Annals of Mathematics **181** (2015), 769–807
http://dx.doi.org/10.4007/annals.2015.181.2.7

# Solution of Leray's problem for stationary Navier-Stokes equations in plane and axially symmetric spatial domains

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**Abstract**

We study the nonhomogeneous boundary value problem for the Navier-Stokes equations of steady motion of a viscous incompressible fluid in arbitrary bounded multiply connected plane or axially-symmetric spatial domains. (For axially symmetric domains, data is assumed to be axially symmetric as well.) We prove that this problem has a solution under the sole necessary condition of zero total flux through the boundary. The problem was formulated by Jean Leray 80 years ago. The proof of the main result uses Bernoulli's law for a weak solution to the Euler equations.

## 1. **Introduction**

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, $n=2,3$, with $C^2$-smooth boundary $\partial\Omega=\cup_{j=0}^N\Gamma_j$ consisting of $N+1$ disjoint components $\Gamma_j$, $j=0,\dots,N$. Consider the stationary Navier-Stokes system with nonhomogeneous boundary conditions

$$\left\{\begin{aligned} -\nu\Delta\mathbf{u}+\big(\mathbf{u}\cdot\nabla\big)\mathbf{u}+\nabla p=\mathbf{f} &\quad \text{in } \Omega,\\ \operatorname{div}\,\mathbf{u}=0 &\quad \text{in } \Omega,\\ \mathbf{u}=\mathbf{a} &\quad \text{on } \partial\Omega. \end{aligned}\right. \tag{1.1}$$

The continuity equation ($1.1_2$) implies the compatibility condition

$$\int\limits_{\partial\Omega}\mathbf{a}\cdot\mathbf{n}\,ds=\sum_{j=0}^N\int\limits_{\Gamma_j}\mathbf{a}\cdot\mathbf{n}\,ds=\sum_{j=0}^N\mathcal{F}_j=0 \tag{1.2}$$

necessary for the solvability of problem (1.1), where $\mathbf{n}$ is a unit outward (with respect to $\Omega$) normal vector to $\partial\Omega$ and $\mathcal{F}_j=\int_{\Gamma_j}\mathbf{a}\cdot\mathbf{n}\,dS$. Condition (1.2) means that the total flux of the fluid through $\partial\Omega$ is zero.

In his famous paper of 1933 [22], Jean Leray proved that problem (1.1) has a solution provided[1]

$$\mathcal{F}_j = \int\limits_{\Gamma_j} \mathbf{a} \cdot \mathbf{n} \, dS = 0, \qquad j = 0, 1, \ldots, N. \tag{1.3}$$

The case when the boundary value $\mathbf{a}$ satisfies only the necessary condition (1.2) was left open by Leray, and the problem whether (1.1), (1.2) admit (or do not admit) a solution is known in the scientific community as *Leray's problem.*

Leray's problem has been studied in many papers. However, in spite of all efforts, the existence of a weak solution $\mathbf{u} \in W^{1,2}(\Omega)$ to problem (1.1) was established only under assumption (1.3) (see, e.g., [22], [20], [21], [33], [13]), or for sufficiently small fluxes $\mathcal{F}_j$ [2] (see, e.g., [8], [9], [11], [12], [2], [30], [29], [18]), or under certain symmetry conditions on the domain $\Omega$ and the boundary value $\mathbf{a}$ and the external force $\mathbf{f}$ (see, e.g., [1], [31], [10], [25], [28], [27]). Recently [17], the existence theorem for (1.1) was proved for a plane domain $\Omega$ with two connected components of the boundary assuming only that the flux through the external component is negative (the inflow condition). A similar result was also obtained for the spatial axially symmetric case [15]. In particular, the existence was established without any restrictions on the fluxes $\mathcal{F}_j$, under the assumption that all components $\Gamma_j$ of $\partial\Omega$ intersect the axis of symmetry. For more detailed historical surveys, one can see the recent papers [17] or [27], [28].

In the present paper we solve Leray's problem for the plane case $n = 2$ and for the axially symmetric domains in $\mathbb{R}^3$. (For axially symmetric spatial domains the boundary value $\mathbf{a}$ and the external force $\mathbf{f}$ are assumed to be axially symmetric as well.) The main result for the plane case is as follows.

THEOREM 1.1. *Assume that $\Omega \subset \mathbb{R}^2$ is a bounded domain with $C^2$-smooth boundary $\partial\Omega$. If $\mathbf{f} \in W^{1,2}(\Omega)$ and $\mathbf{a} \in W^{3/2,2}(\partial\Omega)$ satisfy condition* (1.2), *then problem* (1.1) *admits at least one weak solution* $\mathbf{u}$.

*Remark* 1.1. It is well known (see [21]) that under the hypotheses of Theorem 1.1, every weak solution $\mathbf{u}$ of problem (1.1) is more regular; i.e., $\mathbf{u} \in W^{2,2}(\Omega) \cap W^{3,2}_{\mathrm{loc}}(\Omega)$. Generally speaking, the solution is as regular as the data allow, in particular, $\mathbf{u}$ is $C^\infty$-smooth when $\mathbf{f}$, $\mathbf{a}$, and $\partial\Omega$ are $C^\infty$-smooth.

The proof of the existence theorem is based on an *a priori* estimate that we derive using a *reductio ad absurdum* argument of Leray [22]. The essentially new part in this argument is the use of Bernoulli's law obtained in [14]

[1] Condition (1.3) does not allow the presence of sinks and sources.

[2] This condition does not assumes the norm of the boundary value $\mathbf{a}$ to be small.

for Sobolev solutions to the Euler equations. (The detailed proofs are presented in [17].) The results concerning Bernoulli's law are based on the recent version of the Morse-Sard theorem proved by J. Bourgain, M. Korobkov and J. Kristensen [3]. This theorem implies, in particular, that almost all level sets of a function $\psi \in W^{2,1}(\Omega)$ are finite unions of $C^1$-curves. This allows us to construct suitable subdomains (bounded by smooth stream lines) and to estimate the $L^2$-norm of the gradient of the total head pressure. We use some ideas here that are close (on a heuristic level) to the Hopf maximum principle for the solutions of elliptic PDEs. (For a more detailed explanation, see Section 3.3.1.) Finally, a contradiction is obtained using the Coarea formula.

The paper is organized as follows. Section 2 contains preliminaries. Basically, this section consists of standard facts, except for the results of Section 2.2, where we formulate the recent version [3] of the Morse-Sard Theorem for the space $W^{2,1}(\mathbb{R}^2)$, which plays a key role. In Section 3.1 we briefly recall the elegant *reductio ad absurdum* Leray's argument. In Section 3.2 we discuss properties of the limit solution to the Euler equations, which were known before. (Mainly, we recall some facts from [17].) In Section 3.3 we prove some new properties of this limit solution and get a contradiction. Finally, in Section 4 we adapt these methods to the axially symmetric spatial case.

## 2. Notation and auxiliary results

2.1. *Function spaces and definitions.* By *a domain* we mean a connected open set. In this paper we deal with bounded domains $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, with $C^2$-smooth boundary $\partial\Omega = \cup_{j=0}^N \Gamma_j$ consisting of $N+1$ disjoint components $\Gamma_j$; i.e.,

$$\Omega = \Omega_0 \setminus \Big(\bigcup_{j=1}^N \bar\Omega_j\Big), \quad \bar\Omega_j \subset \Omega_0,\ j = 1, \dots, N, \tag{2.1}$$

where $\Gamma_j = \partial\Omega_j$.

We use standard notation for function spaces: $C^k(\overline{\Omega})$, $C^k(\partial\Omega)$, $W^{k,q}(\Omega)$, $\mathring{W}^{k,q}(\Omega)$, $W^{\alpha,q}(\partial\Omega)$, where $\alpha \in (0,1), k \in \mathbb{N}_0, q \in [1, +\infty]$. In our notation we do not distinguish function spaces for scalar and vector-valued functions; it will be clear from the context whether we use scalar, vector, or tensor-valued function spaces. Denote by $H(\Omega)$ the subspace of all solenoidal vector fields ($\operatorname{div} \mathbf{u} = 0$) from $\mathring{W}^{1,2}(\Omega)$ equipped with the norm $\|\mathbf{u}\|_{H(\Omega)} = \|\nabla \mathbf{u}\|_{L^2(\Omega)}$. Observe that for functions $\mathbf{u} \in H(\Omega)$, the norm $\|\cdot\|_{H(\Omega)}$ is equivalent to $\|\cdot\|_{W^{1,2}(\Omega)}$.

Working with Sobolev functions, we always assume that the "best representatives" are chosen. For $w \in L^1_{\rm loc}(\Omega)$, the best representative $w^*$ is defined as

$$w^*(x) = \begin{cases} \lim\limits_{r\to 0} \fint_{B_r(x)} w(z)dz, & \text{if the finite limit exists,} \\ 0 & \text{otherwise,} \end{cases}$$

where $\fint_{B_r(x)} w(z)dz = \frac{1}{\mathrm{meas}(B_r(x))} \int_{B_r(x)} w(z)dz$ and $B_r(x) = \{y : |y-x| < r\}$ is the ball of radius $r$ centered at $x$.

Below we discuss some properties of the best representatives of Sobolev functions.

LEMMA 2.1 (see, e.g., [7, §4.8, Th. 1 and §4.9.2, Th. 2]). *If $w \in W^{1,s}(\mathbb{R}^2)$, $s \geq 1$, then there exists a set $A_{1,w} \subset \mathbb{R}^2$ with the following properties*:

(i) $\mathfrak{H}^1(A_{1,w}) = 0$;

(ii) *for each $x \in \Omega \setminus A_{1,w}$,*

$$\lim_{r\to 0} \fint_{B_r(x)} |w(z) - w(x)|^2 \, dz = 0;$$

(iii) *for every $\varepsilon > 0$, there exists a set $U \subset \mathbb{R}^2$ with $\mathfrak{H}^1_\infty(U) < \varepsilon$ and $A_{1,w} \subset U$ such that the function $w$ is continuous on $\overline{\Omega} \setminus U$;*

(iv) *for every unit vector $\mathbf{l} \in \partial B_1(0)$ and almost all straight lines $L$ parallel to $\mathbf{l}$, the restriction $w|_L$ is an absolutely continuous function (of one variable).*

Here and henceforth we denote by $\mathfrak{H}^1$ the one-dimensional Hausdorff measure, i.e., $\mathfrak{H}^1(F) = \lim\limits_{t\to 0+} \mathfrak{H}^1_t(F)$, where

$$\mathfrak{H}^1_t(F) = \inf\Big\{ \sum_{i=1}^{\infty} \mathrm{diam} F_i : \mathrm{diam} F_i \leq t, F \subset \bigcup_{i=1}^{\infty} F_i \Big\}.$$

*Remark* 2.1. Property (iii) of Lemma 2.1 means that $f$ is quasicontinuous with respect to the Hausdorff content $\mathfrak{H}^1_\infty$. Really, Theorem 1(iii) of Section 4.8 in [7] asserts that $f \in W^{1,s}(\mathbb{R}^2)$ is quasicontinuous with respect to the $s$-capacity. But it is well known that for $s = 1$, smallness of the 1-capacity of a set $F \subset \mathbb{R}^2$ is equivalent to smallness of $\mathfrak{H}^1_\infty(F)$ (see, e.g., [7, §5.6.3, Th. 3 and its proof]).

*Remark* 2.2. By the Sobolev extension theorem, Lemma 2.1 is true for functions $w \in W^{1,s}(\Omega)$, where $\Omega \subset \mathbb{R}^2$ is a bounded Lipschitz domain. By the trace theorem, each function $w \in W^{1,s}(\Omega)$ is "well defined" for $\mathfrak{H}^1$-almost all $x \in \partial\Omega$. Therefore, we assume that every function $w \in W^{1,s}(\Omega)$ is defined on $\overline{\Omega}$.

2.2. *On the Morse-Sard and Luzin N-properties of Sobolev functions in* $W^{2,1}$. First, let us recall some classical differentiability properties of Sobolev functions.

LEMMA 2.2 (see [6, Prop. 1]). *If $\psi \in W^{2,1}(\mathbb{R}^2)$, then $\psi$ is continuous and there exists a set $A_\psi$ with $\mathfrak{H}^1(A_\psi) = 0$ such that $\psi$ is differentiable (in the classical sense) at all $x \in \mathbb{R}^2 \setminus A_\psi$. Moreover, the classical derivative coincides with $\nabla\psi(x)$, where $\lim\limits_{r\to 0} \fint_{B_r(x)} |\nabla\psi(z) - \nabla\psi(x)|^2\, dz = 0$.*

The theorem below is due to J. Bourgain, M. Korobkov and J. Kristensen [3].

THEOREM 2.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary. If $\psi \in W^{2,1}(\Omega)$, then*

(i) $\mathfrak{H}^1(\{\psi(x) : x \in \overline{\Omega} \setminus A_\psi \ \&\ \nabla\psi(x) = 0\}) = 0$.

(ii) *For every $\varepsilon > 0$, there exists $\delta > 0$ such that $\mathfrak{H}^1(\psi(U)) < \varepsilon$ for any set $U \subset \overline{\Omega}$ with $\mathfrak{H}^1_\infty(U) < \delta$.*

(iii) *For every $\varepsilon > 0$, there exist an open set $V \subset \mathbb{R}$ with $\mathfrak{H}^1(V) < \varepsilon$ and a function $g \in C^1(\mathbb{R}^2)$ such that for each $x \in \overline{\Omega}$ if $\psi(x) \notin V$, then $x \notin A_\psi$ and $\psi(x) = g(x)$, $\nabla\psi(x) = \nabla g(x) \neq 0$.*

(iv) *For $\mathfrak{H}^1$–almost all $y \in \psi(\overline{\Omega}) \subset \mathbb{R}$, the preimage $\psi^{-1}(y)$ is a finite disjoint family of $C^1$-curves $S_j$, $j = 1, 2, \dots, N(y)$. Each $S_j$ is either a cycle in $\Omega$ (i.e., $S_j \subset \Omega$ is homeomorphic to the unit circle $\mathbb{S}^1$) or a simple arc with endpoints on $\partial\Omega$. (In this case $S_j$ is transversal to $\partial\Omega$.)*

2.3. *Some facts from topology.* We shall need some topological definitions and results. By *continuum* we mean a compact connected set. We understand connectedness in the sense of general topology. A subset of a topological space is called *an arc* if it is homeomorphic to the unit interval $[0, 1]$.

Let us shortly present some results from the classical paper of A. S. Kronrod [19] concerning level sets of continuous functions. Let $Q = [0, 1] \times [0, 1]$ be a square in $\mathbb{R}^2$, and let $f$ be a continuous function on $Q$. Denote by $E_t$ a level set of the function $f$, i.e., $E_t = \{x \in Q : f(x) = t\}$. A connected component $K$ of the level set $E_t$ containing a point $x_0$ is a maximal connected subset of $E_t$ containing $x_0$. By $T_f$ denote a family of all connected components of level sets of $f$. It was established in [19] that $T_f$ equipped by a natural topology[3] is a one-dimensional topological tree.[4] Endpoints of this tree[5] are the components $C \in T_f$ that do not separate $Q$; i.e., $Q \setminus C$ is a connected

[3] A system of neighborhoods in this topology is defined as follows. For a component $C \in T_f$ and an open set $U \supset C$, the set $\{B \in T_f : B \subset U\}$ is called a neighborhood of $C$. Accordingly, the convergence in $T_f$ is defined by the following rule: $T_f \ni C_i \to C$ iff $\sup\limits_{x\in C_i} \operatorname{dist}(x, C) \to 0$.

[4] A locally connected continuum $T$ is called *a topological tree* if it does not contain a curve homeomorphic to a circle or, equivalently, if any two different points of $T$ can be joined by a unique arc. This definition implies that $T$ has topological dimension 1.

[5] A point of a continuum $K$ is called an *endpoint of $K$* (resp., *a branching point of $K$*) if its topological index equals 1 (more or equal to 3 resp.). For a topological tree $T$, this definition is

set. Branching points of the tree are the components $C \in T_f$ such that $Q \setminus C$ has more than two connected components (see [19, Th. 5]). By results of [19, Lemma 1] (see also [24] and [26]), the set of all branching points of $T_f$ is at most countable. The main property of a tree is that any two points could be joined by a unique arc. Therefore, the same is true for $T_f$.

LEMMA 2.3 ([19, Lemma 13]). *If $f \in C(Q)$, then for any two different points $A \in T_f$ and $B \in T_f$, there exists a unique arc $J = J(A, B) \subset T_f$ joining $A$ to $B$. Moreover, for every inner point $C$ of this arc, the points $A, B$ lie in different connected components of the set $T_f \setminus \{C\}$.*

We can reformulate the above lemma in the following equivalent form.

LEMMA 2.4. *If $f \in C(Q)$, then for any two different points $A, B \in T_f$, there exists a continuous injective function $\varphi : [0,1] \to T_f$ with the properties*

(i) $\varphi(0) = A$, $\varphi(1) = B$;

(ii) *for any $t_0 \in [0,1]$,*

$$\lim_{[0,1]\ni t \to t_0} \sup_{x \in \varphi(t)} \operatorname{dist}(x, \varphi(t_0)) \to 0;$$

(iii) *for any $t \in (0,1)$, the sets $A, B$ lie in different connected components of the set $Q \setminus \varphi(t)$.*

*Remark* 2.3. If in Lemma 2.4 $f \in W^{2,1}(Q)$, then by Theorem 2.1(iv), there exists a dense subset $E$ of $(0,1)$ such that $\varphi(t)$ is a $C^1$-curve for every $t \in E$. Moreover, $\varphi(t)$ is either a cycle or a simple arc with endpoints on $\partial Q$.

*Remark* 2.4. All results of Lemmas 2.3 and 2.4 remain valid for level sets of continuous functions $f : \overline{\Omega} \to \mathbb{R}$, where $\Omega$ is a multi-connected bounded domain of type (2.1), provided $f \equiv \xi_j = \text{const}$ on each inner boundary component $\Gamma_j$ with $j = 1, \dots, N$. Indeed, we can extend $f$ to the whole $\overline{\Omega}_0$ by putting $f(x) = \xi_j$ for $x \in \overline{\Omega}_j$, $j = 1, \dots, N$. The extended function $f$ will be continuous on the set $\overline{\Omega}_0$ that is homeomorphic to the unit square $Q = [0,1]^2$.

## 3. The plane case

3.1. *Leray's argument reductio ad absurdum.* Consider the Navier-Stokes problem (1.1) in the $C^2$-smooth domain $\Omega \subset \mathbb{R}^2$ defined by (2.1) with $\mathbf{f} \in W^{1,2}(\Omega)$. Without loss of generality, we may assume that $\mathbf{f} = \nabla^{\perp} b$ with $b \in W^{2,2}(\Omega)$[6], where $(x, y)^{\perp} = (-y, x)$. If the boundary value $\mathbf{a} \in W^{3/2,2}(\partial\Omega)$ satisfies condition (1.2), then there exists a solenoidal extension $\mathbf{A} \in W^{2,2}(\Omega)$

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equivalent to the following: a point $C \in T$ is an endpoint of $T$ (resp., a branching point of $T$) if the set $T \setminus \{C\}$ is connected (resp., if $T \setminus \{C\}$ has more than two connected components).

[6] By the Helmholtz-Weyl decomposition, for a $C^2$-smooth bounded domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, every $\mathbf{f} \in W^{1,2}(\Omega)$ can be represented as the sum $\mathbf{f} = \operatorname{curl} \mathbf{b} + \nabla\varphi$ for $n = 3$, and

of $\mathbf{a}$ (see [21], [32], [12]). Using this fact and standard results [21], we can find a weak solution $\mathbf{U} \in W^{2,2}(\Omega)$ to the Stokes problem such that $\mathbf{U} - \mathbf{A} \in H(\Omega) \cap W^{2,2}(\Omega)$ and

$$\nu \int_\Omega \nabla \mathbf{U} \cdot \nabla \boldsymbol{\eta}\, dx = \int_\Omega \mathbf{f} \cdot \boldsymbol{\eta}\, dx \quad \forall\, \boldsymbol{\eta} \in H(\Omega). \tag{3.1}$$

Moreover,

$$\|\mathbf{U}\|_{W^{2,2}(\Omega)} \le c\Big(\|\mathbf{a}\|_{W^{3/2,2}(\partial\Omega)} + \|\mathbf{f}\|_{L^2(\Omega)}\Big). \tag{3.2}$$

By *weak solution* of problem (1.1) we understand a function $\mathbf{u}$ such that $\mathbf{w} = \mathbf{u} - \mathbf{U} \in H(\Omega)$ and

$$\begin{aligned} \nu \int_\Omega \nabla \mathbf{w} \cdot \nabla \boldsymbol{\eta}\, dx - \int_\Omega \Big((\mathbf{w}+\mathbf{U})\cdot\nabla\Big)\boldsymbol{\eta}\cdot\mathbf{w}\, dx - \int_\Omega \Big(\mathbf{w}\cdot\nabla\Big)\boldsymbol{\eta}\cdot\mathbf{U}\, dx \\ = \int_\Omega \Big(\mathbf{U}\cdot\nabla\Big)\boldsymbol{\eta}\cdot\mathbf{U}\, dx \qquad \forall \boldsymbol{\eta} \in H(\Omega). \end{aligned} \tag{3.3}$$

Let us briefly reproduce the contradiction argument of Leray [22] that was later used in many other papers. (See, e.g., [20], [21], [13], [1]; see also [17] for details.) It is well known (see, e.g., [21]) that integral identity (3.3) is equivalent to an operator equation in the space $H(\Omega)$ with a compact operator. Therefore, by the Leray-Schauder theorem, to prove the existence of a weak solution to the Navier-Stokes problem (1.1), it is sufficient to show that all the solutions of the integral identity

$$\begin{aligned} \nu \int_\Omega \nabla \mathbf{w} \cdot \nabla \boldsymbol{\eta}\, dx - \lambda\int_\Omega \Big((\mathbf{w}+\mathbf{U})\cdot\nabla\Big)\boldsymbol{\eta}\cdot\mathbf{w}\, dx - \lambda\int_\Omega \Big(\mathbf{w}\cdot\nabla\Big)\boldsymbol{\eta}\cdot\mathbf{U}\, dx \\ = \lambda\int_\Omega \Big(\mathbf{U}\cdot\nabla\Big)\boldsymbol{\eta}\cdot\mathbf{U}\, dx \qquad \forall\, \boldsymbol{\eta} \in H(\Omega) \end{aligned} \tag{3.4}$$

are uniformly bounded in $H(\Omega)$ (with respect to $\lambda \in [0,1]$). Assume that this is false. Then there exist sequences $\{\lambda_k\}_{k\in\mathbb{N}} \subset [0,1]$ and $\{\widehat{\mathbf{w}}_k\}_{k\in\mathbb{N}} \in H(\Omega)$ such that

$$\begin{aligned} \nu \int_\Omega \nabla \widehat{\mathbf{w}}_k \cdot \nabla \boldsymbol{\eta}\, dx - \lambda_k\int_\Omega \Big((\widehat{\mathbf{w}}_k+\mathbf{U})\cdot\nabla\Big)\boldsymbol{\eta}\cdot\widehat{\mathbf{w}}_k\, dx - \lambda_k\int_\Omega \Big(\widehat{\mathbf{w}}_k\cdot\nabla\Big)\boldsymbol{\eta}\cdot\mathbf{U}\, dx \\ = \lambda_k\int_\Omega \Big(\mathbf{U}\cdot\nabla\Big)\boldsymbol{\eta}\cdot\mathbf{U}\, dx \qquad \forall\, \boldsymbol{\eta} \in H(\Omega) \end{aligned} \tag{3.5}$$

and

$$\lim_{k\to\infty} \lambda_k = \lambda_0 \in [0,1], \qquad \lim_{k\to\infty} J_k = \lim_{k\to\infty} \|\widehat{\mathbf{w}}_k\|_{H(\Omega)} = \infty. \tag{3.6}$$

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$\mathbf{f} = \nabla^\perp b + \nabla\varphi$ for $n = 2$, with $\mathbf{b}, b, \varphi \in W^{2,2}(\Omega)$, and the gradient part is included then into the pressure term (see, e.g., [21]).

Using well-known techniques ([17], [1]), one shows that there exist $\widehat{p}_k$ with[7] $\|\widehat{p}_k\|_{W^{1,q}(\Omega)} \le C(q)J_k^2$, $q \in [1,2)$, such that the pair $\big(\widehat{\mathbf{u}}_k = \widehat{\mathbf{w}}_k + \mathbf{U},\ \widehat{p}_k\big)$ is a solution to the following system:

$$\tag{3.7} \begin{cases} -\nu\Delta\widehat{\mathbf{u}}_k + \lambda_k\big(\widehat{\mathbf{u}}_k\cdot\nabla\big)\widehat{\mathbf{u}}_k + \nabla\widehat{p}_k = \mathbf{f} & \text{in } \Omega,\\ \operatorname{div}\widehat{\mathbf{u}}_k = 0 & \text{in } \ \Omega,\\ \widehat{\mathbf{u}}_k = \mathbf{a} & \text{on } \partial\Omega. \end{cases}$$

Choose $\boldsymbol{\eta} = J_k^{-2}\widehat{\mathbf{w}}_k$ in (3.5) and set $\mathbf{w}_k = J_k^{-1}\widehat{\mathbf{w}}_k$. Taking into account that

$$\int\limits_\Omega \big((\mathbf{w}_k + \mathbf{U})\cdot\nabla\big)\mathbf{w}_k\cdot\mathbf{w}_k\,dx = 0,$$

we have

$$\tag{3.8} \nu\int\limits_\Omega |\nabla\mathbf{w}_k|^2\,dx = \lambda_k\int\limits_\Omega \big(\mathbf{w}_k\cdot\nabla\big)\mathbf{w}_k\cdot\mathbf{U}\,dx + J_k^{-1}\lambda_k\int\limits_\Omega \big(\mathbf{U}\cdot\nabla\big)\mathbf{w}_k\cdot\mathbf{U}\,dx.$$

Since $\|\mathbf{w}_k\|_{H(\Omega)} = 1$, extracting a subsequence (if necessary), we can assume without loss of generality that $\mathbf{w}_k$ converges weakly in $H(\Omega)$ to a vector field $\mathbf{v} \in H(\Omega)$. By the compact embedding

$$H(\Omega) \hookrightarrow L^r(\Omega) \quad \forall\, r \in [1,\infty),$$

the subsequence $\{\mathbf{w}_k\}$ converges strongly in $L^r(\Omega)$. Therefore, letting $k \to \infty$ in equality (3.8), we obtain

$$\tag{3.9} \nu = \lambda_0\int\limits_\Omega \big(\mathbf{v}\cdot\nabla\big)\mathbf{v}\cdot\mathbf{U}\,dx.$$

In particular, $\lambda_0 > 0$, so $\lambda_k$ are separated from zero.

Put $\nu_k = (\lambda_k J_k)^{-1}\nu$. Multiplying identities (3.7) by $\frac{1}{\lambda_k J_k^2} = \frac{\lambda_k\nu_k^2}{\nu^2}$, we see that the pair $\big(\mathbf{u}_k = \frac{1}{J_k}\widehat{\mathbf{u}}_k,\ p_k = \frac{1}{\lambda_k J_k^2}\widehat{p}_k\big)$ satisfies the following system:

$$\tag{3.10} \begin{cases} -\nu_k\Delta\mathbf{u}_k + \big(\mathbf{u}_k\cdot\nabla\big)\mathbf{u}_k + \nabla p_k = \mathbf{f}_k & \text{in } \Omega,\\ \operatorname{div}\mathbf{u}_k = 0 & \text{in } \Omega,\\ \mathbf{u}_k = \mathbf{a}_k & \text{on } \partial\Omega, \end{cases}$$

[7]The uniform estimates for the norms $\|p_k\|_{W^{1,q}(\Omega)}$ follow from well-known results concerning regularity of solutions to the Stokes problem (see [32, Chap. 1, §2.5] or [21]). Observe that in [17] we could have only $p_k \in W^{1,q}_{\mathrm{loc}}(\Omega)$ because $\partial\Omega$ has been assumed to be only Lipschitz. However, for domains $\Omega$ with $C^2$-smooth boundary and $\mathbf{a} \in W^{3/2,2}(\partial\Omega)$, the corresponding estimates hold globally.

where $\mathbf{f}_k = \frac{\lambda_k \nu_k^2}{\nu^2}\mathbf{f}$, $\mathbf{a}_k = \frac{\lambda_k \nu_k}{\nu}\mathbf{a}$, the norms $\|\mathbf{u}_k\|_{W^{1,2}(\Omega)}$ and $\|p_k\|_{W^{1,q}(\Omega)}$ are uniformly bounded for each $q \in [1,2)$, $\mathbf{u}_k \in W^{3,2}_{\rm loc}(\Omega)$, $p_k \in W^{2,2}_{\rm loc}(\Omega)$[8], and $\mathbf{u}_k \rightharpoonup \mathbf{v}$ in $W^{1,2}(\Omega)$. Extracting a subsequence (if necessary), we may assume without loss of generality that, in addition, $p_k \rightharpoonup p$ in $W^{1,q}(\Omega)$ for each $q \in [1,2)$. Then the limit functions $(\mathbf{v}, p)$ satisfy the Euler system

$$\begin{cases} \big(\mathbf{v}\cdot\nabla\big)\mathbf{v} + \nabla p = 0 & \text{in } \Omega, \\ \operatorname{div}\mathbf{v} = 0 & \text{in } \Omega, \\ \mathbf{v} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.11}$$

In conclusion, we can state the following lemma.

LEMMA 3.1. *Assume that $\Omega \subset \mathbb{R}^2$ is a bounded domain of type (2.1) with $C^2$-smooth boundary $\partial\Omega$, $\mathbf{f} = \nabla^\perp b$, $b \in W^{2,2}(\Omega)$, and $\mathbf{a} \in W^{3/2,2}(\partial\Omega)$ satisfies condition (1.2). If there are no weak solutions to (1.1), then there exist $\mathbf{v}, p$ with the following properties:*

(E) $\mathbf{v} \in W^{1,2}(\Omega)$, $p \in W^{1,q}(\Omega)$, $q \in (1,2)$, *and the pair* $\big(\mathbf{v}, p\big)$ *satisfies the Euler system* (3.11).

(E-NS) *Conditions* (E) *are satisfied, and there exist sequences of functions* $\mathbf{u}_k \in W^{1,2}(\Omega)$, $p_k \in W^{1,q}(\Omega)$ *and numbers* $\nu_k \to 0+$, $\lambda_k \to \lambda_0 > 0$ *such that the norms* $\|\mathbf{u}_k\|_{W^{1,2}(\Omega)}$, $\|p_k\|_{W^{1,q}(\Omega)}$ *are uniformly bounded for every* $q \in [1,2)$, *the pairs* $(\mathbf{u}_k, p_k)$ *satisfy* (3.10) *with* $\mathbf{f}_k = \frac{\lambda_k \nu_k^2}{\nu^2}\mathbf{f}$, $\mathbf{a}_k = \frac{\lambda_k \nu_k}{\nu}\mathbf{a}$, *and*

$$\|\nabla \mathbf{u}_k\|_{L^2(\Omega)} \to 1, \quad \mathbf{u}_k \rightharpoonup \mathbf{v} \textit{ in } W^{1,2}(\Omega), \quad p_k \rightharpoonup p \textit{ in } W^{1,q}(\Omega) \quad \forall\, q \in [1,2).$$

*Moreover,* $\mathbf{u}_k \in W^{3,2}_{\rm loc}(\Omega)$, $p_k \in W^{2,2}_{\rm loc}(\Omega)$.

From now on we assume that assumptions (E-NS) are satisfied. Our goal is to prove that they lead to a contradiction. This implies the validity of Theorem 1.1.

3.2. *Some previous results on the Euler equations.* In this subsection we collect the information on the limit solution $\big(\mathbf{v}, p\big)$ to (3.11) obtained in previous papers.

The next statement was proved in [13, Lemma 4] and in [1, Th. 2.2]; see also [17, Rem. 3.2].

[8]The interior regularity of the solution depends on the regularity of $\mathbf{f} \in W^{1,2}(\Omega)$, but not on the regularity of the boundary value $\mathbf{a}$; see [21].

THEOREM 3.1. *If conditions* (E) *are satisfied, then there exist constants* $\widehat{p}_0,\dots,\widehat{p}_N$ *such that*

$$p(x)\equiv\widehat{p}_j \quad \text{for } \mathfrak{H}^1\text{-almost all } x\in\Gamma_j. \tag{3.12}$$

*Remark* 3.1. From Theorem 3.1 and from the classical results of [5] it follows that

$$p\in C(\overline{\Omega})\cap W^{1,2}(\Omega)$$

if conditions (E) are satisfied. (For the accurate proof of this fact, see, e.g., [17, Th. 3.3].) Consequently, the identities (3.12) are valid for all $x\in\Gamma_j$ (instead of "$\mathfrak{H}^1$-almost all").

COROLLARY 3.1. *If conditions* (E-NS) *are satisfied, then*

$$-\frac{\nu}{\lambda_0}=\sum_{j=0}^N\widehat{p}_j\int\limits_{\Gamma_j}\mathbf{a}\cdot\mathbf{n}\,ds=\sum_{j=0}^N\widehat{p}_j\mathcal{F}_j. \tag{3.13}$$

*Proof.* By simple calculations from (3.9) and ($3.11_1$), it follows that

$$\frac{\nu}{\lambda_0}=-\int\limits_\Omega\nabla p\cdot\mathbf{U}\,dx=-\int\limits_\Omega\operatorname{div}(p\mathbf{U})\,dx=-\int\limits_{\partial\Omega}p\,\mathbf{a}\cdot\mathbf{n}\,ds.$$

In virtue of (3.12), this implies (3.13). □

Set $\Phi_k=p_k+\frac12|\mathbf{u}_k|^2$, $\Phi=p+\frac12|\mathbf{v}|^2$. From ($3.11_2$) and ($3.11_3$), it follows that there exists a stream function $\psi\in W^{2,2}(\Omega)$ such that

$$\nabla\psi\equiv\mathbf{v}^\perp \quad \text{in } \overline{\Omega}. \tag{3.14}$$

Here and henceforth we set $(a,b)^\perp=(-b,a)$.

Applying Lemmas 2.1, 2.2 and Remark 2.2 to the functions $\mathbf{v},\psi,\Phi$, we get the following

LEMMA 3.2. *If conditions* (E) *are satisfied, then the stream function* $\psi$ *is continuous on* $\overline{\Omega}$ *and there exists a set* $A_{\mathbf{v}}\subset\overline{\Omega}$ *such that*

(i) $\mathfrak{H}^1(A_{\mathbf{v}})=0$.

(ii) *For all* $x\in\Omega\setminus A_{\mathbf{v}}$,

$$\lim_{r\to0}\fint_{B_r(x)}|\mathbf{v}(z)-\mathbf{v}(x)|^2dz=\lim_{r\to0}\fint_{B_r(x)}|\Phi(z)-\Phi(x)|^2dz=0;$$

*moreover, the function* $\psi$ *is differentiable at* $x$ *and* $\nabla\psi(x)=(-v_2(x),v_1(x))$.

(iii) *For every* $\varepsilon>0$, *there exists a set* $U\subset\mathbb{R}^2$ *with* $\mathfrak{H}^1_\infty(U)<\varepsilon$ *such that* $A_{\mathbf{v}}\subset U$ *and the functions* $\mathbf{v},\Phi$ *are continuous in* $\overline{\Omega}\setminus U$.

The next version of Bernoulli's Law for solutions in Sobolev spaces was obtained in [14, Th. 1]. (See also [17, Th. 3.2] for a more detailed proof.)

THEOREM 3.2. *Let conditions* (E) *be satisfied, and let* $A_{\mathbf{v}} \subset \overline{\Omega}$ *be the set from Lemma 3.2. For any compact connected set* $K \subset \overline{\Omega}$, *the following property holds: if*

$$\psi\Big|_K = \text{const}, \tag{3.15}$$

*then*

$$\Phi(x_1) = \Phi(x_2) \quad \textit{for all } x_1, x_2 \in K \setminus A_{\mathbf{v}}. \tag{3.16}$$

LEMMA 3.3. *If conditions* (E) *are satisfied, then there exist constants* $\xi_0, \dots, \xi_N \in \mathbb{R}$ *such that* $\psi(x) \equiv \xi_j$ *on each component* $\Gamma_j$, $j = 0, \dots, N$.

*Proof.* The assertion follows easily from the fact that $\mathbf{v}$ extended by 0 outside $\Omega$ belongs to the space $H(\mathbb{R}^2) \subset W^{1,2}_{\rm loc}(\mathbb{R}^2)$, and hence the stream function $\psi \in W^{2,2}_{\rm loc}(\mathbb{R}^2)$ is well defined in $\mathbb{R}^2$ with $\nabla\psi = 0$ in $\mathbb{R}^2 \setminus \Omega$. □

For $x \in \overline{\Omega}$, denote by $K_x$ the connected component of the level set $\{z \in \overline{\Omega} : \psi(z) = \psi(x)\}$ containing the point $x$. By Lemma 3.3, $K_x \cap \partial\Omega = \emptyset$ for every $y \in \psi(\overline{\Omega}) \setminus \{\xi_0, \dots, \xi_N\}$ and for every $x \in \psi^{-1}(y)$. Thus, Theorem 2.1(ii) and (iv) imply that for almost all $y \in \psi(\overline{\Omega})$ and for every $x \in \psi^{-1}(y)$, the equality $K_x \cap A_{\mathbf{v}} = \emptyset$ holds and the component $K_x \subset \Omega$ is a $C^1$-curve homeomorphic to the circle. We call such $K_x$ *an admissible cycle.*

The next lemma was obtained in [17, Lemma 3.3].

LEMMA 3.4. *If conditions* (E-NS) *are satisfied, then there exists a subsequence* $\Phi_{k_l}$ *such that* $\Phi_{k_l}|_S$ *converges to* $\Phi|_S$ *uniformly* $\Phi_{k_l}|_S \rightrightarrows \Phi|_S$ *on almost all*[9] *admissible cycles* $S$.

Below we assume (without loss of generality) that the subsequence $\Phi_{k_l}$ coincides with $\Phi_k$. Admissible cycles $S$ from Lemma 3.4 will be called *regular cycles.*

3.3. *Obtaining a contradiction.* We consider the following two cases:

(a) The maximum of $\Phi$ is attained on the boundary $\partial\Omega$:

$$\max_{j=0,\dots,N} \widehat{p}_j = \operatorname*{ess\,sup}_{x\in\Omega} \Phi(x). \tag{3.17}$$

(b) The maximum of $\Phi$ is not attained[10] on $\partial\Omega$:

$$\max_{j=0,\dots,N} \widehat{p}_j < \operatorname*{ess\,sup}_{x\in\Omega} \Phi(x). \tag{3.18}$$

[9] "Almost all cycles" means cycles in preimages $\psi^{-1}(y)$ for almost all values $y \in \psi(\overline{\Omega})$.

[10] The case $\operatorname*{ess\,sup}_{x\in\Omega} \Phi(x) = +\infty$ is not excluded.

3.3.1. *The maximum of $\Phi$ is attained on the boundary $\partial\Omega$.* Let (3.17) hold. Adding a constant to the pressure we can assume, without loss of generality, that

$$\max_{j=0,\dots,N} \widehat{p}_j = \operatorname*{ess\,sup}_{x\in\Omega} \Phi(x) = 0. \tag{3.19}$$

In particular,

$$\Phi(x) \le 0 \quad \text{in } \Omega. \tag{3.20}$$

If $\widehat{p}_0 = \widehat{p}_1 = \cdots = \widehat{p}_N$, then by Corollary 3.1 and the flux condition (1.2), we immediately obtain the required contradiction. Thus, assume that

$$\min_{j=0,\dots,N} \widehat{p}_j < 0. \tag{3.21}$$

Change (if necessary) the numbering of the boundary components $\Gamma_0$, $\Gamma_1$, …, $\Gamma_N$ in such a way that

$$\widehat{p}_j < 0, \quad j = 0,\dots,M, \tag{3.22}$$

$$\widehat{p}_{M+1} = \cdots = \widehat{p}_N = 0. \tag{3.23}$$

First, we introduce the main idea of the proof in a heuristic way. It is well known that every $\Phi_k$ satisfies the linear elliptic equation

$$\Delta\Phi_k = \omega_k^2 + \frac{1}{\nu_k}\operatorname{div}(\Phi_k \mathbf{u}_k) - \frac{1}{\nu_k}\mathbf{f}_k \cdot \mathbf{u}_k. \tag{3.24}$$

If $\mathbf{f}_k = 0$, then by Hopf's maximum principle, in a subdomain $\Omega' \Subset \Omega$ with $C^2$-smooth boundary $\partial\Omega'$, the maximum of $\Phi_k$ is attained at the boundary $\partial\Omega'$, and if $x_* \in \partial\Omega'$ is a maximum point, then the normal derivative of $\Phi_k$ at $x_*$ is strictly positive. It is not sufficient to apply this property directly. Instead we will use some "integral analogs" that lead to a contradiction by using the Coarea formula (see Lemmas 3.8 and 3.9). For $i \in \mathbb{N}$ and sufficiently large $k \ge k(i)$, we construct a set $E_i \subset \Omega$ consisting of level lines of $\Phi_k$ such that $\Phi_k|_{E_i} \to 0$ as $i \to \infty$ and $E_i$ separates the boundary component $\Gamma_N$ (where $\Phi = 0$) from the boundary components $\Gamma_j$ with $j = 0,\dots,M$ (where $\Phi < 0$). On the one hand, the length of each of these level lines is bounded from below by a positive constant (since they separate the boundary components), and by the Coarea formula this implies the estimate from below for $\int_{E_i} |\nabla\Phi_k|$. On the other hand, elliptic equation (3.24) for $\Phi_k$, the convergence $\mathbf{f}_k \to 0$, and boundary conditions ($3.10_3$) allow us to estimate $\int_{E_i} |\nabla\Phi_k|^2$ from above (see Lemma 3.8), and this asymptotically contradicts the previous one.

The main idea of the proof for a general multiply connected domain is the same as in the case of annulus-like domains (when $\partial\Omega = \Gamma_0 \cup \Gamma_1$). The proof has an analytical nature, and unessential differences concern only well-known geometrical properties of level sets of continuous functions of two variables.

First of all, we need some information concerning the behavior of the limit total head pressure $\Phi$ on stream lines. We do not know whether the function $\Phi$ is continuous or not on $\Omega$. But we shall prove that $\Phi$ has some continuity properties on stream lines.

By Remark 2.4 and Lemma 3.3, we can apply Kronrod's results to the stream function $\psi$. Define the total head pressure on the Kronrod tree $T_\psi$ (see Section 2.3 ) as follows. Let $K \in T_\psi$ with $\operatorname{diam} K > 0$. Take any $x \in K \setminus A_{\mathbf{v}}$, and put $\Phi(K) = \Phi(x)$. This definition is correct by Bernoulli's Law (see Theorem 3.2).

Lemma 3.5. *Let* $A, B \in T_\psi$, $\operatorname{diam} A > 0, \operatorname{diam} B > 0$. *Consider the corresponding arc* $[A,B] \subset T_\psi$ *joining* $A$ *to* $B$ (*see Lemmas* 2.3 *and* 2.4). *Then the restriction* $\Phi|_{[A,B]}$ *is a continuous function.*

*Proof.* Put $(A,B) = [A,B] \setminus \{A,B\}$. Let $C_i \in (A,B)$ and $C_i \to C_0$ in $T_\psi$. By construction, each $C_i$ is a connected component of the level set of $\psi$ and the sets $A, B$ lie in different connected components of $\mathbb{R}^2 \setminus C_i$. Therefore,

$$\operatorname{diam}(C_i) \geq \min(\operatorname{diam}(A), \operatorname{diam}(B)) > 0.$$

By the definition of convergence in $T_\psi$, we have

$$\sup_{x \in C_i} \operatorname{dist}(x, C_0) \to 0 \quad \text{as } i \to \infty. \tag{3.25}$$

By Theorem 3.2, there exist constants $c_i \in \mathbb{R}$ such that $\Phi(x) \equiv c_i$ for all $x \in C_i \setminus A_{\mathbf{v}}$, where $\mathfrak{H}^1(A_{\mathbf{v}}) = 0$. Analogously, $\Phi(x) \equiv c_0$ for all $x \in C_0 \setminus A_{\mathbf{v}}$. If $c_i \nrightarrow c_0$, then we can assume, without loss of generality, that

$$c_i \to c_\infty \neq c_0 \quad \text{as } i \to \infty \tag{3.26}$$

and the components $C_i$ converge as $i \to \infty$ in the Hausdorff metric[11] to some set $C_0' \subset C_0$. Clearly, $\operatorname{diam}(C_0') > 0$. Take a straight line $L$ such that the projection of $C_0'$ on $L$ is not a singleton. Since $C_0'$ is a connected set, this projection is a segment. Let $I_0$ be the interior of this segment. For $z \in I_0$, denote by $L_z$ the straight line such that $z \in L_z$ and $L_z \perp L$. From Lemma 3.2(i) and (iii) it follows that $L_z \cap A_{\mathbf{v}} = \emptyset$ for $\mathfrak{H}^1$-almost all $z \in I_0$, and the restriction $\Phi|_{\overline{\Omega} \cap L_z}$ is continuous. Fix a point $z \in I_0$ with above properties. Then by construction, $C_i \cap L_z \neq \emptyset$ for sufficiently large $i$. Now, take a sequence

[11] The Hausdorff distance $d_H$ between two compact sets $A, B \subset \mathbb{R}^n$ is defined as follows: $d_H(A,B) = \max\Big(\sup_{a \in A} \operatorname{dist}(a,B), \sup_{b \in B} \operatorname{dist}(b,A)\Big)$ (see, e.g., [4, §7.3.1]). By Blaschke selection theorem [ibid], for any uniformly bounded sequence of compact sets $A_i \subset \mathbb{R}^n$, there exists a subsequence $A_{i_j}$ that converges to some compact set $A_0$ with respect to the Hausdorff distance. Of course, if all $A_i$ are compact connected sets and $\operatorname{diam} A_i \geq \delta$ for some $\delta > 0$, then the limit set $A_0$ is also connected and $\operatorname{diam} A_0 \geq \delta$.

$y_i \in C_i \cap L_z$, and extract a convergent subsequence $y_{i_j} \to y_0 \in C_0'$. Since $\Phi|_{\overline{\Omega}\cap L_z}$ is continuous, we have $\Phi(y_{i_j}) = c_{i_j} \to \Phi(y_0) = c_0$ as $j \to \infty$. This contradicts (3.26). □

For the velocities $\mathbf{u}_k = (u_k^1, u_k^2)$ and $\mathbf{v} = (v^1, v^2)$, denote by $\omega_k$ and $\omega$ the corresponding vorticities: $\omega_k = \partial_2 u_k^1 - \partial_1 u_k^2$, $\omega = \partial_2 v^1 - \partial_1 v^2 = \Delta\psi$. The following formulas are direct consequences of (3.11) and (3.10):

$$\nabla\Phi \equiv \omega \mathbf{v}^\perp = \omega\nabla\psi, \qquad \nabla\Phi_k \equiv -\nu_k \nabla^\perp \omega_k + \omega_k \mathbf{u}_k^\perp + \mathbf{f}_k \quad \text{in } \Omega. \tag{3.27}$$

We say that a set $\mathcal{Z} \subset T_\psi$ has $T$-measure zero if $\mathfrak{H}^1(\{\psi(C) : C \in \mathcal{Z}\}) = 0$. The function $\Phi|_{T_\psi}$ has some analogs of Luzin's $N$-property.

Lemma 3.6. *Let $A, B \in T_\psi$ with* $\operatorname{diam}(A) > 0$, $\operatorname{diam}(B) > 0$. *If $\mathcal{Z} \subset [A, B]$ has $T$-measure zero, then* $\mathfrak{H}^1(\{\Phi(C) : C \in \mathcal{Z}\}) = 0$.

*Proof.* Recall that the Coarea formula

$$\int_E |\nabla f|\, dx = \int_{\mathbb{R}} \mathfrak{H}^1(E \cap f^{-1}(y))\, dy \tag{3.28}$$

holds for a measurable set $E$ and the best representative (see Lemma 2.1) of any Sobolev function $f \in W^{1,1}(\Omega)$ (see, e.g., [23]).

Now, let $\mathcal{Z} \subset [A, B]$ have $T$-measure zero. Set $E = \cup_{C\in\mathcal{Z}} C$. Then by definition, $\mathfrak{H}^1(\psi(E)) = 0$. Take a Borel set $G \supset \psi(E)$ with $\mathfrak{H}^1(G) = 0$, and put $\mathcal{Z}' = \{C \in [A, B] : \psi(C) \in G\}$, $E' = \cup_{C\in\mathcal{Z}'} C$. Then $E'$ is a Borel set as well and $E' \supset E$. Hence, by Coarea formula (3.28) applied to $\psi|_{E'}$, we see that $\nabla\psi(x) = 0$ for $\mathfrak{H}^2$-almost all $x \in E'$. Then by (3.27), $\nabla\Phi(x) = 0$ for $\mathfrak{H}^2$-almost all $x \in E$. Applying the Coarea formula to $\Phi|_{E'}$, we obtain

$$0 = \int_{E'} |\nabla\Phi|\, dx = \int_{\mathbb{R}} \sum_{C\in\mathcal{Z}' : \Phi(C)=y} \mathfrak{H}^1(C)\, dy.$$

Since $\mathfrak{H}^1(C) \geq \min\big(\operatorname{diam}(A), \operatorname{diam}(B)\big) > 0$ for every $C \in [A, B]$, we have $\mathfrak{H}^1(\{\Phi(C) : C \in \mathcal{Z}'\}) = 0$, and this implies the assertion of Lemma 3.6. □

From Lemmas 3.4 and 3.6 we have

Corollary 3.2. *If $A, B \in T_\psi$ with* $\operatorname{diam}(A) > 0$, $\operatorname{diam}(B) > 0$, *then*

$$\mathfrak{H}^1\Big(\{\Phi(C) : C \in [A, B] \textit{ and } C \textit{ is not a regular cycle}\}\Big) = 0.$$

Denote by $B_0, \dots, B_N$ the elements of $T_\psi$ such that $B_j \supset \Gamma_j$, $j = 0, \dots, N$. By virtue of Lemma 3.3, every element $C \in [B_i, B_j] \setminus \{B_i, B_j\}$ is a connected component of a level set of $\psi$ such that the sets $B_i$, $B_j$ lie in different connected components of $\mathbb{R}^2 \setminus C$.

Put

$$\alpha = \max_{j=0,\dots,M} \min_{C\in[B_j,B_N]} \Phi(C).$$

By (3.22), $\alpha < 0$. Take a sequence of positive values $t_i \in (0,-\alpha)$, $i \in \mathbb{N}$, with $t_{i+1} = \frac{1}{2}t_i$ and such that the implication

$$\Phi(C) = -t_i \Rightarrow C \text{ is a regular cycle}$$

holds for every $j = 0,\dots,M$ and for all $C \in [B_j, B_N]$. The existence of the above sequence follows from Corollary 3.2.

Consider the natural order on the arc $[C_j, B_N]$, namely, $C' < C''$ if $C''$ is closer to $B_N$ than $C'$. (That means, $C'$ and $B_N$ lie in different connected components of the set $T_\psi \setminus \{C''\}$.) For $j = 0,\dots M$ and $i \in \mathbb{N}$, put

$$A_i^j = \max\{C \in [B_j, B_N] : \Phi(C) = -t_i\}.$$

In other words, $A_i^j$ is an element of the set $\{C \in [B_j, B_N] : \Phi(C) = -t_i\}$ that is closest to $\Gamma_N$. By construction, each $A_i^j$ is a regular cycle.[12] (See Figure 1 for the case of annulus type domains ($N = 1$).)

By definition of regular cycles (see the commentary to Lemma 3.4), each set $A_i^j$ is a $C^1$-curve homeomorphic to the unit circle and $A_i^j \subset \Omega$. In particular, for each $i \in \mathbb{N}$, the compact set $\cup_{j=0}^M A_i^j$ is separated from $\partial\Omega$ and $\operatorname{dist}(\cup_{j=0}^M A_i^j, \partial\Omega) > 0$. Then for each $i$ and for sufficiently small $h > 0$ (this smallness depends on $i$), we have the inclusion $\{x \in \Omega : \operatorname{dist}(x, \Gamma_N) < h\} \subset \Omega \setminus \left(\cup_{j=0}^M A_i^j\right)$. Of course, the set $\{x \in \Omega : \operatorname{dist}(x, \Gamma_N) < h\}$ is connected for small $h$. (It is homeomorphic to the open ring.) Hence, for small $h$, this set is included in some connected component of the open set $\Omega \setminus \left(\cup_{j=0}^M A_i^j\right)$. Denote this component by $V_i$. In particular, there holds $\Gamma_N \subset \partial V_i$.

We claim that

$$\Omega \cap \partial V_i = A_i^0 \cup \dots \cup A_i^M. \tag{3.29}$$

(We present this proof as typical for set identities. Below, similar proofs for other set identities are omitted because of their simplicity.) Indeed, by construction, $\Omega \cap \partial V_i \subset A_i^0 \cup \dots \cup A_i^M$ (since $V_i$ is a connected component of the open set $\Omega \setminus \left(\cup_{j=0}^M A_i^j\right)$). Suppose that (3.29) is false; i.e., $A_i^{j_1} \not\subset \partial V_i$ for some $j_1 \in \{0,\dots,M\}$. Then, obviously, $A_i^{j_1} \cap \overline{V}_i = \emptyset$ (since the cycles $A_i^j$ are either disjoint or coincide) and there exists a cycle $A_i^{j_2} \neq A_i^{j_1}$ such that the sets $V_i$ and $A_i^{j_1}$ lie in different connected components of $\mathbb{R}^2 \setminus A_i^{j_2}$. In particular, $A_i^{j_2}$ separates $\Gamma_N$ from $A_i^{j_1}$. But the last assertion contradicts the

[12] Some of these cycles $A_i^j$ could coincide — i.e., equalities of type $A_i^{j_1} = A_i^{j_2}$ are possible (if Kronrod arcs $[B_{j_1}, B_N]$ and $[B_{j_2}, B_N]$ have nontrivial intersection) — but this *a priori* possibility has no influence on our arguments. Note also that by construction, the cycles $A_i^j$ are either disjoint or coincide; i.e., if $A_i^{j_1} \neq A_i^{j_2}$, then $A_i^{j_1} \cap A_i^{j_2} = \emptyset$.

definition of $A_i^j$: indeed, by construction $\Phi > t_i = \Phi(A_i^{j_2})$ at the interior points of the Kronrod arc $[A_i^{j_1}, B_N]$. Consequently, $A_i^{j_2} \notin [A_i^{j_1}, B_N]$. The obtained contradiction finishes the proof of (3.29).

By construction, the sequence of domains $V_i$ is decreasing; i.e., $V_i \supset V_{i+1}$. Hence, the sequence of sets $(\partial\Omega) \cap (\partial V_i)$ is nonincreasing:

$$(\partial\Omega) \cap (\partial V_i) \supseteqq (\partial\Omega) \cap (\partial V_{i+1}). \tag{3.30}$$

Every set $(\partial\Omega) \cap (\partial V_i)$ consists of several components $\Gamma_l$ with $l > M$ (since arcs $\cup_{j=0}^M A_i^j$ separate $\Gamma_N$ from $\Gamma_0, \dots, \Gamma_M$, but not necessary from other $\Gamma_l$). Since there are only finitely many components $\Gamma_l$, using the monotonicity property (3.30) we conclude that for sufficiently large $i$, the set $(\partial\Omega) \cap (\partial V_i)$ is independent of $i$. So we may assume, without loss of generality, that $(\partial\Omega) \cap (\partial V_i) = \Gamma_K \cup \dots \cup \Gamma_N$, where $K \in \{M+1, \dots, N\}$. Therefore,

$$\partial V_i = A_i^0 \cup \dots \cup A_i^M \cup \Gamma_K \cup \dots \cup \Gamma_N. \tag{3.31}$$

From Lemma 3.4 we have the uniform convergence $\Phi_k|_{A_i^j} \rightrightarrows \Phi(A_i^j) = -t_i$ as $k \to \infty$. Thus for every $i \in \mathbb{N}$, there exists $k_i$ such that for all $k \geq k_i$,

$$\Phi_k|_{A_i^j} < -\frac{7}{8}t_i, \quad \Phi_k|_{A_{i+1}^j} > -\frac{5}{8}t_i \quad \forall j = 0, \dots, M. \tag{3.32}$$

Then

$$\forall t \in \Big[\frac{5}{8}t_i, \frac{7}{8}t_i\Big] \; \forall k \geq k_i \quad \Phi_k|_{A_i^j} < -t, \quad \Phi_k|_{A_{i+1}^j} > -t \quad \forall j = 0, \dots, M. \tag{3.33}$$

For $k \geq k_i$, $j = 0, \dots, M$, and $t \in [\frac{5}{8}t_i, \frac{7}{8}t_i]$, denote by $W_{ik}^j(t)$ the connected component of the open set $\{x \in V_i \setminus \overline{V}_{i+1} : \Phi_k(x) > -t\}$ such that $\partial W_{ik}^j(t) \supset A_{i+1}^j$, and put

$$W_{ik}(t) = \bigcup_{j=0}^{M} W_{ik}^j(t), \qquad S_{ik}(t) = (\partial W_{ik}(t)) \cap V_i \setminus \overline{V}_{i+1}.$$

Clearly, $\Phi_k \equiv -t$ on $S_{ik}(t)$. By construction,

$$\partial W_{ik}(t) = S_{ik}(t) \cup A_{i+1}^0 \cup \dots \cup A_{i+1}^M; \tag{3.34}$$

see Figure 1. Since by (E–NS) each $\Phi_k$ belongs to $W^{2,2}_{\mathrm{loc}}(\Omega)$, by the Morse-Sard theorem for Sobolev functions (see Theorem 2.1) we have that for almost all $t \in [\frac{5}{8}t_i, \frac{7}{8}t_i]$, the level set $S_{ik}(t)$ consists of finitely many $C^1$-cycles and $\Phi_k$ is differentiable (in classical sense) at every point $x \in S_{ik}(t)$ with $\nabla\Phi_k(x) \neq 0$. The values $t \in [\frac{5}{8}t_i, \frac{7}{8}t_i]$ having the above property will be called $(k,i)$-*regular*. By construction,

$$\int\limits_{S_{ik}(t)} \nabla\Phi_k \cdot \mathbf{n}\, ds = -\int\limits_{S_{ik}(t)} |\nabla\Phi_k|\, ds < 0, \tag{3.35}$$

where $\mathbf{n}$ is the unit outward (with respect to $W_{ik}(t)$) normal vector to $\partial W_{ik}(t)$.

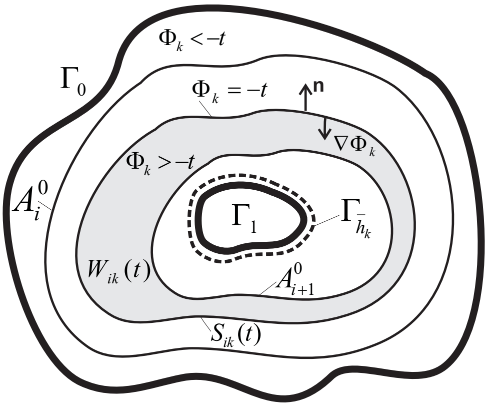


Figure 1. The case of an annulus-like domain ($N = 1$).

For $h > 0$, denote $\Gamma_h = \{x \in \Omega : \mathrm{dist}(x, \Gamma_K \cup \cdots \cup \Gamma_N) = h)\}$, $\Omega_h = \{x \in \Omega : \mathrm{dist}(x, \Gamma_K \cup \cdots \cup \Gamma_N) < h)\}$. By elementary results of analysis, there is a positive constant

$$\delta_0 < \frac{1}{2} \min\{|x - y| : x \in \Gamma_j,\ y \in \Gamma_m,\ j, m \in \{0, \ldots, N\}, j \neq m\}$$

such that for each $h \leq \delta_0$, the set $\Gamma_h$ is a union of $N - K + 1$ $C^1$-smooth curves homeomorphic to the circle, and

$$\mathfrak{H}^1(\Gamma_h) \leq C_0 \quad \forall h \in (0, \delta_0], \tag{3.36}$$

where $C_0 = 3\mathfrak{H}^1(\Gamma_K \cup \cdots \cup \Gamma_N)$ is independent of $h$.

Since $\Phi \neq \mathrm{const}$ on $V_i$, by (3.27) we have $\int\limits_{V_i} \omega^2\, dx > 0$ for each $i$. Hence, from the weak convergence $\omega_k \rightharpoonup \omega$ in $L^2(\Omega)$, it follows

Lemma 3.7. *For any $i \in \mathbb{N}$, there exist constants $\varepsilon_i > 0$, $\delta_i \in (0, \delta_0)$, and $k_i' \in \mathbb{N}$ such that*

$$\overline{\Omega}_{\delta_i} \cap A_i^j = \overline{\Omega}_{\delta_i} \cap A_{i+1}^j = \emptyset, \quad j = 0, \ldots, M,$$

*and* $\int\limits_{V_{i+1} \setminus \Omega_{\delta_i}} \omega_k^2\, dx > \varepsilon_i$ *for all $k \geq k_i'$.*

The key step is the following estimate.

Lemma 3.8. *For any $i \in \mathbb{N}$, there exists $k(i) \in \mathbb{N}$ such that the inequality*

$$\int\limits_{S_{ik}(t)} |\nabla \Phi_k|\, ds < \mathcal{F}t \tag{3.37}$$

*holds for every $k \geq k(i)$ and for almost all $t \in [\frac{5}{8}t_i, \frac{7}{8}t_i]$, where the constant $\mathcal{F}$ is independent of $t, k$ and $i$.*

*Proof.* Fix $i \in \mathbb{N}$, and assume $k \geq k_i$ (see (3.32)). Take a sufficiently small $\sigma > 0$. (The exact value of $\sigma$ will be specified below.) We choose the parameter $\delta_\sigma \in (0, \delta_i]$ (see Lemma 3.7) small enough to satisfy the following conditions:

$$\int\limits_{\Gamma_h} \Phi^2 \, ds < \frac{1}{3}\sigma^2 \quad \forall h \in (0, \delta_\sigma], \tag{3.38}$$

$$-\frac{1}{3}\sigma^2 < \int\limits_{\Gamma_{h'}} \Phi_k^2 \, ds - \int\limits_{\Gamma_{h''}} \Phi_k^2 \, ds < \frac{1}{3}\sigma^2 \quad \forall h', h'' \in (0, \delta_\sigma] \;\; \forall k \in \mathbb{N}. \tag{3.39}$$

The last estimate follows from the fact that for any $q \in (1,2)$, the norms $\|\Phi_k\|_{W^{1,q}(\Omega)}$ are uniformly bounded. Consequently, the norms $\|\Phi_k \nabla \Phi_k\|_{L^q(\Omega)}$ are uniformly bounded as well. In particular, for $q = 6/5$, we have

$$\left| \int\limits_{\Gamma_{h'}} \Phi_k^2 \, ds - \int\limits_{\Gamma_{h''}} \Phi_k^2 \, ds \right| \leq 2 \int\limits_{\Omega_{h''} \setminus \Omega_{h'}} |\Phi_k| \cdot |\nabla \Phi_k| \, dx$$

$$\leq 2 \Bigg( \int\limits_{\Omega_{h''} \setminus \Omega_{h'}} |\Phi_k \nabla \Phi_k|^{6/5} \, dx \Bigg)^{\frac{5}{6}} \operatorname{meas}(\Omega_{h''} \setminus \Omega_{h'})^{\frac{1}{6}} \to 0 \qquad \text{as } h', h'' \to 0.$$

From the weak convergence $\Phi_k \rightharpoonup \Phi$ in the space $W^{1,q}(\Omega)$, $q \in (1,2)$, it follows that[13] $\Phi_k|_{\Gamma_h} \rightrightarrows \Phi|_{\Gamma_h}$ as $k \to \infty$ for almost all $h \in (0, \delta_\sigma)$; see [1], [17].[14] From the last fact and (3.38)–(3.39) we see that there exists $k' \in \mathbb{N}$ such that

$$\int\limits_{\Gamma_h} \Phi_k^2 \, ds < \sigma^2 \quad \forall h \in (0, \delta_\sigma] \;\; \forall k \geq k'. \tag{3.40}$$

Obviously, for a function $g \in W^{2,2}(\Omega)$ and for an arbitrary $C^1$-cycle $S \subset \Omega$, we have

$$\int\limits_S \nabla^\perp g \cdot \mathbf{n} \, ds = \int\limits_S \nabla g \cdot \mathbf{l} \, ds = 0,$$

where $\mathbf{l}$ is the tangent vector to $S$. Consequently, by (3.27),

$$\int\limits_S \nabla \Phi_k \cdot \mathbf{n} \, ds = \int\limits_S \omega_k \mathbf{u}_k^\perp \cdot \mathbf{n} \, ds;$$

recall that by our assumptions, $\mathbf{f} = \nabla^\perp b$.

[13] Really this uniform convergence hold for a subsequence $\Phi_{k_l}$ (Lemma 3.4), which we identify with $\Phi_k$.

[14] In [1] Amick proved the uniform convergence $\Phi_k \rightrightarrows \Phi$ on almost all circles. However, his method can be easily modified to prove the uniform convergence on almost all level lines of every $C^1$-smooth function with nonzero gradient. Such modification was done in the proof of Lemma 3.3 of [17].

Now, fix a sufficiently small $\varepsilon > 0$. (The exact value of $\varepsilon$ will be specified below.) Our next purpose is as follows. For a given sufficiently large $k \geq k'$, find a number $\bar{h}_k \in (0, \delta_\sigma)$ such that the estimates

$$\left| \int\limits_{\Gamma_{\bar{h}_k}} \nabla \Phi_k \cdot \mathbf{n}\, ds \right| = \left| \int\limits_{\Gamma_{\bar{h}_k}} \omega_k \mathbf{u}_k^\perp \cdot \mathbf{n}\, ds \right| < \varepsilon, \tag{3.41}$$

$$\int\limits_{\Gamma_{\bar{h}_k}} |\mathbf{u}_k|^2\, ds < C_\varepsilon \nu_k^2 \tag{3.42}$$

hold, where the constant $C_\varepsilon$ *is independent of* $k$ *and* $\sigma$. For this purpose, take $\Gamma = \Gamma_K \cup \dots \cup \Gamma_N$, and consider the function $g(h) = \int\limits_{\Gamma_h} |\mathbf{u}_k|^2\, ds$. In particular,

$$g(0) = \int\limits_{\Gamma} |\mathbf{u}_k|^2\, ds = \frac{(\lambda_k \nu_k)^2}{\nu^2} \|\mathbf{a}\|^2_{L^2(\Gamma)},$$

where $\lambda_k \in (0, 1]$; see ($3.10_3$). Also denote $f(h) = \int\limits_{\Gamma_h} |\nabla \mathbf{u}_k| \cdot |\mathbf{u}_k|\, ds$. By the classical formula of changing variables in the integral, there exists a $C^1$-smooth function $J : \overline{\Omega}_{\delta_i} \to (0, +\infty)$ (not depending on $k$) such that $J_\Gamma \equiv 1$ and[15]

$$\Big( \int\limits_{\Gamma_h} J |\mathbf{u}_k|^2\, ds \Big)'_h \leq 2 \int\limits_{\Gamma_h} J |\mathbf{u}_k| \cdot |\nabla \mathbf{u}_k|\, ds.$$

Consequently, there are constants $C_1, C_2 > 0$ (not depending on $k, h$) such that for every $h_0 \in (0, \delta_i]$, the following estimate holds:

$$\ln\Big( \frac{C_1 g(h_0)}{\nu_k^2} \Big) \leq C_2 \int\limits_0^{h_0} \frac{f(h)}{g(h)}\, dh. \tag{3.43}$$

Put

$$C_\varepsilon = \frac{1}{C_1} \exp\Big( \frac{2C_2}{\varepsilon} \Big).$$

[15] Here $J(x)$ is the Jacobian of the following mapping: $\varphi : \overline{\Omega}_{\delta_i} \ni x \mapsto \varphi(x) = (\gamma(x), \operatorname{dist}(x, \Gamma)) \in \Gamma \times [0, \delta_i]$, where $\gamma(x) \in \Gamma$ is a metric projection of $x$ onto $\Gamma$: $|x - \gamma(x)| = \operatorname{dist}(x, \Gamma)$. It is well known that $C^2$-smoothness of $\Gamma$ and smallness $\delta_i < \delta_0$ guarantee that the mapping $\varphi$ is $C^1$-smooth diffeomorphism and, in particular, $J(x)$ is separated from zero and infinity by positive constants. Note also that for every $x \in \overline{\Omega}_{\delta_i}$, the segment $[x, \gamma(x)]$ is perpendicular to curves $\Gamma_h$, and $\gamma(y) \equiv \gamma(x)$ for all $y \in [x, \gamma(x)]$. In other words, the mapping $\varphi$ generates an orthogonal curvilinear system whose coordinate lines are curves $\Gamma_h$ and rectilinear segments of type $[x, \gamma(x)]$.

Consider two possible cases:

*Case* 1. $g(h) \le C_\varepsilon \nu_k^2 \ \forall h \in [0, \delta_\sigma]$. Then by the Hölder inequality we obtain

$$\frac{1}{\delta_\sigma} \int\limits_{\Omega_{\delta_\sigma}} |\nabla \mathbf{u}_k| \cdot |\mathbf{u}_k| \, dx \le \frac{1}{\delta_\sigma} \sqrt{\delta_\sigma C_\varepsilon \nu_k^2} \Big( \int\limits_{\Omega_{\delta_\sigma}} |\nabla \mathbf{u}_k|^2 \, dx \Big)^{\frac{1}{2}} < \sqrt{\frac{2C_\varepsilon \nu_k^2}{\delta_\sigma}}. \tag{3.44}$$

Here we have used the estimate

$$\int\limits_{\Omega} |\nabla \mathbf{u}_k|^2 \, dx < 2, \tag{3.45}$$

which is valid for sufficiently large $k$ (because of the convergence $\int\limits_{\Omega} |\nabla \mathbf{u}_k|^2 \, dx \to 1$; see assumptions (E-NS)). Estimate (3.44) implies that there exists $\bar h_k \in (0, \delta_\sigma)$ such that

$$\int\limits_{\Gamma_{\bar h_k}} |\nabla \mathbf{u}_k| \cdot |\mathbf{u}_k| \, ds < \sqrt{\frac{2C_\varepsilon \nu_k^2}{\delta_\sigma}}. \tag{3.46}$$

Then, taking into account that $\nu_k \to 0$ as $k \to \infty$, while $C_\varepsilon, \delta_\sigma$ are independent of $k$, we obtain the required estimates (3.41) and (3.42) for sufficiently large $k$.

*Case* 2. $\sup\limits_{h \in [0,\delta_\sigma]} g(h) > C_\varepsilon \nu_k^2$. Take $h_0 = \min\{h \in [0, \delta_\sigma] : g(h) = C_\varepsilon \nu_k^2\}$. By choice of $C_\varepsilon$ and (3.43),

$$\frac{2}{\varepsilon} \le \int\limits_0^{h_0} \frac{f(h)}{g(h)} \, dh. \tag{3.47}$$

We claim that there exists $\bar h_k \in (0, h_0)$ satisfying (3.41) and (3.42). Suppose the contrary; then $f(h) \ge \varepsilon$ for all $h \in (0, h_0)$. By the Hölder inequality,

$$f^2(h) \le g(h) \cdot \int\limits_{\Gamma_h} |\nabla \mathbf{u}_k|^2 \, ds.$$

Consequently,

$$\int\limits_{\Gamma_h} |\nabla \mathbf{u}_k|^2 \, ds \ge \frac{f^2(h)}{g(h)} \ge \frac{f(h)\varepsilon}{g(h)} \qquad \forall h \in (0, h_0).$$

Hence

$$\int\limits_{\Omega_{h_0}} |\nabla \mathbf{u}_k|^2 \, dx = \int\limits_0^{h_0} dh \int\limits_{\Gamma_h} |\nabla \mathbf{u}_k|^2 \, ds \ge \int\limits_0^{h_0} \frac{f(h)\varepsilon}{g(h)} \, dh \ge 2.$$

(In the last inequality we have used (3.47), and in the first equality we have used the well-known identity $|\nabla \operatorname{dist}(x, \Gamma)| \equiv 1$ on $\Omega_{\delta_i}$.) We have obtained a contradiction with (3.45). This proves the existence of the required $\bar{h}_k \in (0, \delta_\sigma)$ satisfying (3.41) and (3.42) for sufficiently large $k$.

Now, for $(k,i)$-regular value $t \in [\frac{5}{8}t_i, \frac{7}{8}t_i]$, consider the domain

$$\Omega_{i\bar{h}_k}(t) = W_{ik}(t) \cup \overline{V}_{i+1} \setminus \overline{\Omega}_{\bar{h}_k}.$$

By construction, $\partial\Omega_{i\bar{h}_k}(t) = \Gamma_{\bar{h}_k} \cup S_{ik}(t)$ (see Figure 1). Integrating the equation

$$\Delta\Phi_k = \omega_k^2 + \frac{1}{\nu_k}\operatorname{div}(\Phi_k\mathbf{u}_k) - \frac{1}{\nu_k}\mathbf{f}_k \cdot \mathbf{u}_k \tag{3.48}$$

over the domain $\Omega_{i\bar{h}_k}(t)$, we have

$$\begin{aligned}
&\int\limits_{S_{ik}(t)} \nabla\Phi_k \cdot \mathbf{n}\, ds + \int\limits_{\Gamma_{\bar{h}_k}} \nabla\Phi_k \cdot \mathbf{n}\, ds \\
&\qquad = \int\limits_{\Omega_{i\bar{h}_k}(t)} \omega_k^2\, dx - \frac{1}{\nu_k}\int\limits_{\Omega_{i\bar{h}_k}(t)} \mathbf{f}_k \cdot \mathbf{u}_k\, dx \\
&\qquad\quad + \frac{1}{\nu_k}\int\limits_{S_{ik}(t)} \Phi_k\mathbf{u}_k \cdot \mathbf{n}\, ds + \frac{1}{\nu_k}\int\limits_{\Gamma_{\bar{h}_k}} \Phi_k\mathbf{u}_k \cdot \mathbf{n}\, ds \\
&\qquad = \int\limits_{\Omega_{i\bar{h}_k}(t)} \omega_k^2\, dx - \frac{1}{\nu_k}\int\limits_{\Omega_{i\bar{h}_k}(t)} \mathbf{f}_k \cdot \mathbf{u}_k\, dx - t\lambda_k\bar{\mathcal{F}} + \frac{1}{\nu_k}\int\limits_{\Gamma_{\bar{h}_k}} \Phi_k\mathbf{u}_k \cdot \mathbf{n}\, ds,
\end{aligned} \tag{3.49}$$

where $\bar{\mathcal{F}} = \frac{1}{\nu}(\mathcal{F}_0 + \cdots + \mathcal{F}_M)$. In view of (3.35) and (3.41), we can estimate

$$\begin{aligned}
\int\limits_{S_{ik}(t)} |\nabla\Phi_k|\, ds \le t\mathcal{F} + \varepsilon + \frac{1}{\nu_k}\int\limits_{\Omega_{i\bar{h}_k}(t)} \mathbf{f}_k \cdot \mathbf{u}_k\, dx - \int\limits_{\Omega_{i\bar{h}_k}(t)} \omega_k^2\, dx \\
+ \frac{1}{\nu_k}\Bigg(\int\limits_{\Gamma_{\bar{h}_k}} \Phi_k^2\, ds\Bigg)^{\frac{1}{2}}\Bigg(\int\limits_{\Gamma_{\bar{h}_k}} |\mathbf{u}_k|^2\, ds\Bigg)^{\frac{1}{2}},
\end{aligned} \tag{3.50}$$

with $\mathcal{F} = |\bar{\mathcal{F}}|$. By definition, $\frac{1}{\nu_k}\|\mathbf{f}_k\|_{L^2(\Omega)} = \frac{\lambda_k\nu_k}{\nu^2}\|\mathbf{f}\|_{L^2(\Omega)} \to 0$ as $k \to \infty$. Therefore,

$$\Bigg|\frac{1}{\nu_k}\int\limits_{\Omega_{i\bar{h}_k}(t)} \mathbf{f}_k \cdot \mathbf{u}_k\, dx\Bigg| \le \varepsilon$$

for sufficiently large $k$. Using inequalities (3.40) and (3.42), we obtain

$$\begin{aligned}
\int\limits_{S_{ik}(t)} |\nabla\Phi_k|\, ds &\le t\mathcal{F} + 2\varepsilon + \sigma\sqrt{C_\varepsilon} - \int\limits_{\Omega_{i\bar{h}_k}(t)} \omega_k^2\, dx \\
&\le t\mathcal{F} + 2\varepsilon + \sigma\sqrt{C_\varepsilon} - \int\limits_{V_{i+1}\setminus\Omega_{\delta_i}} \omega_k^2\, dx,
\end{aligned} \tag{3.51}$$

where $C_\varepsilon$ is independent of $k$ and $\sigma$. Choosing $\varepsilon = \frac{1}{6}\varepsilon_i$, $\sigma = \frac{1}{3\sqrt{C_\varepsilon}}\varepsilon_i$, and a sufficiently large $k$, from Lemma 3.7 we obtain $2\varepsilon + \sigma\sqrt{C_\varepsilon} - \int_{V_{i+1}\setminus\Omega_{\delta_i}} \omega_k^2\,dx \le 0$. Estimate (3.37) is proved. □

Now, we receive the required contradiction using the Coarea formula.

LEMMA 3.9. *Assume that $\Omega \subset \mathbb{R}^2$ is a bounded domain of type (2.1) with $C^2$-smooth boundary $\partial\Omega$, $\mathbf{f} \in W^{1,2}(\Omega)$, and $\mathbf{a} \in W^{3/2,2}(\partial\Omega)$ satisfies condition (1.2). Then assumptions* (E-NS) *and* (3.17) *lead to a contradiction.*

*Proof.* For $i \in \mathbb{N}$ and $k \ge k(i)$ (see Lemma 3.8), put

$$E_i = \bigcup_{t\in[\frac{5}{8}t_i,\frac{7}{8}t_i]} S_{ik}(t).$$

By the Coarea formula (3.28) (see also [23]), for any integrable function $g : E_i \to \mathbb{R}$, the equality

$$\int\limits_{E_i} g|\nabla\Phi_k|\,dx = \int\limits_{\frac{5}{8}t_i}^{\frac{7}{8}t_i} \int\limits_{S_{ik}(t)} g(x)\,d\mathfrak{H}^1(x)\,dt \tag{3.52}$$

holds. In particular, taking $g = |\nabla\Phi_k|$ and using (3.37), we obtain

$$\int\limits_{E_i} |\nabla\Phi_k|^2\,dx = \int\limits_{\frac{5}{8}t_i}^{\frac{7}{8}t_i} \int\limits_{S_{ik}(t)} |\nabla\Phi_k|(x)\,d\mathfrak{H}^1(x)\,dt \le \int\limits_{\frac{5}{8}t_i}^{\frac{7}{8}t_i} \mathcal{F}t\,dt = \mathcal{F}'t_i^2, \tag{3.53}$$

where $\mathcal{F}' = \frac{3}{16}\mathcal{F}$ is independent of $i$. Now, taking $g = 1$ in (3.52) and using the Hölder inequality, we have

$$\begin{aligned}\int\limits_{\frac{5}{8}t_i}^{\frac{7}{8}t_i} \mathfrak{H}^1\big(S_{ik}(t)\big)\,dt = \int\limits_{E_i} |\nabla\Phi_k|\,dx &\le \Big(\int\limits_{E_i} |\nabla\Phi_k|^2\,dx\Big)^{\frac{1}{2}} \big(\operatorname{meas}(E_i)\big)^{\frac{1}{2}} \\ &\le \sqrt{\mathcal{F}'}t_i\big(\operatorname{meas}(E_i)\big)^{\frac{1}{2}}.\end{aligned} \tag{3.54}$$

By construction, for almost all $t \in [\frac{5}{8}t_i, \frac{7}{8}t_i]$, the set $S_{ik}(t)$ is a finite union of smooth cycles and $S_{ik}(t)$ separates $A_i^j$ from $A_{i+1}^j$ for $j = 0, \dots, M$. Thus, each set $S_{ik}(t)$ separates $\Gamma_j$ from $\Gamma_N$. In particular,

$$\mathfrak{H}^1(S_{ik}(t)) \ge \min\big(\operatorname{diam}(\Gamma_j), \operatorname{diam}(\Gamma_N)\big).$$

Hence, the left integral in (3.54) is greater than $Ct_i$, where $C > 0$ does not depend on $i$. On the other hand, evidently, $\operatorname{meas}(E_i) \le \operatorname{meas}\big(V_i \setminus V_{i+1}\big) \to 0$ as $i \to \infty$. The obtained contradiction finishes the proof of Lemma 3.9. □

3.3.2. *The maximum of $\Phi$ is not attained at $\partial\Omega$.* In this subsection we consider the case (b), when (3.18) holds. Adding a constant to the pressure, we assume, without loss of generality, that

$$\max_{j=0,\dots,N} \widehat{p}_j < 0 < \operatorname*{ess\,sup}_{x\in\Omega} \Phi(x). \tag{3.55}$$

(Here we do not exclude the case $\operatorname*{ess\,sup}_{x\in\Omega} \Phi(x) = +\infty$.) Denote $\sigma = \max_{j=0,\dots,N} \widehat{p}_j < 0$.

As in the previous subsection, we consider the behavior of $\Phi$ on the Kronrod tree $T_\psi$. In particular, Lemmas 3.5 and 3.6 hold.

LEMMA 3.10. *There exists $F \in T_\psi$ such that* $\operatorname{diam} F > 0$, $F \cap \partial\Omega = \emptyset$, *and* $\Phi(F) > \sigma$.

*Proof.* By assumptions, $\Phi(x) \le \sigma$ for every $x \in \partial\Omega \setminus A_{\mathbf{v}}$ and there is a set of a positive measure $E \subset \Omega$ such that $\Phi(x) > \sigma$ at each $x \in E$. In virtue of Lemma 3.2(iii), there exists a straight-line segment $I = [x_0, y_0] \subset \overline{\Omega}$ with $I \cap A_{\mathbf{v}} = \emptyset$, $x_0 \in \partial\Omega$, $y_0 \in E$, such that $\Phi|_I$ is a continuous function. By construction, $\Phi(x_0) \le \sigma$, $\Phi(y_0) \ge \sigma + \delta_0$ with some $\delta_0 > 0$. Take a subinterval $I_1 = [x_1, y_0] \subset \Omega$ such that $\Phi(x_1) = \sigma + \frac{1}{2}\delta_0$ and $\Phi(x) \ge \sigma + \frac{1}{2}\delta_0$ for each $x \in [x_1, y_0]$. Then by Bernoulli's Law (see Theorem 3.2), $\psi \ne \text{const}$ on $I_1$. Hence, we can take $x \in I_1$ such that the preimage $\psi^{-1}(\psi(x))$ consists of a finite union of regular cycles (see Lemma 3.4). Denote by $F$ the regular cycle containing $x$. Then by construction, $\Phi(F) \ge \sigma + \frac{1}{2}\delta_0$, and by definition of regular cycles, $\operatorname{diam} F > 0$ and $F \cap \partial\Omega = \emptyset$. $\square$

Fix $F$ from above lemma and consider the behavior of $\Phi$ on the Kronrod arcs $[B_j, F]$, $j = 0, \dots N$. (Recall that by $B_j$ we denote the elements of $T_\psi$ such that $\Gamma_j \subset B_j$.) The rest part of this subsection is similar to that of Section 3.3.1 with the following difference: $F$ now plays the role that was played before by $B_N$, and the calculations become easier since $F$ lies strictly inside $\Omega$.

Adding a constant to the pressure, we could assume, without loss of generality, that $\Phi(F) = 0$. Then by construction, $0 > \sigma \ge \Phi(B_j)$ for each $j = 0, \dots, N$ So, using Lemmas 3.5, 3.6 and Corollary 3.2 we can find a sequence of positive numbers $t_i \in (0, -\sigma)$, $i \in \mathbb{N}$, with $t_{i+1} = \frac{1}{2}t_i$, and the corresponding regular cycles $A_i^j \in [B_j, F]$, $j = 0, \dots, N$, with $\Phi(A_i^j) = -t_i$ and $\Phi(C) > -t_i$ for all $C \in (A_i^j, F]$. Denote by $V_i$ the connected component of the set $\Omega \setminus (A_i^0 \cup \dots \cup A_i^N)$ containing $F$. By construction, $\overline{V}_i \subset \Omega$, $V_i \supset \overline{V}_{i+1}$, and

$$\partial V_i = A_i^0 \cup \dots \cup A_i^N. \tag{3.56}$$

By definition of regular cycles (see Lemma 3.4), we again obtain estimates (3.32) and (3.33) for $k \ge k_i$. Accordingly, for $k \ge k_i$ and $t \in [\frac{5}{8}t_i, \frac{7}{8}t_i]$, we can define $W^j_{ik}(t)$ as the connected component of the open set

$$\{x \in V_i \setminus \overline{V}_{i+1} : \Phi_k(x) > -t\}$$

with $\partial W^j_{ik}(t) \supset A^j_{i+1}$ and put

$$W_{ik}(t) = \bigcup_{j=0}^{N} W^j_{ik}(t), \qquad S_{ik}(t) = (\partial W_{ik}(t)) \cap V_i \setminus \overline{V}_{i+1}.$$

By construction,

$$\partial W_{ik}(t) = S_{ik}(t) \cup A^0_{i+1} \cup \dots \cup A^N_{i+1},$$

and the set $S_{ik}(t)$ separates $A^0_i \cup \dots \cup A^N_i$ from $A^0_{i+1} \cup \dots \cup A^N_{i+1}$. By the Morse-Sard theorem (see Theorem 2.1) applied to $\Phi_k \in W^{2,2}_{\mathrm{loc}}(\Omega)$, for almost all $t \in [\frac{5}{8}t_i, \frac{7}{8}t_i]$, the level set $S_{ik}(t)$ consists of finitely many $C^1$-cycles. Moreover, by construction,

$$\int\limits_{S_{ik}(t)} \nabla\Phi_k \cdot \mathbf{n}\, ds = -\int\limits_{S_{ik}(t)} |\nabla\Phi_k|\, ds < 0, \tag{3.57}$$

where $\mathbf{n}$ is the unit outward normal vector to $\partial W_{ik}(t)$. As before, we call such values $t \in [\frac{5}{8}t_i, \frac{7}{8}t_i]$ $(k,i)$-*regular*.

Since $\Phi \neq$ const on $V_i$, from (3.27) it follows that $\int\limits_{V_i} \omega^2\, dx > 0$ for each $i$, and taking into account the weak convergence $\omega_k \rightharpoonup \omega$ in $L^2(\Omega)$, we get

LEMMA 3.11. *For every $i \in \mathbb{N}$, there exist constants $\varepsilon_i > 0$, $\delta_i \in (0, \delta_0)$ and $k'_i \in \mathbb{N}$ such that $\int\limits_{V_{i+1}} \omega_k^2\, dx > \varepsilon_i$ for all $k \geq k'_i$.*

Now, we can prove

LEMMA 3.12. *Assume that $\Omega \subset \mathbb{R}^2$ is a bounded domain of type* (2.1) *with $C^2$-smooth boundary $\partial\Omega$, $\mathbf{f} \in W^{1,2}(\Omega)$, and $\mathbf{a} \in W^{3/2,2}(\partial\Omega)$ satisfies condition (1.2). Then assumptions* (E-NS) *and* (3.18) *lead to a contradiction.*

*Proof.* The proof of this lemma is similar to that of Lemma 3.8. However, now the situation is easier since we separate $V_i$ from the whole boundary $\partial\Omega$. Fix $i \in \mathbb{N}$, and assume that $k \geq k_i$ (see (3.32)). For a $(k,i)$-regular value $t \in [\frac{5}{8}t_i, \frac{7}{8}t_i]$, consider the domain

$$\Omega_{ik}(t) = W_{ik}(t) \cup \overline{V}_{i+1}.$$

By construction, $\partial\Omega_{ik}(t) = S_{ik}(t)$. Integrating identity (3.48) over $\Omega_{ik}(t)$, we obtain

$$
\begin{aligned}
0 > \int\limits_{S_{ik}(t)} \nabla\Phi_k \cdot \mathbf{n}\, ds &= \int\limits_{\Omega_{ik}(t)} \omega_k^2\, dx + \frac{1}{\nu_k} \int\limits_{S_{ik}(t)} \Phi_k \mathbf{u}_k \cdot \mathbf{n}\, ds \\
-\frac{1}{\nu_k} \int\limits_{\Omega_{ik}(t)} \mathbf{f}_k \cdot \mathbf{u}_k\, dx &= \int\limits_{\Omega_{ik}(t)} \omega_k^2\, dx - \frac{t}{\nu_k} \int\limits_{S_{ik}(t)} \mathbf{u}_k \cdot \mathbf{n}\, ds \\
-\frac{1}{\nu_k} \int\limits_{\Omega_{ik}(t)} \mathbf{f}_k \cdot \mathbf{u}_k\, dx &= \int\limits_{\Omega_{ik}(t)} \omega_k^2\, dx - \frac{1}{\nu_k} \int\limits_{\Omega_{ik}(t)} \mathbf{f}_k \cdot \mathbf{u}_k\, dx
\end{aligned}
\tag{3.58}
$$

and, as before, we have a contradiction with Lemma 3.11. □

*Proof of Theorem* 1.1. Let the hypotheses of Theorem 1.1 be satisfied. Suppose that its assertion fails. Then, by Lemma 3.1, there exist $\mathbf{v}, p$ and a sequence $(\mathbf{u}_k, p_k)$ satisfying (E-NS), and by Lemmas 3.12 and 3.9 these assumptions lead to a contradiction. □

## 4. Axially symmetric case

First, let us specify some notation. Let $O_{x_1}, O_{x_2}, O_{x_3}$ be coordinate axes in $\mathbb{R}^3$ and $\theta = \operatorname{arctg}(x_2/x_1)$, $r = (x_1^2+x_2^2)^{1/2}$, $z = x_3$ be cylindrical coordinates. Denote by $v_\theta, v_r, v_z$ the projections of the vector $\mathbf{v}$ on the axes $\theta, r, z$.

A function $f$ is said to be *axially symmetric* if it does not depend on $\theta$. A vector-valued function $\mathbf{h} = (h_r, h_\theta, h_z)$ is called *axially symmetric* if $h_r$, $h_\theta$ and $h_z$ do not depend on $\theta$. A vector-valued function $\mathbf{h}' = (h_r, h_\theta, h_z)$ is called *axially symmetric with no swirl* if $h_\theta = 0$ while $h_r$ and $h_z$ do not depend on $\theta$.

The main result of this section is as follows.

THEOREM 4.1. *Assume that $\Omega \subset \mathbb{R}^3$ is a bounded axially symmetric domain of type* (2.1) *with $C^2$-smooth boundary $\partial\Omega$. If $\mathbf{f} \in W^{1,2}(\Omega)$, $\mathbf{a} \in W^{3/2,2}(\partial\Omega)$ are axially symmetric and $\mathbf{a}$ satisfies condition* (1.2), *then* (1.1) *admits at least one weak axially symmetric solution. Moreover, if $\mathbf{f}$ and $\mathbf{a}$ are axially symmetric with no swirl, then* (1.1) *admits at least one weak axially symmetric solution with no swirl.*

Using the "reductio ad absurdum" Leray argument (the main idea is presented in Section 3.1 for the plane case; specific details concerning the axially symmetric case can be found in [15]), it is possible to prove the following

LEMMA 4.1. *Assume that $\Omega \subset \mathbb{R}^3$ is a bounded axially symmetric domain of type* (2.1) *with $C^2$-smooth boundary $\partial\Omega$, $\mathbf{f} = \operatorname{curl} \mathbf{b}$, $\mathbf{b} \in W^{2,2}(\Omega)$, $\mathbf{a} \in W^{3/2,2}(\partial\Omega)$ are axially symmetric, and $\mathbf{a}$ satisfies condition* (1.2). *If the* *assertion of Theorem 4.1 is false, then there exist* $\mathbf{v}, p$ *with the following properties:*

(E-AX) *The axially symmetric functions* $\mathbf{v} \in \mathring{W}^{1,2}(\Omega)$, $p \in W^{1,3/2}(\Omega)$ *satisfy the Euler system* (3.11).

(E-NS-AX) *Condition* (E-AX) *is satisfied, and there exist a sequences of axially symmetric functions* $\mathbf{u}_k \in W^{1,2}(\Omega)$, $p_k \in W^{1,3/2}(\Omega)$ *and numbers* $\nu_k \to 0+$, $\lambda_k \to \lambda_0 > 0$ *such that the norms* $\|\mathbf{u}_k\|_{W^{1,2}(\Omega)}$, $\|p_k\|_{W^{1,3/2}(\Omega)}$ *are uniformly bounded, the pair* $(\mathbf{u}_k, p_k)$ *satisfies* (3.10) *with* $\mathbf{f}_k = \frac{\lambda_k \nu_k^2}{\nu^2}\mathbf{f}$, $\mathbf{a}_k = \frac{\lambda_k \nu_k}{\nu}\mathbf{a}$, *and*

$$\|\nabla \mathbf{u}_k\|_{L^2(\Omega)} \to 1, \quad \mathbf{u}_k \rightharpoonup \mathbf{v} \ \ in \ \ W^{1,2}(\Omega), \quad p_k \rightharpoonup p \ \ in \ \ W^{1,3/2}(\Omega). \tag{4.1}$$

*Moreover,* $\mathbf{u}_k \in W^{3,2}_{\mathrm{loc}}(\Omega)$ *and* $p_k \in W^{2,2}_{\mathrm{loc}}(\Omega)$.

As in the previous section, in order to prove existence Theorem 4.1, we need to show that conditions (E-NS-AX) lead to a contradiction.

Assume that

$$\Gamma_j \cap O_{x_3} \neq \emptyset, \quad j = 0, \dots, M',$$

$$\Gamma_j \cap O_{x_3} = \emptyset, \quad j = M'+1, \dots, N.$$

Let $P_+ = \{(0, x_2, x_3) : x_2 > 0,\ x_3 \in \mathbb{R}\}$, $\mathcal{D} = \Omega \cap P_+$. Obviously, on $P_+$ the coordinates $x_2, x_3$ coincide with the coordinates $r, z$.

For a set $A \subset \mathbb{R}^3$, put $\breve{A} := A \cap P_+$, and for $B \subset P_+$, denote by $\widetilde{B}$ the set in $\mathbb{R}^3$ obtained by rotation of $B$ around $O_z$-axis.

One can easily see that

(S$_1$) $\mathcal{D}$ is a bounded plane domain with Lipschitz boundary. Moreover, $\breve{\Gamma}_j$ ($\breve{A}$ is defined just above) is a connected set for every $j = 0, \dots, N$. In other words, $\{\breve{\Gamma}_j : j = 0, \dots, N\}$ coincides with the family of all connected components of the set $P_+ \cap \partial\mathcal{D}$.

Hence, $\mathbf{v}$ and $p$ satisfy the following system in the plane domain $\mathcal{D}$:

$$\begin{cases} \dfrac{\partial p}{\partial r} - \dfrac{(v_\theta)^2}{r} + v_r \dfrac{\partial v_r}{\partial r} + v_z \dfrac{\partial v_r}{\partial z} & = 0, \\ \dfrac{\partial p}{\partial z} + v_r \dfrac{\partial v_z}{\partial r} + v_z \dfrac{\partial v_z}{\partial z} & = 0, \\ \dfrac{v_\theta v_r}{r} + v_r \dfrac{\partial v_\theta}{\partial r} + v_z \dfrac{\partial v_\theta}{\partial z} & = 0, \\ \dfrac{\partial (r v_r)}{\partial r} + \dfrac{\partial (r v_z)}{\partial z} & = 0 \end{cases} \tag{4.2}$$

(these equations are satisfied for almost all $x \in \mathcal{D}$) and

$$\mathbf{v}(x) = 0 \quad \text{for } \mathfrak{H}^1\text{-almost all } x \in P_+ \cap \partial\mathcal{D}. \tag{4.3}$$

We have the following integral estimates: $\mathbf{v} \in W^{1,2}_{\mathrm{loc}}(\mathcal{D})$,

$$\int_{\mathcal{D}} r|\nabla \mathbf{v}(r,z)|^2\,drdz < \infty \tag{4.4}$$

and, by the Sobolev embedding theorem for three-dimensional domains, $\mathbf{v} \in L^6(\Omega)$, i.e.,

$$\int_{\mathcal{D}} r|\mathbf{v}(r,z)|^6\,drdz < \infty. \tag{4.5}$$

Also, the condition $\nabla p \in L^{3/2}(\Omega)$ can be written as

$$\int_{\mathcal{D}} r|\nabla p(r,z)|^{3/2}\,drdz < \infty. \tag{4.6}$$

4.1. *Some previous results on Euler equations.* The next statement was proved in [13, Lemma 4] and in [1, Th. 2.2].

THEOREM 4.2. *If conditions* (E-AX) *are satisfied, then*

$$\forall j \in \{0,1,\dots,N\}\ \exists\,\widehat{p}_j \in \mathbb{R}: \quad p(x) \equiv \widehat{p}_j \quad \textit{for } \mathfrak{H}^2\textit{-almost all } x \in \Gamma_j. \tag{4.7}$$

*In particular, by axial symmetry,*

$$p(x) \equiv \widehat{p}_j \quad \textit{for } \mathfrak{H}^1\textit{-almost all } x \in \breve{\Gamma}_j. \tag{4.8}$$

The following result was obtained in [15].

THEOREM 4.3. *If conditions* (E-AX) *are satisfied, then* $\widehat{p}_0 = \dots = \widehat{p}_{M'}$, *where* $\widehat{p}_j$ *are the constants from Theorem* 4.2.

We need a weak version of Bernoulli's law for a Sobolev solution $(\mathbf{v}, p)$ to the Euler equations (4.2) (see Theorem 4.4 below).

From the last equality in (4.2) and from (4.4) it follows that there exists a stream function $\psi \in W^{2,2}_{\mathrm{loc}}(\mathcal{D})$ such that

$$\frac{\partial \psi}{\partial r} = -rv_z, \quad \frac{\partial \psi}{\partial z} = rv_r. \tag{4.9}$$

Fix a point $x_* \in \mathcal{D}$. For $\varepsilon > 0$, denote by $\mathcal{D}_\varepsilon$ the connected component of $\mathcal{D} \cap \{(r,z) : r > \varepsilon\}$ containing $x_*$. Since

$$\psi \in W^{2,2}(\mathcal{D}_\varepsilon) \quad \forall \varepsilon > 0, \tag{4.10}$$

by Sobolev embedding theorem, $\psi \in C(\bar{\mathcal{D}}_\varepsilon)$. Hence $\psi$ is continuous at points of $\bar{\mathcal{D}} \setminus O_z = \bar{\mathcal{D}} \setminus \{(0,z) : z \in \mathbb{R}\}$.

LEMMA 4.2 (cf. Lemma 3.3). *If conditions* (E-AX) *are satisfied, then there exist constants* $\xi_0, \dots, \xi_N \in \mathbb{R}$ *such that* $\psi(x) \equiv \xi_j$ *on each curve* $\breve{\Gamma}_j$, $j = 0, \dots, N$.

*Proof.* In virtue of (4.3) and (4.9), we have $\nabla\psi(x) = 0$ for $\mathfrak{H}^1$-almost all $x \in \partial\mathcal{D} \setminus O_z$. Then the Morse-Sard property (see Theorem 2.1) implies that

$$\text{for any connected set } C \subset \partial\mathcal{D} \setminus O_z,\ \exists\, \alpha = \alpha(C) \in \mathbb{R} : \psi(x) \equiv \alpha\ \forall x \in C.$$

Hence, since $\breve{\Gamma}_j$ are connected (see $(\mathrm{S}_1)$), the lemma follows. □

Denote by $\Phi = p + \dfrac{|\mathbf{v}|^2}{2}$ the total head pressure corresponding to the solution $(\mathbf{v}, p)$. Obviously,

$$\int\limits_{\mathcal{D}} r|\nabla\Phi(r,z)|^{3/2}\, drdz < \infty. \tag{4.11}$$

Hence,

$$\Phi \in W^{1,3/2}(\mathcal{D}_\varepsilon) \quad \forall \varepsilon > 0. \tag{4.12}$$

Applying Lemmas 2.1, 2.2, and Remark 2.2 to the functions $\mathbf{v}, \psi, \Phi$, we get the following

LEMMA 4.3. *If conditions* (E-AX) *hold, then there exists a set* $A_{\mathbf{v}} \subset \overline{\mathcal{D}}$ *such that*

(i) $\mathfrak{H}^1(A_{\mathbf{v}}) = 0$.
(ii) *For all* $x = (r,z) \in \mathcal{D} \setminus A_{\mathbf{v}}$,

$$\lim_{\rho\to 0} ⨍_{B_\rho(x)} |\mathbf{v}(y) - \mathbf{v}(x)|^2 dy = \lim_{\rho\to 0} ⨍_{B_\rho(x)} |\Phi(y) - \Phi(x)|^2 dy = 0;$$

*moreover, the function* $\psi$ *is differentiable at* $x$, *and*

$$\nabla\psi(x) = (-rv_z(x), rv_r(x)).$$

(iii) *For every* $\varepsilon > 0$, *there exists a set* $U \subset \mathbb{R}^2$ *with* $\mathfrak{H}^1_\infty(U) < \varepsilon$, $A_{\mathbf{v}} \subset U$, *and such that the functions* $\mathbf{v}, \Phi$ *are continuous on* $\overline{\mathcal{D}} \setminus (U \cup O_z)$.

The next two results were obtained in [15].

THEOREM 4.4 (Bernoulli's Law). *Let conditions* (E-AX) *be valid, and let* $A_{\mathbf{v}}$ *be a set from Lemma 4.3. For any compact connected set* $K \subset \bar{\mathcal{D}} \setminus O_z$, *the following property holds: if*

$$\psi\Big|_K = \text{const}, \tag{4.13}$$

*then*

$$\Phi(x_1) = \Phi(x_2) \quad \textit{for all } x_1, x_2 \in K \setminus A_{\mathbf{v}}. \tag{4.14}$$

We also need the following assertion from [15] concerning the behavior of the total head pressure near the singularity axis $O_z$.

LEMMA 4.4. *Assume that conditions* (E-AX) *are satisfied. Let* $K_i$ *be a sequence of compact sets with the following properties:* $K_i \subset \bar{\mathcal{D}} \setminus O_z$, $\psi|_{K_i} = \text{const}$, *and* $\lim_{i\to\infty} \inf_{(r,z)\in K_i} r = 0$, $\varliminf_{i\to\infty} \sup_{(r,z)\in K_i} r > 0$. *Then* $\Phi(K_i) \to \widehat{p}_0$ *as* $i \to \infty$.

Here we denote by $\Phi(K_i)$ the corresponding constant $c_i \in \mathbb{R}$ such that $\Phi(x) = c_i$ for all $x \in K_i \setminus A_{\mathbf{v}}$ (see Theorem 4.4).

4.2. *Obtaining a contradiction.* We consider three possible cases.

(a) The maximum of $\Phi$ is attained on the boundary component intersecting the symmetry axis:

$$\widehat{p}_0 = \max_{j=0,\dots,N} \widehat{p}_j = \operatorname*{ess\,sup}_{x\in\Omega} \Phi(x). \tag{4.15}$$

(b) The maximum of $\Phi$ is attained on a boundary component that does not intersect the symmetry axis:

$$\widehat{p}_0 < \widehat{p}_N = \max_{j=0,\dots,N} \widehat{p}_j = \operatorname*{ess\,sup}_{x\in\Omega} \Phi(x). \tag{4.16}$$

(c) The maximum of $\Phi$ is not attained on $\partial\Omega$:

$$\max_{j=0,\dots,N} \widehat{p}_j < \operatorname*{ess\,sup}_{x\in\Omega} \Phi(x). \tag{4.17}$$

4.2.1. *The case* $\operatorname*{ess\,sup}_{x\in\Omega} \Phi(x) = \widehat{p}_0$. Let us consider case (4.15). Adding a constant to the pressure $p$, we can assume, without loss of generality, that

$$\widehat{p}_0 = \operatorname*{ess\,sup}_{x\in\Omega} \Phi(x) = 0. \tag{4.18}$$

Since the identity $\widehat{p}_0 = \widehat{p}_1 = \cdots = \widehat{p}_N$ is impossible (see Corollary 3.1, which is valid also for the axial-symmetric case), we have that $\widehat{p}_j < 0$ for some $j \in \{M'+1, \dots, N\}$. (Recall that by Theorem 4.3, $\widehat{p}_0 = \cdots = \widehat{p}_{M'} = 0$.)

Now, we receive a contradiction following the arguments of [15], [16]. For the reader's convenience, we recall these arguments. From equation ($3.11_1$) we obtain the identity

$$\begin{aligned} 0 &= x \cdot \nabla p(x) + x \cdot \big(\mathbf{v}(x) \cdot \nabla\big)\mathbf{v}(x) \\ &= \operatorname{div}\Big[x\,p(x) + \big(\mathbf{v}(x)\cdot x\big)\mathbf{v}(x)\Big] - p(x)\,\operatorname{div} x - |\mathbf{v}(x)|^2 \\ &= \operatorname{div}\Big[x\,p(x) + \big(\mathbf{v}(x)\cdot x\big)\mathbf{v}(x)\Big] - 3\Phi(x) + \frac{1}{2}|\mathbf{v}(x)|^2. \end{aligned} \tag{4.19}$$

Integrating it over $\Omega$ and using (4.18), we derive

$$0 \geq \int\limits_{\Omega} \Big[3\Phi(x) - \frac{1}{2}|\mathbf{v}(x)|^2\Big]\,dx = \int\limits_{\partial\Omega} p(x)\big(x\cdot\mathbf{n}\big)\,ds = \sum_{j=0}^{N} \widehat{p}_j \int\limits_{\Gamma_j} \big(x\cdot\mathbf{n}\big)\,ds$$
$$= -\sum_{j=1}^{N} \widehat{p}_j \int\limits_{\Omega_j} \operatorname{div} x\,dx = -3\sum_{j=1}^{N} \widehat{p}_j|\Omega_j| > 0.$$

The obtained contradiction finishes the proof for case (4.15).

4.2.2. *The case* $\widehat{p}_0 < \widehat{p}_N = \operatorname*{ess\,sup}_{x\in\bar\Omega} \Phi(x)$. Suppose that (4.16) holds. We may assume, without loss of generality, that the maximum value is zero; i.e.,

$$\widehat{p}_0 < \widehat{p}_N = \max_{j=0,\dots,N} \widehat{p}_j = \operatorname*{ess\,sup}_{x\in\bar\Omega} \Phi(x) = 0. \tag{4.20}$$

From Theorem 4.3, we have

$$\widehat{p}_0 = \cdots = \widehat{p}_{M'} < 0. \tag{4.21}$$

Change (if necessary) the numbering of the boundary components $\Gamma_{M'+1}, \dots, \Gamma_{N-1}$ so that

$$\widehat{p}_j < 0, \quad j = 0, \dots, M, \quad M \geq M', \tag{4.22}$$

$$\widehat{p}_{M+1} = \cdots = \widehat{p}_N = 0. \tag{4.23}$$

The first goal is to remove a neighborhood of the singularity line $O_z$ from our considerations. Then we can reduce the proof to the plane case considered in Section 3.3.1.

Take $r_0 > 0$ such that the open set $\mathcal{D}_\varepsilon = \{(r,z) \in \mathcal{D} : r > \varepsilon\}$ is connected for every $\varepsilon \leq r_0$ (i.e., $\mathcal{D}_\varepsilon$ is a domain), and

$$\begin{aligned} &\breve\Gamma_j \subset \overline{D}_{r_0} \quad \text{and} \quad \inf_{(r,z)\in\breve\Gamma_j} r \geq 2r_0, \quad j = M'+1, \dots, N,\\ &\breve\Gamma_j \cap \overline{D}_\varepsilon \text{ is a connected set}\\ &\text{and} \quad \sup_{(r,z)\in\breve\Gamma_j\cap\overline{D}_\varepsilon} r \geq 2r_0, \ \ j = 0, \dots, M', \ \varepsilon \in (0, r_0]. \end{aligned} \tag{4.24}$$

Let a set $C \subset \overline{\mathcal{D}}_\varepsilon$ separate $\breve\Gamma_i$ and $\breve\Gamma_j$ in $\mathcal{D}_\varepsilon$ for some different indexes $i, j \in \{0, \dots, N\}$; i.e., $\breve\Gamma_i \cap \overline{\mathcal{D}}_\varepsilon$ and $\breve\Gamma_j \cap \overline{\mathcal{D}}_\varepsilon$ lie in different connected components of $\overline{\mathcal{D}}_\varepsilon \setminus C$. Obviously, for $\varepsilon \in (0, r_0]$, there exists a constant $\delta(\varepsilon) > 0$ (not depending on $i, j, C$) such that the uniform estimate $\sup_{(r,z)\in C} r \geq \delta(\varepsilon)$ holds (see Figure 2). Moreover, the function $\delta(\varepsilon)$ is nondecreasing. In particular,

$$\delta(\varepsilon) \geq \delta(r_0), \quad \varepsilon \in (0, r_0]. \tag{4.25}$$

By Remark 2.4 and Lemma 4.2, we can apply Kronrod's results to the stream function $\psi|_{\bar{\mathcal{D}}_\varepsilon}$, $\varepsilon \in (0, r_0]$. Accordingly, $T_{\psi,\varepsilon}$ means the corresponding

Kronrod tree for the restriction $\psi|_{\bar{\mathcal{D}}_\varepsilon}$. Define the total head pressure on $T_{\psi,\varepsilon}$ as we did in Section 3.3.1. Then the following analog of Lemma 3.5 holds.

LEMMA 4.5. *Let $A, B \in T_{\psi,\varepsilon}$, where $\varepsilon \in (0, r_0]$, $\operatorname{diam} A > 0$, and $\operatorname{diam} B > 0$. Consider the corresponding arc $[A, B] \subset T_{\psi,\varepsilon}$ joining $A$ to $B$ (see Lemmas 2.3 and 2.4). Then the restriction $\Phi|_{[A,B]}$ is a continuous function.*

The lemma is proved using the argument of Lemma 3.5 and taking into account the above definitions, Theorem 4.4, and the continuity properties of $\Phi$; see Lemma 4.3(iii).

Denote by $B_0^\varepsilon, \ldots, B_N^\varepsilon$ the elements of $T_{\psi,\varepsilon}$ such that $B_j^\varepsilon \supset \breve{\Gamma}_j \cap \bar{\mathcal{D}}_\varepsilon$, $j = 0, \ldots, M'$, and $B_j^\varepsilon \supset \breve{\Gamma}_j$, $j = M'+1, \ldots, N$. By construction, $\Phi(B_j^\varepsilon) < 0$ for $j = 0, \ldots, M$, and $\Phi(B_j^\varepsilon) = 0$ for $j = M+1, \ldots, N$. For $r > 0$, let $L_r$ be the horizontal straight line $L_r = \{(r, z) : z \in \mathbb{R}\}$. We have

LEMMA 4.6. *There exist $r_* \in (0, r_0]$ and $C_j \in [B_j^{r_*}, B_N^{r_*}]$, $j = 0, \ldots, M$, such that $\Phi(C_j) < 0$ and $C \cap L_{r_*} = \emptyset$ for all $C \in [C_j, B_N^{r_*}]$.*

*Proof.* Suppose that the lemma fails for some $j = 0, \ldots, M$. Then it is easy to construct $r_i \to 0$ and $C^i \in [B_j^{r_i}, B_N^{r_i}]$ such that $C^i \cap L_{r_i} \neq \emptyset$ and $\Phi(C^i) \to 0$. Since by (4.22), $\widehat{p}_0 < 0$, we have $\Phi(C^i) \nrightarrow \widehat{p}_0$. By (4.25), $\sup\limits_{(r,z)\in C^i} r \geq \delta(r_0)$. Thus, we have a contradiction with Lemma 4.4, and the result is proved. □

Lemma 4.6 allows us to remove a neighborhood of the singularity line $O_z$ from our argument. Thus, we can apply the approach developed in Section 3.3.1 for the plane case. Put, for simplicity, $T_\psi = T_{\psi,r_*}$ and $B_j = B_j^{r_*}$. Since $\partial\mathcal{D}_{r_*} \subset B_0 \cup \cdots \cup B_N \cup L_{r_*}$ and the set $\{B_0, \ldots, B_N\} \subset T_\psi$ is finite, we can change $C_j$ (if necessary) so that the assertion of Lemma 4.6 takes the following stronger form:

$$\forall j = 0, \ldots, M \quad C_j \in [B_j, B_N], \quad \Phi(C_j) < 0, \tag{4.26}$$

and

$$C \cap \partial\mathcal{D}_{r_*} = \emptyset \quad \forall C \in [C_j, B_N). \tag{4.27}$$

Observe that $\Gamma_j \cap L_{r_*} \neq \emptyset$ for $j = 0, \ldots, M'$. Therefore, if a cycle $C \in T_\psi$ separates $\Gamma_N$ from $\Gamma_0$ and $C \cap \partial\mathcal{D}_{r_*} = \emptyset$, then $C$ separates $\Gamma_N$ from $\Gamma_j$ for all $j = 1, \ldots, M'$. So we can take $C_0 = \cdots = C_{M'}$ (see Figure 2) and consider only the Kronrod arcs $[C_{M'}, B_N]$, …, $[C_M, B_N]$.

Recall that a set $\mathcal{Z} \subset T_\psi$ has $T$-measure zero if $\mathfrak{H}^1(\{\psi(C) : C \in \mathcal{Z}\}) = 0$.

LEMMA 4.7. *For every $j = M', \ldots, M$, $T$-almost all $C \in [C_j, B_N]$ are $C^1$-curves homeomorphic to the circle. Moreover, there exists a subsequence $\Phi_{k_l}$ such that the sequence $\Phi_{k_l}|_C$ converges to $\Phi|_C$ uniformly $\Phi_k|_C \rightrightarrows \Phi|_C$ on $T$-almost all cycles $C \in [C_j, B_N]$.*

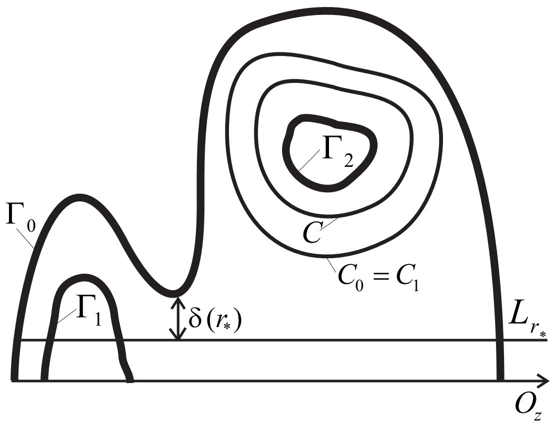


Figure 2. The domain $\mathcal{D}$ for the case $M' = M = 1, N = 2$.

The first assertion of the lemma follows from Theorem 2.1(iv) and (4.27). The validity of the second one for $T$-almost all $C \in [C_j, B_N]$ was proved in [17, Lemma 3.3].

Below we assume (without loss of generality) that the subsequence $\Phi_{k_l}$ coincides with $\Phi_k$. Besides, cycles satisfying the assertion of Lemma 4.7 will be called *regular cycles*.

From Lemmas 4.7 and 3.6 (which is also valid for the axially symmetric case), we obtain

Corollary 4.1. *For each $j = M', \dots, M$, we have*

$$\mathfrak{H}^1\big(\{\Phi(C) : C \in [C_j, B_N] \text{ and } C \text{ is not a regular cycle}\}\big) = 0.$$

As in the plane case (see Section 3.3.1), we can take a sequence of positive values $t_i$ with $t_{i+1} = \frac{1}{2}t_i$, the corresponding regular cycles $A_i^j \in [C_j, B_N]$ with $\Phi(A_i^j) = -t_i$, and the sequence of domains $V_i \subset \mathcal{D}_{r_*}$ with

$$\partial V_i = A_i^{M'} \cup \dots \cup A_i^M \cup \breve{\Gamma}_K \cup \dots \cup \breve{\Gamma}_N, \tag{4.28}$$

where $K \geq M+1$ is independent of $i$.

By the definition of regular cycles, we have again estimates (3.32) and (3.33) for $k \geq k_i$. Accordingly, for $k \geq k_i$ and $t \in [\frac{5}{8}t_i, \frac{7}{8}t_i]$, we can define $W_{ik}^j(t)$ as the connected component of the open set $\{x \in V_i \setminus \overline{V}_{i+1} : \Phi_k(x) > -t\}$ with $\partial W_{ik}^j(t) \supset A_{i+1}^j$ and put

$$W_{ik}(t) = \bigcup_{j=M'}^{M} W_{ik}^j(t), \qquad S_{ik}(t) = (\partial W_{ik}(t)) \cap V_i \setminus \overline{V}_{i+1}.$$

By construction,

$$\partial W_{ik}(t) = S_{ik}(t) \cup A_{i+1}^{M'} \cup \dots \cup A_{i+1}^{M},$$

and the set $S_{ik}(t)$ separates $A_i^{M'} \cup \dots \cup A_i^M$ from $A_{i+1}^{M'} \cup \dots \cup A_{i+1}^M$. Since $\Phi_k \in W^{2,2}_{\mathrm{loc}}(\Omega)$ (see (E-NS-AX)), by the Morse-Sard theorem (see Theorem 2.1), for almost all $t \in [\frac{5}{8}t_i, \frac{7}{8}t_i]$, the level set $S_{ik}(t)$ consists of finitely many $C^1$-cycles and $\Phi_k$ is differentiable (in classical sense) at every point $x \in S_{ik}(t)$ with $\nabla\Phi_k(x) \neq 0$. Therefore, $\widetilde{S}_{ik}(t)$ is a finite union of smooth surfaces (tori), and by construction,

$$\int\limits_{\widetilde{S}_{ik}(t)} \nabla\Phi_k \cdot \mathbf{n}\, dS = -\int\limits_{\widetilde{S}_{ik}(t)} |\nabla\Phi_k|\, dS < 0, \tag{4.29}$$

where $\mathbf{n}$ is the unit outward normal vector to $\partial\widetilde{W}_{ik}(t)$. (Recall that for a set $B \subset P_+$, we denote by $\widetilde{B}$ the set in $\mathbb{R}^3$ obtaining by rotation of $B$ around $O_z$-axis.)

For $h > 0$, denote $\Gamma_h = \{x \in \Omega : \mathrm{dist}(x, \Gamma_K \cup \dots \cup \Gamma_N) = h)\}$, $\Omega_h = \{x \in \Omega : \mathrm{dist}(x, \Gamma_K \cup \dots \cup \Gamma_N) < h)\}$. Since the distance function $\mathrm{dist}(x, \partial\Omega)$ is $C^1$-regular and the norm of its gradient is equal to one in the neighborhood of $\partial\Omega$, there is a constant $\delta_0 > 0$ such that for every $h \leq \delta_0$, the set $\Gamma_h$ is a union of $N - K + 1$ $C^1$-smooth surfaces homeomorphic to the torus, and

$$\mathfrak{H}^2(\Gamma_h) \leq c_0 \quad \forall h \in (0, \delta_0], \tag{4.30}$$

where the constant $c_0 = 3\mathfrak{H}^2(\Gamma_K \cup \dots \cup \Gamma_N)$ is independent of $h$.

By a direct calculation, (4.2) implies

$$\nabla\Phi = \mathbf{v} \times \boldsymbol{\omega} \quad \text{ in } \Omega, \tag{4.31}$$

where $\boldsymbol{\omega} = \mathrm{curl}\, \mathbf{v}$; i.e.,

$$\boldsymbol{\omega} = (\omega_r, \omega_\theta, \omega_z) = \Big(-\frac{\partial v_\theta}{\partial z}, \ \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}, \ \frac{v_\theta}{r} + \frac{\partial v_\theta}{\partial r}\Big).$$

Set $\boldsymbol{\omega}_k = \mathrm{curl}\, \mathbf{u}_k$, $\omega(x) = |\boldsymbol{\omega}(x)|$, $\omega_k(x) = |\boldsymbol{\omega}_k(x)|$. Since $\Phi \neq \mathrm{const}$ on $V_i$, (4.31) implies $\int_{\widetilde{V}_i} \omega^2\, dx > 0$ for every $i$. Hence, from the weak convergence $\boldsymbol{\omega}_k \rightharpoonup \boldsymbol{\omega}$ in $L^2(\Omega)$ it follows

Lemma 4.8. *For any $i \in \mathbb{N}$, there exist constants $\varepsilon_i > 0$, $\delta_i \in (0, \delta_0)$ and $k_i' \in \mathbb{N}$ such that*

$$\overline{\Omega}_{\delta_i} \cap A_i^j = \overline{\Omega}_{\delta_i} \cap A_{i+1}^j = \emptyset, \quad j = M', \dots, M,$$

*and* $\displaystyle\int\limits_{\widetilde{V}_{i+1} \setminus \Omega_{\delta_i}} \omega_k^2\, dx > \varepsilon_i$ *for all* $k \geq k_i'$.

Now we are ready to prove the key estimate.

LEMMA 4.9. *For any $i \in \mathbb{N}$, there exists $k(i) \in \mathbb{N}$ such that for every $k \geq k(i)$ and for almost all $t \in [\frac{5}{8}t_i, \frac{7}{8}t_i]$, the inequality*

$$\int\limits_{\widetilde{S}_{ik}(t)} |\nabla \Phi_k|\, dS < \mathcal{F}t, \tag{4.32}$$

*holds with the constant $\mathcal{F}$ independent of $t, k$ and $i$.*

*Proof.* Since the proof of this lemma is similar to that of Lemma 3.8 for the plane case, we comment only some key steps.

Fix $i \in \mathbb{N}$. Below we always assume that $k \geq k_i$ (see (3.32)). Since we have removed a neighborhood of the singularity line $O_z$, we can use the Sobolev embedding theorem in the plane domain $\mathcal{D}_{r_*}$. In particular, from the uniform estimate $\|\Phi_k\|_{W^{1,3/2}(\mathcal{D}_{r_*})} \leq$ const, we deduce that the norms $\|\Phi_k\|_{L^6(\mathcal{D}_{r_*})}$ are uniformly bounded. Consequently, by the Hölder inequality, $\|\Phi_k \nabla \Phi_k\|_{L^{6/5}(\mathcal{D}_{r_*})} \leq$ const, and this implies

$$\|\Phi_k \nabla \Phi_k\|_{L^{6/5}(\widetilde{\mathcal{D}}_{r_*})} \leq \text{const}. \tag{4.33}$$

Fix a sufficiently small $\sigma > 0$ (the exact value of $\sigma$ will be specified below), and take the parameter $\delta_\sigma \in (0, \delta_i]$ (see Lemma 4.8) small enough to satisfy the following conditions:

$$\Omega_{\delta_\sigma} \cap \widetilde{A}_i^j = \Omega_{\delta_\sigma} \cap \widetilde{A}_{i+1}^j = \emptyset, \quad j = M', \dots, M, \tag{4.34}$$

$$\int\limits_{\Gamma_h} \Phi_k^2\, dS < \sigma^2 \quad \forall h \in (0, \delta_\sigma] \ \ \forall k \geq k'. \tag{4.35}$$

(The last estimate follows from the identity $\Phi|_{\Gamma_K \cup \dots \cup \Gamma_N} \equiv 0$, the weak convergence $\Phi_k \rightharpoonup \Phi$ in the space $W^{1,3/2}(\Omega)$, and (4.33).)

By a direct calculation, (3.10) implies

$$\nabla \Phi_k = -\nu_k \text{curl}\, \boldsymbol{\omega}_k + \boldsymbol{\omega}_k \times \mathbf{u}_k + \mathbf{f}_k = -\nu_k \text{curl}\, \boldsymbol{\omega}_k + \boldsymbol{\omega}_k \times \mathbf{u}_k + \frac{\lambda_k \nu_k^2}{\nu^2} \text{curl}\, \mathbf{b}.$$

By the Stokes theorem, for any $C^1$-smooth closed surface $S \subset \Omega$ and $\mathbf{g} \in W^{2,2}(\Omega)$, we have

$$\int\limits_S \text{curl}\, \mathbf{g} \cdot \mathbf{n}\, dS = 0.$$

So, in particular,

$$\int\limits_S \nabla \Phi_k \cdot \mathbf{n}\, dS = \int\limits_S (\boldsymbol{\omega}_k \times \mathbf{u}_k) \cdot \mathbf{n}\, dS.$$

Now, fix a sufficiently small $\varepsilon > 0$. (The exact value of $\varepsilon$ will be specified below.) For a given sufficiently large $k \geq k'$, we make a special procedure to find a number $\bar{h}_k \in (0, \delta_\sigma)$ such that the estimates

$$\left| \int_{\Gamma_{\bar{h}_k}} \nabla \Phi_k \cdot \mathbf{n} \, dS \right| \leq 2 \int_{\Gamma_{\bar{h}_k}} |\mathbf{u}_k| \cdot |\nabla \mathbf{u}_k| \, dS < \varepsilon, \tag{4.36}$$

$$\int_{\Gamma_{\bar{h}_k}} |\mathbf{u}_k|^2 \, dS \leq C_\varepsilon \nu_k^2 \tag{4.37}$$

hold, where $C_\varepsilon$ is independent of $k$ and $\sigma$. This procedure exactly repeats the argument lines of the proof of Lemma 3.8.

The final part of the proof is identical to that of Lemma 3.8. We have to integrate formula (3.48) (which is valid for the axially symmetric case as well) over the three-dimensional domain $\Omega_{i\bar{h}_k}(t)$ with $\partial\Omega_{i\bar{h}_k}(t) = \Gamma_{\bar{h}_k} \cup \widetilde{S}_{ik}(t)$. This means that we have only to replace the curves $S_{ik}(t)$ by the surfaces $\widetilde{S}_{ik}(t)$ in the corresponding integrals. □

Now, we obtain a contradiction by repeating word-by-word the proof of Lemma 3.9 and replacing the one-dimensional Hausdorff measure by the two-dimensional one, and the curves $S_{ik}(t)$ by the surfaces $\widetilde{S}_{ik}(t)$ in the corresponding integrals.

4.2.3. *The case* $\operatorname{ess\,sup}_{x\in\Omega} \Phi(x) > \max_{j=0,\ldots,N} \widehat{p}_j$. Assume that (4.17) is satisfied, and set $\sigma = \max_{j=0,\ldots,N} \widehat{p}_j$. Then, as in the proof of Lemma 3.10, we can find a compact connected set $F \subset \mathcal{D} \setminus A_{\mathbf{v}}$ such that $\operatorname{diam}(F) > 0$, $\psi|_F = \text{const}$, and $\Phi(F) > \sigma$. Without loss of generality, we may assume that $\sigma < 0$ and $\Phi(F) = 0$. Since now it is more difficult to separate $F$ from $\partial\mathcal{D}$ by regular cycles (than in Lemma 3.10), we have to apply the method of Section 4.2.2. Namely, take a number $r_0 > 0$ such that $F \subset D_{r_0}$, the open set $\mathcal{D}_\varepsilon = \{(r, z) \in \mathcal{D} : r > \varepsilon\}$ is connected for every $\varepsilon \leq r_0$, and conditions (4.24) are satisfied. Then for $\varepsilon \in (0, r_0]$, we can consider the behavior of $\Phi$ on the Kronrod trees $T_{\psi,\varepsilon}$ corresponding to the restrictions $\psi|_{\bar{\mathcal{D}}_\varepsilon}$. Denote by $F^\varepsilon$ the element of $T_{\psi,\varepsilon}$ containing $F$. Using the same procedure as in Section 4.2.2, we can find $r_* \in (0, r_0]$ such that the following lemma holds.

Lemma 4.10. *There exist* $C_j \in [B_j^{r_*}, F^{r_*}]$, $j = 0, \ldots, N$, *such that* $\Phi(C_j) < 0$ *and* $C \cap L_{r_*} = \emptyset$ *for all* $C \in [C_j, F^{r_*}]$.

Set $T_\psi = T_{\psi,r_*}$, $F^* = F^{r_*}$, and $B_j = B_j^{r_*}$, i.e., $B_j \in T_\psi$ and $B_j \supset \breve{\Gamma}_j \cap \overline{\mathcal{D}}_{r_*}$. As above, we can change $C_j$ (if necessary) so that Lemma 4.10 takes the following stronger form:

$$\forall j = 0, \ldots, N \quad C_j \in [B_j, F^*], \quad \Phi(C_j) < 0,$$
$$C \cap \partial\mathcal{D}_{r_*} = \emptyset \quad \forall C \in [C_j, F^*], \qquad \text{and} \qquad C_0 = \cdots = C_{M'}.$$

The rest of the procedure of obtaining a contradiction is done in the same way as in Section 3.3.2. Namely, we need to take positive numbers $t_i = 2^{-i}t_0$, regular cycles $A_i^j \in [C_j, F^*]$ with $\Phi(A_i^j) = -t_i$, and the set $S_{ik}(t)$ with $\Phi_k|_{S_{ik}(t)} \equiv -t$ separating $A_i^{M'} \cup \cdots \cup A_i^N$ from $A_{i+1}^{M'} \cup \cdots \cup A_{i+1}^N$, etc. The only difference is that we have to integrate identity (3.48) over the three-dimensional domains $\Omega_{ik}(t)$ with $\partial\Omega_{ik}(t) = \widetilde{S}_{ik}(t)$.

*Proof of Theorem* 4.1. Let the hypotheses of Theorem 4.1 be satisfied. Suppose that its assertion fails. Then by Lemma 4.1 there exist $\mathbf{v}, p$ and a sequence $(\mathbf{u}_k, p_k)$ satisfying (E-NS-AX). However, in Sections 4.2.1–4.2.3 we have shown that assumptions (E-NS-AX) lead to a contradiction in all possible cases (4.15)–(4.17). This finishes the proof of Theorem 4.1. □

*Remark* 4.1. In Lemma 4.1, let the data $\mathbf{f}$ and $\mathbf{a}$ be axially symmetric with no swirl. If the corresponding assertion of Theorem 4.1 fails, then it can be shown (see [15]) that conditions (E-NS-AX) are satisfied with $\mathbf{u}_k$ axially symmetric with no swirl as well. But since we have proved that assumptions (E-NS-AX) lead to a contradiction in the more general case with possible swirl, we get the validity of both assertions of Theorem 4.1.

*Remark* 4.2. It is well known (see [21]) that under the hypothesis of Theorem 4.1, every weak solution $\mathbf{u}$ of problem (1.1) is more regular; i.e, $\mathbf{u} \in W^{2,2}(\Omega) \cap W^{3,2}_{\mathrm{loc}}(\Omega)$.

*Acknowledgements.* We are deeply indebted to S. M. Nedogibchenko and V. V. Pukhnachev for valuable discussions. We are grateful to the referee for very careful reading of the manuscript and for useful remarks and suggestions.

The research of M. Korobkov was partially supported by the Russian Foundation for Basic Research (project No. 12-01-00390-a), by the Grant of the Russian Federation for the State Support of Researches (Agreement No. 14.B25.31.0029), and by the Grant of the President of Russia for support of young doctors of sciences (grant No. MD-5146.2013.1). The research of K. Pileckas was funded by the Lithuanian-Swiss cooperation programme to reduce economic and social disparities within the enlarged European Union under project agreement No. CH-3-ŠMM-01/01.

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(Received: January 30, 2013)
(Revised: January 16, 2014)

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