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Finite complex reflection arrangements are $K(\pi, 1)$

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Abstract

Let V be a finite dimensional complex vector space and $W \subseteq \mathrm{GL}(V)$ be a finite complex reflection group. Let V^{reg} be the complement in V of the reflecting hyperplanes. We prove that V^{reg} is a $K(\pi, 1)$ space. This was predicted by a classical conjecture, originally stated by Brieskorn for complexified real reflection groups. The complexified real case follows from a theorem of Deligne and, after contributions by Nakamura and Orlik-Solomon, only six exceptional cases remained open. In addition to solving these six cases, our approach is applicable to most previously known cases, including complexified real groups for which we obtain a new proof, based on new geometric objects. We also address a number of questions about $\pi_1(W \backslash V^{\mathrm{reg}})$, the braid group of W. This includes a description of periodic elements in terms of a braid analog of Springer's theory of regular elements.

Contents

Notation

We save the letter i for indexing purposes and denote by $\sqrt{-1}$ a complex square root of -1 fixed once and for all. If n is a positive integer, we denote by ζ_n the standard n-th root of unity $\exp(2\sqrt{-1}\pi/n)$.

Many objects depend on a complex reflection group W, e.g., the braid group $B(W)$. We often drop the explicit mention of W and write B for $B(W)$. When n is an integer, we denote by B_n the braid group on n strings, together with its standard generating set $\boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_{n-1}$; it is isomorphic to the braid group of $\mathfrak{S}_n$ in its permutation reflection representation (see Section 3). The groups B and B_n appear simultaneously and should not be confused.

Introduction

Let V be a finite dimensional complex vector space and $W \subseteq \mathrm{GL}(V)$ be a complex reflection group. (All reflection groups considered here are assumed to be finite.)

Let V^{reg} be the complement in V of the reflecting hyperplanes. In the case when W is a type A reflection group, Fadell and Neuwirth proved in the early 1960s that V^{reg} is a $K(\pi,1)$ space. (This is an elementary use of fibration exact sequences; see [33].) Brieskorn conjectured in 1971 [17] that the $K(\pi,1)$ property holds when W is a complexified real reflection group. It is not clear who first stated the conjecture in the context of arbitrary complex reflection groups. It may be found, for example, in Orlik-Terao's book:

Conjecture 0.1 ([47, pp. 163, 259]). *The universal cover of V^{reg} is contractible.*

Our main result is a proof of this conjecture. It is clearly sufficient to consider the case when W is irreducible, which we assume from now on. Irreducible complex reflection groups have been classified by Shephard-Todd [53].

The complexified real case (i.e., Brieskorn's conjecture) was quickly settled by Deligne [29]. The rank 2 complex case is trivial. The case of the infinite family $G(de,e,n)$ was solved in 1983 by Nakamura [44]. (Here again, the monomiality of the group allows an efficient use of fibrations.) A few other cases immediately follow from the observation by Orlik-Solomon [46] that certain discriminants of nonreal complex reflection groups are isomorphic to discriminants of complexified real reflection groups.

Combining all previously known results, the conjecture remained open for six exceptional types: G_{24}, G_{27}, G_{29}, G_{31}, G_{33} and G_{34}. We complete the proof of the conjecture by dealing with these cases.

Let $d_1 \leq \cdots \leq d_n$ be the degrees of W. Let $d_1^* \geq \cdots \geq d_n^* = 0$ be the codegrees of W. We say that W is a *duality* group if $d_i + d_i^* = d_n$ for all i. (By analogy with the real case, we then say that d_n is the *Coxeter number* of W, denoted by h.) We say that W is *well generated* if it may be generated by n reflections. Orlik-Solomon observed, by inspecting the classification of Shephard-Todd, that

$$W \text{ is a duality group } \Leftrightarrow W \text{ is well generated.}$$

The first ten sections of this article are devoted to the proof of the following theorem.

THEOREM 0.2. *Let W be a well-generated complex reflection group. The universal cover of V^{reg} is contractible.*

The proof relies on combinatorial and geometric objects that are specific to well-generated groups. It is essentially “case-free,” although a few combinatorial lemmas still require some limited use of the Shephard-Todd classification.

Five of the six open cases are well generated: G_{24}, G_{27}, G_{29}, G_{33} and G_{34}. The theorem also applies to the complexified real case, for which we obtain a new proof, not relying on [29].

The remaining case, G_{31}, is not well generated: it is an irreducible complex reflection group of rank 4 that cannot be generated by less than 5 reflections. Fortunately, we may view it as the centralizer of a 4-regular element (in the sense of Springer [55]) in the group G_{37} (the complexification of the real group of type E_8). By refining the geometric and combinatorial tools introduced in the study of the duality case, one obtains a relative version of Theorem 0.2:

THEOREM 0.3. *Let W be a well-generated complex reflection group. Let d be a Springer regular number, let ζ be a primitive complex d-th root of unity, and let $w \in W$ be a ζ-regular element. Let $V' := \ker(w - \zeta)$, and let W' be the centralizer of w in W, viewed as a complex reflection group acting on V' (see* [55]*). Let ${V'}^{\mathrm{reg}}$ be the associated hyperplane complement. The universal cover of ${V'}^{\mathrm{reg}}$ is contractible.*

In particular, this applies to G_{31} and, based on earlier results, completes the proof of the $K(\pi, 1)$ conjecture.

Note that the case $d = 1$ in Theorem 0.3 is precisely Theorem 0.2. We state the two results separately because it better reflects the organization of the paper.

As by-products of our construction, we obtain new cases of several standard conjectures about the braid group of W, defined by

$$B(W) := \pi_1(W \backslash V^{\mathrm{reg}}).$$

Proving Theorem 0.2: *general strategy.* The general architecture of our proof is borrowed from Deligne's original approach, but the details are quite different. Every construction here is an analogue of a construction from [29] but relies on different combinatorial and geometric objects. As in [29], one studies a certain braid monoid M, whose structure expresses properties of reduced decompositions in W, and one proves that it is a lattice for the divisibility order. (This amounts to saying that the monoid is *Garside.*) As in [29], one uses semi-algebraic geometry to construct an open covering of the universal cover of V^{reg}, with the property that nonempty intersections are contractible. This implies that the universal cover is homotopy equivalent to the nerve of the covering. As in [29], one interprets this nerve as a certain flag complex obtained from M. As in [29], the contractibility of the nerve follows from the lattice property for M. However, our proof does not use the classical braid monoid but rather a dual braid monoid ([3], [6]), whose construction is generalized to all well-generated complex reflection groups. The construction of the open covering is the most problematic step: by contrast with the real case, one cannot rely on the notions of *walls* and *chambers*. The idea here is to work in $W\backslash V^{\mathrm{reg}}$ and to use a generalization of the Lyashko-Looijenga morphism. This morphism allows a description of $W\backslash V^{\mathrm{reg}}$ by means of a ramified covering of a type A reflection orbifold. Classical objects like *walls*, *chambers* and *galleries* can somehow be "pulled back," via the Lyashko-Looijenga morphism, to give semi-algebraic objects related to the dual braid monoid.

Lyashko-Looijenga coverings. The quotient map $\pi : V^{\mathrm{reg}} \twoheadrightarrow W\backslash V^{\mathrm{reg}}$ is a regular covering. Once a system of basic invariants $(f_1, \dots, f_n)$ is chosen, the quotient space $W\backslash V^{\mathrm{reg}}$ identifies with the complement in $\mathbb{C}^n$ of an algebraic hypersurface $\mathcal{H}$, the *discriminant*, of equation $\Delta \in \mathbb{C}[X_1, \dots, X_n]$. If W is an irreducible duality complex reflection group, it is possible to choose $(f_1, \dots, f_n)$ such that

$$\Delta = X_n^n + \alpha_2 X_n^{n-2} + \cdots + \alpha_n,$$

where $\alpha_2, \dots, \alpha_n \in \mathbb{C}[X_1, \dots, X_{n-1}]$. Let $Y := \operatorname{Spec} \mathbb{C}[X_1, \dots, X_{n-1}]$, together with the natural map $p : W\backslash V^{\mathrm{reg}} \to Y$. We have an identification $W\backslash V \simeq \mathbb{C}^n \xrightarrow{\sim} Y \times \mathbb{C}$ sending the orbit $\overline{v}$ of $v \in V$ to $(p(\overline{v}), f_n(v))$. The fiber of p over $y \in Y$ is a line L_y that intersects $\mathcal{H}$ at n points (counted with multiplicities). Generically, the n points are distinct. Let $\mathcal{K}$ be the *bifurcation locus*, i.e., the algebraic hypersurface of Y consisting of points y such that the intersection has cardinality $< n$. Classical results from invariant theory of complex reflection groups make it possible (and very easy) to generalize a construction by Looijenga and Lyashko: the map LL (for "Lyashko-Looijenga") sending $y \in Y - \mathcal{K}$ to the subset $\{x_1, \dots, x_n\} \subseteq \mathbb{C}$ such that $p^{-1}(y) \cap \mathcal{H} = \{(y, x_1), \dots, (y, x_n)\}$ is a regular covering of degree $n!h^n/|W|$ of the (centered) configuration space of

n points in $\mathbb{C}$. In particular, $Y - \mathcal{K}$ is a $K(\pi, 1)$. This observation, which is apparently new in the nonreal case, already allows a refinement of our earlier results ([2], [8]) on presentations for the braid group of W.

The dual braid monoid. When W is complexified real, a *dual braid monoid* was constructed in [3] (generalising the construction of Birman-Ko-Lee [11]; similar partial results were independently obtained by Brady and Watt [14]). The construction was later generalized in [6] to the complex reflection group $G(e, e, n)$. Let R be the set of all reflections in a well-generated group W. The idea is that the pair (W, R) has some "Coxeter-like" features. Instead of looking at relations of the type

$$\underbrace{sts\cdots}_{m_{s,t}} = \underbrace{tst\cdots}_{m_{s,t}},$$

one considers relations of the type

$$st = tu,$$

where $s, t, u \in R$. Let $\mathcal{S}$ be the set of all relations of this type holding in W. In general, $B(W) \not\simeq \langle R \,|\, \mathcal{S} \rangle$, but it is possible to find natural subsets $R_c \subseteq R$ and $\mathcal{S}_c \subseteq \mathcal{S}$ such that $B(W) \simeq \langle R_c \,|\, \mathcal{S}_c \rangle$. (If W is complexified real, $R_c = R$.) The elements of $\mathcal{S}_c$ are called *dual braid relations*. The choices of R_c and $\mathcal{S}_c$ are natural once a *Coxeter element* c has been chosen. (The notion of *Coxeter element* generalizes to the nonreal well-generated groups in terms of Springer's theory of regular elements.) Since the relations are positive, one may view the presentation as a monoid presentation, defining a monoid $M(W)$. The crucial property of this monoid is that it is a lattice for the divisibility order or, more precisely, a *Garside monoid*. Following Deligne, Bestvina, T. Brady and Charney-Meier-Whittlesey, the Garside structure provides a convenient simplicial Eilenberg-McLane $K(B(W), 1)$ space ([29], [10], [13], [23]). The earlier results on the dual braid monoid are improved here in two directions:

- The construction is generalized to the few exceptional cases (G_{24}, G_{27}, G_{29}, G_{33} and G_{34}) not covered by [3] and [6].
- A new geometric interpretation is given, via the Lyashko-Looijenga covering. This interpretation is different from the one given in [3, §4].

The second improvement is the most important. It relies (so far) on a counting argument, following and extending a property that, for the complexified real case, was conjectured by Looijenga and proved in a letter from Deligne to Looijenga [30].

Tunnels. The classical theory of real reflection groups combines a "combinatorial" theory (Coxeter systems) and a "geometric" theory (expressed in the language, invented by Tits, of walls, chambers, galleries, buildings ...).

We expect the dual braid monoid approach to eventually provide effective substitutes for much of this classical theory. A first step in this direction is the notion of a *tunnel*, which is a rudimentary geometric object replacing the classical notion of *minimal gallery between two chambers*. An important difference with the classical geometric language is that tunnels are naturally visualized in $W\backslash V$ (instead of V). A tunnel T is a path in $W\backslash V^{\mathrm{reg}}$ drawn inside a single line L_y (for some $y \in Y$) and with constant imaginary part. It represents an element b_T of the dual braid monoid M. An element of M is *simple* if it is represented by a tunnel. This notion coincides with the notion of a simple element associated with the Garside structure. In the classical approach, for any chamber $\mathcal{C}$, there are as many equivalence classes of minimal galleries starting at $\mathcal{C}$ as simple elements. (This number is $|W|$.) Here the situation is different: in a given L_y, not all simple elements are represented. The simple elements represented in different L_y's may be compared thanks to a huge "fat basepoint" $\mathcal{U}$ that is both dense in $W\backslash V^{\mathrm{reg}}$ and contractible.

Proving Theorem 0.3. The strategy is the same as for Theorem 0.2. With the notation of the theorem, the quotient space $W'\backslash V'^{\mathrm{reg}}$ may be identified with $(W\backslash V^{\mathrm{reg}})^{\mu_d}$ for the natural action of the cyclic group μ_d. This action induces an automorphism of $B(W)$ that, unfortunately, does not preserve the dual braid monoid. However, it is possible to replace $B(W)$ by a sort of categorical barycentric subdivision, its *d-divided Garside category* M_d, on which μ_d acts by *diagram automorphisms*. This construction is explained in my separate article [5] and recalled in Appendix B. The fixed subcategory $M_d^{\mu_d}$ is again a Garside category. It should be thought of a *dual braid category for* $B(W')$ and gives rise to a natural simplicial space, whose realization is an Eilenberg-MacLane space. As before, one shows that $(W\backslash V^{\mathrm{reg}})^{\mu_d}$ is homotopy equivalent to this simplicial model by studying the nerve of a certain open covering of a certain model of the universal cover of $(W\backslash V^{\mathrm{reg}})^{\mu_d}$, very similar to the one used for $W\backslash V^{\mathrm{reg}}$ (except that one has to replace the contractible "basepoint" $\mathcal{U}$ by a family of nonoverlapping contractible "basepoints," one for each object of $M_d^{\mu_d}$). This involves replacing tunnels by a suitable notion of *circular tunnels*. Section 11 focuses the geometric aspects of the proof of Theorem 0.3; it is probably fair to say that the true explanation lies in the properties of M_d and in the general theorems about periodic elements in Garside groupoids that are explained in [5].

By-products.

THEOREM 0.4. *Braid groups of well-generated complex reflection groups are Garside groups.*

In the situation of Theorem 0.3, we prove that the braid group $B(W')$ is a weak Garside group, which is almost as good.

In particular, $B(W)$ is torsion-free, admits nice solutions to the word and conjugacy problems, is biautomatic, admits a finite $K(\pi, 1)$ (our construction provides an explicit one), and much more; see [26] for a quite complete reference. None of this was known for the six exceptional groups mentioned above.

THEOREM 0.5 (Theorem 12.8). *The center of the braid group of an irreducible complex reflection group is cyclic.*

Again, the cases of G_{24}, G_{27}, G_{29}, G_{31}, G_{33} and G_{34} are new. This settles a conjecture by Broué-Malle-Rouquier [20].

For $B(G_{29})$, $B(G_{31})$, $B(G_{33})$ and $B(G_{34})$, no presentations were known until now, although some conjectures made in [8] were supported by strong evidence.

THEOREM 0.6. *The conjectural presentations for $B(G_{29})$, $B(G_{31})$, $B(G_{33})$ and $B(G_{34})$ given in* [8] *are correct.*

Combined with [7] and [8], this completes the longstanding task of finding presentations for all generalized braid groups associated with finite complex reflection groups. Theorem 0.6 is much easier than the previously mentioned results and only relies on a minor improvement over [2] and [8]. However, the material presented here allows for a more conceptual proof.

Periodic elements in braid groups. In connection with their work on Deligne-Lusztig varieties (see [18] for more details), Broué-Michel predicted the existence of an analog for braid groups of Springer's theory of regular elements. This amounts to a conjectural description of periodic elements (elements with a central power) and their centralizers. When W is the symmetric group, periodic elements in $B(W)$ may be understood thanks to Kerékjártó's theorem on periodic homeomorphisms of the disk. In the more general setting of spherical type Artin groups, finding a simple description of periodic elements was an open question. We are able to solve these problems when W is well generated: Theorem 12.4 contains a complete description of the roots of the generator of the center of the pure braid group $P(W)$ and of their centralizers.

As for Theorem 0.3, the main conceptual ingredient towards the proof of Theorem 12.4 is a general property of Garside categories, explained in our separate paper [5]. What is done here is the minor step consisting of reinterpreting the general *Kerékjártó theorem for Garside categories* from [5] in terms of the S^1-structure on the regular orbit space $W\backslash V^{\mathrm{reg}}$.

Noncrossing partitions. A combinatorial by-product of our approach is a general construction of *generalized noncrossing partitions*, associated to each type of well-generated complex reflection groups.

In the classical cases A_n, B_n, D_n and, more generally, $G(e, e, n)$, the structure of $M(W)$ is understood in terms of suitable notions of *noncrossing* *partitions* ([1], [6], [7]). The dual braid monoid of an irreducible well-generated complex reflection W gives rise to a lattice of *generalized noncrossing partitions*, whose cardinal is the *generalized Catalan number*

$$\mathrm{Cat}(W) := \prod_{i=1}^{n} \frac{d_i + d_n}{d_i}.$$

(The term "partition" should not be taken too seriously: except for the classical types, lattice elements do not have natural interpretations as actual set-theoretic partitions.) It is likely that this combinatorial object has some representation-theoretic interpretation. In the "badly-generated" case, $\mathrm{Cat}(W)$ may fail to be an integer, and the natural substitute for $\mathrm{NCP}(W)$ is the graph of simple elements of the dual braid category.

1. Complex reflection groups, discriminants, braid groups

Let V be a vector space of finite dimension n. A *reflection group* in $\mathrm{GL}(V)$ is a subgroup W generated by (*generalized*) *reflections*, i.e., elements whose fixed subspace is a hyperplane. When the base field of V is $\mathbb{C}$, we say that W is a *complex reflection group.* We are only interested in finite reflection groups and will always assume finiteness, unless otherwise specified.

Let $W \subseteq \mathrm{GL}(V)$ be a complex reflection group. A *system of basic invariants* for W is an n-tuple $f = (f_1, \ldots, f_n)$ of homogeneous generators of $\mathcal{O}_V^W$, the algebra of W-invariant polynomial functions on V. A classical theorem of Shephard-Todd [53] asserts that such tuples exist and that they consist of algebraically independent terms. Set $d_i := \deg f_i$; these numbers are the *degrees* of W. Up to reordering, we may assume that $d_1 \leq d_2 \leq \cdots \leq d_n$. The sequence $(d_1, \ldots, d_n)$ is then independent of the choice of f.

Choosing a system of basic invariants f amounts to choosing a graded algebra isomorphism $\mathcal{O}_V^W \simeq \mathbb{C}[X_1, \ldots, X_n]$, $f_i \mapsto X_i$, where the indeterminate X_i is declared homogeneous with degree d_i. Geometrically, this isomorphism identifies the categorical quotient $W \backslash V$ with the affine space $\mathbb{C}^n$.

Further features of the invariant theory of complex reflection groups involve invariant vector fields and invariant differential forms on V.

THEOREM 1.1 ([47, Lemma 6.48]). *The $\mathcal{O}_V^W$-modules $(\mathcal{O}_V \otimes V)^W$ and $(\mathcal{O}_V \otimes V^*)^W$ are free of rank n.*

If $f = (f_1, \ldots, f_n)$ is a system of basic invariants, $df := (df_1, \ldots, df_n)$ is a $\mathcal{O}_V^W$-basis for $(\mathcal{O}_V \otimes V^*)^W$. Being homogeneous, the module $(\mathcal{O}_V \otimes V)^W$ admits a homogeneous basis.

Definition 1.2. A system of basic derivations for W is a homogeneous $\mathcal{O}_V^W$-basis $\xi = (\xi_1, \ldots, \xi_n)$ of $(\mathcal{O}_V \otimes V)^W$, with $\deg(\xi_1) \geq \deg(\xi_2) \geq \cdots \geq \deg(\xi_n)$.

The sequence $(d_1^*, \ldots, d_n^*) := (\deg(\xi_1), \ldots, \deg(\xi_n))$ is the sequence of *codegrees* of W. (It does not depend on the choice of ξ.)

Note that, as in [2], we label codegrees in decreasing order, which is slightly unusual. When W is a complexified real reflection group, we have $V \simeq V^*$ as W-modules; thus $d_i^* = d_{n-i+1} - 2$ for all i. This relation is specific to the real situation and is not relevant here.

The Euler vector field on V is invariant and of degree 0. Thus $d_n^* = 0$.

Invariant vector fields define vector fields on the quotient variety. Let f be a system of basic invariants and ξ be a system of basic derivations. For $j \in \{1, \ldots, n\}$, the vector field ξ_j defines a vector field $\overline{\xi_j}$ on $W \backslash V$. Since $\frac{\partial}{\partial f_1}, \ldots, \frac{\partial}{\partial f_n}$ is a $\mathcal{O}_V^W$-basis of the module of polynomial vector fields on $W \backslash V$, we have

$$\overline{\xi_j} = \sum_{i=1}^{n} m_{i,j} \frac{\partial}{\partial f_i},$$

where the $m_{i,j}$ are uniquely defined elements of $\mathcal{O}_V^W$.

Definition 1.3. The discriminant matrix of W (with respect to f and ξ) is $M := (m_{i,j})_{i,j}$.

By weighted homogeneity, one has

LEMMA 1.4. *For all i, j, $\mathrm{wt}(m_{i,j}) = d_i + d_j^*$.*

The vector space V decomposes as a direct sum $\bigoplus_i V_i$ of irreducible representations of W. Denote by W_i the irreducible reflection group in $\mathrm{GL}(V_i)$ generated by (the restriction of) the reflections in W whose hyperplanes contain $\bigoplus_{j \neq i} V_j$. We have $W \simeq \prod_i W_i$. Viewing W and the W_i's as *reflection groups*, i.e., groups endowed with a reflection representation, it is natural to actually write $W = \bigoplus_i W_i$.

We denote by $\mathcal{A}$ the arrangement of W, i.e., the set of reflecting hyperplanes of reflections in W. We set

$$V^{\mathrm{reg}} := V - \bigcup_{H \in \mathcal{A}} H.$$

Denote by p the quotient map $V \twoheadrightarrow W \backslash V$. Choose a basepoint $v_0 \in V^{\mathrm{reg}}$.

Definition 1.5 ([20]). The *braid group* of W is $B(W) := \pi_1(W \backslash V^{\mathrm{reg}}, p(v_0))$.

Later on, when working with well-generated reflection groups, we will slightly upgrade this definition by replacing the basepoint by a convenient contractible subspace of $W \backslash V^{\mathrm{reg}}$ (see Definition 6.4).

To write explicit equations, one chooses a system of basic invariants f. The *discriminant* $\Delta(W, f) \in \mathbb{C}[X_1, \ldots, X_n]$ is the reduced equation of $p(\bigcup_{H \in \mathcal{A}} H)$ via the identification $\mathcal{O}_V^W \simeq \mathbb{C}[X_1, \ldots, X_n]$.

One easily sees that $B(W) \simeq \prod_i B(W_i)$. More generally, all objects studied here behave "semi-simply," and we may restrict our attention to irreducible complex reflection groups.

Since $B(W)$ is the fundamental group of the complement of an algebraic hypersurface, it is generated by particular elements called *generators-of-the-monodromy* or *meridians*. (See, for example, [20] or [2].) They map to reflections under the natural epimorphism $B(W) \to W$. The diagrams given in [20] symbolize presentations whose generators are generators-of-the-monodromy (except for the six exceptional types for which no presentation was known).

Definition 1.6. The generators-of-the-monodromy of $B(W)$ are called *braid reflections*.

This terminology was suggested by Broué. It is actually tempting to simply call them *reflections*: since they generate $B(W)$, the braid group appears to be some sort of (infinite) "reflection group." This guiding intuition is quite effective.

Another natural feature of $B(W)$ is the existence of a natural length function, which is the unique group morphism

$$l : B(W) \to \mathbb{Z}$$

such that, for all braid reflection $s \in B(W)$, $l(s) = 1$.

Consider the intersection lattice $\mathcal{L}(\mathcal{A}) := \{\bigcap_{H\in A} H | A \subseteq \mathcal{A}\}$. Elements of $\mathcal{L}(\mathcal{A})$ are called *flats*. It is standard to endow $\mathcal{L}(\mathcal{A})$ with the reversed-inclusion partial ordering:

$$\forall L, L' \in \mathcal{L}(\mathcal{A}), L \leq L' :\Leftrightarrow L \supseteq L'.$$

For $L \in \mathcal{L}(\mathcal{A})$, we denote by L^0 the complement in L of the flats strictly included in L. The $(L^0)_{L\in\mathcal{L}(\mathcal{A})}$ form a stratification $\mathcal{S}$ of V. We consider the partial ordering on $\mathcal{S}$ defined by $L^0 \leq L'^0 :\Leftrightarrow L \leq L'$. This is a degeneracy relation:

$$\forall L \in \mathcal{L}(\mathcal{A}), \overline{L^0} = L = \bigcup_{S\in\mathcal{S}, L^0 \leq S} S.$$

Since W acts on $\mathcal{A}$, it acts on $\mathcal{L}(\mathcal{A})$ and we obtain a quotient stratification $\overline{\mathcal{S}}$ of $W\backslash V$ called *discriminant stratification*.

PROPOSITION 1.7 ([47, Cor. 6.114]). *Let $v \in V$. The tangent space to the stratum of $\mathcal{S}$ containing v is spanned by the vectors $\xi_1(v), \ldots, \xi_n(v)$. The tangent space to the stratum of $\overline{\mathcal{S}}$ containing $\overline{v}$ is spanned by the vectors $\overline{\xi}_1(v), \ldots, \overline{\xi}_n(v)$.*

Another chapter of the classical invariant theory of complex reflection groups is Springer's theory of regular elements:

Definition 1.8. Let $W \subseteq \mathrm{GL}(V)$ be a complex reflection group. Let ζ be a complex root of unity. An element $w \in W$ is *ζ-regular* if $\ker(w-\zeta) \cap V^{\mathrm{reg}} \neq \varnothing$. The eigenvalue ζ is then called a *regular eigenvalue* for W and its order called a *regular number* for W.

Note that, since W acts freely on V^{reg}, a ζ-regular element must have the same order as ζ. The regularity of ζ only depends on its order d since the k-th power of a ζ-regular element is ζ^k-regular.

The following theorem compiles some of the main features.

THEOREM 1.9. *Let $W \subseteq \mathrm{GL}(V)$ an irreducible complex reflection group, with degrees $d_1, \ldots, d_n$ and codegrees $d_1^*, \ldots, d_n^*$.*

(1) *Let d be a positive integer. Set*

$$A(d) := \{i = 1, \ldots, n \mid d \mid d_i\} \quad \textit{and} \quad B(d) := \{i = 1, \ldots, n \mid d \mid d_i^*\}.$$

Then $|A(d)| \leq |B(d)|$, and d is regular if and only if $|A(d)| = |B(d)|$.

(2) *Let w be a ζ-regular element of order d. Let $V' := \ker(w - \zeta)$. The centralizer $W' := C_W(w)$, viewed in its natural representation in $\mathrm{GL}(V')$, is a complex reflection group with degrees $(d_i)_{i \in A(d)}$ and codegrees $(d_i^*)_{i \in B(d)}$.*

(3) *Let w be ζ-regular element of order d. Then $W' \backslash V' \simeq (W \backslash V)^{\mu_d}$ and $W' \backslash V'^{\mathrm{reg}} \simeq (W \backslash V^{\mathrm{reg}})^{\mu_d}$ (where the μ_d-action is the quotient action of the scalar multiplication on V).*

Statement (2) was proved by Springer in his seminal paper [55] (except for the part about codegrees, first observed by Denef-Loeser [31] and then conceptually proved by Broué [19, 5.19 (4)]). Statement (3) was proved independently by Lehrer and Denef-Loeser. Statement (1) was initially observed by Lehrer-Springer on a case-by-case basis; a conceptual proof was given by Lehrer-Michel [40].

Example 1.10. Let $W := G_{37} = W(E_8)$. It is a well-generated complex reflection group in $\mathrm{GL}_8(\mathbb{C})$, whose degrees are

$$2, 8, 12, 14, 18, 20, 24, 30.$$

By duality, the codegrees are

$$0, 6, 10, 12, 16, 18, 22, 28.$$

The integer 4 is regular. The centralizer W' is a complex reflection group of type G_{31} . Its degrees are

$$8, 12, 20, 24,$$

while the codegrees are

$$0, 12, 16, 28.$$

It is not a duality group, and it is not well generated (see next section).

2. Well-generated complex reflection groups

Irreducible complex reflection groups were classified fifty years ago by Shephard and Todd [53]. There is an infinite family $G(de, e, n)$, where d, e, n are positive integers, and 34 exceptions $G_4, \dots, G_{37}$. Let us distinguish three subclasses of complex reflection groups:

- (*complexified*) *real reflection groups*, obtained by scalar extension from reflection groups of real vector spaces;
- 2-*reflection groups*, generated by reflections of order 2;
- *well-generated reflection groups*, complex reflection groups $W \subseteq \mathrm{GL}(V)$ that can be generated by $\dim_{\mathbb{C}} V/V^W$ reflections, where $V^W := \{v \in V | \forall w \in W, wv = v\}$.

Real reflection groups are both 2-reflection groups and well generated. For nonreal groups, any combination of the other two properties may hold.

As far as the $K(\pi, 1)$ conjecture and properties of braid groups are concerned, it is enough to restrict one's attention to 2-reflection groups:

Definition 2.1. Let $W \subseteq \mathrm{GL}(V)$ and $W' \subseteq \mathrm{GL}(V')$ be complex reflection groups. We say that W and W' are *isodiscriminantal* if one may find systems of basic invariants f (resp. f') for W (resp. W') such that $\Delta(W, f) = \Delta(W', f')$.

When this happens, $W \backslash V^{\mathrm{reg}} \simeq W' \backslash V'^{\mathrm{reg}}$ and $B(W) \simeq B(W')$.

THEOREM 2.2. *Any complex reflection group is isodiscriminantal to a complex* 2*-reflection group.*

Proof. This may be observed on the classification and was certainly known to experts. In [20], Broué-Malle-Rouquier associate to each complex reflection group a diagram symbolizing a presentation by generators and relations; they notice that the degrees and codegrees are invariants of the underlying braid diagram (removing torsion relations from the presentation). Actually, the braid diagrams are invariants of isodiscriminantality classes. (Compare [20] with [46].) So the theorem can be rephrased as: for any diagram in the tables of Broué-Malle-Rouquier, the diagram with the same braid relations but where all torsion relations have order 2 is also in the tables; this is an easy check. □

Remark 2.3. The work of Couwenberg-Heckman-Looijenga [25] could possibly be adapted to provide a direct argument. Let us sketch a conjectural way to proceed. All references and notation are from [25]. Assume that W is not a 2-reflection group. For each $H \in \mathcal{A}$, let e_H be the order of the pointwise stabilizer W_H and set $\kappa_H := 1 - e_H/2$. Consider the Dunkl connection ∇ with connection form $\sum_{H \in \mathcal{A}} \omega_H \otimes \kappa_H \pi_H$, as in Example 2.5. Since $e_H \geq 2$, we have $\kappa_H \leq 0$. In particular, $\kappa_0 = 1/n \sum_{H \in \mathcal{A}} \kappa_H \leq 0$ (Lemma 2.13) and we are in the situation of loc. cit., Section 5. In many cases, this suffices to conclude.

The problem is that, even though at least some e_H have to be > 2, it is possible that $\mathcal{A}$ contains several orbits, some of them with $e_H = 2$. To handle this, one has to enlarge the "Schwarz symmetry group" of loc. cit., Section 4.

The importance of the distinction between well-generated and "badly-generated" groups was first pointed out by Orlik-Solomon, who observed in [45] a coincidence with invariant-theoretical aspects. Their observations may be refined and completed as follows.

THEOREM 2.4. *Let W be an irreducible complex reflection group. The following assertions are equivalent*:

(i) *W is well generated.*
(ii) *For all $i \in \{1, \dots, n\}$, $d_i + d_i^* = d_n$.*
(iii) *For all $i \in \{1, \dots, n\}$, $d_i + d_i^* \leq d_n$.*
(iv) *For any system of basic invariants f, there exists a system of basic derivations ξ such that the discriminant matrix decomposes as $M = M_0 + X_n M_1$, where M_0, M_1 are matrices with coefficients in $\mathbb{C}[X_1, \dots, X_{n-1}]$ and M_1 is lower triangular with nonzero scalars on the diagonal.*
(v) *For any system of basic invariants f, we have $\frac{\partial^n \Delta(W,f)}{(\partial X_n)^n} \in \mathbb{C}^\times$. (That is, $\Delta(W,f)$, viewed as a polynomial in X_n with coefficients in $\mathbb{C}[X_1, \dots, X_{n-1}]$, is monic of degree n.)*

The matrix M_1 from assertion (iv) is an analogue of the matrix J^* from [52, p. 10]. Assertion (iv) itself generalizes the nondegeneracy argument for J^*, which is an important piece of the construction of Saito's "flat structure."

Proof. (i) $\Rightarrow$ (ii) was observed in [45] inspecting the classification. We still have no good explanation.

(ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (v). Let $h := d_n$. A first step is to observe that, under assumption (iii), h is a regular number. Indeed, condition (iii) implies, for any $i = 1, \dots, n-1$, that $0 < d_i < h$ and $0 < d_i^* < h$, thus that exactly one degree (d_n) and one codegree (d_n^*) are multiples of h, thus h is regular (Theorem 1.9(1)). Since h is regular and divides only one degree, we may use [2, Lemma 1.6(ii)] to obtain assertion (v): the discriminant is X_n-monic, and by weighted-homogeneity it must be of degree n.

(iii) $\Rightarrow$ (iv) is a refinement of the previous discussion. Each entry $m_{i,j}$ of the matrix M is weighted-homogeneous of weight $d_i + d_j^* \leq d_n + d_1^* = 2h - d_1 < 2h$; since X_n has weight h, $\deg_{X_n} m_{i,j} \leq 1$. This explains the decomposition $M = M_0 + X_n M_1$, where M_0 and M_1 have coefficients in $\mathcal{O}_Y$. If $i < j$ and $d_i < d_j$, then $d_i + d_j^* < d_j + d_j^* = h$; thus $\deg_{X_n} m_{i,j} = 0$. The matrix M_1 is *almost lower triangular*, i.e., lower triangular except that there could be nonzero terms above the diagonal in square diagonal blocks corresponding to successive equal degrees. (Successive degrees may indeed be equal, as in the example of type D_4, where the degrees are $2,4,4,6$.)

Let $i_0 < j_0$ such that $d_{i_0} = d_{i_0+1} = \cdots = d_{j_0}$. (Looking at the classification, one may observe that this forces $j = i+1$; this observation is not used in the argument below.) For all $i,j \in \{i_0,\ldots,j_0\}$, we have $d_i + d_j^* = h$. By weighted homogeneity, this implies that the corresponding square block of M_1 consists of scalars. The basic derivations $\xi_{i_0},\ldots,\xi_{j_0}$ all have the same degree; thus one is allowed to perform Gaussian elimination on the corresponding columns of M. Thus, up to replacing ξ by another system of basic derivations ξ', we may assume that M_1 is lower triangular.

The diagonal terms of M_1 must be scalars, once again by weighted homogeneity. Assuming (iii), we already know that (v) holds. The determinant of M is $\Delta(X,f)$; (v) implies that the coefficient of X_n^n is nonzero. This coefficient is the product of the diagonal terms of M_1. We have proved (iv).

(iv) $\Rightarrow$ (v) is trivial.

(v) $\Rightarrow$ (i) follows from the main result in [2]. □

The following notion was considered in [50] for real reflection groups.

Definition 2.5. A system of basic derivations is *flat* (with respect to f) if the discriminant matrix may be written $M = M_0 + X_n \,\mathrm{Id}$, where M_0 is a matrix with coefficients in $\mathbb{C}[X_1,\ldots,X_{n-1}]$.

COROLLARY 2.6. *Let W be a well-generated irreducible reflection group, together with a system of basic invariants f. There exists a flat system of basic derivations.*

Proof. Let ξ be any system of basic derivations. Write $M = M_0 + X_n M_1$, as in characterization (iv) from the above theorem. The matrix M_1 is invertible in $\mathrm{GL}_n(\mathbb{C}[X_1,\ldots,X_{n-1}])$. The matrix $M_1^{-1}M = M_1^{-1}M_0 + X_n \,\mathrm{Id}$ represents a flat system of basic derivations. (Weighted homogeneity is preserved by the Gaussian elimination procedure.) □

Contrary to what happens with real reflection groups, we may not use the identification $V \simeq V^*$ to obtain a “flat system of basic invariants.”

Irreducible groups that are not well generated may always be generated by $\dim V + 1$ reflections. This fact has been observed long ago, by case-by-case inspection, but no general argument is known. In some sense, these badly-generated groups should be thought of as affine groups. The simplest example of a non-well-generated group is the group $G(4,2,2)$, generated by

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Among high-dimensional exceptional complex reflection groups, only G_{31} is badly generated (see Example 1.10).

2-reflection groups		Other well-generated groups
Real series	$G(1,1,n)$ (A_{n-1})	G_4, G_8, G_{16}, G_{25}, G_{32}
	$G(2,1,n)$ (B_{n-1})	$G(d,1,n)$, G_5, G_{10}, G_{18}, G_{26}
Real/complex series	$G(e,e,n)$ (contains D_n and $I_2(e)$)	G_6, G_9, G_{14}, G_{17}, G_{20}, G_{21}
Real exceptions	G_{23} (H_3), G_{28} (F_4), G_{30} (F_4) G_{35} (E_6), G_{36} (E_7), G_{37} (E_8)	
Complex exceptions	G_{24}, G_{27}, G_{29}, G_{33}, G_{34}	

Table 1. Irreducible well-generated complex reflection groups.

In the sequel, several arguments are case-by-case. The list of irreducible well-generated complex reflection groups is given in Table 1. The actual number of cases to consider depends on the type of result:

- Thanks to Theorem 2.2, for any statement about $W\backslash V^{\mathrm{reg}}$ and its topology, one may restrict one's attention to groups generated by involutive reflections, which are listed in the first column of the table.
- Some results involve the actual structure of W (e.g., Proposition 7.6), and groups with higher order reflections have to be considered. These cases are listed in the second column of the table, on the same line as the corresponding 2-reflection groups (see Theorem 2.2).

LEMMA 2.7. *Let* $W \subseteq \mathrm{GL}(V)$ *be a well-generated complex reflection group. Let* $v \in V$. *Let*

$$V_v := \bigcap_{H\in\mathcal{A},\, v\in H} H,$$

$$W_v := \{w \in W | wv = v\}.$$

Then W_v *may be generated by* $\dim_{\mathbb{C}} V/V_v$ *reflections. In particular*, W_v *is again a well-generated complex reflection group.*

Proof. The fact that W_v is again a complex reflection group is a classical theorem due to Steinberg. The fact that W_v is again well generated is easy to check on the classification. (It follows, for example, from Broué-Malle-Rouquier's observation that their diagrams in [20] provide generating systems for representatives of all conjugacy classes of W_v; when W is real, no case-by-case is needed, since W_v is again real and thus well generated.) $\square$

3. Symmetric groups, configurations spaces and classical braid groups

This section introduces some basic terminology and notation. Everything here is classical and elementary.

Let n be a positive integer. The symmetric group $\mathfrak{S}_n$ may be viewed as a reflection group, acting on $\mathbb{C}^n$ by permuting the canonical basis. This representation is not irreducible. Let H be the hyperplane of equation $\sum_{i=1}^n X_i = 0$ (where $X_1, \dots, X_n$ is the dual canonical basis of $\mathbb{C}^n$). It is preserved by $\mathfrak{S}_n$, which acts on it as an irreducible complex reflection group.

We have $\mathbb{C}[X_1, \dots, X_n]^{\mathfrak{S}_n} = \mathbb{C}[\sigma_1, \dots, \sigma_n]$ and $\mathcal{O}_H^{\mathfrak{S}_n} = \mathbb{C}[\sigma_1, \dots, \sigma_n]/\sigma_1$, where $\sigma_1, \dots, \sigma_n$ are the elementary symmetric functions on $X_1, \dots, X_n$. Set

$$\overline{E}_n := \mathfrak{S}_n \backslash \mathbb{C}^n = \operatorname{Spec} \mathbb{C}[\sigma_1, \dots, \sigma_n]$$

and

$$E_n := \mathfrak{S}_n \backslash H = \operatorname{Spec} \mathbb{C}[\sigma_1, \dots, \sigma_n]/\sigma_1.$$

These spaces have more convenient descriptions in terms of multisets.

Recall that a *multiset* is a set S (the *support* of the multiset) together with a map $m : S \to \mathbb{Z}_{\geq 1}$ (the *multiplicity*). The *cardinality* of such a multiset is $\sum_{s \in S} m(s)$. (It lies in $\mathbb{Z}_{\geq 0} \cup \{\infty\}$.) If (S, m) and (S', m') are two multisets and if S, S' are subsets of a common ambient set, then we may define a *multiset* (*disjoint*) *union* $(S, m) \cup (S', m')$, whose support is $S \cup S'$ and whose multiplicity is $m + m'$ (where m, resp. m', is extended by 0 outside S, resp S'). If $(s_1, \dots, s_n)$ is a sequence of elements of a given set S, we use the notation $\boldsymbol{\{} s_1, \dots, s_n \boldsymbol{\}}$ (with brackets in bold font) to refer to the multiset $\bigcup_{i=1}^n (\{s_i\}, 1)$, i.e., the multiset consisting of the s_i's "taken with multiplicities."

Let $(x_1, \dots, x_n) \in \mathbb{C}^n$. The associated $\mathfrak{S}_n$-orbit is uniquely determined by $\boldsymbol{\{} x_1, \dots, x_n \boldsymbol{\}}$. This identifies E_n' with the set of multisets of cardinality n with support in $\mathbb{C}$. (Such multisets are called *configurations of n points in* $\mathbb{C}$.) The subvariety E_n, defined by $\sigma_1 = 0$, consists of *centered* configurations, i.e., configurations $\boldsymbol{\{} x_1, \dots, x_n \boldsymbol{\}}$ satisfying $\sum_{i=1}^n x_i = 0$. The natural inclusion $E_n \subseteq E_n'$ admits the retraction ρ defined by

$$\rho(\boldsymbol{\{} x_1, \dots, x_n \boldsymbol{\}}) := \left\{ x_1 - \sum_{i=1}^n x_i/n, \dots, x_n - \sum_{i=1}^n x_i/n \right\}.$$

Algebraically, this corresponds to the identification of $\mathbb{C}[\sigma_1, \dots, \sigma_n]/\sigma_1$ with $\mathbb{C}[\sigma_2, \dots, \sigma_n]$.

We find it convenient to use configurations in E_n' to represent elements of E_n, implicitly working through ρ. For example, in the proof of Proposition 9.3, it makes sense to describe a deformation retraction of a subspace of E_n to a point in terms of arbitrary configurations because the construction, which only implies the relative values of the x_i's, is compatible with ρ. We adopt this viewpoint from now on, without further justifications. (Compatibility will always be obvious.)

Consider the lexicographic total ordering of $\mathbb{C}$: if $z, z' \in \mathbb{C}$, we set

$$z \leq z' :\Leftrightarrow \begin{cases} \mathrm{re}(z) < \mathrm{re}(z') & \text{or} \\ \mathrm{re}(z) = \mathrm{re}(z') \text{ and } \mathrm{im}(z) \leq \mathrm{im}(z'). \end{cases}$$

Definition 3.1. The *ordered support* of an element of E_n is the unique sequence $(x_1, \ldots, x_k)$ such that the set $\{x_1, \ldots, x_k\}$ is the support and $x_1 < x_2 < \cdots < x_k$.

We may uniquely represent an element of E_n by its ordered support $(x_1, \ldots, x_k)$ and the sequence $(n_1, \ldots, n_k)$ of multiplicities at $x_1, \ldots, x_k$.

The regular orbit space $E_n^{\mathrm{reg}} := \mathfrak{S}_n \backslash H^{\mathrm{reg}}$ consists of those multisets whose support has cardinality n (or, equivalently, whose multiplicity is constantly equal to 1). More generally, the strata of the discriminant stratification of E_n are indexed by partitions of n: the stratum S_λ associated with a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, where the λ_j's are integers with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$ and $\sum_{j=1}^k \lambda_j = n$, consists of configurations whose supports have cardinality k and whose multiplicity functions take the values $\lambda_1, \ldots, \lambda_k$ (with multiplicities).

The braid group B_n associated with $\mathfrak{S}_n$ is the usual braid group on n strings. We need to be more precise about our choice of basepoint. For this purpose, we define

$$E_n^{\mathrm{gen}}$$

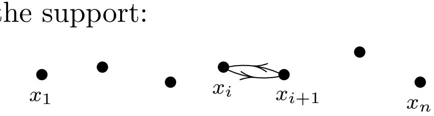
as the subset of E_n^{reg} consisting of configurations of n points with distinct real parts. (This is the first in a series of definitions of semi-algebraic nature.) It is clear that

Lemma 3.2. *E_n^{gen} is contractible.*

Using our topological conventions, we set

$$B_n := \pi_1(E_n^{\mathrm{reg}}, E_n^{\mathrm{gen}}).$$

This group admits a *standard generating set* (the one considered by Artin), consisting of braid reflections $\boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_{n-1}$ defined as follows. (We used bold fonts to avoid confusion with the elementary symmetric functions.) Let $(x_1, \ldots, x_n)$ be the ordered support of a point in E_n^{gen}. Then $\boldsymbol{\sigma}_i$ is represented by the following motion of the support:



Artin's presentation for B_n is

$$B_n = \langle \boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_{n-1} \,|\, \boldsymbol{\sigma}_i \boldsymbol{\sigma}_{i+1} \boldsymbol{\sigma}_i = \boldsymbol{\sigma}_{i+1} \boldsymbol{\sigma}_i \boldsymbol{\sigma}_{i+1}, \boldsymbol{\sigma}_i \boldsymbol{\sigma}_j = \boldsymbol{\sigma}_j \boldsymbol{\sigma}_i \text{ if } |i-j| > 1 \rangle .$$

The following definition requires a compatibility condition, which is a classical elementary consequence of the above presentation.

Definition 3.3. Let G be a group. The (*right*) *Hurwitz action of* B_n *on* G^n is defined by

$$(g_1, \dots, g_{i-1}, g_i, g_{i+1}, g_{i+2}, \dots, g_n) \cdot \boldsymbol{\sigma}_i \\ := (g_1, \dots, g_{i-1}, g_{i+1}, g_{i+1}^{-1} g_i g_{i+1}, g_{i+2}, \dots, g_n)$$

for all $(g_1, \dots, g_n) \in G^n$ and all $i \in \{1, \dots, n-1\}$.

This action preserves the fibers of the product map $G^n \to G, (g_1, \dots, g_n) \mapsto g_1 \cdots g_n$.

4. The affine Van Kampen method

This sections contains some generalities about Zariski-Van Kampen techniques. Let $P \in \mathbb{C}[X_1, \dots, X_n]$ be a reduced polynomial. What we have in mind is that P is the discriminant of a complex reflection group, but it does not cost more to work in a general context. Let $\mathcal{H}$ be the hypersurface of $\mathbb{C}^n$ defined by $P = 0$. Van Kampen's method is a strategy for computing a presentation of

$$\pi_1(\mathbb{C}^n - \mathcal{H}).$$

We assume that P actually involves X_n and write

$$P = \alpha_0 X_n^d + \alpha_1 X_n^{d-1} + \alpha_2 X_n^{d-2} + \cdots + \alpha_d,$$

where d is a positive integer (the *degree in* X_n) and $\alpha_0, \dots, \alpha_d$ are elements of $\mathbb{C}[X_1, \dots, X_{n-1}]$, with $\alpha_0 \neq 0$. We say that P is X_n*-monic* if α_0 is a scalar; when one is only interested in the hypersurface defined by P, it is convenient to then renormalize P to have $\alpha_0 = 1$. We denote by $\mathrm{Disc}_{X_n}(P)$ the *discriminant* of P with respect to X_n, i.e., the resultant of P and $\frac{\partial P}{\partial X_n}$.

This discriminant is a nonzero element of $\mathbb{C}[X_1, \dots, X_{n-1}]$ and defines a hypersurface $\mathcal{K}$ in $Y := \operatorname{Spec} \mathbb{C}[X_1, \dots, X_{n-1}] \simeq \mathbb{C}^{n-1}$. Let p be the natural projection $\operatorname{Spec} \mathbb{C}[X_1, \dots, X_n] \twoheadrightarrow Y$.

Definition 4.1. The *bifurcation locus* of P (with respect to the projection p) is the algebraic hypersurface $\mathcal{K} \subseteq Y$ defined by the equation $\mathrm{Disc}_{X_n}(P) = 0$.

Definition 4.2. A point $y \in Y$ is said to be *generic* if it is not in $\mathcal{K}$; a *generic line of direction* X_n is the fiber L_y of p over a generic point $y \in Y$.

We represent points of $\mathbb{C}^n$ by pairs (y, z), where z is the value of the X_n coordinate and y is the image of the point under p. Let $E := p^{-1}(Y - \mathcal{K}) \cap (\mathbb{C}^n - \mathcal{H})$. The projection p restricts to a locally trivial fibration $E \to Y - \mathcal{K}$, whose fibers are complex lines with d points removed. Choose a basepoint $(y, z) \in E$, and let F be the fiber containing (y, z).

The following basic lemma was brought to my attention by Deligne and should certainly have been included in my earlier paper [2].

LEMMA 4.3. *If P is X_n-monic, the fibration $p : E \to Y - \mathcal{K}$ is split.*

Proof. Assume $\alpha_0 = 1$. Consider $\phi : Y \to \mathbb{C}, y \mapsto 1 + \sum_{i=1}^{d} |\alpha_i(y)|$. Then the map $Y - \mathcal{K} \to E, y \mapsto (y, \phi(y))$ is a splitting since by construction $P(y, \phi(y))$ is always nonzero. $\square$

As a consequence, the fibration long exact sequence breaks into split short exact sequences. Consider the commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & \pi_1(F,(y,z)) \arrow[r, "\iota_*"] \arrow[dr, "\alpha"'] & \pi_1(E,(y,z)) \arrow[r, "p_*"] \arrow[d, "\beta"] & \pi_1(Y-\mathcal{K},y) \arrow[r] & 1 \\ & & \pi_1(\mathbb{C}^n - \mathcal{H},(y,z)), & & \end{tikzcd}$$

whose first line is a split exact sequence (the end of the fibration exact sequence); β comes from the inclusion of spaces.

We are in the context of [2, Th. 2.5], from which we conclude that α is surjective. We can actually be more precise and write a presentation for $\pi_1(\mathbb{C}^n - \mathcal{H}, (y, z))$. The semi-direct product structure of $\pi_1(E, (y, z))$ defines a morphism $\Phi : \pi_1(Y - \mathcal{K}, y) \to \operatorname{Aut}(\pi_1(F, (y, z)))$.

THEOREM 4.4 (Van Kampen presentation). *Let $f_1, \ldots, f_d$ be generators of $\pi_1(F, (y, z)) \simeq F_d$ (the free group on d generators). Let $g_1, \ldots, g_m$ be generators of $\pi_1(Y - \mathcal{K}, y)$ with associated automorphisms $\phi_j := \Phi(g_j)$. We have*

$$\pi_1(\mathbb{C}^n - \mathcal{H}, (y, z)) \simeq \langle f_1, \ldots, f_d \,|\, f_i = \phi_j(f_i),\ 1 \le i \le d,\ 1 \le j \le m \rangle .$$

Proof. Using the semi-direct product structure, we have the presentation

$$\pi_1(E, (y, z)) \simeq \langle f_1, \ldots, f_d, g_1, \ldots, g_m \,|\, g_j f_i = \phi_j(f_i) g_j,\ 1 \le i \le d,\ 1 \le j \le m \rangle .$$

One concludes by observing that $g_1, \ldots, g_m$ may be chosen to be meridians ("generators-of-the-monodromy") around the irreducible components of $p^{-1}(\mathcal{K})$; by [2, Lemma 2.1.(ii)], $\ker \beta$ is generated as a normal subgroup by those meridians. $\square$

COROLLARY 4.5 (The explicit Zariski 2-plane section). *Let P be a reduced polynomial in $\mathbb{C}[X_1, \ldots, X_n]$. Let $a_1, \ldots, a_{n-2} \in \mathbb{C}$. Assume that*

(i) *P is X_n-monic.*
(ii) *The coefficients of $\operatorname{Disc}_{X_n}(P)$ viewed as polynomials with variable X_{n-1} and coefficients in $\mathbb{C}[X_1, \ldots, X_{n-2}]$ are altogether coprime. (In particular, this holds when $\operatorname{Disc}_{X_n}(P)$ is X_{n-1}-monic.)*
(iii) $\operatorname{Disc}_{X_{n-1}}(\operatorname{Disc}_{X_n}(P))(a_1, \ldots, a_{n-2}) \neq 0$.

Let $\mathcal{H}$ be the hypersurface of $\mathbb{C}^n$ with equation $P = 0$, and let Π be the affine 2-plane in $\mathbb{C}^n$ defined by $X_1 = a_1, \ldots, X_{n-2} = a_{n-2}$. Then the map $\Pi \cap (\mathbb{C}^n - \mathcal{H}) \hookrightarrow \mathbb{C}^n - \mathcal{H}$ is a π_1-isomorphism.

Proof. Consider the affine line $p(\Pi) \subseteq Y$. Condition (iii) expresses that $p(\Pi)$ is a *generic line of direction* X_{n-1} in Y and, under condition (ii), [2, Th. 2.5], the injection $p(\Pi) \cap (Y - \mathcal{K}) \hookrightarrow Y - \mathcal{K}$ is π_1-surjective. In particular, in Theorem 4.4, we can choose loops drawn in $\Pi \cap E$ to represent the lifted generators of $\pi_1(Y - \mathcal{K})$. □

Combined with our earlier work with Jean Michel, this corollary suffices to prove Theorem 0.6, since the 2-plane sections described in [8] satisfy conditions (i)–(iii). Presentations for $\pi_1(\Pi \cap (\mathbb{C}^n - \mathcal{H}))$ were obtained in [8], using our software package VKCURVE. At this stage, Theorem 0.6 relies on brutal computations. The sequel will provide a much more satisfying approach at least for $W \neq G_{31}$.

5. Lyashko-Looijenga coverings

Let W be an irreducible well-generated complex reflection group, together with a system of basic invariants f and a flat system of basic derivations ξ with discriminant matrix $M_0 + X_n \,\mathrm{Id}$. Expanding the determinant, we observe that

$$\Delta_f = \det(M_0 + X_n \,\mathrm{Id}) = X_n^n + \alpha_2 X_n^{n-2} + \alpha_3 X_n^{n-3} + \cdots + \alpha_n,$$

where $\alpha_i \in \mathbb{C}[X_1, \ldots, X_{n-1}]$. Since Δ_f is weighted homogeneous of total weight nh for the system of weights $\mathrm{wt}(X_i) = d_i$, each α_i is weighted homogeneous of weight ih.

Definition 5.1. The (generalized) *Lyashko-Looijenga morphism* is the morphism LL from $Y = \operatorname{Spec} \mathbb{C}[X_1, \ldots, X_{n-1}]$ to $E_n \simeq \operatorname{Spec} \mathbb{C}[\sigma_2, \ldots, \sigma_n]$ defined by $\sigma_i \mapsto (-1)^i \alpha_i$.

This is of course much better understood in the following geometric terms. Via the choice of a system of basic invariants, we have chosen an isomorphism

$$W \backslash V \simeq Y \times \mathbb{C}.$$

Let $v \in V$. The orbit $\overline{v} \in W \backslash V$ is represented by a pair

$$(y, z) \in Y \times \mathbb{C},$$

where $z = f_n(v)$ and y is the point in Y with coordinates $(f_1(v), \ldots, f_{n-1}(v))$. This encoding of points in $W \backslash V$ will be used throughout this article. As in the previous section, we study the space $W \backslash V \simeq Y \times \mathbb{C}$ according to the fibers of the the projection $p : W \backslash V \to Y, (y, z) \mapsto y$.

Definition 5.2. For any point y in Y, we denote by L_y the fiber of the projection $p : W \backslash V \to Y$ over y.

For any $y \in Y$, the affine line L_y intersects the discriminant $\mathcal{H}$ in n points (counted with multiplicities), whose coordinates are

$$(y, x_1), \ldots, (y, x_n),$$

where $\{x_1, \dots, x_n\}$ is the multiset of solutions in X_n of $\Delta_f = 0$ where each α_i has been replaced by its value at y. We have

$$\mathrm{LL}(y) = \{x_1, \dots, x_n\}.$$

The bifurcation locus $\mathcal{K} \subseteq Y$ (see the previous section) corresponds precisely to those y such that $\mathrm{LL}(y)$ contains multiple points.

The main theorem of this section generalizes earlier results from [42]:

THEOREM 5.3. *The polynomials* $\alpha_2, \dots, \alpha_n \in \mathbb{C}[X_1, \dots, X_{n-1}]$ *are algebraically independent, and* $\mathbb{C}[X_1, \dots, X_{n-1}]$ *is a free graded* $\mathbb{C}[\alpha_2, \dots, \alpha_n]$-*module of rank* $n!h^n/|W|$. *As a consequence,* LL *is a finite morphism. It restricts to an unramified covering* $Y - \mathcal{K} \twoheadrightarrow E_n^{\mathrm{reg}}$ *of degree* $n!h^n/|W|$.

I thank Eduard Looijenga for precious help with the theorem. A prior version of this text contained a gap (the key Lemma 5.6), and the very nice argument below is due to him.

LEMMA 5.4. *Let* $v \in V$ *with image* $\overline{v} \in W\backslash V$. *Let*

$$V_v := \bigcap_{H \in \mathcal{A},\, v \in H} H.$$

The multiplicity of $\mathcal{H}$ *at* $\overline{v}$ *is* $\dim_{\mathbb{C}} V/V_v$.

Proof. At $v=0$, the multiplicity is the valuation of Δ_f, which is indeed n.

When $v \neq 0$, we consider the parabolic subgroup $W_v := \{w \in W | wv = v\}$. By Lemma 2.7, W_v is again a well-generated complex reflection group. The quotient map $\bigcup_{H\in\mathcal{A}} H \twoheadrightarrow \mathcal{H} = W\backslash(\bigcup_{H\in\mathcal{A}} H)$ factors through $\bigcup_{H\in\mathcal{A}} H \twoheadrightarrow W_v\backslash(\bigcup_{H\in\mathcal{A}} H)$. Because $W_v\backslash(\bigcup_{H\in\mathcal{A}} H) \twoheadrightarrow W\backslash(\bigcup_{H\in\mathcal{A}} H)$ is unramified over $\overline{v}$, the multiplicity of $\mathcal{H}$ at $\overline{v}$ coincides with the multiplicity of $W_v\backslash(\bigcup_{H\in\mathcal{A}} H)$ at the image $\tilde{v}$ of v. Around $\tilde{v}$, $W_v\backslash(\bigcup_{H\in\mathcal{A}} H)$ is the same as $W_v\backslash(\bigcup_{H\in\mathcal{A},\, v\in H} H)$. This hypersurface is a direct product of V_v with the discriminant of W_v. The multiplicity at $\tilde{v}$ is the multiplicity at the origin of the discriminant of W_v. After reduction to the irreducible case, we apply the already solved case: the multiplicity is the rank of W_v or, in other words, $\dim_{\mathbb{C}} V/V_v$. $\square$

Remark 5.5. As was suggested by Referee #4, Lemma 5.4 is actually a characterization of well-generated reflection groups: as pointed out in [2, Prop. 4.2], the minimum number of reflections needed to generate an irreducible complex reflection group is equal to the valuation of the discriminant (i.e., the degree of the smallest degree monomial), which is the same as the multiplicity at 0: so when W is not well generated, the multiplicity at 0 is greater than $\dim_{\mathbb{C}}(V)$.

LEMMA 5.6. $\mathrm{LL}^{-1}(0) = \{0\}$.

In this statement, the 0 on the left denotes the multiset with n copies of 0.

Proof. Let C be the tangent cone to $\mathcal{H}$ in $\mathbb{C}^n \simeq Y \times \mathbb{C}$. It is a closed subvariety of the tangent bundle to $\mathbb{C}^n$. Because $\mathcal{H}$ is quasi-homogeneous and for all $i \in \{1, \ldots, n-1\}, \mathrm{wt}(X_n) > \mathrm{wt}(X_i)$, the cone C is "horizontal" at 0. In particular, the (fiber over 0 of the tangent cone to the) "vertical" line L_0 of equation $X_1 = \cdots = X_{n-1} = 0$ is not in C.

Let $y = (x_1, \ldots, x_{n-1}) \in \mathrm{LL}^{-1}(0)$, i.e., such that the line L_y with $X_1 = x_1, \ldots, X_{n-1} = x_{n-1}$ intersects $\mathcal{H}$ in only one point, $(y, 0) \in Y \times \mathbb{C}$. We want to prove that $y = 0$. Because $\mathcal{H}$ is quasi-homogeneous, it is enough to work in a neighborhood of the origin. In particular, we may assume that y is close enough to 0 for L_y to still be outside C. Using a refined Bézout theorem ([34, Cor. 12.4]), we have

$$i((y,0), L_y.\mathcal{H}; \mathbb{C}^n) = m_{(y,0)}(\mathcal{H}),$$

where

- $i((y,0), L_y.\mathcal{H}; \mathbb{C}^n)$ is the intersection multiplicity of $\mathcal{H}$ and L_y at $(y,0)$. This is the order of 0 as a root of the polynomial $\Delta_f|_{X_1 = x_1, \ldots, X_{n-1} = x_{n-1}}$. By assumption, this is n.
- $m_{(y,0)}(\mathcal{H})$ is the multiplicity of $\mathcal{H}$ at $(y,0)$. Let v be a preimage of $(y,0)$ in V. By Lemma 5.4, $m_{(y,0)}(\mathcal{H}) = \dim_{\mathbb{C}} V/V_v$.

Thus $\dim_{\mathbb{C}} V/V_v = n$, $V_v = 0$, $W_v = W$ and $v = 0$. □

Because LL is quasi-homogeneous, Lemma 5.6 implies that LL is a finite (= quasi-finite and proper) morphism or, in other words, that $\mathbb{C}[X_1, \ldots, X_{n-1}]$ is a finite graded $\mathbb{C}[\alpha_2, \ldots, \alpha_n]$-module. In particular, $\alpha_2, \ldots, \alpha_n$ are algebraically independent.

Because $\mathbb{C}[X_1, \ldots, X_{n-1}]$ is Cohen-Macaulay and finite over $\mathbb{C}[\alpha_2, \ldots, \alpha_n]$, it is a free $\mathbb{C}[\alpha_2, \ldots, \alpha_n]$-module. The rank may be computed by comparing Hilbert series. Since each X_i has weight d_i, the Hilbert series of $\mathbb{C}[X_1, \ldots, X_{n-1}]$ is $\prod_{i=1}^{n-1} \frac{1}{1-t^{d_i}}$. Since each α_i has weight ih, the Hilbert series of $\mathbb{C}[\alpha_2, \ldots, \alpha_n]$ is $\prod_{i=2}^{n} \frac{1}{1-t^{ih}}$. The rank is the limit at $t \to 1$ of the quotient of these series, equal to

$$\prod_{i=1}^{n-1} \frac{(i+1)h}{d_i} = \frac{n! h^{n-1}}{d_1 \ldots d_{n-1}} = \frac{n! h^n}{|W|}.$$

Theorem 5.3 now follows from the following generalization of [42, Th. 1.4].

LEMMA 5.7. LL *is étale on* $Y - \mathcal{K}$.

Proof. As mentioned in [42, (1.5)], the result will follow if we prove that for all $y \in Y - \mathcal{K}$ with $\mathrm{LL}(y) = \{x_1, \ldots, x_n\}$, the hyperplanes $H_1, \ldots, H_n$ tangent to $\mathcal{H}$ at the n distinct points $(y, x_1), \ldots, (y, x_n)$ are in general position.

To prove this, we use Proposition 1.7. Each H_i is spanned by $\overline{\xi}_1(y, x_i), \ldots, \overline{\xi}_n(y, x_i)$. Let $(\varepsilon_1, \ldots, \varepsilon_n)$ be the basis of $(W \backslash V)^*$ dual to $(\frac{\partial}{\partial X_1}, \ldots, \frac{\partial}{\partial X_n})$. Let

$l_i = \sum_j \lambda_{j,i}\varepsilon_j$ be a nonzero vector in $(W\backslash V)^*$ orthogonal to H_i. This amounts to taking a nonzero column vector $(\lambda_{j,i})_{j=1,\dots,n}$ in the kernel of $M(y, x_i)$ or, equivalently, an eigenvector of $M_0(y)$ associated to the eigenvalue $-x_i$. By assumption, the x_i are distinct. The eigenvectors are linearly independent. $\square$

The theorem has the following corollary (which we will not use).

COROLLARY 5.8. *The space* $Y - \mathcal{K}$ *is a* $K(\pi, 1)$.

For the sake of clarity, let us also mention

COROLLARY 5.9. *Let* $v \in V$. *Denote by* $(y, z) \in Y \times \mathbb{C}$ (*identified with* $W\backslash V$) *the image of* v. *The following integers coincide*:

(i) *the multiplicity of* z *in* $\mathrm{LL}(y)$;
(ii) *the intersection multiplicity of* L_y *with* $\mathcal{H}$ *at* (y, z);
(iii) *the multiplicity of* $\mathcal{H}$ *at* (y, z);
(iv) *the rank* $\dim_{\mathbb{C}} V/V_v$ *of the parabolic subgroup* W_v.

Proof. The integer defined by (i) and (ii) are the same by their very definition. The identity between (iii) and (iv) is Lemma 5.4. The argument used to prove Lemma 5.6 also shows the identity between (ii) and (iii). $\square$

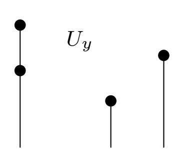
The discriminant stratification of E_n yields a natural stratification of Y: when λ is a partition of n, the stratum Y_λ consists of points y such that the multiplicities of $\mathrm{LL}(y)$ are distributed according to λ. Applying the corollary, one sees that the stratification $Y = \bigsqcup_{\lambda \models n} Y_\lambda$ is the "shadow" of the discriminant stratification restricted to $\mathcal{H}$.

6. Tunnels, labels and the Hurwitz rule

Let W be an irreducible well-generated complex reflection group. We keep the notation from the previous section. Let $y \in Y$. Let U_y be the complement in L_y of the vertical imaginary half-lines below the points of $\mathrm{LL}(y)$ or, in more formal terms,

$$U_y := \{(y, z) \in L_y | \forall x \in \mathrm{LL}(y), \mathrm{re}(z) = \mathrm{re}(x) \Rightarrow \mathrm{im}(z) > \mathrm{im}(x)\}.$$

Here is an example where the support of $\mathrm{LL}(y)$ consists of four points and U_y is the complement of three half-lines:



"Generically," U_y is the complement of n vertical half-lines. We have to be careful about what "generically" means here. A prerequisite is that $\mathrm{LL}(y)$

should consist of n distinct points, which amounts to $y \in Y - \mathcal{K}$ or, equivalently,

$$\mathrm{LL}(y) \in E_n^{\mathrm{reg}},$$

but this is not enough: one needs these points to be on distinct vertical lines or, equivalently, that

$$\mathrm{LL}(y) \in E_n^{\mathrm{gen}}.$$

Definition 6.1. We set $Y^{\mathrm{gen}} := \mathrm{LL}^{-1}(E_n^{\mathrm{gen}})$.

This space, being the "fiber" of the covering LL over the "basepoint" of the basespace E_n^{reg}, is equipped with a Galois action of $B_n := \pi_1(E_n^{\mathrm{reg}}, E_n^{\mathrm{gen}})$.

Definition 6.2. The *fat basepoint* of $W \backslash V^{\mathrm{reg}}$ is the subset $\mathcal{U}$ defined by

$$\mathcal{U} := \bigcup_{y \in Y} U_y$$

or, equivalently, by

$$\mathcal{U} := \{(y, z) \in Y \times \mathbb{C} | \forall x \in \mathrm{LL}(y), \mathrm{re}(z) = \mathrm{re}(x) \Rightarrow \mathrm{im}(z) > \mathrm{im}(x)\}.$$

This definition calls for a few comments:

- First, as we will see if the next lemma and Definition 6.4 below, the subset $\mathcal{U}$ can and will be used "as if" it was a genuine basepoint.
- Note also that the definition implicitly relies on the choice of a preferred direction in the complex line. The fat basepoint is a truly semi-algebraic object and, geometrically, all constructions below depend on the choice of a preferred element in the unit circle S^1.

LEMMA 6.3. *The fat basepoint $\mathcal{U}$ is dense in $W \backslash V^{\mathrm{reg}}$, open and contractible.*

Proof. The first two statements are clear. Define a continuous function $\beta : Y \to \mathbb{R}$ by

$$\beta(y) := \max\{\mathrm{im}(x) | x \in \mathrm{LL}(y)\} + 1.$$

Points of $W \backslash V$ are represented by pairs $(y, z) \in Y \times \mathbb{C}$, or equivalently by triples $(y, a, b) \in Y \times \mathbb{R} \times \mathbb{R}$, where $a = \mathrm{re}(z)$ and $b = \mathrm{im}(z)$. For $t \in [0, 1]$, define $\phi_t : W \backslash V \to W \backslash V$ by

$$\phi_t(y, a, b) := \begin{cases} (y, a, b) & \text{if } b \geq \beta(y), \\ (y, a, b + t(\beta(y) - b)) & \text{if } b \leq \beta(y). \end{cases}$$

Each ϕ_t preserves $\mathcal{U}$, and the homotopy ϕ restricts to a deformation retraction of $\mathcal{U}$ to

$$\bigcup_{y \in Y} \{(y, z) \in L_y | \mathrm{im}(z) \geq \beta(y)\}.$$

The latter is a locally trivial bundle over the contractible space Y, with contractible fibers. (The fibers are half-planes.) Thus it is contractible. $\square$

As explained in Appendix A, we may (and will) use $\mathcal{U}$ as "basepoint" for $W\backslash V^{\mathrm{reg}}$ and refine our definition of $B(W)$:

Definition 6.4. The *braid group of* W is $B(W) := \pi_1(W\backslash V^{\mathrm{reg}}, \mathcal{U})$.

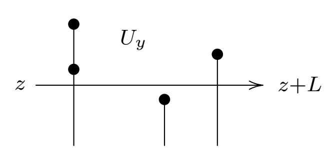
As for other notions actually depending on W, we often write B instead of $B(W)$, since most of the time we implicitly refer to a given W.

Remark 6.5. We will need to consider a natural projection $\pi : B \to W$. Recall that such a morphism is part of the fibration exact sequence

$$1 \longrightarrow \pi_1(V^{\mathrm{reg}}) \longrightarrow \pi_1(W\backslash V^{\mathrm{reg}}) \xrightarrow{\pi} W \longrightarrow 1.$$

For this exact sequence to be well defined, one has to make consistent choices of basepoints in V^{reg} and in $W\backslash V^{\mathrm{reg}}$. We have already described our "basepoint" $\mathcal{U}$ in $W\backslash V^{\mathrm{reg}}$. Choose $u \in \mathcal{U}$, and choose a preimage $\tilde{u}$ of u in V^{reg}. If $u' \in \mathcal{U}$ is another choice and if γ is a path in $\mathcal{U}$ from u to u', then γ lifts to a unique path $\tilde{\gamma}$ starting at $\tilde{u}$; since $\mathcal{U}$ is contractible, the fixed-endpoint homotopy class of $\tilde{\gamma}$ (and, in particular, its final point) does not depend on γ. In other words, once we have chosen a preimage of one point of $\mathcal{U}$, we have a natural section $\tilde{\mathcal{U}}$ of $\mathcal{U}$ in V^{reg}, as well as a transitive system of isomorphisms between $(\pi_1(V^{\mathrm{reg}}, \tilde{u}))_{\tilde{u} \in \tilde{\mathcal{U}}}$. From now on, we assume we have made such a choice, and we define the pure braid group as $\pi_1(V^{\mathrm{reg}}, \tilde{\mathcal{U}})$. This selects one particular morphism π. (The $|W|$ possible choices yield conjugate morphisms.)

Definition 6.6. A *semitunnel* is a triple $T = (y, z, L) \in Y \times \mathbb{C} \times \mathbb{R}_{\geq 0}$ such that $(y, z) \in \mathcal{U}$ and the affine segment $[(y, z), (y, z + L)]$ lies in $W\backslash V^{\mathrm{reg}}$. The path γ_T associated with T is the path $t \mapsto (y, z + tL)$. The semitunnel T is a *tunnel* if in addition $(y, z + L) \in \mathcal{U}$.



The distinction between tunnels and semitunnels should be understood in light of our topological conventions: if T is a tunnel, γ_T represents an element

$$b_T \in \pi_1(W\backslash V^{\mathrm{reg}}, \mathcal{U}),$$

while semitunnels will be used to represent points of the universal cover

$$(\mathrm{UniCover}(W\backslash V^{\mathrm{reg}}, \mathcal{U}))_{\mathcal{U}};$$

see Section 10.

Definition 6.7. An element $b \in B$ is *simple* if $b = b_T$ for some tunnel T. The set of simple elements in B is denoted by S.

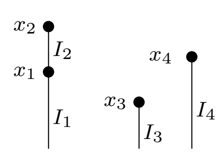
Remark 6.8. Later on (Corollary 7.9), we will show that S is finite. This might be disconcerting at first sight, as S is stable under particular conjugacy operations and one might be misled into believing that S is a union of conjugacy classes, which it is not. The finiteness of S and its direct description in terms of the combinatorics of the finite group W (Proposition 8.5) is a key ingredient of this paper.

Each tunnel lives in a single L_y, which is isomorphic to a complex line, and where the tunnel may be represented by the constant imaginary part affine segment $[z, z+L]$. The triple (y, z, L) may be uniquely recovered from $[(y,z),(y,z+L)]$. A frequent abuse of terminology will consist of using the term *tunnel* (or *semitunnel*) to designate either the triple (y,z,L), or the segment $[(y,z),(y,z+L)]$, or the pair $(y,[z,z+L])$, depending on the context. (In particular, when intersecting tunnels with geometric objects, the tunnels should be understood as affine segments.)

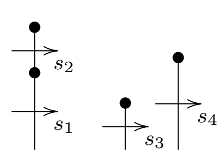
Let $y \in Y$. Let $(x_1, \dots, x_k)$ be the ordered support of $\mathrm{LL}(y)$. The space $p_x((L_y \cap W \backslash V^{\mathrm{reg}}) - U_y)$ is a union of k disjoint open affine intervals $I_1, \dots, I_k$, where

$$I_i := \begin{cases} (x_i - \sqrt{-1}\infty, x_i) & \text{if } i = 1 \text{ or } (i > 1 \text{ and } \mathrm{re}(x_{i-1}) < \mathrm{re}(x_i)), \\ (x_{i-1}, x_i) & \text{otherwise.} \end{cases}$$

(By $(x_i - \sqrt{-1}\infty, x_i)$, we mean the open vertical half-line below x_i.) In the first case (when I_i is not bounded), we say that x_i is *deep*. In the picture below, there are three deep points, x_1, x_3 and x_4.



Choose a *system of elementary tunnels* for y. By this, we mean the choice, for each $i = 1, \dots, k$, of a small tunnel T_i in L_y crossing I_i and not crossing the other intervals; let $s_i := b_{T_i}$ be the associated element of B.



These elements depend only on y and not on the explicit choice of elementary tunnels.

Definition 6.9. The sequence $\mathrm{lbl}(y) := (s_1, \ldots, s_k)$ is the *label of* y. Let $i_1, i_2, \ldots, i_l$ be the indices of the successive deep points of $\mathrm{LL}(y)$. The *deep label* of y is the subsequence $(s_{i_1}, \ldots, s_{i_l})$.

In the above example, the deep label is (s_1, s_3, s_4). The length of the label is n if and only if $y \in Y - \mathcal{K}$. In this case, the deep label coincides with the label if and only if $y \in Y^{\mathrm{gen}}$.

Later on, it will appear that the pair $(\mathrm{LL}(y), \mathrm{lbl}(y))$ uniquely determines y (Theorem 7.20).

Remark 6.10. When y is generic, $\mathrm{lbl}(y)$ is an n-tuple of braid reflections. (Because an elementary tunnel crossing the interval below a point in $\mathrm{LL}(y)$ is essentially the same as a small circle around this point; when y is generic, the points in $\mathrm{LL}(y)$ correspond to smooth points of the discriminant, and the elementary tunnels represent generators of the monodromy.)

Consider the case $y = 0$ (given by the equations $X_1 = 0, \ldots, X_{n-1} = 0$). The multiset $\mathrm{LL}(y)$ has support $\{0\}$ with multiplicity n.

Definition 6.11. We denote by δ the simple element such that $\mathrm{lbl}(0) = (\delta)$.

This element plays the role of Deligne's element Δ. Choose $v \in V^{\mathrm{reg}}$ such that the W-orbit $\overline{v}$ lies in L_0. Broué-Malle-Rouquier consider the element (denoted by $\boldsymbol{\pi}$, [20, Notation 2.3]) in the pure braid group $P(W)$ represented by the loop

$$\begin{aligned} [0,1] &\longrightarrow V^{\mathrm{reg}} \\ t &\longmapsto v \exp(2\sqrt{-1}\pi t). \end{aligned}$$

We prefer a different notation:

Definition 6.12. We call this element of $P(W)$ *full-twist* and denote it by τ.

They observe that this element lies in the center of B ([20, Th. 2.24]) and conjecture that it generates the center of P.

Since X_n has weight h, δ^h coincides with τ. More precisely, δ is represented by the loop that is the image in $W \backslash V^{\mathrm{reg}}$ of the path in V^{reg}

$$\begin{aligned} [0,1] &\longrightarrow V^{\mathrm{reg}} \\ t &\longmapsto v \exp(2\sqrt{-1}\pi t/h). \end{aligned}$$

In particular,

Lemma 6.13. *The element $\tau = \delta^h$ is central in $B(W)$ and lies in $P(W)$. The image of δ in W is ζ_h-regular, in the sense of Springer* (*see Definition* 1.8).

See Theorem 12.3 for a description of the center of $B(W)$. Each tunnel lives in a single fiber L_y. Let us now explain how simple elements represented by tunnels living in different fibers may be compared. The idea is that, since being a tunnel is an open condition, one may perturb y without affecting the simple element:

Definition 6.14. Let $T = (y, z, L)$ be a tunnel. A *T-neighborhood* of y is a path-connected neighborhood Ω of y in Y such that, for all $y' \in \Omega$, $T' := (y', z, L)$ is a tunnel.

Such neighborhoods clearly exist for all $y \in Y$.

LEMMA 6.15 (The Hurwitz rule). *Let $T = (y, z, L)$ be a tunnel, representing a simple element s. Let Ω be a T-neighborhood of y. For all $y' \in \Omega$, $T' := (y', z, L)$ represents s.*

Proof. This simply expresses that the tunnels (y', z, L) and (y, z, L) represent homotopic paths, which is clear by definition of Ω. □

Remark 6.16. Let $T_1, \ldots, T_k$ be a system of elementary tunnels for y. Let Ω_i be a T_i-neighborhood for y. A *standard neighborhood* of y could be defined as a path-connected neighborhood Ω of y inside $\cap_{i=1}^k \Omega_i$. These standard neighborhoods form a basis for the topology of Y. A consequence of the Hurwitz rule is that the label of y may uniquely be recovered once we know the label of a single $y' \in \Omega$. Distinct $y' \in \Omega$ correspond to different "desingularizations" of y, and their labels are obtained by further factorizing terms in the label of y. Among them are full desingularizations (corresponding to factorizations in n terms), corresponding to choosing y' in the nonempty intersection $\Omega \cap Y^{\mathrm{gen}}$.

The remainder of this section consists of various consequences of the Hurwitz rule.

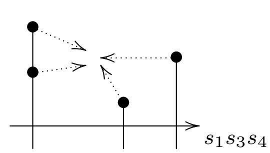
COROLLARY 6.17. *Let $y \in Y$. Let $(x_1, \ldots, x_k)$ be the ordered support of $\mathrm{LL}(y)$, and let $(n_1, \ldots, n_k)$ be the multiplicities. For any i, the natural length of s_i is given by*

$$l(s_i) = \sum_{j \text{ s.t. } \mathrm{re}(x_i) = \mathrm{re}(x_j) \text{ and } \mathrm{im}(x_i) \leq \mathrm{im}(x_j)} n_j.$$

Proof. The case $y \in Y^{\mathrm{gen}}$ is a consequence of Remark 6.10. The general case follows by perturbing and applying the Hurwitz rule. □

COROLLARY 6.18. *Let $y \in Y$. Let $(s_{i_1}, \ldots, s_{i_l})$ be the deep label of y. We have $s_{i_1} \cdots s_{i_l} = \delta$.*

Proof. Any tunnel T deep enough and long enough represents $s_{i_1} \cdots s_{i_l}$.



The origin $0 \in Y$ lies in a T-neighborhood of y. To conclude, apply the Hurwitz rule. □

Let us recall the following standard notion.

Definition 6.19 (The Hurwitz action). Let G be a group, and let B_n be the braid group on n strings with its usual system of generators $\boldsymbol{\sigma}_1, \dots, \boldsymbol{\sigma}_{n-1}$. The *Hurwitz action* of B_n on G^n, denoted as a multiplication on the right, is the unique group right action such that

$$(g_1, \dots, g_n) \cdot \boldsymbol{\sigma}_i = (g_1, \dots, g_{i-1}, g_i g_{i+1} g_i^{-1}, g_i, g_{i+2}, \dots, g_n).$$

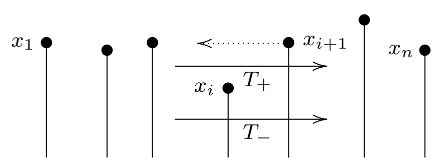
In the following corollary, the notation $y \cdot \beta$ refers to the covering action of $\pi_1(E_n^{\mathrm{reg}}, x)$ on $\mathrm{LL}^{-1}(x)$.

COROLLARY 6.20. *Let $x \in E_n^{\mathrm{gen}}$. Let $\beta \in \pi_1(E_n^{\mathrm{reg}}, x)$, $y \in \mathrm{LL}^{-1}(x)$ and $y' := y \cdot \beta$. Let $(b_1, \dots, b_n)$ be the label of y and $(b'_1, \dots, b'_n)$ be the label of y'. Then*

$$(b'_1, \dots, b'_n) = (b_1, \dots, b_n) \cdot \beta,$$

where β acts by right Hurwitz action.

Proof. It suffices to prove this for a standard generator $\boldsymbol{\sigma}_i$. Let $(x_1, \dots, x_n)$ be the ordered support of x. By the Hurwitz rule, we may adjust the imaginary parts of the x_i's without affecting the label; in particular, we may assume that $\mathrm{im}(x_i) < \mathrm{im}(x_{i+1})$. We may find tunnels $T_- = (y, z_-, L)$ and $T_+ = (y, z_+, L)$ as in the picture below:



The path in E_n^{reg} where x_{i+1} moves along the dotted arrow and all other points are fixed represents $\boldsymbol{\sigma_i}$. Applying the Hurwitz rule to T_+, we obtain $b'_i = b_{i+1}$; applying the Hurwitz rule to T_-, we obtain $b'_i b'_{i+1} = b_i b_{i+1}$. The result follows. □

COROLLARY 6.21. *Let $y \in Y^{\mathrm{gen}}$. The cardinality of the Hurwitz orbit $\mathrm{lbl}(y) \cdot B_n$ is at most $n!h^n/|W|$, and there is an equivalence between*

(i) $|\,\mathrm{lbl}(y) \cdot B_n| = n!h^n/|W|$;
(ii) *the orbits $y \cdot B_n$ and $\mathrm{lbl}(y) \cdot B_n$ are isomorphic as B_n-sets*;
(iii) *the map $Y^{\mathrm{gen}} \to E_n \times B^n, y \mapsto (\mathrm{LL}(y), \mathrm{lbl}(y))$ is injective.*

In the next section, we will prove that conditions (i)–(iii) actually hold. This is not a trivial statement.

Proof. Let G be a group and Ω, Ω' be two G-sets, together with a G-set morphism $\rho : \Omega \to \Omega'$. Assume that Ω' is transitive. Then ρ is surjective. Assume in addition that Ω is finite. Then Ω' is finite, $|\Omega'| \leq |\Omega|$ and

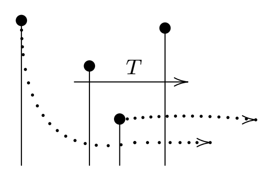
$$|\Omega'| = |\Omega| \Leftrightarrow \rho \text{ is injective} \Leftrightarrow \rho \text{ is an isomorphism.}$$

By the previous corollary, one has $\mathrm{lbl}(y \cdot \beta) = \mathrm{lbl}(y) \cdot \beta$. In other words, the map lbl extends to a B_n-set morphism $y \cdot B_n \to \mathrm{lbl}(y) \cdot B_n$. We apply our discussion to $G := B_n$, $\Omega := y \cdot B_n$ and $\Omega' := \mathrm{lbl}(y) \cdot B_n$. Both B_n-sets are clearly transitive. Since $y \in Y - \mathcal{K}$, we have $|\mathrm{LL}^{-1}(\mathrm{LL}(y))| = |y \cdot B_n| = n!h^n/|W|$ (Theorem 5.3). We deduce that $|\,\mathrm{lbl}(y) \cdot B_n| \leq n!h^n/|W|$ and (i) $\Leftrightarrow$ (ii).

Assertion (iii) amounts to saying that, for all $y \in Y^{\mathrm{gen}}$, lbl is injective on the fiber of LL containing y. This fiber is precisely the orbit $y \cdot B_n$. Under this rephrasing, it is clear that (ii) $\Leftrightarrow$ (iii). □

Corollary 6.22. *Let s be a simple element. There exist $y \in Y^{\mathrm{gen}}$ and $i \in \{1, \ldots, n\}$ such that $s = s_1 \cdots s_i$, where $(s_1, \ldots, s_n) := \mathrm{lbl}(y)$.*

Proof. Let T be a tunnel representing s. Any T-neighborhood of y contains generic points. Up to perturbing y, we may assume that $y \in Y^{\mathrm{gen}}$. The picture below explains, on an example, how to move certain points (following the dotted paths) of the underlying configuration to reach a suitable y':



This path in E_n^{reg} lifts, via LL, to a path in a T-neighborhood of y whose final point y' satisfies the conditions of the lemma. □

7. **Reduced decompositions of Coxeter elements**

7.1. *From braids to elements of W.*

Definition 7.1. Let W be an irreducible well-generated complex reflection group. An element $c \in W$ is a (*generalized*) *Coxeter element* if it is ζ_h-regular. More generally, if W is a well-generated complex reflection group decomposed as a sum $W = \bigoplus_i W_i$ of irreducible groups, a Coxeter element in W is a product $c = \prod_i c_i$ of Coxeter elements in each W_i.

Lemma 7.2. *When W is irreducible, a Coxeter element c in W has no nontrivial fixed point.*

Proof. As shown by Springer [55], the eigenvalues of a ζ-regular element are $\zeta^{1-d_1}, \ldots, \zeta^{1-d_n}$. Applying this to $\zeta = \zeta_h$ and noting that $0 \leq d_1 \leq \cdots \leq d_n = h$, we obtain the desired result. □

When W is irreducible, we may use the constructions from the previous section and the morphism $\pi : B(W) \twoheadrightarrow W$ to obtain typical Coxeter elements (see Definition 6.11):

LEMMA 7.3. *When W is irreducible, the element $c := \pi(\delta)$ is a Coxeter element in W.*

Proof. This is a rephrasing of Lemma 6.13. $\square$

The other Coxeter elements, which are conjugates of c, appear when considering other basepoints over $\mathcal{U}$ (see Remark 6.5).

More generally, we have

LEMMA 7.4. *Let $y \in Y$. Let $(x_1, \ldots, x_k)$ be the ordered support of* $\mathrm{LL}(y)$, *and let $(n_1, \ldots, n_k)$ be the multiplicities. Assume that* $\mathrm{re}(x_1) < \cdots < \mathrm{re}(x_k)$. *Let $(s_1, \ldots, s_k) :=$* $\mathrm{lbl}(y)$.

For all i, set $c_i := \pi(s_i)$. Then there exists a preimage $v_i \in V$ of $(y, x_i) \in Y \times \mathbb{C} \simeq W\backslash V$ such that c_i is a Coxeter element in the parabolic subgroup W_{v_i}. In particular, if $n_i = 1$, then c_i is a reflection.

Proof. When $n_i = n$ (thus $i = k = 1$), the result is Lemma 7.3. When $n_i = 1$, as pointed out in Remark 6.10, the element s_i is represented by a small loop around a smooth point in the discriminant; thus it is a braid reflection and maps to a reflection in W.

The general case is similar: locally near (y, x_i), the discriminant is a direct product of V_{v_i} with the discriminant of W_{v_i}. (See the proof of Lemma 5.4.) This local structure provides a specific morphism $B(W_{v_i}) \to B(W)$ such that the element "δ_i" in $B(W_{v_i})$ (the product of the δ's associated with each irreducible components of W_{v_i}) maps to s_i. The lemma follows. $\square$

The assumption $\mathrm{re}(x_1) < \cdots < \mathrm{re}(x_k)$ may be removed at the cost of replacing $\pi(s_i)$ by $\pi(s_{i-1})^{-1}\pi_(s_i)$ when $\mathrm{re}(x_{i-1}) = \mathrm{re}(x_i)$. This is behind Definition 7.14 below.

Let R be the set of all reflections in W. As in [3], for all $w \in W$, we denote by $\mathrm{Red}_R(w)$ the set of *reduced R-decompositions of* w, i.e., minimal length sequences of elements of R with product w. Since R is closed under conjugacy, $\mathrm{Red}_R(c)$ is stable under Hurwitz action. We also consider the length function $l_R : W \to \mathbb{Z}_{\geq 0}$, whose value at w is the common length of the elements of $\mathrm{Red}_R(w)$, and two partial orderings of W defined as follows. For all $w, w' \in W$, we set

$$w \preccurlyeq_R w' :\Leftrightarrow l_R(w) + l_R(w^{-1}w') = l_R(w')$$

and

$$w' \succcurlyeq_R w :\Leftrightarrow l_R(w'w^{-1}) + l_R(w) = l_R(w').$$

Since R is invariant by conjugacy, we have $w \preccurlyeq_R w' \Leftrightarrow w' \succcurlyeq_R w$.

Let $y \in Y^{\mathrm{gen}}$, let $(r_1, \ldots, r_n) := (\pi(s_1), \ldots, \pi(s_n)) = \pi^n(\mathrm{lbl}(y))$, and let $(s_1, \ldots, s_n) := \mathrm{lbl}(y)$. By Lemma 7.4, since all multiplicities are 1, the factorization $(r_1, \ldots, r_n)$ expressed c as a product of n reflections. Because the fixed-point set of c is trivial (Lemma 7.2), c cannot expressed as the product of less than n reflections, so the factorization has minimal length and it lies in $\mathrm{Red}_R(c)$.

The key result of this section is

THEOREM 7.5. *Let $y \in Y^{\mathrm{gen}}$. The maps*

$$y \cdot B_n \xrightarrow{\mathrm{lbl}} \mathrm{lbl}(y) \cdot B_n \xrightarrow{\pi^n} \pi^n(\mathrm{lbl}(y)) \cdot B_n$$

are isomorphisms of B_n-sets, where $y \cdot B_n$ is the Galois orbit of y and where $\mathrm{lbl}(y) \cdot B_n$ and $\pi^n(\mathrm{lbl}(y)) \cdot B_n$ are Hurwitz orbits.

The theorem implies that conditions (i)–(iii) from Corollary 6.21 actually hold.

In the real case, this was initially conjectured by Looijenga [42, (3.5)] and proved in a letter from Deligne to Looijenga (crediting discussions with Tits and Zagier) [30]; an equivalent property ([3, Fact 2.2.4]) was independently used in our earlier construction of the dual braid monoid.

These proofs for the real case are based on case-by-case numerology: because the B_n-sets are transitive, it suffices to prove that the cardinality of $y \cdot B_n$ (which is by construction the degree of LL) coincides with that of $(r_1, \ldots, r_n) \cdot B_n$. This is an enumeration problem in W and may be tackled by case-by-case analysis. (The infinite family are easy to deal with, computers can take care of the exceptional types.)

This enumerative approach carries on to our setting: Theorem 7.5 immediately follows from the following propositino.

PROPOSITION 7.6. *Let W be a well-generated complex reflection group. Let c be a Coxeter element in W. The Hurwitz action is transitive on $\mathrm{Red}_R(c)$. When W is irreducible, one has $|\mathrm{Red}_R(c)| = n!h^n/|W|$.*

Proof. The proposition clearly reduces to the case when W is irreducible: in the reducible case, reduced decompositions of Coxeter elements are "shuffles" of reduced decompositions of the Coxeter elements of the irreducible summands.

We prove the result case-by-case. (See Table 1 in Section 2 for the list of cases to be considered.) The complexified real case is studied in [30]. (Transitivity is easy and does not require case-by-case; see, e.g., [3, Prop. 1.6.1].) The $G(e, e, r)$ case combines two results from [6]: Proposition 6.1 (transitivity) and Theorem 8.1 (cardinality).

The case of $G(d, 1, r)$ goes as follows. For all integers i, j with $1 \leq i < j \leq n$, denote by $\tau_{i,j}$ the permutation matrix associated with the transposition $(i\ j) \in \mathfrak{S}_n = G(1, 1, n) \hookrightarrow G(d, 1, r)$. For all $\zeta \in \mu_d$ and all $i \in \{1, \ldots, n\}$, denote by $\rho_{i,\zeta}$ the diagonal matrix $\mathrm{Diag}(1, \ldots, 1, \zeta, 1, \ldots, 1)$, where ζ is in i-th position. There are two types of reflections in $G(d,1,n)$: *long* reflections are elements of the form $\rho_{i,\zeta}$, with $\zeta \neq 1$; *short* reflections are elements of the form $\tau_{i,j}^{\zeta} := \rho_{i,\zeta}^{-1}\tau_{i,j}\rho_{i,\zeta}$, with $\zeta \in \mu_d$ and $1 \leq i < j \leq n$. A typical Coxeter element is $c := \rho_{1,\zeta_d}\tau_{1,2}\tau_{2,3}\cdots\tau_{n-1,n}$. Since Coxeter elements form a single conjugacy class, and since R is invariant under conjugacy, it suffices to prove the claims for this particular c. Let $(r_1,\dots,r_n) \in \mathrm{Red}_R(c)$. Let us prove that it is Hurwitz equivalent to $(\rho_{1,\zeta_d},\tau_{1,2},\tau_{2,3},\dots,\tau_{n-1,n})$. Consider the morphism $G(d,1,n) \to G(1,1,n), g \mapsto \overline{g}$ sending a monomial matrix to the underlying permutation matrix. This map sends $\tau_{i,j}^{\zeta}$ to $\tau_{i,j}$ and $\rho_{i,\zeta}$ to 1. The element $\overline{c}$ is a Coxeter element in $G(1,1,n)$. One deduces that there is a unique long reflection r_{i_0} among $r_1,\dots,r_n$ and that $(\overline{r_1},\dots,\widehat{\overline{r_{i_0}}},\dots,\overline{r_n})$ is a reduced R-decomposition of $\overline{c}$ in $G(1,1,n)$. Up to applying suitable Hurwitz moves, we may assume that $i_0 = 1$. Using the transitivity result already known in the type $G(1,1,n)$ case, we see that $(r_1,\dots,r_n)$ is Hurwitz equivalent to $(\rho_{i,\zeta},\tau_{1,2}^{\alpha_1},\dots,\tau_{n-1,n}^{\alpha_{n-1}})$, where $i \in \{1,\dots,n\}$, $\zeta \in \mu_e - \{1\}$ and $\alpha_1,\dots,\alpha_{n_1} \in \mu_d$. By considering the determinant, we see that $\zeta = \zeta_d$. A direct computation shows that, if $i > 1$,

$$\rho_{i,\zeta}\tau_{1,2}^{\alpha_1}\cdots\tau_{i-1,i}^{\alpha_{i-1}} = \tau_{1,2}^{\alpha_1}\cdots\tau_{i-1,i}^{\alpha_{i-1}}\rho_{i-1,\zeta}.$$

One may use this relation to construct an explicit sequence of Hurwitz moves showing that $(r_1,\dots,r_n)$ is equivalent to $(\rho_{1,\zeta_d},\tau_{1,2}^{\alpha'_1},\dots,\tau_{n-1,n}^{\alpha'_{n-1}})$. One concludes by observing that $\tau_{1,2}^{\alpha'_1}\cdots\tau_{n-1,n}^{\alpha'_{n-1}} = \tau_{1,2}\cdots\tau_{n-1,n}$ forces $\alpha'_1 = \alpha'_2 = \cdots = \alpha'_{n-1} = 1$. The claim about cardinality is not difficult once it is observed that an element of $\mathrm{Red}_R(c)$ is uniquely determined by (1) the position of the long reflection $\rho_{i,\zeta}$, (2) the integer i, (3) a reduced R-decomposition of the Coxeter element $\overline{c}$.

There remains a finite number of exceptional types that are treated by computer. □

Until the end of this section, we assume that W is irreducible.

LEMMA 7.7. *Let $y \in Y^{\mathrm{gen}}$, with label $(s_1,\dots,s_n)$. Let $i \in \{1,\dots,n\}$. Then*

$$l_R(\pi(s_1\cdots s_i)) = i.$$

Proof. Since $y \in Y^{\mathrm{gen}}$, each s_j is a braid reflection, mapped under π to a reflection $r_j \in R$. Thus $l_R(\pi(s_1\cdots s_i)) \leq i$ and $l_R(\pi(s_{i+1}\cdots s_n)) \leq n-i$. Since $\pi(s_1\cdots s_i)\pi(s_{i+1}\cdots s_n)$ is a Coxeter element (Lemma 7.3), it has length n. This forces both inequalities to be equalities. □

LEMMA 7.8. *The restriction of π to the set S of simple elements is injective.*

Proof. Let s and s' be simple elements such that $\pi(s) = \pi(s')$. By Corollary 6.22, we may find $y, y' \in Y^{\mathrm{gen}}$, with $\mathrm{lbl}(y) = (s_1,\dots,s_n)$ and $\mathrm{lbl}(y') = (s'_1,\dots,s'_n)$, and $i,j \in \{0,\dots,n\}$ such that $s = s_1\cdots s_i$ and $s' = s'_1\cdots s'_j$. Both

$\mathrm{LL}(y)$ and $\mathrm{LL}(y')$ consists of n distinct points with distinct real parts; the naive affine homotopy from $\mathrm{LL}(y)$ to $\mathrm{LL}(y')$ can be lifted to a homotopy from y to y'' such that $\mathrm{LL}(y'') = \mathrm{LL}(y')$ and, thanks to the Hurwitz rule, $\mathrm{lbl}(y) = \mathrm{lbl}(y'')$. So, up to replacing y with y'', we can assume that $\mathrm{LL}(y) = \mathrm{LL}(y')$. By Lemma 7.7, we must have $i = j$. Applying Proposition 7.6 to $w = \pi(s)$, we may find $\beta \in B_i$ such that $(\pi(s_1), \ldots, \pi(s_i)) \cdot \beta = (\pi(s'_1), \ldots, \pi(s'_i))$. Similarly, we find $\beta' \in B_{n-i}$ such that $(\pi(s_{i+1}), \ldots, \pi(s_n)) \cdot \beta' = (\pi(s'_{i+1}), \ldots, \pi(s'_n))$. View $B_i \times B_{n-i}$ as a subgroup of B_n (the first factor braids the first i strings, the second factors braids the $n-i$ last strings), and set $\beta'' := (\beta, \beta') \in B_n$. We have $\pi^n(\mathrm{lbl}(y)) \cdot \beta'' = \pi^n(\mathrm{lbl}(y'))$. Applying Theorem 7.5, this implies that $\mathrm{lbl}(y) \cdot \beta'' = \mathrm{lbl}(y')$. Clearly, β'' does not modify the product of the first i terms of the labels. Thus $s = s'$. □

COROLLARY 7.9. *The set of simple elements S is finite.*

7.2. *Simplicial Hurwitz structures.*

Definition 7.10. Let k be a positive integer. We set

$$\mathrm{D}_k(\delta) := \{(s_1, \ldots, s_k) \in S^k | \delta = s_1 \cdots s_k\},$$
$$\mathrm{D}_\bullet(\delta) := (\mathrm{D}_k(\delta))_{k \in \mathbb{Z}_{\geq 0}},$$
$$\mathrm{D}_k(c) := \{(w_1, \ldots, w_k) \in W^k | c = w_1 \ldots w_k \text{ and } l_R(c) = \sum_i l_R(w_i)\},$$
$$\mathrm{D}_\bullet(c) := (\mathrm{D}_k(c))_{k \in \mathbb{Z}_{\geq 0}}.$$

The definition of $\mathrm{D}_\bullet(\delta)$ is a particular case of Definition B.15, but this anticipates what will be discussed in the next section (dual braid monoid Garside structure).

As often with graded objects, it is convenient to view $\mathrm{D}_\bullet(\delta)$ and $\mathrm{D}_\bullet(c)$ as disjoint unions of their graded components.

Let $t = (t_1, \ldots, t_k)$ be a sequence in either $\mathrm{D}_\bullet(\delta)$ or $\mathrm{D}_\bullet(c)$. We may consider

- *faces* of t, sequences of the form
$$(t_1, \ldots, t_{i-1}, t_i t_{i+1}, t_{i+2}, \ldots, t_k);$$
- *degeneracies* of t, sequences of the form
$$(t_1, \ldots, t_i, 1, t_{i+1}, \ldots, t_k).$$

This equips both $\mathrm{D}_\bullet(\delta)$ and $\mathrm{D}_\bullet(c)$ with a *simplicial set* structure (see Appendix B). But there is an additional structure on both sets, provided by “graded Hurwitz action”: each B_k acts on D_k.

Remark 7.11. There are obvious compatibility rules between graded Hurwitz action and simplicial structure. The two structures combine in a fantastic algebraic package — I do not have any good name for it ("simplicial Hurwitz structure," "stratified Hurwitz set"?) — that faithfully encodes both the monodromy theory and the ramification theory of the Lyashko-Looijenga covering. By just considering the action of the "Coxeter element" braid in B_k, combined with the simplicial structure, we obtain the "helicoidal" structure of Remark B.17 in Appendix B (a generalization of *cyclic* structures, in the sense of Connes [24]). But the simplicial Hurwitz structure provides more than that. (In a way, it is a "parabolically helicoidal structure.")

THEOREM 7.12. *The projection maps* $(s_1, \ldots, s_k) \mapsto (\pi(s_1), \ldots, \pi(s_k))$ *induce an isomorphism of simplicial sets*

$$\mathrm{D}_\bullet(\delta) \xrightarrow{\sim} \mathrm{D}_\bullet(c).$$

Proof. Clearly, the map is well defined and compatible with both faces and degeneracies. Injectivity is an obvious consequence of Lemma 7.8.

Surjectivity. Any $(c_1, \ldots, c_k) \in \mathrm{D}_\bullet(c)$ can be obtained by applying a sequences of face maps and degeneracy maps starting from an element in $\mathrm{Red}_R(c)$. (Start by concatenating reduced decompositions of nontrivial c_i's; it is obvious how to get from there back to $(c_1, \ldots, c_k)$.) By compatibility, it is enough to show that elements of $\mathrm{Red}_R(c)$ are in the image. As Hurwitz action is transitive on $\mathrm{Red}_R(c)$ (Proposition 7.6) and compatible with projection, it is enough to show that *one* element of $\mathrm{Red}_R(c)$ is in the image, which is obvious: just take a generic y; the projection of $\mathrm{lbl}(y)$ lies in $\mathrm{Red}_R(c)$. $\square$

Remark 7.13. It is very tempting, and very convenient too, to identify $\mathrm{D}_\bullet(\delta)$ with $\mathrm{D}_\bullet(c)$ and S with its image in W. In particular, we will say that $w \in W$ is a *simple element* if it lies in the image of S. Very often, when considering labels, we will consider those as factorizations of c in W. This viewpoint helps remembering that computations involving labels are about combinatorics in a finite group and that small examples can be worked out by hand.

7.3. *Reduced labels and trivialization of* Y. This following variation on the notion of label was introduced by Vivien Ripoll [49] after he noticed unnecessary complications in earlier versions of the current paper.

Definition 7.14. Let $y \in Y$ with label $(s_1, \ldots, s_k)$. Let $(x_1, \ldots, x_k)$ be the ordered support of $\mathrm{LL}(y)$. The *reduced label* of x is the sequence $\mathrm{rlbl}(y) = (s'_1, \ldots, s'_k)$ defined by

$$s'_i := \begin{cases} s_i & \text{if } i = k \text{ or } \mathrm{re}(x_i) < \mathrm{re}(x_{i+1}), \\ s_i s_{i+1}^{-1} & \text{if } i < k \text{ and } \mathrm{re}(x_i) = \mathrm{re}(x_{i+1}). \end{cases}$$

One way to geometrically understand reduced labels is to see them as braids represented by small loops around points in the support:

Alternately, the reduced label of y can be viewed as the label of a generic y' obtained by applying a small clockwise rotation to y:

LEMMA 7.15. *For all* $y \in Y$, *there exists a real number* $\alpha > 0$, *such that for all* ε *such that* $0 < \varepsilon < \alpha$, *the reduced label of* y *coincides with the label of* $e^{-\sqrt{-1}\pi\varepsilon}y$. *As a consequence*, *the reduced label of* y *is a sequence of simple elements*, *with product* δ.

Note. This result is better understood in light of the general study of the $\mathbb{C}^{\times}$-action; see Section 11 and, in particular, the basic Lemma 11.1.

Proof. Obvious consequence of the Hurwitz rule. □

Rephrasing Corollary 6.18 in terms of reduced labels, we also get

LEMMA 7.16. *For all* $y \in Y$, $\mathrm{rlbl}(y) \in \mathrm{D}_{\bullet}(c)$.

(As announced in Remark 7.13, we choose to work in $\mathrm{D}_{\bullet}(c)$ rather than $\mathrm{D}_{\bullet}(\delta)$.)

Definition 7.17. Let $x \in E_n$ with ordered support $(x_1, \ldots, x_k)$ and ordered multiplicity $(n_1, \ldots, n_k)$. Let $\sigma = (s_1, \ldots, s_l) \in \mathrm{D}_{\bullet}(c)$ be a factorization of c into l simple elements.

Both $(n_1, \ldots, n_k)$ and $(l_R(s_1), \ldots, l_R(s_k))$ are *compositions of* n (finite integral sequences that add up to n). If $(n_1, \ldots, n_k) = (l_R(s_1), \ldots, l_R(s_k))$, we say that x and σ are *compatible*.

We denote by $E_n \boxtimes \mathrm{D}_{\bullet}(c)$ the set of compatible pairs. In other words, it is the pullback

$$\begin{tikzcd}
E_n \boxtimes \mathrm{D}_{\bullet}(c) \arrow[r] \arrow[d] & \mathrm{D}_{\bullet}(c) \arrow[d] \\
E_n \arrow[r] & \{\text{compositions of } n\}.
\end{tikzcd}$$

Clearly, if x and σ are compatible, then σ must be nondegenerate. So the pullback map $E_n \boxtimes \mathrm{D}_{\bullet}(c) \to \mathrm{D}_{\bullet}(c)$ is not surjective. People finding this annoying may want to introduce a specific notation for the set of nondegenerate factorizations (factorizations not containing the trivial element).

Definition 7.18. If $\sigma, \tau \in \mathrm{D}_\bullet(c)$, we say that τ *is a face of* σ, and we write

$$\sigma \vdash \tau$$

if each term in τ is the partial product of consecutive terms in σ. (In other words, τ is obtained from σ by consecutive simplicial face operators.)

For all $\sigma \in \mathrm{D}_\bullet(c)$, we set

$$F_\sigma := \{\rho \in \mathrm{Red}_R(c) | \rho \vdash \sigma\}.$$

Lemma 7.19. *If* $\sigma = (c_1, \dots, c_k) \in \mathrm{D}_\bullet(c)$ *is nondegenerate, then* $F_\sigma \neq \varnothing$ *and consists of a single Hurwitz orbit for the natural subgroup* $B_{l(c_1)} \times \cdots \times B_{l(c_k)}$ *of the braid group* B_n.

Proof. Elements of F_σ are obtained by concatenating reduced decompositions of $c_1, \dots, c_k$. Each c_i is a parabolic Coxeter element (Lemma 7.4) whose factorizations form a single $B_{l(c_i)}$-orbit (Proposition 7.6). $\square$

Theorem 7.20 (trivialization of Y). *The map* $\mathrm{LL} \times \mathrm{rlbl}$ *induces a bijection*

$$\mathrm{LL} \times \mathrm{rlbl} : Y \xrightarrow{\sim} E_n \boxtimes \mathrm{D}_\bullet(c).$$

Proof. That the image of rlbl lies in $\mathrm{D}_\bullet(c)$ is Lemma 7.16. That the image of $\mathrm{LL} \times \mathrm{rlbl}$ lies in $E_n \boxtimes \mathrm{D}_\bullet(c)$ follows from Corollary 6.17. By Theorem 7.5, $\mathrm{LL} \times \mathrm{rlbl}$ restricts to a bijection $Y^{\mathrm{reg}} \xrightarrow{\sim} E_n^{\mathrm{reg}} \boxtimes \mathrm{D}_\bullet(c)$. What remains at stake is the behavior in the singular part of the covering.

Surjectivity of $\mathrm{LL} \times \mathrm{rlbl}$. Let $(e, \sigma) \in E_n \boxtimes \mathrm{D}_\bullet(c)$. Let $(e_m)_{m \in \mathbb{Z}_{\geq 0}}$ be a sequence of points in E_n^{gen} converging to e. (Because E_n^{gen} is dense in E_n, such a sequence exists.) Let $\sigma' \in F_\sigma$ (which is nonempty by Lemma 7.19). By Theorem 7.5, there exists a unique sequence $(y_m(\sigma'))_{m \in \mathbb{Z}_{\geq 0}}$ of points in Y such that, for all m, $\mathrm{LL}(y_m) = e_m$ and $\mathrm{rlbl}(y_m(\sigma')) = \sigma'$. This sequence lies in $\mathrm{LL}^{-1}(\{e\} \cup \bigcup_m \{e_m\})$, a compact subset of Y (a finite morphism, LL is proper; the pre-image of a compact subset under a proper morphism is compact) and admits an adherence value y such that $\mathrm{LL}(y) = e$. Applying the Hurwitz rule, one observes that $\mathrm{rlbl}(y) = \sigma$.

Note that the above argument also shows that the ramification degree of LL at y is $|F_\sigma|$, as each $\mathrm{LL}^{-1}(e_m)$ consists of the $|F_\sigma|$ distinct points $\{y_m(\sigma') | \sigma' \in F_\sigma\}$.

Injectivity of $\mathrm{LL} \times \mathrm{rlbl}$. Let $e \in E_n$. As LL is a finite morphism of degree $|\mathrm{Red}_R(c)|$, the ramification formula over e says that

$$|\mathrm{Red}_R(c)| = \sum_{y \in LL^{-1}(e)} d(y),$$

where $d(y)$ is the ramification degree at y. (See, for example, [34, Exam. 4.3.7].) By grouping according to $\mathrm{rlbl}(y)$, we get

$$|\operatorname{Red}_R(c)| = \sum_{\substack{\sigma \in \mathrm{D}_\bullet(c) \\ \sigma \text{ compatible with } e}} |LL^{-1}(e) \cap \mathrm{rlbl}^{-1}(\sigma)| \cdot |F_\sigma|.$$

Clearly, each element of $\operatorname{Red}_R(c)$ lies in F_σ for exactly one $\sigma \in \mathrm{D}_\bullet(c)$ such that σ is compatible with e. (This σ is obtained by multiplying consecutive terms according to the multiplicity pattern defined by e.) So

$$|\operatorname{Red}_R(c)| = \sum_{\substack{\sigma \in \mathrm{D}_\bullet(c) \\ \sigma \text{ compatible with } e}} |F_\sigma|.$$

To conclude, we observe that the surjectivity part of theorem implies that, for each σ compatible with e, $|LL^{-1}(e) \cap \mathrm{rlbl}^{-1}(\sigma)| \geq 1$. This forces each $|LL^{-1}(e) \cap \mathrm{rlbl}^{-1}(\sigma)|$ to be equal to 1. □

We can equip $E_n \boxtimes \mathrm{D}_\bullet(c)$ with a natural topology that turns the bijection of Theorem 7.20 into a homeomorphism. This topology is as follows. Let $(x, (c_1, \ldots, c_m)) \in E_n \boxtimes \mathrm{D}_\bullet(c)$. Choose a system of elementary tunnels $(T_1, \ldots, T_m)$ (as introduced above Definition 6.9). Neighborhoods for $(x, (c_1, \ldots, c_m))$ are obtained by considering all compatible pairs $(x', (d_1, \ldots, d_l))$ as follows:

(1) We choose Ω, a small enough neighborhood of x in E_n such that (the projections in $\mathbb{C}$ of) the tunnels $T_1, \ldots, T_m$ do not intersect any point in any configuration $x'' \in \Omega$. That such neighborhoods exist and form a basis for the topology of E_n is obvious (and is the analog in E_n of the notion of T-neighborhood from Definition 6.14).
(2) We allow x' to be any point in Ω.
(3) Combining (1) and (2), we get Hurwitz rule equations for $T_1, \ldots, T_m$, expressing relations between the d_j's to the c_i's that must be satisfied.

Remark 7.21. In the previous section, we used the homotopy lifting property of the unramified part of LL to lift paths in E_n^{reg} to paths in Y. In general, if γ is a path $[0,1] \to E_n$, there may be more than one way to lift γ to a path $\tilde{\gamma}$ such that $\mathrm{LL} \circ \tilde{\gamma} = \gamma$, even if one fixes the initial point $\tilde{\gamma}(0)$. However, a consequence of Theorem 7.20 and the above discussion is that if γ has *non-decreasing ramification* (i.e., if points can be merged but not unmerged when t increases), then there exists a unique continuous lift $\tilde{\gamma}$ once $\tilde{\gamma}(0)$ has been fixed. This will be very useful for constructing explicit retractions in the next sections.

Remark 7.22. Charting the various geometric constructions, we get

$$\begin{tikzcd}
\bigcup_{H\in\mathcal{A}} H \arrow[r] \arrow[d] & V \arrow[d, "W\backslash"'] & & \\
\mathcal{H} \arrow[r] & W\backslash V \arrow[r, equal] & Y\times\mathbb{C} \arrow[d, "\text{Saito}"] & \\
 & \mathcal{K} \arrow[r] \arrow[d] & Y \arrow[r, "\text{rlbl}"] \arrow[d, "\text{LL}"'] \arrow[dr, equal, "7.20"] & \mathrm{D}_\bullet(c) \\
 & \mathcal{H}_{A_{n-1}} \arrow[r] & E_n & E_n \boxtimes \mathrm{D}_\bullet(c) \arrow[l] \arrow[u] \\
 & \bigcup_{1\le i<j\le n} H_{i,j} \arrow[r] \arrow[u] & \mathbb{C}^n. \arrow[u, "\mathfrak{S}_n\backslash"'] &
\end{tikzcd}$$

Here is what we have done so far. We set out to study the higher homotopy groups of $V^{\mathrm{reg}} = V - \bigcup_{H\in\mathcal{A}} H$. Because W has no ramification on V^{reg}, we may work in the quotient $W\backslash V^{\mathrm{reg}}$, which we view as a singular fibration over Y. Individually, the generic fibers (outside $\mathcal{K}$) and degenerated fibers (above $\mathcal{K}$) are fairly easy to control: they all are punctured complex lines.

Most of the hard work happens in the base space Y, which controls how fibers are glued together (degeneracy, monodromy, etc.). To visualize Y and perform computations in it, we compare it via LL with a classical configuration space E_n. The trivialization Theorem 7.20 gives a neat description, in terms of the combinatorics of W, of both generic and singular fibers of LL.

One way to see $\mathrm{D}_\bullet(c)$ is think of it as the "Galois group" of LL — except that LL is *not* a Galois covering. Metaphorically, $\mathrm{LL} : Y \to E_n$ is a *virtual reflection group*: like the quotient map $V \to W\backslash V$, it is a finite algebraic morphism between two affine spaces; when such a map is Galois, the theorem of Chevalley-Shephard-Todd says that it must be the quotient map of a complex reflection group.

7.4. *A variation*: *trivializing* $W\backslash V$ *and* $W\backslash V^{\mathrm{reg}}$. The way from $W\backslash V^{\mathrm{reg}}$ to $E_n \boxtimes \mathrm{D}_\bullet(c)$, as summarized in Remark 7.22, is a bit long and complicated. This can be simplified thanks to variations on the definitions of LL and E_n:

Definition 7.23 (extended Lyashko-Looijenga morphism). Let (y,z) be a point in $W\backslash V \simeq Y\times\mathbb{C}$. We denote by $\overline{\mathrm{LL}}((y,z))$ the configuration $\mathrm{LL}(y) - z$, obtained by shifting by $-z$ all points in $\mathrm{LL}(y)$.

Note that the image of $\overline{\mathrm{LL}}$ lies in $\overline{E}_n$ (the space of not necessarily centered configurations) rather than just E_n (the space of centered configurations).

Definition 7.24. We denote by $\overline{E}_n^{\circ}$ the subspace of $\overline{E}_n$ consisting of configurations not containing 0.

For $(y,z) \in W\backslash V$, we also set $\mathrm{rlbl}((y,z)) := \mathrm{rlbl}(y)$. The notion of *compatible pairs* in $E_n \times \mathrm{D}_\bullet(c)$ carries on to $\overline{E}_n \times \mathrm{D}_\bullet(c)$, and we define $\overline{E}_n \boxtimes \mathrm{D}_\bullet(c)$.

THEOREM 7.25. *The map*

$$\begin{aligned} \overline{\mathrm{LL}} \times \mathrm{rlbl} : W\backslash V &\longrightarrow \overline{E}_n \times \mathrm{D}_\bullet(c) \\ (y,z) &\longmapsto (LL(y) - z, \mathrm{rlbl}(y)) \end{aligned}$$

is a homeomorphism. It induces by restriction a homeomorphism

$$\overline{\mathrm{LL}} \times \mathrm{rlbl} : W\backslash V^{\mathrm{reg}} \xrightarrow{\sim} \overline{E}_n^{\circ} \boxtimes \mathrm{D}_\bullet(c).$$

Proof. Consider the map

$$\begin{aligned} W\backslash V &\longrightarrow \mathbb{C} \times (E_n \boxtimes \mathrm{D}_\bullet(c)) \\ (y,z) &\longmapsto (z, LL(y), \mathrm{rlbl}(y)). \end{aligned}$$

Using Theorem 7.20 and the subsequent discussion, we see that it is an homeomorphism. Now observe that

$$\begin{aligned} \mathbb{C} \times E_n &\longrightarrow \overline{E} \\ (z, \{x_1, \dots, x_n\}) &\longmapsto \{x_1 - z, \dots, x_n - z\} \end{aligned}$$

is a homeomorphism: indeed, z can be recovered from $\{x_1 - z, \dots, x_n - z\}$ as its barycenter. The theorem follows easily. □

The next three sections rely on the trivialization of Theorem 7.20, but it is possible to rephrase them using Theorem 7.25 instead. Theorem 7.25 will be especially useful in Section 11, where we will focus on rotational motions (rather than the translational motions used in the next three sections).

8. **The dual braid monoid**

Here again, W is an irreducible well-generated complex reflection group. For simplicity, we further assume, in this section and in the following ones, that W is generated by 2-reflections: by Theorem 2.2, this suffices to address the $K(\pi,1)$ conjecture. We restrict ourselves to 2-reflection groups not because the construction would otherwise fail (it does work), but because some case-by-case arguments (especially Lemma 8.6) would need to be more extensively detailed.

We keep the notation from the previous sections. Recall that an element $b \in B$ is *simple* if $b = b_T$ for some tunnel T and that the set of simple elements is denoted by S.

Definition 8.1. The *dual braid monoid* is the submonoid M of B generated by S.

Consider the binary relation $\preccurlyeq$ defined on S as follows. Let s and s' be simple elements. We write $s \preccurlyeq s'$ if and only if there exist $(y,z) \in W\backslash V^{\mathrm{reg}}$, $L, L' \in \mathbb{R}_{\geq 0}$ with $L \leq L'$ such that (y,z,L) is a tunnel representing s and (y,z,L') is a tunnel representing s'.

We write $s \prec s'$ when $s \preccurlyeq s'$ and $s \neq s'$.

This section is devoted to the proof of

THEOREM 8.2. *The monoid M is a Garside monoid, with a set of simple elements S and Garside element δ. The relation $\preccurlyeq$ defined above on S is the restriction to S of the left divisibility order in M.*

The monoid M generates B, which inherits a structure of Garside group.

A survey of Garside theory is provided in Appendix B.

Remark 8.3. The theorem blends two results of distinct natures: one is about the Garside structure of a certain monoid $\mathbf{M}(P_c)$ (see Lemma 8.8 for a presentation of $\mathbf{M}(P_c)$); the other one identifies M with $\mathbf{M}(P_c)$. The latter essentially amounts to writing a presentation for B. It is a substitute for Brieskorn's presentation theorem [16] except that our presentation involves *dual braid relations* instead of Artin-Tits braid relations.

Several results mentioned in the introduction follow from this theorem. Theorem 0.4 does not assume irreducibility, but follows immediately from the irreducible case since direct products of Garside groups are Garside groups. Another important consequence of Theorem 8.2 is that one obtains a nice simplicial complex, $\mathrm{gar}(G,\Sigma)$ (see Definition B.11), that is both contractible (Theorem B.14) and acted on by B. This complex will serve as a simplicial model for the universal cover of $W\backslash V^{\mathrm{reg}}$. The strategy of proof of Theorem 8.2 is very similar to the one in [6].

PROPOSITION 8.4. *For all $s \in S$, we have $l(s) = l_R(\pi(s))$. For all $s, s' \in S$, the following statements are equivalent*:

(i) $s \preccurlyeq s'$;

(ii) $\exists s'' \in S, ss'' = s'$;

(iii) $\pi(s) \preccurlyeq_R \pi(s')$.

Proof. The first statement follows from Lemma 7.7.

(i) $\Rightarrow$ (ii). Assume that $s \preccurlyeq s'$, and choose $(y,z) \in W\backslash V^{\mathrm{reg}}$ and $L \leq L'$ such that (y,z,L) is a tunnel representing s and (y,z,L') is a tunnel representing s'. Then $(y, z+L, L'-L)$ is a tunnel representing $s'' \in S$ such that $ss'' = s'$.

(ii) $\Rightarrow$ (iii). The natural length function l is additive on B. Thus, under (ii), we have $l(s) + l(s'') = l(s')$. On the other hand, for all $\sigma \in S$, $l(\sigma) =$

$l_R(\pi(\sigma))$. Thus $l_R(\pi(s)) + l_R(\pi(s'')) = l_R(\pi(s'))$. Since $\pi(s'') = \pi(s)^{-1}\pi(s')$, this implies that $\pi(s) \preccurlyeq_R \pi(s')$.

(iii) $\Rightarrow$ (i). We may find $y, y' \in Y^{\mathrm{gen}}$, with $\mathrm{lbl}(y) = (s_1, \ldots, s_n)$ and $\mathrm{lbl}(y') = (s'_1, \ldots, s'_n)$ such that $s = s_1 s_2 \cdots s_{l(s)}$ and $s' = s'_1 s'_2 \cdots s'_{l(s')}$. Set $w := \pi(s)$, $w' := \pi(s')$ and, for $i = 1, \ldots, n$, set $r_i := \pi(s_i)$ and $r'_i := \pi(s'_i)$. Assuming (iii), we may find $r''_1, \ldots, r''_{l(s')-l(s)} \in R$ such that $r_1 \cdots r_{l(s)} r''_1 \cdots r''_{l(s')-l(s)} = r'_1 \cdots r'_{l(s')}$. The sequences

$$(r_1, \ldots, r_{l(s)}, r''_1, \ldots, r''_{l(s')-l(s)})$$

and

$$(r'_1, \ldots, r'_{l(s')})$$

both lie in $\mathrm{Red}_R(w')$. Since $w' \preccurlyeq c$, both sequences lie in the same Hurwitz orbit (Proposition 7.6). Thus

$$(r'_1, \ldots, r'_n)$$

and

$$(r_1, \ldots, r_{l(s)}, r''_1, \ldots, r''_{l(s')-l(s)}, r'_{l(s')+1}, \ldots, r'_n)$$

are transformed one onto the other by Hurwitz action of a braid $\beta \in B_n$ only braiding the first $l(s')$ strands. The Hurwitz transform of

$$\mathrm{lbl}(y') = (s'_1, \ldots, s'_n)$$

by β is the label

$$(s''_1, \ldots, s''_n)$$

of some $y'' \in Y^{\mathrm{gen}}$. Since the braid only involves the first $l(s')$ strands, $s''_1 \cdots s''_{l(s')} = s'_1 \cdots s'_{l(s')} = s'$. One has $\pi(s''_i) = r_i$ for $i = 1, \ldots, l(s)$; thus $\pi(s''_1 \cdots s''_{l(s)}) = \pi(s_1 \cdots s_{l(s)})$. By Lemma 7.8, this implies that $s''_1 \cdots s''_{l(s)} = s_1 \cdots s_{l(s)} = s$. For x with small enough real and imaginary parts, one may find real numbers L, L' with $0 < L \leq L'$ such that (y'', x, L) is a tunnel with associated simple $s''_1 \cdots s''_{l(s)} = s$ and (y'', x, L') is a tunnel with associated simple $s''_1 \cdots s''_{l(s')} = s'$. $\square$

PROPOSITION 8.5. *The map π restricts to an isomorphism $(S, \preccurlyeq) \xrightarrow{\sim} ([1, c], \preccurlyeq_R)$. In particular, $\preccurlyeq$ is an order relation on S.*

Proof. The previous proposition, applied to $s' = \delta$, proves that $\pi(S) \subseteq [1, c]$. It also proves that π induces a morphism of sets with binary relations $(S, \preccurlyeq) \to ([1, c], \preccurlyeq_R)$. The injectivity is Lemma 7.8.

Surjectivity: Choose $y \in Y^{\mathrm{gen}}$. Let $(s_1, \ldots, s_n) := \mathrm{lbl}(y)$. Let $r_i := \pi(s_i)$. We have $(r_1, \ldots, r_n) \in \mathrm{Red}_R(c)$. Let $w \in [1, c]$. We may find $(r'_1, \ldots, r'_n) \in \mathrm{Red}_R(c)$ such that $r'_1 \cdots r'_{l_T(w)} = w$. By Proposition 7.6, $(r'_1, \ldots, r'_n)$ is a Hurwitz transformed of $(r_1, \ldots, r_n)$; thus there exists $y' \in Y^{\mathrm{gen}}$ such that $\pi_*(\mathrm{lbl}(y')) = (r'_1, \ldots, r'_n)$. The simple element that is the product of the first $l(w)$ terms of $\mathrm{lbl}(y')$ is in $\pi^{-1}(w)$. $\square$

We have the following key lemma.

Lemma 8.6. *The poset* $([1,c], \preccurlyeq_R)$ *is a lattice.*

Proof. The real case is done in [3]; a recent beautiful case-free argument has been found by Brady-Watt [15]. The case of $G(e,e,r)$ is done in [6]. The remaining cases have been checked by computer. □

Definition 8.7. The *dual braid relations* with respect to W and c are all the formal relations of the form

$$rr' = r'r'',$$

where $r, r', r'' \in R$ are such that $r \neq r'$, $rr' \in [1,c]$, and the relation $rr' = r'r''$ holds in W.

Clearly, dual braid relations only involve reflections in $R \cap [1,c]$. When W is complexified real, $R \subseteq [1,c]$. This does not hold in general. (See the tables at the end of the article.)

As in [3] (see also [5, §1] and Appendix B), one endows $[1,c]$ with a partial product and obtains a monoid $\mathbf{M}(P_c)$.

From Lemma 8.6, we will deduce that $\mathbf{M}(P_c)$ is a Garside monoid. We will also identify $\mathbf{M}(P_c)$ with M. The following lemma generalizes [3, Th. 2.1.4].

Lemma 8.8. *Let* $R_c := R \cap [1,c]$. *The monoid* $\mathbf{M}(P_c)$ *admits the monoid presentation*

$$\mathbf{M}(P_c) \simeq \langle R_c \mid \textit{dual braid relations} \rangle .$$

Remark 8.9. When viewed as a group presentation, the presentation of the lemma is a presentation for $\mathbf{G}(P_c)$. As soon as we prove Theorem 8.2, Lemma 8.8 will give an explicit presentation for B. A way to reprove Theorem 0.6 for groups different from G_{31} is by simplifying the (redundant) presentation given by the lemma. This does not involve any computer-assisted monodromy computation.

Proof of Lemma 8.8. By definition, $\mathbf{M}(P_c)$ admits the presentation with generators R_c and a relation $r_1 \cdots r_k = r'_1 \cdots r'_k$ for each pair $(r_1, \dots, r_k)$, $(r'_1, \dots, r'_k)$ of reduced R-decompositions of the same element $w \in [1,c]$. Call these relations *Hurwitz relations*. By transitivity of the Hurwitz action on $\mathrm{Red}_R(c)$ (Proposition 7.6), the Hurwitz relations are consequences of the dual braid relations. The dual braid relations clearly hold in $\mathbf{M}(P_c)$. (To see this, complete (r, r') to an element $(r, r', r_3, \dots, r_n) \in \mathrm{Red}_R(c)$.) This proves the lemma. □

Proof of Theorem 8.2. Set $R_c := R \cap [1,c]$. We are in the situation of Section 0.4 in [3]: (W, R_c) is a *generated group* and c is *balanced*. (One first observes that $\{w | w \preccurlyeq_{R_c} c\} = \{w | w \preccurlyeq_R c\}$ and $\{w | c \succcurlyeq_{R_c} w\} = \{w | c \succcurlyeq_R w\}$; one concludes noting that c is balanced with respect to (W, R), which is immediate since R is invariant by conjugacy.) By Lemma 8.6 and [3, Th. 0.5.2], the premonoid $P_c := [1, c]$ (together with the natural partial product) is a Garside premonoid. We obtain a Garside monoid $\mathbf{M}(P_c)$.

By Proposition 8.5, the restriction of π is a bijection from S to P_c. Let ϕ be the inverse bijection. Let $w, w' \in P_c$. Assume that the product $w.w'$ is defined in P_c. Let w'' be the value of this product. One has $w \preccurlyeq_R w''$, thus $\phi(w) \preccurlyeq \phi(w'')$ (using again Proposition 8.5), and we may find b' in S such that $\phi(w)b' = \phi(w'')$ (Proposition 8.4).

Claim. $b' = \phi(w')$. Indeed, b' and $\phi(w')$ are two simple elements whose image by π is $w^{-1}w''$; one concludes using Lemma 7.8. This proves that ϕ induces a premonoid morphism $P_c \to S$ (where S is equipped with the restriction of the monoid structure) and thus induces a monoid morphism $\mathbf{M}(P_c) \to M$ and a group morphism $\Phi : \mathbf{G}(P_c) \to B$.

Let use prove that Φ is an isomorphism. Choose a basepoint $y \in Y^{\mathrm{gen}}$. Let $\gamma_1, \ldots, \gamma_n$ be generators of $\pi_1(Y - \mathcal{K}, y)$. Let $(s_1, \ldots, s_n)$ be the label of y. Let us reinterpret the presentation from Theorem 4.4 in terms of Hurwitz action. Since $\mathrm{LL} : Y - \mathcal{K} \to E_n^{\mathrm{reg}}$ is a covering, $\pi_1(Y - \mathcal{K}, y)$ may be identified with a subgroup $H \subseteq B_n$. The generators $f_1, \ldots, f_n$ in Theorem 4.4 may be chosen to be $s_1, \ldots, s_n$, and the monodromy automorphism $\phi_1, \ldots, \phi_m$ are obtained by Hurwitz action on $s_1, \ldots, s_n$. Let $h \in H$. Let $(s'_1, \ldots, s'_n) := h(s_1, \ldots, s_n)$; by this, we mean the Hurwitz action of h on the free group generated by $s_1, \ldots, s_n$; the s'_i's are words in the s_i's. Call *Van Kampen relations* the relations of the type $s'_i = s_i$, for any $i \in \{1, \ldots, n\}$, $h \in H$, and s'_i obtained as above. We have

$$B \simeq \langle s_1, \ldots, s_n \mid \text{Van Kampen relations} \rangle.$$

The map π induces a bijection from $A := \pi^{-1}(R_c) \subseteq S$ to R_c. Let $r_1, \ldots, r_n$ be the images of $s_1, \ldots, s_n$. By transitivity of Hurwitz action on $\mathrm{Red}_R(c)$, the group $\mathbf{G}(P_c)$ is generated by $r_1, \ldots, r_n$, the remaining generators in the presentation of Lemma 8.8 appearing as conjugates of $r_1, \ldots, r_n$ (by successive use of dual braid relations). Our generating sets are compatible, and the morphism

$$\begin{aligned} \Phi : \mathbf{G}(P_c) \simeq \langle R_c \mid & \text{ dual braid relations} \rangle \\ & \to B \simeq \langle s_1, \ldots, s_n \mid \text{Van Kampen relations} \rangle \end{aligned}$$

is defined by $r_i \mapsto s_i$. Add to the presentation of B formal generators indexed by

$$A - \{s_1, \ldots, s_n\}$$

as well as the dual braid relations $\pi(r)\pi(r') = \pi(r')\pi(r'')$. Since the relations already hold in $\mathbf{G}(P_c)$, they hold in B, and we obtain a new presentation:

$$B \simeq \langle A \mid \text{Van Kampen relations on } \{s_1, \ldots, s_n\}, \text{ dual braid relations on } A \rangle .$$

To conclude that Φ is an isomorphism, it is enough to observe that the dual braid relations encode the full Hurwitz action of B_n of $(s_1, \dots, s_n)$, while the Van Kampen relations encode the action of $H \subseteq B_n$: thus Van Kampen relations are consequences of dual braid relations, and $\mathbf{G}(P_c)$ and B are given by equivalent presentations.

Since Φ is an isomorphism and $\mathbf{M}(P_c)$ naturally embeds in $\mathbf{G}(P_c)$ (this is a crucial property of Garside monoids [26]), $\mathbf{M}(P_c)$ is isomorphic to its image M in B. The rest of theorem is clear. □

As mentioned earlier, one may view B as a "reflection group," generated by the set $\mathcal{R}$ of all braid reflections. An element of M is in $\mathcal{R}$ if and only if it has length 1 for the natural length function or, equivalently, if it is an *atom* (i.e., an element that has no strict divisor in M except the unit). By Proposition 7.6 and Theorem 7.5, there is a bijection between $\mathrm{Red}_R(c)$ and the image of $\mathrm{lbl} : Y^{\mathrm{gen}} \to \mathcal{R}^n$. This image is clearly contained in $\mathrm{Red}_{\mathcal{R}}(\delta)$. The conjecture below claims an analogue in B of the transitivity of the Hurwitz action on $\mathrm{Red}_R(c)$. (δ is the natural substitute for a Coxeter element in B.) It implies that any element in $\mathrm{Red}_{\mathcal{R}}(\delta)$ is the label of some $y \in Y^{\mathrm{gen}}$.

Conjecture 8.10. *The Hurwitz action of B_n on* $\mathrm{Red}_{\mathcal{R}}(\delta)$ *is transitive.*

9. Chains of simple elements

Here again, W is an irreducible well-generated complex reflection group, and the notation from the previous section is still in use. For $k = 0, \dots, n$, we denote by $\mathcal{C}_k$ the set of (strict) chains in $S - \{1\}$ of cardinal k, i.e., the set of k-tuples $(c_1, \dots, c_k)$ in S^k such that

$$1 \prec c_1 \prec \cdots \prec c_k$$

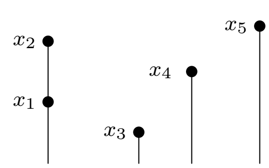
or, equivalently, the set of k-tuples $(c_1, \dots, c_k)$ in M^k such that

$$1 \prec c_1 \prec \cdots \prec c_k \preccurlyeq \delta.$$

It is convenient to write $\{1 \prec c_1 \prec \cdots \prec c_k\}$ instead of $(c_1, \dots, c_k)$.

We set $\mathcal{C} := \bigsqcup_{k=0}^{n} \mathcal{C}_k$. Let $C := \{1 \prec c_1 \prec \cdots \prec c_k\} \in \mathcal{C}$. We say that $y \in Y$ *represents* C if there exist $x \in U_y$ and real numbers $L_1, \dots, L_k$ such that $0 < L_1 < \cdots < L_k$ and, for $i = 1, \dots, k$, (y, x, L_i) is a tunnel representing C_i.

Example. If $\mathrm{LL}(y)$ is as in the illustration below and $(s_1, \dots, s_5) = \mathrm{lbl}(y)$, y represents $1 \prec s_1 \prec s_1s_3 \prec s_1s_3s_4 \prec s_1s_3s_4s_5$, $1 \prec s_2 \prec s_2s_4 \prec s_2s_4s_5$, $1 \prec s_2 \prec s_2s_5$ and their subchains, but does not represent $1 \prec s_2 \prec s_2s_3$ nor $1 \prec s_3 \prec s_3s_5$.



Definition 9.1. For all $C \in \mathcal{C}$, we set $Y_C := \{y \in Y | y \text{ represents } C\}$.

To illustrate this notion, we observe that implication (ii) $\Rightarrow$ (i) from Proposition 8.4 expresses that for all $C \in \mathcal{C}_2$, Y_C is nonempty. Based on the results from the previous sections, this easily generalizes to

Lemma 9.2. *For all $C \in \mathcal{C}$, the space Y_C is nonempty.*

The goal of this section is to prove

Proposition 9.3. *For all $C \in \mathcal{C}$, the space Y_C is contractible.*

This technical result will be used in Section 10, when studying the nerve of an open covering of the universal cover of $W \backslash V^{\mathrm{reg}}$: we will need to prove that certain nonempty intersections of open sets are contractible, and these intersections will appear as fiber bundles over some Y_C, with contractible fibers.

The proposition is not very deep nor difficult but somehow *inconvenient* to prove since the retraction will be described via LL, through ramification points. The following particular cases are easier to obtain:

- If C is the chain $1 \prec \delta$, then $Y_C = Y \simeq \mathbb{C}^{n-1}$.
- More significantly, let W be a complex reflection group of type A_2. Up to renormalization, the discriminant is $X_2^2 + X_1^3$. Identify Y with $\mathbb{C}$. For all $y \in \mathbb{C}$, $\mathrm{LL}(y) = \{\pm(-y)^{3/2}\}$. In particular, $\mathrm{LL}(1) = \{\pm\sqrt{-1}\}$. Let s be the simple element represented by the tunnel with $y = 1$, $x = -1$ and $L = 2$. Let C be the chain $1 \prec s$. Then Y_C is the open cone consisting of nonzero elements of $\mathbb{C}$ with argument in the open interval $(-2\pi/3, 2\pi/3)$.
- Assume that $C \in \mathcal{C}_n$. All points in Y_C are generic. Consider the map $Y^{\mathrm{gen}} \to \mathcal{C}_n \times E_n$ sending y to the pair $(\{1 \prec s_1 \prec s_1 s_2 \prec \cdots \prec s_1 s_2 \cdots s_n = \delta\}, \mathrm{LL}(y))$, where $(s_1, \ldots, s_n) = \mathrm{lbl}(y)$. This map is a homeomorphism. The $(Y_C)_{C \in \mathcal{C}_n}$ are the connected components of Y^{gen}. Each of these components is homeomorphic to E_n^{gen}, which is contractible. These $(Y_C)_{C \in \mathcal{C}_n}$ are some analogues of chambers.

In the following proposition, if $A \subseteq \mathrm{LL}(y)$ is a submultiset, the *deep label* of A is the sequence $(t_1, \ldots, t_p)$ of labels (with respect to y) of points in the support of A that are deep with respect to A. (Since these points may not be deep in $\mathrm{LL}(y)$, the deep label of A is not necessarily a subsequence of the deep label of y.)

Lemma 9.4. *Let $y \in Y$. Let $T = (y, z, L)$ and $T' = (y, z', L')$ be two tunnels in L_y such that $b_T = b_{T'}$. Then T and T' cross the same intervals among $I_1, \ldots, I_n$. (In other words, T and T' are homotopic as tunnels drawn in L_y.)*

Proof. Up to perturbing y, we may assume that $y \in Y^{\mathrm{gen}}$. Let $(s_1, \ldots, s_n) := \mathrm{lbl}(y)$ and $(r_1, \ldots, r_n) := \pi^n(\mathrm{lbl}(y))$. Let $i_1, \ldots, i_l$ (resp. $j_1, \ldots, j_m$) be the successive indices of the intervals among $I_1, \ldots, I_n$ crossed by T (resp. T'). We have $b_T = s_{i_1} \cdots s_{i_l}$ and $b_{T'} = s_{j_1} \cdots s_{j_m}$. Assuming that $b_T = b_{T'}$, we obtain $s_{i_1} \cdots s_{i_l} = s_{j_1} \cdots s_{j_m}$ and $r_{i_1} \cdots r_{i_l} = r_{j_1} \cdots r_{j_m}$. Let $w := r_{i_1} \cdots r_{i_l}$. By Lemma 7.7, $l = m$; both $(r_{i_1}, \ldots, r_{i_l})$ and $(r_{i_1}, \ldots, r_{i_l})$ are reduced decompositions of w and we have

$$(*) \qquad \ker(w-1) = \bigcap_{k=1}^{l} \ker(r_{i_k} - 1) = \bigcap_{k=1}^{l} \ker(r_{j_k} - 1).$$

Assume that $(i_1, \ldots, i_l) \neq (j_1, \ldots, j_l)$. We may find $j \in \{j_1, \ldots, j_l\}$ such that, for example, $i_1 < \cdots < i_k < j < i_{k+1} < \ldots i_l$. Noting that the element $r_{i_1} \cdots r_{i_k} r_j r_{i_{k+1}} \cdots r_{i_l}$ is a parabolic Coxeter element, we deduce that $\ker(r_j - 1) \not\supseteq \ker(w-1)$. This contradicts $(*)$. ☐

Proposition 9.5. *Let $C = \{1 \prec c_1 \prec c_2 \prec \cdots \prec c_m\}$ be a chain in $\mathcal{C}_m$. Let $y \in Y$, let $(x_1, \ldots, x_k)$ be the ordered support of* $\mathrm{LL}(y)$ *and $(s_1, \ldots, s_k)$ be the label of y. The following assertions are equivalent*:

(i) $y \in Y_C$.

(ii) *There exists a partition of* $\mathrm{LL}(y)$ *into $m+1$ submultisets $A_0, \ldots, A_m$ such that*

  (a) *For all $i, j \in \{1, \ldots, m\}$ with $i < j$, for all $x \in A_i$ and all $x' \in A_j$, we have* $\mathrm{re}(x) < \mathrm{re}(x')$;

  (b) *for all $x \in A_0$, one has*

$$\mathrm{re}(x) < \min_{x' \in A_1 \cup \cdots \cup A_m} \mathrm{re}(x')$$

  or

$$\mathrm{im}(x) < \min_{x' \in A_1 \cup \cdots \cup A_m} \mathrm{im}(x')$$

  or

$$\mathrm{re}(x) > \max_{x' \in A_1 \cup \cdots \cup A_m} \mathrm{re}(x');$$

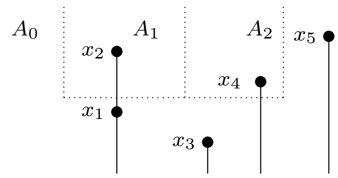
  (c) *for all $i \in \{1, \ldots, m\}$, the product of the deep label of A_i is $c_{i-1}^{-1} c_i$ (where one sets $c_0 = 1$).*

Moreover, in situation (ii), *the partition* $\mathrm{LL}(y) = A_0 \sqcup \cdots \sqcup A_m$ *is uniquely determined by y and C.*

The picture below illustrates the proposition for particular y and C. Here $\mathrm{lbl}(y) = (s_1, \ldots, s_5)$ and the considered chain is

$$C = \{1 \prec s_2 \prec s_2 s_4\}.$$

We have chosen tunnels $T_1 = (y, x, L_1)$ and $T_2 = (y, x, L_2)$, with $L_1 < L_2$, such that $b_{T_1} = s_2$ and $b_{T_2} = s_2 s_4$. The dotted lines represent these tunnels as well as the vertical half-lines above x, $x + L_1$ and $x + L_2$. They partition the complex line into three connected components; the partition $A_0 \sqcup A_1 \sqcup A_2$ is the associated partition of $\mathrm{LL}(y)$. It is clear that the possibility of drawing the tunnels is subject precisely to the conditions on $A_0 \sqcup A_1 \sqcup A_2$ expressed in the proposition.



Proof. (i) ⇒ (ii): Let $(y, z, L_1), \ldots, (y, z, L_m)$ be tunnels representing the successive nontrivial terms of $C = \{1 \prec c_1 \prec \cdots \prec c_m\}$. Set $z_0 := z$ and, for $i = 1, \ldots, m$, $z_i := z + L_i$. We have

$$\operatorname{re}(z_0) < \operatorname{re}(z_1) < \cdots < \operatorname{re}(z_m)$$

and

$$\operatorname{im}(z_0) = \operatorname{im}(z_1) = \cdots = \operatorname{im}(z_m).$$

For $i \in \{1, \ldots, m\}$, let A_i be the submultiset of $\mathrm{LL}(y)$ consisting of points x such that $\operatorname{re}(z_{i-1}) < \operatorname{re}(x) < \operatorname{re}(z_i)$ and $\operatorname{im}(x) > \operatorname{im}(z_0)$. Let A_0 be the complement in $\mathrm{LL}(y)$ of $A_1 \cup \cdots \cup A_m$. One easily checks (a), (b) and (c).

(ii) ⇒ (i): Conversely, assume we are given a partition $A_0 \sqcup A_1 \sqcup \cdots \sqcup A_m$ satisfying conditions (a) and (b).

One may recover tunnels $(y, z, L_1), \ldots, (y, z, L_m)$ such that the above construction yields the partition $A_0 \sqcup A_1 \sqcup \cdots \sqcup A_m$. Condition (c) then implies that the tunnels represent the elements of C and thus that $y \in Y_C$.

Uniqueness of the partition. This is a consequence of condition (c) and Lemma 9.4. □

LEMMA 9.6. *Let $C = \{1 \prec c_1 \prec c_2 \prec \cdots \prec c_m\} \in \mathcal{C}$. Define Y_C^0 as the subspace of Y_C consisting of points y whose associated partition $A_0, \ldots, A_m$ (from Proposition 9.5(ii)) satisfies the following conditions*:

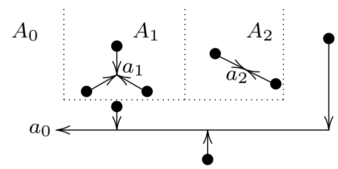
- *for $i = 0, \ldots, m$, the support of A_i is a singleton $\{a_i\}$; and*
- $\operatorname{re}(a_0) = \min_{i=1,\ldots,m} \operatorname{re}(a_i) - 1$ *and* $\operatorname{im}(a_0) = \min_{i=1,\ldots,m} \operatorname{im}(a_i) - 1$.

Then Y_C^0 is contractible.

Proof. Let $y \in Y_C^0$. The support $(x_0, \ldots, x_m)$ of $\mathrm{LL}(y)$ is generic; thus the label $(s_0, \ldots, s_m)$ of y coincides with the deep label, and we have $s_0 \cdots s_m = \delta$ (Corollary 6.18). By Proposition 9.5, Condition (ii)(c), we have, for $i = 1, \ldots, m$, $s_i = c_{i-1}^{-1} c_i$ (where $c_0 = 1$). Thus $s_0 = \delta c_m^{-1}$. We have proved that the label of any $y \in Y_C^0$ must be $(\delta c_m^{-1}, c_0^{-1} c_1, \ldots, c_{m-1}^{-1} c_m)$. A consequence of Theorem 7.20 is that the map $Y_C^0 \to E_m^{\mathrm{gen}}$ sending y to $(x_1, \ldots, x_m)$ is a homeomorphism. One concludes with Lemma 3.2. □

We may now proceed to the proof of Proposition 9.3. Let $C = \{1 \prec c_1 \prec \cdots \prec c_m\} \in \mathcal{C}$. Let $y \in Y_C$. Let $(x_1, \ldots, x_k)$ be the ordered support of $\mathrm{LL}(y)$, and $(n_1, \dots, n_k)$ be the multiplicities. Let $A_0, \dots, A_m$ be the partition of $\mathrm{LL}(y)$ described in Proposition 9.5(ii).

The picture below gives an idea of the retraction of Y_C onto Y_C^0 that will be explicitly constructed. It illustrates the motion of a given point $y \in Y_C$; the black dots indicate the support of $\mathrm{LL}(y)$, and the arrows indicate how this support moves during the retraction.



For $i = 1, \dots, m$, consider the multiset mass center

$$a_i := \frac{\sum_{x \in A_i} x}{|A_i|}.$$

(In this expression, A_i is viewed as a multiset: each x_j in A_i is taken n_j times, and $|A_i|$ is the multiset cardinal, i.e., the sum of the n_j such that $x_j \in A_i$.)

For each $t \in [0,1]$, let

$$\gamma_y(t) := A_0 \cup \bigcup_{i=1}^{m} \{(1-t)x + ta_i | x \in A_i\}.$$

(Here again, we consider the multiset union; in particular, the multicardinal of $\gamma_y(t)$ is constant, equal to n — i.e., $\gamma_y(t) \in E_n$.) This defines a path in E_n.

As explained in Remark 7.21, the path γ_y uniquely lifts to a path $\tilde{\gamma}_y$ in Y such that $\tilde{\gamma}_y(0) = y$. An easy consequence of Proposition 9.5 is that $\tilde{\gamma}_y$ is actually drawn in Y_C.

Let $y' := \tilde{\gamma}_y(1)$. Let

$$R := \min_{i=1,\dots,m} \mathrm{re}(a_i) = \mathrm{re}(a_1)$$

and

$$I := \min_{i=1,\dots,m} \mathrm{im}(a_i).$$

For all $x \in A_0$, we have $\mathrm{re}(x) < R$ or $\mathrm{im}(x) < I$ or $\mathrm{re}(x) > \mathrm{re}(a_m)$; denote by x' the complex number with the same real part as x and imaginary part $I-1$; let $a_0 := R - 1 + \sqrt{-1}(I-1)$. Consider the path $\beta_x : [0,1] \to \mathbb{C}$ defined by

$$\beta_x(t) := \begin{cases} (1-2t)x + 2tx' & \text{if } t \leq 1/2, \\ (2-2t)x' + (2t-1)a_0 & \text{if } t \geq 1/2. \end{cases}$$

For all $t \in [0,1]$, one has $\mathrm{re}(\beta_x(t)) < R$ or $\mathrm{im}(\beta_x(t)) < I$ or $\mathrm{re}(\beta_x(t)) > \mathrm{re}(a_m)$.

The path $\gamma'_y : [0,1] \to E_n$ defined by

$$\gamma'_y(t) := \{\underbrace{a_1, \dots, a_1}_{|A_1| \text{ times}}\} \cup \dots \cup \{\underbrace{a_m, \dots, a_m}_{|A_m| \text{ times}}\} \cup \bigcup_{x \in A_0} \beta_x(t)$$

lifts to a unique path $\tilde{\gamma}'_y$ in Y such that $\tilde{\gamma}'_y(0) = y'$. Once again, a direct application of Proposition 9.5 ensures that $\tilde{\gamma}'_y$ is actually in Y_C. The endpoint $y'' := \tilde{\gamma}'_y(1)$ lies in the subspace Y_C^0 of Corollary 9.6.

The map

$$\begin{aligned} \varphi : Y_C \times [0,1] &\longrightarrow Y_C \\ (y,t) &\longmapsto (\tilde{\gamma}'_y \circ \tilde{\gamma}_y)(t) \end{aligned}$$

is a retraction of Y_C onto its contractible subspace Y_C^0. Thus Y_C is contractible.

10. The universal cover of $W\backslash V^{\mathrm{reg}}$

The main result of this section is

THEOREM 10.1. *The universal cover of* $W\backslash V^{\mathrm{reg}}$ *is homotopy equivalent to* $\mathrm{gar}(B,S)$.

Combined with Theorem B.14, this proves our main result Theorem 0.2 in the irreducible case. The reducible case follows.

To prove this theorem, we construct an open covering $(\widehat{\mathcal{U}_b})_{b\in B}$ such that intersections of $(\widehat{\mathcal{U}_b})_{b\in B}$ are either empty or contractible (Proposition 10.7). Under these assumptions, a standard theorem from algebraic topology ([36, 4G.3]) shows that the universal cover is homotopy equivalent to the *nerve*, i.e., the simplicial space determined by nonempty intersections. By showing that the nerve is $\mathrm{gar}(B,S)$ (Proposition 10.6), we obtain the desired result.

As explained in Definition A.5, our "basepoint" $\mathcal{U}$ provides us with a model denoted

$$\mathrm{UniCover}(W\backslash V^{\mathrm{reg}}, \mathcal{U})$$

for the universal cover of $W\backslash V^{\mathrm{reg}}$. Recall that, to any semitunnel T, one associates a path γ_T whose source is in $\mathcal{U}$ and thus a point in $\mathrm{UniCover}(W\backslash V^{\mathrm{reg}}, \mathcal{U})$. Moreover, we have a left action of $B = \pi_1(W\backslash V^{\mathrm{reg}}, \mathcal{U})$ on $\mathrm{UniCover}(W\backslash V^{\mathrm{reg}}, \mathcal{U})$. With these conventions, our open covering is very easy to define:

Definition 10.2. The set $\widehat{\mathcal{U}_1}$ is the subset of $\mathrm{UniCover}(W\backslash V^{\mathrm{reg}}, \mathcal{U})$ of elements represented by semitunnels. For all $b \in B$, we set $\widehat{\mathcal{U}_b} := b\widehat{\mathcal{U}_1}$.

Two semitunnels represent the same point in $\mathrm{UniCover}(W\backslash V^{\mathrm{reg}}, \mathcal{U})$ if and only if they are equivalent in the following sense:

Definition 10.3. Two semitunnels $T = (y,x,L)$ and $T' = (y',x',L')$ are *equivalent* if and only if $y = y'$, $x + L = x' + L'$ and the affine segment $[(y,x),(y,x')]$ is included in $\mathcal{U}$.

Let $T = (y,x,L)$ be a semitunnel. The point of $\widehat{\mathcal{U}_1}$ determined by T, or in other words the equivalence class of T, is uniquely determined by y, $x + L$

and

$$\lambda(T) := \inf \{\lambda' \in [0,L] \,|\, [(y,x),(y,x+L-\lambda')] \subseteq \mathcal{U}\}.$$

The number λ is the infimum of the length of semitunnels in the equivalence class of T. This infimum may not be a minimum, since $(y, x+L-\lambda(T), \lambda(T))$ may not be a semitunnel (unless T is included in $\mathcal{U}$).

We consider the following subsets of $\widehat{\mathcal{U}}_1$:

($\mathcal{U}_1$) If $\lambda(T) = 0 = \min\{\lambda' \in [0,L] \,|\, [x, x+L-\lambda'] \subseteq U_y\}$, then T is a tunnel, equivalent to $(y, x+L, 0)$. Elements of $\widehat{\mathcal{U}}_1$ represented by such tunnels of length 0 form an open subset denoted by $\mathcal{U}_1$. This subset is actually a sheet over $\mathcal{U}$ of the universal covering, corresponding to the trivial lift of the "basepoint."

($\overline{\mathcal{U}}_1$) We denote by $\overline{\mathcal{U}}_1$ the subset consisting of points represented by semitunnels with $\lambda(T) = 0$ (but without requiring that the "inf" is actually a "min"). We obviously have $\mathcal{U}_1 \subseteq \overline{\mathcal{U}}_1$, and $\overline{\mathcal{U}}_1$ is contained in the closure of $\mathcal{U}_1$. It is readily seen that, whenever $\lambda(T) = 0$, then the semi-open interval $[x, x+L)$ lies in U_y, and that $x + L \in U_y$ if and only if the associated point lies in $\mathcal{U}_1$.

Similarly, for all $b \in B$, we set

$$\mathcal{U}_b := b\mathcal{U}_1, \quad \overline{\mathcal{U}}_b := b\overline{\mathcal{U}}_1.$$

Also, for any $y \in Y$, we denote by $\widehat{\mathcal{U}}_{b,y}$ (resp. $\mathcal{U}_{b,y}$, resp. $\overline{\mathcal{U}}_{b,y}$) the intersection of $\widehat{\mathcal{U}}_b$ (resp. $\mathcal{U}_b$, resp. $\overline{\mathcal{U}}_b$) with the fiber over y of the composed map $\mathrm{UniCover}(W\backslash V^{\mathrm{reg}}, \mathcal{U}) \to W\backslash V^{\mathrm{reg}} \to Y$.

LEMMA 10.4. *The family* $(\overline{\mathcal{U}}_b)_{b\in B}$ *is a partition of* $\mathrm{UniCover}(W\backslash V^{\mathrm{reg}}, \mathcal{U})$.

Proof. We first prove that $\overline{\mathcal{U}}_b \cap \overline{\mathcal{U}}_1 \neq \varnothing \Rightarrow b = 1$. Let $p \in b\overline{\mathcal{U}}_1 \cap \overline{\mathcal{U}}_1$. Let $T = (y, x, L)$ be a semitunnel with $\lambda(T) = 0$ representing p. Let $T' = (y', x', L')$ be a semitunnel with $\lambda(T') = 0$ representing $b^{-1}(p)$. Let γ be a path from y to y' representing b. The paths T and $\gamma T'$ represent the same point p in the universal cover; hence they are homotopic. By looking at the projection onto the base space, we see that $x + L = x' + L'$ and $y = y'$.

- If $L = 0$ or $L' = 0$, then $x + L = x' + L'$ lies in $U_y = U_{y'}$. This is implies that both $[x, x+L] \subseteq U_y$ and $[x', x'+L'] \subseteq U_{y'}$, thus γ is homotopic to the trivial path and $b = 1$.
- If both $L > 0$ and $L' > 0$, then for ε small enough (i.e., such that $0 < \varepsilon < \min(L, L')$),
  - T is the concatenation of $T_0 = (y, x, L - \varepsilon)$ and $T_\varepsilon = (y, x + L - \varepsilon, \varepsilon)$;
  - T' is the concatenation of $T'_0 = (y', x', L' - \varepsilon)$ and $T'_\varepsilon = (y', x' + L' - \varepsilon, \varepsilon) = T_\varepsilon$.

Since $\lambda(T) = \lambda(T') = 0$, T_0 and T'_0 are tunnels representing the trivial braid. From $T \sim \gamma T'$, we deduce $T_0 T_\varepsilon \sim \gamma T'_0 T_\varepsilon$ and $T_0 \sim \gamma T'_0$, which shows that $b = 1$.

Using the B-action, this implies that $\overline{\mathcal{U}}_b \cap \overline{\mathcal{U}}_{b'} \neq \varnothing \Rightarrow b = b'$.

To conclude, it suffices to show that the projection $\overline{\mathcal{U}}_1 \to W\backslash V^{\mathrm{reg}}$ is bijective. Any point in $z \in W\backslash V^{\mathrm{reg}}$ is the target of a unique equivalence class of semitunnels T with $\lambda(T) = 0$; depending on whether $z \in \mathcal{U}$ or not, the associated point will be in $\mathcal{U}_1$ or in $\overline{\mathcal{U}}_1 - \mathcal{U}_1$. □

In particular, $(\widehat{\mathcal{U}_b})_{b\in B}$ is a covering of UniCover$(W\backslash V^{\mathrm{reg}}, \mathcal{U})$.

LEMMA 10.5. *For all $b \in B$, $\widehat{\mathcal{U}_b}$ is open and contractible.*

Proof. It is enough to deal with $b = 1$. That $\widehat{\mathcal{U}_1}$ is open is easy. Let $\mathcal{T}$ be the space of semitunnels and $\sim$ the equivalence relation. As a set, $\widehat{\mathcal{U}_1} \simeq \mathcal{T}/\sim$. Consider the map

$$\begin{aligned} \phi : \mathcal{T} \times [0,1] &\longrightarrow \mathcal{T} \\ (T = (y,x,L), t) &\longmapsto (y, x, L - t\lambda(T)). \end{aligned}$$

If $T \sim T'$, then for all $t, \phi(T,t) \sim \phi(T',t)$. Thus ϕ induces a map $\overline{\phi} : \widehat{\mathcal{U}_1} \times [0,1] \to \widehat{\mathcal{U}_1}$. This map is continuous (this follows from the fact that λ induces a continuous function on $\widehat{\mathcal{U}_1}$) and $\overline{\phi}(T,t) = T$ if $T \in \overline{\mathcal{U}}_1$ or if $t = 0$. We have proved that $\widehat{\mathcal{U}_1}$ retracts to $\overline{\mathcal{U}}_1$. We are left with having to prove that $\overline{\mathcal{U}}_1$ is contractible. Inside $\overline{\mathcal{U}}_1$ lies $\mathcal{U}_1$ that is contractible since it is a standard lift of the contractible space $\mathcal{U}$.

We conclude by observing that $\mathcal{U}_1$ and $\overline{\mathcal{U}}_1$ are homotopy equivalent. There is probably a standard theorem from semialgebraic geometry applicable here, but I was unable to find a proper reference. Below is a "bare-hand" argument: it explains how $\overline{\mathcal{U}}_1$ may be "locally retracted" inside $\mathcal{U}_1$. (Constructing a global retraction seems difficult.)

Both spaces have homotopy type of CW-complexes, and to prove homotopy equivalence it is enough to prove that any continuous map $f : S^k \to \overline{\mathcal{U}}_1$ (where S^k is a sphere) may be homotoped to a map $S^k \to \mathcal{U}_1$. Assume that there is a semitunnel $T = (y, z, L)$ such that $(T/\sim) \in f(S^k) \cap (\overline{\mathcal{U}}_1 - \mathcal{U}_1)$. We have $L > 0$. For any $\varepsilon > 0$, let $B_\varepsilon(y)$ be the open ball of radius ε in Y around y, and let $I_\varepsilon(z)$ be the affine interval $(z - \sqrt{-1}\varepsilon, z + \sqrt{-1}\varepsilon)$. For ε small enough, there is a unique continuous function $L_\varepsilon : B_\varepsilon(y) \times I_\varepsilon(z) \to \mathbb{R}$ such that $L_\varepsilon(y,z) = L$ and for all $(y', z') \in B_\varepsilon(y) \times I_\varepsilon(z)$, $(y', z', L_\varepsilon(y', z'))$ represents a point in $\overline{\mathcal{U}}_1 - \mathcal{U}_1$. The "half-ball"

$$H_\varepsilon := \{(y', z', L') \in B_\varepsilon(y) \times I_\varepsilon(z) \times \mathbb{R} | 0 \leq L' \leq L_\varepsilon(y', z')\}$$

is a neighborhood of $T/\sim$ in $\overline{\mathcal{U}_1}$. Working inside this neighborhood, one may homotope f to f' such that $f'(S^k) \cap (\overline{\mathcal{U}}_1 - \mathcal{U}_1) \subseteq f(S^k) \cap (\overline{\mathcal{U}}_1 - \mathcal{U}_1) - \{T/\sim\}$.

Compactness of $f(S^k)\cap(\overline{\mathcal{U}}_1-\mathcal{U}_1)$ guarantees that one can iterate this process a finite number of times to get rid of all $f(S^k)\cap(\overline{\mathcal{U}}_1-\mathcal{U}_1)$. $\square$

PROPOSITION 10.6. *The nerve of* $(\widehat{\mathcal{U}_b})_{b\in B}$ *is* $\operatorname{gar}(B,S)$.

Proof. Let $b,b'\in B$ such that $\widehat{\mathcal{U}_b}\cap\widehat{\mathcal{U}_{b'}}\neq\varnothing$. Let $T=(y,x,L)$ and $T'=(y',x',L')$ be semitunnels, representing points z and z' in $\widehat{\mathcal{U}_1}$, such that $bz=b'z'$. The image of z (resp. z') in $W\backslash V^{\mathrm{reg}}$ is $(y,x+L)$ (resp. $(y',x'+L')$). Thus $x+L=x'+L'$ and $y=y'$. Up to permuting b and b', we may assume that $L\geq L'$. Since $x'=x+L-L'$ is in U_y, $T'':=(y,x,L-L')$ is a tunnel, representing a simple element b''. The tunnel T is a concatenation of T'' and T'. This implies that $z=b''z'$ and $b'z'=bz=bb''z'$. By faithfulness of the B-action on the orbit of z, we conclude that $bb''=b'$.

We have proved that the 1-skeletons of the nerve and of $\operatorname{gar}(B,S)$ coincide. To conclude, it remains to check that the nerve is a flag complex. Let $C\subseteq B$ be such that for all $b,b'\in C$, either $b^{-1}b'$ or $b'^{-1}b$ is simple. Note that, unless $b=b'$, $b^{-1}b'$ and $b'^{-1}b$ cannot both be simple (because Garside monoids are cancellative); in this setup, we write $b\preccurlyeq b'$ for $b^{-1}b'\in S$. This defines a total ordering on C that, when both b and b' are in M, coincides with the previously defined left prefix ordering.

We have to prove that $\bigcup_{b\in C}\widehat{\mathcal{U}_b}\neq\varnothing$. Let $c_0,\dots,c_m$ be the elements of C, numbered according to the total ordering on C induced by $\preccurlyeq$:

$$c_0\prec c_1\prec c_2\prec\cdots\prec c_m\preccurlyeq c_0\delta.$$

Up to left-dividing each term by c_0, we may assume that $c_0=1$. Let $y\in Y_C$. We may find $x,L_1,\dots,L_m$ such that (y,x,L_i) represents c_i for all i. The point of the universal cover represented by (y,x,L_m) belongs to $\bigcup_{b\in C}\widehat{\mathcal{U}_b}$. $\square$

PROPOSITION 10.7. *Let C be a subset of B such that* $\bigcap_{b\in C}\widehat{\mathcal{U}_b}\neq\varnothing$. *Then* $\bigcap_{b\in C}\widehat{\mathcal{U}_b}$ *is contractible.*

Proof. As in the previous proof, we write $C=\{c_0,c_1,\dots,c_m\}$ with

$$c_0\prec c_1\prec c_2\prec\cdots\prec c_m\preccurlyeq c_0\delta$$

and assume without loss of generality that $c_0=1$.

The case $m=0$ is Lemma 10.5.

Assume that $m\geq 1$. Let $T=(y,z,L)$. The point represented by T lies in $\bigcap_{b\in C}\widehat{\mathcal{U}_b}$ if and only if there exists $L_1,\dots,L_m$ with $0<L_1<\cdots<L_m<L$ such that, for all i, $T_i:=(y,z,L_i)$ is a tunnel representing c_i. Given $y\in Y$, it is possible to find such $L_1,\dots,L_m$ if and only if $y\in Y_C$. This justifies

$$\bigcap_{b\in C}\widehat{\mathcal{U}}_{b,y}\neq\varnothing\Leftrightarrow y\in Y_C.$$

For a given $y \in Y_C$, let us study the intersection $\bigcap_{b\in C}\widehat{\mathcal{U}}_{b,y}$. Let $(x_1,\dots,x_k)$ be the ordered support of $\mathrm{LL}(y)$. Let $A_0,\dots,A_m$ be the associated partition of $\mathrm{LL}(y)$, defined in Proposition 9.5. Let

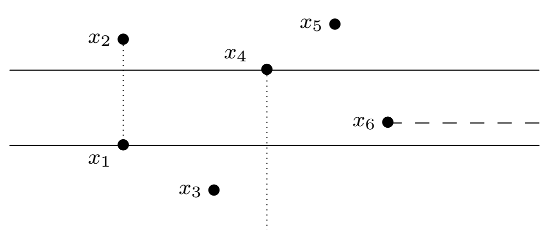
$$\begin{aligned}R_+(y,C) &:= \max\{\mathrm{re}(z) | z \in A_m\},\\ R_-(y,C) &:= \min\{\mathrm{re}(z) | z \in A_1\},\\ I_+(y,C) &:= \min\{\mathrm{im}(z) | z \in A_1 \cup \dots \cup A_m\},\end{aligned}$$

and

$$I_-(y,C) = \sup\{\mathrm{im}(z) | z \in A_0 \text{ and } R_- \le \mathrm{re}(z) \le R_+\}.$$

We have $I_-(y,C) < I_+$. It may happen that $I_-(y,C) = -\infty$.

We illustrate this in a picture. In the example, the support of $\mathrm{LL}(y)$ is $x_1,\dots,x_6$ and $C = \{1 \prec s_2 \prec s_2s_4\}$, where $(s_1,\dots,s_6) = \mathrm{lbl}(y)$. The support of A_1 is x_2, and the support of A_2 is x_4. The remaining points are in A_0. The lines $\mathrm{im}(z) = I_-(y,C)$ and $\mathrm{im}(z) = I_+(y,C)$ are represented by full lines. A semitunnel (y,z,L) representing a point in $\bigcap_{b\in C}\widehat{\mathcal{U}}_{b,y}$ must cross the intervals I_2 and I_4 represented by dotted lines. One must have $\mathrm{re}(z) < R_-(y,C)$ and $I_-(y,C) < \mathrm{im}(z) < I_+(y,C)$; the final point $z+L$ must satisfy $\mathrm{re}(z+L) > R_+(y,C)$. This final point may be any complex number z' in the rectangle $\mathrm{re}(z') > R_+(y,C)$ and $I_-(y,C) < \mathrm{im}(z') < I_+(y,C)$, except the points on the closed horizontal half-line to the right of x_6 (indicated by a dashed line), which cannot be reached.



Generalising the example, one shows that $\bigcap_{b\in C}\widehat{\mathcal{U}}_b$ may be identified with

$$\bigcup_{y\in Y_C} E(y,C),$$

where $E(y,C)$ is the open rectangle of $\mathbb{C}$ defined by

$$\mathrm{re}(z) > R_+(y,C), \quad I_-(y,C) < \mathrm{im}(z) < I_+(y,C)$$

from which have been removed the possible points of A_0 and the horizontal half-lines to their rights.

Let $\overline{E}(y,C)$ be the rectangle

$$\mathrm{re}(z) \ge R_+(y,C), \quad I_-(y,C) < \mathrm{im}(z) < I_+(y,C)$$

from which have been removed the possible points of A_0 and the horizontal half-lines to their rights.

A homotopy argument similar to the one used in the proof of Lemma 10.5 shows that $\bigcup_{y\in Y_C} E(y,C)$ and $\bigcup_{y\in Y_C} \overline{E}(y,C)$ are homotopy equivalent. (There might be a nicer argument from semialgebraic geometry.) The latter may be retracted to the union of open intervals

$$\bigcup_{y\in Y_C} (I_-(y,C), I_+(y,C));$$

on each rectangle, the retraction is

$$\overline{E}(y,C)\times[0,1] \to \overline{E}(y,C), (z,t) \mapsto R_+(y,C)+t(\mathrm{re}(z)-R_+(y,C))+\sqrt{-1}\,\mathrm{im}(z).$$

The union

$$\bigcup_{y\in Y_C} (I_-(y,C), I_+(y,C))$$

may be retracted to

$$\bigcup_{y\in Y_C} [I_0(y,C), I_+(y,C)),$$

where $I_0(y,C) := \max(\frac{I_-(y,C)+I_+(y,C)}{2}, I_+(y,C)-1)$. The latter space is a fiber bundle over Y_C, with fibers intervals. Since the basespace Y_C is contractible (Proposition 9.3), this fiber bundle is contractible. So is $\bigcap_{b\in C} \widehat{\mathcal{U}_b}$. $\square$

11. Centralizers of regular elements

This section is devoted to the proof of Theorem 0.3: if W' is the centralizer of a d-regular element in an irreducible well-generated complex reflection group W, then the hyperplane complement of W' is $K(\pi,1)$. To prove this, we develop a *relative version* of the tools and constructions presented in the previous sections, following the same generic pattern of proof, but with a categorical twist.

Let m be a positive integer. As explained in Appendix B (Theorem B.22), starting from any Garside structure — we will start from the dual braid monoid M, with its Garside element δ and Garside automorphism ϕ — there exists a groupoid $\mathcal{G}_m$ equivalent as a category to B, and admitting a Garside structure (M_m, δ_m, ϕ_m) where the order of ϕ_m is m times the order of ϕ. Because ϕ_m is compatible with the Garside structure, the fixed subgroupoid $\mathcal{G}_m^{\phi_m}$ admits a Garside structure $(M_m^{\phi_m}, \delta_m, \phi_m)$.

Our main result so far is that $W\backslash V^{\mathrm{reg}}$ is a $K(B,1)$. While μ_d naturally acts on $W\backslash V^{\mathrm{reg}}$, the dual braid monoid M may not have an automorphism of order d. By replacing M by M_d, or by any M_m where m is such that $d|mh$, we can obtain a Garside structure with a symmetry of order d. So the categorical viewpoint provides the algebraic structure that we need.

As B and $\mathcal{G}_m$ are equivalent, a $K(B,1)$ space is homotopy equivalent to a $K(\mathcal{G}_m,1)$ space. If one could view $W\backslash V^{\mathrm{reg}}$ as a $K(\mathcal{G}_m,1)$ in a μ_d-equivariant fashion, then the relative $K(\pi,1)$ property should follow by simply considering fixed points. Although we do not attempt to systematically investigate this intuition (the connection between the cyclic/helicoidal structure, the nonpositively curved aspects, and geometric constructions such as the Milnor fiber, would be especially worth studying), it is the true explanation behind the miracles of the current section.

11.1. *A Garside category with symmetry of order d.* Since $W\backslash V$ and Y are defined by graded subalgebras of $\mathcal{O}_V$, these varieties are equipped with quotient actions of $\mathbb{C}^\times$. There is also a natural $\mathbb{C}^\times$-action on the configuration spaces E_n and $\overline{E}_n^\circ$, defined by

$$\lambda \cdot \{z_1, \ldots, z_n\} := \{\lambda z_1, \ldots, \lambda z_n\}.$$

LEMMA 11.1. *For all $(y,z) \in W\backslash V$ and for all $\lambda \in \mathbb{C}^\times$, we have $\overline{\mathrm{LL}}(\lambda(y,z)) = \lambda^h \overline{\mathrm{LL}}((y,z))$.*

Proof. Elementary. □

If d divides h, then $(W\backslash V^{\mathrm{reg}})^{\mu_d}$ is again the regular orbit space of a well-generated complex reflection group. (Because of Theorem 1.9 (2), the centralizer W' is again a duality group and thus is well generated.) Because we have already proved the $K(\pi,1)$ property in this case, it would be sufficient to focus on the case when d does not divide h.

Convention 11.2. Until the end of Section 11, we work under the following assumptions and notation:

- d is a fixed regular number;
- we set
$$h' := \frac{h}{h \wedge d} \quad \text{and} \quad d' := \frac{d}{h \wedge d}$$
and, in particular, we have
$$h'd = hd' = h'd'(h \wedge d) = h \vee d.$$

Note that we do not assume that $d' > 1$. Actually, taking $d' = d = 1$, we will obtain an alternate proof that the arrangement of W is $K(\pi,1)$. This alternate proof is geometrically simpler, but a bit more abstract, compared to the argument provided in the previous two sections.

LEMMA 11.3. *We have $d' \mid n$.*

Proof. Because d is regular, $(W\backslash V^{\mathrm{reg}})^{\mu_d} \neq \varnothing$. Let $(y,z) \in (W\backslash V^{\mathrm{reg}})^{\mu_d}$. Applying Lemma 11.1 to $\lambda = \zeta_d$, we see that $\overline{\mathrm{LL}}((y,z))$ is ζ_d^h-invariant. Note that ζ_d^h is a primitive d'-th root of unity; orbits under multiplication by primitive d'-th roots of unity have either cardinal d' or consists only of 0. The orbit $\{0\}$ is not permitted as $\overline{\mathrm{LL}}((y,z)) \in \overline{E}_n^\circ$. Thus n, the cardinality of the multiset $\overline{\mathrm{LL}}(y)$, is a multiple of d'. □

Remark 11.4. In the above proof, another consequence of the fact that $\overline{\mathrm{LL}}((y,z)) = LL(y) - z$ is ζ_d^h-invariant is that its barycenter $-z$ must be 0. We have $\overline{\mathrm{LL}}((y,z)) = \mathrm{LL}(y)$, and $(W\backslash V^{\mathrm{reg}})^{\mu_d}$ can be identified with its image in Y.

To construct a Garside category with a symmetry of order d, we start with the dual braid monoid M, together with its Garside element δ, Garside automorphism ϕ and set of simple elements S.

As explained in Appendix B, all the structure is nicely encoded in the Garside set $\mathrm{D}_\bullet(\delta)$ of factorizations of δ. Through the isomorphism of Theorem 7.12, we can work instead in $\mathrm{D}_\bullet(c)$, whose degree k component is

$$\mathrm{D}_k(c) = \{(c_1, \ldots, c_k) \in W^k | c_1 \cdots c_k = c \text{ and } l_R(c_1) + \cdots + l_R(c_k) = l_R(c)\}.$$

By reminding us that this is just about combinatorics in a finite group, working in $\mathrm{D}_\bullet(c)$ helps reducing the cognitive load. The automorphism ϕ acts on $\mathrm{D}_k(c)$ by

$$(c_1, \ldots, c_k) \cdot \phi = (c_2, \ldots, c_m, c_1^c).$$

To distinguish it from the topologically-defined braid group B, we denote by G the group of fractions of M. (A consequence of Theorem 8.2 is that $G \simeq B$.) As explained in Appendix B, the components of $\mathrm{D}_\bullet(c)$ whose degrees are multiples of d' form, collectively, the Garside set

$$\sqrt[d']{\mathrm{D}_\bullet}(c)$$

of a Garside category $M_{d'}$, whose groupoid of fractions $G_{d'}$ is equivalent to G and whose Garside automorphism $\phi_{d'}$, has order d' times the order of ϕ.

Definition 11.5 (relative dual Garside set). We set

$$\mathrm{D}'_\bullet(c) := \left(\sqrt[d']{\mathrm{D}_\bullet}(c)\right)^{\phi_{d'}^{h'}}.$$

The *dual Garside category* for W' relative to the pair (W, W') is the fixed subcategory

$$M' := M_{d'}^{\phi_{d'}^{h'}}.$$

We denote by G' the groupoid obtained by formally inverting the morphisms in M'.

By functoriality, we have the following commutative diagram:

$$\begin{tikzcd}
M' \arrow[r, hook] \arrow[d, hook] & M_{d'} \arrow[r, two heads, "\kappa_m"] \arrow[d, hook] & M \arrow[d, hook] \\
G' \arrow[r] & G_{d'} \arrow[r, "\sim"] & G \arrow[d, "\simeq"] \\
& & B.
\end{tikzcd}$$

The "monomorphism" arrows denote functors that are faithful (injective on morphisms.) The functor κ_m, introduced in Definition B.23, is *full* (surjective on morphisms), as simple elements are in its image. The "$\sim$" arrows denote equivalence of categories. The morphism from G to B is a true isomorphism (Theorem 8.2.)

In Definition 11.23 below, we will define topological groupoids to complete the bottom line of this diagram.

THEOREM 11.6. *The category M' is a Garside category, with Garside set $\mathrm{D}'_\bullet(c)$. As a consequence, G' is a Garside groupoid.*

Proof. See Appendix B and [5]: $M_{d'}$ is a Garside category, and the fixed subcategory by an automorphism of a Garside category is a Garside category (whose Garside set is the fixed subset for the automorphism acting on the original Garside set). $\square$

As explained in Remark B.18, one can use the low degree components of $\mathrm{D}'_\bullet(c)$ to write a presentation by generators and relations for M' and G'.

- There is just one element in $\mathrm{D}'_0(c)$; this element corresponds to the unique object in M and does not have much algebraic significance for M'.
- Elements in $\mathrm{D}'_1(c)$ are $\phi_{d'}^{h'}$-invariant factorizations $(c_1, \ldots, c_{d'})$ of c. They correspond to objects in M'.
- Elements in $\mathrm{D}'_2(c)$ are $\phi_{d'}^{h'}$-invariant factorizations $(c_1, \ldots, c_{2d'})$ of c. They correspond to simple elements in M'.

The centralizer W' (see Theorem 1.9) may be badly generated, but the category M' provides a substitute for the dual braid monoid.

Example 11.7. In the situation of Example 1.10 ($W = G_{37}$, $d = 4$, $W' = G_{31}$), we have $d' = 2$ and $h' = 15$. The category M_2 consists of factorizations of c as a product $c = uv$ of two elements. The automorphism ϕ_2^{15} acts on such pairs by

$$(u, v) \mapsto (v^{c^7}, u^{c^8}).$$

The object set $\mathrm{D}'_1(c)$ of M', the relative dual Garside category for G_{31}, is indexed by the solutions (u, v) to the following system of equations:

$$\begin{aligned} uv &= c, \\ l_R(u) + l_R(v) &= l_R(c) = 8, \\ u &= v^{c^7}, \\ v &= u^{c^8} \end{aligned}$$

or, equivalently (since c^{15} is central, the last two equations are equivalent),

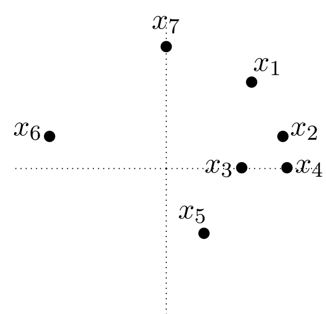
$$uu^{c^8} = c,$$
$$l_R(u) = 4.$$

Using a computer to sift through all elements of length 4, one finds 88 solutions.

The Garside automorphism ϕ_2 acts on $\mathrm{D}'_1(c)$ by $(u, u^{c^8}) \mapsto (u^{c^8}, u^c)$. The object set $\mathrm{D}'_1(c)$ decomposes into ϕ_2-orbits: one orbit of size 3, 2 orbits of size 5 and 5 orbits of size 15.

Example 11.7 is the only case needed in our proof of the $K(\pi, 1)$ conjecture, but it does no harm to study the relative situation in full generality.

11.2. *Cyclic labels.* Departing from the earlier convention initiated in Definition 3.1, we now enumerate the distinct points of a configuration not containing 0 clockwise, starting from 12 o'clock plus ε seconds, and for points with identical argument, we enumerate them by increasing modulus, as in the following example:



Definition 11.8. Let $x \in \overline{E}_n^\circ$. The above defined sequence $(x_1, \ldots, x_m)$ of distinct points in x is the *cyclic support* of x. For all i, we denote by θ_i the unique real number such that $0 < \theta_i \leq 2\pi$ and

$$e^{\sqrt{-1}\theta_i} x_i = \sqrt{-1}|x_i|.$$

(In other words, a rotation with angle θ_i sends x_i to the positive imaginary half-line.) The nondecreasing sequence $(\theta_1, \ldots, \theta_m)$ is the *cyclic argument* of x. The sequence $(n_1, \ldots, n_m)$, where n_i is the multiplicity of x_i, is the *cyclic multiplicity* of x.

The cyclic support (resp. argument, multiplicity) of $(y, z) \in W\backslash V^{\mathrm{reg}}$ is, by extension, defined as that of $\overline{\mathrm{LL}}((y, z))$.

In the above picture, we have $\theta_3 = \theta_4 = \pi/2$ and $\theta_7 = 2\pi$.

Let $x \in (W\backslash V^{\mathrm{reg}})^{\mu_d}$ with associated cyclic support $(x_1, \ldots, x_m)$ and cyclic argument $(\theta_1, \ldots, \theta_m)$. Let $\varepsilon > 0$. By Lemma 11.1, the point $e^{\sqrt{-1}(\theta_i - \varepsilon)} x_i$ lies in $\overline{\mathrm{LL}}(e^{\sqrt{-1}(\theta_i - \varepsilon)/h} x)$; thus it can be associated a simple element $s_{i,\varepsilon}$ that is part of the label of $e^{\sqrt{-1}(\theta_i-\varepsilon)/h}x$ (in the sense of Definition 6.9). As mentioned before, in this section we prefer to view simple elements in W (via Proposition 8.5).

Using the Hurwitz rule, one readily sees that $s_{i,\varepsilon}$ does not depends on the choice of a small enough ε. For example, note that for $i=1$, any value of ε with $0<\varepsilon\leq\theta_1$ is suitable and yields the same simple element. In particular, we have a well-defined sequence

$$(c_1,\ldots,c_m) := (s_{1,\varepsilon},\ldots,s_{m,\varepsilon}).$$

Definition 11.9. The *cyclic label* of $(y,z)\in W\backslash V^{\mathrm{reg}}$ is the sequence $\mathrm{clbl}(y,z) := (c_1,\ldots,c_m)$.

LEMMA 11.10. *For all* $x\in W\backslash V^{\mathrm{reg}}$, $\mathrm{clbl}(y,z)\in \mathrm{D}_\bullet(c)$.

Proof. Consider the path: $t\mapsto e^{\sqrt{-1}\frac{2\pi}{h}t}x$. As explained after Definition 6.11, this loop represents the element $\delta\in B$.

This path is the concatenation of topologically trivial paths and of paths of the form

$$\begin{aligned}\gamma_{y,i,\varepsilon} : [\theta_i-\varepsilon,\theta_i+\varepsilon] &\to W\backslash V^{\mathrm{reg}}\\ t &\mapsto e^{\sqrt{-1}\frac{tL}{h}}x,\end{aligned}$$

each of which, by the Hurwitz rule, represents the simple element in B corresponding to c_i.

The product of these simple elements is δ; thus , after projecting to W, $c_1\cdots c_m = c$. ☐

LEMMA 11.11. *Let* $x\in W\backslash V^{\mathrm{reg}}$ *with cyclic support* $(x_1,\ldots,x_m)$, *cyclic argument* $(\theta_1,\ldots,\theta_m)$ *and cyclic label* $(c_1,\ldots,c_m)$. *Let* $l\leq m$ *be the largest integer such that* $\theta_1=\theta_2=\cdots=\theta_l$. *Then*

(i) *Let* $(d_1,\ldots,d_m)$ *be the reduced label* $\mathrm{rlbl}(x)$. *Let* $\sigma\in\mathfrak{S}_m$ *be the unique permutation such that the ordered support of* $\overline{\mathrm{LL}}(x)$ (*in the sense of Definition* 3.1) *is* $(x_{\sigma(1)},\ldots,x_{\sigma(m)})$. *Then, for all integers* i *from* 1 *to* l, *we have* $c_i=d_{\sigma^{-1}(i)}$.

(ii) $e^{\sqrt{-1}\frac{\theta_1}{h}}x$ *has cyclic argument* $(\theta_{l+1}-\theta_1,\ldots,\theta_m-\theta_1,2\pi,\ldots,2\pi)$.

(iii) $e^{\sqrt{-1}\frac{\theta_1}{h}}x$ *has cyclic label* $(c_{l+1},\ldots,c_m,c_1^c,\ldots,c_l^c)$.

Proof. (i): as θ_1 is the first element of the cyclic argument, no point within $(x_1,\ldots,x_l)$ is passed above by any other point in the support through a rotation of angle $\theta_1-\varepsilon$; using the Hurwitz rule, we see that reduced labels of those points are preserved throughout the motion used to define the first l terms of the cyclic label.

(ii) is obvious by construction.

(iii): By construction and using (ii), we see that the last l terms of the cyclic label of $e^{\sqrt{-1}\frac{\theta_1}{h}}(y,z)$ are the first l terms of the cyclic label of $e^{\sqrt{-1}\frac{2\pi-\varepsilon}{h}}e^{\sqrt{-1}\frac{\theta_1}{h}}(y,z) = e^{\sqrt{-1}\frac{\theta_1-\varepsilon}{h}}e^{\sqrt{-1}\frac{2\pi}{h}}(y,z)$. We conclude by observing that, since the path $t \mapsto e^{\sqrt{-1}\frac{2\pi}{h}t}(y,z)$ represents c, the cyclic label of of $e^{\sqrt{-1}\frac{2\pi}{h}}(y,z)$ is the conjugate label $(c_1^c,\dots,c_m^c)$. □

Definition 11.12. A configuration $x \in \overline{E}_n^\circ$ is *cyclically compatible* with $(c_1,\dots,c_m) \in \mathrm{D}_\bullet(c)$ if the cyclic multiplicity of x is $(l_R(c_1),\dots,l_R(c_m))$. We denote by

$$\overline{E}_n^\circ \boxdot \mathrm{D}_\bullet(c)$$

the subspace of $\overline{E}_n^\circ \times \mathrm{D}_\bullet(c)$ consisting of cyclically compatible pairs.

PROPOSITION 11.13. *The map* $(\overline{\mathrm{LL}}, \mathrm{clbl})$ *induces a bijection*

$$\begin{aligned} W\backslash V^{\mathrm{reg}} &\longrightarrow \overline{E}_n^\circ \boxdot \mathrm{D}_\bullet(c) \\ x &\longmapsto (\overline{\mathrm{LL}}(x), \mathrm{clbl}(x)). \end{aligned}$$

Proof. That the image lies in $\overline{E}_n^\circ \boxdot \mathrm{D}_\bullet(c)$ is obvious by construction. Let $x \in W\backslash V^{\mathrm{reg}}$ with cyclic argument $(\theta_1,\dots,\theta_m)$. Consecutive applications of Lemma 11.11 to

$$y, e^{\sqrt{-1}\frac{\theta_1}{h}}x, \dots, e^{\sqrt{-1}\frac{\theta_{m-1}}{h}}x$$

show how to recover $\mathrm{rlbl}(x)$ from $\mathrm{clbl}(x)$, and vice-versa, via a sequence of permutations and c-conjugacies. (Note that the sequence of operations to get from $\mathrm{rlbl}(c)$ to $\mathrm{clbl}(c)$ depends on $\overline{\mathrm{LL}}(x)$, and it is much simpler to define it recursively using Lemma 11.11 than it is to write down an explicit formula.) This provides a natural bijection

$$\Phi : \overline{E}_n^\circ \boxtimes \mathrm{D}_\bullet(c) \to \overline{E}_n^\circ \boxdot \mathrm{D}_\bullet(c)$$

that fits in a commutative diagram

$$\begin{tikzcd} & \overline{E}_n^\circ \boxtimes \mathrm{D}_\bullet(c) \arrow[dd, "\Phi"] \\ W\backslash V^{\mathrm{reg}} \arrow[ur, "(\overline{\mathrm{LL}},\mathrm{rlbl})"] \arrow[dr, "(\overline{\mathrm{LL}},\mathrm{clbl})"'] & \\ & \overline{E}_n^\circ \boxdot \mathrm{D}_\bullet(c). \end{tikzcd}$$

Since the restriction of $(\overline{\mathrm{LL}}, \mathrm{rlbl})$ is bijective (Theorem 7.25), the restriction of $(\overline{\mathrm{LL}}, \mathrm{clbl})$ is the composition of two bijections; hence it is a bijection. □

In the above proof, the transition bijection $\Phi : \overline{E}_n^\circ \boxtimes \mathrm{D}_\bullet(c) \to \overline{E}_n^\circ \boxdot \mathrm{D}_\bullet(c)$ is a mere $\overline{E}_n^\circ$-dependent change of notation on the $\mathrm{D}_\bullet(c)$-component. We equip

$\overline{E}_n^\circ \boxdot \mathrm{D}_\bullet(c)$ with the topology induced via Φ by that on $\overline{E}_n^\circ \boxtimes \mathrm{D}_\bullet(c)$. The trivialization of Proposition 11.13 is a homeomorphism.

LEMMA 11.14. *Let $x \in W\backslash V^{\mathrm{reg}}$. Assume that $\overline{\mathrm{LL}}(x) \in (\overline{E}_n^\circ)^{\mu_{d'}}$. Then* $\mathrm{clbl}(x) \in \sqrt[d']{\mathrm{D}_\bullet}(c)$ *and* $\mathrm{clbl}(\zeta_{d'h}x) = \mathrm{clbl}(x) \cdot \phi_{d'}$.

Proof. The path $t \mapsto e^{\sqrt{-1}\frac{2\pi}{d'h}t}x$ connects x to $\zeta_{d'h}x$. Since $\overline{\mathrm{LL}}(x) \in (\overline{E}_n^\circ)^{\mu_{d'}}$, $\overline{\mathrm{LL}}(x)$ consists of $d'k$ distinct points for some integer k. The first k points have cyclic argument less that $2\pi/d'$. We have

$$\mathrm{clbl}(x) = (c_1, \ldots, c_{d'k})$$

and, by the very construction of clbl, it is obvious that the first $d'(k-1)$ terms of $\mathrm{clbl}(\zeta_{d'h}x)$ are $(c_{k+1}, c_{k+2}, \ldots, c_{d'k})$. The determination of the final k terms of $\mathrm{clbl}(\zeta_{d'h}x)$, and the equality $\mathrm{clbl}(\zeta_{d'h}x) = \mathrm{clbl}(x) \cdot \phi_{d'}$, is an easy exercise using the Hurwitz rule and Lemma 11.10. □

LEMMA 11.15. *For all $x \in W\backslash V^{\mathrm{reg}}$, the following assertions are equivalent*:

(i) $x \in (W\backslash V^{\mathrm{reg}})^{\mu_d}$,
(ii) $\overline{\mathrm{LL}}(x) \in (\overline{E}_n^\circ)^{\mu_{d'}}$ *and* $\mathrm{clbl}(x) \in \mathrm{D}'_\bullet(c)$.

Proof. Assume (i): we have $x = \zeta_d x$. By Lemma 11.1, $\overline{\mathrm{LL}}(\zeta_d x) = \zeta_d^h \overline{\mathrm{LL}}(x)$, so we have $\zeta_d^h \overline{\mathrm{LL}}(x) = \overline{\mathrm{LL}}(x)$. As ζ_d^h is a primitive root of order d', we deduce that $\overline{\mathrm{LL}}(x) \in (\overline{E}_n^\circ)^{\mu_{d'}}$. We also have $\mathrm{clbl}(x) = \mathrm{clbl}(\zeta_d x) = \mathrm{clbl}(\zeta_{d'h}^{h'} x)$. Using Lemma 11.14, we deduce that $\mathrm{clbl}(x) = \mathrm{clbl}(x) \cdot \phi_{d'}^{h'}$.

Conversely, assuming (ii), we conclude that x and $\zeta_d x$ satisfy $\overline{\mathrm{LL}}(x) = \overline{\mathrm{LL}}(\zeta_d x)$ and, using Lemma 11.14, that $\mathrm{clbl}(x) = \mathrm{clbl}(\zeta_d x)$. Using Proposition 11.13, we obtain (i). □

Let us already note that, as a consequence, we obtain a very strong combinatorial property of regular numbers:

COROLLARY 11.16. *The object set $\mathrm{D}'_1(c)$ is nonempty.*

By combining Proposition 11.13 and Lemma 11.15, we obtain

PROPOSITION 11.17. *The map $(\overline{\mathrm{LL}}, \mathrm{clbl})$ induces a bijection*

$$\begin{aligned}(W\backslash V^{\mathrm{reg}})^{\mu_d} &\longrightarrow (\overline{E}_n^\circ)^{\mu_{d'}} \boxdot \mathrm{D}'_\bullet(c) \\ y &\longmapsto (\overline{\mathrm{LL}}(y), \mathrm{clbl}(y)).\end{aligned}$$

11.3. *Basepoints and groupoids.* As explained in Definition 6.2, the "fat basepoint" $\mathcal{U}$ is the set of pairs $(y, z) \in Y \times \mathbb{C}$ such that "z is not below a point in $\mathrm{LL}(y)$" or, equivalently, such that $\overline{\mathrm{LL}}((y, z))$ does not contain any point in the closed half-line $\sqrt{-1}\mathbb{R}_{\geq 0}$:

$$\mathcal{U}^{\mu_d} \simeq \{x \in (W\backslash V^{\text{reg}})^{\mu_d} | \overline{\mathrm{LL}}(x) \cap \sqrt{-1}\mathbb{R}_{\geq 0} = \varnothing\}.$$

This formula fails to capture the obvious symmetry constraints deduced from Lemma 11.1: if x is μ_d invariant, then $\overline{\mathrm{LL}}(x)$ must be $(\mu_d)^h$-invariant and hence $\mu_{d'}$-invariant. Hence,

$$\mathcal{U}^{\mu_d} \simeq \left\{ x \in (W\backslash V^{\text{reg}})^{\mu_d} | \overline{\mathrm{LL}}(x) \cap \bigcup_{\zeta \in \mu_{d'}} \zeta\sqrt{-1}\mathbb{R}_{\geq 0} = \varnothing \right\}.$$

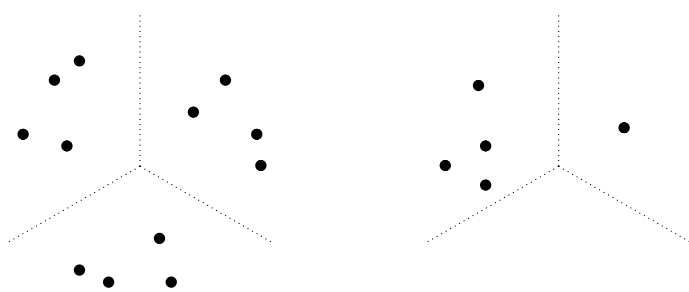
We also introduce

$$\mathcal{U}_{d'} := \left\{ x \in W\backslash V^{\text{reg}} | \overline{\mathrm{LL}}(x) \cap \bigcup_{\zeta \in \mu_{d'}} \zeta\sqrt{-1}\mathbb{R}_{\geq 0} = \varnothing \right\}.$$

Clearly,

$$\mathcal{U}^{\mu_d} \subseteq \mathcal{U}_{d'} \subseteq \mathcal{U}.$$

An illustration (with $d' = 3$) of what $\overline{\mathrm{LL}}(x)$ may look like for some $x \in \mathcal{U}^{\mu_d}$ (left) and for some $x \in \mathcal{U}_{d'} - \mathcal{U}^{\mu_d}$ (right):



By contrast with $\mathcal{U}$, neither $\mathcal{U}^{\mu_d}$ or $\mathcal{U}_{d'}$ is connected when $d' > 1$. However, both happen to have contractible connected components (see Lemma 11.22 below), which makes them suitable as "fat groupoid basepoints" (Definition A.4).

Let $x \in \mathcal{U}_{d'}$. The support of $\overline{\mathrm{LL}}(x)$ can be partitioned into d' (possibly empty) groups $B_1, \ldots, B_{d'}$, according to which $2\pi/d'$-sector they lie in: the group B_j consists of m_j points $x_{\alpha_j}, \ldots, x_{\alpha_j+m_j}$ with cyclic argument in $(\frac{2\pi}{d'}(j-1), \frac{2\pi}{d'}j)$.

Definition 11.18. The *d'-cyclic content* of $x \in \mathcal{U}_{d'}$ is the sequence

$$\mathrm{cc}_{d'}(x) := (c_{\alpha_1} \ldots c_{\alpha_1+m_1}, \ldots, c_{\alpha_{d'}} \ldots c_{\alpha_{d'}+m_j}) \in \sqrt[d']{\mathrm{D}}_1(c)$$

Note that some of the m_j's may be 0. Contrary to the cyclic label, which is always nondegenerate, the cyclic content may contain trivial terms.

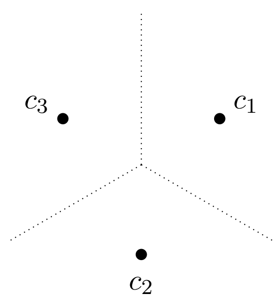
LEMMA 11.19. *Let $x \in \mathcal{U}_{d'}$. Consider the path $[0,1] \to W\backslash V^{\mathrm{reg}}, t \mapsto e^{\sqrt{-1}\frac{2\pi}{d'h}t}x$. This path represents in B the simple element associated with the first terms in $\mathrm{cc}_{d'}(x)$.*

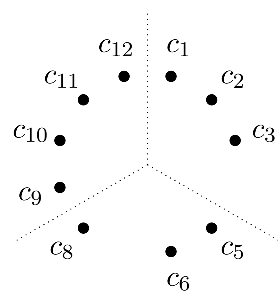
Proof. Obvious by construction of clbl and cc. □

In the following definition, the existence and uniqueness of the standard image are guaranteed by Proposition 11.13.

Definition 11.20 (standard image x_σ). Let $\sigma = (c_1, \ldots, c_{d'k}) \in \sqrt[d']{\mathrm{D}_\bullet}(c)$, with $k \geq 1$. The *standard image* x_σ is the unique element of $\mathcal{U}_{d'}$ such that, for all j such that $c_j \neq 1$, the point $e^{\sqrt{-1}\pi(\frac{1}{2}-\frac{2j-1}{d'k})}$ is in $\overline{\mathrm{LL}}(x_\sigma)$ and the corresponding term in $\mathrm{clbl}(x_\sigma)$ is c_j.

Here are two examples with $d' = 3$, first with $k = 1$ and (c_1, c_2, c_3) not containing any trivial term, and then with $k = 4$ and the trivial terms in $(c_1, \ldots, c_{12})$ being c_4 and c_7:





LEMMA 11.21. *Let $\sigma = (c_1, \ldots, c_{d'k}) \in \sqrt[d']{\mathrm{D}_\bullet}(c)$, with $k \geq 1$. We have*

(i) $\mathrm{clbl}(x_\sigma)$ *is obtained by removing trivial terms in* σ;

(ii) $\mathrm{cc}_{d'}(x_\sigma) = (c_1 \cdots c_k, \ldots, c_{k(d'-1)+1} \cdots c_{kd'})$;

(iii) *if $\sigma \in \mathrm{D}'_\bullet(c)$, then $x_\sigma \in \mathcal{U}^{\mu_d}$.*

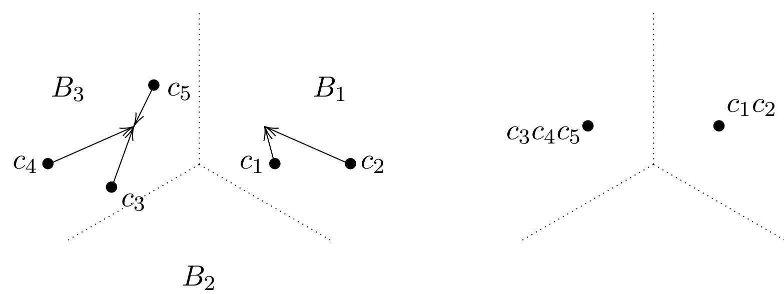
Proof. Obvious by construction. □

LEMMA 11.22.

(i) *Let C be a connected component of $\mathcal{U}_{d'}$ (resp. $\mathcal{U}^{\mu_d}$). The map $\mathrm{cc}_{d'}$ is constant on C.*
(ii) *Let C, C' be connected components of $\mathcal{U}_{d'}$ (resp. $\mathcal{U}^{\mu_d}$). If $C \neq C'$, then $\mathrm{cc}_{d'}(C) \neq \mathrm{cc}_{d'}(C')$.*
(iii) *The map $\mathrm{cc}_{d'}$ restricts to bijections $\pi_0(\mathcal{U}_{d'}) \simeq \sqrt[d']{\mathrm{D}_1}(c)$ and $\pi_0(\mathcal{U}^{\mu_d}) \simeq \mathrm{D}'_1(c)$.*
(iv) *The connected components of $\mathcal{U}_{d'}$ (resp. $\mathcal{U}^{\mu_d}$) are contractible.*

Proof. (i): Let x and x' be two points in the same connected component of $\mathcal{U}_{d'}$. Starting from any path connecting x to x' within $\mathcal{U}_{d'}$, Lemma 11.19 produces a homotopy showing that the first term of $\mathrm{cc}_{d'}(x)$ coincides with the first term of $\mathrm{cc}_{d'}(x')$. Applying the same argument to ζx and $\zeta x'$ for $\zeta \in \mu_d$, we see that $\mathrm{cc}_{d'}(x) = \mathrm{cc}_{d'}(x')$.

(ii) and (iv): Let $x \in \mathcal{U}_{d'}$. Consider the standard element $x_{\mathrm{cc}_{d'}(x)}$. We construct a path $\gamma : [0,1] \to \mathcal{U}_{d'}$ from x to $x_{\mathrm{cc}_{d'}(x)}$ by sliding the points in $\overline{\mathrm{LL}}(x)$ in each region B_j affinely towards the corresponding point in $\overline{\mathrm{LL}}(x_{\mathrm{cc}_{d'}(x)})$, as in the following picture:



This proves that x and $x_{\mathrm{cc}_{d'}(x)}$ lie in the same connected component of $\mathcal{U}_{d'}$ (resp. $\mathcal{U}^{\mu_d}$, since if x is μ_d-invariant, so is the path γ). (ii) follows: if x, x' are such that $\mathrm{cc}_{d'}(x) = \mathrm{cc}_{d'}(x')$, then x and x' must lie in the same connected component. Using (i), we also notice that the path γ is part of a deformation-retraction of the full connected component of x onto the single point $x_{\mathrm{cc}_{d'}(x)}$. This proves (iv).

(iii): combining (i) and (ii), we see that $\mathrm{cc}_{d'}$ induces a bijection from $\pi_0(\mathcal{U}_{d'})$ onto its image in $\sqrt[d']{\mathrm{D}_1}(c)$ (resp. from $\pi_0(\mathcal{U}^{\mu_d})$ onto its image in $\mathrm{D}'_1(c)$). To prove (iii), it suffices to check that, for any decomposition in $\sigma \in \sqrt[d']{\mathrm{D}_1}(c)$ (resp. $\mathrm{D}'_1(c)$), there exists a point x in $\mathcal{U}_{d'}$ (resp. $\mathcal{U}^{\mu_d}$) such that $\mathrm{cc}_{d'}(x) = \sigma$; the standard image x_σ provides a particular example of such a point. $\square$

A consequence of Lemma 11.22(iv) is that we can use $\mathcal{U}_{d'}$ and $\mathcal{U}^{\mu_d}$ as a "fat groupoid basepoint" (see Definition A.4):

Definition 11.23. The relative braid category associated with W' is the groupoid

$$B' := \pi_1((W\backslash V^{\mathrm{reg}})^{\mu_d}, \mathcal{U}^{\mu_d}).$$

We also set

$$B_{d'} := \pi_1(W\backslash V^{\mathrm{reg}}, \mathcal{U}_{d'}).$$

By functoriality of π_1, we have natural functors

$$B' \longrightarrow B_{d'} \xrightarrow{\sim} B \; .$$

Note that the functor $B_{d'} \to B$ is an *equivalence of categories* and not an isomorphism: it is not injective on objects.

11.4. *Circular tunnels and semitunnels.*

Definition 11.24. A *circular semitunnel* is an element $T = (x, L)$ in

$$\mathcal{U}_{d'} \times \left[0, \frac{2\pi}{d'h}\right].$$

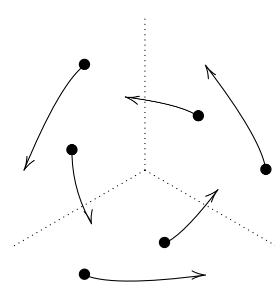
We say that L is the length of the semitunnel.

The path γ_T associated with T is the path $[0,1] \to (W \backslash V^{\mathrm{reg}})^{\mu_d}, t \mapsto e^{\sqrt{-1}t}x$.

The circular semitunnel T is a *circular tunnel* if it satisfies the additional condition

$$e^{\sqrt{-1}L} x \in \mathcal{U}_{d'}.$$

Let $T = (x, L)$ be a circular (semi)tunnel. Because $\overline{\mathrm{LL}}$ has degree h, the angular rotation of $\mathrm{LL}(\gamma_T)$ throughout the motion is hL. Here is an example of tunnel of length $\frac{\pi}{3h}$:



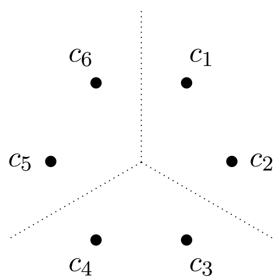
Definition 11.25. Let $\sigma = (c_1, \ldots, c_{2d'}) \in \sqrt[d']{\mathrm{D}_2}(c)$. The circular tunnel

$$\left(x_\sigma, \frac{\pi}{d'h}\right)$$

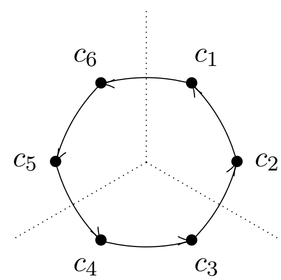
defines an element of $B_{d'}$, which we denote by b_σ.

If $\sigma \in \mathrm{D}_2'(c)$, then the circular tunnel also represents an element of B', which we denote by b'_σ.

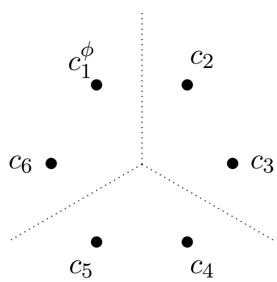
Later on, we will prove that the natural map $B' \to B_{d'}$ is injective, which will allow us to identify b'_σ with b_σ, but at this stage we are not supposed to know this.



source of b'_σ



b'_σ



target of b'_σ

LEMMA 11.26.

(i) *The map* $\sqrt[d']{\mathrm{D}}_2(c) \to B_{d'}, \sigma \mapsto b_\sigma$ *extends to a groupoid morphism*

$$\psi : G_{d'} \longrightarrow B_{d'}.$$

(ii) *The map* $\mathrm{D}'_2(c) \to B', \sigma \mapsto b'_\sigma$ *extends to a groupoid morphism*

$$\psi : G' \longrightarrow B'.$$

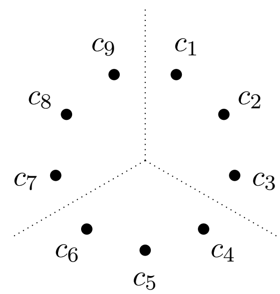
Proof. (i): The groupoid $G_{d'}$ has a presentation with generators indexed by $\sqrt[d']{\mathrm{D}}_2(c)$ and relations indexed by $\sqrt[d']{\mathrm{D}}_3(c)$. Let $\rho = (c_1, \dots, c_{3d'}) \in \sqrt[d']{\mathrm{D}}_3(c)$. It corresponds to the relation $\sigma\tau = \mu$, where

$$\sigma = (c_1, c_2c_3, c_4, c_5c_6, \dots, c_{3d'-2}, c_{3d'-1}c_{3d'}),$$

$$\tau = (c_2, c_3c_4, c_5, c_6c_7, \dots, c_{3d'-1}, c_{3d'}c_1^\phi),$$

$$\mu = (c_1c_2, c_3, c_4c_5, c_6, \dots, c_{3d'-2}c_{3d'-1}, c_{3d'}).$$

Consider the standard element x_ρ:



The needed relation $b_\sigma b_\tau = b_\mu$ follows from the following easy observations:

- the circular tunnel $T_\sigma := (x_\rho, \frac{2\pi}{3d'h})$ represents b_σ;
- the circular tunnel $T_\tau := (\zeta_{3d'h} x_\rho, \frac{2\pi}{3d'h})$ represents b_τ;
- the circular tunnel $T_\mu := (x_\rho, \frac{4\pi}{3d'h})$ represents b_μ;
- T_μ is the composition of T_σ and T_μ.

(ii). In the above construction, if $\rho \in \mathrm{D}'_3(c)$, then $\sigma, \tau, \mu \in \mathrm{D}'_2(c)$ and the same argument applies. $\square$

LEMMA 11.27. *Let* $\sigma = (c_1, \ldots, c_{2d'}) \in \sqrt[d']{\mathrm{D}_2}(c)$. *Then the image of* b_σ *via the natural functor* $B_{d'} \to B$ *is the simple element corresponding to* c_1. (*Using the notation introduced in Definition* B.23; *it is the image of* $\kappa_m(\sigma)$ *under the natural embedding* $M \hookrightarrow B$.)

Proof. This is obvious by definition of clbl. $\square$

We have a commutative diagram of functors:

$$\begin{tikzcd}
M' \arrow[r, hook] \arrow[d, hook] & M_{d'} \arrow[r, two heads, "\kappa_m"] \arrow[d, hook] & M \arrow[d, hook] \\
G' \arrow[r, hook] \arrow[d, "\psi'"] & G_{d'} \arrow[r, "\sim"] \arrow[d, "\psi"] & G \arrow[d, "\simeq"] \\
B' \arrow[r] & B_{d'} \arrow[r, "\sim"] & B.
\end{tikzcd}$$

THEOREM 11.28. *The morphism* $\psi : G_{d'} \to B_{d'}$ *is a groupoid isomorphism. It restricts to a groupoid isomorphism* $\psi' : G' \to B'$.

Proof. Let o, o' be objects in $G_{d'}$. Since we have category equivalences $G_{d'} \sim G$ and $B_{d'} \sim B$, we have a commutative diagram of set-theoretic maps:

$$\begin{tikzcd} \mathrm{Hom}_{G_{d'}}(o,o') \arrow[r, "\simeq"] \arrow[d, "\psi"] & G \arrow[d, "\simeq"] \\ \mathrm{Hom}_{B_{d'}}(\psi(o),\psi(o')) \arrow[r, "\simeq"] & B, \end{tikzcd}$$

which shows that ψ is an equivalence of categories.

By Lemma 11.22(iii), both ψ and ψ' are bijective on objects. In particular, ψ is an isomorphism of categories.

The subdiagram

$$\begin{tikzcd} G' \arrow[r, hook] \arrow[d, "\psi'"] & G_{d'} \arrow[d, "\simeq"] \\ B' \arrow[r] & B_{d'} \end{tikzcd}$$

shows that ψ' is faithful.

We are left with having to prove that ψ' is full. This follows from a generic position argument. Let b' a morphism in B'. It can be represented by a path $\gamma : [0,1] \to (W\backslash V^{\mathrm{reg}})^{\mu_d}$, and we can assume that at any given $t \in [0,1]$, at most one point in $\overline{\mathrm{LL}}(\gamma(t))$ lies on the vertical half-line $\sqrt{-1}\mathbb{R}_{\geq 0}$ (as points not satisfying this form a subspace of real codimension 2 in $(W\backslash V^{\mathrm{reg}})^{\mu_d}$); this expresses γ as a concatenation of paths homotopic to circular tunnel paths in B'. $\square$

Remark 11.29. Using Theorem 11.28, one can write down a presentation by generators and relations for B' and deduce a presentation and relations for the braid group of W'.

11.5. *The universal cover.* Let us choose a base object $o \in \mathrm{D}'_1(c)$. By Lemma 11.22(iii), it corresponds to a connected component of $\mathcal{U}^{\mu_d}$ which, by abuse of notation, we still denote by o. By Lemma 11.22(iv), this component is contractible and, using Definition A.5, we get a model $\mathrm{UniCover}((W\backslash V^{\mathrm{reg}})^{\mu_d}, o)$ for the universal cover of $(W\backslash V^{\mathrm{reg}})^{\mu_d}$.

Definition 11.30. Let $g \in \mathrm{obj}(o \downarrow B')$ or, in other words, let g be a morphism in B' with source o. Let o' be the target of g. Let γ be a path $(W\backslash V^{\mathrm{reg}})^{\mu_d}$ representing g. Let $T = (x, L)$ be a circular semitunnel such that x lies in the connected component (corresponding to) o'. Then γ and γ_T can be composed (up to homotopically trivial glue binding them) and $\gamma\gamma_T$ represents a point $p \in \mathrm{UniCover}((W\backslash V^{\mathrm{reg}})^{\mu_d}, o)$. We denote by

$$\mathcal{V}_g$$

the subspace of $\operatorname{UniCover}((W\backslash V^{\mathrm{reg}})^{\mu_d}, o)$ consisting of points p that can be obtained this way, by concatenating a path representing g with a circular semitunnel.

The following result is a variation on Proposition 9.3.

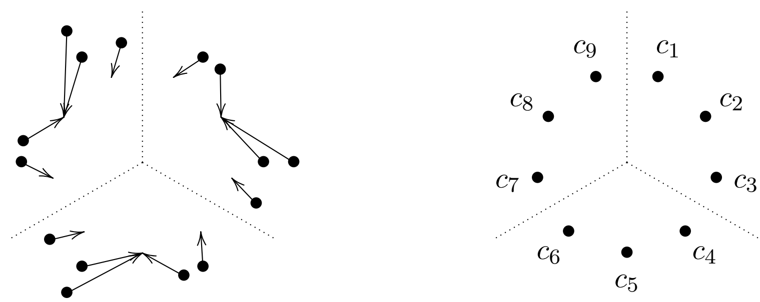
Lemma 11.31. *Let σ be a nondegenerate element of $\mathrm{D}'_\bullet(c)$. Then the subspace $U_\sigma \subseteq \mathcal{U}^{\mu_d}$ consisting of points x such that $\operatorname{clbl}(x) \vdash \sigma$ is contractible.*

The notation $\vdash$ was introduced in Definition 7.18.

Proof. Write $\sigma = (c_1, \ldots, c_{md'})$. Clearly, $x_\sigma \in U_\sigma$, so $U_\sigma \neq \varnothing$.

Let $x \in U_\sigma$. We have $\operatorname{clbl}(x) = (d_1, \ldots, d_{pd'})$, with $p \geq m$. Each c_j is the product of consecutive $d_{\alpha(j)} \ldots d_{\alpha(j)+\beta(j)}$. This defines a mapping $\{1, \ldots, pd'\} \to \{1, \ldots, md'\}$, mapping each i to the unique j such that $\alpha(j) \leq i \leq \alpha(j) + \beta(j)$. By moving the point i-th point in $\overline{\mathrm{LL}}(x)$ continuously towards the j-th point in $\overline{\mathrm{LL}}(x_\sigma)$, one obtains a deformation-retraction of U_σ onto the single point x_σ.

In the illustration below, we have chosen $p = 5$ and $m = 3$:



x $\qquad\qquad$ x_σ

∎

Theorem 11.32.

(1) *For all $g \in \operatorname{obj}(o \downarrow B')$, the space $\mathcal{V}_g$ is open.*
(2) *We have*
$$\operatorname{UniCover}((W\backslash V^{\mathrm{reg}})^{\mu_d}, o) = \bigcup_{g \in \operatorname{obj}(o \downarrow B')} \mathcal{V}_g.$$
(3) *The nerve of the open covering $(\mathcal{V}_g)_{g \in \operatorname{obj}(o \downarrow B')}$ is $\operatorname{gar}(B', S', o)$.*
(4) *For all simplex $\{g_0, \ldots, g_k\} \in \operatorname{gar}(B', S', o)$, the intersection $\bigcap_i \mathcal{V}_{g_i}$ is contractible.*

The complex gar is introduced in Definition B.11. Before proving the theorem, we observe that it implies Theorem 0.3 and, by addressing the remaining case of G_{31}, completes the proof of the $K(\pi, 1)$ conjecture.

Corollary 11.33. *The space $(W\backslash V^{\mathrm{reg}})^{\mu_d}$ is a $K(\pi, 1)$ space.*

Proof of Corollary 11.33. Using [36, 4G.3], we deduce from Theorem 11.32 that

$$\mathrm{UniCover}((W\backslash V^{\mathrm{reg}})^{\mu_d}, o)$$

is homotopy equivalent to $|\,\mathrm{gar}(B', S', o)|$. By Theorem B.14, $|\,\mathrm{gar}(B', S', o)|$ is contractible. □

Proof of Theorem 11.32. (1): Consider a point in $\mathcal{V}_g$ associated with γ and γ_T, where $T = (x, L)$ is a circular semitunnel; for a neighborhood Ω of x in $\mathcal{U}^{\mu_d}$, concatenating γ with the circular semitunnels $((x', L))_{x' \in \Omega}$ yields a neighborhood of the original point in $\mathcal{V}_g$.

(2): Consider a point p in $\mathrm{UniCover}((W\backslash V^{\mathrm{reg}})^{\mu_d}, o)$, associated with a path γ in $(W\backslash V^{\mathrm{reg}})^{\mu_d}$ with source in o. If $\gamma(1) \in \mathcal{U}^{\mu_d}$, then taking T the trivial circular semitunnel of length 0 starting at $\gamma(1)$ shows that $p \in \mathcal{V}_g$, where g is the element associated with γ. In the general case, we may find $\varepsilon > 0$ arbitrarily small such that $e^{\sqrt{-1}\varepsilon}\gamma(1)$ lies in $\mathcal{U}^{\mu_d}$; p is then represented by $e^{\sqrt{-1}\varepsilon}\gamma$ concatenated with the path associated with a small circular semitunnel T_ε: we see that $p \in \mathcal{V}_g$, where g is the element associated with $e^{\sqrt{-1}\varepsilon}\gamma$.

(3) (rank 2): Assume that $\mathcal{V}_g \cap \mathcal{V}_{g'} \neq \varnothing$. Then $p \in \mathcal{V}_g \cap \mathcal{V}_{g'}$ can be represented by both $\gamma\gamma_T$ (where γ represents g and $T = (x, L)$ is a circular semitunnel) and $\gamma'\gamma_{T'}$ (where γ' represents g' and $T' = (x', L')$ is a circular semitunnel). Up to exchanging g and g', we may assume that $L \geq L'$. Let T'' be the shortened circular semitunnel $(x, L - L')$, and let T''' be the remaining chunk $(e^{\sqrt{-1}(L-L')}x, L')$. The paths $\gamma\gamma_T$ and $\gamma'\gamma_{T'}$ are homotopic; so are γ_T and $\gamma_{T''}\gamma_{T'''}$. Because they have the same target and because they both are scalar rotations, the paths $\gamma_{T'''}$ and $\gamma_{T'}$ are homotopic. We conclude that $\gamma\gamma_{T''}$ and γ' are homotopic. In particular, T'' is a circular tunnel, representing a simple element s'', and we have $gs'' = g'$. Conversely, it is clear that if s' is a simple element such that $gs'' = g'$, we can explicitly construct a point in $\mathcal{V}_g \cap \mathcal{V}_{g'}$.

(3) (higher rank): The same argument still works, after noting that a nerve simplex $\{g_0, \ldots, g_k\}$ can be ordered in such a way that each g_i is represented by $\gamma_i\gamma_{T_i=(x_i,L_i)}$ and $L_0 \geq \cdots \geq L_k$. Considering for each $i \leq j$ the truncated circular semitunnel $T''_{i,j} = (x_i, L_i - L_j)$ and remaining chunk $T'''_{i,j} = (e^{\sqrt{-1}(L_i-L_j)}x_i, L_j)$, we see that $T''_{i,j}$ is actually a circular tunnel, representing a simple element $s''_{i,j}$ such that $g_i s''_{i,j} = g_j$.

(4) First, we consider individual $\mathcal{V}_g$. By retracting the circular semitunnel part to length 0, we easily see that each $\mathcal{V}_g$ is contractible. (This is the analog of Lemma 10.5.)

Now we have to prove the analog of Proposition 10.7, and we need to consider nonempty nontrivial intersections $\bigcap_{j=0}^{n} \mathcal{V}_{g_j}$. The proof of (3) provides the basis for an explicit description of these. We keep the same conventions and notation. A point p in $\bigcap_{j=0}^{n} \mathcal{V}_{g_j}$ can be represented by $\gamma_0\gamma_T$, where $T = (x, L)$ is a circular semitunnel such that each truncation $T''_{0,j} = (x_0, L_0 - L_j)$ is a circular tunnel representing $s''_{0,j} := g_0^{-1} g_j$. Interestingly, changing γ_0 to another path representing g_0 yields the same p, because there is only one homotopy class of such paths. In other words, $\bigcap_{j=0}^{n} \mathcal{V}_{g_j}$ is indexed by circular semitunnels $T = (x, L)$ such that there exists a nondecreasing sequence $0 \leq L'_1 \leq \cdots \leq L'_k \leq L$ such that, for each j, the truncation (x, L'_j) represents $g_0^{-1} g_j$. When T satisfies this condition, we say that it *represents* the sequence $(g_0^{-1} g_1, \ldots, g_0^{-1} g_k)$. Actually, $\bigcap_{j=0}^{n} \mathcal{V}_{g_j}$ is homotopy equivalent to the space of circular semitunnels representing $(g_0^{-1} g_1, \ldots, g_0^{-1} g_k)$, equipped with the product topology. (This is the analog of Proposition 9.5.)

We say that $x \in \mathcal{U}^{\mu_d}$ represents $(g_0^{-1} g_1, \ldots, g_0^{-1} g_k)$ if there exists a nondecreasing sequence $0 \leq L'_1 \leq \ldots L'_k \leq \frac{2\pi}{d'h}$ such that, for each j, the truncation (x, L'_j) represents $g_0^{-1} g_j$. An obvious retraction argument (onto the particular choice $L = \frac{2\pi}{d'h}$ for the length) shows that the space of circular semitunnels representing $(g_0^{-1} g_1, \ldots, g_0^{-1} g_k)$ is homotopy-equivalent to the space of points $x \in \mathcal{U}^{\mu_d}$ representing $(g_0^{-1} g_1, \ldots, g_0^{-1} g_k)$.

To conclude, we are down to proving that the space of points $x \in \mathcal{U}^{\mu_d}$ representing

$$(g_0^{-1} g_1, \ldots, g_0^{-1} g_k)$$

is contractible. This is Lemma 11.31. □

CONJECTURE 11.34. *The methods of this section can be adapted to work in the context of Lehrer-Springer theory* (*an extension of Springer theory*; *see* [41].) *The combinatorics of the associated categorical Garside structure follow Armstrong's variant of the cyclic sieving phenomenon.* (*See* [39, Th. 2, p. 204].)

12. **Periodic elements in** $B(W)$

As before, W is an irreducible well-generated complex reflection group, $\tau \in P(W)$ is the full-twist and $\delta \in B(W)$ is the Garside element of the dual braid monoid $M(W)$. The image of δ in W is a Coxeter element c.

Definition 12.1. An element of $B(W)$ is *periodic* if it admits a central power.

The goal of this section is to prove that the center $ZB(W)$ is cyclic, and to establish a correspondence between periodic elements in $B(W)$ and regular elements in W. As with the previous section, the real substance of the arguments lies more in the algebraic tools from [5] than in the easy geometric interpretation.

LEMMA 12.2. *The intersection of the subgroup $\langle c\rangle \subseteq W$ with the interval $[1, c]$ is $\{1, c\}$.*

Proof. [32, Prop. 4.2] gives an argument for the real case that, as pointed out by J. Michel, applies verbatim. We include it for the convenience of the reader. Let $2 \leq d_1, \ldots, d_n = h$ be the reflection degrees. Because c is regular, its eigenvalues are $\zeta_d^{1-d_i}$ ([55, Th. 4.2(v)]). Assume that some power c^k lies in $[1, c]$ and thus that

$$l_R(c^k) + l_R(c^{1-k}) = n.$$

By Proposition 8.4 and Lemma 7.4, $n - l_R(c^k)$ is the number of eigenvalues of c^k distinct from 1; thus

$$l_R(c^k) = n - \#\{i|(1 - d_i)k \equiv 0[h]\}.$$

Using the same formula for c^{1-k}, we deduce that

$$\#\{i|(1 - d_i)k \equiv 0[h]\} + \#\{i|(1 - d_i)(1 - k) \equiv 0[h]\} = n.$$

For a given i, $(1 - d_i)k$ and $(1 - d_i)(1 - k)$ cannot simultaneously divide h. The identity then forces that, for all i, either $(1 - d_i)k$ or $(1 - d_i)(1 - k)$ is a multiple of h. But, when $2 \leq k \leq h - 1$, neither $(1 - h)k$ nor $(1 - h)(1 - k)$ is a multiple of h. $\square$

THEOREM 12.3. *Let $h' := \frac{h}{\gcd(d_1,\ldots,d_n)}$. The centers of $B(W)$ and W are cyclic, generated respectively by $\delta^{h'}$ and $c^{h'}$.*

Proof. That ZW is cyclic of order $\gcd(d_1, \ldots, d_n)$ is classical; any central element is regular; because $c^{h'}$ is regular and has the right order, it must generate ZW.

Let us study the conjugacy action of δ. Write a given $b \in B(W)$ in Garside normal form $b = \delta^k s_1 \cdots s_l$, where $k \in \mathbb{Z}$ and $s_1, \ldots, s_l \in [1, c] - \{1, c\}$. The normal form of b^δ is $\delta^k s_1^c \cdots s_l^c$.

Because $c^{h'}$ is the smallest central power of c in W, $\delta^{h'}$ is the smallest central power of δ in $B(W)$. Moreover, if $b \in ZB(W)$, then it must commute with δ and any simple term s_i in its normal form $\delta^k s_1 \cdots s_l$ must commute with c. Because c is Coxeter element, the centralizer of c in W is $\langle c\rangle$. (This follows from Theorem 1.9 and, actually, extends Corollary 4.4 in [55].) Combining this with Lemma 12.2, we see that $l = 0$, i.e., that $b \in \langle\delta\rangle$. $\square$

Because of Theorem 12.3, an element $\gamma \in B(W)$ is periodic if and only if it is *commensurable* with τ (or δ), i.e., if there exists p, q such that

$$\gamma^q = \tau^p.$$

To simplify notation, we restrict our attention to the situation where

$$\gamma^d = \tau$$

and call such a periodic γ a *d-th root of* τ. The theory works the same way for other (p,q).

When d is regular and x_0 is a ζ_d-regular eigenvector, we may consider the *standard d-th root of* τ represented by

$$\begin{aligned} [0,1] &\longrightarrow W\backslash V^{\mathrm{reg}} \\ t &\longmapsto e^{2\pi\sqrt{-1}\frac{t}{d}}x_0 \end{aligned}$$

and denoted by $\sqrt[d]{\tau}$. This of course involves choosing a particular basepoint, but because the statements below are "up to conjugacy," one should not worry too much about this.

A particular case is

$$\sqrt[h]{\tau} = \delta.$$

THEOREM 12.4 (Springer theory in braid groups). *Let d be a positive integer.*

(i) *There exist d-th roots of τ if and only if d is regular.*
(ii) *When d is regular, there is a single conjugacy class of d-th roots of τ in $B(W)$. In particular, all d-th roots of τ are conjugate to $\sqrt[d]{\tau}$.*
(iii) *Let ρ be a d-th root of τ. Let w be the image of ρ in W. Then w is ζ_d-regular, and the centralizer $C_{B(W)}(\rho)$ is isomorphic to the braid group $B(W')$ of the centralizer $W' := C_W(w)$.*

Proof. (i) A consequence of [5, Cor. 10.4] is that, if $\tau = \delta^h$ admits d-th roots, then $M_d^{\phi_d^h}$ is nonempty. By Lemma 11.15, this implies that d is regular. The converse is obvious. (We may consider the particular root $\sqrt[d]{\tau}$.)

(ii) Using [5, Cor. 10.4] and [5, Prop. 9.8], we see that conjugacy classes of d-th roots of τ are in one-to-one correspondence with connected components of the category G'. Because it is equivalent to a group, this category is connected.

(iii) That w is regular follows from (ii) because it is conjugate to the image in W of $\sqrt[d]{\tau}$, whose image in W has a x_0 as ζ_d-regular eigenvector. The assertion about the centralizer follows from its categorical rephrasing in M_d, where it is trivial (the conjugacy action being a power of the diagram automorphism of the Garside structure). $\square$

Remark 12.5. This answers many questions and conjectures by Broué, Michel and others. (See [18] for more details.) Particular cases of (i) were obtained by Broué-Michel and, independently, by Shvartsman, [21], [54]. Assertion (ii) can be viewed either as a Kerékjártó type theorem ("*all periodic elements are conjugate to a rotation*"; see [5]) or as a Sylow type theorem ("*all $C_{B(W)}(\rho)$ are conjugate*"). The type A case of (ii) actually follows from Kerékjártó's theorem on periodic homeomorphism of the disk ([37]; see also [5] for a more complete bibliography). The type A case of (iii) was proved in [7].

Of course, the most natural interpretation is to view the theorem as providing a braid analog of Theorem 1.9.

Remark 12.6. Let W' be the centralizer of a regular element in W. Let W'' be the centralizer of a regular element in W'. In terms of orbit varieties, $W'\backslash V' = (W\backslash V)^{\mu_d}$ and $W''\backslash V'' = (W\backslash V)^{\mu_{de}}$. Regular elements of W' are regular in W. It should be possible, by applying Theorem 12.4 to the pair (W, W''), to generalize the result to the pair (W', W'').

COROLLARY 12.7. *The center of the braid group $B(G_{31})$ is cyclic.*

Proof. Let W be G_{37}, the well-generated reflection group of type E_8. The degrees are

$$2, 8, 12, 14, 18, 20, 24, 30,$$

and the codegrees are

$$0, 6, 10, 12, 16, 18, 22, 28.$$

The number 4 is regular, with centralizer W' of type G_{31}.

By Theorem 1.9 (1), we see that 24 is also regular. Let ρ be a 24-th root of τ. The centralizer is the rank 1 reflection group of type A_1, with braid group $\mathbf{Z}$.

Applying Theorem 12.4 to ρ^6, we recognize the braid group of G_{31} as a centralizer in $B(W)$:

$$B(G_{31}) \simeq C_{B(W)}(\rho^6).$$

Applying Theorem 12.4 to ρ, we see that

$$C_{B(W)}(\rho) \simeq B(A_1) \simeq \mathbf{Z}.$$

Clearly, $\rho \in C_{B(W)}(\rho^6)$. In particular, any $z \in ZB(G_{31})$ must commute with ρ and hence lie in $C_{B(W)}(\rho) \simeq \mathbf{Z}$. □

Combining Theorem 12.3 and Corollary 12.7, we complete the proof of

THEOREM 12.8 (Theorem 0.5). *The center of the braid group of an irreducible complex reflection group is cyclic.*

Indeed, Broué-Malle-Rouquier conjectured this in [20], and proved it for all cases but six exceptional types: five of these exceptions are well-generated and covered by Theorem 12.3, the remaining case being G_{31}.

13. Generalized noncrossing partitions

Here again, W is an irreducible well-generated complex reflection group generated by reflections of order 2.

When W is of type A_{n-1}, the lattice $(S, \preccurlyeq)$ is isomorphic to the lattice of noncrossing partitions of a regular n-gon ([7], [13]). Following [48], [3] and [6], we call *lattice of generalized noncrossing partitions of type* W the lattice

$$(S, \preccurlyeq)$$

and *Catalan number of type* W the number

$$\mathrm{Cat}(W) := \prod_{i=1}^{n} \frac{d_i + h}{d_i}.$$

The operation sending $s \preccurlyeq t$ to $s^{-1}t$ is an analogue of the Kreweras complement operation. The map $s \mapsto s^{-1}\delta$ is an anti-automorphism of the lattice.

In the Coxeter case, Chapoton (see [22]) discovered a general formula for the number $Z_W(N)$ of weak chains $s_1 \preccurlyeq s_2 \preccurlyeq \cdots \preccurlyeq s_{N-1}$ of length $N-1$ in $(S, \preccurlyeq)$ or, equivalently (by Lemma B.16), for the cardinality of $\mathrm{D}_N(c)$. This formula continues to hold, though we are only able to prove this case-by-case. (See [1] and [22] for the Coxeter types; the $G(e,e,n)$ case was done in [6]; the remaining types are done by computer.)

PROPOSITION 13.1. *We have, for all* N,

$$Z_W(N) = \prod_{i=1}^{n} \frac{d_i + (N-1)h}{d_i}.$$

COROLLARY 13.2. *We have* $|S| = \mathrm{Cat}(W)$.

Another interesting numerical invariant is the *Poincaré polynomial*

$$\mathrm{Poin}(S) := \sum_{s \in S} t^{l(s)}.$$

The numerical data for the exceptional types (real and nonreal) is summarised in Table 2. The coefficient of t in the Poincaré polynomial is the cardinal of R_c. One observes that

$$R = R_c \Leftrightarrow W \text{ is real.}$$

In the Weyl group case, $\mathrm{Poin}(S)$ may be interpreted as the Poincaré polynomial of the cohomology of a toric variety related to cluster algebras ([22]).

When W' is the (not necessarily well-generated) centralizer of a d-regular element in a well-generated W, the natural substitute for $Z_W(N)$ is the cardinality $Z'_{W'}(N)$ of $\mathrm{D}'_N(c)$. In a joint work with Vic Reiner, we conjectured that the μ_d-action exhibits a *cyclic sieving phenomenon*: the number of fixed points should be the value at ζ_d of a q-analog of the number of chains; the conjecture has now been proved by Krattenthaler-Müller:

W	degrees	$\lvert R\rvert$	$\mathrm{Cat}(W)$	$\mathrm{Poin}(S)$	$\lvert \mathrm{Red}_R(c)\rvert$
G_{23} (H_3)	$2,6,10$	15	32	$1+15t+15t^2+t^3$	50
G_{24}	$4,6,14$	21	30	$1+14t+14t^2+t^3$	49
G_{27}	$6,12,30$	45	42	$1+20t+20t^2+t^3$	75
G_{28} (F_4)	$2,6,8,12$	24	105	$1+24t+55t^2+24t^3+t^4$	432
G_{29}	$4,8,12,20$	40	112	$1+25t+60t^2+25t^3+t^4$	500
G_{30} (H_4)	$2,12,20,30$	60	280	$1+60t+158t^2+60t^3+t^4$	1350
G_{33}	$4,6,10,12,18$	45	308	$1+30t+123t^2$ $+123t^3+30t^4+t^5$	4374
G_{34}	$6,12,18,$ $24,30,42$	126	1584	$1+56t+385t^2+700t^3$ $+385t^4+56t^5+t^6$	100842
G_{35} (E_6)	$2,5,6,$ $8,9,12$	36	833	$1+36t+204t^2+351t^3$ $+204t^4+36t^5+t^6$	41472
$G_{36}(E_7)$	$2,6,8,10,$ $12,14,18$	63	4160	$1+63t+546t^2+1470t^3$ $+1470t^4+546t^5+63t^6+t^7$	1062882
G_{37} (E_8)	$2,8,12,14,$ $18,20,24,30$	120	25080	$1+120t+1540t^2$ $+6120t^3+9518t^4+6120t^5$ $+1540t^6+120t^7+t^8$	37968750

Table 2. Numerical invariants of generalized noncrossing partitions.

Theorem 13.3 (Conjecture 6.5 in [9], proved in [39]). *Let q be an indeterminate. For any $a \in \mathbb{Z}_{\geq 1}$, set $[a]_q := 1 + q + \cdots + q^{a-1}$. Then*

$$\prod_{i=1}^{n} \frac{[d_i + (Nd' - 1)h]_q}{[d_i]_q}$$

is a polynomial in q whose value at $q = \zeta_d$ is $Z'_{W'}(N)$.

When $N = 1$, the formula gives the number of objects in M'. For G_{31}, we have 88 objects (see Example 11.7).

Appendix A. The fat basepoint trick

A.1. *Fundamental groupoids.* Let E be a topological space. Let γ be a path in E, i.e., a continuous map $[0,1] \to E$. We say that γ is a *path from* $\gamma(0)$ *to* $\gamma(1)$ — or that $\gamma(0)$ is the *source* and $\gamma(1)$ the *target*. The concatenation rule is as follows. If γ, γ' are paths such that $\gamma(1) = \gamma'(0)$, the product $\gamma\gamma'$ is the path mapping $t \leq 1/2$ to $\gamma(2t)$ and $t \geq 1/2$ to $\gamma'(2t-1)$.

We denote by $\pi_1(E)$ the fundamental groupoid of E: its elements are homotopy classes of paths in E, with composition rule as above. As a category, its object set is E.

For any "basepoint" $e \in E$, the *fundamental group of E with respect to e* is

$$\pi_1(E,e) := \mathrm{Hom}_{\pi_1(E)}(e,e).$$

When E is a *path-connected* space or, equivalently, when $\pi_1(E)$ is a *connected* groupoid, all fundamental groups of E, with respect to all possible basepoints, are isomorphic. **However, they are not canonically isomorphic:** any element $g \in \mathrm{Hom}_{\pi_1(E)}(e,e')$ yields an isomorphism:

$$\begin{aligned} \phi_g : \pi_1(E,e) &\xrightarrow{\sim} \pi_1(E,e') \\ f &\longmapsto g^{-1} f g, \end{aligned}$$

but there is no natural way to make consistent choices of such g's and build a *transitive systems of isomorphisms* connecting

$$(\pi_1(E,e))_{e\in E}.$$

In other words, there is no legitimate way to drop the reference to a specific basepoint and talk about *the* fundamental group of E.

Let $A \subseteq E$. Consider the natural functor $\iota : \pi_1(A) \to \pi_1(E)$.

LEMMA A.1. *When A is simply connected, then*

$$(\phi_{\iota(g)})_{g\in\pi_1(A)}$$

is a transitive system of isomorphisms connecting $(\pi_1(E,a))_{a\in A}$.

Proof. The space A is simply connected if and only if the category $\pi_1(A)$ is equivalent to the trivial category. In a category equivalent to the trivial category, transitivity comes for free: $(\phi_g)_{g\in\pi_1(A)}$ is transitive, and by functoriality so is $(\phi_{\iota(g)})_{g\in\pi_1(A)}$. □

This legitimates the following definition.

Definition A.2 (fat basepoint trick, group version). The fundamental group of E with respect to a simply-connected subspace A is the transitive limit

$$\pi_1(E,A) = \varinjlim_{a\in A} \pi_1(E,a)$$

with respect to the transitive system of isomorphisms $(\phi_{\iota(g)})_{g\in\pi_1(A)}$.

Remark A.3. Instead of constructing the group $\pi_1(E,A)$ as a transitive limit of fundamental groups, one can choose to equip the set of relative homotopy classes with a group structure; there is no difference, except in language.

Practically speaking, $\pi_1(E,A)$ should be thought of as any $\pi_1(E,a)$, for some $a \in A$, together with an unambiguous recipe thanks to which any path in A from any $a' \in A$ to any $a'' \in A$ represents a unique element of $\pi_1(E,a)$;

moreover, one may forget about which $a \in A$ was chosen and change it at our convenience.

In Section 11, we use an extended version of the trick. Let C and C' be path-connected components of A. For any $g \in \mathrm{Hom}_C(c_1, c_2)$ and $g' \in \mathrm{Hom}_{C'}(c'_1, c'_2)$, we have an isomorphism

$$\begin{aligned} \psi(g, g') : \pi_1(E, c_1, c'_1) &\xrightarrow{\sim} \pi_1(E, c_2, c'_2) \\ f &\longmapsto g^{-1} f g'. \end{aligned}$$

When both C and C' are simply connected, this provides a transitive system of isomorphisms $(\psi(g, g'))_{(g,g') \in \pi_1(C) \times \pi_1(C')}$ thanks to which we can define

$$\mathrm{Hom}_E(C, C') := \varinjlim_{(c,c') \in C \times C'} \mathrm{Hom}_E(c, c').$$

Definition A.4 (fat basepoint trick, groupoid version). The fundamental groupoid of E with respect to a subspace A whose path-connected components are simply connected is the groupoid

$$\pi_1(E, A)$$

whose objects are path-connected components of A and such that

$$\mathrm{Hom}_{\pi_1(E,A)}(C, C') = \mathrm{Hom}_E(C, C').$$

When A is connected, we recover the group version.

A.2. *Universal covers.* Assume that E is path connected. By Galois theory, there is a correspondence between subgroups of the fundamental group and topological coverings of E.

Universal covers can be constructed as soon as E is reasonably healthy; e.g., it is enough to assume that E is locally simply connected (which is trivially satisfied by all spaces considered here).

A good reference is Hatcher [36, §1.3]. The construction starts with the choice of a basepoint $e \in E$. As a set, the universal cover $\widetilde{E}_e$ has one point per element in $\pi_1(E)$ with source e. The fundamental groupoid $\pi_1(\widetilde{E}_e)$ coincides with the category $(e \downarrow \pi_1(E))$ of *objects under* e *in* $\pi_1(E)$, in the sense of Mac Lane [43, II.6, Comma Categories]. This interpretation actually clarifies why $\widetilde{E}_e$ is simply connected: the category $(e \downarrow \pi_1(E))$ is equivalent to the trivial category. (This is the categorical way of being contractible.)

The universal cover construction can thus be rephrased as follows. The object set of $(e \downarrow \pi_1(E))$ can be equipped with a natural topology. (This is where the locally simple-connectedness of E comes into play.)

Actually, when $\mathcal{G}$ is a groupoid and o is an object of $\mathcal{G}$, then $(o \downarrow \mathcal{G})$ should be viewed as a categorical universal cover for $\mathcal{G}$.

When $A \subseteq E$ is simply connected, there is a natural transitive system of isomorphisms connecting

$$((a \downarrow \pi_1(E)))_{a \in A}$$

and thus a transitive system of bijections connecting

$$(\widetilde{E}_a)_{a \in A}.$$

It is not hard to check that these bijections are homeomorphisms and are compatible with the covering maps.

Definition A.5 (fat basepoint trick, universal cover version). The universal cover of E with respect to a simply-connected subspace A is

$$\mathrm{UniCover}(E, A) := \varinjlim_{a \in A} \widetilde{E}_a.$$

Clearly, $\mathrm{UniCover}(E, a) \simeq \widetilde{E}_a$.

LEMMA A.6. *The natural left-action of* $\pi_1(E, a)$ *on* $\widetilde{E}_a$ *is compatible with the transitive system and gives rise to a left-action of* $\pi_1(E, A)$ *on the universal cover* $\mathrm{UniCover}(E, A)$.

In real life, this means: up to topologically trivial paths within A, any path connecting two points of A (representing an element of $\pi_1(E, A)$) can be concatenated with any path with source in A (representing a point in $\mathrm{UniCover}(E, A)$) to unambiguously yield another point in $\mathrm{UniCover}(E, A)$. We do not care about which exact points in A were chosen, we do not have to give them names and we can forget about them.

When A is not contractible but has simply-connected components, the *groupoid cover* version, blending Definitions A.4 and A.5, is a bit more tedious to formulate:

- we get a model $\mathrm{UniCover}(E, o)$ for each connected component $o \subseteq A$;
- any g in $\pi_1(E, A)$ with source o and target o' induces a homeomorphism from

$$\mathrm{UniCover}(E, o) \to \mathrm{UniCover}(E, o')$$

  subject to obvious compatibility rules.

The latter data is what deserves to be called a *groupoid action on* the collection $(\mathrm{UniCover}(E, o))_{o \in \pi_0(A)}$. This situation is implicit behind Definition 11.30.

Appendix B. Garside structures

B.1. *Cohomology of groups and groupoids.* A groupoid is a (small) category where all morphisms are invertible. A group is a groupoid with a single object.

A *simplicial complex* X is a family of subsets of an ambient space S, such that whenever $A \in X$, any subset $B \subseteq A$ also lies in X. Simplicial complexes form a category, equipped with a *geometric realization* functor to the category of topological spaces.

As explained in [43], a *simplicial set* is a contravariant functor from the simplicial category to the category of sets or, equivalently, a collection $(X_n)_{n \in \mathbf{Z}_{\geq 0}}$ of sets, together with *face maps* and *degeneracy maps* respectively shifting dimensions by -1 and $+1$, and subject to obvious compatibility rules.

Simplicial complexes naturally give rises to simplicial sets, but not all simplicial sets can be obtained that way.

The *nerve* of a (small) category $\mathcal{C}$ is a simplicial set $\mathcal{NC}$ whose 0-skeleton is the object set of $\mathcal{C}$ and whose n-simplices, $n \geq 1$, are composable sequences $(f_1, \ldots, f_n)$ of $\mathcal{C}$-morphisms:

$$x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} x_2 \xrightarrow{f_3} x_3 \cdots\cdots\to x_{n-1} \xrightarrow{f_n} x_n \ .$$

In the simplicial structure on $\mathcal{NC}$, face maps correspond to removing objects (and composing or dropping morphisms accordingly) and degeneracy maps correspond to inserting identity morphisms at a given object. An element of $\mathcal{NC}$ is *nondegenerate* if it does not contain any identity morphism.

Simplicial sets naturally form a category, and the nerve construction is functorial from the category **Cat** of small categories to the category **SimpSet** of simplicial sets. There is a standard *geometric realization* functor **SimpSet** $\to$ **Top**, $X \mapsto |X|$, where **Top** category of topological spaces. See, for example, [36, appendix, Simplicial CW structures]. The construction actually provides us, for each *abstract* simplex $x \in X$, with a singular simplex in $|X|$, i.e., a continuous map from a standard affine simplex to X. Actually, the 0-skeleton X_0 is mapped injectively into $|X|$. We can use X_0 as a groupoid fat basepoint (Definition A.4).

Definition B.1. Let $\mathcal{G}$ be a groupoid. A *simplicial* $K(\mathcal{G}, 1)$ is a simplicial set X such that $\pi_1(|X|, X_0) \simeq \mathcal{G}$ and such that connected components of $|X|$ have no higher homotopy groups.

In particular, X_0 must be in bijection with the object set of $\mathcal{G}$.

THEOREM B.2 ("bar" resolution, quotient version). *The nerve $\mathcal{NG}$ of a groupoid $\mathcal{G}$ is a simplicial $K(\mathcal{G}, 1)$.*

The beauty of the categorical viewpoint is that the theorem comes as a mostly free by-product of the observation that the *nerve realization* functor **Cat** $\to$ **Top** extends to a functor of 2-categories (mapping natural transformations to homotopies). See Lemma 7.1 and Proposition 7.3 in [5]. The underlying combinatorics coincide with that of the standard "bar" resolution of group cohomology:

Definition B.3 ("bar" simplicial set). Let o be an object of $\mathcal{G}$. We use the bar symbol $og_0[g_1|\cdots|g_k]$, or simply $g_0[g_1|\cdots|g_k]$, to express that $(g_0, g_1, \ldots, g_k)$ is a sequence of composable morphisms in $\mathcal{G}$ (i.e., an element of $\mathcal{N}_{k+1}\mathcal{G}$) such that the source of g_0 is o. We denote by

$$\mathrm{bar}(\mathcal{G}, o)$$

the simplicial set whose k-skeleton consists of bar symbols $g_0[g_1|\ldots|g_k]$, subject to the faces and degeneracy maps of $\mathcal{N}_{k+1}\mathcal{G}$ (except those involving g_0, as we view $g_0[g_1|\ldots|g_k]$ as a k-simplex, not a $(k+1)$-simplex).

Theorem B.4 ("bar" resolution, universal cover version). *Let $\mathcal{G}$ be a groupoid, and let o be an object of $\mathcal{G}$. The geometric realization of* $\mathrm{bar}(\mathcal{G}, o)$ *is contractible.*

Theorems B.2 and B.4 express two flavors of the same result:

$$\mathrm{UniCover}(|\mathcal{N}\mathcal{G}|, o) \simeq |\,\mathrm{bar}(\mathcal{G}, o)|.$$

Note that we do not have to assume that $\mathcal{G}$ is connected: Theorem B.2 expresses something about each connected component, while Theorem B.4 sees only one connected component per choice of o.

B.2. *Garside structures.* Garside's approach [35] to the word and conjugacy problem in the classical braid group B_n was a key ingredient in Deligne's paper [29]. It was later axiomatized as a generic combinatorial group theory notion [28], rephrased with a geometric group theory viewpoint [10], [23] and generalized to groupoids [38].

There are many ways to tell the story, and the most general setup involves quite a lot of technicalities. Under favorable conditions, typically when there is a natural homogeneous length function (which is the case here), the whole story could probably fit in a 100 pages graduate-level textbook. In this absence of this yet-to-be-written account, the only detailed reference at hand is the much longer book [27], which focuses on word-theoretic axiomatic aspects and does not cover all aspects explained here.

As far as the current paper is concerned, the notation and results listed in [5] are more than sufficient — especially, we only consider Garside structure that are homogeneous, which removes a lot of the technicalities addressed in [27].

Let $\mathcal{C}$ be a (small) category equipped with

- an endofunctor ϕ, which we write with *right-conjugacy* notation $x \mapsto x^\phi$, $f \mapsto f^\phi$;
- a natural transformation Δ from the identity functor to ϕ.

Example B.5. When $\mathcal{C}$ is a monoid M, this simply means that $\Delta \in M$ and ϕ is the right conjugacy action $f \mapsto f^\phi = \Delta^{-1} f \Delta$.

In general, there is a morphism $\Delta_x : x \to x^\phi$ for each object x, and for each morphism f from x to y, the following diagram is commutative:

$$\begin{tikzcd} x \arrow[r, "\Delta_x"] \arrow[d, "f"'] & x^\phi \arrow[d, "f^\phi"] \\ y \arrow[r, "\Delta_y"'] & y^\phi. \end{tikzcd}$$

As Δ is a collection of morphisms, one for each source object, it makes sense to write "$f \in \Delta$" instead of "f is the morphism in Δ whose source is the source of f." It is even tempting to write "$f = \Delta$" instead, an abusive yet convenient notation inspired by the monoid case (where Δ consists of a single element), just like we write "$f = 1$" instead of "f is the identity morphism whose source is the source of f."

Definition B.6. An element of $f \in \mathcal{C}$ is *simple* with respect to Δ if there exists $g \in \mathcal{C}$ such that $fg = \Delta$.

An *atom* is an element $a \in \mathcal{C}$ such that, for all $f, g \in \mathcal{C}$, $a = fg \Rightarrow f = 1$ or $g = 1$.

The category $\mathcal{C}$ is *homogeneous* if there exists a functor l from $\mathcal{C}$ to the monoid $(\mathbb{Z}_{\geq 0}, +)$ such that for all $f \in \mathcal{C}, l(f) = 0 \Rightarrow f = 1$.

The category $\mathcal{C}$ is *cancellative* if for all $f, g, h \in \mathcal{C}, (fh = gh$ or $hg = hf)$ $\Rightarrow f = g$.

The category $\mathcal{C}$ is a *lattice* if it admits pullbacks and pushouts.

Pushouts and pullbacks are classical concepts from category theory. (See, for example, [43].) The existence of pushouts means that any two morphisms f, g with common source admit a right least common multiple. In poset language, this means that f and g admit a least upper bound for the prefix ordering. Pullback is the dual concept.

Definition B.7 (Definition 2.4 in [5]). A *Garside structure* is a triple $(\mathcal{C}, \Delta, \phi)$ satisfying

(i) $\mathcal{C}$ is a category, ϕ an automorphism of $\mathcal{C}$ and Δ a natural transformation from the identity functor to ϕ;
(ii) $\mathcal{C}$ is homogeneous and cancellative;
(iii) all atoms are simple with respect to Δ;
(iv) $\mathcal{C}$ is a lattice.

This axiom set is more restrictive than necessary, but it was chosen in [5] to be on the safe side when claiming that existing proofs in the context of Garside monoids still worked in the categorical contexts.

The crux is axiom (iv). This is where concrete examples of Garside structures encapsulate deep geometric/topological/combinatorial miracles (such as

Lemma 8.6 or, in Deligne's paper, properties of galleries that are specific to *simplicial* arrangements).

Let $(\mathcal{C}, \Delta, \phi)$ be a Garside structure. The set $\mathcal{S}$ of simple elements with respect to Δ, seen either as an abstract set together with a partial product structure obtained by restricting the category structure to $\mathcal{S}$ (a Garside *germ*, in the sense of [5]), or as a subset of $\mathcal{C}$ (a Garside *family*, in the sense of [27]), is enough to recover the whole structure $(\mathcal{C}, \Delta, \phi)$.

As explained in [5], [27], the axiom set can be rewritten in terms of axioms involving only $\mathcal{S}$. Another interesting invariant of Garside structures is the Garside set $\mathrm{D}_\bullet$ introduced below.

Definition B.8. A *Garside category* is a category $\mathcal{C}$ that can be equipped with a Garside structure. A *Garside groupoid* is a groupoid $\mathcal{G}$ that is the groupoid of fractions of a Garside category.

Note that Garside groupoids can appear as groupoids of fractions of several Garside categories that are not equivalent. (For example, finite type Artin groups admit both the classical and dual Garside structures.)

Let $\mathcal{C}$ be a Garside category, with groupoid of fractions $\mathcal{G}$. Key properties include

- the natural function $\mathcal{C} \to \mathcal{G}$ is faithful;
- the word and conjugacy problems in $\mathcal{C}$ and $\mathcal{G}$ can be solved;
- $\mathcal{G}$ has finite cohomological dimension (Theorem B.10 below);
- in particular, $\mathcal{G}$ is torsion-free.

B.3. *Cohomology of Garside groups and groupoids.*

Definition B.9 (Garside nerve). A simplex $(f_1, \dots, f_n) \in \mathcal{NC}$ is *simple* with respect to Δ if the product $f_1 \cdots f_n$ is simple. The set of simple simplices in $\mathcal{NC}$ forms a simplicial set that we denote by $\mathcal{NS}$, the *Garside nerve of* $\mathcal{C}$ *with respect to* $\mathcal{S}$.

THEOREM B.10 ("gar" resolution, quotient version). *Let $\mathcal{G}$ be a groupoid, and let $\mathcal{S}$ be a Garside structure on $\mathcal{G}$. The Garside nerve $\mathcal{NS}$ is a simplicial $K(\mathcal{G}, 1)$.*

Definition B.11 ("gar" flag complex). Let $\mathcal{G}$ be a groupoid, and let $\mathcal{S}$ be a Garside structure on $\mathcal{G}$, and let o be an object of $\mathcal{G}$. We denote by

$$\mathrm{gar}(\mathcal{G}, \mathcal{S}, o)$$

the simplicial complex with underlying space $\mathrm{obj}(o \downarrow \mathcal{G})$ (i.e., the set of morphisms in $\mathcal{G}$ with source o) and such that $\{g_0, \dots, g_k\}$ is a k-simplex if and only if, for all i, j, $g_i^{-1} g_j \in \mathcal{S}$ or $g_j^{-1} g_i \in \mathcal{S}$.

When $\mathcal{G}$ is a group, we shorten the notation to $\mathrm{gar}(\mathcal{G}, \mathcal{S})$ as there is only one possible choice for o.

In particular, a subset of $\mathcal{G}$ spans a simplex if and only if every subpair spans an edge. Such a simplicial complex is called a *flag complex*. A flag complex is uniquely determined by its 1-skeleton.

A variant of Definition B.11 that more closely resembles Definition B.3 is as follows.

Definition B.12 ("gar" simplicial set). We denote by $\mathrm{gar}'(\mathcal{G}, \mathcal{S}, o)$ the subcomplex of $\mathrm{bar}(\mathcal{G}, o)$ consisting of bar symbols $g_0[g_1|\cdots|g_k]$ such that

$$\forall i > 0, g_i \in \mathcal{S} \quad \text{and} \quad g_1 \cdots g_k \in \mathcal{S}.$$

LEMMA B.13. *The map $g_0[g_1|\cdots|g_k] \mapsto \{g_0, g_0g_1, g_0g_1g_2, \ldots, g_0 \ldots g_k\}$ induces a bijection between nondegenerate simplices in $\mathrm{gar}'(\mathcal{G}, \mathcal{S}, o)$ and simplices in $\mathrm{gar}(\mathcal{G}, \mathcal{S}, o)$.*

Proof. Because of the natural length function on $\mathcal{S}$, when $g, h \in \mathcal{G}$ are distinct, we cannot have both $g^{-1}h \in \mathcal{S}$ and $h^{-1}g \in \mathcal{S}$. Thus any simplex $A \subseteq \mathcal{G}$ of $\mathrm{gar}(\mathcal{G}, \mathcal{S}, o)$ admits a unique ordering $A = \{h_0, \ldots, h_k\}$ such that $i \leq j \Leftrightarrow h_i^{-1}h_j \in \mathcal{S}$. The only bar symbol in the preimage of A is $h_0[h_0^{-1}h_1|h_1^{-1}h_2|\cdots|h_{k-1}^{-1}h_k]$. $\square$

By comparing Definitions B.3 and B.11, we see that Garside structures allow a twofold gain:

- By contrast with $\mathrm{bar}(\mathcal{G}, o)$, the complex $\mathrm{gar}(\mathcal{G}, \mathcal{S}, o)$ is finite-dimensional: Garside structures on groupoids bound their cohomological dimension.
- We can replace the abstract simplicial set by a very concrete simplicial complex (that is actually a flag complex): in this paper, proving the $K(\pi, 1)$ property involves interpreting this simplicial complex as the nerve of an open covering.

THEOREM B.14 ("gar" resolution, universal cover version). *Let $\mathcal{G}$ be a groupoid, and let $\mathcal{S}$ be a Garside structure on $\mathcal{G}$, and let o be an object of $\mathcal{G}$. The geometric realization of $\mathrm{gar}(\mathcal{G}, \mathcal{S}, o)$ is contractible.*

We have slightly departed from results and phrasings that can be found in the literature, but Theorems B.10 and B.14 are easy categorical variants of the main results in [23], which themselves are Garside group versions of results by Bestvina about Artin groups [10], [13]; these variants can be proved using the same exact strategy.

B.4. *Garside sets.*

Definition B.15. Let $(\mathcal{C}, \Delta, \phi)$ be a Garside structure with set of simple elements $\mathcal{S}$. The associated *Garside set* is the collection

$$\mathrm{D}_\bullet(\Delta) := (\mathrm{D}_k(\Delta))_{k \in \mathbb{Z}_{\geq 0}},$$

where

$$\mathrm{D}_k(\Delta) := \{(f_1, \dots, f_k) \in \mathcal{S}^k | f_1 \cdots f_k = \Delta\},$$

together with, for all k, the following structure:

- the k face maps $d_1, \dots, d_k : \mathrm{D}_k(\Delta) \to \mathrm{D}_{k-1}(\Delta)$, such that for all $\sigma = (f_1, \dots, f_k) \in \mathrm{D}_k(\Delta)$, we have $d_1(\sigma) = (f_1 f_2, f_3, \dots, f_k)$, $\dots$, $d_{k-1}(\sigma) = (f_1, \dots, f_{k-2}, f_{k-1} f_k)$ and $d_k(\sigma) = (f_2, \dots, f_k f_1^{\phi})$;
- the k degeneracy maps $s_1, \dots, s_k : \mathrm{D}_{k-1}(\Delta) \to \mathrm{D}_k(\Delta)$ obtained by inserting identities at the k-possible locations in $(f_1, \dots, f_{k-1})$;
- the "screwdriver" map $\rho : \mathrm{D}_k(\Delta) \to \mathrm{D}_k(\Delta), (f_1, \dots, f_k) \mapsto (f_2, \dots, f_k, f_1^{\phi})$.

When there is no ambiguity on Δ, we write $\mathrm{D}_\bullet$ instead of $\mathrm{D}_\bullet(\Delta)$.

Note the analogy with $\mathrm{D}_\bullet(c)$ (Definition 7.10) that , by Theorem 7.12, happens to be isomorphic to the Garside set of the dual braid monoid.

Together, the face and degeneracy maps form a (degree-shifted) simplicial structure, which is isomorphism to that on $\mathcal{NS}$ via the following trivial lemma.

LEMMA B.16 (Kreweras map). *For all $k \geq 1$, the map*

$$\begin{aligned} (\mathcal{NS})_{k-1} &\longrightarrow \mathrm{D}_k(\Delta) \\ (f_1, \dots, f_{k-1}) &\longmapsto (f_1, \dots, f_{k-1}, (f_1 \cdots f_{k-1})^{-1}\Delta) \end{aligned}$$

is bijective.

Remark B.17 (Helicoidal structure). The are obvious compatibility axioms between the screwdriver map ρ and the simplicial structure. If ϕ were to act trivially on simple elements, these axioms would be that of a *cyclic set*, in the sense of [24]. Bökstedt-Hsiang-Madsen provide variant axioms for the case when ρ has finite order ($|\rho|$-*cyclic sets* [12]). As it is interesting to consider Garside structures where ρ has infinite order (see for example [4]), imposing conditions on the order of ρ seems a bit artificial. Relaxing the finite order condition yields a natural notion of *helicoidal set*, which I have not found in the literature. (Maybe I did not search well enough.) The geometric realization of a helicoidal set is equipped with a natural $\mathbf{R}$-action, which for cyclic sets and d-cyclic sets, factors through a natural S^1-action. This S^1-action and its compatibility with the scalar action on V is the true explanation for the miracles of Section 11.

Remark B.18 (Garside structure from Garside set). Note that the whole Garside structure can be recovered from $\mathrm{D}_\bullet$. Consider the following category presentation:

(1) The set D_1 is, literally, the Garside family Δ. It contains one element per object in $\mathcal{C}$, and serves as an abstract object set.
(2) The set D_2 is in bijection with $\mathcal{S}$. We take D_2 as a formal set of generators.

(3) The elements in D_3 are the defining relations of the presentation: the triple (f_1, f_2, f_3) expresses the relation $(f_1, f_2 f_3)(f_2, f_3 f_1^{\phi}) = (f_1 f_2, f_3)$. (Indeed, via the Kreweras bijection between D_2 and $\mathcal{S}$, this is the relation $f_1 \cdot f_2 = f_1 f_2$.)

This presentation defines an abstract category $\mathcal{C}(\mathrm{D}_\bullet)$ that is isomorphic to $\mathcal{C}$. In other words, the category $\mathcal{C}$, together with Δ and $\mathcal{S}$, can be functorially retrieved from the helicoidal set $\mathrm{D}_\bullet$. Note that higher degree elements in $\mathrm{D}_\bullet$ express further syzygies but, because of Theorem B.10, they only contain homotopically trivial stuff.

Exercise B.19. Write down the axioms for a helicoidal set.

Question B.20. Is there a pleasant way to phrase axioms for *abstract Garside sets* (helicoidal sets H such that $\mathcal{C}(H)$ is a Garside category)?

B.5. *Divided Garside structures.* The main construction in [5] is a kind of "barycentric subdivision" functor for Garside categories. At the level of cyclic sets, it coincides with an earlier construction by Bökstedt-Hsiang-Madsen, [12]. Thanks to our index-shifting Lemma B.16, we can describe this construction in remarkably simple terms. (Note how the Kreweras map allows much simpler notation compared to those of [12] and [5].)

Definition B.21 (divided Garside set). Let $(\mathcal{C}, \Delta, \phi)$ be a Garside structure. Let m be a positive integer. The *m-divided Garside set* is the graded set

$$\sqrt[m]{\mathrm{D}}_\bullet(\Delta) = (\sqrt[m]{\mathrm{D}}_k(\Delta))_{k \in \mathbb{Z}_{\geq 0}},$$

where

$$\sqrt[m]{\mathrm{D}}_k(\Delta) := \mathrm{D}_{mk}(\Delta),$$

equipped with

- faces $d'_1, \ldots, d'_k : \sqrt[m]{\mathrm{D}}_k(\Delta) \to \sqrt[m]{\mathrm{D}}_{k-1}(\Delta)$ defined by
$$d'_i = d_i d_{i+k} \ldots d_{i+(d'-1)k}$$
(composed from right to left);
- degeneracy maps $s'_1, \ldots, s'_k : \sqrt[m]{\mathrm{D}}_{k-1}(\Delta) \to \sqrt[m]{\mathrm{D}}_k(\Delta)$ defined by
$$s'_i = s_i s_{s+k} \cdots s_{i+(d'-1)k}$$
(composed from right to left);
- the screwdriver map is the restriction of that of $\mathrm{D}_\bullet$.

Theorem B.22 (after Section 9 in [5]). *Let $(\mathcal{C}, \Delta, \phi)$ be a Garside structure with a set of simple elements $\mathcal{S}$. Let m be a positive integer. Then $\sqrt[m]{\mathrm{D}}_\bullet(\Delta)$ is the Garside set of Garside structure $(\mathcal{C}_m, \Delta_m, \phi_m)$ such that $\mathcal{G}_m$, the groupoid of fractions of $\mathcal{C}_m$, is equivalent as a category to $\mathcal{G}$, the groupoid of fractions of $\mathcal{C}$.*

Following Remark B.18, the Garside structure $(\mathcal{C}_m, \Delta_m, \phi_m)$ is uniquely determined by its Garside set $\sqrt[m]{\mathrm{D}}_\bullet(\Delta)$, and we can write presentations by generators and relations for $\mathcal{C}_m$ and $\mathcal{G}_m$ as follows.

The underlying object is $\sqrt[m]{\mathrm{D}}_1(\Delta) = \mathrm{D}_m(\Delta)$, and the set of simple elements $\mathcal{S}_m$ is in bijection with $\sqrt[m]{\mathrm{D}}_2(\Delta) = \mathrm{D}_{2m}(\Delta)$. It is better to understand everything in terms of commutative diagrams. An object $(f_1, \dots, f_m)$ is viewed as a commutative diagram

$$\begin{tikzcd} \bullet \arrow[r, "f_1"] \arrow[rrrr, bend right, "\Delta"'] & \bullet \arrow[r, "f_2"] & \bullet \arrow[r, dotted] & \bullet \arrow[r, "f_m"] & \bullet, \end{tikzcd}$$

where all arrows are in $\mathcal{S}$, whereas a simple morphism $(f_1, \dots, f_{2m})$ in $\mathcal{S}_m \subseteq \mathcal{C}_m$ is viewed as a commutative diagram

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "f_1"'] \arrow[rrrr, bend left, "\Delta"] & \bullet \arrow[r] \arrow[d, "f_3"'] & \bullet \arrow[r, dotted, no head] \arrow[d, "f_5"'] & \bullet \arrow[r] \arrow[d, "f_{2m-1}"'] & \bullet \arrow[d, "f_1^\phi"] \\ \bullet \arrow[r] \arrow[ur, "f_2"] \arrow[rrrr, bend right, "\Delta"'] & \bullet \arrow[r] \arrow[ur, "f_4"] & \bullet \arrow[r, dotted, no head] & \bullet \arrow[r] \arrow[ur, "f_{2m}"] & \bullet, \end{tikzcd}$$

where all arrows are in S; its source is the object $(f_1f_2, f_3f_4, \dots, f_{2m-1}f_{2m})$, and its target is the object $(f_2f_3, f_4f_5, \dots, f_{2m}f_1^\phi)$:

$$\begin{tikzcd} \bullet \arrow[r, "f_1f_2"] \arrow[rrr, bend right, "\Delta"'] & \bullet \arrow[r, dotted] & \bullet \arrow[r, "f_{2m-1}f_{2m}"] & \bullet \end{tikzcd} \xrightarrow{(f_1,f_2,\dots,f_{2m})} \begin{tikzcd} \bullet \arrow[r, "f_2f_3"] \arrow[rrr, bend right, "\Delta"'] & \bullet \arrow[r, dotted] & \bullet \arrow[r, "f_{2m}f_1^\phi"] & \bullet. \end{tikzcd}$$

These morphisms are subject to defining relations indexed by $\sqrt[m]{\mathrm{D}}_3(\Delta) = \mathrm{D}_{3m}(\Delta)$. The relation $(f_1, \dots, f_{3m})$ can be visualized as

$$\begin{tikzcd} \bullet \arrow[d, "f_1"'] \arrow[rrrr, bend left, "\Delta"] & \bullet \arrow[d, "f_4"] & \bullet \arrow[r, dotted, no head] \arrow[d, "f_7"] & \bullet \arrow[d] & \bullet \arrow[d, "f_1^\phi"] \\ \bullet \arrow[d, "f_2"'] & \bullet \arrow[d, "f_8"'] & \bullet \arrow[r, dotted, no head] \arrow[d, "f_5"'] & \bullet \arrow[d] & \bullet \arrow[d, "f_2^\phi"] \\ \bullet \arrow[uur, "f_3"] & \bullet \arrow[uur, "f_6"] & \bullet \arrow[r, dotted, no head] & \bullet \arrow[uur, "f_{3m}"] & \bullet \end{tikzcd}$$

and expresses the defining relation

$$(f_1, f_2f_3, f_4, f_5f_6, \dots) \cdot (f_2, f_3f_4, f_5, f_6f_7, \dots) = (f_1f_2, f_3, f_4f_5, f_6, \dots).$$

The automorphism ϕ_m acts on objects by

$$(f_1, f_2, \dots, f_m) \mapsto (f_2, \dots, f_m, f_1^\phi)$$

and on simple morphisms by

$$(f_1, f_2, f_3, f_4, \dots, f_{2m-1}, f_{2m}) \mapsto (f_3, f_4, \dots, f_{2m-1}, f_{2m}, f_1^\phi, f_2^\phi).$$

The Garside element Δ_m with source $(f_1, f_2, \dots, f_{m-1}, f_m)$ has target $(f_2, f_3 \dots, f_m, f_1^{\phi})$ and corresponds to the commutative diagram

$$\begin{tikzcd}
\bullet \arrow[r, "f_1"] \arrow[d, "f_1"'] \arrow[dr, phantom, "1" near start] & \bullet \arrow[r, "f_2"] \arrow[d, "f_2"'] \arrow[dr, phantom, "1" near start] & \bullet \arrow[r, dotted, no head] \arrow[d, "f_3"'] & \bullet \arrow[r, "f_m"] \arrow[d, "f_m"'] \arrow[dr, phantom, "1" near start] & \bullet \arrow[d, "f_1^{\phi}"] \\
\bullet \arrow[r, "f_2"'] \arrow[ur] & \bullet \arrow[r, "f_3"'] \arrow[ur] & \bullet \arrow[r, dotted, no head] & \bullet \arrow[r, "f_1^{\phi}"'] \arrow[ur] & \bullet .
\end{tikzcd}$$

Definition B.23. The *collapse map* is

$$\begin{aligned} \kappa_m : \sqrt[m]{\mathrm{D}_2(\Delta)} &\longrightarrow \mathrm{D}_2(\Delta) \\ (f_1, f_2, \dots, f_{2m}) &\longmapsto (f_1, f_2 \cdots f_{2m}). \end{aligned}$$

By inspecting the defining relations of $\mathcal{C}$ (Remark B.18) and $\mathcal{C}_m$ (just above), one sees that κ_m extends to a collapse functor $\mathcal{C}_m \to \mathcal{C}$. In [5, §9]a less trivial functor $\Theta_m : \mathcal{C} \to \mathcal{C}_m$ is defined .

THEOREM B.24. *The functors $\kappa_m : \mathcal{C}_m \to \mathcal{C}$ and $\Theta_m : \mathcal{C} \to \mathcal{C}_m$ are such that $\kappa_m \circ \Theta_m = 1_{\mathcal{C}}$. They induce equivalences of categories $\kappa_m : \mathcal{G}_m \to \mathcal{G}$ and $\Theta_m : \mathcal{G} \to \mathcal{G}_m$.*

Proof. That $\kappa_m \circ \Theta_m = 1_{\mathcal{C}}$ is obvious by construction. That $\Theta_m : \mathcal{G} \to \mathcal{G}_m$ is an equivalence of categories is [5, Th. 9.5]. □

Acknowledgements (2007). Work on this project started during stimulating visits to KIAS (Seoul) and RIMS (Kyoto), in the summer of 2003, when I realized that the dual braid monoid construction could be generalized to well-generated groups. I thank Sang Jin Lee and Kyoji Saito for their hospitality and for their interests in discussing these topics. After this initial progress, I remained stuck for many months, trying to construct the open covering using convex geometry in V^{reg} (refining [3, §4]). Two observations were crucial to figuring out that working in the quotient space was more appropriate. First, Kyoji Saito pointed out that the starting point for the construction of the *flat structure* (or *Frobenius manifold structure*; see [52]) on real reflection orbifolds was precisely the duality between degrees and codegrees. The intuition that the flat structure has something to do with the $K(\pi, 1)$ property is explicitly mentioned as a motivation for [52] (see also [51]). The second useful discussion was with Frédéric Chapoton, who pointed out the numerological coincidence, in the Coxeter case, between the degree of the Lyashko-Looijenga covering and the number of maximal chains in the lattice of noncrossing partitions.

I thank Pierre Deligne, Eduard Looijenga, Jean Michel, Vic Reiner and Vivien Ripoll for comments, critiques and suggestions.

Theorem 12.4 is an overdue answer to a question Michel Broué asked me in 1997, when I was a graduate student under his supervision. Most of my

work on braid groups was motivated by this problem. I am glad to have been able to solve it just in time for his sixtieth birthday.

Postscript (2013). This article, based on results obtained between 2003 and 2006, was first circulated as a preprint in October 2006. Having spent three years focused on a single theorem, I liked the idea of writing the proof as a single article. That was naive, and I now understand that a series of smaller papers would have been a much wiser approach to publication.

More than five years had passed between the initial submission, in April 2007, and the beginning of the revision work. In the meantime, I had quit academic life and engaged in a new project that was both very demanding and impossible to put on hold. The revision work, from August 2012 to September 2013, took place during weekends and a few dedicated day offs. A consequence is the inevitable Harlequin pattern of styles, not just because the proof borrows from different areas of mathematics, but because it was written over so many years.

The introduction, as well as Sections 1–6, 8–10 and 12–13, are from 2007 (with due corrections, naturally). Section 11 and Appendix B are brand new from 2013. None of the referees could get through the old Section 11, and and my advice is to burn any remaining copies; the new Section 11 follows the same argument, but in a cleaned-up, clarified and hopefully intelligible manner. It actually handles the well-generated case too ($d = 1$), so Sections 9–10 could be removed without logical harm. Appendix A is expanded from an old "Notation" section. As for Section 7, it retains its original content, complemented with a much clarifying factorization theorem (Theorem 7.20) and an even better one (Theorem 7.25). Suffice it to say, the old sections now look incredibly clumsy and immature.

I thank the four referees for their time and effort, especially Referee #3, for his enthusiastic comments, and even more so Referee #4: his careful and detailed 2012 report allowed the editorial process to resume, effectively saving the paper. I thank David Gabai for his editorial patience and tenacity.

Two letters commenting my initial draft, sent by Pierre Deligne in 2006, remained for many years the only sign that *at least one person seemed to believe my proof.* I thank him for his generosity. I also thank Jean Michel, who demonstrated an indefatigable curiosity for this paper, challenging me until he could understand it.

I would have renounced trying to get this work published if it was not for the comforting support of my mathematical friends, especially Emmanuel Breuillard, Michel Broué and Raphaël Rouquier.

Index

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