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Truncations of level 1 of elements in the loop group of a reductive group

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Abstract

The aim of this article is to define and study a new invariant of elements of loop groups that is invariant under σ-conjugation by a hyperspecial maximal open subgroup and that we call the truncation of level 1. We classify truncations of level 1 and describe their specialization behavior. Furthermore, we prove group-theoretic conditions for the set of σ-conjugacy classes obtained from elements of a given truncation of level 1 and in particular for the generic σ-conjugacy class in any given truncation stratum. In the last section we relate our invariant to the Ekedahl-Oort stratification of the Siegel moduli space and to generalizations to other PEL Shimura varieties.

1. Introduction

Let k be an algebraically closed field of characteristic $p > 0$. Let L be either $k((t))$ or $\mathrm{Quot}(W(k))$, and let $\mathcal{O}$ be the valuation ring. Here $W(k)$ is the ring of Witt vectors of k. We denote by $\sigma : x \mapsto x^q$ the Frobenius of k over $\mathbb{F}_q$ for some fixed $q = p^r$ and also the Frobenius of L over $F = \mathbb{F}_q((t))$ respectively $F = \mathbb{Q}_q = \mathrm{Quot}(W(\mathbb{F}_q))$. Let $\mathcal{O}_F$ be the valuation ring of F. We denote the uniformizer t or p of $\mathcal{O}_F$ by ε.

Let G be a connected reductive group over $\mathcal{O}_F$. Then G is quasi-split and split over an unramified extension of $\mathcal{O}_F$ (compare Section 2.1). Let B be a Borel subgroup of G, and let T be a maximal torus contained in B. Let $K = G(\mathcal{O})$, and let I be the inverse image of $B(k)$ under the projection $K \to G(k)$. Let K_1 be the kernel of the projection $K \to G(k)$.

For $b \in G(L)$, we call $\{g^{-1}b\sigma(g) \mid g \in K\}$ the K-σ-conjugacy class of b and $[b] = \{g^{-1}b\sigma(g) \mid g \in G(L)\}$ the σ-conjugacy class of b. In [Kot85] Kottwitz studies σ-conjugacy classes of elements of $G(L)$ and classifies them by two invariants, the Newton point and the Kottwitz point; cf. Section 1.2. In particular, he obtains a discrete invariant on the set of K-σ-conjugacy classes. The aim of this article is to study a second invariant of K-σ-conjugacy classes, namely the truncation of level 1, which we define as the associated K-σ-conjugacy class in $K_1\backslash G(L)/K_1$.

Note that for L of mixed characteristic, both K-σ-conjugacy classes and σ-conjugacy classes occur naturally in the study of the reduction of Shimura varieties of PEL type, i.e., for moduli spaces of abelian varieties or p-divisible groups with extra structure consisting of a *p*olarization, *e*ndomorphisms, and a *l*evel structure. For example, p-divisible groups of height h over an algebraically closed field k of characteristic p are classified by their Dieudonné modules. The Dieudonné module is a pair $(\mathbf{M}, F)$ where $\mathbf{M}$ is a free $W(k)$-module of rank h and where $F : \mathbf{M} \to \mathbf{M}$ is a σ-linear homomorphism satisfying $F(\mathbf{M}) \supseteq p\mathbf{M}$. Here σ denotes the Frobenius of $W(k)$ over $\mathbb{Z}_p$. Choosing a basis for $\mathbf{M}$ we can write $F = b\sigma$ for some $b \in \mathrm{GL}_h(W(k)[1/p])$. A change of the basis amounts to σ-conjugating b by an element of $\mathrm{GL}_h(W(k)) = K$. Thus the isomorphism class of the p-divisible group corresponds to the K-σ-conjugacy class of b. Isogeny classes of p-divisible groups are likewise in bijection with rational Dieudonné modules, which are described by the σ-conjugacy classes of the corresponding elements $b \in \mathrm{GL}_h(W(k)[1/p])$. In the function field case $L = k((t))$ a similar interpretation relates K-σ-conjugacy classes and conjugacy classes of elements of $G(L)$ to isomorphism classes and isogeny classes of local G-shtukas, respectively.

1.1. *Classification of truncations of level* 1. Let us first introduce some notation. Let $W = N_T(L)/T(L)$ denote the (absolute) Weyl group of T in G where N_T denotes the normalizer of T. Let $\widetilde{W} = N_T(L)/T(\mathcal{O}) \cong W \ltimes X_*(T)$ denote the extended affine Weyl group. For each $w \in W$ we choose a representative in $N_T(\mathcal{O})$. We denote this representative by the same letter as the element itself. If M is a Levi subgroup of G containing T, let W_M be the Weyl group of M and denote by ${}^MW$ respectively ${}^M\widetilde{W}$ the set of elements x of W respectively $\widetilde{W}$ that are shortest representatives of their coset W_Mx. Similarly, W^M denotes the set of elements x that are the shortest representatives of their cosets xW_M and accordingly for $\widetilde{W}$. For a dominant $\mu \in X_*(T)$, let M_μ be the centralizer of μ and let ${}^\mu W = \sigma^{-1}({}^{M_\mu}W)$. Let $P_\mu = M_\mu B$, a standard parabolic with Levi subgroup M_μ. Let $x_\mu = w_0w_{0,\mu}$, where w_0 denotes the longest element of W and where $w_{0,\mu}$ is the longest element of W_{M_μ}. Let $\tau_\mu = x_\mu\varepsilon^\mu$, where ε^μ is the image of ε under $\mu : \mathbb{G}_m \to T$. Then τ_μ is the shortest element of $W\varepsilon^\mu W$.

The classification of K-σ-conjugacy classes of elements of $K_1\backslash G(L)/K_1$ is given by the following theorem which we prove in Section 3. The second part of the theorem establishes a relation between the subdivisions of $K_1\backslash G(L)/K_1$ according to K-σ-conjugacy classes and according to Iwahori-double cosets.

Theorem 1.1.

(1) *Let* $\mathcal{T} = \{(w,\mu) \in W \times X_*(T)_{\text{dom}} \mid w \in {}^{\mu}W\}$. *Then the map assigning to* (w,μ) *the* K-σ*-conjugacy class of* $K_1 w\tau_\mu K_1$ *is a bijection between* $\mathcal{T}$ *and the set of* K-σ*-conjugacy classes in* $K_1\backslash G(L)/K_1$.
(2) *Let* $\mu \in X_*(T)_{\text{dom}}$ *and* $w \in {}^{\mu}W$. *Then each element of* $Iw\tau_\mu I$ *is* I-σ*-conjugate to an element of* $K_1 w\tau_\mu K_1$.

Definition 1.2. We denote by tr the map $G(L) \to \mathcal{T}$ assigning to each b the element of $\mathcal{T}$ corresponding to its K-σ-conjugacy class in $K_1\backslash G(L)/K_1$ under the bijection in Theorem 1.1. The pair $\text{tr}(b) \in W \times X_*(T)$ is called the *truncation of level* 1 of b.

Let $L = k((t))$. In this case we can also study the variation of the truncation of level 1 in families. Let LG be the loop group of $G_{\mathbb{F}_q}$, i.e., the group ind-scheme representing the functor on $\mathbb{F}_q$-algebras $R \mapsto G(R((t)))$; compare [Fal03, Def. 1]. We show in Section 4 that for each $(w,\mu) \in \mathcal{T}$, the set of $b \in G(L)$ with $\text{tr}(b) = (w,\mu)$ is the set of k-valued points of a bounded locally closed subscheme of the loop group LG of $G_{\mathbb{F}_q}$. For the notion of boundedness, see Section 2.

Definition 1.3. Let $(w,\mu) \in \mathcal{T}$, and assume that $\text{char}(F) = p$. Let $S_{w,\mu}$ be the reduced subscheme of the loop group of $G_{\mathbb{F}_q}$ such that $S_{w,\mu}(k)$ consists of those $g \in G(k((t)))$ with $\text{tr}(g) = (w,\mu)$.

The closure of a stratum $S_{w,\mu}$ in LG is a union of finitely many strata (see Lemma 4.1).

Theorem 1.4. *Let* $S_{w',\mu'}, S_{w,\mu} \subseteq$ LG *be two truncation strata. Then* $S_{w',\mu'} \subseteq \overline{S_{w,\mu}}$ *if and only if there is a* $\tilde{w} \in W$ *with* $\tilde{w}w'\tau_{\mu'}\sigma(\tilde{w})^{-1} \leq w\tau_\mu$ *with respect to the Bruhat order.*

For $F = \mathbb{Q}_q$, it is not clear how to define an ind-scheme having $G(L)$ as its set of k-valued points. However one can study the stratifications induced on the reduction modulo p of certain Shimura varieties. The main part of this paper is concerned with elements of $G(L)$ for both cases or, whenever a scheme structure is involved, the equicharacteristic case. The applications of our theory to Shimura varieties are detailed in Section 7.

1.2. *Truncations of level* 1 *and* σ*-conjugacy classes.* A second major goal of this article is to compare the stratification of LG by truncations of level 1 to the stratification by σ-conjugacy classes. More precisely, we study when a given truncation stratum intersects a given σ-conjugacy class nontrivially. Our main result in this context (Theorem 1.5) is a necessary condition for nonemptiness of these intersections that determines, in particular, the generic σ-conjugacy class in each trunction stratum.

We first review Kottwitz's classification [Kot85] of the set $B(G)$ of σ-conjugacy classes of elements $b \in G(L)$ that generalizes the notion of Newton polygons. (Compare also [RR96, §1] for a more complete review of these results.) Each σ-conjugacy class is determined by two invariants. One of them is given by a map $\kappa_G : B(G) \to \pi_1(G)_\Gamma$, where $\pi_1(G)$ is the quotient of $X_*(T)$ by the coroot lattice and where Γ is the absolute Galois group of F. There is the following explicit description of κ_G. Let $b \in G(L)$, and let $\mu \in X_*(T)$ be such that $b \in K\varepsilon^\mu K$; compare Section 2.3. Then $\kappa_G(b)$ is the image of μ under the canonical projection from $X_*(T)$ to $\pi_1(G)_\Gamma$. The second invariant is the so-called Newton point $\nu = \nu_b$ of b, an element of $(X_*(T)_{\mathbb{Q}}/W)^\Gamma$, the set of Γ-invariant W-orbits on $X_*(T) \otimes \mathbb{Q}$. We usually consider the dominant representative of ν, an element of $X_*(T)_{\mathbb{Q}}^\Gamma$ which we denote by the same letter ν. This invariant is the direct analog of the usual Newton polygon classifying F-isocrystals over an algebraically closed field. The images of ν_b and $\kappa(b)$ in $\pi_1(G)_\Gamma \otimes \mathbb{Q}$ coincide. Note that Kottwitz's original article only considers the case of mixed characteristic, but the other case can be treated in exactly the same way. Furthermore, the two invariants ν and κ lie in groups that are independent of the choice of L; compare Remark 6.9.

We further need the partial order on $B(G)$ defined by Rapoport and Richartz in [RR96]. It is given by $[b] \preceq [b']$ if and only if $\kappa_G(b) = \kappa_G(b')$ and $\nu_b \preceq \nu_{b'}$. Here the second condition means that $\tilde{\nu}_{b'} - \tilde{\nu}_b$ is a linear combination of positive coroots with coefficients in $\mathbb{Q}_{\geq 0}$ where $\tilde{\nu}_{b'}$ and $\tilde{\nu}_b$ are dominant representatives of the two orbits (compare Lemma 2.2 of loc. cit.). Their Theorem 3.6 shows that for each $[b]$, the union of all σ-conjugacy classes that are less or equal to $[b]$ is closed in the loop group. More precisely, they show a corresponding statement over a field F of mixed characteristic. The function field analog can be shown in a similar, but slightly easier way using properties of the affine Grassmannian; compare [HV11, Th. 7.3]. For split groups G, [Vie13] shows that $\preceq$ describes the precise closure relations of the classes $[b] \subset \mathrm{LG}$.

Let $[b] \in B(G)$. Let M be the centralizer of the dominant Newton point ν_b of b, the Levi component of a standard parabolic subgroup defined over $\mathcal{O}_F$. In Section 6 we define $[b]$-short elements as elements x of length 0 in $\widetilde{W}_M$ with M-dominant Newton point ν_b and $\kappa_G(x) = \kappa_G(b)$. In particular, $[b]$-short elements are contained in $[b]$. The following theorem is a necessary condition for nonemptiness of intersections of truncation strata and σ-conjugacy classes. It is equivalent to nonemptiness of the intersection of a σ-conjugacy class with the closure of a truncation stratum and can (contrary to the definition of nonemptiness itself) be effectively checked in finite time.

THEOREM 1.5. *Let* $b \in G(k((t)))$, *and let* $(w, \mu) = \mathrm{tr}(b)$. *Then there is a* $[b]$*-short element* $x \in \overline{S_{w,\mu}}$.

We call an element $x \in \widetilde{W}$ short if it is $[b]$-short for some $[b] \in B(G)$. Note that each stratum $S_{w,\mu}$ in the loop group is irreducible (Lemma 4.1), hence it contains a unique generic σ-conjugacy class. From Theorem 1.5, one deduces the following corollary, which characterizes this σ-conjugacy class.

COROLLARY 1.6. *Let $[b]$ be the generic σ-conjugacy class in $S_{w,\mu} \subseteq \mathrm{LG}$ for some $w \in {}^{\mu}W$. Then $[b]$ is equal to the unique maximal element in the set of σ-conjugacy classes of short elements $x \in \widetilde{W}$ such that $x \in \overline{S_{w,\mu}}$. This is also the same as the maximal class $[x]$ among all $x \in \widetilde{W}$ with $x \leq w\tau_{\mu}$ in the Bruhat order.*

1.3. *Comparison between equal and mixed characteristic.* In Section 6 we prove the following theorem that allows to translate results between the function field case and the arithmetic case without having to repeat proofs. It uses that the set $B(G)$ of σ-conjugagcy classes of elements of $G(k((t)))$ can be canonically identified with that for $G(W(k)[1/p])$ using the invariants ν and κ (Remark 6.9).

THEOREM 1.7. *Let $(w, \mu) \in \mathcal{T} \subseteq W \times X_*(T)$. Then a σ-conjugacy class in $\mathrm{LG}(k)$ contains an element of truncation type (w, μ) if and only if the corresponding σ-conjugacy class in $G(W(k)[1/p])$ contains an element of truncation type (w, μ).*

Using this comparison and Theorem 1.4 we obtain the following analog of Theorem 1.5 in the arithmetic context.

THEOREM 1.8. *Let $(w, \mu) \in \mathcal{T}$, and let $b \in G(W(k)[1/p])$ with $\mathrm{tr}(b) = (w, \mu)$. Then there is a $[b]$-short element x satisfying the following condition. Let $\mathrm{tr}(x) = (w', \mu')$. Then there is a $\tilde{w} \in W$ with $\tilde{w}w'\tau_{\mu'}\sigma(\tilde{w})^{-1} \leq w\tau_{\mu}$.*

1.4. *Comparison with Ekedahl-Oort strata.* Let X be a p-divisible group over an algebraically closed field k of characteristic p. Let $(\mathbf{M}, F)$ be its Dieudonné module, and write $F = b\sigma$ with $b \in \mathrm{GL}_h(W(k)[1/p])$ with respect to some trivialization of $\mathbf{M}$. As $p\mathbf{M} \subseteq F(\mathbf{M}) \subseteq \mathbf{M}$, we have $b \in Kp^{\mu}K$ for some minuscule $\mu \in X_*(T)$.

In [Oor01], Oort shows that one obtains a discrete invariant of X (the so-called Ekedahl-Oort invariant) by considering the isomorphism class of the p-torsion points $X[p]$, or equivalently by studying the reduction modulo p of the Dieudonné module $\mathbf{M}$ together with the two maps induced by $F : \mathbf{M} \to \mathbf{M}$ and $V = pF^{-1} : \mathbf{M} \to \mathbf{M}$. Reformulating this invariant in terms of the element b, it corresponds to considering the K_1-double coset. In other words, we can apply our theory in the special case $G = \mathrm{GL}_h$ and μ minuscule for $\mathcal{O} = W(k)$ to study the Ekedahl-Oort invariant of p-divisible groups. Likewise, truncations of level 1 for other groups yield classifications of Ekedahl-Oort invariants of p-divisible groups with extra structure by a polarization or endomorphisms.

In Section 7 we further study the relation between truncation strata in loop groups and Ekedahl-Oort strata in PEL Shimura varieties. Using Theorem 1.7 we obtain a direct comparison for nonemptiness of intersections between truncation strata and σ-conjugacy classes on the one hand and between Ekedahl-Oort strata and isogeny classes of p-divisible groups on the other hand. It allows us to deduce a nonemptiness criterion for Shimura varieties that is analogous to Theorem 1.5 and generalizes a result of Harashita that proved a conjecture by Oort.

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2. **Reductive groups over local rings**

In this section we summarize some facts about reductive groups over local rings that are used frequently in the paper.

2.1. Let G be a connected reductive group over $\mathcal{O}_F$. Then G is quasi-split and split over an unramified extension of $\mathcal{O}_F$. Indeed, let k_F be the residue field of $\mathcal{O}_F$. Then G_{k_F} is quasi-split and split over a finite extension of k_F. Furthermore, a Borel subgroup over k_F and a split maximal torus over a finite extension of k_F can be lifted to a Borel subgroup and a split maximal torus over $\mathcal{O}_F$ respectively over the corresponding unramified extension of $\mathcal{O}_F$; compare [VW13, A.4].

2.2. The extended affine Weyl group $\widetilde{W}$ has a decomposition $\widetilde{W} \cong \Omega \ltimes W_{\rm aff}$. Here Ω is the subset of elements of $\widetilde{W}$ that fix the chosen Iwahori subgroup I of $G(L)$. The second factor $W_{\rm aff}$ is the affine Weyl group of G. In terms of the decomposition $\widetilde{W} \cong W \ltimes X_*(T)$, it has the following description. Let $G_{\rm sc}$ be the simply connected cover of G, and let $T_{\rm sc}$ the inverse image of T in $G_{\rm sc}$. Then $W_{\rm aff} \cong W \ltimes X_*(T_{\rm sc})$ and $\Omega \cong X_*(T)/X_*(T_{\rm sc})$. The affine Weyl group of G is an infinite Coxeter group. It is generated by the simple reflections s_i associated with the simple roots of T in G together with the simple affine root. The choice of I also induces an ordering on $\widetilde{W}$, the Bruhat ordering. It is defined as follows. Let $x, y \in \widetilde{W}$, and let $x = \omega_x x'$ and $y = \omega_y y'$ be their decompositions into elements of Ω and W_{aff}. Then $x \leq y$ if and only if $\omega_x = \omega_y$ and if there are reduced expressions $x' = s_{i_1} \cdots s_{i_n}$ and $y' = s_{j_1} \cdots s_{j_m}$ for x' and y' such that $(s_{i_1}, \dots, s_{i_n})$ is a subsequence of $(s_{j_1}, \dots, s_{j_m})$.

Recall the morphism $\kappa_G : G(L) \to \pi_1(G)_\Gamma$ where $\pi_1(G)$ is the quotient of $X_*(T)$ by the coroot lattice. It induces a surjection $\kappa_G : \widetilde{W} \cong W \ltimes X_*(T) \to \pi_1(G)_\Gamma$. The subgroup W_{aff} of $\widetilde{W}$ is in the kernel of κ_G. On the subgroup Ω of the extended affine Weyl group, κ_G induces the canonical projection $\Omega \cong X_*(T)/X_*(T_{\text{sc}}) \xrightarrow{\sim} \pi_1(G) \to \pi_1(G)_\Gamma$.

2.3. We have the following decompositions. More details can, for example, be found in [Tit79].

Iwasawa decomposition. Let P be a parabolic subgroup of G. Then $G(L) = P(L)K$.

Bruhat-Tits decomposition. $G(L) = \coprod_{x \in \widetilde{W}} IxI$. In the function field case the double cosets are locally closed subschemes of LG. The closure of IxI is equal to the union of all $Ix'I$, where $x' \leq x$ in the Bruhat order.

Cartan decomposition. $G(L) = \coprod_{\mu \in X_*(T)_{\text{dom}}} K\varepsilon^\mu K$, where $X_*(T)_{\text{dom}}$ denotes the set of dominant elements of $X_*(T)$ and where ε^μ is defined to be the image of ε under $\mu : \mathbb{G}_m \to T$. In the function field case the double cosets are locally closed subschemes of LG. The closure of $K\varepsilon^\mu K$ is equal to the union of all $K\varepsilon^{\mu'} K$ where $\mu' \preceq \mu$. Here $\mu' \preceq \mu$ if $\mu - \mu'$ is a nonnegative integral linear combination of positive coroots.

Iwahori decomposition. Let P be a standard parabolic subgroup of G, and let N be its unipotent radical and M the Levi factor containing T. Let $\overline{N}$ be the unipotent radical of the opposite parabolic. Let $I_M = I \cap M(L)$ and analogously for $N, \overline{N}$. Then $I = I_N I_M I_{\overline{N}}$.

2.4. A subset of the loop group LG is called bounded if it is contained in a finite union of double cosets $K\varepsilon^\mu K$. For $n \in \mathbb{N}$ let $K_n = \{g \in K \mid g \equiv 1 \pmod{\varepsilon^n}\}$. Then a subscheme S of LG is called admissible if there is an $n_S \in \mathbb{N}$ with $SK_{n_S} = S$. Let $S \subseteq$ LG be a bounded and admissible subscheme, and let n_S be as above. Let $\mathcal{B}$ be a finite union of double cosets containing S. Then S can be studied by considering the image in $\mathcal{B}/K_{n_S}$, which is a scheme of finite type. For example, S is called locally closed if the same holds for its image in $\mathcal{B}/K_{n_S}$. The closure of S is defined to be the inverse image under LG $\to$ LG$/K_{n_S}$ of the closure of S in LG$/K_{n_S}$. The subscheme S is called smooth or irreducible if the same holds for its image in $\mathcal{B}/K_{n_S}$. Note that these notions do not depend on the choice of $\mathcal{B}$ and of n_S provided that they are large enough.

2.5. The following lemma is a variant of the theorem of Lang-Steinberg for the infinite-dimensional group schemes that we want to consider.

LEMMA 2.1. *Let $H \subseteq K$ be a subgroup of K. For all $n \in \mathbb{Z}_{\geq 0}$ let $H_n = \{h \in H \mid h \equiv 1 \pmod{\varepsilon^n}\}$. We assume that H/H_n and H_{n-1}/H_n are connected linear algebraic groups for all n. Let $g \in G(L)$ with $g^{-1}H_n g \subseteq \sigma(H_n)$ for all n. Then the morphism $H \to H$ with $h \mapsto \sigma^{-1}(g^{-1}h^{-1}g)h$ is surjective.*

Proof. Let $h \in H$ and $n \in \mathbb{N}$. By the Theorem of Lang-Steinberg there is an $h_n \in H/H_n$ with $\sigma^{-1}(g^{-1}h_n^{-1}g)h_n \in hH_n$. We want to show that we can lift h_n to an element $h_{n+1} \in H/H_{n+1}$ with $\sigma^{-1}(g^{-1}h_{n+1}^{-1}g)h_{n+1} \in hH_{n+1}$. Let $f_{n+1} \in H/H_{n+1}$ be an arbitrary lift of h_n. We now apply the Theorem of Lang-Steinberg to the morphism $H_n/H_{n+1} \to H_n/H_{n+1}$ with $\psi \mapsto h^{-1}\sigma^{-1}(g^{-1}\psi^{-1}f_{n+1}^{-1}g)f_{n+1}\psi$. Note that H_{n+1} is a normal subgroup of H for all n. Hence this is indeed a well-defined element of H_n/H_{n+1}. Let ψ_{n+1} be an inverse image of the identity element under this morphism. Then $h_{n+1} = f_{n+1}\psi_{n+1}$ is as claimed. Using induction and passing to the limit we obtain an element $h_\infty \in K$ with $\sigma^{-1}(g^{-1}h_\infty^{-1}g)h_\infty = h$. □

2.6. Let P be a standard parabolic subgroup of G, i.e., $B \subseteq P$. We denote by M the Levi factor containing T and by N its unipotent radical. Let $\mu \in X_*(T)$. Let α be a root and U_α the corresponding root subgroup. Then $\varepsilon^\mu U_\alpha(x)\varepsilon^{-\mu} = U_\alpha(\varepsilon^{\langle\alpha,\mu\rangle}x)$. In particular, we have $\varepsilon^\mu N(\mathcal{O})\varepsilon^{-\mu} \subseteq N(\mathcal{O}) \cap K_1$ if $\langle\alpha,\mu\rangle > 0$ for all roots α of T in N. This is, for example, the case if μ is dominant and M contains the centralizer of μ.

3. Truncations of level 1

The goal of this section is to prove Theorem 1.1; in particular, we allow both the function field case and the case of mixed characteristic. The proof follows a strategy by Bédard [Béd85].

Proof of Theorem 1.1. Let $b \in G(L)$. By the Cartan decomposition there is a unique dominant $\mu \in X_*(T)$ with $b \in K\varepsilon^\mu K$ and b is K-σ-conjugate to an element of the form $b_0 x_\mu \varepsilon^\mu = b_0\tau_\mu$ with $b_0 \in K$.

To show (1) we have to prove that there is a unique $w \in {}^\mu W$ such that $b_0\tau_\mu$ is K-σ-conjugate to an element of $K_1 w\tau_\mu K_1$. We use induction on i to show that there exist a sequence of elements $u_i \in W$, two sequences of standard Levi subgroups M_i, M_i' of G, and a sequence of elements $b_i \in M_i'(\mathcal{O})$ with the following properties:

(a) $M_0 = M_0' = G$,
$M_1' = x_\mu M_\mu x_\mu^{-1}$ and $M_1 = \sigma^{-1}(M_\mu)$,
$M_i = M_{i-1}' \cap u_{i-1}^{-1}\sigma^{-1}(M_\mu)u_{i-1}$, and
$M_i' = M_1' \cap x_\mu\sigma(u_{i-1}M_{i-1}'u_{i-1}^{-1})x_\mu^{-1} = x_\mu\sigma(u_{i-1}M_i u_{i-1}^{-1})x_\mu^{-1}$ for $i > 1$.

(b) $u_0 = 1$ and
$u_i = u_{i-1}\delta_i$ for $i > 0$ for some $\delta_i \in W_{M_{i-1}'}$ that is the shortest representative

of $W_{M_i}\delta_i W_{M'_i}$ and
u_i is the shortest representative of $W_{M_1}u_iW_{M'_i}$.

(c) b is K-σ-conjugate to an element of $K_1u_ib_i\tau_\mu K_1$.

(d) $u_iW_{M'_i} \subseteq W$ is uniquely determined by the K-σ-conjugacy class of b in $K_1\backslash G(L)/K_1$.

(e) $u_ib'_i\tau_\mu$ with $b'_i \in M'_i(\mathcal{O})$ is in the K-σ-conjugacy class of b in $K_1\backslash G(L)/K_1$ if and only if the images in $G(k) = G(\mathcal{O})/K_1$ of the two elements b_i, b'_i are in the same $M_{i+1}(k)$-orbit in $U_{P_{i+1}}(k)\backslash M'_i(k)/U_{\overline{P}'_{i+1}}(k)$ under the action

$$M_{i+1}(\mathcal{O}) \times U_{P_{i+1}}(k)\backslash M'_i(k)/U_{\overline{P}'_{i+1}}(k) \to U_{P_{i+1}}(k)\backslash M'_i(k)/U_{\overline{P}'_{i+1}}(k)$$
$$(g, m) \mapsto g^{-1}mx_\mu\sigma(u_igu_i^{-1})x_\mu^{-1}.$$

Here
$P_0 = P'_0 = G$,
$P'_1 = x_\mu P_\mu x_\mu^{-1}$ and $P_1 = \sigma^{-1}(P_\mu)$,
$P_i = M'_{i-1} \cap u_{i-1}^{-1}\sigma^{-1}(P_\mu)u_{i-1}$ and
$P'_i = M'_1 \cap x_\mu\sigma(u_{i-1}P'_{i-1}u_{i-1}^{-1})x_\mu^{-1}$ for $i > 1$ are parabolic subgroups of M'_{i-1}.
Furthermore, U_P denotes the unipotent radical of a linear algebraic group P.

Before we begin the proof, let us show that some of the conditions automatically follow from the others. We first check inductively that (a) and (b) imply that $M'_i \supseteq P'_{i+1}$ and that M'_{i+1} is the Levi subgroup of P'_{i+1} containing T for all i. For $i = 0$ this is obvious. The second statement is also clear by induction. From (b) and the induction hypothesis $P'_i \subseteq M'_{i-1}$ we see that $P'_{i+1} = M'_1 \cap x_\mu\sigma(u_iP'_iu_i^{-1})x_\mu^{-1}$ is contained in $M'_1 \cap x_\mu\sigma(u_{i-1}M'_{i-1}u_{i-1}^{-1})x_\mu^{-1} = M'_i$.

Results of Bédard [Béd85], or Lusztig ([Lus04], (a)–(d) in the proof of Proposition 2.4) show that the last condition in (b) follows automatically using induction, using the condition on δ_i and the definition of M_i and M'_i.

Note that if $g \in M_{i+1}(\mathcal{O})$, then $x_\mu\sigma(u_igu_i^{-1})x_\mu^{-1} \in M'_{i+1}(\mathcal{O})$, so the action in (e) is well defined.

Claim. *Conditions* (a) *and* (b) *above imply that the M_i and M'_i are standard Levi subgroups.*

We show this claim using induction. The Levi subgroups M_μ and $\sigma^{-1}(M_\mu)$ are standard as μ is dominant and as B is invariant under σ. Now we use the following fact: If M is a standard Levi, if α is a simple root that lies in M, and if $x \in W^M$ then $x(\alpha)$ is again positive. If each such $x(\alpha)$ is again simple then xMx^{-1} is again a standard Levi subgroup. For $x = x_\mu = w_0w_{0,\mu}$ this implies that $x_\mu M_\mu x_\mu^{-1}$ is standard. For the induction step we show that if M, M' are standard and x is the shortest representative of $W_{M'}xW_M$, then $M' \cap xMx^{-1}$ is also standard. By the above fact it is enough to show that for every simple root α in M such that $x(\alpha)$ is a root in M', this root is also simple. We have $1+\ell(x)=\ell(xs_\alpha)=\ell(xs_\alpha x^{-1})+\ell(x)$ where the last equality uses that $xs_\alpha x^{-1}\in W_{M'}$, compare [DDPW08, Lemma 4.17]. Hence $\ell(xs_\alpha x^{-1})=1$, and $x(\alpha)$ is simple. We apply this first to $u_{i-1}^{-1}\in W^{M_1}\cap {}^{M'_{i-1}}W$, M_1 and M'_{i-1} to obtain inductively that $M_i=M'_{i-1}\cap u_{i-1}^{-1}M_1u_{i-1}$ is standard. For M'_i the properties of σ and x_μ already used above imply that it is enough to show that $M_1\cap u_{i-1}M'_{i-1}u_{i-1}^{-1}$ is standard. This follows in the same way as before.

We now carry out the induction to show (a)–(e). For $i=0$, (d) is obvious and (c) has been shown above. For (e) Section 2.6 implies that on K_1-double cosets, the effect of σ-conjugation of $b_0\tau_\mu$ by $g\in U_{P_1}(\mathcal{O})$ is the same as left multiplication of b_0 by g. Similarly one sees $\varepsilon^{-\mu}U_{\overline{P}_\mu}(\mathcal{O})\varepsilon^\mu\subset K_1$. Hence right multiplication of b_0 by an element of $U_{\overline{P}'_1}(\mathcal{O})$ does not change the class $b_0\tau_\mu K_1$. The effect of the action of $M_1(\mathcal{O})$ on b_0 corresponds to σ-conjugation of $b_0\tau_\mu$ and thus it does not change the K-σ-conjugacy class of $b_0\tau_\mu$. For the other direction, if an element $g\in K$ conjugates $b_0\tau_\mu$ into $M'_0(\mathcal{O})\tau_\mu=K\varepsilon^\mu$, then $\sigma(g)\in K\cap\varepsilon^{-\mu}K\varepsilon^\mu$. In particular g is contained in the parahoric subgroup of K of elements whose image in $G(k)$ is in $P_1(k)$. Using the analog of the Iwahori decomposition for this subgroup and the fact that $\sigma(g)\in\varepsilon^{-\mu}K\varepsilon^\mu$ we obtain a decomposition of g into factors in $U_{P_1}(\mathcal{O})$, $M_1(\mathcal{O})$ and $\varepsilon^{-\sigma^{-1}(\mu)}U_{\overline{P}_1}(\mathcal{O})\varepsilon^{\sigma^{-1}(\mu)}=\sigma^{-1}(\varepsilon^{-\mu}U_{\overline{P}_\mu}(\mathcal{O})\varepsilon^\mu)\subset K_1$. This shows (e) and finishes the argument for $i=0$.

We have to show that (a)–(e) for some i imply the same properties for $i+1$. Let b_i be as in (c). We decompose b_i using the Bruhat decomposition to obtain that $b_i\in K_1P_{i+1}(\mathcal{O})\delta_{i+1}\overline{P}'_{i+1}(\mathcal{O})$ for some δ_{i+1} as in (b) and with P_{i+1} and P'_{i+1} as in (e). We may assume that the factor in K_1 is trivial. By (e) we may further assume that the factors in $P_{i+1}(\mathcal{O})$ and $\overline{P}'_{i+1}(\mathcal{O})$ lie in $M_{i+1}(\mathcal{O})$ and $M'_{i+1}(\mathcal{O})$, respectively. We obtain a decomposition $u_ib_i\tau_\mu\in\big(u_iM_{i+1}(\mathcal{O})u_i^{-1}\big)u_i\delta_{i+1}M'_{i+1}(\mathcal{O})\tau_\mu$. After σ-conjugating $u_ib_i\tau_\mu$ with the factor in $u_iM_{i+1}(\mathcal{O})u_i^{-1}$ and using that $\sigma(u_iM_{i+1}(\mathcal{O})u_i^{-1})=x_\mu^{-1}M'_{i+1}x_\mu\subseteq x_\mu^{-1}M'_1x_\mu=M_\mu$ we obtain (c) for $i+1$. Property (d) follows from the uniqueness of the Bruhat decomposition together with (d) and (e) for i. It remains to show (e). If we replace b_{i+1} by βb_{i+1} where β is an element of $U_{P_{i+2}}(\mathcal{O})$, this has the effect that the product $u_{i+1}b_{i+1}\tau_\mu$ is multiplied on the left with an element $\delta(\beta)$ of $u_{i+1}U_{P_{i+2}}(\mathcal{O})u_{i+1}^{-1}=U_{P_1}(\mathcal{O})\cap u_{i+1}M'_{i+1}(\mathcal{O})u_{i+1}^{-1}$. This does not change the K-σ-conjugacy class in $K_1\backslash G(L)/K_1$. Indeed, by Section 2.6 right multiplication by elements of $\sigma(U_{P_1}(\mathcal{O})\cap u_{i+1}M'_{i+1}(\mathcal{O})u_{i+1}^{-1})$ does not change the coset $K_1u_{i+1}b_{i+1}\tau_\mu$, and hence $K_1\delta(\beta)u_{i+1}b_{i+1}\tau_\mu=K_1\delta(\beta)u_{i+1}b_{i+1}\tau_\mu\sigma(\delta(\beta)^{-1})$.

Now we want to show that replacing b_{i+1} by $b_{i+1}\beta$ with $\beta\in U_{\overline{P}'_{i+2}}(\mathcal{O})$ also does not change the K-σ-conjugacy class of $u_{i+1}b_{i+1}\tau_\mu$. As $M'_{i+1}\subseteq x_\mu M_\mu x_\mu^{-1}$ this replacement has the effect that $u_{i+1}b_{i+1}\tau_\mu=u_{i+1}b_{i+1}x_\mu\varepsilon^\mu$ is multiplied on the right with $\tau_\mu^{-1}\beta\tau_\mu = x_\mu^{-1}\beta x_\mu$. We σ-conjugate with the element

$$\sigma^{-1}(x_\mu^{-1}\beta x_\mu)^{-1} \in \sigma^{-1}(x_\mu^{-1}U_{\overline{P}'_{i+2}}(\mathcal{O})x_\mu) = \sigma^{-1}(M_\mu(\mathcal{O})) \cap u_{i+1}U_{\overline{P}'_{i+1}}(\mathcal{O})u_{i+1}^{-1}$$

(which is in particular in K). Then we obtain an element of

$$u_{i+1}U_{\overline{P}'_{i+1}}(\mathcal{O})b_{i+1}x_\mu\varepsilon^\mu.$$

As $b_{i+1} \in M'_{i+1}(\mathcal{O})$, the element lies in $u_{i+1}b_{i+1}U_{\overline{P}'_{i+1}}(\mathcal{O})x_\mu\varepsilon^\mu$. Using induction we obtain that this is contained in the same class as $u_{i+1}b_{i+1}x_\mu\varepsilon^\mu$. Finally, the effect of the action of $M_{i+2}(\mathcal{O})$ on b_{i+1} corresponds to σ-conjugation of $u_{i+1}b_{i+1}\tau_\mu$ by elements of $u_{i+1}M_{i+2}(\mathcal{O})u_{i+1}^{-1}$, and thus it leaves the K-σ-conjugacy class of $u_{i+1}b_{i+1}\tau_\mu$ stable. It remains to show the other direction of (e). So assume that $u_{i+1}b'_{i+1}\tau_\mu = u_i(\delta_{i+1}b'_{i+1})\tau_\mu$ with $b'_{i+1} \in M'_{i+1}(\mathcal{O})$ is in the K-σ-conjugacy class of $u_{i+1}b_{i+1}\tau_\mu = u_i(\delta_{i+1}b_{i+1})\tau_\mu$ in $K_1\backslash G(L)/K_1$. Using induction for $\delta_{i+1}b'_{i+1}, \delta_{i+1}b_{i+1} \in M'_i(\mathcal{O})$ we obtain elements $g \in M_{i+1}(\mathcal{O})$, $a \in U_{P_{i+1}}(\mathcal{O})$ and $a' \in U_{\overline{P}'_{i+1}}(\mathcal{O})$ with

$$g^{-1}\delta_{i+1}b_{i+1}x_\mu\sigma(u_igu_i^{-1})x_\mu^{-1} = a\delta_{i+1}b'_{i+1}a'.$$

Let $h = \delta_{i+1}^{-1}g\delta_{i+1}$ and $\tilde{a} = b'_{i+1}a'(b'_{i+1})^{-1}$. Then $\tilde{a} \in U_{\overline{P}'_{i+1}}$, and

$$h^{-1}b_{i+1}x_\mu\sigma(u_{i+1}hu_{i+1}^{-1})x_\mu^{-1} = \delta_{i+1}^{-1}a\delta_{i+1}\tilde{a}b'_{i+1}. \tag{1}$$

Notice that if P, Q are connected linear algebraic subgroups containing T such that P is parabolic and if we denote $P = MU_P$ the decomposition into the Levi subgroup containing T and the unipotent radical, then

$$P \cap Q = (U_P \cap Q)(M \cap Q). \tag{2}$$

Indeed, both sides contain T and the same root subgroups, and are generated by these subgroups.

We have $b_{i+1}, b'_{i+1}, x_\mu\sigma(u_{i+1}hu_{i+1}^{-1})x_\mu^{-1} \in M'_{i+1}(\mathcal{O})$. Thus (1) implies that

$$h\delta_{i+1}^{-1}a\delta_{i+1}\tilde{a} \in M'_{i+1}(\mathcal{O}). \tag{3}$$

Hence $h\delta_{i+1}^{-1}a\delta_{i+1} \in \overline{P}'_{i+1}(\mathcal{O})$, and (as it is equal to $\delta_{i+1}^{-1}ga\delta_{i+1}$), it is also contained in $(\delta_{i+1}^{-1}P_{i+1}\delta_{i+1})(\mathcal{O})$. Using this and (2) we obtain that

$$\begin{aligned}
\tilde{a} &\in (U_{\overline{P}'_{i+1}} \cap (M'_{i+1} \cdot (\overline{P}'_{i+1} \cap \delta_{i+1}^{-1}P_{i+1}\delta_{i+1})))(\mathcal{O}) \\
&= (U_{\overline{P}'_{i+1}} \cap (M'_{i+1} \cdot (U_{\overline{P}'_{i+1}} \cap \delta_{i+1}^{-1}P_{i+1}\delta_{i+1})))(\mathcal{O}) \\
&= (U_{\overline{P}'_{i+1}} \cap \delta_{i+1}^{-1}P_{i+1}\delta_{i+1})(\mathcal{O}) \\
&= \big((U_{\overline{P}'_{i+1}} \cap \delta_{i+1}^{-1}U_{P_{i+1}}\delta_{i+1})(U_{\overline{P}'_{i+1}} \cap \delta_{i+1}^{-1}M_{i+1}\delta_{i+1})\big)(\mathcal{O}).
\end{aligned}$$

We write $\tilde{a} = a_1 a_2$ for this decomposition. Replacing $\tilde{a}$ by a_2 and a by $a\delta_{i+1}a_1\delta_{i+1}^{-1} \in U_{P_{i+1}}(\mathcal{O})$ the right hand side of (1) does not change. Hence we may assume that $\tilde{a} \in (U_{\overline{P}'_{i+1}} \cap \delta_{i+1}^{-1}M_{i+1}\delta_{i+1})(\mathcal{O})$.

In particular, as now $\tilde{a} \in (\delta_{i+1}^{-1}M_{i+1}\delta_{i+1})(\mathcal{O})$ we have $\tilde{a}^{-1}\delta_{i+1}^{-1}a\delta_{i+1}\tilde{a} \in (\delta_{i+1}^{-1}U_{P_{i+1}}\delta_{i+1})(\mathcal{O})$. Equation (1) is equivalent to

$$\tilde{a}(h\tilde{a})^{-1}b_{i+1}x_\mu\sigma(u_{i+1}h\tilde{a}u_{i+1}^{-1})x_\mu^{-1}(x_\mu\sigma(u_{i+1}\tilde{a}^{-1}u_{i+1}^{-1})x_\mu^{-1}) = \tilde{a}(\tilde{a}^{-1}\delta_{i+1}^{-1}a\delta_{i+1}\tilde{a})b'_{i+1}.$$

Replacing h by $h\tilde{a} \in (\delta_{i+1}^{-1}M_{i+1}\delta_{i+1})(\mathcal{O})$ and $\delta_{i+1}^{-1}a\delta_{i+1}$ by $\tilde{a}^{-1}\delta_{i+1}^{-1}a\delta_{i+1}\tilde{a}$ we obtain the equivalent equation (using these new variables)

$$h^{-1}b_{i+1}x_\mu\sigma(u_{i+1}hu_{i+1}^{-1})x_\mu^{-1} = \delta_{i+1}^{-1}a\delta_{i+1}b'_{i+1}\zeta$$

where

$$\zeta = x_\mu\sigma(u_{i+1}\tilde{a}u_{i+1}^{-1})x_\mu^{-1} \in (M'_{i+1} \cap x_\mu\sigma(u_{i+1}U_{\overline{P}'_{i+1}}u_{i+1}^{-1})x_\mu^{-1})(\mathcal{O}) = U_{\overline{P}'_{i+2}}(\mathcal{O}).$$

As we may multiply b'_{i+1} on the right by elements in $U_{\overline{P}'_{i+2}}(\mathcal{O})$ we may assume that $\zeta = 1$, which corresponds to (1) for $\tilde{a} = 1$. In particular, (3) yields $h\delta_{i+1}^{-1}a\delta_{i+1} \in M'_{i+1}(\mathcal{O})$, and as before it is also an element of $(\delta_{i+1}^{-1}P_{i+1}\delta_{i+1})(\mathcal{O})$. Thus by (2) we have

$$h\delta_{i+1}^{-1}a\delta_{i+1} \in ((\delta_{i+1}^{-1}M_{i+1}\delta_{i+1} \cap M'_{i+1})(\delta_{i+1}^{-1}U_{P_{i+1}}\delta_{i+1} \cap M'_{i+1}))(\mathcal{O}).$$

As $h \in \delta_{i+1}^{-1}M_{i+1}\delta_{i+1}$ and $a \in U_{P_{i+1}}$ this implies that

$$\begin{aligned} h &\in (\delta_{i+1}^{-1}M_{i+1}\delta_{i+1} \cap M'_{i+1})(\mathcal{O}), \\ \delta_{i+1}^{-1}a\delta_{i+1} &\in (\delta_{i+1}^{-1}U_{P_{i+1}}\delta_{i+1} \cap M'_{i+1})(\mathcal{O}). \end{aligned} \tag{4}$$

We obtain

$$\begin{aligned} h &\in (\delta_{i+1}^{-1}M_{i+1}\delta_{i+1} \cap M'_{i+1})(\mathcal{O}) \\ &= (u_{i+1}^{-1}\sigma^{-1}(x_\mu^{-1}M'_{i+1}x_\mu)u_{i+1} \cap M'_{i+1})(\mathcal{O}) \\ &\subseteq (u_{i+1}^{-1}\sigma^{-1}(x_\mu^{-1}M'_1x_\mu)u_{i+1} \cap M'_{i+1})(\mathcal{O}) \\ &= (u_{i+1}^{-1}M_1u_{i+1} \cap M'_{i+1})(\mathcal{O}) = M_{i+2}(\mathcal{O}). \end{aligned}$$

By definition, $u_{i+1}\delta_{i+1}^{-1}U_{P_{i+1}}\delta_{i+1}u_{i+1}^{-1} = u_iU_{P_{i+1}}u_i^{-1} \subseteq U_{P_1}$. Thus (4) implies

$$\delta_{i+1}^{-1}a\delta_{i+1} \in (u_{i+1}^{-1}U_{P_1}u_{i+1} \cap M'_{i+1}))(\mathcal{O}) = U_{P_{i+2}}(\mathcal{O}).$$

Altogether this means that via the elements $h \in M_{i+2}(\mathcal{O})$, $\delta_{i+1}^{-1}a\delta_{i+1} \in U_{P_{i+2}}(\mathcal{O})$, and $\zeta \in U_{\overline{P}'_{i+2}}(\mathcal{O})$, we proved that the two elements $b_{i+1}, b'_{i+1} \in M'_{i+1}(\mathcal{O})$ are in the same $M_{i+2}(\mathcal{O})$-orbit in $U_{P_{i+2}}(k)\backslash M'_{i+1}(k)/U_{\overline{P}'_{i+2}}(k)$. This finishes the induction step for (e) and completes the induction.

The M'_i form a decreasing family of Levi subgroups and thus become constant after finitely many steps. Thus for n sufficiently large, $x_\mu\sigma(u_nM'_nu_n^{-1})x_\mu^{-1}$

$= M'_n = M'_{n+1}$, and $M_n = M'_n$. As $M'_n = M'_{n+1}$ we obtain $\sigma(u_n M'_n u_n) \subseteq x_\mu^{-1} M'_n x_\mu \subseteq x_\mu^{-1} M'_1 x_\mu = M_\mu$. Thus we can apply Lemma 2.1 to obtain that each element of $u_n M'_n(\mathcal{O})\tau_\mu$ is $u_n M'_n(\mathcal{O})u_n^{-1}$-$\sigma$-conjugate to $u_n\tau_\mu$. Then by the last assertion in (b), $w = u_n$ is as desired.

We now prove (2). Each element of $Iw\tau_\mu I$ is obviously I-σ-conjugate to some element $g \in w\tau_\mu I$. We have the Iwahori decomposition $I = N_\mu(\mathcal{O}) I_{M_\mu} K_1$ where $I_{M_\mu} = I \cap M_\mu(\mathcal{O})$ and where N_μ is the unipotent radical of $P_\mu = M_\mu B$. We apply this to the last factor of $g \in w\tau_\mu I$. Now we use Section 2.6 in the form $\varepsilon^\mu N_\mu(\mathcal{O}) \subseteq (N_\mu(\mathcal{O}) \cap K_1)\varepsilon^\mu$ and see that we can multiply g by elements of K_1 on both sides to replace it by an element $g \in w\tau_\mu I_{M_\mu}$. Thus it is I_{M_1}-σ-conjugate to an element of $I_{M_1} w\tau_\mu = I_{M_1} w x_\mu \varepsilon^\mu$. As $w \in {}^\mu W = {}^{M_1}W$, conjugation by w maps positive roots in M_1 to positive roots (not necessarily in M_1). Hence we have $w^{-1} I_{M_1} w \subset I \cap (w^{-1} M_1(\mathcal{O}) w)$. Conjugation by w_0 maps all positive roots to negative roots, conjugation by $w_{0,\mu}$ maps positive roots in M_μ to negative roots in M_μ (and vice versa) and leaves positive roots in N_μ positive. Hence

$$\begin{aligned} I_{M_1} w x_\mu \varepsilon^\mu &= w x_\mu ((w x_\mu)^{-1} I_{M_1} w x_\mu)\varepsilon^\mu \\ &\subseteq K_1 w x_\mu (I_{M_\mu} \cap (w x_\mu)^{-1} I_{M_1} w x_\mu)) \overline{N}_\mu(\mathcal{O})\varepsilon^\mu \\ &\subseteq K_1 w x_\mu (I_{M_\mu} \cap (w x_\mu)^{-1} I_{M_1} w x_\mu))\varepsilon^\mu K_1. \end{aligned}$$

Iterating this argument we see that the element g is I-σ-conjugate to an element of $K_1 w x_\mu I_\infty \varepsilon^\mu K_1$ where $I_\infty = I \cap \bigcap_{i\geq 0}(\mathrm{Ad}_{(wx_\mu)^{-1}}\sigma^{-1})^i(M_\mu(\mathcal{O}))$ and where $\mathrm{Ad}_{(wx_\mu)^{-1}}$ denotes conjugation with the given element. As

$$\bigcap_{i\geq 0}(\mathrm{Ad}_{(wx_\mu)^{-1}}\sigma^{-1})^i(M_\mu)$$

is an intersection of Levi subgroups, it is equal to $\bigcap_{i=0}^n(\mathrm{Ad}_{(wx_\mu)^{-1}}\sigma^{-1})^i(M_\mu)$ for each sufficiently large n. Thus for the preceeding step it is in fact sufficient to σ-conjugate g by finitely many elements. As I_∞ commutes with ε^μ and satisfies $(wx_\mu)^{-1}\sigma^{-1}(I_\infty)(wx_\mu) = I_\infty$ we can apply Lemma 2.1 to obtain that each element of $K_1 w x_\mu I_\infty \varepsilon^\mu K_1$ is I_∞-σ-conjugate to an element in $K_1 w\tau_\mu K_1$. □

4. Closure relations

In this section we assume that $L = k((t))$; i.e., we consider the function field case. Recall that $S_{w,\mu}$ is the locus in LG where the truncation of level 1 is equal to (w, μ).

Lemma 4.1.

(1) *Each $S_{w,\mu}$ is bounded and admissible.*
(2) *The closure $\overline{S_{w,\mu}}$ of $S_{w,\mu}$ is a union of finitely many strata $S_{w',\mu'}$.*

(3) *$S_{w,\mu}$ is locally closed, smooth and irreducible.*
(4) *$g_0 \in G(L)$ is in $\overline{S_{w,\mu}}$ if and only if it is K-σ-conjugate to an element of $\overline{Iw\tau_\mu I}$.*

Proof. The stratum is bounded because it is contained in $Kt^\mu K$, and admissible because it is invariant under K_1. For the second assertion note that $\overline{S_{w,\mu}}$ is invariant under K-σ-conjugation and under multiplication by K_1 on both sides. Thus it is a union of strata. The union is finite because $\overline{S_{w,\mu}} \subseteq \overline{Kt^\mu K}$, hence each of the strata $S_{w',\mu'}$ in the closure has to satisfy $\mu' \preceq \mu$. The first assertion of (3) follows from (1) and (2). The other two assertions of (3) follow as $S_{w,\mu}/K_1$ is the orbit under the σ-conjugation action of K of the subscheme $K_1 w\tau_\mu K_1$. In (4) the second condition implies the first by Theorem 1.1(2). Now let $g_0 \in \overline{S_{w,\mu}}$. Thus there is a $g \in G(k[[z]]((t)))$ such that its reduction modulo z is equal to g_0 and such that its image g_η in $G(k((z))((t)))$ is in $S_{w,\mu}(k((z)))$. Hence there is an $h \in G(k((z))^{\mathrm{alg}}[[t]])$ with $h^{-1}g_\eta\sigma(h) \in K_{1,k((z))^{\mathrm{alg}}}w\tau_\mu K_{1,k((z))^{\mathrm{alg}}}$. Here $k((z))^{\mathrm{alg}}$ denotes an algebraic closure of $k((z))$. Replacing h by a suitable element of $hK_{1,k((z))^{\mathrm{alg}}}$ we may assume that it is defined over a finite extension of $k((z))$. We may replace $k((z))$ by that totally ramified extension and thus assume that h is defined over $k((z))$ itself. As $K/I \cong G(k)/B(k)$ is proper, there is a $k[[z]]$-valued point of K/I such that the induced $k((z))$-valued point coincides with the image of h in $K/I(k((z)))$. Let $\tilde{h} \in G(k[[z,t]])$ be a lift of that point. Such a lift exists because $k[[z]]$ is local, the map $G \to G/B$ has local sections, and we have the section $G(k((z))) \hookrightarrow K(k((z))) = G(k((z))[[t]])$ of the projection morphism $K \to G$. Denote by $\tilde{h}_0$ and $\tilde{h}_\eta$ the images of $\tilde{h}$ in $G(k[[t]])$ and $G(k((z))[[t]])$, respectively. As the generic points of h and $\tilde{h}$ coincide up to an element of $I_{k((z))}$, we obtain that $\tilde{h}_\eta^{-1}g_\eta\sigma(\tilde{h}_\eta) \in Iw\tau_\mu I$. Hence $\tilde{h}_0^{-1}g_0\tilde{h}_0 \in \overline{Iw\tau_\mu I}$ which proves (4). □

Before proving Theorem 1.4 we need some preparations. They are on the lines of [He07, §3] where similar results are shown for finite Weyl groups and without the σ-action (but allowing disconnected groups).

Remark 4.2. If $x, y, z \in \widetilde{W}$ with $x \in IyIzI$ then $x = y'z$ for some $y' \leq y$. Indeed, this follows by induction from $Is_iIzI \subseteq IzI \cup Is_izI$ for each (finite or affine) simple reflection s_i.

Let $\overline{N}$ be the unipotent radical of the Borel subgroup opposite of B. Let $\mathcal{N}^-$ be the inverse image of $\overline{N}$ in $G(k[t^{-1}]) \subset G(k((t)))$, compare [Fal03, §2].

LEMMA 4.3. *Let $x, y \in \widetilde{W}$. The subset $\{x'y \mid x' \leq x\}$ of $\widetilde{W}$ contains a unique minimal element z. We have $l(z) = l(y) - l(zy^{-1})$ and $\overline{IxIy\mathcal{N}^-} = \overline{Iz\mathcal{N}^-}$. In particular, $z \leq x'y'$ for every $x' \leq x$ and $y' \geq y$.*

Proof. For any $x' \leq x$, $Ix' \subseteq \overline{IxI}$. Thus $\overline{Ix'y\mathcal{N}^-} \subseteq \overline{IxIy\mathcal{N}^-}$. We choose an increasing sequence S_i of irreducible bounded subschemes of $\mathcal{N}^-$ with $\mathcal{N}^- = \bigcup_i S_i$. Recall from [Fal03, §3] that $I\backslash \mathrm{LG}$ is the disjoint union of the $\mathcal{N}^-$-orbits of the elements $x \in \widetilde{W}$ and that $Ix_1\mathcal{N}^- \subseteq \overline{Ix_2\mathcal{N}^-}$ if and only if $x_2 \leq x_1$. Note that $\overline{IxIyS_i}$ is an irreducible bounded and admissible subscheme of LG. Let $y_i \in \widetilde{W}$ be the element whose orbit contains the generic point of $IxIyS_i$. Then $y_i \geq y_{i+1}$ for all i, hence $y_i = y_{i+1}$ for all sufficiently large i. Let y_∞ be this element of $\widetilde{W}$. Then $\overline{IxIy\mathcal{N}^-} = \overline{Iy_\infty\mathcal{N}^-}$. As $Ix'y\mathcal{N}^- \subseteq \overline{Iy_\infty\mathcal{N}^-}$ we have that $x'y \geq y_\infty$ for all $x' \leq x$.

It remains to show that $y_\infty = x_\infty y$ for some $x_\infty \leq x$ with $l(x_\infty y) = l(y) - l(x_\infty)$. We use induction on the length of x. If $l(x) = 0$, the statement is clear. Assume that $l(x) > 0$. Let s_i be a simple reflection with $s_i x < x$, and set $\xi = s_i x$. We have

$$\overline{IxIy\mathcal{N}^-} = \overline{Is_iI\xi Iy\mathcal{N}^-} = \overline{Is_i\overline{I\xi Iy\mathcal{N}^-}}.$$

By induction there is a $\xi' \leq \xi$ such that $l(\xi' y) = l(y) - l(\xi')$ and $\overline{I\xi Iy\mathcal{N}^-} = \overline{I\xi' y\mathcal{N}^-}$. Thus

$$\overline{Is_i\overline{I\xi Iy\mathcal{N}^-}} = \overline{Is_i\overline{I\xi' y\mathcal{N}^-}} = \overline{Is_iI\xi' y\mathcal{N}^-} = \begin{cases} \overline{I\xi' y\mathcal{N}^-} & \text{if } s_i\xi' y > \xi' y, \\ \overline{Is_i\xi' y\mathcal{N}^-} & \text{if } s_i\xi' y < \xi' y. \end{cases}$$

We have $\xi' \leq s_i x < x$, thus $s_i\xi' \leq x$. If $s_i\xi' y > \xi' y$ we can choose $x_\infty = \xi'$. If $s_i\xi' y < \xi' y$ then $l(s_i\xi' y) = l(\xi' y) - 1 = l(y) - l(\xi') - 1$. Thus $l(s_i\xi') = l(\xi') + 1$ and $l(s_i\xi' y) = l(y) - l(s_i\xi')$, and we can choose $x_\infty = s_i\xi'$. Thus the assertion holds for x. □

LEMMA 4.4.

(1) *If $a, b \in \widetilde{W}$ and $x \leq ab$, then there exist $a' \leq a$ and $b' \leq b$ with $a'b' = x$ and $l(a') + l(b') = l(x)$.*

(2) *Let M and M' be standard Levi subgroups, $w \in \widetilde{W}^M \cap {}^{M'}\widetilde{W}$ and $v \in W_M$. Then $wv \in {}^{M'}\widetilde{W}$ if and only if $v \in {}^K W$, where $K = M \cap w^{-1}M'w$.*

Proof. For the first assertion, the general statement follows from the special case $x = ab$. This in its turn is a consequence of the exchange property of Coxeter groups using induction: If $a = \omega s_{i_1} \cdots s_{i_r}$ with $\omega \in \Omega$ is a reduced expression for a and s_i a simple affine reflection then either $l(as_i) = l(a) + 1$ or $as_i = \omega s_{i_1} \cdots \hat{s}_{i_j} \cdots s_{i_r}$ for some j. For a proof of the second statement see for example [DDPW08, Th. 4.18]. □

LEMMA 4.5. *Let M and M' be standard Levi subgroups, $w \in \widetilde{W}^M \cap {}^{M'}\widetilde{W}$ and $v \in W_M$. Let $K = M \cap w^{-1}M'w$. Then $wv = xwy$ for some $x \in W_{wKw^{-1}}$ and $y \in W_M \cap {}^K\widetilde{W}$.*

Proof. By [DDPW08, Th. 4.18] we have $wv = xwy$ for some $x \in W_{M'}$ and $y \in W_M \cap {}^K\widetilde{W}$. But then $x = wvy^{-1}w^{-1} \in wW_Mw^{-1} \cap W_{M'} = wW_{M\cap w^{-1}M'w}w^{-1} = W_{wKw^{-1}}$. □

Lemma 4.6. *Let M be a standard Levi subgroup of G and let $x \in {}^M\widetilde{W}$. Let $y \in \widetilde{W}$. Then $y \geq wx\sigma(w)^{-1}$ for some $w \in W_M$ if and only if there are $u, v \in W_M$ with $v \leq u$ and $y \geq ux\sigma(v)^{-1}$.*

Proof. Let $y \in \widetilde{W}$ and let $u, v \in W_M$ with $v \leq u$ and $y \geq ux\sigma(v)^{-1}$. We have to show that $y \geq wx\sigma(w)^{-1}$ for some $w \in W_M$. We use induction on the size of the Levi subgroup and thus may assume that the statement is true for all $M' \subsetneq M$. We use a second induction on the length $l(u)$. We write $x = ab$ with $a \in {}^M\widetilde{W} \cap \widetilde{W}^{\sigma(M)}$ and $b \in W_{\sigma(M)}$. Setting $M' = M \cap a\sigma(M)a^{-1}$ we decompose u as u_1u_2 with $u_1 \in W^{M'}$ and $u_2 \in W_{M'}$. Together with $v \leq u$ this induces a decomposition $v = v_1v_2$ with $v_i \leq u_i$ and $l(v) = l(v_1) + l(v_2)$. Note that our choice of a implies that M' is again the Levi factor of a standard parabolic subgroup. We consider two cases:

Case 1: $u_1 = v_1 = 1$. In this case u and v are in $W_{M'}$, and $x \in {}^{M'}\widetilde{W}$. If $M' \neq M$, then the assertion follows from the induction hypothesis. If $M' = M = a\sigma(M)a^{-1}$, then since $ab \in {}^M\widetilde{W}$, we have that $b = 1$. Thus $ux\sigma(v)^{-1} \geq x$ which implies the assertion.

Case 2: $u_1 \neq 1$. In this case $l(u_2) < l(u)$. By induction hypothesis, there is an $x' = u'x\sigma(u')^{-1} \leq u_2x\sigma(v_2)^{-1}$. Let $v_3 \leq v_1$ be such that $x'\sigma(v_3)^{-1}$ is the unique element of minimal length in $\{x'\sigma(v')^{-1} \mid v' \leq v_1\}$ (see Lemma 4.3). Then the last assertion of Lemma 4.3 implies that

$$x'\sigma(v_3)^{-1} \leq (u_2x\sigma(v_2)^{-1})\sigma(v_1)^{-1} = u_2x\sigma(v)^{-1}.$$

By Lemma 4.5 we can write

$$x\sigma(v)^{-1} = a(b\sigma(v)^{-1}) \in ({}^M\widetilde{W} \cap \widetilde{W}^{\sigma(M)})W_{\sigma(M)}$$

as $\alpha a\delta$ with $\alpha \in W_{M'}$ and $\delta \in W_{\sigma(M)} \cap {}^{a^{-1}M'a}W$. By Lemma 4.4(2), $\beta = a\delta \in {}^M\widetilde{W}$. Thus

$$\begin{aligned} l(u_1u_2x\sigma(v)^{-1}) &= l(u_1u_2\alpha\beta) = l(u_1u_2\alpha) + l(\beta) = l(u_1) + l(u_2\alpha) + l(\beta) \\ &= l(u_1) + l(u_2\alpha\beta) = l(u_1) + l(u_2x\sigma(v)^{-1}). \end{aligned}$$

As $x'\sigma(v_3)^{-1} \leq u_2x\sigma(v)^{-1}$ and $v_3 \leq v_1 \leq u_1$, this implies that

$$(v_3u')x\sigma(v_3u')^{-1} = v_3x'\sigma(v_3)^{-1} \leq ux\sigma(v)^{-1} \leq y.$$ □

Proof of Theorem 1.4. We have to show that (w', μ') is the truncation of level 1 of an element of IyI for some $y \leq w\tau_\mu$ if and only if it is of the form in the theorem. The *if* part is obvious. For the other direction we use an approach which is similar to the proof of Theorem 1.1 to compute the truncations of level 1 occurring in the cosets IyI for y as above. We decompose y as $w_1\tau_{\mu'}w'_1$ with $w_1, w'_1 \in W$, μ' dominant and such that the lengths of the three elements add up to that of y. Each truncation of an element of IyI already occurs in $Iy = Iw_1\tau_{\mu'}w'_1$, and thus also in $\sigma^{-1}(w'_1)Iw_1\tau_{\mu'}$. By Remark 4.2, each such element is contained in $I\sigma^{-1}(\tilde{w}'_1)w_1\tau_{\mu'}I$ for some $\sigma^{-1}(\tilde{w}'_1) \leq \sigma^{-1}(w'_1)$. This is equivalent to $\tilde{w}'_1 \leq w'_1$ as I is invariant under σ. Using Lemma 4.4(1) for $\sigma^{-1}(\tilde{w}'_1)w_1$ and replacing y by a smaller element we see that we may assume that $\tilde{w}'_1 = w'_1$ and that $l(\sigma^{-1}(w'_1)w_1\tau_{\mu'}) = l(w_1\tau_{\mu'}w'_1) = l(w_1) + l(\tau_{\mu'}) + l(w'_1)$. We have to consider the truncation types occurring in $\sigma^{-1}(w'_1)w_1\tau_{\mu'}I$. It is enough to show that for each such type (w', μ') there is a $u \in W$ with $uw'\tau_{\mu'}\sigma(u)^{-1} \leq \sigma^{-1}(w'_1)w_1\tau_{\mu'}$. Indeed, by Lemma 4.4(1) this implies that there is a $v_1 \leq \sigma^{-1}(w'_1)$ such that $v_1^{-1}uw'\tau_{\mu'}\sigma(u)^{-1}\sigma(v_1) \leq w_1\tau_{\mu'}w'_1$. By Lemma 4.6, it is furthermore enough to show the following claim.

Claim. *Let (w', μ') be the truncation of level 1 of an element $g \in Ix\tau_{\mu'}I$ for some $x \in W$. Then there are $v \leq u \in \sigma^{-1}(W_{M_{\mu'}})$ with $uw'\tau_{\mu'}\sigma(v)^{-1} = x\tau_{\mu'}$.*

By σ-conjugating with the first factor of g we may assume that it is contained in $x\tau_{\mu'}I$. Changing g within its K_1-double coset we may assume that the factor in I is in fact contained in $I \cap B(\mathcal{O}) \cap \overline{P}_{\mu'}(\mathcal{O}) \subseteq I \cap M_{\mu'}(\mathcal{O})$. A second σ-conjugation then implies that we may assume that $g \in (I \cap M_1(\mathcal{O}))x\tau_{\mu'}$ where $M_1 = \sigma^{-1}(M_{\mu'})$ is as in the proof of Theorem 1.1. Note that for the groups defined in that proof we have $M'_i \subseteq M'_1$ and hence $u_iM_{i+1}(\mathcal{O})u_i^{-1} \subseteq M_1(\mathcal{O})$. In particular, the construction in this proof implies for the element $g \in (I \cap M_1(\mathcal{O}))x\tau_{\mu'}$ that there is an $f \in M_1(\mathcal{O})$ with $f^{-1}g\sigma(f) \in K_1w'\tau_{\mu'}K_1$. We decompose f as $f = i_1ui_2 \in IW_{\sigma^{-1}M_{\mu'}}I$. Then $i_1ui_2w'\tau_{\mu'}\sigma(i_1ui_2)^{-1} \in Ix\tau_{\mu'}I$. Recall that $\tau_{\mu'}$ is the shortest representative of its W-double coset and $w' \in {}^{\sigma^{-1}(M_{\mu'})}W$. Thus $w'\tau_{\mu'} \in {}^{\sigma^{-1}(M_{\mu'})}\widetilde{W}$. Hence $Ix\tau_{\mu'}I \subseteq IuIw'\tau_{\mu'}I\sigma(u)^{-1}I = Iuw'\tau_{\mu'}I\sigma(u)^{-1}I$. Thus there is a $v \leq u$ with $uw'\tau_{\mu'}\sigma(v)^{-1} = x\tau_{\mu'}$. □

The following corollary to the theorem which considers the special case $\mu = \mu'$ is analogous to results by He [He07] and Wedhorn [Wed].

Corollary 4.7. *$S_{w',\mu} \subseteq \overline{S_{w,\mu}}$ if and only if there is a $\tilde{w} \in \sigma^{-1}(W_{M_\mu})$ with $\tilde{w}^{-1}w'x_\mu\sigma(\tilde{w})x_\mu^{-1} \leq w$.*

Proof. Recall that τ_μ is the unique shortest element of the extended affine Weyl group lying in $Wt^\mu W$. Especially, $y \leq w\tau_\mu$ with $y \in Wt^\mu W$ if and only if $y = w_y\tau_\mu$ for some $w_y \leq w$ in W. From the theorem we obtain that $S_{w',\mu} \subseteq \overline{S_{w,\mu}}$ if and only if there is a $\tilde{w} \in W$ such that $\tilde{w}^{-1}w'\tau_\mu\sigma(\tilde{w}) = w_y\tau_\mu$ for some w_y as above. Thus $\tau_\mu\sigma(\tilde{w}) = v\tau_\mu$ for some $v \in W$. As $\tau_\mu = x_\mu t^\mu$ we obtain $v = x_\mu\sigma(\tilde{w})x_\mu^{-1}$ and $\sigma(\tilde{w}) \in W_{M_\mu}$. □

5. Nonemptiness of intersections of truncation strata with σ-conjugacy classes

For the discussion of short elements we allow both possible cases for F.

Definition 5.1. Let $[b] \in B(G)$ and let $\nu \in X_*(T)_{\mathbb{Q}}^{\Gamma}$ be its dominant Newton point. Let M_ν be the centralizer of ν in G. Then $x \in \widetilde{W}$ is called $[b]$-short if $x \in \Omega_{M_\nu} \subseteq \widetilde{W}_{M_\nu}$, the M_ν-dominant Newton point of x is equal to ν, and $\kappa_G(x) = \kappa_G(b)$.

An element $x \in \widetilde{W}$ is called short if it is $[b]$-short for some $b \in B(G)$.

Remark 5.2. From the classification of $B(G)$ we obtain that all $[b]$-short elements are contained in $[b]$.

Lemma 5.3. *Each $[b] \in B(G)$ contains a $[b]$-short element. If G is split, this element is unique.*

Proof. Let $\nu \in X_*(T)_{\mathbb{Q}}^{\Gamma}$ be the dominant Newton point of b and let M be the centralizer of ν in G. Then there is an element b_0 of $M(L) \cap [b]$ whose M-dominant Newton point is equal to ν [Kot85, Prop. 6.2]. Let $\mu_0 \in X_*(T)$ be M-dominant with $b_0 \in M(\mathcal{O})\varepsilon^{\mu_0}M(\mathcal{O})$. Let ω be the image of μ_0 in $\pi_1(M)$. Note that its image under the projection to $\pi_1(G)_\Gamma$ is equal to $\kappa_G(b)$. Let $x \in \Omega_M$ be the unique element whose image under the isomorphism to $\pi_1(M)$ is equal to ω. Then x is basic in M with $\kappa_M(x) = \kappa_M(b_0)$, hence with M-dominant Newton point ν. In particular, x is $[b]$-short.

For split G, we have $\pi_1(G) = \pi_1(G)_\Gamma$. The kernel of the projection $\pi_1(M) \to \pi_1(G)$ is torsion free. Hence $\omega \in \pi_1(M)$ is the unique element whose image in $\pi_1(G)$ is equal to $\kappa(b)$ and whose image in $\pi_1(M) \otimes \mathbb{Q}$ is equal to the image of ν under the projection to $\pi_1(M) \otimes \mathbb{Q}$. Each element b' of $[b] \cap M(L)$ whose M-dominant Newton point is equal to ν has to satisfy $\kappa_M(b') = \omega$. Thus, there is a unique such element which lies in Ω_M. □

For the rest of this section let $L = k((t))$, i.e., we consider the function field case.

Remark 5.4. By Theorem 1.1 (2), $S_{w,\mu}$ has nonempty intersection with some σ-conjugacy class $[b]$ if and only if $[b] \cap Iw\tau_\mu I \neq \emptyset$. By the Grothendieck specialization theorem [RR96, Th. 3.6], the generic σ-conjugacy class in $S_{w,\mu}$ respectively the generic class in $Iw\tau_\mu I$ are the largest classes (with respect to $\preceq$) whose intersections with $S_{w,\mu}$ respectively $Iw\tau_\mu I$ are nonempty. Hence also these generic classes coincide.

Proposition 5.5. *Let $b \in G(L)$ and let M be the centralizer of its dominant Newton point. Let $x \in \widetilde{W}$ with $b \in IxI$. Then there is a $[b]$-short element $x_b \in \widetilde{W}$ and a $w \in {}^MW$ with $w^{-1}x_b\sigma(w) \leq x$ in the Bruhat order.*

Proof. Let $P = BM$ be the standard parabolic subgroup of G with Levi component M, and let N be its unipotent radical. We fix a $[b]$-short element $y \in \widetilde{W}_M$. Let $g \in G(L)$ with $g^{-1}y\sigma(g) = b \in IxI$. Using the Iwasawa decomposition and the Bruhat decomposition we write $g = nmi_1wi_2$ with $n \in N(L)$, $m \in M(L)$, $i_1, i_2 \in I$, and $w \in W$. By the Iwahori decomposition, $i_1 \in P(\mathcal{O})K_1$. As $w^{-1}K_1w \subseteq I$, we may assume that $i_1 = \mathrm{id}$. Furthermore, we can replace g by gi_2^{-1} without changing the property $g^{-1}y\sigma(g) \in IxI$. Thus we may assume that $g = nmw$ with $g^{-1}y\sigma(g) \in IxI$. Finally we may assume that w is of minimal length in its coset $W_M w$.

The next step is to show that we may assume that $n = 1$, i.e., that $w^{-1}m^{-1}y\sigma(mw) \in \overline{IxI}$. We have

$$g^{-1}y\sigma(g) = w^{-1}m^{-1}\left[n^{-1}y\sigma(n)y^{-1}\right]y\sigma(mw). \tag{5}$$

We abbreviate the expression in the bracket, which is in $N(L)$, by $\tilde{n}$. We want to construct a family of elements of $\overline{IxI}$ over $\mathbb{A}^1_k$ such that its fiber over 1 is $g^{-1}y\sigma(g)$, and that the fiber over 0 is $w^{-1}m^{-1}y\sigma(mw)$. Let LN be the loop group associated with N over k, i. e. the group ind-scheme representing the functor on k-algebras $R \mapsto N(R((t)))$. Let $\chi \in X_*(T)$ be central in M and such that $\langle\alpha, \chi\rangle > 0$ for every simple root α of T in N. Let

$$\begin{aligned}\phi : \mathbb{A}^1_k \setminus \{0\} &\to LN\\ a &\mapsto \chi(a)\tilde{n}\chi(a)^{-1}.\end{aligned}$$

Let α be a root of T in N and let U_α denote the corresponding root subgroup. Conjugation by $\chi(a)$ maps $U_\alpha(y)$ to $U_\alpha(a^jy)$ where $j = \langle\alpha, \chi\rangle > 0$. Especially, ϕ has an extension to a morphism $\phi : \mathbb{A}^1_k \to LN$ that maps 0 to id. As $\chi(a)$ is central in M,

$$w^{-1}m^{-1}\phi(a)y\sigma(mw) = (w^{-1}\chi(a)w)w^{-1}m^{-1}\tilde{n}y\sigma(mw)(\sigma(w)^{-1}\chi(a)^{-1}\sigma(w))$$

for every $a \neq 0$. Using (5), we obtain that this is in IxI. Hence

$$w^{-1}m^{-1}\phi(0)y\sigma(mw) = w^{-1}m^{-1}y\sigma(mw) \in \overline{IxI}.$$

It remains to show that $w^{-1}m^{-1}y\sigma(mw) \in \overline{IxI}$ implies that $w^{-1}x_b\sigma(w) \in \overline{IxI}$ for some $[b]$-short element x_b. Let $I_M = I \cap M(L)$. The minimality property of w implies that for any positive root α of T in M the root β with $w^{-1}U_\alpha w = U_\beta$ is also positive (although not necessarily in M). As I and M are defined over $\mathcal{O}_F$, the same holds for $\sigma(w)$. Thus

$$w^{-1}I_Mm^{-1}y\sigma(m)I_M\sigma(w) \subseteq Iw^{-1}m^{-1}y\sigma(mw)I \subseteq \overline{IxI}.$$

Using the Cartan decomposition for M we have $m^{-1}y\sigma(m) \in M(\mathcal{O}_L)\varepsilon^{\mu'}M(\mathcal{O}_L)$ for some M-dominant $\mu' \in X_*(T)$. Let $x_b \in \Omega_M \subseteq \widetilde{W}_M$ be the unique element whose image under the projection $pr_M : \widetilde{W}_M \to \pi_1(M)$ agrees with the image of μ'. In particular, this implies that $\kappa_M(x_b) = \kappa_M(m^{-1}y\sigma(m)) = \kappa_M(y) \in \pi_1(M)_\Gamma$. As x_b is basic in M, the M-dominant Newton polygons of x_b and

y agree. Thus x_b is a $[b]$-short element. As $x_b \in \Omega_M$ we have that $I_M x_b I_M$ is the unique closed I_M-double coset of the form $I_M h I_M$ with $h \in \widetilde{W}_M$ and $pr_M(h) = pr_M(x_b)$. It is contained in the closure of any other such double coset. Applying this to $I_M m^{-1} y\sigma(m) I_M$ we obtain

$$w^{-1} I_M x_b I_M \sigma(w) \subseteq \overline{IxI}.$$

This implies $w^{-1} x_b \sigma(w) \in \overline{IxI}$. □

If G is split, the first (and largest) part of the proof of Proposition 5.5 can also be deduced from a nonemptiness criterion by Görtz, Haines, Kottwitz, and Reuman, [GHKR10, Cor. 12.1.2] using the relation between short elements and fundamental alcoves in Lemma 6.11.

COROLLARY 5.6. *Let $[b_x]$ be the generic σ-conjugacy class in IxI for some $x \in \widetilde{W}$. Then $[b_x]$ is the unique largest (with respect to the order described in the introduction) among the classes $[y]$ where $y \in \widetilde{W}$ with $y \leq x$ in the Bruhat order. It is also equal to the largest among the $[y]$ where $y \leq x$ is in addition of the form $y = w^{-1} z\sigma(w)$ where z is $[y]$-short and where $w \in {}^{M_y}W$ for the centralizer M_y of the dominant Newton point of y.*

Proof. The generic σ-conjugacy classes of IxI and $\overline{IxI}$ coincide. By the Grothendieck specialization theorem [RR96, Th. 3.6], the generic class of $\overline{IxI}$ is the unique largest (with respect to the order described in the introduction) among the classes $[g]$ with $g \in \overline{IxI}$. Hence the assertion follows from Proposition 5.5. □

Proof of Theorem 1.5 and Corollary 1.6. Theorem 1.5 and Corollary 1.6 follow from Proposition 5.5 and Corollary 5.6 by Remark 5.4. □

PROPOSITION 5.7. *Let (w, μ) be the truncation type of a $[b]$-short element for some σ-conjugacy class $[b]$. If a σ-conjugacy class $[b']$ is contained in $\overline{[b]}$ then there exists a $[b']$-short element x' such that $S_{w',\mu'} \subseteq \overline{S_{w,\mu}}$ where $(w', \mu') = \mathrm{tr}(x')$. If $[b] = [b_{w\tau_\mu}]$ then the converse also holds. This is in particular always the case if G is split.*

The closure of $[b]$ is a union of σ-conjugacy classes. By [RR96, Th. 3.6] a necessary condition for $[b'] \subseteq \overline{[b]}$ is that $[b'] \preceq [b]$, i.e., $\kappa_G(b) = \kappa_G(b')$ and $\nu_{b'} \preceq \nu_b$. In [Vie13] it is shown that for split G this condition is also sufficient.

Proof. Assume that $[b'] \subseteq \overline{[b]}$. Let $g \in G(k[[z]]((t)))$ such that $g_{k((z))} \in [b]$ and $g_k \in [b']$. Let $h \in G(k((z))^{\mathrm{alg}}((t)))$ with $h^{-1} g_{k((z))}\sigma(h) \in Iw\tau_\mu I$. Here $k((z))^{\mathrm{alg}}$ denotes an algebraic closure of $k((z))$. The closed Schubert cell in LG/I containing h is a scheme of finite type. Thus replacing h by some representative of $hI_{k((z))^{\mathrm{alg}}}$ we may assume that h is defined over a finite extension of $k((z))$. Replacing $k[[z]]$ by its integral closure in that extension we may assume $h \in G(k((z))((t)))$. Also, as the closed Schubert cell is a proper subscheme of LG/I, the class hI contains an element of $\mathrm{LG}/I(k[[z]])$. As $k[[z]]$ is local for the étale topology, [HV12, Lemma 2.3] shows that we obtain an element of $\mathrm{LG}(k[[z]]) = G(k[[z]]((t)))$ in the inverse image. We denote this element again by h. Then $\tilde{g} = h^{-1}g\sigma(h) \in G(k[[z]]((t)))$ with $\tilde{g}_{k((z))} \in Iw\tau_\mu I \subseteq S_{w,\mu}$ and $\tilde{g}_k \in [b']$. Hence $\overline{S_{w,\mu}}$ contains an element of $[b']$ and thus by Theorem 1.5 also some stratum $S_{w',\mu'}$.

Let now $[b] = [b_{w\tau_\mu}]$ and assume that there exists a $[b']$-short element x' such that $S_{w',\mu'} \subseteq \overline{S_{w,\mu}}$. Then $[b'] \cap \overline{[b]} \neq \emptyset$, hence $[b'] \subseteq \overline{[b]}$.

It remains to show that for split G we always have $[b] = [b_{w\tau_\mu}]$. Let $[b'] = [b_{w\tau_\mu}]$. Then $[b] \cap \overline{[b']} \neq \emptyset$, hence $[b] \subseteq \overline{[b']}$. Let (w', μ') be the truncation type of the unique $[b']$-short element (compare Lemma 5.3). Then by the first assertion of this proposition, we have $S_{w,\mu} \subseteq \overline{S_{w',\mu'}}$. On the other hand, $[b'] \cap Iw\tau_\mu I \neq \emptyset$, thus by Proposition 5.5 $S_{w',\mu'} \subseteq \overline{S_{w,\mu}}$. Thus $w = w'$, $\mu = \mu'$, and $[b] = [b']$. □

Remark 5.8. Essentially the same proof also shows the following statement. Let $b, b' \in G(L)$ and let $x \in \widetilde{W}$ such that $IxI \subseteq [b]$ (for example a P-fundamental alcove contained in $[b]$ as in Theorem 6.5). Then $[b'] \subseteq \overline{[b]}$ if and only if $[b'] \cap \overline{IxI} \neq \emptyset$.

6. Comparison between the arithmetic case and the function field case

In this section we consider both cases $L = k((t))$ and $L = \mathrm{Quot}(W(k))$, and compare between them.

Definition 6.1.

(1) For $x \in G(L)$ let $\phi_x : G(L) \to G(L)$ with $g \mapsto \sigma(xgx^{-1})$.
(2) Let P be a semistandard parabolic subgroup of G, i.e., a parabolic subgroup containing T but not necessarily B. Let N be its unipotent radical and M the Levi factor containing T. Let $\overline{N}$ be the unipotent radical of the opposite parabolic. Then an element $x \in \widetilde{W}$ is called P-fundamental if $\phi_x(I_M) = I_M$, $\phi_x(I_N) \subseteq I_N$, and $\phi_x(I_{\overline{N}}) \supseteq I_{\overline{N}}$.

Definition 6.1 is a generalization of Görtz, Haines, Kottwitz, and Reuman's notion of fundamental P-alcoves for split groups from [GHKR10, 13]. Also, Lemma 6.4, Theorem 6.5 and Proposition 6.10(1) are generalizations to unramified groups of corresponding results of [GHKR10]. However, for our main theorem in this context (Theorem 6.5) one needs a different proof than the one used for split groups.

Remark 6.2. Let $x \in \widetilde{W}$. Let $r > 0$ be such that G is split over an unramified extension of $\mathcal{O}_F$ of degree r. Hence σ^r acts trivially on $\widetilde{W}$. Let $x' := \sigma(x)\sigma^2(x) \cdots \sigma^r(x)$. Let $P = MN$ be a semistandard parabolic subgroup with $\phi_x(P) = P$. Then $x'P(x')^{-1} = \phi_x^r(P) = P$, hence $x' \in \widetilde{W}_M$. We denote the M-dominant Newton point of the σ^r-conjugacy class of x' by $\nu_{r,M}$.

LEMMA 6.3. *Let $x \in [b] \cap \widetilde{W}$ be P-fundamental for some $P = MN$. Let r and x' be as in Remark 6.2. Then x is Q-fundamental for a semistandard parabolic $Q = M_Q N_Q$ if and only if*

(i) $\phi_x(Q) = Q$,
(ii) $\nu_{r,M}$ *is central in* M_Q, *and*
(iii) $\langle \nu_{r,M}, \alpha \rangle \geq 0$ *for each root* α *of* T *in* N_Q.

Proof. Let $P = MN$ where M is the Levi factor of P containing T. As x is P-fundamental, ϕ_x stabilizes M and N, hence (i) holds for P. Besides, $I_M = \phi_x^r(I_M) = x' I_M (x')^{-1}$. Hence $x' \in \Omega_M$, and the Newton point $\nu_{r,M}$ is central in M. Similarly, $\phi_x(I_N) \subseteq I_N$ implies condition (iii) for P. Indeed, let $r' > 0$ be such that $(x')^{r'} \in X_*(T) \subseteq \widetilde{W}$. For example, r' can be chosen to be the order of the factor in W of $x' \in \widetilde{W} \cong W \ltimes X_*(T)$. Then $(x')^{r'} = r'\nu_{r,M}$, and (iii) follows using Section 2.6. To prove the converse we first consider a special case. Let M' be the centralizer of $\nu_{r,M}$. Then $M \subseteq M'$. Let P' be the parabolic subgroup generated by M' and N. Let N' be its unipotent radical. Then $\phi_x(P') = P'$, and $P' \supseteq P$ and $N' \subseteq N$. Thus in order to show that x is P'-fundamental it is enough to verify $\phi_x(I_{M'}) = I_{M'}$. We consider the decomposition $I_{M'} = I_M I_{N \cap M'} I_{\overline{N} \cap M'}$. Each of the subgroups $M, N \cap M', \overline{N} \cap M'$ is stable under ϕ_x, so we can consider each factor of $I_{M'}$ separately. For I_M the assertion is just the assumption. For $I_{N \cap M'}$ we have $\phi_x^i(I_{N\cap M'}) \subseteq I_{N \cap M'}$ for every $i > 0$. For $i = rr'$ (with r' as above) we have equality in the above containment. Indeed, $\phi_x^{rr'}(g) = \sigma^{rr'}((x')^{r'} g (x')^{-r'})$ and $(x')^{r'} \in X_*(T)$ with $\langle \alpha, (x')^{r'} \rangle = \langle \alpha, r'\nu_{r,M} \rangle = 0$ for every root of T in M'. Considering the whole chain of containments we obtain equality for every i. A similar argument applies to $\overline{N} \cap M'$. It remains to show that if $Q \subseteq P'$ satisfies (i)–(iii), then x is Q-fundamental, but this is obvious. □

LEMMA 6.4. *If x is P-fundamental then every element of IxI is I-σ-conjugate to x.*

Proof. Each element of IxI is I-σ-conjugate to an element of xI. By Lemma 6.3 we may assume that P is maximal with the property that x is P-fundamental, i.e., that $\langle \nu_{r,M}, \alpha \rangle > 0$ for each root α of T in N. In particular ϕ_x is topologically nilpotent on I_N, see Section 2.6.

Let $g \in I$. We apply the Iwahori decomposition to g to obtain $g = g_N g_M g_{\overline{N}} \in I_N I_M I_{\overline{N}}$. Note that $x g_N x^{-1} \in \sigma^{-1}(I) = I$. Thus

$$xg = (x g_N x^{-1}) x g_M g_{\overline{N}}$$

is I-σ-conjugate to $x g_M g_{\overline{N}} \phi_x(g) \in x(I \cap \phi_x(I))$. By the Iwahori decomposition $I \cap \phi_x(I) = \phi_x(I_N) I_M I_{\overline{N}}$. Using this to decompose $g_M g_{\overline{N}} \phi_x(g)$ and iterating we obtain that xg is I-σ-conjugate to an element of $x(I\cap\phi_x^n(I))$ for every n. Note that in the nth iteration we only σ-conjugate by an element of $x\phi_x^n(I_N)x^{-1}$. The morphism ϕ_x is topologically nilpotent on I_N. Hence the product of these elements exists and in the limit we obtain that xg is I-σ-conjugate to an element of $x\left(\bigcap_{n\geq 0}\phi_x^n(I)\right) = xI_MI_{\overline{N}}$. We write this element as $xg'_Mg'_{\overline{N}}$. It is I-σ-conjugate to $x(x^{-1}\sigma^{-1}(g'_{\overline{N}})x)g'_M = xg'_M((g'_M)^{-1}\phi_x^{-1}(g'_{\overline{N}})g'_M)\in x(I\cap\phi_x^{-1}(I)\cap M\overline{N})$. A similar iteration as above shows that $xg'_Mg'_{\overline{N}}$ is I-σ-conjugate to an element of xI_M. By our assumption $h\mapsto x^{-1}\sigma^{-1}(h^{-1})xh = \phi_x^{-1}(h^{-1})h$ defines a morphism $I_M\to I_M$. By Lemma 2.1 it is surjective, hence for every $g\in I_M$ there is a $\sigma^{-1}(h)\in\sigma^{-1}(I_M)\subseteq I$ which σ-conjugates x to xg. □

THEOREM 6.5. *For every $[b]\in B(G)$ there exists an $x\in\widetilde{W}$ such that $x\in[b]$ is P-fundamental for some semi-standard P.*

Note that it is not easy to give an explicit description of a P-fundamental alcove contained in a given $[b]$. In general, neither $[b]$-short elements nor the representatives $w\tau_\mu$ of their truncation types (w,μ) are P-fundamental for any P.

For the proof of the theorem we need the following three lemmas.

LEMMA 6.6. *Let P be a semistandard parabolic subgroup of G. Let $\tilde{I}$ be an Iwahori subgroup of* LG *containing $T(\mathcal{O})$. Let $\tilde{I}_M=\tilde{I}\cap M$ and similarly for $\tilde{I}_N$ and $\tilde{I}_{\overline{N}}$. Let $x\in\widetilde{W}$ with $\phi_x(\tilde{I}_M)=\tilde{I}_M$. Then $\phi_x(\tilde{I}_N)\subseteq\tilde{I}_N$ if and only if $\phi_x(\tilde{I}_{\overline{N}})\supseteq\tilde{I}_{\overline{N}}$.*

Note that in this lemma we do not assume $\tilde{I}$ or P to be fixed by σ.

Proof. As $\tilde{I}$ contains $T(\mathcal{O})$, the group $\tilde{I}_{\overline{N}}$ is a product of its intersections with the root subgroups for roots of T in $\overline{N}$. Let U_α be such a root subgroup. We write $x=\varepsilon^{\mu_x}w_x$ with $\mu_x\in X_*(T)$ and $w_x\in W$. Let $\psi(\alpha)$ be the root of T in $\overline{N}$ with $\sigma(w_xU_{\psi(\alpha)}w_x^{-1})=U_\alpha$. The assertion on $\tilde{I}_{\overline{N}}$ is equivalent to $U_\alpha\cap\tilde{I}_{\overline{N}}\subseteq\sigma(x\tilde{I}_{\overline{N}}x^{-1})$ for all α. This is equivalent to $U_\alpha\cap\tilde{I}_{\overline{N}}\subseteq\sigma(x(U_{\psi(\alpha)}\cap\tilde{I}_{\overline{N}})x^{-1})$. Note that $U_\alpha\cong\mathbb{G}_a$, hence we can identify $U_\alpha\cap\tilde{I}$ with $\varepsilon^{\phi_\alpha}k[[\varepsilon]]$ for some $\phi_\alpha\in\mathbb{Z}$. As $x=\varepsilon^{\mu_x}w_x$ the above inclusion holds if and only if

$$\langle\mu_x,\sigma^{-1}(\alpha)\rangle+\phi_{\psi(\alpha)}\leq\phi_\alpha \tag{6}$$

for all α. As $\tilde{I}$ is an Iwahori subgroup we have $\phi_\alpha+\phi_{-\alpha}=1$ for all α. Hence (6) is equivalent to

$$\langle\mu_x,\sigma^{-1}(-\alpha)\rangle+\phi_{-\psi(\alpha)}\geq\phi_{-\alpha}.$$

Note that $-\psi(\alpha)=\psi(-\alpha)$. Hence this last inequality is equivalent to the inclusion $\sigma(x\tilde{I}_Nx^{-1})\subseteq\tilde{I}_N$. □

Lemma 6.7. *Let $[b] \in B(G)$, and let M_ν be the centralizer of its dominant Newton point. Let P_ν be the associated parabolic subgroup and let N_ν be its unipotent radical. Then $[b]$ contains a P-fundamental alcove x if and only if there is an Iwahori subgroup $\tilde{I}$ with $\tilde{I} \cap M_\nu = I \cap M_\nu$ and an element $b_0 \in \Omega_{M_\nu} \cap [b]$ with $\sigma(b_0(\tilde{I} \cap N_\nu)b_0^{-1}) \subseteq \tilde{I} \cap N_\nu$.*

Proof. Assume first that there is an Iwahori subgroup $\tilde{I}$ and a b_0 as above. As $b_0 \in \Omega_{M_\nu}$ we have $\phi_{b_0}(I_{M_\nu}) = \sigma(I_{M_\nu}) = I_{M_\nu}$. By Lemma 6.6, $\phi_{b_0}(\tilde{I} \cap \overline{N}_\nu) \supseteq \tilde{I} \cap \overline{N}_\nu$. Note that $\tilde{I} \cap M_\nu = I \cap M_\nu$ implies that $T(\mathcal{O}) \subset \tilde{I}$. Let $y \in \widetilde{W}$ with $y^{-1}\tilde{I}y = I$. Let $M = y^{-1}M_\nu y$ and $P = y^{-1}P_\nu y$ with unipotent radical N and opposite $\overline{N}$. Then $y^{-1}\tilde{I}_{M_\nu}y = I_M$. Let $x = \sigma^{-1}(y)^{-1}b_0 y \in [b]$. We have

$$\begin{aligned}\sigma(xI_Mx^{-1}) &= \sigma(xy^{-1}\tilde{I}_{M_\nu}yx^{-1})\\ &= y^{-1}\sigma(b_0\tilde{I}_{M_\nu}b_0^{-1})y\\ &= I_M\end{aligned}$$

and similar translations for N and $\overline{N}$. Hence x is a P-fundamental alcove in $[b]$. For the other direction let x be a P-fundamental alcove and let $w \in {}^MW^{M_\nu}$ with $w^{-1}Mw = M_\nu$. Then $w^{-1}(I \cap M)w = I \cap M_\nu$. A similar translation as above shows that $\tilde{I} = w^{-1}Iw$ and $b_0 = \sigma^{-1}(w)^{-1}xw$ satisfy $\phi_{b_0}(P_\nu) = P_\nu = \sigma(P_\nu)$, hence $b_0 \in \widetilde{W}_{M_\nu}$. Furthermore, $\phi_{b_0}(I_{M_\nu}) = I_{M_\nu} = \sigma(I_{M_\nu})$, whence $b_0 I_{M_\nu} b_0^{-1} = I_{M_\nu}$ and $b_0 \in \Omega_{M_\nu}$. □

Lemma 6.8. *Let M be the Levi factor containing T of a standard parabolic subgroup P, and let N be the unipotent radical of P. Let I_1 and I_2 be two Iwahori subgroups containing $I \cap M$ where I is the standard Iwahori. Then there is a unique Iwahori subgroup I' containing $I \cap M$, $I_1 \cap N$, and $I_2 \cap N$ and minimizing the intersection $I' \cap N$.*

Proof. The Iwahori subgroups we are interested in correspond to alcoves in the apartment corresponding to T in the Bruhat-Tits building of G. Note that an Iwahori subgroup $\tilde{I}$ containing $I \cap M$ satisfies $\tilde{I} \cap M = I \cap M$. The Iwahori subgroups J containing $\tilde{I} \cap P$ correspond to the alcoves in the intersection of the half-spaces of the apartment corresponding to the conditions $J \cap U_\alpha \supseteq \tilde{I} \cap U_\alpha$ for each root α in P. We denote this subset of the apartment by $\mathcal{P}_{\tilde{I}}$. Note that $\mathcal{P}_{\tilde{I}} = \mathcal{P}_{\tilde{I}'}$ implies that $\tilde{I} \cap P = \tilde{I}' \cap P$ and thus $\tilde{I} = \tilde{I}'$. It is thus enough to show that $\mathcal{P}_{I_1} \cap \mathcal{P}_{I_2} = \mathcal{P}_{I'}$ for some I'. Note that our assumption $I_1 \cap M = I_2 \cap M$ implies that $\mathcal{P}_{I_1} \cap \mathcal{P}_{I_2}$ is nonempty. Let $\mathfrak{a}_1$ and $\mathfrak{a}_2$ denote the alcoves corresponding to I_1 and I_2. To prove the assertion above we use induction on the minimal distance in the building between $\mathfrak{a}_1$ and an alcove in $\mathcal{P}_{I_1} \cap \mathcal{P}_{I_2}$. If this distance is 0, then $\mathfrak{a}_1 \in \mathcal{P}_{I_1} \cap \mathcal{P}_{I_2}$, hence $\mathcal{P}_{I_1} \cap \mathcal{P}_{I_2} = \mathcal{P}_{I_1}$. Assume now that $\mathfrak{a}_1 \notin \mathcal{P}_{I_1} \cap \mathcal{P}_{I_2}$. Then there is an affine hyperplane bounding $\mathfrak{a}_1$ and with the property that $\mathfrak{a}_1$ and $\mathcal{P}_{I_1} \cap \mathcal{P}_{I_2}$ lie on different sides of this hyperplane. Let us denote this hyperplane by H. Let $\mathfrak{a}_1'$ be the alcove obtained from $\mathfrak{a}_1$ by reflection at H and let I_1' be the corresponding Iwahori subgroup. Then by definition the minimal distance of $\mathfrak{a}_1'$ to an element of $\mathcal{P}_{I_1} \cap \mathcal{P}_{I_2}$ is 1 less than the corresponding distance for $\mathfrak{a}_1$. Thus by induction it is enough to show that $\mathcal{P}_{I_1} \cap \mathcal{P}_{I_2} = \mathcal{P}_{I_1'} \cap \mathcal{P}_{I_2}$. For all affine hyperplanes $H' \neq H$, the two alcoves $\mathfrak{a}_1'$ and $\mathfrak{a}_1$ lie on the same side of H'. Let S be the half-space bounded by H and containing $\mathfrak{a}_1'$. Then $\mathcal{P}_{I_1'} = \mathcal{P}_{I_1} \cap S$. On the other hand we chose H such that $\mathcal{P}_{I_1} \cap \mathcal{P}_{I_2} \subseteq S$. Hence $\mathcal{P}_{I_1} \cap \mathcal{P}_{I_2} = \mathcal{P}_{I_1'} \cap \mathcal{P}_{I_2}$. □

For a precise description of the sets $\mathcal{P}_I$ for the standard Iwahori compare [GHKR10, 3].

Proof of Theorem 6.5. Let $\nu \in X_*(T)_{\mathbb{Q},\mathrm{dom}}$ be the dominant Newton point of $[b]$. Let P_ν be the associated parabolic subgroup and $P_\nu = M_\nu N_\nu$ the decomposition into the Levi factor containing T and the unipotent radical. Recall that ν, P_ν, M_ν and N_ν are σ-invariant. Let $b_0 \in \widetilde{W}_{M_\nu}$ be a $[b]$-short element. By Lemma 6.7 it is enough to prove that there is an Iwahori subgroup $\tilde{I}$ of LG with $\tilde{I} \cap M_\nu = I_{M_\nu}$ and such that $\phi_{b_0}(\tilde{I} \cap N) \subseteq \tilde{I} \cap N$. Let $r > 0$ be such that G is split over some unramified extension of $\mathcal{O}_F$ of degree r. In particular, σ^r then acts trivially on the root system of G and on $\widetilde{W}$. Let $c := \sigma(b_0)\sigma^2(b_0)\cdots\sigma^r(b_0) \in \widetilde{W}$. Applying the decomposition $\widetilde{W} = W \ltimes X_*(T)$ we obtain $c = w_c\mu_c$. Let n_c be the order of w_c in W. Replacing r by $n_c r$ and using $\sigma(b_0)\sigma^2(b_0)\cdots\sigma^{rn_c}b_0 = c^{n_c} \in X_*(T)$ we may assume that we already have $c \in X_*(T)$. Note that as b_0 is $[b]$-short, we have $c = \nu_{r,M_\nu} = r\nu$ as elements of $X_*(T)_{\mathbb{Q}}$. In particular, c is dominant and central in M_ν. Using Section 2.6 and the fact that σ^r acts trivially on $\widetilde{W}$ we obtain that $\sigma^r(cI_{M_\nu}c^{-1}) = cI_{M_\nu}c^{-1} = I_{M_\nu}$ and $\sigma^r(cI_{N_\nu}c^{-1}) = cI_{N_\nu}c^{-1} \subseteq I_{N_\nu}$. Hence c itself is a P_ν-fundamental alcove for the σ^r-conjugacy class of c in G. Let $\tilde{I}$ with $\tilde{I} \cap M_\nu = I_{M_\nu}$ be unique Iwahori subgroup of LG such that $\tilde{I} \cap N_\nu$ is minimal containing $I_{N_\nu}, \phi_{b_0}(I_{N_\nu}), \dots, \phi_{b_0}^{r-1}(I_{N_\nu})$, cf. Lemma 6.8. Then $\phi_{b_0}(\tilde{I})$ is again an Iwahori subgroup. It satisfies $\phi_{b_0}(\tilde{I}) \cap M_\nu = \phi_{b_0}(\tilde{I} \cap M_\nu) = I_{M_\nu}$ and the analogous minimality property for $\phi_{b_0}(I_{N_\nu}), \dots, \phi_{b_0}^r(I_{N_\nu})$. We have $\phi_{b_0}^r(I_{N_\nu}) = \sigma^r(cI_{N_\nu}c^{-1}) \subseteq I_{N_\nu}$. Thus $\phi_{b_0}(\tilde{I} \cap N_\nu) \subseteq \tilde{I} \cap N_\nu$, and $\tilde{I}$ is as claimed. □

Remark 6.9. Denote for the moment by $B(G)_{k((t))}$ the set of σ-conjugacy classes in $G(k((t)))$, and denote by $B(G)_{W(k)[1/p]}$ the corresponding set in $G(W(k)[1/p])$. Kottwitz's classification maps both sets injectively to $X_*(T)_{\mathbb{Q}} \times \pi_1(G)_\Gamma$. Note that $X_*(T)_{\mathbb{Q}} \times \pi_1(G)_\Gamma$ only depends on $G_{\mathbb{F}_q}$ but not on k or on the choice of the arithmetic or the function field case. Furthermore, the images of $B(G)_{k((t))}$ and $B(G)_{W(k)[1/p]}$ in $X_*(T)_{\mathbb{Q}} \times \pi_1(G)_\Gamma$ can also be described in terms of $G_{\mathbb{F}_q}$, and are independent of the choice of L. In particular we obtain canonical bijections $B(G)_{k((t))} \cong B(G)_{W(k)[1/p]}$ and $B(G)_{k((t))} \cong B(G)_{k'((t))}$ for all algebraically closed fields k' of characteristic p. From now on we use these bijections to identify the sets of σ-conjugacy classes and write again $B(G)$ for all of them.

The main reason to introduce fundamental alcoves in this paper is the following proposition which yields a direct comparison between nonemptiness of intersections of σ-conjugacy classes and Iwahori double cosets in the function field case and in the arithmetic case.

PROPOSITION 6.10.

(1) *Let* $L = k((t))$ *or* $\mathrm{Quot}(W(k))$. *Let* $[b] \in B(G)$ *and let* x_b *be a* P*-fundamental alcove contained in* $[b]$. *Then*

$$\{x \in \widetilde{W} \mid IxI \cap [b] \neq \emptyset\} = \{x \in \widetilde{W} \mid x \in Iy^{-1}Ix_bI\sigma(y)I \text{ for some } y \in \widetilde{W}\}.$$

(2) *Let* $x \in \widetilde{W}$. *Then a* σ*-conjugacy class in* $G(W(k)[1/p])$ *contains an element of* IxI *(for* I *defined with respect to* $W(k)$*) if and only if the corresponding* σ*-conjugacy class in* $G(k((t)))$ *contains an element of* IxI *(where* I *is now a subgroup of* $\mathrm{LG}(k)$*).*

For split groups the first assertion is [GHKR10, Prop. 13.3.1]. Our statement follows using the same proof. As it is very short we repeat it for the reader's convenience.

Proof. Let $g \in IxI \cap [b]$. Then there is an $h \in \mathrm{LG}$ with $h^{-1}x_b\sigma(h) = g$. Let $y \in \widetilde{W}$ with $h \in IyI$. Then $x \in IxI = IgI \subset Iy^{-1}Ix_bI\sigma(y)I$. For the other direction let $x \in Iy^{-1}Ix_bI\sigma(y)I$ for some $y \in \widetilde{W}$. Then $IxI \cap y^{-1}Ix_bI\sigma(y) \neq \emptyset$. Recall that every element of Ix_bI is of the form $i^{-1}x_b\sigma(i)$ for some $i \in I$ (Lemma 6.4). Thus IxI contains an element of the form $y^{-1}i^{-1}x_b\sigma(iy) \in [b]$.

From (1) together with Theorem 6.5 we see that both conditions in (2) can be translated into the same condition in terms of the combinatorics of $\widetilde{W}$ which is independent of the choice of L. Thus (2) follows. □

In particular, we can now easily deduce Theorem 1.7.

Proof of Theorem 1.7. This follows from Proposition 6.10(2) together with Theorem 1.1(2). □

We finish our discussion of fundamental alcoves by a comparison to short elements.

LEMMA 6.11. *Let* G *be split. Then every* P*-fundamental alcove in a given* σ*-conjugacy class* $[b]$ *is* W*-conjugate to the unique* $[b]$*-short element.*

Note that for split groups, W-conjugation coincides with W-σ-conjugation.

Proof. Let b_0 be P-fundamental and contained in $[b]$. We write $P = MN$ for the unipotent radical N of P and the Levi factor M containing T. Let x be the $[b]$-short element and let M_ν be the centralizer of the dominant Newton point ν of b. By definition $x \in \widetilde{W}$ is an element of length 0 in $\widetilde{W}_{M_\nu}$. By Lemma 6.3 we may assume that M is equal to the centralizer of the M-dominant Newton point of b_0. As $[b_0] = [b]$, their Newton points coincide and $\kappa_G(b_0) = \kappa_G(b)$. As in the proof of Lemma 5.3 the Newton point of b_0 together with $\kappa_G(b_0)$ determines $\kappa_M(b_0)$. Hence there is a $w \in W$ that conjugates M to M_ν and also $\kappa_M(b_0)$ to the element $\psi_b \in \pi_1(M_\nu)$ defined in the proof of Lemma 5.3. Choosing w of minimal length in its coset $W_M w W_{M_\nu}$ it also conjugates I_M to I_{M_ν}. Now $w^{-1}b_0w$ and x are in $\widetilde{W}_{M_\nu}$ and both have length 0 as elements of $\widetilde{W}_{M_\nu}$. Thus they are in the subgroup Ω_{M_ν}. As G is split, we have that $\kappa_{M_\nu} : \Omega_{M_\nu} \to \pi_1(M_\nu)$ is an isomorphism. As the images of x and $w^{-1}b_0w$ under κ_{M_ν} coincide, the elements have to be equal. Hence b_0 is W-conjugate to the $[b]$-short element x. □

7. Applications

In this section we consider the case $F = \mathbb{Q}_p$. We review some of the theory of Ekedahl-Oort strata and relate it to our notion of truncations of level 1. We concentrate on the example of the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties of dimension g, and briefly indicate possible generalizations to other Shimura varieties of PEL type. For more details on this general theory of Ekedahl-Oort strata we refer to [VW13].

The Ekedahl-Oort stratification is the stratification of $\mathcal{A}_g$ according to the p-torsion $(A,\lambda)[p]$ of the principally polarized abelian varieties (A,λ) associated with the points of $\mathcal{A}_g$. It was first defined and studied by Oort in [Oor01]. Oort classifies the p-torsion $(A,\lambda)[p]$ by a finite combinatorial invariant, so-called elementary sequences. A second description of the Ekedahl-Oort invariant (and more generally of $G(k)$-orbits on a certain variety associated with a reductive group G over $\mathbb{Z}_p$ together with a fixed Levi subgroup that is also defined over $\mathbb{Z}_p$) has been given by Moonen and Wedhorn in [MW04]. They use a description by so-called F-zips and identify the index set for the Ekedahl-Oort stratification of $\mathcal{A}_g$ with ${}^\mu W$ where μ is the minuscule dominant element given by the Shimura datum and where W is the Weyl group of GSp_{2g}. Another related theory is Vasiu's classification of so-called Shimura F-crystals in [Vas10, Main Theorem C].

In the language of truncations of level 1 the Ekedahl-Oort invariant on $\mathcal{A}_g$ can be studied as follows. From an element of $\mathcal{A}_g(k)$ we obtain a polarized p-divisible group $(A,\lambda)[p^\infty]$. The polarization equips its Dieudonné module with a symplectic form $\langle\cdot,\cdot\rangle$. In the same way as in Section 1.4 we trivialize its Dieudonné module and obtain that the Frobenius is given by an element b

of $\mathrm{GSp}_{2g}(W(k)[1/p])$, well-defined up to σ-conjugation with $\mathrm{GSp}_{2g}(W(k)) = K(k)$. It satisfies that $b \in K(k)\mu(p)K(k)$ where μ is as above, i.e., $\mu(p)$ is the diagonal matrix with entries p and 1, each with multiplicity g. The Ekedahl-Oort stratification is then nothing but the stratification that one obtains by considering the truncations of level 1 of the elements b. The index set for truncations of level 1 of elements of $K\mu(p)K$ is equal to ${}^{\mu}W$ (Theorem 1.1(1)). This identification coincides with the one used in the classification by Moonen and Wedhorn.

For $w \in {}^{\mu}W$ let S_w be the reduced subscheme of the reduction of $\mathcal{A}_g$ given by the condition that $(A, \lambda)[p]$ has Ekedahl-Oort invariant w. Oort proves that each stratum S_w is locally closed, and the closure $\overline{S_w}$ is a union of strata. The set of strata that are contained in $\overline{S_w}$ is determined in [Wed] together with (6.4) of loc. cit. It is given by the same formula as the closure relations between the corresponding strata $S_{w,\mu}$ in the loop group of $G = \mathrm{GSp}_{2g}$ (that we compute in Corollary 4.7).

Recall that in Theorem 1.1(2) we established a comparison between the stratification by truncations of level 1 and the subdivision of LG into Iwahori double cosets. Relations between the Ekedahl-Oort stratification and the subdivision into Iwahori-double cosets are also used in the theory of moduli spaces of abelian varieties, see for example [EvdG09, Cor. 8.4(iii)] or [GY12, 9].

We now compare Oort's minimal p-divisible groups (see [Oor05]) to our notion of short elements. Let X be a p-divisible group over an algebraically closed field k and let $(\mathbf{M}, F)$ be its Dieudonné module. Let $\mathbf{N} = \mathbf{M} \otimes_{W(k)} \mathrm{Quot}(W(k))$. By definition there is a unique isomorphism class of minimal p-divisible groups in each isogeny class of p-divisible groups (see [Oor05]). Explicitly, if X is minimal, its Dieudonné module is isomorphic to a Dieudonné module of the following form. There is a decomposition of the rational Dieudonné module into simple summands $\mathbf{N} = \bigoplus_{i=1}^{l} \mathbf{N}_l$ such that $\mathbf{M} = \bigoplus_{i=1}^{l} \mathbf{M} \cap \mathbf{N}_i$. Let $\lambda_i = n_i/h_i$ with $(n_i, h_i) = 1$ be the slope of $\mathbf{N}_i$. Then $\mathbf{M} \cap \mathbf{N}_i$ has a basis $e_1^i, \dots, e_{h_i}^i$ such that $F(e_j^i) = e_{j+n_i}^i$. Here we use the notation $e_{j+h_i}^i = pe_j^i$. Equivalently, X is minimal if the endomorphisms of $(\mathbf{M}, F)$ are a maximal order in the endomorphisms of $(\mathbf{N}, F)$.

Let now $f_j^i = e_{h_i+1-j}^i$. Let $h = \dim \mathbf{N}$. One easily checks that if we write $F = b\sigma$ for $b \in \mathrm{GL}_h(L)$ with respect to the basis $f_1^1, \dots, f_{h_1}^1, f_1^2, \dots$, then b is contained in the Levi subgroup M given by the decomposition $\mathbf{N} = \bigoplus_{i=1}^{l} \mathbf{N}_i$. Furthermore, if μ denotes the M-dominant Hodge polygon of b (with respect to the choice of the upper triangular matrices as Borel subgroup), then $\mu \in \{0,1\}^h$ is minuscule and $b = \tau_{\mu,M}$ satisfies $bI_M b^{-1} = I_M$. Hence a p-divisible group is minimal if and only if the K-σ-conjugacy class of the element determining the Frobenius on the Dieudonné module contains a short element, or equivalently (by Lemma 6.11) a P-fundamental alcove.

Corollary 7.1. *Let X be a minimal p-divisible group over k and let Y be a p-divisible group with $X[p] \cong Y[p]$. Then $X \cong Y$. An analogous assertion holds for polarized p-divisible groups.*

This reproves the main theorem of [Oor05].

Proof. We use Dieudonné theory and trivialize the Dieudonné modules of X and Y to reformulate the assertion. Let $G = \mathrm{GL}_h$ respectively GSp_h where h is the height of X. Let $b_X \in G(W(k)[1/p])$ be the element describing the Frobenius on the Dieudonné module of X. As X is minimal we can choose the trivialization in such a way that b_X is a P-fundamental alcove for some P. Let $b_Y \in G(W(k)[1/p])$ be the element describing the Frobenius on the Dieudonné module of Y. As $X[p] \cong Y[p]$ we can choose the trivialization in such a way that $b_Y \in K_1 b_X K_1$. By Lemma 6.4, b_X and b_Y are I-σ-conjugate to each other. In particular, they are K-σ-conjugate which implies that $X \cong Y$. □

It would be interesting to construct a generalization of minimal p-divisible groups for all good reductions of PEL Shimura varieties. In particular, one would be interested in a representative of a given isogeny class of p-divisible groups with endomorphisms and polarization satisfying the analogue of Corollary 7.1. Although we constructed P-fundamental alcoves for all σ-conjugacy classes of elements of $G(L)$ for all G, our theory does not imply the existence of such minimal p-divisible groups with extra structure for nonsplit G. The reason is that we did not study whether there exist P-fundamental alcoves in a given σ-conjugacy class which in addition lie in the prescribed K-double coset given by μ. A weaker generalization of the notion of minimality would be to call a p-divisible group with PEL structure minimal if the K-σ-conjugacy class of the element determining the Frobenius on the Dieudonné module contains a short element. Our theory implies the existence of such elements in each isogeny class, compare the discussion after Corollary 7.2.

One interesting open question about Ekedahl-Oort strata is to determine which Newton polygons occur in a given Ekedahl-Oort stratum. Our theory for loop groups (in particular, Theorem 1.5) together with the comparison results of the preceding section yield the following necessary condition.

Corollary 7.2. *Let x be a k-valued point of S_w for some w. Let $x_0 \in \mathcal{A}_g(k)$ be a point corresponding to the minimal p-divisible group in the isogeny class corresponding to x. Then $x_0 \in \overline{S_w}$.*

Proof. Let μ be the Hodge vector associated with $\mathcal{A}_g$. Let $[b] \in B(G)$ be the class corresponding to the isogeny class of the p-divisible group corresponding to x. By Theorem 1.7 the corresponding class $[b]$ in LG intersects the truncation stratum $S_{w,\mu} \subseteq \mathrm{LG}$. Let $b_0 \in \widetilde{W}$ be a $[b]$-short element. Then the representative of b_0 in $\mathrm{GSp}_{2g}(W(k)[1/p])$ describes the Dieudonné module of the minimal p-divisible group in $[b]$. Let (w_0, μ) be its truncation type. From Theorem 1.5 we obtain that $S_{w_0,\mu} \subseteq \overline{S_{w,\mu}}$ in LG. Recall that the closure relations between the strata S_w are known to coincide with those between the corresponding strata $S_{w,\mu}$ in the loop group of $G = \mathrm{GSp}_{2g}$. Thus the corollary follows. □

For the Siegel moduli space $\mathcal{A}_g$ this has been conjectured by Oort [Oor04], Conjecture 6.9 and has been shown previously by Harashita in [Har07], [Har09], [Har10] using different methods. While this article was being finished, Harashita published a preprint [Har12] in which he proves an analog of Corollary 7.2 for some catalog of p-divisible groups in the nonpolarized case (without endomorphisms). Our approach to prove Corollary 7.2 also leads to variants without polarization, and/or with endomorphisms: Let S_w denote the truncation strata in a moduli space of abelian varieties associated with a PEL Shimura variety with good reduction at p. The same proof as above then shows that $x \in S_w(k)$ for some $w \in {}^{\mu}W$ implies that there is an element x_0 whose p-divisible group (with extra structure) is isogenous to the one corresponding to x, such that the associated element $b_{x_0} \in G(W(k)[1/p])$ is short, and such that $x_0 \in \overline{S_w}$. This element is in general (for nonsplit G) not uniquely defined by the isogeny class (compare Lemma 5.3). For more details we refer to [VW13].

For the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties of dimension g in characteristic $p > 2$ we know by [EvdG09, Th. 11.5] that each Ekedahl-Oort stratum which is not contained in the supersingular locus is irreducible. In particular, there is a unique generic Newton polygon in each Ekedahl-Oort stratum S_w of $\mathcal{A}_g$. Then in the same way as for the loop group we can use the above result to determine this Newton polygon.

Corollary 7.3. *Let ν be the generic Newton polygon in $S_w \subseteq \mathcal{A}_g$ for some $w \in {}^{\mu}W$. Then ν is the maximal element in the set of Newton polygons of short elements x such that $x \in \overline{S_w}$.*

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