

Cyclic extensions and the local lifting problem

By ANDREW OBUS and STEFAN WEWERS

Abstract

The local Oort conjecture states that, if Γ is cyclic and k is an algebraically closed field of characteristic p , then all Γ -extensions of $k[[t]]$ should lift to characteristic zero. We prove a critical case of this conjecture. In particular, we show that the conjecture is always true when $v_p(|\Gamma|) \leq 3$ and is true for arbitrarily highly p -divisible cyclic groups Γ when a certain condition on the higher ramification filtration is satisfied.

1. Introduction

1.1. *The local lifting problem.* Let \bar{Y} be a smooth, projective, connected curve over an algebraically closed field k of characteristic $p > 0$. Results in deformation theory going back to Grothendieck show that \bar{Y} can always be lifted to characteristic zero. Specifically, one can always find a discrete valuation ring (DVR) R in characteristic zero, with residue field k , such that there exists a smooth relative R -curve Y with $Y \times_R k \cong \bar{Y}$.

In [24], Oort asked the natural question: can one lift a *Galois cover* of curves to characteristic zero? That is, if Γ is a finite group and $\bar{f} : \bar{Y} \rightarrow \bar{X}$ is a Γ -Galois cover of smooth, projective, connected curves, is there a Γ -Galois cover $f : Y \rightarrow X$ of smooth relative curves over a DVR R in characteristic zero whose special fiber is $\bar{f} : \bar{Y} \rightarrow \bar{X}$? Clearly, the answer is not always “yes.” For instance, the group $\Gamma = \mathbb{Z}/p \times \mathbb{Z}/p$ acts faithfully on $\bar{Y} = \mathbb{P}_k^1$ via an embedding into $\mathbb{G}_a(k)$. If $\bar{X} = \bar{Y}/\Gamma$, then the generic fiber of any lift $f : Y \rightarrow X$ of $\bar{f} : \bar{Y} \rightarrow \bar{X}$ must be a Γ -Galois cover $\mathbb{P}_K^1 \rightarrow \mathbb{P}_K^1$, where $K = \text{Frac}(R)$. But if $p \geq 3$, then Γ cannot act faithfully on \mathbb{P}^1 in characteristic zero, so such a lift does not exist. However, Oort conjectured ([25]) that Γ -covers should always lift when Γ is cyclic. Our main result (Theorem 1.4) proves the Oort conjecture whenever $v_p(|\Gamma|) \leq 3$, and in many cases for arbitrarily large cyclic groups Γ . Specific statements are in Section 1.3. The case we prove is critical, as Pop ([26]) is able to reduce the conjecture to the case we have proven, thus proving the entire conjecture. See Remark 1.5.

Although this was not fully known at the time of [24], it turns out that the nature of lifting Galois covers of curves is completely local. Let B be the branch locus of the Γ -Galois cover $\bar{f} : \bar{Y} \rightarrow \bar{X}$. If, for each $x \in B$, one can lift the cover \bar{f} when restricted to a formal neighborhood \hat{x} of x , then one can lift f as well. This is known as the *local-global principle*. If Γ is abelian, then techniques already used in [27] should give a proof. In any case, proofs for arbitrary Γ have been given by Bertin and Mézard ([3]), Green and Matignon ([15]), and Garuti ([12]).

The restriction $\bar{f}|_{\hat{x}}$ is a disjoint union of covers of the form $\text{Spec } k[[z]] \rightarrow \text{Spec } k[[t]]$. Thus, the study of lifting Galois covers can be reduced to the following *local lifting problem*:

Problem 1.1 (The local lifting problem). Let k be an algebraically closed field of characteristic p and Γ a finite group. Let $k[[z]]/k[[t]]$ be a Γ -Galois extension (i.e., an integral extension of integrally closed domains that is Γ -Galois on the level of fraction fields). Does there exist a DVR R of characteristic zero with residue field k and a Γ -Galois extension $R[[Z]]/R[[T]]$ that reduces to $k[[z]]/k[[t]]$? That is, does the Γ -action on $R[[Z]]$ reduce to that on $k[[z]]$ if we assume that Z (resp. T) reduces to z (resp. t)?

Remark 1.2. If $L/k[[t]]$ is any Γ -Galois extension, then the structure theorem for complete DVR's shows that L is always abstractly isomorphic to $k[[z]]$. So in the local lifting problem, we may as well talk about general Γ -Galois extensions of $k[[t]]$.

In light of the local-global principle, one translates the conjecture of Oort above into the local context.

CONJECTURE 1.3 (Local Oort conjecture). *The local lifting problem can always be solved when Γ is cyclic.*

Note that a consequence of the local Oort conjecture and the local-global principle is that any Galois cover of k -curves with cyclic *inertia* groups (not just cyclic Galois group) lifts to characteristic zero.

The first author's paper [23] is a detailed exposition of many aspects of the local lifting problem.

1.2. *Previously known results.* It is easy to prove that if Conjecture 1.3 is true for a given \mathbb{Z}/p^n -extension $L_n/k[[t]]$, then it is also true for any \mathbb{Z}/rp^n -extension ($p \nmid r$) whose unique \mathbb{Z}/p^n -subextension is $L_n/k[[t]]$. (See, e.g., the proof of [23, Prop. 6.3].) Thus, we may reduce to the case where $\Gamma \cong \mathbb{Z}/p^n$.

If $\Gamma \cong \mathbb{Z}/p$ or \mathbb{Z}/p^2 , then Conjecture 1.3 has already been shown to be true. The original proof of the case $\Gamma \cong \mathbb{Z}/p$ was given by Sekiguchi, Oort, and Suwa in [27]. Their proof was given in the global context for any \mathbb{Z}/p -cover

of smooth, projective curves (although local-global techniques were implicit in the proof).

Later, Green and Matignon reproved Conjecture 1.3 for $\Gamma = \mathbb{Z}/p$, and they also proved it for $\Gamma = \mathbb{Z}/p^2$ [15, II, Ths. 4.1 and 5.5]. Their lifts were given by explicit Kummer extensions. The form of these extensions was inspired by *Sekiguchi-Suwa Theory* (or *Kummer-Artin-Schreier-Witt Theory*), although Green and Matignon developed their proofs independently. This theory gives (in principle) explicit equations for group schemes classifying unramified \mathbb{Z}/p^n -extensions of flat local R -algebras, where R is a complete discrete valuation ring in mixed characteristic $(0, p)$. When $n \leq 2$, it is manageable to write down these equations explicitly, and Green and Matignon were able to exploit this to write down their lifts. See [28] for an overview of the general theory, [29] for an expanded version with proofs included, [30] for a detailed account of the case $n \leq 2$, or [32] for a briefer overview of this case.

Unfortunately, when $n \geq 3$, the equations involved in Sekiguchi-Suwa theory become extremely complicated and extraordinarily difficult to work with. No one has been able to use the method of [15] to prove Conjecture 1.3 for *any* \mathbb{Z}/p^n -extension with $n \geq 3$. Indeed, prior to this paper, Conjecture 1.3 for such extensions was only known to be true for sporadic examples arising from Lubin-Tate formal group laws ([16], [14]).

However, there has long been evidence for the truth of Conjecture 1.3, in the sense that all of the main known obstructions to lifting (such as the Bertin/Katz-Gabber-Bertin obstructions of [2], [10] and the Hurwitz tree obstruction of [9], [8]) vanish for cyclic extensions.

1.3. *Main result.* To state our main result, Theorem 1.4, we recall that a \mathbb{Z}/p^n -extension $L_n/k[[t]]$ gives rise to a higher ramification filtration $\Gamma_{s \geq 0}^s$ on the group $\Gamma = \mathbb{Z}/p^n$ for the upper numbering ([31, IV]). The breaks in this filtration (i.e., the values i for which $\Gamma^i \not\supseteq \Gamma^j$ for all $j > i$) will be denoted by (m_1, m_2, \dots, m_n) . One knows that $m_i \in \mathbb{N}$ and

$$(1) \quad m_i \geq p m_{i-1}$$

for $i = 2, \dots, n$ (see, e.g., [13]).

THEOREM 1.4. *Let $L_n/k[[t]]$ be a \mathbb{Z}/p^n -extension with upper ramification breaks (m_1, m_2, \dots, m_n) . Suppose, for $3 \leq i \leq n - 1$, that there is no integer a_i such that*

$$(2) \quad \frac{m_i}{p} - m_{i-1} < a_i \leq \left(\frac{m_i}{m_i - m_{i-1}} \right) \left(\frac{m_i}{p} - m_{i-1} \right).$$

Then Conjecture 1.3 holds for $L_n/k[[t]]$.

Remark 1.5.

- (i) It is not hard to see that the condition in Theorem 1.4 is satisfied whenever $L_n/k[[t]]$ has no “essential ramification,” i.e., that $m_i < p m_{i-1} + p$

for $2 \leq i \leq n$. Pop's proof of the Oort conjecture in [26] reduces the (local) Oort conjecture to the case where there is no essential ramification. Assuming no essential ramification does not seem to simplify the proof of Theorem 1.4.

- (ii) The condition in Theorem 1.4 is vacuous for $n = 3$, so Conjecture 1.3 for $\Gamma \cong \mathbb{Z}/p^3$ is an immediate consequence.
- (iii) The condition (2) for a given i is equivalent to saying that if $m_i = pm_{i-1} + pr_i - \eta_i$, with r_i and η_i integers such that $0 \leq \eta_i < p$, then $0 \leq r_i \leq \eta_i$. This equivalence is proved in [26, Rem. 4.12]. While the phrasing in Theorem 1.4 is arguably more complicated, it is a natural byproduct of the proof.
- (iv) One can conjecture further that if $\Gamma \cong \mathbb{Z}/p^n$, one should be able to take $R = W(k)[\zeta_{p^n}]$ in Conjecture 1.3, where ζ_{p^n} is a p^n -th root of unity. This is known when $n \leq 2$. Unfortunately, our proof gives no effective bounds on R . The proof of [26] gives, in theory, some effective bounds on R , but they are much weaker than what is expected.

Remark 1.6. It would be interesting to investigate the local lifting problem when $\Gamma \cong \mathbb{Z}/p^n \rtimes \mathbb{Z}/m$, with $p \nmid m$. One can show that a necessary condition to lift a Γ -extension $L/k[[t]]$ to characteristic zero is that the action of \mathbb{Z}/m on \mathbb{Z}/p^n is either faithful or trivial and that if it is faithful, then the upper ramification breaks (m_1, \dots, m_n) of the \mathbb{Z}/p^n -subextension are all congruent to $-1 \pmod{m}$. In light of [26], one can ask if this is the only restriction. For instance, should all D_{p^n} -extensions lift for odd p ?

1.4. *Outline of the paper.* We start with a short section (Section 2) over-viewing the basics of the Artin-Schreier-Witt theory, giving an explicit characterization of the \mathbb{Z}/p^n -extensions of $k[[t]]$. In Section 3, we set up our induction on n and show how it proves Theorem 1.4. We prove the base cases $n = 1$ and $n = 2$ in Section 4. The paper begins in earnest with Section 5. In Sections 5.1 and 5.2 we introduce the language of *characters*, which will often be more convenient than the language of extensions for expressing our results. In Section 5.3, we introduce Kato's *Swan conductor* in the situation relevant to us. The Swan conductor serves several purposes in this paper, most notably giving us a way to measure how bad the reduction of a cover is. In Sections 5.4 and 5.5, we examine the particular case of \mathbb{Z}/p -extensions in great detail. This is important, as \mathbb{Z}/p -extensions are the building blocks of our inductive process.

In Section 6, we give the main proofs. Unlike in [15], we do not try to write down a lift of a given \mathbb{Z}/p^n -extension explicitly. In particular, we do not use the Sekiguchi-Suwa theory at all, except in the relatively trivial case of \mathbb{Z}/p -extensions (i.e., Kummer-Artin-Schreier theory). Instead, we write down what the form of the equations should be in order that we might lift *some* \mathbb{Z}/p^n -extension. Then, we show that if equations in this form do not reduce

to a Galois extension, then they can be deformed to yield something that comes closer to reducing to a Galois extension, and this deformation process eventually terminates. This proves Conjecture 1.3 for some \mathbb{Z}/p^n -extension. We then show that, given a solution for some particular extension, we can find solutions to many more. A more detailed outline of Section 6 is given in Section 6.1.

The proofs of several key technical results are postponed to Section 7. This is partially because proving these results would disrupt the continuity of Section 6 and partially because the proofs share much notation, and thus are more easily read together.

1.5. Conventions. The letter K will always be a field of characteristic zero that is complete with respect to a discrete valuation $v : K^\times \rightarrow \mathbb{Q}$. We assume that the residue field k of K is algebraically closed of characteristic p . We also assume that the valuation v is normalized such that $v(p) = 1$. The ring of integers of K will be denoted R . We fix an algebraic closure \overline{K} of K , and whenever necessary, we will replace K by a suitable finite extension within \overline{K} , without changing the above notation. The maximal ideal of R will be denoted \mathfrak{m} . Furthermore, for each $r \in \mathbb{N}$, we fix once and for all a compatible system of r -th roots $p^{1/r}$ in \overline{K} such that if $ab = r$, then $(p^{1/r})^a = p^{1/b}$. Thus $p^q \in \overline{K}$ is well defined for any $q \in \mathbb{Q}$. The greatest integer function of x is written $[x]$.

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2. Artin-Schreier-Witt theory

If $L_n/k[[t]]$ is a \mathbb{Z}/p^n -extension, then so is the extension $M_n/k((t))$, where $M_n = \text{Frac}(L_n)$. The classical Artin-Schreier-Witt theory states that $M_n/k((t))$ is given by an Artin-Schreier-Witt equation

$$\wp(x_1, \dots, x_n) = (f_1, \dots, f_n),$$

where (f_1, \dots, f_n) lies in the ring $W_n(k((t)))$ of truncated Witt vectors, F is the Frobenius morphism on $W_n(k((t)))$, and $\wp(x) := F(x) - x$ is the Artin-Schreier-Witt isogeny. Adding a truncated Witt vector of the form $\wp(y)$ to (f_1, \dots, f_n) does not change the extension, and we obtain a group isomorphism $H^1(k((t)), \mathbb{Z}/p^n) \cong W_n(k((t)))/\wp(W_n(k((t))))$. Since we can add $\wp(y)$ to a Witt vector without changing the extension, we may assume that the f_i are polynomials in t^{-1} , all of whose terms have prime-to- p degree. In this case, if

$$(3) \quad m_i := \max\{p^{i-j} \deg_{t^{-1}}(f_j) \mid j = 1, \dots, i\},$$

then the m_i are exactly the breaks in the higher ramification filtration of $M_n/k((t))$ ([13, Th. 1.1]). From this, one easily sees that $p \nmid m_1$, that $m_i \geq pm_{i-1}$ for $2 \leq i \leq n$, and that if $p \mid m_i$, then $m_i = pm_{i-1}$.

For more details, see [35] or the exercises on page 330 of [20].

3. The induction process

Let $L_n/k[[t]]$ be a \mathbb{Z}/p^n -extension. A theorem of Harbater-Katz-Gabber ([17], [19]) shows that (after possibly changing the uniformizer t of $k[[t]]$) there exists a unique cover $\bar{Y}_n \rightarrow \bar{X} := \mathbb{P}_k^1$ that is étale outside $t = 0$, totally ramified above $t = 0$, and such that the formal completion of $\bar{Y}_n \rightarrow \bar{X}$ at $t = 0$ yields the extension $L_n/k[[t]]$. The local-global principle thus shows that solvability of the local lifting problem from $L_n/k[[t]]$ is equivalent to the following claim, which will be more convenient to work with.

Claim 3.1. Given a Γ -Galois extension $L_n/k[[t]]$, with $\Gamma \cong \mathbb{Z}/p^n$, then after possibly changing the uniformizer t of $k[[t]]$, there exists a Γ -Galois cover $Y_n \rightarrow X := \mathbb{P}_K^1$ (where K is the fraction field of some DVR R as above) with the following properties:

- (i) The cover $Y_n \rightarrow X$ has good reduction with respect to the standard model \mathbb{P}_R^1 of X and reduces to a Γ -Galois cover $\bar{Y}_n \rightarrow \bar{X} = \mathbb{P}_k^1$ that is totally ramified above the point $t = 0$ and étale everywhere else.
- (ii) The completion of $\bar{Y}_n \rightarrow \bar{X}$ at $t = 0$ yields $L_n/k[[t]]$.

Remark 3.2. Let T be a coordinate of \mathbb{P}_R^1 reducing to t . Then conditions (i) and (ii) in Claim 3.1 can be reformulated as follows:

- (i) The cover $Y_n \rightarrow X$ is étale outside the open disk

$$D := \{T \mid |T| < 1\}.$$

- (ii) The inverse image of D in Y_n is an open disk.
- (iii) If $A = R[[T]]\{T^{-1}\}$ is the ring

$$\left\{ \sum_{j \in \mathbb{Z}} a_j T^j \mid a_j \in R, a_j \rightarrow 0 \text{ as } j \rightarrow -\infty \right\},$$

the cover $Y_n \rightarrow X$ is unramified when base changed to $\text{Spec } A$ (which corresponds to the “boundary” of the disk D). The extension of residue fields is isomorphic to the extension of fraction fields coming from $L_n/k[[t_0]]$.

If R is a characteristic zero DVR with residue field k and fraction field K , set $D(r) = \{T \in \overline{K} \mid |T| < |p|^r\}$, using the non-archimedean absolute value on K induced from the valuation.

We prove Theorem 1.4 (in the context of Claim 3.1) by induction using the following base case (Lemma 3.3) and induction step (Theorem 3.4).

LEMMA 3.3. *If $n = 1$ (resp. $n = 2$), let $L_n/k[[t]]$ be a \mathbb{Z}/p^n -extension with upper ramification break m_1 (resp. breaks (m_1, m_2)). Then there exists a \mathbb{Z}/p^n -cover $Y_n \rightarrow X$ satisfying Claim 3.1 for $L_n/k[[t]]$, which is étale outside the open disk $D(r_n)$, where $r_n = 1/m_n(p - 1)$.*

THEOREM 3.4.

- (i) *Suppose $n > 1$, and let $L_n/k[[t]]$ be a \mathbb{Z}/p^n -extension with upper ramification breaks (m_1, \dots, m_n) and \mathbb{Z}/p^{n-1} -subextension $L_{n-1}/k[[t]]$. Suppose there exists a \mathbb{Z}/p^{n-1} -cover $Y_{n-1} \rightarrow X$ satisfying Claim 3.1 for $L_{n-1}/k[[t]]$, which is furthermore étale outside the open disk $D(r_{n-1})$, where $r_{n-1} = 1/m_{n-1}(p - 1)$. Then there is a \mathbb{Z}/p^n -cover $Y_n \rightarrow X$ satisfying Claim 3.1 for $L_n/k[[t]]$.*
- (ii) *If $L_n/k[[t]]$ is as in part (i) and there is no integer a satisfying*

$$\frac{m_n}{p} - m_{n-1} < a \leq \left(\frac{m_n}{m_n - m_{n-1}} \right) \left(\frac{m_n}{p} - m_{n-1} \right),$$

then the \mathbb{Z}/p^n -cover $Y_n \rightarrow X$ in part (i) can be chosen to be étale outside $D(r_n)$, where $r_n = 1/m_n(p - 1)$.

Theorem 1.4 now follows easily.

Proof of Theorem 1.4. Let $L_n/k[[t]]$ be in the form of Theorem 1.4. Note that, for any $n' \leq n$, the unique $\mathbb{Z}/p^{n'}$ -subextension $L_{n'}/k[[t]]$ of $L_n/k[[t]]$ has upper ramification breaks $(m_1, \dots, m_{n'})$ ([31, IV, Prop. 14]). Using Lemma 3.3, Theorem 3.4, and induction, it follows that there is a \mathbb{Z}/p^{n-1} -cover $Y_{n-1} \rightarrow X$ satisfying Claim 3.1 for $L_{n-1}/k[[t]]$, which is étale outside $D(r_{n-1})$. Then Theorem 3.4(ii) shows that Claim 3.1, thus Conjecture 1.3, holds for $L_n/k[[t]]$. □

4. The base case

The proof of Lemma 3.3 is straightforward, using the explicit equations given in [15].

Proof of Lemma 3.3 for \mathbb{Z}/p -extensions. By Artin-Schreier theory, any \mathbb{Z}/p -extension of $k((t))$ is given by an equation $y^p - y = f_1$, where $f_1 \in k((t))$

is unique up to adding elements of the form $a^p - a$, with $a \in k((t))$. Thus, we may assume that $f_1 = \sum_{i=1}^{m_1} a_i t^{-i} \in k[t^{-1}]$ is a polynomial in t^{-1} such that $a_i = 0$ for $i \equiv 0 \pmod{p}$. Then the break in the ramification filtration is m_1 , which is prime to p . Since $\sqrt[m_1]{1/f_1}$ is a uniformizer of $k((t))$, we may make a change of variables and assume $f_1 = t^{-m_1}$. So we assume our equation is given by $y^p - y = t^{-m_1}$. In [15, II, Th. 4.1], a \mathbb{Z}/p -cover $Y \rightarrow \mathbb{P}^1$ satisfying Claim 3.1 is given by the Kummer extension

$$(4) \quad Z^p = G_1(T) := 1 + \lambda^p T^{-m_1},$$

where T reduces to t and $\lambda = \zeta_p - 1$ for ζ_p a primitive p -th root of unity. (This equation was already used in [27].) In particular, one makes the substitution $Z = 1 + \lambda Y$, which leads to the equation

$$Y^p - Y + o(1) = T^{-m_1},$$

giving the correct reduction. (Here $o(1)$ refers to terms with coefficients of positive valuation.) The zeroes of G_1 all have valuation

$$\frac{p}{m_1(p-1)} > \frac{1}{m_1(p-1)} =: r_1,$$

and the unique pole is at $T = 0$. Since the branch points all have valuation greater than r_1 , the lemma is proved. \square

Proof of Lemma 3.3 for \mathbb{Z}/p^2 -extensions. Let $L_2/k((t))$ be a \mathbb{Z}/p^2 -extension with upper ramification breaks (m_1, m_2) , and let $r_i = 1/m_i(p-1)$ for $i \in \{1, 2\}$. After a possible change of variables, [15, II, Th. 5.5] gives a \mathbb{Z}/p^2 -cover $Y \rightarrow \mathbb{P}^1$ satisfying Claim 3.1, in the form of a Kummer extension

$$Z^{p^2} = G_1(T)G_2(T)^p.$$

Here G_1 is as in (4) and G_2 is a polynomial in T^{-1} , which is called

$$(5) \quad G(T^{-1}) + p\mu^p \sum_{s=1}^{m_1(p-1)} A_s T^{-s}$$

in [15, II, Th. 5.5]. Also, $v(\mu) = 1/p(p-1)$ and $v(A_s) \geq 0$ for all s . It is clear from the expression for $G(T^{-1})$ given in loc. cit. that the coefficient of each nonconstant term of $G(T^{-1})$ has valuation at least $1/(p-1)$. The same holds for the $p\mu^p A_s$ by inspection. Furthermore, the proof of [15, II, Th. 5.5] shows that each term in (5) has degree less than m_2 in T^{-1} . (m_2 is called d in loc. cit.)

Now, we have already seen that the zeroes and poles of G_1 have valuation greater than $r_1 > r_2$. The only pole of G_2 is at $T = 0$, so it suffices to show that the zeroes of G_2 have valuation greater than r_2 , or equivalently (by the theory of Newton polygons), that the coefficient of $T^{-\ell}$ in $G_2 - 1$ has valuation greater than $\ell r_2 = \ell/m_2(p-1)$. But this is true because $\ell < m_2$ and the valuation is at least $1/(p-1)$. \square

5. Characters and Swan conductors

In this section we introduce the general geometric setup (characters) and our most important tools (Swan conductors). As laid out in the introduction, our goal is to construct p^n -cyclic covers $Y \rightarrow X = \mathbb{P}_K^1$ of the projective line reducing to a given cover $\bar{Y} \rightarrow \bar{X} = \mathbb{P}_k^1$, which is étale outside the origin. For technical reasons it is more convenient to work with the corresponding character of the Galois group of the function field of X .

5.1. *Geometric setup.* Let X be a smooth, projective, and absolutely irreducible curve over K . We write \mathbb{K} for the function field of X . We assume that X has good reduction, and we fix a smooth R -model X_R . We let $\bar{X} := X_R \otimes_R k$ denote the special fiber of X_R . We also fix a K -rational point x_0 on X and write $\bar{x}_0 \in \bar{X}$ for the specialization of x_0 with respect to the model X_R . In our main example, we have $X = \mathbb{P}_K^1$, with $X_R = \mathbb{P}_R^1$ and $x_0 = 0$, but we will not assume this in Section 5.

We let X^{an} denote the rigid analytic space associated to X . The *residue class* of x_0 with respect to the model X_R ,

$$D :=]\bar{x}_0[_{X_R} \subset X^{\text{an}},$$

is the set of points of X^{an} specializing to $\bar{x}_0 \in \bar{X}$ ([5]). It is an open subspace of X^{an} , isomorphic to the open unit disk. To make this isomorphism explicit we choose an element $T \in \mathcal{O}_{X_R, \bar{x}_0}$ with $T(x_0) = 0$ and whose restriction to the special fiber generates the maximal ideal of $\mathcal{O}_{\bar{X}, \bar{x}_0}$. (This is possible because $X_R \rightarrow \text{Spec } R$ is smooth.) Then $\hat{\mathcal{O}}_{X_R, \bar{x}_0} = R[[T]]$, and T induces an isomorphism of rigid analytic spaces

$$D \cong \{x \in (\mathbb{A}_K^1)^{\text{an}} \mid v(x) > 0\}$$

that sends the point x_0 to the origin. We call T a *parameter* for the open disk D with *center* x_0 . The choice of T having been made, we identify D with the above subspace of $(\mathbb{A}_K^1)^{\text{an}}$.

For $r \in \mathbb{Q}_{\geq 0}$, we define

$$D[r] := \{x \in D \mid v(x) \geq r\}.$$

We have $D[0] = D$. For $r > 0$, the subset $D[r] \subset D$ is an affinoid subdomain. Let $v_r : \mathbb{K}^\times \rightarrow \mathbb{Q}$ denote the “Gauss valuation” with respect to $D[r]$. This is a discrete valuation on \mathbb{K} that extends the valuation v on K and has the property $v_r(T) = r$. It corresponds to the supremum norm on the open subset $D[r] \subset X^{\text{an}}$.

Let κ_r denote the residue field of \mathbb{K} with respect to the valuation v_r . For $r = 0$, we have that κ_0 is the function field of \bar{X} . We let $\text{ord}_{\bar{x}_0} : \kappa_0^\times \rightarrow \mathbb{Z}$ denote the normalized valuation corresponding to the point $\bar{x}_0 \in \bar{X}$.

Now suppose that $r > 0$. Then after replacing K by a finite extension (which depends on r) we may assume that $p^r \in K$. Then $D[r]$ is isomorphic to a closed unit disk over K with parameter $T_r := p^{-r}T$. Moreover, the residue field κ_r is the function field of the canonical reduction $\overline{D}[r]$ of the affinoid $D[r]$. In fact, $\overline{D}[r]$ is isomorphic to the affine line over k with function field $\kappa_r = k(t)$, where t is the image of T_r in κ_r . For a closed point $\bar{x} \in \overline{D}[r]$, we let $\text{ord}_{\bar{x}} : \kappa_r^\times \rightarrow \mathbb{Z}$ denote the normalized discrete valuation corresponding to the specialization of \bar{x} on $\overline{D}[r]$. We let ord_∞ denote the unique normalized discrete valuation on κ_r corresponding to the “point at infinity.”

Notation 5.1. For $F \in \mathbb{K}^\times$ and $r \in \mathbb{Q}_{\geq 0}$, we let $[F]_r$ denote the image of $p^{-v_r(F)}F$ in the residue field κ_r .

5.2. *Characters.* We fix $n \geq 1$ and assume that K contains a primitive p^n -th root of unity ζ_{p^n} . (This is true after a finite extension of K .) For an arbitrary field L , we set

$$H_{p^n}^1(L) := H^1(L, \mathbb{Z}/p^n\mathbb{Z}).$$

In the case of \mathbb{K} , we have

$$H_{p^n}^1(\mathbb{K}) := H^1(\mathbb{K}, \mathbb{Z}/p^n\mathbb{Z}) \cong \mathbb{K}^\times / (\mathbb{K}^\times)^{p^n}.$$

(The latter isomorphism depends on the choice of ζ_{p^n} .) Elements of $H_{p^n}^1(\mathbb{K})$ are called *characters* on X . Given an element $F \in \mathbb{K}^\times$, we let $\mathfrak{K}_n(F) \in H_{p^n}^1(\mathbb{K})$ denote the character corresponding to the class of F in $\mathbb{K}^\times / (\mathbb{K}^\times)^{p^n}$.

For $i = 1, \dots, n$, the homomorphism

$$\mathbb{Z}/p^i\mathbb{Z} \rightarrow \mathbb{Z}/p^n\mathbb{Z}, \quad a \mapsto p^{n-i}a,$$

induces an injective homomorphism $H_{p^i}^1(\mathbb{K}) \hookrightarrow H_{p^n}^1(\mathbb{K})$. Its image consists of all characters killed by p^i . We consider $H_{p^i}^1(\mathbb{K})$ as a subgroup of $H_{p^n}^1(\mathbb{K})$ via this embedding.

A character $\chi \in H_{p^n}^1(\mathbb{K})$ gives rise to a (possibly branched) Galois cover $Y \rightarrow X$. If $\chi = \mathfrak{K}_n(F)$ for some $F \in \mathbb{K}^\times$, then Y is a connected component of the smooth projective curve given generically by the Kummer equation $y^{p^n} = F$. If χ has order p^i as element of $H_{p^n}^1(\mathbb{K})$, then the Galois group of $Y \rightarrow X$ is the unique subgroup of $\mathbb{Z}/p^n\mathbb{Z}$ of order p^i .

A point $x \in X$ is called a *branch point* for the character $\chi \in H_{p^n}^1(\mathbb{K})$ if it is a branch point for the cover $Y \rightarrow X$. The *branching index* of x is the order of the inertia group for some point $y \in Y$ above x . The set of all branch points is called the *branch locus* of χ and is denoted by $\mathbb{B}(\chi)$.

Definition 5.2. A character $\chi \in H_{p^n}^1(\mathbb{K})$ is called *admissible* if its branch locus $\mathbb{B}(\chi)$ is contained in the open disk D .

5.2.1. *Reduction of characters.* Let $\chi \in H_{p^n}^1(\mathbb{K})$ be an admissible character of order p^n , and let $Y \rightarrow X$ be the corresponding cyclic Galois cover. Let $\Gamma \cong \mathbb{Z}/p^n\mathbb{Z}$ denote the Galois group of $Y \rightarrow X$. Let Y_R be the normalization of X_R in Y . Then Y_R is a normal R -model of Y and we have $X_R = Y_R/\Gamma$.

After enlarging our ground field K , we may assume that the character χ is *weakly unramified* with respect to the valuation v_0 ; see [11]. By definition, this means that for all extensions w of v_0 to the function field of Y , the ramification index $e(w/v_0)$ is equal to 1. It then follows that the special fiber $\bar{Y} := Y_R \otimes_R k$ is reduced; see, e.g., [1, §2.2].

Definition 5.3. We say that the character χ has *étale reduction* if the map $\bar{Y} \rightarrow \bar{X}$ is generically étale.

In terms of Galois cohomology the definition can be rephrased as follows. The character χ has étale reduction if and only if the restriction of χ to the completion $\hat{\mathbb{K}}_0$ of \mathbb{K} with respect to v_0 is *unramified*. The latter means that $\chi|_{\hat{\mathbb{K}}_0}$ lies in the image of the cospecialization morphism

$$H_{p^n}^1(\kappa_0) \rightarrow H_{p^n}^1(\hat{\mathbb{K}}_0)$$

(which is simply the restriction morphism induced by the projection $\text{Gal}_{\hat{\mathbb{K}}_0} \rightarrow \text{Gal}_{\kappa_0}$). Since the cospecialization morphism is injective, there exists a unique character $\bar{\chi} \in H_{p^n}^1(\kappa_0)$ whose image in $H_{p^n}^1(\hat{\mathbb{K}}_0)$ is $\chi|_{\hat{\mathbb{K}}_0}$. By construction, the Galois cover of \bar{X} corresponding to $\bar{\chi}$ is isomorphic to an irreducible component of the normalization of \bar{Y} .

Definition 5.4. If χ has étale reduction, we call $\bar{\chi}$ the *reduction* of χ and χ a *lift* of $\bar{\chi}$.

Remark 5.5. Assume that χ has étale reduction. Then the condition that χ is admissible implies that the cover $\bar{Y} \rightarrow \bar{X}$ corresponding to the reduction $\bar{\chi}$ is étale over $\bar{X} - \{\bar{x}_0\}$. (The proof uses Purity of Branch Locus; see, e.g., [22].) It follows that \bar{Y} is smooth outside the inverse image of \bar{x}_0 .

Definition 5.6. Let $\chi \in H_{p^n}^1(\mathbb{K})$ be an admissible character of order p^n . We say that χ has *good reduction* if it has étale reduction and the cover $\bar{Y} \rightarrow \bar{X}$ corresponding to the reduction $\bar{\chi}$ of χ is smooth.

Note that a \mathbb{Z}/p^n -cover of \mathbb{P}_k^1 , unramified outside x_0 , is uniquely determined by its germ above the branch point (see, e.g., [19]). Thus, with the above notation, the local Oort conjecture (more specifically, Claim 3.1) may be reformulated as follows.

CONJECTURE 5.7. *Suppose that $X = \mathbb{P}_K^1$. Let $\bar{\chi} \in H_{p^n}^1(\kappa_0)$ be a character of order p^n , unramified outside of \bar{x}_0 . Then (after replacing K by a finite*

extension, if necessary) there exists an admissible character $\chi \in H_{p^n}^1(\mathbb{K})$ with good reduction lifting $\bar{\chi}$.

5.3. Swan conductors.

5.3.1. Fix $r \in \mathbb{Q}_{\geq 0}$. We assume that $p^r \in K$. Let $\hat{\mathbb{K}}_r$ denote the completion of \mathbb{K} with respect to the valuation v_r . Let $\chi \in H_{p^n}^1(\mathbb{K})$ be a character of order $\leq p^n$. By Epp’s theorem ([11]) we may assume that the restriction $\chi|_{\hat{\mathbb{K}}_r}$ is weakly unramified. Under this condition, we can define three types of invariants that measure in some way the ramification of χ with respect to the valuation v_r .

First of all, we have the *depth Swan conductor*

$$\delta_\chi(r) := \text{sw}(\chi|_{\hat{\mathbb{K}}_r}) \in \mathbb{Q}_{\geq 0};$$

see [34, Def. 3.3]. By definition, $\delta_\chi(r) = 0$ if and only if χ is unramified with respect to v_r . If this is the case, then the reduction $\bar{\chi}_r \in H_{p^n}^1(\kappa_r)$ is well defined. (See the previous subsection on the case $r = 0$.)

Let us now assume that $\delta_\chi(r) > 0$. Then we can define the *differential Swan conductor* of χ with respect to v_r :

$$\omega_\chi(r) := \text{dsw}(\chi|_{\hat{\mathbb{K}}_r}) \in \Omega_{\kappa_r}^1;$$

see [34, Def. 3.9].

Finally, let $\text{ord}_{\bar{x}} : \kappa_r^\times \rightarrow \mathbb{Z}$ be a normalized discrete valuation whose restriction to k is trivial. Of course, $\text{ord}_{\bar{x}}$ corresponds either to a closed point \bar{x} on the canonical reduction of the affinoid $D[r]$, or it corresponds to the point at infinity, $\bar{x} = \infty$. Then the composite of v_r with $\text{ord}_{\bar{x}}$ is a valuation on \mathbb{K} of rank two, which we denote by $\eta(r, \bar{x}) : \mathbb{K}^\times \rightarrow \mathbb{Q} \times \mathbb{Z}$. (See, e.g., [36, §10, p. 43]; the group $\mathbb{Q} \times \mathbb{Z}$ is equipped with the lexicographic ordering.) By definition, we have

$$\eta(r, \bar{x})(F) = (v_r(F), \text{ord}_{\bar{x}}([F]_r))$$

for $F \in \mathbb{K}^\times$. In [18], Kato defines a Swan conductor $\text{sw}_\chi^K(r, \bar{x}) \in \mathbb{Q}_{\geq 0} \times \mathbb{Z}$ of χ with respect to $\eta(r, \bar{x})$. (See [18, Defs. 2.4 and 3.10]; note that we have $\varepsilon := (0, 1)$.) By definition, the first component of $\text{sw}_\chi^K(r, \bar{x})$ is equal to $\delta_\chi(r)$. We define the *boundary Swan conductor*

$$\text{sw}_\chi(r, \bar{x}) \in \mathbb{Z}$$

as the second component of $\text{sw}_\chi^K(r, \bar{x})$.

Remark 5.8. The invariant $\text{sw}_\chi(r, \bar{x})$ is determined by the invariants $\delta_\chi(r)$ and $\omega_\chi(r)$, as follows:

(i) If $\delta_\chi(r) = 0$, then

$$\text{sw}_\chi(r, \bar{x}) = \text{sw}_{\bar{\chi}_r}(\bar{x}).$$

Here $\bar{\chi}_r$ is the reduction of χ with respect to v_r (well defined because χ is unramified at v_r) and $\text{sw}_{\bar{\chi}_r}(\bar{x})$ is the usual Swan conductor of $\bar{\chi}_r$ with respect to the valuation $\text{ord}_{\bar{x}}$ (one less than the Artin conductor for nontrivial characters; see [31, VI, §2]). This formula follows easily from the definitions. As a consequence we see that $\text{sw}_{\chi}(r, \bar{x}) \geq 0$ and that $\text{sw}_{\chi}(r, \bar{x}) = 0$ if and only if $\bar{\chi}_r$ is unramified with respect to $\text{ord}_{\bar{x}}$.

(ii) If $\delta_{\chi}(r) > 0$, then we have

$$\text{sw}_{\chi}(r, \bar{x}) = -\text{ord}_{\bar{x}}(\omega_{\chi}(r)) - 1.$$

This follows from [18, Cor. 4.6].¹

PROPOSITION 5.9. *Let $\chi_1, \chi_2 \in H_{p^n}^1(\mathbb{K})$, and let $\chi_3 = \chi_1\chi_2$. For $i \in \{1, 2, 3\}$ and $r \in \mathbb{Q}_{\geq 0}$, set $\delta_i = \delta_{\chi_i}(r)$. If $\delta_i > 0$, then we set $\omega_i := \omega_{\chi_i}(r)$. If $\delta_i = 0$, then $\bar{\chi}_i \in H_{p^n}^1(\kappa_r)$ denotes the reduction of χ_i with respect to v_r .*

- (i) *If $\delta_1 \neq \delta_2$, then $\delta_3 = \max(\delta_1, \delta_2)$. If $\delta_1 > \delta_2$, then $\omega_3 = \omega_1$.*
- (ii) *Assume $\delta_1 = \delta_2 > 0$. Then*

$$\omega_1 + \omega_2 \neq 0 \quad \Rightarrow \quad \delta_1 = \delta_2 = \delta_3, \quad \omega_3 = \omega_1 + \omega_2$$

and

$$\omega_1 + \omega_2 = 0 \quad \Rightarrow \quad \delta_3 < \delta_1.$$

(iii) *Assume $\delta_1 = \delta_2 = 0$. Then $\delta_3 = 0$ and $\bar{\chi}_3 = \bar{\chi}_1\bar{\chi}_2$.*

Proof. Parts (i)–(ii) follow from [34, Prop. 3.10]. Part (iii) is clear, because the cospecialization map $H_{p^n}^1(\kappa_r) \rightarrow H_{p^n}^1(\hat{\mathbb{K}}_r)$ is a homomorphism. \square

5.3.2. The finite extension of K that was necessary in order to define the invariants $\delta_{\chi}(r)$, $\omega_{\chi}(r)$, and $\text{sw}_{\chi}(r, \bar{x})$ depends on r . However, the values $\delta_{\chi}(r)$ and $\omega_{\chi}(r)$ do not depend on the choice of this extension. Therefore, it makes sense to consider δ_{χ} , ω_{χ} , and sw_{χ} as functions in $r \in \mathbb{Q}_{\geq 0}$ and \bar{x} .

PROPOSITION 5.10. *δ_{χ} extends to a continuous, piecewise linear function*

$$\delta_{\chi} : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}.$$

Furthermore,

- (i) *For $r \in \mathbb{Q}_{>0}$, the left (resp. right) derivative of δ_{χ} at r is $-\text{sw}_{\chi}(r, \infty)$ (resp. $\text{sw}_{\chi}(r, 0)$).*
- (ii) *If r is a kink of δ_{χ} (meaning that the left and right derivatives do not agree), then $r \in \mathbb{Q}$.*

Proof. See, e.g., [33, Prop. 2.9]. A more direct proof of a special case of the proposition can be derived from Section 5.4 below. \square

¹The proof in [18] uses class field theory and works only if the residue field of K is quasi-finite. For a much more direct and elementary proof, see [7].

COROLLARY 5.11. *If $r \geq 0$ and $\delta_\chi(r) > 0$, then the left and right derivatives of δ_χ at r are given by $\text{ord}_\infty(\omega_\chi(r)) + 1$ and $-\text{ord}_0(\omega_\chi(r)) - 1$, respectively.*

Proof. Immediate from Proposition 5.10(i) and Remark 5.8(ii). □

5.3.3. We are going to characterize the case when χ has good reduction in terms of the function δ_χ . Our main tool for this is a certain “local vanishing cycles formula.” As a special case, we recover the criterion for good reduction from [15, §3.4]. We fix an admissible character $\chi \in H_{p^n}^1(\mathbb{K})$ of order p^n and let $Y \rightarrow X$ denote the corresponding Galois cover. Let us also fix $r \in \mathbb{Q}_{\geq 0}$ and assume that $p^r \in K$.

Suppose first that $r > 0$. Then the affinoid subdomain $D[r] \subset X^{\text{an}}$ gives rise to an admissible blowup $X'_R \rightarrow X_R$ with the following properties ([1, §3.5] and [6]): Firstly, X'_R is a semistable curve whose special fiber $\overline{X}' := X'_R \otimes_R k$ consists of two smooth irreducible components that meet in exactly one point. The first component is the strict transform of \overline{X} , which we may identify with \overline{X} . The second component is the exceptional divisor \overline{Z} of the blowup $X'_R \rightarrow X_R$, which is isomorphic to the projective line over k and intersects \overline{X} in the distinguished point \bar{x}_0 . By construction, the complement $\overline{Z}^\circ := \overline{Z} \setminus \{\bar{x}_0\}$ is identified with the canonical reduction $\overline{D}[r]$ of the affinoid $D[r]$. In particular, this means that the discrete valuation on \mathbb{K} corresponding to the prime divisor $\overline{Z} \subset X'_R$ is equivalent to the valuation v_r and that the residue field κ_r may be identified with the function field of \overline{Z} .

Let Y'_R denote the normalization of X'_R in Y . We obtain a commutative diagram

$$\begin{array}{ccc} Y'_R & \longrightarrow & Y_R \\ \downarrow & & \downarrow \\ X'_R & \longrightarrow & X_R \end{array}$$

in which the vertical maps are finite Γ -covers and each horizontal map is the composition of an admissible blowup with a normalization. Let $\overline{W} \subset Y'_R$ be the exceptional divisor of $Y'_R \rightarrow Y_R$. After enlarging the ground field K we may assume that \overline{W} is reduced. Note that this holds if and only if the character χ is weakly unramified with respect to the valuation v_r and that this is exactly the condition we need to define $\delta_\chi(r)$, $\omega_\chi(r)$, and $\text{sw}_\chi(r, \bar{x})$. We now choose a closed point $\bar{x} \in \overline{Z}^\circ = \overline{D}[r]$ and a point $\bar{y} \in \overline{W}$ lying over \overline{Z} . We let

$$U(r, \bar{x}) :=]\bar{x}[_{D[r]}$$

denote the residue class of \bar{x} on the affinoid $D[r]$. Clearly, $U(r, \bar{x})$ is isomorphic to the open unit disk. Finally, we let $q : \widetilde{\overline{W}} \rightarrow \overline{W}$ denote the normalization of \overline{W} and set

$$\delta_{\bar{y}} := \dim_k (q_* \mathcal{O}_{\widetilde{\overline{W}}} / \mathcal{O}_{\overline{W}})_{\bar{y}}.$$

Then $\delta_{\bar{y}} \geq 0$ and we have $\delta_{\bar{y}} = 0$ if and only if $\bar{y} \in \overline{W}$ is a smooth point.

The above notation extends to the case $r = 0$ as follows. If $r = 0$, then we let $\bar{Z} := \bar{X}$ denote the special fiber of the smooth model X_R of X and $\bar{W} := \bar{Y}$ the special fiber of Y . We set $\bar{x} := \bar{x}_0$ and choose an arbitrary point $\bar{y} \in \bar{W}$ above \bar{x}_0 . The residue class $U(r, \bar{x})$ is now equal to the open disk D , and the invariant $\delta_{\bar{y}}$ is defined in the same way as for $r > 0$.

PROPOSITION 5.12. *With the notation introduced above, we have*

$$\text{sw}_\chi(r, \bar{x}) = |\mathbb{B}(\chi) \cap U(r, \bar{x})| - 1 - 2\delta_{\bar{y}}.$$

Proof. This follows from [18, Th. 6.7]. To see this, note that the left-hand side of the formula in loc. cit. (the “vanishing cycles”) remains invariant if the sheaf \mathcal{F} is pulled back to a Galois cover of $\text{Spec}(A)$ on which \mathcal{F} becomes constant. In our situation we take for $A := \mathcal{O}_{X'_R, \bar{x}}^h$ the henselian local ring of \bar{x} on the scheme X'_R and for \mathcal{F} the étale sheaf corresponding to the character χ . Then the ring extension $B := \mathcal{O}_{Y'_R, \bar{y}}^h/A$ gives rise to a Galois cover that trivializes \mathcal{F} . To prove Proposition 5.12, one applies the formula from loc. cit. to \mathcal{F} and its pullback. After equating the left-hand side of both formulas and tracing back the definitions of all terms appearing in the right-hand side, one obtains the desired result. \square

As a first consequence of the above proposition we reprove the following important criterion for good reduction from [15, §3.4].

COROLLARY 5.13.

(i) *Let $\chi \in H_{p^n}^1(\mathbb{K})$ be an admissible character of order p^n . Then*

$$|\mathbb{B}(\chi)| \geq \text{sw}_\chi(0, \bar{x}_0) + 1.$$

Also, χ has good reduction if and only if $\delta_\chi(0) = 0$ and equality holds above.

(ii) *Suppose χ has good reduction with upper ramification breaks (m_1, \dots, m_n) . If $1 \leq i \leq n$, then*

$$|\{x \in \mathbb{B}(\chi) \mid \text{ramification index of } x \text{ is exactly } p^{n-i+1}\}| = m_i - m_{i-1},$$

where we set $m_0 = -1$.

Proof. The inequality in part (i) follows immediately from Proposition 5.12 since $\mathbb{B}(\chi) \subset D = U(0, \bar{x}_0)$ by assumption. Now, by definition, χ has good reduction if and only if $\delta_\chi(0) = 0$ and $\bar{Y} = W$ is smooth in any point \bar{y} above the distinguished point \bar{x}_0 . The latter condition is equivalent to $\delta_{\bar{y}} = 0$. Thus, the rest of part (i) also follows from Proposition 5.12.

In the situation of part (ii), the character $\chi_i := \chi_n^{p^{n-i}} \in H_{p^i}^1(\mathbb{K})$ is an admissible character with reduction $\bar{\chi}_i$ of order p^i with upper ramification breaks (m_1, \dots, m_i) . Thus the Swan conductor of $\bar{\chi}_i$ is m_i ([31, Cor. 2 to Th. 1], noting that the Swan conductor is one less than the Artin conductor).

By part (i), $|\mathbb{B}(\chi_i)| = m_i + 1$. Since elements of $\mathbb{B}(\chi_i)$ correspond exactly to elements of $\mathbb{B}(\chi)$ with ramification index at least p^{n-i+1} , part (ii) follows. \square

COROLLARY 5.14. *Let $\chi \in H_{p^n}^1(\mathbb{K})$ be an admissible character of order p^n , let $r \in \mathbb{Q}_{>0}$, and let \bar{x} a point on the canonical reduction of $D[r]$. Then*

$$\text{sw}_\chi(r, \bar{x}) \leq |\mathbb{B}(\chi) \cap U(r, \bar{x})| - 1.$$

Moreover, if χ has good reduction, then equality holds.

Proof. The inequality follows immediately from Proposition 5.12. To prove the second statement we note that if χ has good reduction, then in the situation of Proposition 5.12 the point \bar{y} is a nonsingular point of \bar{W} . This is because the curve \bar{W} is the exceptional divisor of the modification $Y'_R \rightarrow Y_R$. If χ has good reduction, then Y_R is smooth over R and hence regular. It follows from Castelnuovo's criterion (see, e.g., [21, Th. 9.3.8]) that \bar{W} is smooth. \square

COROLLARY 5.15. *In the situation of Corollary 5.14, if $\delta_\chi(r) > 0$, we have $\text{ord}_{\bar{x}}(\omega_\chi(r)) \geq -|\mathbb{B}(\chi) \cap U(r, \bar{x})|$, with equality if χ has good reduction.*

Proof. Immediate from Corollary 5.14 and Remark 5.8(ii). \square

Remark 5.16.

- (i) If χ has good reduction, then Corollaries 5.11 and 5.15 show that δ_χ is a piecewise linear, weakly concave down function. Moreover, the position of the kinks of δ_χ correspond to the valuations of the ramification points in $\mathbb{B}(\chi)$. If $r > 0$ is a kink, then the number of ramification points of χ with valuation r is precisely the difference between the left and the right derivative of δ_χ at r .
- (ii) Now assume that $\mathbb{B}(\chi) \subset D(r_0)$ for some $r_0 \in \mathbb{Q}_{>0}$. Then it follows from Remark 5.8, Corollary 5.11, and Corollary 5.15 that the restriction of δ_χ to the interval $[0, r_0]$ is weakly concave up. Together with (i) this shows that if χ has good reduction, then $\delta_\chi|_{[0, r_0]}$ is linear.

5.4. Characters of order p .

5.4.1. We will now describe in the special case $n = 1$ how to determine the function δ_χ explicitly in terms of a suitable element $F \in \mathbb{K}^\times$ corresponding to the character $\chi \in H_p^1(\mathbb{K}) \cong \mathbb{K}^\times / (\mathbb{K}^\times)^p$.

PROPOSITION 5.17. *Let $F \in \mathbb{K}^\times \setminus (\mathbb{K}^\times)^p$, let $\chi := \mathfrak{R}_1(F) \in H_p^1(\mathbb{K})$, and let $r \in \mathbb{Q}_{\geq 0}$. Suppose that $v_r(F) = 0$. Suppose, moreover, that χ is weakly unramified with respect to v_r .*

(i) *We have*

$$\delta_\chi(r) = \frac{p}{p-1} - \max_H v_r(F - H^p),$$

where H ranges over all elements of \mathbb{K} .

(ii) The maximum of $v_r(F - H^p)$ in (i) is achieved if and only if

$$g := [F - H^p]_r \notin \kappa_r^p.$$

If this is the case, and $\delta_\chi(r) > 0$, then

$$\omega_\chi(r) = \begin{cases} dg/g & \text{if } \delta_\chi(r) = p/(p-1), \\ dg & \text{if } 0 < \delta_\chi(r) < p/(p-1). \end{cases}$$

If, instead, $\delta_\chi(r) = 0$, then $\bar{\chi}$ corresponds to the Artin-Schreier extension given by the equation $y^p - y = g$.

Proof. The assumption $g \notin \kappa_r^p$ shows that $v_r(H) \geq 0$. If $v_r(H) > 0$, then $g = [F]_r$. In this case, [34, Prop. 5.3] says that $\delta_\chi(r) = p/(p-1)$ and $\omega_\chi(r) = dg/g$. Otherwise,

$$FH^{-p} = 1 + p^s G,$$

where $s := v_r(F - H^p) > 0$ and $G \in \mathbb{K}$ is an element with $v_r(G) = 0$ and residue class g . By [34, Prop. 5.3], we now have $\delta_\chi(r) = p/(p-1) - s$. Moreover, if $\delta_\chi(r) > 0$, then $\omega_\chi(r) = dg$.

If $\delta_\chi(r) = 0$, then $s = p/(p-1)$. Make a change of variable $1 + \lambda Y = FH^{-p}$, where $\lambda \in K$ is the unique element satisfying $\lambda^{p-1} = -p$ and

$$v(1 + \lambda/p^{1/(p-1)}) > 0.$$

Then the equation $(1 + \lambda Y)^p = 1 + p^s G$ yields $Y^p - Y = G + o(1)$, which reduces to the desired Artin-Schreier extension. \square

5.5. *Detecting the slope of δ_χ .* Let $\chi \in H_p^1(\mathbb{K})$ be an admissible character of order p , giving rise to a branched cover $Y \rightarrow X$. Let $m > 1$ be a prime-to- p integer. We assume that the following conditions hold:

- (a) The branch locus of χ is contained in the closed disk $D[r_0]$, for some $r_0 > 0$, and $T = 0$ is one of the branch points.
- (b) For all $r \in (0, r_0]$, the left derivative of δ_χ at r is $\leq m$. (Equivalently, by Proposition 5.10, $\text{sw}_\chi(r, \infty) \geq -m$.)
- (c) For all $r \in (0, r_0]$, we have $\delta_\chi(r) > 0$.

Because of condition (a) we can represent χ as the Kummer class of a power series

$$F = 1 + \sum_{i=1}^{\infty} a_i T^{-i},$$

with $a_i \in R$ and $v(a_i) \geq r_0 i$. We wish to find a polynomial H in T^{-1} whose p -th power approximates F well enough to use Proposition 5.17 simultaneously for all r in an interval $(0, s] \cap \mathbb{Q}$ for some $0 < s < r_0$. We will then get explicit expressions for the slopes of δ_χ on the interval $[0, r_0]$.

For any $N \geq 1$, set

$$H := 1 + \sum_{j=1}^N b_j T^{-j}.$$

Here we consider the b_j for the moment as indeterminates. Write

$$F - H^p = \sum_{k=1}^{\infty} c_k T^{-k},$$

where c_k is a polynomial in $b_1, \dots, b_{\min(k, N)}$. Note that $c_k = a_k \in R$ for $k > pN$.

LEMMA 5.18. *Assuming condition (a), after replacing K by some finite extension, there exist $b_1, \dots, b_N \in R$ such that*

- (i) $v(c_k) \geq r_0 k$ for all k , and
- (ii) $c_{kp} = 0$ for all $k \leq N$.

Proof. For (ii) to hold, we can solve the equations $c_{pN} = c_{p(N-1)} = \dots = c_p = 0$ inductively:

$$\begin{aligned} c_{pN} &= b_N^p - a_{Np} = 0, \\ c_{p(N-1)} &= b_{N-1}^p + \dots = 0, \\ &\vdots \quad \quad \quad \vdots \end{aligned}$$

One easily checks that these solutions verify $v(b_j) \geq jr_0$ for all j . So we have $v_{r_0}(F), v_{r_0}(H) \geq 0$. Thus we get $v_{r_0}(F - H^p) \geq 0$, which is equivalent to (i). \square

Remark 5.19. The proof above shows that there are only finitely many solutions for the b_j and that they vary analytically as the a_i do.

PROPOSITION 5.20. *Assume conditions (a), (b), and (c) hold. Choose $s \in (0, r_0) \cap \mathbb{Q}$ and $N \in \mathbb{N}$ such that*

$$(6) \quad pN \geq \frac{p}{(p-1)(r_0 - s)}.$$

Let b_1, \dots, b_N be as in Lemma 5.18. Define $\lambda_m(\chi) \in [0, r_0]$ by

$$\lambda_m(\chi) := \max(\{r \in (0, r_0] \mid \text{sw}_{\chi}(r, \infty) > -m\} \cup \{0\}).$$

Set

$$\mu_m(\chi) := \max(\{\frac{v(c_m) - v(c_k)}{m - k} \mid 1 \leq k < m\} \cup \{0\}).$$

Then

- (i) For all $r \in (0, s] \cap \mathbb{Q}$, we have

$$[F - H^p]_r \notin \kappa_r^p.$$

Therefore,

$$\delta_\chi(r) = p/(p-1) - v_r(F - H^p), \quad \text{sw}_\chi(r, \infty) = -\text{ord}_\infty[F - H^p]_r.$$

(ii) We have

$$\lambda_m(\chi) < s \quad \Leftrightarrow \quad \mu_m(\chi) < s.$$

(iii) If $\lambda_m(\chi) < s$, then $\lambda_m(\chi) = \mu_m(\chi)$.

Remark 5.21. Note that if $\lambda_m(\chi) \neq r_0$, then Proposition 5.10 implies that $\lambda_m(\chi)$ is the largest value in $(0, r_0]$ where δ_χ has a kink.

Proof of Proposition 5.20. Fix $r \in (0, s] \cap \mathbb{Q}$, and set $M := \text{ord}_\infty[F - H^p]_r$. By definition and by Lemma 5.18(i), we have

$$(7) \quad v_r(F - H^p) = v(c_M) - rM \geq M(r_0 - r) \geq M(r_0 - s).$$

On the other hand, condition (c) shows that $\delta_\chi(r) > 0$ and Proposition 5.17 show that

$$(8) \quad v_r(F - H^p) < \frac{p}{p-1}.$$

Using (7), (8), and the choice of N , we obtain the inequality

$$(9) \quad M < \frac{p}{(p-1)(r_0 - s)} \leq Np.$$

If M were divisible by p , then (9) and Lemma 5.18 (ii) would show that $c_M = 0$, which contradicts the definition of M . Therefore, M is prime to p , and part (i) of the lemma follows from Proposition 5.17 and Remark 5.8(ii).

In order to prove (ii) and (iii) we note that, by condition (b), $\lambda_m(\chi) < s$ is equivalent to $\text{sw}_\chi(s, \infty) = -m$. Let \mathcal{N} be the Newton polygon of the power series $F - H^p = \sum_{k=1}^{\infty} c_k T^{-k}$. By (i), the Swan conductor of χ on $(0, s]$ is determined by \mathcal{N} . In particular, we have $\text{sw}_\chi(s, \infty) = -m$ for some $r \in (0, s]$ if and only if the point $(m, v(c_m))$ is a vertex of \mathcal{N} and $s_1 \leq s < s_2$, where s_1 (resp. s_2) is the slope of the edge to the left (resp. to the right) of the vertex $(m, v(c_m))$. Furthermore, in this case we have $\mu_m(\chi) = s_1 < s$ and $\lambda_m(\chi) = s_1$, which proves (iii).

To prove (ii), it remains to show that if $\mu_m(\chi) < s$, then $\lambda_m(\chi) < s$. If $(m, v(c_m))$ is a vertex of \mathcal{N} , then as in the paragraph above we have $\lambda_m(\chi) = \mu_m(\chi)$, so $\lambda_m(\chi) < s$. If $(m, v(c_m))$ is not a vertex of \mathcal{N} , then there exists a line segment of \mathcal{N} with slope $s' < \mu_m(\chi) < s$ connecting two points $(i, v(c_i))$ and $(j, v(c_j))$, where $i < m < j$. But this means that $\text{sw}_\chi(s, \infty) \geq -j$, which contradicts condition (b). This proves (ii). \square

Proposition 5.20 will be the key to Section 6.4.

6. Proof of Theorem 3.4

6.1. *Plan of the proof.* We continue with the notation of Section 5, and for the rest of the paper, we set $X \cong \mathbb{P}^1$ and $x_0 = 0$. Recall that D is the unit disk in $(\mathbb{A}_K^1)^{\text{an}}$ centered at 0. For $r \in \mathbb{Q}_{\geq 0}$, we set

$$D(r) = \{T \in (\mathbb{A}_K^1)^{\text{an}} \mid |T| < |p|^r\} \subseteq D.$$

We are given a character $\bar{\chi}_n \in H_{p^n}^1(\kappa_0)$ of order exactly p^n , with upper ramification breaks (m_1, m_2, \dots, m_n) . We further assume that $n \geq 2$. For $1 \leq i \leq n$, set $r_i = 1/m_i(p - 1)$. Recall that $p \nmid m_1$ and

$$m_i \geq p m_{i-1}$$

for $i = 2, \dots, n$. Moreover, if the inequality above is strict, then $(m_i, p) = 1$. For $i = 1, \dots, n$, we set $\bar{\chi}_i := \bar{\chi}_n^{p^{n-i}} \in H_{p^i}^1(\kappa_0)$. By assumption, for each $1 \leq i < n$, there is a character χ_i lifting $\bar{\chi}_i$. We assume that $\mathbb{B}(\chi_{n-1})$ lies in the disk $D(r_{n-1})$, and we may further assume that $T = 0$ is a branch point of order p^{n-1} . In order to prove Theorem 3.4, we must show that there exists a character $\chi_n \in H_{p^n}^1(\mathbb{K})$ with (good) reduction $\bar{\chi}_n$. Furthermore, we must have $\mathbb{B}(\chi_n) \subseteq D$, and if there is no integer a satisfying

$$(10) \quad \frac{m_n}{p} - m_{n-1} < a \leq \left(\frac{m_n}{m_n - m_{n-1}} \right) \left(\frac{m_n}{p} - m_{n-1} \right),$$

then we must even have $\mathbb{B}(\chi_n) \subseteq D(r_n)$. We will construct χ_n such that $\chi_n^p = \chi_{n-1}$.

We may assume that χ_{n-1} corresponds to an extension of \mathbb{K} given by a system of Kummer equations

$$y_i^p = y_{i-1} G_i, \quad i = 1, \dots, n - 1,$$

with $y_0 := 1$ and $G_i \in \mathbb{K}$. Any $\chi \in H_{p^n}^1(\mathbb{K})$ such that $\chi^p = \chi_{n-1}$ is given by an additional equation

$$(11) \quad y_n^p = y_{n-1} G.$$

Since we must have $\mathbb{B}(\chi) \subseteq D$, we will search for $G \in 1 + T^{-1}\mathfrak{m}[T^{-1}]$, where \mathfrak{m} is the maximal ideal of R . In particular,

$$(12) \quad G = \prod_{i=1}^N (1 - x_i T^{-1})^{a_i},$$

where $a_i \in \mathbb{N}$, $(a_i, p) = 1$, and $x_i \in \mathfrak{m}$ are pairwise distinct. We will say that the polynomial G gives rise to the character χ . If x_i is a branch point of χ_{n-1} , then we may also transfer the term $(1 - x_i t^{-1})^{a_i}$ into G_{n-1} . Therefore, we may assume that none of the x_i is a branch point of χ_{n-1} . If this is the case, then

Corollary 5.13(ii) shows that a necessary condition for good reduction of χ is that $N = m_n - m_{n-1}$. We assume this.

Our proof that a choice G_n for G exists giving rise to a character χ_n whose (good) reduction is $\bar{\chi}_n$ will be done in two parts:

(Part A) We prove that there exists a polynomial $G_{\min} \in 1 + T^{-1}\mathfrak{m}[T^{-1}]$ giving rise to a character χ_{\min} with good reduction $\bar{\chi}_{\min}$ having upper ramification breaks $(m_1, \dots, m_{n-1}, pm_{n-1})$ at the ramified point.

(Part B) We construct the desired polynomial G_n by modifying G_{\min} .

Furthermore, we show that if there is no a satisfying (10), and if G_n gives rise to χ_n , then $\mathbb{B}(\chi_n) \subseteq D(r_n)$. This will complete the proof of Theorem 3.4.

We remark that the basic strategy for our proof is adapted from the proof of the case $n = 2$ by Green and Matignon, [15]. Essentially, Part (A) corresponds to Lemmas 5.2 and 5.3 in loc. cit., whereas Part (B) corresponds to Lemma 5.4.

The proof of Part (A) will be done in three steps. The first step (Section 6.2) is to find an appropriate family of candidate polynomials for G_{\min} , which we will call \mathcal{G}_n . This family is defined in Definition 6.7. The second step (Section 6.3) is to show that if a polynomial in \mathcal{G}_n yields a character with bad reduction, it can be altered (within \mathcal{G}_n) to obtain a new polynomial whose reduction is “closer” to being good (i.e., the depth Swan conductor of the corresponding character is lower). The key result here is Proposition 6.13. Lastly (Section 6.4), we show there must exist a polynomial in \mathcal{G}_n that is “closest” to having good reduction (i.e., the depth Swan conductor of the corresponding character is minimal). This is the content of Proposition 6.15. Combining these steps shows that there must exist $G_{\min} \in \mathcal{G}_n$ giving a character with good reduction.

Proposition 6.21 proves Part (B) and is found in Section 6.5.

6.2. A family of candidate polynomials.

6.2.1. We continue with the setup of Section 6.1. In particular, recall that χ_i is a lift of $\bar{\chi}_i$ for $1 \leq i < n$, and χ is the character arising from G , as in (11). If $r \in \mathbb{Q}_{\geq 0}$, then to simplify the notation we will write $\delta_i(r)$ and $\omega_i(r)$ instead of $\delta_{\chi_i}(r)$ and $\omega_{\chi_i}(r)$ for the depth and differential Swan conductors of χ_i (Section 5.3). Furthermore, we write $\delta_n(r)$ and $\omega_n(r)$ instead of $\delta_{\chi}(r)$ and $\omega_{\chi}(r)$. As will become apparent in Proposition 6.4 and its proof, it will be very important to control $\omega_n(r_{n-1})$. The polynomials G that give our desired $\omega_n(r_{n-1})$ will comprise our candidate family \mathcal{G}_n .

LEMMA 6.1. *Assume $1 \leq i < n$. Then for $r \in [0, r_i]$, we have*

$$(13) \quad \delta_i(r) = m_i \cdot r.$$

Moreover, for $0 < r \leq r_i$, we have

$$(14) \quad \omega_i(r) = \frac{c_i dt}{t^{m_i+1}}.$$

Here $c_i \in k^\times$ is a constant depending on i and r .

Proof. By hypothesis, χ_i has good reduction $\bar{\chi}_i$. Therefore, $\delta_i(0) = 0$. On the other hand, the hypothesis that χ_i has good reduction and that all of its $m_i + 1$ branch points are contained in the disk $D(r_i)$ implies that

$$\text{ord}_0(\omega_i(r)) = -m_i - 1, \quad \text{ord}_{\bar{x}}(\omega_i(r)) = 0$$

for all $r \in (0, r_i]$ and $\bar{x} \neq 0, \infty$, using Corollary 5.15. So (13) follows, using Corollary 5.11. But now the same corollary shows that

$$\text{ord}_\infty(\omega_i(r)) = m_i - 1,$$

and (14) follows as well. □

Remark 6.2. Suppose $1 < i < n$ and $0 < r < r_i$. Lemma 6.1 shows that

$$p\delta_{i-1}(r) \leq \delta_i(r) < \frac{1}{p-1}.$$

Moreover, the first inequality is an equality if and only if $m_i = pm_{i-1}$. It follows from [34, Th. 4.3 (ii)] that $\mathcal{C}(\omega_i(r)) = 0$ if and only if $m_i > pm_{i-1}$, where \mathcal{C} is the Cartier operator. This is consistent with (14). On the other hand, if $m_i = pm_{i-1}$, then

$$\mathcal{C}(\omega_i(r)) = \omega_{i-1}(r).$$

In particular, we have $c_i = c_{i-1}^p$ in (14).

6.2.2. We now focus on the critical radius r_{n-1} . To further simplify the notation we will, until the end of Section 6.2, write ω_i (resp. δ_i) instead of $\omega_i(r_{n-1})$ (resp. $\delta_i(r_{n-1})$). By Lemma 6.1 we have $\delta_{n-1} = 1/(p-1)$. So [34, Th. 4.3] says that

$$(15) \quad \delta_n = \frac{p}{p-1}$$

and

$$(16) \quad \mathcal{C}(\omega_n) = \omega_n + \omega_{n-1}.$$

Let m be the minimal upper ramification break m_i such that m_{n-1} is a power of p times m_i . Thus m is prime to p . Set $\nu = n - 1 - i$. Thus $0 \leq \nu \leq n - 2$ and $m_{n-1} = mp^\nu$.

By Lemma 6.1 and Remark 6.2, we have

$$\omega_i = \frac{c dt}{t^{m+1}}, \dots, \omega_{n-1} = \frac{c^{p^\nu} dt}{t^{p^\nu m+1}}$$

for some $c \in k^\times$. After a change of parameter we may assume that $c = m$, viewed as an element of k^\times . Note that $m^p = m$. Set

$$(17) \quad \eta := -(\omega_i + \cdots + \omega_{n-1}) = -m \sum_{j=0}^{\nu} t^{-mpj-1} dt.$$

LEMMA 6.3. *Let $g = [G]_{r_{n-1}}$. Then*

$$\omega_n = \eta + \frac{dg}{g}.$$

Proof. One easily checks that $\mathcal{C}(\eta) = \eta + \omega_{n-1}$. Using (16) we conclude that

$$\omega_n = \eta + \frac{dh}{h}$$

for some $h \in \kappa_{r_{n-1}}^\times$.

Let us first assume that $G = 1$. Then $\mathbb{B}(\chi_n)$ lies in the disk $D(r_{n-1})$. It follows from Corollary 5.15 that the differential ω_n has no poles outside $t = 0$. Since η has no poles outside $t = 0$, this can happen only if $dh/h = 0$. This shows that the lemma is true if $G = 1$.

To prove the general case, we note that multiplying G by an element $H \in \mathbb{K}^\times$ has the effect of adding the character $\mathfrak{R}_1(H) \in H_p^1(\mathbb{K})$ to χ . We may assume that $v_{r_{n-1}}(H) = 0$ and let $h \in \kappa_{r_{n-1}}$ denote the residue of H . Then we have

$$\delta_{\mathfrak{R}_1(H)} = \frac{p}{p-1} \quad \text{and} \quad \omega_{\mathfrak{R}_1(H)} = \frac{dh}{h}$$

if and only if $h \notin \kappa_{r_{n-1}}^p$, by Proposition 5.17. If $h \in \kappa_{r_{n-1}}^p$, then $\delta_{\mathfrak{R}_1(H)} < p/(p-1)$. In both cases, Proposition 5.9 shows that multiplying G by H has the effect of adding dh/h to ω_n . The lemma follows. \square

The following proposition is not strictly necessary for the proof of Theorem 3.4, but it helps to narrow our search for the correct G_n . Recall that we assume G_n to be in the form (12) and that $N = m_n - m_{n-1}$.

PROPOSITION 6.4. *If χ has good reduction, then the following hold:*

- (i) *For all i , we have $v(x_i) \leq r_{n-1} = 1/m_{n-1}(p-1)$.*
- (ii) *For i, j with $v(x_i) = v(x_j) = r_{n-1}$, we have $\bar{x}_i \neq \bar{x}_j$ (where \bar{x}_i denotes the reduction of $x_i p^{-r_{n-1}}$).*
- (iii) *Write $N = N_1 + N_2$, where N_1 is the number of x_i 's with $v(x_i) = r_{n-1}$. We may assume that $v(x_i) < r_{n-1}$ for $i = N_1 + 1, \dots, n$. Then*

$$\sum_{i=N_1+1}^N a_i \equiv 0 \pmod{p}.$$

In particular, if $a_i = 1$ for all i , then $N_2 \equiv 0 \pmod{p}$.

(iv) If $m_n = pm_{n-1}$, then $N_1 = m_{n-1}(p - 1)$ and $N_2 = 0$. Otherwise, $N_1 < m_{n-1}(p - 1)$ and $N_2 > 0$.

Proof. By Lemma 6.3, we have

$$(18) \quad \omega_n = \frac{dg}{g} - m \sum_{j=0}^{\nu} t^{-mp^j-1} dt,$$

where

$$g := [G]_{r_{n-1}} \in k[t^{-1}].$$

It follows that $\text{ord}_0(\omega_n) = -m_{n-1} - 1$. So if χ has good reduction, then Corollary 5.15 shows that the number of branch points specializing to 0 (i.e., with valuation $> r_{n-1}$) must be equal to $m_{n-1} + 1$. Since χ_{n-1} has exactly $m_{n-1} + 1$ branch points with valuation $> r_{n-1}$, none of the new branch points can have this property. This proves (i).

By (18), ω_n can have at most a simple pole at any point $\bar{x} \neq 0$, and then good reduction and Corollary 5.15 imply that branch points with radius r_{n-1} have to lie in distinct residue classes. This proves (ii). It follows similarly from Corollary 5.15 that ω_n has no zeroes outside $t = \infty$, so the fact that $\text{div}(\omega_n)$ has degree -2 means that $\text{ord}_\infty(\omega_n) = m_{n-1} + N_1 - 1 \geq 0$. But it is easy to see that

$$\text{ord}_\infty(g) = \sum_{i=N_1+1}^N a_i,$$

and (18) shows that $\text{ord}_\infty(\omega_n) \geq -1$ if and only if $\text{ord}_\infty g \equiv 0 \pmod{p}$. This proves (iii).

On the other hand, we have $\mathcal{C}(\omega_n) = \omega_n + \omega_{n-1}$ and $\text{ord}_\infty(\omega_{n-1}) = m_{n-1} - 1$, which implies, by an easy calculation, that

$$\text{ord}_\infty \omega_n \leq pm_{n-1} - 1.$$

It follows that $N_1 \leq m_{n-1}(p - 1)$.

Now suppose that $m_n = pm_{n-1}$. Then Corollaries 5.11 and 5.15 show that the right derivative of δ_n is at most $m_{n-1} + N = pm_{n-1}$ on $[0, r_{n-1})$. Since $\delta_n(r_{n-1}) = p/(p - 1)$ by (15), and good reduction requires $\delta_n(0) = 0$, this slope must be pm_{n-1} on the entire interval. Thus

$$\text{ord}_\infty \omega_n = m_n - 1 = pm_{n-1} - 1,$$

by Corollary 5.11. Hence $N_1 = N = m_{n-1}(p - 1)$ and $N_2 = 0$. Otherwise, if $m_n > pm_{n-1}$, then the condition that δ_n is weakly concave down (Remark 5.16) and has right derivative m_n at $r = 0$ (Proposition 5.10(i)) implies that $\delta_n(r) > pm_{n-1}r$ for $0 < r < r_{n-1}$. But this means that $\text{ord}_\infty \omega_n + 1 < pm_{n-1}$ at $r = r_{n-1}$, hence $N_1 < m_{n-1}(p - 1)$. It follows that

$$N_2 = m_n - m_{n-1} - N_1 > m_n - pm_{n-1} > 0.$$

This completes the proof of the proposition. □

It follows from Proposition 6.4 that, up to a constant factor that we may eliminate by rescaling t , we have

$$(19) \quad [G]_{r_{n-1}} = g = t^{a_0} \prod_{i=1}^{N_1} (1 - \bar{x}_i t^{-1})^{a_i},$$

where

$$a_0 := - \sum_{i=N_1+1}^N a_i \equiv 0 \pmod{p}.$$

Hence

$$(20) \quad \frac{dg}{g} = \sum_{i=1}^{N_1} \frac{a_i \bar{x}_i t^{-2} dt}{1 - \bar{x}_i t^{-1}}.$$

COROLLARY 6.5. *In the notation of Proposition 6.4, if χ has good reduction, then*

$$(21) \quad \omega_n = \frac{dg}{g} - m \sum_{j=0}^{\nu} t^{-mp^j-1} dt = \frac{c dt}{t^{m_{n-1}+1} \prod_{i=1}^{N_1} (t - \bar{x}_i)},$$

where $c := (-1)^{N_1+1} m (\prod_{i=1}^{N_1} \bar{x}_i)$ is a nonzero constant. In particular, $\text{ord}_{\infty} \omega_n = m_{n-1} + N_1 - 1$.

Proof. The middle expression is the expression deduced for ω_n in (18). We have seen in the proof of Proposition 6.4 that ω_n has simple poles at the \bar{x}_i , a pole of order $m_{n-1} + 1$ at 0, and no zero outside ∞ . It follows that ω_n is equal to the right-hand side of (21) times a constant. To determine this constant, one computes the Laurent series representation in t of both sides. \square

The next theorem, showing that we can often find a g satisfying the conditions of 6.5, is critical.

THEOREM 6.6. *Suppose $m|m_n$ (equivalently, $m|N = m_n - m_{n-1}$). Then, under the assumption that $a_i = 1$ for $i \geq 1$, there is a solution $g \in k[t^{-1}]$ to (21) as in (19) of degree N .*

Proof. This is contained in Corollary 7.8. \square

Definition 6.7. If g is the solution to (21) guaranteed by Theorem 6.6, then we define \mathcal{G}_n to be the subset of all G of the form (12) with $[G]_{r_{n-1}} = g$ (see Notation 5.1).

Remark 6.8. Corollary 7.8 also shows that if $a_i = 1$ for all $i \geq 1$ and N_1 is as in (19), then we have $m|N_1$ and

$$m_{n-1}(p-1) - mp < N_1 \leq m_{n-1}(p-1), \quad N_1 \equiv N \pmod{p}.$$

This determines N_1 uniquely. This means that if G is as in (12), has all $a_i = 1$, and gives rise to χ with good reduction, then the number of zeroes of G with valuation r_{n-1} is fixed. The importance of this condition is illustrated in Example 6.9.

Example 6.9. Assume that $p = 5$, set $n = 3$, and suppose the upper ramification breaks of $\bar{\chi}_3$ are $(m_1, m_2, m_3) = (1, 5, 34)$. If G_3 is in the form of (12) with all $a_i = 1$, and if G_3 gives rise to a character χ_3 with good reduction $\bar{\chi}_3$, then $N = 29$. Remark 6.8 shows that χ_3 gives rise to a cover with exactly $N_1 = 19$ branch points at radius $r_2 = 1/20$ and $N_2 = 10$ branch points at radius $< 1/20$. We know from (15) that $\delta_3(1/20) = 5/4$. By Corollaries 5.11 and 5.15, the (left) slope of δ_3 at $r \leq 1/20$ is equal to $5 + N(r)$, where $N(r)$ is the number of branch points with valuation $\geq r$. Since $\delta(0) = 0$, one can show that there must be at least one branch point with valuation $\leq 1/200$. But $r_3 = 1/136 > 1/200$, so $\mathbb{B}(\chi_3)$ cannot lie in the disk $D(r_3)$.

See Remark 6.22 and Example 7.18 for such an example with 10 branch points with valuation exactly $1/200$. Note that this does not contradict Theorem 3.4(ii), as we can take $a = 2$ there.

Remark 6.10. If we do not assume $a_i = 1$ for all i , then (21) can still sometimes be solved. In particular, in light of Example 6.9, it would be nice to find solutions to (21) with arbitrary a_i and N_1 not satisfying Remark 6.8. We might then have some hope of finding a lift χ_n of $\bar{\chi}_n$ with $\mathbb{B}(\chi_n)$ lying in the disk $D(r_n)$, even when the condition in Theorem 3.4(ii) does not hold. However, even when such solutions to (21) exist, it seems as if our current techniques are often insufficient to turn them into lifts. For further discussion, see Remark 7.12.

6.3. Reducing the depth Swan conductor. We maintain the notation of Section 6.2, and we assume further that $m_n = pm_{n-1}$. Then, by Proposition 6.4, we have $N = N_1 = m_{n-1}(p - 1)$.

Recall that any $G \in \mathcal{G}_n$ (Definition 6.7) gives rise to a character χ of order p^n lifting χ_{n-1} as in (11), by adjoining the equation $y_n^p = y_{n-1}G$. By Corollaries 6.5 and 5.11, we know that the left derivative of δ_n at r_{n-1} is m_n . Recall also from (15) that $\delta_n(r_{n-1}) = p/(p - 1) = m_n r_{n-1}$. It follows that there exists $0 \leq \lambda < r_{n-1}$ such that

$$\delta_n(s) = s \cdot m_n = psm_{n-1} = p\delta_{n-1}(s)$$

for all $s \in [\lambda, r_{n-1}]$ (the last equality following from Lemma 6.1). Let $\lambda(G)$ be the minimal value of λ with this property. In other words, $\lambda(G)$ is the largest kink of the function δ_n on the open interval $(0, r_{n-1})$ (or is zero if δ_n is linear on $[0, r_{n-1}]$). Note that $\delta_n(\lambda) = m_n\lambda < p/(p - 1)$.

PROPOSITION 6.11. *If $G \in \mathcal{G}_n$ satisfies $\lambda(G) = 0$, then the corresponding character χ has good reduction.*

Proof. By definition, $\lambda(G) = 0$ implies $\delta_\chi(0) = 0$. Now, if χ lifts χ_{n-1} and has $\delta_\chi(0) = 0$, then $\text{sw}_\chi(0, \bar{x}_0) \geq pm_{n-1}$, as the Swan conductor of a p^n -cyclic extension must be at least p times the Swan conductor of its index p subextension (because each upper ramification break of a p^n -cyclic extension must be at least p times the previous one). On the other hand, by construction, $\mathbb{B}(\chi)$ has exactly $N + m_{n-1} + 1 = pm_{n-1} + 1$ branch points. The proposition then follows by Corollary 5.13(i). \square

Thus, in order to prove Part (A), it suffices to show that $\lambda(G) = 0$ for some $G \in \mathcal{G}_n$. Proposition 6.13 will show that $\lambda(G) = 0$ is the only possible minimal value of $\lambda(G)$, as G ranges over \mathcal{G}_n . In Section 6.4, we will show that this minimum is realized. First, we state a lemma.

LEMMA 6.12. *Let $G \in \mathcal{G}_n$ and $r \in [0, r_{n-1})$. Let $f \in t^{-1}k[t^{-1}]$ be a polynomial of degree $< m_n$ without constant term, which we regard as an element of κ_r . Set $s := p/(p - 1) - m_n r$. Then, after a possible finite extension of K , there exists $G' \in \mathcal{G}_n$ and $F \in \mathbb{K}$ such that $v_r(F) = 0$, $[F]_r = f$, and*

$$\frac{G'}{G} \equiv 1 - p^s F \pmod{(\mathbb{K}^\times)^p}.$$

Proof. The proof is given as Corollary 7.11. \square

PROPOSITION 6.13. *Suppose $G \in \mathcal{G}_n$ with $\lambda(G) > 0$. Then there exists $G' \in \mathcal{G}_n$ with $\lambda(G') < \lambda(G)$.*

Proof. If $\lambda := \lambda(G) > 0$, then by Corollary 5.11 we have that $\text{ord}_\infty(\omega_n(\lambda)) + 1$ is the left derivative of $\delta_n(r)$. Since δ_n is concave up at λ (Remark 5.16(ii)), we conclude that

$$\text{ord}_\infty(\omega_n(\lambda)) < m_n - 1.$$

By hypothesis we have $p\delta_{n-1}(\lambda) = \delta_n(\lambda) = m_n\lambda < p/(p - 1)$. Therefore [34, Prop. 4.3 (ii)] shows that

$$\mathcal{C}(\omega_n(\lambda)) = \omega_{n-1}(\lambda) = \frac{c dt}{t^{m_{n-1}+1}},$$

for some $c \in k$. It follows that

$$\omega_n(\lambda) = \frac{c^p dt}{t^{m_n+1}} + df$$

for some $f \in \kappa_\lambda$. Note that by Corollaries 5.11 and 5.15,

$$\text{ord}_0(\omega_n(\lambda)) = -m_n - 1, \quad \text{ord}_{\bar{x}}(\omega_n(\lambda)) \geq 0 \quad \forall \bar{x} \neq 0.$$

We may therefore assume that f is a polynomial in t^{-1} of degree $< m_n$ and without constant term.

By Lemma 6.12, there exists $G' \in \mathcal{G}_n$ such that

$$\frac{G'}{G} \equiv 1 - p^s F \pmod{(\mathbb{K}^\times)^p},$$

where $v_\lambda(F) = 0$, where $[F]_\lambda = f$, and where $s := p/(p-1) - m_n\lambda$. Replacing G by the polynomial G' has the effect of adding $\psi := \mathfrak{K}_1(G'/G)$ to χ . Using Proposition 5.17, we see that

$$\delta_\psi(\lambda) = \frac{p}{p-1} - s = \delta_n(\lambda).$$

Therefore, Proposition 5.9 shows that the effect on $\omega_n(\lambda)$ is addition of $-df$, and the result is that

$$\text{ord}_\infty(\omega_n(\lambda)) = m_n + 1.$$

We conclude, using Corollary 5.11, that $\lambda(G') < \lambda(G)$. \square

Remark 6.14. An important reason why we must assume that $\mathbb{B}(\chi_{n-1}) \in D(r_{n-1})$ is to ensure that no branch point of χ_{n-1} has valuation less than λ . If there were such a branch point, then $\text{ord}_\infty(\omega_n(\lambda))$ above could be negative, which would allow f not to be a polynomial in t^{-1} , which would prevent us from applying Lemma 6.12.

6.4. The minimal depth Swan conductor. We continue with the notation of Section 6.3, as well as the assumption that $m_n = pm_{n-1}$ and all $a_i = 1$ for $i \geq 1$. To finish the proof of Part (A) from Section 6.1, we must show that the function $\lambda : \mathcal{G}_n \rightarrow \mathbb{Q}_{\geq 0}$ defined in Section 6.3 takes the value 0 for some $\chi \in \mathcal{G}_n$. By Proposition 6.13, the existence of such a χ is established by the following proposition.

PROPOSITION 6.15. *The function $\chi \mapsto \lambda(\chi)$ takes a minimal value on \mathcal{G}_n .*

The rest of Section 6.4 is devoted to the proof of this proposition.

6.4.1. A lemma from rigid analysis. The following lemma, which is an easy consequence of the *maximum modulus principle*, is a crucial ingredient in the proof of Proposition 6.15.

LEMMA 6.16. *Let $X = \text{Spm}(A)$ be an affinoid domain over K and $f_1, \dots, f_n \in A$ analytic functions on X . Then the function*

$$\phi : X \rightarrow \mathbb{R}, \quad \phi(x) := \max_{1 \leq i \leq n} \sqrt[i]{|f_i(x)|}$$

takes a minimal value. Equivalently, the function

$$x \mapsto \min_i \frac{v(f_i(x))}{i}$$

takes a maximal value on X .

Proof. Let B/A be a finite ring extension that contains elements $g_i \in B$ such that $g_i^i = f_i$ for $i = 1, \dots, n$. Then B is again an affinoid K -algebra, and the induced morphism $q : Y := \text{Spm}(B) \rightarrow X$ is finite and surjective. For any point $y \in Y$, we have

$$\phi(q(y)) = \max_{1 \leq i \leq n} |g_i(x)|.$$

So by [4, Lemma 7.3.4/7], the function $\phi \circ q$ takes its minimal value on Y . Since q is surjective, this shows that ϕ takes its minimal value on X . \square

6.4.2. *An affinoid containing \mathcal{G}_n .* Let \mathcal{G} be the set of all polynomials of the form

$$G = \prod_{i=1}^{N_1} (1 - x_i T^{-1}),$$

where $x_i \in \overline{K}$ has valuation r_{n-1} and where the residue classes of $x_i/p^{r_{n-1}}$ in k are all distinct. Given $G \in \mathcal{G}$, we may assume (after passing to a finite extension of K) that $x_1, \dots, x_{N_1} \in K$. In this way, we can consider \mathcal{G} as a subset of affine N_1 -space over K via the coordinates x_i and G as a K -rational point. It is clear that \mathcal{G} is an affinoid subdomain of $(\mathbb{A}_K^{N_1})^{\text{an}}$. We identify elements of \mathcal{G} with the characters χ that they give rise to (Section 6.1).

Recall that it is a consequence of Proposition 6.4(iv) that $\mathcal{G}_n \subseteq \mathcal{G}$. In particular,

$$\mathcal{G}_n = \{G \in \mathcal{G} \mid [G]_{r_{n-1}} = g\},$$

where g is the unique solution of equation (21) guaranteed by Theorem 6.6. As a rigid analytic space, \mathcal{G}_n is isomorphic to the open unit polydisk. The idea of the proof is to show that $\lambda(\mathcal{G})$ takes a minimal value on \mathcal{G} and that the point where this minimum is achieved must lie in \mathcal{G}_n .

6.4.3. Let $\phi_{n-1} : Y_{n-1} \rightarrow X$ be the Galois cover corresponding to the character χ_{n-1} . By our induction hypothesis, it has good reduction and is totally ramified above $T = 0$. It follows that the rigid analytic subspace $C := \phi_{n-1}^{-1}(D) \subseteq Y_{n-1}$ is an open disk and contains the unique point $y_{n-1} \in Y_{n-1}$ above $T = 0$. We choose a parameter \tilde{T} for the disk C such that $\tilde{T}(y_{n-1}) = 0$. Then

$$T = \tilde{T}^{p^{n-1}} u(\tilde{T}), \quad \text{with } u(\tilde{T}) \in R[[\tilde{T}]]^\times.$$

We conclude that for $r > 0$, the inverse image of the closed disk $D(r) \subset D$ defined by the condition $v(T) \geq r$ is the closed disk $C(\tilde{r})$ defined by $v(\tilde{T}) = \tilde{r} := p^{-n+1}r$. Set $\tilde{r}_{n-1} := p^{-n+1}r_{n-1}$. Let \mathbb{K}_{n-1} denote the function field of Y_{n-1} .

Let us fix, for the moment, $\chi \in \mathcal{G}$ such that $\chi^p = \chi_{n-1}$. Let $\tilde{\chi} := \chi|_{\mathbb{K}_{n-1}} \in H_p^1(\mathbb{K}_{n-1})$ denote the restriction of χ to \mathbb{K}_{n-1} . If χ corresponds to a cover $Y \rightarrow X$, then $\tilde{\chi}$ corresponds to the cover $Y \rightarrow Y_{n-1}$. If $\chi \in \mathcal{G}_n$, then in

analogy to $\lambda(\chi)$, we write $\lambda(\tilde{\chi})$ for the minimum $\tilde{r} \in [0, \tilde{r}_{n-1}]$ such that $\delta_{\tilde{\chi}}$ is linear on $[\tilde{r}, \tilde{r}_{n-1}]$. If $\chi \in \mathcal{G} \setminus \mathcal{G}_n$, then we define $\lambda(\tilde{\chi}) = \tilde{r}_{n-1}$.

LEMMA 6.17.

- (i) For $\chi \in \mathcal{G}_n$, we have $\lambda(\tilde{\chi}) = p^{-n+1}\lambda(\chi)$.
- (ii) Let

$$\tilde{m} = p^n m_{n-1} - \sum_{i=1}^{n-1} m_i (p-1) p^{i-1}.$$

Then $p \nmid \tilde{m}$ and the character $\tilde{\chi} \in H_p^1(\mathbb{K}_{n-1})$ satisfies the conditions (a), (b), and (c) of Section 5.5 (with respect to \tilde{m} , the open disk $C \subset Y_{n-1}^{\text{an}}$ and the family of subdisks $C(\tilde{r})$ for $\tilde{r} \in [0, \tilde{r}_{n-1}]$).

- (iii) If $\lambda_{\tilde{m}}(\tilde{\chi})$ is as in Proposition 5.20 and if we set r_0 in Proposition 5.20 equal to \tilde{r}_{n-1} , then $\lambda(\tilde{\chi}) = \lambda_{\tilde{m}}(\tilde{\chi})$ for all $\chi \in \mathcal{G}$.

Proof. For $r > 0$, we systematically use the notation $\tilde{r} := p^{-n+1}r$. Then the valuation $v_{\tilde{r}}$ on \mathbb{K}_{n-1} (corresponding to the Gauss norm on $C(\tilde{r})$) is the unique extension of v_r . By [34, §7.1] we have

$$\begin{aligned} \delta_{\tilde{\chi}}(\tilde{r}) &= \psi_{\mathbb{K}_{n-1}/\mathbb{K}}(\delta_{\chi}(r)) \\ &= \delta_{\chi}(r) - \left(\delta_1(r) \frac{p-1}{p^{n-1}} + \dots + \delta_{n-1}(r) \frac{p-1}{p} \right), \end{aligned}$$

where ψ is the inverse Herbrand function ([31, IV, §3]). Since all the characters χ_i ($1 \leq i < n$) have good reduction and their branch points are contained in $D(r_{n-1})$, it follows from Remark 5.16(ii) that each δ_i ($1 \leq i < n$) is linear of slope m_i on the interval $[0, r_{n-1}]$. Therefore, we have

$$\delta_{\tilde{\chi}}(\tilde{r}) = \delta_{\chi}(r) + \left(\frac{\tilde{m}}{p^{n-1}} - pm_{n-1} \right) r.$$

Thus, the left slope of $\delta_{\tilde{\chi}}$ at \tilde{r} is equal to $p^{n-1}c + \tilde{m} - p^n m_{n-1}$, where c is the left-slope of δ_{χ} at r . Part (i) follows immediately. Part (ii) follows from the fact that $c \leq m_n = pm_{n-1}$. Part (iii) also follows from this fact, along with Proposition 5.10 and the fact that $\text{sw}_{\chi}(r_{n-1}, \infty) = pm_{n-1}$ if and only if $\chi \in \mathcal{G}_n$. □

Explicitly, the character $\tilde{\chi}$ is the Kummer class of the element

$$F := G_1^{1/p^{n-1}} G_2^{1/p^{n-2}} \cdot \dots \cdot G_{n-1}^{1/p} G \in \mathbb{K}_{n-1}^{\times}.$$

We write F as a power series in the parameter \tilde{T} :

$$F = 1 + \sum_{\ell=1}^{\infty} a_{\ell} \tilde{T}^{-\ell}.$$

Note that since G_1, \dots, G_n are fixed, F is uniquely determined by the choice of G . So we may consider the coefficients a_{ℓ} as functions on the space \mathcal{G} . It is

easy to see that the a_ℓ are analytic functions on \mathcal{G} that are bounded by 1. In fact, a_ℓ is a polynomial in the coordinates x_i with coefficients in R .

6.4.4. We continue with the proof of Proposition 6.15. Let $\chi_0 \in \mathcal{G}_n$ be an arbitrary lift lying in the residue class determined by the reduction g . From the discussion at the beginning of Section 6.3, it follows that $\lambda(\chi_0) < r_{n-1}$. We may therefore choose a rational number $s \in (\lambda(\chi_0), r_{n-1})$. Recall from Section 6.4.3 that $\tilde{\chi}$ is the restriction of χ to the function field \mathbb{K}_{n-1} of Y_{n-1} . Then by Lemma 6.17, we have

$$\lambda(\tilde{\chi}) < \tilde{s} := p^{1-n}s < \tilde{r}_{n-1}.$$

We also choose an integer N such that

$$Np \geq \frac{p}{(p-1)(\tilde{r}_{n-1} - \tilde{s})}.$$

Compare with (6).

LEMMA 6.18. *There exist a finite cover $\mathcal{G}' \rightarrow \mathcal{G}$ and analytic functions b_1, \dots, b_N on \mathcal{G}' with the following property. Set*

$$H := 1 + \sum_{j=1}^N b_j \tilde{T}_1^{-j},$$

and write

$$F - H^p = \sum_{\ell=1}^{\infty} c_\ell \tilde{T}_1^{-\ell},$$

where the c_ℓ are now analytic functions on \mathcal{G}' . Then

- (i) for all $\ell \geq 1$ and all points $x \in \mathcal{G}'$, we have $v(c_\ell(x)) \geq r_0\ell$;
- (ii) we have $c_{p\ell} = 0$ for $\ell \leq N$.

Proof. By Lemma 5.18 and Remark 5.19, there are finitely many solutions for the b_j at each point in \mathcal{G} and the solutions vary analytically as the a_ℓ vary in \mathcal{G} . This proves the lemma. \square

6.4.5. We can now complete the proof of Proposition 6.15. Let \tilde{m} be as in Lemma 6.17, and define the function $\mu_{\tilde{m}} : \mathcal{G}' \rightarrow \mathbb{R}$ by the formula

$$\mu_{\tilde{m}}(x) := \max \left(\left\{ \frac{v(c_{\tilde{m}}(x)) - v(c_\ell(x))}{\tilde{m} - \ell} \mid 1 \leq \ell < \tilde{m} \right\} \cup \{0\} \right).$$

Let $\chi \in \mathcal{G}$, write $\tilde{\chi} := \chi|_{\mathbb{K}_{n-1}}$ for its restriction to the function field of Y_{n-1} , and let $x \in \mathcal{G}'$ be an arbitrary point above χ . By Lemma 6.18, we can apply Proposition 5.20 to compare $\mu_{\tilde{m}}(x)$ to $\lambda_{\tilde{m}}(\tilde{\chi})$, which by Lemma 6.17 is equal to $\lambda(\tilde{\chi})$. We conclude that $\mu_{\tilde{m}}(x) < \tilde{s}$ if and only if $\lambda(\tilde{\chi}) < \tilde{s}$. Moreover, if this is the case, then we have $\mu_{\tilde{m}}(x) = \lambda(\tilde{\chi})$. Note also that in any case we have $\lambda(\chi) = p^{n-1}\lambda(\tilde{\chi})$ when $\chi \in \mathcal{G}_n$.

We apply these arguments twice. Firstly, let $\chi_0 \in \mathcal{G}_n$ be the character with $\lambda(\chi_0) < s$ from the beginning of Section 6.4.4. Let $\chi'_0 \in \mathcal{G}'$ be a point above χ_0 . Then $\mu_{\tilde{m}}(\chi'_0) < \tilde{s}$.

It follows from Lemma 6.16 that the function μ_m takes a minimum on \mathcal{G}' . Let $x \in \mathcal{G}'$ be a point where this minimum is achieved, and let $\chi \in \mathcal{G}$ be the corresponding lift. We have $\mu_{\tilde{m}}(x) \leq \mu_{\tilde{m}}(\chi'_0) < \tilde{s}$. Since

$$\lambda_{\tilde{m}}(\tilde{\chi}) = \mu_{\tilde{m}}(x) < s < \tilde{r}_{n-1},$$

we see that $\chi \in \mathcal{G}_n$. Applying the above arguments a second time, we conclude that $\lambda(\chi) = p^{n-1}\mu_{\tilde{m}}(x)$ and that this is actually the minimal value of the function $\lambda : \mathcal{G}_n \rightarrow \mathbb{R}$. This completes the proof of Proposition 6.15. \square

Combining Propositions 6.11, 6.13, and 6.15 finishes the proof of Part (A) from Section 6.1. So there is a polynomial $G_{\min} \in \mathcal{G}_n$ giving rise to a character χ_{\min} with good reduction and upper ramification breaks $(m_1, \dots, m_{n-1}, pm_{n-1})$ at the ramification point.

6.5. *Beyond minimality.* We now prove Part (B) from Section 6.1. Maintain the notation of the previous parts of Section 6. Let $G_{\min} \in \mathcal{G}_n$ be such that its corresponding character χ_{\min} has good reduction $\bar{\chi}_{\min}$. Such a G_{\min} exists by Part (A). Note that G_{\min} is a polynomial in $1 + T^{-1}\mathfrak{m}[T^{-1}]$ of degree $m_{n-1}(p - 1)$.

Recall that $\bar{\chi}_n$ is our original character, with upper ramification breaks (m_1, \dots, m_n) , and that m_n is *not* necessarily equal to pm_{n-1} . Furthermore, we saw in Section 2 that $\bar{\chi}_n$ corresponds (upon completion at $t = 0$) to a (truncated) Witt vector $w_n := (f_1, \dots, f_n) \in W_n(k((t)))$, and we may assume that each f_i is a polynomial in $k[t^{-1}]$, all of whose terms have prime-to- p degree. Then (3) shows that $m_n = \max(pm_{n-1}, \deg(f_n))$. On the other hand, $\bar{\chi}_{\min}$ has upper ramification breaks $(m_1, \dots, m_{n-1}, pm_{n-1})$ and corresponds to a Witt vector $w_{\min} := (f_1, \dots, f_{n-1}, f_{\min})$, where $f_{\min} \in k[t^{-1}]$ has degree $< pm_{n-1}$ and only terms of prime-to- p degree. Subtracting Witt vectors yields

$$(22) \quad w_n - w_{\min} = (0, \dots, 0, f_n - f_{\min}).$$

Let $f = f_n - f_{\min}$, which has degree $\leq m_n$, and let $F \in T^{-1}R[T^{-1}]$ be such that $\deg F = \deg f$, the valuation $v_0(F) = 0$, and $[F]_0 = f$.

PROPOSITION 6.19. *After a possible finite extension of K , there exists $\varepsilon \in \mathbb{Q}_{>0}$, as well as $G_n \in 1 + T^{-1}\mathfrak{m}[T^{-1}]$ of degree at most $m_n - m_{n-1}$ and $H \in 1 + T^{-1}\mathfrak{m}[T^{-1}]$ such that (H, G_n) is a solution to*

$$G_{\min}H^p - G_n \equiv -p^{p/(p-1)}F \pmod{p^{p/(p-1)+\varepsilon}R[T^{-1}]}.$$

Proof. This follows immediately from Corollary 7.16, substituting m_n and r_n for m'_n and r'_n . \square

PROPOSITION 6.20. *Suppose that there is no integer a satisfying*

$$(23) \quad \frac{m_n}{p} - m_{n-1} < a \leq \left(\frac{m_n}{m_n - m_{n-1}} \right) \left(\frac{m_n}{p} - m_{n-1} \right).$$

Then we can find H and G_n as in Corollary 7.16 such that $v_{r_n}(G_n - 1) > 0$, where $r_n = 1/m_n(p - 1)$. Thus all zeroes of G_n lie in the open disk $D(r_n)$.

Proof. This follows immediately from Proposition 7.17, substituting m_n and r_n for m'_n and r'_n . □

PROPOSITION 6.21. *There is a character χ_n whose (good) reduction is $\bar{\chi}_n$ such that $\mathbb{B}(\chi_n) \subseteq D$. If there is no $a \in \mathbb{Z}$ satisfying (23), then $\mathbb{B}(\chi_n) \subseteq D(r_n)$, where $r_n = 1/m_n(p - 1)$.*

Proof. The character χ_n will correspond to G_n , in the notation of Proposition 6.19. From that proposition, we have

$$(24) \quad \frac{G_n}{G_{\min}} H^{-p} = 1 + p^{p/(p-1)} \frac{F}{G_{\min} H^p} =: \tilde{F}.$$

Now, $v_0(\tilde{F} - 1) = p/(p - 1)$ and $[\tilde{F} - 1]_0 = f$, which is not a p -th power. Thus, Proposition 5.17 shows that if $\chi_{\tilde{F}} = \mathfrak{K}_1(\tilde{F}) \in H_p^1(\mathbb{K})$, then $\delta_{\chi_{\tilde{F}}}(0) = 0$ and the reduction $\bar{\chi}_{\tilde{F}}$ corresponds to the Artin-Schreier extension given by $y^p - y = f$. Thus if $\chi'_{\tilde{F}} = \mathfrak{K}_n(\tilde{F}^{p^{n-1}}) \in H_{p^n}^1(\mathbb{K})$, then we also have $\delta_{\chi'_{\tilde{F}}}(0) = 0$ and the reduction $\bar{\chi}'_{\tilde{F}}$ corresponds to the same extension, which is encoded by the Witt vector $(0, \dots, 0, f)$.

On the other hand, note that χ_{\min} corresponds to the element

$$G_1 G_2^p \cdots G_{n-1}^{p^{n-2}} G_{\min}^{p^{n-1}} \in \mathbb{K}^\times / (\mathbb{K}^\times)^{p^n},$$

whereas the character χ_n coming from G_n corresponds to the element

$$G_1 G_2^p \cdots G_{n-1}^{p^{n-2}} G_n^{p^{n-1}} \in \mathbb{K}^\times / (\mathbb{K}^\times)^{p^n}.$$

By (24), we have $\chi_{\min} \chi'_{\tilde{F}} = \chi_n$ as elements of $H_{p^n}^1(\mathbb{K})$. By Proposition 5.9(iii), we have that $\bar{\chi}_{\min} \bar{\chi}'_{\tilde{F}}$ is the reduction of χ_n . Since $\bar{\chi}'_{\tilde{F}}$ corresponds to the Witt vector $(0, \dots, 0, f)$, it follows from (22) that the reduction of χ_n is, in fact, $\bar{\chi}_n$. Since G_n is a polynomial of degree $\leq m_n - m_{n-1}$ in T^{-1} , we have that $|\mathbb{B}(\chi_n)| \leq m_n + 1$. By Corollary 5.13(i), we have equality, and thus χ_n has good reduction $\bar{\chi}_n$, proving the first assertion of the proposition. The second assertion then follows immediately from Proposition 6.20. □

Proposition 6.21 completes the proof of Part (B), and Theorem 3.4 follows immediately.

Remark 6.22. Example 7.18 shows that it is possible for the result of Proposition 6.20 not to hold when there is an $a \in \mathbb{Z}$ satisfying (23). (In

particular, we take $p = 5$, with $n = 3$, the ramification jumps $(m_1, m_2, m_3) = (1, 5, 34)$, and $a = 2$.) When this is the case, the branch locus of χ_n generated above does not lie in the disk $D(r_n)$.

7. Proofs of technical results

In this section, we give the proofs of Theorem 6.6, Lemma 6.12, and Propositions 6.19 and 6.20. In fact, we will prove Theorem 6.6 and Lemma 6.12 in somewhat more generality. All the proofs are related to each other and will share much notation.

Throughout Section 7, we will use notation parallel to that used in Section 6. Let (m_1, \dots, m_n) be a sequence of positive integers such that $p \nmid m_1$, that $m_i \geq pm_{i-1}$ for $1 \leq i \leq n$, and that if $p|m_i$, then $m_i = pm_{i-1}$. For $1 \leq i \leq n$, set $r_i = 1/m_i(p - 1)$. Write $N = m_n - m_{n-1}$. Let N_1 and N_2 be nonnegative integers such that $N_1 + N_2 = N$ and $p|N_2$. Lastly, let m be the minimal m_i such that m_{n-1} is a p -th power times m_i . Set $\nu = n - 1 - i$. Thus, $m_{n-1} = mp^\nu$.

7.1. *Arbitrary types.* We work under two helpful assumptions.

Assumption 7.1. There exist $a_0, a_1, \dots, a_{N_1} \in \mathbb{Z}$ and $\bar{x}_1, \dots, \bar{x}_{N_1} \in k^\times$ with $p|a_0$, with $0 < a_i < p$ for $i \geq 1$, and with $\bar{x}_i \neq \bar{x}_j$, such that the differential form

$$\omega := \frac{dg}{g} - m \sum_{j=0}^{\nu} t^{-mp^j-1} dt, \quad \text{with } g := t^{a_0} \prod_{i=1}^{N_1} (1 - \bar{x}_i t^{-1})^{a_i},$$

satisfies $\text{ord}_\infty(\omega) = m_n - N_2 - 1 = N_1 + m_{n-1} - 1$ (cf. (18)).

To formulate the second assumption we need more notation. We have, by (20), that

$$\frac{dg}{g} = \sum_{i=1}^{N_1} \frac{a_i \bar{x}_i t^{-2} dt}{1 - \bar{x}_i t^{-1}} = \sum_{\ell=0}^{\infty} \left(\sum_{i=1}^{N_1} a_i \bar{x}_i^{\ell+1} \right) t^{-\ell-2} dt.$$

Assumption 7.1 is therefore equivalent to the system of equations

$$\sum_i a_i \bar{x}_i^{\ell+1} = \begin{cases} m, & \ell = mp^j - 1, \quad 0 \leq j \leq \nu, \\ 0, & \text{otherwise} \end{cases}$$

for $\ell = 0, \dots, m_n - N_2 - 2 = N_1 + m_{n-1} - 2$. The Jacobi matrix of this system of equations at the point (\bar{x}_i) is the $(N_1 + m_{n-1} - 1, N_1)$ -matrix

$$\left((\ell + 1) a_i \bar{x}_i^\ell \right)_{\ell, i}$$

over k . The rows of this matrix corresponding to an index ℓ with $\ell \equiv -1 \pmod{p}$ vanish. Crossing out these trivial rows we obtain the matrix

$$\left((\ell + 1)a_i \bar{x}_i^\ell \right)_{\ell \not\equiv -1 \pmod{p}, i}$$

For our key result, Lemma 7.4, we need this matrix to be invertible; this is the case if and only if the matrix

$$(25) \quad A := \left(\bar{x}_i^\ell \right)_{\ell \not\equiv -1 \pmod{p}, i}$$

is invertible.

Assumption 7.2. The matrix A in (25) is invertible.

Remark 7.3. Note that a trivial necessary condition for Assumption 7.2 is that A is a square matrix. This is not in general true. For instance, if $m = 1$, so that $m_{n-1} = p^{n-2}$, then A is square if and only if

$$p^{n-1} - p^{n-2} - p < N_1 \leq p^{n-1} - p^{n-2}.$$

Fix g as in Assumption 7.1, and assume Assumption 7.2. In the field $K(T)$, set $\tilde{T} = p^{-r_{n-1}}T$. Let \mathcal{G}_n denote the family of polynomials in $R[T^{-1}]$ of the form

$$G = \prod_{i=1}^{N_1} (1 - x_i \tilde{T}^{-1})^{a_i}$$

such that x_i reduces to \bar{x}_i . In particular, for any $G \in \mathcal{G}_n$, we have $[G]_{r_{n-1}} = t^{-a_0}g$. Lastly, let $m' = N_1 + m_{n-1}$.

LEMMA 7.4. *Under Assumptions 7.1 and 7.2, let $G \in \mathcal{G}_n$, and let $J \in 1 + \tilde{T}^{-1}\mathfrak{m}[\tilde{T}^{-1}]$.*

- (i) *There exist a unique $G' \in \mathcal{G}_n$ and a unique polynomial $I \in 1 + \tilde{T}^{-p}\mathfrak{m}[\tilde{T}^{-p}]$ of degree $\leq m' - 1$ in \tilde{T}^{-1} such that*

$$\frac{G'}{G} I \equiv J \pmod{\tilde{T}^{-m'}}.$$

If $J \equiv 1 \pmod{p^\sigma, \tilde{T}^{-m'}}$ for $\sigma \in \mathbb{Q}_{>0}$, then $v_{r_{n-1}}(G'/G - 1) \geq \sigma$ and $v_{r_{n-1}}(I - 1) \geq \sigma$.

- (ii) *Let $0 < s < p/(p-1)$ be a rational number. After a possible finite extension of K , there exist $G' \in \mathcal{G}_n$ and a polynomial $H \in 1 + \tilde{T}^{-1}\mathfrak{m}[\tilde{T}^{-1}]$ such that we have*

$$\frac{G'}{G} H^p \equiv J \pmod{p^s, \tilde{T}^{-m'}}.$$

If $J \equiv 1 \pmod{p^\sigma, \tilde{T}^{-m'}}$ for some $0 < \sigma < p/(p-1)$, then we can choose G' and H such that $v_{r_{n-1}}(G'/G - 1) \geq \sigma$ and $v_{r_{n-1}}(H^p - 1) \geq \sigma$.

Proof. By assumption we have

$$G = \prod_{i=1}^{N_1} (1 - x_i \widetilde{T}^{-1})^{a_i},$$

where $x_i \in R$ is a lift of \bar{x}_i . We set

$$G' = \prod_{i=1}^{N_1} (1 - x'_i \widetilde{T}^{-1})^{a_i}, \quad x'_i := x_i + z_i,$$

where the z_i are for the moment considered as indeterminates. We also set

$$I := 1 + \sum_{j=1}^{[(m'-1)/p]} b_j \widetilde{T}^{-pj}$$

for another system of indeterminates b_j . Write

$$\frac{G'}{G} I = 1 + \sum_{\ell=1}^{\infty} c_\ell \widetilde{T}^{-\ell},$$

where c_ℓ is a formal power series in (z_i, b_j) . A simple computation shows that

$$(26) \quad \begin{aligned} \frac{\partial c_\ell}{\partial z_i} \Big|_{z_i=b_j=0} &= a_i x_i^{\ell-1}, \\ \frac{\partial c_\ell}{\partial b_j} \Big|_{z_i=b_j=0} &= \begin{cases} 1, & \ell = pj, \\ 0, & \ell \neq pj. \end{cases} \end{aligned}$$

The congruence

$$(27) \quad \frac{G'}{G} I \equiv J \pmod{\widetilde{T}^{-m'}}$$

corresponds to a system of $m' - 1$ equations (one equation for each monomial $c_\ell \widetilde{T}^{-\ell}$, $\ell = 1, \dots, m' - 1$) in the indeterminates (z_i, b_j) . The Jacobi matrix of this system of equations is invertible over R if and only if its reduction is invertible over k . From (26) it is easy to see that this is true if and only if the matrix A from (25) is invertible, which is the case by Assumption 7.2. We conclude that (27) has a unique solution with $z_i, b_j \in \mathfrak{m}$. In fact, by the effective Hensel's Lemma, $v(z_i)$ and $v(b_j)$ are all at least as large as $v_{r_{n-1}}(J-1)$. This proves (i).

To prove (ii), we will build G' and H through successive approximation. Let G'_1 and

$$I_1 = 1 + \sum_{j=1}^{[(m'-1)/p]} b_{j,1} \widetilde{T}^{-pj}$$

be the unique solution guaranteed by (i). So $(G'_1/G)I_1 \equiv J \pmod{\widetilde{T}^{-m'}}$. Set

$$H_1 := 1 + \sum_{j=1}^{[(m'-1)/p]} b_{j,1}^{1/p} \widetilde{T}^{-j}$$

for any choice of p -th roots, and set $J_1 := (G'_1/G)H_1^p$. Since $H_1^p \equiv I_1 \pmod{p}$, we have that $J/J_1 \equiv 1 \pmod{p, \widetilde{T}^{-m'}}$. Now, let G'_2 and

$$I_2 = 1 + \sum_{j=1}^{[(m'-1)/p]} b_{j,2} \widetilde{T}^{-pj}$$

be the unique solution to

$$\frac{G'_2}{G'_1} I_2 \equiv \frac{J}{J_1} \pmod{\widetilde{T}^{-m'}}$$

guaranteed by (i). Note that since the coefficients of $J/J_1 \pmod{\widetilde{T}^{-m'}}$ have valuation at least 1, part (i) gives that $v(b_{j,2}) \geq 1$ for all j . Let

$$H_2 := 1 + \sum_{j=1}^{[(m'-1)/p]} b_{j,2}^{1/p} \widetilde{T}^{-j}.$$

Then $H_2^p \equiv I_2 \pmod{p^{1+1/p}}$. Thus

$$(28) \quad \frac{G'_2}{G} (H_1 H_2)^p = \frac{G'_2}{G'_1} H_2^p \cdot \frac{G'_1}{G} H_1^p \equiv \frac{J}{J_1} \cdot J_1 \equiv J \pmod{p^{1+1/p}, \widetilde{T}^{-m'}}.$$

We can repeat this process for all $i \in \mathbb{N}$, letting G'_i and I_i be the unique solution to

$$\frac{G'_i}{G'_{i-1}} I_i \equiv \frac{J}{J_{i-1}} \pmod{\widetilde{T}^{-m'}}$$

guaranteed by (i), and constructing H_i from I_i in the same manner as H_1 and H_2 . Set

$$J_i = \frac{G'_i}{G} (H_1 \cdots H_i)^p.$$

If $\gamma_i = \sum_{j=1}^{i-1} 1/p^j$, then

$$\frac{G'_i}{G'_{i-1}} H_i^p \equiv \frac{J}{J_{i-1}} \pmod{p^{\gamma_i}}.$$

Analogously to (28), one derives

$$\frac{G'_i}{G} (H_1 \cdots H_i)^p = \frac{G'_i}{G'_{i-1}} H_i^p \cdot \frac{G'_{i-1}}{G} (H_1 \cdots H_{i-1})^p \equiv J \pmod{p^{\gamma_i}, \widetilde{T}^{-m'}}.$$

Since $s < p/(p-1)$, we have that $\gamma_i > s$ for large enough i . Setting $G' = G'_i$ and $H = (H_1 \cdots H_i)$ for such an i gives the desired solution to part (ii).

To check the last statement of part (ii), we show that $v_{r_{n-1}}(H_i^p - 1) \geq \sigma$ for all $i \in \mathbb{N}$. Since $v_{r_{n-1}}(I_1 - 1) \geq \sigma$ by (i), we have $v_{r_{n-1}}(H_1^p - 1) \geq \sigma$. (Here

we use that $\sigma < p/(p - 1)$ to estimate the cross terms.) Since part (i) also shows $v_{r_{n-1}}(G'_1/G - 1) \geq \sigma$, we see that $v_{r_{n-1}}(J_1 - 1) \geq \sigma$. An easy induction now shows that $v_{r_{n-1}}(H_i^p - 1) \geq \sigma$ for all i . \square

COROLLARY 7.5. *Under Assumptions 7.1 and 7.2, let $G \in \mathcal{G}_n$, let $r \in \mathbb{Q} \cap (0, r_{n-1})$, and let $f \in t^{-1}k[t^{-1}]$ be a polynomial of degree $< m' = m_{n-1} + N_1$ without constant term. Fix some $s \in \mathbb{Q}$ with*

$$m'(r_{n-1} - r) \leq s \leq \frac{p}{p - 1}.$$

Then there exists $G' \in \mathcal{G}_n$, a polynomial $H \in 1 + T^{-1}R[T^{-1}]$, and $F \in \mathbb{K}$ such that $(G'/G)H^p = 1 - p^s F$ with $v_r(F) = 0$ and $[F]_r = f$.

Proof. Let F' be a polynomial in T^{-1} of the same degree as f such that $v_r(F') = 0$ and $[F']_r = f$. Now, $v_{r_{n-1}}(p^s F') = s - \deg(f)(r_{n-1} - r)$, which is positive by our assumptions. Choose σ such that $s - (r_{n-1} - r) < \sigma < p/(p - 1)$. Then Lemma 7.4(ii) yields G' and H such that

$$\frac{G'}{G} H^p \equiv 1 - p^s F' \pmod{p^\sigma, \widetilde{T}^{-m'}}.$$

Lemma 7.4(ii) also allows us to assume that

$$v_{r_{n-1}}\left(\frac{G'}{G} H^p - 1\right) \geq s - \deg(f)(r_{n-1} - r).$$

If F is such that $(G'/G)H^p = 1 - p^s F$, then it suffices to show that $v_r(F - F') > 0$. Write $p^s F - p^s F'$ as a power series $\sum_{j=0}^\infty \alpha_j \widetilde{T}^{-j}$. For $j < m'$, we have $v(\alpha_j) \geq \sigma$. For $j \geq m'$, we have $v(\alpha_j) \geq s - \deg(f)(r_{n-1} - r)$. In both cases, we have $v(\alpha_j) + j(r_{n-1} - r) > s$, which shows that $v_r(p^s F - p^s F') > s$, and thus $v_r(F - F') > 0$. \square

7.2. Specialization to the context of Theorem 6.6 and Lemma 6.12. Maintain the notation of Section 7.1. In this section, we show that Assumptions 7.1 and 7.2 are satisfied in the situations of Section 6. In particular, we want to show that there is a g satisfying Assumption 7.1 in the form

$$(29) \quad g = t^{-N_2} \prod_{i=1}^{N_1} (1 - \bar{x}_i t^{-1}).$$

Such a solution (where $a_0 = -N_2$ and $a_i = 1$ for $i > 1$) is called a solution with *simple type*. In the situation where we only look for solutions with simple type, (21) can be rewritten as

$$(30) \quad \omega := \frac{dg}{g} - m \sum_{j=0}^\nu t^{-mp^j - 1} dt = \frac{cdt}{t^{N+m_{n-1}+1}g},$$

where $c \neq 0$. Recall that $N = m_n - m_{n-1} \geq m_{n-1}(p - 1)$ and that $m_n \not\equiv 0 \pmod{p}$ if the inequality is strict.

LEMMA 7.6. *Suppose g is of simple type as in (29). Then g satisfies (30) (and thus (21)) if and only if it satisfies Assumption 7.1.*

Proof. The “only if” direction is immediate from the right-hand side of (30). To prove the “if” direction, suppose that g is of simple type and satisfies Assumption 7.1. Then ω has a zero of order $m_{n-1} + N_1 - 1$ at ∞ , simple poles at all the N_1 zeroes of g , a pole of order $m_{n-1} + 1$ at 0, and no other poles. Thus, ω can have no other zeroes. So the differential forms on each side of (30) have the same divisor, and we can choose c to get our desired equality. \square

THEOREM 7.7. *Assume that $m = 1$; i.e.,*

$$(m_1, \dots, m_{n-1}) = (1, p, \dots, p^{n-2} = p^\nu).$$

Then (30) has a unique solution $g \in k(t)$. This solution is of the form (29), with pairwise distinct $\bar{x}_i \in k$, and where

$$p^{\nu+1} - p^\nu - p < N_1 \leq p^{\nu+1} - p^\nu, \quad N_1 \equiv N \pmod{p}.$$

Proof. We rewrite (30) as the nonhomogenous linear differential equation

$$(31) \quad dg - g \cdot \left(\sum_{j=0}^{\nu} t^{-p^j-1} dt \right) = ct^{-N-p^\nu-1} dt,$$

and we first look for solutions in $k((t))$ of the form $g = \sum_{i=-N}^{\infty} \alpha_i t^i$. We obtain a system of linear equations in the coefficients α_i . In degree $-N - p^\nu - 1$, we obtain $\alpha_{-N} = -c$, and for $i \geq 1$, we get a linear expression for α_{-N+i} in terms of $\alpha_{-N}, \dots, \alpha_{-N+i-1}$. In other words, (31) has a unique solution of the form $g = -ct^{-N} + \dots \in k((t))$. We also observe that the linear equations become periodic in i , as soon as $i \geq p^\nu$, which means that the coefficients of g (which only take values in \mathbb{F}_p) are also eventually periodic. This means that g is actually a *rational* function in t , i.e., $g \in k(t)$ (as if P is the period of the coefficients of g , then one can write an equation relating g and $t^P g$). Now we can use (30) to see that

$$(32) \quad \text{ord}_0(g) = -N, \quad \text{ord}_{\bar{x}}(g) \in \{0, 1\}, \quad \bar{x} \neq 0, \infty.$$

It follows that $g \in k[t, t^{-1}]$ is a Laurent polynomial with leading term $-ct^{-N}$ with only simple zeroes outside of $t = 0, \infty$. Set

$$(33) \quad N_2 := \text{ord}_{\infty}(g)$$

and $N_1 := N - N_2$. Then g is of the form (29). It remains to see that N_1 is as claimed in the theorem.

Let ω denote the differential form in (30). By (33) we have

$$(34) \quad \text{ord}_{\infty}(\omega) = p^\nu - 1 + N_1 \geq p^\nu - 1 \geq 0,$$

which implies that $\text{res}_\infty(\omega) = \text{res}_\infty(dg/g) = 0$. We conclude that $N_2 \equiv 0 \pmod{p}$ and hence $N_1 \equiv N \pmod{p}$. It remains to prove the inequality for N_1 stated in the theorem.

First assume $\nu = 0$. Then $\text{ord}_\infty(\omega) = N_1$. Also, $\mathcal{C}(\omega) = dg/g = \omega + t^{-2}dt$. Thus, either $N_1 = 0$ or $\text{ord}_\infty \mathcal{C}(\omega) = 0$. But it is easy to see that if $\text{ord}_\infty \mathcal{C}(\omega) = 0$ and $\text{ord}_\infty(\omega) \geq 0$, then $\text{ord}_\infty \omega < p$. Thus $0 \leq N_1 < p$, which is exactly the condition of the theorem.

Now assume $\nu \geq 1$. Set

$$h := t^{-N_2}g^{-1} = \prod_{i=1}^{N_1} (1 - \bar{x}_i t^{-1})^{-1}.$$

Since $N_2 \equiv 0 \pmod{p}$, h satisfies the differential equation

$$(35) \quad dh + h \cdot \left(\sum_{j=0}^{\nu} t^{-p^j-1} dt \right) = ct^{-N_1-p^\nu-1} h^2 dt,$$

which is derived from (31). If we write $h = 1 + b_1 t^{-1} + b_2 t^{-2} + \dots$ and plug this into (35), we see that the coefficients b_1, \dots, b_{p-1} satisfy the simple linear equations

$$b_1 = 1, \quad 2b_2 = b_1, \dots, \quad (p-1)b_{p-1} = b_{p-2}.$$

We conclude that $b_i = 1/i! \neq 0$ for $i = 0, 1, \dots, p-1$. Write $N_1 = pM - a$ with $0 \leq a < p$ and $M \in \mathbb{Z}$. Then

$$\omega = ct^{-N_1-p^\nu-1} h dt = c(t^{-M-p^{\nu-1}})^p (t^a + b_1 t^{a-1} + \dots) \frac{dt}{t},$$

from which we see that

$$\mathcal{C}(\omega) = c^{1/p} \cdot t^{-M-p^{\nu-1}} (b_a^{1/p} + b_{p+a}^{1/p} t^{-1} + \dots) \frac{dt}{t}.$$

Since $b_a \neq 0$, it follows that

$$(36) \quad \text{ord}_\infty \mathcal{C}(\omega) = M + p^{\nu-1} - 1.$$

In particular, this and (34) show that $\text{ord}_\infty \mathcal{C}(\omega) < \text{ord}_\infty \omega$. Therefore, the equality

$$\mathcal{C}(\omega) = \omega + t^{-p^\nu-1} dt$$

shows that

$$(37) \quad \text{ord}_\infty \mathcal{C}(\omega) = p^\nu - 1.$$

Combining (36) and (37), we conclude that $M = p^\nu - p^{\nu-1}$ and hence

$$N_1 = pM - a = p^{\nu+1} - p^\nu - a.$$

This completes the proof of the theorem. □

COROLLARY 7.8. *If $m|N$, then (30) has a unique solution $g(t) \in k(t)$. In this case, the solution is given by $g(t) = h(t^m)$, where $h(t)$ is the solution to (31) with N replaced by $N' := N/m$ and c replaced by $c' := c/m$. In particular, $g(t) \in k[t^{-1}]$ is of simple type and of degree N in t^{-1} . Furthermore, $m|N_1$ and we have*

$$m_{n-1}(p-1) - mp < N_1 \leq m_{n-1}(p-1), \quad N_1 \equiv N \pmod{p}.$$

Proof. Equation (30) can be rewritten as

$$(38) \quad dg - g \cdot m \cdot \left(\sum_{j=0}^{\nu} t^{-mp^j-1} dt \right) = ct^{-N-m_{n-1}-1} dt.$$

Since $g(t) = h(t^m)$, we have $dg = mt^{m-1}dh(t^m)$. By definition, h satisfies

$$(39) \quad dh - h \cdot \left(\sum_{j=0}^{\nu} t^{-p^j-1} dt \right) = c't^{-N'-p^{\nu}-1} dt.$$

Multiplying (39) by t , substituting t^m for t on both sides, and then multiplying by m/t yields (30). This shows that $g(t)$ is as in the corollary. Then g is unique because its Laurent series coefficients can be calculated by recursion. The properties of N_1 follow easily. □

Remark 7.9. Suppose $g \in k[t^{-1}]$ is a solution to (30), thus (38). Let \bar{c}_i be the coefficient of t^{-i} in g (with $\bar{c}_i = 0$ if $i < 0$). Then (38) in degree $-i-1$ gives the equation

$$(40) \quad i\bar{c}_i + m(\bar{c}_{i-m} + \bar{c}_{i-pm} + \dots + \bar{c}_{i-p^{\nu}m}) = \begin{cases} -c & i = N + m_{n-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 7.8 proves Theorem 6.6. Since $m|m_{n-1}$ by definition and $N = m_n - m_{n-1}$, we have that $m_n = pm_{n-1}$ implies $m|N$ and thus that the solution g to (30) guaranteed by Corollary 7.8 satisfies Assumption 7.1.

LEMMA 7.10. *If g is a solution to (30) with simple type, and we further assume that $m_n = pm_{n-1}$, then Assumption 7.2 holds as well.*

Proof. Corollary 7.8 shows that $N_1 = m_{n-1}(p-1)$. One then sees that the matrix A in (25) is square. We show that $\ker(A) = 0$. Suppose $\vec{v} \in \ker(A)$ is nonzero, and let \vec{x} be the vector $(\bar{x}_1, \dots, \bar{x}_n)$. Then, if $\varepsilon^2 = 0$, the vector $\vec{x} + \varepsilon\vec{v}$ must also satisfy Assumption 7.1, and thus, by Lemma 7.6, $\vec{x} + \varepsilon\vec{v}$ must satisfy (38) (equivalent to (30)). Now, we claim that replacing \vec{x} with $\vec{x} + \varepsilon\vec{v}$ replaces g with $g + \varepsilon h(\vec{v})$, where $h(\vec{v})$ is a nonzero polynomial in t^{-1} . Given the claim, equation (38) then implies that $dh - h \cdot m \cdot (\sum_{j=0}^{\nu} t^{-mp^j-1} dt) = 0$

or that

$$\frac{dh}{h} = m \left(\sum_{j=0}^{\nu} t^{-mp^j - 1} dt \right).$$

But the right-hand side is not logarithmic, as it has a multiple pole at $t = 0$. This is a contradiction.

To prove the claim, we must show that $h(\vec{v})$ is nonzero. If \vec{e}_j is the j -th standard basis vector, then $h(\vec{e}_j) = t^{-N_2 - 1} \left(\prod_{i=1}^{N_1} (1 - \bar{x}_i t^{-1}) \right) / (1 - \bar{x}_j t^{-1})$. In particular, \bar{x}_i is a zero of $h(\vec{e}_j)$ for all $i \neq j$, but not when $i = j$. But then no $h(\vec{e}_j)$ can be a linear combination of the $h(\vec{e}_i)$ with $i \neq j$, because that would imply that \bar{x}_j is a zero of $h(\vec{e}_j)$. Thus $h(\vec{v}) \neq 0$ for $\vec{v} \neq 0$. \square

COROLLARY 7.11. *Lemma 6.12 holds.*

Proof. If $m_n = pm_{n-1}$, then Corollary 7.8 and Lemma 7.10 guarantee that there is a unique g satisfying (30) (thus Assumption 7.1) and Assumption 7.2, and that $N_2 = 0$. By Lemma 7.6, the family \mathcal{G}_n from Section 7.1 (in particular, Corollary 7.5) is the same as the family defined in Definition 6.7. Then Lemma 6.12 follows from Corollary 7.5, noting that since $N = N_1 = m_{n-1}(p - 1)$, we have $m' = m_n = pm_{n-1}$ in Corollary 7.5. \square

Remark 7.12. Suppose $m = 1$, so that $m_{n-1} = p^{\nu-2}$. Then the condition $p^{n-1} - p^{n-2} - p < N_1 \leq p^{n-1} - p^{n-2}$ from Theorem 7.7 is exactly the same as the necessary condition for the matrix A in (25) to be square (Remark 7.3), and thus it is necessary in order for Assumption 7.2 to be satisfied. So if g satisfies Assumption 7.1, even if g does not have simple type, we still must have the same bounds on N_1 in order to proceed with any proof that uses Assumption 7.2, which is essential for our proof of Lemma 6.12. The importance of bounds on N_1 was shown in Example 6.9 and Remark 6.10.

7.3. *An invertible matrix.* We give a result (Lemma 7.14) that will be used in Section 7.4. Assume that $m_n = pm_{n-1}$ and that g is a simple type solution to (30). By Corollary 7.8, we have $N_1 = N = m_{n-1}(p - 1)$ and $N_2 = 0$. Maintain the notation \mathcal{G}_n from Section 7.1. Recall that $\tilde{T} = p^{-r_{n-1}}T$. Recall also that if $\Omega \in \mathcal{G}_n$, then $\Omega \in 1 + \tilde{T}^{-1}R[\tilde{T}^{-1}]$ has degree $m_{n-1}(p - 1)$ and simple zeroes.

Let $V \cong R^{m_{n-1}}$ be the free R -module of polynomials of the form

$$\sum_{j=1}^{m_{n-1}} b_j \tilde{T}^{-pj} \quad (b_j \in R),$$

and let $W \cong R^{m_{n-1}}$ be the free R -module of polynomials of the form

$$\sum_{j=m_{n-1}(p-1)+1}^{pm_{n-1}} a_j \tilde{T}^{-j} \quad (a_j \in R).$$

For $\Omega \in \mathcal{G}_n$, consider the linear map $L : V \rightarrow W$ defined by

$$L_\Omega(J) = (\Omega J)^{\text{tr}},$$

where $(\cdot)^{\text{tr}}$ truncates a polynomial in \widetilde{T}^{-1} to preserve only the terms of degree $m_{n-1}(p-1) + 1$ through pm_{n-1} , inclusive. Let $A_\Omega \in M_{m_{n-1}}(R)$ be the matrix of this linear map, relative to the input basis $\widetilde{T}^{-p}, \dots, \widetilde{T}^{-pm_{n-1}}$ and the output basis $\widetilde{T}^{-(m_{n-1}(p-1)+1)}, \dots, \widetilde{T}^{-pm_{n-1}}$. If c_i is the coefficient of \widetilde{T}^{-i} in Ω (with $c_i = 0$ for $i < 0$), then the matrix A_Ω is given by

$$(41) \quad A_\Omega = \begin{pmatrix} c_{m_{n-1}(p-1)-p+1} & c_{m_{n-1}(p-1)-2p+1} & \cdots & c_{-m_{n-1}+1} \\ c_{m_{n-1}(p-1)-p+2} & c_{m_{n-1}(p-1)-2p+2} & \cdots & c_{-m_{n-1}+2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{pm_{n-1}-p} & c_{pm_{n-1}-2p} & \cdots & c_0 \end{pmatrix}.$$

LEMMA 7.13. *For any $\Omega \in \mathcal{G}_n$, the matrix A_Ω above lies in $\text{GL}_{m_{n-1}}(R)$.*

Proof. We note that A_Ω lying in $\text{GL}_{m_{n-1}}(R)$ is equivalent to the reduction \overline{A}_Ω of A_Ω lying in $\text{GL}_{m_{n-1}}(k)$, which is equivalent to the linear transformation given by \overline{A}_Ω being surjective. Since \overline{A}_Ω does not depend on the choice of $\Omega \in \mathcal{G}_n$, it suffices to show that, for any $w \in W$, there exists *some* $\Omega \in \mathcal{G}_n$ and some $v \in V$ such that $L_\Omega(v) = w$. Consequently, it suffices to exhibit an R -basis $w_1, \dots, w_{m_{n-1}}$ of W such that each w_i is in the image of some L_Ω .

Take any $G \in \mathcal{G}_n$, and let

$$w'_i = (\widetilde{T}^{-(m_{n-1}(p-1)+i)}G)^{\text{tr}}, \quad 1 \leq i \leq m_{n-1}.$$

One easily sees that the w'_i form an R -basis of W . If $i = m_{n-1}$, take $w_{m_{n-1}} = w'_{m_{n-1}} = \widetilde{T}^{-pm_{n-1}}$. Then, $w_i = L_\Omega(\widetilde{T}^{-pm_{n-1}})$, regardless of Ω .

Now, suppose $1 \leq i < m_{n-1}$. For any small $\varepsilon \in \mathbb{Q}_{>0}$, let $r \in \mathbb{Q}$ be such that

$$(m_{n-1} - i)(r_{n-1} - r) = \varepsilon.$$

(We choose ε small enough so that $r > 0$.) If $s = pm_{n-1}(r_{n-1} - r)$, then the monomial

$$F = p^{(m_{n-1}(p-1)+i)(r-r_{n-1})}\widetilde{T}^{-(m_{n-1}(p-1)+i)}$$

satisfies $[F]_r = t^{-(m_{n-1}(p-1)+i)}$ and $(p^sFG)^{\text{tr}} = p^\varepsilon w'_i$. By Lemma 7.4(i) with $J = 1 + p^sF$, there is a unique solution (G', I) to

$$G'I - G \equiv p^sFG \pmod{\widetilde{T}^{-pm_{n-1}}},$$

where $G' \in \mathcal{G}_n$ and I is of the form $1 + \sum_{j=1}^{m_{n-1}-1} b_j \widetilde{T}^{-pj}$, with all $b_j \in \mathfrak{m}$. Then $L_{G'}(I - 1) = p^\varepsilon w'_i + cw_{m_{n-1}}$, with $c \in R$ and $I - 1 \in V$. By continuity and the linearity of $L_{G'}$, it follows that there exists $c' \in R$ and $v \in V$ such that

$L_{G'}(v) = w'_i + c'w_{m_{n-1}}$. If $w_i = w'_i + c'w_{m_{n-1}}$, then w_i is in the image of $L_{G'}$. Since $w_1, \dots, w_{m_{n-1}}$ form a basis of W , we are done. \square

LEMMA 7.14. *Let $\alpha = (m_n - p[m_n/p])$. Let $C \in M_{m_{n-1}}(R)$ be the matrix with $C_{ij} = c_{m_{n-1}(p-1)-pj+i+\alpha}$. Then $C \in \text{GL}_{m_{n-1}}(R)$.*

Proof. It suffices to show that the reduction \bar{C} of C to $M_{m_{n-1}}(k)$ is invertible. For all j , let \bar{c}_j be the reduction of c_j to k . These are the coefficients of t^{-j} in g (see Remark 7.9).

By Lemma 7.13, the reduction \bar{A} of A_Ω in (41) to $M_{m_{n-1}}(k)$ is invertible. Notice that \bar{A} satisfies $\bar{A}_{ij} = \bar{c}_{m_{n-1}(p-1)-pj+i}$. Thus, up to reordering rows, the matrix \bar{C} is derived from the matrix \bar{A} by replacing each of the first α rows R_1, \dots, R_α of \bar{A} with rows R'_1, \dots, R'_α such that each entry \bar{c}_ℓ in R_i is replaced by $\bar{c}_{\ell+m_{n-1}}$ in R'_i .

Now, setting $i = l + m_{n-1}$ in (40), we obtain that the c_l satisfy the recursion

$$(42) \quad \bar{c}_{\ell+m_{n-1}} = -\frac{m}{\ell + m_{n-1}}(\bar{c}_{\ell+m_{n-1}-m} + \bar{c}_{\ell+m_{n-1}-pm} + \dots + \bar{c}_{\ell+m_{n-1}-p^\nu m}),$$

so long as $p \nmid \ell + m_{n-1}$. Since $0 \leq \alpha < p$, we see that in the first α rows of \bar{A} , no \bar{c}_l appears with $p \mid \ell + m_{n-1}$, so (42) holds. All entries of R_1 are of the form \bar{c}_ℓ , with $\ell \equiv 1 - m_{n-1} \pmod{p}$, so (42) shows that R'_1 is a linear combination of the rows of \bar{A} , where the coefficient of R_1 is the unit $-m/1$. So replacing R_1 with R'_1 gives a matrix \bar{A}_1 such that $\det(\bar{A}_1) = -m \det(\bar{A})$. In particular, \bar{A}_1 is invertible. For the same reasons, replacing the row R_2 of \bar{A}_1 with R'_2 gives a matrix \bar{A}_2 that is again invertible. Repeating this process a total of α times yields an invertible matrix \bar{A}_α . Since \bar{C} is obtained from \bar{A}_α by reordering rows, it is also invertible. \square

7.4. *Proof of Propositions 6.19 and 6.20.* Maintain the notation and assumptions of Section 7.3. Let $G_{\min} \in \mathcal{G}_n$. Note that $G_{\min} \in R[\widetilde{T}^{-1}]$ has degree $m_n - m_{n-1} = m_{n-1}(p-1)$. Let $m'_n \geq m_n = pm_{n-1}$, with m'_n prime to p unless $m'_n = m_n$. Let $F \in T^{-1}R[T^{-1}]$ have degree $\leq m'_n$.

In order to prove Proposition 6.19, we begin by trying to find polynomials I and G'_n in $R[T]$ such that

$$(43) \quad G_{\min}I - G'_n \equiv -p^{p/(p-1)}F \pmod{p^{p/(p-1)+\varepsilon}R[T^{-1}]}$$

for some $\varepsilon > 0$, with I having only monomials of degree divisible by p and G'_n having degree at most $m'_n - m_{n-1}$. Consider instead the congruence

$$(44) \quad G_{\min}I - G'_n \equiv -p^{p/(p-1)}F \pmod{\widetilde{T}^{-m'_n-1}K[\widetilde{T}^{-1}]}.$$

Since F has degree at most m'_n in \tilde{T}^{-1} , we look for I of the form

$$I = \sum_{i=0}^{\lfloor m'_n/p \rfloor} b_i \tilde{T}^{-pi}.$$

Being able to choose G'_n affords us some freedom; we may choose the coefficients b_i for $i \leq m'_n/p - m_{n-1}$ at will. This is because the terms in $G_{\min}I$ to which these b_i contribute have degree at most $m'_n - m_{n-1}$. As in Section 7.3, write $G_{\min} = \sum_{i=0}^{m_{n-1}(p-1)} c_i \tilde{T}^{-i}$, with $c_i = 0$ for i not between 0 and $m_{n-1}(p-1)$.

PROPOSITION 7.15. *After a possible finite extension of K , there exists a solution (I, G'_n) to the equation*

$$G_{\min}I - G'_n \equiv -p^{p/(p-1)}F \pmod{\tilde{T}^{-m'_n-1}K[\tilde{T}^{-1}]}$$

as described above with coefficients in K . Furthermore, a solution exists with $b_0 = 1$, $b_i = 0$ for $1 \leq i \leq m'_n/p - m_{n-1}$, and

$$v(b_i) \geq \frac{p}{p-1} - \frac{m'_n}{m_{n-1}(p-1)}$$

for all other i .

Proof. We take $b_0 = 1$ and $b_i = 0$ for $1 \leq i \leq m'_n/p - m_{n-1}$. We rewrite (43) as

$$G_{\min}(I - 1) - G'_n \equiv p^{p/(p-1)}F - G_{\min} \pmod{\tilde{T}^{-m'_n-1}K[\tilde{T}^{-1}]}.$$

This is a system of $m'_n + 1$ equations (corresponding to the terms of degree 0 through m'_n) in $m'_n + 1$ variables (corresponding to the coefficients c_0 through $c_{m'_n - m_{n-1}}$ of G'_n and $b_{\lfloor m'_n/p \rfloor - m_{n-1} + 1}$ through $b_{\lfloor m'_n/p \rfloor}$ of I). Let E be the matrix representing this system. We will show that $E \in \text{GL}_{m'_n+1}(R)$. Let the columns of E correspond to the variables just listed, and let the rows of E correspond to degrees 0 through m'_n , in order. Then E can be written as a block matrix,

$$(45) \quad E = \left(\begin{array}{c|c} -I_{m'_n - m_{n-1} + 1} & B \\ \hline 0 & C \end{array} \right),$$

where $I_{m'_n - m_{n-1} + 1}$ is the identity matrix of size $m'_n - m_{n-1} + 1$. Since the entries of B and C lie in R , being coefficients of G_{\min} , we see that showing that $E \in \text{GL}_{m'_n+1}(R)$ is equivalent to showing that $C \in \text{GL}_{m_{n-1}}(R)$. Now, the ij -th entry of C is c_ℓ , with

$$\begin{aligned} \ell &= m'_n - m_{n-1} + i - p \left(\left\lfloor \frac{m'_n}{p} \right\rfloor - m_{n-1} + j \right) \\ &= m_{n-1}(p-1) - pj + i + \left(m'_n - p \left\lfloor \frac{m'_n}{p} \right\rfloor \right). \end{aligned}$$

By Lemma 7.14, the matrix C lies in $\text{GL}_{m_{n-1}}(R)$, thus so does E .

Lastly, we show that we have

$$v(b_i) \geq \frac{p}{p-1} - \frac{m'_n}{m_{n-1}(p-1)}$$

for all $i > m'_n/p - m_{n-1}$. Since $F \in R[T^{-1}]$, the coefficient of \tilde{T}^{-i} in $-p^{p/(p-1)}F$ has valuation at least $p/(p-1) - i/m_{n-1}(p-1)$. Since F has degree at most m'_n in \tilde{T}^{-1} , all coefficients of $-p^{p/(p-1)}F$ (in terms of \tilde{T}) have valuation at least

$$\frac{p}{p-1} - \frac{m'_n}{m_{n-1}(p-1)}.$$

The b_i in question are given as entries of the column vector $E^{-1}\vec{v}$, where \vec{v} is the column vector whose entries are the coefficients of $1, \tilde{T}^{-1}, \dots, \tilde{T}^{-m'_n}$ in $-p^{p/(p-1)}F$. Since E^{-1} has entries in R , we are done. \square

We now prove Proposition 6.19.

COROLLARY 7.16. *Let*

$$I = \sum_{i=0}^{\lfloor m'_n/p \rfloor} b_i \tilde{T}^{-pi}$$

be the solution to (44) found in Proposition 7.15. If

$$H = \sum_{i=0}^{\lfloor m'_n/p \rfloor} b_i^{1/p} \tilde{T}^{-i},$$

for any choice of p -th roots, then there exists (after a possible finite extension of K) $G_n \in R[T^{-1}]$ of degree at most $m'_n - m_{n-1}$ such that (H, G_n) is a solution to

$$G_{\min}H^p - G_n \equiv -p^{p/(p-1)}F \pmod{p^{p/(p-1)+\varepsilon}R[T^{-1}]}$$

for some $\varepsilon \in \mathbb{Q}_{>0}$. Furthermore, H and G_n lie in $1 + T^{-1}\mathfrak{m}[T^{-1}]$.

Proof. Working in terms of \tilde{T} , we have that $H^p - I = \sum_{i=1}^{\lfloor m'_n/p \rfloor} pa_i \tilde{T}^{-i}$, where for all i ,

$$v(a_i) \geq \min_j(v(b_j)) \geq \frac{p}{p-1} - \frac{m'_n}{m_{n-1}(p-1)}.$$

Since $G_{\min} \in R[\tilde{T}^{-1}]$, it follows that

$$G_{\min}(H^p - I) \equiv \sum_{i=1}^{m'_n} e_i \tilde{T}^{-i} \pmod{\tilde{T}^{-m'_n-1}R[\tilde{T}^{-1}]},$$

where

$$v(e_i) \geq 1 + \frac{p}{p-1} - \frac{m'_n}{m_{n-1}(p-1)}$$

for all i . Set $G_n = G'_n + \sum_{i=1}^{m'_n - m_{n-1}} e_i \widetilde{T}^{-i}$, where G'_n is as in Proposition 7.15. We must show that $G_{\min} H^p - G_n \equiv -p^{p/(p-1)} F \pmod{p^{p/(p-1)+\varepsilon} R[T^{-1}]}$ for small enough $\varepsilon \in \mathbb{Q}_{>0}$.

If $0 \leq i \leq m'_n - m_{n-1}$, then the terms involving \widetilde{T}^{-i} (and thus T^{-i}) agree exactly for $G_{\min} H^p - G_n$ and $-p^{p/(p-1)} F$ by construction.

If $m'_n - m_{n-1} + 1 \leq i \leq m'_n$, then the terms involving \widetilde{T}^{-i} (or T^{-i}) agree exactly for $G_{\min} I$ and $-p^{p/(p-1)} F$. (G_n and G'_n have no terms of these degrees.) So the coefficient of \widetilde{T}^{-i} in $G_{\min} H^p - G_n + p^{p/(p-1)} F$ is e_i , which has valuation at least

$$1 + \frac{p}{p-1} - \frac{m'_n}{m_{n-1}(p-1)}.$$

Thus the coefficient of T^{-i} in $G_{\min} H^p - G_n + p^{p/(p-1)} F$ has valuation at least

$$1 + \frac{p}{p-1} - \frac{m'_n}{m_{n-1}(p-1)} + \frac{i}{m_{n-1}(p-1)} > \frac{p}{p-1} + \frac{p-2}{p-1} \geq \frac{p}{p-1},$$

so the desired congruence holds for these terms.

For $i > m'_n$, the coefficient of \widetilde{T}^{-i} in $G_{\min} H^p - G_n + p^{p/(p-1)} F$ (which is the same as in $G_{\min} H^p$, as F is of degree $\leq m'_n$) has valuation at least $p/(p-1) - m'_n/m_{n-1}(p-1)$. So the corresponding coefficient of T^{-i} has valuation at least

$$\frac{p}{p-1} - \frac{m'_n}{m_{n-1}(p-1)} + \frac{i}{m_{n-1}(p-1)} > \frac{p}{p-1},$$

proving the desired congruence.

To prove the last assertion, note that any nonconstant coefficient of H is of the form $b_i^{1/p} \widetilde{T}^i$, with

$$v(b_i) \geq \frac{p}{p-1} - \frac{m'_n}{m_{n-1}(p-1)}$$

and $i > m'_n/p - m_{n-1}$. It follows easily that $v(b_i^{1/p}) + ir_{n-1} > 0$. Thus, when H is written in terms of T^{-1} , all nonconstant coefficients have positive valuation. The same is then true for G_n . □

We examine our solution above in greater detail to prove Proposition 6.20.

PROPOSITION 7.17. *Let G_n and H be as in Corollary 7.16. Suppose that there is no integer a satisfying*

$$(46) \quad \frac{m'_n}{p} - m_{n-1} < a \leq \left(\frac{m'_n}{m'_n - m_{n-1}} \right) \left(\frac{m'_n}{p} - m_{n-1} \right).$$

Then $v_{r'_n}(H - 1) > 0$ and $v_{r'_n}(G_n - 1) > 0$, where $r'_n = 1/m'_n(p-1)$. Thus all zeroes of G_n lie in the open disk $D(r'_n)$.

Proof. Since $v_{r'_n}(G_{\min} - 1) > 0$, we need only show that $v_{r'_n}(H - 1) > 0$. Let $T' = p^{-r'_n}T$. Writing $H - 1$ as $\sum_i \gamma_i(T')^{-i}$, we see from Proposition 7.15 and the definition of H that γ_i is nonzero only when

$$[m'_n/p] - m_{n-1} + 1 \leq i \leq [m'_n/p].$$

In particular, $i > m'_n/p - m_{n-1}$. Furthermore, if b_i is the coefficient of \tilde{T}^{-pi} in I (Proposition 7.15), then by the definition of H (Corollary 7.16), we have

$$\begin{aligned} v(\gamma_i) &= \frac{v(b_i)}{p} + i \left(\frac{1}{m_{n-1}(p-1)} - \frac{1}{m'_n(p-1)} \right) \\ &\geq \frac{1}{p-1} \left(1 - \frac{m'_n}{pm_{n-1}} + \frac{i}{m_{n-1}} - \frac{i}{m'_n} \right). \end{aligned}$$

If

$$i > \left(\frac{m'_n}{m'_n - m_{n-1}} \right) \left(\frac{m'_n}{p} - m_{n-1} \right),$$

then $v(\gamma_i) > 0$, and we are done. □

Example 7.18. We give a counterexample to Proposition 7.17 when there is an a satisfying (46). Let $p = 5$, with $n = 3$, the ramification jumps $(m_1, m_2, m_3) = (1, 5, 34)$, and $F = T^{-34}$. Take $G_{\min} \in \mathcal{G}_3$ as above. Then the matrix C from (45) becomes

$$\begin{pmatrix} c_{20} & c_{15} & c_{10} & c_5 & c_0 \\ c_{21} & c_{16} & c_{11} & c_6 & c_1 \\ c_{22} & c_{17} & c_{12} & c_7 & c_2 \\ c_{23} & c_{18} & c_{13} & c_8 & c_3 \\ c_{24} & c_{19} & c_{14} & c_9 & c_4 \end{pmatrix}.$$

Now, based on (40) and the fact that c_0 must be 1, one can calculate that the reduction of this matrix to $M_5(k)$ is

$$\bar{C} = \begin{pmatrix} 4 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 4 & 2 & 3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}$$

and \bar{C}^{-1} has a nonzero entry in the upper right-hand corner (easily checked using the adjoint formula for the inverse).

Now, if E is as in (45), then

$$E^{-1} = \left(\begin{array}{c|c} -I_{30} & BC^{-1} \\ \hline 0 & C^{-1} \end{array} \right).$$

We calculate the coefficient b_2 of \tilde{T}^{-10} in I (from Proposition 7.15). It is the entry of $E^{-1}\vec{v}$ (see end of proof of Proposition 7.15) corresponding to the top row \vec{w} of C^{-1} (i.e., the row $(0|\vec{w})$ of E^{-1}). Here \vec{v} is a vector whose only nonzero entry is in the last position, and is the coefficient of \tilde{T}^{-34} in $5^{5/4}F$, which has valuation $5/4 - 34(1/5(4)) = -9/20$. Since C^{-1} has an entry of valuation zero in the upper right-hand corner, we see that $v(b_2) = -9/20$. If $T' = 5^{-r_3}T = 5^{r_2-r_3}\tilde{T}$, then if we write I in terms of T' , the coefficient of $(T')^{-10}$ has valuation $-9/20 + 10/20 - 10/34(4) = 1/20 - 10/136 < 0$. The coefficient of $(T')^{-10}$ in H^p has the same valuation, as does the coefficient of $(T')^{-10}$ in $G_{\min}H^p$, as does the coefficient of $(T')^{-10}$ in G_3 . So the zeroes of G_3 do not all lie in the disk $D(r_3)$. In fact, using the Newton polygon, one can show that 10 zeroes of G_3 have valuation $1/200$.

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UNIVERSITY OF VIRGINIA, CHARLOTTESVILLE, VA
E-mail: obus@virginia.edu

UNIVERSITÄT ULM, ULM, GERMANY
E-mail: stefan.wewers@uni-ulm.de