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Calabi flow, geodesic rays, and uniqueness of constant scalar curvature Kähler metrics

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Abstract

We prove that constant scalar curvature Kähler metric "adjacent" to a fixed Kähler class is unique up to isomorphism. The proof is based on the study of a fourth order evolution equation, namely, the *Calabi flow*, from a new geometric perspective, and on the geometry of the space of Kähler metrics.

1. **Introduction**

The Kempf-Ness theorem relates complex quotient to symplectic reduction. Suppose a compact connected group G acts on a compact Kähler manifold X. We assume the action preserves the Kähler structure, with a moment map $\mu : X \to \mathfrak{g}^*$. Then the action extends to a holomorphic action of the complexified group $G^{\mathbb{C}}$. Under proper hypothesis the notion of stability could be defined. Then the Kempf-Ness theorem says that as sets,

$$X^{\mathrm{ss}}/G^{\mathbb{C}} \simeq \mu^{-1}(0)//G.$$

To be more precise,

(1) A $G^{\mathbb{C}}$-orbit is poly-stable if and only if it contains a zero of the moment map. The zeroes within it form a unique G-orbit.
(2) A $G^{\mathbb{C}}$-orbit is semi-stable if and only if its closure contains a zero of the moment map. We call such a zero a *destabilizer* of the original $G^{\mathbb{C}}$-orbit. The destabilizers all lie in the unique poly-stable orbit in the closure of the original orbit.

In Kähler geometry, according to S. Donaldson [17] (see also [25]) the problem of finding cscK (constant scalar curvature Kähler) metrics formally fits into a similar picture. However the spaces involved are infinite dimensional. Given a compact Kähler manifold (M, ω, J), denote by $\mathcal{G}$ the group of Hamiltonian diffeomorphisms of (M, ω) and by $\mathcal{J}$ the space consisting of

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almost complex structures on M that are compatible with ω. $\mathcal{J}$ admits a natural Kähler structure which is invariant under the action of $\mathcal{G}$. The moment map is given by the Hermitian scalar curvature. The complexification of $\mathcal{G}$ may not exist since $\mathcal{G}$ is infinite dimensional. Nevertheless, it still makes sense talking about the orbits of $\mathcal{G}^{\mathbb{C}}$—it is simply the leaf of the foliation obtained by complexifying the infinitesimal actions of $\mathcal{G}$. Then the $\mathcal{G}^{\mathbb{C}}$ leaf of an integrable complex structure can be viewed as a principal $\mathcal{G}$-bundle over the Kähler class $[\omega]$. Thus an analogue to the Kempf-Ness theorem should relate the stability of the leaves to the existence of cscK metrics in the corresponding Kähler class. This was made more precise as the *Yau-Tian-Donaldson conjecture* (see [46]). The notion of "stability" in this case is the so-called "K-stability", see [47], [21]. There are also other related notion of stability; see, for example, [40], [38], etc.

Note that the Kempf-Ness theorem consists of both the existence and uniqueness part. It is known that the existence of cscK metrics implies various kinds of stability, however the converse is fairly difficult, due to the appearance of fourth order nonlinear partial differential equations. Recently Donaldson [22] proved a general result that the conjecture is true for toric surfaces. The uniqueness part corresponding to the poly-stable case is known by

THEOREM 1.1 (Donaldson [19], Chen-Tian [15]). *Constant scalar curvature Kähler metric in a fixed Kähler class, if it exists, is unique up to holomorphic isometry.*

Remark 1.2. When the manifold is Fano, the uniqueness of Kähler-Einstein metrics was previously proved by Bando-Mabuchi [2], and it was later generalized to the case of Kähler-Ricci solitons by Tian-Zhu [49]. The uniqueness of cscK metrics was first proved by the first author in the case when $c_1(X) \leq 0$ (cf. [8]).

The purpose of this paper is to prove the uniqueness in the semi-stable case.

THEOREM 1.3. *If there are two* cscK *structures* J_1 *and* J_2 *both lying in the* (C^∞) *closure of the* $\mathcal{G}^{\mathbb{C}}$ *leaf of a complex structure* $J \in \mathcal{J}^{\text{int}}$, *then there is a symplectic diffeomorphism* f *such that* $f^*J_1 = J_2$.

Defintion 1.4. Let (M, ω, J) be a Kähler manifold, and let $\mathcal{H}$ be the space of Kähler metrics in the Kähler class of ω. We say another Kähler structure (ω', J') on M is adjacent to $\mathcal{H}$ if there is a sequence of Kähler metrics $\omega_i \in \mathcal{H}$ and diffeomorphisms f_i of M such that

$$f_i^*\omega_i \to \omega', f_i^*J \to J'$$

in the C^∞ sense. So, in particular, the corresponding sequence of Riemannian metrics g_i converges to g' in the Cheeger-Gromov sense. Similarly, let (M, J)

be a Fano manifold. We say another complex structure J' on M is adjacent to J if there is a sequence of diffeomorphisms f_i such that

$$f_i^* J \to J'.$$

Remark 1.5. The above definition is related to the "jumping" phenomenon of complex structures; i.e., the space of isomorphism classes of complex structures on a fixed manifold is in general not Hausdorff. As a simple example, we can consider the blowup of $\mathbb{P}^2$ at three points p_1, p_2, and p_3. The underlying differential manifold is fixed, and a choice of the three points defines a complex structure. A choice of three points in a general position gives rise to the same complex structure, while a choice of three points on a line provides an example of an adjacent complex structure.

It follows from Theorem 1.3 that

THEOREM 1.6. *Let (M, ω, J) be a Kähler manifold. Assume $[\omega]$ is integral. Suppose there are two* csc *Kähler structures (ω_1, J_1) and (ω_2, J_2) both adjacent to the Kähler class of (ω, J); then they are isomorphic.*

COROLLARY 1.7. *Let (M, J) be a Fano manifold. Suppose there are two complex structures J_1 and J_2 both adjacent to J and both admitting Kähler-Einstein metrics; then (M, J_1) and (M, J_2) are bi-holomorphic.*

In terms of algebro-geometric language, Theorem 1.6 can be rephrased as (see also Question 9.1)

COROLLARY 1.8. *Let (M, L) be a polarized manifold. Suppose there are two test configurations $(\mathcal{M}_1, \mathcal{L}_1)$ and $(\mathcal{M}_2, \mathcal{L}_2)$ for (M, L) so that the central fibers are smooth and admit* cscK *metrics; then the two central fibers are isomorphic as polarized manifolds.*

Remark 1.9. After finishing this paper, we learned that our Theorem 1.6 and Corollary 1.7 partially confirmed a conjecture of G. Tian [48] in the case of constant scalar curvature Kähler metric.

Now we briefly outline the strategy of our proof of the main theorem. First, we will detect an adjacent cscK metric by *Calabi flow*. This is usually viewed as a fourth order parabolic equation on the space of Kähler potentials, and in general little is known about this. Our new perspective is to view the Calabi flow also as a gradient flow on the space $\mathcal{J}$ of almost complex structures. By proving a Łojasiewicz type inequality, we establish the global existence and convergence of the Calabi flow near a cscK metric. Suppose now we have two cscK metrics adjacent to a fixed Kähler class. Then there are two Calabi flows in the neighborhoods of the corresponding cscK metrics. Since the Calabi flow decreases geodesic distance, we get a bound on the two Calabi flows in terms of geodesic distance. It is not known whether this bound implies a C^0 bound automatically. Here we get around this difficulty by showing that in our case the Calabi flow is *asymptotic* to a smooth geodesic ray. This involves a local study of the infinite dimensional Hamiltonian action of $\mathcal{G}$, which is the main technical part of this paper. We are also lead to first look at the analogous finite dimensional problem, which concerns the asymptotic behavior of the Kempf-Ness flow and has an independent interest. Finally, using the theory of space of Kähler metrics, we derive a C^0 bound for the two parallel geodesic rays, which enables us to prove the main theorem.

The organization of this paper is as follows. In Section 2, we review Donaldson's infinite dimensional moment map picture in Kähler geometry and recall some known results for our later use. In Section 3, we state the Łojasiewicz inequality and "Łojasiewicz arguments" for the gradient flow of a real analytic function. In Section 4, we prove that in the finite dimensional case, the Kempf-Ness flow for a semi-stable point is asymptotic to a rational geodesic ray (optimal degeneration). In Section 5, we study stability of the Calabi flow near a cscK metric when the complex structure is deformed. In Section 6, we generalize the arguments in Section 4 to the infinite dimensional setting by considering the "reduced" Calabi flow. In Section 7, the relative C^0 bound for two smooth parallel geodesic rays tamed by bounded geometry is derived. In Section 8, we prove the main theorems. In Section 9, we shall discuss some further problems related to this study. The appendix contains the proof of some technical lemmas used in Sections 4 and 6.

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2. **The space of Kähler structures**

Here we review the infinite dimensional moment map picture discovered by Fujiki [25] and Donaldson [17]. Let (M, ω, J_0) be a compact Kähler manifold. Denote by $\mathcal{J}$ the space of almost complex structures on M that are compatible with ω and by $\mathcal{J}^{\text{int}}$ the subspace of $\mathcal{J}$ consisting of integrable almost complex structures compatible with ω. Then $\mathcal{J}$ is the space of smooth sections of an $\mathrm{Sp}(2n)/U(n)$ bundle over M, so it carries a natural Kähler structure. Indeed, there is a global holomorphic coordinate chart if we use the ball model of the Siegel upper half space in the usual way. J_0 determines a splitting $TM \otimes \mathbb{C} = T^{1,0} \oplus T^{0,1}$ such that ω induces a positive definite Hermitian inner product on $T^{1,0}$. Then $\mathcal{J}$ could be identified with the space

$$\Omega_S^{0,1}(T^{1,0}) = \{\mu \in \Omega^{0,1}(T^{1,0}) | \mathcal{A}(\mu) = 0, \mathrm{Id} - \bar{\mu} \circ \mu > 0\},$$

where $\mathcal{A}$ is the composition $\Omega^{0,p}(T^{1,0}) \to \Omega^{0,p}(T^{*0,1}) \to \Omega^{0,p+1}$. An element μ corresponds to an almost complex structure J whose corresponding $(1,0)$ tangent space consists of vectors of the form $X - \bar{\mu}(X)(X \in T^{1,0})$. Near J_0, $\mathcal{J}^{\mathrm{int}}$ is a subvariety of $\mathcal{J}$ cut out by quadratic equations:

$$N(\mu) = \bar{\partial}\mu + [\mu, \mu] = 0.$$

Denote by $\mathcal{G}$ the group of Hamiltonian diffeomorphisms of (M, ω). Its Lie algebra is $C_0^\infty(M; \mathbb{R})$. $\mathcal{G}$ will be the infinite dimensional analogue of a compact group, though the exponential map is not locally surjective for $\mathcal{G}$. $\mathcal{G}$ acts naturally on $\mathcal{J}$, keeping $\mathcal{J}^{\mathrm{int}}$ invariant. A. Fujiki [25] and S. Donaldson [17] independently discovered that the $\mathcal{G}$ action has a moment map given by the Hermitian scalar curvature functional $S - \underline{S}$[1], which can be viewed as an element in $(C_0^\infty(M; \mathbb{R}))^*$ through the L^2 inner product with respect to the measure $d\mu = \omega^n$. When J is integrable, $S(J)$ is simply the Riemannian scalar curvature of the Riemannian metric induced by ω and J. We say $J_0 \in \mathcal{J}$ is cscK if J_0 is integrable and (ω, J_0) has constant scalar curvature. So in the symplectic theory we are naturally lead to consider cscK metrics.

In the complex story, we need to look at $\mathcal{G}^{\mathbb{C}}$. Since $\mathcal{G}$ is infinite dimensional, there may not exist a genuine complexification $\mathcal{G}^{\mathbb{C}}$. Nevertheless, we can still define the $\mathcal{G}^{\mathbb{C}}$ leaf of an integral complex structure J_0 as follows. The infinitesimal action of $\mathcal{G}$ at a point $J \in \mathcal{J}$ is given by

$$\mathcal{D}_J : C_0^\infty(M; \mathbb{R}) \to \Omega_S^{0,1}(T^{1,0}); \phi \to \bar{\partial}_J X_\phi.$$

This operator can be naturally complexified to an operator from $C_0^\infty(M; \mathbb{C}) = C_0^\infty(M; \mathbb{R}) \oplus \sqrt{-1} C_0^\infty(M; \mathbb{R})$ to $\Omega_S^{0,1}(T^{1,0})$. Then a complex structure J is on the $\mathcal{G}^{\mathbb{C}}$ leaf of J_0 if there is a smooth path $J_t \in \mathcal{J}^{\mathrm{int}}$ such that $\dot{J}_t$ lies in the image of $\mathcal{D}_{J_t}$. $\mathcal{G}$ acts on the leaf naturally and the quotient is the space of Kähler metrics cohomologous to $[\omega]_{J_0}$. So the latter could be viewed as "$\mathcal{G}^{\mathbb{C}}/\mathcal{G}$". We define the space of Kähler potentials

$$\mathcal{H} = \{\phi \in C^\infty(M; \mathbb{R}) | \omega + \sqrt{-1}\partial\bar{\partial}\phi > 0\}.$$

[1]Here $\underline{S}$ is the average of scalar curvature, which indeed depends only on $[\omega]$ and $c_1(\omega)$, not on the choice of any compatible J.

Then $\mathcal{H}/\mathbb{R}$ is formally the "dual" symmetric space of $\mathcal{G}$. This was made more precise by Mabuchi [34], Semmes [41] and Donaldson [18]. Define a Weil-Petersson type Riemannian metric on $\mathcal{H}$ by

$$(\psi_1, \psi_2)_\phi = \int_M \psi_1 \psi_2 d\mu_\phi$$

for $\psi_1, \psi_2 \in T_\phi \mathcal{H}$. It can be shown that the Riemannian curvature tensor is co-variantly constant and the sectional curvature is nonpositive. A path $\phi(t)$ in $\mathcal{H}$ is a *geodesic* if it satisfies the equation

$$\ddot{\phi}(t) - |\nabla_{\phi(t)} \dot{\phi}(t)|^2_{\phi(t)} = 0.$$

The first named author [8] proved the existence of a unique $C^{1,1}$ geodesic segment connecting any two points in $\mathcal{H}$, and consequently that $\mathcal{H}$ is a metric space with the distance given by the length of the $C^{1,1}$ geodesics. It is proved in [4] that under this metric $\mathcal{H}$ is nonpositively curved in the sense of Alexanderov. So far the best regularity for the Dirichlet problems of the geodesic equation was obtained by Chen-Tian [15]. The initial value problem for the geodesic equation is in general not well posed. But by the nonpositiveness of the curvature of $\mathcal{H}$, there should be lots of geodesic rays in $\mathcal{H}$. In [9], the first author proved the following general theorem, which we shall use later:

THEOREM 2.1. *Given a smooth geodesic ray $\phi(t)$ in $\mathcal{H}$ that is tamed by a bounded geometry, there is a unique relative $C^{1,1}$ geodesic ray $\psi(t)$ emanating from any point ψ in $\mathcal{H}$ such that*

$$|\phi(t) - \psi(t)|_{C^{1,1}} \leq C.$$

Remark 2.2. For the precise definition of "tameness" we refer to [9]. But we point out that this is merely a technical condition imposed on the behavior of $\phi(t)$ at infinity so that the analysis on noncompact manifolds work. In our later applications where the geodesic ray $\phi(t)$ arises naturally from a test configuration with smooth total space, this assumption is always satisfied.

Defintion 2.3. Two geodesic rays $\phi(t)$ and $\psi(t)$ in $\mathcal{H}$ are said to be *parallel* if

$$d_{\mathcal{H}}(\phi(t), \psi(t)) \leq C.$$

Hence it is clear by definition that if $|\phi(t) - \psi(t)|_{C^0} \leq C$, then ϕ and ψ are parallel.

Analogous to the finite dimensional Kempf-Ness setting, there is a relevant functional E defined on $\mathcal{H}$, called the Mabuchi *K-energy*. It is the anti-derivative of the following closed one-form:

$$dE_\phi(\psi) = -\int_M (S(\phi) - \underline{S})\psi d\mu_\phi. \tag{1}$$

So the norm square of the gradient of E is the Calabi energy:

$$\mathrm{Ca}(\phi) = \int_M (S(\phi) - \underline{S})^2 d\mu_\phi.$$

By a direct calculation, along a smooth geodesic $\phi(t)$, we have

$$\frac{d^2}{dt^2}E(\phi(t)) = \int_M |\mathcal{D}_t\dot{\phi}(t)|^2 d\mu_{\phi(t)} \geq 0.$$

According to [7], E can be extended to a continuous function on all $C^{1,1}$ potentials in $\mathcal{H}$. However, it is not clear why E is still convex. The first author proved some weak versions of convexity. In the case when $[\omega]$ is integral, we gave simplified proofs in [14] using quantization (see also [3]). We recall them for our later purpose.

Lemma 2.4 ([9], [14]). *Given any ϕ_0, $\phi_1 \in \mathcal{H}$, we have*

$$E(\phi_1) - E(\phi_0) \leq \sqrt{\mathrm{Ca}(\phi_1)} \cdot d(\phi_0, \phi_1).$$

Lemma 2.5 ([9], [14]). *Given any ϕ_0, $\phi_1 \in \mathcal{H}$, let $\phi(t)$ be the $C^{1,1}$ geodesic connecting them. Then the derivatives of $E(\phi(t))$ at the end-points are well defined and they satisfy the following inequality*:

$$\frac{d}{dt}|_{t=0}E(\phi(t)) \leq \frac{d}{dt}|_{t=1}E(\phi(t)).$$

This lemma implies that

Lemma 2.6 ([4]). *The Calabi flow on $\mathcal{H}$ decreases geodesic distance.*

3. Łojasiewicz inequality

In this section we recall Łojasiewicz's theory for the structure of a real analytic function. The following fundamental structure theorem for real analytic functions is well known:

Theorem 3.1 (Łojasiewicz inequality). *Suppose f is a real analytic function defined in a neighborhood U of the origin in $\mathbb{R}^n$. If $f(0)=0$ and $\nabla f(0)=0$, then there exist constants $C>0$ and $\alpha \in [\frac{1}{2}, 1)$, and shrinking U if necessary, depending on n and f, such that for any $x \in V$, it holds that*

$$|\nabla f(x)| \geq C \cdot |f(x)|^{\alpha}. \tag{2}$$

This type of inequality is crucial in controlling the behavior of the gradient flow. If $\alpha = \frac{1}{2}$, then we get exponential convergence. If $\alpha > \frac{1}{2}$, then we can obtain polynomial convergence:

Corollary 3.2. *Suppose f is a nonnegative real-analytic function defined in a neighborhood U of the origin in $\mathbb{R}^n$ with $f(0)=0$ and with Łojasiewicz exponent $\alpha \in (\frac{1}{2}, 1)$. Then there exists a neighborhood $V \subset U$ of the origin such that for any $x_0 \in V$, the downward gradient flow of f,*

$$\begin{cases} \frac{d}{dt}x(t) = -\nabla f(x(t)), \\ x(0) = x_0, \end{cases}$$

converges uniformly to a limit $x_\infty \in U$ *with* $f(x_\infty) = 0$. *Moreover, we have the following estimates*:

(1) $f(x(t)) \leq C \cdot t^{-\frac{1}{2\alpha-1}}$;
(2) $d(x(t), x(\infty)) \leq C \cdot t^{-\frac{1-\alpha}{2\alpha-1}}$.

Proof. The proof is quite standard, and we call it "*Łojasiewicz arguments*" for later reference. Denote

$$V_\delta = \{x \in \mathbb{R}^n | |x| \leq \delta\},$$

and fix $\delta > 0$ small so that inequality (2) holds for $x \in V_\delta$. In our calculation the constant C may vary from line to line. If $x(t) \in V_\delta$ for $t \in [0, T]$, then we compute

$$\frac{d}{dt} f^{1-\alpha}(x(t)) = -(1-\alpha) \cdot f^{-\alpha}(x(t)) \cdot |\nabla f(x(t))|^2 \leq -C \cdot |\dot{x}(t)|,$$

and thus for any $T > 0$,

$$\int_0^T |\dot{x}(t)| dt \leq \frac{1}{C} \cdot f^{1-\alpha}(x_0).$$

For any $\varepsilon \leq \frac{\delta}{2}$ small, we choose $\delta_2 \leq \delta$ small such that $f(x) \leq (C \cdot \varepsilon)^{\frac{1}{1-\alpha}}$ for $x \in V_{\delta_2}$, and $\delta_1 = \min\{\varepsilon, \delta_2\}$. Then the flow initiating from any point $x_0 \in V_{\delta_1}$ will stay in $V_{2\varepsilon}$. So the Łojasiewicz inequality holds for all $x(t)$. Now

$$\frac{d}{dt} f^{1-2\alpha}(x(t)) = -(1-2\alpha) \cdot f^{-2\alpha}(x(t)) \cdot |\nabla f(x(t))|^2 \geq (2\alpha - 1) \cdot C^2,$$

so

$$f(x(t)) \leq C \cdot t^{-\frac{1}{2\alpha-1}}.$$

For any $T_1 \leq T_2$, we get

$$d(x(T_1), x(T_2)) \leq \int_{T_1}^{T_2} |\dot{x}(t)| dt \leq C \cdot T_1^{-\frac{1-\alpha}{2\alpha-1}}.$$

Therefore we obtain polynomial convergence and the required estimates. □

4. **Finite dimensional case**

4.1. *Kempf-Ness theorem.* Let (M, ω, J) be a compact Kähler manifold, and assume there is an action of a compact connected group G on M that preserves the Kähler structure, with moment map μ. This induces a holomorphic action of the complexified group $G^{\mathbb{C}}$. Let $\mathfrak{g}$ and $\mathfrak{g}^{\mathbb{C}}$ be the Lie algebra of G and $G^{\mathbb{C}}$ respectively, and let I be the natural complex structure on $\mathfrak{g}^{\mathbb{C}}$. Then the Kempf-Ness theorem relates the complex quotient by $G^{\mathbb{C}}$ to the symplectic reduction by G ([23]).

Theorem 4.1 (Kempf-Ness). *A $G^{\mathbb{C}}$-orbit contains a zero of the moment map if and only if it is poly-stable. It is unique up to the action of G. A $G^{\mathbb{C}}$-orbit is semi-stable if and only if its closure contains a zero of the moment map; this zero is in the unique poly-stable orbit contained in the closure of the original orbit.*

In this paper we are only interested in the uniqueness problem. We will first give a proof in the finite dimensional case, using an analytic approach. An essential ingredient in the proof of the Kempf-Ness theorem is the existence of a function E, called the *Kempf-Ness function*. Given a point $x \in M$, one can define a one-form α on $G^{\mathbb{C}}$ as

$$\alpha_g(R_g\xi) = -\langle \mu(g.x), I\xi \rangle,$$

where R_g is the right translation by g and $\xi \in \mathfrak{g}_{\mathbb{C}}$. It is easy to check that α is closed and invariant under the left G-action. Then α is the pullback of a closed one-form $\bar{\alpha}$ from $G^{\mathbb{C}}/G$. It is well known that $G^{\mathbb{C}}/G$ is always contractible, so α gives rise to a function E, up to an additive constant. Notice that if the G action is linearizable, this coincides with the usual definition given by the logarithm of the length of a vector on the induced line bundle. In any case, we call a $G^{\mathbb{C}}$-orbit *poly-stable* (*semi-stable*) if the corresponding function E is proper (bounded below) on $G^{\mathbb{C}}/G$. It is a standard fact that E is geodesically convex; i.e., $\bar{\alpha}$ is monotone along geodesics in $G^{\mathbb{C}}/G$. The critical points of E consist exactly of the zeroes of μ in the given $G^{\mathbb{C}}$-orbit. So any $G^{\mathbb{C}}$-orbit contains at most one zero of the moment map, up to the action of G. In the semi-stable case, we consider the function $f(x) = |\mu(x)|^2$ on M and its downward gradient flow $x(t)$. The flow line is tangent to the $G^{\mathbb{C}}$-orbit and the induced flow in $G^{\mathbb{C}}/G$ is exactly the downward gradient flow of E. We call either flow the *Kempf-Ness flow*. As we will see more explicitly later, a theorem of Duistermaat [33] says that for $x(0)$ close to a zero of μ, the flow $x(t)$ converges polynomially fast to a limit in $\mu^{-1}(0)$. Now suppose x is semi-stable and x_1, x_2 are two poly-stable points in $\overline{G^{\mathbb{C}}.x}$. Without loss of generality, we can assume $\mu(x_1) = \mu(x_2) = 0$. Take $y_1, y_2 \in G^{\mathbb{C}}.x$ such that y_i is close to x_i. Then the gradient flows $x_i(t)$ converges to a point $z_i \in \mu^{-1}(0)$ near x_i. Denote by $\gamma_i(t)$ the corresponding flow in $G^{\mathbb{C}}/G$. Since the gradient flow of a geodesically convex function decreases the geodesic distance, $d(\gamma_1(t), \gamma_2(t))$ is uniformly bounded. By compactness, we conclude that z_1 and z_2 must be in the same $G^{\mathbb{C}}$-orbit, and by the uniqueness in the poly-stable case, we see that z_1 and z_2 must lie in the same G-orbit. By choosing y_i arbitrarily close to x_i, we conclude that x_1 and x_2 are in the same G-orbit.

The above argument proves the uniqueness of poly-stable orbit in the closure of a semi-stable orbit. There are technical difficulties extending this argument to the infinite dimensional setting, due to the loss of compactness.

As a result, we need to investigate the gradient flow in the finite dimensional case more carefully. Our goal in the next subsections is to prove Theorem 4.6; that is, for a semi-stable point, the gradient flow is asymptotic to an "optimal" geodesic ray at infinity. We first introduce the notion of a "asymptotic geodesic ray."

Defintion 4.2. We say a curve $\gamma(t)(t \in [0, \infty))$ in a simply-connected non-positively curved space is asymptotic to a geodesic ray $\chi(t)$ if for any fixed $s > 0$, $d(\gamma_t(s), \chi(s))$ tends to zero as t tends to ∞, where γ_t is the geodesic connecting $\chi(0)$ and $\gamma(t)$ that is parametrized by arc-length. In other words, $\chi(t)$ is the point in the sphere at infinity induced by $\gamma(t)$ as $t \to \infty$.

It follows from the definition that any two geodesic rays $\chi_1(t)$ and $\chi_2(t)$ that are both asymptotic to a given curve $\gamma(t)$ must be *parallel*; that is, $d(\chi_1(t), \chi_2(t))$ is uniformly bounded.

4.2. *Standard case.* Let (V, J_0, g_0) be an n dimensional unitary representation of a compact connected Lie group G, so we have a group homomorphism $G \to U(n)$. Then V is naturally a representation of the complexified group $G^{\mathbb{C}}$. Denote by Ω_0 the induced Kähler form on V. It is easy to see that the G action admits a natural moment map $\mu : V \to \mathfrak{g}^* \simeq \mathfrak{g}$, where we have identified $\mathfrak{g}$ with $\mathfrak{g}^*$ by fixing an invariant metric. It is defined as

$$\langle \mu(v), \xi \rangle = \frac{1}{2}\Omega_0(\xi.v, v). \tag{3}$$

For any $v \in V$, denote the infinitesimal action of G at v by

$$L_v : \mathfrak{g} \to V; \xi \mapsto \xi.v.$$

Then it is easy to see that

$$\mu(v) = -\frac{1}{2}L_v^*(J_0 v). \tag{4}$$

L_v can also be viewed as a map from $\mathfrak{g}^{\mathbb{C}}$ to V, and then $\mu(v) = \frac{1}{2}IL_v^* v$.

Now consider the Kempf-Ness flow, i.e., the downward gradient flow of the function $f : V \to \mathbb{R}; v \mapsto |\mu(v)|^2$,

$$\frac{d}{dt}v = -\nabla f(v) = -J_0 L_v(\mu(v)). \tag{5}$$

Since f is a homogeneous polynomial, and thus real analytic, the Łojasiewicz inequality holds for f; i.e., there exist a constant $C > 0$ and $\alpha \in [\frac{1}{2}, 1)$ such that for v close to zero,

$$|\nabla f(v)| \geq C \cdot |f(v)|^\alpha.$$

The previous Łojasiewicz arguments show that for v close to 0, the flow (5) starting from v converges polynomially fast to a critical point of f.

From now on we assume 0 destabilizes v, i.e., $0 \in \overline{G^{\mathbb{C}}.v} \setminus G^{\mathbb{C}}.v$. Thus the gradient flow (5) converges to the origin by Theorem 4.1. For our purpose we need a sharp estimate of the order of convergence. Since everything is homogeneous, we can study the induced flow on $\mathbb{P}(V)$. The action of G is then holomorphic and Hamiltonian with respect to the Fubini-Study metric on $\mathbb{P}(V)$, with moment map $\hat{\mu} : \mathbb{P}(V) \to \mathfrak{g}$. It is then easy to see that

$$\hat{\mu}([v]) = \frac{\mu(v)}{|v|^2}.$$

Let $\hat{f} = |\hat{\mu}|^2$. Then we can study the downward gradient flow of $\hat{f}$ on $\mathbb{P}(V)$:

$$\frac{d}{ds}[v] = -\nabla \hat{f}([v]) = -J_0 L_{[v]}(\hat{\mu}([v])). \tag{6}$$

Let $\pi : V \to \mathbb{P}(V)$ be the quotient map. Then clearly

$$\pi_*(\nabla f(v)) = |v|^2 \nabla \hat{f}([v]).$$

So the flow (6) is just a re-parametrization of the image under π of the flow (5): if $v(t)$ satisfies (5), then $[v(s)]$ satisfies (6), with $\frac{ds}{dt} = |v(t)|^2$. Since $\hat{f}$ is also real analytic, the flow $[v(s)]$ converges polynomially fast to a unique limit $[v]_\infty$.

Lemma 4.3. $\hat{\mu}([v]_\infty) \neq 0$.

Proof. Otherwise $[v]$ is semi-stable with respect to the action of $G^{\mathbb{C}}$ on $\mathbb{P}(V)$, thus the corresponding Kempf-Ness function $\log |g.v|^2$ is bounded below on $G^{\mathbb{C}}$. This contradicts the assumption that $0 \in \overline{G^{\mathbb{C}}.v}$. □

Thus we know that

$$\frac{\mu(v(s))}{|v(s)|^2} = \hat{\mu}([v]_\infty) + O(s^{-\gamma})(\gamma > 0)$$

is bounded away from zero when s is large enough. So for t sufficiently large, we have

$$|\nabla f(v(t))| \geq C \cdot |f(v(t))|^{\frac{3}{4}}. \tag{7}$$

The Łojasiewicz arguments then ensure that $v(t)$ actually converges to 0 in the order $O(t^{-\frac{1}{2}})$. So we obtain $s \leq C \cdot \log t$.

Now since the gradient flow of f is tangent to the $G^{\mathbb{C}}$ orbit, it can also be viewed as a flow on $G^{\mathbb{C}}/G$. This is given by a path $\gamma(t) = [g(t)]$, where $g(t) \in G^{\mathbb{C}}$ satisfies

$$\dot{\gamma}(t) = \dot{g}(t)g(t)^{-1} = -I\mu(g(t).v).$$

Note the Łojasiewicz exponent $\frac{3}{4}$ in equation (7) is optimal, since if we have a Łojasiewicz inequality for $\alpha > 3/4$, then $|v(t)| = O(t^{-\frac{1}{2}-\delta})$ for some $\delta > 0$ and this would imply the length of $\gamma(t)$ is bounded as $t \to \infty$, which contradicts the fact that $0 \notin G^{\mathbb{C}}.v$.

The re-parametrized equation corresponding to (6) is

$$\dot{\gamma}(s) = \dot{g}(s)g(s)^{-1} = -I\hat{\mu}(g(s).[v]),$$

and

$$\dot{\gamma}(s) = -I\hat{\mu}([v]_\infty) + O(s^{-\gamma}).$$

Here we have identified the tangent space at a point in $G^{\mathbb{C}}/G$ naturally with $I\mathfrak{g} = \sqrt{-1}\mathfrak{g}$. In the following we shall use this re-parametrization because $|\dot{\gamma}(s)|$ has a lower bound as $s \to \infty$, which makes it more convenient to analyze the asymptotic behavior.

Proposition 4.4. *γ is asymptotic to a geodesic ray χ in $G^{\mathbb{C}}/G$ that is rational (i.e., generates a $\mathbb{C}^*$ action) and also degenerates v to 0. Moreover, the direction of χ is conjugate to $\frac{I\hat{\mu}([v]_\infty)}{|\hat{\mu}([v]_\infty)|}$ under the adjoint action of G.*

Proof. We already know $\dot{\gamma}(s)$ is getting close to $\hat{\mu}([v]_\infty)$, but this is not immediate to conclude that γ is asymptotic to a geodesic ray with direction $\hat{\mu}([v]_\infty)$. We shall analyze this more carefully, by elementary geometry. First it is easy to see that

$$|\ddot{\gamma}(s)| = |L^*_{[v](s)}L_{[v](s)}\hat{\mu}([v](s))|,$$

where $L_{[v](s)}$ denotes the infinitesimal action of $\mathfrak{g}$ at $[v](s)$. Since $[v](s) \to [v]_\infty$ as $s \to \infty$, by Corollary 3.2 we get

$$\int_T^\infty |\ddot{\gamma}(s)|ds \le C\int_T^\infty |L_{[v](s)}\hat{\mu}([v](s))|ds = C\int_T^\infty |\nabla \hat{f}(s)|ds \le C\cdot T^{-\beta},$$

where $\beta = \frac{1-\alpha}{2\alpha-1} > 0$. Notice that here α is the exponent appearing in the Łojasiewicz inequality for $\hat{f}$, not the original f. From the above we know $\lim_{s\to\infty}|\dot{\gamma}(s)| = |\hat{\mu}([v]_\infty)| > 0$, so if we parametrize γ by arc-length and denote the resulting path by $\widetilde{\gamma}(u)$, then we have

$$|\ddot{\widetilde{\gamma}}(u)| = |\dot{\gamma}(s)|^{-2}|\ddot{\gamma}(s) - \frac{\langle\ddot{\gamma}(s),\dot{\gamma}(s)\rangle}{|\dot{\gamma}(s)|^2}\dot{\gamma}(s)| \le C\cdot|\ddot{\gamma}(s)|.$$

Therefore,

$$\int_T^\infty |\ddot{\widetilde{\gamma}}(u)|du \le C\cdot T^{-\beta}.$$

Now for any $u > 0$, let $\widetilde{\gamma}_u(v)(v \in [0,1])$ be the geodesic in $G^{\mathbb{C}}/G$ connecting $\widetilde{\gamma}(0)$ and $\widetilde{\gamma}(u)$. Denote by $L_u(v)(v \in [0,u])$ the distance between $\widetilde{\gamma}(v)$ and $\widetilde{\gamma}_u(v)$. Then $L_u(0) = L_u(u) = 0$ and a standard calculation of the second variation of length (using the nonpositivity of the sectional curvature of $G^{\mathbb{C}}/G$) gives

$$\frac{d^2}{dv^2}L_u(v) \ge -|\ddot{\widetilde{\gamma}}(v)|.$$

Now define the function

$$f_u(v) = \int_0^v \int_w^\infty |\ddot{\tilde{\gamma}}(r)| dr dw - \frac{v}{u} \int_0^u \int_w^\infty |\ddot{\tilde{\gamma}}(r)| dr dw.$$

This is well defined by the above decay estimate of $|\ddot{\tilde{\gamma}}|$, and $f_u(0) = f_u(u) = 0$ and

$$\frac{d^2}{dv^2} f_u(v) = -|\ddot{\tilde{\gamma}}(v)|.$$

Thus by the maximum principle, $L_u(v) \leq f_u(v)$ for all $u > 0$ and $v \in [0, u]$. Fixing v we see that

$$\sup_{u \geq v} L_u(v) \leq \int_0^v \int_w^\infty |\ddot{\tilde{\gamma}}(r)| dr dw \leq C \cdot v^{1-\beta}.$$

Moreover, for any $u_2 > u_1 \gg 1$, by comparison argument the angle between $\widetilde{\gamma}_{u_1}$ and $\widetilde{\gamma}_{u_2}$ is bounded by $d(\widetilde{\gamma}_{u_1}(u_1), \widetilde{\gamma}_{u_2}(u_1))/u_1 = L_{u_2}(u_1)/u_1$, which is controlled by $C \cdot u_1^{-\beta}$. Thus we conclude that the direction of $\widetilde{\gamma}_u$ is converging uniformly to some limit direction, and so $\widetilde{\gamma}$ (and thus γ) is asymptotic to a geodesic ray χ starting from $\gamma(0)$. Now for any $s > 0$, in the same way we get a geodesic ray χ_s starting from $\gamma(s)$ that is asymptotic to γ. So the rays χ_s are all asymptotic to each other and it follows that they are all parallel, and then $\dot{\chi}_s(0)$ are all conjugate to each other under the adjoint action of G. On the other hand, if we denote by $\gamma_{s_1,s_2}(u)$ ($u \in [0, 1]$) the geodesic connecting $\gamma(s_1)$ and $\gamma(s_2)$ for $s_1 < s_2$, then again by second variation,

$$\frac{d}{ds_2} \langle \frac{\dot{\gamma}(s_2)}{|\dot{\gamma}(s_2)|}, \frac{\dot{\gamma}_{s_1,s_2}(1)}{|\dot{\gamma}_{s_1,s_2}(1)|} \rangle \geq -\frac{|\ddot{\gamma}(s_2)|}{|\dot{\gamma}(s_2)|} \geq -C|\ddot{\gamma}(s_2)|.$$

So we get

$$\left\langle \frac{\dot{\gamma}(s_2)}{|\dot{\gamma}(s_2)|}, \frac{\dot{\gamma}_{s_1,s_2}(1)}{|\dot{\gamma}_{s_1,s_2}(1)|} \right\rangle \geq 1 - \int_{s_1}^{s_2} |\ddot{\gamma}(u)| du \geq 1 - C \cdot s_1^{-\beta}.$$

We know $\dot{\gamma}(s_2) = I\hat{\mu}([v]_\infty) + O(s_2^{-\alpha})$, and when we fix s_1, as $s_2 \to \infty$ up to the adjoint action of G, we have $\frac{\dot{\gamma}_{s_1,s_2}(1)}{|\dot{\gamma}_{s_1,s_2}(1)|} \to \dot{\chi}_{s_1}(0)$. So $\langle \dot{\chi}_{s_1}(0), I\hat{\mu}([v]_\infty) \rangle \geq 1 - C \cdot s_1^{-\beta}$. Let $s_1 \to \infty$. We see $\dot{\chi}(0)$ is conjugate to $\frac{I\hat{\mu}([v]_\infty)}{|\hat{\mu}([v]_\infty)|}$ under the adjoint action of G.

It is now not hard to see that $\chi(s)$ also degenerates v to the origin since the path $v(t)$ is of order $O(t^{-\frac{1}{2}}) = O(e^{-C \cdot s})$. By Kempf [30] and Ness [35] , the direction $I\hat{\mu}([v]_\infty)$ is indeed rational; i.e., it generates an algebraic one-parameter subgroup $\lambda : \mathbb{C}^* \to G^{\mathbb{C}}$. Moreover, the direction $I\hat{\mu}([v]_\infty)$ is the unique (up to the adjoint action of G) optimal direction for v in the sense of Kirwan [31] and Ness [35]. $\square$

4.3. *Linear case.* Now we suppose G acts linearly on $(V = \mathbb{C}^n, \Omega, J_0)$, where J_0 is the standard complex structure on $\mathbb{C}^n$ and Ω is a real-analytic symplectic form compatible with J_0. Then the action has a real-analytic moment map μ with $\mu(0) = 0$. μ is not necessarily standard but the Łojasiewicz inequality still holds for $f = |\mu|^2$. Suppose $0 \in \overline{G^{\mathbb{C}}.v}$. Then the downward gradient flow $v(t)$ of $f(v) = |\mu(v)|^2$ converges to the origin polynomially fast.

PROPOSITION 4.5. *$v(t)$ converges to 0 in the order $O(t^{-\frac{1}{2}})$. Moreover, let $\gamma(t)$ be the corresponding flow in $G^{\mathbb{C}}/G$. Then there exists a rational geodesic ray χ in $G^{\mathbb{C}}/G$ that also degenerates v to 0 and is asymptotic to γ.*

We prove this by comparing with the Kempf-Ness flow for the standard moment map. Let $\hat{v}(t)$ be the downward gradient flow of $\hat{f}(v) = |\hat{\mu}(v)|^2$, where $\hat{\mu}$ is the moment map for the linearized G action on $(V = T_0V, \Omega_0, J_0)$. By Proposition 4.4 in the previous subsection, $\hat{v}(t)$ converges to zero in the order $O(t^{-\frac{1}{2}})$ and the corresponding flow $\hat{\gamma}(t)$ is asymptotic to a rational geodesic ray $\chi(t)$. Let $\gamma(t)$ in $G^{\mathbb{C}}/G$ be the flow corresponding to $v(t)$. We want to show $\gamma(t)$ is also asymptotic to $\chi(t)$. It suffices to bound the distance $L(t)$ between $\gamma(t)$ and $\hat{\gamma}(t)$. Let $\psi_t(s)(s \in [0,1])$ be the geodesic connecting $\gamma(t)$ and $\hat{\gamma}(t)$. Then

$$
\begin{aligned}
(8)\quad L(t)\frac{d}{dt}L(t) &= \langle \dot{\psi}_t(1), -I\hat{\mu}(\hat{v}(t))\rangle - \langle \dot{\psi}_t(0), -I\mu(v(t))\rangle \\
&= (\langle I\dot{\psi}_t(1), \mu(\hat{v}(t))\rangle - \langle I\dot{\psi}_t(0), \mu(v(t))\rangle) + \langle \dot{\psi}_t(1), -I\hat{\mu}(\hat{v}(t)) + I\mu(\hat{v}(t))\rangle \\
&\leq L(t)|\hat{\mu}(\hat{v}(t)) - \mu(\hat{v}(t))|,
\end{aligned}
$$

where we used the fact that the Kempf-Ness function is geodesically convex. To estimate the last term, notice that since the G action is linear, for any $\xi \in \mathfrak{g}$ and $v \in V$, we have

$$\langle \mu(v), \xi\rangle = \int_0^1 \Omega_{tv}(\xi.tv, v)dt = \frac{1}{2}\Omega_0(\xi.v, v) + O(|v|^3) = \langle \hat{\mu}(v), \xi\rangle + O(|v|^3).$$

Since $\hat{v}(t) = O(t^{-\frac{1}{2}})$, we obtain $\frac{d}{dt}L(t) \leq C \cdot t^{-\frac{3}{2}}$, and so $L(t)$ is uniformly bounded. Therefore, we conclude Proposition 4.5.

4.4. *General case.* Let (M, ω, J, G, μ) be a real analytic Hamiltonian G-action on a compact real analytic Kähler manifold. Fix a bi-invariant metric on $\mathfrak{g}$, and identify $\mathfrak{g}$ with $\mathfrak{g}^*$ as usual. Suppose $y \in M$ is semi-stable but not poly-stable. Denote by $y(t)$ the Kempf-Ness flow starting from y. Since $f = |\mu|^2$ is a real-analytic function on M, as before by the Łojasiewicz arguments, $y(t)$ converges polynomially fast to a unique limit x with $\mu(x) = 0$. By assumption we know $x \in \overline{G.y} \setminus G.y$.

THEOREM 4.6. *The corresponding flow $\gamma(t)$ in $G^{\mathbb{C}}/G$ is asymptotic to a geodesic ray $\chi(t)$ that is rational and also degenerates y to x.*

Remark 4.7. Here we could define $\chi(t)$ as the "optimal" degeneration of y, generalizing the usual definition in the linear case.

The proof of this theorem follows a similar pattern as that in the previous subsection, but the analysis is more involved. The main difficulty is that we cannot linearize the problem since the isotropy group G_0 of x may be strictly smaller than G, and we have to compare $\gamma(t)$ with the Kempf-Ness flow for the group G_0. We first denote by $G_0^{\mathbb{C}}$ the isotropy group of x in $G^{\mathbb{C}}$. The following well-known lemma justifies the abuse of notation:

Lemma 4.8. *$G_0^{\mathbb{C}}$ is the complexification of G_0 (hence is reductive).*

Proof. In the Lie algebra level, we just need to show that if $\xi.x + J\eta.x = 0$ for some $\xi, \eta \in \mathfrak{g}$, then $\xi.x = \eta.x = 0$. This follows easily from the definition of the moment map:

$$\omega(\eta.x, J\eta.x) = (d\mu(J\eta.x), \eta) = (d\mu(J\eta.x + \xi.x), \eta) - (\mathrm{Ad}^*_\xi \mu(x), \eta) = 0.$$

Hence $\eta.x = 0$ and $\xi.x = 0$. □

Let $\mathfrak{g}_0$ be the Lie algebra of G_0. The bi-invariant product on $\mathfrak{g}$ allows a G_0 invariant splitting $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{m}$. The action of G_0 on M is also Hamiltonian, with moment map $\hat{\mu}$ given by the orthogonal projection of μ to $\mathfrak{g}_0$.

Proposition 4.9. *There is a point in the $G^{\mathbb{C}}$-orbit of y, say $\hat{y}$, so that $x \in \overline{G_0^{\mathbb{C}}.\hat{y}}$.*

Given this, let $\tilde{y}(t)$ and $\hat{y}(t)$ be the Kempf-Ness flow starting from $\hat{y}$ with respect to the group G and G_0 respectively, and let $\tilde{\gamma}(t)$ and $\hat{\gamma}(t)$ be the corresponding path in $G^{\mathbb{C}}/G$ and $G_0^{\mathbb{C}}/G_0$ respectively. Since G_0 fixed x, locally we can holomorphically linearize the action; i.e., there is a G_0-equivariant holomorphic equivalence Φ from a neighborhood V of 0 in T_xM to M so that $\Phi(0) = x$. Then we are reduced to the linear case considered in the previous subsection, and by Proposition 4.5, $\hat{y}(t)$ converges to x in the order $O(t^{-\frac{1}{2}})$ and $\hat{\gamma}(t)$ is asymptotic to a rational geodesic ray $\chi(t)$ that also degenerates $\hat{y}$ to x. On the other hand, $G_0^{\mathbb{C}}/G_0$ is naturally a totally geodesic submanifold of $G^{\mathbb{C}}/G$, and we have

Proposition 4.10. *The distance between $\tilde{\gamma}(t)$ and $\hat{\gamma}(t)$ in $G^{\mathbb{C}}/G$ is uniformly bounded.*

Since by the distance decreasing property of the Kempf-Ness flow the distance between $\gamma(t)$ and $\tilde{\gamma}(t)$ is uniformly bounded, Theorem 4.6 then follows easily.

Now we prove Propositions 4.9 and 4.10. Denote by N the orthogonal complement of $\mathfrak{g}^{\mathbb{C}}.x = \mathfrak{g}.x \oplus J\mathfrak{g}.x$ in T_xM. Then N is a G_0-invariant subspace of T_xM, endowed with the induced Kähler structure (J_N, Ω_N). The linear G_0 action on N has a canonical moment map $\mu_N : N \to \mathfrak{g}_0$ (cf. (4)).

The proof of Proposition 4.9 makes use of the normal form for the complex group action, which is $G^{\mathbb{C}} \times_{G_0^{\mathbb{C}}} N$. More precisely, we can choose a G_0 equivariant embedding $\Psi : T_xM \to M$, with $\Psi(0) = x$ and $d\Psi(0) = \mathrm{Id}$. Then we define $\Phi : G^{\mathbb{C}} \times_{G_0^{\mathbb{C}}} N \to M; [(g, v)] \to g.\Psi(v)$. This is a local diffeomorphism around $[(\mathrm{Id}, 0)]$. So for any y close to x, there is a $(g, v) \in G^{\mathbb{C}} \times N$ with $|v| \le Cd(x, y)$ such that $y = g.\Phi(\mathrm{Id}, v)$. By assumption, the Kempf-Ness flow $y(t)$ starting from y converges to x. Choosing a subsequence $t_j \to \infty$, we obtain $(g(t_j), v(t_j)) \in G^{\mathbb{C}} \times N$ such that $y(t_j) = g(t_j).\Phi(\mathrm{Id}, v(t_j))$ and $\lim_{j\to\infty} v(t_j) = 0$. Since $y(t_j) \in G^{\mathbb{C}}.y$, we have $v(t_j) \in G_0^{\mathbb{C}}.v$. Let $\hat{y} = \Phi(\mathrm{Id}, v) = g^{-1}.y$; then $x \in \overline{G_0^{\mathbb{C}}.\hat{y}}$.

To prove Proposition 4.10 we need to use the Marle-Guillemin-Sternberg normal form for the Hamiltonian group action; cf. [28], [37]. The local model Y is the symplectic reduction by G_0 of the product $T^*G \times N$, where T^*G is endowed with the canonical symplectic form Ω_0, and the G_0 action on T^*G is induced from the right multiplication on G. To be more explicit, we can trivialize $T^*G \simeq G \times \mathfrak{g}^* \simeq G \times \mathfrak{g}$ by left translation. Then at (g, ρ), for two tangent vectors $(\xi_1, \rho_1), (\xi_2, \rho_2) \in \mathfrak{g} \oplus \mathfrak{g}$, we have

$$\Omega_0((\xi_1, \rho_1), (\xi_2, \rho_2)) = \langle \rho_2, \xi_1 \rangle - \langle \rho_1, \xi_2 \rangle + \langle \rho, [\xi_1, \xi_2] \rangle.$$

The action of G_0 on $T^*G \times N$ is given by $h.(g, \rho, v) = (gh^{-1}, \mathrm{Ad}_h\rho, h.v)$, with moment map

$$\mu_0(g, \rho, v) = -\pi_{\mathfrak{g}_0}\rho + \mu_N(v).$$

Then the symplectic reduction $Y = G \times_{G_0} (\mathfrak{m} \oplus N)$. A tangent vector at $[g, \rho, v] \in Y$ is given by $[\xi_1, \rho_1, v_1] \in (\mathfrak{g} \oplus \mathfrak{m} \oplus N)/\mathfrak{g}_0$. Then the symplectic form on Y is given by

$$\begin{aligned}\Omega_Y([\xi_1, \rho_1, v_1], [\xi_2, \rho_2, v_2]) &= \langle \rho_2 + d_v\mu_N(v_2), \xi_1 \rangle - \langle \rho_1 + d_v\mu_N(v_1), \xi_2 \rangle \\ &\quad + \langle \rho + \mu_N(v), [\xi_1, \xi_2] \rangle + \Omega_N(v_1, v_2),\end{aligned}$$

where $d_v\mu_N$ is the differential of μ_N at v. The left G action on Y is Hamiltonian with moment map $\mu_Y([g, \rho, v]) = \mathrm{Ad}_g(\rho + \mu_N(v))$. Then Y is the local model for the Hamiltonian action. More precisely,

LEMMA 4.11 (Marle-Guillemin-Sternberg). *There exists a local G equivariant real-analytic symplectic diffeomorphism Φ from a neighborhood U of $[\mathrm{Id}, 0, 0] \in Y$ onto a neighborhood of x in M so that $\Phi^*\mu = \mu_Y$ and $\Phi([\mathrm{Id}, 0, 0]) = x$.*

For our purpose we also need to control the complex structure. Let $L : \mathfrak{m} \to \mathfrak{m}$ be the linear map such that

$$\langle L(\xi), \eta\rangle = \Omega_x(X_\xi, JX_\eta), \tag{9}$$

where X_ξ is the infinitesimal action of ξ at x. Let J_0 be the almost complex structure at $[\mathrm{Id}, 0, 0]$ that, viewed as linear endomorphism of $\mathfrak{m}\oplus\mathfrak{m}\oplus N$, sends (ξ, ρ, v) to $(-L^{-1}(\rho), L(\xi), J_N.v)$. We denote also by J_0 the extension to a neighborhood of $[\mathrm{Id}, 0, 0]$—which we identify with an open subset of $\mathfrak{m}\oplus\mathfrak{m}\oplus N$ by the exponential map on G. The new ingredient that we need is

Lemma 4.12. *We may choose Φ so that $J_Y := \Phi^*J$ is equal to J_0 at $[\mathrm{Id}, 0, 0]$ and $\Phi^*J - J_0 = O(r^2)$ on N.*

This follows from the proof of the Lemma 4.11. The proof of these two lemmas will be given in the appendix.

We continue to prove Proposition 4.10. Using the above two lemmas, we may work on (U, Ω_Y, J_Y). We denote by $\psi_t(s)(s \in [0, 1])$ the geodesic in $G^{\mathbb{C}}/G$ connecting $\tilde{\gamma}(t)$ and $\hat{\gamma}(t)$ and by $L(t)$ the length of ψ_t. Then similar to (8), using the convexity of Kempf-Ness function on $G^{\mathbb{C}}/G$, we have

$$\frac{d}{dt}L(t) \le |\mu(\hat{y}(t)) - \hat{\mu}(\hat{y}(t))|.$$

In our situation we can write $\hat{y}(t) = [g(t), \rho(t), v(t)]$, where $g(t)$ are $\rho(t)$ are uniquely determined by the choice at $t = 0$ if we require $g(t)^{-1}\dot{g}(t) \in \mathfrak{m}$. Then $\mu(\hat{y}(t)) - \hat{\mu}(\hat{y}(t)) = \mathrm{Ad}_{g(t)}\rho(t)$. Let $\hat{f} = |\hat{\mu}|^2$.

$$\begin{aligned}\nabla\hat{f} = J_Y.([0, ad_{\mu_N(v)}\rho, \mu_N(v).v]) &= [-L^{-1}(ad_{\mu_N(v)}\rho), 0, J_0\cdot(\mu_N(v).v)]\\ &\quad + (J_Y - J_0)ad_{\mu_N(v)}\rho + (J_Y - J_0)\mu_N(v).v.\end{aligned}$$

By considering only the second factor, we obtain

$$|\dot{\hat{y}}(t)| = |\nabla\hat{f}(\hat{y}(t))| \le C\cdot(d(\hat{y}(t), x)|\mu_N(v(t))||\rho(t)| + d(\hat{y}(t), x)^2|\mu_N(v(t)).v(t)|).$$

Since $d(\hat{y}(t), x), |\rho(t)|, |v(t)| = O(t^{-\frac{1}{2}})$, we have

$$|\dot{\hat{y}}(t)| \le C\cdot(t^{-\frac{3}{2}}|\rho(t)| + t^{-\frac{5}{2}}).$$

Since $\rho(\infty) = 0$, it is then easy to obtain $|\rho(t)| \le C\cdot t^{-\frac{3}{2}}$. So $L(t)$ is uniformly bounded.

5. Stability of the Calabi flow

We first recall the definition of the Calabi flow. It is an infinite dimensional analogue of the previously mentioned Kempf-Ness flow. Let (M, ω, J_0) be a Kähler manifold. As before, we have the group $\mathcal{G}$ acting on $\mathcal{J}$ and preserving $\mathcal{J}^{\mathrm{int}}$. The action of $\mathcal{G}$ on $\mathcal{J}$ has a moment map given by the Hermitian scalar curvature functional $S - \underline{S} : \mathcal{J} \to C_0^\infty(M;\mathbb{R})$. Its norm is called the *Calabi functional*:

$$\mathrm{Ca}(J) = \int_M (S(J) - \underline{S})^2 d\mu_\omega.$$

The gradient of Ca under the natural metric on $\mathcal{J}$ is given by

$$\nabla \mathrm{Ca}(J) = \frac{1}{2} J \mathcal{D}_J S(J).^2$$

The *Calabi flow* is the downward gradient flow of Ca on $\mathcal{J}^{\mathrm{int}}$. Its equation is given by

$$\frac{d}{dt} J(t) = -\frac{1}{2} J(t) \mathcal{D}_{J(t)} S(J(t)). \tag{10}$$

As in the finite dimensional space, the Calabi flow can be lifted to $\mathcal{G}^{\mathbb{C}}/\mathcal{G}$, which in this case is just the space of Kähler metrics

$$\mathcal{H}_J = \{\phi \in C_0^\infty(M;\mathbb{R}) | \omega + \sqrt{-1}\partial_J \bar{\partial}_J \phi > 0\}.$$

The equation reads

$$\frac{d}{dt}\phi(t) = S(\phi(t)) - \underline{S}. \tag{11}$$

By (1), this is also the downward gradient flow of the Mabuchi functional E. The two equations (10) and (11) are essentially equivalent.

Lemma 5.1. *Any solution of* (11) *naturally gives rise to a solution of* (10); *any solution $J(t)$ of* (10) *induces a solution of* (11) *if $J(t)$ all lie in $\mathcal{J}^{\mathrm{int}}$.*

Proof. Given a path $\phi(t) \in \mathcal{H}$, we consider the time-dependent vector fields $X(t) = -\frac{1}{2}\nabla_{\phi(t)}\dot{\phi}(t)$. Let f_t be the family of diffeomorphisms generated by $X(t)$. Then $f_t^*(\omega + \sqrt{-1}\partial\bar{\partial}\phi(t)) = \omega$. Let $J(t) = f_t^* J$. Then

$$\frac{d}{dt} J(t) = -\frac{1}{2} J(t) \mathcal{D}_{J(t)} \dot{\phi}(t).$$

This proves the first half of the lemma. For the second half, we assume $J(t)$ is a solution to (10). Then we consider the vector fields $X(t) = \frac{1}{2}\nabla_{J(t)} S(J(t))$ and the induced diffeomorphisms f_t. Then $f_t^* J(t) = J(0)$ since $J(t) \in \mathcal{J}^{\mathrm{int}}$, and $f_t^*\omega = \omega + \sqrt{-1} dJ(0) d\phi(t)$, with $\frac{d}{dt}\phi(t) = S(\phi(t)) - \underline{S}$. □

Equation (10) is not parabolic, due to the $\mathcal{G}$ invariance. But (11) is parabolic, and we have the following estimates:

[2] The factor comes from the fact that the metric we choose on $\mathcal{J}$ is $(\mu_1, \mu_2)_J := 2Re \int_M \langle \mu_1, \mu_2 \rangle_J \omega^n$.

LEMMA 5.2 (see [12]). *Suppose there are constants* $C_1, C_2 > 0$ *such that along the Calabi flow*

$$\begin{cases} \frac{\partial \phi}{\partial t} = S - \underline{S}, \\ \phi(0) = \phi_0, \end{cases} \tag{12}$$

we have

$$||\mathrm{Rm}(g(t))||_{L^\infty(g(t))} \leq C_1,$$

and the Sobolev constant of $g(t)$ *is bounded by* C_2 *for all* $t \in [0, T)$. *Then for any* $l > 0$ *and* $t \in [1, T)$, *we have*

$$||\nabla_t^l \mathrm{Rm}(g(t))||_{L^\infty(g(t))} \leq C,$$

where $C > 0$ *depends only* C_1, C_2, l, n.

The Calabi flow equation in the form (11) was first proposed by E. Calabi [5], [6] to find extremal metrics in a fixed Kähler class. The short time existence was established by Chen-He [11]. They also proved the global existence assuming Ricci curvature bound.

The equation (10) also has its own advantage. Namely, when the space $\mathcal{H}$ does not admit any cscK metric, the solution of equation (11) must diverge when $t \to \infty$. However, it is still possible that the corresponding $J(t)$ still converges in the bigger ambient space $\mathcal{J}$. In this section we are interested in the Calabi flow (10) starting from an integrable complex structure in a neighborhood of a cscK metric. We shall prove the following theorem:

THEOREM 5.3. *Suppose that* $J_0 \in \mathcal{J}$ *is* cscK. *Then there exists a small* $C^{k,\lambda}(k \gg 1)$ *neighborhood* $\mathcal{U}$ *of* J_0 *in* $\mathcal{J}^{\mathrm{int}}$ *such that the Calabi flow* $J(t)$ *starting from any* $J \in \mathcal{U}$ *exists globally and converges polynomially fast to a* cscK *metric* $J_\infty \in \mathcal{J}$ *in* $C^{k,\lambda}$ *topology. Up to a Hamiltonian diffeomorphism we can assume* J_∞ *is smooth. Then the convergence is also in* C^∞.

Remark 5.4. When J lies on the leaf of J_0, i.e., the corresponding Kähler metrics are in the same Kähler classes, this was proved in [11] and the convergence is indeed exponential. In general, the convergence is exponential if and only if J_0 and J_∞ are on the same $\mathcal{G}^{\mathbb{C}}$ leaf.

Remark 5.5. There are also studies of stability of other geometrical flows (such as Kähler-Ricci flow) in Kähler geometry when the complex structure is deformed; see, for example, [13], [50]. We believe the idea in this section could also apply to other settings. In a sequel to this paper [43], the second author and Y-Q. Wang proved a similar stability theorem for the Kähler-Ricci flow on Fano manifolds. We should mention that two alternative approaches in the study of the stability of Kähler-Ricci flow have been announced by C. Arezzo-G. La Nave and G. Tian-X. Zhu.

In general this type of stability result is based on a very rough *a priori* estimate of the length of the flow and the parabolicity. Here the key ingredient is the following Łojasiewicz type inequality, which yields the required *a priori* estimate:

THEOREM 5.6. *Suppose* $J_0 \in \mathcal{J}^{\text{int}}$ *is* cscK. *Then there exist an* $L^2_k (k \gg 1)$ *neighborhood* $\mathcal{U}$ *of* J_0 *in* $\mathcal{J}^{\text{int}}$ *and constants* $C > 0$, $\alpha \in [\frac{1}{2}, 1)$ *such that for any* $J \in \mathcal{U}$, *the following inequality holds*:

$$||\mathcal{D}_J S(J)||_{L^2} \geq C \cdot ||S(J) - \underline{S}||^{2\alpha}_{L^2}, \tag{13}$$

where $\mathcal{D}_J\phi = \bar{\partial}_J X_\phi + \bar{X}_\phi . N_J$. *When* J *is integrable*, $\mathcal{D}_J\phi = \bar{\partial}_J X_\phi$ *is the Lichnerowicz operator.*

Remark 5.7. The Łojasiewicz inequality was first used by L. Simon [42] in the study of convergence of parabolic partial differential equations. Råde [39] used Simon's idea to study the convergence of the Yang-Mills flow on two or three dimensional manifold. It also appeared in the study of asymptotic behavior in Floer theory in [20]. Here we follow [39] closely.

We begin the proof by reducing the problem to a finite dimensional one and then use Łojasiewicz's inequality(Theorem 3.1).

To simplify the notation, we assume the function spaces appearing below consist of normalized functions, i.e., functions with average zero. We have the elliptic complex at J_0 (see [27]):

$$L^2_{k+2}(M;\mathbb{C}) \xrightarrow{\mathcal{D}_0} T_{J_0}\mathcal{J} = L^2_k(\Omega^{0,1}_S(T^{1,0})) \xrightarrow{\bar{\partial}_0} L^2_{k-1}(\Omega^{0,2}_S(T^{1,0})),$$

where $\Omega^{0,p}_S(T^{1,0})$ is the kernel of the operator $\mathcal{A}$ in Section 3. So we have an L^2 orthogonal decomposition:

$$\Omega^{0,1}_S(T^{1,0}) = \operatorname{Im}\mathcal{D}_0 \oplus \operatorname{Ker}\mathcal{D}_0^*. \tag{14}$$

On the other hand, the infinitesimal action of the gauge group $\mathcal{G}$ is just the restriction of $\mathcal{D}_0$ to $L^2_{k+2}(M;\mathbb{R})$, which we denote by Q_0. Since J_0 is cscK, $\mathcal{D}_0^*\mathcal{D}_0$ is a real operator. Thus

$$\operatorname{Im}(\mathcal{D}_0) = \mathcal{D}_0(L^2_{k+2}(M;\mathbb{R})) \oplus \mathcal{D}_0(L^2_{k+2}(M;\sqrt{-1}\mathbb{R}))$$

is an L^2 orthogonal decomposition, and so

$$L^2_k(\Omega^{0,1}_S(T^{1,0})) = \operatorname{Im}Q_0 \oplus \operatorname{Ker}Q_0^*,$$

where explicitly, $Q_0^*\mu = \operatorname{Re}\mathcal{D}_0^*\mu$.

Now as in Section 2, we identify an L^2_k neighborhood of J_0 with an open set in the Hilbert space $L^2_k(\Omega^{0,1}_S(T^{1,0}))$. By the implicit function theorem, any integrable complex structure $J = J_0 + \mu \in \mathcal{J}^{\text{int}}$ with $||\mu||_{L^2_k}$ small is in the $\mathcal{G}$

orbit of an integrable complex structure $J_0 + \nu$ with $\nu \in \operatorname{Ker} Q_0^*$ and $||\nu||_{L_k^2}$ small. Since both sides of (13) are invariant under the action of $\mathcal{G}$, it suffices to prove it for $\mu \in \operatorname{Ker} Q_0^*$.

We still need to fix another gauge. The reason is that the Hessian of the Calabi functional restricted to $\operatorname{Ker} Q_0^*$ is essentially the operator $\mathcal{D}_0\mathcal{D}_0^*$, which from the above complex is not elliptic. This is also the reason that in the statement of the theorem we have to restrict to integrable complex structures. Recall that $\mathcal{J}^{\mathrm{int}}$ is the subvariety of $\mathcal{J}$ cut out by the equation:

$$N(\mu) = \bar{\partial}_0\mu + [\mu, \mu] = 0.$$

We would like to linearize this space to $\operatorname{Ker} \bar{\partial}_0$. Let $W = \operatorname{Ker} Q_0^* \cap \operatorname{Ker} \bar{\partial}_0$. Consider the operator

$$\Phi : (W \cap L_k^2(\Omega_S^{0,1}(T^{1,0}))) \times (\operatorname{Im} \bar{\partial}_0 \cap L_{k+1}^2(\Omega_S^{0,2}(T^{1,0}))) \to \operatorname{Im} \bar{\partial}_0 \cap L_{k-1}^2(\Omega_S^{0,2}(T^{1,0}))$$

by sending (μ, α) to the orthogonal projection to $\operatorname{Im} \bar{\partial}_0$ of $N(\mu + \bar{\partial}_0^*\alpha)$. We have the linearization $D\Phi_0(\nu, \beta) = \bar{\partial}_0\bar{\partial}_0^*\beta$, whose second component is an isomorphism. So by the implicit function theorem, for any $\nu \in W \cap L_k^2(\Omega_S^{0,1}(T^{1,0}))$ with $||\nu||_{L_k^2}$ small, there exists a unique $\alpha = \alpha(\nu) \in \operatorname{Im} \bar{\partial}_0 \subset L_{k+1}^2(\Omega_S^{0,2}(T^{1,0}))$ with $||\alpha||_{L_{k+1}^2}$ small such that $\mu = \nu + \bar{\partial}_0^*\alpha$ satisfies $\Phi(\mu) = 0$. Furthermore, we have

$$||\alpha(\nu)||_{L_{k+1}^2} \leq C \cdot ||\nu||_{L_k^2}^2.$$

Define a map L from $B_{\varepsilon_1}(W \cap L_k^2(\Omega_S^{0,1}(T^{1,0})))$ to $\operatorname{Ker} Q_0^* \cap L_k^2(\Omega_S^{0,1}(T^{1,0}))$ by sending ν to μ. Then L is real analytic and a neighborhood of J_0 in $\mathcal{J}^{\mathrm{int}} \cap \operatorname{Ker} Q_0^* \cap L_k^2(\Omega_S^{0,1}(T^{1,0}))$ is contained in the image of L. Moreover, we have that for all $\nu \in B_{\varepsilon_1} W \cap L_k^2(\Omega_S^{0,1}(T^{1,0}))$ and $\lambda \in W \cap L_l^2(\Omega_S^{0,1}(T^{1,0}))$(for any $l \leq k$),

$$c_l \cdot ||\lambda||_{L_l^2} \leq ||(DL)_\nu(\lambda)||_{L_l^2} \leq C_l \cdot ||\lambda||_{L_l^2} \tag{15}$$

and

$$c_l \cdot ||\lambda||_{L_l^2} \leq ||(DL)_\nu^*(DL)_\nu(\lambda)||_{L_l^2} \leq C_l \cdot ||\lambda||_{L_l^2}. \tag{16}$$

To be explicit, the differential of α at ν is given by

$$(D\alpha)_\nu(\lambda) = (D\Phi)_{L(\nu)}(0, -)^{-1} \circ (D\Phi)_{L(\nu)}(\lambda, 0).$$

So if we denote $\mu = L(\nu)$ and $\beta = (D\alpha)_\nu(\lambda)$, then β satisfies

$$\bar{\partial}_0\bar{\partial}_0^*\beta + \Pi_{Im\bar{\partial}_0}[\mu, \bar{\partial}_0^*\beta] = \bar{\partial}_0\lambda + \Pi_{Im\bar{\partial}_0}[\mu, \lambda] = \Pi_{Im\bar{\partial}_0}[\mu, \lambda].$$

Thus by ellipticity we obtain for ν small that

$$||(D\alpha)_\nu(\lambda)||_{L_{l+1}^2} \leq C \cdot ||\nu||_{L_k^2} \cdot ||\lambda||_{L_l^2}. \tag{17}$$

(15) follows from (17), and similarly we can prove (16).

Now consider the Hilbert space $W \cap L^2_k(\Omega^{0,1}_S(T^{1,0}))$ with the constant L^2 metric defined by J_0. Define the functional $\widetilde{\mathrm{Ca}}$ on a small neighborhood of the origin in $W \cap L^2_k(\Omega^{0,1}_S(T^{1,0}))$ by pulling back Ca through L; i.e.,

$$\widetilde{\mathrm{Ca}}(\nu) = \frac{1}{2}\mathrm{Ca}(L(\nu)) = \frac{1}{2}\int (S(L(\nu)) - \underline{S})^2\omega^n.$$

It is easy to see that

$$\delta_\lambda S(L(\nu)) = 2\,\mathrm{Im}\,\mathcal{D}^*_{L(\nu)}((DL)_\nu(\lambda)).$$

So the gradient is

$$\nabla\widetilde{\mathrm{Ca}} = (DL)^*_\nu(J\mathcal{D}_{L(\nu)}S(L(\nu))).$$

We first prove that in a neighborhood of 0 in W,

$$||\nabla\widetilde{\mathrm{Ca}}(\nu)||_{L^2} \geq C\cdot(\widetilde{\mathrm{Ca}}(\nu))^\alpha. \tag{18}$$

The linearization of the gradient is the Hessian

$$H_0 := \delta.\nabla\widetilde{\mathrm{Ca}} : L^2_k(W) \to L^2_{k-4}(W); \lambda \mapsto 2J_0\mathcal{D}_0\mathcal{D}_0^*\lambda.$$

H_0 is an elliptic operator, so it has a finite dimensional kernel W_0 consisting of smooth elements, and W has the following decomposition:

$$W = W_0 \oplus W',$$

where H_0 restricts to invertible operators from $L^2_k(W')$ to $L^2_{k-4}(W')$. So there exists a $c > 0$ such that for any $\mu' \in W'$, we have

$$||H_0(\mu')||_{L^2_{k-4}} \geq C\cdot||\mu'||_{L^2_k}.$$

By the implicit function theorem, for any $\mu_0 \in W_0$ with $||\mu_0||_{L^2}$[3] small, there exists a unique element $\mu' = G(\mu_0) \in W'$ with $||\mu'||_{L^2_k}$ small such that $\nabla\widetilde{\mathrm{Ca}}(\mu_0 + \mu') \in W_0$. Moreover, the map $G : B_{\varepsilon_1}W_0 \to B_{\varepsilon_2}W'$ is real analytic. Now consider the function

$$f : W_0 \to \mathbb{R}; \mu_0 \mapsto \widetilde{\mathrm{Ca}}(\mu_0 + G(\mu_0)).$$

By construction, this is a real analytic function. For any $\mu_0 \in W_0$, it is easy to see that $\nabla f(\mu_0) = \nabla\widetilde{\mathrm{Ca}}(\mu_0 + G(\mu_0)) \in W_0$.

Now we shall estimate the two sides of inequality (18) separately. For any $\mu \in W$ with $||\mu||_{L^2_k} \leq \varepsilon$, we can write $\mu = \mu_0 + G(\mu_0) + \mu'$, where $\mu_0 \in W_0$, $\mu' \in W'$, and

$$||\mu_0||_{L^2_k} \leq c\cdot||\mu||_{L^2_k}; \quad ||G(\mu_0)||_{L^2_k} \leq c\cdot||\mu||_{L^2_k}; \quad ||\mu'||_{L^2_k} \leq c\cdot||\mu||_{L^2_k}.$$

[3]Since W_0 is finite dimensional, any two norms on it are equivalent. We use the L^2 norm for our later purpose.

For the left-hand side of (18), we have

$$\begin{aligned}\nabla\widetilde{\mathrm{Ca}}(\mu) &= \nabla\widetilde{\mathrm{Ca}}(\mu_0+G(\mu_0)+\mu') \\ &= \nabla\widetilde{\mathrm{Ca}}(\mu_0+G(\mu_0)) + \int_0^1 \delta_{\mu'}\nabla\widetilde{\mathrm{Ca}}(\mu_0+G(\mu_0)+s\mu')ds \\ &= \nabla f(\mu_0) + \delta_{\mu'}\nabla\widetilde{\mathrm{Ca}}(0) \\ &\quad + \int_0^1 (\delta_{\mu'}\nabla\widetilde{\mathrm{Ca}}(\mu_0+G(\mu_0)+s\mu') - \delta_{\mu'}\nabla\widetilde{\mathrm{Ca}}(0))ds.\end{aligned}$$

The first two terms are L^2 orthogonal to each other. For the second term, we have

$$||\delta_{\mu'}\nabla\widetilde{\mathrm{Ca}}(0)||_{L^2}^2 = ||H_0(\mu')||_{L^2}^2 \geq C\cdot ||\mu'||_{L_4^2}^2.$$

For the last term, we have

$$||\delta_{\mu'}\nabla\widetilde{\mathrm{Ca}}(\mu_0+G(\mu_0)+s\mu') - \delta_{\mu'}\nabla\widetilde{\mathrm{Ca}}(0)|| \leq C\cdot||\mu||_{L_k^2}||\mu'||_{L_4^2} \leq C\cdot\varepsilon\cdot||\mu'||_{L_4^2}.$$

Therefore, we have

$$||\nabla\widetilde{\mathrm{Ca}}(\mu)||_{L^2}^2 \geq |\nabla f(\mu_0)|_{L^2}^2 + C\cdot||\mu'||_{L_4^2}^2. \tag{19}$$

For the right-hand side of (18), we have

$$\begin{aligned}\widetilde{\mathrm{Ca}}(\mu) &= \widetilde{\mathrm{Ca}}(\mu_0+G(\mu_0)+\mu') \\ &= \widetilde{\mathrm{Ca}}(\mu_0+G(\mu_0)) + \int_0^1 \nabla\widetilde{\mathrm{Ca}}(\mu_0+G(\mu_0)+s\mu')\mu'ds \\ &= f(\mu_0) + \nabla f(\mu_0)\mu' + \int_0^1\int_0^1 \delta_{\mu'}\nabla\widetilde{\mathrm{Ca}}(\mu_0+G(\mu_0)+st\mu')\mu'dtds \\ &= f(\mu_0) + H_0(\mu')\mu' + \int_0^1\int_0^1 (\delta_{\mu'}\nabla\widetilde{\mathrm{Ca}}(\mu_0+G(\mu_0)+st\mu') \\ &\quad - \delta_{\mu'}\nabla\widetilde{\mathrm{Ca}}(0))\mu'dtds.\end{aligned}$$

So

$$\widetilde{\mathrm{Ca}}(\mu) \leq |f(\mu_0)|_{L^2} + C\cdot||\mu'||_{L_4^2}^2. \tag{20}$$

Now we apply the Łojasiewicz inequality to f and obtain that

$$|\nabla f(\mu_0)|_{L^2} \geq C\cdot|f(\mu_0)|^\alpha$$

for some $\alpha\in[\frac{1}{2},1)$. Together with (19) and (20) we have proved (18).

To prove (13), we need to compare $||\nabla\mathrm{Ca}(L(\nu))||_{L^2}$ and $||\nabla\widetilde{\mathrm{Ca}}(\nu)||_{L^2}$, i.e., we want

$$||(DL)_\nu^*(\mathcal{D}_{L(\nu)}S(L(\nu)))||_{L^2} \leq C\cdot||\mathcal{D}_{L(\nu)}S(L(\nu))||_{L^2}. \tag{21}$$

We can take the L^2 decomposition

$$\mathcal{D}_{L(\nu)}S(L(\nu)) = (DL)_\nu\lambda + \beta,$$

where $\lambda \in W$ and $\beta \in \operatorname{Ker}(DL)^*_\nu$. So we just need to prove

$$||(DL)^*_\nu(DL)_\nu\lambda||_{L^2} \leq C \cdot ||(DL)_\nu\lambda||_{L^2}$$

for any λ. This follows from (15) and (16).

Now we follow the Łojasiewicz arguments. Suppose we have a Calabi flow $J(t)$ along an integral leaf staying in an L^2_k neighborhood of J_0. Then by (10),

$$\frac{d}{dt}\mathrm{Ca}(J)^{1-\alpha} = -(1-\alpha)\mathrm{Ca}(J)^{-\alpha}||\nabla\mathrm{Ca}(J)||^2_{L^2(t)} \leq -C \cdot ||\nabla\mathrm{Ca}(J)||_{L^2(t)}.$$

Thus

$$\int_0^t ||\dot{J}||_{L^2(s)}ds = \int_0^t ||\nabla\mathrm{Ca}(J(s))||_{L^2(s)}ds \leq C \cdot \mathrm{Ca}(J(0))^{1-\alpha}. \tag{22}$$

So we get an L^2 length estimate for the Calabi flow in terms of the initial Calabi energy. For γ slightly bigger than α, we have for $\beta = 2 - \frac{\gamma}{\alpha} < 1$,

$$\frac{d}{dt}\mathrm{Ca}(J)^{1-\gamma} = -(1-\gamma)\mathrm{Ca}(J)^{-\gamma}||\nabla\mathrm{Ca}(J)||^2_{L^2(t)} \leq -C \cdot ||\nabla\mathrm{Ca}(J)||^\beta_{L^2(t)}.$$

So for $\beta \in (2 - \frac{1}{\alpha}, 1)$, we have

$$\int_0^t ||\dot{J}(s)||^\beta_{L^2(s)}ds = \int_0^t ||\nabla\mathrm{Ca}(J(s))||^\beta_{L^2(s)}ds \leq C(\beta) \cdot \mathrm{Ca}(J(0))^{1-(2-\beta)\alpha}. \tag{23}$$

Also we have polynomial decay:

$$\frac{d}{dt}\mathrm{Ca}(t)^{1-2\alpha} \geq C > 0,$$

so

$$\mathrm{Ca}(J(t)) \leq C \cdot (t+1)^{-\frac{1}{2\alpha-1}}. \tag{24}$$

Now we define

$$\mathcal{U}^\delta_k = \{J \in C^{k,\lambda}(\mathcal{J}^{\mathrm{int}}) \mid ||\mu_J||_{C^{k,\lambda}} \leq \delta\},$$

where again we identify J close to J_0 with $\mu_J \in \Omega^{0,1}_S(T^{1,0})$. Notice that if $\delta \ll 1$, then for any tensor ξ, the $C^{k,\lambda}_J$ norms defined by (J,ω) are equivalent for any $J \in \mathcal{U}^\delta_k$. We omit the subscript J if $J = J_0$. Also for k sufficiently large, the Sobolev constant is uniformly bounded in $\mathcal{U}^\delta_k$.

THEOREM 5.8. *Suppose J_0 is a* cscK *metric in $\mathcal{J}^{\mathrm{int}}$. Then there exist $\delta_2 > \delta_1 > 0$ such that for any $J(0) \in \mathcal{U}^{\delta_1}_{k+10}$, the Calabi flow $J(t)(t > 0)$ starting from $J(0)$ will stay in $\mathcal{U}^{\delta_2}_k$ all the time.*

Proof. Choose $\delta > 0$ such that the previous *a priori* estimates hold in $\mathcal{U}^\delta_k$. If suffices to prove that there exists $\delta_1 < \delta_2 < \delta$ such that for any Calabi flow $J(t)$ with $J(0) \in \mathcal{U}^{\delta_1}_{k+10}$, if $J(t) \in \mathcal{U}^\delta_k$ for $t \in [0,T)$, then $J(T) \in \mathcal{U}^{\delta_2}_k$. By Lemma 5.2, for $t \geq 1$ and l, we have

$$||\mathrm{Rm}(J(t))||_{C^{l,\lambda}_t} \leq C(l).$$

Now fix $\beta \in (2-\frac{1}{\alpha}, 1)$. For any p, there is an $N(p)$(independent of $t \geq 1$) such that the following interpolation inequality holds:

$$||\dot{J}(t)||_{L^2_p(t)} \leq C(p) \cdot ||\dot{J}(t)||^{\beta}_{L^2(t)} \cdot ||\mathcal{D}_J S(J)||^{1-\beta}_{L^2_{N(p)}(t)} \leq C(p) \cdot ||\dot{J}(t)||^{\beta}_{L^2(t)}.$$

So by (23), we have

$$\begin{aligned}\int_1^T ||\dot{J}(t)||_{L^2_p(t)} dt &\leq C(p) \cdot \mathrm{Ca}(J(1))^{1-(2-\beta)\alpha} \\ &\leq C(p) \cdot \mathrm{Ca}(J(0))^{1-(2-\beta)\alpha} \leq C(p) \cdot \varepsilon(\delta_1).\end{aligned}$$

Since the Sobolev constant is uniformly bounded in $\mathcal{U}^{\delta}_k$, for any l, we obtain

$$\int_1^T ||\dot{J}(t)||_{C^{l,\lambda}_t} dt \leq C(l) \cdot \varepsilon(\delta_1).$$

Therefore,

$$||J(T) - J(1)||_{C^{k,\lambda}} \leq \int_1^T ||\dot{J}(t)||_{C^{k,\lambda}} dt \leq \varepsilon(\delta_1).$$

Now we use Lemma 5.1 and the finite time stability of the parabolic equation (10). We have

$$||J(1) - J_0||_{C^{k,\lambda}} = \varepsilon(\delta_1).$$

Note here that we have to lose derivatives because of the gauge transformation involved. Thus

$$||J(T) - J_0||_{C^{k,\lambda}} \leq \varepsilon(\delta_1).$$

Now choose $\delta_2 = \frac{\delta}{2}$ and $\varepsilon(\delta_1) \leq \delta_2$. Then the theorem is concluded. □

From Theorem 5.8, we know the Calabi flow exists globally in $C^{k,\lambda}$ and thus by sequence converges to J_∞ in $C^{k,\beta}$ for $\beta < \alpha$. Now, again by the Łojasiewicz arguments, we see the limit must be unique and the convergence is in a polynomial rate in $C^{k,\lambda}$.

Now we assume that $J_\infty = J_0$ is smooth. Then we can prove smooth convergence. We first use the ellipticity to obtain *a priori* estimates in $\mathcal{U}^{\delta}_k$ for $k \gg 1$. Any $\mu \in \mathcal{U}^{\delta}_k$ satisfies the following elliptic system:

$$\begin{cases} \operatorname{Im} \mathcal{D}^*_0 \mu = S(\mu) + O(||\mu||^2_{L^2_2}), \\ \operatorname{Re} \mathcal{D}^*_0 \mu = Q^*_0(\mu), \\ \bar{\partial}\mu + [\mu, \mu] = 0. \end{cases} \tag{25}$$

So we have the following *a priori* estimate:

$$||\mu||_{C^{l+2,\alpha}} \leq C \cdot (||\mu||_{C^{l,\lambda}} + ||S(\mu)||_{C^{l,\lambda}} + ||Q^*_0(\mu)||_{C^{l,\lambda}}). \tag{26}$$

From the proof of Theorem 5.8, we know that $||\mu(t)||_{C^{k,\lambda}}$ and $||S(\mu(t))||_{C^{k,\lambda}}$ are uniformly bounded. Since

$$||Q^*_0(\mu(t))||_{C^{k,\lambda}} \leq \int_t^\infty ||Q^*_0(\dot{\mu}(s))||_{C^{k,\lambda}_s} ds \leq \varepsilon(\mathrm{Ca}(J(s)))$$

is bounded, we obtain $||\mu(t)||_{C^{k+2,\alpha}}$ bound, so we can derive smooth convergence by a bootstrapping argument. This finishes the proof of Theorem 5.3.

Theorem 5.3 has its own interest. This yields a purely analytical proof of an extension of a theorem due to Chen [10] and Székelyhidi [44]. This is inspired by an observation of Tosatti [51]. One of the advantages of this new proof is that we do not require the Kähler class to be integral.

THEOREM 5.9 ([10]). *For any $J \in \mathcal{U}$, the Mabuchi functional E on the space of Kähler metrics compatible with J is bounded below, and the lower bound is achieved by the infimum along the Calabi flow initiating from J.*

Proof. From the proof of Theorem 5.3 we know the Calabi flow $J(t) \in \mathcal{J}^{\mathrm{int}}$ starting from J converges to a limit J_∞ with the estimate

$$\mathrm{Ca}(J(t)) \leq C \cdot (t+1)^{-\frac{1}{2\alpha-1}}.$$

By Lemma 5.1, this is equivalent to the Calabi flow $\phi(t)$ in the space of Kähler metrics compatible with J. Then

$$\begin{aligned} E(\phi(t)) &= E(\phi(0)) - \int_0^t \mathrm{Ca}(\phi(s))ds \\ &\geq E(\phi(0)) - C \cdot \frac{2\alpha-1}{2\alpha-2} \cdot [1-(t+1)^{\frac{2\alpha-2}{2\alpha-1}}] \geq -C'. \end{aligned}$$

For any other Kähler potential ϕ, we have by Lemma 2.4 that

$$E(\phi) \geq E(\phi(t)) - \sqrt{\mathrm{Ca}(\phi(t))} \cdot d(\phi, \phi(t)).$$

Since

$$\begin{aligned} d(\phi, \phi(t)) &\leq d(\phi, \phi(0)) + d(\phi(0), \phi(t)) \\ &\leq C + \int_0^t \sqrt{\mathrm{Ca}(\phi(s))} ds \leq C \cdot [1 + (t+1)^{\frac{4\alpha-3}{4\alpha-2}}], \end{aligned}$$

we have

$$E(\phi) \geq \liminf_{t\to\infty} E(\phi(t)) - C \cdot (t+1)^{-\frac{1}{4\alpha-2}} \cdot [1+(t+1)^{\frac{4\alpha-3}{4\alpha-2}}] = \lim_{t\to\infty} E(\phi(t))$$

is bounded below. □

6. Reduced Calabi flow

In this section we shall discuss a reduced finite dimensional problem. The usual Kuranishi method provides a local slice as follows. Assume J_0 is cscK. As before, we have the following elliptic complex:

$$C_0^\infty(M;\mathbb{C}) \xrightarrow{\mathcal{D}_0} T_{J_0}\mathcal{J} = \Omega_S^{0,1}(T^{1,0}) \xrightarrow{\bar\partial_0} \Omega_S^{0,2}(T^{1,0}).$$

Let $\square_0 = \mathcal{D}_0\mathcal{D}_0^* + (\bar\partial_0^*\bar\partial_0)^2$ and $H^1 = \mathrm{Ker}\,\square_0$. Let G be the isotropy group of J_0, which is the group of Hamiltonian isometries of (M, ω, J_0), with Lie algebra

$\mathfrak{g} = \operatorname{Ker} \mathcal{D}_0 \cap C_0^\infty(M;\mathbb{R})$. By the classical Matsushima-Lichnerowicz theorem, $\operatorname{Ker} \mathcal{D}_0$ is the complexification $\mathfrak{g}^{\mathbb{C}}$ of $\mathfrak{g}$, and so the complexification $G^{\mathbb{C}}$ of G is a subgroup of the group of holomorphic transformations of (M, J_0), with Lie algebra $\mathfrak{g}^{\mathbb{C}} = \operatorname{Ker} \mathcal{D}_0$. Then the linear G action on H^1 extends to an action of $G^{\mathbb{C}}$. For convenience, we include a proof of the following standard fact:

LEMMA 6.1 (Kuranishi). *There exist a neighborhood B of 0 in H^1 and a G-equivariant holomorphic embedding*

$$\Phi : B \to \mathcal{J}$$

such that

(1) $\Phi(0) = J_0$.
(2) *If v_1 and v_2 in B are in the same $G^{\mathbb{C}}$-orbit and $\Phi(v_1)$ is integrable, then $\Phi(v_2)$ is integrable, and $\Phi(v_1)$ and $\Phi(v_2)$ are in the same $\mathcal{G}^{\mathbb{C}}$ leaf. Conversely, if $\Phi(v)$ is integrable and $(d\Phi)_v(u)$ is tangent to the $\mathcal{G}^{\mathbb{C}}$ leaf at $\Phi(v)$, then u is tangent to the $G^{\mathbb{C}}$-orbit at v.*
(3) *Any integrable J sufficiently close to J_0 lies in the $\mathcal{G}^{\mathbb{C}}$ leaf of some element in the image of Φ.*

Proof. We can identify any J close to J_0 with an element μ in $\Omega_S^{0,1}(T^{1,0})$, and J is integrable if and only if

$$N(\mu) = \bar{\partial}_0 \mu + [\mu, \mu] = 0.$$

We can first choose a G-equivariant holomorphic embedding Ψ from a ball in $\Omega_S^{0,1}(T^{1,0})$ into $\mathcal{J}$ with $d\Psi_0 = \mathrm{Id}$, by using an "average trick." Let

$$V = \{\mu \in \Omega_S^{0,1}(T^{1,0}) | \mathcal{D}_0^* \mu = 0\}$$

and

$$U = \{\mu \in \Omega_S^{0,1}(T^{1,0}) | N(\mu) = 0, \mathcal{D}_0^* \mu = 0\}.$$

Denote by G the Green operator for $\Box_0$ and $H : \Omega_S^{0,1}(T^{1,0}) \to H^1$ the orthogonal projection. Then for any $\mu \in U$, we have

$$\mu = G\Box_0 \mu + H\mu = -G\bar{\partial}_0^* \bar{\partial}_0 \bar{\partial}_0^* [\mu, \mu] + H\mu.$$

Define a G-equivariant map

$$F : \Omega_S^{0,1}(T^{1,0}) \to \Omega_S^{0,1}(T^{1,0}); \mu \mapsto \mu + G\bar{\partial}_0^* \bar{\partial}_0 \bar{\partial}_0^* [\mu, \mu],$$

where both spaces are endowed with the Sobolev L_k^2 norm. Its derivative at 0 is the identity map, so by the implicit function theorem, there is an inverse holomorphic map $F^{-1} : V_1 (\subset \Omega_S^{0,1}(T^{1,0})) \to V_2 (\subset \Omega_S^{0,1}(T^{1,0}))$. Let Q be restriction of F^{-1} on $B = V_1 \cap H^1$ and Φ be the composition

$$\Phi : B \to \mathcal{J}; v \mapsto \Psi \circ Q(v).$$

Since H^1 consists of smooth elements, the image of Φ also consists of smooth elements.

Now we check Φ is the desired map. For any $v \in B$, we have

$$\mathcal{D}_0^* Q(v) = -\mathcal{D}_0^* G \bar{\partial}_0^* \bar{\partial}_0 \bar{\partial}_0^* [Q(v), Q(v)] = 0$$

and

$$\begin{aligned} N(Q(v)) &= -\bar{\partial}_0 G \bar{\partial}_0^* \bar{\partial}_0 \bar{\partial}_0^* [Q(v), Q(v)] + [Q(v), Q(v)] \\ &= G(\bar{\partial}_0^* \bar{\partial}_0)^2 [Q(v), Q(v)] - H[Q(v), Q(v)] \\ &= 2G \bar{\partial}_0^* \bar{\partial}_0 \bar{\partial}_0^* [N(Q(v)), Q(v)] - H[Q(v), Q(v)]. \end{aligned}$$

So $N(Q(v)) = 0$ if and only if $H[Q(v), Q(v)] = 0$, as in [32]. Therefore a neighborhood of 0 in U is an analytic set contained in the image of Q. Since both Ψ and F are G-equivariant and holomorphic, the first part of (2) is true.

Now we prove (3). Given $Y \in L^2_{k+1}(M, TM)$, we define an L^2_{k+1} diffeomorphism F_Y by setting $F_Y(x) = \exp_x(Y(x))$, where exp is defined using a fixed metric g. For $\sigma > 0$, we let L_σ be the σ ball in $L^2_{k+1}(M, TM)$ and B_σ the σ ball in B. Consider the map $\Sigma : L_\sigma \times B_\sigma \to \operatorname{Im} \partial_0 \subset L^2_{k+1}(\Omega^{2,0}_{J_0})$, which sends (Y, v) first to the $(2,0)$ component (with respect to $J_v = \Phi(v)$) of $(F_Y^{-1})^* \omega - \omega$. Then L^2 project to $\operatorname{Im} \partial_0$ using the metric (ω, J_0). Since $\Phi(v)$ depends smoothly on v, one sees that Σ is smooth and if σ is small, then $(Y, v) \in (L_\sigma \times B_\sigma) \cap \Sigma^{-1}(0)$ implies $(F_Y^{-1})^* \omega$ is compatible with J_v, i.e., $F_Y^* J_v \in \mathcal{J}$. Furthermore, $d\Sigma|_0(Y, v) = -\partial_0(\iota_Y \omega)^{1,0}$ is surjective with a bounded right inverse. For $\phi = \phi_1 + i\phi_2 \in L^2_{k+2}(M; \mathbb{C})$, the vector field $X_\phi = X_{\phi_1} + J_v X_{\phi_2}$ lies in the kernel of $d\Sigma|_{(0,v)}$. So by implicit function theorem, for σ small there is an open set W in $L^2_{k+2}(M; \mathbb{C})$ and a smooth embedding $R : W \times B_\sigma \to L^2_{k+2}(M; TM)$ so that $R(0, v) = 0$, $dR|_{(0,v)}(\phi, 0) = X_\phi$ and $F^*_{R(\phi,v)} J_v \in \mathcal{J}$. So for any v with $J_v \in \mathcal{J}^{\text{int}}$, $F^*_{R(\phi,v)} J_v$ lies in the $\mathcal{G}^{\mathbb{C}}$ leaf of J_v. Let $\mu(\phi, v) \in L^2_k(\Omega^{0,1}_S(T^{1,0}))$ be the element representing $F^*_{R(\phi,v)} J_v$. Notice $N(\mu)$ differs from the Nijenhuis tensor of μ by an algebraic projection (which relates the different Dolbeault decompositions with respect to μ and J_0). Since $Q(v)$ satisfies the above equation, we see $\mu(\phi, v)$ satisfies an equation of the form

$$\Box_0 T(\phi, v, N(\mu)) = 2\bar{\partial}_0^* \bar{\partial}_0 \bar{\partial}_0^* [T(\phi, v, N(\mu)), S(\phi, v, \mu)]. \tag{27}$$

Here T, S depend continuously on (ϕ, v) and smoothly on the last component, with $S(0, v, \mu) = \mu$, $T(0, v, N) = N$, and T depends linearly on N. Now given any $J \in \mathcal{J}^{\text{int}}$ close to J_0 in L^2_k, the corresponding μ satisfies $N(\mu) = 0$, so it also satisfies (27) for all (ϕ, v). Denote by Π_1 and Π_2 the projection from $L^2_k(\Omega^{0,1}_S(T^{1,0}))$ to $\operatorname{Im} \mathcal{D}_0$ and $\operatorname{Ker} \Box_0$ respectively. Then it follows from standard elliptic theory that if we have two μ_1, μ_2 with L^2_k norm sufficiently small and both satisfy (27) for some (ϕ, v) small, then $\Pi_1(\mu_1) = \Pi_1(\mu_2)$ and $\Pi_2(\mu_1) = \Pi_2(\mu_2)$ imply $\mu_1 = \mu_2$. Now we define a map $P : W \times B_\sigma \to \operatorname{Im} \mathcal{D}_0 \times H^1$ by sending (ϕ, v) to $(\Pi_1(F^*_{R(\phi,v)} J_v), \Pi_2(F^*_{R(\phi,v)} J_v))$. This is a smooth map, and

$dP_{(0,0)}(\phi, v) = (\mathcal{D}_0\phi, v)$ is surjective with finite dimensional kernel. By implicit function theorem and the above discussion, any $\mu \in \mathcal{J}^{\mathrm{int}}$ that is L^2_k close to J_0 is in the $\mathcal{G}^{\mathbb{C}}$ leaf of some element in the image of Φ. So (3) is proved.

It suffices to prove the last statement in (2). Suppose $\mu = \Phi(v)$, and $\nu = (d\Phi)_v(u)$ is tangent to the $\mathcal{G}^{\mathbb{C}}$ leaf; i.e., $\nu = \mathcal{D}_\mu\phi$ for some complex valued function ϕ. This implies that $dP_{(0,v)}(\phi, 0) \in \operatorname{Im}\mathcal{D}_\mu \cap H^1 = 0$. By the semi-continuity of the dimension of kernel of $dP_{(0,v)}$, we see ϕ must be in $\mathfrak{g}^{\mathbb{C}}$ and u is tangent to the $G^{\mathbb{C}}$ orbit of v. $\square$

By [17], the action of $\mathcal{G}$ on $\mathcal{J}$ has a moment map given by the scalar curvature functional $\mu = S - \underline{S} : \mathcal{J} \to C^\infty_0(M;\mathbb{R})$. The downward gradient flow of $|\mu|^2$ is just the Calabi flow. Now we reduce this flow to a finite dimensional flow. Note that G as a subgroup of $\mathcal{G}$ acts on $\mathcal{J}$ with induced moment map $\bar{\mu} = \Pi_{\mathfrak{g}}(S - \underline{S})$. It is the L^2 projection of μ to $\mathfrak{g}$ with respect to the natural volume form. We can consider the gradient flow of $|\bar{\mu}|^2$, whose equation reads

$$\frac{d}{dt}J = -\frac{1}{2}J\mathcal{D}_J\bar{\mu}(J). \tag{28}$$

If we have a solution to equation (28) such that J_t is integrable for all $t \in [0, T]$, then we can translate it to a flow in $\mathcal{H}$ given by

$$\frac{d}{dt}\phi = \Pi_{f_t^*\mathfrak{g}}(S(\phi) - \underline{S}), \tag{29}$$

where f_t is the family of diffeomorphism satisfying

$$\frac{d}{dt}f_t = -\frac{1}{2}J_t X_{S(J_t)},$$

and the projection is taken with respect to the volume form of $f_t^*\omega$. We will study the relation between this flow and the Calabi flow later on. Let us call the flow (28) or (29) the *reduced Calabi flow*. It is the gradient flow of the norm squared of the moment map of a finite dimensional compact group action.

Now we can pull back the Kähler structure on $\mathcal{J}$ to B, denoted by $(\tilde{\Omega}, \tilde{J})$. By the previous lemma, we know G acts on $(B, \tilde{\Omega}, \tilde{J})$ holomorphically and isometrically, with moment map $\tilde{\mu}$ equal to $\Phi^*\bar{\mu}$. We can then study the reduced Calabi flow on a finite dimensional ambient space B. Let J be an integrable complex structure J close to J_0 such that the Calabi flow $J(t)$ converges to J_0. Suppose J_0 is not in the $\mathcal{G}^{\mathbb{C}}$ leaf of J. By property (3) in Lemma 6.1, we can smoothly perturb $J(t)$ to $\bar{J}(t)$ in the $\mathcal{G}^{\mathbb{C}}$-orbit such that $\bar{J}(t) = \Phi(v(t))$ for $v(t) \to 0 \in B$. Since $\dot{\bar{J}}(t)$ is tangent to the $\mathcal{G}^{\mathbb{C}}$ leaf, by property (2) in Lemma 6.1, we see that $\dot{v}(t)$ is tangent to the $G^{\mathbb{C}}$-orbit. So v is destabilized by 0 in B under the $G^{\mathbb{C}}$ action. By our previous study of the finite dimensional case, the reduced Calabi flow starting from v exists for all time and converges to 0 in the order $O(t^{-\frac{1}{2}})$, and the corresponding flow $\hat{J}(t)$ in $G^{\mathbb{C}}/G$

is asymptotic to a rational geodesic ray χ that also degenerates v to zero. We can view χ as a geodesic ray in $\mathcal{H}$ as well, and it is then natural to expect that the reduced Calabi flow in $\mathcal{H}$ is asymptotic to a smooth geodesic ray with the same degeneration limit. This requires slightly more clarification. Now we define a map F from an open set in $G^{\mathbb{C}}/G$ to $\mathcal{H}$ as follows. This open set is a geodesic convex open set $\mathcal{U}$ in $G^{\mathbb{C}}/G$ such that $[g].v$ still lies in the previously constructed Kuranishi slice. For $[g] \in \mathcal{U}$, choose an arbitrary smooth path $g(t)$ in $G^{\mathbb{C}}/G$ with $g(0) = \mathrm{Id}$, $g(1) = g$ and $[g(t)] \in \mathcal{U}$. Let $v(t) = g(t).v$ and $J(t) = \Phi(v(t))$. Write $\dot{g}(t) \cdot g(t)^{-1} = \xi(t) + \sqrt{-1}\eta(t) \in \mathfrak{g} \oplus \sqrt{-1}\mathfrak{g}$. Then $J(t)$ is integrable for each t, and

$$\frac{d}{dt}J(t) = \mathcal{D}_{J(t)}\xi(t) + J(t)\mathcal{D}_{J(t)}\eta(t),$$

where $\xi(t)$ and $\eta(t)$ are viewed as functions on M through the inclusion $\mathfrak{g} \subset C_0^\infty(M;\mathbb{R})$. Choose an isotopy of Hamiltonian diffeomorphisms f_t such that $\frac{d}{dt}f_t = -X_{\xi(t)}$. Then $\widetilde{J}(t) = f_t^* J(t)$ satisfies

$$\frac{d}{dt}\widetilde{J}(t) = \widetilde{J}(t)\mathcal{D}_{\widetilde{J}(t)}\widetilde{\eta}(t),$$

where $\widetilde{\eta}(t) = f_t^*\eta(t)$. In fact, $\widetilde{J}(t) = \Phi(h(t)g(t).v)$, where $h(t)$ is a path in G so that $\dot{h}(t)h(t)^{-1} + \mathrm{Ad}_{h(t)}\xi(t) = 0$. By Lemma 5.1, if we choose an isotopy of diffeomorphisms k_t with $\frac{d}{dt}k_t = -\nabla_{\widetilde{J}(t)}\widetilde{\eta}(t)$, then $k_t^*\widetilde{J}(t) = J$ and $k_t^*\omega = \omega_t = \omega + \sqrt{-1}\partial\bar{\partial}\phi(t)$. We define $F([g])$ to be $\phi(1)$. Of course we need to show this is well defined. It suffices to show the definition is independent of the path chosen in $G^{\mathbb{C}}/G$. Since $G^{\mathbb{C}}/G$ is always simply connected, we only need to show it is invariant under based homotopy. For this, we choose a two parameter family $g_{s,t}$ in $G^{\mathbb{C}}$ such that $g_{s,0}$ is equal to identity, and $g_{s,1} = g$. Correspondingly we have $h(s,t)$ in G with $h(s,0)$ equal to identity. Let $\widetilde{g}_{s,t} = h_{s,t} \cdot g_{s,t}$. Then we have

$$\frac{\partial}{\partial t}\widetilde{g}_{s,t} \cdot \widetilde{g}_{s,t}^{-1} = \sqrt{-1}\eta(s,t) \in \sqrt{-1}\mathfrak{g}.$$

Also we have

$$\frac{\partial}{\partial s}\widetilde{g}_{s,t} \cdot \widetilde{g}_{s,t}^{-1} = \xi(s,t) + \sqrt{-1}\zeta(s,t) \in \mathfrak{g} \oplus \sqrt{-1}\mathfrak{g}.$$

So we have the relation

$$\sqrt{-1}\frac{\partial}{\partial s}\eta(s,t) = \frac{\partial}{\partial t}\xi(s,t) + \sqrt{-1}\frac{\partial}{\partial t}\zeta(s,t) - [\sqrt{-1}\eta(s,t), \xi(s,t) + \sqrt{-1}\zeta(s,t)].$$

In particular,

$$\frac{\partial}{\partial s}\eta(s,t) = \frac{\partial}{\partial t}\zeta(s,t) - [\eta(s,t), \xi(s,t)].$$

Also $\xi(s,0) = \zeta(s,0) = \xi(s,1) = \zeta(s,1) = 0$. Let $J_{s,t} = \Phi(\tilde{g}_{s,t}.v)$, and let $k_{s,t}$ be the two parameter family of diffeomorphisms obtained by fixing s and integrate along the t direction as before. In particular, $k(s,0)$ is equal to identity for all s. Using Cartan formula it is straightforward to calculate that

$$\frac{\partial}{\partial s}\frac{\partial}{\partial t}k_{s,t}^{*}\omega = -\frac{\partial}{\partial s}k_{s,t}^{*}dJ_{s,t}d\eta(s,t) = -\frac{\partial}{\partial s}dJdk_{s,t}^{*}\eta(s,t) = -dJd\frac{\partial}{\partial t}k_{s,t}^{*}\zeta(s,t).$$

Thus

$$\frac{\partial}{\partial s}|_{t=1}k_{s,t}^{*}\omega = -dJd\left(\int_0^1 \frac{\partial}{\partial t}k_{s,t}^{*}\zeta(s,t)dt\right) = -dJd(k_{s,1}^{*}\zeta(s,1)) = 0.$$

So the map F depends only on the point $[g]$, not on the path chosen. Therefore F is a well-defined smooth map. Similarly, one can show that F is a local isometric embedding; in particular, the image is totally geodesic. Since the flow $\hat{J}(t)$ is asymptotic to a rational geodesic ray $\mathbb{C}^*$ in $G^{\mathbb{C}}/G$ that also degenerates v to 0, under the map F we see the corresponding reduced Calabi flow in $\mathcal{H}$ is also asymptotic to a smooth rational geodesic ray χ with the same degeneration limit. Then it follows from general theory (see, for example, [44]) that χ is tamed by a smooth test configuration, so it is tamed by a bounded geometry in the sense of [9].

To prove that the Calabi flow is asymptotic to the reduced Calabi flow, we need to generalize Lemmas 4.11 and 4.12 to the infinite dimensional case. Then by the same argument as before, together with Lemma 2.5 that the Mabuchi functional is weakly convex, one can show

Lemma 6.2. *Let $\hat{J}(t)$ be the reduced Calabi flow as before, and let $\hat{\phi}(t)$ be the corresponding flow in $\mathcal{H}$. Then for any Calabi flow path $\phi(t) \in \mathcal{H}$, there is a constant $C > 0$ so that for all t, $d(\phi(t), \hat{\phi}(t)) \leq C$.*

The proof will be given in the appendix. Combining all these we arrive at the following theorem:

Theorem 6.3. *Let (M, ω_0, J_0) be a* csc *Kähler manifold. Let J be a complex structure in $\mathcal{J}$ close to J_0. The Calabi flow starting from J converges to J_0 at the infinity. Suppose J_0 is not in the $\mathcal{G}^{\mathbb{C}}$ leaf of J. Then there is a smooth geodesic ray $\phi(t)$ in the space of Kähler metrics $\mathcal{H}_{\omega,J}$ that is tamed by bounded geometry and degenerates J to J_0 in the space $\mathcal{J}$. Furthermore, $\phi(t)$ is asymptotic to the Calabi flow with respect to the Mabuchi-Semmes-Donaldson metric, in the sense of Definition* 4.2.

7. **Relative bound for parallel geodesic rays**

The goal of this section is to obtain a C^0 estimate of potentials from geodesic distance bound. In general this seems to be very difficult, but we will show such an estimate holds if we are along two parallel geodesic rays tamed by bounded geometry. It is well known that in a Riemannian manifold with nonpositive curvature, the distance between two geodesics is a convex function. We first justify this property for the infinite dimensional space $\mathcal{H}$.

Lemma 7.1. *Let $\phi_1(t)$ and $\phi_2(t)$ be two $C^{1,1}$ geodesics in $\mathcal{H}$. Then the function $d(\phi_1(t), \phi_2(t))$ is a convex function of t.*

Proof. We first assume both geodesics are C^∞. Let $\gamma_\varepsilon(t,s)$ be the ε-geodesic connecting $\gamma_1(t)$ and $\gamma_2(t)$ (see [8]). Then

$$\frac{d^2}{dt^2}L(\gamma_\varepsilon(t)) = \int_0^1 \frac{1}{|\gamma_{\varepsilon,s}|}\{|\gamma_{\varepsilon,ts}^\perp|^2 - R(\gamma_{\varepsilon,s},\gamma_{\varepsilon,t})\}ds + \frac{1}{|\gamma_{\varepsilon,s}|}\langle\gamma_{\varepsilon,s},\gamma_{\varepsilon,tt}\rangle|_0^1 - \int_0^1 \frac{\langle\gamma_{\varepsilon,ss},\gamma_{\varepsilon,tt}\rangle}{|\gamma_{\varepsilon,s}|} + \frac{\langle\gamma_{\varepsilon,s},\gamma_{\varepsilon,ss}\rangle\langle\gamma_{\varepsilon,s},\gamma_{\varepsilon,tt}\rangle}{|\gamma_{\varepsilon,s}|^3}ds.$$

Along the ε-geodesics, we have

$$|\gamma_{\varepsilon,ss}| = \sqrt{\int_0^1 (\phi_{\varepsilon,ss} - \nabla_{\phi_{\varepsilon,s}}\phi_{\varepsilon,s})^2\omega_{\phi_\varepsilon}^n} \le C(t)\sqrt{\varepsilon},$$

where $C(t)$ is uniformly bounded if t varies in a bounded interval. Also $|\gamma_{\varepsilon,tt}| \le C(t)$, and $|\gamma_{\varepsilon,s}| \to L_t$ uniformly for $s\in[0,1]$ and t bounded. Therefore, we have

$$\frac{d^2}{dt^2}L(\gamma_\varepsilon(t)) \ge -C(t)\sqrt{\varepsilon},$$

so for any $a \le b$,

$$L_\varepsilon(ta+(1-t)b) \le tL_\varepsilon(a) + (1-t)L_\varepsilon(b) + C\sqrt{\varepsilon}(t-a)(b-t).$$

Let $\varepsilon \to 0$. Then we have

$$L(ta+(1-t)b) \le tL(a) + (1-t)L(b).$$

So $L(t)$ is still a convex function, and the argument of the lemma yields the same conclusion.

In the general case we need to define the distance between two $C^{1,1}$ potentials, which is just the infimum of the length of all $C^{1,1}$ paths connecting the two points. Clearly the distance between any two points is always nonnegative.

Now we assume ϕ_1 and ϕ_2 are $C^{1,1}$ but $\phi_i(0)$ and $\phi_i(1)$ are smooth. We want to prove that for $t\in[0,1]$,

$$L(t) \le (1-t)L(0) + tL(1). \tag{30}$$

To prove this, choose a δ-geodesic ϕ_δ^i approximating ϕ_i with endpoints fixed. Let $\phi_{\varepsilon,\delta}(t,s)$ be the geodesic connecting $\phi_\delta^1(t)$ and $\phi_\delta^2(t)$, and let $L_{\varepsilon,\delta}(t)$ be its length. Then similar calculation shows that

$$\frac{d^2}{dt^2}L_{\varepsilon,\delta}(t) \ge -C\sqrt{\delta} - C(\delta,t)\sqrt{\varepsilon}.$$

So

$$L_{\varepsilon,\delta}(t) \le (1-t)L_{\varepsilon,\delta}(0) + tL_{\varepsilon,\delta}(1) + \frac{1}{2}(C\sqrt{\delta} + C(\delta,t)\sqrt{\varepsilon})t(1-t).$$

Let $\varepsilon\to 0$, we have

$$L_\delta(t) \le (1-t)L_\delta(0) + tL_\delta(1) + C\sqrt{\delta}.$$

Let $\delta \to 0$. Then we get the desired inequality. So the theorem is true in this case.

If $\phi_i(0)$ and $\phi_i(1)$ are not assumed to be smooth, we can approximate them weakly in $C^{1,1}$ by smooth potentials $\phi_i^\varepsilon(0)$, $\phi_i^\varepsilon(1)$ respectively. Let $\phi_i^\varepsilon(t)$ be the geodesic connecting $\phi_i^\varepsilon(0)$ and $\phi_i^\varepsilon(1)$. Then we know $d(\phi_1^\varepsilon(t), \phi_2^\varepsilon(t))$ is a convex function. By maximum principle for the Monge-Ampère equations, we know

$$|\phi_i^\varepsilon(t) - \phi_i(t)|_{C^0} \leq \max(|\phi_i^\varepsilon(0) - \phi_i(0)|_{C^0}, |\phi_i^\varepsilon(1) - \phi_i(1)|_{C^0}).$$

Hence $|\phi_i^\varepsilon(t) - \phi_i(t)|_{C^0} \to 0$; in particular, $d(\phi_i^\varepsilon(t), \phi_i(t)) \to 0$. Therefore, $d(\phi_1^\varepsilon(t), \phi_2^\varepsilon(t))$ converges uniformly to $d(\phi_1(t), \phi_2(t))$. So the latter is also convex. □

Lemma 7.2. *If ϕ_1 is in $\mathcal{H}$ (i.e., ϕ_1 is smooth and ω_1 is positive) and ϕ_2 is $C^{1,1}$, then $d(\phi_1, \phi_2) = 0$ if and only if $\phi_1 = \phi_2$.*

Proof. We can choose C^∞ potential ϕ_2^ε converging to ϕ_2 weakly in $C^{1,1}$ as $\varepsilon \to 0$. Then by [8],

$$d(\phi_1, \phi_2^\varepsilon) \geq \max\left(\int_{\phi_1 \geq \phi_2^\varepsilon} (\phi_1 - \phi_2^\varepsilon)\omega_{\phi_1}^n, \int_{\phi_2^\varepsilon \geq \phi_1} (\phi_2^\varepsilon - \phi_1)\omega_{\phi_2^\varepsilon}^n\right).$$

Let $\varepsilon \to 0$. Then we get

$$d(\phi_1, \phi_2) \geq \max\left(\int_{\phi_1 \geq \phi_2} (\phi_1 - \phi_2)\omega_1^n, \int_{\phi_2 \geq \phi_1} (\phi_2 - \phi_1)\omega_2^n\right).$$

So if $d(\phi_1, \phi_2) = 0$, then

$$\int_{\phi_1 \geq \phi_2} (\phi_1 - \phi_2)\omega_1^n = \int_{\phi_2 \geq \phi_1} (\phi_2 - \phi_1)\omega_2^n = 0.$$

The first equation implies $\phi_1 \leq \phi_2$. The second equation implies that

$$\int_{\phi_2 > \phi_1} \omega_2^n = 0.$$

Let $\Omega = \{x \in M | \phi_2(x) > \phi_1(x)\}$. Then by Stokes' formula,

$$\begin{aligned}
\int_\Omega \omega_1^n &= \int_\Omega \omega_1^n - \omega_2^n \\
&= \int_\Omega \sqrt{-1}\partial\bar{\partial}(\phi_1 - \phi_2) \cdot \sum_{j=0}^{n-1} \omega_1^j \wedge \omega_2^{n-1-j} \\
&= \int_{\partial\Omega} \sqrt{-1}\bar{\partial}(\phi_1 - \phi_2) \cdot \sum_{j=0}^{n-1} \omega_1^j \wedge \omega_2^{n-1-j} \\
&= 0.
\end{aligned}$$

So Ω is empty. Thus $\phi_1 = \phi_2$. □

Corollary 7.3. *Let ϕ_1 be a geodesic ray tamed by bounded geometry (see [9]), and let ϕ_2 be another geodesic ray parallel to ϕ_1 with $\phi_2(0)$ smooth. Then $\phi_1 - \phi_2$ has a uniform relative $C^{1,1}$ bound (with respect to ω_{ϕ_1}).*

Proof. By [9], there is a $C^{1,1}$ geodesic ray ϕ_3 emanating from $\phi_2(0)$ such that $|\phi_3(t) - \phi_1(t)|_{C^{1,1}_{\phi_1}} \leq C$. Thus $d(\phi_2(t), \phi_3(t))$ is uniformly bounded. Since $\phi_2(0) = \phi_3(0)$, by Lemma 7.1, $d(\phi_2(t), \phi_3(t)) = 0$. Lemma 7.2 then implies $\phi_2(t) = \phi_3(t)$. So $|\phi_2(t) - \phi_1(t)|_{C^{1,1}_{\phi_1}} \leq C$. □

Corollary 7.4. *Let $\gamma_1(t)$ and $\gamma_2(t)$ be two smooth paths in $\mathcal{H}$ with $d(\gamma_1(t), \gamma_2(t))$ uniformly bounded. Suppose $\phi(t)$ is a smooth geodesic ray in $\mathcal{H}$ asymptotic to γ_1. Then it is also asymptotic to γ_2.*

Proof. Let $\gamma_i(t,s)$ be the geodesic connecting $\phi(0)$ and $\gamma_i(t)$ parametrized by arc-length. Fix s. By assumption, $d(\gamma_1(t,s), \phi(s)) \to 0$ as $t \to \infty$. So, in particular, $d(\phi(0), \gamma_1(t)) \to \infty$. Suppose $d(\gamma_1(t), \gamma_2(t)) \leq C$. Choose T large enough so that $d(\phi(0), \gamma_1(T)) \gg s+C$. Then $d(\gamma_1(T, T-C), \gamma_2(T, T-C)) \leq 4C$. By Lemma 7.1, as $T \to \infty$,

$$d(\gamma_1(T,s), \gamma_2(T,s)) \leq \frac{s}{T} \cdot 4C \to 0.$$

By definition, $\phi(t)$ is asymptotic to γ_2. □

The following follows similarly, and we omit the details.

Corollary 7.5. *Let $\gamma(t)$ be a smooth path in $\mathcal{H}$ that is asymptotic to two smooth geodesic rays $\phi_1(t)$ and $\phi_2(t)$. Then ϕ_1 and ϕ_2 are parallel; i.e., $d(\phi_1(t), \phi_2(t))$ is uniformly bounded. If we assume one of them is tamed by bounded geometry, say ϕ_1, then by Corollary 7.3, $|\phi_1(t) - \phi_2(t)|_{C^{1,1}_{\phi_1}} \leq C$.*

8. Proof of the main theorems

Now we proceed to prove the main theorems.

Lemma 8.1. *Suppose g_i is a sequence of Riemmanian metrics on a manifold M. If there are two sequences f_i and h_i of diffeomorphism of M such that $f_i^* g_i \to g_1$, and $h_i^* g_i \to g_2$ in C^∞, then $f_i \circ h_i^{-1}$ converges by subsequence to a diffeomorphism f in C^∞ with $f^* g_2 = g_1$.*

The proof is standard using compactness; see, for example, [26]. We omit it here.

Corollary 8.2. *The quotient $\mathcal{J}/\mathcal{G}$ is Hausdorff in the C^∞ topology.*

Lemma 8.3 (The C^0 bound implies no Kähler collapsing). *Suppose there are two sequences ϕ_i, $\psi_i \in \mathcal{H}$ converging in the Cheeger-Gromov sense, i.e.,* *there are two sequences of diffeomorphisms* f_i, h_i *such that*

$$f_i^*(J, \omega_{\phi_i}) \to (J_1, \omega_1)$$

and

$$h_i^*(J, \omega_{\psi_i}) \to (J_2, \omega_2)$$

in the C^∞ *topology. If* $|\phi_i - \psi_i|_{C^0} \leq C$, *then* $|\phi_i - \psi_i|_{C^k_{\omega_{\phi_i}}}$ *is bounded for all* k, *and there is a subsequence* k_i *such that* $f_{k_i}^{-1} \circ h_{k_i}$ *converges in* C^∞ *to a diffeomorphism* f *with* $f^*J_1 = J_2$ *and* $f^*\omega_1 = \omega_2 + \sqrt{-1}\partial_{J_2}\bar{\partial}_{J_2}\phi$.

The proof is quite standard now, given the volume estimates in [11]. We refer the readers to [11] for detailed proof.

Proof of Theorem 1.3. We may assume J_1 and J_2 are not in the $\mathcal{G}^{\mathbb{C}}$ leaf of J, the proof in the other case is similar. We proceed by contradiction. Suppose J_1 and J_2 were not in the same $\mathcal{G}$-orbit. Then by Corollary 8.2 we can assume there are disjoint $\mathcal{G}$ invariant neighborhoods $\mathcal{U}_1$, $\mathcal{U}_2$ of J_1, J_2 respectively. Pick J_i' in the intersection of $\mathcal{U}_i$ with $\mathcal{G}^{\mathbb{C}}$ leaf of J. Now by Theorem 5.3, we know that the Calabi flow $J_i(t)$ starting from J_i' exits globally and converges to $J_i(\infty) \in \mathcal{U}_i$. So $J_1(\infty)$ and $J_2(\infty)$ are not in the same $\mathcal{G}$-orbit either. By Theorem 6.3, the corresponding Calabi flow $\phi_i(t)$ in the space of Kähler metrics is asymptotic to a smooth geodesic ray that also degenerates some other $\hat{J}_i$ to $J_i(\infty)$. Since J_1' and J_2' are both in the $\mathcal{G}^{\mathbb{C}}$ leaf of J, we can pull everything back to J, and then we have two Calabi flows $\phi_i(t)$ each asymptotic to a smooth geodesic ray $\chi_i(t)$ tamed by bounded geometry. By [4], $d(\phi_1(t), \phi_2(t))$ is decreasing, so by Corollary 7.4, $\phi_1(t)$ is also asymptotic to $\chi_2(t)$. By Corollaries 7.3 and 7.5,

$$|\chi_1(t) - \chi_2(t)|_{C^{1,1}} \leq C.$$

So Lemma 8.3 implies there is no Kähler collapsing, and there is a diffeomorphism f with $f^*J_1(\infty) = J_2(\infty)$, and $f^*\omega = \omega + \sqrt{-1}\partial\bar{\partial}\phi$. Since$(f^*\omega, J_2(\infty))$ and $(\omega, J_2(\infty)$ are both csc Kähler structures in the same Kähler class, by Theorem 1.1, there is a diffeomorphism h with $h^*J_2(\infty) = J_2(\infty)$ and $h^*f^*\omega = \omega$, so $(f \circ h)^*(\omega, J_1(\infty)) = (\omega, J_2(\infty))$. Contradiction. □

Proof of Theorem 1.6. Suppose $f_i^*(\omega_{\phi_i}, J) \to (\omega_1, J_1)$ and $h_i^*(\omega_{\psi_i}, J) \to (\omega_2, J_2)$. Since $[\omega]$ is integral, we see that $[f_i^*\omega_{\phi_i}] = [\omega_1]$ for i large enough, so we can further assume that $f_i^*\omega_{\phi_i} = \omega_1$ and $h_i^*\omega_{\psi_i} = \omega_2$. Then we can follow the proof of Theorem 1.3. □

Proof of Corollary 1.7. Suppose $f_i^*J \to J_1$. Since $c_1(J_1) > 0$, we have $c_1(f_i^*J) > 0$, and we can choose a sequence of Kähler metrics ω_i in $c_1(J)$ such that $f_i^*\omega_i \to \omega_1$. Then we can apply Theorem 1.6. □

9. Further discussion

There are also some further interesting questions.

(1) One important question would be a general notion of optimal degenerations and its relation to the Calabi flow. For example, one may like to generalize the theorem to the uniqueness of some "canonical" objects in the closure, allowing the occurrence of singularities. On the other hand, inspired by the Yau-Tian-Donaldson conjecture, one would like to know if there is an algebraic-geometric counterpart of Theorem 1.3. That is, given a polarized variety (M, L), suppose (M_1, L_1) and (M_2, L_2) are K-polystable (possibly singular) varieties that are both central fibers of some test configurations for (M, L), then we would like to know whether $(M_1, L_1) \simeq (M_2, L_2)$.

(2) *The quantization approach* [20], [24]. In the case of discrete automorphism group, Donaldson [20] proved that the existence of a cscK metric implies asymptotic Chow stability. Theorem 1.1 in this case follows from the proof. One may try to extend this approach to prove Theorem 1.3 too. However, this cannot be straightforward. The reason is that for an adjacent csc Kähler structure whose underlying complex structure is different from the original one, the automorphism group cannot be finite; and it is known that the existence of a cscK metric (or even KE metric) does not necessarily imply asymptotic Chow poly-stability; see the recent counterexample in [36], [16]. It seems to the authors that more delicate work is required to proceed by the quantization method.

(3) Our result partially confirmed Tian's conjecture [48] for cscK metrics. (The original conjecture allows mild singularities.) In the case of general extremal metrics, the original statement in [48] has to be modified. This can be easily seen in the corresponding finite dimensional analogue. In that case any gradient flow can be reversed and we can get critical points in the limit along both directions of the flow. Clearly they are not in the same G-orbit and therefore "adjacent" critical point is not necessarily unique. In our infinite dimensional case, the naive uniqueness also fails for adjacent extremal metrics. Such examples were already implicit in Calabi's seminal paper [6]. Namely, we consider the blowup of $\mathbb{P}^2$ at three distinct points p_1, p_2 and p_3(denoted by $\mathrm{Bl}_{p_1,p_2,p_3}(\mathbb{P}^2)$). Then by [1], the class $\pi^*[\omega_{FS}] - \varepsilon^2([E_1] + [E_2] + [E_3])$ contains extremal metrics for ε small enough. If p_1, p_2 and p_3 are in general position(i.e., they do not lie on a line), then $\mathrm{Bl}_{p_1,p_2,p_3}(\mathbb{P}^2)$ are all bi-holomorphic and by [6] the classes $\pi^*[\omega_{FS}] - \varepsilon^2([E_1]+[E_2]+[E_3])$ have vanishing Futaki invariant; thus the extremal metrics are cscK. If p_1, p_2 and p_3 lie on a line, Calabi pointed out in [6] that there is no cscK metric due to the Lichnérowicz-Matsushima theorem. It is easy to see that for a fixed Kähler class $\pi^*[\omega_{FS}] - \varepsilon^2([E_1]+[E_2]+[E_3])$, the extremal metrics in the case where p_1, p_2 and p_3 lie on a line are adjacent

to the cscK metrics in the case p_1, p_2, p_3 are in general position. So we can find proper extremal metrics even adjacent to cscK metrics. C. LeBrun also pointed out to us another example, where we can look at the Hirzebruch surfaces F_{2n} of even degree. If $n > m$, then with appropriate polarization, F_{2n} is adjacent to F_{2m}, while in [5], Calabi explicitly constructed extremal metrics in any Kähler classes.

The reason why the uniqueness fails can also be seen from the fact that our proof depends on the Calabi flow in an essential way. Since the Calabi flow can only detect destabilizing extremal metrics, we might want to consider only the uniqueness of destabilizing (i.e., energy minimizing) extremal metrics, as a modification of Tian's conjecture. This idea of destabilizing extremal metrics has already been implicitly discussed in [9].

(4) The integrality assumption in Theorem 1.6 is used only for fixing the symplectic form. It seems possible to remove this assumption.

Appendix A. Marle-Guillemin-Sternberg normal form

In this appendix, we prove Lemmas 4.11, 4.12 and 6.2.

A.1. *Model case.* Suppose ω is a Kähler metric defined in a neighborhood of 0 in $\mathbb{C}^n$. In general, we cannot trivialize both the complex structure and the symplectic structure simultaneously; however, we can make either of them standard, with appropriate control on the other. First, by the Kähler condition, one can choose local holomorphic coordinate such that

$$\omega = \omega_0 + O(|z|^2),$$

where ω_0 is the usual symplectic form on $\mathbb{C}^n$. In this way the complex structure is made standard, while the error on the symplectic form is quadratic.

We can also make the symplectic form standard, by applying the usual Moser trick. Let $\alpha = \omega - \omega_0$. Then we can write $\alpha = f_1^*\alpha - f_0^*\alpha = d\theta$, where $\theta = \int_0^1 f_t^*(X \lrcorner \alpha)dt$, and f_t is the isotopy generated by the radial vector field X. So $\theta = O(|z|^3)$. Let $\omega_t = (1-t)\omega + t\omega_0$. Then $\phi_t^*\omega_t = \omega_0$, where ϕ_t is the isotopy generated by the vector fields Y_t satisfying $Y_t \lrcorner \omega_t = -\theta$. It is easy to see that $Y_t = O(|z|^3)$, and so $\phi_t(z) = z + O(|z|^3)$ and

$$\phi_t^* J_0 - J_0 = O(|z|^2).$$

So we can ensure a quadratic error on the complex structure.

A.2. *Proof of Lemmas* 4.11 *and* 4.12. We adopt the notation in Section 4.4. Since Ω is Kähler, we can choose local adapted holomorphic coordinates near x and identify a neighborhood V of x with an open set in T_xM, and such that $\Omega - \Omega_0 = O(r^2)$. Here $\Omega_0 = \Omega|_{T_xM}$. Let $\exp_x$ be the exponential map with respect to the metric induced by J and Ω. Then $\exp_x$ is

G_0-equivariant, and

$$\exp_x(\rho + v) = \rho + v + O(r^3). \tag{31}$$

Recall we have a decomposition $T_xM = \mathfrak{g}.x \oplus J\mathfrak{g}.x \oplus N$. We may identify $\xi \in \mathfrak{m}$ with $X \in \mathfrak{g}.x$ by the formula $\langle \xi, \eta \rangle = \Omega(X_\eta, JX)$. Then we may view $T_xM = \mathfrak{m} \oplus \mathfrak{m} \oplus N$. Consider the G_0 equivariant map

$$\Psi : G \times_{G_0} (\mathfrak{m} \oplus N) \to M; [g, \rho, v] \mapsto g. \exp_x(\rho + v).$$

The derivative at $[\mathrm{Id}, 0, 0]$ is given by the linear isomorphism

$$d\Psi : \mathfrak{m} \oplus \mathfrak{m} \oplus N \to T_xM = \mathfrak{m} \oplus \mathfrak{m} \oplus N; (\xi, \rho, v) \mapsto (L(\xi), \rho,$$

where L is the map defined in (9). So we may assume Ψ is a G equivariant diffeomorphism from a neighborhood U of $[\mathrm{Id}, 0, 0]$ onto $V_1 \subset V$. Denote $\Omega' = \exp^* \Omega$ and $J' = \exp^* J$. Then

$$J'_{[\mathrm{Id},0,0]}(\xi, \rho, v) = (-L^{-1}(\rho), L(\xi), J_N.v).$$

Thus we have $J' = J_Y$ at $[\mathrm{Id}, 0, 0]$.

On T_xM we denote by ∂_z the coordinate vector field on N, $(\partial_q, \partial_p = J\partial_q)$ on $\mathfrak{g}.x \oplus J\mathfrak{g}.x \simeq \mathfrak{m} \oplus \mathfrak{m}$. These can be viewed naturally as vector fields on V_1 and thus on U by the map Ψ. On U, let $(\partial_\rho, \partial_v)$ be the vector fields corresponding to (∂_p, ∂_z) on $\mathfrak{m} \oplus N$ through the projection $G \times (\mathfrak{m} \oplus N) \to Y$, and let ∂_ξ be the vector field induced by left translation of $L^{-1}(\partial_q)$. It is easy to check, using (31), that at $[\mathrm{Id}, \rho, v]$, we have

$$\partial_z = \partial_v + O(r^2); \partial_p = \partial_\rho + O(r^2); \partial_q = L(\partial_\xi) + O(r).$$

Then it is straightforward to check that at $[\mathrm{Id}, \rho, v]$,

$$\Omega_Y(X_1, X_2) = \Omega'(X_1, X_2) + O(r^2),$$

for $X_1, X_2 \in \{\partial_z, \partial_p\}$, and at $[\mathrm{Id}, 0, 0]$, $\Omega_Y = \Omega'$. Therefore, we obtain

$$\alpha = \Omega' - \Omega_Y = O(r^2)(dzdz + dzdp + dpdp) + O(r)(dzdq + dpdq + dqdq).$$

Now let $f_t : U \to U; [g, \rho, v] \to [g, t\rho, tv]$. Then we have $\alpha = d\theta$, and at $[\mathrm{Id}, \rho, v]$,

$$\theta = \int_0^1 f_t^*(X_t \lrcorner \alpha) dt = O(r^2)dq + O(r^3).$$

Let $\Omega_t = (1-t)\Omega_Y + t\Omega'$. Then $\phi_t^* \Omega_t = \Omega_Y$, where $\dot{\phi}_t = Y_t$ satisfies $Y_t \lrcorner \Omega_t = \theta$. At $[\mathrm{Id}, \rho, v]$, we have

$$Y_t = O(r^2)\partial_p + O(r^3) = O(r^2)\partial_\rho + O(r^3).$$

Since Y_t is G-invariant, this is also true at $[g, \rho, v] \in U$. Thus the integral curve of Y_t satisfies $v_t = v + O(r^3)$; $\rho_t = \rho + O(r^2)$. Therefore,

$$(\phi_t^* J')\partial_v = J_Y \partial_v + O(r^2).$$

Let $\Phi = \phi_1$. Then we have $\Phi^*\Omega' = \Omega_Y$. We get the required estimate that

$$\Phi^* J' - J_Y = O(r),$$

and

$$\Phi^* J' \cdot X - J_Y \cdot X = O(r^2)|X|$$

for $X \in N$. This proves Lemmas 4.11 and 4.12.

A.3. *Proof of Lemma* 6.2. Now we proceed to our infinite dimensional problem, following the same route as in the finite dimensional setting. However, there are a few more technical issues, as we shall see below. Suppose (M, ω, J_0) is a constant scalar curvature Kähler manifold. Then the relevant group $\mathcal{G}$ is the group of Hamiltonian diffeomorphisms of (M, ω), which acts on the space $\mathcal{J}$ of almost complex structures compatible with ω. Here in order to apply the implicit function theorem, we shall put C^∞ topology on these infinite dimensional objects, which makes them into *tame Fréchet manifolds* in the sense of [29]. Here we use the collection of norms $|\cdot|_k$, the C^k norm on tensors with respect to the fixed background metric. The fact that $\mathcal{J}$ is a smooth tame space follows easily from the definition, and we also have

LEMMA A.1. *$\mathcal{G}$ is a smooth tame Lie group.*

Proof. We first prove it is a smooth tame space. We can identify a Hamiltonian diffeomorphism H with an exact Lagrangian graph G_H in $\mathcal{M} = M \times M$; i.e.,

$$G_H = \{(x, H(x)) | x \in M\}.$$

Here $\mathcal{M}$ is endowed with a canonical symplectic form $\omega' = \pi_1^*\omega - \pi_2^*\omega$, where π_i is the projection map to the i-th factor. A Lagrangian graph is called *exact* if it can be deformed by exact Lagrangian isotopies to the identity. We can construct local charts for $\mathcal{G}$ as follows. Given any $H \in \mathcal{G}$, by Weinstein's Lagrangian neighborhood theorem [52], we can choose a symplectic diffeomorphism between a tubular neighborhood $\mathcal{U}$ of G_H in $\mathcal{M}$ and a tubular neighborhood $\mathcal{V}$ of 0 section in the cotangent bundle T^*M. Then locally any Hamiltonian diffeomorphism close to H is represented by the graph of an exact one-form, i.e., the differential of some real valued function on M. So locally $\mathcal{U}$ can be identified with an open subset of $C_0^\infty(M; \mathbb{R})$. Thus $\mathcal{G}$ is modelled on $C_0^\infty(M; \mathbb{R})$. To check the transition function is smooth tame, we notice that in our case, locally between any two charts there is a symplectic diffeomorphism of the cotangent bundle $F : T^*M \to T^*M$ that is identity on the zero section. Then the induced transition map is smooth tame, by observing that the C^k distance between the graph of exact one-forms $d\phi_1$ and $d\phi_2$ is equivalent to the C^{k+1} distance between ϕ_1 and ϕ_2. Similarly we can prove that the group multiplication and inverses are both smooth tame. $\square$

$\mathcal{J}$ inherits a natural (weak) Kähler structure (Ω, I) from the original Kähler manifold M. As recalled in Section 2, there is a global chart which identifies $\mathcal{J}$ with the space $\Omega^{0,1}_S(T^{1,0})$. Using this, the Kähler form satisfies

$$|\Omega_\mu - \Omega_0|_k \leq C_k |\mu|_k^2$$

for $\mu \in \Omega^{0,1}_S(T^{1,0})$ close to zero, where Ω_0 is the constant Kähler form on $\Omega^{0,1}_S(T^{1,0})$. Let Ψ be the exponential map on $\mathcal{J}$ with respect to the natural Riemannian metric. This is well defined since it is simply the fiberwise exponential map on the symmetric space $\mathrm{Sp}(2n)/U(n)$. It is also clear that

$$|\Psi(\mu) - \mu|_k \leq C_k |\mu|_k^3.$$

The action of $\mathcal{G}$ on $\mathcal{J}$ preserves the Kähler structure and has a moment map given by the Hermitian scalar curvature functional $m(J) = S(J) - \underline{S}$. Denote by G the identity component of the holomorphic isometry group of (M, ω, J_0). Let $\mathfrak{g}$ and $\mathfrak{g}_0$ be the Lie algebra of $\mathcal{G}$ and G respectively. Then we have an L^2 orthogonal decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{m}$, where $\mathfrak{m}$ is the image of $Q^* = \operatorname{Re} \mathcal{D}_0^*$ (cf. (14)). The local model space we consider is

$$Y = \mathcal{G} \times_G (\mathfrak{m} \oplus N),$$

where $\mathcal{G}$ acts adjointly on $\mathfrak{g}$ by $f.\phi = f^*\phi$. N is the orthogonal complement of the image of $\mathcal{D}_0$ in $\Omega^{0,1}(T^{1,0})$, and G acts on N by pulling back: $g.\mu = g^*\mu$. Since G acts smooth tame and freely on $\mathcal{G} \times (\mathfrak{m} \oplus N)$, we know that Y is a tame space with a smooth tame left $\mathcal{G}$ action.

The action of G on N is Hamiltonian with moment map given by

$$m_N : N \to \mathfrak{g}_0; (m_N(v), \xi) = \frac{1}{2}\Omega(\xi.v, v).$$

Since $\mathfrak{g}_0$ is finite dimensional, $m_N(v)$ is a well-defined tame map. As in the finite dimensional case, there is a canonically defined (weak) symplectic form on Y given by

$$\begin{aligned}
\Omega_Y|_{[g,\rho,v]}&((L_g\xi_1, \rho_1, v_1), (L_g\xi_2, \rho_2, v_2)) \\
&:= \langle \rho_2 + d_v\mu_N(v_2), \xi_1 \rangle - \langle \rho_1 + d_v\mu_N(v_1), \xi_2 \rangle \\
&\quad + \langle \rho + \mu_N(v), [\xi_1, \xi_2] \rangle + \Omega_0(v_1, v_2) \\
&= \langle \xi_1, \rho_2 \rangle - \langle \rho_1 + d_v\mu_N(v_1) - [\rho + \mu_N(v), \xi_1], \xi_2) \\
&\quad + \Omega_0(v_1, v_2) + (\xi_1, d_v\mu_N(v_2)).
\end{aligned}$$

The left G action on Y is Hamiltonian with moment map given by

$$m_Y : [g, \rho, v] = g^*(\rho + m_N(v)).$$

Now, as before, we identify $\xi \in \mathfrak{m}$ with $X \in \mathfrak{g}.J_0$ by $\langle \xi, \eta \rangle = \Omega_0(\mathcal{D}_0\eta, JX)$. More precisely, using the operator $L : \mathfrak{m} \to \mathfrak{m}; \xi \to \mathcal{D}_0^*\mathcal{D}_0\xi$, we identify $\rho \in \mathfrak{m}$

with $\sqrt{-1}\mathcal{D}_0(L^{-1}\rho) \in I\mathfrak{g}.J_0 \subset \Omega_S^{0,1}(T^{1,0})$. We then define

$$\Phi : Y \to \mathcal{J}; [g, \rho, v] \mapsto g^*\Psi(\rho + v).$$

LEMMA A.2. *Φ is smooth tame with a local smooth tame inverse around $[\mathrm{Id}, 0, 0]$.*

Proof. It is clear by definition that the map is smooth and tame. The k-th derivative of Φ is tame of degree $k+1$. To apply the implicit function theorem, we need to study the derivative of Φ near $[\mathrm{Id}, 0, 0]$. At $\delta = [g, \rho, v]$, we denote $\mu = \Phi(\delta)$. Then we have

$$\begin{aligned} D_\delta\Phi : \mathfrak{m} \oplus \mathfrak{m} \oplus N \to \Omega_S^{0,1}(T^{1,0}); [\phi, \psi, u] \\ \mapsto -(I + \mu(I - \bar{\mu}\mu)^{-1}\bar{\mu})^{-1}(I - \bar{\mu})^{-1}[Q_\mu(\phi) \\ + g^* D\Psi|_{\rho+v}(\sqrt{-1}Q_0(L^{-1}\psi) + u)](I - \mu). \end{aligned}$$

To find the inverse to D_δ, we need to first decompose $\Omega_S^{0,1}(T^{1,0})$ into the direct sum of $D\Psi^{-1} \circ \mathrm{Im}\, Q_\mu|_{\mathfrak{m}}$ and $\mathrm{Ker}\, Q_0^*$ with estimate. This can be done using elliptic theory. We then obtain

$$\nu = (D\Psi)^{-1} \circ Q_\mu\phi + \sqrt{-1}Q_0L^{-1}\psi + \eta,$$

where $\eta \in \mathrm{Ker}\, D_0^*$. Take the map $P_\mu : \nu \mapsto (\phi, \sqrt{-1}Q_0L^{-1}\psi + \eta)$. Then it is smooth tame again by elliptic estimates. Since the inverse of $D_\delta\Phi$ is the combination of P_μ with some other smooth tame operator, it is also smooth tame. Moreover, $(D_\delta\Phi)^{-1}$ is also tame in the δ variable. Then we can apply the Nash-Moser implicit function theorem [29] to conclude the lemma. □

By the above lemma we can pull back the symplectic form Ω and the complex structure I to a neighborhood $\mathcal{U}$ of $[\mathrm{Id}, 0, 0]$, which we denote by Ω' and I' respectively. There is also a canonical almost complex structure I_0 on $\mathcal{U}$ defined by

$$I_0 : \mathfrak{m} \oplus \mathfrak{m} \oplus N \to \mathfrak{m} \oplus \mathfrak{m} \oplus N; (\xi, \rho, v) \to (-(\mathcal{D}_0^*\mathcal{D}_0)^{-1}\rho, \mathcal{D}_0^*\mathcal{D}_0\xi, I(0)(v)).$$

It is easy to see that $I' = I_0$ at $[\mathrm{Id}, 0, 0]$.

PROPOSITION A.3. *There are neighborhoods $\mathcal{U}_i, \mathcal{V}_i (i = 1, 2)$ $(\mathcal{U}_2 \subset \mathcal{U}_1)$ of $[\mathrm{Id}, 0, 0]$ in Y and two $\mathcal{G}$-equivariant smooth tame maps*

$$\Sigma_1 : \mathcal{U}_1 \to \mathcal{V}_1; \quad \Sigma_2 : \mathcal{V}_2 \to \mathcal{U}_2,$$

that fix the $\mathcal{G}$-orbit of $[\mathrm{Id}, 0, 0]$ such that $\Sigma_1 \circ \Sigma_2$ is equal to the identity and such that

$$\Sigma_1^*\tilde{\Omega} = \Omega'; \quad \Sigma_2^*\Omega' = \tilde{\Omega}.$$

Furthermore, there exists k and s large so that for any $X \in N$ and $[g, \rho, v] \in \mathcal{V}_2$,

$$|(D\Sigma_1) \circ I' \circ (D\Sigma_2)(X) - I_0(X)|_k \leq C(k, s)|X|_{k+s} r_{k+s}^2,$$

and at $[\mathrm{Id}, 0, 0]$,

$$(D\Sigma_1) \circ I' \circ (D\Sigma_2) = I_0.$$

Here the estimate is only in the tame sense; i.e., the norm on the left-hand side might be weaker than that on the right, r_{k+s} *is the distance between* $[g, \rho, v]$ *and* $[\mathrm{Id}, 0, 0]$, *measured by the* C^{k+s} *norm with respect to the background metric.*

Proof. The idea of the proof is the same as the finite dimensional case. The main difficulty is to solve an ordinary differential equation in infinite dimension. Once this is established, then everything else will follow formally. First we have ($\mu = \Phi([\mathrm{Id}, L^{-1}\rho, v])$):

$$\begin{aligned}
&\Omega'_{[\mathrm{Id},L^{-1}\rho,v]}((\xi_1, L^{-1}\rho_1, v_1), (\xi_2, L^{-1}\rho_2, v_2))\\
&\quad= \Omega_\mu(\mathcal{D}_\mu\xi_1 + d\Psi_*(\sqrt{-1}\mathcal{D}_0\rho_1 + v_1), \mathcal{D}_\mu\xi_2 + d\Psi_*(\sqrt{-1}\mathcal{D}_0\rho_2 + v_2))\\
&\quad= -\mathrm{Im}(\mathcal{D}_\mu\xi_1 + d\Psi_*(\sqrt{-1}\mathcal{D}_0\rho_1 + v_1), \mathcal{D}_\mu\xi_2 + d\Psi_*(\sqrt{-1}\mathcal{D}_0\rho_2 + v_2))_{L^2}\\
&\quad= (-\operatorname{Im}\mathcal{D}_\mu^*\mathcal{D}_\mu\xi_1 - \operatorname{Re}\mathcal{D}_\mu^* \circ d\Psi_*(\mathcal{D}_0\rho_1 - \sqrt{-1}v_1), \xi_2)_{L^2}\\
&\qquad+ (\operatorname{Re}\mathcal{D}_0^* \circ (d\Psi_*)^t(\mathcal{D}_\mu\xi_1) + \operatorname{Im}\mathcal{D}_0^* \circ (d\Psi_*)^t d\Psi_*(\mathcal{D}_0\rho_1 - \sqrt{-1}v_1), \rho_2)_{L^2}\\
&\qquad+ ((d\Psi_*)^t(\mathcal{D}_\mu\xi_1 + d\Psi_*(\sqrt{-1}\mathcal{D}_0\rho_1 + v_1)), v_2)_{L^2}.
\end{aligned}$$

Let $\Omega_t = (1-t)\Omega_Y + t\Omega'$. Then $\Omega_t = \Omega_Y = \Omega'$ at $[\mathrm{Id}, 0, 0]$. The isotopy $f_t : [g, L^{-1}\rho, v] \to [g, tL^{-1}\rho, tv]$ gives rise to time-independent vector field $X_t(f_t([g, L^{-1}\rho, v])) = [0, L^{-1}\rho, v]$. We first need to solve for the time-dependent vector field Y_t through the following relation:

$$\Omega_{t\ [g,L^{-1}\rho,v]}(Y_t, Z) = \int_0^1 (\Omega' - \tilde{\Omega})_{[g,sL^{-1}\rho,sv]}((0, L^{-1}\rho, v), f_{s*}Z)ds. \tag{32}$$

Notice that Y_t is $\mathcal{G}$-invariant, so we can assume $g = \mathrm{Id}$. Let $Y_t = (\xi_1, L^{-1}\rho_1, v_1) \in \mathfrak{m} \oplus \mathfrak{m} \oplus N$ and $Z = (\xi_2, L^{-1}\rho_2, v_2)$. By choosing Z arbitrarily, we get the following system of equations:

(33)

$$\begin{aligned}
&- t\operatorname{Im}\mathcal{D}_\mu^*\mathcal{D}_\mu\xi_1 - t\operatorname{Re}\mathcal{D}_\mu^* \circ d\Psi_*(\mathcal{D}_0\rho_1 - \sqrt{-1}v_1)\\
&- (1-t)\mathcal{D}_0^*\mathcal{D}_0\rho_1 - (1-t)d_v\mu_N(v_1) + (1-t)[\mathcal{D}_0^*\mathcal{D}_0\rho + \mu_N(v), \xi_1]\\
= &\int_0^1 \operatorname{Re}\mathcal{D}_{\mu_s}^* \circ d\Psi_{s*}(s\mathcal{D}_0\rho + \sqrt{-1}sv) + \mathcal{D}_0^*\mathcal{D}_0(s\rho) - d_v\mu_N(sv)ds \mod \mathfrak{g}_0,
\end{aligned}$$

$$\begin{aligned}
&t\operatorname{Re}\mathcal{D}_0^* \circ (d\Psi_*)^t\mathcal{D}_\mu\xi_1\\
&\quad+ t\operatorname{Im}\mathcal{D}_0^* \circ (d\Psi_*)^t \circ (d\Psi_*)(\mathcal{D}_0\rho_1 - \sqrt{-1}v_1) + (1-t)\mathcal{D}_0^*\mathcal{D}_0\xi_1\\
&= -\int_0^1 \operatorname{Im}\mathcal{D}_0^* \circ (d\Psi_s)_*^t \circ (d\Psi_s)_*(s\mathcal{D}_0\rho - \sqrt{-1}sv)ds \mod \mathfrak{g}_0,
\end{aligned} \tag{34}$$

and
(35)

$$t(d\Psi_*)^t \circ (\mathcal{D}_\mu \xi_1 + d\Psi_*(\sqrt{-1}\mathcal{D}_0\rho_1 + v_1)) + (1-t)v_1 + (1-t)(d_v\mu_N)^*(\xi_1)$$
$$= -\int_0^1 (d\Psi_{s*})^t \circ d\Psi_{s*}(s\sqrt{-1}\mathcal{D}_0\rho + sv) - svds \text{ mod } \operatorname{Im}\mathcal{D}_0.$$

We claim this system has a unique smooth solution for $[\rho, v]$ small in C^∞. For this, we first notice that when $[\rho, v] = [0, 0]$, the operators are simply $[\mathcal{D}_0^*\mathcal{D}_0\rho_1, \mathcal{D}_0^*\mathcal{D}_0\xi_1, v_1]$ and the system with any prescribed right-hand side can be solved by elliptic theory (Hodge theory). Then for $[\rho, v]$ small, the unique solvability follows from an easy application of the implicit function theorem (for Banach manifolds), and again by ellipticity the solutions are smooth and with estimates.

Next we prove that there are two neighborhoods $\mathcal{N}_1$, $\mathcal{N}_2$ of 0 in $\mathfrak{m} \times N$, and a smooth tame map F from $\mathcal{N}_1$ to $C^\infty([0,1], \mathfrak{m} \times N)$ such that the time 1 evaluation of the image of F is a smooth tame map from $\mathcal{N}_1$ to $\mathcal{N}_2$ and for any $(\rho, v) \in \mathcal{N}_1$,

$$\begin{cases} \frac{d}{dt}F_t(\rho, v) = (\rho_1(t), v_1(t)), & t \in [0,1], \\ F_0(\rho, v) = (\rho, v). \end{cases} \tag{36}$$

To prove this claim, we shall exploit Hamilton's implicit function theorem again. Define a map

$$H : C^\infty([0,1], \mathfrak{m} \times N) \to (\mathfrak{m} \times N) \times C^\infty([0,1], \mathfrak{m} \times N)$$

that sends $(\rho(t), v(t))$ to $(\rho(0), v(0)) \times (\dot\rho(t) - \rho_1(t), \dot v(t) - v_1(t))$. It is clear that H is a smooth tame map and $H(0) = 0$. We need to show that for $x = (\rho(t), v(t))$ close to zero, the derivative of H at x is invertible and its inverse is smooth tame. Let $\delta x = (\tilde\rho(t), \tilde v(t))$. Then the derivative of H along δx is given by $(\tilde\rho(0), \tilde v(0)) \times (\dot{\tilde\rho} - \delta\rho_1(\tilde\rho), (\dot{\tilde v} - \delta v_1(\tilde v))$. So the invertibilty of dH is equivalent to the solvability of the Cauchy problem of the following linear system along $(\rho(t), v(t))$:

$$\begin{cases} \frac{d}{dt}(\alpha, u) = (\delta\rho_1(\alpha), \delta v_1(u)) + (\beta, q), & t \in [0,1], \\ (\alpha(0), u(0)) = (\tilde\rho(0), \tilde v(0)). \end{cases} \tag{37}$$

Thus we need to linearize equations (33) and (35). By collecting highest order terms, we get the following:
(38)

$$\begin{cases} \dot\alpha(t) = A_1(\rho(t), v(t))\alpha(t) + B_{-2}(\rho(t), v(t))u(t) + \beta(t), \\ \dot u(t) = C_2(\rho(t), v(t))\dot\alpha(t) + D_2(\rho(t), v(t))\alpha(t) + E_0(\rho(t), v(t))u(t) + q(t), \\ (\alpha(0), u(0)) = (\tilde\rho(0), \tilde v(0)), \end{cases}$$

where the A_i's are pseudo-differential operators of order i whose coefficients depend smoothly on $(\rho(t), v(t))$. The first operator A_1 is of order 1 due to the bracket term $(1-t)[\mathcal{D}_0^*\mathcal{D}_0\rho, \xi_1]$. Let $w(t) = u(t) - C_2(\rho(t), v(t))\alpha(t)$. Then the systems of equations for $(\alpha(t), w(t))$ become symmetric hyperbolic, for which the Cauchy problem is always solvable with estimates; see [45, §0.8]. One problem is that here the operator A_1, as the composition of an inverse of a fourth order elliptic operator and a fifth order differential operator, is not precisely in the Hörmander class OPS^1. But this can be taken care by using the parametrix of the elliptic operator. From the proof we can check that the solution depends tamely on $(\rho(t), v(t))$, $(\beta(t), q(t))$ and the initial condition $(\tilde{\rho}(0), \tilde{v}(0))$. Moreover, we can ensure that the solution $(\alpha(t), u(t))$ still lies in $\mathfrak{m} \times N$. So by the Nash-Moser implicit function theorem [29], H has a local smooth tame inverse. Let $F = H^{-1}(-, 0)$; the claim is then proved. Now for any $[g, \rho, v]$ close to $[\mathrm{Id}, 0, 0]$, we obtain a path $(\rho(t), v(t)) = F_t(\rho, v)$. Then we can solve the ordinary differential equation $\dot{g}(t) = L_{g(t)}\xi_1(t)$, where $\xi_1(t)$ is determined by $(\rho(t), v(t))$. Then $[g(t), \rho(t), v(t)]$ is an integral curve of Y_t by the $\mathcal{G}$-invariancy. Now we define

$$\Sigma_2 : \mathcal{V}_2 \to \mathcal{U}_2; [g, \rho, v] \mapsto [g(1), F_1(\rho, v)].$$

Then from the previous arguments we know that Σ_2 is smooth tame and fixes $\mathcal{G}.[\mathrm{Id}, 0, 0]$. Moreover, $\Sigma_2^*\Omega' = \Omega_Y$. It follows from equations (33), (34) and (35) that we have a tame estimate

$$|v_1(t)|_k \leq C(k) \cdot (|\rho(t)|_{k+s_1} + |v(t)|_{k+s_1})^3.$$

Since $|(v(t), \rho(t))|_{k+s_1} \leq C(k) \cdot |(v(0), \rho(0))|_{k+s}$ for some $s > s_1$, we obtain

$$|v(1) - v(0)|_k \leq C \cdot (|\rho(0)|_{k+s} + |v(0)|_{k+s})^3.$$

By symmetry, we can obtain the map Σ_1. Then one can check that the required estimates hold. □

Now to prove Lemma 6.2, we just need to apply the previous proposition to the path $\hat{J}(t)$ and use exactly the same argument as in the proof of Theorem 4.6.

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