Annals of Mathematics **179** (2014), 1175–1183
http://dx.doi.org/10.4007/annals.2014.179.3.8

On the quantitative distribution of polynomial nilsequences – erratum

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Abstract

This is an erratum to the paper *The quantitative behaviour of polynomial orbits on nilmanifolds* by the authors, published as Ann. of Math. (2) **175** (2012), no. 2, 465–540. The proof of Theorem 8.6 of that paper, which claims a distribution result for multiparameter polynomial sequences on nilmanifolds, was incorrect. We provide two fixes for this issue here. First, we deduce the "equal sides" case $N_1 = \cdots = N_t = N$ of this result from the 1-parameter results in the paper. This is the same basic mode of argument we attempted originally, though the details are different. The equal sides case is the only one required in applications such as the proof of the inverse conjectures for the Gowers norms due to the authors and Ziegler. To remove the equal sides condition one must rerun the entire argument of our paper in the context of multiparameter polynomial sequences $g : \mathbb{Z}^t \to G$ rather than 1-parameter sequences $g : \mathbb{Z} \to G$ as is currently done: a more detailed sketch of how this may be done is available online.

Contents

1. Introduction

We quote from [GT12] and use its notation without any further comment. The problematic part of that paper is Section 8, in which a "multiparameter quantitative Leibman theorem," [GT12, Th. 8.6] is established: results such as [GT12, Th. 1.19] and [GT12, Th. 2.9], which involve only one variable polynomial maps, are not affected.

In [GT12, §8] we attempted to deduce a multiparameter result from the 1-parameter version, [GT12, Th. 2.9]. Unfortunately the deduction is erroneous: the problem comes with the line "By switching the indices $i_1, \dots, i_t$ if necessary..." towards the end of the proof. The problem is that the horizontal character η defined towards the start of the proof may change when this is done, and this invalidates the argument. We thank Bryna Kra and Wenbo Sun for drawing this oversight to our attention and for further drawing our attention to an error in the first version of this erratum.

Our aim is to correct this oversight in the "equal parameters" case of [GT12, Th. 8.6] in which $N_1 = \dots = N_d = N$, which is the only one required for all applications we know of. To lift this restriction seems to require running the entire argument of [GT12] in the context of multiparameter maps from $\mathbb{Z}^t$ to G. In [GT] we provide a guide to doing this, of necessity extremely dependent on [GT12]. Unfortunately the changes required in the multivariate case propagate right back to the most basic result in [GT12], Proposition 3.1, which must be proven in a multivariate setting.

The problematic result [GT12, Th. 8.6] was required in Sections 9 and 10 of [GT12], and as a consequence those results are restricted to the equal parameter case if one only uses the fix contained in this erratum. By following [GT], one could remove this restriction.

Finally, in Section 4, we list some additional minor errata to [GT12], which we take the opportunity to record here.

Let us briefly summarise the subsequent publications depending on [GT12, Th. 8.6] that we are aware of.

- In [GTZ12], the proof of the GI(s) conjectures, the appeal to [GT12] occurs in Appendix D, specifically Theorem D.2. In this application we have $N_1 = \dots = N_t = N$.
- In [GT10], the appeal to [GT12] occurs in the proof of the counting lemma. A slightly modified version of the problematic [GT12, Th. 8.6] is required, which is stated as [GT10, Th. 3.6]. The proof of this is given in [GT10, App. B], where it may be confirmed that again we only require the case $N_1 = \dots = N_t = N$.
- [GSa], [GSb] Whilst these papers do state results depending on [GT12, Th. 8.6] in which the equal sides condition is not assumed, the authors have confirmed to us that the main results of these papers, and in particular the results used subsequently in [FKS13], only require the equal sides case.

Let us recall the precise statement of [GT12, Th. 8.6] in the equal sides case.

THEOREM [GT12, Th. 8.6]. *Let $0 < \delta < 1/2$, and let $m, t, N, d \geqslant 1$ be integers. Suppose that G/Γ is an m-dimensional nilmanifold equipped with a*

$\frac{1}{\delta}$*-rational Mal'cev basis* $\mathcal{X}$ *adapted to some filtration* $G_\bullet$ *of degree* d *and that* $g \in \mathrm{poly}(\mathbb{Z}^t, G_\bullet)$. *Then either* $(g(\vec{n})\Gamma)_{\vec{n}\in[N]^t}$ *is* δ*-equidistributed, or else there is some horizontal character* η *with* $0 < \|\eta\| \ll \delta^{-O_{d,m,t}(1)}$ *such that*

$$\|\eta \circ g\|_{C^\infty[N]^t} \ll \delta^{-O_{d,m,t}(1)}.$$

Notation. For the definition of the smoothness norm $C^\infty[N]^t$, see the start of the next section. We will not explicitly indicate the dependence of constants C or implied constants $O()$ on the parameters m, t and d, which will remain fixed throughout this erratum. We will write $\mathscr{I}$ for the set of multi-indices $\vec{i} = (i_1, \dots, i_t)$ of total degree at most d, that is to say tuples of nonnegative integers with $i_1 + \cdots + i_t \leqslant d$. Write $|\vec{i}| := i_1 + \cdots + i_t$.

2. Some results on polynomials

In this section we record some useful distribution results on polynomials which we will need in both of the proofs of [GT12, Th. 8.6].

We start with some remarks about Taylor coefficients and smoothness norms. If $f : \mathbb{Z}^t \to \mathbb{R}$ is a polynomial map, then in [GT12, Def. 8.2] we defined the Taylor coefficients of f by writing

$$f(\vec{n}) = \sum_{\vec{i}} \alpha_{\vec{i}} \binom{\vec{n}}{\vec{i}}. \tag{2.1}$$

We then define the smoothness norm

$$\|f\|_{C^\infty[N]^t} := \sup_{\vec{i}\neq 0} N^{|\vec{i}|} \|\alpha_{\vec{i}}\|_{\mathbb{R}/\mathbb{Z}}.$$

Here, however it is more convenient to use the conventional Taylor expansion

$$f(\vec{n}) = \sum_{\vec{i}} \beta_{\vec{i}} \vec{n}^{\vec{i}} \tag{2.2}$$

and to consider the variant smoothness norm

$$\|f\|_{C^\infty_*[N]^t} := \sup_{\vec{i}\neq 0} N^{|\vec{i}|} \|\beta_{\vec{i}}\|_{\mathbb{R}/\mathbb{Z}}.$$

Lemma 2.1. *Suppose that* $\|f\|_{C^\infty_*[N]^t} \leqslant M$. *Then there is some* $r = O(1)$ *such that* $\|rf\|_{C^\infty[N]^t} \ll M$.

Proof. This follows from the fact that $\alpha_{\vec{i}} = \sum_{\vec{j}\in\mathscr{I}} M_{\vec{i},\vec{j}} \beta_{\vec{j}}$ with each $M_{\vec{i},\vec{j}}$ rational with height $O(1)$ and $M_{\vec{i},\vec{j}} = 0$ when $|\vec{j}| < |\vec{i}|$. □

We turn now to the following statement, which is actually the special case $G/\Gamma = \mathbb{R}/\mathbb{Z}$ of the problematic result [GT12, Th. 8.6].

Proposition 2.2. *Suppose that $g : \mathbb{Z}^t \to \mathbb{R}$ is a polynomial of total degree d, and let $0 < \delta < \frac{1}{2}$. Then either $(g(n) (\mathrm{mod}\ \mathbb{Z}))$ is δ-equidistributed, or else there is some $q \in \mathbb{Z}$, $0 < |q| \ll \delta^{-O(1)}$ such that $\|qg\|_{C^\infty[N]^t} \ll \delta^{-O(1)}$.*

Proof. Slightly amusingly, the attempted argument of [GT12, Th. 8.6] is actually valid in this case. We run through the details briefly, referring the reader to the aforementioned argument if further clarification is required. A simple averaging argument confirms that, for $\gg \delta^{O(1)} N^{t-1}$ values of $(n_2, \ldots, n_t) \in [N]^{t-1}$, the polynomial sequence $(g_{n_2,\ldots,n_t}(n) (\mathrm{mod}\ \mathbb{Z}))_{n \in [N]}$ is not $\delta^{O(1)}$-equidistributed, where here we set $g_{n_2,\ldots,n_t}(n) := g(n, n_2, \ldots, n_t)$. For each such tuple, [GT12, Th. 2.9] implies that there is an integer $\eta_{n_2,\ldots,n_t}$ with $0 < |\eta_{n_2,\ldots,n_t}| \ll \delta^{-O(1)}$ such that $\|\eta_{n_2,\ldots,n_t} g_{n_2,\ldots,n_t}\|_{C^\infty[N]} \ll \delta^{-O(1)}$. By pigeonholing in the $\delta^{-O(1)}$ possible values of $\eta_{n_2,\ldots,n_t}$ and passing to a thinner set of tuples $(n_2, \ldots, n_t)$, we may assume that $\eta_{n_2,\ldots,n_t} = \eta$ does not depend on $(n_2, \ldots, n_t)$. Writing $p := \eta g$, and continuing to argue as in the proof of [GT12, Th. 8.6] as far as (8.2), we deduce that for all $\vec{i}$ with $i_1 > 0$, there is some $q_{\vec{i}} \ll \delta^{-O(1)}$ such that $\|q_{\vec{i}} p_{\vec{i}}\|_{\mathbb{R}/\mathbb{Z}} \ll \delta^{-O(1)}/N^{|\vec{i}|}$, where $p_{\vec{i}}$ is the $\vec{i}$th Taylor coefficient of p. Hence, defining $\tilde{q}_{\vec{i}} := \eta q_{\vec{i}}$, we have $\|\tilde{q}_{\vec{i}} g_{\vec{i}}\|_{\mathbb{R}/\mathbb{Z}} \ll \delta^{-O(1)}/N^{|\vec{i}|}$. A similar argument holds whenever there is some index j with $\vec{i}_j > 0$, that is to say whenever $\vec{i} \neq 0$. Taking $q := \prod_{\vec{i} \in \mathscr{I}} \tilde{q}_{\vec{i}}$, the result follows. (Note that in the attempted argument of [GT12, Th. 8.6] we would obtain different horizontal characters η_j for each j, which cannot be combined by simple multiplication to give a horizontal character independent of j as we did here.) □

We use the above proposition to obtain a generalisation of [GT12, Lemma 4.5] to polynomials of several variables.

Proposition 2.3. *Suppose that $g : \mathbb{Z}^t \to \mathbb{R}$ is a polynomial such that $\|g(\vec{n})\|_{\mathbb{R}/\mathbb{Z}} \leqslant \varepsilon$ for at least δN^t values of $\vec{n} \in [N]^t$, where $\varepsilon < \delta/10$. Then there is some $Q \ll \delta^{-O(1)}$ such that $\|Qg\|_{C^\infty[N]^t} \ll \delta^{-O(1)}\varepsilon$. In particular, $\|Qg(\vec{0})\|_{\mathbb{R}/\mathbb{Z}} \ll \delta^{-O(1)}\varepsilon$.*

Proof. This is essentially the same as the proof of [GT12, Lemma 4.5]. We include the argument for convenience. If $\varepsilon \gg \delta^C$, then the result follows immediately from Proposition 2.2, so assume this is not the case. Expand

$$g(\vec{n}) = \sum_{\vec{i} \in \mathscr{I}} \alpha_{\vec{i}} \binom{\vec{n}}{\vec{i}}$$

as a Taylor series. It follows from the assumption that none of the polynomials λg, $\lambda \leqslant \delta/2\varepsilon$, is $\delta^{O(1)}$-equidistributed on $[N]^t$. Thus by Proposition 2.3 we see that for each $\lambda \leqslant \delta/2\varepsilon$, there is $q_\lambda \ll \delta^{-O(1)}$ such that

$\|q_\lambda \lambda \alpha_{\vec{i}}\|_{\mathbb{R}/\mathbb{Z}} \ll \delta^{-O(1)}/N^{|\vec{i}|}$ for all $\vec{i} \in \mathscr{I}$. Pigeonholing in the possible values of q_λ we see that there is $q \ll \delta^{-O(1)}$ such that for $\gg \delta^{O(1)}/\varepsilon$ values of $\lambda \leqslant \delta/2\varepsilon$, we have $\|\lambda q \alpha_{\vec{i}}\|_{\mathbb{R}/\mathbb{Z}} \ll \delta^{-O(1)}/N^{|\vec{i}|}$ for all $\vec{i} \in \mathscr{I}$. It follows from [GT12, Lemma 3.2] that for each $\vec{i} \in \mathscr{I}$, there is $q_{\vec{i}} \ll \delta^{-O(1)}$ such that $\|q_{\vec{i}} \alpha_{\vec{i}}\|_{\mathbb{R}/\mathbb{Z}} \ll \varepsilon\delta^{-O(1)}/N^{|\vec{i}|}$. Writing $Q := \prod_{\vec{i} \in \mathscr{I}} q_{\vec{i}}$, we see that $Q \ll \delta^{-O(1)}$ and that $\|Q\alpha_{\vec{i}}\|_{\mathbb{R}/\mathbb{Z}} \ll \varepsilon\delta^{-O(1)}/N^{|\vec{i}|}$ for all $\vec{i} \in \mathscr{I}$.

To get the final conclusion, note that

$$\|Q(g(\vec{n}) - g(\vec{0}))\|_{\mathbb{R}/\mathbb{Z}} \ll \delta^{-O(1)}\varepsilon$$

whenever $\vec{n} \in [N]^t$. Since there is at least one value of $\vec{n}$ such that $\|g(\vec{n})\|_{\mathbb{R}/\mathbb{Z}} \leqslant \varepsilon$, and $Q \ll \delta^{-O(1)}$, the result follows. □

We will need the following lemma of Schwartz-Zippel type.

LEMMA 2.4 (Schwartz-Zippel type lemma). *Let $f : \mathbb{Z}^t \to \mathbb{R}$ be a nonzero polynomial of degree d. Then the number of zeros of f in $[L]^t \subset \mathbb{Z}^t$ is bounded by $O_{d,t}(L^{t-1})$.*

Proof. We proceed by induction on t, the result being clear when $t = 1$. Expand

$$f(n_1, \dots, n_t) = c_d(n_1, \dots, n_{t-1})n_t^d + \cdots + c_0(n_1, \dots, n_{t-1}).$$

For at least one value of i, the polynomial $c_i(n_1, \dots, n_{t-1})$ is not identically zero, and hence it has $O_{d,t}(L^{t-2})$ roots $(n_1, \dots, n_{t-1}) \in [L]^{t-1}$ by the inductive hypothesis. However if $(n_1, \dots, n_{t-1})$ is not one of these roots, then f is nontrivial as a polynomial in n_t, and hence it is satisfied by no more than d values of n_t. □

3. Proof of [GT12, Th. 8.6] in the equal parameters case

Let L be a positive integer parameter to be specified later (it will be δ^{-C} for some large C), and write $\vec{L} := (L, \dots, L)$. Let the notation be as in [GT12, Th. 8.6], as repeated above. The first step is to cover the cube $[N]^t$ by 1-parameter progressions of length N/L^2 pointing in various directions. More precisely, we have

LEMMA 3.1. *Suppose that $(g(\vec{n})\Gamma)_{\vec{n} \in [N]^t}$ fails to be δ-equidistributed. Suppose that $\vec{q} \in [\vec{L}] = [L]^t$. Suppose that $N > L^2$ and that $L > C/\delta$ for some large C. Then $(g(\vec{x} + \vec{q}n)\Gamma)_{n \in [N/L^2]}$ fails to be $\frac{1}{2}\delta$-equidistributed for at least $\frac{1}{4}\delta N^t$ tuples $\vec{x} \in [N]^t$.*

Proof. Since $(g(\vec{n})\Gamma)_{\vec{n}\in[N]^t}$ is not δ-equidistributed, there is some Lipschitz function $F : G/\Gamma \to \mathbb{C}$, $\int_{G/\Gamma} F = 0$ such that

$$|\mathbb{E}_{\vec{n}\in[N]^t} F(g(\vec{n}))| \geqslant \delta \|F\|_{\mathrm{Lip}}.$$

However, introducing an additional averaging, the left-hand side is equal to

$$\mathbb{E}_{\vec{x}\in[N]^t}\mathbb{E}_{n\in[N/L^2]} F(g(\vec{x}+\vec{q}n)) + O\left(\frac{1}{L}\|F\|_{\mathrm{Lip}}\right).$$

In particular, if $L > C/\delta$ with C large enough, then we have

$$\mathbb{E}_{\vec{x}\in[N]^t}\mathbb{E}_{n\in[N/L^2]} F(g(\vec{x}+\vec{q}n)) \geqslant \frac{3}{4}\delta\|F\|_{\mathrm{Lip}}.$$

It follows that for at least $\frac{1}{4}\delta N^d$ tuples $\vec{x}$ we have

$$\mathbb{E}_{n\in[N/L^2]} F(g(\vec{x}+\vec{q}n)) \geqslant \frac{1}{2}\delta\|F\|_{\mathrm{Lip}},$$

and this implies the result. □

Write $p(\vec{n}) := \pi(g(\vec{n}))$, where π is projection onto the horizontal torus $(G/\Gamma)_{\mathrm{ab}}$. Recall that the horizontal torus has dimension m_{ab}, so p takes values in $\mathbb{R}^{m_{\mathrm{ab}}}$. The total degree (highest degree of any monomial) of p is at most d. Expand

$$p(\vec{x}+\vec{q}n) = \sum_{i=1}^{d} \sum_{\vec{i}\in\mathscr{I}:|\vec{i}|=i} c_{\vec{i}}(\vec{x})\vec{q}^{\,\vec{i}} n^i. \tag{3.1}$$

Here, the $c_{\vec{i}} : \mathbb{Z}^t \to \mathbb{R}^{m_{\mathrm{ab}}}$ are polynomials of total degree at most d.

Now we claim that the map from $\mathbb{Z}$ to G defined by $n \mapsto g(\vec{x}+\vec{q}n)$ lies in $\mathrm{poly}(\mathbb{Z}, G_\bullet)$. Indeed the map from $\mathbb{Z}^t$ to G given by $\vec{n} \mapsto g(\vec{x}+\vec{q}\cdot\vec{n})$ lies in $\mathrm{poly}(\mathbb{Z}^t, G_\bullet)$ by [GT12, Cor. 6.8], and so it suffices to check that if $h(\vec{n}) \in \mathrm{poly}(\mathbb{Z}^t, G_\bullet)$, then the diagonal map $h^\Delta(n) := h(n, n, \dots, n)$ lies in $\mathrm{poly}(\mathbb{Z}, G_\bullet)$. But this is obvious from the definition, [GT12, Def. 6.1].

Suppose that $(g(\vec{x}+\vec{q}n)\Gamma)_{n\in[N/L^2]}$ fails to be $\frac{1}{2}\delta$-equidistributed. By Lemma 3.1, for every $\vec{q} \in [\vec{L}]$ this is so for at least $\frac{1}{4}\delta N^t$ values of $\vec{x} \in [N]^t$. By [GT12, Th. 2.9], which is applicable by the claim in the preceding paragraph, the following is therefore true. For all $\vec{q}$ with $0 \leqslant q_i < L$, there are at least $\frac{1}{4}\delta N^t$ choices of $\vec{x} \in [N]^t$ such that there is some $\xi(\vec{q},\vec{x}) \in \mathbb{Z}^{m_{\mathrm{ab}}}$, $0 < |\xi(\vec{q},\vec{x})| \ll \delta^{-O(1)}$ such that

$$\Big\|\xi(\vec{q},\vec{x}) \cdot \sum_{\vec{i}:|\vec{i}|=i} c_{\vec{i}}(\vec{x})\vec{q}^{\,\vec{i}}\Big\|_{\mathbb{R}/\mathbb{Z}} \ll (L/\delta)^{O(1)} N^{-i}$$

for all $i = 1, \dots, d$.

By the pigeonhole principle there is some $\xi \in \mathbb{Z}^{m_{\mathrm{ab}}}$, $0 < |\xi| \ll \delta^{-O(1)}$, together with a subset $\mathscr{S} \subset [\vec{L}] \times [N]^t$, $|\mathscr{S}| \gg \delta^{O(1)}(LN)^t$, such that $\xi(\vec{q},\vec{x}) = \xi$ whenever $(\vec{q},\vec{x}) \in \mathscr{S}$. For each $\vec{q} \in [\vec{L}]$, write $X_q := \{\vec{x} \in [N]^t : (\vec{q},\vec{x}) \in \mathscr{S}\}$.

Then for $\gg \delta^{O(1)} L^t$ values of $\vec{q} \in [\vec{L}]$, we have $|X_q| \gg \delta^{O(1)} N^t$. Let $\mathscr{Q}$ be the set of such $\vec{q}$.

Thus

$$\Big\| \xi \cdot \sum_{\vec{i}:|\vec{i}|=i} c_{\vec{i}}(\vec{x}) \vec{q}^{\,\vec{i}} \Big\|_{\mathbb{R}/\mathbb{Z}} \ll (L/\delta)^{O(1)} N^{-i} \tag{3.2}$$

whenever $\vec{x} \in X_{\vec{q}}$, and for all $i = 1, \ldots, d$, and if $\vec{q} \in \mathscr{Q}$, then $|X_{\vec{q}}| \gg \delta^{C'} N^t$.

Now we apply Proposition 2.3, with $g : \mathbb{Z}^t \to \mathbb{R}$ given by

$$g(\vec{x}) = g_{i,\vec{q}}(\vec{x}) := \xi \cdot \sum_{\vec{i}:|\vec{i}|=i} c_{\vec{i}}(\vec{x}) \vec{q}^{\,\vec{i}}.$$

If $N > (L/\delta)^C$ for C large enough, then $\varepsilon := (L/\delta)^{O(1)} N^{-i}$ is small enough that $\varepsilon < \frac{1}{10}\delta^{C'}$ and so the proposition applies.

We conclude that for each $\vec{q} \in \mathscr{Q}$ and for each $i = 1, \ldots, d$, there is some $Q_i = Q_i(\vec{q})$, $Q_i(\vec{q}) \ll \delta^{-O(1)}$ such that

$$\Big\| Q_i(\vec{q}) \xi \cdot \sum_{\vec{i}:|\vec{i}|=i} c_{\vec{i}}(\vec{0}) \vec{q}^{\,\vec{i}} \Big\|_{\mathbb{R}/\mathbb{Z}} \ll (L/\delta)^{O(1)} N^{-i}.$$

Since there are $\ll \delta^{-O(1)}$ possibilities for $(Q_1(\vec{q}), \ldots, Q_d(\vec{q}))$, we may pass to a set $\mathscr{Q}' \subset \mathscr{Q}$, $|\mathscr{Q}'| \gg \delta^{O(1)} L^d$, such that $Q_i(\vec{q}) = Q_i$ is independent of $\vec{q}$ as $\vec{q}$ ranges over $\mathscr{Q}'$. Setting $\tilde{\xi} := Q_1 \ldots Q_d \xi$, we then have

$$\| \tilde{\xi} \cdot \sum_{\vec{i}:|\vec{i}|=i} c_{\vec{i}}(\vec{0}) \vec{q}^{\,\vec{i}} \|_{\mathbb{R}/\mathbb{Z}} \ll (L/\delta)^{O(1)} N^{-i} \tag{3.3}$$

for all $\vec{q} \in \mathscr{Q}'$ and for all $i = 1, \ldots, d$.

We claim that if $L = \delta^{-C}$ with C big enough, then as a consequence of (3.3) we have

$$\Big\| \tilde{\tilde{\xi}} \cdot c_{\vec{i}}(\vec{0}) \Big\|_{\mathbb{R}/\mathbb{Z}} \ll (L/\delta)^{O(1)} N^{-i} \ll \delta^{-O(1)} N^{-i} \tag{3.4}$$

for all $\vec{i} \in \mathscr{I}$, where $\tilde{\tilde{\xi}} = \tilde{Q}\tilde{\xi}$ with $|\tilde{Q}| \ll \delta^{-O(1)}$.

Leaving the proof of this claim aside for the moment, setting $\vec{x} = \vec{0}$ and $n = 1$ in (3.1) reveals that

$$p(\vec{n}) = \sum_{\vec{i} \in \mathscr{I}} c_{\vec{i}}(\vec{0}) \vec{n}^{\vec{i}},$$

and so (3.4) implies that $\|\tilde{\tilde{\xi}} \cdot p\|_{C^\infty_*[N]^t} \ll \delta^{-O(1)}$. By Lemma 2.1, there is some $r = O(1)$ such that $\|r\tilde{\tilde{\xi}} \cdot p\|_{C^\infty[N]^t} \ll \delta^{-O(1)}$. Defining the horizontal character η to be $r\tilde{\tilde{\xi}} \cdot \pi$, this concludes the proof of [GT12, Th. 8.6] in the equal parameters case.

It remains to check the claim (3.4). We do this by taking linear combinations of (3.3) for different $\vec{q} \in \mathscr{Q}'$ in order to isolate each individual Taylor coefficient $c_{\vec{i}}(\vec{0})$. The key input is the following lemma.

LEMMA 3.2. *Let $\mathscr{Q} \subset [L]^t$ be a set of size εL^t, and to each $\vec{q} \in \mathscr{Q}$, associate the vector $v_{\vec{q}} := (\vec{q}^{\,\vec{i}})_{\vec{i} \in \mathscr{I}} \in \mathbb{Q}^{\mathscr{I}}$. Then, provided $L > C/\varepsilon$, the $v_{\vec{q}}$ span $\mathbb{Q}^{\mathscr{I}}$.*

Proof. If not, there is some $w \in \mathbb{Q}^{\mathscr{I}}$ such that $w \cdot v_{\vec{q}} = 0$ for all $\vec{q} \in \mathscr{Q}$. Thus

$$\sum_{\vec{i}} w_{\vec{i}} \vec{q}^{\,\vec{i}} = 0$$

whenever $\vec{q} \in \mathscr{Q}$. This is a polynomial equation of total degree i in $q_1, \dots, q_t$, and it is not the trivial polynomial. Therefore by Lemma 2.4 this equation has $O(L^{t-1}) < |\mathscr{Q}|$ solutions, contrary to assumption. □

Returning to our proof of the claim (3.4), take $L = \delta^{-C}$ large enough that Lemma 3.2 applies (with $\varepsilon := |\mathscr{Q}'|/N^t$). Then for each $\vec{i} \in \mathscr{I}$ we may select $\vec{q}_1, \dots, \vec{q}_{|\mathscr{I}|} \in \mathscr{Q}'$ and rationals γ_m such that

$$\mathbf{1}_{\vec{i}=\vec{j}} = \sum_{m=1}^{|\mathscr{I}|} \gamma_m \vec{q}_m^{\,\vec{j}}.$$

Inverting these linear relations using the adjoint formula for the inverse (or by using Siegel's lemma), we see that the γ_m are all rationals of height $\ll \delta^{-O(1)}$. Taking $\tilde{Q}$ to be the product of the denominators of all these γ_m, across all values of $\vec{i} \in \mathscr{I}$, we have $\tilde{Q} \ll \delta^{-O(1)}$ and now

$$\tilde{Q} \mathbf{1}_{\vec{i}=\vec{j}} = \sum_{m=1}^{|\mathscr{I}|} \gamma'_m \vec{q}_m^{\,\vec{j}}$$

with the γ'_m being *integers* of size at most $\delta^{-O(1)}$. We may now take appropriate linear combinations of (3.3) to get the claim (3.4), thereby concluding the argument.

4. Minor errata

We take the opportunity to correct some further small points in [GT12].

- In the proof of [GT12, Lemma 3.2], k and q are the same.
- The invocation of [GT12, Lemma 3.2] in the proof of [GT12, Prop. 5.3] is only valid in the regime $\sup_i |\zeta_i| \leqslant \frac{\delta}{2(1+m)}$, due to the hypothesis $\varepsilon \leqslant \delta/2$ in [GT12, Lemma 3.2]. However, the conclusion $|\alpha| \ll_m \sup_i |\zeta_i| \delta^{-C}/N$ is trivially valid in the remaining case $\sup_i |\zeta_i| > \frac{\delta}{2(1+m)}$, due to the hypothesis $|\alpha| \leqslant 1/\delta N$.
- After (7.6), g_h should be $\tilde{g}_h$.

- The last lines of the proof of Proposition 7.2 are valid for the case $j \geqslant 1$. For the $j = 0$ case, one needs to replace G_{j+1} by G_2, that is to say one needs to verify $g_i^{\binom{n+h}{i}} = g_i^{\binom{n}{i}} (\text{mod } G_2)$. For $i \geqslant 2$, this is clear; for $i = 0, 1$, one can verify that g_i is trivial since $g_2(0) = g_2(1) = \text{id}_G$.
- The proof of Lemma 7.8 is not correct as it stands, because we failed to check that $[G'_1, G'_1] \subseteq G'_2$. For this to hold we need $\ker \tilde{\eta}_2 \subset [G, G]$, which is not in general true. However, earlier arguments in Section 7 complete the analysis unless we are in this case, by the remarks at the bottom of page 512. Note that there is a further small misprint on page 512, line -3: this should state that η_2 annihilates $[G, G]$.

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(Received: October 16, 2013)
(Revised: January 27, 2014)

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