}_{j+1} = j$, and thus $(c^{[\xi_1]}_{j+2} r_1 - c^{[\xi_1]}_{j+1}\omega_1) = 2$ and $(c^{[\xi_1]}_{w+2} r_1 - c^{[\xi_1]}_{w+1}\omega_1) = 2$. It therefore suffices to show that

$$\frac{2A_{n_1-1;1}}{A_{n_1-2;1}} > \frac{2(w-j)A_{n_1;1}}{(w+1)A_{n_1;1} - 2A_{n_1-1;1}},$$

but this is equivalent to

$$(w+1)A_{n_1;1}A_{n_1-1;1} > (w-j)A_{n_1;1}A_{n_1-2;1} + 2A_{n_1-1;1}A_{n_1-1;1},$$

which holds true for $j \geq 1$, by Lemma 3.14. This completes the proof in the case $j \geq 2$.

Case 2. Suppose that $j = 1$. Let $\widehat{\xi_1} = \lim_{w\to\infty} \frac{c^{[\xi_1]}_{w+1}}{c^{[\xi_1]}_{w}}$; in other words, $\widehat{\xi_1} = \frac{\xi_1 + \sqrt{\xi_1^2 - 4}}{2}$.

Case 2-1. Suppose that

$$\frac{A_{n_1-1;1}}{A_{n_1-2;1}}\omega_1 - \frac{(\xi_1 r_1 - \omega_1)\widehat{\xi_1}(\omega_1 A_{n_1-1;1} - \xi_1 A_{n_1-2;1})A_{n_1-1;1}}{A^2_{n_1-2;1}} \geq \xi_1. \tag{55}$$

If $r_1 = 2$, then one can check (50) directly. Suppose $r_1 \geq 3$. Then $\omega_1 c^{[r_1]}_0 - \xi_1 c^{[r_1]}_{-1} = -w_1 + \xi_1 r_1$ is the number of arrows between e and d' in the seed t^*, and $\omega_1 c^{[r_1]}_{n-1} - \xi_1 c^{[r_1]}_{n-2}$ is the number of arrows between some pair of vertices in the seed t_{w_E}. In particular, these numbers are positive. Then by Lemma 5.9, we have $(\omega_1 c^{[r_1]}_0 - \xi_1 c^{[r_1]}_{-1})(\omega_1 c^{[r_1]}_{n-1} - \xi_1 c^{[r_1]}_{n-2}) > c^{[r_1]}_{n-2}$ for $n \in \{n_1-2, n_1-1\}$. So

$$(\xi_1 r_1 - \omega_1)(\omega_1 c^{[r_1]}_{n-1} - \xi_1 c^{[r_1]}_{n-2}) > c^{[r_1]}_{n-2},$$

hence

$$(\xi_1 r_1 - \omega_1)(\omega_1 A_{n_1-1;1} - \xi_1 A_{n_1-2;1}) > A_{n_1-2;1},$$

because $A_{i;1}$ is a positive linear combination of $c^{[r_1]}_i$ and $c^{[r_1]}_{i+1}$.

Then (55) implies that

$$\begin{aligned}
&\left(\omega_1-\widehat{\xi_1}\right)\frac{A_{n_1-1;1}}{A_{n_1-2;1}}\geq \xi_1\\
\Longrightarrow\quad &\left(\omega_1-\frac{2}{\xi_1 r_1-\omega_1}\right)\frac{A_{n_1-1;1}}{A_{n_1-2;1}}\geq \xi_1\\
\Longleftrightarrow\quad &\omega_1-\frac{2}{\xi_1 r_1-\omega_1}\geq \xi_1\frac{A_{n_1-2;1}}{A_{n_1-1;1}}\\
\Longleftrightarrow\quad &\omega_1-\xi_1\frac{A_{n_1-2;1}}{A_{n_1-1;1}}\geq \frac{2}{\xi_1 r_1-\omega_1}\\
\Longleftrightarrow\quad &(\xi_1 r_1-\omega_1)(\omega_1-\xi_1\frac{A_{n_1-2;1}}{A_{n_1-1;1}})\geq 2,
\end{aligned}$$

and we can use the argument following (54).

LEMMA 5.9. *Fix a positive integer $r\geq 3$, and let $c_i=c_i^{[r]}$. Let ω,ξ,a,b be integers such that $b-a$, $\omega c_a-\xi c_{a-1}$ and $\omega c_b-\xi c_{b-1}$ are positive. Then $(\omega c_a-\xi c_{a-1})(\omega c_b-\xi c_{b-1})>c_{b-a-1}$.*

Proof. Suppose that $\omega^2-r\omega\xi+\xi^2<0$. If $b-a\leq 2$, then $c_{b-a-1}\leq 0$, so there is nothing to show. Thanks to Lemma 3.1, we get

$$(\omega c_a-\xi c_{a-1})(\omega c_{b+1}-\xi c_b)-(\omega c_{a+1}-\xi c_a)(\omega c_b-\xi c_{b-1})=-c_{b-a+1}(\omega^2-r\omega\xi+\xi^2),$$

and the desired statement follows from induction on $b-a$.

Suppose that $\omega^2-r\omega\xi+\xi^2>0$. Since $c_i=-c_{2-i}$ for $i\in\mathbb{Z}$, we assume $(\omega c_a-\xi c_{a-1})<(\omega c_b-\xi c_{b-1})$ without loss of generality. Then there exists an integer $e<a$ such that $\omega c_{e+1}-\xi c_e>0$ and $\omega c_e-\xi c_{e-1}\leq 0$. Then again using $c_i=-c_{2-i}$ and Lemma 3.1, we get

$$\omega c_b-\xi c_{b-1}=(\omega c_{e+1}-\xi c_e)c_{b-e+1}+|\omega c_e-\xi c_{e-1}|c_{b-e},$$

which is clearly greater than c_{b-a-1}. □

Case 2-2. Suppose that

$$\frac{A_{n_1-1;1}}{A_{n_1-2;1}}\omega_1-\frac{(\xi_1 r_1-\omega_1)\widehat{\xi_1}(\omega_1 A_{n_1-1;1}-\xi_1 A_{n_1-2;1})A_{n_1-1;1}}{A^2_{n_1-2;1}}<\xi_1.$$

Then

$$\begin{aligned}
\xi_1\frac{A_{n_1-2;1}}{A_{n_1-1;1}}&>\omega_1-\frac{(\xi_1 r_1-\omega_1)\widehat{\xi_1}(\omega_1 A_{n_1-1;1}-\xi_1 A_{n_1-2;1})}{A_{n_1-2;1}}\\
\Longleftrightarrow\quad&\frac{(\xi_1 r_1-\omega_1)\widehat{\xi_1}(\omega_1 A_{n_1-1;1}-\xi_1 A_{n_1-2;1})}{A_{n_1-2;1}}\\
&\qquad+(\xi_1 r_1-\omega_1)>\xi_1 r_1-\xi_1\frac{A_{n_1-2;1}}{A_{n_1-1;1}}
\end{aligned}$$

$$\overset{\text{Lemma 3.10}(a)}{\Longleftrightarrow} \quad \frac{(\xi_1 r_1 - \omega_1)\widehat{\xi_1}(\omega_1 A_{n_1-1;1} - \xi_1 A_{n_1-2;1})}{A_{n_1-2;1}} + (\xi_1 r_1 - \omega_1) > \xi_1 \frac{A_{n_1;1}}{A_{n_1-1;1}}$$

$$\Longleftrightarrow \quad (\xi_1 r_1 - \omega_1) A_{n_1-1;1} > \frac{\xi_1 A_{n_1;1}}{\frac{\widehat{\xi_1}(\omega_1 A_{n_1-1;1} - \xi_1 A_{n_1-2;1})}{A_{n_1-2;1}} + 1}$$

$$\Longleftrightarrow \quad \frac{(\xi_1 r_1 - \omega_1) A_{n_1-1;1}}{A_{n_1-2;1}} > \frac{\xi_1 A_{n_1;1}}{\widehat{\xi_1}(\omega_1 A_{n_1-1;1} - \xi_1 A_{n_1-2;1}) + A_{n_1-2;1}}$$

$$\overset{\text{Lemma 5.10}}{\Longrightarrow} \quad \frac{(\xi_1 r_1 - \omega_1) A_{n_1-1;1}}{A_{n_1-2;1}} > \frac{\xi_1 c_w^{[\xi_1]} A_{n_1;1}}{c_{w+1}^{[\xi_1]}(\omega_1 A_{n_1-1;1} - \xi_1 A_{n_1-2;1}) + c_w^{[\xi_1]} A_{n_1-2;1}}$$

$$\overset{\text{by recursive definition of } c_w^{[\xi_1]}}{\Longleftrightarrow} \quad \frac{(\xi_1 r_1 - \omega_1) A_{n_1-1;1}}{A_{n_1-2;1}} > \frac{\xi_1 c_w^{[\xi_1]} A_{n_1;1}}{c_{w+1}^{[\xi_1]} \omega_1 A_{n_1-1;1} - c_{w+2}^{[\xi_1]} A_{n_1-2;1}}$$

$$\overset{\text{Lemma 3.14}}{\Longleftrightarrow} \quad \frac{(\xi_1 r_1 - \omega_1) A_{n_1-1;1}}{A_{n_1-2;1}} > \frac{\xi_1 c_w^{[\xi_1]} A_{n_1;1}}{c_{w+2}^{[\xi_1]} A_{n_1;1} - (c_{w+2}^{[\xi_1]} r_1 - c_{w+1}^{[\xi_1]} \omega_1) A_{n_1-1;1}},$$

which is inequality (50) for $j = 1$.

LEMMA 5.10. *We have*

$$\frac{\xi_1 c_{w+1}^{[\xi_1]} A_{n_1;1}}{c_{w+2}^{[\xi_1]}(\omega_1 A_{n_1-1;1} - \xi_1 A_{n_1-2;1}) + c_{w+1}^{[\xi_1]} A_{n_1-2;1}} > \frac{\xi_1 c_w^{[\xi_1]} A_{n_1;1}}{c_{w+1}^{[\xi_1]}(\omega_1 A_{n_1-1;1} - \xi_1 A_{n_1-2;1}) + c_w^{[\xi_1]} A_{n_1-2;1}}.$$

In particular,

$$\lim_{w\to\infty} \frac{\xi_1 c_w^{[\xi_1]} A_{n_1;1}}{c_{w+1}^{[\xi_1]}(\omega_1 A_{n_1-1;1} - \xi_1 A_{n_1-2;1}) + c_w^{[\xi_1]} A_{n_1-2;1}} > \frac{\xi_1 c_w^{[\xi_1]} A_{n_1;1}}{c_{w+1}^{[\xi_1]}(\omega_1 A_{n_1-1;1} - \xi_1 A_{n_1-2;1}) + c_w^{[\xi_1]} A_{n_1-2;1}}.$$

Proof. The statement is equivalent to

$$\frac{\xi_1 c_{w+1}^{[\xi_1]} A_{n_1;1}}{c_{w+2}^{[\xi_1]}\omega_1 A_{n_1-1;1} - c_{w+3}^{[\xi_1]} A_{n_1-2;1}} > \frac{\xi_1 c_{w}^{[\xi_1]} A_{n_1;1}}{c_{w+1}^{[\xi_1]}\omega_1 A_{n_1-1;1} - c_{w+2}^{[\xi_1]} A_{n_1-2;1}}$$

$$\iff \quad c_{w+1}^{[\xi_1]} c_{w+1}^{[\xi_1]}\omega_1 A_{n_1-1;1} - c_{w+1}^{[\xi_1]} c_{w+2}^{[\xi_1]} A_{n_1-2;1}$$

$$> c_{w}^{[\xi_1]} c_{w+2}^{[\xi_1]}\omega_1 A_{n_1-1;1} - c_{w}^{[\xi_1]} c_{w+3}^{[\xi_1]} A_{n_1-2;1}$$

$$\overset{\text{Lemma 3.1}}{\iff} \quad \omega_1 A_{n_1-1;1} - \xi_1 A_{n_1-2;1} > 0. \qquad \square$$

This completes the proof of Proposition 5.2 modulo the following lemmas, which are proved in [23].

LEMMA 5.11 ([23, Lemma 4.8]). *Let a, b, c be any nonnegative integers. Then there are nonnegative integers $d_0, \dots, d_c$ such that*

$$\binom{aX+b}{c} = \sum_{i=0}^{c} d_i \binom{X}{i}$$

for all nonnegative integers X.

LEMMA 5.12 ([23, Lemma 4.9]). *Let a, b be any nonnegative integers. Then there are nonnegative integers $e_0, \dots, e_{a+b}$ such that*

$$\binom{X}{a}\binom{X}{b} = \sum_{i=0}^{a+b} e_i \binom{X}{i}$$

for all nonnegative integers X.

LEMMA 5.13 ([23, Lemma 4.5]). $\left\lfloor (A_{n_1-1;1} - \varsigma)\frac{A_{n_1-1;1}}{A_{n_1;1}} \right\rfloor - s_{n_1-2;1} \geq 0.$

LEMMA 5.14 ([23, Lemma 4.6]). *Let I be a subset of the integers, and let $q : I \to \mathbb{R}$ be a function. Suppose that a polynomial of x of the form*

$$\sum_{m \in I} q(m) x^{b-m}$$

is divisible by $(1+x)^g$ and its quotient has nonnegative coefficients. Let

$$p(m) = \sum_{i=0}^{h} d_i \binom{b-m}{i}$$

be a polynomial of m with $d_i \geq 0$. Then

$$\sum_{m \in I} p(m) q(m) x^{b-m}$$

is divisible by $(1+x)^{g-h}$ and its quotient has nonnegative coefficients.

5.3. *Proof of Proposition 5.5.*

Proof. By Theorem 3.24 and replacing x_1 by $x_{d,t_{w_Z}}^{-1}$, we have that

$$\sum_{\substack{\tau_{0;1},\tau_{1;1},\ldots,\tau_{n_1-2;1}\\ s_{n_1-1;1}=A_{n_1-1;1}-\varsigma}} \left(\prod_{w=0}^{n_1-2} \begin{bmatrix} A_{w+1;1}-r_1 s_{w;1} \\ \tau_{w;1} \end{bmatrix}\right) x_{d;t_{\omega_Z}}{}^{A_{n_1-1;1}-r_1 s_{n_1-2;1}}$$

is divisible by $(1+x_{d;t_{\omega_Z}}{}^{r_1})^{r_1(A_{n_1-1;1}-\varsigma)-A_{n_1;1}}$ in $\mathbb{Z}[x_{d;t_{\omega_Z}}{}^{\pm 1}]$, and the resulting quotient has nonnegative coefficients. Multiplying the sum with

$$x_{d;t_{\omega_Z}}{}^{r_1\left\lfloor (A_{n_1-1;1}-\varsigma)\frac{A_{n_1-1;1}}{A_{n_1;1}}\right\rfloor - A_{n_1-1;1}}$$

shows that

$$\sum_{\substack{\tau_{0;1},\tau_{1;1},\ldots,\tau_{n_1-2;1}\\ s_{n_1-1;1}=A_{n_1-1;1}-\varsigma}} \left(\prod_{w=0}^{n_1-2} \begin{bmatrix} A_{w+1;1}-r_1 s_{w;1} \\ \tau_{w;1} \end{bmatrix}\right) \times (x_{d;t_{\omega_Z}}{}^{r_1})^{\left\lfloor (A_{n_1-1;1}-\varsigma)\frac{A_{n_1-1;1}}{A_{n_1;1}}\right\rfloor - s_{n_1-2;1}} \tag{56}$$

is also divisible by $(1+x_{d;t_{\omega_Z}}{}^{r_1})^{r_1(A_{n_1-1;1}-\varsigma)-A_{n_1;1}}$, and the resulting quotient has nonnegative coefficients. Moreover, Lemma 5.13 above implies that the exponents in the expression (56) are nonnegative and, since the divisor has constant term 1, this shows that the quotient is a polynomial.

Note that the statement about the divisibility of (56) also holds when we replace $(x^r_{d;t_{\omega_Z}})$ with any other expression X. We can write the second sum of (15) as follows:

$$\sum_{\theta>0} x_{e;t'}^{-\theta} \sum_{\varsigma\geq 0} \sum_{\mathfrak{r}} \lambda_{\mathfrak{r}} \mathfrak{r} \sum_{b,m} q(m)p(m)X^{b-m},$$

where

$$q(m) = \prod_{w=0}^{n_1-2} \begin{bmatrix} A_{w+1;1}-r_1 s_{w;1} \\ \tau_{w;1} \end{bmatrix},$$

$$p(m) = \sum_{j=0}^{\sum_{w=1}^{n_2-3}\tau_{w,2}} d_j \binom{\left\lfloor (A_{n_1-1;1}-\varsigma)\frac{A_{n_1-1;1}}{A_{n_1;1}}\right\rfloor - s_{n_1-2;1}}{j},$$

$$b = \left\lfloor (A_{n_1-1;1}-\varsigma)\frac{A_{n_1-1;1}}{A_{n_1;1}}\right\rfloor,$$

$$m = s_{n_1-2;1},$$

$$X = \left(\frac{\prod_i x_{i;t'}^{[b_{i,e}^{t'}]_+}}{\prod_i x_{i;t'}^{[-b_{i,e}^{t'}]_+}}\right)^{\operatorname{sgn}(2b_{d,e}^{t'}+1)}.$$

Moreover, we can replace the upper bound $\sum_{w=1}^{n_2-3}\tau_{w,2}$ of the sum in $p(m)$ by the larger integer $A_{n_1-2;1}-r_1\varsigma-\theta$ and setting $d_j=0$, whenever $j>\sum_{w=1}^{n_2-3}\tau_{w,2}$. The fact that $\sum_{w=1}^{n_2-3}\tau_{w,2}\le A_{n_1-2;1}-r_1\varsigma-\theta$ follows from equation (44).

Using Lemma 5.14 with $g=r_1(A_{n_1-1;1}-\varsigma)-A_{n_1;1}$ and $h=A_{n_1-2;1}-r_1\varsigma-\theta$, we get that the second sum in the expression in (15) is divisible by

$$\left(1+\left(\frac{\prod_i x_{i;t'}^{[b_{i,e}^{t'}]_+}}{\prod_i x_{i;t'}^{[-b_{i,e}^{t'}]_+}}\right)^{\operatorname{sgn}(2b_{d,e}^{t'}+1)}\right)^{r_1(A_{n_1-1;1}-\varsigma)-A_{n_1;1}-(A_{n_1-2;1}-r_1\varsigma-\theta)}$$

$$\stackrel{\text{Lemma 3.14}}{=}\left(1+\left(\frac{\prod_i x_{i;t'}^{[b_{i,e}^{t'}]_+}}{\prod_i x_{i;t'}^{[-b_{i,e}^{t'}]_+}}\right)^{\operatorname{sgn}(2b_{d,e}^{t'}+1)}\right)^{\theta}, \tag{57}$$

and the resulting quotient has nonnegative coefficients. Finally, dividing (57) by $x_{e;t'}^{\theta}$ and using the fact that

$$\widetilde{\widetilde{x_{e;t}}}=\left(\prod_i x_{i;t'}^{[b_{i,e}^{t'}]_+}+\prod_i x_{i;t'}^{[-b_{i,e}^{t'}]_+}\right)\Big/ x_{e;t'},$$

we see that the second sum in (15) is divisible by $\widetilde{\widetilde{x_{e;t}}}^{\theta}$. This completes the proof of Proposition 5.5(1). Part (2) can be proved in a similar way using Proposition 5.4. □

6. Application to quiver Grassmannians

Let (Q,S) be a quiver with potential, and let M be an indecomposable representation of (Q,S) that is obtained by a mutation sequence starting from a negative simple representation; see [8, §5]. Let $\mathrm{Gr}_{\mathbf{e}}M$ denote the Grassmannian of subrepresentations of M of dimension vector $\mathbf{e}$.

THEOREM 6.1. *The Euler-Poincaré characteristic of* $\mathrm{Gr}_{\mathbf{e}}M$ *is nonnegative.*

Proof. It has been shown in [8] that M corresponds to a cluster variable whose F-polynomial is equal to a sum of monomials whose coefficients are given by the Euler-Poincaré characteristic of $\mathrm{Gr}_{\mathbf{e}}(M)$. Theorem 4.2 implies that these coefficients are nonnegative. □

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