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Kodaira dimension and zeros of holomorphic one-forms

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Abstract

We show that every holomorphic one-form on a smooth complex projective variety of general type must vanish at some point. The proof uses generic vanishing theory for Hodge modules on abelian varieties.

Introduction

1. In this article, we use results about Hodge modules on abelian varieties [PS13] to prove two conjectures about zeros of holomorphic one-forms on smooth complex projective varieties. One conjecture was formulated by Hacon and Kovács [HK05] and by Luo and Zhang [LZ05]; the former partly attribute the question to Carrell [Car74] and also explain why it is natural to consider varieties of general type.

Conjecture 1.1 (Hacon-Kovács, Luo-Zhang). *If X is a smooth complex projective variety of general type, then the zero locus*

$$Z(\omega) = \left\{ x \in X \;\middle|\; \omega(T_x X) = 0 \right\}$$

of every global holomorphic one-form $\omega \in H^0(X, \Omega_X^1)$ is nonempty.

The statement is a tautology in the case of curves; for surfaces, it was proved by Carrell [Car74], and for threefolds, it was proved by Luo and Zhang [LZ05]. It was also known to be true if the canonical bundle of X is ample [Zha97], and more generally when X is minimal [HK05]. For varieties that are not necessarily of general type, Luo and Zhang proposed the following more general version of the conjecture, in terms of the Kodaira dimension $\kappa(X)$, and proved it in the case of threefolds [LZ05].

Conjecture 1.2 (Luo-Zhang). *Let X be a smooth complex projective variety, and let $W \subseteq H^0(X, \Omega_X^1)$ be a linear subspace such that $Z(\omega)$ is empty* *for every nonzero one-form* $\omega \in W$. *Then the dimension of* W *can be at most* $\dim X - \kappa(X)$.

2. Since every holomorphic one-form on X is the pullback of a holomorphic one-form from the Albanese variety $\operatorname{Alb} X$, it is natural to consider an arbitrary morphism from X to an abelian variety and to ask under what conditions the pullback of a holomorphic one-form must have a zero at some point of X. Our main result is the following theorem, which implies both of the above conjectures.

Theorem 2.1. *Let* X *be a smooth complex projective variety, and let* $f\colon X \to A$ *be a morphism to an abelian variety. If* $H^0\big(X, \omega_X^{\otimes d} \otimes f^*L^{-1}\big) \neq 0$ *for some integer* $d \geq 1$ *and some ample line bundle* L *on* A, *then* $Z(\omega)$ *is nonempty for every* ω *in the image of* $f^*\colon H^0(A, \Omega_A^1) \to H^0(X, \Omega_X^1)$.

The condition in the theorem is equivalent to requiring that, up to birational equivalence, the morphism f should factor through the Iitaka fibration of (X, K_X).

3. One consequence of Conjecture 1.2 is the following.

Corollary 3.1. *If* $f\colon X \to A$ *is a smooth morphism from a smooth complex projective variety onto an abelian variety, then* $\dim A \leq \dim X - \kappa(X)$.

Proof. Since f is smooth, $Z(f^*\omega)$ is empty for every nonzero $\omega \in H^1(A, \Omega_A^1)$. The desired inequality then follows from Conjecture 1.2 because f^* is injective. □

In particular, there are no nontrivial smooth morphisms from a variety of general type to an abelian variety; this was proved by Viehweg and Zuo [VZ01] when the base is an elliptic curve and then by Hacon and Kovács [HK05] in general. Together with subadditivity, Corollary 3.1 has the following application.

Corollary 3.2. *If* $f\colon X \to A$ *is a smooth morphism onto an abelian variety, and if all fibers of* f *are of general type, then* f *is birationally isotrivial.*

Proof. We have to show that X becomes birational to a product after a generically finite base-change; it suffices to prove that $\operatorname{Var}(f) = 0$, in Viehweg's terminology. Since the fibers of f are of general type, we know from the main result of [Kol87] that

$$\kappa(F) + \operatorname{Var}(f) \leq \kappa(X).$$

On the other hand, $\kappa(X) \leq \dim F$ by Corollary 3.1, and so we are done. □

Successive versions of this result go back to Migliorini [Mig95] (for smooth families of minimal surfaces of general type over an elliptic curve), Kovács

[Kov96] (for smooth families of minimal varieties of general type over an elliptic curve), and Viehweg-Zuo [VZ01] (for smooth families of varieties of general type over an elliptic curve). Over a base of arbitrary dimension, Kovács [Kov97] and Zhang [Zha97] have shown that if the canonical bundle of the fiber is assumed to be ample, the morphism f must actually be isotrivial.

4. Before we get into the details of the proof, we explain how Theorem 2.1 implies Conjecture 1.2 — and hence also Conjecture 1.1, which is a special case.

The statement of the conjecture is vacuous when $\kappa(X) = -\infty$, and so we shall assume from now on that $\kappa(X) \geq 0$. Let $\mu\colon X' \to X$ be a birational modification of X such that $g\colon X' \to Z$ is a smooth model for the Iitaka fibration. The general fiber of g is then a smooth projective variety of dimension $\delta(X) = \dim X - \kappa(X)$ and Kodaira dimension zero, and therefore it maps surjectively to its own Albanese variety [Kaw81, Th. 1]. By a standard argument [Kaw81, Proof of Th. 13], the image in $\operatorname{Alb} X$ of every fiber of g must be a translate of a single abelian variety. Letting A denote the quotient of $\operatorname{Alb} X$ by this abelian variety, we obtain $\dim A \geq \dim H^0(X, \Omega_X^1) - \delta(X)$ and the following commutative diagram:

$$\begin{array}{ccc} X' & \xrightarrow{\mu} & X \\ \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle f} \\ Z & \longrightarrow & A. \end{array} \tag{4.1}$$

By construction, Z dominates $f(X)$, and so we conclude from Theorem 2.1 that all holomorphic one-forms in the image of $f^*\colon H^0(A, \Omega_A^1) \to H^0(X, \Omega_X^1)$ have a nonempty zero locus. Since this subspace has codimension at most $\delta(X)$ in the space of all holomorphic one-forms, we see that Conjecture 1.2 must be true.

Proof of the theorem

5. Let $f\colon X \to A$ be a morphism from a smooth projective variety to an abelian variety. We define $V = H^0(A, \Omega_A^1)$ and introduce the set

$$Z_f = \Big\{ (x, \omega) \in X \times V \;\Big|\; (f^*\omega)(T_x X) = 0 \Big\}.$$

To prove Theorem 2.1, we have to show that the projection from Z_f to V is surjective; what we will actually show is that $(f \times \mathrm{id})(Z_f) \subseteq A \times V$ maps onto V.

6. In our proof, we use techniques from generic vanishing theory and from Saito's theory of Hodge modules. To motivate the use of generic vanishing theory, suppose for a moment that one could take $d = 1$ in Theorem 2.1; in other words, suppose that there was a nontrivial morphism

$$f^* L \to \omega_X.$$

By one of the generic vanishing results due to Green and Lazarsfeld [GL87, Th. 3.1], the existence of a holomorphic one-form $f^*\omega$ without zeros implies that $H^0(X, \omega_X \otimes f^*P_\alpha) = 0$ for general $\alpha \in \operatorname{Pic}^0(A)$. This leads to the conclusion that

$$H^0(A, L \otimes P_\alpha \otimes f_*\mathscr{O}_X) = 0, \tag{6.1}$$

which contradicts the fact that L is ample, because a general translate of an ample divisor does not contain $f(X)$.

In reality, only some large power of $\omega_X \otimes f^*L^{-1}$ will have a section — after an étale cover that makes L sufficiently divisible — and so we are forced to work on a resolution of singularities of a d-fold branched covering of X. Now the additional singularities in the morphism to A prevent the simple argument from above from going through. To fix this problem, we use an extension of generic vanishing theory to Hodge modules, developed in [PS13]. With the help of this tool, we can still obtain (6.1) when there is a holomorphic one-form $f^*\omega$ without zeros.

7. Our method owes a lot to the paper [VZ01], in which Viehweg and Zuo show, among other things, that there are no smooth morphisms from a complex projective variety X of general type to an elliptic curve E. To orient the reader, we shall briefly recall a key step in their proof [VZ01, Lemma 3.1]; a similar technique also appears in the work of Kovács [Kov96], [Kov02]. Using a carefully chosen resolution of singularities of a certain branched covering of X, Viehweg and Zuo produce a logarithmic Higgs bundle $\bigoplus \mathcal{E}^{p,q}$ on the elliptic curve E, whose Higgs field

$$\theta^{p,q} \colon \mathcal{E}^{p,q} \to \mathcal{E}^{p-1,q+1} \otimes \Omega_E^1(\log S')$$

has logarithmic poles along a divisor S'; by construction, S' contains the set of points S where the morphism from X to E is singular. They also construct a Higgs subbundle $\bigoplus \mathcal{F}^{p,q}$ with two properties: the restriction of the Higgs field satisfies

$$\theta^{p,q}\big(\mathcal{F}^{p,q}\big) \subseteq \mathcal{F}^{p-1,q+1} \otimes \Omega_E^1(\log S),$$

and $\mathcal{F}^{r,0}$ is an ample line bundle (where $r = \dim X - 1$). Since any subbundle of $\ker \theta^{p,q}$ has degree ≤ 0, it follows that S cannot be empty: otherwise, the image of $\mathcal{F}^{r,0}$ under some iterate of the Higgs field would be an ample subbundle of $\ker \theta^{p,q}$.

8. The associated graded of a Hodge module may be thought of as a generalization of a Higgs bundle. Recall that a Hodge module on a smooth complex algebraic variety X is a very special kind of filtered $\mathscr{D}$-module $(\mathcal{M}, F)$; in particular, $\mathcal{M}$ is a regular holonomic left $\mathscr{D}_X$-module, and $F_\bullet\mathcal{M}$ is a good

increasing filtration by $\mathscr{O}_X$-coherent subsheaves. The associated graded

$$\operatorname{gr}_\bullet^F \mathcal{M} = \bigoplus_{k \in \mathbb{Z}} F_k\mathcal{M} \big/ F_{k-1}\mathcal{M}$$

is coherent over the symmetric algebra $\operatorname{Sym} \mathscr{T}_X$. It therefore determines a coherent sheaf $\operatorname{gr}^F \mathcal{M}$ on the cotangent bundle T^*X, whose support is the so-called characteristic variety $\operatorname{Ch}(\mathcal{M})$ of the $\mathscr{D}$-module. A longer summary of Saito's theory can be found in [PS13, §§2.1–2] and of course in the introduction to [Sai88].

9. For later use, we point out a connection between Theorem 2.1 and Hodge modules, having to do with the properties of the set $(f \times \mathrm{id})(Z_f)$. The structure sheaf $\mathscr{O}_X$ is naturally a left $\mathscr{D}_X$-module; the direct image functor for $\mathscr{D}$-modules takes it to a complex $f_+\mathscr{O}_X$ of regular holonomic $\mathscr{D}_A$-modules. According to Kashiwara's estimate for the behavior of the characteristic variety,

$$\operatorname{Ch}\bigl(f_+\mathscr{O}_X\bigr) \subseteq (f \times \mathrm{id})\bigl(df^{-1}(0)\bigr) = (f \times \mathrm{id})(Z_f), \tag{9.1}$$

where the notation is as in the following diagram:

$$\begin{tikzcd} X \times V \arrow[r, "df"] \arrow[d, "f \times \mathrm{id}"] & T^*X \\ A \times V. \end{tikzcd} \tag{9.2}$$

One consequence of Saito's theory is that $\mathscr{O}_X$, equipped with the obvious filtration ($\operatorname{gr}_0^F \mathscr{O}_X = \mathscr{O}_X$), is actually a Hodge module. Because f is projective, $f_+(\mathscr{O}_X, F)$ (with the induced filtration) is then a complex of Hodge modules on A, and Saito's decomposition theorem [Sai88, Th. 5.3.1] gives us a noncanonical splitting

$$f_+(\mathscr{O}_X, F) \simeq \bigoplus_{i \in \mathbb{Z}} \mathcal{H}^i f_+(\mathscr{O}_X, F)[-i] \tag{9.3}$$

in the derived category of filtered $\mathscr{D}$-modules. This means the set $(f \times \mathrm{id})(Z_f)$ contains the characteristic varieties of the Hodge modules $\mathcal{H}^i f_+(\mathscr{O}_X, F)$.

10. The above considerations suggest a way to generalize the construction in [VZ01] to the setting of Theorem 2.1. Namely, suppose that we manage to find a Hodge module $(\mathcal{M}, F)$ on the abelian variety A, and a graded $\operatorname{Sym} \mathscr{T}_A$-submodule

$$\mathscr{F}_\bullet \subseteq \operatorname{gr}_\bullet^F \mathcal{M}. \tag{10.1}$$

Denote by $\mathscr{F}$ and $\operatorname{gr}^F \mathcal{M}$ the associated coherent sheaves on $T^*A = A \times V$, and suppose that the following three conditions are satisfied:

(1) There is a morphism $h\colon Y \to A$ from a smooth projective variety such that $(\mathcal{M}, F)$ is a direct summand of some $\mathcal{H}^i h_+(\mathscr{O}_Y, F)$.

(2) The support of $\mathscr{F}$ is contained in the set $(f \times \mathrm{id})(Z_f) \subseteq A \times V$.
(3) For some $k \in \mathbb{Z}$, the sheaf $\mathscr{F}_k$ is isomorphic to $L \otimes f_* \mathscr{O}_X$, where L is an ample line bundle on A.

If that is the case, we can use the generic vanishing theory for Hodge modules that we developed in [PS13] to show that Z_f projects onto V.

Proposition 10.2. *If a Hodge module $(\mathcal{M}, F)$ and a graded $\operatorname{Sym} \mathscr{T}_A$-module $\mathscr{F}_\bullet$ with the above properties exist, then the projection from Z_f to V must be surjective.*

Proof. Let P be the normalized Poincaré bundle on $A \times \hat{A}$. Using its pull-back to $A \times \hat{A} \times V$ as a kernel, we define the Fourier-Mukai transform of $\operatorname{gr}^F \mathcal{M}$ to be the complex of coherent sheaves

$$E = \mathbf{R}\Phi_P(\operatorname{gr}^F \mathcal{M}) = \mathbf{R}(p_{23})_* \Big(p_{13}^* \big(\operatorname{gr}^F \mathcal{M} \big) \otimes p_{12}^* P \Big)$$

on $\hat{A} \times V$. When the Hodge module $(\mathcal{M}, F)$ comes from a morphism to an abelian variety as in (1), the complex E has special properties [PS13, §4.5]:

(a) E is a *perverse coherent sheaf*, meaning that its cohomology sheaves $\mathcal{H}^\ell E$ are zero for $\ell < 0$ and are supported in codimension at least 2ℓ otherwise.
(b) The union of the supports of all the higher cohomology sheaves of E is a finite union of translates of triple tori in $\hat{A} \times V$.
(c) The dual complex $\mathbf{R}\mathcal{H}om(E, \mathscr{O})$ has the same properties.

Here a triple torus, in Simpson's terminology, means a subset of the form

$$\operatorname{im}\Big(\varphi^* \colon \hat{B} \times H^0(B, \Omega_B^1) \to \hat{A} \times H^0(A, \Omega_A^1) \Big),$$

where $\varphi \colon A \to B$ is a morphism to another abelian variety. Roughly speaking, (a) is derived using a vanishing theorem by Saito [PS13, Lemma 2.5], whereas (b) is derived using a result by Arapura and Simpson about cohomology support loci of Higgs bundles [Ara92], with the help of the decomposition in (9.3).

It follows that the 0-th cohomology sheaf of the complex E is locally free outside a finite union of translates of triple tori of codimension at least two. Indeed, the locus where $\mathcal{H}^0 E$ is not locally free is

$$\bigcup_{\ell \geq 1} \operatorname{Supp} R^\ell \mathcal{H}om\big(\mathcal{H}^0 E, \mathscr{O} \big) \subseteq \bigcup_{\ell \geq 1} \operatorname{Supp} R^\ell \mathcal{H}om(E, \mathscr{O}) \cup \bigcup_{\ell \geq 1} \operatorname{Supp} \mathcal{H}^\ell E,$$

which is of the asserted kind because $\mathbf{R}\mathcal{H}om(E, \mathscr{O})$ also satisfies (a) and (b). Since the higher cohomology sheaves of E are supported in a finite union of translates of triple tori of codimension at least two, the restriction of the complex E to the subspace $\{\alpha\} \times V$ is a single locally free sheaf for general $\alpha \in \hat{A}$.

Translating this back into a statement on $A \times V$ involves a few simple manipulations with the formula for the Fourier-Mukai transform; the result is that

$$p_{2*}\bigl(p_1^* P_\alpha \otimes \operatorname{gr}^F \mathcal{M}\bigr)$$

is a locally free sheaf on V. Here $p_1 \colon A \times V \to A$ and $p_2 \colon A \times V \to V$ are the projections to the two factors, and P_α is the line bundle corresponding to $\alpha \in \hat{A}$.

Now suppose that the projection from Z_f to V was *not* surjective. Because the support of $\mathscr{F}$ is contained in $(f \times \mathrm{id})(Z_f)$ by (2), the subsheaf

$$p_{2*}\bigl(p_1^* P_\alpha \otimes \mathscr{F}\bigr) \subseteq p_{2*}\bigl(p_1^* P_\alpha \otimes \operatorname{gr}^F \mathcal{M}\bigr)$$

would then be torsion and therefore zero. As V is a vector space, we are forced to the conclusion that the corresponding graded $\operatorname{Sym} V^*$-module

$$\bigoplus_{k \in \mathbb{Z}} H^0\bigl(A, P_\alpha \otimes \mathscr{F}_k\bigr)$$

is also zero. But this contradicts our assumption in (3) that $\mathscr{F}_k \simeq L \otimes f_* \mathscr{O}_X$ for some $k \in \mathbb{Z}$. In fact, if $P_\alpha \otimes L \otimes f_* \mathscr{O}_X$ has no global sections, then every global section of $P_\alpha \otimes L$ has to vanish set-theoretically along $f(X)$; but then $f(X)$ is contained in a general translate of an ample divisor, which is absurd. $\square$

11. The remainder of the paper is devoted to constructing the two objects in (10.1) under the assumptions of Theorem 2.1. From now on, we let X be a smooth complex projective variety of dimension n and Kodaira dimension $\kappa(X) \geq 0$. We also assume that we have a morphism $f \colon X \to A$ to an abelian variety such that $\omega_X^{\otimes d} \otimes f^* L^{-1}$ has a section for some $d \geq 1$ and some ample line bundle L on A. Using the geometry of abelian varieties, this can be improved as follows.

LEMMA 11.1. *After a finite étale base change on A, we can find an ample line bundle L such that the d-th power of $\omega_X \otimes f^* L^{-1}$ has a section for some $d \geq 1$.*

Proof. Fix an ample line bundle L_1 on A, and choose $d \geq 1$ such that $\omega_X^{\otimes d} \otimes f^* L_1^{-1}$ has a section. Let $[2d] \colon A \to A$ denote multiplication by $2d$; then $[2d]^* L_1 \simeq L^{\otimes d}$ for some ample line bundle L on A, which clearly does the job. $\square$

Since the conclusion of Theorem 2.1 is unaffected by finite étale morphisms, we may assume for the remainder of the argument that the d-th power of the line bundle $B = \omega_X \otimes f^* L^{-1}$ has a nontrivial section; for the sake of convenience, we shall take $d \geq 1$ to be the smallest integer with this property.

12. A nontrivial section $s \in H^0\bigl(X, B^{\otimes d}\bigr)$ defines a branched covering $\pi\colon X_d \to X$ of degree d, unramified outside the divisor $Z(s)$; see [EV92, §3] for details. Since d is minimal, X_d is irreducible; let $\mu\colon Y \to X_d$ be a resolution of singularities that is an isomorphism over the complement of $Z(s)$, and define $\varphi = \pi \circ \mu$ and $h = f \circ \varphi$. The following commutative diagram shows all the relevant morphisms:

(12.1)
$$\begin{tikzcd} Y \arrow[r, "\mu"] \arrow[rr, bend left, "\varphi"] \arrow[drr, bend right, "h"'] & X_d \arrow[r, "\pi"] & X \arrow[d, "f"] \\ & & A. \end{tikzcd}$$

By construction, X_d is embedded in the total space of the line bundle B, and so the pullback $\pi^* B$ has a tautological section; the induced morphism $\varphi^* B^{-1} \to \mathscr{O}_Y$ is an isomorphism over the complement of $Z(s)$. After composing it with $\varphi^* \Omega_X^k \to \Omega_Y^k$, we obtain for every $k = 0, 1, \dots, n$ an injective morphism

$$\varphi^*\bigl(B^{-1} \otimes \Omega_X^k\bigr) \to \Omega_Y^k,$$

which is actually an isomorphism over the complement of $Z(s)$. Pushing forward to X, and using the fact that $\mathscr{O}_X \to \varphi_* \mathscr{O}_Y$ is injective, we find that the morphisms

$$B^{-1} \otimes \Omega_X^k \to \varphi_* \Omega_Y^k \tag{12.2}$$

are also injective.

13. Let $S = \operatorname{Sym} V^*$ be the symmetric algebra on the dual of $V = H^0(A, \Omega_A^1)$, and consider the complex of graded $\mathscr{O}_X \otimes S$-modules

$$C_{X,\bullet} = \Bigl[\mathscr{O}_X \otimes S_{\bullet - g} \to \Omega_X^1 \otimes S_{\bullet - g + 1} \to \cdots \to \Omega_X^n \otimes S_{\bullet - g + n}\Bigr],$$

placed in cohomological degrees $-g, \dots, 0$, where $g = \dim A$ and $n = \dim X$. The differential in the complex is induced by the evaluation morphism $V \otimes \mathscr{O}_X \to \Omega_X^1$. Concretely, let $\omega_1, \dots, \omega_g \in V$ be a basis, and denote by $s_1, \dots, s_g \in S_1$ the dual basis; then the formula for the differential is

$$\Omega_X^p \otimes S_{\bullet - g + p} \to \Omega_X^{p+1} \otimes S_{\bullet - g + p + 1}, \quad \theta \otimes s \mapsto \sum_{i=1}^{g} \bigl(\theta \wedge f^* \omega_i\bigr) \otimes s_i s.$$

We use similar notation on Y as well.

Lemma 13.1. *There is a morphism of complexes of graded $\mathscr{O}_A \otimes S$-modules*

$$\mathbf{R} f_* \bigl(B^{-1} \otimes C_{X,\bullet}\bigr) \to \mathbf{R} h_* C_{Y,\bullet},$$

induced by the individual morphisms in (12.2).

Proof. The morphisms in (12.2) commute with the differentials in the two complexes because $\varphi^*\big(f^*\omega\big) = h^*\omega$ for every $\omega \in V$. □

14. We denote by C_X the complex of coherent sheaves on $X \times V$ associated with the complex of graded $\mathscr{O}_X \otimes S$-modules $C_{X,\bullet}$.

LEMMA 14.1. *The support of C_X is equal to $Z_f \subseteq X \times V$.*

Proof. Let $p_1 \colon X \times V \to X$ denote the first projection; then

$$C_X = \Big[p_1^*\mathscr{O}_X \to p_1^*\Omega_X^1 \to \cdots \to p_1^*\Omega_X^n\Big],$$

with differential induced by the tautological section of $p_1^*\Omega_X^1$. This shows that C_X is equal to the pullback of the Koszul resolution for the structure sheaf of the zero section in T^*X via the morphism $df \colon X \times V \to T^*X$. In particular, $\operatorname{Supp} C_X$ is equal to $df^{-1}(0) = Z_f$, in the notation of (9.2). □

15. The reason for introducing $C_{X,\bullet}$ is that it is closely related to the direct image $f_+(\mathscr{O}_X, F)$. In fact, we have the following refinement of Kashiwara's estimate (9.1).

LEMMA 15.1. *The associated graded of the Hodge module $\mathcal{H}^i f_+(\mathscr{O}_X, F)$ is*

$$\operatorname{gr}_\bullet^F\Big(\mathcal{H}^i f_+(\mathscr{O}_X, F)\Big) \simeq R^i f_* C_{X,\bullet}.$$

Proof. This is proved in [PS13, Prop. 2.11]. In a nutshell, a result by Laumon gives the isomorphism $\operatorname{gr}_\bullet^F f_+(\mathscr{O}_X, F) \simeq \mathbf{R}f_* C_{X,\bullet}$ in the derived category of graded $\mathscr{O}_A \otimes S$-modules; to deduce the assertion about individual cohomology sheaves, one has to use the fact that the complex $f_+(\mathscr{O}_X, F)$ is strict [Sai88, Th. 5.3.1]. This ensures that every $\mathcal{H}^i f_+(\mathscr{O}_X, F)$ is again a filtered $\mathscr{D}$-module; a priori, it is only an object of a larger abelian category, due to the fact that the derived category of filtered $\mathscr{D}$-modules is the derived category of an exact category. Loosely speaking, this means that, for Hodge modules, taking the associated graded commutes with direct images; arbitrary filtered $\mathscr{D}$-modules do not have this property. □

16. We are now in a position to carry out the construction suggested in Section 10. For the Hodge module $(\mathcal{M}, F)$, we take $\mathcal{H}^0 h_+(\mathscr{O}_Y, F)$; it is really a Hodge module because the morphism h is projective [Sai88, Th. 5.3.1]. As explained in Lemma 15.1, one has an isomorphism

$$\operatorname{gr}_\bullet^F \mathcal{M} \simeq R^0 h_* C_{Y,\bullet}$$

as graded modules over $\operatorname{Sym} \mathscr{T}_A = \mathscr{O}_A \otimes S$. Now define $\mathscr{F}_\bullet$ to be the image of $R^0 f_*\big(B^{-1} \otimes C_{X,\bullet}\big)$ in $R^0 h_* C_{Y,\bullet}$, using the morphism provided by Lemma 13.1. In this way, we get a graded $\operatorname{Sym} \mathscr{T}_A$-submodule of $\operatorname{gr}_\bullet^F \mathcal{M}$ as in (10.1).

17. It remains to check that $(\mathcal{M}, F)$ and $\mathscr{F}_\bullet$ have all the required properties.

PROPOSITION 17.1. *$(\mathcal{M}, F)$ and $\mathscr{F}_\bullet$ satisfy the conditions in (1)–(3).*

Proof. It is obvious from the construction that (1) holds. To prove (2), note that $\mathscr{F}$ is by definition a quotient of the coherent sheaf

$$R^0(f \times \mathrm{id})_*\big(p_1^* B^{-1} \otimes C_X\big).$$

The complex in parentheses is supported in the set Z_f by Lemma 14.1; consequently, the support of $\mathscr{F}$ is contained in $(f \times \mathrm{id})(Z_f)$, as required. To finish up, we shall argue that (3) is true when $k = g - n$. We clearly have $C_{X,k} = \omega_X$ and $C_{Y,k} = \omega_Y$, and the morphism $f^*L = B^{-1} \otimes \omega_X \to \varphi_*\omega_Y$ in (12.2) is injective. After pushing forward to the abelian variety A, we find that the resulting morphism

$$L \otimes f_*\mathscr{O}_X \simeq f_*f^*L \to h_*\omega_Y$$

is still injective. But $\mathrm{gr}_k^F \mathcal{M} \simeq h_*\omega_Y$, and so $\mathscr{F}_k$ is isomorphic to $L \otimes f_*\mathscr{O}_X$. □

Now Proposition 10.2 shows that Z_f projects onto V; this means that every holomorphic one-form in the image of $f^* \colon H^0(A, \Omega_A^1) \to H^0(X, \Omega_X^1)$ has a nonempty zero locus. We have proved Theorem 2.1, Conjecture 1.2, and Conjecture 1.1.

18. To illustrate how our proof works, we shall briefly compare it with the argument by Viehweg and Zuo (sketched in Section 7); either method shows that there are no smooth morphisms $f \colon X \to E$ from a variety of general type to an elliptic curve. It is easy to see that $(f \times \mathrm{id})(Z_f)$ dominates $V = H^0(E, \Omega_E^1)$ if and only if f has at least one singular fiber, so let us quickly see how the assumption that f is smooth leads to a contradiction in both cases.

In the case of Viehweg and Zuo, f smooth implies that $S = \emptyset$, and hence that the iterates under the Higgs field

$$\big(\theta^{p,q} \circ \cdots \circ \theta^{r-1,1} \circ \theta^{r,0}\big)(\mathscr{F}^{r,0}) \subseteq \mathscr{E}^{p-1,q+1}$$

of the ample line bundle $\mathscr{F}^{r,0}$ are all nonzero (because subbundles of $\ker \theta^{p,q}$ have degree ≤ 0); but this is not possible because $\theta^{0,r} = 0$. In other words, smoothness of f and positivity of $\mathscr{F}^{r,0}$ would prevent the Higgs field from being nilpotent, contradicting the fact that it is always nilpotent.

In our case, let $\partial \in H^0(E, \mathscr{T}_E)$ be a nonzero global vector field. The method used in Proposition 10.2 produces a nonzero graded $\mathbb{C}[\partial]$-submodule

$$\bigoplus_{k \geq g-n} H^0(X, P_\alpha \otimes \mathscr{F}_k) \subseteq \bigoplus_{k \geq g-n} H^0\big(X, P_\alpha \otimes \mathrm{gr}_k^F \mathcal{M}\big),$$

which is ∂-torsion if $(f \times \mathrm{id})(Z_f) = E \times \{0\}$; but this is not possible because the ambient module is free over $\mathbb{C}[\partial]$. In other words, smoothness of f and positivity of $\mathscr{F}_{g-n}$ would create a nontrivial torsion submodule, contradicting the fact that the larger module is always free.

More generally, our branched covering construction is nearly identical to the one in [VZ01], except that we do not need to be as careful in choosing the initial section of $B^{\otimes d}$ or the resolution of singularities; this is due to the power of Saito's theory. The idea of having two objects — one with good properties coming from Hodge theory, the other encoding the singularities of the morphism f — also comes from Viehweg and Zuo. But the actual mechanism behind the proof of Proposition 10.2 based on generic vanishing theory is different, because Higgs bundles and Hodge modules have different properties.

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References

[Ara92] D. Arapura, Higgs line bundles, Green-Lazarsfeld sets, and maps of Kähler manifolds to curves, *Bull. Amer. Math. Soc.* **26** (1992), 310–314. MR 1129312. Zbl 0759.14016. http://dx.doi.org/10.1090/S0273-0979-1992-00283-5.

[Car74] J. B. Carrell, Holomorphic one forms and characteristic numbers, *Topology* **13** (1974), 225–228. MR 0348147. Zbl 0305.32016. http://dx.doi.org/10.1016/0040-9383(74)90015-9.

[EV92] H. Esnault and E. Viehweg, *Lectures on Vanishing Theorems*, *DMV Seminar* **20**, Birkhäuser, Basel, 1992. MR 1193913. Zbl 0779.14003. http://dx.doi.org/10.1007/978-3-0348-8600-0.

[GL87] M. Green and R. Lazarsfeld, Deformation theory, generic vanishing theorems, and some conjectures of Enriques, Catanese and Beauville, *Invent. Math.* **90** (1987), 389–407. MR 0910207. Zbl 0659.14007. http://dx.doi.org/10.1007/BF01388711.

[HK05] C. D. Hacon and S. J. Kovács, Holomorphic one-forms on varieties of general type, *Ann. Sci. École Norm. Sup.* **38** (2005), 599–607. MR 2172952. Zbl 1083.14015. http://dx.doi.org/10.1016/j.ansens.2004.12.002.

[Kaw81] Y. Kawamata, Characterization of abelian varieties, *Compositio Math.* **43** (1981), 253–276. MR 0622451. Zbl 0471.14022. Available at http://www.numdam.org/item?id=CM_1981__43_2_253_0.

[Kol87] J. Kollár, Subadditivity of the Kodaira dimension: fibers of general type, in *Algebraic Geometry* (Sendai, 1985), *Adv. Stud. Pure Math.* **10**, North-Holland, Amsterdam, 1987, pp. 361–398. MR 0946244. Zbl 0659.14024.

[Kov96] S. J. Kovács, Smooth families over rational and elliptic curves, *J. Algebraic Geom.* **5** (1996), 369–385. MR 1374712. Zbl 0861.14033.

[Kov97] S. J. Kovács, Families over a base with a birationally nef tangent bundle, *Math. Ann.* **308** (1997), 347–359. MR 1464907. Zbl 0922.14024. http://dx.doi.org/10.1007/s002080050079.

[Kov02] S. J. Kovács, Logarithmic vanishing theorems and Arakelov-Parshin boundedness for singular varieties, *Compositio Math.* **131** (2002), 291–317. MR 1905025. Zbl 1048.14006. http://dx.doi.org/10.1023/A:1015592420937.

[LZ05] T. Luo and Q. Zhang, Holomorphic forms on threefolds, in *Recent Progress in Arithmetic and Algebraic Geometry*, *Contemp. Math.* **386**, Amer. Math. Soc., Providence, RI, 2005, pp. 87–94. MR 2182772. Zbl 1216.14013. http://dx.doi.org/10.1090/conm/386/07219.

[Mig95] L. Migliorini, A smooth family of minimal surfaces of general type over a curve of genus at most one is trivial, *J. Algebraic Geom.* **4** (1995), 353–361. MR 1311355. Zbl 0834.14021.

[PS13] M. Popa and C. Schnell, Generic vanishing theory via mixed Hodge modules, *Forum Math. Sigma* **1** (2013), e1, 60pp. MR 3090229. Zbl 06209103. http://dx.doi.org/10.1017/fms.2013.1.

[Sai88] M. Saito, Modules de Hodge polarisables, *Publ. Res. Inst. Math. Sci.* **24** (1988), 849–995 (1989). MR 1000123. Zbl 0691.14007. http://dx.doi.org/10.2977/prims/1195173930.

[VZ01] E. Viehweg and K. Zuo, On the isotriviality of families of projective manifolds over curves, *J. Algebraic Geom.* **10** (2001), 781–799. MR 1838979. Zbl 1079.14503.

[Zha97] Q. Zhang, Global holomorphic one-forms on projective manifolds with ample canonical bundles, *J. Algebraic Geom.* **6** (1997), 777–787. MR 1487236. Zbl 0922.14008.

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