

# The Oort Conjecture on lifting covers of curves

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## Abstract

We show that the conjecture of Oort on lifting covers of curves is true. The main ingredients in the proof are a deformation argument in characteristic  $p$  and (a special case of) a very recent result by Obus–Wewers. A kind of boundedness result is given as well.

## 1. Introduction

The aim of this note is to present a proof of the (classical) Oort Conjecture, which is a question about lifting Galois covers of curves from characteristic  $p > 0$  to characteristic zero. In one form or the other, this kind of question might well be considered math folklore, and it was also well known that in general the lifting is not possible. The problem was systematically addressed and formulated by Oort [16] in the 1980’s, but see rather [17]. The general context of the lifting question/problem is as follows: Let  $k$  be an algebraically closed field of characteristic  $p > 0$ , and let  $W(k)$  be the ring of Witt vectors over  $k$ . Let  $Y \rightarrow X$  be a possibly ramified  $G$ -cover of complete smooth  $k$ -curves, where  $G$  is a finite group. The *lifting problem* for the  $G$ -cover  $Y \rightarrow X$  is whether there exists a finite extension  $R$  of  $W(k)$  and a  $G$ -cover  $\mathcal{Y}_R \rightarrow \mathcal{X}_R$  of complete smooth  $R$ -curves whose special fiber is the given  $G$ -cover  $Y \rightarrow X$ . If such a  $G$ -cover  $\mathcal{Y}_R \rightarrow \mathcal{X}_R$  exists, we say that the  $G$ -cover  $Y \rightarrow X$  has a *smooth lifting*, or that the lifting problem for the  $G$ -cover  $Y \rightarrow X$  is *solvable*. The lifting problem in general is not solvable, because over  $k$  there are curves of genus  $g > 1$  with huge automorphism groups (see, e.g., Roquette [19]), whereas in characteristic zero, one has the Hurwitz bound  $84(g - 1)$  for the order of the automorphism group. The Oort conjecture on lifting curve covers asserts roughly that the general obstruction to the solvability of the lifting problem originates from the nature of  $G$ , and its simplest form is the following:

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OORT CONJECTURE. *The lifting problem is solvable for all cyclic covers.*

As we will see later, the above conjecture is equivalent to the following more general statement, which is a generalization of Grothendieck's lifting theorem for *tame covers*:

(GENERAL) OORT CONJECTURE. *The lifting problem is solvable for all  $G$ -covers  $Y \rightarrow X$  whose inertia groups are cyclic.*

There is also the *local Oort conjecture*, which asserts that every finite cyclic extension  $k[[t]] \hookrightarrow k[[u]]$  has a smooth lifting; i.e., it is canonically the reduction of a cyclic extension  $R[[T]] \hookrightarrow R[[U]]$  for some finite extension  $R$  of  $W(k)$ . The local and global Oort conjectures are related as follows; see Fact 4.13, where further equivalent forms of these conjectures are given: Let  $R$  be a finite extension of  $W(k)$ , and let  $\mathcal{X}_R$  be a complete smooth curve with special fiber  $X$ . Then a given  $G$ -cover  $Y \rightarrow X$ ,  $y \mapsto x$ , with cyclic inertia groups lifts to a  $G$ -cover  $\mathcal{Y}_R \rightarrow \mathcal{X}_R$  of complete smooth  $R$ -curves if and only if (the maximal  $p$ -power subextension of) the local extension  $k[[t_x]] := \widehat{\mathcal{O}}_{X,x} \hookrightarrow \widehat{\mathcal{O}}_{Y,y} =: k[[t_y]]$  has smooth liftings over  $R$  for all  $y \mapsto x$ ,  $x \in X$ .

*Notation.* Let  $\deg_p(\mathfrak{D})$  be the different degree of the maximal  $p$ -power subextension of  $k[[t]] \hookrightarrow k[[u]]$ , and let  $\deg_p(\mathfrak{D}_x)$  be correspondingly defined for  $k[[t_x]] \hookrightarrow k[[t_y]]$  at each  $y \mapsto x$ .

THEOREM 1.1. *The (General) Oort Conjecture holds. Moreover, for every positive integer  $\delta$ , there exists an algebraic integer  $\pi_\delta \in \overline{\mathbb{Z}}$  such that for every prime number  $p$  and every algebraically closed field  $k$  with  $\text{char}(k) = p$ , in the above notation the following hold:*

- (1) *Every cycle  $k[[t]] \hookrightarrow k[[u]]$  with  $\deg_p(\mathfrak{D}) \leq \delta$  has a smooth lifting over  $R = W(k)[\pi_\delta]$ .*
- (2) *Let  $Y \rightarrow X$  be a  $G$ -cover with cyclic inertia groups and  $\deg_p(\mathfrak{D}_x) \leq \delta$  for all  $x \in X$ . Then the  $G$ -cover  $Y \rightarrow X$  has a smooth lifting over  $R = W(k)[\pi_\delta]$ .*

*Historical note.* The first evidence for the Oort Conjecture is the fact that the conjecture holds for  $G$ -covers  $Y \rightarrow X$  that have tame ramification only, i.e.,  $G$ -covers whose inertia groups are cyclic of the form  $\mathbb{Z}/m$  with  $(p, m) = 1$ . Indeed, the lifting of such  $G$ -covers follows from the famous *Grothendieck's specialization theorem* for the tame fundamental group; see, e.g., SGA I. The first result that involved typical wild ramification was Oort–Sekiguchi–Suwa [21], which tackled the case of  $\mathbb{Z}/p$ -covers. It was followed by a quite intensive research; see the survey article Obus [13] as well as the bibliography list at the end of this note. Garuti [5] and [9] contain a lot of foundational work and beyond that show that every  $G$ -cover  $Y \rightarrow X$  has possibly nonsmooth liftings but with a well-understood geometry. This aspect of the problem was revisited

by Saidi [20], where among other things a systematic discussion of the (equivalent) forms of the Oort Conjecture is given. The paper Green–Matignon [7] contains further foundational work and gives a positive answer to the Oort Conjecture in the case of inertia groups of the form  $\mathbb{Z}/mp^e$  with  $(p, m) = 1$  and  $e \leq 2$ , relying on/using ideas from the Sekiguchi–Suwa theory [22], [23]. In Corry [4], a completely new ( $p$ -adic) approach to the local Oort Conjecture was proposed. The paper Bertin–Mézard [1] addresses the deformation theory for covers, whereas Chinburg–Guralnick–Harbater [2], [3] initiated the study of the so-called *Oort groups* and showed that the class of Oort groups is quite restrictive. Last but not least, the very recent result by Obus–Wewers [15] solves the Oort Conjecture in the case the inertia groups are of the form  $\mathbb{Z}/mp^e$  for  $(p, m) = 1$  and  $e \leq 3$ , and a case *critical for the method of this note*, when the upper ramification jumps are subject to some explicit restrictions; see the explanations in Remark 4.17 for details.

*About the proof.* Concerning technical tools, we freely use a few of the foundational results from the papers mentioned above. The main novel tools for the proof are Key Lemma 3.2 and its global form Theorem 3.6, and second, Key Lemma 4.15, which is actually a special case of the main result Obus–Wewers [15]; see Section 4 for precise details and references. All these results will be used as “black boxes” in the proof of the Oort Conjecture, given in Section 4. Concerning the idea of the proof, there is little to say: The point is to first deform a given  $G$ -cover  $Y \rightarrow X$  to a cover  $\mathcal{Y}_\mathfrak{o} \rightarrow \mathcal{X}_\mathfrak{o}$  over  $\mathfrak{o} = k[[\varpi]]$  in such a way that the ramification of the deformed cover has no *essential upper jumps* as defined/introduced at the beginning of Section 3, then apply the local-global principles, etc. I should also mention that the first variant of the proof (January, 2012) was shorter, but it relied on model theoretical tools and did not give any kind of bounds for the degree of the extension  $W(k) \hookrightarrow R$ .

*Acknowledgements.* If my recollection is correct, during an MFO Workshop in 2003 or so, someone asked what should be the “characteristic  $p$  Oort Conjecture,” but it seems that nobody ever followed up (successfully) on that idea. My special thanks go to the referee for the careful reading of my manuscript and his several comments and suggestions.

## 2. Reviewing well-known facts

Throughout this section,  $k$  is an algebraically closed field with positive characteristic  $\text{char}(k) = p > 0$ . All the other fields will be field extensions of  $k$ ; in particular, they will be fields of characteristic  $p$ .

2.A. *Reviewing higher ramification for cyclic extensions.* (See, e.g., Serre [24, IV].) Let  $\mathcal{O}_E \subset E$  be a discrete valuation ring with valuation  $v$ . Let  $F|E$  be a finite Galois extension with Galois group  $\text{Gal}(F|E)$  such that the

prolongations  $w$  of  $v$  to  $F$  have separable residue field extensions  $\kappa(w)|\kappa(v)$ . If  $T \subset Z$  denote the inertia, respectively decomposition, groups of  $w|v$ , and  $E \hookrightarrow F^Z \hookrightarrow F^T \hookrightarrow F$  denote the corresponding decomposition, respectively inertia, subfields of  $F|E$ , then one has that  $[F : E] = [F^Z : E] f(w|v) e(w|v)$ , where  $f(w|v) = [\kappa(w) : \kappa(v)] = (Z : T)$  and  $e(w|v) = (wF^\times : vE^\times) = (T : 1)$  are the residue degree, respectively the ramification index, of  $w|v$ . Finally, we suppose that  $wF^\times = \mathbb{Z}$ .

Recall that the lower ramification groups of  $w|v$  are defined as follows: For every  $j \geq -1$ , we set  $G_j := \{\sigma \in \text{Gal}(F|E) \mid w(z) \geq 0 \Rightarrow w(\sigma z - z) > j\}$  and call it the  $j^{\text{th}}$  lower ramification group of  $F|E$  at  $w|v$ . Then  $G_j = 1$  for  $j$  sufficiently large, and  $G_{-1} = Z$ ,  $G_0 = T$ , and  $G_1$  is the wild ramification group of  $w|v$ , i.e., the Sylow  $p$ -group of  $T$ , where  $p = \text{char}(\kappa(v)) > 0$ .

Let  $\mathcal{O}_F$  be the integral closure of  $\mathcal{O}_E$  in  $F$ . Then  $\mathcal{O}_F$  is a principal ideal domain whose localizations at its maximal ideals  $\mathfrak{q} \subset \mathcal{O}_F$  are precisely the valuation rings of the prolongations  $w|v$  of  $v$  to  $F$ . The first important fact about the lower ramification groups  $(G_j)_j$  is *Hilbert's different formula*, which gives the degree of the different  $\mathfrak{D}_v := \mathfrak{D}_{\mathcal{O}_F|\mathcal{O}_E}$  of  $F|E$  at  $v$  in terms of the orders of the lower ramification groups (see, e.g., Serre [24, IV]):

$$\text{deg}(\mathfrak{D}_v) = [F^T : E] \sum_{j=0}^{\infty} (|G_j| - 1).$$

We denote by  $j_\rho$  the lower jumps for  $w|v$ , as being the numbers satisfying  $G_{j_\rho} \neq G_{j_\rho+1}$ . In particular, setting  $j_{-1} = -1$  and  $j_0 = 0$ , and denoting the lower jumps for  $L|K$  by  $j_0 \leq j_1 \leq \dots \leq j_r$ , one has that  $j_r = \max\{j \mid G_j \neq 1\}$ .

Now suppose that  $T = \mathbb{Z}/p^{e_T}$  is cyclic, where  $p = \text{char}(\kappa(v))$ . Then  $G_0 = G_1$  and every subgroup of  $T$  is a lower ramification group for  $F|E$  at  $w|v$ ; see Serre, [24, IV]. Hence all the nontrivial subgroups  $G_{j_1} \geq \dots \geq G_{j_{e_T}}$  of  $T$  are higher ramification subgroups, and there are  $e_T$  lower ramification jumps  $j_1 \leq \dots \leq j_{e_T}$ . Finally, the Hilbert's different formula becomes

$$\begin{aligned} \text{deg}(\mathfrak{D}_v) &= [F^T : E] \left( p^{e_T} - 1 + \sum_{\rho=1}^{e_T} (j_\rho - j_{\rho-1})(|G_{j_\rho}| - 1) \right) \\ &= [F^T : E] \left( p^{e_T} - 1 + \sum_{\rho=1}^{e_T} (j_\rho - j_{\rho-1})(p^{e_T - (\rho-1)} - 1) \right). \end{aligned}$$

We recall that the lower ramification subgroups behave functorially in the base field, i.e., if  $E \subseteq E' \subset F$ , for every  $j$  one has that  $G'_j = G_j \cap \text{Gal}(F|E')$ . On the other hand, the lower ramification groups do not behave functorially with respect to Galois sub-extensions. Therefore, one introduces the *upper ramification groups*  $G^i$  for  $i \geq -1$  of  $F|E$  at  $w|v$ , which behave functorially under taking Galois sub-extensions; see Serre [24, IV].

At least in the case that the inertia group  $T$  is cyclic  $T = \mathbb{Z}/p^{e_T}$  with  $p = \text{char}(\kappa(v))$  as above, the formula that relates the lower ramification groups  $G_j$  to the upper ramification groups  $G^i$  is explicit via *Herbrand's formula*; see, e.g., Serre [24, IV, §3]. Namely, if  $\iota_0 := 0$  and  $\iota_1 \leq \dots \leq \iota_{e_T}$  are the *upper ramification jumps* at  $w|v$  in  $F|E$ , then one has

$$j_\rho - j_{\rho-1} = p^{\rho-1}(\iota_\rho - \iota_{\rho-1}), \quad \rho = 1, \dots, e_T.$$

Thus in the case  $T = \mathbb{Z}/p^{e_T}$  with  $p = \text{char}(\kappa(v))$  as above, one gets

$$\begin{aligned} \deg(\mathfrak{D}_v) &= [F^T : E] \left( p^{e_T} - 1 + \sum_{\rho=1}^{e_T} (\iota_\rho - \iota_{\rho-1}) p^{\rho-1} (p^{e_T - (\rho-1)} - 1) \right) \\ &= [F^T : E] \sum_{\rho=1}^{e_T} (\iota_\rho + 1) (p^\rho - p^{\rho-1}). \end{aligned}$$

We conclude this subsection by recalling that in the above situation, i.e., if the residue field extension  $\kappa(w)|\kappa(v)$  is separable, the groups  $G_j$  and  $G^i$  do not change under completion, and the ones for  $j, i \geq 0$  do not change under unramified extensions.

2.B. *Explicit formulas via Artin–Schreier–Witt Theory.* Recall that the Artin–Schreier–Witt theory gives a description of the cyclic  $p$ -power extensions of a field  $E$  with  $\text{char}(E) = p > 0$  via finite length Witt vectors as follows; see, e.g., Lang [12], or Serre [24, IV]. Let  $\mathcal{T}$  be an integrally closed domain over  $\mathbb{F}_p$ , and let  $W_e(\mathcal{T}) = \{\mathbf{a} := (a_1, \dots, a_e) \mid a_1, \dots, a_e \in \mathcal{T}\}$  be the Witt vectors of length  $e$  over  $\mathcal{T}$ . Then the Frobenius morphism  $\text{Frob}$  of  $\mathcal{T}$  lifts to the *Frobenius morphism*  $\text{Frob}_e$  of  $W_e(\mathcal{T})$ , and one defines the *Artin–Schreier–Witt operator*  $\wp_e := \text{Frob}_e - \text{Id}$  of  $\mathcal{T}$ . If  $\mathcal{T} \hookrightarrow \mathcal{T}^{\text{nr}}$  is an ind-étale universal cover of  $\mathcal{T}$ , one has the Artin–Schreier–Witt exact sequence

$$0 \rightarrow W_e(\mathbb{F}_p) = \mathbb{Z}/p^e \longrightarrow W_e(\mathcal{T}^{\text{nr}}) \xrightarrow{\wp_e} W_e(\mathcal{T}^{\text{nr}}) \rightarrow 0$$

of sheaves on  $\text{Et}(\mathcal{T})$ . In particular, if  $\text{Pic}(\mathcal{T}) = 0$ , one gets a canonical isomorphism

$$W_e(\mathcal{T})/\text{im}(\wp_e) \rightarrow \text{Hom}(\pi_1(\mathcal{T}), \mathbb{Z}/p^e),$$

which gives rise to a canonical bijection correspondence between the cyclic subgroups  $\langle \mathbf{a} \rangle \subset W_e(\mathcal{T})/\text{im}(\wp_e)$  and the integral étale cyclic extensions  $\mathcal{T} \hookrightarrow \mathcal{T}_{\mathbf{a}}$  with Galois group a quotient of  $\mathbb{Z}/p^e$  by via

$$\langle \mathbf{a} \rangle \mapsto \mathcal{T}_{\mathbf{a}} := \mathcal{T}[\mathbf{x}], \quad \text{where } \mathbf{x} = (x_1, \dots, x_e) \text{ and } \wp_e(\mathbf{x}) = \mathbf{a}.$$

*Fact 2.1.* In the above context, let  $\mathcal{T} := E$  be a field with  $\text{char}(E) = p$ .

(1) For  $\mathbf{a} = (a_1, \dots, a_e)$  an arbitrary Witt vector of length  $e$  over  $E$ , and  $E_{\mathbf{a}}|E$  as above, one has:  $[E_{\mathbf{a}} : E] = p^n$  if and only if  $a_1 \notin \text{im}(\wp_1)$ ; further, if  $a_1 \in \text{im}(\wp_1)$ , then  $[E_{\mathbf{a}} : E] = p^m$  with  $m$  nonnegative and minimal such that  $(a_1, \dots, a_{e-m}) \in \text{im}(\wp_{e-m})$ .

Precisely, if  $0 \leq m < n$  and  $(a_1, \dots, a_{e-m}) = \wp_{e-m}(c_1, \dots, c_{e-m})$ , then choosing  $\mathbf{b} = (b_1, \dots, b_m)$  such that

$$(0, \dots, 0, b_1, \dots, b_m) = (a_1, \dots, a_e) - \wp_e(c_1, \dots, c_{e-m}, 0, \dots, 0),$$

one has that  $E_{\mathbf{a}}|E$  is actually the cyclic extension  $E_{\mathbf{b}}|E$  of degree  $p^m$  of  $E$ .

(2) Let  $v$  be a valuation of  $E$  with valuation ring  $\mathcal{O}$ , residue field  $\mathcal{O} \rightarrow \kappa(v)$ , and value groups  $v(E)$ . And for  $a \in E$ , let  $v(a)$  be its valuation, and if  $a \in \mathcal{O}$ , let  $\bar{a} \in \kappa(v)$  be the residue of  $a$ . Given  $\mathbf{a} = (a_1, \dots, a_e)$ , and some  $m \leq e$ , set  $\mathbf{a}_m := (a_1, \dots, a_m)$ . Then the behavior of  $v$  in the cyclic field extension  $E \hookrightarrow E_{\mathbf{a}}$  satisfies the following:

- (a) Suppose that  $v(a_1), \dots, v(a_m) \geq 0$ . Then  $v$  is not ramified in  $E \hookrightarrow E_{\mathbf{a}_m}$ , and  $v$  is totally split in  $E \hookrightarrow E_{\mathbf{a}_m}$  if and only if  $(\bar{a}_1, \dots, \bar{a}_m) \in \wp_m(\kappa(v))$ .
- (b) Let  $m$  be minimal such that  $v(a_m) < 0$  and  $v(a_m)$  is not divisible by  $p$  in  $vE^\times$ . Then the (wild) inertia field of  $v$  is strictly contained in  $E_{\mathbf{a}_m}$ .

We next have a closer look at the relationship between higher ramification and Artin–Schreier–Witt theory. Let  $E, v$  be a discrete valued field with  $\text{char}(E) = p > 0$ . Let  $F|E$  be a finite Galois extension and  $w|v$  a prolongation of  $v$  to  $F$  with inertia/decomposition groups  $T \leq Z$ . Suppose that the residue field extension  $\kappa(w)|\kappa(v)$  is *separable*, or equivalently,  $|T| = e(w|v)$ , where  $e(w|v)$  is the ramification index.

Then the classical Hilbert ramification theory works for  $w|v$ ; i.e., the lower/upper ramification groups and indices and the Hilbert different formula are as above. Further, they are invariant under completions; i.e., if  $\widehat{F}|\widehat{E}$  is the completion of  $F|E$  at  $w|v$ , then the lower/upper ramification groups and indices of  $w|v$  in  $F|E$  and  $\widehat{F}|\widehat{E}$  are the same.

Since  $\text{char}(E) = p > 0$ , the completion  $\widehat{E}$  of  $E$  is a Laurent power series field  $\widehat{E} = \lambda((t))$ , where  $\lambda$  is any maximal subfield of  $\widehat{E}$  on which  $v$  is trivial, and  $t$  is any uniformizing parameter of  $v$ , hence  $\kappa(v) \cong \lambda$  (noncanonically). Further, the same is true correspondingly for  $F$ ; i.e.,  $\widehat{F} = \lambda_F((z))$ , etc. On the other hand, after fixing an identification  $\widehat{E} = \lambda((t))$ , let  $\lambda'|\lambda$  be the finite separable extension of  $\lambda$  that is isomorphic to  $\kappa(w)|\kappa(v)$  under the field isomorphism  $\kappa(v) \cong \lambda$ . Then using Hensel's Lemma, it follows that  $\lambda'$  can be embedded in  $\widehat{F}$ , and finally  $\widehat{F}|\widehat{E}$  becomes isomorphic to  $\lambda'((z))|\lambda((t))$ , where  $t, z$  are uniformizing parameters of  $v$ , respectively  $w$ , and  $\lambda'|\lambda$  is isomorphic to  $\kappa(w)|\kappa(v)$ .

We thus conclude that the lower/upper ramification groups  $G_j$  and  $G^j$  of  $w|v$  are the same as the ones of  $\lambda'((z))|\lambda((t))$ , and the ones for  $j, i \geq 0$  are the same as the ones of  $\bar{\lambda}((z))|\bar{\lambda}((t))$ .

We next have a closer look at the case where  $F|E$  is a  $\mathbb{Z}/p^e$ -extension. Then  $F = E_{\mathbf{q}}$ , where  $\mathbf{q} = (q_1, \dots, q_e) \in W_e(E)$  is some Witt vector of length

$e$  over  $E$ , and  $T = \mathbb{Z}/p^{e_T}$  for some  $0 \leq e_T \leq e$ . (Formally correct it would be  $p^{e-e_T}\mathbb{Z}/p^e\mathbb{Z}$ .) We will do the following:

- First, find sufficient conditions on  $\mathbf{q}$  that imply  $\kappa(w)|\kappa(v)$  is separable.
- Second, relate the upper ramification jumps to some (almost uniquely determined) Witt vectors over  $\lambda[t^{-1}]$  called equivalent (*quasi*) *standard forms* of  $\mathbf{q}$ .

*Definition/Remark 2.2.* Let  $\lambda$  be a fixed field of characteristic  $p > 0$ .

- (1) We say that a Witt vector  $\mathbf{p} = (p_1, \dots, p_e) \in W_e([\lambda[t^{-1}]])$  is in *standard form* if each  $p_i = \sum_{ij} a_{ij}t^{-j} \in \lambda[t^{-1}]$  satisfies:  $p \mid j$  implies  $a_{ij} = 0$  for all  $1 \leq i \leq e$  and all  $j$ .
- (2) Given  $\mathbf{q} \in W_e(\lambda((t)))$ , an equivalent *quasi standard form* of  $\mathbf{q}$  is any Witt vector of the form  $\mathbf{p} + \mathbf{a}$  with  $\mathbf{p} \in W_e([\lambda[t^{-1}]])$  in standard form and  $\mathbf{a} \in W_e(\lambda)$  such that  $\mathbf{q}$  and  $\mathbf{p} + \mathbf{a}$  are equivalent modulo  $\text{im}(\wp_e)$ .
- (3) We notice that the equivalent quasi standard form of  $\mathbf{q} \in W_e(\lambda((t)))$  is unique modulo  $\wp_e(W_e(\lambda))$  in the following sense: Let  $\mathbf{p} + \mathbf{a}$  and  $\mathbf{p}' + \mathbf{a}'$  be equivalent quasi standard forms of  $\mathbf{q}$ . Then  $\mathbf{p} = \mathbf{p}'$  and  $\mathbf{a} - \mathbf{a}' \in \wp_e(W_e(\lambda))$ .

Using the remarks above and the estimate for  $\text{deg}(\mathfrak{D}_v)$  at the end of Section 2.A, following Garuti [6, Th. 1.1] and Thomas [25, Prop. 4.2] (see also Obus–Priess [14] for assertions concerning the upper jumps  $\iota_1 \leq \dots \leq \iota_{e_T}$ ), one has the following:

*Fact 2.3.* For  $\mathbf{q} = (q_1, \dots, q_e) \in W_e(E)$ , let  $F := E_{\mathbf{q}}$  be the corresponding  $p$ -power cyclic extension and  $w|v$  be a prolongation of  $v$  to  $F$ . Let  $\widehat{E} = \lambda((t))$  and  $\widehat{F} = \lambda_F((z))$ , with  $\lambda_F = \lambda'$  if the residue field extension  $\kappa(w)|\kappa(v)$  is separable, be as above. Then one has

- (1) Set  $\delta_1 := \max\{-v(q_1), 0\}$ , and define inductively  $\delta_i := \max\{p\delta_{i-1}, -v(q_i)\}$  for  $1 < i \leq e$ . Let  $\delta_0 = \lfloor \delta_e/(p-1) \rfloor$ , and let  $\lambda_0 := \text{Frob}^{\delta_0}(\lambda)$  be the image of  $\lambda$  under the  $\delta_0$  power of the Frobenius morphism. If  $\mathbf{q}$  is defined over  $\lambda_0((t))$ , the following hold:
  - (a) The residue field extension  $\kappa(w)|\kappa(v)$  of  $F|E$  is separable, and the Hilbert higher ramification theory works for  $F|E$  as indicated in Section 2.A.
  - (b) There exists a quasi standard equivalent form  $\mathbf{p} + \mathbf{a} \in W_e([\lambda[t^{-1}]])$  of  $\mathbf{q}$ , and moreover, the entries  $p_i \in \lambda[t^{-1}]$  of  $\mathbf{p} = (p_1, \dots, p_e)$  satisfy the following:
    - If  $\delta_i = 0$ , then  $p_i = 0$ .
    - Let  $\delta_i > 0$ . If  $p \mid \delta_i$ , then  $\text{deg}(p_i(t^{-1})) < \delta_i$  (possibly,  $p_i = 0$ ). And if  $p \nmid \delta_i$ , then  $\text{deg}(p_i(t^{-1})) = \delta_i = -v(q_i)$  and  $q_i$  and  $p_i$  have equal leading terms.
- (2) Suppose that  $\kappa(w)|\kappa(v)$  is separable and  $\mathbf{q}$  has a quasi standard form  $\mathbf{p} + \mathbf{a} \in W_e([\lambda[t^{-1}]])$  over  $\lambda((t))$ . Setting  $\mathbf{p} =: (p_1, \dots, p_e)$ , we define  $r_T = e + 1$  if

$p_i = 0$  for all  $1 \leq i \leq e$  and  $r_T := \min\{i \mid p_i(t^{-1}) \neq 0\}$  otherwise. Then setting  $e_T := e - r_T + 1$  and letting  $F_{r_T-1}|E$  be the unique subextension of  $F|E$  of degree  $p^{r_T-1}$ , one has

- (a)  $w|v$  is unramified in  $E \hookrightarrow F_{r_T-1}$  and  $\widehat{F}_{r_T-1} = \lambda'((t))$  inside  $\widehat{F} = \lambda'((z))$ .
- (b)  $w|v$  is totally wildly ramified in  $F_{r_T-1} \hookrightarrow F$  and has upper ramification jumps

$$i_\alpha = \max \{ p i_{\alpha-1}, \deg(p_{\alpha+r_T-1}) \}, \quad \alpha = 1, \dots, e_T \quad (\text{where } i_0 = 0).$$

- (c) The degree of the different at  $w|v$  in  $E \hookrightarrow F$  is given by

$$\deg(\mathfrak{D}_v) = p^{r_T-1} \sum_{\alpha=1}^{e_T} (i_\alpha + 1)(p^\alpha - p^{\alpha-1}).$$

*2.C. Local criterion for good reduction.* Let  $k$  be an algebraically closed field of characteristic  $p > 0$ , and let  $\mathfrak{o}$  be a complete discrete valuation ring with quotient field  $\hat{\kappa} = \text{Quot}(\mathfrak{o})$  and residue field  $k$ . Let  $\mathcal{A} = \mathfrak{o}[[T]]$  be the power series ring in the variable  $T$  over  $\mathfrak{o}$ ; hence  $\mathcal{A}$  is a two-dimensional complete regular ring with maximal ideal  $(\pi, T)$  and residue field  $\mathcal{A} \rightarrow \mathcal{A}/(\pi, T) = k$ . Further,  $\mathcal{R} := \mathcal{A}[\pi^{-1}] = \mathcal{A} \otimes_{\mathfrak{o}} \hat{\kappa}$  is the ring of power series in  $T$  over  $\hat{\kappa}$  having  $v_{\hat{\kappa}}$ -bounded below coefficients and satisfying:  $\mathcal{R}$  is a Dedekind ring with  $\text{Spec}(\mathcal{R}) = \text{Spec}(\mathcal{A}) \setminus V(\pi)$  in bijection with the points of the open rigid disc  $\mathfrak{X}$  of radius 1 over the complete valued field  $\hat{\kappa}$ . Finally,  $A := \mathcal{A}/(\pi) = k[[t]]$  is the power series ring in the variable  $t := T \pmod{(\pi)}$ , thus a complete discrete valuation ring.

Let  $\mathcal{K} := \text{Quot}(\mathcal{A}) = \text{Quot}(\mathcal{R})$  and  $K := \text{Quot}(A) = k((t))$  be the fraction fields of  $\mathcal{A}$ , respectively  $A$ . Let  $\mathcal{K} \hookrightarrow \mathcal{L}$  be a finite separable field extension, and let  $\mathcal{B} \subset \mathcal{S}$  be the integral closures of  $\mathcal{A} \subset \mathcal{R}$  in the field extension  $\mathcal{K} \hookrightarrow \mathcal{L}$ . Since  $\mathcal{K} \hookrightarrow \mathcal{L}$  is finite separable, it follows that  $\mathcal{B}$  is a finite  $\mathcal{A}$ -module, and  $\mathcal{S}$  is a finite  $\mathcal{R}$ -module, in particular, a Dedekind ring; see e.g., Serre [24, I] for this finiteness assertion.

Next let  $\mathfrak{r}_1, \dots, \mathfrak{r}_r$  be the prime ideals of  $\mathcal{B}$  above  $(\pi)$ , say with residue fields  $\kappa(\mathfrak{r}_i) \mid \kappa(\pi)$ . Then each  $\mathfrak{r}_i$  has height one, and the localizations  $\mathcal{B}_{\mathfrak{r}_i}$  are precisely the valuation rings of  $\mathcal{L}$  above the discrete valuation ring  $\mathcal{A}_{(\pi)}$  of  $\mathcal{K}$ . And since  $\mathcal{B}$  is a finite  $\mathcal{A}$ -module, thus  $\mathcal{B}_{(\pi)}$  is a finite  $\mathcal{A}_{(\pi)}$  module as well, the fundamental equality holds (see e.g., Serre [24, I]):

$$[\mathcal{L} : \mathcal{K}] = \sum_{i=1}^r e(\mathfrak{r}_i|\pi) \cdot f(\mathfrak{r}_i|\pi),$$

where  $e(\mathfrak{r}_i|\pi)$  and  $f(\mathfrak{r}_i|\pi) = [\kappa(\mathfrak{r}_i) : \kappa(\pi)]$  is the ramification index, respectively the residual degree at  $\mathfrak{r}_i|\pi$ . Hence if  $v_\pi$  is the discrete valuation of  $\mathcal{K}$  with valuation ring  $\mathcal{A}_{(\pi)}$ , one has  $K = \text{Quot}(\mathcal{A}/(\pi)) = \kappa(\pi) = \kappa(v_\pi)$ , and the following are equivalent:

- (i) There exists a prolongation  $v_{\mathcal{L}}$  of  $v_{\pi}$  to  $\mathcal{L}$  such that  $[\mathcal{L} : \mathcal{K}] = [L : K]$ , where  $L := \kappa(v_{\mathcal{L}})$ .
- (ii) The ideal  $\mathfrak{r} := \pi\mathcal{B}$  is a prime ideal of  $\mathcal{B}$ , or equivalently,  $\pi$  is a prime element of  $\mathcal{B}$ .

If the above equivalent conditions (i) and (ii) hold, then  $\mathcal{B}_{\mathfrak{r}}$  is the valuation ring of  $v_{\mathcal{L}}$ , and one has  $\kappa(\mathfrak{r}) = \text{Quot}(\mathcal{B}/(\pi)) = \kappa(v_{\mathcal{L}}) = L$ . Further,  $v_{\mathcal{L}}$  is the unique prolongation of  $v_{\pi}$  to  $\mathcal{L}$ , and  $\mathfrak{r} = \pi\mathcal{B}$  is the unique prime ideal of  $\mathcal{B}$  above the ideal  $\pi\mathcal{A}$  of  $\mathcal{A}$ . Finally, we say that  $v_{\pi}$  is *totally inert* in  $\mathcal{K} \hookrightarrow \mathcal{L}$  if  $\kappa(v_{\mathcal{L}})|\kappa(v_{\pi})$  is separable and conditions (i) and (ii) hold.

One has the following criterion for good reduction, which is a special case of the theory developed in Kato [11, §5]; see Green–Matignon [7, §3, especially 3.4] for details.

*Fact 2.4.* In the above notation, suppose that  $v_{\pi}$  is totally inert in  $\mathcal{K} \hookrightarrow \mathcal{L}$ . Let  $A \hookrightarrow B$  be the integral closure of  $A = k[[t]] = \mathcal{A}/(\pi)$  in the field extension  $K \hookrightarrow L$ . Let  $\deg(\mathfrak{D}_{\mathcal{S}|\mathcal{R}})$  and  $\deg(\mathfrak{D}_{B|A})$  be the different degrees of the extensions of Dedekind rings  $\mathcal{R} \hookrightarrow \mathcal{S}$ , respectively  $A \hookrightarrow B$ . Then one always has  $\deg(\mathfrak{D}_{\mathcal{S}|\mathcal{R}}) \geq \deg(\mathfrak{D}_{B|A})$ , and the following are equivalent:

- (i)  $\deg(\mathfrak{D}_{\mathcal{S}|\mathcal{R}}) = \deg(\mathfrak{D}_{B|A})$ .
- (ii)  $\mathcal{B} = \mathfrak{o}[[Z]]$  for some  $Z \in \mathcal{B}$  and  $B = \mathcal{B}/(\pi) = k[[z]]$ , where  $z = Z \pmod{(\pi)}$ .

### 3. The characteristic $p$ Oort Conjecture

In this section,  $k$  is a fixed algebraically closed field with  $\text{char}(k) = p > 0$ .

*Remark/Definition 3.1.* (1) In the context and notation from Section 2.A, suppose that  $\text{char}(E) = p > 0$ , and the finite Galois extension  $F|E$  together with the discrete valuations  $w|v$  satisfy conditions (i) and (ii) from that section. Then the Hilbert ramification theory applies to  $F|E$  endowed with  $w|v$ . Moreover, supposing that the inertia group  $T$  of  $w|v$  is cyclic, say  $T = T_1 \times T_0$  with  $T_0 = \mathbb{Z}/p^{e_T}$  and  $T_1 = \mathbb{Z}/m$  of order prime to  $p$ , it follows that  $F^{T_1}|F^T$  is a cyclic  $\mathbb{Z}/p^{e_T}$ -extension, which is at the same time the unique maximal totally wildly ramified subextension of  $F|F^T$ . Then setting  $\iota_0 = 0$ , by Fact 2.3(2), one has that the higher ramification jumps  $\iota_1 \leq \dots \leq \iota_{e_T}$  of  $w|v$  in  $F^{T_1}|F^T$  satisfy

$$p\iota_{\rho-1} \leq \iota_{\rho} \text{ for } 0 < \rho \leq e_T, \text{ and } p\iota_{\rho-1} < \iota_{\rho} \text{ if and only if } p \nmid \iota_{\rho}.$$

Hence for each  $\rho = 1, \dots, e_T$ , the division of  $\iota_{\rho} - p\iota_{\rho-1}$  by  $p$  gives

$$\iota_{\rho} - p\iota_{\rho-1} = pq_{\rho} + \varepsilon_{\rho},$$

with  $0 \leq q_{\rho}$ ,  $0 \leq \varepsilon_{\rho} < p$ . Thus  $0 < \varepsilon_{\rho}$  if and only if  $(p, \iota_{\rho}) = 1$  if and only if  $p\iota_{\rho-1} < \iota_{\rho}$ . We call  $q_{\rho}$  the *essential part* of the upper jump for  $w|v$  at  $\rho$ , and if  $0 < q_{\rho}$ , we say that  $\iota_{\rho}$  is an *essential upper jump* for  $w|v$  in  $L|K$  and that  $\rho$  is an *essential upper index* for  $w|v$  in  $L|K$ .

(2) Let  $\mathcal{R} \hookrightarrow \mathcal{S}$  be any generically finite Galois extension of Dedekind  $k$ -algebras with cyclic inertia groups, and let  $\mathcal{K} := \text{Quot}(\mathcal{R}) \hookrightarrow \text{Quot}(\mathcal{S}) =: \mathcal{L}$  be the corresponding Galois extension of their quotient fields. For a maximal ideal  $\mathfrak{p} \in \text{Spec}(\mathcal{R})$  and  $\mathfrak{q} \in \text{Spec}(\mathcal{S})$  above  $\mathfrak{p}$ , let  $w_{\mathfrak{q}}$  and  $v_{\mathfrak{p}}$  be the valuations of  $\mathcal{L}$ , respectively  $\mathcal{K}$  defined by the local rings of, respectively  $\mathfrak{p}$ . Thus  $w_{\mathfrak{q}}$  is a prolongation of  $v_{\mathfrak{p}}$  to  $\mathcal{L}$ . We will say that  $\mathcal{R} \hookrightarrow \mathcal{S}$  has (no) essential ramification jumps at  $\mathfrak{p}$  if the following hold:  $\kappa(\mathfrak{q})|\kappa(\mathfrak{p})$  is separable and there are (no) essential upper ramification jumps for  $w_{\mathfrak{q}}|v_{\mathfrak{p}}$  in  $\mathcal{K} \hookrightarrow \mathcal{L}$ . And we say that  $\mathcal{R} \hookrightarrow \mathcal{S}$  has no essential ramification if there is no essential ramification at any maximal ideal  $\mathfrak{p} \in \text{Spec}(\mathcal{R})$ . Otherwise we will say that  $\mathcal{R} \hookrightarrow \mathcal{S}$  has essential ramification.

In the remaining part of this subsection, we will work in a special case of the situation presented in Section 2.C, which is as follows: Let  $\mathfrak{o} := k[[\varpi]]$  be the power series ring in the variable  $\varpi \neq t$  over  $k$ , and  $\mathfrak{m} := \varpi \mathfrak{o}$  its valuation ideal, and  $\hat{\kappa} := k((\varpi)) = \text{Quot}(\mathfrak{o})$ . Let further  $\mathcal{A} := k[[\varpi, t]] = \mathfrak{o}[[t]]$  and  $\mathcal{K} := k((\varpi, t)) := \text{Quot}(\mathcal{A})$  be its field of fractions; thus, in particular,  $A = \mathcal{A}/(\varpi) = k[[t]]$ . We further consider  $\mathcal{R} := \mathcal{A}[\varpi^{-1}] = \mathcal{A} \otimes_{\mathfrak{o}} \hat{\kappa}$ , the ring of power series in  $t$  over  $\hat{\kappa}$  having  $v_{\hat{\kappa}}$ -bounded coefficients, and notice that  $\mathcal{R}$  is a Dedekind ring having  $\text{Spec}(\mathcal{R}) = \text{Spec}(\mathcal{A}) \setminus V(\varpi)$  in bijection with the points of the open rigid disc of radius 1 over the complete valued field  $\hat{\kappa}$ . In particular, the elements  $x \in \mathfrak{m}$  will be interpreted as  $\hat{\kappa}$ -rational points of  $\text{Spec } \mathcal{R}$ . Finally, for a finite separable field extension  $\mathcal{K} \hookrightarrow \mathcal{L}$ , we let  $\mathcal{B} \subset \mathcal{S}$  be the integral closures of  $\mathcal{A} \subset \mathcal{R}$  in the finite field extension  $\mathcal{K} \hookrightarrow \mathcal{L}$ . Hence  $\mathcal{B}$  is finite  $\mathcal{A}$ -module, and  $\mathcal{S}$  is a finite  $\mathcal{R}$ -module, in particular, a Dedekind ring.

In the above context, let  $A = k[[t]] \hookrightarrow k[[z]] =: B$  be a cyclic  $\mathbb{Z}/p^e$ -extension with upper ramification jumps  $\iota_1 \leq \dots \leq \iota_e$ . Let  $e_0$  be the number of essential upper jumps, which could be zero. We set  $r_0 := 1$ , and if there exist essential upper jumps, i.e.,  $0 < e_0$ , let  $r_1 \leq \dots \leq r_{e_0}$  be the essential upper indices for  $L|K$ . Thus we get two finite increasing sequences: first,  $(r_i)_{0 \leq i \leq e_0}$  with  $r_0 := 1$  and  $r_{e_0} \leq e$ , and second,  $(d_i)_{0 \leq i \leq e_0}$  with  $d_0 := 1$  and  $d_i := d_{i-1} + q_i$  for  $0 < i \leq e_0$ .

**KEY LEMMA 3.2** (Characteristic  $p$  local Oort conjecture). *In the above notation, let  $A = k[[t]] \hookrightarrow k[[z]] =: B$  be a cyclic  $\mathbb{Z}/p^e$ -extension with upper ramification jumps  $\iota_1 \leq \dots \leq \iota_e$  and  $\delta_0 := \lfloor \iota_e / (p - 1) \rfloor$ . In the above notation, set  $N := 1 + q_1 + \dots + q_e$  and let  $x_1, \dots, x_N \in \mathfrak{m}_{\mathfrak{o}}$  be distinct elements that are  $p^{\delta_0}$ -powers. Then there exists a cyclic  $\mathbb{Z}/p^e$ -extension  $\mathcal{K} \hookrightarrow \mathcal{L}$  such that the integral closures  $\mathcal{A} \hookrightarrow \mathcal{B}$  of  $\mathcal{A}$ , respectively  $\mathcal{R} \hookrightarrow \mathcal{S}$  of  $\mathcal{R}$ , in the field extension  $\mathcal{K} \hookrightarrow \mathcal{L}$  satisfy*

- (1)  $\mathcal{B} = k[[\varpi, Z]]$  for some  $Z \in \mathcal{B}$ , and  $\mathcal{A}/(\varpi) \hookrightarrow \mathcal{B}/(\varpi)$  is  $\mathbb{Z}/p^e$ -isomorphic to  $A \hookrightarrow B$ .

- (2) The canonical morphism  $\mathcal{R} \hookrightarrow \mathcal{S}$  has no essential ramification and is ramified only at the points  $y_\mu \in \text{Spec } \mathcal{S}$  above the points  $x_\mu \in \text{Spec } \mathcal{R}$ ,  $1 \leq \mu \leq N$ .
- (3) For each  $1 < \mu$ , consider  $d_i$  such that  $d_{i-1} < \mu \leq d_i$ . Set  $e_1 = e$  and  $e_\mu = e - r_i + 1$  for  $1 < \mu$ . Then the order of the inertia group  $T_\mu$  at  $y_\mu \mapsto x_\mu$  is  $|T_\mu| = p^{e_\mu}$  and the upper ramification jumps  $(\iota_{\mu\alpha})_{1 \leq \rho \leq e_\mu}$  are given by
  - (i)  $\iota_{1\alpha} = p\iota_{1,\alpha-1} + \varepsilon_\alpha$  for  $1 \leq \alpha \leq e_1$ ;
  - (ii)  $\iota_{\mu\alpha} = p^\alpha - 1$  for  $1 < \mu \leq N$  and  $1 \leq \alpha \leq e_\mu$ .
- (4) In particular, the upper ramification jumps  $\mathbf{\iota}_\mu := (\iota_{\mu 1}, \dots, \iota_{\mu, e_\mu})$  at each  $y_\mu \mapsto x_\mu$  depend only on the initial upper ramification jumps  $\mathbf{\iota} := (\iota_1, \dots, \iota_e)$ ,  $1 \leq \mu \leq N$ .

The proof of Key Lemma 3.2 will be carried out in Sections 3.A, 3.B, and 3.C. We begin by recalling that in the notation from Section 2.B, there exists  $\mathbf{p} = (p_1(t^{-1}), \dots, p_e(t^{-1}))$ , say in standard form, such that  $L = K_{\mathbf{p}}$ . The integral closure  $A \hookrightarrow B$  of  $A = k[[t]]$  in the field extension  $K \hookrightarrow L$  is of the form  $B = k[[z]]$  for any uniformizing parameter  $z$  of  $L = \text{Quot}(B)$ . And the degree of the different  $\mathfrak{D}_v := \mathfrak{D}_{B|A}$  is  $\text{deg}(\mathfrak{D}_v) = \sum_{\rho=1}^e (\iota_\rho + 1)(p^\rho - p^{\rho-1})$ .

3.A. *Combinatorics of the upper jumps.* Given the sequence of upper ramification jumps  $\mathbf{\iota} = (\iota_1, \dots, \iota_e)$ , recall the notation introduced before Key Lemma 3.2; namely,  $e_0$  is the number of essential upper jumps, which could be zero. We set  $r_0 := 1$ , and if  $e_0 > 0$ , letting  $r_1, \dots, r_{e_0}$  be the essential upper jumps, we get an increasing sequence  $(r_i)_{0 \leq i \leq e_0}$ . For technical reasons (to simplify notation) we set  $r_{e_0+1} := e + 1$  and call it the *improper upper index*. Note that if  $e_0 = 0$ , then  $r_{e_0+1}$  would become  $r_1 := e + 1$ . Further, recalling the finite strictly increasing sequence  $(d_i)_{0 \leq i \leq e_0}$ , defined by  $d_0 := 1$ ,  $d_i := d_{i-1} + q_{r_i}$  for  $0 < i \leq e_0$ , we notice that  $N := 1 + q_1 + \dots + q_e = d_{e_0}$ .

*Construction 3.3.* First, for technical reasons, set  $\theta_{\mu 0} = 0$  for  $1 \leq \mu \leq N$ . In the above notation/context we construct an  $N \times e$  matrix  $(\theta_{\mu\rho})_{1 \leq \mu \leq N, 1 \leq \rho \leq e}$  as follows:

- If  $e_0 = 0$ , then  $N = 1$ , and we define the  $1 \times e$  matrix by  $\theta_{1\rho} := \iota_\rho$ ,  $1 \leq \rho \leq e$ .
- If  $e_0 > 0$ , thus  $N > 1$ , we define
  - (a)  $\theta_{1\rho} = p\theta_{1,\rho-1} + \varepsilon_\rho$  for  $1 \leq \rho \leq e$ ;
  - (b) for  $i = 1, \dots, e_0$  and  $d_{i-1} < \mu \leq d_i$ , define
    - $\theta_{\mu\rho} = 0$  for  $1 \leq \rho < r_i$ ,
    - $\theta_{\mu\rho} = p\theta_{\mu,\rho-1} + p - 1 = p^{\rho-r_i+1} - 1$  for  $r_i \leq \rho \leq e$ .

Notice that in the case  $e_0 > 0$ , one has: Let  $\rho$  with  $1 \leq \rho \leq e$  be given. Consider the unique  $1 \leq i \leq e_0$  such that  $r_i \leq \rho < r_{i+1}$ . (Recall the if  $r_i = e$ ,

then  $r_{i+1} := e + 1$  by the convention above!) Then for all  $\mu$  with  $1 \leq \mu \leq N$ , one has:  $\theta_{\mu\rho} \neq 0$  if and only if  $\mu \leq d_i$ .

The fundamental combinatorial property of  $(\theta_{\mu\rho})_{1 \leq \mu \leq N, 1 \leq \rho \leq e}$  is given by the following:

LEMMA 3.4. *For  $0 \leq i \leq e_0$  and  $r_i \leq \rho < r_{i+1}$ , the following equality holds:  $\iota_\rho + 1 = \sum_{1 \leq \mu \leq d_i} (\theta_{\mu\rho} + 1)$ .*

*Proof.* The proof follows by induction on  $\rho = 1, \dots, e$ . Indeed, if  $e_0 = 0$ , then  $N = 1$ , and there is nothing to prove. Thus supposing that  $e_0 > 0$ , one argues as follows:

- The assertion holds for  $\rho = 1$ : First, suppose that  $r_1 = 1$ , thus  $\iota_1$  is an essential upper jump. Since  $\rho = 1$ , it follows that  $i = 1$  is the unique index  $i$  for which  $r_i \leq \rho < r_{i+1}$ . Hence the sum in the right-hand side is taken over  $1 \leq \mu \leq d_1$ , where  $d_1 = 1 + q_1$ . Further, by the definitions one has:  $\theta_{11} = \varepsilon_1$  and  $\theta_{\mu 1} = p - 1$  for  $1 < \mu \leq d_1$ , and  $\iota_1 = pq_1 + \varepsilon_1$ . Thus  $\iota_1 + 1 = pq_1 + \varepsilon_1 + 1 = (\varepsilon_1 + 1) + \sum_{1 < \mu \leq d_1} ((p - 1) + 1) = \sum_{1 \leq \mu \leq d_1} (\theta_{\mu 1} + 1)$ . Second, suppose that  $1 < r_1$ . Then  $\iota_1$  is not an essential upper jump, i.e.,  $\iota_1 = \varepsilon_1 = \theta_{11}$ , and  $i = 0$  is the unique index  $i$  for which  $r_i \leq \rho < r_{i+1}$ . Hence  $d_i = d_0 = 1$ , and the right-hand sum is  $\sum_{1 \leq \mu \leq d_0} (\theta_{\mu 1} + 1) = \theta_{11} + 1$ , and the equality of the Lemma 3.4 becomes obvious in this case.

- If the assertion of Lemma 3.4 holds for  $\rho < e$ , the assertion also holds for  $\rho + 1$ ; indeed, let  $i$  be such that  $r_i \leq \rho < r_{i+1}$ .

*Case 1:*  $\rho + 1 < r_{i+1}$ . Then  $r_i \leq \rho + 1 < r_{i+1}$  and, in particular,  $\rho + 1$  is not an essential jump index. Hence by definitions one has that  $\iota_{\rho+1} = p\iota_\rho + \varepsilon_{\rho+1}$  with  $0 \leq \varepsilon_{\rho+1} < p$ . On the other hand, by the induction hypothesis we have that  $\iota_\rho = \theta_{1\rho} + \sum_{1 < \mu \leq d_i} (\theta_{\mu\rho} + 1)$ . Hence taking into account the definitions of  $\theta_{\mu, \rho+1}$ , we conclude the proof in Case 1 as follows:

$$\begin{aligned}
\iota_{\rho+1} + 1 &= p\iota_\rho + \varepsilon_{\rho+1} + 1 \\
&= p \sum_{1 \leq \mu \leq d_i} (\theta_{\mu\rho} + 1) + \varepsilon_{\rho+1} + 1 \\
&= p \left( \theta_{1\rho} + \sum_{1 < \mu \leq d_i} (\theta_{\mu\rho} + 1) \right) + \varepsilon_{\rho+1} + 1 \\
&= (p\theta_{1\rho} + \varepsilon_{\rho+1} + 1) + \sum_{1 < \mu \leq d_i} ((p\theta_{\mu\rho} + p - 1) + 1) \\
&= (\theta_{1, \rho+1} + 1) + \sum_{1 < \mu \leq d_i} (\theta_{\mu, \rho+1} + 1) \\
&= \sum_{1 \leq \mu \leq d_i} (\theta_{\mu, \rho+1} + 1), \text{ as claimed.}
\end{aligned}$$

*Case 2:*  $\rho + 1 = r_{i+1}$ . Then  $\rho + 1$  is an essential jump index; thus by definitions one has:  $\nu_{\rho+1} = p\nu_{\rho} + pq_{\rho+1} + \varepsilon_{\rho+1}$  with  $0 < q_{\rho+1}$  and  $0 < \varepsilon_{\rho+1} < p$ ,  $d_{i+1} = d_i + q_{\rho+1}$ ,  $r_{i+1} \leq \rho + 1 < r_{i+2}$ . On the other hand, by the induction hypothesis one has  $\nu_{\rho} = \theta_{1\rho} + \sum_{1 < \mu \leq d_i} (\theta_{\mu\rho} + 1)$ . Therefore, using the definitions of  $\theta_{\mu, \rho+1}$  and reasoning as above, we get

$$\begin{aligned} \nu_{\rho+1} + 1 &= p\nu_{\rho} + pq_{\rho+1} + \varepsilon_{\rho+1} + 1 \\ &= p\left(\theta_{1\rho} + \sum_{1 < \mu \leq d_i} (\theta_{\mu\rho} + 1)\right) + pq_{\rho+1} + \varepsilon_{\rho+1} + 1 \\ &= (p\theta_{1\rho} + \varepsilon_{\rho+1} + 1) + \sum_{1 < \mu \leq d_i} (p\theta_{\mu\rho} + p) + pq_{\rho+1} \\ &= (\theta_{1, \rho+1} + 1) + \sum_{1 < \mu \leq d_i} ((p\theta_{\mu\rho} + p - 1) + 1) + \sum_{d_i < \mu \leq d_{i+1}} ((p - 1) + 1) \\ &= (\theta_{1, \rho+1} + 1) + \sum_{1 < \mu \leq d_i} (\theta_{\mu, \rho+1} + 1) + \sum_{d_i < \mu \leq d_{i+1}} (\theta_{\mu, \rho+1} + 1) \\ &= \sum_{1 \leq \mu \leq d_{i+1}} (\theta_{\mu, \rho+1} + 1), \text{ as claimed.} \end{aligned}$$

This completes the proof of Lemma 3.4. □

**3.B. Deforming ramification.** In the context of Key Lemma 3.2, we introduce notation as follows: Let  $\hat{\kappa} \hookrightarrow \mathbf{l}$  be an algebraic closure of  $\hat{\kappa}$  and  $\mathfrak{o}_{\mathbf{l}}$  its valuation ring, and let  $v$  be the prolongation of the canonical valuation of  $\hat{\kappa}$  to  $\mathbf{l}$ . Hence  $v(\hat{\kappa}) = \mathbb{Z}$  and  $v(\mathbf{l}) = \mathbb{Q}$ . Further, recall that  $\delta_0 := \lfloor \nu_e / (p - 1) \rfloor$ , and let  $\mathfrak{o}_0 := k[[t^{p^{\delta_0}}]]$ , and  $\mathfrak{m}_0$  its valuation ideal, and  $\hat{\kappa}_0 = \text{Quot}(\mathfrak{o}_0) = k((t^{p^{\delta_0}}))$ . In particular, since  $k$  is algebraically closed, it follows that  $\hat{\kappa}_0 = \text{Frob}^{\delta_0}(\hat{\kappa})$ . Since each  $x_{\mu} \in \mathfrak{m}_0$  is a  $p^{\delta_0}$ -power, it follows that actually one has  $x_{\mu} \in \mathfrak{m}_0 \subset \mathfrak{m}$ ; in particular,  $v(x_{\mu}) \geq p^{\delta} > 0$ .

Let  $k[[t]] \hookrightarrow k[[z]]$  be the  $\mathbb{Z}/p^e$ -cyclic extension given in Key Lemma 3.2 with upper ramification jumps  $\nu_1 \leq \dots \leq \nu_e$ . Then setting  $K = k((t))$  and  $L = k((z))$ , in the notation introduced in Section 2.B, there exists a Witt vector  $\mathbf{p} = (p_1(t^{-1}), \dots, p_e(t^{-1}))$  over  $k((t))$  such that  $L = K_{\mathbf{p}}$ , and since  $k$  is algebraically closed, hence  $W_e(k) = \wp_e(W_e(k))$ , we can/will suppose that  $\mathbf{p}$  is in *standard form* as introduced and discussed before Fact 2.3; hence

$$\nu_{\rho} = \max\{p\nu_{\rho-1}, \deg(p_{\rho}(t^{-1}))\}, \quad \rho = 1, \dots, e.$$

Next recall that in the notation an context from Construction 3.3, for every  $1 \leq \rho \leq e$ , there exists a unique  $r_i$  such that  $r_i \leq \rho < r_{i+1}$ . And for such a  $\rho$ , by Lemma 3.4, one has that

$$\nu_{\rho} = \theta_{1\rho} + \sum_{1 < \mu \leq d_i} (\theta_{\mu\rho} + 1).$$

Since  $k$  is algebraically closed, thus every  $p_\rho(t^{-1})$  is a product of distinct linear factors, one can write each  $p_\rho(t^{-1})$  in  $k[t^{-1}]$  (nonuniquely) as a product  $p_\rho(t^{-1}) = \prod_{1 \leq \mu \leq N} p_{\mu\rho}(t^{-1})$  whose degrees satisfy  $\deg(p_{1\rho}(t^{-1})) \leq \theta_{1\rho}$ ,  $\deg(p_{\mu\rho}(t^{-1})) \leq \theta_{\mu\rho} + 1$  for  $1 < \mu \leq d_i$ , and  $p_\rho = 1$  for  $d_i < \mu \leq N$ . Notice also that if  $p_{\nu\rho-1} < \nu_\rho$ , then  $\nu_\rho = \deg(p_\rho(t^{-1}))$ . Hence by Lemma 3.4 it follows that in this situation all of the above inequalities are actually equalities; i.e.,  $\deg(p_{1\rho}(t^{-1})) = \theta_{1\rho}$ ,  $\deg(p_{\mu\rho}(t^{-1})) = \theta_{\mu\rho} + 1$  for  $1 < \mu \leq d_i$ .

Coming back to the context of Key Lemma 3.2, we set  $t_\mu := t - x_\mu \in \mathfrak{o}_0[t]$  for every  $\mu = 1, \dots, N$  and consider *permissible liftings*  $Q_{\mu\rho}(t_\mu^{-1}) \in \mathfrak{o}_0[t_\mu^{-1}]$  of  $p_{\mu\rho}(t^{-1}) \in k[t^{-1}]$  under the specialization homomorphism  $\mathfrak{o}_0[t_\mu^{-1}] \rightarrow k[t^{-1}]$ , which means that  $Q_{\mu\rho}(t_\mu^{-1})$  satisfy

$$(\dagger) \quad \deg(Q_{1\rho}) \leq \theta_{1\rho}, \deg(Q_{\mu\rho}) \leq \theta_{\mu\rho} + 1 \text{ for } 1 < \mu \leq d_i, \quad Q_{\mu\rho} = 1 \text{ for } d_i < \mu \leq N.$$

We further consider the following resulting Witt vector of length  $e$  over  $\mathcal{K}$ :

$$(\dagger\dagger) \quad \mathbf{Q} := (Q_1, \dots, Q_e) \in W_e(\mathcal{K}), \quad \text{where } Q_\rho := \prod_\mu Q_{\mu\rho} \text{ for } \rho = 1, \dots, e$$

and consider the corresponding cyclic field extension  $\mathcal{L} := \mathcal{K}_{\mathbf{Q}}$ . Let  $\mathcal{A} \hookrightarrow \mathcal{B}$  be the normalization of  $\mathcal{A}$  in  $\mathcal{K} \hookrightarrow \mathcal{L}$ . Since  $\mathcal{A} = k[[\varpi, t]]$  is Noetherian and  $\mathcal{K} \hookrightarrow \mathcal{L}$  is separable, it follows that  $\mathcal{B}$  is a finite  $\mathcal{A}$ -algebra, thus Noetherian. And since  $\mathcal{A}$  is local and complete, so is  $\mathcal{B}$ .

We next take a closer look at the branching in the finite extension of  $\sigma$ -algebras  $\mathcal{A} \hookrightarrow \mathcal{B}$ , where  $\mathfrak{o} := k[[\varpi]]$ . In order to do so, we introduce geometric language as follows:  $\mathcal{X} = \text{Spec } \mathcal{A}$  and  $\mathcal{Y} = \text{Spec } \mathcal{B}$ , hence  $\mathcal{A} \hookrightarrow \mathcal{B}$  defines a finite  $\sigma$ -morphism  $\mathcal{Y} \rightarrow \mathcal{X}$ . Further, we denote by  $\mathcal{Y}_\eta := \text{Spec } \mathcal{S} \rightarrow \text{Spec } \mathcal{R} =: \mathcal{X}_\eta$  and  $\mathcal{Y}_s := \text{Spec } \mathcal{B}/(\varpi) \rightarrow \text{Spec } \mathcal{A}/(\varpi) =: \mathcal{X}_s$  the generic fiber, respectively the special fiber, of  $\mathcal{Y} \rightarrow \mathcal{X}$ . In particular,  $\mathcal{X}_s = \text{Spec } \mathcal{A}$  and  $\mathcal{Y}_s \rightarrow \mathcal{X}_s$  is a finite morphism.

We notice that since  $\mathcal{A} = k[[\varpi, t]]$  is a two-dimensional local regular ring,  $\mathcal{X}$  is a two-dimensional regular scheme. Therefore, the branch locus of  $\mathcal{Y} \rightarrow \mathcal{X}$  is of pure co-dimension one. Thus to describe the branching behavior of  $\mathcal{Y} \rightarrow \mathcal{X}$ , one has to describe the branching at the generic point  $(\varpi)$  of the special fiber  $\mathcal{X}_s \hookrightarrow \mathcal{X}$  of  $\mathcal{X}$  and at the closed points  $x$  of the generic fiber  $\mathcal{X}_\eta \hookrightarrow \mathcal{X}$  of  $\mathcal{X}$ .

*The branching at  $(\varpi)$ .* Recall that  $\mathcal{K} \hookrightarrow \mathcal{L}$  is defined as a cyclic extension by  $\mathbf{Q} := (Q_1, \dots, Q_e)$ , where each  $Q_\rho$  is of the form  $Q_\rho = \prod_\mu Q_{\mu\rho}(t_\mu^{-1})$  with  $Q_{\mu\rho}(t_\mu^{-1}) \in \mathfrak{o}[t_\mu^{-1}] \subset \mathcal{A}[t_{x_1}^{-1}, \dots, t_{x_N}^{-1}]$ . Since  $\varpi$  does not divide  $t_\mu = t - x_\mu$  in the factorial ring  $\mathcal{A}$ , it follows that  $t_\mu^{-1} \in \mathcal{A}_{(\varpi)}$ , hence  $\mathcal{A}[t_{x_1}^{-1}, \dots, t_{x_N}^{-1}] \hookrightarrow \mathcal{A}_{(\varpi)}$ . Thus,  $Q_\rho \in \mathcal{A}_{(\varpi)}$  for  $1 \leq \rho \leq e$ , hence  $\varpi$  is not branched in  $\mathcal{K} \hookrightarrow \mathcal{L}$  by the first part of Fact 2.1(2)(a). Moreover, since  $(Q_1, \dots, Q_e) \mapsto (p_1, \dots, p_e)$  under the specialization homomorphism  $\mathcal{A}[[t_{x_1}^{-1}, \dots, t_{x_N}^{-1}]] \hookrightarrow \mathcal{A}_{(\varpi)} \rightarrow \kappa(\varpi) = K$  and  $p_1(t^{-1})$  does not lie in  $\wp(K)$ , it follows by the second part of Fact 2.1(2)(a) that

$\text{Gal}(\mathcal{L}|\mathcal{K})$  is contained in the decomposition group of  $v_\varpi$ . In other words,  $\varpi$  is totally inert in  $\mathcal{K} \hookrightarrow \mathcal{L}$ . In particular,  $\mathcal{Y} \rightarrow \mathcal{X}$  is étale above  $\varpi$ , and moreover, the special fiber  $\mathcal{Y}_s \rightarrow \mathcal{X}_s$  of  $\mathcal{Y} \rightarrow \mathcal{X}$  is reduced, irreducible, and generically it is the  $\mathbb{Z}/p^e$ -cyclic extension  $K \hookrightarrow L$ .

*The branching at the points  $x$  of the generic fiber  $\mathcal{X}_\eta = \text{Spec } \mathcal{R}$ .* First, let  $x \in \mathcal{X}_\eta$  be a closed point different from  $x_1, \dots, x_N$ , and let  $\mathcal{O}_x$  be the local ring of  $\mathcal{X}_\eta$  at  $x$ . Then  $t_\mu := t - x_\mu \in \mathcal{O}_x^\times$ , because  $x \neq x_\mu$ , hence  $t_\mu^{-1} \in \mathcal{O}_x$  as well. Therefore  $Q_{\mu\rho} \in \mathcal{O}_x$  for all  $\mu$  and  $\rho$ , thus finally  $Q_1, \dots, Q_e \in \mathcal{O}_x$ . Hence by the first part of Fact 2.1(2)(a), it follows that  $x$  is not branched in the field extension  $\mathcal{K} \hookrightarrow \mathcal{L}$ .

It is therefore left to analyze the branching behavior of  $\mathcal{Y}_\eta \rightarrow \mathcal{X}_\eta$  at the closed points  $x_1, \dots, x_N \in \mathcal{X}_\eta$ . In this process we will also compute the contribution of the different of  $\mathcal{K} \hookrightarrow \mathcal{L}$  at the point  $x_\mu$  to the total different  $\mathfrak{D}_{\mathcal{S}|\mathcal{R}}$  for  $\mu = 1, \dots, N$ . Let  $v_\mu$  be the  $t_\mu$ -adic valuation of  $\mathcal{K}$ , and let  $w_\mu|v_\mu$  be a fixed prolongation of  $v_\mu$  to  $\mathcal{L}$ . Finally let  $\mathcal{K}_\mu := \hat{\kappa}((t_\mu))$  be the completion of  $\mathcal{K}$  at  $v_\mu$ . For fixed  $\mu$ , setting  $a_\nu := x_\mu - x_\nu$ , one has that  $a_\nu \in \mathfrak{m}_0$  and  $a \neq 0$ , and  $t_\nu = t_\mu - a_\nu$ . Hence  $t_\nu$  is a  $v_\mu$ -unit. In particular,  $t_\nu^{-1} \in \hat{\kappa}_0[[t_\mu]]$  and  $\hat{\kappa}_0[t_\nu^{-1}] \subset \hat{\kappa}_0[[t_\mu]]$ . Hence since  $Q_{\nu\rho}(t_\nu^{-1}) \in \hat{\kappa}_0[t_\nu^{-1}]$ , it follows that  $Q_{\nu\rho}(t_\nu^{-1}) \in \hat{\kappa}_0[[t_\mu]]$  for every  $\nu \neq \mu$ . Thus  $\eta_{\mu\rho} := \prod_{\nu \neq \mu} Q_{\nu\rho}(t_\nu^{-1}) \in \hat{\kappa}_0[[t_\mu]]$  is itself a power series in  $t_\mu$  over  $\hat{\kappa}_0$  such that  $Q_\rho = \eta_{\mu\rho} Q_{\mu\rho}$  inside  $\hat{\kappa}_0((t_\mu))$ . Thus we conclude that

$$(*) \quad \mathbf{Q} = (Q_1, \dots, Q_e) \in W_e(\hat{\kappa}_0((t_\mu))).$$

Further, since  $v_\mu(\eta_{\mu\rho}) \geq 0$  and  $v_\mu(Q_{\mu\rho}) = -\text{deg}(Q_{\mu\rho}(t_\mu^{-1}))$ , we see that the Witt vector  $\mathbf{Q} = (Q_1, \dots, Q_e)$  viewed over  $\mathcal{K}_\mu = \hat{\kappa}((t_\mu))$  has entries  $Q_\rho$  that for  $1 \leq \rho \leq e$  satisfy

$$(**) \quad -v_\mu(Q_\rho) = -v(\eta_{\mu\rho} Q_{\mu\rho}) = -v_\mu(\eta_{\mu\rho}) - v_\mu(Q_{\mu\rho}) \leq \text{deg}(Q_{\mu\rho}(t_\mu^{-1})).$$

Thus setting  $\delta_{\mu 1} := \max\{-v_\mu(Q_1), 0\}$  and  $\delta_{\mu\rho} := \max\{-v_\mu(Q_\rho), p\delta_{\mu, \rho-1}\}$  for  $1 < \rho \leq e$ , it follows by assertion (†) above and the properties of the  $(\theta_{\mu\rho})_\rho$  (precisely, that  $0 < \theta_{1,1} < p$  and  $p\theta_{1, \rho-1} \leq \theta_{1\rho}$ , and  $0 < \theta_{\mu, \rho-1}$  implies  $\theta_{\mu\rho} + 1 = p(\theta_{\mu, \rho-1} + 1)$  for  $\mu > 1$ ) that  $\delta_{1e} \leq \theta_{1e}$  and  $\delta_{\mu e} \leq \theta_{\mu e} + 1$  for  $1 < \mu \leq N$ . Thus  $\delta_\mu := \lfloor \delta_{\mu e} / (p - 1) \rfloor \leq \lfloor \nu_e / (p - 1) \rfloor =: \delta_0$ ; hence one gets

$$\hat{\kappa}_0 = \text{Frob}^{\delta_0}(\hat{\kappa}) \subseteq \text{Frob}^{\delta_\mu}(\hat{\kappa}) =: \hat{\kappa}_\mu \subseteq \hat{\kappa},$$

and thus the Witt vector  $\mathbf{Q} \in W_e(\mathcal{K})$  is defined over  $\hat{\kappa}_\mu((t_\mu)) \subset \hat{\kappa}((t_\mu)) = \mathcal{K}_\mu$ .

Therefore recalling that  $w_\mu|v_\mu$  is a prolongation of  $v_\mu$  of  $\mathcal{L}$ , it follows by Fact 2.3(1) that the residue field extension  $\kappa(w_\mu)|\kappa(v_\mu)$  is separable,<sup>1</sup> and  $\mathbf{Q}$  has quasi standard forms  $\mathbf{P}_\mu + \mathbf{a}_\mu$  with  $\mathbf{P}_\mu \in W_e(\hat{\kappa}[t_\mu^{-1}])$  and  $\mathbf{a}_\mu \in W_e(\hat{\kappa})$ . Further, setting  $\mathbf{P}_\mu := (P_{\mu 1}, \dots, P_{\mu e})$ , it follows that  $\text{deg}(P_{\mu\rho}) \leq -v_\mu(Q_\rho)$  for

<sup>1</sup>Actually,  $\kappa(v_\mu) = \hat{\kappa}$  by the fact that  $x_\mu$  is  $\hat{\kappa}$ -rational point of  $\mathcal{X}_\eta$ .

all  $\rho$ . Thus using (\*\*) above, it follows that  $\deg(P_{\mu\rho}(t_\mu^{-1})) \leq \deg(Q_{\mu\rho}(t_\mu^{-1}))$  for all  $\rho = 1, \dots, e$ .

In order to announce the conclusion of these preparations, set  $r_\mu := e + 1$  if all  $P_{\mu\rho}(t_\mu^{-1})$  are constant,  $r_\mu = 1$  if  $P_{\mu 1}(t_\mu^{-1})$  is nonconstant, and otherwise, let  $1 < r_\mu \leq e$  be minimal such that  $P_{\mu\rho}(t_\mu^{-1})$  is constant for all  $1 \leq \rho < r_\mu$ . Finally, let  $\mathcal{K} \hookrightarrow \mathcal{L}_{r_\mu-1}$  be the unique subextension of  $\mathcal{K} \hookrightarrow \mathcal{L}$  of degree  $p^{r_\mu-1}$ .

LEMMA 3.5. *In the above notation,  $w_\mu$  is not ramified in  $\mathcal{K} \hookrightarrow \mathcal{L}_{r_\mu-1}$  and is totally ramified in  $\mathcal{L}_{r_\mu-1} \hookrightarrow \mathcal{L}$  having upper ramification jumps  $(\iota_{\mu\alpha})_{1 \leq \alpha \leq e-r_\mu+1}$  satisfying  $\iota_{\mu\alpha} \leq \theta_{\mu,\alpha+r_\mu-1}$ .*

*Proof.* Applying Fact 2.3(2) to the quasi standard form  $\mathbf{P}_\mu + \mathbf{a}_\mu$  of  $\mathbf{Q}$ , the only assertion left to be proved is that  $\iota_{\mu\alpha} \leq \theta_{\mu,\alpha+r_\mu-1}$  holds for all  $1 \leq \alpha \leq e - r_\mu + 1$ . For that proof, recall that the degrees of  $P_{\mu\rho}$  and  $Q_{\mu\rho}$  are related by  $\deg(P_{\mu\rho}(t_\mu^{-1})) \leq \deg(Q_{\mu\rho}(t_\mu^{-1}))$ .

First, suppose that  $\mu = 1$ . Then, by (†), one has  $\deg(Q_{1\rho}(t_1^{-1})) \leq \theta_{1\rho}$  for all  $\rho = 1, \dots, e$ . Thus  $\deg(P_{1\rho}(t_1^{-1})) \leq \deg(Q_{1\rho}(t_1^{-1})) \leq \theta_{1\rho}$ , and we are done.

Second, in the notation from Construction 3.3, suppose that  $d_{i-1} < \mu \leq d_i$ . Then  $Q_{\mu\rho} = 1$  for  $\rho < r_i$ , and  $\deg(Q_{\mu\rho}(t_\mu^{-1})) \leq \theta_{\mu\rho} + 1 = p^{\rho-r_i+1}$  for  $r_i \leq \rho \leq e$ . Thus for  $r_i \leq \rho \leq e$ , we have: First,  $\deg(P_{\mu\rho}(t_\mu^{-1})) \leq \deg(Q_{\mu\rho}(t_\mu^{-1})) \leq p^{\rho-r_i+1}$  and second,  $\deg(P_{\mu\rho}(t_\mu^{-1}))$  is prime to  $p$ , because  $\mathbf{P}_\mu := (P_{\mu 1}, \dots, P_{\mu e}) + \mathbf{a}_\mu$  is a quasi standard form of  $\mathbf{Q}$ . Hence at least one of the last two inequalities is strict; thus  $\deg(P_{\mu\rho}(t_\mu^{-1})) \leq p^{\rho-r_i+1} - 1 = \theta_{\mu\rho}$  for  $r_i \leq \rho \leq e$ . This concludes the proof of the Lemma 3.5.  $\square$

3.C. *Finishing the proof of Key Lemma 3.2.* Recall that  $t_\mu$  is the canonical uniformizing parameter at  $x_\mu$ , and let  $v_\mu$  be the  $t_\mu$ -adic valuation of  $\mathcal{K}$ . Let further  $\mathfrak{D}_{v_\mu}$  be the local part at  $v_\mu$  of the global different  $\mathfrak{D}_{\mathcal{S}|\mathcal{R}}$  of the extension of Dedekind rings  $\mathcal{R} \hookrightarrow \mathcal{S}$ . Then by the Hilbert ramification theory one has

$$\deg(\mathfrak{D}_{\mathcal{S}|\mathcal{R}}) = \sum_{\mu=1}^N \deg(\mathfrak{D}_{v_\mu}).$$

In order to compute  $\deg(\mathfrak{D}_{v_\mu})$ , recall that if  $T_\mu \subseteq \mathbb{Z}/p^e$  is the inertia group at some prolongation of  $v_\mu$  to  $\mathcal{L}$ , in the notation from Lemma 3.5, one has:  $|T_\mu| = p^{e-r_\mu+1}$  and therefore  $r_\mu = e + 1$  if and only if  $|T_\mu| = 1$  if and only if  $v_\mu$  is unramified in  $\mathcal{K} \hookrightarrow \mathcal{L}$  if and only if  $\deg(\mathfrak{D}_{v_\mu}) = 0$ . Equivalently,  $r_\mu \leq e$  if and only if  $v_\mu$  has ramification in  $\mathcal{K} \hookrightarrow \mathcal{L}$ , and if so,  $v_\mu$  is unramified in  $\mathcal{K} \hookrightarrow \mathcal{L}_{r_\mu-1}$ , and  $v_\mu$  is totally (wildly) ramified in  $\mathcal{L}_{r_\mu-1} \hookrightarrow \mathcal{L}$  with upper ramification jumps  $(\iota_{\mu\alpha})_{1 \leq \alpha \leq e-r_\mu+1}$  satisfying  $\iota_{\mu\alpha} \leq \theta_{\mu,\alpha+r_\mu-1}$ . We further notice that in the notation from Construction 3.3, one has: Consider  $d_i$  such that  $d_{i-1} < \mu \leq d_i$ , and consider the corresponding  $r_i$  (which satisfies  $\theta_{\mu\rho} = 0$

for  $\rho < r_i$  and  $0 < \theta_{\mu\rho}$  for  $r_i \leq \rho \leq e$ . Then by the property (†) above of  $\deg(Q_{\mu\rho})$  and the inequality  $\deg(P_{\mu\rho}(t_\mu^{-1})) \leq \deg(Q_{\mu\rho}(t_\mu^{-1}))$ , one gets that  $r_i \leq r_\mu$ . Finally, using the formula for  $\deg(\mathfrak{D}_{v_\mu})$  given in Section 2(A), we obtain the following estimates:

$$\begin{aligned}
\deg(\mathfrak{D}_{v_\mu}) &= [\mathcal{L}^{T_\mu} : \mathcal{K}] \sum_{\alpha=1}^{e-r_\mu+1} (\iota_{\mu\alpha} + 1)(p^\alpha - p^{\alpha-1}) \\
&= p^{r_\mu-1} \sum_{\alpha=1}^{e-r_\mu+1} (\iota_{\mu\alpha} + 1)(p^\alpha - p^{\alpha-1}) \\
&= \sum_{\alpha=1}^{e-r_\mu+1} (\iota_{\mu\alpha} + 1)(p^{r_\mu+\alpha-1} - p^{r_\mu+\alpha-2}) \\
&\leq \sum_{\alpha=1}^{e-r_\mu+1} (\theta_{\mu,\alpha+r_\mu-1} + 1)(p^{r_\mu+\alpha-1} - p^{r_\mu+\alpha-2}) \\
&= \sum_{\rho=r_\mu}^e (\theta_{\mu\rho} + 1)(p^\rho - p^{\rho-1}) \\
&\leq \sum_{\rho=r_i}^e (\theta_{\mu\rho} + 1)(p^\rho - p^{\rho-1}).
\end{aligned}$$

Thus recalling that  $N = 1 + q_1 + \cdots + q_{e_0}$  and  $d_0 = 1$ ,  $d_i = d_{i-1} + q_i$  for  $1 \leq i \leq e_0$ , and further, by Lemma 3.4 that  $\sum_{1 \leq \mu \leq d_i} (\theta_{\mu\rho} + 1) = \iota_\rho + 1$ , we get

$$\begin{aligned}
\deg(\mathfrak{D}_{S|\mathcal{R}}) &= \sum_{1 \leq \mu \leq N} \deg(\mathfrak{D}_{v_\mu}) = \deg(\mathfrak{D}_{v_1}) + \sum_{1 \leq i \leq e_0} \sum_{d_{i-1} < \mu \leq d_i} \deg(\mathfrak{D}_{v_\mu}) \\
&\leq \sum_{1 \leq \rho \leq e} (\theta_{1\rho} + 1)(p^\rho - p^{\rho-1}) + \sum_{1 \leq i \leq e_0} \sum_{d_{i-1} < \mu \leq d_i} \sum_{r_i \leq \rho \leq e} (\theta_{\mu\rho} + 1)(p^\rho - p^{\rho-1}) \\
&= \sum_{1 \leq \rho \leq e} (\theta_{1\rho} + 1)(p^\rho - p^{\rho-1}) + \sum_{1 \leq i \leq e_0} \sum_{r_i \leq \rho < r_{i+1}} \sum_{d_0 < \mu \leq d_i} (\theta_{\mu\rho} + 1)(p^\rho - p^{\rho-1}) \\
&= \sum_{0 \leq i \leq e_0} \sum_{r_i \leq \rho < r_{i+1}} \left( (\theta_{1\rho} + 1) + \sum_{1 < \mu \leq d_i} (\theta_{\mu\rho} + 1) \right) (p^\rho - p^{\rho-1}) \\
&= \sum_{0 \leq i \leq e_0} \sum_{r_i \leq \rho < r_{i+1}} \sum_{1 \leq \mu \leq d_i} (\theta_{\mu\rho} + 1)(p^\rho - p^{\rho-1}) \\
&= \sum_{0 \leq i \leq e_0} \sum_{r_i \leq \rho < r_{i+1}} (\iota_\rho + 1)(p^\rho - p^{\rho-1}) \\
&= \sum_{1 \leq \rho \leq e} (\iota_\rho + 1)(p^\rho - p^{\rho-1}) \\
&= \deg(\mathfrak{D}_{B|A}),
\end{aligned}$$

thus concluding that  $\deg(\mathfrak{D}_{S|\mathcal{R}}) \leq \deg(\mathfrak{D}_{B|A})$ . Since by Fact 2.4, the inequality  $\deg(\mathfrak{D}_{S|\mathcal{R}}) \geq \deg(\mathfrak{D}_{B|A})$  always holds, it follows that  $\deg(\mathfrak{D}_{S|\mathcal{R}}) = \deg(\mathfrak{D}_{B|A})$ , and we conclude the proof of Key Lemma 3.2 by applying the local criterion for good reduction Fact 2.4.

*Remark.* We also notice that all the involved inequalities above are in fact equalities. In particular, for all  $i = 0, \dots, e$ , one has  $r_i = r_\mu$ , and further:  $v_{1\alpha} = \theta_{1\alpha}$  for  $1 \leq \alpha \leq e$ , and for  $1 < \mu \leq d_i$  one has that  $v_{\mu\alpha} = \theta_{\mu, \alpha - r_i + 1}$  for  $1 \leq \alpha \leq e - r_i + 1$ . Thus, in particular,  $\deg(\tilde{P}_{1\rho}) = \theta_{1\rho}$  if  $\theta_{1\rho}$  is prime to  $p$ , and  $\deg(\tilde{P}_{\mu\rho}) = \theta_{\mu\rho}$  for all  $\mu > 1$  and all  $\rho$ . This could be shown directly through quite tedious computations and not employing Fact 2.4.

3.D. *Characteristic  $p$  global Oort Conjecture.*

**THEOREM 3.6** (Characteristic  $p$  global Oort conjecture). *In the notation from Key Lemma 3.2, let  $Y \rightarrow X$  be a  $G$ -cover of complete smooth  $k$ -curves having only cyclic groups as inertia groups, and set  $\mathcal{X}_\mathfrak{o} := X \times_k \mathfrak{o}$ . Then there exists a  $G$ -cover of complete smooth  $\mathfrak{o}$ -curves  $\mathcal{Y}_\mathfrak{o} \rightarrow \mathcal{X}_\mathfrak{o}$  with special fiber the  $G$ -cover  $Y \rightarrow X$  such that the generic fiber  $\mathcal{Y}_{\hat{\kappa}} \rightarrow \mathcal{X}_{\hat{\kappa}}$  of  $\mathcal{Y}_\mathfrak{o} \rightarrow \mathcal{X}_\mathfrak{o}$  has no essential ramification.*

*Proof.* The proof is similar to the proof of the fact that the general Oort conjecture is equivalent to the local Oort conjecture for cyclic  $p$ -power covers (see, e.g., Garuti [5, §3], and Saidi [20, §1.2]), the emphasis in our situation being on deforming the ramification in order have no essential ramification. The main arguments are as follows.

As in Theorem 3.6, let  $Y \rightarrow X$  be given, and let  $Y_\mathfrak{o} \rightarrow X_\mathfrak{o} = \mathcal{X}_\mathfrak{o}$  be the base changes of  $Y \rightarrow X$  under the embedding  $k \hookrightarrow \mathfrak{o}$  and  $Y_{\hat{\kappa}} \rightarrow X_{\hat{\kappa}}$  be the generic fiber of  $Y_\mathfrak{o} \rightarrow X_\mathfrak{o}$ . Then  $Y_\mathfrak{o} \rightarrow X_\mathfrak{o}$  and  $Y_{\hat{\kappa}} \rightarrow X_{\hat{\kappa}}$  are  $G$ -covers of projective smooth  $\mathfrak{o}$ -curves, respectively  $\hat{\kappa}$ -curves, and let  $Y_{\hat{\kappa}}^{\text{an}} \rightarrow X_{\hat{\kappa}}^{\text{an}}$  be the (rigid) analytification of  $Y_{\hat{\kappa}} \rightarrow X_{\hat{\kappa}}$ . Then  $Y_{\hat{\kappa}}^{\text{an}} \rightarrow X_{\hat{\kappa}}^{\text{an}}$  is a  $G$ -cover of projective smooth rigid  $\hat{\kappa}$ -curves. We will deform  $Y_{\hat{\kappa}}^{\text{an}} \rightarrow X_{\hat{\kappa}}^{\text{an}}$  to the rigid analytification  $\mathcal{Y}^{\text{an}} \rightarrow \mathcal{X}^{\text{an}} = X_{\hat{\kappa}}^{\text{an}}$  of the generic fiber  $\mathcal{Y}_{\hat{\kappa}} \rightarrow \mathcal{X}_{\hat{\kappa}} = X_{\hat{\kappa}}$  of a  $G$ -cover  $\mathcal{Y}_\mathfrak{o} \rightarrow \mathcal{X}_\mathfrak{o}$  of projective smooth  $\mathfrak{o}$ -curves such that  $\mathcal{Y}_{\hat{\kappa}} \rightarrow \mathcal{X}_{\hat{\kappa}}$  has no essential ramification.

For  $y \mapsto x$  under  $Y \rightarrow X$ , let  $t$  and  $u$  be uniformizing parameters at  $x$ , respectively  $y$ , and  $k[[t]] \hookrightarrow k[[u]]$  be the local  $T$ -extension at  $y \mapsto x$ , where  $T$  is the inertia group at  $y \mapsto x$ . The minimal polynomial of  $u$  over  $k((t))$  is an Eisenstein polynomial of the form  $p(U) = U^n + a_{n-1}(t)U^{n-1} + \dots + a_1(t)U + a_0(t)$ , where  $a_1(t), \dots, a_{n-1}(t) \in tk[[t]]$  and  $a_0(t) = a_{01}t + a_{02}t^2 + \dots \in k[[t]]$  with  $a_{01} \neq 0$ . Here  $|T| = n$ .

*Step 1: The behavior of  $Y_{\hat{\kappa}} \rightarrow X_{\hat{\kappa}}$  above small disks around  $y \mapsto x$ .* Let  $|\cdot|$  be the absolute value of  $\hat{\kappa}$ , and consider the rigid analytic subspaces  $\mathcal{E} \subset Y_{\hat{\kappa}}^{\text{an}}$  defined by  $|u(y')| \leq |\varpi|$  and  $\mathcal{D} \subset X_{\hat{\kappa}}^{\text{an}}$  defined by  $|t(x')| \leq |\eta|$ , where  $\varpi^n =: \eta$ .

Then  $\mathcal{E}$  is  $T$ -invariant, and  $\mathcal{E} \rightarrow \mathcal{D}$  under the canonical morphism  $Y_{\hat{\kappa}}^{\text{an}} \rightarrow X_{\hat{\kappa}}^{\text{an}}$ . Further, if  $\mathcal{E}_Y \subset Y_{\hat{\kappa}}^{\text{an}}$  is the preimage of  $\mathcal{D} \subset X_{\hat{\kappa}}^{\text{an}}$ , there is a  $G$ -equivariant isomorphism of rigid analytic spaces

$$\mathcal{E}_Y \rightarrow \text{Ind}_T^G \mathcal{E}.$$

Further, letting  $Y_x \subset Y$  be the fiber of  $Y_{\hat{\kappa}}^{\text{an}} \rightarrow X_{\hat{\kappa}}^{\text{an}}$  at  $x$ , it follows that  $\mathcal{E}_Y^* := \mathcal{E}_Y \setminus Y_x$  is the preimage of  $\mathcal{D}^* := \mathcal{D} \setminus \{x\}$  under  $\mathcal{E}_Y \rightarrow \mathcal{D}$ . Further,  $\mathcal{E}_Y^* \rightarrow \mathcal{D}^*$  is an étale  $G$ -cover of rigid analytic spaces, and the ring of rigid analytic functions  $\mathcal{O}_{\mathcal{E}^*}$  on  $\mathcal{E}^*$  has a quite concrete description in terms of rigid analytic functions  $\mathcal{O}_{\mathcal{D}^*}$  on  $\mathcal{D}^*$  as follows. Let  $T = \mathbb{Z}/p^e \times \mathbb{Z}/m$  with  $(p, m) = 1$ . Let  $k[[t]] \hookrightarrow k[[t_1]]$  and  $k[[t]] \hookrightarrow k[[z]]$  be the  $\mathbb{Z}/m$ -subextension, respectively the  $\mathbb{Z}/p^e$  subextension of  $k[[t]] \hookrightarrow k[[u]]$ . Since  $k$  is algebraically closed, we can choose  $t_1$  such that  $t_1^m = t$ . Further, the  $\mathbb{Z}/p^e$  subextension is defined by a Witt vector  $\mathbf{p} = (p_1, \dots, p_e)$  over  $k[t^{-1}]$ , which is in standard form. Thus if  $\wp_e(\mathbf{z}) = \mathbf{p}$ , then one has  $k((u)) = k((t))[t_1, \mathbf{z}]$ . Hence  $k[[u]]$  is the integral closure of  $k[[t]]$  in the field of fractions of  $k[[t]][t_1, \mathbf{z}]$ . Thus we conclude that  $\mathcal{O}_{\mathcal{E}^*}$  is the normalization of  $\mathcal{O}_{\mathcal{D}^*}[t_1, \mathbf{z}]$  in its total ring of fractions (which is actually a field), and therefore we have

- (\*) The  $T$ -cover  $\mathcal{E} \rightarrow \mathcal{D}$  is obtained as the normalization of  $\mathcal{D}$  in the extension of the total ring of fractions of the ring extension  $\mathcal{O}_{\mathcal{D}^*} \hookrightarrow \mathcal{O}_{\mathcal{D}^*}[t_1, \mathbf{z}]$ .

*Step 2: The boundary morphisms  $\partial\mathcal{E} \rightarrow \partial\mathcal{D}$  and  $\partial\mathcal{E}_Y \rightarrow \partial\mathcal{D}$ .* In the above notation, let  $\partial\mathcal{D} \subset \mathcal{D}^*$  and  $\partial\mathcal{E} \subset \mathcal{E}^*$  be the “boundaries” of  $\mathcal{D}$ , respectively  $\mathcal{E}$ , which means that  $\partial\mathcal{D}$  is the affinoid defined by the Tate algebra  $\hat{\kappa}\langle t_\eta, t_\eta^{-1} \rangle$ , where  $t_\eta = t/\eta$ , respectively,  $\partial\mathcal{E}$  is defined by the Tate algebra  $\hat{\kappa}\langle t_\varpi, t_\varpi^{-1} \rangle$ , where  $t_\varpi = t/\varpi$ . Since  $p(U) = U^n + a_{n-1}(t)U^{n-1} + \dots + a_1(t)U + a_0(t)$  and  $k[[t]] \hookrightarrow k[[u]]$  is a  $T$ -extension, one checks easily that  $\partial\mathcal{E}$  is  $T$ -invariant, and  $\mathcal{E} \rightarrow \mathcal{D}$  gives rise by restriction to a morphism  $\partial\mathcal{E} \rightarrow \partial\mathcal{D}$ . Furthermore, the following hold:

- (a) The  $T$ -cover  $\partial\mathcal{E} \rightarrow \partial\mathcal{D}$  is obtained as the normalization of  $\partial\mathcal{D}$  in the extension of the total ring of fractions of the ring extension  $\mathcal{O}_{\partial\mathcal{D}} \hookrightarrow \mathcal{O}_{\partial\mathcal{D}}[t_1, \mathbf{z}]$ , where  $t_1$  and  $\mathbf{z}$  are as at (\*) in Step 1 above.
- (b) Let  $\partial\mathcal{E}_Y$  be the preimage of  $\partial\mathcal{D}$  under  $Y_{\hat{\kappa}}^{\text{an}} \rightarrow X_{\hat{\kappa}}^{\text{an}}$ . Then there is a  $G$ -equivariant isomorphism

$$\partial\mathcal{E}_Y \rightarrow \text{Ind}_T^G \partial\mathcal{E}.$$

*Step 3: Using Key Lemma 3.2.* Recall that  $k[[t]] \hookrightarrow k[[z]]$  is the  $\mathbb{Z}/p^e$ -subextension of  $k[[t]] \hookrightarrow k[[u]]$  and  $\mathbf{p} = (p_1, \dots, p_e)$  was the Witt vector in standard form over  $k[t^{-1}]$  defining  $k[[t]] \hookrightarrow k[[z]]$ . In the notation and context from Key Lemma 3.2, we consider *nonzero* elements  $x_1, \dots, x_N \in \mathfrak{m}_0$  and permissible preimages  $Q_{\mu\rho}$  of  $p_{\mu\rho}$  with  $|x_\mu|$  and  $|Q_{\mu\rho} - p_{\mu\rho}|$  (coefficient-wise)

sufficiently small<sup>2</sup> for all  $\mu, \rho$ . We consider the resulting  $\mathbb{Z}/p^e$  ring extension  $\mathcal{A} = k[[\varpi, t]] \hookrightarrow k[[\varpi, Z]] = \mathcal{B}$  and its fiber  $\mathcal{R} = k((\varpi))\langle t \rangle \hookrightarrow k((\varpi))\langle Z \rangle = \mathcal{S}$  over  $k((\varpi))$ . Thus  $\mathcal{R} \hookrightarrow \mathcal{S}$  is the integral closure of  $\mathcal{R}$  in the  $\mathbb{Z}/p^e$ -cyclic field extension of  $\mathcal{K} \hookrightarrow \mathcal{K}_{\mathbf{Q}}$ , where  $\mathbf{Q} = (Q_1, \dots, Q_e)$  satisfies for each  $\rho = 1, \dots, e$  the following:

- (i)  $Q_\rho \in \mathfrak{o}_0[t_1^{-1}, \dots, t_N^{-1}]$ , where  $t_\mu = t - x_\mu$  for  $\mu = 1, \dots, N$ .
- (ii)  $Q_\rho \mapsto p_\rho$  under the specialization  $\mathfrak{o}_0[t_1^{-1}, \dots, t_N^{-1}] \rightarrow k[t^{-1}]$ .

Let  $\mathfrak{Y} \rightarrow \mathfrak{X}$  be the rigid analytification of  $\text{Spec } \mathcal{S} \rightarrow \text{Spec } \mathcal{R}$ . Recalling that  $n := |T|$  and  $n = p^e m$  with  $(p, m) = 1$ , and  $\varpi^n =: \eta$ , let  $\mathcal{D} \subset \mathfrak{X}$  be the closed ball of radius  $|\eta|$  with parameter  $t/\eta$  and  $\partial\mathcal{D} \subset \mathcal{D}$  be its boundary. Finally let  $\mathcal{E}_0 \rightarrow \mathcal{D}$  and  $\partial\mathcal{E}_0 \rightarrow \partial\mathcal{D}$  be the preimages of  $\mathcal{D}$ , respectively  $\partial\mathcal{D}$ , under  $\mathfrak{Y} \rightarrow \mathfrak{X}$ . Then the ring of rigid analytic functions  $\mathcal{O}_{\mathcal{E}_0}$  on  $\mathcal{E}_0$  is the integral closure of  $\mathcal{O}_{\mathcal{D}}[\mathbf{Z}]$  in its total ring of fractions (which is a field), where  $\mathbf{Z} := (Z_1, \dots, Z_e)$  satisfies  $\wp_e(\mathbf{Z}) = \mathbf{Q}$ . Correspondingly,  $\mathcal{O}_{\partial\mathcal{E}_0}$  is the normalization of  $\mathcal{O}_{\partial\mathcal{D}}[\mathbf{Z}]$  in its total ring of fractions.

On the other hand, since each  $|x_\mu|$  is sufficiently small, we can set  $x_\mu = \eta x'_\mu$  with  $|x'_\mu|$  sufficiently small for each  $\mu = 1, \dots, N$ . Thus one has

$$(t - x_\mu)^{-1} = t^{-1}(1 - x'_\mu \eta/t)^{-1} = t^{-1} + \sum_{n=2}^{\infty} a_\mu \eta^n t^{-n} = t^{-1} + \sum_{n=2}^{\infty} a_{\mu n} t_\eta^{-n}$$

for some  $a_{\mu n} \in \mathfrak{o}$  with  $|a_{\mu n}|$  sufficiently small as well. Hence each  $Q_\rho$  is of the form

$$Q_\rho = p_\rho(t^{-1}) + b_\rho q_\rho \quad \text{with } q_\rho \in \hat{\kappa}\langle t_\eta, t_\eta^{-1} \rangle, |b_\rho| \text{ sufficiently small.}$$

Since  $|b_\rho|$  are sufficiently small for  $1 \leq \rho \leq e$ , it follows that  $\mathbf{Q}$  satisfies

$$\mathbf{Q} \in \mathfrak{p} + \wp_e(\mathcal{O}_{\partial\mathcal{D}}),$$

and therefore,  $\mathcal{O}_{\partial\mathcal{D}}[\mathbf{Z}] = \mathcal{O}_{\partial\mathcal{D}}[\mathbf{z}]$ , where  $\wp_e(\mathbf{z}) = \mathfrak{p}$ . Thus the sheaf of rigid analytic functions  $\mathcal{O}_{\partial\mathcal{E}_0}$  is the normalization of  $\mathcal{O}_{\partial\mathcal{D}}[\mathbf{z}]$  in its (total) ring of fractions.

Finally let  $\tilde{\mathcal{E}}$  and  $\partial\tilde{\mathcal{E}}$  be the normalizations of  $\mathcal{E}_0$ , respectively  $\partial\mathcal{E}_0$  in the (total) ring of fractions of  $\mathcal{O}_{\mathcal{E}_0}[t_1]$ , respectively  $\mathcal{O}_{\partial\mathcal{E}_0}[t_1]$ , where  $t_1^m = t$ . One gets  $T = \mathbb{Z}/m \times \mathbb{Z}/p^e$ -covers

$$\tilde{\mathcal{E}} \rightarrow \mathcal{D}, \quad \partial\tilde{\mathcal{E}} \rightarrow \partial\mathcal{D},$$

which satisfy the following:

- (1) The  $T$ -cover  $\tilde{\mathcal{E}} \rightarrow \mathcal{D}$  factors as  $\tilde{\mathcal{E}} \rightarrow \mathcal{E}_0 \rightarrow \mathcal{D}$ , where  $\mathcal{E}_0 \rightarrow \mathcal{D}$  is a  $\mathbb{Z}/p^e$ -cover,  $\tilde{\mathcal{E}} \rightarrow \mathcal{E}_0$  is a  $\mathbb{Z}/m$ -cover, and the following hold:
  - (a)  $\mathcal{E}_0 \rightarrow \mathcal{D}$  is branched at  $x_1, \dots, x_N \in \mathfrak{X}$  only and has no essential ramification jumps;
  - (b)  $\tilde{\mathcal{E}} \rightarrow \mathcal{D}$  is branched at  $t = 0$ , and it is tamely branched there.

<sup>2</sup>One can give an explicit permissible upper bound for  $|x_\mu|$  and  $|Q_{\mu\rho} - p_{\mu\rho}|$ .

- (2) The  $T$ -cover  $\partial\tilde{\mathcal{E}} \rightarrow \partial\mathcal{D}$  factors as  $\partial\tilde{\mathcal{E}} \rightarrow \partial\mathcal{E}_0 \rightarrow \partial\mathcal{D}$ , where  $\partial\mathcal{E}_0 \rightarrow \partial\mathcal{D}$  is a  $\mathbb{Z}/p^e$ -cover and  $\partial\tilde{\mathcal{E}} \rightarrow \partial\mathcal{E}_0$  is a  $\mathbb{Z}/m$ -cover, and the following hold:
- (a)  $\partial\mathcal{E}_0 \rightarrow \partial\mathcal{D}$  is the normalization of  $\partial\mathcal{D}$  in the extension of the total ring of fractions of the extension  $\mathcal{O}_{\partial\mathcal{D}} \hookrightarrow \mathcal{O}_{\partial\mathcal{D}}[\mathbf{Z}] = \mathcal{O}_{\partial\mathcal{D}}[\mathbf{z}]$ ;
  - (b)  $\partial\tilde{\mathcal{E}} \rightarrow \partial\mathcal{D}$  is the normalization of  $\partial\mathcal{D}$  in the extension of the total ring of fractions of the ring extension  $\mathcal{O}_{\partial\mathcal{D}} \hookrightarrow \mathcal{O}_{\partial\mathcal{D}}[t_1, \mathbf{Z}] = \mathcal{O}_{\partial\mathcal{D}}[t_1, \mathbf{z}]$ .

We thus conclude that the  $T$ -cover  $\partial\mathcal{E} \rightarrow \partial\mathcal{D}$  defined in Step 2 and the  $T$ -cover  $\partial\tilde{\mathcal{E}} \rightarrow \partial\mathcal{D}$  defined above are  $T$ -equivariantly  $\partial\mathcal{D}$ -isomorphic. Thus one has  $\partial\mathcal{D}$ -isomorphic  $G$ -covers:

$$(**) \quad \text{Ind}_T^G \partial\mathcal{E} \rightarrow \partial\mathcal{D}, \quad \text{Ind}_T^G \partial\tilde{\mathcal{E}} \rightarrow \partial\mathcal{D}.$$

*Step 4: Finishing the proof of Theorem 3.6.* Let  $\Sigma \subset X$  be the finitely many branch points of  $Y \rightarrow X$ . For every  $x \in \Sigma$  and a fixed  $y \in Y_x$ , let  $T_x$  be the inertia groups at  $y$  and  $\mathcal{E}_{Y,x} = \text{Ind}_{T_x}^G \mathcal{E}_x \rightarrow \mathcal{D}_x$  be the  $G$ -cover of affinoid spaces constructed at Step 1, and let  $\partial\mathcal{E}_{Y,x} = \text{Ind}_{T_x}^G \partial\mathcal{E}_x \rightarrow \partial\mathcal{D}_x$  be the corresponding “boundary”  $G$ -cover of affinoid spaces constructed in Step 2. Finally, consider the  $T_x$ -covers of affinoids  $\tilde{\mathcal{E}}_x \rightarrow \mathcal{D}_x$  and  $\partial\tilde{\mathcal{E}}_x \rightarrow \partial\mathcal{D}_x$  constructed in Step 3 and the resulting induced  $G$ -covers  $\tilde{\mathcal{E}}_{Y,x} := \text{Ind}_{T_x}^G \tilde{\mathcal{E}}_x \rightarrow \mathcal{D}_x$  and its “boundary”  $G$ -cover  $\partial\tilde{\mathcal{E}}_{Y,x} := \text{Ind}_{T_x}^G \partial\tilde{\mathcal{E}}_x \rightarrow \partial\mathcal{D}_x$ . Since by  $(**)$  one has a  $\partial\mathcal{D}_x$ -isomorphism of  $G$ -covers  $\text{Ind}_T^G \partial\mathcal{E}_x \rightarrow \text{Ind}_T^G \partial\tilde{\mathcal{E}}_x$ , a standard gluing procedure leads to a  $G$ -cover of separated rigid analytic spaces  $\mathcal{Y}^{\text{an}} \rightarrow \mathcal{X}^{\text{an}} = X_{\hat{\kappa}}^{\text{an}}$  which has the properties: First, above  $\mathcal{D}_x$  it is isomorphic to  $\partial\tilde{\mathcal{E}}_{Y,x}$  thus, in particular, has no essential ramification above  $\mathcal{D}_x$ , and second, it is isomorphic to  $Y_{\hat{\kappa}}^{\text{an}}$  above the complement of  $\cup_{x \in \Sigma} \mathcal{D}_x$  hence, in particular, is étale over the complement of  $\cup_{x \in \Sigma} \mathcal{D}_x$ . By the rigid/formal GAGA,  $\mathcal{Y}^{\text{an}} \rightarrow \mathcal{X}^{\text{an}}$  is the analytification of the generic fiber  $\mathcal{Y}_{\hat{\kappa}} \rightarrow \mathcal{X}_{\hat{\kappa}}$  of a unique  $G$ -cover of projective smooth  $\mathfrak{o}$ -curves  $\mathcal{Y}_0 \rightarrow \mathcal{X}_0$ . This completes the proof of Theorem 3.6.  $\square$

## 4. Proof of Theorem 1.1

### 4.A. Generalities about covers of $\mathbb{P}^1$ .

*Notation 4.1.* We begin by introducing notation concerning families of covers of curves that will be used throughout this section. Let  $S$  be a separated, integral normal scheme, e.g.,  $S = \text{Spec } A$  with  $A$  an integrally closed domain, and  $\mathbf{k} := \kappa(S)$  its field of rational functions. Let  $\mathbf{k}(t) \hookrightarrow F$  be a finite extension of  $\mathbf{k}(t)$ .

- (1)  $\mathbb{P}_{t,S}^1 = \text{Proj } \mathbb{Z}[t_0, t_1] \times S$  is the  $t$ -projective line over  $S$ , where  $t = t_1/t_0$  is the canonical parameter on  $\mathbb{P}_{t,S}^1$ . In particular,  $\mathbb{P}_{t,S}^1$  is the gluing of its canonical affine lines over  $S$ , namely  $\mathbb{A}_{t,S}^1 := \text{Spec } \mathbb{Z}[t] \times S$  and  $\mathbb{A}_{t^{-1},S}^1 := \text{Spec } \mathbb{Z}[t^{-1}] \times S$ .

(2) Let  $\mathcal{Y}_{t,S} \rightarrow \mathbb{A}_{t,S}^1$  and  $\mathcal{Y}_{t-1,S} \rightarrow \mathbb{A}_{t-1,S}^1$  be the corresponding normalizations in  $\mathbf{k}(t) \hookrightarrow F$ . Then the normalization  $\mathcal{Y}_S \rightarrow \mathbb{P}_{t,S}^1$  of  $\mathbb{P}_{t,S}^1$  in  $\mathbf{k}(t) \hookrightarrow F$  is nothing but the gluing of the two affine  $S$ -curves  $\mathcal{Y}_{t,S} \rightarrow \mathbb{A}_{t,S}^1$  and  $\mathcal{Y}_{t-1,S} \rightarrow \mathbb{A}_{t-1,S}^1$ .

(3) For  $\mathfrak{p} \in S$ , let  $S(\mathfrak{p}) \hookrightarrow S$  be the Zariski closure of  $\mathfrak{p}$  in  $S$  (endowed with the reduced scheme structure). We denote by  $\mathcal{O}_{\mathfrak{p}} := \mathcal{O}_{S,\mathfrak{p}}$  the local ring at  $\mathfrak{p} \in S$ . We set  $S_{\mathfrak{p}} := \text{Spec } \mathcal{O}_{\mathfrak{p}}$  and consider the canonical morphism  $S_{\mathfrak{p}} \hookrightarrow S$ . We notice that  $\mathfrak{p} \hookrightarrow S$  is both the generic fiber of  $S(\mathfrak{p}) \hookrightarrow S$  and the special fiber of  $S_{\mathfrak{p}} \hookrightarrow S$  at  $\mathfrak{p}$ . We get corresponding base changes:

$$\mathcal{Y}_{S(\mathfrak{p})} \rightarrow \mathbb{P}_{t,S(\mathfrak{p})}^1, \quad \mathcal{Y}_{S_{\mathfrak{p}}} \rightarrow \mathbb{P}_{t,S_{\mathfrak{p}}}^1, \quad \mathcal{Y}_{\mathfrak{p}} \rightarrow \mathbb{P}_{t,\mathfrak{p}}^1.$$

We also notice that  $\mathcal{Y}_{\mathfrak{p}} \rightarrow \mathbb{P}_{t,\mathfrak{p}}^1$  is both the generic fiber of  $\mathcal{Y}_{S(\mathfrak{p})} \rightarrow \mathbb{P}_{t,S(\mathfrak{p})}^1$  and the special fiber of  $\mathcal{Y}_{S_{\mathfrak{p}}} \rightarrow \mathbb{P}_{t,S_{\mathfrak{p}}}^1$ .

(4) Finally, affine schemes will be sometimes replaced by the corresponding rings. Concretely, if  $S = \text{Spec } A$  and  $\mathbf{k} = \text{Quot}(A)$ , for a finite extension  $\mathbf{k}(t) \hookrightarrow F$ , one has/denotes

- (a) The  $t$ -projective line over  $A$  is  $\mathbb{P}_{t,A}^1 = \text{Spec } A[t] \cup \text{Spec } A[t^{-1}]$ , and the normalization  $\mathcal{Y}_A \rightarrow \mathbb{P}_{t,A}^1$  of  $\mathbb{P}_{t,A}^1$  in  $\mathbf{k}(t) \hookrightarrow F$  is obtained as the gluing of  $\text{Spec } \mathcal{R}_t \rightarrow \text{Spec } A[t]$  and  $\text{Spec } \mathcal{R}_{t-1} \rightarrow \text{Spec } A[t^{-1}]$ , where  $\mathcal{R}_t$ , respectively  $\mathcal{R}_{t-1}$ , are the integral closures of  $A[t]$ , respectively of  $A[t^{-1}]$ , in the field extension  $\mathbf{k}(t) \hookrightarrow F$ .
- (b) For  $\mathfrak{p} \in \text{Spec}(A)$ , one has/denotes:  $\mathcal{Y}_{A/\mathfrak{p}} \rightarrow \mathbb{P}_{t,A/\mathfrak{p}}^1$  and  $\mathcal{Y}_{A_{\mathfrak{p}}} \rightarrow \mathbb{P}_{t,A_{\mathfrak{p}}}^1$  are the base changes of  $\mathcal{Y}_A \rightarrow \mathbb{P}_{t,A}^1$  under  $A \rightarrow A/\mathfrak{p}$ , respectively  $A \hookrightarrow A_{\mathfrak{p}}$ . And finally, the special fiber  $\mathcal{Y}_{\kappa_{\mathfrak{p}}} \rightarrow \mathbb{P}_{t,\kappa_{\mathfrak{p}}}^1$  of  $\mathcal{Y}_A \rightarrow \mathbb{P}_{t,A}^1$  at  $\mathfrak{p}$  is both the special fiber of  $\mathcal{Y}_{A_{\mathfrak{p}}} \rightarrow \mathbb{P}_{t,A_{\mathfrak{p}}}^1$  and the generic fiber of  $\mathcal{Y}_{A/\mathfrak{p}} \rightarrow \mathbb{P}_{t,A/\mathfrak{p}}^1$ .

In the above notation, let  $A = \mathcal{O}$  be a local ring with maximal ideal  $\mathfrak{m}$  and residue field  $\kappa_{\mathfrak{m}}$ . Suppose that  $\mathcal{O}_v$  is a (Krull) valuation ring of  $\mathbf{k}$  that dominates  $\mathcal{O}$  and has  $\kappa_{\mathfrak{m}} = \kappa_v$ . (Recall that there always exist Krull valuation rings  $\mathcal{O}_v$  dominating  $\mathcal{O}$  and having  $\kappa_{\mathfrak{m}} \subseteq \kappa_v \subseteq \bar{\kappa}_{\mathfrak{m}}$ , but usually  $\kappa_{\mathfrak{m}} \subset \kappa_v$  strictly.) We denote by  $\mathcal{Y}_{\mathcal{O}} \rightarrow \mathbb{P}_{t,\mathcal{O}}^1$  and  $\mathcal{Y}_{\mathcal{O}_v} \rightarrow \mathbb{P}_{t,\mathcal{O}_v}^1$  the normalizations of the corresponding projective lines. The canonical morphism  $\text{Spec } \mathcal{O}_v \rightarrow \text{Spec } \mathcal{O}$  gives canonically commutative diagrams of dominant morphisms:

$$\begin{array}{ccccc} \mathcal{Y}_{\mathcal{O}_v} & \rightarrow & \mathbb{P}_{t,\mathcal{O}_v}^1 & & \mathcal{Y}_{\mathbf{k}} \rightarrow \mathcal{Y}_{\mathcal{O}_v} \leftarrow \mathcal{Y}_v \\ \downarrow & & \downarrow & & \downarrow \cong \quad \downarrow \quad \downarrow \\ \mathcal{Y}_{\mathcal{O}} & \rightarrow & \mathbb{P}_{t,\mathcal{O}}^1 & & \mathcal{Y}_{\mathbf{k}} \rightarrow \mathcal{Y}_{\mathcal{O}} \leftarrow \mathcal{Y}_{\mathfrak{m}}. \end{array}$$

We denote by  $\eta_{\mathfrak{m}} \in \mathbb{P}_{t,\mathfrak{m}}^1$  the generic point of the special fiber of  $\mathbb{P}_{t,\mathcal{O}}^1$ , and by  $\mathcal{Y}_{\mathfrak{m},i} \in \mathcal{Y}_{\mathfrak{m}}$  the irreducible components the special fiber of  $\mathcal{Y}_{\mathfrak{m}}$  of  $\mathcal{Y}_{\mathcal{O}}$ , and by  $\eta_{\mathfrak{m},i} \in \mathcal{Y}_{\mathfrak{m},i}$  their generic points. Correspondingly,  $\eta_v \in \mathbb{P}_{t,v}^1$  is the generic point

of the special fiber of  $\mathbb{P}_{t, \mathcal{O}_v}^1$ , and  $\mathcal{Y}_{v,j} \in \mathcal{Y}_v$  are the irreducible components of the special fiber of  $\mathcal{Y}_v$  of  $\mathcal{Y}_{\mathcal{O}_v}$ , and by  $\eta_{v,i} \in \mathcal{Y}_{v,i}$  their generic points. We notice that  $\mathbb{P}_{t,v}^1 \rightarrow \mathbb{P}_{t,m}^1$  is an isomorphism (because  $\kappa_m = \kappa_v$ ). The local ring  $\mathcal{O}_{\eta_v}$  of  $\eta_v \in \mathbb{P}_{t, \mathcal{O}_v}^1$  is the valuation ring of the so-called *Gauss valuation*  $v_t$  of  $\mathbf{k}(t)$ , thus  $\mathcal{O}_{\eta_{v,j}}$  are the valuation rings of the prolongations  $v_j$  of  $v_t$  to  $F$ . Hence by the valuation criterion for completeness, for every  $\eta_{m,i}$  there exists some  $\eta_{v,j}$  such that  $\eta_{v,j} \mapsto \eta_{m,i}$  under  $\mathcal{Y}_v \rightarrow \mathcal{Y}_m$ . Therefore,  $\mathcal{Y}_v \rightarrow \mathcal{Y}_m$  maps  $\mathcal{Y}_{v,j}$  dominantly onto  $\mathcal{Y}_{m,i}$ .

Finally, for every complete  $k$ -curve  $C$  we denote by  $g_C$  the geometric genus of  $C$ . And to fix notation, let  $\mathcal{Y}_{v,1} \subseteq \mathcal{Y}_v$  and  $\mathcal{Y}_{m,1} \subseteq \mathcal{Y}_m$  be irreducible components such that  $\eta_{v,1} \mapsto \eta_{m,1}$  under  $\mathcal{Y}_{\mathcal{O}_v} \rightarrow \mathcal{Y}_{\mathcal{O}}$ , thus the latter map defines a dominant morphism of curves  $\mathcal{Y}_{v,1} \rightarrow \mathcal{Y}_{m,1}$ .

LEMMA 4.2. *In the above notation, let  $Y_{m,1} \rightarrow \mathcal{Y}_{m,1}$  be the normalization of  $\mathcal{Y}_{m,1}$ . Suppose that  $[\kappa(\eta_{m,1}) : \kappa_m(t)] \geq [F : \mathbf{k}(t)]$  and that  $\mathcal{Y}_{\mathbf{k}}$  is smooth. Then the following hold:*

- (1)  $\mathcal{Y}_m$  and  $\mathcal{Y}_v$  are irreducible and reduced, and  $[\kappa(\eta_{m,1}) : \kappa_m(t)] = [F : \mathbf{k}(t)]$ .
- (2) If  $g_{Y_{m,1}} \geq g_{\mathcal{Y}_{\mathbf{k}}}$ , then  $\mathcal{Y}_{\mathcal{O}_v} \rightarrow \mathbb{P}_{t, \mathcal{O}_v}^1$  is a cover of smooth  $\mathcal{O}_v$ -curves, and  $g_{Y_{m,1}} = g_{\mathcal{Y}_{\mathbf{k}}}$ .

*Proof.* For (1): One has  $\kappa(\eta_m) = \kappa_m(t) = \kappa_v(t) = \kappa(\eta_v)$  and a canonical embedding  $\kappa(\eta_{m,1}) \hookrightarrow \kappa(\eta_{v,1})$ . Thus first using the hypothesis and next using the fundamental (in)equality, one gets that

$$\begin{aligned} [\kappa(\eta_{v,1}) : \kappa_v(t)] &\geq [\kappa(\eta_{m,1}) : \kappa_m(t)] \geq [F : \mathbf{k}(t)] \\ &\geq \sum_j [\kappa(v_j) : \kappa_v(t)] e(v_j|v_t) \delta(v_j|v_t), \end{aligned}$$

where  $e(\cdot|\cdot)$  is the ramification index and  $\delta(\cdot|\cdot)$  is the Ostrowski defect. Thus  $v_1$  is the only prolongation of  $v_t$  to  $F$ , and  $e(v_1|v_t) = 1 = \delta(v_1|v_t)$ . In particular,  $\mathcal{Y}_{v,1}$  is the unique irreducible component of  $\mathcal{Y}_v$ , and moreover,  $\mathcal{Y}_v = \mathcal{Y}_{v,1}$  is reduced. In particular, we also have that  $\mathcal{Y}_{m,1} \subset \mathcal{Y}_m$  is the unique irreducible component of  $\mathcal{Y}_m$ , etc.

For (2): Since  $\mathcal{Y}_v$  is reduced and irreducible, by Roquette [18, Satz I], it follows that the Euler characteristics of the special fiber  $\mathcal{Y}_v$  and that of the generic fiber  $\mathcal{Y}_{\mathbf{k}}$  of  $\mathcal{Y}_{\mathcal{O}_v}$  are equal:

$$\chi(\mathcal{Y}_{\mathbf{k}}|\mathbf{k}) = \chi(\mathcal{Y}_v|\kappa_v).$$

Recall that one has dominant morphism  $\mathcal{Y}_v = \mathcal{Y}_{v,1} \rightarrow \mathcal{Y}_{m,1}$ , which give rise canonically to a dominant morphism of the normalizations  $Y_v \rightarrow Y_{m,1}$ ; thus one has  $g_{Y_{m,1}} \leq g_{Y_v}$ . Therefore using, first, the hypothesis of assertion (2), by the definition of  $\chi(\cdot|\cdot)$ , we have

$$1 - g_{Y_{m,1}} \leq 1 - g_{\mathcal{Y}_{\mathbf{k}}} =: \chi(\mathcal{Y}_{\mathbf{k}}|\mathbf{k}) = \chi(\mathcal{Y}_v|\kappa_v) \leq \chi(Y_v|\kappa_v) \leq \chi(Y_{m,1}|\kappa_m) = 1 - g_{Y_{m,1}}.$$

Hence all the above inequalities are in fact equalities and  $\chi(\mathcal{Y}_v|\kappa_v) = \chi(Y_v|\kappa_v)$ . But then the normalization  $Y_v \rightarrow \mathcal{Y}_v$  is an isomorphism, and  $\mathcal{Y}_{\mathcal{O}_v}$  is smooth, etc.  $\square$

In the context above, let  $A = \mathcal{O}$  be the *valuation ring* of a valuation  $v$ , and suppose that  $v$  is the valuation theoretical composition  $v := v_0 \circ v_1$  of two valuations, say with valuation rings  $\mathcal{O}_1 \subset \mathbf{k}$ , respectively  $\mathcal{O}_0 \subset \mathbf{k}_0$ , where  $\mathbf{k}_0 := \mathbf{k}v_1$  is the residue field of  $v_1$ . Then  $\mathbf{k}v := \mathbf{k} := \mathbf{k}_0v_0$  is the residue field of both  $v$  and  $v_0$ . Let  $v_t$  and  $v_{1,t}$  be the Gauss valuations of  $\mathbf{k}(t)$  defined by  $v$ , respectively  $v_1$ . Finally letting  $t_0 := t \pmod{\mathfrak{m}_{v_{1,t}}}$  be the residue of  $t$  with respect to  $v_{1,t}$ , we consider the Gauss valuation  $v_{0,t_0}$  of  $\kappa_{v_1}(t_0)$ . Then  $v_t$  is actually the valuation theoretical composition  $v_t = v_{0,t_0} \circ v_{1,t}$ . Suppose that the following hold:

- (i) The special fiber  $\mathcal{Y}_{1,s}$  of the normalization  $\mathcal{Y}_1 \rightarrow \mathbb{P}_{t,\mathcal{O}_1}^1$  of  $\mathbb{P}_{t,\mathcal{O}_1}^1$  in  $\mathbf{k}(t) \hookrightarrow F$  is irreducible; hence  $v_{1,t}$  has a unique prolongation  $w_1$  to  $F$ , and the residue field  $Fw_1$  of  $w_1$  satisfies  $F_0 := Fw_1 = \kappa(\mathcal{Y}_{1,s})$ .
- (ii) The special fiber  $\mathcal{Y}_{0,s}$  of the normalization  $\mathcal{Y}_0 \rightarrow \mathbb{P}_{t,\mathcal{O}_0}^1$  of  $\mathbb{P}_{t,\mathcal{O}_0}^1$  in the field extension  $\mathbf{k}_0(t_0) \hookrightarrow F_0$  is irreducible; hence  $v_{0,t_0}$  has a unique prolongation  $w_0$  to  $F_0$ , and  $F_0w_0 = \kappa(\mathcal{Y}_{0,s})$ .

LEMMA 4.3 (Transitivity of smooth covers). *In the above notation, suppose that the hypotheses (i), (ii) are satisfied. Set  $w := w_0 \circ w_1$ , and let  $\mathcal{Y} \rightarrow \mathbb{P}_{t,\mathcal{O}}^1$  be the normalization of  $\mathbb{P}_{t,\mathcal{O}}^1$  in  $\mathbf{k}(t) \hookrightarrow F$ . Then  $w$  is the unique prolongation of  $v_t$  to  $F$ , and the following hold:*

- (1) *The base change of  $\mathcal{Y}$  under  $\mathcal{O} \hookrightarrow \mathcal{O}_{v_1}$  is  $\mathcal{Y}_1 = \mathcal{Y} \times_{\mathcal{O}} \mathcal{O}_{v_1}$  canonically; thus  $\mathcal{Y}_{1,s} = \mathcal{Y}_{\mathfrak{m}_1}$  is the fiber of  $\mathcal{Y}$  at the valuation ideal  $\mathfrak{m}_1 \in \text{Spec } \mathcal{O}$  of  $v_1$ .*
- (2) *Let  $\mathcal{Y}_{\mathcal{O}_0} \rightarrow \mathbb{P}_{t,\mathcal{O}_0}^1$  be the base change of  $\mathcal{Y} \rightarrow \mathbb{P}_{t,\mathcal{O}}^1$  under the  $\mathcal{O} \twoheadrightarrow \mathcal{O}_0$ . Then  $\mathcal{Y}_{\mathfrak{m}_1}$  is the generic fiber of  $\mathcal{Y}_{\mathcal{O}_0}$  and  $\mathcal{Y}_0 \rightarrow \mathbb{P}_{t,\mathcal{O}_0}^1$  is the normalization of  $\mathcal{Y}_{\mathcal{O}_0} \rightarrow \mathbb{P}_{t,\mathcal{O}_0}^1$ .*

*In particular,  $\mathcal{Y}$  is a smooth  $\mathcal{O}$ -curve if and only if  $\mathcal{Y}_i$  is a smooth  $\mathcal{O}_i$ -curve for  $i = 0, 1$ .*

*Proof.* Clear, by the discussion above and Roquette [18, Satz I], combined with the fact that a projective curve is smooth if and only if its arithmetic and geometric genera are equal.  $\square$

4.B. *A specialization result.* We begin by recalling the following two well-known facts. The first one is by Harbater [8] for  $p$ -power covers and by Katz–Gabber [11] in general: Let  $k$  be an algebraically closed field with  $\text{char}(k) = p$ . Then the localization at  $t = 0$  gives rise to a bijection between the finite Galois covers of  $\mathbb{P}_{t,k}^1$  unramified outside  $t = 0, \infty$  and tamely ramified at  $t = \infty$ , and the finite Galois extensions  $k[[t]] \hookrightarrow k[[z]]$ . Moreover, this bijection preserves

the ramification data at  $t = 0$ . Thus given a cyclic  $\mathbb{Z}/p^e$ -cover  $k[[t]] \hookrightarrow k[[z]]$ , there exists a unique cyclic  $\mathbb{Z}/p^e$ -cover of complete smooth curves  $Y_k \rightarrow \mathbb{P}_{t,k}^1$  that is branched only (and totally branched) at  $t = 0$  such that  $k[[t]] \hookrightarrow k[[z]]$  is the extension of local rings of  $Y \rightarrow \mathbb{P}_{t,k}^1$  above  $t = 0$ . We will say that  $Y \rightarrow \mathbb{P}_{t,k}^1$  is the HKG-cover for  $k[[t]] \hookrightarrow k[[z]]$ . The second fact is the local-global principle for the Oort Conjecture; see, e.g., Garuti [5, §3], Saidi [20, §1.2, especially Prop. 1.2.4], which among other things imply

**LGP 4.4.** *Let  $k[[t]] \hookrightarrow k[[z]]$  be a  $\mathbb{Z}/p^e$ -extension and  $Y_k \rightarrow \mathbb{P}_{t,k}^1$  be its HKG-cover. Further let  $W(k) \hookrightarrow R$  be a finite extension of DVR's. Then the  $\mathbb{Z}/p^e$ -extension  $k[[t]] \hookrightarrow k[[z]]$  has a smooth lifting over  $R$  if and only if the  $\mathbb{Z}/p^e$ -cover  $Y_k \rightarrow \mathbb{P}_{t,k}^1$  has a smooth lifting over  $R$ .*

Next let  $e$  be a fixed positive integer, and consider finite sequences of positive numbers  $\mathbf{i} := (i_1 \leq \dots \leq i_e)$  satisfying:  $1 \leq i_1$  is prime to  $p$ , and  $i_{\nu+1} = p i_\nu + \varepsilon$  with  $\varepsilon \geq 0$  and  $\varepsilon$  prime to  $p$  if  $\varepsilon > 0$ . For such a sequence  $\mathbf{i}$ , we consider the sequence  $\mathbf{P}_i = (P_1, \dots, P_e)$  of generic polynomials in standard form  $P_\nu = P_\nu(t^{-1}) = \sum_{i=1}^{i_\nu} a_{\nu,i} t^{-i}$  in the variable  $t^{-1}$  over  $\mathbb{F} := \overline{\mathbb{F}}_p$ , which by definition means the following:

- (i) For all  $\nu, i$  one has:  $\deg(p_\nu(t^{-1})) \leq i_\nu$ , and  $p \mid i$  implies  $a_{\nu,i} = 0$ .
- (ii) The system  $(a_{\nu,i})_{\nu,i}$  of nonzero coefficients  $a_{\nu,i}$  is algebraically independent over  $\mathbb{F} := \overline{\mathbb{F}}_p$ .

We let  $|\mathbf{i}| = \sum_{\nu=1}^{i_e} (i_\nu - \lfloor i_\nu/p \rfloor)$  be the number of nonzero coefficients  $a_{\nu,i}$  (where  $\lfloor a \rfloor$  is the integer part of  $a$ ). We denote by  $A_i := \mathbb{F}[(a_{\nu,i})_{\nu,i}]$  the corresponding polynomial ring in  $|\mathbf{i}|$  variables, and we let  $\mathbb{A}^{|\mathbf{i}|} = \text{Spec } A_i$  be the  $|\mathbf{i}|$ -dimensional affine space over  $\mathbb{F}$ .

For every  $x \in \mathbb{A}^{|\mathbf{i}|}$ , let  $k_x$  be an algebraically closed field extension of  $\mathbb{F}$  that contains  $\kappa_x$ , say via an  $\mathbb{F}$ -embedding  $\phi_x : \kappa_x \hookrightarrow k_x$ , and let  $\bar{x} \in \mathbb{A}^{|\mathbf{i}|}(k_x)$  be the  $k_x$ -rational point of  $\mathbb{A}^{|\mathbf{i}|}$  defined by  $\phi_x$ . Let  $\mathbf{p}_{i,x} = (p_{1,x}, \dots, p_{e,x})$  and  $\mathbf{p}_{i,\bar{x}} = (p_{1,\bar{x}}, \dots, p_{e,\bar{x}})$  be the images of  $\mathbf{P}_i$  over  $\kappa_x$ , respectively  $k_x$ . Then one has virtually by definition that  $p_{\nu,\bar{x}} = \phi_x(p_{\nu,x})$ ; thus  $\mathbf{p}_{i,\bar{x}} = \phi_x(\mathbf{p}_{i,x})$ . In particular, if  $\deg(P_\nu(t^{-1})) = \deg(p_{\nu,x}(t^{-1}))$  for all  $\nu$  with  $(p, i_\nu) = 1$ , then  $\mathbf{p}_{i,x}$  gives rise via the Artin–Schreier–Witt theory mentioned in Section 2.B to a cyclic extension  $k_x[[t]] \hookrightarrow k_x[[z_x]]$  of degree  $p^e$  with upper jumps  $\mathbf{i} = (i_1, \dots, i_e)$ , therefore canonically to the corresponding HKG-cover  $Y_{k_x} \rightarrow \mathbb{P}_{t,k_x}^1$ .

**Definition 4.5.** For  $k_x$  as above, let  $k_x[[t]] \hookrightarrow k_x[[z_x]]$  be a cyclic  $\mathbb{Z}/p^e$ -extension and  $Y_{k_x} \rightarrow \mathbb{P}_{t,k_x}^1$  be its HKG-cover. We say that  $k_x[[t]] \rightarrow k_x[[z_x]]$  is an  $\mathbf{i}$ -extension at  $x \in \mathbb{A}^{|\mathbf{i}|}$  and that  $Y_{k_x} \rightarrow \mathbb{P}_{t,k_x}^1$  is an  $\mathbf{i}$ -HKG-cover at  $x \in \mathbb{A}^{|\mathbf{i}|}$  if  $k_x[[t]] \hookrightarrow k_x[[z_x]]$  has  $\mathbf{i} = (i_1, \dots, i_e)$  as upper ramification jumps.

*Notation 4.6.* We denote by  $\Sigma_{\mathbf{z}} \subseteq \mathbb{A}^{|\mathbf{z}|}$  the set of all  $x \in \mathbb{A}^{|\mathbf{z}|}$  that satisfy: There exists some mixed characteristic otherwise general Krull valuation ring  $R_x$  with residue field  $k_x$  such that some  $\mathbf{z}$ -HKG-cover  $Y_{k_x} \rightarrow \mathbb{P}_{t,k_x}^1$  has a smooth lifting over  $R_x$ .

**PROPOSITION 4.7.** *In Notation 4.6, suppose that  $\Sigma_{\mathbf{z}} \subseteq \mathbb{A}^{|\mathbf{z}|}$  is Zariski dense. Then there exist algebraic integers  $\pi_{\mathbf{z}} \in \overline{\mathbb{Z}}$  such that for every algebraically closed field  $k$  with  $\text{char}(k) = p$ , one has: Every  $\mathbf{z}$ -HKG-cover  $Y_k \rightarrow \mathbb{P}_{t,k}^1$  has a smooth lifting over  $W(k)[\pi_{\mathbf{z}}]$ .*

*Proof.* The proof is quite involved, and after the following Lemma 4.8, it has two main steps as detailed below.

**LEMMA 4.8.** *The generic point  $\eta_{\mathbf{z}} \in \mathbb{A}^{|\mathbf{z}|}$  lies in  $\Sigma_{\mathbf{z}}$ .*

*Proof.* Let  $\mathfrak{U}$  be an ultrafilter on  $\Sigma_{\mathbf{z}}$  that contains all the Zariski open subsets of  $\Sigma_{\mathbf{z}}$ . (Since  $\Sigma_{\mathbf{z}}$  is Zariski dense in the irreducible scheme  $\mathbb{A}^{|\mathbf{z}|}$ , any Zariski open subset of  $\Sigma_{\mathbf{z}}$  is dense as well; thus ultrafilter  $\mathfrak{U}$  exist.) Let  $v_x$  be the valuation of  $R_x$ , and let  $k_x \rightarrow \Theta_x \subset R_x$  be any set of representatives for  $k_x$  in  $R_x$ . Consider the following ultraproducts indexed by  $\Sigma_{\mathbf{z}}$ :

$$\begin{aligned} {}^*k &:= (\prod_x k_x) / \mathfrak{U} \rightarrow {}^*\Theta \\ &:= (\prod_x \Theta_x) / \mathfrak{U} \subset {}^*W := (\prod_x W(k_x)) / \mathfrak{U} \hookrightarrow (\prod_x R_x) / \mathfrak{U} =: {}^*R. \end{aligned}$$

By general model theoretical principles, it follows that  ${}^*R$  is a valuation ring having residue field equal to  ${}^*k$ , and  ${}^*\Theta \subset {}^*R$  is a system of representatives for the residue field  ${}^*k$  of  ${}^*R$ .

Next, coming to geometry, by general model theoretical principles, it follows that the family of  $\mathbb{Z}/p^e$ -covers  $Y_x \rightarrow \mathbb{P}_{t,k_x}^1$  with upper ramification jumps  $\mathbf{z} = (z_1, \dots, z_f)$  gives rise to a  $\mathfrak{U}$ -generic  $\mathbb{Z}/p^e$ -cover  $Y_{*k} \rightarrow \mathbb{P}_{t,*k}^1$  of complete smooth  ${}^*k$ -curves with upper ramification jumps  $\mathbf{z}$ . In terms of finite cyclic extensions and their HKG-covers that means precisely the following: Recall that  $Y_x \rightarrow \mathbb{P}_{t,k_x}^1$  is the HKG-cover whose local behavior at  $t = 0$  is given by the system of polynomials  $\mathbf{p}_{\mathbf{z},\bar{x}} = (p_{1,\bar{x}}, \dots, p_{e,\bar{x}}) = (\phi_x(P_1), \dots, \phi_x(P_e)) = \phi_x(\mathbf{P}_{\mathbf{z}})$ , where  $P_{\nu} = \sum_{i=0}^{z_{\nu}} a_{\nu,i} t^{-i}$  is the generic polynomial of degree  $z_{\nu}$  over  $\mathbb{F}$ . Then setting

$${}^*a_{\nu,i} := (\phi_x(a_{\nu,i})_x) / \mathfrak{U} \in {}^*k \quad \text{and} \quad {}^*p_{\nu}(t^{-1}) := \sum_{i=0}^{z_{\nu}} {}^*a_{\nu,i} t^{-i} \quad \text{for } 1 \leq \nu \leq e,$$

we get a system of polynomials  ${}^*\mathbf{p}_{\mathbf{z}} := ({}^*p_1, \dots, {}^*p_e)$  from  ${}^*k[t^{-1}]$  that defines the local extension  ${}^*k[[t]] \hookrightarrow {}^*k[[z]]$  with HKG-cover  $Y_{*k} \rightarrow \mathbb{P}_{t,*k}^1$ . Moreover, since  $\mathfrak{U}$  contains a basis of the Zariski open subsets of  $\Sigma_{\mathbf{z}}$  and  $\Sigma_{\mathbf{z}}$  is Zariski dense in  $\mathbb{A}^{|\mathbf{z}|}$ , the  $\mathbb{F}$ -homomorphism

$$\mathbb{F}[(a_{\nu,i})_{\nu,i}] \rightarrow {}^*k, \quad a_{\nu,i} \mapsto {}^*a_{\nu,i}$$

is injective. That means that the  $*k$ -rational point of  $\mathbb{A}^{|\mathfrak{z}|}$  defined above factors through the generic point  $\eta_{\mathfrak{z}} \in \mathbb{A}^{|\mathfrak{z}|}$  and, in particular,  $*\mathbf{p}_{\mathfrak{z}} = *\phi_{\eta_{\mathfrak{z}}}(\mathbf{P}_{\mathfrak{z}})$ .

Again, by general model theoretical principles for ultraproducts of ( $G$ -covers of) complete smooth curves over rings, one has: The family of the  $\mathbb{Z}/p^e$ -covers of complete smooth curves  $\mathcal{Y}_{R_x} \rightarrow \mathbb{P}_{t,R_x}^1$  with special fiber  $Y_x \rightarrow \mathbb{P}_{t,k_x}^1$  gives rise to a  $\mathbb{Z}/p^e$ -cover  $\mathcal{Y}_{*R} \rightarrow \mathbb{P}_{t,*R}^1$  of complete smooth  $*R$ -curves, with  $Y_{*k} \rightarrow \mathbb{P}_{t,*k}^1$  as special fiber.

On the other hand, the local behavior at  $t = 0$  of the  $\mathfrak{z}$ -HKG-cover  $Y_{*k} \rightarrow \mathbb{P}_{t,*k}^1$  is given by  $*\mathbf{p}_{\mathfrak{z}} = *\phi_{\eta_{\mathfrak{z}}}(\mathbf{P}_{\mathfrak{z}})$ . Therefore,  $\Sigma_{\mathfrak{z}}$  contains the generic point  $\eta_{\mathfrak{z}}$  of  $\mathbb{A}^{|\mathfrak{z}|}$ .  $\square$

*Step 1: Constructing a parameter space for  $\mathfrak{z}$ -HKG-covers.* Let  $\mathbb{A}^{|\mathfrak{z}|} \hookrightarrow \mathbb{P}^{|\mathfrak{z}|}$  be the canonical embedding of the affine  $\mathbb{F}$ -space  $\mathbb{A}^{|\mathfrak{z}|} := \text{Spec } \mathbb{F}[(a_{\nu,i})_{\nu,i}]$  into the corresponding projective  $\mathbb{F}$ -space  $\mathbb{P}^{|\mathfrak{z}|} := \text{Proj } \mathbb{F}[t_0, (t_{\nu,i})_{\nu,i}]$  via the  $t_0$ -dehomogenization  $a_{\nu,i} = t_{\nu,i}/t_0$ . Letting  $\mathbf{Z}_0 := \mathbb{Z}_p^{\text{nr}}$  be the maximal unramified extension and

$$\mathbb{A}_{\mathbf{Z}_0}^{|\mathfrak{z}|} = \text{Spec } \mathbf{Z}_0[(a_{\nu,i})_{\nu,i}] \quad \text{and} \quad \mathbb{P}_{\mathbf{Z}_0}^{|\mathfrak{z}|} = \text{Proj } \mathbf{Z}_0[t_0, (t_{\nu,i})_{\nu,i}],$$

the embedding  $\mathbb{A}^{|\mathfrak{z}|} \hookrightarrow \mathbb{P}^{|\mathfrak{z}|}$  is the special fiber of  $\mathbb{A}_{\mathbf{Z}_0}^{|\mathfrak{z}|} \hookrightarrow \mathbb{P}_{\mathbf{Z}_0}^{|\mathfrak{z}|}$ . Notice that  $*\phi_{\eta_{\mathfrak{z}}} : A_{\mathfrak{z}} \rightarrow *k$  gives rise via  $*k \rightarrow *\Theta$  canonically to an embedding of  $\mathbf{Z}_0$ -algebras defined by

$$(*) \quad *\phi_{\mathbf{Z}_0} : A_{\mathfrak{z},\mathbf{Z}_0} := \mathbf{Z}_0[(a_{\nu,i})_{\nu,i}] \hookrightarrow *R, \quad a_{\nu,i} \mapsto *\phi_{\eta_{\mathfrak{z}}}(a_{\nu,i}).$$

Let  $R_0 \subset *R$  be a  $\mathbf{Z}_0$ -algebra of finite type containing  $\mathbf{Z}_0[(a_{\nu,i})_{\nu,i}]$  such that the  $\mathbb{Z}/p^e$ -cover  $\mathcal{Y}_{*R} \rightarrow \mathbb{P}_{t,*R}^1$  is defined over  $R_0$ ; i.e., there exists a  $\mathbb{Z}/p^e$ -cover  $\mathcal{Y}_{R_0} \rightarrow \mathbb{P}_{t,R_0}^1$  of complete smooth  $R_0$ -curves such that  $\mathcal{Y}_{*R} \rightarrow \mathbb{P}_{t,*R}^1$  is the base change of  $\mathcal{Y}_{R_0} \rightarrow \mathbb{P}_{t,R_0}^1$  under  $R_0 \hookrightarrow *R$ .

Let  $*\mathfrak{m} \subset *R$  be the valuation ideal of  $*R$  and  $*v$  be the valuation of  $*R$ . Then  $\mathfrak{q} := *\mathfrak{m} \cap R_0$  is the center of  $*v$  in  $U := \text{Spec } R_0$ ; thus, in particular, the center of  $*v$  in  $U$  is nonempty.

Since  $U = \text{Spec } R_0$  is an (integral) affine  $\mathbf{Z}_0$ -scheme of finite type, there are embeddings  $U \hookrightarrow \mathbb{A}_{\mathbf{Z}_0}^n \hookrightarrow \mathbb{P}_{\mathbf{Z}_0}^n$  for sufficiently large  $n$ . Thus the closure  $\bar{U}$  of  $U$  in  $\mathbb{P}_{\mathbf{Z}_0}^n$  is a projective  $\mathbf{Z}_0$ -scheme, and the point  $\mathfrak{q} \in U \subset \bar{U}$  is the center of  $*v$  on  $\bar{U}$ . Next, the canonical  $\mathbf{Z}_0$ -embedding  $A_{\mathfrak{z},\mathbf{Z}_0} \hookrightarrow R_0$  gives rise to a rational dominant map  $\bar{U} \dashrightarrow \mathbb{P}_{\mathbf{Z}_0}^{|\mathfrak{z}|}$  defined over  $\mathbf{Z}_0$ . Thus considering the blowup  $\tilde{U} \rightarrow \bar{U}$  of the indeterminacy locus of the rational map  $\bar{U} \dashrightarrow \mathbb{P}_{\mathbf{Z}_0}^{|\mathfrak{z}|}$ , the resulting rational map  $\tilde{U} \rightarrow \bar{U} \dashrightarrow \mathbb{P}_{\mathbf{Z}_0}^{|\mathfrak{z}|}$  is everywhere defined on  $\tilde{U}$ ; thus one gets a well-defined dominant, and hence surjective,  $\mathbf{Z}_0$ -morphism  $\tilde{U} \rightarrow \mathbb{P}_{\mathbf{Z}_0}^{|\mathfrak{z}|}$ . Moreover,  $\bar{U}$  is a scheme of finite type over  $\mathbf{Z}_0$ , thus so is  $\tilde{U}$ , hence  $\tilde{U}$  is a projective  $\mathbf{Z}_0$ -scheme. We also notice that since  $\bar{U} \dashrightarrow \mathbb{P}_{\mathbf{Z}_0}^{|\mathfrak{z}|}$  is defined on  $U \subset \bar{U}$ , the morphism  $\tilde{U} \rightarrow \bar{U}$  is an isomorphism above  $U$ , thus at the center  $\mathfrak{q} \in U$  of  $*v$  on  $\bar{U}$ . Thus letting

$\tilde{q} \in \tilde{U}$  be the preimage of  $q$  in  $\tilde{U}$ , it follows that  $\mathcal{O}_{\tilde{q}} = \mathcal{O}_q$ , thus  $\tilde{q}$  is the center of  ${}^*v$  on  $\tilde{U}$  under the canonical embedding  $\kappa(\tilde{U}) = \kappa(U) \hookrightarrow \text{Quot}({}^*R)$ . Finally, let  $V \rightarrow \tilde{U}$  be the normalization of  $\tilde{U}$ . Then one has: First,  $V$  is a projective  $\mathbf{Z}_0$ -scheme. Second, since valuation rings are integrally closed, it follows that the canonical morphism  $\text{Spec } {}^*R \rightarrow \text{Spec } \mathcal{O}_{\tilde{q}} \rightarrow \tilde{U}$  factors canonically through  $V \rightarrow \tilde{U}$ . We denote by  $\mathfrak{p} \in V$  the center of  ${}^*v$  on  $V$  under the canonical morphism  $\text{Spec } {}^*R \rightarrow V$ . In other words, if  $\mathcal{O}_{\mathfrak{p}}$  is the local ring of  $\mathfrak{p} \in V$ , then  ${}^*R$  dominates  $\mathcal{O}_{\mathfrak{p}}$ . We thus have the following:

*Conclusion 4.9.* There exists a projective normal  $\mathbf{Z}_0$ -scheme  $V$  whose function field  $\kappa(V)$  is  $\mathbf{Z}_0$ -embeddable in  $\text{Quot}({}^*R)$ , say  $\kappa(V) \hookrightarrow \text{Quot}({}^*R)$ , satisfying the following:

- (1) Let  $\mathfrak{p} \in V$  be the center of  ${}^*v$  on  $V$  induced by  $\kappa(V) \hookrightarrow \text{Quot}({}^*R)$ , and let  $\mathcal{O}_{\mathfrak{p}}$  be the local ring of  $\mathfrak{p}$ . Then the  $\mathbb{Z}/p^e$ -cover of complete smooth  ${}^*R$ -curves  $\mathcal{Y}_{*R} \rightarrow \mathbb{P}_{t, *R}^1$  is defined over  $\mathcal{O}_{\mathfrak{p}}$ . That means that there exists a  $\mathbb{Z}/p^e$ -cover of complete smooth  $\mathcal{O}_{\mathfrak{p}}$ -curves  $\mathcal{Y}_{\mathcal{O}_{\mathfrak{p}}} \rightarrow \mathbb{P}_{t, \mathcal{O}_{\mathfrak{p}}}^1$  such that  $\mathcal{Y}_{*R} \rightarrow \mathbb{P}_{t, *R}^1$  is the base change of  $\mathcal{Y}_{\mathcal{O}_{\mathfrak{p}}} \rightarrow \mathbb{P}_{t, \mathcal{O}_{\mathfrak{p}}}^1$  under  $\mathcal{O}_{\mathfrak{p}} \hookrightarrow {}^*R$ .
- (2) The image of  ${}^*\phi_{\mathbf{Z}_0} : A_{\mathbf{z}, \mathbf{Z}_0} \rightarrow {}^*R$  is contained in the image of the canonical embedding  $\kappa(V) \hookrightarrow \text{Quot}({}^*R)$ , and the resulting  $\mathbf{Z}_0$ -embedding  $A_{\mathbf{z}, \mathbf{Z}_0} \hookrightarrow \kappa(V)$  is defined by some proper morphism

$$V \rightarrow \mathbb{P}_{\mathbf{Z}_0}^{|\mathbf{z}|}.$$

We notice that condition (1) implies the following: If  $\mathcal{O}_{\mathfrak{p}} \rightarrow \kappa_{\mathfrak{p}}$  is the residue field of  $\mathcal{O}_{\mathfrak{p}}$ , then the special fiber  $Y_{\mathfrak{p}} \rightarrow \mathbb{P}_{t, \mathfrak{p}}^1$  of  $\mathcal{Y}_{\mathcal{O}_{\mathfrak{p}}} \rightarrow \mathbb{P}_{t, \mathcal{O}_{\mathfrak{p}}}^1$  is a  $\mathbb{Z}/p^e$ -cover of complete smooth  $\kappa_{\mathfrak{p}}$ -curves whose base change under  $\kappa_{\mathfrak{p}} \hookrightarrow {}^*k$  is canonically isomorphic to  $Y_{*k} \rightarrow \mathbb{P}_{t, *k}^1$ . In other words, the embedding  ${}^*\phi_{\eta_{\mathbf{z}}} : A_{\mathbf{z}} \hookrightarrow {}^*k$  defined by (\*) above factors through  $A_{\mathbf{z}} \hookrightarrow \kappa_{\eta_{\mathbf{z}}} \hookrightarrow \kappa_{\mathfrak{p}}$ , and  $\mathfrak{p} \in V$  is mapped to the generic point  $\eta_{\mathbf{z}} \in \mathbb{P}^{|\mathbf{z}|}$  of the special fiber  $\mathbb{P}^{|\mathbf{z}|} \hookrightarrow \mathbb{P}_{\mathbf{Z}_0}^{|\mathbf{z}|}$  under  $V \rightarrow \mathbb{P}_{\mathbf{Z}_0}^{|\mathbf{z}|}$ .

Let  $V(\mathfrak{p}) \subset V$  be the Zariski closure of  $\mathfrak{p}$  in  $V$  viewed as a closed  $\mathbf{Z}_0$ -subscheme of  $V$  endowed with the reduced scheme structure. Since  $\mathfrak{p} \mapsto \eta_{\mathbf{z}}$ , one has that  $\kappa_{\eta_{\mathbf{z}}} \hookrightarrow \kappa_{\mathfrak{p}}$ , hence  $\kappa_{\mathfrak{p}}$  has characteristic  $p$ , and  $\mathfrak{p}$  lies in the special fiber  $V_{\mathbb{F}}$  of  $V$ . We conclude that  $V(\mathfrak{p}) \subset V_{\mathbb{F}}$ .

Next, if  $V(\mathfrak{p})$  has codimension  $> 0$  in  $V_{\mathbb{F}}$ , let  $\tilde{V} \rightarrow V$  be the normalization of the blowup of  $V$  along the closed reduced  $\mathbf{Z}_0$ -subscheme  $V(\mathfrak{p})$ . Let  $E_1, \dots, E_r \subset \tilde{V}$  be the finitely many irreducible components of the preimage of the exceptional divisor of the blowup. Then the generic points  $\mathfrak{r}_i$  of the  $E_i$ ,  $i = 1, \dots, r$ , are precisely the points of codimension one of  $\tilde{V}$  that map to  $\mathfrak{p}$  under  $\tilde{V} \rightarrow V$ , and  $\cup_i E_i$  is the preimage of  $V(\mathfrak{p})$  in  $V$ .

Further, if  $\mathfrak{r} = \mathfrak{r}_i$  is fixed,  $\mathcal{O}_{\mathfrak{r}}$  is the local ring of  $\mathfrak{r} \in \tilde{V}$ , and  $\kappa_{\mathfrak{r}}$  is its residue field, it follows that  $\mathcal{O}_{\mathfrak{p}} \hookrightarrow \mathcal{O}_{\mathfrak{r}}$  and  $\kappa_{\mathfrak{p}} \hookrightarrow \kappa_{\mathfrak{r}}$  canonically. Recall that by the property (1) above,  $\mathcal{Y}_{\mathcal{O}_{\mathfrak{p}}} \rightarrow \mathbb{P}_{t, \mathcal{O}_{\mathfrak{p}}}^1$  is a  $\mathbb{Z}/p^e$ -cover of smooth  $\mathcal{O}_{\mathfrak{p}}$ -curves with special fiber  $Y_{\mathfrak{p}} \rightarrow \mathbb{P}_{t, \mathfrak{p}}^1$  whose base change under  $\kappa_{\mathfrak{p}} \hookrightarrow {}^*k$  is  $Y_{*k} \rightarrow \mathbb{P}_{t, *k}^1$ .

Therefore, the base change  $\mathcal{Y}_{\mathcal{O}_\tau} \rightarrow \mathbb{P}_{t, \mathcal{O}_\tau}^1$  of  $\mathcal{Y}_{\mathcal{O}_\mathfrak{p}} \rightarrow \mathbb{P}_{t, \mathcal{O}_\mathfrak{p}}^1$  defined by  $\mathcal{O}_\mathfrak{p} \hookrightarrow \mathcal{O}_\tau$  is a  $\mathbb{Z}/p^e$  cover of proper smooth  $\tilde{\mathcal{O}}$ -curves whose special fiber  $Y_\tau \rightarrow \mathbb{P}_{t, \tau}^1$  is the base change of  $Y_\mathfrak{p} \rightarrow \mathbb{P}_{t, \mathfrak{p}}^1$  under  $\kappa_\mathfrak{p} \hookrightarrow \kappa_\tau$ . Hence choosing any  $\kappa_\mathfrak{p}$ -embedding  $\kappa_\tau \hookrightarrow {}^*k$ , we get that the special fiber  $Y_\tau \rightarrow \mathbb{P}_{t, \tau}^1$  becomes  $Y_{*k} \rightarrow \mathbb{P}_{t, *k}^1$  under the base change defined by  $\kappa_\tau \hookrightarrow {}^*k$ .

Hence by replacing  $V$  by  $\tilde{V}$  if necessary, we can suppose that  $\mathfrak{p} \in V$  has codimension one in  $V$ , or equivalently, that  $V(\mathfrak{p}) \subset V_{\mathbb{F}}$  is an irreducible component of  $V_{\mathbb{F}}$ .

By de Jong’s theory of alterations [10, Th. 6.5 and the discussion thereafter], one has the following:

*Fact 4.10.* In the above context, there exists a finite extension of discrete valuation rings  $\mathbf{Z}_0 \hookrightarrow \mathbf{Z}$  and a dominant generically finite proper morphism  $W \rightarrow V \times_{\mathbf{Z}_0} \mathbf{Z}$  of projective  $\mathbf{Z}$ -schemes with  $W$  strictly semi-stable over  $\mathbf{Z}$ ; i.e., the generic fiber of  $W$  is a smooth projective variety over  $\text{Quot}(\mathbf{Z})$ , and the special fiber  $W_{\mathbb{F}}$  is reduced and satisfies: If  $W_{\mathbb{F}, j}$  with  $j \in J$  is any set of  $|J|$  distinct irreducible components of  $W_{\mathbb{F}}$ , then  $\cap_j W_{\mathbb{F}, j}$  is a smooth subscheme of  $W$  of codimension  $|J|$ .

We also notice that  $\mathbf{Z}_0$  has only ramified extension, thus  $\mathbf{Z} = \mathbf{Z}_0[\pi_0]$ , where  $\pi_0$  is any uniformizing parameter of  $\mathbf{Z}$ . We claim that one can choose  $\pi_0$  to be an algebraic integer, i.e., integral over  $\mathbb{Z}$ . Indeed, let  $\mu'_\infty$  be the group of all the roots of unity of order prime to  $p$ . Then  $\mathbb{Z}[\mu'_\infty] \subset \mathbf{Z}_0$  is  $p$ -adically dense in  $\mathbf{Z}_0$ . Hence the minimal polynomial  $p_{\pi_0}(T) \in \mathbf{Z}_0[T]$  of  $\pi_0$  over  $\mathbf{Z}_0$  can be approximated arbitrarily close by monic polynomials over  $\mathbb{Z}[\mu'_\infty]$ , etc.

Consider the sequence  $W \rightarrow V \rightarrow \mathbb{P}_{\mathbf{Z}_0}^{|\mathfrak{q}|}$  of dominant, thus surjective, morphisms of projective integral  $\mathbf{Z}_0$ -schemes, and let  $\mathfrak{q} \in W$  be a fixed preimage of  $\mathfrak{p}$ . Since the first morphism is generically finite, it follows that  $\mathfrak{q}$  has codimension one because  $\mathfrak{p} \in V$  has codimension one. And since  $\mathfrak{p} \in V_{\mathbb{F}}$ , it follows that  $\mathfrak{q} \in W_{\mathbb{F}}$ . Thus the Zariski closure  $W(\mathfrak{q}) \subseteq W_{\mathbb{F}}$  is an irreducible component of the special fiber  $W_{\mathbb{F}} \subset W$  of  $W$ . Hence  $W \rightarrow V \rightarrow \mathbb{P}_{\mathbf{Z}_0}^{|\mathfrak{q}|}$  gives rise to the sequence of surjective morphisms of projective  $\mathbf{Z}_0$ -schemes

$$W(\mathfrak{q}) \rightarrow V(\mathfrak{p}) \rightarrow \mathbb{P}^{|\mathfrak{q}|},$$

each of which is an irreducible component of the corresponding ambient  $\mathbf{Z}_0$ -scheme.

*Definition 4.11.* We say that  $W(\mathfrak{q}) \rightarrow V(\mathfrak{p}) \rightarrow \mathbb{P}^{|\mathfrak{q}|}$  is a *parameter space* for  $\mathfrak{v}$ -HKG-covers.

The reason for the above terminology is as follows (see especially Lemma 4.12):

- First, recall that by the Conclusion 4.9 above,  $\mathcal{Y}_{\mathcal{O}_\mathfrak{p}} \rightarrow \mathbb{P}_{t, \mathcal{O}_\mathfrak{p}}^1$  is a  $\mathbb{Z}/p^e$ -cover of smooth  $\mathcal{O}_\mathfrak{p}$ -curves with special fiber  $Y_\mathfrak{p} \rightarrow \mathbb{P}_{t, \mathfrak{p}}^1$  whose base change under

$\kappa_{\mathfrak{p}} \hookrightarrow {}^*k$  is  $Y_{*k} \rightarrow \mathbb{P}_{t, *k}^1$ . Let  $\mathcal{Y}_{\mathcal{O}_q} \rightarrow \mathbb{P}_{t, \mathcal{O}_q}^1$  be the base change of  $\mathcal{Y}_{\mathcal{O}_{\mathfrak{p}}} \rightarrow \mathbb{P}_{t, \mathcal{O}_{\mathfrak{p}}}^1$  under  $\mathcal{O}_{\mathfrak{p}} \hookrightarrow \mathcal{O}_q$ . Then  $\mathcal{Y}_{\mathcal{O}_q} \rightarrow \mathbb{P}_{t, \mathcal{O}_q}^1$  is a  $\mathbb{Z}/p^e$ -cover of proper smooth  $\mathcal{O}_q$ -curves whose special fiber  $Y_q \rightarrow \mathbb{P}_{t, q}^1$  is the base change of  $Y_{\mathfrak{p}} \rightarrow \mathbb{P}_{t, \mathfrak{p}}^1$  under  $\kappa_{\mathfrak{p}} \hookrightarrow \kappa_q$ . Again, choosing any  $\kappa_{\mathfrak{p}}$ -embedding of  $\kappa_q \hookrightarrow {}^*k$ , one gets that the base change of the special fiber  $Y_q \rightarrow \mathbb{P}_{t, q}^1$  under  $\kappa_q \hookrightarrow {}^*k$  becomes  $Y_{*k} \rightarrow \mathbb{P}_{t, *k}^1$ . This means that the embedding  ${}^*\phi_{\eta_{\mathfrak{z}}} : A_{\mathfrak{z}} \hookrightarrow {}^*k$  defined at (\*) in the beginning of Step 1 factors through  $A_{\mathfrak{z}} \hookrightarrow \kappa_{\eta_{\mathfrak{z}}} \hookrightarrow \kappa_{\mathfrak{p}} \hookrightarrow \kappa_q$ , reflecting the fact that  $\mathfrak{q} \mapsto \mathfrak{p} \mapsto \eta_{\mathfrak{z}}$ . In other words, there exists a  ${}^*k$ -rational point  ${}^*\phi_{\mathfrak{q}} : \kappa_{\mathfrak{q}} \rightarrow {}^*k$  such that the given  ${}^*k$ -rational point  ${}^*\phi_{\eta_{\mathfrak{z}}} : \kappa_{\eta_{\mathfrak{z}}} \rightarrow {}^*k$  defined by  ${}^*\phi_{\eta_{\mathfrak{z}}} : A_{\mathfrak{z}} \rightarrow {}^*k$  is of the form

$$(**) \quad {}^*\phi_{\eta_{\mathfrak{z}}} = {}^*\phi_{\mathfrak{q}} \circ (\kappa_{\eta_{\mathfrak{z}}} \hookrightarrow \kappa_{\mathfrak{q}}).$$

• Second, let  $\lambda := \kappa(W)$  be the function field of  $W$ , and let  $F := \kappa(\mathcal{Y}_{\mathcal{O}_q})$  be the function field of  $\mathcal{Y}_{\mathcal{O}_q}$ . Then  $\mathcal{Y}_{\mathcal{O}_q} \rightarrow \mathbb{P}_{t, \mathcal{O}_q}^1$  has as generic fiber a  $\mathbb{Z}/p^e$ -cover of complete smooth  $\lambda$ -curves  $\mathcal{Y}_{\lambda} \rightarrow \mathbb{P}_{t, \lambda}^1$  and gives rise to a  $\mathbb{Z}/p^e$  extension of function field in one variable  $\lambda(t) \hookrightarrow F$ . Since  $\mathcal{O}_q$  is a (discrete) valuation ring and  $\mathcal{Y}_{\mathcal{O}_q} \rightarrow \mathbb{P}_{t, \mathcal{O}_q}^1$  is a cover of smooth  $\mathcal{O}_q$ -curves, it follows by the discussion in Section 4.A that  $\mathcal{Y}_{\mathcal{O}_q} \rightarrow \mathbb{P}_{t, \mathcal{O}_q}^1$  is precisely the normalization of  $\mathbb{P}_{t, \mathcal{O}_q}^1$  in the function field extension  $\lambda(t) \hookrightarrow F$ . Notice that  $\mathbb{P}_{t, \lambda}^1$  is the generic fiber of  $\mathbb{P}_{t, W}^1$ , and consider

$$\mathcal{Y}_W \rightarrow \mathbb{P}_{t, W}^1$$

the normalization of  $\mathbb{P}_{t, W}^1$  in the field extension  $\lambda(t) \hookrightarrow F$ . Then by definition one has that the base change of  $\mathcal{Y}_W \rightarrow \mathbb{P}_{t, W}^1$  under  $\text{Spec } \mathcal{O}_q \hookrightarrow W$  is precisely  $\mathcal{Y}_{\mathcal{O}_q} \rightarrow \mathbb{P}_{t, \mathcal{O}_q}^1$ .

LEMMA 4.12. *Let  $x \in \mathbb{A}^{|\mathfrak{z}|}$  be such that the image  $\mathbf{p}_{\mathfrak{z}, x} = (p_{1,x}, \dots, p_{e,x})$  of  $\mathbf{P}_{\mathfrak{z}} = (P_1, \dots, P_e)$  under  $A_{\mathfrak{z}} \rightarrow \kappa_x$  satisfies  $\deg(p_{\nu,x}) = \deg(P_{\nu})$  for all  $\nu = 1, \dots, e$ . Let  $y \in W(\mathfrak{q})$  be a preimage of  $x$  under  $W(\mathfrak{q}) \rightarrow V(\mathfrak{p}) \rightarrow \mathbb{P}_{\mathfrak{z}_0}^{|\mathfrak{z}|}$ , and let  $\mathcal{O}_v$  be a valuation ring dominating  $\mathcal{O}_y$  with  $\kappa_v = \kappa_y$ . Let  $\mathcal{Y}_{\mathcal{O}_v} \rightarrow \mathbb{P}_{t, \mathcal{O}_v}^1$  be the base change of  $\mathcal{Y}_W \rightarrow \mathbb{P}_{t, W}^1$  under  $\text{Spec } \mathcal{O}_v \hookrightarrow W$ . Then  $\mathcal{Y}_{\mathcal{O}_v} \rightarrow \mathbb{P}_{t, \mathcal{O}_v}^1$  is a  $\mathbb{Z}/p^e$ -cover of projective smooth  $\mathcal{O}_v$ -curves.*

*Proof.* Let  $\mathcal{Y}_{\mathcal{O}_y} \rightarrow \mathbb{P}_{t, \mathcal{O}_y}^1$  be the base change of  $\mathcal{Y}_W \rightarrow \mathbb{P}_{t, W}^1$  under the canonical embedding  $\text{Spec } \mathcal{O}_y \hookrightarrow W$ ; thus, in particular,  $\mathcal{Y}_{\mathcal{O}_y} \rightarrow \mathbb{P}_{t, \mathcal{O}_y}^1$  is the normalization of  $\mathbb{P}_{t, \mathcal{O}_y}^1$  in the field extension  $\lambda(t) \hookrightarrow F$ . Since  $y \in W(\mathfrak{q})$  and the geometric fiber  $\mathcal{Y}_{\mathfrak{q}} \rightarrow \mathbb{P}_{t, \mathfrak{q}}^1$  of  $\mathcal{Y}_{W(\mathfrak{q})} \rightarrow \mathbb{P}_{t, W(\mathfrak{q})}^1$  is a  $\mathbb{Z}/p^e$ -cover of smooth complete curves, the same holds correspondingly if one replaces the local ring  $\mathcal{O}_y = \mathcal{O}_{W,y}$  by  $\mathfrak{o}_y := \mathcal{O}_{W(\mathfrak{q}),y} = \mathcal{O}_y/\mathfrak{q}$  and  $\lambda(t) \hookrightarrow F$  by  $\kappa_{\mathfrak{q}}(t) \hookrightarrow F_{\mathfrak{q}}$ , where  $F_{\mathfrak{q}} := \kappa(\mathcal{Y}_{\mathfrak{q}})$  is viewed as function field over  $\kappa_{\mathfrak{q}}$ . Recall that the local extension  $\kappa_{\mathfrak{q}}[[t]] \hookrightarrow \kappa_{\mathfrak{q}}[[z_{\mathfrak{q}}]]$  of  $\mathcal{Y}_{\mathfrak{q}} \rightarrow \mathbb{P}_{t, \mathfrak{q}}^1$  at  $t = 0$  is defined by the image  $\mathbf{p}_{\mathfrak{z}, \mathfrak{q}}$  of  $\mathbf{P}_{\mathfrak{z}}$  under the canonical embedding  $A_{\mathfrak{z}} \hookrightarrow \kappa_{\eta_{\mathfrak{z}}} \hookrightarrow \kappa_{\mathfrak{q}}$ . On the other hand, if  $\mathfrak{o}_x$  denotes the local ring of  $x \in \mathbb{A}^{|\mathfrak{z}|} \subset \mathbb{P}^{|\mathfrak{z}|}$ , then  $A_{\mathfrak{z}} \subset \mathfrak{o}_x$  and  $\mathfrak{o}_y$  dominates  $\mathfrak{o}_x$ . Hence

$A_{\mathbf{i}} \hookrightarrow \mathfrak{o}_y$ , and therefore,  $\mathbf{p}_{\mathbf{i},\mathfrak{q}}$  is defined over  $\mathfrak{o}_y$ . Further, by the commutativity of the diagrams

$$\begin{array}{ccc} A_{\mathbf{i}} & \hookrightarrow & \mathfrak{o}_y \\ \downarrow & & \downarrow \\ \kappa_x & \hookrightarrow & \kappa_y, \end{array} \quad \begin{array}{ccc} \mathbf{P}_{\mathbf{i}} & \mapsto & \mathbf{p}_{\mathbf{i},\mathfrak{q}} \\ \downarrow & & \downarrow \\ \mathbf{p}_{\mathbf{i},x} & \mapsto & \mathbf{p}_{\mathbf{i},y}, \end{array}$$

it follows that the image of  $\mathbf{p}_{\mathbf{i},\mathfrak{q}}$  under the residue homomorphism  $\mathfrak{o}_y \rightarrow \kappa_y$  equals the image of  $\mathbf{p}_{\mathbf{i},x}$  under  $\kappa_x \hookrightarrow \kappa_y$ . Thus by the functoriality of the Artin–Schreier–Witt theory, it follows that every irreducible component of the special fiber of  $\mathbb{P}_{t,\mathfrak{o}_y}^1$  dominates the  $\mathbf{i}$ -HKG-cover of  $\mathbb{P}_{t,\kappa_y}^1$  defined by  $\mathbf{p}_{\mathbf{i},y}$ . Since  $\deg(p_{\nu,y}) = \deg(p_{\nu,x}) = \deg P_{\nu}$  for all  $\nu$ , the latter cover must have degree  $p^e$  and upper ramification jumps  $\mathbf{i} = (i_1, \dots, i_e)$ . Hence we can apply Lemma 4.2 and conclude that the special fibers  $\mathcal{Y}_y$  and  $\mathcal{Y}_v$  are reduced and irreducible.

In order to conclude, we notice that by the discussion above, the normalization  $Y_y \rightarrow \mathcal{Y}_y$  dominates the  $\mathbf{i}$ -HKG-cover of  $\mathbb{P}_{t,y}^1$  defined by  $\mathbf{p}_{\mathbf{i},y}$ . Since every  $\mathbf{i}$ -HKG-cover has as genus a constant depending on  $\mathbf{i}$  only, thus including the generic fiber, it follows that  $g_{Y_y} \geq g_{\mathcal{Y}_y}$ . We thus conclude the proof of Lemma 4.12 by applying Lemma 4.2.  $\square$

*Step 2: Finishing the proof of Proposition 4.7.* Coming back to the proof of Proposition 4.7 we proceed as follows. Let  $k$  be any algebraically closed field with  $\text{char}(k) = p$ , and let  $Y_k \rightarrow \mathbb{P}_{t,k}^1$  be an  $\mathbf{i}$ -HKG-cover, say with local ring extension  $k[[t]] \hookrightarrow k[[z]]$  at  $t = 0$  defined by  $\mathbf{p}_{\mathbf{i}} = (p_1, \dots, p_e)$ . Let  $x \in \mathbb{A}^{|\mathbf{i}|}$  be the point defined by the specialization map  $\phi_x : A_{\mathbf{i}} \rightarrow k$ ,  $\mathbf{P}_{\mathbf{i}} \mapsto \mathbf{p}_{\mathbf{i}}$ . Since  $W(\mathfrak{q}) \rightarrow V(\mathfrak{p}) \rightarrow \mathbb{P}^{|\mathbf{i}|}$  are surjective  $\mathbb{F}$ -morphisms, there exists a preimage  $y \in W(\mathfrak{q})$  of  $x$  such that  $\kappa_x \hookrightarrow \kappa_y$  is finite. Since  $k$  is algebraically closed, there is a  $\kappa_x$ -embedding  $\phi_y : \kappa_y \hookrightarrow k$  such that  $\phi_x = \phi_y \circ (\kappa_x \hookrightarrow \kappa_y)$ . In particular, if  $\mathbf{p}_{\mathbf{i},y}$  is the image of  $\mathbf{p}_{\mathbf{i},x}$  under  $\kappa_x \hookrightarrow \kappa_y$ , then  $\mathbf{p}_{\mathbf{i}} = \phi_y(\mathbf{p}_{\mathbf{i},y})$ .

Let  $W_{\mathbb{F},j}$  with  $1 \leq j \leq n_y$  be the irreducible components of  $W_{\mathbb{F}}$  that contain  $y$ , and set  $W_y := \bigcap_j W_{\mathbb{F},j}$ . Then by Fact 4.10 mentioned above,  $W_y$  is a smooth  $\mathbb{F}$ -subvariety  $W_J \subset W_{\mathbb{F}}$ , and the following hold (see, e.g., de Jong [10, §2.16 and explanations thereafter]): The local ring  $\mathcal{O}_y$  of  $y \in W$  has a system of regular parameters  $(u_1, \dots, u_{N_y})$  that satisfy

- (i)  $u_j$  defines locally at  $y$  the equation of  $W_{\mathbb{F},j}$  and  $\pi_0 = u_1 \dots u_{n_y}$  with  $\pi_0$  as in Fact 4.10.
- (ii)  $(u_j)_{n_y < j \leq N_y}$  give rise to a regular system of parameters at  $y \in W_y$  in  $\mathcal{O}_y / (u_1, \dots, u_{n_y})$ .

Consider the ideal  $\mathfrak{r} := (\varepsilon_i u_1 - u_i)_{1 < i \leq N_y} \subset \mathcal{O}_y$ , where  $\varepsilon_i \in \mathcal{O}_y^\times$  are arbitrary. Then  $\mathfrak{r}$  is a regular point having  $(\varepsilon_i u_1 - u_i)_{1 < i \leq N_y}$  as a regular system of local parameters, and

$$\mathbf{Z}_y := \mathcal{O}_y / \mathfrak{r}$$

is a discrete valuation ring with uniformizing parameter  $\pi_y := u_1 \pmod{\mathfrak{r}}$ , and  $\varepsilon_y \pi_y^{n_y} = \pi_0$ , where  $\varepsilon_y := \varepsilon_2 \dots \varepsilon_{n_y} \pmod{\mathfrak{r}}$  lies in  $\mathbf{Z}_y^\times$ . Hence if  $\varepsilon_y = 1$ , we get that  $\pi_y^{n_y} = \pi_0$ . Thus choosing  $\varepsilon_2, \dots, \varepsilon_{n_y} = 1$  and  $\pi_0$  to be an algebraic integer,  $\pi_y$  is an algebraic integer as well.

Since  $\mathfrak{r} \in \text{Spec } \mathcal{O}_y$  is a regular point, there exists a valuation  $v_1$  of  $\text{Quot}(\mathcal{O}_y)$  with center  $\mathfrak{r}$  and residue field equal to  $\kappa_{\mathfrak{r}} = \text{Quot}(\mathbf{Z}_y)$ . Further, let  $v_0$  be the canonical valuation of the discrete valuation ring  $\mathbf{Z}_y$ . Then the valuation ring  $\mathcal{O}_v$  of the valuation theoretical composition  $v := v_0 \circ v_1$  dominates  $\mathcal{O}_y$  and has  $\kappa_v = \kappa_y$ . Hence by Lemma 4.12, it follows that  $\mathcal{Y}_{\mathcal{O}_v} \rightarrow \mathbb{P}_{t, \mathcal{O}_v}^1$  is a  $\mathbb{Z}/p^e$ -cover of complete smooth  $\mathcal{O}_v$ -curves. Hence by Lemma 4.3, it follows that  $\mathcal{Y}_{\mathbf{Z}_y} \rightarrow \mathbb{P}_{t, \mathbf{Z}_y}^1$  is a  $\mathbb{Z}/p^e$ -cover of complete smooth  $\mathbf{Z}_y$ -curves.

Now let  $x \in \Sigma_{\mathfrak{z}} \subset \mathbb{A}^{|\mathfrak{z}|}$  vary, and for every such  $x$ , consider some preimage  $y \in W(\mathfrak{q})$  under  $W(\mathfrak{q}) \rightarrow V(\mathfrak{p}) \rightarrow \mathbb{P}^{|\mathfrak{z}|}$  such that  $\kappa_x \hookrightarrow \kappa_y$  is a finite extension. Then performing the above construction, we get the corresponding  $n_y$ . Notice that  $n_y < N_y \leq \dim(W)$ , by the fact that  $N_y = \text{Krull.dim}(\mathcal{O}_y)$ , which is the codimension of  $y$  in  $W$ .

Let  $n_{\mathfrak{z}} = \text{l.c.m.}(n_y)_{y \in W(\mathfrak{q})}$ , and notice that by the discussion above,  $n_{\mathfrak{z}}$  is bounded by  $n!$ , where  $n = \dim(W) - 1$ . Choose fixed algebraic integers  $\pi_0, \pi_{\mathfrak{z}} \in \overline{\mathbb{Z}}$  such that  $\mathbf{Z} = \mathbf{Z}_0[\pi_0]$  and  $\pi_{\mathfrak{z}}^{n_{\mathfrak{z}}} = \pi_0$ . Then there are canonical embeddings  $\mathbf{Z}_y \hookrightarrow W(\overline{\kappa}_y)[\pi_y] \hookrightarrow W(k)[\pi_{\mathfrak{z}}]$ , and the base change of  $\mathcal{Y}_{\mathbf{Z}_y} \rightarrow \mathbb{P}_{t, \mathbf{Z}_y}^1$  under  $\mathbf{Z}_y \hookrightarrow W(k)[\pi_{\mathfrak{z}}]$  is a  $\mathbb{Z}/p^e$ -cover of smooth  $W(k)[\pi_{\mathfrak{z}}]$ -curves

$$\mathcal{Y}_{W(k)[\pi_{\mathfrak{z}}]} \rightarrow \mathbb{P}_{t, W(k)[\pi_{\mathfrak{z}}]}^1$$

with special fiber the  $\mathfrak{z}$ -HKG-cover  $Y_k \rightarrow \mathbb{P}_{t, k}^1$  attached to the given cyclic  $\mathbb{Z}/p^e$ -extension  $k[[t]] \hookrightarrow k[[z]]$ . This concludes the proof of Proposition 4.7.  $\square$

4.C. *The strategy of proof for Theorem 1.1.* We begin by recalling that there are several forms of the Oort Conjecture (OC), which are all equivalent; see, e.g., Bertin–Mézard [1], Garuti [5, §3], Chinburg–Guralnick–Harbater [2], or Saidi [20, §3.1].

Let  $k$  be an algebraically closed field with  $\text{char}(k) = p > 0$ . Let  $W(k)$  be the ring of Witt vectors over  $k$ , and let  $W(k) \hookrightarrow R$  denote a finite extension of discrete valuation rings. We consider the following two situations, which are related to two variants of OC:

- (a)  $Y \rightarrow X$  is a finite (ramified)  $G$ -cover of complete smooth  $k$ -curves such that the inertia groups at all closed points  $y \in Y$  are cyclic.
- (b)  $\mathcal{X}_R$  is a complete smooth  $R$ -curve with special fiber  $X$ , and  $Y \rightarrow X$  is a (ramified)  $G$ -cover of complete smooth curves as in case (a) above.

We say that OC *holds over*  $R$  in case (a) or (b) if there exists a  $G$ -cover of complete smooth  $R$ -curves  $\mathcal{Y}_R \rightarrow \mathcal{X}_R$ , with  $\mathcal{X}_R$  the given one in case (b), having the  $G$ -cover  $Y \rightarrow X$  as special fiber. And given a cyclic extension

$k[[t]] \hookrightarrow k[[u]]$ , we say that the *local OC holds over  $R$*  for  $k[[t]] \hookrightarrow k[[u]]$  if there exists a smooth lifting  $R[[T]] \hookrightarrow R[[U]]$  of  $k[[t]] \hookrightarrow k[[u]]$ .

*Fact 4.13.* Let  $W(k) \hookrightarrow R$  be a finite extension of DVR's. Then the following hold:

- (1) *Local global principle for OC.* Let  $Y \rightarrow X$ ,  $y \mapsto x$ , be a finite  $G$ -cover of projective  $k$ -curves with cyclic inertia groups, and let  $k[[t_x]] \hookrightarrow k[[t_y]]$  be the corresponding extensions of local rings for  $y \mapsto x$ . Let  $\mathcal{X}_R$  be some complete smooth  $R$ -curve with special fiber  $X$ . Then the following are equivalent:
  - (i) There is a  $G$ -cover of  $R$ -curves  $\mathcal{Y}_R \rightarrow \mathcal{X}_R$  as above with special fiber  $Y \rightarrow X$ .
  - (ii) The local extensions  $k[[t_x]] \hookrightarrow k[[t_y]]$  have smooth liftings over  $R$  for all  $y \mapsto x$ .
- (2) *Equivalent forms of OC.* Let  $k[[t]] \hookrightarrow k[[u]]$  be a cyclic extension, and let  $k[[t]] \hookrightarrow k[[z]]$  be its maximal  $p$ -power subextension. The following assertions are equivalent:
  - (i) the local OC holds over  $R$  for  $k[[t]] \hookrightarrow k[[u]]$ ;
  - (ii) the local OC holds over  $R$  for  $k[[t]] \hookrightarrow k[[z]]$ ;
  - (iii) the OC holds over  $R$  for the HKG-cover of  $k[[t]] \hookrightarrow k[[z]]$ .

Thus in order to prove Theorem 1.1 from introduction, we can proceed as follows: Let  $Y \rightarrow X$  be a given  $G$ -cover of projective smooth  $k$ -curves, with branch locus  $\Sigma \subset X$ . Then for a given algebraic integer  $\pi \in \overline{\mathbb{Z}}$  such that  $R := W(k)[\pi]$  is a DVR, and a smooth  $R$ -curve  $\mathcal{X}_R$  with special fiber  $X$ , one has: The OC holds for  $Y \rightarrow X$  over  $R$  if and only if the local OC holds for the local cyclic extension  $k[[t_x]] \hookrightarrow k[[t_y]]$  over  $R$  for all  $x \in X$ ; and the local OC holds for a fixed local cyclic extension  $k[[t_x]] =: k[[t]] \hookrightarrow k[[u]] := k[[t_y]]$  over  $R$  if and only if the local OC holds over  $R$  for the cyclic  $p$ -power subextension  $k[[t]] \hookrightarrow k[[z]]$  of  $k[[t_x]] \hookrightarrow k[[t_y]]$ . Finally, the latter is equivalent to the fact that the HKG-cover  $Y \rightarrow \mathbb{P}_{t,k}^1$  of  $k[[t]] \hookrightarrow k[[z]]$  is smoothly liftable over  $R$ . Thus we conclude that assertions (1) and (2) of Theorem 1.1 are in fact equivalent, and equivalent as well to assertion (1) for  $p$ -power cyclic extensions and to assertion (2) for HKG-covers  $Y \rightarrow \mathbb{P}_{t,k}^1$  of  $p$ -power cyclic extensions.

To tackle the latter two equivalent cases of Theorem 1.1, we first notice that given a positive integer  $\delta$ , one has: Let  $k$  be an algebraically closed field with  $\text{char}(k) = p$ , and let  $\mathbf{v}_p = (\nu_1, \dots, \nu_e)$  be the upper ramification jumps sequence of a  $\mathbb{Z}/p^e$ -extension  $k[[t]] \hookrightarrow k[[z]]$  having  $\text{deg}(\mathfrak{D}) \leq \delta$ . (Notice that for  $p$ -power cyclic extensions  $k[[t]] \hookrightarrow k[[z]]$ , one has that  $\text{deg}_p(\mathfrak{D}) = \text{deg}(\mathfrak{D})$  merely by definition.) By the Hilbert different formula, it follows that  $p$ ,  $e$ , and  $\nu_1, \dots, \nu_e$  are bounded by  $\delta$ . Hence there are only finitely many choices for  $p$  and the upper ramification jumps sequences  $\mathbf{v}_p = (\nu_1, \dots, \nu_e)$  defined by cyclic

$p$ -power extensions  $k[[t]] \hookrightarrow k[[z]]$  having  $\deg(\mathfrak{D}) \leq \delta$ . Suppose the following holds:

(†) For each  $p$  and each upper ramification jumps sequence  $\mathbf{v} = (v_1, \dots, v_e)$  in characteristic  $p$ , there exists an algebraic integer  $\pi_{\mathbf{v}} \in \overline{\mathbb{Z}}$  such that for all algebraically closed fields  $k$  with  $\text{char}(k) = p$ , one has: All  $\mathbf{v}$ -HKG-covers  $Y \rightarrow \mathbb{P}_{t,k}^1$ , or equivalently, all  $\mathbb{Z}/p^e$ -cyclic  $\mathbf{v}$ -extensions  $k[[t]] \hookrightarrow k[[z]]$ , have smooth liftings over  $W(k)[\pi_{\mathbf{v}}]$ .

Then choosing an algebraic integer  $\pi_{\delta} \in \overline{\mathbb{Z}}$  such that  $\mathbb{Z}_p[\pi_{\mathbf{v}_p}] \subset \mathbb{Z}_p[\pi_{\delta}]$  for each  $p$  and  $\mathbf{v}_p$ , it follows that  $W(k)[\pi_{\mathbf{v}_p}] \subseteq W(k)[\pi_{\delta}]$  for all  $p, \mathbf{v}_p$  and  $k$  as above. Thus all the  $\mathbf{v}_p$ -HKG-covers over  $k$ , or equivalently, all the  $\mathbb{Z}/p^e$ -cyclic  $\mathbf{v}_p$ -extensions  $k[[t]] \hookrightarrow k[[z]]$ , have smooth liftings over  $W(k)[\pi_{\delta}]$ . Hence to prove Theorem 1.1, it is sufficient to prove assertion (†) above.

In order to prove assertion (†), let prime number  $p$  and an upper ramification jumps sequence  $\mathbf{v} = (v_1, \dots, v_e)$  in characteristic  $p$  be given. For  $\mathbf{v} = (v_1, \dots, v_e)$ , set  $\delta_0 := \lfloor v_e/(p-1) \rfloor$ , and recall  $N, e_0, (d_i)_{0 \leq i \leq e}$  and the matrix  $(\theta_{\mu\rho})_{\mu,\rho}$ , as defined in the beginning of Section 2.A, and Construction 3.3. Further, for every  $1 \leq \mu \leq N$ , let  $1 \leq i \leq e_0$  be such that  $d_{i-1} < \mu \leq d_i$ , and set  $e_{\mu} := e - r_i + 1$ . Finally, set  $v_{\mu,\alpha} := \theta_{\mu,r_i-1+\alpha}$  for all  $1 \leq \alpha \leq e_{\mu}, 1 \leq \mu \leq N$ , and consider the resulting  $N$  sequences  $\mathbf{v}_{\mu} = (v_{\mu 1}, \dots, v_{\mu e_{\mu}})$ .

Let  $k$  be an algebraically closed field with  $\text{char}(k) = p$ . We view  $\mathfrak{o} := k[[\varpi]]$  as a DVR with valuation ideal  $\mathfrak{m}_{\mathfrak{o}}$ , and we let  $x_1, \dots, x_N \in \mathfrak{m}_{\mathfrak{o}}$  be distinct elements which are  $p^{\delta_0}$ -powers.

- (1) Consider any  $\mathbb{Z}/p^e$ -extension  $k[[t]] \hookrightarrow k[[z]]$  with upper ramification jumps sequence  $\mathbf{v} = (v_1, \dots, v_e)$ , and let  $Y \rightarrow \mathbb{P}_{t,k}^1$  be the corresponding  $\mathbf{v}$ -HKG-cover.
- (2) By Theorem 3.6 combined with Key Lemma 3.2, there exists a  $\mathbb{Z}/p^e$ -cover of complete smooth  $\mathfrak{o}$ -curves  $\mathcal{Y}_{\mathfrak{o}} \rightarrow \mathbb{P}_{t,\mathfrak{o}}^1$  such that the following hold:
  - (a) the special fiber of  $\mathcal{Y}_{\mathfrak{o}} \rightarrow \mathbb{P}_{t,\mathfrak{o}}^1$  is the given  $\mathbf{v}$ -HKG-cover  $Y \rightarrow \mathbb{P}_{t,k}^1$ ;
  - (b) the generic fiber  $\mathcal{Y}_{\hat{k}} \rightarrow \mathbb{P}_{t,\hat{k}}^1$  of  $\mathcal{Y}_{\mathfrak{o}} \rightarrow \mathbb{P}_{t,\mathfrak{o}}^1$  is branched above  $x_1, \dots, x_N$  only;<sup>3</sup>
  - (c) the upper ramification jumps above  $x_{\mu}$  are  $\mathbf{v}_{\mu} := (v_{\mu 1}, \dots, v_{\mu e_{\mu}})$  for every  $\mu = 1, \dots, N$ . Thus  $\mathcal{Y}_{\hat{k}} \rightarrow \mathbb{P}_{t,\hat{k}}^1$  has no essential ramification.

In the above context, let  $\hat{k} \hookrightarrow \mathfrak{l}$  be an algebraic closure of  $\hat{k}$ . For every  $\mu = 1, \dots, N$ , consider the  $\mathbf{v}_{\mu}$ -HKG-cover  $Y_{\mu} \rightarrow \mathbb{P}_{t,\mathfrak{l}}^1$  of the local  $\mathbb{Z}/p^{e_{\mu}}$ -extension  $\mathfrak{l}[[t_{\mu}]] \hookrightarrow \mathfrak{l}[[z_{\mu}]]$ , where  $z_{\mu}$  and  $t_{\mu}$  are local parameters at  $y_{\mu} \mapsto x_{\mu}$ .

**HYPOTHESIS 4.14.** *In Notation 4.6, suppose that the subset  $\Sigma_{\mathbf{v}_{\mu}} \subset \mathbb{A}^{|\mathbf{v}_{\mu}|}$  is Zariski dense for every  $\mathbf{v}_{\mu} = (v_{\mu 1}, \dots, v_{\mu e_{\mu}})$ ,  $\mu = 1, \dots, N$ .*

<sup>3</sup>N.B.,  $\mathcal{Y}_{\hat{k}} \rightarrow \mathbb{P}_{t,\hat{k}}^1$  is in general not an  $\mathbf{v}$ -HKG-cover because it has  $N$  branched points!

Since  $\Sigma_{\mathbf{v}_\mu} \subset \mathbb{A}^{|\mathbf{v}_\mu|}$  is Zariski dense, we can apply Proposition 4.7 for each  $\mathbf{v}_\mu = (v_{\mu 1}, \dots, v_{\mu e_\mu})$  and get: There exists some algebraic integer  $\pi_\mu$  such that the  $\mathbf{v}_\mu$ -HKG-cover  $Y_\mu \rightarrow \mathbb{P}_{t, \mathbf{l}}^1$  has a smooth lifting over  $W(\mathbf{l})[\pi_\mu]$ . Thus by the local-global principle Fact 4.13, it follows that if  $\pi_{\mathbf{v}}$  is an algebraic integer such that  $\pi_\mu \in W(\mathbf{l})[\pi_{\mathbf{v}}]$  for all  $\mu = 1, \dots, N$ , then  $Y_{\mathbf{l}} \rightarrow \mathbb{P}_{t, \mathbf{l}}^1$  has a smooth lifting  $\mathcal{Y}_{\mathcal{O}_1} \rightarrow \mathbb{P}_{t, \mathcal{O}_1}^1$  over  $\mathcal{O}_1 := W(\mathbf{l})[\pi_{\mathbf{v}}]$ .

Let  $v_1$  be the canonical valuation of  $\mathcal{O}_1$ , and let  $v_0$  be the (unique) prolongation of the valuation of  $\mathfrak{o}$  to  $\mathbf{l}$ , say having valuation ring  $\mathcal{O}_0$ . Then the base change  $\mathcal{Y}_{\mathcal{O}_0} \rightarrow \mathbb{P}_{t, \mathcal{O}_0}^1$  of  $\mathcal{Y}_{\mathfrak{o}} \rightarrow \mathbb{P}_{t, \mathfrak{o}}^1$  defined by  $\mathfrak{o} \hookrightarrow \mathcal{O}_0$  is a  $\mathbb{Z}/p^e$ -cover of complete smooth  $\mathcal{O}_0$ -curves with generic fiber  $\mathcal{Y}_{\mathbf{l}} \rightarrow \mathbb{P}_{t, \mathbf{l}}^1$ . And the  $\mathbb{Z}/p^e$ -cover of complete smooth  $\mathbf{l}$ -curves  $\mathcal{Y}_{\mathbf{l}} \rightarrow \mathbb{P}_{t, \mathbf{l}}^1$  is the special fiber of the  $\mathbb{Z}/p^e$ -cover of smooth  $\mathcal{O}_1$ -curves  $\mathcal{Y}_{\mathcal{O}_1} \rightarrow \mathbb{P}_{t, \mathcal{O}_1}^1$ . Then setting  $v := v_0 \circ v_1$  and letting  $\mathcal{O}$  be the valuation ring of  $v$ , it follows by Lemma 4.3 that there exists a smooth lifting of the  $\mathbf{v}$ -HKG-cover  $Y \rightarrow \mathbb{P}_{t, k}^1$  to a  $\mathbb{Z}/p^e$ -cover of smooth  $\mathcal{O}$ -curves  $\mathcal{Y}_{\mathcal{O}} \rightarrow \mathbb{P}_{t, \mathcal{O}}^1$ .

Finally, let  $\mathbf{p}_{\mathbf{v}} = (p_1, \dots, p_e)$  with  $p_\rho(t^{-1}) \in k[t^{-1}]$  be a standard system of polynomials defining the  $\mathbb{Z}/p^e$ -extension  $k[[t]] \hookrightarrow k[[z]]$  we started with at point (3) above. In the context of Notation 4.6, let  $x \in \mathbb{A}^{|\mathbf{v}|}$  be the point defined by the specialization map  $\mathbf{P}_{\mathbf{v}} \mapsto \mathbf{p}_{\mathbf{v}}$ . Since the above (Krull) valuation ring  $\mathcal{O}$  is a mixed characteristic valuation ring with residue field  $k$ , it follows by the definition of  $\Sigma_{\mathbf{v}} \subset \mathbb{A}^{|\mathbf{v}|}$  that  $x \in \Sigma_{\mathbf{v}}$ . On the other hand, the  $\mathbb{Z}/p^e$ -extension  $k[[t]] \hookrightarrow k[[z]]$  we started with at point (3) above was arbitrary. Therefore the set  $\Sigma_{\mathbf{v}} \subset \mathbb{A}^{|\mathbf{v}|}$  is Zariski dense. Hence using Proposition 4.7, we conclude that

*Hypothesis 4.14 implies the existence of an algebraic integer  $\pi_{\mathbf{v}}$  such that every  $\mathbf{v}$ -HKG-cover  $Y \rightarrow \mathbb{P}_{t, k}^1$  has a smooth lifting over  $W(k)[\pi_{\mathbf{v}}]$ .*

4.D. *Concluding the proof of the Oort Conjecture.* By the observation above, the proof of the Oort Conjecture is reduced to showing that Hypothesis 4.14 holds for every system of upper ramification indices  $\mathbf{v} = (v_1, \dots, v_e)$  that has no essential jump indices; i.e.,  $p v_{\rho-1} \leq v_\rho < p v_{\rho-1} + p$  for  $\rho = 1, \dots, e$ . Via the local-global principle Fact 4.13, this fact is equivalent to a (very) special case of the local Oort Conjecture, which follows from a result recently announced by Obus–Wewers [15, Th. 1.4]. The special case we need is

**KEY LEMMA 4.15** (Special case of Obus–Wewers). *In the notation and context as above, let  $k[[t]] \hookrightarrow k[[z]]$  be cyclic extension of degree  $p^e$  that has no essential ramification. Then the local Oort Conjecture holds for  $k[[t]] \hookrightarrow k[[z]]$ ; i.e.,  $k[[t]] \hookrightarrow k[[z]]$  has a smooth lifting over some finite extension  $R$  of  $W(k)$  to a smooth cyclic cover  $R[[T]] \hookrightarrow R[[Z]]$ .*

*Proof.* Recall that Theorem 1.4 from Obus–Wewers [15] asserts that the local Oort conjecture holds for cyclic extensions  $k[[t]] \hookrightarrow k[[z]]$  of degree  $p^e$ , provided the upper ramification jumps  $v_1 \leq \dots \leq v_e$  satisfy the following:

HYPOTHESIS 4.16. *For every  $3 \leq \nu < e$ , there is no  $a \in \mathbb{Z}$  such that*

$$\iota_{\nu+1} - p\iota_\nu < pa \leq (\iota_{\nu+1} - p\iota_\nu) \frac{\iota_{\nu+1}}{\iota_{\nu+1} - \iota_\nu}.$$

Notice that if  $k[[t]] \hookrightarrow k[[z]]$  has no proper essential ramification jumps, thus by definition  $\iota_{\nu+1} \leq p\iota_\nu + p - 1$  for all  $1 \leq \nu < e$ , then hypothesis (\*) above is satisfied. Indeed, if  $a$  satisfies both inequalities above, then  $p\iota_\nu \leq \iota_{\nu+1}$  implies that  $a$  must be positive. Hence setting  $\varepsilon := \iota_{\nu+1} - p\iota_\nu$  and  $u := \iota_\nu$ , the second inequality becomes  $pa(\varepsilon + (p-1)u) \leq \varepsilon(\varepsilon + pu)$ , which is equivalent to  $p(p-1)au \leq \varepsilon(\varepsilon + pu - pa)$ . Hence taking into account that  $\varepsilon \leq p-1$ , we get  $p(p-1)au \leq (p-1)(p-1 + pu - ap)$ ; thus dividing by  $(p-1)$ , we get  $pa u \leq p-1 + pu - ap$ , or equivalently,  $p(a-1)(u+1) + 1 \leq 0$ , which does not hold for any positive integer  $a$ .  $\square$

This concludes the proof of Theorem 1.1.

*Remark 4.17.* To see how restrictive Hypothesis 4.16 is, proceed as follows: Setting  $\varepsilon := \iota_{\nu+1} - p\iota_\nu$  and  $u := \iota_\nu$ , we have that  $\iota_{\nu+1} = \varepsilon + pu$  for some  $0 \leq \varepsilon$ , and if  $0 < \varepsilon$ , then  $p \nmid \varepsilon$ . Hence writing  $\varepsilon = rp - \eta$  with  $1 \leq \eta \leq p-1$ , Hypothesis 4.16 for  $a := r$  implies that

*At least one of the inequalities  $\varepsilon < rp \leq \varepsilon(\varepsilon + pu)/(\varepsilon + (p-1)u)$   
does not hold.*

Since the first inequality holds, the second inequality must not hold. Therefore we must have  $rp > \varepsilon(\varepsilon + pu)/(\varepsilon + (p-1)u)$ . Since the right-hand side equals  $\varepsilon + \varepsilon u/(\varepsilon + (p-1)u)$ , and  $rp - \varepsilon = \eta$ , the above inequality is equivalent to  $\eta > \varepsilon u/(\varepsilon + (p-1)u)$ , hence to  $\eta\varepsilon + \eta(p-1)u > \varepsilon u$ , and finally equivalent to  $\eta^2(p-1) > (\varepsilon - \eta(p-1))(u - \eta)$ . Thus since  $\varepsilon - \eta(p-1) = (r - \eta)p$ , the last inequality becomes  $\eta^2(p-1) > p(r - \eta)(u - \eta)$ . Since  $p > p-1$ , the last inequality implies  $\eta^2 > (r - \eta)(u - \eta)$ . On the other hand, recalling that  $u = \iota_\nu$  and  $\nu \geq 3$ , it follows that  $u \geq \iota_3 \geq p\iota_2 \geq p^2\iota_1 \geq p^2$ . On the other hand, since  $1 \leq \eta < p$ , one has  $u - \eta \geq p^2 - (p-1) > (p-1)^2 \geq \eta^2$ . Therefore, in order to satisfy the inequality  $\eta^2 > (r - \eta)(u - \eta)$ , one must have  $r - \eta \leq 0$ , or equivalently,  $r \leq \eta$ . One concludes that the Hypothesis 4.16 is equivalent to

HYPOTHESIS 4.16'. *Setting  $\iota_{\nu+1} - p\iota_\nu = pr_\nu - \eta_\nu$  with  $0 \leq \eta_\nu < p$ , one has  $0 \leq r_\nu \leq \eta_\nu$ .*

It seems to me that this is a better/easier formulation than the original form of Hypothesis 4.16. Using this reformulation, we also can see how restrictive the Hypothesis 4.16 is. Namely, if  $\iota_{\nu+1} \equiv -1 \pmod{p}$ , then one must have  $\iota_{\nu+1} = p\iota_\nu + p - 1$ . In general, one has  $\iota_{\nu+1} \leq p\iota_\nu + (p-1)^2$  and this upper bound can be reached only if  $\iota_{\nu+1} \equiv 1 \pmod{p}$ . This shows that for  $3 < e$ , the Hypothesis 4.16 becomes quite restrictive indeed.

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