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A general regularity theory for stable codimension 1 integral varifolds

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Abstract

We give a necessary and sufficient geometric structural condition, which we call the α-Structural Hypothesis, for a stable codimension 1 integral varifold on a smooth Riemannian manifold to correspond to an embedded smooth hypersurface away from a small set of generally unavoidable singularities. The α-Structural Hypothesis says that no point of the support of the varifold has a neighborhood in which the support is the union of three or more embedded $C^{1,\alpha}$ hypersurfaces-with-boundary meeting (only) along their common boundary. We establish that whenever a stable integral n-varifold on a smooth $(n+1)$-dimensional Riemannian manifold satisfies the α-Structural Hypothesis for some $\alpha \in (0, 1/2)$, its singular set is empty if $n \leq 6$, discrete if $n = 7$ and has Hausdorff dimension $\leq n-7$ if $n \geq 8$; in view of well-known examples, this is the best possible general dimension estimate on the singular set of a varifold satisfying our hypotheses. We also establish compactness of mass-bounded subsets of the class of stable codimension 1 integral varifolds satisfying the α-Structural Hypothesis for some $\alpha \in (0, 1/2)$.

The α-Structural Hypothesis on an n-varifold for any $\alpha \in (0, 1/2)$ is readily implied by either of the following two hypotheses: (i) the varifold corresponds to an absolutely area minimizing rectifiable current with no boundary, (ii) the singular set of the varifold has vanishing $(n-1)$-dimensional Hausdorff measure. Thus, our theory subsumes the well-known regularity theory for codimension 1 area minimizing rectifiable currents and settles the long standing question as to which weakest size hypothesis on the singular set of a stable minimal hypersurface guarantees the validity of the above regularity conclusions.

An optimal strong maximum principle for stationary codimension 1 integral varifolds follows from our regularity and compactness theorems.

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1. **Introduction**

Here we study regularity properties of stable critical points of the n-dimensional area functional in a smooth $(n+1)$-dimensional Riemannian manifold, addressing, among a number of other things, the following basic question:

> *When is a stable critical point V of the n-dimensional area functional in a smooth $(n+1)$-dimensional Riemannian manifold made-up of pairwise disjoint, smooth, embedded, connected hypersurfaces each of which is itself a critical point of area?*

Without further hypothesis, V need not satisfy the stated property; this is illustrated by (a sufficiently small region of) any stationary union of three or more hypersurfaces-with-boundary meeting along a common $(n–1)$-dimensional submanifold (e.g., a pair of transverse hyperplanes in a Euclidean space). In each of these examples, the connected components of the regular part of the union are not individually critical points of area (in the sense of having vanishing first variation with respect to area for deformations by compactly supported smooth vector fields of the ambient space; see precise definition in Section 3).

We give a geometrically optimal answer to the above question by establishing a precise version (given as Corollary 1) of the following assertion:

> *Presence of a region of V where three or more hypersurfaces-with-boundary meet along their common boundary is the only obstruction for V to correspond to a locally finite union of pairwise disjoint, smooth,*

embedded, connected hypersurfaces each of which is itself a critical point of area.

This follows directly from our main theorem (the Regularity and Compactness Theorem) which establishes a precise version of the following regularity statement:

Presence of a region of V where three or more hypersurfaces-with-boundary meet along their common boundary is the only obstruction to complete regularity of V in low dimensions and to regularity of V away from a small, quantifiable, set of generally unavoidable singularities in general dimensions.

In proving these results, we shall first work in the context where the ambient manifold is an open subset of $\mathbf{R}^{n+1}$ with the Euclidean metric. The differences that arise in the proof in replacing Euclidean ambient space by a general smooth $(n+1)$-dimensional Riemannian manifold amount to "error terms" in various identities and inequalities that are valid in the Euclidean setting, and they can be handled in a straightforward manner. We shall discuss this further in the penultimate section of the paper.

Here, a *critical point of the n-dimensional area* means a *stationary* integral n-varifold; i.e., an integral n-varifold having zero first variation with respect to area under deformation by the flow generated by any compactly supported C^1 vector field of the ambient space (see hypothesis ($\mathcal{S}1$) in Section 3).

For a varifold V, let $\operatorname{reg} V$ denote its regular part, i.e., the smoothly embedded part (of the support of the weight measure $\|V\|$ associated with V), and let $\operatorname{sing} V$ denote its singular set, i.e., the complement of $\operatorname{reg} V$ (in the support of $\|V\|$); see Section 2 for the precise definitions of these terms.

A stationary integral varifold V is *stable* if $\operatorname{reg} V$ is stable in the sense that V has nonnegative second variation with respect to area under deformation by the flow generated by any C^1 ambient vector field that is compactly supported away from $\operatorname{sing} V$ and that, on $\operatorname{reg} V$, is normal to $\operatorname{reg} V$. In our codimension 1 setting and for Euclidean ambient space, stability of V whenever $\operatorname{reg} V$ is orientable is equivalent to requiring that $\operatorname{reg} V$ satisfies the following *stability inequality* ([Sim83, §9]):

$$\int_{\operatorname{reg} V} |A|^2\zeta^2\, d\mathcal{H}^n \leq \int_{\operatorname{reg} V} |\nabla \zeta|^2\, d\mathcal{H}^n \quad \forall\, \zeta \in C^1_c(\operatorname{reg} V);$$

here A denotes the second fundamental form of $\operatorname{reg} V$, $|A|$ the length of A, ∇ the gradient operator on $\operatorname{reg} V$ and $\mathcal{H}^n$ is the n-dimensional Hausdorff measure on $\mathbf{R}^{n+1}$. (In fact a slightly weaker form of the stability hypothesis suffices for the proofs of all of our theorems here, and as a result of that, orientability of $\operatorname{reg} V$ for the varifolds V considered here is a conclusion rather than a hypothesis; see hypothesis ($\mathcal{S}2$) in Section 3 and Corollary 3.2.)

By a *stable integral n-varifold* we mean a stable, stationary integral n-varifold. For $\alpha \in (0,1)$ and V an integral varifold on a smooth Riemannian manifold, we state the following condition (hypothesis ($\mathcal{S}3$) in Section 3), which we shall refer to often throughout the rest of the introduction:

α-Structural Hypothesis. *No singular point of V has a neighborhood in which V corresponds to a union of embedded $C^{1,\alpha}$ hypersurfaces-with-boundary meeting (only) along their common $C^{1,\alpha}$ boundary (and with multiplicity a constant positive integer on each of the constituent hypersurfaces-with-boundary).*

Our main theorem (Theorem 18.1; for Euclidean ambient space, Theorem 3.1) can now be stated as follows:

Regularity and Compactness Theorem. *A stable integral n-varifold V on a smooth $(n+1)$-dimensional Riemannian manifold corresponds to an embedded hypersurface with no singularities when $1 \leq n \leq 6$; to one with at most a discrete set of singularities when $n = 7$; and to one with a closed set of singularities having Hausdorff dimension at most $n-7$ when $n \geq 8$ (and with multiplicity, in each case, a constant positive integer on each connected component of the hypersurface), provided V satisfies the α-Structural Hypothesis above for some $\alpha \in (0, 1/2)$.*

Furthermore, for any given $\alpha \in (0,1/2)$, each mass-bounded subset of the class of stable codimension 1 integral varifolds satisfying the α-Structural Hypothesis is compact in the topology of varifold convergence.

In case V corresponds to an absolutely area minimizing codimension 1 rectifiable current, the regularity conclusion of this theorem is well known and is the result of combined work of E. De Giorgi [DG61], R. Reifenberg [Rei60], W. Fleming [Fle62], F. Almgren [Alm66], J. Simons [Sim68] and H. Federer [Fed70]. While our work uses ideas and results from some of these pioneering works, it does not rely upon the fact that the conclusions hold in the area minimizing case; it is interesting to note that the above theorem indeed subsumes the regularity theory for codimension 1 area minimizing rectifiable currents for the following simple reason: If T is a rectifiable current on an open ball and if T has no boundary in the interior of the ball and is supported on a union of three or more embedded hypersurfaces-with-boundary meeting only along their common boundary, then T cannot be area minimizing.

Let V be a stationary integral n-varifold on a Riemannian manifold N. Once we know that the singular set of V is sufficiently small—in fact, as small as having vanishing $(n-1)$-dimensional Hausdorff measure—it is not difficult to check that the multiplicity 1 varifold associated with each connected component of the regular part of V is itself stationary in N. Thus we deduce from the Regularity and Compactness Theorem the following:

Corollary 1. *The α-Structural Hypothesis* (*see above*) *for some* $\alpha \in (0, 1/2)$ *is necessary and sufficient for a stable codimension* 1 *integral varifold* V *on a smooth Riemannian manifold* N *to have the following "local decomposability property": For each open* $\Omega \subset N$ *with compact closure in* N, *there exist a finite number of pairwise disjoint, smooth, embedded, connected hypersurfaces* $M_1, M_2, \ldots, M_k$ *of* Ω (*possibly with a nonempty interior singular set* $\operatorname{sing} M_j = (\overline{M}_j \setminus M_j) \cap \Omega$ *for each* $j = 1, 2, \ldots, k$) *and positive integers* $q_1, q_2, \ldots, q_k$ *such that the multiplicity* 1 *varifold* $|M_j|$ *defined by* M_j *is stationary in* Ω *for each* $j = 1, 2, \ldots, k$ *and* $V \llcorner \Omega = \sum_{j=1}^{k} q_j |M_j|$.

In 1981, R. Schoen and L. Simon ([SS81]) proved that the conclusions of the Regularity and Compactness Theorem hold for the n-dimensional stable minimal hypersurfaces (viz. embedded hypersurfaces that are stationary and stable as multiplicity 1 varifolds) satisfying, in place of the α-Structural Hypothesis, the (much more restrictive) property that the singular sets have locally finite $(n-2)$-dimensional Hausdorff measure. Since then, it has remained an open question as to what the *weakest* size hypothesis (in terms of Hausdorff measure) on the singular sets is that would guarantee the validity of the same conclusions. Since vanishing of the $(n-1)$-dimensional Hausdorff measure of the singular set trivially implies the α-Structural Hypothesis, we have the following immediate corollary of the Regularity and Compactness Theorem, which settles this question:

Corollary 2. *The conclusions of the Regularity and Compactness Theorem hold for the n-dimensional stable minimal hypersurfaces with singular sets of vanishing* $(n-1)$-*dimensional Hausdorff measure. In fact, a stable codimension* 1 *integral n-varifold* V *satisfies the α-Structural Hypothesis for some* $\alpha \in (0, 1/2)$ *if and only if its singular set has vanishing* $(n-1)$-*dimensional Hausdorff measure.*

A union of two transversely intersecting hyperplanes in a Euclidean space shows that for no $\gamma > 0$ can the singular set hypothesis in Corollary 2 be weakened to vanishing of the $(n-1+\gamma)$-dimensional Hausdorff measure.

In contrast to our α-Structural Hypothesis, the singular set hypothesis of [SS81] (i.e., the hypothesis that $\mathcal{H}^{n-2}(\operatorname{sing} V \cap K) < \infty$ for each compact subset K of the ambient space), together with stability away from the singular set, *a priori* implies, by a straightforward argument, that the singularities are "removable for the stability inequality"— that is to say, the above stability inequality is valid for the larger class of test functions ζ that are the restrictions to the hypersurface of compactly supported smooth functions of the ambient space (that are not required to vanish near the singular set). The techniques employed in [SS81] in the proof of the regularity theorems therein relied on this fact in an essential way. Interestingly, the α-Structural Hypothesis or, for that matter, vanishing of the $(n-1)$-dimensional Hausdorff measure of the singular set, does not seem to imply *a priori* even local finiteness of total curvature, viz. $\int_{\mathrm{reg}\, V\cap K} |A|^2 < \infty$ for each compact subset K of the ambient space (whereas the singular set hypothesis of [SS81] does, in view of the strengthening of the stability inequality just mentioned). This means that in our proof we cannot use the stability inequality in a direct way over arbitrary regions of the varifolds. (Of course *a posteriori* we can strengthen the stability inequality in the manner described above so, in particular, it is true under our hypotheses that $\int_{\mathrm{reg}\, V\cap K} |A|^2 < \infty$ for each compact subset K of the ambient space.) Nevertheless, at several stages our proof makes indispensable use of the work of Schoen and Simon—specifically, Theorem 3.5 below; indeed, application of Theorem 3.5 in regions where we have sufficient control over the singular set is a principal way in which the stability hypothesis enters our proof.

The Regularity and Compactness Theorem is optimal in several ways. A key aspect of the theorem is that it *requires no hypothesis concerning the size of the singular sets*; nor does it require any hypothesis concerning the generally-difficult-to-control set of points where some tangent cone is a plane of multiplicity 2 or higher. What suffices is the α-Structural Hypothesis, which is easier to check in principle. As mentioned before, stationary unions of half-hyperplanes of a Euclidean space meeting along common axes illustrate that the α-Structural Hypothesis is a sharp condition needed for the regularity conclusions of the theorem.

In view of well-known examples of 7-dimensional stable hypercones with isolated singularities (e.g., the cone over $\mathbf{S}^3(1/\sqrt{2}) \times \mathbf{S}^3(1/\sqrt{2}) \subset \mathbf{R}^8$), the Regularity and Compactness Theorem is also optimal with regard to its conclusions in the sense that it gives, in dimensions ≥ 7, the optimal general estimate on the Hausdorff dimension of the singular sets.

It remains an open question as to what one can say about the size of the singular sets if the stability hypothesis in the theorem is removed. Obviously in this case one cannot draw the same conclusions in view of the fact that there are embedded nonequatorial minimal surfaces of $\mathbf{S}^3$ (e.g., $\mathbf{S}^1(1/\sqrt{2}) \times \mathbf{S}^1(1/\sqrt{2}) \subset \mathbf{S}^3$), the cones over which provide examples of stationary (unstable) hypercones in $\mathbf{R}^4$ with isolated singularities. There are no 2-dimensional singular stationary hypercones satisfying the α-Structural Hypothesis; however, it is not known whether there is a singular 2-dimensional stationary integral varifold V in $\mathbf{R}^3$ such that V either satisfies the α-Structural Hypothesis or has a singular set of vanishing 1-dimensional Hausdorff measure or has an isolated singularity. It also remains largely open what one can say concerning stable integral varifolds of codimension > 1. Again, the same conclusions as in our theorem cannot be made in this case due to the presence of branch point singularities, as illustrated by 2-dimensional holomorphic varieties with isolated branch points. See the remark following the statement of the Regularity and Compactness Theorem in Section 3 (Theorem 3.1) for a further discussion on optimality of our results here.

For a general stationary integral varifold, a point where some tangent cone is a plane of multiplicity 2 or higher may or may not be a regular point. Our "Sheeting Theorem" (Theorem 3.3 below) implies that if the varifold satisfies the hypotheses of the Regularity and Compactness Theorem, then such a point is a regular point. (As is well known, a point where there is a multiplicity 1 tangent plane is always a regular point, for any stationary integral varifold, by the regularity theorem of W. K. Allard; see [All72, §8] and also [Sim83, Th. 23.1].) Indeed, the Sheeting Theorem is one of the two principal ingredients of the proof of the Regularity and Compactness Theorem; the other is the "Minimum Distance Theorem" (Theorem 3.4), which implies that no tangent cone to a varifold satisfying the hypotheses of the Regularity and Compactness Theorem can be supported by a union of three or more half-hyperplanes meeting along a common $(n-1)$-dimensional axis.

A direct consequence of Allard's regularity theorem is that the regular part of a stationary integral varifold is a nonempty—in fact a dense—subset of its support [All72, §8.1]. Thus, given stationarity of the varifold, our stability hypothesis, which concerns only the regular part of the varifold, is never vacuously true. However, an open, dense subset could have arbitrarily small (positive) measure, and in fact, as mentioned above, under the stationarity hypothesis alone no general result whatsoever is known concerning the Hausdorff measure of the singular sets. Closely related to this is the point made before that from the hypotheses of the Regularity and Compactness Theorem, not even local finiteness of total curvature seems to follow *a priori*. In light of these considerations that indicate that our hypotheses are rather mild, it is somewhat surprising that our hypotheses imply optimal regularity of the hypersurfaces.

We may summarise all of the various regularity results discussed above and established in subsequent sections of the paper as follows:

THEOREM. *Let V be a stable integral n-varifold on a smooth $(n+1)$-dimensional Riemannian manifold N. The following statements concerning V are equivalent*:

(a) *For some $\alpha \in (0, 1/2)$, V satisfies the α-Structural Hypothesis*, *viz. no singular point of V has a neighborhood in which V corresponds to a union of $C^{1,\alpha}$ embedded hypersurfaces-with-boundary meeting* (*only*) *along their common boundary*, *with multiplicity a constant positive integer on each constituent hypersurface-with-boundary.*

(b) $\operatorname{sing} V = \emptyset$ *if* $1 \leq n \leq 6$, $\operatorname{sing} V$ *is discrete if* $n = 7$ *and* $\mathcal{H}^{n-7+\gamma}(\operatorname{sing} V) = 0$ *for each* $\gamma > 0$ *if* $n \geq 8$.

(c) $\mathcal{H}^{n-1}(\operatorname{sing} V) = 0$.
(d) *V has the local decomposability property* (*defined in Corollary* 1 *above*), *viz. for each open $\Omega \subset N$ with compact closure in N, there exist a finite number of pairwise disjoint, smooth, embedded, connected hypersurfaces $M_1, M_2, \ldots, M_k$ of Ω* (*possibly with $(\overline{M}_j \setminus M_j) \cap \Omega$ nonempty for each $j = 1, 2, \ldots, k$*) *and positive integers $q_1, q_2, \ldots, q_k$ such that the multiplicity* 1 *varifold $|M_j|$ defined by M_j is stationary in Ω for each $j = 1, 2, \ldots, k$ and $V \llcorner \Omega = \sum_{j=1}^{k} q_j |M_j|$.*
(e) *No tangent cone of V corresponds to a union of three or more half-hyperplanes meeting along a common $(n-1)$-dimensional subspace, with multiplicity a constant positive integer on each constituent half-hyperplane.*
(f) *V satisfies the α-Structural Hypothesis for each $\alpha \in (0, 1/2)$.*

Finally, we mention another direct implication of the Regularity and Compactness Theorem, namely, the following optimal strong maximum principle (Theorem 19.1) for codimension 1 stationary integral varifolds:

Varifold Maximum principle. *Let N be a smooth $(n+1)$-dimensional Riemannian manifold, and let Ω_1, Ω_2 be open subsets of N such that $\Omega_1 \subset \Omega_2$. Let $M_i = \partial\, \Omega_i$ for $i = 1, 2$. If for $i = 1, 2$, M_i is connected, $\mathcal{H}^{n-1}(\operatorname{sing} M_i) = 0$ and $V_i \equiv |M_i|$ is stationary in N, then either* $\operatorname{spt} \|V_1\| = \operatorname{spt} \|V_2\|$ *or* $\operatorname{spt} \|V_1\| \cap \operatorname{spt} \|V_2\| = \emptyset$. *Here* $\operatorname{sing} M_i = M_i \setminus \operatorname{reg} M_i$, *where* $\operatorname{reg} M_i$ *is the set of points $X \in M_i$ such that M_i is a smooth, embedded submanifold near X.*

See Section 2 for an explanation of notation used here. If the varifolds V_1 and V_2 are both free of singularities, the theorem is easily seen to follow from the Hopf maximum principle. B. Solomon and B. White [SW89] proved the theorem assuming only that one of V_1 or V_2 is free of singularities (allowing the other to be arbitrary with no restriction on its singular set). M. Moschen [Mos77] and independently L. Simon [Sim87] established the result in case V_1 and V_2 correspond to area minimizing integral currents, both possibly singular. Using the Schoen-Simon regularity theory [SS81], some key ideas from [Sim87] as well as the Solomon–White theorem, T. Ilmanen [Ilm96] established the theorem (for stationary V_1, V_2) subject to the stronger condition $\mathcal{H}^{n-2}(\operatorname{sing} M_i) < \infty$ for $i = 1, 2$. The version above follows directly from the argument in [Ilm96], in view of the fact that we may use Corollary 2 in places where the argument in [Ilm96] depended on the Schoen-Simon theory. This version is optimal in the sense that larger singular sets cannot generally be allowed.

Outline of the method. Here we give a brief description of the proof of the Regularity and Compactness Theorem. Fix any $\alpha \in (0, 1)$, and let $\mathcal{S}_\alpha$ denote the family of stable integral n-varifolds of the open ball $B_2^{n+1}(0) \subset \mathbf{R}^{n+1}$ satisfying the α-Structural Hypothesis. The proof of the Regularity and

Compactness Theorem is based on establishing the fact that no tangent cone at a singular point of a varifold belonging to the varifold closure of $\mathcal{S}_\alpha$ can be supported by (a) a hyperplane or (b) a union of half-hyperplanes meeting along an $(n-1)$-dimensional subspace. Once this is established, it is not difficult to reach the conclusions of the theorem with standard arguments.

The assertion in case (a) is implied by the following regularity result (Theorem 3.3):

Sheeting Theorem. *Whenever a varifold in $\mathcal{S}_\alpha$ is weakly close to a given hyperplane $\mathbf{P}_0$ of constant positive integer multiplicity, it must break up in the interior into disjoint, embedded smooth graphs ("sheets") of small curvature over $\mathbf{P}_0$.*

The assertion in case (b) is a consequence of the following (Theorem 3.4):

Minimum Distance Theorem. *No varifold in $\mathcal{S}_\alpha$ can be weakly close to a given stationary integral hypercone $\mathbf{C}_0$ corresponding to a union of three or more half-hyperplanes meeting along an $(n-1)$-dimensional subspace (and with constant positive integer multiplicity on each half-hyperplane).*

Our strategy is to prove both the Sheeting Theorem and the Minimum Distance Theorem simultaneously by an inductive argument, inducting on the multiplicity q of $\mathbf{P}_0$ for the Sheeting Theorem and on the density $\Theta(\|\mathbf{C}_0\|, 0)$ ($=$ q or $q + 1/2$) of $\mathbf{C}_0$ at the origin for the Minimum Distance Theorem, where q is an integer ≥ 1. Approaching both theorems inductively and *simultaneously* in this manner makes it possible to establish, for varifolds in $\mathcal{S}_\alpha$ (satisfying appropriate "small excess" hypotheses in accordance with the theorems) and for their "blow-ups," many of the necessary *a priori* estimates that seem inaccessible via an approach (inductive or otherwise) aimed at proving the two theorems separately.

The main general idea in the argument is the following: Let q be an integer ≥ 2, and assume by induction the validity of the Sheeting Theorem when $\mathbf{P}_0$ has multiplicity $\in \{1, \ldots, q-1\}$ and of the Minimum Distance Theorem when $\Theta(\|\mathbf{C}_0\|, 0) \in \{3/2, \ldots, q - 1/2, q\}$. Then, in a region of a varifold in $\mathcal{S}_\alpha$ where no singular point has density $\geq q$, we may apply the induction hypotheses together with a theorem of J. Simons ([Sim68]; see also [Sim83, App. B]) and the "generalised stratification of stationary integral varifolds" due to F. J. Almgren Jr. [Alm00, Th. 2.26 and Rem. 2.28] to reduce the dimension of the singular set to a low value. This permits effective usage of the stability hypothesis, including applicability of the Schoen-Simon version ([SS81, Th. 2]; also Theorem 3.5 below) of the Sheeting Theorem, in such a region. On the other hand, in the presence of singularities of density $\geq q$ (and whenever the density ratio of the varifold at scale 1 is close to q), it is possible to make good use of the monotonicity formula; most notable among its consequences in the present context are versions (Theorem 10.1 and Corollary 10.2), for a varifold in $\mathcal{S}_\alpha$ with small "height excess" relative to a hyperplane and lower order height excess relative to certain cones, of L. Simon's [Sim93] *a priori* L^2-estimates, and an analogous, new, "nonconcentration-of-tilt-excess" estimate (Theorem 7.1(b)) giving control of the amount of its "tilt-excess" relative to the hyperplane in regions where there is a high concentration of points of density $\geq q$.

Combining these techniques, we are able to fully analyse, under the induction hypotheses, the "coarse blow-ups," namely, the compact class $\mathcal{B}_q \subset W^{1,2}_{\text{loc}}(B_1; \mathbf{R}^q) \cap L^2\ (B_1; \mathbf{R}^q)$ ($B_1 =$ the open unit ball in $\mathbf{R}^n$) consisting of ordered q-tuples of functions produced by blowing up sequences of varifolds in $\mathcal{S}_\alpha$ converging weakly to a multiplicity q hyperplane. (See the precise definition of $\mathcal{B}_q$ at the end of Section 5.) One of the key properties that needs to be established for $\mathcal{B}_q$ is that it does not contain an element H whose graph is the union of q half-hyperplanes in one half-space of $\mathbf{R}^{n+1}$ and q half-hyperplanes in the complementary half-space, with all half-hyperplanes meeting along a common $(n-1)$-dimensional subspace and at least two of them distinct on one side or the other. (This is a Minimum Distance Theorem for $\mathcal{B}_q$, analogous to the Minimum Distance Theorem for $\mathcal{S}_\alpha$.) Establishing this property takes considerable effort and occupies a significant part (Sections 9 through 14) of our work. It is achieved as follows:

First we rule out (in Section 9), by a first variation argument utilizing the nonconcentration-of-tilt-excess estimate of Theorem 7.1(b), the possibility that there is such $H \in \mathcal{B}_q$ with its graph having all q half-hyperplanes on one side coinciding (but not on the other).

The second, more involved step is to rule out the existence of such an element in $\mathcal{B}_q$ (call it H') with its graph having at least two distinct half-hyperplanes on each side. To this end we assume, arguing by contradiction, that there is such $H' \in \mathcal{B}_q$ and use the induction hypotheses to implement a "fine blow-up" procedure (see the definition at the end of Section 11), where certain sequences of varifolds in $\mathcal{S}_\alpha$ are blown up by their height excess (the "fine excess") relative to appropriate unions of half-hyperplanes (corresponding to "vertical" scalings of H' by the coarse excess of the varifolds giving rise to H'). We use first variation arguments (in particular, Simon's L^2-estimates and the nonconcentration-of-tilt-excess estimate of Theorem 7.1(b)) and the standard $C^{1,\beta}$ boundary regularity theory for harmonic functions to prove a uniform interior continuity estimate (Theorem 12.2) for the first derivatives of the fine blow-ups, and we use it, via an excess improvement argument, to show that our assumption $H' \in \mathcal{B}_q$ must contradict one of the induction hypotheses, namely, that the Minimum Distance Theorem is valid when $\Theta(\|\mathbf{C}_0\|, 0) = q$. This enables us to conclude that the coarse blow-up class $\mathcal{B}_q$ has the asserted property, viz. that the only elements in $\mathcal{B}_q$ that are given by linear functions on either of two complementary half-spaces are the ones given by q copies of a single linear function everywhere.

Equipped with this fact and a number of other key properties that we establish for the coarse blow-ups (see items ($\mathcal{B}1$)–($\mathcal{B}7$) of Section 4 for a complete list), we ultimately obtain (in Theorems 14.3 and 4.1), subject to the induction hypotheses, interior C^1 regularity of coarse blow-ups and consequently, that any coarse blow-up is an ordered set of q harmonic functions (a Sheeting Theorem for $\mathcal{B}_q$, analogous to the Sheeting Theorem for $\mathcal{S}_\alpha$); furthermore, we show that these harmonic functions all agree if infinitely many members of a sequence of varifolds giving rise to the blow-up contain, in the interior, points of density $\geq q$.

The preceding result is the key to completion of the induction step for the Sheeting Theorem. Together with the Schoen-Simon version of the Sheeting Theorem, it enables us to prove a De Giorgi type lemma (Lemma 15.1), the iterative application of which leads us to the following conclusion: Let $\mathbf{P}_0$ be a hyperplane with multiplicity q, and suppose that V is a varifold in $\mathcal{S}_\alpha$ lying weakly close to $\mathbf{P}_0$ in a unit cylinder over $\mathbf{P}_0$. Let D be the region of $\mathbf{P}_0$ inside a cylinder slightly smaller than the unit cylinder. Then (i) there is a closed subset of D over each point of which the support of V consists of a single point; furthermore, at this point, V has a unique multiplicity q tangent hyperplane almost parallel to $\mathbf{P}_0$, and relative to this tangent hyperplane, the height excess of V satisfies a uniform decay estimate; and (ii) over the complementary open set, V corresponds to embedded graphs of q ordered, analytic functions of small gradient solving the minimal surface equation. Facts (i), (ii) and elliptic estimates imply, by an elementary general argument (Lemma 4.3), that the varifold corresponds to q ordered graphs over all of D and that each graph satisfies a uniform $C^{1,\beta}$ estimate (Theorem 15.2) for some fixed $\beta \in (0,1)$, completing the induction step for the Sheeting Theorem.

The final step of the argument is to complete induction for the Minimum Distance Theorem, which requires showing that the Minimum Distance Theorem holds whenever $\Theta(\|\mathbf{C}_0\|, 0) \in \{q + 1/2, q + 1\}$, where $\mathbf{C}_0$ is a stationary cone as in the theorem. Since we may now assume the validity of the Sheeting Theorem for multiplicity up to and including q, we have all the necessary ingredients to establish (in Theorem 16.1) that given such $\mathbf{C}_0$, if there is a varifold $V \in \mathcal{S}_\alpha$ weakly close to $\mathbf{C}_0$, then it must in the interior be made up of $C^{1,\alpha}$ embedded hypersurfaces-with-boundary meeting along their common boundary; this directly contradicts the α-Structural Hypothesis and proves the Minimum Distance Theorem, subject to the induction hypotheses, when $\Theta(\|\mathbf{C}_0\|, 0) \in \{q + 1/2, q + 1\}$. Our argument also establishes the Minimum Distance Theorem when $\Theta(\|\mathbf{C}_0\|, 0) \in \{3/2, 2\}$, since in this case we have, in place of the induction hypotheses, Allard's Regularity Theorem, which implies the Sheeting Theorem when $q = 1$.

This completes the outline of the proof.

Acknowledgements. Several technical aspects of this work owe a large mathematical debt to the ideas of L. Simon contained in his seminal work [Sim93] on asymptotics for minimal submanifolds near singularities. A crucial first step needed for the theory developed here is an *a priori* estimate, due to R. Schoen and L. Simon [SS81], for sufficiently regular stable minimal hypersurfaces. Other key previously known results used here include: an integral estimate (the Hardt–Simon inequality in Section 4) for (coarse) blow-ups of stationary varifolds, first established in the work of R. Hardt and L. Simon ([HS79]); F. J. Almgren's multiple-valued approximate graph decomposition theorem [Alm00]; Almgren's generalised stratification principle for stationary integral varifolds [Alm00]; and J. Simons' theorem concerning nonexistence of low dimensional singular stable minimal hypercones [Sim68]. The basic "harmonic approximation" idea, used at several places in this work in various forms and levels of sophistication, has its origin in the work of E. De Giorgi [DG61], and is found in a form closer to the way it is used here in the work of W. K. Allard [All72].

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2. **Notation**

The following notation will be used throughout the paper:

- n is a fixed positive integer ≥ 2, $\mathbf{R}^{n+1}$ denotes the $(n+1)$-dimensional Euclidean space and $(x^1, x^2, y^1, y^2, \ldots, y^{n-1})$, which we shall sometimes abbreviate as (x^1, x^2, y), denotes a general point in $\mathbf{R}^{n+1}$. We shall identify $\mathbf{R}^n$ with the hyperplane $\{x^1 = 0\}$ of $\mathbf{R}^{n+1}$ and $\mathbf{R}^{n-1}$ with the subspace $\{x^1 = x^2 = 0\}$.
- For $Y \in \mathbf{R}^{n+1}$ and $\rho > 0$, $B^{n+1}_\rho(Y) = \{X \in \mathbf{R}^{n+1} : |X - Y| < \rho\}$.

- For $Y \in \mathbf{R}^n$ and $\rho > 0$, $B_\rho(Y) = \{X \in \mathbf{R}^n : |X - Y| < \rho\}$. We shall often abbreviate $B_\rho(0)$ as B_ρ.
- For $Y \in \mathbf{R}^{n+1}$ and $\rho > 0$, $\eta_{Y,\rho} : \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ is the map defined by $\eta_{Y,\rho}(X) = \rho^{-1}(X - Y)$, and η_ρ abbreviates $\eta_{0,\rho}$.
- $\mathcal{H}^k$ denotes the k-dimensional Hausdorff measure in $\mathbf{R}^{n+1}$, and $\omega_n = \mathcal{H}^n(B_1(0))$.
- For $A, B \subset \mathbf{R}^{n+1}$, $\mathrm{dist}_{\mathcal{H}}(A, B)$ denotes the Hausdorff distance between A and B.
- For $X \in \mathbf{R}^{n+1}$ and $A \subset \mathbf{R}^{n+1}$, $\mathrm{dist}(X, A) = \inf_{Y \in A} |X - Y|$.
- For $A \subset \mathbf{R}^{n+1}$, $\overline{A}$ denotes the closure of A.
- G_n denotes the space of hyperplanes of $\mathbf{R}^{n+1}$.

For an n-varifold V ([All72]; see also [Sim83, Ch. 8]) on an open subset Ω of $\mathbf{R}^{n+1}$, an open subset $\widetilde{\Omega}$ of Ω, a Lipschitz mapping $f : \Omega \to \mathbf{R}^{n+1}$ and a countably n-rectifiable subset M of Ω with locally finite $\mathcal{H}^n$-measure, we use the following notation:

- $V \llcorner \widetilde{\Omega}$ abbreviates the restriction $V \llcorner (\widetilde{\Omega} \times G_n)$ of V to $\widetilde{\Omega} \times G_n$.
- $\|V\|$ denotes the weight measure on Ω associated with V.
- $\mathrm{spt}\,\|V\|$ denotes the support of $\|V\|$.
- $f_\# V$ denotes the image varifold under the mapping f.
- $|M|$ denotes the multiplicity 1 varifold on Ω associated with M.
- For $Z \in \mathrm{spt}\,\|V\| \cap \Omega$, $\mathrm{VarTan}(V, Z)$ denotes the set of tangent cones to V at Z.
- $\mathrm{reg}\, V$ denotes the (interior) regular part of $\mathrm{spt}\,\|V\|$. Thus, $X \in \mathrm{reg}\, V$ if and only if $X \in \mathrm{spt}\,\|V\| \cap \Omega$ and there exists $\rho > 0$ such that $\overline{B^{n+1}_\rho(X)} \cap \mathrm{spt}\,\|V\|$ is a smooth, compact, connected, embedded hypersurface-with-boundary, with its boundary contained in $\partial B^{n+1}_\rho(X)$.
- $\mathrm{sing}\, V$ denotes the interior singular set of $\mathrm{spt}\,\|V\|$. Thus, $\mathrm{sing}\, V = (\mathrm{spt}\,\|V\| \setminus \mathrm{reg}\, V) \cap \Omega$.

3. Statement of the main theorems

The Class $\mathcal{S}_\alpha$. Fix any $\alpha \in (0, 1)$. Denote by $\mathcal{S}_\alpha$ the collection of all integral n-varifolds V on $B^{n+1}_2(0)$ with $0 \in \mathrm{spt}\,\|V\|$, $\|V\|(B^{n+1}_2(0)) < \infty$ and satisfying the following conditions:

($\mathcal{S}1$) STATIONARITY: V has zero first variation with respect to the area functional in the following sense:

For any given vector field $\psi \in C^1_c(B^{n+1}_2(0); \mathbf{R}^{n+1})$, $\varepsilon > 0$ and C^2 map $\varphi : (-\varepsilon, \varepsilon) \times B^{n+1}_2(0) \to B^{n+1}_2(0)$ such that

(i) $\varphi(t, \cdot) : B^{n+1}_2(0) \to B^{n+1}_2(0)$ is a C^2 diffeomorphism for each $t \in (-\varepsilon, \varepsilon)$ with $\varphi(0, \cdot)$ equal to the identity map on $B^{n+1}_2(0)$,

(ii) $\varphi(t, x) = x$ for each $(t, x) \in (-\varepsilon, \varepsilon) \times \left(B^{n+1}_2(0) \setminus \mathrm{spt}\,\psi\right)$, and

(iii) $\partial\,\varphi(t, \cdot)/\partial\, t|_{t=0} = \psi$

(the flow generated by ψ for instance gives rise to such a family $\varphi(t,\cdot)$), we have that

$$\frac{d}{dt}\bigg|_{t=0} \|\varphi(t,\cdot)_{\#}\, V\|(B_2^{n+1}(0)) = 0;$$

equivalently (see [Sim83, §39]),

$$\int_{B_2^{n+1}(0)\times G_n} \operatorname{div}_S \psi(X)\, dV(X,S) = 0 \tag{3.1}$$

for every vector field $\psi \in C^1_c(B_2^{n+1}(0); \mathbf{R}^{n+1})$.

($\mathcal{S}2$) STABILITY: For each open ball $\Omega \subset B_2^{n+1}(0)$ such that $\operatorname{sing} V \cap \Omega = \emptyset$ in case $2 \leq n \leq 6$ or $\mathcal{H}^{n-7+\gamma}(\operatorname{sing} V \cap \Omega) = 0$ for every $\gamma > 0$ in case $n \geq 7$, we have that

$$\int_{\operatorname{reg} V\cap\Omega} |A|^2\zeta^2\, d\mathcal{H}^n \leq \int_{\operatorname{reg} V\cap\Omega} |\nabla\,\zeta|^2\, d\mathcal{H}^n \quad \forall\, \zeta \in C^1_c(\operatorname{reg} V \cap \Omega), \tag{3.2}$$

where A denotes the second fundamental form of $\operatorname{reg} V$, $|A|$ the length of A and ∇ denotes the gradient operator on $\operatorname{reg} V$; equivalently (see [Sim83, §9]), for each such Ω, V has nonnegative second variation with respect to area for normal deformations compactly supported in $\Omega \setminus \operatorname{sing} V$, in the following sense: for any given vector field $\psi \in C^1_c(\Omega \setminus \operatorname{sing} V; \mathbf{R}^{n+1})$ with $\psi(X) \perp T_X \operatorname{reg} V$ for each $X \in \operatorname{reg} V \cap \Omega$,

$$\frac{d^2}{dt^2}\bigg|_{t=0} \|\varphi(t,\cdot)_{\#}\, V\|(B_2^{n+1}(0)) \geq 0,$$

where $\varphi(t,\cdot)$, $t \in (-\varepsilon, \varepsilon)$, are the C^2 diffeomorphisms of $B_2^{n+1}(0)$ associated with ψ, described in ($\mathcal{S}1$) above.

($\mathcal{S}3$) α-STRUCTURAL HYPOTHESIS: For each given $Z \in \operatorname{sing} V$, there exists *no* $\rho > 0$ such that $\operatorname{spt}\|V\| \cap B_\rho^{n+1}(Z)$ is equal to the union of a finite number of embedded $C^{1,\alpha}$ hypersurfaces-with-boundary of $B_\rho^{n+1}(Z)$, all having a common $C^{1,\alpha}$ boundary in $B_\rho^{n+1}(Z)$ containing Z and no two intersecting except along their common boundary.

Remarks. (1) Note that the stability hypothesis ($\mathcal{S}2$) concerns only the regular part $\operatorname{reg} V$, and by Allard's regularity theorem, $\operatorname{reg} V \neq \emptyset$—in fact, $\operatorname{reg} V$ is an open, dense subset of $\operatorname{spt}\|V\|$—whenever V is stationary ([All72, §8.1]). Thus given hypothesis ($\mathcal{S}1$), hypothesis ($\mathcal{S}2$) is never vacuously true. However an open, dense subset can have arbitrarily small positive measure, so it is not at all obvious whether hypothesis ($\mathcal{S}2$) is sufficiently strong to give any control over the singular set. By our main theorem (Theorem 3.1 below) however, we conclude that for $V \in \mathcal{S}_\alpha$, $\operatorname{sing} V$ must in fact be very low dimensional.

(2) The hypothesis $\mathcal{H}^{n-1}(\operatorname{sing} V) = 0$ trivially implies ($\mathcal{S}3$), so all of our theorems concerning the class $\mathcal{S}_\alpha$ in particular apply to the class of stable minimal hypersurfaces M of $B_2^{n+1}(0)$ (that is, smooth embedded hypersurfaces M of $B_2^{n+1}(0)$ with their associated multiplicity 1 varifolds $V = |M|$ satisfying ($\mathcal{S}1$) and ($\mathcal{S}2$)) with no removable singularities (thus, if $X \in \overline{M} \cap B_2^{n+1}(0)$ and $\overline{M}$ is a smooth, embedded hypersurface near X, then $X \in M$) and with

$$\mathcal{H}^{n-1}(\operatorname{sing} M) = 0,$$

where $\operatorname{sing} M = (\overline{M} \setminus M) \cap B_2^{n+1}(0)$. In fact, by Theorem 3.1, these two classes, modulo multiplicity, are the same.

(3) By the Hopf boundary point lemma, Hypothesis ($\mathcal{S}3$) is satisfied if no tangent cone to V at a singular point is supported by a union of three or more distinct n-dimensional half-hyperplanes meeting along an $(n-1)$-dimensional subspace. By Theorem 3.4 below, for stable codimension 1 integral varifolds, this condition on the tangent cones is in fact equivalent to hypothesis ($\mathcal{S}3$).

Our main theorem concerning the varifolds in $\mathcal{S}_\alpha$ is the following:

THEOREM 3.1 (Regularity and Compactness Theorem). *Let $\alpha \in (0,1)$. Let $\{V_k\} \subset \mathcal{S}_\alpha$ be a sequence with*

$$\limsup_{k\to\infty} \|V_k\|(B_2^{n+1}(0)) < \infty.$$

There exist a subsequence $\{k'\}$ of $\{k\}$ and a varifold $V \in \mathcal{S}_\alpha$ with

$$\mathcal{H}^{n-7+\gamma}(\operatorname{sing}\ V \cap B_2^{n+1}(0)) = 0$$

for each $\gamma > 0$ if $n \geq 7$, $\operatorname{sing} V \cap B_2^{n+1}(0)$ discrete if $n = 7$ and $\operatorname{sing} V \cap B_2^{n+1}(0) = \emptyset$ if $1 \leq n \leq 6$ such that $V_{k'} \to V$ as varifolds on $B_2^{n+1}(0)$ and also $\operatorname{spt} \|V_{k'}\| \to \operatorname{spt} \|V\|$ smoothly (i.e., in the C^m topology for every m) locally in $B_2^{n+1}(0) \setminus \operatorname{sing} V$.

In particular, if $W \in \mathcal{S}_\alpha$, then $\mathcal{H}^{n-7+\gamma}(\operatorname{sing} W \cap B_2^{n+1}(0)) = 0$ for each $\gamma > 0$ if $n \geq 7$, $\operatorname{sing} W \cap B_2^{n+1}(0)$ is discrete if $n = 7$ and $\operatorname{sing} W \cap B_2^{n+1}(0) = \emptyset$ if $2 \leq n \leq 6$.

Note that we do not *a priori* assume orientability of $\operatorname{reg} V$ for $V \in \mathcal{S}_\alpha$; indeed, by virtue of low dimensionality of $\operatorname{sing} V$ guaranteed by Theorem 3.1, orientability of $\operatorname{reg} V$ *follows* if $V \in \mathcal{S}_\alpha$:

COROLLARY 3.2. *If $V \in \mathcal{S}_\alpha$, then $\operatorname{reg} V$ is orientable.*

Our proof of Theorem 3.1 will be based on the following two theorems:

THEOREM 3.3 (Sheeting Theorem). *Let $\alpha \in (0,1)$. Corresponding to each $\Lambda \in [1,\infty)$ and $\theta \in (0,1)$, there exists a number $\varepsilon_0 = \varepsilon_0(n,\Lambda,\alpha,\theta) \in (0,1)$ such* *that if* $V \in \mathcal{S}_\alpha$, $(\omega_n 2^n)^{-1}\|V\|(B_2^{n+1}(0)) \leq \Lambda$ *and*

$$\mathrm{dist}_{\mathcal{H}}(\mathrm{spt}\,\|V\| \cap (\mathbf{R} \times B_1), \{0\} \times B_1) < \varepsilon_0,$$

then

$$V \llcorner (\mathbf{R} \times B_\theta) = \sum_{j=1}^{q} |\mathrm{graph}\, u_j|$$

for some integer q, *where* $u_j \in C^{1,\beta}(B_\theta)$ *for each* $j = 1, 2, \ldots, q$; $u_1 \leq u_2 \leq \cdots \leq u_q$;

$$\sup_{B_\theta}(|u_j| + |Du_j|) + \sup_{X_1, X_2 \in B_\theta,\, X_1 \neq X_2} \frac{|Du_j(X_1) - Du_j(X_2)|}{|X_1 - X_2|^\beta} \leq C\left(\int_{\mathbf{R} \times B_1} |x^1|^2\, d\|V\|(X)\right)^{1/2}.$$

Furthermore, u_j *solves the minimal surface equation weakly on* B_θ *and hence in fact* $u_j \in C^\infty(B_\theta)$ *for each* $j = 1, 2, \ldots, q$. *Here* $C = C(n, \Lambda, \alpha, \theta) \in (0, \infty)$ *and* $\beta = \beta(n, \Lambda, \alpha, \theta) \in (0, 1)$.

Theorem 3.4 (Minimum Distance Theorem). *Let* $\alpha \in (0,1)$. *Let* $\delta \in (0,1/2)$, *and let* $\mathbf{C}_0$ *be an* n*-dimensional stationary cone in* $\mathbf{R}^{n+1}$ *such that* spt $\|\mathbf{C}_0\|$ *is equal to a finite union of at least three distinct* n*-dimensional half-hyperplanes of* $\mathbf{R}^{n+1}$ *meeting along an* $(n-1)$*-dimensional subspace. There exists* $\varepsilon = \varepsilon(n, \alpha, \delta, \mathbf{C}_0) \in (0,1)$ *such that if* $V \in \mathcal{S}_\alpha$, $\Theta(\|V\|, 0) \geq \Theta(\|\mathbf{C}_0\|, 0)$ *and* $(\omega_n 2^n)^{-1}\|V\|(B_2^{n+1}(0)) \leq \Theta_{\mathbf{C}_0}(0) + \delta$, *then*

$$\mathrm{dist}_{\mathcal{H}}(\mathrm{spt}\,\|V\| \cap B_1^{n+1}(0), \mathrm{spt}\,\|\mathbf{C}_0\| \cap B_1^{n+1}(0)) \geq \varepsilon.$$

The proofs of Theorems 3.1, 3.3 and 3.4 will be given in Sections 17, 15 and 16 respectively.

Remark. Theorems 3.1, 3.3 and 3.4 are optimal in several ways:

(a) Examples such as pairs of transverse hyperplanes or a union of three half-hyperplanes meeting at 120° angles along a common axis show that Theorems 3.3, 3.4 and 3.1 do not hold if the structural hypothesis ($\mathcal{S}3$) is removed (or replaced by the condition $\mathcal{H}^{n-1+\gamma}(\mathrm{sing}\, V) = 0$ for any $\gamma > 0$). Stable *branched* minimal hypersurfaces (e.g., those constructed in [SW07] or in [Ros10]) show that in the absence of hypothesis ($\mathcal{S}3$), even when $n = 2$, there is no hope of proving regularity of stable codimension 1 integral varifolds away from the set of points near which the varifold has the structure ruled out by hypothesis ($\mathcal{S}3$). Thus hypothesis ($\mathcal{S}3$) can, in particular, be viewed as a geometric condition that implies nonexistence of branch points in stable codimension 1 integral varifolds.

(b) Appropriate rescalings of a standard 2-dimensional Catenoid in $\mathbf{R}^3$ show that Theorem 3.3 does not hold without the stability hypothesis ($\mathcal{S}2$).

Similarly, rescalings of a Scherk's second surface show that Theorem 3.4 does not hold without ($\mathcal{S}2$). However it is an open question, even when $n = 2$, whether some form of Theorem 3.1 giving a bound on the singular set holds without ($\mathcal{S}2$). In fact, it remains open whether 2 dimensional stationary integral varifolds in $\mathbf{R}^3$ must be regular almost everywhere, even subject to a condition such as ($\mathcal{S}3$).

(c) There are many examples provided by complex algebraic varieties demonstrating that Theorems 3.3 and 3.1 do not hold in codimension > 1 even if the stability hypothesis ($\mathcal{S}2$) (where the corresponding higher codimensional stability inequality takes a different form from (3.2); see [Sim83, §9]) is replaced by the (stronger) absolutely area minimizing hypothesis. For instance, the holomorphic varieties $V_t = \{(z,w) : z^2 = tw^3 + tw\} \cap B_1^4(0) \subset \mathbf{C} \times \mathbf{C} \equiv \mathbf{R}^4$, $t \in \mathbf{R}$, which are smooth, embedded area minimizing submanifolds lying close to the plane $\{z = 0\} \cap B_1^4(0)$ for small $|t| \neq 0$, show that Theorem 3.3 does not hold in codimension > 1. Those holomorphic varieties with branch point singularities such as $V = \{(z,w) : z^2 = w^3\} \cap B_1^4(0) \subset \mathbf{C} \times \mathbf{C}$ show that even in 2 dimension, C^2 regularity, and hence Theorem 3.1, is false if codimension > 1. (For area minimizing currents of dimension n and arbitrary codimension, Almgren's theorem ([Alm00]) gives the optimal bound on the Hausdorff dimension of the interior singular sets; namely, $n - 2$.) Since the cone $\mathbf{C}_0$ in Theorem 3.4 is not area minimizing, there are no area minimizing examples nearby, but a given transverse pair of planes in $\mathbf{R}^3 \times \{0\} \subset \mathbf{R}^4$, for instance, can be perturbed in $\mathbf{R}^4$ into a union of two planes intersecting only at the origin, and the latter union is of course stable and satisfies ($\mathcal{S}3$), showing that Theorem 3.4 is false in codimension > 1.

Our theorems generalise the regularity and compactness theory of R. Schoen and L. Simon [SS81], which established Theorems 3.3 and 3.1 for stable codimension 1 integral varifolds V on $B_2^{n+1}(0)$ under the hypothesis $\mathcal{H}^{n-2}(\operatorname{sing} V \cap K) < \infty$ for each compact $K \subset B_2^{n+1}(0)$ in place of our hypothesis ($\mathcal{S}3$). (Under this more stringent hypothesis on the singular set, Theorem 3.4 is a straightforward consequence of Theorem 3.3 and inequality (3.2).) Our proofs of Theorem 3.3 and Theorem 3.4 however rely on the Schoen-Simon version of Theorem 3.3 in an essential way; in fact, what we need is the following slightly weaker version of their theorem:

Theorem 3.5 ([SS81, special case of Th. 2]). *Let V be an integral n-varifold on $B_2^{n+1}(0)$ and in place of* ($\mathcal{S}3$), *assume the* (*stronger*) *condition that $\mathcal{H}^{n-7+\gamma}(\operatorname{sing} V) = 0$ for every $\gamma > 0$ in case $n \geq 7$ and $\operatorname{sing} V = \emptyset$ in case $2 \leq n \leq 6$. Let all other hypotheses be as in Theorem* 3.3. *Then the conclusions of Theorem* 3.3 *hold.*

Remark. It suffices to prove Theorem 3.3 for $\theta = 1/8$ and arbitrary $\Lambda \in [1,\infty)$. To see this, suppose that case $\theta = 1/8$ of the theorem is true, with $\varepsilon' = \varepsilon'(n,\alpha,\Lambda) \in (0,1)$ corresponding to ε_0. Let $\theta \in (1/8,1)$ and let the hypotheses be as in the theorem with $\varepsilon_0 = \varepsilon_0(n,\alpha,\Lambda,\theta) \in (0,1)$ satisfying $\varepsilon_0 < \left(\frac{1-\theta}{8}\right)\varepsilon'(n,\alpha,3^n\Lambda)$. We may then apply the case $\theta = 1/8$ of the theorem with $3^n\Lambda$ in place of Λ and with $\widetilde{V} = \left(\eta_{Z,(1-\theta)/2}\right)_{\#} V \in \mathcal{S}_\alpha$ in place of V, where $Z \in \operatorname{spt}\|V\| \cap (\mathbf{R}\times B_\theta)$ is arbitrary; since we may cover $\operatorname{spt}\|V\|\cap(\mathbf{R}\times B_\theta)$ by a collection of balls $B^{n+1}_{(1-\theta)/2}(Z_j)$, $j = 1,2,\ldots,N$, with $Z_j \in \operatorname{spt}\|V\|\cap(\mathbf{R}\times B_\theta)$ and $N = N(n,\Lambda,\theta)$, the required estimate follows.

So assume $\theta = 1/8$, and let the hypotheses be as in Theorem 3.3. It follows from Allard's integral varifold compactness theorem ([All72, Th. 6.4]) and the Constancy Theorem for stationary integral varifolds ([Sim83, Th. 41.1]) that if $\varepsilon_0 = \varepsilon_0(n,\Lambda) \in (0,1)$ is sufficiently small, then $q - 1/2 \le (\omega_n R^n)^{-1}\|V\|(\mathbf{R}\times B_R) < q+1/2$ for some integer $q \in [1,\Lambda+1)$ and $R \in \{1/3, 2/3\}$. Then $V_1 \equiv \eta_{0,1/3\,\#} V$ satisfies $(\omega_n 2^n)^{-1}\|V_1\|(B^{n+1}_2(0)) < q+1/2$ and $q - 1/2 \le \omega_n^{-1}\|V_1\|(\mathbf{R}\times B_1) < q+1/2$. Thus in order to prove the special case $\theta = 1/8$ of Theorem 3.3 (and therefore the general version), it suffices to establish the following:

Theorem 3.3′ (Sheeting Theorem). Let $\alpha \in (0,1)$. Let q be any integer ≥ 1. There exists a number $\varepsilon_0 = \varepsilon_0(n,\alpha,q) \in (0,1)$ such that if $V \in \mathcal{S}_\alpha$, $(\omega_n 2^n)^{-1}\|V\|(B^{n+1}_2(0)) < q+1/2$, $q - 1/2 \le \omega_n^{-1}\|V\|(\mathbf{R}\times B_1) < q+1/2$ and $\operatorname{dist}_{\mathcal{H}}(\operatorname{spt}\|V\|\cap(\mathbf{R}\times B_1), \{0\}\times B_1) < \varepsilon_0$, then

$$V \llcorner (\mathbf{R}\times B_{3/8}) = \sum_{j=1}^{q} |\operatorname{graph} u_j|,$$

where $u_j \in C^{1,\beta}(B_{3/8})$ for each $j = 1,2,\ldots,q$; $u_1 \le u_2 \le \cdots \le u_q$;

$$\sup_{B_{3/8}}(|u_j| + |Du_j|) + \sup_{X_1,X_2\in B_{3/8},\, X_1\ne X_2} \frac{|Du_j(X_1) - Du_j(X_2)|}{|X_1 - X_2|^\beta} \le C\left(\int_{\mathbf{R}\times B_1} |x^1|^2\, d\|V\|(X)\right)^{1/2};$$

and u_j solves the minimal surface equation (weakly) on $B_{3/8}$. Here $C = C(n,q,\alpha) \in (0,\infty)$ and $\beta = \beta(n,q,\alpha) \in (0,1)$.

Finally, we note that in the absence of the α-Structural Hypothesis ($\mathcal{S}3$), Theorems 3.1, 3.3 and the upper semi-continuity of density of stationary integral varifolds readily imply the following:

Corollary 3.6. *Let V be a stable integral n-varifold on $B^{n+1}_2(0)$ (in the sense that V satisfies* (3.1) *and* (3.2)). *If $Z \in \operatorname{sing} V$ and one of the tangent cones to V at Z is (the varifold associated with) a hyperplane with multiplicity $q \in \{2,3,\ldots\}$, then for any $\alpha \in (0,1)$, there exist a sequence of points $Z_j \in$*

$\mathrm{sing}\, V$ with $Z_j \neq Z$, $Z_j \to Z$ and a sequence of numbers σ_j with $0 < \sigma_j < |Z_j - Z|$ such that for each $j = 1, 2, 3, \ldots$, $\mathrm{spt}\, \|V\| \cap B^{n+1}_{\sigma_j}(Z_j)$ is the union of at least 3 and at most $2q$ embedded $C^{1,\alpha}$ hypersurfaces-with-boundary meeting only along an $(n-1)$-dimensional $C^{1,\alpha}$ submanifold of $B^{n+1}_{\sigma_j}(Z_j)$ containing Z_j.

In fact, if $Z \in \mathrm{sing}\, V$ is such that one tangent cone $\mathbf{C}$ to V at Z has the form, after a rotation, $\mathbf{C} = \mathbf{C}' \times \mathbf{R}^{n-k}$ for some $k \in \{0, 1, \ldots, \min\{6, n\}\}$, then for any $\alpha \in (0,1)$, there exist a sequence of points $Z_j \in \mathrm{sing}\, V$ with $Z_j \neq Z$, $Z_j \to Z$ and a sequence of numbers σ_j with $0 < \sigma_j < |Z_j - Z|$ such that for each $j = 1, 2, 3, \ldots$, $\mathrm{spt}\, \|V\| \cap B^{n+1}_{\sigma_j}(Z_j)$ is the union of at least 3 and at most $2\Theta(\|V\|, Z)$ embedded $C^{1,\alpha}$ hypersurfaces-with-boundary meeting only along an $(n-1)$-dimensional $C^{1,\alpha}$ submanifold of $B^{n+1}_{\sigma_j}(Z_j)$ containing Z_j.

4. **Proper blow-up classes**

Fix an integer $q \geq 1$ and a constant $C \in (0, \infty)$. Consider a family $\mathcal{B}$ of functions $v = (v^1, v^2, \ldots, v^q) : B_1 \to \mathbf{R}^q$ satisfying the following properties:

($\mathcal{B}1$) $\mathcal{B} \subset W^{1,2}_{\mathrm{loc}}(B_1; \mathbf{R}^q) \cap L^2(B_1; \mathbf{R}^q)$.

($\mathcal{B}2$) If $v \in \mathcal{B}$, then $v^1 \leq v^2 \leq \cdots \leq v^q$.

($\mathcal{B}3$) If $v \in \mathcal{B}$, then $\Delta\, v_a = 0$ in B_1, where $v_a = q^{-1} \sum_{j=1}^q v^j$.

($\mathcal{B}4$) For each $v \in \mathcal{B}$ and each $z \in B_1$, either ($\mathcal{B}4\,$I) or ($\mathcal{B}4\,$II) below is true:

($\mathcal{B}4\,$I) The *Hardt-Simon inequality*

$$\sum_{j=1}^{q} \int_{B_{\rho/2}(z)} R_z^{2-n} \left(\frac{\partial\left((v^j - v_a(z))/R_z\right)}{\partial\, R_z} \right)^2 \leq C\, \rho^{-n-2} \int_{B_\rho(z)} |v - \ell_{v,\,z}|^2$$

holds for each $\rho \in (0, \frac{3}{8}(1 - |z|)]$, where $R_z(x) = |x - z|$, $\ell_{v,\,z}(x) = v_a(z) + Dv_a(z) \cdot (x - z)$ and $v - \ell_{v,\,z} = (v^1 - \ell_{v,\,z}, v^2 - \ell_{v,\,z}, \ldots, v^q - \ell_{v,\,z})$.

($\mathcal{B}4\,$II) There exists $\sigma = \sigma(z) \in (0, 1 - |z|]$ such that $\Delta\, v = 0$ in $B_\sigma(z)$.

($\mathcal{B}5$) If $v \in \mathcal{B}$, then

($\mathcal{B}5\,$I) $\widetilde{v}_{z,\sigma}(\cdot) \equiv \|v(z + \sigma(\cdot))\|^{-1}_{L^2(B_1(0))} v(z + \sigma(\cdot)) \in \mathcal{B}$ for each $z \in B_1$ and $\sigma \in (0, \frac{3}{8}(1 - |z|)]$ whenever $v \not\equiv 0$ in $B_\sigma(z)$;

($\mathcal{B}5\,$II) $v \circ \gamma \in \mathcal{B}$ for each orthogonal rotation γ of $\mathbf{R}^n$; and

($\mathcal{B}5\,$III) $\|v - \ell_v\|^{-1}_{L^2(B_1(0))} (v - \ell_v) \in \mathcal{B}$ whenever $v - \ell_v \not\equiv 0$ in B_1, where $\ell_v(x) = v_a(0) + Dv_a(0) \cdot x$ for $x \in \mathbf{R}^n$ and $v - \ell_v = (v^1 - \ell_v, v^2 - \ell_v, \ldots, v^q - \ell_v)$.

($\mathcal{B}6$) If $\{v_k\}_{k=1}^\infty \subset \mathcal{B}$, then there exist a subsequence $\{k'\}$ of $\{k\}$ and a function $v \in \mathcal{B}$ such that $v_{k'} \to v$ locally in $L^2(B_1)$ and locally weakly in $W^{1,2}(B_1)$.

($\mathcal{B}7$) If $v \in \mathcal{B}$ is such that for each $j = 1, 2, \ldots, q$, there exist linear functions $L_1^j, L_2^j : \mathbf{R}^n \to \mathbf{R}$ with $v^j(x^2, y) = L_1^j(x^2, y)$ if $x^2 > 0$, $v^j(x^2, y) = L_2^j(x^2, y)$ if $x^2 \leq 0$ and $L_1^j(0, y) = L_2^k(0, y)$ for $1 \leq j, k \leq q$, $y \in \mathbf{R}^{n-1}$, then $v^1 = v^2 = \cdots = v^q = L$ for some linear function $L : \mathbf{R}^n \to \mathbf{R}$.

We shall refer to any such class $\mathcal{B}$ as a *proper blow-up class.*

Our main result in this section (Theorem 4.1 below) is that *functions in any proper blow-up class are harmonic*. Subsequently, we shall prove that the collection of functions arising as “coarse blow-ups” (see Section 5 for the definition) of mass-bounded sequences of varifolds in $\mathcal{S}_\alpha$ converging weakly to a hyperplane is a proper blow-up class for a suitable constant C depending only on n and the mass bound.

Remark. The first use of the inequality in ($\mathcal{B}4\,$I) in the context of regularity theory for minimal submanifolds is due to R. Hardt and L. Simon ([HS79]).

Let $\mathcal{B}$ be a proper blow-up class. There exists a constant $\tau = \tau(\mathcal{B}) \in (0, 1/4)$ such that if $v \in \mathcal{B}$, $v_a(0) = 0$ and property ($\mathcal{B}4\,$I) holds with $z = 0$, then

$$\int_{B_1 \setminus B_\tau} |v|^2 \geq \frac{1}{2} \int_{B_1} |v|^2. \tag{4.1}$$

To see this, note that since every weakly convergent sequence in $W^{1,2}(B_{2/3})$ is bounded in $W^{1,2}(B_{2/3})$, it follows from the compactness property ($\mathcal{B}6$) and property ($\mathcal{B}5\,$I) that there exists a constant $C_1 = C_1(\mathcal{B}) \in (0, \infty)$ such that $\int_{B_{1/4}} |Dv|^2 \leq C_1 \int_{B_1} |v|^2$ for every $v \in \mathcal{B}$. Hence by property ($\mathcal{B}4\,$I) with $z = 0$ and $\rho = 3/8$, we see that if $v_a(0) = 0$, then

$$\int_{B_{3/16}} \frac{|v|^2}{R^2} \leq 2(C_2 + C_1) \int_{B_1} |v|^2,$$

where $C_2 = C_2(C, n)$ and we have used the fact that, since v_a is harmonic, $|\ell_{v,0}(x)|^2 = |Dv_a(0)|^2|x|^2 \leq C_3 \int_{B_{1/4}} |Dv|^2 \leq C_3 C_1 \int_{B_1} |v|^2$ for every $x \in B_{3/8}$, with $C_3 = C_3(n, q)$. This readily implies that for each $\tau \in (0, 3/16)$, $\int_{B_\tau} |v|^2 \leq 2(C_2 + C_1)\tau^2 \int_{B_1} |v|^2$, and choosing $\tau = \tau(\mathcal{B}) \in (0, 3/16)$ such that $2(C_2 + C_1)\tau^2 < 1/2$, we deduce (4.1).

THEOREM 4.1. *If $\mathcal{B}$ is a proper blow-up class for some $C \in (0, \infty)$, then each $v \in \mathcal{B}$ is harmonic in B_1. Furthermore, if $v \in \mathcal{B}$ and there is a point $z \in B_1$ such that ($\mathcal{B}4\,I$) is satisfied, then $v^1 = v^2 = \cdots = v^q$.*

The proof of this Theorem will be based on the following proposition:

PROPOSITION 4.2. *Let $\mathcal{B}$ be a proper blow-up class, and let $\tau = \tau(\mathcal{B}) \in (0, 1/4)$ be the constant as in* (4.1). *If $v \in \mathcal{B}$ satisfies property* ($\mathcal{B}4\,$I) *with $z = 0$ and if v is homogeneous of degree* 1 *in the annulus $B_1 \setminus B_\tau$, viz. $\frac{\partial(v/R)}{\partial R} = 0$ almost everywhere in $B_1 \setminus B_\tau$, then $v^j = L$ in B_1 for some linear function L and all $j \in \{1, 2, \ldots, q\}$.*

For the proofs of Theorem 4.1, Proposition 4.2 and subsequently, we shall need the following general principle:

LEMMA 4.3. *Let $w \in L^2(B_1; \mathbf{R}^q)$. Suppose there is a closed subset $\Gamma \subset B_1$ and numbers $\beta, \beta_1, \beta_2 \in (0, \infty)$, $\mu \in (0,1)$ and $\varepsilon \in (0, 1/4)$ such that the following hold: For each $z \in \Gamma \cap B_{3/4}$, there is an affine function $\ell_z : \mathbf{R}^n \to \mathbf{R}^q$ with $\sup_{B_1} |\ell_z| \le \beta$ such that*

$$\sigma^{-n-2} \int_{B_\sigma(z)} |w - \ell_z|^2 \le \beta_1 \left(\frac{\sigma}{\rho}\right)^\mu \rho^{-n-2} \int_{B_\rho(z)} |w - \ell_z|^2$$

for all $0 < \sigma \le \rho/2 \le \varepsilon/2$ and for each $z \in B_{3/4} \setminus \Gamma$, there is an affine function $\ell_z : \mathbf{R}^n \to \mathbf{R}^q$ such that

$$\sigma^{-n-2} \int_{B_\sigma(z)} |w - \ell_z|^2 \le \beta_2 \left(\frac{\sigma}{\rho}\right)^\mu \rho^{-n-2} \int_{B_\rho(z)} |w - \ell|^2$$

for each affine function $\ell : \mathbf{R}^n \to \mathbf{R}^q$ and all $0 < \sigma \le \rho/2 < \frac{1}{2} \min\{1/4, \operatorname{dist}(z, \Gamma)\}$. Then $w \in C^{1,\lambda}(B_{1/2})$ for some $\lambda = \lambda(n, q, \beta_1, \beta_2, \varepsilon, \mu) \in (0,1)$ with

$$\sup_{B_{1/2}} (|w| + |Dw|) + \sup_{x,y \in B_{1/2}, x \ne y} \frac{|Dw(x) - Dw(y)|}{|x-y|^\lambda} \le C \left(\beta^2 + \int_{B_1} |w|^2\right)^{1/2},$$

where $C = C(n, q, \beta_1, \beta_2, \varepsilon) \in (0, \infty)$.

Remark. In our applications of the lemma, the component functions of w, in $B_1 \setminus \Gamma$, will either be harmonic or smooth functions with small gradient solving the minimal surface equation; the second estimate in the hypotheses, with $\ell_z(x) = w(z) + Dw(z) \cdot (x - z)$ and β_2 depending only on n, follows in these cases from standard interior estimates for second derivatives of harmonic functions and solutions to uniformly elliptic equations.

Proof. Consider an arbitrary point $y \in B_{3/4}$ and a number $\rho \in (0, \varepsilon)$. With $\gamma = \gamma(n, \beta_1, \varepsilon, \mu) \in (0, 1/8)$ to be chosen, if there is a point $z \in \Gamma \cap \overline{B_{\gamma\rho}(y)}$, then by the given condition with $\rho - |z-y|$ in place of ρ and $\sigma = \gamma\rho + |z-y|$,

$$\begin{aligned}
&(\gamma\rho)^{-n-2} \int_{B_{\gamma\rho}(y)} |w - \ell_z|^2 \\
&\quad\le \left(1 + \frac{|z-y|}{\gamma\rho}\right)^{n+2} (\gamma\rho + |z-y|)^{-n-2} \int_{B_{\gamma\rho + |z-y|}(z)} |w - \ell_z|^2 \\
&\quad\le 2^{n+2} \beta_1 \left(\frac{\gamma\rho + |z-y|}{\rho - |z-y|}\right)^\mu (\rho - |z-y|)^{-n-2} \int_{B_{\rho - |z-y|}(z)} |w - \ell_z|^2 \\
&\quad\le 4^{n+2} \beta_1 \left(\frac{2\gamma}{1-\gamma}\right)^\mu \rho^{-n-2} \int_{B_\rho(y)} |w - \ell_z|^2.
\end{aligned}$$

Choosing $\gamma = \gamma(n, \beta_1, \varepsilon, \mu) \in (0, \varepsilon)$ such that $4^{n+2}\beta_1 \left(\frac{2\gamma}{1-\gamma}\right)^\mu < 1/4$, we see from this that

$$(\gamma\rho)^{-n-2} \int_{B_{\gamma\rho}(y)} |w - \ell_z|^2 \le 4^{-1} \rho^{-n-2} \int_{B_\rho(y)} |w - \ell_z|^2$$

for any $y \in B_{3/4}$ and $\rho \in (0, \varepsilon)$ provided there is a point $z \in \Gamma \cap \overline{B_{\gamma\rho}(y)}$. In particular, if $z^\star \in \Gamma$ is such that $|y - z^\star| = \text{dist}(y, \Gamma)$, then

$$(\gamma\rho)^{-n-2} \int_{B_{\gamma\rho}(y)} |w - \ell_{z^\star}|^2 \leq 4^{-1}\rho^{-n-2} \int_{B_\rho(y)} |w - \ell_{z^\star}|^2 \tag{4.2}$$

for each $\rho \in (0, \varepsilon)$ such that $\gamma\rho \geq |y - z^\star|$. On the other hand, if $\Gamma \cap \overline{B_{\gamma\rho}(y)} = \emptyset$, then again by the given condition we know that for any affine function ℓ,

$$(\sigma\gamma\rho)^{-n-2} \int_{B_{\sigma\gamma\rho}(y)} |w - \ell_y|^2 \leq \beta_2 \sigma^\mu (\gamma\rho)^{-n-2} \int_{B_{\gamma\rho}(y)} |w - \ell|^2 \tag{4.3}$$

for all $\sigma \in (0, 1/2]$. Iterating inequality (4.2) with $\rho = \gamma^j$, $j = 1, 2, \ldots$ and using inequality (4.3), we see that for each $y \in B_{3/4} \setminus \Gamma$, there is an integer $j^\star \geq 1$, an affine function $\ell_\star$ $(= \ell_{z^\star})$ with $\sup_{B_1} |\ell_\star| \leq \beta$ and an affine function ℓ_y such that

$$(\sigma\gamma^{j^\star+1})^{-n-2} \int_{B_{\sigma\gamma^{j^\star+1}}(y)} |w - \ell_y|^2 \leq \beta_2 \sigma^\mu (\gamma^{j^\star+1})^{-n-2} \int_{B_{\gamma^{j^\star+1}}(y)} |w - \ell|^2 \tag{4.4}$$

for each affine function ℓ and each $\sigma \in (0, 1/2]$; and

$$\begin{aligned}(\gamma^j)^{-n-2} \int_{B_{\gamma^j}(y)} |w - \ell_\star|^2 &\leq 4^{-1}(\gamma^{j-1})^{-n-2} \int_{B_{\gamma^{j-1}}(y)} |w - \ell_\star|^2 \\ &\leq 4^{-(j-1)}\gamma^{-n-2} \int_{B_\gamma(y)} |w - \ell_\star|^2 \quad \text{for each } j = 1, 2, \ldots, j^\star.\end{aligned} \tag{4.5}$$

By taking $\ell = \ell_\star$, $\sigma = 1/2$ in (4.4) and $j = j^\star$ in (4.5), and using the triangle inequality, we see that

$$\left(\frac{1}{2}\gamma^{j^\star+1}\right)^{-n-2} \int_{B_{\frac{1}{2}\gamma^{j^\star+1}}(y)} |\ell_y - \ell_\star|^2 \leq C4^{-(j^\star-1)} \int_{B_\gamma(y)} |w - \ell_\star|^2$$

which, in particular, implies

$$\left(\gamma^j\right)^{-n-2} \int_{B_{\gamma^j}(y)} |\ell_y - \ell_\star|^2 \leq C4^{-j} \int_{B_\gamma(y)} |w - \ell_\star|^2 \tag{4.6}$$

for $j = 1, 2, \ldots, j^\star$, where $C = C(n, \beta_2, \mu, \gamma) \in (0, \infty)$.

By (4.5) and (4.6), we conclude that

$$(\gamma^j)^{-n-2} \int_{B_{\gamma^j}(y)} |w - \ell_y|^2 \leq C4^{-(j-1)} \int_{B_\gamma(y)} |w - \ell_\star|^2 \tag{4.7}$$

for each $j = 1, 2, \ldots, j^\star$. Thus if $y \in B_{3/4} \setminus \Gamma$, we deduce that

$$\rho^{-n-2} \int_{B_\rho(y)} |w - \ell_y|^2 \leq C\rho^\lambda \int_{B_\gamma(y)} |w - \ell_\star|^2 \tag{4.8}$$

for all $\rho \in (0, \gamma/2]$, by considering, for any given $\rho \in (0, \gamma/2]$, the two alternatives:

(i) $2\rho \leq \gamma^{j^\star+1}$, in which case $\rho = \sigma\gamma^{j^\star+1}$ for some $\sigma \in (0, 1/2]$ and we use (4.4) provided $\gamma = \gamma(n, q, \beta_1, \beta_2, \mu, \varepsilon)$ is chosen to satisfy $\gamma^\mu < 1/4$ also, or

(ii) $\gamma^{j+1} < 2\rho \leq \gamma^j$ for some $j \in \{1, 2, \ldots, j^\star\}$, in which case we use (4.7).

In view of (4.8) (in case $y \in B_{3/4} \setminus \Gamma$) and the given condition (in case $y \in B_{3/4} \cap \Gamma$), we conclude that for each $y \in B_{3/4}$, there exists an affine function ℓ_y such that for all $\rho \in (0, \gamma/2]$,

$$\rho^{-n-2} \int_{B_\rho(y)} |w - \ell_y|^2 \leq C\rho^\lambda \left(\beta^2 + \int_{B_1} |w|^2\right), \tag{4.9}$$

where $C = C(n, q, \beta_1, \beta_2, \mu, \varepsilon) \in (0, \infty)$ and $\lambda = \lambda(n, q, \beta_1, \beta_2, \mu, \varepsilon) \in (0, 1)$. It is standard that from this the assertions of the lemma follow. □

In the proofs of Proposition 4.2, Theorem 4.1 and subsequently, for $v \in \mathcal{B}$, we let

$$\Gamma_v = \{z \in B_1 \setminus \Omega_v : (\mathcal{B}4\,\mathrm{I}) \text{ holds}\},$$

where

$$\Omega_v = \{z \in B_1 : \exists \rho \in (0, 1 - |z|] \text{ such that}$$
$$v^1(x) = v^2(x) = \cdots = v^q(x) (= v_a(x)) \text{ for a.e. } x \in B_\rho(z)\}.$$

Remark. Note that it follows directly from property ($\mathcal{B}4$) that Γ_v is a relatively closed subset of B_1 and on $B_1 \setminus \Gamma_v$, v^j is almost everywhere equal to a harmonic function for each $j = 1, 2, \ldots, q$.

Proof of Proposition 4.2. Let $\tau = \tau(\mathcal{B}) \in (0, 1/4)$ be as in (4.1). Note first that if $v \in \mathcal{B}$ is homogeneous of degree 1 in any annulus $B_1 \setminus B_{\tau'}$, $\tau' \in (0, 1)$, viz. v satisfies $\frac{\partial(v/R)}{\partial R} = 0$ almost everywhere in $B_1 \setminus B_{\tau'}$, then, since $v_a = q^{-1}\sum_{j=1}^q v^j$ is harmonic in B_1 by property ($\mathcal{B}3$), it follows that v_a is a linear function in B_1.

Let $\mathcal{H}$ denote the collection of all homogeneous of degree 1 functions $\widetilde{v} : \mathbf{R}^n \to \mathbf{R}^q$ such that $\widetilde{v}|_{B_1 \setminus B_\tau} \equiv v|_{B_1 \setminus B_\tau}$ for some $v \in \mathcal{B}$ satisfying property ($\mathcal{B}4\,\mathrm{I}$) with $z = 0$. For any given $\widetilde{v} \in \mathcal{H}$, let $T(\widetilde{v}) = \{z \in \mathbf{R}^n : \widetilde{v}(x + z) = \widetilde{v}(x)$ for almost every $x \in \mathbf{R}^n\}$. Using homogeneity of $\widetilde{v}$, it is standard to verify that $T(\widetilde{v})$ is a linear subspace of $\mathbf{R}^n$.

For $k = 0, 1, 2, \ldots, n$, let $\mathcal{H}_k = \{\widetilde{v} \in \mathcal{H} : \dim T(\widetilde{v}) = n - k\}$ so that $\mathcal{H} = \bigcup_{k=0}^n \mathcal{H}_k$. Clearly $\mathcal{H}_0 = \{0\}$. Let $\widetilde{v} \in \mathcal{H}_1$, and let v be any element $\in \mathcal{B}$ that is homogeneous of degree 1 in $B_1 \setminus B_\tau$ such that v satisfies property ($\mathcal{B}4\,\mathrm{I}$) with $z = 0$ and v agrees with $\widetilde{v}$ on $B_1 \setminus B_\tau$. We wish to show that there exists a linear function L such that $v^j = L$ in B_1 for each $j \in \{1, \ldots, q\}$. This is true if $v^j = v_a$ on B_1 for each $j \in \{1, \ldots, q\}$, so suppose $v - v_a = (v^1 - v_a, \ldots, v^q - v_a) \not\equiv 0$ in B_1, and let $w = \|v - v_a\|^{-1}(v - v_a)$. Then $w \in \mathcal{B}$ by property ($\mathcal{B}5\,\mathrm{III}$), $w \not\equiv 0$, $w_a \equiv 0$ and property ($\mathcal{B}4\,\mathrm{I}$) is satisfied with w in place of v and $z = 0$, and hence by (4.1), $w \not\equiv 0$ in $B_1 \setminus B_\tau$. By the definition of $\mathcal{H}_1$ and property

($\mathcal{B}5$ II) (of v), we may assume that $T(\widetilde{v}) = \{0\} \times \mathbf{R}^{n-1}$, and by homogeneity of w in $B_1 \setminus B_\tau$, it then follows that there exist constants $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_q$, $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_q$, with $\sum_{j=1}^q \lambda_j = \sum_{j=1}^q \mu_j = 0$ such that, for each $j \in \{1, \ldots, q\}$, $w^j(x^2, y) = \lambda_j x^2$ for each $(x^2, y) \in (B_1 \setminus B_\tau) \cap \{x^2 < 0\}$ and $w^j(x^2, y) = \mu_j x^2$ for each $(x^2, y) \in (B_1 \setminus B_\tau) \cap \{x^2 > 0\}$. Moreover, since $w \not\equiv 0$ in $B_1 \setminus B_\tau$, we must have some $j_0 \in \{1, \ldots, q-1\}$ such that either $\lambda_{j_0} > \lambda_{j_0+1}$ or $\mu_{j_0} < \mu_{j_0+1}$. Thus, taking any point $(0, y_1) \in (B_1 \setminus B_\tau) \cap (\{0\} \times \mathbf{R}^{n-1})$ and any number σ_1 with $0 < \sigma_1 < \min\{1 - |y_1|, |y_1| - \tau\}$ and setting $\widetilde{w} = \|w((0, y_1) + \sigma_1(\cdot))\|^{-1}_{L^2(B_1)} w((0, y_1) + \sigma_1(\cdot))$, we produce an element $\widetilde{w} \in \mathcal{B}$ whose existence contradicts the fact that $\mathcal{B}$ satisfies property ($\mathcal{B}7$). Hence it must be that $v - v_a = 0$ in B_1, and $\mathcal{H}_1$ consists of linear functions.

Now let k_1 be the smallest integer $\in \{2, 3, \ldots, n\}$ such that $\mathcal{H}_{k_1} \neq \emptyset$. Consider any $\widetilde{v} \in \mathcal{H}_{k_1}$, and let v be any element $\in \mathcal{B}$ such that v satisfies property ($\mathcal{B}4$ I) with $z = 0$ and v agrees with $\widetilde{v}$ on $B_1 \setminus B_\tau$. By property ($\mathcal{B}5$ II) (of v), we may assume that $T(\widetilde{v}) = \{0\} \times \mathbf{R}^{n-k_1}$. If $\Gamma_v \cap (B_1 \setminus \overline{B_\tau}) \subseteq \{0\} \times \mathbf{R}^{n-k_1}$, then by the remark immediately following the definition of Γ_v, v^j is harmonic in $(B_1 \setminus \overline{B_\tau}) \setminus \left(\{0\} \times \mathbf{R}^{n-k_1}\right)$ for each $j \in \{1, 2, \ldots, q\}$, whence by homogeneity, $\widetilde{v}$ is harmonic on $B_1 \setminus \left(\{0\} \times \mathbf{R}^{n-k_1}\right)$. Since $\widetilde{v}^j \in W^{1,2}_{\mathrm{loc}}(\mathbf{R}^n)$ and independent of the last $(n - k_1)$ variables, it follows that $\widetilde{v}^j$ is harmonic in all of $\mathbf{R}^n$. By homogeneity of $\widetilde{v}$ again and property ($\mathcal{B}2$) of v, it follows that $\widetilde{v}^1 = \widetilde{v}^2 = \cdots = \widetilde{v}^q = L$ for some linear function L, contrary to the assumption that $\widetilde{v} \in \mathcal{H}_{k_1}$ for $k_1 \geq 2$. So we must have that $\Gamma_v \cap (B_1 \setminus \overline{B_\tau}) \setminus \left(\{0\} \times \mathbf{R}^{n-k_1}\right) \neq \emptyset$. We shall contradict this also.

Let K be any compact subset of $(B_1 \setminus \overline{B_\tau}) \setminus \left(\{0\} \times \mathbf{R}^{n-k_1}\right)$. We claim that there exists $\varepsilon = \varepsilon(v, K, \mathcal{B}) \in (0, 1)$ such that for each $z \in K \cap \Gamma_v$ and each ρ with $0 < \rho \leq \varepsilon$,

(4.10)
$$\sum_{j=1}^q \int_{B_\rho(z) \setminus B_{\tau\rho}(z)} R_z^{2-n} \left(\frac{\partial\left((v^j - v_a)/R_z\right)}{\partial R_z}\right)^2 \geq \varepsilon \rho^{-n-2} \sum_{j=1}^q \int_{B_\rho(z)} |v^j - v_a|^2.$$

(Recall that v_a is a linear function.) If this were false, then there would exist points $z, z_i \in K \cap \Gamma_v$, $i = 1, 2, 3, \ldots$, with $z_i \to z$, and radii $\rho_i \to 0$ such that $v - v_a \not\equiv 0$ in $B_{\rho_i}(z_i)$ for each $i = 1, 2, 3, \ldots$ and

(4.11)
$$\sum_{j=1}^q \int_{B_{\rho_i}(z_i) \setminus B_{\tau\rho_i}(z_i)} R_{z_i}^{2-n} \left(\frac{\partial\left((v^j - v_a)/R_{z_i}\right)}{\partial R_{z_i}}\right)^2 < \varepsilon_i \rho_i^{-n-2} \sum_{j=1}^q \int_{B_{\rho_i}(z_i)} |v^j - v_a|^2,$$

where $\varepsilon_i \to 0^+$. By property ($\mathcal{B}5$ III), we have that $w \equiv \|v - v_a\|^{-1}_{L^2(B_1)}(v - v_a) \in \mathcal{B}$ so that, by property ($\mathcal{B}5$ I), $w_i \equiv w_{z_i,\rho_i} = \|w(z_i + \rho_i(\cdot))\|^{-1}_{L^2(B_1)} w(z_i + \rho_i(\cdot))$ also belongs to $\mathcal{B}$ for each sufficiently large i, and hence, by property ($\mathcal{B}6$), there exists $w_\star \in \mathcal{B}$ such that after passing to a subsequence, $w_i \to w_\star$ locally in $L^2(B_1)$

and locally weakly in $W^{1,2}(B_1)$. Since $\|w_i\|_{L^2(B_1)} = 1$, it follows from (4.11) that $\|w_i\|_{L^2(B_{3/4})} > c$ for sufficiently large i, where $c = c(n) > 0$. Hence $w_\star \not\equiv 0$ in B_1. In view of the strong convergence $w_i \to w_\star$ locally in $L^2(B_1)$ and the weak convergence $Dw_i \to Dw_\star$ locally in $L^2(B_1)$ (which, in particular, implies that $\int_{B_{1-\varepsilon}(0)\setminus B_{\varepsilon'}(0)} |x|^{-n-2}(Dw_\star \cdot x)^2 \leq \liminf_{i\to\infty} \int_{B_{1-\varepsilon}(0)\setminus B_{\varepsilon'}(0)} |x|^{-n-2}(Dw_i \cdot x)^2$ for any $\varepsilon, \varepsilon' \in (0, 1/4)$), it follows from (4.11) that $w_\star$ is homogeneous of degree 1 in $B_1 \setminus B_\tau$, and since property ($\mathcal{B}4\,\mathrm{I}$) is satisfied with w_i in place of v and $z = 0$, that it is also satisfied with $w_\star$ in place of v and $z = 0$. Thus if $\widetilde{w}_\star$ denotes the homogeneous of degree 1 extension of $w_\star|_{B_1\setminus B_\tau}$ to all of $\mathbf{R}^n$, then $\widetilde{w}_\star \in \mathcal{H}$. Note also that $\{0\} \times \mathbf{R}^{n-k_1} \subseteq T(\widetilde{w}_\star)$.

Now by homogeneity of v in $B_1 \setminus B_\tau$, we have that for each $y \in B_1$, sufficiently small $\sigma > 0$ and sufficiently large i,

$$\begin{aligned}
\sigma^{-n}\int_{B_\sigma(y)} w_i(x+z)\,dx &= \varepsilon_i^{-1}\sigma^{-n}\int_{B_\sigma(y)} w(z_i + \rho_i(x+z))\,dx \\
&= (1+\rho_i)\varepsilon_i^{-1}\sigma^{-n}\int_{B_\sigma(y)} w(z_i + (1+\rho_i)^{-1}\rho_i(z-z_i) + (1+\rho_i)^{-1}\rho_i x)\,dx \\
&= (1+\rho_i)^{n+1}\varepsilon_i^{-1}\sigma^{-n}\int_{B_{(1+\rho_i)^{-1}\sigma}((1+\rho_i)^{-1}(z-z_i+y))} w(z_i+\rho_i x)\,dx \\
&= (1+\rho_i)^{n+1}\sigma^{-n}\int_{B_{(1+\rho_i)^{-1}\sigma}((1+\rho_i)^{-1}(z-z_i+y))} w_i(x)\,dx,
\end{aligned}$$

where $\varepsilon_i = \|w(z_i + \rho_i(\cdot))\|_{L^2(B_1)}$, so first letting $i \to \infty$ in this (noting that $z_i \to z$) and then letting $\sigma \to 0$, we conclude that $\widetilde{w}_\star(y+z) = \widetilde{w}_\star(y)$ for almost every y; i.e., $z \in T(\widetilde{w}_\star)$. But $z \in B_1 \setminus \left(\{0\} \times \mathbf{R}^{n-k_1}\right)$ (since $z \in K$), and therefore we must have $\dim T(\widetilde{w}_\star) > n - k_1$. On the other hand, note that by the definition of k_1, either $k_1 = 2$ or (in case $k_1 \geq 3$) $\mathcal{H}_k = \emptyset$ for all $k = 2, \ldots, (k_1 - 1)$ so that, in either case, whenever $\dim T(\widetilde{v}) > n - k_1$ for some $\widetilde{v} \in \mathcal{H}$, it follows that $\widetilde{v} \in \mathcal{H}_1$. Thus we have shown that $\widetilde{w}_\star \in \mathcal{H}_1$ and hence that $\widetilde{w}_\star^1 = \widetilde{w}_\star^2 = \cdots = \widetilde{w}_\star^q = L$ for some linear function L. But since $(w_i)_a \equiv 0$ for each $j = 1, 2, \ldots$, it follows that $L = (\widetilde{w}_\star)_a = 0$, which is a contradiction. Thus (4.10) must hold for some $\varepsilon = \varepsilon(v, K, \mathcal{B}) \in (0, 1)$ and all $z \in K$, $\rho \in (0, \varepsilon]$ as claimed.

Combining (4.10) with property ($\mathcal{B}4\,\mathrm{I}$), we then have that

$$\begin{aligned}
\sum_{j=1}^q \int_{B_\rho(z)\setminus B_{\tau\rho}(z)} & R_z^{2-n}\left(\frac{\partial\left((v^j - v_a)/R_z\right)}{\partial R_z}\right)^2 \\
&\geq \frac{\varepsilon}{C}\sum_{j=1}^q \int_{B_{\tau\rho}(z)} R_z^{2-n}\left(\frac{\partial\,(v^j - v_a)/R_z)}{\partial R_z}\right)^2,
\end{aligned}$$

which implies that
(4.12)
$$\sum_{j=1}^{q}\int_{B_{\tau\rho}(z)} R_z^{2-n}\left(\frac{\partial\left((v^j-v_a)/R_z\right)}{\partial R_z}\right)^2 \leq \theta \sum_{j=1}^{q}\int_{B_{\rho}(z)} R_z^{2-n}\left(\frac{\partial\left(v^j-v_a\right)/R_z)}{\partial R_z}\right)^2$$
for all $z\in K\cap\Gamma_v$ and $\rho\in(0,\varepsilon]$, where $\theta=\theta(v,K,\mathcal{B})\in(0,1)$. Iterating this (for fixed $z\in K\cap\Gamma_v$) with $\tau^i\rho$, $i=1,2,3,\ldots$ in place of ρ, we see that
$$\sum_{j=1}^{q}\int_{B_{\tau^i\rho}(z)} R_z^{2-n}\left(\frac{\partial\left((v^j-v_a)/R_z\right)}{\partial R_z}\right)^2 \leq \theta^i \sum_{j=1}^{q}\int_{B_{\rho}(z)} R_z^{2-n}\left(\frac{\partial\left(v^j-v_a\right)/R_z)}{\partial R_z}\right)^2$$
for $i=0,1,2,3\ldots$, which readily implies that
$$\sum_{j=1}^{q}\int_{B_{\sigma}(z)} R_z^{2-n}\left(\frac{\partial\left((v^j-v_a)/R_z\right)}{\partial R_z}\right)^2 \leq \beta\left(\frac{\sigma}{\rho}\right)^{\mu} \sum_{j=1}^{q}\int_{B_{\rho}(z)} R_z^{2-n}\left(\frac{\partial\left(v^j-v_a\right)/R_z)}{\partial R_z}\right)^2$$
for any $z\in K\cap\Gamma_v$ and all $0<\sigma\leq\rho/2\leq\varepsilon/2$, where the constants $\beta=\beta(v,K,\mathcal{B})\in(0,\infty)$ and $\mu=\mu(v,K,\mathcal{B})\in(0,1)$ are independent of z. By property ($\mathcal{B}4\,$I) and inequality (4.10), this yields the estimate
(4.13)
$$\sum_{j=1}^{q}\sigma^{-n-2}\int_{B_{\sigma}(z)}|v^j-v_a|^2 \leq 2^{-n-2}\varepsilon^{-1}C\beta\left(\frac{\sigma}{\rho}\right)^{\mu}\rho^{-n-2}\sum_{j=1}^{q}\int_{B_{\rho}(z)}|v^j-v_a|^2$$
for each $z\in K\cap\Gamma_v$ and $0<\sigma\leq\rho/2\leq\varepsilon/4$. Since property ($\mathcal{B}4\,$II) and the definition of Γ_v imply that v is harmonic in $\mathbf{R}^n\setminus\Gamma_v$, we deduce from Lemma 4.3, the remark immediately following Lemma 4.3 and the arbitrariness of K that $v\in C^1\left((B_1\setminus\overline{B_\tau})\setminus\left(\{0\}\times\mathbf{R}^{n-k_1}\right)\right)$.

Now by property ($\mathcal{B}4\,$I), $\Gamma_v\cap(B_1\setminus\overline{B_\tau})\setminus\left(\{0\}\times\mathbf{R}^{n-k_1}\right)\subset$ the zero set of $u^j\equiv\left(v^j-v^{j-1}\right)\big|_{B_1\setminus\overline{B_\tau}}$ for each $j=2,\ldots,q$. Since u^j is nonnegative and C^1 in $(B_1\setminus\overline{B_\tau})\setminus\left(\{0\}\times\mathbf{R}^{n-k_1}\right)$, it follows that $Du^j(z)=0$ for any $z\in\Gamma_v\cap(B_1\setminus\overline{B_\tau})\setminus\left(\{0\}\times\mathbf{R}^{n-k_1}\right)$. Also, by property ($\mathcal{B}4\,$II) and the definition of Γ_v, u^j is harmonic in $(B_1\setminus\overline{B_\tau})\setminus\left(\Gamma_v\cup\left(\{0\}\times\mathbf{R}^{n-k_1}\right)\right)$. In order to derive a contradiction, pick any point $z_1\in\Gamma_v\cap(B_1\setminus\overline{B_\tau})\setminus\left(\{0\}\times\mathbf{R}^{n-k_1}\right)$ and let $\rho_1=\frac{1}{4}\operatorname{dist}\left(z_1,\partial B_1\cup\partial B_\tau\cup\{0\}\times\mathbf{R}^{n-k_1}\right)$. If $u^j(z)>0$ for some $z\in B_{\rho_1}(z_1)$, then there exists $\rho\in(0,\rho_1)$ such that $u^j>0$ in $B_\rho(z)$ and $\partial B_\rho(z)\cap\left(\Gamma_v\cap(B_1\setminus B_\tau)\setminus\left(\{0\}\times\mathbf{R}^{n-k_1}\right)\right)\neq\emptyset$, contradicting the Hopf boundary point lemma. It follows that $u^j\equiv 0$ in $B_{\rho_1}(z_1)$ for each $j=2,\ldots,q$. But since $z_1 \in \Gamma_v$, this is impossible by the definition of Γ_v, so we see that the assumption $\Gamma_v \cap (B_1 \setminus \overline{B_\tau}) \setminus \left(\{0\} \times \mathbf{R}^{n-k_1}\right) \neq \emptyset$ leads to a contradiction. Thus $\mathcal{H}_k = \emptyset$ for each $k = 2, \ldots, n$, and the proposition is proved. □

Proof of Theorem 4.1. The main point is to prove that $\mathcal{B} \subseteq C^1(B_1)$. For if this is true, then, by exactly the same argument as in the last paragraph of the proof of Proposition 4.2, we see that $\Gamma_v = \emptyset$ for each $v \in \mathcal{B}$, from which the first assertion of the theorem follows immediately.

In view of Lemma 4.3, property ($\mathcal{B}4\,\mathrm{II}$) and property ($\mathcal{B}5\,\mathrm{I}$), to prove that $\mathcal{B} \subseteq C^1(B_1)$, it suffices to establish that there are fixed constants $\beta = \beta(\mathcal{B}) \in (0, \infty)$ and $\mu = \mu(\mathcal{B}) \in (0,1)$ such that for each $v \in \mathcal{B}$, $z \in \Gamma_v \cap B_{3/4}$ and $0 < \sigma \leq \rho/2 \leq 1/8$,

$$\sigma^{-n-2} \sum_{j=1}^{q} \int_{B_\sigma(z)} |v^j - \ell_z|^2 \leq \beta \left(\frac{\sigma}{\rho}\right)^{\mu} \rho^{-n-2} \sum_{j=1}^{q} \int_{B_\rho(z)} |v^j - \ell_z|^2, \tag{4.14}$$

where ℓ_z is the affine function given by $\ell_z(x) = v_a(z) + Dv_a(z) \cdot (x - z)$, $x \in \mathbf{R}^n$. This estimate follows by exactly the same hole-filling argument used in the proof of Proposition 4.2. Specifically, we may first prove, by arguing by contradiction and using Proposition 4.2, that there exists a fixed constant $\varepsilon = \varepsilon(\mathcal{B}) > 0$ such that if $v \in \mathcal{B}$, $0 \in \Gamma_v$, $v_a(0) = 0$ and $Dv_a(0) = 0$, then

$$\sum_{j=1}^{q} \int_{B_{1/4}(0) \setminus B_{\tau/4}(0)} R^{2-n} \left(\frac{\partial\,(v^j/R)}{\partial\,R}\right)^2 \geq \varepsilon \sum_{j=1}^{q} \int_{B_{1/4}(0)} |v^j|^2,$$

where $\tau = \tau(\mathcal{B}) \in (0, 1/4)$ is the constant as in (4.1). It follows from this and property ($\mathcal{B}4\,\mathrm{I}$) (by arguing as in the proof of (4.13)) that if $v \in \mathcal{B}$, $0 \in \Gamma_v$, $v_a(0) = 0$ and $Dv_a(0) = 0$, then

$$\rho^{-n-2} \int_{B_\rho(0)} |v|^2 \leq \beta \rho^{\mu} \int_{B_{1/2}} |v|^2 \qquad \forall \rho \in (0, 1/2],$$

where $\beta = \beta(\mathcal{B}) \in (0, \infty)$ and $\mu = \mu(\mathcal{B}) \in (0, 1)$. In view of properties ($\mathcal{B}5\,\mathrm{I}$) and ($\mathcal{B}5\,\mathrm{III}$), the estimate (4.14) follows from this.

Since finiteness of $\sum_{j=1}^{q} \int_{B_\rho(z)} R_z^{2-n} \left(\frac{\partial((v^j - v_a(z))/R_z)}{\partial R_z}\right)^2$ implies that $v^1(z) = v^2(z) = \cdots = v^q(z)$ $(= v_a(z))$, the second assertion of the theorem follows from the first, property ($\mathcal{B}2$) and the maximum principle. □

5. **Lipschitz approximation and coarse blow-ups**

Here we recall (in Theorem 5.1 below) some facts concerning approximation of a stationary integral varifold weakly close to a hyperplane by the graph of a Lipschitz function over the hyperplane. These results were established by Almgren ([Alm00]), adapting, for the higher multiplicity setting, the corresponding result of Allard ([All72]) for multiplicity 1 varifolds. We shall use these facts to blow up mass-bounded sequences of varifolds weakly converging to a hyperplane.

First note the following elementary fact, which we shall need here and subsequently: If V is a stationary integral n-varifold on $B_2^{n+1}(0)$, then

$$\int_{B_2^{n+1}(0)} |\nabla^V x^1|^2 \widetilde{\zeta}^2 \, d\|V\|(X) \leq 4 \int_{B_2^{n+1}(0)} |x^1|^2 |\nabla^V \widetilde{\zeta}|^2 \, d\|V\|(X) \tag{5.1}$$

for each $\widetilde{\zeta} \in C_c^1(B_2^{n+1}(0))$. This is derived simply by taking $\psi(X) = x^1 \widetilde{\zeta}^2(X) e^1$ in the first variation formula (3.1).

Let $\rho \in (0,1)$, and suppose that $\operatorname{spt} \|V\| \cap (\mathbf{R} \times B_{(1+\rho)/2}) \subset \{|x^1| < 1\}$. Choosing $\widetilde{\zeta}$ in (5.1) such that $\widetilde{\zeta}(x^1, x') = \zeta(x')$ in a neighborhood of $\operatorname{spt} \|V\| \cap (\mathbf{R} \times B_1)$, where $\zeta \in C_c^1(B_{(1+\rho)/2})$ is such that $\zeta \equiv 1$ on B_ρ, $0 \leq \zeta \leq 1$ and $|D\zeta| \leq C$ for some constant $C = C(\rho)$ (e.g., $\widetilde{\zeta}(x^1, x') = \eta(x^1)\zeta(x')$ where $\eta \in C_c^1(-3/2, 3/2)$ with $\eta \equiv 1$ on $[-1,1]$), we deduce from (5.1) that for each $\rho \in (0,1)$,

$$\int_{\mathbf{R} \times B_\rho} |\nabla^V x^1|^2 d\|V\|(X) \leq C \hat{E}_V^2, \tag{5.2}$$

where $C = C(n, \rho) \in (0, \infty)$, and $\hat{E}_V = \sqrt{\int_{\mathbf{R} \times B_1} |x^1|^2 \, d\|V\|(X)}$.

THEOREM 5.1 ([Alm00, Cor. 3.11]). *Let q be a positive integer and $\sigma \in (0,1)$. There exist numbers $\varepsilon_0 = \varepsilon_0(n, q, \sigma) \in (0, 1/2)$ and $\xi = \xi(n,q) \in (0,1/2)$ such that the following holds: Let V be a stationary integral n-varifold on $B_2^{n+1}(0)$ with*

$$(\omega_n 2^n)^{-1} \|V\|(B_2^{n+1}(0)) < q + 1/2, \qquad q - 1/2 \leq \omega_n^{-1} \|V\|(\mathbf{R} \times B_1) < q + 1/2$$

and

$$\hat{E}_V^2 \equiv \int_{\mathbf{R} \times B_1} |x^1|^2 d\|V\|(X) \leq \varepsilon_0.$$

Let

$$\Sigma = \pi\, \widetilde{\Sigma}_1 \cup \pi\, \widetilde{\Sigma}_2 \cup \pi\, \widetilde{\Sigma}_3 \cup \Sigma',$$

where $\pi : \mathbf{R}^{n+1} \to \{0\} \times \mathbf{R}^n$ is the orthogonal projection,

$$\widetilde{\Sigma}_1 = \Big\{ Y \in \operatorname{spt} \|V\| \cap (\mathbf{R} \times B_\sigma) \, : \, \rho^{-n} \int_{\mathbf{R} \times B_\rho(\pi Y)} |\nabla^V x^1|^2 \, d\|V\|(X) \geq \xi \\ \text{for some } \rho \in (0, 1-\sigma) \Big\},$$

$$\widetilde{\Sigma}_2 = \{ Y \in \operatorname{spt} \|V\| \cap (\mathbf{R} \times B_\sigma) \, : \, \text{either } \operatorname{Tan}(\operatorname{spt} \|V\|, Y) \neq \operatorname{Tan}^n(\|V\|, Y) \\ \text{or } \operatorname{Tan}(\operatorname{spt} \|V\|, Y) \notin G_n \text{ or } \Theta(\|V\|, Y) \text{ is not a positive integer} \},$$

where $\operatorname{Tan}(\operatorname{spt} \|V\|, Y)$ *denotes the tangent cone of* $\operatorname{spt} \|V\|$ *at* Y ([Fed69, 3.1.21]) *and* $\operatorname{Tan}^n(\|V\|, Y)$ *denotes the* $(\|V\|, n)$ *approximate tangent cone of*

$\|V\|$ at Y ([Fed69, 3.2.16]),

$$\widetilde{\Sigma}_3 = \left\{Y \in \operatorname{spt}\|V\| \cap (\mathbf{R} \times B_\sigma) \setminus \widetilde{\Sigma}_2 : 1 - (e_1 \cdot \nu(Y))^2 \geq 1/4\right\},$$

where $\nu(Y)$ is the unit normal to $\operatorname{Tan}(\operatorname{spt}\|V\|, Y)$, *and*

$$\Sigma' = \left\{Y \in B_\sigma \setminus (\pi\,\widetilde{\Sigma}_1 \cup \pi\,\widetilde{\Sigma}_2 \cup \pi\,\widetilde{\Sigma}_3) : \Theta(\|\pi_{\#}\,V\|, Y) \leq q-1\right\}.$$

Then

(a) $\mathcal{H}^n(\Sigma) + \|V\|(\mathbf{R} \times \Sigma) \leq C\hat{E}_V^2$, *where* $C = C(n, q, \sigma) \in (0, \infty)$.

(b) *There are Lipschitz functions* $u^j : B_\sigma \to \mathbf{R}$, *with* $\operatorname{Lip} u^j \leq 1/2$ *for each* $j \in \{1, 2, \ldots, q\}$ *such that* $u^1 \leq u^2 \leq \cdots \leq u^q$ *and*

$$\operatorname{spt}\|V\| \cap (\mathbf{R} \times (B_\sigma \setminus \Sigma)) = \textstyle\bigcup_{j=1}^q \operatorname{graph} u^j \cap (\mathbf{R} \times (B_\sigma \setminus \Sigma)).$$

(c) *For each $x \in B_\sigma \setminus \Sigma$ and each $Y \in \operatorname{spt}\|V\| \cap \pi^{-1}(x)$, $\Theta(\|V\|, Y)$ is a positive integer and*

$$\sum_{Y \in \operatorname{spt}\|V\| \cap \pi^{-1}(x)} \Theta(\|V\|, Y) = q.$$

Proof. In view of the Constancy Theorem ([Sim83, Th. 41.1]), the estimate (5.2), and the easily verifiable fact that in the present codimension 1 setting, the "unordered distance" is the same as the "ordered distance" (that is, if $a_j, b_j \in \mathbf{R}$ are such that $a_1 \leq a_2 \leq \cdots \leq a_q$ and $b_1 \leq b_2 \leq \cdots \leq b_q$, then $\mathcal{G}(\{a_1, \ldots, a_q\}, \{b_1, \ldots, b_q\}) \equiv \inf\left\{\sqrt{\sum_{j=1}^q (a_j - b_{\sigma(j)})^2} : \sigma \text{ is a permutation of } \{1, \ldots, q\}\right\} = \sqrt{\sum_{j=1}^q (a_j - b_j)^2}$), the theorem follows immediately from [Alm00, Cor. 3.11], which in turn is a fairly straightforward adaptation of the corresponding argument in [All72] for the case $q = 1$. $\square$

Remark. It is an easy consequence of the monotonicity of mass ratio ([Sim83, §17.5]) that for each $\sigma \in (0, 1)$, there exists $\varepsilon = \varepsilon(n, \sigma) \in (0, 1)$ such that if V is a stationary integral n-varifold on $\mathbf{R} \times B_1$ with $\hat{E}_V^2 = \int_{\mathbf{R} \times B_1} |x^1|^2 \, d\|V\|(X) < \varepsilon$, then

$$\sup_{X \in (\mathbf{R} \times B_\sigma) \cap \operatorname{spt}\|V\|} |x^1| \leq C\hat{E}_V^{1/n},$$

where $C = C(n) \in (0, \infty)$. In particular, under the hypotheses of Theorem 5.1, we have that

$$\sup_{x \in B_\sigma} |u(x)| \leq C\hat{E}_V^{1/n},$$

where $C = C(n) \in (0, \infty)$.

Let q be a positive integer. Let $\{V_k\}$ be a sequence of n-dimensional stationary integral varifolds of $B_2^{n+1}(0)$ such that

(5.3)

$$(\omega_n 2^n)^{-1}\|V_k\|(B_2^{n+1}(0)) < q + 1/2; \quad q - 1/2 \leq \omega_n^{-1}\|V_k\|(\mathbf{R} \times B_1) < q + 1/2$$

for each $k = 1, 2, 3, \ldots$, and $\hat{E}_k \to 0$, where

$$\hat{E}_k^2 \equiv \hat{E}_{V_k}^2 = \int_{\mathbf{R}\times B_1} |x^1|^2 \, d\|V_k\|(X). \tag{5.4}$$

Let $\sigma \in (0,1)$. By Theorem 5.1, for all sufficiently large k, there exist Lipschitz functions $u_k^j : B_\sigma \to \mathbf{R}$, $j = 1, 2, \ldots, q$, with $u_k^1 \leq u_k^2 \leq \cdots \leq u_k^q$ and

$$\operatorname{Lip} u_k^j \leq 1/2 \quad \text{for each} \quad j \in \{1, 2, \ldots, q\} \tag{5.5}$$

such that

$$\operatorname{spt} \|V_k\| \cap (\mathbf{R} \times (B_\sigma \setminus \Sigma_k)) = \textstyle\bigcup_{j=1}^q \operatorname{graph} u_k^j \cap (\mathbf{R} \times (B_\sigma \setminus \Sigma_k)), \tag{5.6}$$

where Σ_k is the measurable subset of B_σ that corresponds to Σ in Theorem 5.1 when V is replaced by V_k; thus by Theorem 5.1,

$$\|V_k\|(\mathbf{R} \times \Sigma_k) + \mathcal{H}^n(\Sigma_k) \leq C\hat{E}_k^2, \tag{5.7}$$

where $C = C(n, q, \sigma) \in (0, \infty)$. Set $v_k^j(x) = \hat{E}_k^{-1} u_k^j(x)$ for $x \in B_\sigma$, and write $v_k = (v_k^1, v_k^2, \ldots, v_k^q)$. Then v_k is Lipschitz on B_σ; and by (5.7) and (5.6),

$$\int_{B_\sigma} |v_k|^2 \leq C, \quad C = C(n, q, \sigma) \in (0, \infty). \tag{5.8}$$

Furthermore,

$$\begin{aligned}\int_{B_\sigma} (1 + |Du_k|^2)^{-1/2} |Du_k|^2 &= \int_{B_\sigma \setminus \Sigma_k} (1 + |Du_k|^2)^{-1/2} |Du_k|^2 \\ &\quad + \int_{B_\sigma \cap \Sigma_k} (1 + |Du_k|^2)^{-1/2} |Du_k|^2 \\ &\leq \int_{\mathbf{R} \times B_\sigma} |\nabla^{V_k} x^1|^2 \, d\|V_k\|(X) + C_1 \hat{E}_k^2 \leq C_2 \hat{E}_k^2,\end{aligned}$$

where $C_1 = C_1(n, q, \sigma) \in (0, \infty)$, $C_2 = C_2(n, q, \sigma) \in (0, \infty)$ and we have used (5.5) in the first inequality and (5.2) in the second. By (5.5) again, this implies that

$$\int_{B_\sigma} |Dv_k|^2 \leq C, \quad C = C(n, q, \sigma) \in (0, \infty). \tag{5.9}$$

In view of the arbitrariness of $\sigma \in (0, 1)$, by (5.8), (5.9), the preceding remark, Rellich's theorem and a diagonal sequence argument, we obtain a function $v \in W^{1,2}_{\mathrm{loc}}(B_1; \mathbf{R}^q) \cap L^2(B_1; \mathbf{R}^q)$ and a subsequence $\{k_j\}$ of $\{k\}$ such that $v_{k_j} \to v$ as $j \to \infty$ in $L^2(B_\sigma; \mathbf{R}^q)$ and weakly in $W^{1,2}(B_\sigma; \mathbf{R}^q)$ for every $\sigma \in (0, 1)$.

Definitions.

(1) *Coarse blow-ups.* Let $v \in W^{1,2}_{\mathrm{loc}}(B_1; \mathbf{R}^q) \cap L^2(B_1; \mathbf{R}^q)$ correspond, in the manner described above, to (a subsequence of) a sequence $\{V_k\}$ of stationary integral n-varifolds of $B_2^{n+1}(0)$ satisfying (5.3) and with $\hat{E}_k \to 0$, where $\hat{E}_k$ is as in (5.4). We shall call v a *coarse blow-up* of the sequence $\{V_k\}$.

(2) *The Class $\mathcal{B}_q$.* Denote by $\mathcal{B}_q$ the collection of all coarse blow-ups of sequences of varifolds $\{V_k\} \subset \mathcal{S}_\alpha$ satisfying (5.3) and for which $\hat{E}_k \to 0$, where $\hat{E}_k$ is as in (5.4).

6. An outline of the proof of the main theorems

Note that if $\mathbf{C}_0$ is a stationary cone as in Theorem 3.4, then $\Theta_{\mathbf{C}_0}(0) = q - 1/2$ or $\Theta_{\mathbf{C}_0}(0) = q$ for some integer $q \geq 2$. We prove both Theorem 3.3′ and Theorem 3.4 simultaneously by induction on q. The case $q = 1$ of Theorem 3.3′ is a consequence of Allard's Regularity Theorem. (Note however that setting $q = 1$ in the proofs of Lemma 15.1 and Theorem 15.2 given below reproduces Allard's argument proving Theorem 3.3′ in case $q = 1$.) Validity of the cases $\Theta(\|\mathbf{C}_0\|, 0) = 3/2$ and $\Theta(\|\mathbf{C}_0\|, 0) = 2$ of Theorem 3.4 will be justified at the end of Section 16.

Let q be an integer ≥ 2 and consider the following:

Induction Hypotheses.

(H1) Theorem 3.3′ *holds with* $1, \ldots, (q-1)$ *in place of* q.

(H2) Theorem 3.4 *holds whenever* $\Theta_{\mathbf{C}_0}(0) \in \{3/2, 2, 5/2, \ldots, q\}$.

The inductive proof of Theorems 3.3′ and 3.4 is obtained by completing, assuming (H1), (H2), the steps below in the order they are listed:

Step 1: Prove that $\mathcal{B}_q$ is a proper blow-up class (Sections 7–14).

Step 2: Prove Theorem 3.3′ (Section 15).

Step 3: Prove Theorem 3.4 when $\Theta(\|\mathbf{C}_0\|, 0) = q + 1/2$ (Section 16).

Step 4: Prove Theorem 3.4 when $\Theta(\|\mathbf{C}_0\|, 0) = q + 1$ (Section 16).

Remarks. (1) Let $m \in \{1, 2, \ldots, n\}$. Suppose that $\mathbf{C}$ is an m-dimensional stationary integral cone in $\mathbf{R}^{n+1}$. Let $L_{\mathbf{C}} = \{Y \in \operatorname{spt}\|\mathbf{C}\| : \Theta(\|\mathbf{C}\|, Y) = \Theta(\|\mathbf{C}\|, 0)\}$. It is a well-known consequence of the monotonicity formula that $L_{\mathbf{C}}$ is a linear subspace of $\mathbf{R}^{n+1}$ of dimension $\leq m$ and that $Y \in L_{\mathbf{C}}$ if and only if $T_{Y\,\#}\,\mathbf{C} = \mathbf{C}$, where $T_Y : \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ is the translation $T_Y(X) = X - Y$. Let $d_{\mathbf{C}} = \dim L_{\mathbf{C}}$. Then, if $\Gamma_{\mathbf{C}}$ is a rotation of $\mathbf{R}^{n+1}$ such that $\Gamma_{\mathbf{C}}(L_{\mathbf{C}}) = \{0\} \times \mathbf{R}^{d_{\mathbf{C}}}$, we have that $\Gamma_{\mathbf{C}\,\#}\,\mathbf{C} = \mathbf{C}' \times \mathbf{R}^{d_{\mathbf{C}}}$, where $\mathbf{C}'$ is a stationary integral cone in $\mathbf{R}^{n+1-d_{\mathbf{C}}}$. Here, given an integer $d \in \{0, 1, 2, \ldots, n\}$ and a rectifiable varifold V' of $\mathbf{R}^{n+1-d}$, we use the notation $V' \times \mathbf{R}^d$ to denote the rectifiable varifold V of $\mathbf{R}^{n+1}$ with $\operatorname{spt}\|V\| = \operatorname{spt}\|V'\| \times \mathbf{R}^d$ and the multiplicity function θ_V defined by $\theta_V(x, y) = \theta_{V'}(x)$ for $(x, y) \in \operatorname{spt}\|V'\| \times \mathbf{R}^d$, where $\theta_{V'}$ is the multiplicity function of V'.

(2) Let q be an integer ≥ 2, and suppose that the induction hypotheses (H1), (H2) hold. Let $V \in \mathcal{S}_\alpha$. Then we have the following:

(a) If $2 \leq n \leq 6$, then $\operatorname{sing} V \cap \{Z \in \operatorname{spt}\|V\| : \Theta(\|V\|, Z) < q\} = \emptyset$.

(b) If $n \geq 7$, $Z \in \operatorname{sing} V$ and $\Theta(\|V\|, Z) < q$, then $d_{\mathbf{C}} \leq n - 7$ for any $\mathbf{C} \in \operatorname{Var\,Tan}(V, Z)$.

To see this, suppose either (a) or (b) is false. Then we have either

(a′) $n \in \{2, 3, \ldots, 6\}$ and there exist a varifold $V \in \mathcal{S}_\alpha$ and a point $Z \in \operatorname{sing} V$ such that $\Theta(\|V\|, Z) < q$, or

(b′) $n \geq 7$ and there exist a varifold $V \in \mathcal{S}_\alpha$ and a point $Z \in \operatorname{sing} V$ with $\Theta(\|V\|, Z) < q$ such that $d_{\mathbf{C}} > n - 7$ for some $\mathbf{C} \in \operatorname{Var\,Tan}(V, Z)$.

If (a′) holds, fix any $\mathbf{C} \in \operatorname{Var\,Tan}(V, Z)$. In either case (a′) or (b′), the induction hypothesis (H1) implies that $d_{\mathbf{C}} \neq n$; for if $d_{\mathbf{C}} = n$, then $\mathbf{C} = q'|P|$ for some integer $q' \in \{1, 2, \ldots, q-1\}$ and some hyperplane P that we may take without loss of generality to be $\{0\} \times \mathbf{R}^n$, whence by the definition of tangent cone and the fact that weak convergence of stationary integral varifolds implies convergence of mass and convergence in Hausdorff distance of the supports of the associated weight measures, for any given $\varepsilon > 0$, there exists $\sigma \in (0, 1 - |Z|/2)$ such that $\operatorname{dist}_{\mathcal{H}}(\operatorname{spt} \|\eta_{Z,\sigma\,\#} V\| \cap (\mathbf{R} \times B_1), \{0\} \times B_1) < \varepsilon$, $q' - 1/2 \leq \omega_n^{-1} \|\eta_{Z,\sigma\,\#} V\|(\mathbf{R} \times B_1) < q' + 1/2$ and $(\omega_n 2^n)^{-1} \|\eta_{Z,\sigma\,\#} V\|(B_2^{n+1}(0)) < q' + 1/2$. Choosing $\varepsilon = \varepsilon_0(n, \alpha, q')$ where ε_0 is as in Theorem 3.3′, by (H1), we may apply Theorem 3.3′ to deduce that near Z, V corresponds to an embedded graph of a $C^{1,\alpha}$ function over P solving the minimal surface equation, and hence $\operatorname{spt} \|V\|$ near Z is an embedded analytic hypersurface, contradicting our assumption that $Z \in \operatorname{sing} V$. Thus $d_{\mathbf{C}} < n$.

Again in either case, the induction hypothesis (H2) implies that $d_{\mathbf{C}} \neq n-1$; for if $d_{\mathbf{C}} = n - 1$, then $\operatorname{spt} \|\mathbf{C}\|$ is the union of at least three half-hyperplanes meeting along an $(n-1)$-dimensional subspace, and since $\Theta(\|\mathbf{C}\|, 0) < q$, we must have that $\Theta(\|\mathbf{C}\|, 0) \in \{3/2, 2, 5/2, \ldots, q - 1/2\}$. Again by the definition of tangent cone, we have that for any given $\varepsilon_1 > 0$, a number $\sigma \in (0, 1 - |Z|/2)$ such that $|(\omega_n 2^n)^{-1} \|\eta_{Z,\sigma\,\#} V\|(B_2^{n+1}(0)) - \Theta(\|\mathbf{C}\|, 0)| < 1/8$ and $\operatorname{dist}_{\mathcal{H}}(\operatorname{spt} \|\eta_{Z,\sigma\,\#} V\| \cap B_1^{n+1}(0), \operatorname{spt} \|\mathbf{C}\| \cap B_1^{n+1}(0)) < \varepsilon_1$, so choosing $\varepsilon_1 = \frac{1}{2}\varepsilon(\alpha, \frac{1}{8}, \mathbf{C})$ where ε is as in Theorem 3.4, we see by hypothesis (H2) that we have a contradiction to Theorem 3.4.

Thus $d_{\mathbf{C}} \leq n - 2$. Assume now without loss of generality that $L_{\mathbf{C}} = \{0\} \times \mathbf{R}^{d_{\mathbf{C}}}$. Then $\mathbf{C} = \mathbf{C}' \times \mathbf{R}^{d_{\mathbf{C}}}$, where $\mathbf{C}'$ is an $(n - d_{\mathbf{C}})$-dimensional stationary integral cone of $\mathbf{R}^{n+1-d_{\mathbf{C}}}$ with $0 \in \operatorname{sing} \mathbf{C}'$. Note that since $\Theta(\|\mathbf{C}\|, Y) < q$ for each $Y \in \operatorname{spt} \|\mathbf{C}\|$, in view of hypothesis (H1), it follows from Theorem 3.3′ that $\operatorname{reg} \mathbf{C}$ satisfies the stability inequality; viz., $\int_{\operatorname{reg} \mathbf{C}} |A_{\mathbf{C}}|^2 \zeta^2 \leq \int_{\operatorname{reg} \mathbf{C}} |\nabla^{\mathbf{C}} \zeta|^2$ for each $\zeta \in C_c^1(\operatorname{reg} \mathbf{C})$, where $A_{\mathbf{C}}$ denotes the second fundamental form of $\operatorname{reg} \mathbf{C}$.

Now by a theorem of J. Simons [Sim68] (see [Sim83, App. B] for a shorter proof), we know that if $2 \leq n \leq 6$, there does not exist, in $\mathbf{R}^{n+1}$, a minimal hypercone with an isolated singularity and satisfying the stability inequality. Applying this to $\mathbf{C}'$, we conclude that if $\operatorname{sing} \mathbf{C} = \{0\} \times \mathbf{R}^{d_{\mathbf{C}}}$, then, in either of the cases (a′) or (b′), we have a contradiction. Hence there is a point $Z_1 \in \operatorname{sing} \mathbf{C} \setminus \{0\} \times \mathbf{R}^{d_{\mathbf{C}}}$.

Let $\mathbf{C}_1 \in \mathrm{Var\ Tan}(\mathbf{C}, Z_1)$. Then $\{tZ_1 : t \in \mathbf{R}\} \times \mathbf{R}^{d_{\mathbf{C}}} \subseteq L_{\mathbf{C}_1}$ so that $d_{\mathbf{C}_1} \geq d_{\mathbf{C}} + 1$. Since $\mathbf{C}_1 \llcorner B_1^{n+1}(0) = \lim_{k\to\infty} V_k$ for some sequence of varifolds $\{V_k\} \subset \mathcal{S}_\alpha$ (indeed, $V_k = \eta_{\widetilde{Z}_k, \sigma_k\,\#} V$ for some sequence of points $\widetilde{Z}_k$ and a sequence of positive numbers σ_k converging to 0) and $\Theta(\|\mathbf{C}_1\|, 0) = \Theta(\|\mathbf{C}\|, Z_1) < q$, by reasoning as above, we see that $d_{\mathbf{C}_1} \leq n-2$ and that $\mathrm{reg}\,\mathbf{C}_1$ satisfies the stability inequality. Thus $d_{\mathbf{C}} \leq n-3$, and hence, in particular, $n \geq 3$.

By Simons' theorem again, there exists a point $Z_2 \in \mathrm{sing}\,\mathbf{C}_1 \setminus L_{\mathbf{C}_1}$, which implies (by reasoning as above considering a cone $\mathbf{C}_2 \in \mathrm{Var\ Tan}(\mathbf{C}_1, Z_2)$) that $d_{\mathbf{C}} \leq n-4$ and $n \geq 4$. Repeating this argument twice more in case (a′), we produce a cone contradicting Simons' theorem, and three times more in case (b′), we reach the conclusion $d_{\mathbf{C}} \leq n-7$ contrary to the assumption. Thus both claims (a) and (b) must hold.

(3) By Remark (2) above and, in case $n \geq 7$, Almgren's generalised stratification of stationary integral varifolds ([Alm00, p. 224, Th. 2.26 and Rem. 2.28]; see [Sim96, §3.4] for a concise presentation of the argument in the context of energy minimizing maps), we have the following:

Let q be an integer ≥ 2. If the induction hypotheses (H1), (H2) *hold, $V \in \mathcal{S}_\alpha$, $\Omega \subseteq B_2^{n+1}(0)$ is open and $\Theta(\|V\|, Z) < q$ for each $Z \in \mathrm{spt}\,\|V\| \cap \Omega$, then $\mathcal{H}^{n-7+\gamma}(\mathrm{sing}\,V \llcorner \Omega) = 0$ for each $\gamma > 0$ if $n \geq 7$ (with $\mathrm{sing}\,V \llcorner \Omega$ discrete if $n = 7$) and $\mathrm{sing}\,V \llcorner \Omega = \emptyset$ if $2 \leq n \leq 6$.*

We shall now begin, and end in Section 14, the central part of our work, namely, the proof that *for any integer $q \geq 1$, the class of functions $\mathcal{B}_q$ (as defined at the end of Section* 5) *is a proper blow-up class (as defined in Section* 4).

7. Nonconcentration of tilt-excess

The main result of this section is the estimate of Theorem 7.1(b), which says that for a stationary integral n-varifold on an open ball in $\mathbf{R}^{n+1}$ having small height excess relative to a hyperplane, concentration of points of "top density" near an $(n-1)$-dimensional subspace L implies nonconcentration, near L, of the tilt-excess of the varifold relative to the hyperplane. This estimate will play a crucial role in the proof that $\mathcal{B}_q$ (see the definition at the end of Section 5) is a proper blow-up class—specifically, in establishing property ($\mathcal{B}7$) (see Section 4) for $\mathcal{B}_q$. No stability hypothesis is required for the results of this section.

THEOREM 7.1. *Let q be a positive integer, $\tau \in (0, 1/16)$ and $\mu \in (0,1)$. There exists a number $\varepsilon_1 = \varepsilon_1(n, q, \tau, \mu) \in (0, 1/2)$ such that if V is a stationary integral n-varifold of $B_2^{n+1}(0)$ with*

$$(\omega_n 2^n)^{-1}\|V\|(B_2^{n+1}(0)) < q + 1/2, \quad q - 1/2 \leq \omega_n^{-1}\|V\|(\mathbf{R} \times B_1) < q + 1/2$$

and

$$\int_{\mathbf{R}\times B_1} |x^1|^2 d\|V\|(X) \leq \varepsilon_1,$$

then the following hold:

(a) *For each point* $Z = (z^1, z') \in \operatorname{spt}\|V\| \cap (\mathbf{R}\times B_{9/16})$ *with* $\Theta(\|V\|, Z) \geq q$,

$$|z^1|^2 \leq C \int_{\mathbf{R}\times B_1} |x^1|^2\, d\|V\|(X),$$

where $C = C(n, q) \in (0, \infty)$.

(b) *If* L *is an* $(n-1)$*-dimensional subspace of* $\{0\}\times\mathbf{R}^n$ *such that*

$$L \cap B_{1/2} \subset \left(\{Z \in \operatorname{spt}\|V\| : \Theta(\|V\|, Z) \geq q\}\right)_\tau,$$

then

$$\int_{(L)_\tau\cap(\mathbf{R}\times B_{1/2})} |\nabla^V x^1|^2 d\|V\|(X) \leq C\tau^{1-\mu} \int_{\mathbf{R}\times B_1} |x^1|^2 d\|V\|(X),$$

where $C = C(n, q, \mu) \in (0, \infty)$. *Here for a subset* A *of* $\mathbf{R}^{n+1}$, *we use the notation* $(A)_\tau = \{X \in \mathbf{R}^{n+1} : \operatorname{dist}(X, A) \leq \tau\}$.

Remarks. (1) Since Theorem 5.1 holds with tilt-excess

$$\int_{\mathbf{R}\times B_1} |\nabla^V x^1|^2\, d\|V\|(X)$$

in place of the height excess $\hat{E}_V^2$ (see [Alm00, Cor. 3.11]), an examination of the proof below in fact shows that for any $\mu \in (0, 1)$, the more refined estimate

$$\int_{(L)_\tau\cap(\mathbf{R}\times B_{1/2})} |\nabla^V x^1|^2 d\|V\|(X)$$
$$\leq C\tau^{1-\mu} \int_{\mathbf{R}\times B_1} |\nabla^V x^1|^2 d\|V\|(X), \quad C = C(n, q, \mu) \in (0, \infty)$$

holds under the hypotheses of Theorem 7.1(b). We do not however need it here.

(2) A similar estimate for height excess relative to certain minimal cones was established in a "multiplicity 1 setting" in [Sim93]. Indeed, we shall later need a version of that as well (see Corollaries 10.8 and 16.5).

Proof. The proof is based on the monotonicity formula [Sim83, 17.5], which implies that, for any $Z \in \operatorname{spt}\|V\| \cap (\mathbf{R}\times B_{9/16})$,

$$(7.1) \qquad \frac{1}{\omega_n}\int_{B^{n+1}_{3/8}(Z)} \frac{|(X-Z)^\perp|^2}{|X-Z|^{n+2}} d\|V\|(X) = \frac{\|V\|(B^{n+1}_{3/8}(Z))}{\omega_n (3/8)^n} - \Theta(\|V\|, Z).$$

Write $\hat{E}_V = \sqrt{\int_{\mathbf{R}\times B_1} |x^1|^2\, d\|V\|(X)}$. Assuming $\varepsilon_1 = \varepsilon_1(n, q) \in (0, \infty)$ is sufficiently small to guarantee the validity of its conclusions, Theorem 5.1 with

$\sigma = 15/16$ implies that

$$\begin{aligned}\|V\|(B^{n+1}_{3/8}(Z)) &\leq \|V\|(\mathbf{R}\times B_{3/8}(z')) \\ &= \|V\|(\mathbf{R}\times(B_{3/8}(z')\setminus\Sigma)) + \|V\|(\mathbf{R}\times(B_{3/8}(z')\cap\Sigma)) \\ &\leq \sum_{j=1}^{q}\int_{B_{3/8}(z')\setminus\Sigma}\sqrt{1+|Du^j|^2}d\mathcal{H}^n + \|V\|(\mathbf{R}\times\Sigma) \\ &\leq \sum_{j=1}^{q}\int_{B_{3/8}(z')}\sqrt{1+|Du^j|^2}d\mathcal{H}^n + C\hat{E}_V^2,\end{aligned}$$

where $C = C(n,q) \in (0,\infty)$, and u^j, $j = 1, 2, \ldots, q$, Σ are as in Theorem 5.1; if, additionally, $\Theta(\|V\|, Z) \geq q$, it follows that

(7.2)

$$\begin{aligned}&\frac{\|V\|(B^{n+1}_{3/8}(Z))}{\omega_n(3/8)^n} - \Theta(\|V\|,Z) \leq \frac{\|V\|(B^{n+1}_{3/8}(Z))}{\omega_n(3/8)^n} - q \\ &\leq \sum_{j=1}^{q}\frac{1}{\omega_n(3/8)^n}\int_{B_{3/8}(z')}\left(\sqrt{1+|Du^j|^2}-1\right)d\mathcal{H}^n + C\hat{E}_V^2 \\ &\leq C\sum_{j=1}^{q}\int_{B_{3/8}(z')}|Du^j|^2\,d\mathcal{H}^n + C\hat{E}_V^2 \\ &\leq C\sum_{j=1}^{q}\int_{B_{3/8}(z')\setminus\Sigma}|Du^j|^2\,d\mathcal{H}^n + C\sum_{j=1}^{q}\int_{B_{3/8}(z')\cap\Sigma}|Du^j|^2\,d\mathcal{H}^n + C\hat{E}_V^2 \\ &\leq C\sum_{j=1}^{q}\int_{B_{3/8}(z')\setminus\Sigma}|Du^j|^2\,d\mathcal{H}^n + C\hat{E}_V^2 \\ &\leq C\int_{\mathbf{R}\times B_{3/8}(z')}|\nabla^V x^1|^2\,d\|V\|(X) + C\hat{E}_V^2 \leq C\hat{E}_V^2,\end{aligned}$$

where $C = C(n,q) \in (0,\infty)$, and in the last inequality we have used (5.2). Thus we deduce from (7.1) that

$$\int_{B^{n+1}_{3/8}(Z)}\frac{|(X-Z)^{\perp}|^2}{|X-Z|^{n+2}}d\|V\|(X) \leq C\hat{E}_V^2 \tag{7.3}$$

for each $Z \in \operatorname{spt}\|V\|\cap(\mathbf{R}\times B_{9/16})$ with $\Theta(\|V\|,Z) \geq q$, where $C = C(n,q) \in (0,\infty)$.

To prove the assertion of part (a) of the theorem, we estimate the left-hand side of (7.1) from below as follows:

(7.4)

$$
\begin{aligned}
&\int_{B^{n+1}_{1/4}(Z)} \frac{|(X-Z)^{\perp}|^2}{|X-Z|^{n+2}} d\|V\|(X) \\
&\quad \geq 4^{n+2} \int_{B^{n+1}_{1/4}(Z)} \left| \sum_{j=2}^{n+1} ((x^j - z^j) e_j^{\perp} + (x^1 - z^1) e_1^{\perp} \right|^2 d\|V\|(X) \\
&\quad \geq \frac{1}{2} 4^{n+2} \int_{B^{n+1}_{1/4}(Z)} |x^1 - z^1|^2 |e_1^{\perp}|^2 d\|V\|(X) - 4^n \int_{B^{n+1}_{1/4}(Z)} \sum_{j=2}^{n+1} |e_j^{\perp}|^2 d\|V\|(X) \\
&\quad = \frac{1}{2} 4^{n+2} \int_{B^{n+1}_{1/4}(Z)} |x^1 - z^1|^2 |e_1^{\perp}|^2 d\|V\|(X) - 4^n \int_{B^{n+1}_{1/4}(Z)} |\nabla^V x^1|^2 d\|V\|(X) \\
&\quad \geq \frac{1}{2} 4^{n+2} \int_{B^{n+1}_{1/4}(Z)} |x^1 - z^1|^2 |e_1^{\perp}|^2 d\|V\|(X) - C\hat{E}_V^2 \\
&\quad \geq 4^{n+1} |z^1|^2 \int_{B^{n+1}_{1/4}(Z)} |e_1^{\perp}|^2 d\|V\|(X) - C\hat{E}_V^2 \\
&\quad \geq 4^{n+1} |z^1|^2 \sum_{j=1}^{q} \int_{B_{1/8}(z') \setminus \Sigma} (1 + |Du^j|^2)^{-1} d\mathcal{H}^n - C\hat{E}_V^2 \geq C|z^1|^2 - C\hat{E}_V^2,
\end{aligned}
$$

where for $\|V\|$, almost every $X \in \operatorname{spt}\|V\|$, $e_j^{\perp}(X)$ is the orthogonal projection of e_j onto the orthogonal complement of the approximate tangent plane $\operatorname{Tan}(\|V\|, X)$ and $C = C(n,q) \in (0,\infty)$. Note that we have used the fact that $|Du^j| \leq 1/2$ almost everywhere and $\mathcal{H}^n(B_{1/8}(z') \setminus \Sigma) \geq \frac{1}{2}\mathcal{H}^n(B_{1/8}(z')) = \frac{1}{2}\omega_n(\frac{1}{8})^n$, which hold by Theorem 5.1 provided $\varepsilon_1 = \varepsilon_1(n,q) \in (0,1/2)$ is sufficiently small. The estimate of (a) readily follows from this and (7.3).

To see (b), let $Z = (z^1, z') \in \operatorname{spt}\|V\| \cap (\mathbf{R} \times B_{9/16})$ be an arbitrary point and choose $\zeta \in C^1_c(\mathbf{R}^{n+1})$ such that $\zeta \equiv 1$ on $B^{n+1}_{3/8}(0)$, $\zeta \equiv 0$ in $\mathbf{R}^{n+1} \setminus B^{n+1}_{1/2}(0)$, $0 \leq \zeta \leq 1$ and $|D\zeta| \leq 16$ everywhere. For $\mu \in (0,1)$, taking

$$\psi(X) = \zeta^2(X-Z)|X-Z|^{-n-2+\mu}|x^1 - z^1|^2(X-Z)$$

in the first variation formula (3.1) (a valid choice as shown by an easy cut-off function argument) and computing and estimating as in [Sim93, p. 616], we deduce that

$$
\begin{aligned}
&\int_{B^{n+1}_{3/8}(Z)} \frac{|x^1 - z^1|^2}{|X-Z|^{n+2-\mu}} d\|V\|(X) \\
&\quad \leq C \int \left(\zeta^2(X-Z) \frac{|(X-Z)^{\perp}|^2}{|X-Z|^{n+2-\mu}} + \frac{|x^1 - z^1|^2}{|X-Z|^{n-\mu}} |\nabla^V \zeta(X-Z)|^2 \right) d\|V\|(X),
\end{aligned}
$$

where $C = C(n,\mu) \in (0,\infty)$. Since $\operatorname{spt} D\zeta \subset B^{n+1}_{1/2}(0) \setminus B^{n+1}_{3/8}(0)$, this together with (7.3) and part (a) implies that

$$\int_{B^{n+1}_{3/8}(Z)} \frac{|x^1 - z^1|^2}{|X - Z|^{n+2-\mu}} d\|V\|(X) \leq C \int_{\mathbf{R}\times B_1} |x^1|^2 d\|V\|(X)$$

for every $Z = (z^1, z') \in \operatorname{spt}\|V\| \cap (\mathbf{R} \times B_{9/16})$ with $\Theta(\|V\|, Z) \geq q$, where $C = C(n, q, \mu) \in (0,\infty)$; in particular,

$$\int_{B^{n+1}_{4\tau}(Z)} |x^1 - z^1|^2 d\|V\|(X) \leq C\tau^{n+2-\mu} \int_{\mathbf{R}\times B_1} |x^1|^2 d\|V\|(X)$$

for each $Z = (z^1, z') \in \operatorname{spt}\|V\| \cap (\mathbf{R} \times B_{9/16})$ with $\Theta(\|V\|, Z) \geq q$ and each $\tau \in (0, 1/16)$. In view of the hypothesis

$$L \cap B_{1/2} \subset (\{Z \in \operatorname{spt}\|V\| : \Theta(\|V\|, Z) \geq q\})_\tau\,,$$

the preceding estimate implies that for each $Y \in L \cap B_{1/2}$, there exists $z^1 \in \mathbf{R}$ such that

$$\int_{B^{n+1}_{2\tau}(Y)} |x^1 - z^1|^2 d\|V\|(X) \leq C\tau^{n+2-\mu} \int_{\mathbf{R}\times B_1} |x^1|^2 d\|V\|(X).$$

This in turn implies by (5.1) (applied with $\eta_{(z^1,0),1\,\#}V$ in place of V and a choice of appropriate test function $\widetilde{\zeta}$) that for each $Y \in L \cap B_{1/2}$,

$$\int_{B^{n+1}_{3\tau/2}(Y)} |\nabla^V x^1|^2 d\|V\|(X) \leq C\tau^{n-\mu} \int_{\mathbf{R}\times B_1} |x^1|^2 d\|V\|(X).$$

Since we may cover the set $(L)_\tau \cap (\mathbf{R} \times B_{1/2})$ by N balls $B^{n+1}_{3\tau/2}(Y_j)$ with $Y_j \in L \cap B_{1/2}$ for $j = 1, 2, \ldots, N$ and with $N \leq C\tau^{1-n}$, $C = C(n)$, it follows that

$$\int_{(L)_\tau \cap (\mathbf{R}\times B_{1/2})} |\nabla^V x^1|^2 d\|V\|(X) \leq C\tau^{1-\mu} \int_{\mathbf{R}\times B_1} |x^1|^2 d\|V\|(X)$$

with $C = C(n, q, \mu) \in (0,\infty)$, as required. □

8. **Properties of coarse blow-ups: Part I**

Recall from Section 4 the defining properties ($\mathcal{B}1$)–($\mathcal{B}7$) of a proper blow-up class $\mathcal{B}$, and note that it follows from the discussion in Section 5 that the class $\mathcal{B} = \mathcal{B}_q$ satisfies properties ($\mathcal{B}1$) and ($\mathcal{B}2$). In this section, we verify that $\mathcal{B}_q$ also satisfies properties ($\mathcal{B}3$)–($\mathcal{B}6$).

Let $v \in \mathcal{B}_q$ be arbitrary. By the definition of $\mathcal{B}_q$, there exists, for each $k = 1, 2, 3, \ldots$, a stationary integral varifold $V_k \in \mathcal{S}_\alpha$ such that the following are true: $(\omega_n 2^n)^{-1}\|V_k\|(B^{n+1}_2(0)) < q + 1/2$; $q - 1/2 \leq \omega_n^{-1}\|V_k\|(\mathbf{R}\times B_1) < q + 1/2$; $\hat{E}_k^2 \equiv \int_{\mathbf{R}\times B_1} |x^1|^2 d\|V_k\|(X) \to 0$ as $k \to \infty$; for each $\sigma \in (0,1)$ and each sufficiently large k depending on σ, if $u^j_k : B_\sigma \to \mathbf{R}$ are the functions corresponding to u^j, $j = 1, 2, \ldots, q$, and $\Sigma_k \subset B_\sigma$ is the measurable set corresponding to Σ

in Theorem 5.1 taken with V_k in place of V, then, $u_k^1 \leq u_k^2 \leq \cdots \leq u_k^q$; u_k^j is Lipschitz with

$$\operatorname{Lip} u_k^j \leq 1/2 \quad \text{for each } j \in \{1, 2, \ldots, q\}; \tag{8.1}$$

$$\operatorname{spt} \|V_k\| \cap (\mathbf{R} \times (B_\sigma \setminus \Sigma_k)) = \textstyle\bigcup_{j=1}^q \operatorname{graph} u_k^j \cap (\mathbf{R} \times (B_\sigma \setminus \Sigma_k));$$

$$\|V_k\|(\mathbf{R} \times \Sigma_k) + \mathcal{H}^n(\Sigma_k) \leq C\hat{E}_k^2, \tag{8.2}$$

where $C = (n, q, \sigma) \in (0, \infty)$; and $\hat{E}_k^{-1} u_k^j \to v^j$ for each $j = 1, 2, \ldots, q$, where the convergence is in $L^2(B_\sigma)$ and weakly in $W^{1,2}(B_\sigma)$.

To verify that v satisfies property $(\mathcal{B}3)$, note that by (3.1), for each k and each function $\zeta \in C_c^1(B_\sigma)$, we have that

$$\int \nabla^{V_k} x^1 \cdot \nabla^{V_k} \widetilde{\zeta} \, d\|V_k\|(X) = 0, \tag{8.3}$$

where $\widetilde{\zeta}$ is any function in $C_c^1(\mathbf{R} \times B_\sigma)$ such that $\widetilde{\zeta} \equiv \zeta_1$ in a neighborhood of $\operatorname{spt} \|V_k\| \cap (\mathbf{R} \times B_\sigma)$, where $\zeta_1(X)$ is defined for $X = (x^1, x') \in \mathbf{R} \times B_\sigma$ by $\zeta_1(x^1, x') = \zeta(x')$. Since $x^1 = \widetilde{u}_k^j(X)$ for $\|V_k\|$ almost every $X = (x^1, x') \in \operatorname{graph} u_k^j \cap \operatorname{spt} \|V_k\|$, where $\widetilde{u}_k^j(x^1, x') = u_k^j(x)$ for $(x^1, x') \in \mathbf{R} \times B_\sigma$, we deduce from (8.3) that

$$\sum_{j=1}^q \int_{B_\sigma} (1 + |Du_k^j|^2)^{-1/2} \, Du_k^j \cdot D\zeta = -\int_{\mathbf{R} \times (B_\sigma \cap \Sigma_k)} \nabla^{V_k} x^1 \cdot \nabla^{V_k} \widetilde{\zeta} \, d\|V_k\|(X) + \sum_{j=1}^q \int_{B_\sigma \cap \Sigma_k} (1 + |Du_k^j|^2)^{-1/2} \, Du_k^j \cdot D\zeta,$$

which can be rewritten as

$$\sum_{j=1}^q \int_{B_\sigma} Du_k^j \cdot D\zeta = -\int_{\mathbf{R} \times (B_\sigma \cap \Sigma_k)} \nabla^{V_k} x^1 \cdot \nabla^{V_k} \widetilde{\zeta} \, d\|V_k\|(X) + \sum_{j=1}^q \int_{B_\sigma \cap \Sigma_k} (1 + |Du_k^j|^2)^{-1/2} \, Du_k^j \cdot D\zeta \; + \; F_k, \tag{8.4}$$

where

(8.5)

$$\begin{aligned} |F_k| &= \left| \sum_{j=1}^q \int_{B_\sigma} (1 + |Du_k^j|^2)^{-1/2} (1 + (1 + |Du_k^j|^2)^{1/2})^{-1} |Du_k^j|^2 Du_k^j \cdot D\zeta \right| \\ &\leq \sup |D\zeta| \int_{\mathbf{R} \times B_\sigma} |\nabla^{V_k} x^1|^2 \, d\|V_k\|(X) \\ &\quad + \sup |D\zeta| \sum_{j=1}^q \int_{B_\sigma \cap \Sigma_k} (1 + |Du_k^j|^2)^{-1/2} (1 + (1 + |Du_k^j|^2)^{1/2})^{-1} |Du_k^j|^3 \\ &\leq \sup |D\zeta| \left(C\hat{E}_k^2 + q\mathcal{H}^n(\Sigma_k) \right). \end{aligned}$$

The last inequality in (8.5), where $C = C(n,\sigma) \in (0,\infty)$, follows from (5.2) and (8.1).

Dividing both sides of (8.4) by $\hat{E}_k$ and letting $k \to \infty$, we deduce, using (8.1), (8.2) and (8.5), that

$$\sum_{j=1}^{q} \int_{B_\sigma} Dv^j \cdot D\zeta = 0$$

for any $\zeta \in C^1_c(B_\sigma)$. Since $\sigma \in (0,1)$ is arbitrary, this implies that $\Delta\, v_a = 0$ in B_1, establishing property ($\mathcal{B}3$) for $\mathcal{B}_q$.

Next we verify that $\mathcal{B}_q$ satisfies properties ($\mathcal{B}5\,\mathrm{I}$), ($\mathcal{B}5\,\mathrm{II}$), ($\mathcal{B}6$) and ($\mathcal{B}5\,\mathrm{III}$), in that order.

Let $z \in B_1$, $\sigma \in (0,(1-|z|)]$ and γ be an orthogonal rotation of $\mathbf{R}^n$, and note that $\widetilde{v}_{z,\sigma} \equiv \|v(z+\sigma(\cdot))\|^{-1}_{L^2(B_1)} v(z+\sigma(\cdot))$ is the coarse blow-up of the sequence $\{\eta_{(0,z),\sigma\,\#}\, V_k\}$, and $v \circ \gamma$ is the coarse blow-up of the sequence $\{\widetilde{\gamma}_{\#}\, V_k\}$, where $\widetilde{\gamma} : \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ is the orthogonal rotation defined by $\widetilde{\gamma}(x^1,x') = (x^1,\gamma(x'))$. Thus $\mathcal{B}_q$ satisfies properties ($\mathcal{B}5\,\mathrm{I}$) and ($\mathcal{B}5\,\mathrm{II}$).

To verify that $\mathcal{B}_q$ satisfies property ($\mathcal{B}6$), let $\{v_\ell\}_{\ell=1}^{\infty}$ be a sequence of elements in $\mathcal{B}_q$, and for each $\ell = 1,2,\ldots$, let $\{V^\ell_k\}_{k=1}^{\infty} \subset \mathcal{S}_\alpha$ be a sequence whose coarse blow-up is v_ℓ. Choose, for each $\ell = 1,2,\ldots$, a positive integer k_ℓ such that $k_1 < k_2 < k_3 < \cdots$, $\hat{E}_{V^\ell_{k_\ell}} < \min\{\ell^{-1}, \varepsilon_0(n,q,1-\ell^{-1})\}$, where ε_0 is as in Theorem 5.1, and $\|\hat{E}^{-1}_{V^\ell_{k_\ell}} u_{\ell,k_\ell} - v_\ell\|_{L^2(B_{1-\ell^{-1}})} < \ell^{-1}$, where $u_{\ell,k_\ell} = (u^1_{\ell,k_\ell}, u^2_{\ell,k_\ell}, \ldots, u^q_{\ell,k_\ell}) : B_{1-\ell^{-1}} \to \mathbf{R}^q$ is the Lipschitz function (with Lipschitz constant of each component function $\leq 1/2$) corresponding to $u = (u^1,u^2,\ldots,u^q)$ of Theorem 5.1 taken with $V^\ell_{k_\ell}$ in place of V and with $\sigma = 1-\ell^{-1}$. That such a choice exists follows from the definition of coarse blow-up. Note also that it follows from (5.8) and (5.9) that for each $\sigma \in (0,1)$ and all sufficiently large ℓ, $\int_{B_\sigma} |v_\ell|^2 + |Dv_\ell|^2 < C$, where $C = C(n,q,\sigma) \in (0,\infty)$ is independent of ℓ. Let $v \in \mathcal{B}_q$ be the coarse blow-up of an appropriate subsequence $\{V^{\ell'}_{k_{\ell'}}\}$ of the sequence $\{V^\ell_{k_\ell}\}$. It is then straightforward to check, after passing to a subsequence of $\{\ell'\}$ without changing notation, that for each $\sigma \in (0,1)$, $v_{\ell'} \to v$ in $L^2(B_\sigma)$ and weakly in $W^{1,2}(B_\sigma)$.

In order to verify that $\mathcal{B}_q$ satisfies property ($\mathcal{B}5\,\mathrm{III}$), note first that if $y \in \mathbf{R}$ is a constant and $v - y \not\equiv 0$ in B_1, then $\|v-y\|^{-1}_{L^2(B_1)}(v-y) \in \mathcal{B}_q$, where we have used the notation $v - y = (v^1-y, v^2-y, \ldots, v^q-y)$. To check this, note that $v(\sigma(\cdot)) - y \not\equiv 0$ for all sufficiently large $\sigma \in (0,1)$ and that for any such σ, $\|v(\sigma(\cdot))-y\|^{-1}_{L^2(B_1)}((v(\sigma(\cdot))-y)$ is the coarse blow-up of the sequence $\{\tau_{k\,\#}\,\eta_{\sigma\,\#}\, V_k\}$, where $\tau_k : \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ is the translation $X \mapsto X - (\hat{E}_k y, 0)$. Thus $\|v(\sigma(\cdot)) - y\|^{-1}_{L^2(B_1)}\,(v(\sigma(\cdot)) - y) \in \mathcal{B}_q$ for all sufficiently large $\sigma \in (0,1)$, and hence it follows from property ($\mathcal{B}6$) that $\|v-y\|^{-1}_{L^2(B_1)}(v-y) \in \mathcal{B}_q$ as claimed.

Next note that if $L : \mathbf{R}^n \to \mathbf{R}$ is a linear function and $v - L \not\equiv 0$ in B_1, then $\|v - L\|^{-1}_{L^2(B_1)}(v - L) \in \mathcal{B}_q$, where, $v - L = (v^1 - L, v^2 - L, \ldots, v^q - L)$. To check this, assume without loss of generality (in view of property ($\mathcal{B}5\, II$)) that $L(x) = \lambda x^2$ for some $\lambda \in \mathbf{R}$, and note that for sufficiently large $\sigma \in (0,1)$, $\|v(\sigma(\cdot)) - \sigma L\|^{-1}_{L^2(B_1)}\,(v(\sigma(\cdot)) - \sigma L)$ is the coarse blow-up of the sequence $\{\Gamma_{k\,\#}\,\eta_{\sigma\,\#}\,V_k\}$, where $\Gamma_k : \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ is the rotation fixing $\{0\} \times \mathbf{R}^{n-1}$ pointwise and mapping the unit normal $\nu_k = \left(1 + \hat{E}_k^2\lambda^2\right)^{-1/2}\left(1, -\hat{E}_k\lambda, 0\right)$ to the hyperplane $P_k \equiv \operatorname{graph} \hat{E}_k L$ to e^1. Thus $\|v(\sigma(\cdot)) - \sigma L\|^{-1}_{L^2(B_1)}\,(v(\sigma(\cdot)) - \sigma L)$ $\in \mathcal{B}_q$ for all sufficiently large $\sigma \in (0,1)$, and it follows from property ($\mathcal{B}6$) that $\|v - L\|_{L^2(B_1)}(v - L) \in \mathcal{B}_q$ as claimed. We deduce that $\mathcal{B}_q$ satisfies property ($\mathcal{B}5\,$III) by applying the above facts with $y = v_a(0)$ and with the linear function L defined by $L(x) = \|v - v_a(0)\|^{-1}_{L^2(B_1)} Dv_a(0) \cdot x$ for $x \in \mathbf{R}^n$. (Note that $v - v_a(0) \not\equiv 0$ in B_1 or else $v - \ell_v \equiv 0$ in B_1, contrary to the hypothesis of ($\mathcal{B}5\,$III), where ℓ_v is as in the statement of ($\mathcal{B}5\,$III).) Note that our argument shows more generally that

$$v \in \mathcal{B}_q,\;\; v - \ell_{v,\,z} \not\equiv 0 \;\text{ in }\; B_1 \quad \Longrightarrow \quad \|v - \ell_{v,\,z}\|^{-1}_{L^2(B_1)}\,(v - \ell_{v,\,z}) \in \mathcal{B}_q \tag{8.6}$$

for each $z \in B_1$, where $\ell_{v,\,z}(x) = v_a(z) + Dv_a(z) \cdot (x - z)$ and $v - \ell_{v,\,z} = (v^1 - \ell_{v,\,z}, \ldots, v^q - \ell_{v,\,z})$.

Finally in this section, we verify that $\mathcal{B}_q$ satisfies property ($\mathcal{B}4$) with a constant $C = C(n,q) \in (0,\infty)$ to be specified momentarily. First note that for any stationary integral n-varifold V on $B_2^{n+1}(0)$ with $\hat{E}_V$ sufficiently small and satisfying the hypotheses of Theorem 5.1 taken with $\sigma = 15/16$ and for any $Z = (z^1, z') \in \operatorname{spt}\|V\| \cap B^{n+1}_{1/8}(0)$ with $\Theta(\|V\|, Z) \geq q$, we have that

$$\sum_{j=1}^{q} \int_{B_{1/2}(z')\setminus\Sigma} \left(\frac{R_z^2}{(u^j - z^1)^2 + R_z^2}\right)^{\frac{n+2}{2}} R_z^{2-n} \cdot \left(\frac{\partial\left((u^j - z^1)/R_z\right)}{\partial\, R_z}\right)^2 d\mathcal{H}^n(x) \leq C_2 \hat{E}_V^2, \tag{8.7}$$

where $R_z(x) = |x - z|$ for $x \in \mathbf{R}^n$ and $C_2 = C_2(n,q) \in (0,\infty)$; the set $\Sigma \subset B_{15/16}$ here and the functions u_j, $j = 1, 2, \ldots, q$ are as in Theorem 5.1 taken with $\sigma = 15/16$. To see this, note that by estimating as in (7.3), it follows that

$$\int_{B^{n+1}_{3/4}(Z)} \frac{|(X - Z)^{\perp}|^2}{|X - Z|^{n+2}} d\|V\|(X) \leq C_2 \hat{E}_V^2,\;\; C_2 = C_2(n,q) \in (0,\infty),$$

while

$$\begin{aligned}
&\int_{B^{n+1}_{3/4}(Z)} \frac{|(X-Z)^{\perp}|^2}{|X-Z|^{n+2}} d\|V\|(X) \\
&\geq \int_{\mathbf{R}\times(B_{1/2}(z')\setminus\Sigma)} \frac{|(X-Z)^{\perp}|^2}{|X-Z|^{n+2}} d\|V\|(X) \\
&\geq \frac{1}{2}\sum_{j=1}^{q}\int_{B_{1/2}(z')\setminus\Sigma} \frac{\left((x'-z')\cdot Du^j(x') - (u^j(x')-z^1)\right)^2}{\left((u^j(x')-z^1)^2+|x'-z'|^2\right)^{\frac{n+2}{2}}} d\mathcal{H}^n(x') \\
&= \frac{1}{2}\sum_{j=1}^{q}\int_{B_{1/2}(z')\setminus\Sigma} \left(\frac{R^2_{z'}}{(u^j-z^1)^2+R^2_{z'}}\right)^{\frac{n+2}{2}} R^{2-n}_{z'} \left(\frac{\partial\left((u^j-z^1)/R_{z'}\right)}{\partial R_{z'}}\right)^2 d\mathcal{H}^n(x').
\end{aligned}$$

Now let $v \in \mathcal{B}_q$, and let $z \in B_1$ be such that ($\mathcal{B}4\,\mathrm{I}$) with $C = C_2$, where $C_2 = C_2(n,q)$ is as in (8.7), fails. By (8.6), $\widetilde{v} \equiv \|v-\ell_{v,z}\|^{-1}_{L^2(B_1)}(v-\ell_{v,z}) \in \mathcal{B}_q$. Let $V_k \in \mathcal{S}_\alpha$ be such that $\widetilde{v}$ is the coarse blow-up of $\{V_k\}$. We claim that then there exists $\sigma_1 > 0$ such that for all sufficiently large k,

$$Z \in \operatorname{spt}\|V_k\| \cap (\mathbf{R}\times B_{\sigma_1}(z)) \implies \Theta(\|V_k\|, Z) < q. \tag{8.8}$$

If not, then there would exist, for each positive integer ℓ, a positive integer $\{k_\ell\}$ with $k_1 < k_2 < k_3 < \cdots$ and a point $Z_\ell = (z^1_\ell, z'_\ell) \in \operatorname{spt}\|V_{k_\ell}\| \cap (\mathbf{R}\times B_{1/\ell}(z))$ such that $\Theta(\|V_{k_\ell}\|, Z_\ell) \geq q$. Fix any $\rho \in (0, \frac{3}{8}(1-|z|)]$. Applying (8.7) with $\eta_{Z_\ell,\rho\,\#}V_{k_\ell}$ in place of V and 0 in place of Z, we then have, after changing variables, that for all sufficiently large ℓ,

$$\begin{aligned}
&\sum_{j=1}^{q}\int_{B_{\rho/2}(z'_\ell)\setminus\Sigma_{k_\ell}} \left(\frac{R^2_{z'_\ell}}{(u^j_{k_\ell}-z^1_\ell)^2+R^2_{z'_\ell}}\right)^{\frac{n+2}{2}} R^{2-n}_{z'_\ell}\left(\frac{\partial\left((u^j_{k_\ell}-z^1_\ell)/R_{z'_\ell}\right)}{\partial R_{z'_\ell}}\right)^2 d\mathcal{H}^n(x) \\
&\qquad \leq C_2\,\rho^{-n-2}\int_{\mathbf{R}\times B_\rho(z'_\ell)} |x^1|^2 d\|V_{k_\ell}\|(X).
\end{aligned} \tag{8.9}$$

Now for all sufficiently large ℓ depending on ρ, $\|V_{k_\ell}\|(\mathbf{R}\times B_{\rho/16}(z'_\ell)) \geq C\rho^n$ for a suitable constant $C = C(n) \in (0,\infty)$, so there exists a point $Y_\ell = (y^1_\ell, y'_\ell) \in \operatorname{spt}\|V_{k_\ell}\| \cap (\mathbf{R}\times(B_{\rho/16}(z'_\ell))$ such that

$$|y^1_\ell|^2 \leq C\rho^{-n}\int_{\mathbf{R}\times B_{\rho/16}(z'_\ell)} |x^1|^2\, d\|V_{k_\ell}\|(X), \tag{8.10}$$

where $C = C(n) \in (0,\infty)$. Applying Theorem 7.1(a) with $\widetilde{V} = \eta_{Y_\ell,\rho/2\,\#}V_{k_\ell}$ in place of V and $\widetilde{Z} = (\rho/2)^{-1}(Z_\ell - Y_\ell)$ in place of Z (noting that $\widetilde{Z} \in \operatorname{spt}\|\widetilde{V}\| \cap (\mathbf{R}\times B_{1/8})$ with $\Theta(\|\widetilde{V}\|, \widetilde{Z}) \geq q$), we deduce, using also (8.10), that

$$|z^1_\ell|^2 \leq C\rho^{-n}\int_{\mathbf{R}\times B_{3\rho/4}(z'_\ell)} |x^1|^2 d\|V_{k_\ell}\|(X) \tag{8.11}$$

for all sufficiently large ℓ, where $C = C(n,q) \in (0,\infty)$. Dividing both sides of (8.9) by $\hat{E}^2_{k_\ell}$, and letting $\ell \to \infty$, we conclude, using (8.11) and the fact that $\sup_{X \in \mathrm{spt}\,\|V_{k_\ell}\| \cap (\mathbf{R} \times B_{3/4}))} |x^1| \to 0$ as $\ell \to \infty$, that

$$\sum_{j=1}^{q} \int_{B_{\rho/2}(z)} R_z^{2-n} \left(\frac{\partial\, ((\widetilde{v}^j - y)/R_z)}{\partial\, R_z} \right)^2 \leq C_2\, \rho^{-n-2} \int_{B_\rho(z)} |\widetilde{v}|^2 \tag{8.12}$$

for some $y \in \mathbf{R}$ and each $\rho \in (0, \frac{3}{8}(1-|z|)]$. (Note that in justifying the above, we have also used the fact that

$$\int_{B_{\rho/2}(z)} \frac{(D\widetilde{v}^j \cdot (x-z))^2}{|x-z|^{n+2}} \leq \liminf_{\ell \to \infty} \int_{B_{\rho/2}(z'_{k_\ell}) \setminus \Sigma_{k_\ell}} \left(\frac{R^2_{z'_\ell}}{(u^j_{k_\ell} - z^1_\ell)^2 + R^2_{z'_\ell}} \right)^{\frac{n+2}{2}} \frac{(Dv^j_{k_\ell} \cdot (x-z))^2}{|x-z|^{n+2}}$$

for each $j = 1, 2, \ldots, q$. To see this, note that $Dv^j_{k_\ell} \to D\widetilde{v}^j$ locally weakly in L^2, which implies that $g_\ell Dv^j_{k_\ell} \to D\widetilde{v}^j$ weakly in $L^2(B_{\rho/2}(z))$ for any sequence of bounded measurable functions g_ℓ with $g_\ell \to 1$ almost everywhere on $B_{\rho/2}(z)$; thus for any $\tau \in (0, \rho/4)$,

$$\int_{B_{\rho/2}(z) \setminus B_\tau(z)} \frac{(D\widetilde{v}^j \cdot (x-z))^2}{|x-z|^{n+2}} = \lim_{\ell \to \infty} \int_{B_{\rho/2}(z) \setminus B_\tau(z)} g_\ell \frac{Dv^j_{k_\ell} \cdot (x-z)(D\widetilde{v}^j \cdot (x-z))}{|x-z|^{n+2}};$$

taking $g_\ell = \left(\frac{R^2_{z'_\ell}}{(u^j_{k_\ell} - z^1_\ell)^2 + R^2_{z'_\ell}} \right)^{\frac{n+2}{4}} \sqrt{\chi_{G_\ell}}$ where $G_\ell = B_{\rho/2}(z'_\ell) \setminus \Sigma_{k_\ell}$ and using Cauchy–Schwarz inequality and letting $\tau \to 0$, we deduce the desired inequality from this.) Since by the triangle inequality (8.12) implies that $\int_{B_{\rho/2}(z)} R_z^{2-n} \left(\frac{\partial\, ((\widetilde{v}_a - y)/R_z)}{\partial\, R_z} \right)^2 < \infty$, it follows that $y = \widetilde{v}_a(z) = 0$. But this contradicts our assumption that property ($\mathcal{B}4$ I) fails for v, leading us to the conclusion that (8.8) must hold for all sufficiently large k.

By Remark 3 of Section 6 and (8.8), it follows that for all sufficiently large k, $\mathcal{H}^{n-7+\gamma}(\mathrm{sing}\, V_k \cap (\mathbf{R} \times B_{\sigma_1}(z))) = 0$ for every $\gamma > 0$ if $n \geq 7$ and $\mathrm{sing}\, V_k \cap (\mathbf{R} \times B_{\sigma_1}(z)) = \emptyset$ if $2 \leq n \leq 6$, so we may apply Theorem 3.5 and standard elliptic theory to conclude that

$$V_k \llcorner (\mathbf{R} \times B_{\sigma_1/2}(z)) = \sum_{j=1}^{q} |\mathrm{graph}\, u^j_k|,$$

where $u_k^j : B_{\sigma_1/2}(z) \to \mathbf{R}$ are C^2 functions satisfying

$$\sup_{B_{\sigma_1/2}(z)} \sum_{j=1}^q |Du_k^j| + |D^2 u_k^j| \leq C\hat{E}_k$$

and solving the minimal surface equation on $B_{\sigma_1/2}(z)$, where $C = C(n, q, \sigma) \in (0, \infty)$. This readily shows that $\Delta \widetilde{v}^j = 0$ on $B_{\sigma_1/2}(z)$ for each $j = 1, 2, \ldots, q$, establishing property $(\mathcal{B}4)$ for $\mathcal{B}_q$.

Remarks. (1) The argument leading to (8.12) proves the following:

Let Ω be an open subset of $B_{3/4}$. If $v \in \mathcal{B}_q$ and $\{V_k\} \subset \mathcal{S}_\alpha$ is a sequence whose coarse blow-up is v (in the sense described in Section 5) and if for infinitely many k, there are points $Z_k \in \operatorname{spt} \|V_k\| \cap (\mathbf{R} \times \Omega)$ with $\Theta(\|V_k\|, Z_k) \geq q$, then there exists a point $z \in \overline{\Omega}$ such that

$$\sum_{j=1}^q \int_{B_{\rho/2}(z)} R_z^{2-n} \left(\frac{\partial \left((v^j - v_a(z))/R_z \right)}{\partial R_z} \right)^2 \leq C_2 \, \rho^{-n-2} \int_{B_\rho(z)} |v|^2$$

for each $\rho \in (0, \frac{3}{8}(1 - |z|)]$.

(2) *Let q be an integer ≥ 2. There exist constants $\eta' = \eta'(n, q, \alpha) \in (0, 1)$ and $\delta' = \delta'(n, q, \alpha) \in (0, 1)$ such that the following is true: If the induction hypotheses* (H1), (H2) *hold, $V \in \mathcal{S}_\alpha$, $(\omega_n 2^n)^{-1} \|V\|(B_2^{n+1}(0)) < q + 1/2$, $\omega_n^{-1} \|V\|(\mathbf{R} \times B_1) < q + 1/2$, $\int_{\mathbf{R} \times B_1} \operatorname{dist}^2(X, \mathbf{P}) \, d\|V\|(X) < \delta'$ for some union $\mathbf{P} \subset \mathbf{R}^{n+1}$ of finitely many (distinct) affine hyperplanes disjoint in $\mathbf{R} \times B_1$ with $\operatorname{dist}_{\mathcal{H}}(\mathbf{P} \cap (\mathbf{R} \times B_1), \{0\} \times B_1) < \delta'$ and, writing $\mathcal{A}$ for the set of affine hyperplanes of $\mathbf{R}^{n+1}$, if*

$$\int_{\mathbf{R} \times B_1} \operatorname{dist}^2(X, \mathbf{P}) \, d\|V\|(X) < \eta' \inf_{L \in \mathcal{A}} \int_{\mathbf{R} \times B_1} \operatorname{dist}^2(X, L) \, d\|V\|(X),$$

then

(a) *$\mathbf{P}$ consists of at least two affine hyperplanes;*
(b) *$\{Z \in \operatorname{spt} \|V\| \cap (\mathbf{R} \times B_{3/4}) : \Theta(\|V\|, Z) \geq q\} = \emptyset$;*
(c) *there exist an integer p with $2 \leq p \leq q$, positive integers $a_j \leq q - 1$, affine hyperplanes $P_j^i \subset \mathbf{P}$, C^2 functions $u_j^i : P_j^1 \cap (\mathbf{R} \times B_{3/4}) \to (P_j^1)^\perp$ with $u_j^1 \cdot e_1 \leq \cdots \leq u_j^{a_j} \cdot e_1$ for $1 \leq j \leq p$, $1 \leq i \leq a_j$ and $u_{j-1}^{a_{j-1}} \cdot e_1 < u_j^1 \cdot e_1$ for $2 \leq j \leq p$ such that $\|u_j^i\|^2_{C^2(P_j^1 \cap (\mathbf{R} \times B_{3/4}))} < C \int_{\mathbf{R} \times B_1} \operatorname{dist}^2(X, \mathbf{P}) \, d\|V\|(X)$, $V \llcorner (\mathbf{R} \times B_{5/8}) = \sum_{j=1}^p V_j$, where $V_j = \sum_{i=1}^{a_j} |\operatorname{graph} u_j^i \cap (\mathbf{R} \times B_{5/8})|$, and*

$$\int_{\mathbf{R} \times B_{5/8}} \operatorname{dist}^2(X, \mathbf{P}) \, d\|V\|(X) = \sum_{j=1}^p \int_{\mathbf{R} \times B_{5/8}} \operatorname{dist}^2(X, \mathbf{P}_j) \, d\|V_j\|(X),$$

where $\mathbf{P}_j = \bigcup_{i=1}^{a_j} P_j^i$. Here $\operatorname{graph} u_j^i = \{X + u_j^i(X) : X \in P_j \cap (\mathbf{R} \times B_{3/4})\}$.

To see this, argue by contradiction: Were the assertion false, we can find a sequence $V_k \in \mathcal{S}_\alpha$ with $(\omega_n 2^n)^{-1}\|V_k\|(B_2^{n+1}(0)) < q+1/2$, $\omega_n^{-1}\|V_k\|(\mathbf{R}\times B_1) < q+1/2$ and for each k, affine hyperplanes $P_k^1, \ldots, P_k^{n_k}$ with $P_k^i \cap P_k^j \cap (\mathbf{R}\times B_1) = \emptyset$ for $1 \leq i < j \leq n_k$ and $\mathrm{dist}_{\mathcal{H}}(\mathbf{P}_k \cap (\mathbf{R}\times B_1), \{0\}\times B_1) \to 0$ as $k \to \infty$ where $\mathbf{P}_k = \bigcup_{j=1}^{n_k} P_k^j$, such that $\int_{\mathbf{R}\times B_1} \mathrm{dist}^2(X, \mathbf{P}_k)\, d\|V_k\|(X) \to 0$ and

(8.13)

$$\left(\inf_{L\in\mathcal{A}} \int_{\mathbf{R}\times B_1} \mathrm{dist}^2(X, L)\, d\|V_k\|(X)\right)^{-1} \int_{\mathbf{R}\times B_1} \mathrm{dist}^2(X, \mathbf{P}_k)\, d\|V_k\|(X) \to 0$$

and yet, at least one of the conclusions (a)–(c) with V_k in place of V and $\mathbf{P}_k$ in place of $\mathbf{P}$ fails. Note that $\inf_{L\in\mathcal{A}} \int_{\mathbf{R}\times B_1} \mathrm{dist}^2(X, L)\, d\|V_k\|(X) \to 0$, and choose $L_k \in \mathcal{A}$ such that

$$\int_{\mathbf{R}\times B_1} \mathrm{dist}^2(X, L_k)\, d\|V_k\|(X) < \frac{3}{2} \inf_{L\in\mathcal{A}} \int_{\mathbf{R}\times B_1} \mathrm{dist}^2(X, L)\, d\|V_k\|(X).$$

Noting then that $L_k \to \{0\}\times \mathbf{R}^n$, choose rigid motions $\Gamma_k : \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ such that $\Gamma_k \to$ Identity and $\Gamma_k(L_k) = \{0\}\times\mathbf{R}^n$, and let $v = (v^1, \ldots, v^\ell) \in W^{1,2}_{\mathrm{loc}}(B_1; \mathbf{R}^p) \cap L^2(B_1; \mathbf{R}^p)$, with $v^1 \leq v^2 \cdots \leq v^\ell$, be the coarse blow-up, as described in Section 5, of (a suitable subsequence of) the sequence $\{\widetilde{V}_k = \eta_{0,13/16\,\#}\, \Gamma_{k\,\#}\, V_k\}$ relative to $\{0\}\times\mathbf{R}^n$, where ℓ is a positive integer $\leq q$. Let $p \leq \ell$ be the number of distinct functions in the set $\{v^1, \ldots, v^\ell\}$, denoted $\widetilde{v}^1, \ldots, \widetilde{v}^p$ with the labelling so chosen that $\widetilde{v}^1 \leq \cdots \leq \widetilde{v}^p$. Then by (8.13), for each k, there exists $\{\widetilde{P}_k^1, \widetilde{P}_k^2, \ldots, \widetilde{P}_k^p\} \subset \{P_k^1, P_k^2, \ldots, P_k^{n_k}\}$ such that, writing $\Gamma_k \widetilde{P}_k^i = \mathrm{graph}\, \widetilde{p}_k^i$ for an affine function $\widetilde{p}_k^i : \mathbf{R}^n \to \mathbf{R}$ with labelling so chosen that $\widetilde{p}_k^1 < \cdots < \widetilde{p}_k^p$ in $\mathbf{R}\times B_1$, we have that $\widetilde{v}^j = \lim_{k\to\infty} \left(\hat{E}_k\right)^{-1} \widetilde{p}_k^j$ for $1 \leq j \leq p$. Thus each v^j is affine, and by (8.13) again, $p \geq 2$ and $\widetilde{v}^p > \widetilde{v}^1$ in B_1. It then follows from Remark (1) above (taken with ℓ in place of q) that $\{Z \in \mathrm{spt}\,\|V_k\| \cap (\mathbf{R}\times B_{3/4}) : \Theta(\|V_k\|, Z) \geq \ell\} = \emptyset$ for sufficiently large k. The rest of the conclusions with V_k in place of V and $\mathbf{P}_k$ in place of $\mathbf{P}$ now follow, for all sufficiently large k, from Remark 3 of Section 6, Theorem 3.5 and standard elliptic estimates, contrary to the assumption that at least one of those conclusions must fail for each k.

(3) *Let q be an integer ≥ 2. There exists a constant $\delta = \delta(n, q, \alpha) \in (0,1)$ such that the following is true*: *If the induction hypotheses* (H1), (H2) *hold*, $V \in \mathcal{S}_\alpha$, $(\omega_n 2^n)^{-1}\|V\|(B_2^{n+1}(0)) < q + 1/2$, $\omega_n^{-1}\|V\|(\mathbf{R}\times B_1) < q+1/2$ *and*

$$\int_{\mathbf{R}\times B_1} \mathrm{dist}^2(X, \mathbf{P})\, d\|V\|(X) < \delta$$

for some union $\mathbf{P} \subset \mathbf{R}^{n+1}$ of at most q affine hyperplanes disjoint in $\mathbf{R}\times B_1$ with $\mathrm{dist}_{\mathcal{H}}(\mathbf{P}\cap(\mathbf{R}\times B_1), \{0\}\times B_1) < \delta$, *then either*

(a) $\{Z \in \mathrm{spt}\,\|V\| \cap (\mathbf{R}\times B_{7/8}) : \Theta(\|V\|, Z) \geq q\} = \emptyset$ *and there exist a positive integer ℓ with $1 \leq \ell \leq q$, distinct affine hyperplanes $P_1, P_2, \ldots, P_\ell \subset \mathbf{P}$,* *positive integers* $q_1, q_2, \ldots, q_\ell$ *with* $\sum_{k=1}^{\ell} q_k \leq q$ *and* C^2 *functions* $u_k^j : P_k \cap (\mathbf{R} \times B_{3/4}) \to P_k^\perp$ *with*

$$\sup_{P_k \cap (\mathbf{R} \times B_{3/4})} |u_k^j|^2 + |Du_k^j|^2 \leq C \int_{\mathbf{R} \times B_1} \operatorname{dist}^2(X, \mathbf{P}) \, d\|V\|(X)$$

for $1 \leq k \leq \ell$, $1 \leq j \leq q_k$ *where* $C = C(n)$, *such that*

$$V \llcorner (\mathbf{R} \times B_{1/2}) = \sum_{k=1}^{\ell} \sum_{j=1}^{q_k} |\operatorname{graph} u_k^j \cap (\mathbf{R} \times B_{1/2})|;$$

or

(b) $\{Z \in \operatorname{spt} \|V\| \cap (\mathbf{R} \times B_{7/8}) : \Theta(\|V\|, Z) \geq q\} \neq \emptyset$, $\omega_n^{-1} \|V\|(\mathbf{R} \times B_1) \geq q - 1/2$, *and there exist an affine hyperplane* $P \subset \mathbf{P}$, *a measurable subset* $\Sigma \subset P \cap (\mathbf{R} \times B_{13/28})$ *Lipschitz functions* $u_1, u_2, \ldots, u_q : P \cap (\mathbf{R} \times B_{13/28}) \to P^\perp$ *with* $\operatorname{Lip}(u_j) \leq 9/16$ *for each* $j \in \{1, 2, \ldots, q\}$ *such that*

$$\mathcal{H}^n(\Sigma) + \|V\|(\mathcal{C}_P(\Sigma)) + \sum_{j=1}^{q} \int_{P \cap (\mathbf{R} \times B_{13/28}) \setminus \Sigma} |u_j|^2 + |Du_j|^2 \leq C \int_{\mathbf{R} \times B_1} \operatorname{dist}^2(X, \mathbf{P}) \, d\|V\|(X)$$

and

$$V \llcorner ((\mathbf{R} \times B_{13/28}) \setminus \mathcal{C}_P(\Sigma)) = \sum_{j=1}^{q} |\operatorname{graph} u_j \cap ((\mathbf{R} \times B_{13/28}) \setminus \mathcal{C}_P(\Sigma))|,$$

where $\mathcal{C}_P(\Sigma) = \{X \in \mathbf{R}^{n+1} : \pi_P(X) \in \Sigma\}$ *with* π_P *denoting the orthogonal projection of* $\mathbf{R}^{n+1}$ *onto* P; *furthermore, in this case we have that for each* $j \in \{1, 2, \ldots, q\}$,

$$\sup_{B_{13/28}} |u_j| \leq C \delta^{1/2n},$$

where $C = C(n) \in (0, \infty)$.

To see this, let $\eta' = \eta'(n, q, \alpha) \in (0, 1)$ and $\delta' = \delta'(n, q, \alpha) \in (0, 1)$ be the constants as in Remark (2) above. Let $\varepsilon_0 = \varepsilon_0(n, q, \alpha, 3/4) \in (0, 1)$ be the constant as in Theorem 5.1. Let the hypotheses of the assertion of Remark (3) be satisfied for sufficiently small $\delta \in (0, \eta' \delta' \varepsilon_0]$, and note that it follows from the Constancy Theorem ([Sim83, Th. 41.1]) that if $\delta = \delta(n, q, \alpha) \in (0, 1)$ is sufficiently small, then there exists an integer m with $1 \leq m \leq q$ such that $\omega_n^{-1} \|V\|(B_1^{n+1}(0)) < m + 1/2$ and $m - 1/2 \leq \omega_n^{-1} 2^n \|V\|(\mathbf{R} \times B_{1/2}) < m + 1/2$. Consider the two alternatives:

(A) $\int_{\mathbf{R} \times B_1} \operatorname{dist}^2(X, \mathbf{P}) \, d\|V\|(X) < \eta' \inf_{L \in \mathcal{A}} \int_{\mathbf{R} \times B_1} \operatorname{dist}^2(X, L) \, d\|V\|(X)$.

(B) $\int_{\mathbf{R} \times B_1} \operatorname{dist}^2(X, \mathbf{P}) \, d\|V\|(X) \geq \eta' \inf_{L \in \mathcal{A}} \int_{\mathbf{R} \times B_1} \operatorname{dist}^2(X, L) \, d\|V\|(X)$.

In case of alternative (B), choose $\widetilde{L} \in \mathcal{A}$ such that

$$\int_{\mathbf{R}\times B_1} \mathrm{dist}^2(X, \widetilde{L})\, d\|V\|(X) < \frac{3}{2} \inf_{L\in\mathcal{A}} \int_{\mathbf{R}\times B_1} \mathrm{dist}^2(X, L)\, d\|V\|(X)$$

and note, by Theorem 5.1, that if $\delta = \delta(n, q, \alpha) \in (0, 1)$ is sufficiently small, then $\mathrm{dist}^2_{\mathcal{H}}(\widetilde{L} \cap (\mathbf{R} \times B_1), P \cap (\mathbf{R} \times B_1)) \leq C \int_{\mathbf{R}\times B_1} \mathrm{dist}^2(X, \mathbf{P})\, d\|V\|(X)$ for some affine hyperplane $P \subset \mathbf{P}$, where $C = C(n) \in (0, \infty)$. Now if $m = q$ and $\{Z \in \mathrm{spt}\, \|V\| \cap (\mathbf{R} \times B_{3/4}) : \Theta(\|V\|, Z) \geq q\} \neq \emptyset$ (in case (B)), the assertion with conclusion (b) follows, for sufficiently small $\delta = \delta(n, q, \alpha) \in (0, 1)$, by applying Theorem 5.1 (with $\eta_{1/2\,\#} V$ in place of V) and using the estimate (5.2) as well as the estimate of the remark following Theorem 5.1, whereas if $m = q$ and $\{Z \in \mathrm{spt}\, \|V\| \cap (\mathbf{R} \times B_{3/4}) : \Theta(\|V\|, Z) \geq q\} = \emptyset$, the assertion with conclusion (a) with $\ell = 1$ and $q_1 = q$ follows from Remark 3 of Section 6, Theorem 3.5 and standard elliptic estimates; if $m \leq q - 1$, hypothesis (H1) implies that conclusion (a) holds.

In case of alternative (A), we argue by induction on q to see that the assertion with conclusion (a) holds: If $q = 2$, the desired conclusion follows directly from Remark (2)(c) above. For general q, let V_j, $\mathbf{P}_j$, a_j be as in Remark 2(c) and note that $a_j \leq q - 1$. For each fixed j, consider the same two alternatives (A) and (B) as above but with V_j, $\mathbf{P}_j$ in place of V, $\mathbf{P}$. In case alternative (B) holds (with V_j, $\mathbf{P}_j$ in place of V, $\mathbf{P}$), we see by elliptic estimates that conclusion (a) (with V_j in place of V and $\ell = 1$) must hold, whereas in case of alternative (A), we may assume by induction the validity of conclusion (a) (with V_j in place of V and suitable ℓ_j in place of ℓ).

9. **Properties of coarse blow-ups: Part II**

Fix an integer $q \geq 2$, and suppose that the induction hypotheses (H1) and (H2) hold. In this section we begin the proof that the coarse blow-up class $\mathcal{B}_q$ satisfies property ($\mathcal{B}7$); we shall complete the proof in Section 14.

Suppose

(†) $v_\star = (v^1_\star, v^2_\star, \ldots, v^q_\star) \in \mathcal{B}_q$ is such that for each $j = 1, 2, \ldots, q$, there exist two linear functions $L^j_1, L^j_2 : \mathbf{R}^n \to \mathbf{R}$ with $L^j_1(0, y) = L^j_2(0, y) = 0$ for each $y \in \mathbf{R}^{n-1}$, $v^j_\star(x^2, y) = L^j_1(x^2, y)$ if $x^2 < 0$ and $v^j_\star(x^2, y) = L^j_2(x^2, y)$ if $x^2 \geq 0$.

In order to show that $\mathcal{B}_q$ satisfies property ($\mathcal{B}7$), we need to prove that $v^1_\star = v^2_\star = \cdots = v^q_\star = L$ for some linear function $L : \mathbf{R}^n \to \mathbf{R}$. We shall do this by establishing the assertions in each of the following two cases:

Case 1: There exists no $v_\star \in \mathcal{B}_q$ as in (†) above such that $L^1_1 = L^2_1 = \cdots = L^q_1$ but $L^j_2 \neq L^{j+1}_2$ for some $j \in \{1, 2, \ldots, q - 1\}$.

Case 2: There exists no $v_\star \in \mathcal{B}_q$ as in (†) above such that $L_1^i \neq L_1^{i+1}$ for some $i \in \{1, 2, \ldots, q-1\}$ and $L_2^j \neq L_2^{j+1}$ for some $j \in \{1, 2, \ldots, q-1\}$.

We prove the assertion of Case 1 in Lemma 9.1 below and complete the proof that $\mathcal{B}_q$ satisfies property ($\mathcal{B}7$) (by proving the assertion of Case 2) in Corollary 14.2; the latter requires a number of preliminary results that we shall establish in Sections 10–14.

LEMMA 9.1. *Let $v_\star$ and L_i^j, $i \in \{1, 2\}$, $j \in \{1, 2, \ldots, q\}$, be as in* (†) *above. If $L_1^1 = L_1^2 = \cdots = L_1^q$, then* (i) *$L_2^1 = L_2^2 = \cdots = L_2^q$ and* (ii) *$v_\star^j = L$ for some linear function L and all $j = 1, 2, \ldots, q$.*

Proof. The assertion of (ii) follows from that of (i) since the average $(v_\star)_a = q^{-1}\sum_{j=1}^q v_\star^j$ is harmonic and hence is a linear function under the hypotheses of the lemma.

Suppose, contrary to the assertion of (i), that $L_2^j \neq L_2^{j+1}$ for some $j \in \{1, 2, \ldots, q-1\}$. By property ($\mathcal{B}5\,\mathrm{III}$), $\frac{v_\star - (v_\star)_a}{\|v_\star - (v_\star)_a\|} \in \mathcal{B}_q$, so we may assume without loss of generality that $L_1^j = 0$ for each $j = 1, 2, \ldots, q$. For $k = 1, 2, \ldots$, let $V_k \in \mathcal{S}_\alpha$ with $(\omega_n 2^n)^{-1}\|V_k\|(B_2^{n+1}(0)) < q+1/2$, $q-1/2 \leq \omega_n^{-1}\|V_k\|(\mathbf{R}\times B_1) < q+1/2$ and $\hat{E}_k^2 = \int_{\mathbf{R}\times B_1} |x^1|^2 d\|V_k\|(X) \to 0$ be such that the coarse blow-up of the sequence V_k, obtained as described in Section 5, is $v_\star$. Let the notation be as in Section 5. Thus for each $\sigma \in (0, 1)$ and each sufficiently large k (depending on σ), there exist Lipschitz functions $u_k^j : B_\sigma \to \mathbf{R}$, $j = 1, 2, \ldots, q$, with $\mathrm{Lip}\, u_k^j \leq 1/2$ for each $j \in \{1, 2, \ldots, q\}$, such that

$$v_\star^j = \lim_{k\to\infty} \hat{E}_k^{-1} u_k^j,$$

where the convergence is in $L^2(B_\sigma)$ and weakly in $W^{1,2}(B_\sigma)$, and

$$\operatorname{spt}\|V_k\| \cap \pi^{-1}(B_\sigma \setminus \Sigma_k) = \textstyle\bigcup_{j=1}^q \operatorname{graph} u_k^j \cap \pi^{-1}(B_\sigma \setminus \Sigma_k), \tag{9.1}$$

where $\Sigma_k \subset B_\sigma$ is the set corresponding to Σ in Theorem 5.1 when V is replaced with V_k so that, in particular,

$$\|V_k\|(\mathbf{R}\times\Sigma_k) + \mathcal{H}^n(\Sigma_k) \leq C\hat{E}_k^2, \tag{9.2}$$

where $C = C(n, q, \sigma) \in (0, \infty)$.

In what follows, we take $\sigma \in [15/16, 1)$ to be fixed. Fix any $\tau \in (0, 1/16)$. Since

$$\begin{aligned}\int_{(\mathbf{R}\times B_{9/16})\cap\{x^2\leq-\tau/2\}} &|x^1|^2 d\|V_k\|(X)\\ &= \sum_{j=1}^q \int_{(B_{9/16}\setminus\Sigma_k)\cap\{x^2\leq-\tau/2\}} \sqrt{1+|Du_k^j|^2}\,|u_k^j|^2 d\mathcal{H}^n\\ &\quad+ \int_{(\mathbf{R}\times(B_{9/16}\cap\Sigma_k))\cap\{x^2\leq-\tau/2\}} |x^1|^2 d\|V_k\|(X),\end{aligned}$$

$\hat{E}_k^{-1} u_k \to 0$ in L^2 on $B_{9/16} \cap \{x^2 \leq -\tau/2\}$ and

$$\sup_{X=(x^1,x')\in \operatorname{spt}\|V_k\|\cap(\mathbf{R}\times B_{9/16})} |x^1| \to 0,$$

it follows from (9.2) that

$$\hat{E}_k^{-2} \int_{(\mathbf{R}\times B_{9/16})\cap\{x^2\leq -\tau/2\}} |x^1|^2 d\|V_k\|(X) \to 0$$

and consequently, by (5.2), that

$$\hat{E}_k^{-2} \int_{(\mathbf{R}\times B_{1/2})\cap\{x^2\leq -\tau\}} |\nabla^{V_k} x^1|^2 d\|V_k\|(X) \to 0. \tag{9.3}$$

We claim that for all sufficiently large k,

$$\Theta(\|V_k\|, Z) < q \ \text{ for all } \ Z \in \operatorname{spt}\|V_k\| \cap (\mathbf{R}\times B_{5/8}) \cap \{x^2 > \tau/8\}. \tag{9.4}$$

If this were false, then there would exist a subsequence $\{k'\}$ of $\{k\}$ and for each k', a point $Z_{k'} = (z^1_{k'}, z'_{k'}) \in \operatorname{spt}\|V_{k'}\| \cap (\mathbf{R}\times B_{5/8}) \cap \{x^2 > \tau/8\}$ with $\Theta(\|V_{k'}\|, Z_{k'}) \geq q$; by the reasoning as in the remark at the end of Section 8, this fact yields

$$\sum_{j=1}^q \int_{B_{1/4}(z')} R_{z'}^{2-n} \left(\frac{\partial\left((v^j_\star - y)/R_{z'}\right)}{\partial\, R_{z'}}\right)^2 d\mathcal{H}^n \leq C$$

for some $z' \in \overline{B}_{5/8} \cap \{x^2 \geq \tau/8\}$ and some $y \in \mathbf{R}$, which implies that $v^j_\star(z') = y$ for all $j = 1, 2, \ldots, q$. But this contradicts our hypothesis that $L_2^j \neq L_2^{j+1}$ for some $j \in \{1, 2, \ldots, q-1\}$, so (9.4) must hold for all sufficiently large k.

With the help of Remark 3 of Section 6, we deduce from (9.4) that for all sufficiently large k, $\mathcal{H}^{n-7+\gamma}(\operatorname{sing} V_k \cap (\mathbf{R}\times B_{5/8}) \cap \{x^2 > \tau/8\}) = 0$ for each $\gamma > 0$ if $n \geq 7$ and $\operatorname{sing} V_k \cap (\mathbf{R}\times B_{5/8}) \cap \{x^2 > \tau/8\} = \emptyset$ if $2 \leq n \leq 6$. We may therefore apply Theorem 3.5 and elliptic theory to deduce that, for all sufficiently large k, $\Sigma_k \cap B_{9/16} \cap \{x^2 > \tau/4\} = \emptyset$;

$$V_k \llcorner ((\mathbf{R}\times B_{9/16}) \cap \{x^2 > \tau/4\}) = \sum_{j=1}^q |\operatorname{graph} u_k^j| \llcorner ((\mathbf{R}\times B_{9/16}) \cap \{x^2 > \tau/4\}); \tag{9.5}$$

and that u_k^j are C^2 on $B_{9/16} \cap \{x^2 > \tau/4\}$, solve the minimal surface equation there and satisfy

$$\sup_{B_{1/2}\cap\{x^2>\tau/4\}} |D^\ell\, u_k|^2 \leq C_\tau \hat{E}_k^2 \tag{9.6}$$

for $\ell = 0, 1, 2$, where C_τ is a constant depending only on n and τ, and D^ℓ denotes the order ℓ differentiation.

We next claim that for all sufficiently large k,

$$(\{0\}\times \mathbf{R}^{n-1}) \cap B_{1/2} \subset (\{Z \in \operatorname{spt}\|V_k\| : \Theta(\|V_k\|, Z) \geq q\})_\tau\,. \tag{9.7}$$

If this were false, then there would exist a point $(0, y) \in \{0\} \times \mathbf{R}^{n-1} \cap \overline{B}_{1/2}$ and a subsequence $\{k'\}$ of $\{k\}$ such that for each k',

$$B^{n+1}_{3\tau/4}((0, y)) \cap \{Z \in \operatorname{spt} \|V_{k'}\| : \Theta(\|V_{k'}\|, Z) \geq q\} = \emptyset.$$

Since $\operatorname{spt} \|V_k\| \cap (\mathbf{R} \times B_{3/4}) \to \{0\} \times B_{3/4}$ in Hausdorff distance, it follows that for each k' and each $Z \in \operatorname{spt} \|V_{k'}\| \cap (\mathbf{R} \times B_{\tau/2}((0, y)))$, we must have $\Theta(\|V_{k'}\|, Z) < q$. Arguing exactly as for (9.5) and (9.6), we conclude that for all sufficiently large k', $\Sigma_{k'} \cap B_{\tau/4}(0, y) = \emptyset$;

$$\operatorname{spt} \|V_{k'}\| \cap (\mathbf{R} \times B_{\tau/4}(0, y)) = \textstyle\bigcup_{j=1}^{q} \text{graph } u^j_{k'}\Big|_{B_{\tau/4}(0,y)};$$

and that $u^j_{k'}$ are C^2 functions on $B_{\tau/4}(0, y)$, satisfy

$$\sum_{j=1}^{q} \sup_{B_{\tau/4}(0,y)} |Du^j_{k'}| + |D^2 u^j_{k'}| \leq C\hat{E}_{k'}, \quad C = C(n, \tau) \in (0, \infty)$$

and solve the minimal surface equation on $B_{\tau/4}(0, y)$. Consequently, $v^j_\star|_{B_{\tau/4}(0,y)}$ must be harmonic for each $j = 1, 2, \ldots, q$, which is however impossible since by hypothesis, $L^j_1 = 0$ for each $j = 1, 2, \ldots, q$ while $L^j_2 \neq L^{j+1}_2$ for some $j \in \{1, 2, \ldots, q - 1\}$. This contradiction establishes (9.7) for all sufficiently large k.

We now proceed to derive the contradiction needed for the proof of the lemma. By taking $\psi(X) = \widetilde{\zeta}(X)e^2$ in the first variation formula (3.1), we deduce that

$$\int \nabla^{V_k} x^2 \cdot \nabla^{V_k} \widetilde{\zeta}(X) d\|V_k\|(X) = 0 \tag{9.8}$$

for each $k = 1, 2, \ldots$ and each $\widetilde{\zeta} \in C^1_c(\mathbf{R} \times B_1)$. Choosing $\widetilde{\zeta}$ to agree with $\zeta'(x^1, x') = \zeta(x')$ in a neighborhood of $\operatorname{spt} \|V\| \cap (\mathbf{R} \times B_{1/4})$, where $\zeta \in C^1_c(B_{1/4})$ is arbitrary, we deduce from this that

$$\sum_{j=1}^{q} \int_{B_{1/4}} \sqrt{1 + |Du^j_k|^2} \left(D_2\zeta - \frac{D_2 u^j_k (D\zeta \cdot Du^j_k)}{1 + |Du^j_k|^2} \right) = F_k, \quad \text{where} \tag{9.9}$$

$$\begin{aligned} F_k = &- \int_{\mathbf{R} \times (B_{1/4} \cap \Sigma_k)} \nabla^{V_k} x^2 \cdot \nabla^{V_k} \widetilde{\zeta}(X) d\|V_k\|(X) \\ &+ \sum_{j=1}^{q} \int_{B_{1/4} \cap \Sigma_k} \sqrt{1 + |Du^j_k|^2} \left(D_2\zeta - \frac{D_2 u^j_k (D\zeta \cdot Du^j_k)}{1 + |Du^j_k|^2} \right). \end{aligned}$$

Since $\int_{B_{1/4}} D_2\zeta = 0$, it follows from (9.9) that

$$\sum_{j=1}^{q} \int_{B_{1/4}} \frac{|Du^j_k|^2}{1 + \sqrt{1 + |Du^j_k|^2}} D_2\zeta - \frac{D_2 u^j_k (D\zeta \cdot Du^j_k)}{\sqrt{1 + |Du^j_k|^2}} = F_k. \tag{9.10}$$

In view of (9.5) and (9.6), it follows from the definition of Σ_k (see Theorem 5.1) that

$$B_{1/4} \cap \Sigma_k \subset B_{1/4} \cap \{x^2 < \tau/2\}. \tag{9.11}$$

We claim also that for all sufficiently large k,

$$\|V_k\|(\mathbf{R}\times(B_{1/4}\cap\Sigma_k))+\mathcal{H}^n(B_{1/4}\cap\Sigma_k) \leq C\int_{(\mathbf{R}\times B_{1/2})\cap\{x^2<\tau\}}|\nabla^{V_k} x^1|^2 d\|V_k\|(X), \tag{9.12}$$

where $C \in (0,\infty)$ is a fixed constant depending only on n and q. To see this, let $\widetilde{\Sigma}_k^{(j)}$, $j=1,2,3$, correspond to the set $\widetilde{\Sigma}_j$ in Theorem 5.1 when V is replaced by V_k, and let Σ'_k correspond to Σ'. Since for each k, $\rho \in (\tau/4, 1/16)$ and $Y \in \operatorname{spt}\|V_k\| \cap (\mathbf{R}\times B_{1/2})$, we trivially have that

$$\begin{aligned}\rho^{-n}\int_{\mathbf{R}\times B_\rho(\pi Y)}&|\nabla^{V_k} x^1|^2 d\|V_k\|(X)\\ &\leq 4^n\tau^{-n}\int_{\mathbf{R}\times B_{3/4}}|\nabla^{V_k} x^1|^2 d\|V_k\|(X) \leq 4^n\tau^{-n}\hat{E}_k^2,\end{aligned} \tag{9.13}$$

and since by definition,

$$\begin{aligned}\widetilde{\Sigma}_k^{(1)} = \{&Y \in \operatorname{spt}\|V_k\| \cap (\mathbf{R}\times B_\sigma) :\\ &\rho^{-n}\int_{\mathbf{R}\times B_\rho(\pi Y)}|\nabla^{V_k} x^1|^2 d\|V_k\|(X) \geq \xi \text{ for some } \rho \in (0,(1-\sigma))\},\end{aligned}$$

where $\xi = \xi(n,q) \in (0,1/2)$ is as in Theorem 5.1, it follows that for all sufficiently large k (depending on τ), $Y \in \widetilde{\Sigma}_k^{(1)}$ if and only if $Y \in \operatorname{spt}\|V_k\|\cap(\mathbf{R}\times B_\sigma)$ and $\rho^{-n}\int_{\mathbf{R}\times B_\rho(\pi Y)}|\nabla^{V_k} x^1|^2 d\|V_k\|(X) \geq \xi$ for some $\rho \in (0,\tau/4]$. Also, by part 3 of the proof of [Alm00, Th. 3.8], we have that for each $x \in B_\sigma$ and each k,

$$\sum_{Y\in\operatorname{spt}\|V_k\|\cap\pi^{-1}(x)\setminus\left(\widetilde{\Sigma}_k^{(1)}\cup\widetilde{\Sigma}_k^{(2)}\right)}\Theta\left(\|V_k\|,Y\right) \leq q. \tag{9.14}$$

In view of (9.11), it follows from the Besicovitch covering lemma and (9.14) that

$$\begin{aligned}\|V_k\|(\mathbf{R}\times(B_{1/4}\cap\pi\,\widetilde{\Sigma}_k^{(j)}))&+\mathcal{H}^n(B_{1/4}\cap\pi\,\widetilde{\Sigma}_k^{(j)})\\ &\leq C\int_{(B_{1/2}\times\mathbf{R})\cap\{x^2<\tau\}}|\nabla^{V_k} x^1|^2 d\|V_k\|(X)\end{aligned}$$

for $j=1$, where $C = C(n,q) \in (0,\infty)$. Since $\|V_k\|(\widetilde{\Sigma}_k^{(2)}) = 0$ (see part 2 of the proof of [Alm00, Th. 3.8]), this estimate also follows for $j=2$ in view of (9.14); it follows for $j=3$, directly from the definition of $\widetilde{\Sigma}_k^{(3)}$, (9.11) and (9.14); it also holds with Σ'_k in place of $\pi\,\widetilde{\Sigma}_k^{(j)}$, by (9.11) and part 5 of the proof of [Alm00, Th. 3.8]. Thus the estimate (9.12), with the constant C depending only on n and q (in particular independent of τ), holds.

By (9.12), Theorem 7.1(b) (with $\mu = 1/2$) and (9.3) we deduce that, since the integrands in both integral expressions in F_k are bounded,

$$\hat{E}_k^{-2}|F_k| \le C \sup |D\zeta| \tau^{1/2} \tag{9.15}$$

for all sufficiently large k, where $C = C(n,q) \in (0,\infty)$.

Abbreviating $w_k = \sum_{j=1}^q \frac{|Du_k^j|^2}{1+\sqrt{1+|Du_k^j|^2}} D_2\zeta - \frac{D_2 u_k^j (D\zeta \cdot Du_k^j)}{\sqrt{1+|Du_k^j|^2}}$, note that

$$\int_{B_{1/4}\setminus \Sigma_k \cap \{x^2 \le \tau\}} |w_k| \le C \sup |D\zeta| \int_{(\mathbf{R}\times B_{1/2})\cap\{x^2\le\tau\}} |\nabla^{V_k} x^1|^2 d\|V_k\|(X),$$

and by (9.12),

$$\int_{B_{1/4}\cap\Sigma_k} |w_k| \le C \sup |D\zeta| \int_{(\mathbf{R}\times B_{1/2})\cap\{x^2\le\tau\}} |\nabla^{V_k} x^1|^2 d\|V_k\|(X),$$

where $C = C(n)$, so that again by Theorem 7.1(b) with $\mu = 1/2$ and (9.3),

$$\hat{E}_k^{-2}\left(\int_{B_{1/4}\setminus\Sigma_k\cap\{x^2\le\tau\}} |w_k| + \int_{B_{1/4}\cap\Sigma_k} |w_k|\right) \le C\sup|D\zeta|\tau^{1/2} \tag{9.16}$$

for all sufficiently large k, where $C = C(n)$. Finally, by (9.6),

$$\lim_{k\to\infty} \hat{E}_k^{-2} \int_{B_{1/4}\cap\{x^2\ge\tau\}} w_k = -\frac{1}{2}\sum_{j=1}^q \int_{B_{1/4}\cap\{x^2\ge\tau\}} |D_2 v_\star^j|^2 D_2\zeta, \tag{9.17}$$

where we have used the fact that $D_i v_\star^j \equiv 0$ for $i = 3,\ldots,(n+1)$ and $j = 1,2,\ldots,q$. Dividing (9.10) by $\hat{E}_k^2$ and first letting $k\to\infty$ and then letting $\tau\to 0$, we conclude from (9.15), (9.16) and (9.17) that

$$\sum_{j=1}^q \int_{B_{1/4}\cap\{x^2\ge 0\}} |D_2 v_\star^j|^2 D_2\zeta = 0$$

for any $\zeta \in C_c^1(B_{1/4})$. Since $v_\star^j = L_2^j$ on $\{x^2 \ge 0\}$, this contradicts (for any choice of $\zeta \in C_c^1(B_{1/4})$ with $\int_{B_{1/4}\cap\{x^2\ge0\}} D_2\zeta \ne 0$) our assumption that $L_2^j \ne L_2^{j+1}$ for some $j \in \{1,2,\ldots,q-1\}$. □

Remark. It follows from Lemma 9.1 and the compactness property ($\mathcal{B}6$) that there exists a constant $c = c(n,q) \in (0,\infty)$ with the following property: If $v \in \mathcal{B}_q$ is such that, for each $j = 1,2,\ldots,q$, $v^j(x^2,y) = \ell_j x^2$ for $x^2 < 0$; $v^j(x^2,y) = m_j x^2$ for $x^2 \ge 0$, where ℓ_j, m_j are constants; and $v^j \not\equiv v_a$ for some $j \in \{1,2,\ldots,q\}$, where $v_a \equiv q^{-1}\sum_{j=1}^q v^j$, then $|\ell_1 - \ell_q|^2 \ge c\sum_{j=1}^q \|v^j - v_a\|^2_{L^2(B_1)}$ and $|m_1 - m_q|^2 \ge c\sum_{j=1}^q \|v^j - v_a\|^2_{L^2(B_1)}$. (Of course once we have completed the proof that $\mathcal{B}_q$ satisfies property ($\mathcal{B}7$), we will have ruled out the existence of such $v \in \mathcal{B}_q$.)

10. Parametric L^2-estimates in terms of fine excess

This section and all of the subsequent sections up to and including Section 14 will be devoted to the proof of the assertion of Case 2 set forth at the beginning of Section 9. Crucial to our proof are the L^2-estimates, given in Theorem 10.1 and Corollary 10.2 below, for a varifold $V \in \mathcal{S}_\alpha$ with small coarse excess (relative to a hyperplane) and lower order "fine excess" relative to an appropriate union of half-hyperplanes meeting along an $(n-1)$-dimensional axis (see Hypotheses 10.1(5) below). These results are adaptations to the present "higher multiplicity" setting of those proved in [Sim93] in the context of "multiplicity 1 classes" of minimal submanifolds.

Notation. (1) Let $\mathcal{C}_q$ denote the set of hypercones $\mathbf{C}$ of $\mathbf{R}^{n+1}$ such that $\mathbf{C} = \sum_{j=1}^{q} |H_j| + |G_j|$, where for each $j \in \{1, 2, \ldots, q\}$, H_j is the half-hyperplane defined by

$$H_j = \{(x^1, x^2, y) \in \mathbf{R}^{n+1} : x^2 < 0 \ \text{ and } \ x^1 = \lambda_j x^2\}$$

and G_j is the half-hyperplane defined by

$$G_j = \{(x^1, x^2, y) \in \mathbf{R}^{n+1} : x^2 > 0 \ \text{ and } \ x^1 = \mu_j x^2\},$$

with λ_j, μ_j constants, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_q$ and $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_q$. Note that we do *not* assume cones in $\mathcal{C}_q$ are stationary in $\mathbf{R}^{n+1}$.

(2) For $p \in \{2, 3, \ldots, 2q\}$, let $\mathcal{C}_q(p)$ denote the set of hypercones $\mathbf{C} = \sum_{j=1}^{q} |H_j| + |G_j| \in \mathcal{C}_q$ as defined above such that the number of *distinct* half-hyperplanes in the set $\{H_1, \ldots, H_q, G_1, \ldots, G_q\}$ is p. Then $\mathcal{C}_q = \bigcup_{p=2}^{2q} \mathcal{C}_q(p)$.

(3) For $V \in \mathcal{S}_\alpha$ and $\mathbf{C} \in \mathcal{C}_q$, define a height excess ("fine excess") $Q_V(\mathbf{C})$ of V relative to $\mathbf{C}$ by

$$Q_V(\mathbf{C}) = \left(\int_{\mathbf{R} \times (B_{1/2} \setminus \{|x^2| < 1/16\})} \operatorname{dist}^2(X, \operatorname{spt} \|V\|)\, d\|\mathbf{C}\|(X) \right.$$
$$\left. + \int_{\mathbf{R} \times B_1} \operatorname{dist}^2(X, \operatorname{spt} \|\mathbf{C}\|)\, d\|V\|(X) \right)^{1/2}.$$

(4) For $q \geq 2$ and $p \in \{4, \ldots, 2q\}$, let

$$Q_V^\star(p) = \inf_{\mathbf{C} \in \bigcup_{k=4}^{p} \mathcal{C}_q(k)} Q_V(\mathbf{C}).$$

Let $\alpha \in (0,1)$, and let q be an integer ≥ 2. In Theorem 10.1, Corollary 10.2 and Lemma 10.8 below and subsequently, we shall consider the following set of hypotheses for appropriately small $\varepsilon, \gamma \in (0,1)$ to be determined depending only on n, q and α:

Hypotheses 10.1.

(1) $V \in \mathcal{S}_\alpha$, $\Theta(\|V\|, 0) \geq q$, $(\omega_n 2^n)^{-1} \|V\|(B_2^{n+1}(0)) < q + 1/2$, $\omega_n^{-1} \|V\|(\mathbf{R} \times B_1) < q + 1/2$.

(2) $\mathbf{C} = \sum_{j=1}^{q} |H_j| + |G_j| \in \mathcal{C}_q$, where for each $j \in \{1, 2, \ldots, q\}$, H_j is the half-hyperplane defined by $H_j = \{(x^1, x^2, y) \in \mathbf{R}^{n+1} : x^2 < 0 \text{ and } x^1 = \lambda_j x^2\}$ and G_j is the half-hyperplane defined by $G_j = \{(x^1, x^2, y) \in \mathbf{R}^{n+1} : x^2 > 0$ and $x^1 = \mu_j x^2\}$, with λ_j, μ_j constants, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_q$ and $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_q$.

(3) $\hat{E}_V^2 \equiv \int_{\mathbf{R}\times B_1} |x^1|^2 d\|V\|(X) < \varepsilon$.

(4) $\{Z : \Theta(\|V\|, Z) \geq q\} \cap \left(\mathbf{R} \times (B_{1/2} \setminus \{|x^2| < 1/16\})\right) = \emptyset$.

(5) $Q_V^2(\mathbf{C}) < \gamma \hat{E}_V^2$.

Remark. There exists $\varepsilon = \varepsilon(n, q) \in (0, 1)$ such that if Hypotheses 10.1 above hold with any $\gamma \in (0, 1)$, and the induction hypotheses (H1), (H2) hold, then

$$\max\{|\lambda_1|, |\lambda_q|\} \leq c_1 \hat{E}_V \quad \text{and} \quad \max\{|\mu_1|, |\mu_q|\} \leq c_1 \hat{E}_V, \tag{10.1}$$

where $c_1 = c_1(n) \in (0, \infty)$. These bounds follow from Hypotheses 10.1(5) in view of the fact that (by Hypotheses 10.1(4), Remark 3 of Section 6 and Theorem 3.5), under Hypotheses 10.1, $V \llcorner (\mathbf{R} \times (B_{1/4} \setminus \{|x^2| < 1/8\})) = \sum_{j=1}^{q} |\text{graph}\, \widetilde{u}_j| + |\text{graph}\, \widetilde{w}_j|$ where, for $j = 1, 2, \ldots, q$, $\widetilde{u}_j \in C^2(B_{1/4} \cap \{x^2 < -1/8\})$, $\widetilde{w}_j \in C^2(B_{1/4} \cap \{x^2 > 1/8\})$ with $\sup_{B_{1/4}\cap\{x^2<-1/8\}} |\widetilde{u}_j| \leq C\hat{E}_V$ and $\sup_{B_{1/4}\cap\{x^2>1/8\}} |\widetilde{w}_j| \leq C\hat{E}_V$, $C = C(n) \in (0, \infty)$.

Let $c_1 = c_1(n)$ be the constant as in (10.1) above, and define a constant $M_0 = M_0(n, q) \in (0, \infty)$ by

$$M_0 = \max\left\{\frac{3}{2}, \frac{2^{2n+8}\omega_n^2(2q+1)^2 c_1^2}{\overline{C}_1}, \frac{2^{2n+8}\omega_n(2q+1)}{\overline{C}_1}\right\},$$

where $\overline{C}_1 = \int_{B_{1/2}\cap\{x^2>1/16\}} |x^2|^2 \, d\mathcal{H}^n(x^2, y)$. We shall use this constant at several places below.

For V as in Hypotheses 10.1, we shall also assume the following for suitable values of $M > 1$:

Hypothesis ($\star$).

$$\hat{E}_V^2 < M \inf_{\{P=\{x^1=\lambda x^2\}\in G_n:\lambda\in\mathbf{R}\}} \int_{\mathbf{R}\times B_1} \operatorname{dist}^2(X, P), d\|V\|(X).$$

Remarks. (1) *If Hypotheses* 10.1 *and Hypothesis* ($\star$) *hold with sufficiently small* $\varepsilon = \varepsilon(n, q) \in (0, 1)$, $\gamma = \gamma(n, q) \in (0, 1)$ *and with* $M = \frac{3}{2}M_0^4$, *then*

$$\begin{aligned} &c\hat{E}_V \leq \max\{|\lambda_1|, |\lambda_q|\}, \quad c\hat{E}_V \leq \max\{|\mu_1|, |\mu_q|\} \quad \text{and} \\ &\min\{|\lambda_1 - \lambda_q|, |\mu_1 - \mu_q|\} \geq 2c\hat{E}_V \end{aligned} \tag{10.2}$$

for some constant $c = c(n, q) \in (0, \infty)$. Indeed, the triangle inequality (in the form $\operatorname{dist}^2(X, P) \leq 2\operatorname{dist}^2(X, \operatorname{spt}\|\mathbf{C}\|) + 2\operatorname{dist}_{\mathcal{H}}^2(P \cap (\mathbf{R}\times B_1), \operatorname{spt}\|\mathbf{C}\| \cap (\mathbf{R}\times B_1))$ for $X \in \mathbf{R} \times B_1$, applied with $P = \{x^1 = \frac{1}{2}(\lambda_1 + \lambda_q)x^2\}$ or $P = \{x^1 =$

$\frac{1}{2}(\mu_1 + \mu_2)x^2\}$), Hypothesis ($\star$) (with $M = \frac{3}{2}M_0^4$) and Hypotheses 10.1 (with sufficiently small $\varepsilon = \varepsilon(n,q) \in (0,1)$ and $\gamma = \gamma(n,q) \in (0,1)$) imply that $|\lambda_1 - \lambda_q| + |\mu_1 - \mu_q| \geq \widetilde{c}\hat{E}_V$ for some $\widetilde{c} = \widetilde{c}(n,q) \in (0,\infty)$. Lemma 9.1 then implies that $\min\{|\lambda_1 - \lambda_q|, |\mu_1 - \mu_q|\} \geq 2c\hat{E}_V$, $c = c(n,q) \in (0,1)$; the first two inequalities of (10.2) follow readily from this.

(2) It follows from the last inequality of (10.2) that if Hypotheses 10.1 and Hypothesis ($\star$) hold with $\varepsilon = \varepsilon(n,q)$, $\gamma = \gamma(n,q) \in (0,1)$ sufficiently small and $M = \frac{3}{2}M_0^4$, then $\mathbf{C} \in \mathcal{C}_q(p)$ for some $p \in \{4, 5, \ldots, 2q\}$.

Finally, for $\mathbf{C}$, V as in Hypotheses 10.1 and appropriately small $\beta \in (0, 1/2)$ (to be determined depending only on n, q and α), we will also need to consider the following:

Hypothesis ($\star\star$). *Either*

(i) $\mathbf{C} \in \mathcal{C}_q(4)$, *or*

(ii) $q \geq 3$, $\mathbf{C} \in \mathcal{C}_q(p)$ *for some* $p \in \{5, \ldots, 2q\}$ *and* $Q_V^2(\mathbf{C}) < \beta\left(Q_V^\star(p-1)\right)^2$.

Remarks. (1) Let $\mathbf{C}$ be as in Hypothesis 10.1(2). If $V \in \mathcal{S}_\alpha$, $\mathbf{C}$ satisfy Hypothesis 10.1(1), Hypothesis ($\star\star$)(ii) with $\beta \in (0, 1/4)$ and if $\lambda_1 = \lambda_1' > \lambda_2' > \cdots > \lambda_{p_1}' = \lambda_q$ are the distinct elements of the set $\{\lambda_1, \ldots, \lambda_q\}$ and $\mu_1 = \mu_1' < \mu_2' < \cdots < \mu_{p_2}' = \mu_q$ are the distinct elements of $\{\mu_1, \ldots, \mu_q\}$ (notation as in Hypothesis 10.1(2)), then it follows from Hypothesis ($\star\star$) and the triangle inequality that

$$\lambda_{i+1}' - \lambda_i' \geq 2c'Q_V^\star(p-1), \quad \mu_{j+1}' - \mu_j' \geq 2c'Q_V^\star(p-1) \tag{10.3}$$

for some constant $c' = c'(n,q) \in (0,\infty)$ and all $i = 1, 2, \ldots, p_1 - 1$ and $j = 1, 2, \ldots, p_2 - 1$.

(2) Suppose $V \in \mathcal{S}_\alpha$, $\mathbf{C} \in \mathcal{C}_q$ satisfy Hypotheses 10.1, Hypothesis ($\star$) and Hypothesis ($\star\star$) for some $\varepsilon, \gamma, \beta \in (0, 1/2)$. If $\mathbf{C}' \in \mathcal{C}_q$ is any other cone with $\mathrm{spt}\,\|\mathbf{C}'\| = \mathrm{spt}\,\|\mathbf{C}\|$, then Hypotheses 10.1, Hypothesis ($\star$) and Hypothesis ($\star\star$) will continue to be satisfied with $\mathbf{C}'$ in place of $\mathbf{C}$ provided γ, β are replaced by $2q\gamma$, $2q\beta$ respectively.

Theorem 10.1. *Let q be an integer ≥ 2, $\alpha \in (0,1)$, $\tau \in (0, 1/8)$ and $\mu \in (0,1)$. There exist numbers $\varepsilon_0 = \varepsilon_0(n,q,\alpha,\tau) \in (0,1)$, $\gamma_0 = \gamma_0(n,q,\alpha,\tau) \in (0,1)$ and $\beta_0 = \beta_0(n,q,\alpha,\tau) \in (0,1)$ such that the following is true*: *Let $V \in \mathcal{S}_\alpha$, $\mathbf{C} \in \mathcal{C}_q$ satisfy Hypotheses* 10.1, *Hypothesis* ($\star$) *and Hypothesis* ($\star\star$) *with $M = \frac{3}{2}M_0^4$ and ε_0,γ_0, β_0 in place of ε, γ, β respectively. Suppose also that the induction hypotheses* (H1), (H2) *hold. Write $\mathbf{C} = \sum_{j=1}^q |H_j| + |G_j|$ where for each $j \in \{1, 2, \ldots, q\}$, H_j is the half-space defined by $H_j = \{(x^1, x^2, y) \in \mathbf{R}^{n+1} : x^2 < 0 \ \text{ and } \ x^1 = \lambda_j x^2\}$, G_j is the half-space defined by $G_j = \{(x^1, x^2, y) \in \mathbf{R}^{n+1} : x^2 > 0 \ \text{ and } \ x^1 = \mu_j x^2\}$, with λ_j, μ_j constants, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_q$ and $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_q$; for $(x^2, y) \in \mathbf{R}^n$ and $j = 1, 2, \ldots, q$, define $h_j(x^2, y) =$*

$\lambda_j x^2$ and $g_j(x^2, y) = \mu_j x^2$. Then, after possibly replacing $\mathbf{C}$ *with another cone* $\mathbf{C}' \in \mathcal{C}_q$ *with* $\operatorname{spt}\|\mathbf{C}'\| = \operatorname{spt}\|\mathbf{C}\|$ *and relabelling* $\mathbf{C}'$ *as* $\mathbf{C}$ (*see the preceding Remark* (2)), *the following must hold*:

(a) $V \llcorner (\mathbf{R} \times (B_{3/4} \setminus \{|x^2| < \tau\})) = \sum_{j=1}^{q} |\mathrm{graph}(h_j + u_j)| + |\mathrm{graph}(g_j + w_j)|$ *where, for each* $j = 1, 2, \ldots, q$,
  - (i) $u_j \in C^2(B_{3/4} \cap \{x^2 < -\tau\})$; $w_j \in C^2(B_{3/4} \cap \{x^2 > \tau\})$;
  - (ii) $h_j + u_j$ *and* $g_j + w_j$ *solve the minimal surface equation on their respective domains*;
  - (iii) $h_1 + u_1 \leq h_2 + u_2 \leq \cdots \leq h_q + u_q$;
  - (iv) $g_1 + w_1 \leq g_2 + w_2 \leq \cdots \leq g_q + w_q$;
  - (v) $\mathrm{dist}((h_j(x^2, y) + u_j(x^2, y), x^2, y), \operatorname{spt}\|\mathbf{C}\|) = (1 + \lambda_j^2)^{-1/2}|u_j(x^2, y)|$, $(x^2, y) \in B_{3/4} \cap \{x^2 < -\tau\}$;
  - (vi) $\mathrm{dist}((g_j(x^2, y) + w_j(x^2, y), x^2, y), \operatorname{spt}\|\mathbf{C}\|) = (1 + \mu_j^2)^{-1/2}|w_j(x^2, y)|$, $(x^2, y) \in B_{3/4} \cap \{x^2 > \tau\}$.

(b) $\int_{B_{5/8}^{n+1}(0)} \frac{|X^\perp|^2}{|X|^{n+2}}\, d\|V\|(X) \leq C \int_{\mathbf{R} \times B_1} \mathrm{dist}^2(X, \operatorname{spt}\|\mathbf{C}\|)\, d\|V\|(X)$.

(c) $\sum_{j=3}^{n+1} \int_{B_{5/8}^{n+1}(0)} |e_j^\perp|^2\, d\|V\|(X) \leq C \int_{\mathbf{R} \times B_1} \mathrm{dist}^2(X, \operatorname{spt}\|\mathbf{C}\|)\, d\|V\|(X)$.

(d) $\int_{B_{5/8}^{n+1}(0)} \frac{\mathrm{dist}^2(X, \operatorname{spt}\|\mathbf{C}\|)}{|X|^{n+2-\mu}}\, d\|V\|(X) \leq \widetilde{C} \int_{\mathbf{R} \times B_1} \mathrm{dist}^2(X, \operatorname{spt}\|\mathbf{C}\|)\, d\|V\|(X)$.

Here $e_j^\perp(X)$ *denotes the orthogonal projection of* e_j *onto* $(T_X \operatorname{spt}\|V\|)^\perp$ *and* $C = C(n, q, \alpha) \in (0, \infty)$, $\widetilde{C} = \widetilde{C}(n, q, \alpha, \mu) \in (0, \infty)$. (*In particular*, C, $\widetilde{C}$ *do not depend on* τ.)

Proof. We first establish conclusion (a). Let $\lambda_1 = \lambda_1' > \lambda_2' > \cdots > \lambda_{p_1}' = \lambda_q$ be the distinct elements of the set $\{\lambda_1, \ldots, \lambda_q\}$ and $\mu_1 = \mu_1' < \mu_2' < \cdots < \mu_{p_2}' = \mu_q$ be the distinct elements of $\{\mu_1, \ldots, \mu_q\}$ so that $p_1, p_2 \leq q$ and $p_1 + p_2 = p$. By (10.2), provided $\varepsilon = \varepsilon(n, q), \gamma = \gamma(n, q) \in (0, 1)$ are sufficiently small, we have that $p_1, p_2 \geq 2$. By Remark (1) at the end of Section 8, Remark (3) of Section 6 and Theorem 3.5, it follows that if $\varepsilon = \varepsilon(n, q, \alpha, \tau), \gamma = \gamma(n, q, \alpha, \tau) \in (0, 1)$ are sufficiently small, then

$$V \llcorner (\mathbf{R} \times (B_{3/4} \setminus \{|x^2| < \tau\})) = \sum_{j=1}^{q} |\mathrm{graph}\, \widetilde{u}_j| + |\mathrm{graph}\, \widetilde{w}_j|, \tag{10.4}$$

where $\widetilde{u}_j \in C^2(B_{3/4} \setminus \{x^2 > -\tau\})$, $\widetilde{w}_j \in C^2(B_{3/4} \setminus \{x^2 < \tau\})$ are functions with small gradient solving the minimal surface equation and with $\widetilde{u}_1 \leq \widetilde{u}_2 \leq \cdots \leq \widetilde{u}_q$ and $\widetilde{w}_1 \leq \widetilde{w}_2 \leq \cdots \leq \widetilde{w}_q$.

If $p = 4$, then $p_1 = p_2 = 2$ and by (10.2), provided $\varepsilon = \varepsilon(n, q), \gamma = \gamma(n, q) \in (0, 1)$ are sufficiently small,

$$c\hat{E}_V \leq \max\{|\lambda_1'|, |\lambda_2'|\} \leq c_1 \hat{E}_V, \quad c\hat{E}_V \leq \max\{|\mu_1'|, |\mu_2'|\} \leq c_1 \hat{E}_V$$

and

$$\min\{|\lambda_1' - \lambda_2'|, |\mu_1' - \mu_2'|\} \geq 2c\hat{E}_V,$$

where $c_1 = c_1(n), c = c(n, q) \in (0, \infty)$ are as in (10.1) and (10.2). Conclusion (a) follows in this case from Hypothesis 10.1(5) and elliptic estimates. Now suppose $\mathbf{C} \in \mathcal{C}_q(p)$ for some $p \in \{5, 6, \ldots, 2q\}$ and assume by induction the following:

(A$_1$) There exist $\widetilde{\varepsilon} = \widetilde{\varepsilon}(n, q, \alpha, \tau)$, $\widetilde{\gamma} = \widetilde{\gamma}(n, q, \alpha, \tau)$ and $\widetilde{\beta} = \widetilde{\beta}(n, q, \alpha, \tau) \in (0, 1)$ such that if Hypotheses 10.1, Hypothesis ($\star$) and Hypothesis ($\star\star$) are satisfied with $M = M_0^4$, $\widetilde{\varepsilon}$, $\widetilde{\gamma}$, $\widetilde{\beta}$ in place of ε, γ, β respectively, and with $V \in \mathcal{S}_\alpha$ and any cone $\widetilde{\mathbf{C}} \in \bigcup_{k=4}^{p-1} \mathcal{C}_q(k)$ in place of $\mathbf{C}$, and if the induction hypotheses (H1), (H2) hold, then conclusion (a) with $\widetilde{\mathbf{C}}$ in place of $\mathbf{C}$ holds.

By (10.3),

$$|\lambda'_{i+1} - \lambda'_i| \geq 2c' Q_V^\star(p-1), \quad |\mu'_{j+1} - \mu'_j| \geq 2c' Q_V^\star(p-1) \tag{10.5}$$

for some constant $c' = c'(n, q) \in (0, \infty)$ and all $i = 1, 2, \ldots, p_1 - 1$ and $j = 1, 2, \ldots, p_2 - 1$. So if

$$(Q_V^\star(p-1))^2 \geq \left(\frac{2}{3}\widetilde{\beta}\right)^{2q} \widetilde{\gamma} \hat{E}_V^2,$$

then it follows from (10.1), (10.4), (10.5) and elliptic estimates that conclusion (a) holds provided $\varepsilon = \varepsilon(n, q, \alpha, \tau)$, $\gamma = \gamma(n, q, \alpha, \tau) \in (0, 1)$ are sufficiently small. If on the other hand

$$(Q_V^\star(p-1))^2 < \left(\frac{2}{3}\widetilde{\beta}\right)^{2q} \widetilde{\gamma} \hat{E}_V^2, \tag{10.6}$$

then we argue as follows: Choose $\mathbf{C}_1 \in \bigcup_{k=4}^{p-1} \mathcal{C}_q(k)$ such that

$$Q_V^2(\mathbf{C}_1) \leq \frac{3}{2} \left(Q_V^\star(p-1)\right)^2. \tag{10.7}$$

If Hypothesis ($\star\star$) is satisfied with $\mathbf{C}_1$ in place of $\mathbf{C}$ and $\widetilde{\beta}$ in place of β, then it follows from assumption (A$_1$) (taken with $\widetilde{\mathbf{C}} = \mathbf{C}_1$), (10.7), Hypothesis ($\star\star$), (10.5) and elliptic estimates that conclusion (a) holds provided $\varepsilon = \varepsilon(n, q, \alpha, \tau)$, $\beta = \beta(n, q, \alpha, \tau) \in (0, 1)$ are sufficiently small; on the other hand, if Hypothesis ($\star\star$) is not satisfied with $\mathbf{C}_1$ in place of $\mathbf{C}$ and $\widetilde{\beta}$ in place of β, then $q \geq 3$, $p \geq 6$, $\mathbf{C}_1 \in \mathcal{C}_q(k_1)$ for some $k_1 \in \{5, \ldots, p-1\}$, and

$$Q_V^2(\mathbf{C}_1) \geq \widetilde{\beta} \left(Q_V^\star(k_1 - 1)\right)^2. \tag{10.8}$$

In this case, choose a cone $\mathbf{C}_2 \in \bigcup_{k=4}^{k_1-1} \mathcal{C}_q(k)$ such that

$$Q_V^2(\mathbf{C}_2) \leq \frac{3}{2} \left(Q_V^\star(k_1 - 1)\right)^2 \tag{10.9}$$

and note that by (10.7), (10.8) and (10.6), we have that

$$Q_V^2(\mathbf{C}_2) \leq \widetilde{\gamma} \hat{E}_V^2; \tag{10.10}$$

by (10.7), (10.5) and (10.8), we have that

$$(10.11)\qquad |\lambda'_{i+1} - \lambda'_i| \geq \frac{4}{3} c' \widetilde{\beta} Q^{\star}_V(k_1 - 1), \quad |\mu'_{j+1} - \mu'_j| \geq \frac{4}{3} c' \widetilde{\beta} Q^{\star}_V(k_1 - 1)$$

for each $i = 1, 2, \ldots, p_1 - 1$ and $j = 1, 2, \ldots, p_2 - 1$; and since $Q^{\star}_V(p-1) \leq Q^{\star}_V(k_1 - 1)$, Hypothesis ($\star\star$) implies that

$$(10.12)\qquad Q^2_V(\mathbf{C}) \leq \beta \left(Q^{\star}_V(k_1 - 1)\right)^2.$$

So again, if Hypothesis ($\star\star$) is satisfied with $\mathbf{C}_2$ in place of $\mathbf{C}$ and $\widetilde{\beta}$ in place of β, it follows from (A_1) (taken with $\widetilde{\mathbf{C}} = \mathbf{C}_2$), (10.11), (10.12) and elliptic estimates that conclusion (a) holds provided $\varepsilon = \varepsilon(n, q, \alpha, \tau)$, $\beta = \beta(n, q, \alpha, \tau) \in (0, 1)$ are sufficiently small; if on the other hand Hypothesis ($\star\star$) is not satisfied with $\mathbf{C}_2$ in place of $\mathbf{C}$ and $\widetilde{\beta}$ in place of β, then we may repeat the above argument in the obvious way. It is clear that at most p repetitions of the argument are necessary to reach conclusion (a).

Now we prove conclusions (b) and (c). Let $\psi : \mathbf{R} \to [0, 1]$ be a decreasing C^2 function with $\psi(t) \equiv 1$ for $t \leq 13/16$, $\psi(t) \equiv 0$ for $t \geq 29/32$, $|\psi'(t)| \leq 32$ and $|\psi''(t)| \leq 1025$. For $\widetilde{X} = (\widetilde{x}^1, \widetilde{x}^2, \widetilde{y}) \in \mathbf{R} \times \mathbf{R} \times \mathbf{R}^{n-1}$, let $\widetilde{R}(\widetilde{X}) = |\widetilde{X}|$ and $\widetilde{r}(\widetilde{X}) = |(\widetilde{x}^1, \widetilde{x}^2, 0)|$. We then have by the inequalities (2), (3) of the proof of Lemma 3.4 of [Sim93] that

$$(10.13)\qquad \int_{B^{n+1}_{5/8}(0)} \frac{|\widetilde{X}^{\perp}|^2}{\widetilde{R}^{n+2}} \, d\|V\|(\widetilde{X}) \leq C \left(\int_{B^{n+1}_1(0)} \psi^2(\widetilde{R}) \, d\|V\|(\widetilde{X}) - \int_{B^{n+1}_1(0)} \psi^2(\widetilde{R}) \, d\|\mathbf{C}\|(\widetilde{X}) \right)$$

and

$$(10.14)\qquad \int_{B^{n+1}_1(0)} \left(1 + \sum_{j=3}^{n+1} |e_j^{\perp}|^2 \right) \psi^2(\widetilde{R}) \, d\|V\|(\widetilde{X}) \leq C \int_{B^{n+1}_1(0)} |(\widetilde{x}^1, \widetilde{x}^2, 0)^{\perp}|^2 (\psi^2(\widetilde{R}) + (\psi'(\widetilde{R}))^2) \, d\|V\|(\widetilde{X}) - 2 \int_{B^{n+1}_1(0)} \widetilde{r}^2 \widetilde{R}^{-1} \psi(\widetilde{R}) \psi'(\widetilde{R}) \, d\|V\|(\widetilde{X}),$$

where $C = C(n) \in (0, \infty)$ and for $\|V\|$-a.e. $\widetilde{X} \in \operatorname{spt} \|V\|$, the expression $(\widetilde{x}^1, \widetilde{x}^2, 0)^{\perp}$ denotes the orthogonal projection of $(\widetilde{x}^1, \widetilde{x}^2, 0)$ onto $(T_{\widetilde{X}} \operatorname{spt} \|V\|)^{\perp}$. Also by the identity (6) of the same proof in [Sim93], we have that

$$(10.15)\qquad \int_{B^{n+1}_1(0)} \psi^2(\widetilde{R}) \, d\|\mathbf{C}\|(\widetilde{X}) = -2 \int_{B^{n+1}_1(0)} \widetilde{r}^2 \widetilde{R}^{-1} \psi(\widetilde{R}) \psi'(\widetilde{R}) \, d\|\mathbf{C}\|(\widetilde{X}).$$

Let δ be a small positive constant to be chosen depending only on n, q and α, let $\pi : \mathbf{R}^{n+1} \to \{0\} \times \mathbf{R}^n$ be the orthogonal projection and let $\mathcal{Y} = B_{15/16} \cap$

$\{|\widetilde{x}^2| < 1/28\} \cap \pi \operatorname{spt} \|V\| \setminus (\{0\} \times \mathbf{R}^{n-1})$. Denote by (x, y) a general point in $\mathbf{R}^n = \{\widetilde{x}^1 = 0\}$ where $x \in \mathbf{R}$ and $y \in \mathbf{R}^{n-1}$. Write

$$\mathcal{Y} = \mathcal{U} \cup \mathcal{W},$$

where $\mathcal{U}$ is the set of points $(x, y) \in \mathcal{Y}$ such that

$$(15|x|/16)^{-n-2} \int_{\mathbf{R} \times B_{15|x|/16}(x,y)} \operatorname{dist}^2(\widetilde{X}, \operatorname{spt} \|\mathbf{C}\|) d\|V\|(\widetilde{X}) < \delta$$

and $\mathcal{W}$ is the set of points $(x, y) \in \mathcal{Y}$ such that

$$(15|x|/16)^{-n-2} \int_{\mathbf{R} \times B_{15|x|/16}(x,y)} \operatorname{dist}^2(\widetilde{X}, \operatorname{spt} \|\mathbf{C}\|) d\|V\|(\widetilde{X}) \geq \delta.$$

Note that if $(x, y) \in \mathcal{Y}$, then $\pi^{-1}(x, y) \cap \operatorname{spt} \|V\| \neq \emptyset$, so it follows from monotonicity of mass ratio that $\|V\|(\mathbf{R} \times B_{|x|/16}(x, y)) \geq \omega_n (|x|/16)^n$. Consequently, for each point $(x, y) \in \mathcal{U}$, there is a point $Z^{(x,y)} \in \operatorname{spt} \|V\| \cap (\mathbf{R} \times B_{|x|/16}(x, y))$ with $\operatorname{dist}(Z^{(x,y)}, \operatorname{spt} \|\mathbf{C}\|) \leq \sqrt{2^{4n+1} \omega_n^{-1} \delta} |x|$ and satisfying, by (10.1),

$$\operatorname{dist}_{\mathcal{H}}(\eta_{Z^{(x,y)}, 7|x|/8} \operatorname{spt} \|\mathbf{C}\| \cap (\mathbf{R} \times B_1), \{0\} \times B_1) < C\sqrt{\delta}$$

provided $\varepsilon_0 = \varepsilon_0(\delta)$ is sufficiently small. Here $C = C(n) \in (0, \infty)$. It also follows from Remark (1) at the end of Section 8, (10.2) and monotonicity of mass ratio that for any $\tau' \in (0, 1)$, we may ensure, by choosing $\varepsilon_0 = \varepsilon_0(n, q, \alpha, \tau'), \gamma_0 = \gamma_0(n, q, \alpha, \tau') \in (0, 1)$ sufficiently small, that $\{Z \in \operatorname{spt} \|V\| \cap (\mathbf{R} \times B_{15/16}) : \Theta(\|V\|, Z) \geq q\} \subset \{(\widetilde{x}^1, \widetilde{x}^2, \widetilde{y}) \in \mathbf{R}^{n+1} : |\widetilde{x}^2| < \tau'\}$ and $\|V\|((\mathbf{R} \times B_{15/16}) \cap \{(\widetilde{x}^1, \widetilde{x}^2, \widetilde{y}) \in \mathbf{R}^{n+1} : |\widetilde{x}^2| < \tau'\}) < C\tau'$, where $C = C(n, q) \in (0, \infty)$. Using these facts with sufficiently small $\tau' = \tau'(n, q) \in (0, 1)$ together with Remark (3) of Section 6 and Theorem 3.5, we find that $\omega_n^{-1} (1/16)^{-n} \|V\|(B^{n+1}_{1/16}(Z)) < q + 1/4$ for any $Z \in \mathbf{R} \times B_{14/16}$ and hence, in particular, that for each $(x, y) \in \mathcal{U}$,

$$\omega_n^{-1} (7|x|/4)^{-n} \|V\|(B^{n+1}_{7|x|/4}(Z^{(x,y)})) < q + 1/4.$$

Furthermore, writing $\Gamma^+ = (\mathbf{R} \times B_{7|x|/8}(\pi\, Z^{(x,y)})) \cap \{|\widetilde{x}^1 - e_1 \cdot Z^{(x,y)}| \geq \frac{3}{4}|x|\}$ and $\Gamma^- = (\mathbf{R} \times B_{7|x|/8}(\pi\, Z^{(x,y)})) \cap \{|\widetilde{x}^1 - e_1 \cdot Z^{(x,y)}| < \frac{3}{4}|x|\}$, we have, for sufficiently small $\delta = \delta(n) \in (0, 1)$ and any $(x, y) \in \mathcal{U}$, that

$$\begin{aligned}
&(7|x|/8)^{-n-2} \int_{\mathbf{R} \times B_{7|x|/8}(\pi\, Z^{(x,y)})} \operatorname{dist}^2 (\widetilde{X}, Z^{(x,y)} + \{0\} \times \mathbf{R}^n)\, d\|V\|(\widetilde{X}) \\
&\leq (7|x|/8)^{-n-2} \int_{\Gamma^-} \operatorname{dist}^2 (\widetilde{X}, Z^{(x,y)} + \{0\} \times \mathbf{R}^n)\, d\|V\|(\widetilde{X}) \\
&\quad + (7|x|/8)^{-n-2} \int_{\Gamma^+} \operatorname{dist}^2 (\widetilde{X}, Z^{(x,y)} + \{0\} \times \mathbf{R}^n)\, d\|V\|(\widetilde{X}) \\
&\leq c|x|^{-n-2} \int_{\mathbf{R} \times B_{|x|}(x,y)} \operatorname{dist}^2(\widetilde{X}, \operatorname{spt} \|\mathbf{C}\|)\, d\|V\|(\widetilde{X}) + c|x|^{-n-2} \|V\|(B^{n+1}_{5|x|/4}(Z^{(x,y)}))\, D
\end{aligned}$$

where $D = \operatorname{dist}^2_{\mathcal{H}}\,(Z^{(x,y)} + \{0\} \times B_{7|x|/8}(0), \operatorname{spt}\|\mathbf{C}\| \cap (\mathbf{R} \times B_{7|x|/8}(\pi\, Z^{(x,y)}))$, $c = c(n) \in (0,\infty)$ and we have used the pointwise inequality

$$\operatorname{dist}(\widetilde{X}, Z^{(x,y)} + \{0\} \times \mathbf{R}^n) \leq 2\operatorname{dist}(\widetilde{X}, \operatorname{spt}\|\mathbf{C}\|)$$

for $\widetilde{X} \in (\mathbf{R} \times B_{7|x|/8}(\pi\, Z^{(x,y)})) \cap \{|\widetilde{x}^1 - Z_1^{(x,y)}| \geq \frac{3}{4}|x|\}$, valid if $\delta = \delta(n) \in (0,1)$ and $\varepsilon_0 = \varepsilon_0(n,q,\alpha) \in (0,1)$ are sufficiently small. Thus provided $\varepsilon_0 = \varepsilon_0(n,q,\delta) \in (0,1)$ is sufficiently small,

(10.16)

$$(7|x|/8)^{-n-2} \int_{\mathbf{R} \times B_{7|x|/8}(\pi\, Z^{(x,y)})} \operatorname{dist}^2(\widetilde{X}, Z^{(x,y)} + \{0\} \times \mathbf{R}^n)\, d\|V\|(\widetilde{X}) < C\delta,$$

where $C = C(n,q) \in (0,\infty)$. In particular, $\|V\|((\mathbf{R} \times B_{7|x|/8}(\pi\, Z^{(x,y)})) \cap \{\widetilde{X} : \operatorname{dist}(\widetilde{X}, Z^{(x,y)} + \{0\} \times \mathbf{R}^n) \geq \delta^{1/4}|x|\}) \leq C\sqrt{\delta}|x|^n$ where $C = C(n,q) \in (0,\infty)$, and consequently,

$$\begin{aligned}
&\omega_n^{-1}(7|x|/8)^{-n}\|V\|(\mathbf{R} \times B_{7|x|/8}(\pi\, Z^{(x,y)})) \\
&\qquad \leq C\sqrt{\delta} + \omega_n^{-1}(7|x|/8)^{-n}\|V\|(B^{n+1}_{(7/8+\delta^{1/4})|x|}(Z^{(x,y)})) < q + 1/2
\end{aligned}$$

provided $\delta = \delta(n,q,\alpha) \in (0,1)$ is sufficiently small. Note also that (10.16) implies that $\operatorname{spt}\|V\| \cap (\mathbf{R} \times B_{3|x|/4}(\pi\, Z^{(x,y)})) \subset \{\widetilde{X} \in \mathbf{R}^{n+1} : \operatorname{dist}(\widetilde{X}, Z^{(x,y)} + \{0\} \times \mathbf{R}^n) < |x|/2\}$ provided $\delta = \delta(n,q,\alpha) \in (0,1)$ is sufficiently small. By applying Remark (3) of Section 8 (with $\eta_{Z^{(x,y)},7|x|/8\,\#}\,V$, $\eta_{Z^{(x,y)},7|x|/8}\operatorname{spt}\|\mathbf{C}\|$ in place of V,$\mathbf{P}$) we deduce that for each $(x,y) \in \mathcal{U}$, there exists a hyperplane $H_{(x,y)}$ with $H_{(x,y)} \cap \{\widetilde{x}^2 > 0\} \in \{G_1, \ldots, G_q\}$ (in case $x > 0$) or $H_{(x,y)} \cap \{\widetilde{x}^2 < 0\} \in \{H_1, \ldots, H_q\}$ (in case $x < 0$), and an $\mathcal{H}^n$-measurable subset $\Sigma_{(x,y)} \subset H_{(x,y)} \cap \operatorname{spt}\|\mathbf{C}\| \cap (\mathbf{R} \times B_{|x|/4}(x,y))$ (where $\Sigma_{(x,y)} = \emptyset$ if Remark (3)(a) applies, and $\Sigma_{(x,y)}$ corresponds to the set Σ as in Remark (3)(b) otherwise) such that

$$\begin{aligned}
&\int_{(\mathbf{R} \times (B_{|x|/4}(x,y)) \cap \{|\widetilde{x}^1| \leq |x|\} \setminus \mathcal{C}_{H_{(x,y)}}(\Sigma_{(x,y)})} |(\widetilde{x}^1, \widetilde{x}^2, 0)^\perp|^2\, d\|V\|(\widetilde{X}) \\
&\qquad + \int_{(\mathbf{R} \times B_{|x|/4}(x,y)) \cap \mathcal{C}_{H_{(x,y)}}(\Sigma_{(x,y)})} |\widetilde{x}^2|^2\, d\|V\|(\widetilde{X}) \\
&\qquad \leq C \int_{\mathbf{R} \times B_{15|x|/16}(x,y)} \operatorname{dist}^2(\widetilde{X}, \operatorname{spt}\|\mathbf{C}\|)\, d\|V\|(\widetilde{X}),
\end{aligned}$$

where $C = C(n,q,\alpha) \in (0,\infty)$ and $\mathcal{C}_H(A) = \{X \in \mathbf{R}^{n+1} : \pi_H(X) \in A\}$. Since the pointwise inequality $|\widetilde{x}^1| \leq 2\operatorname{dist}(\widetilde{X}, \operatorname{spt}\|\mathbf{C}\|)$ holds whenever $\widetilde{X} = (\widetilde{x}^1, \widetilde{x}^2, \widetilde{y}) \in F \equiv (\mathbf{R} \times B_{|x|/4}(x,y)) \cap \{|\widetilde{x}^1| > |x|\}$, we also have that

$$\begin{aligned}
&\int_{F \setminus \mathcal{C}_{H_{(x,y)}}(\Sigma_{(x,y)})} |(\widetilde{x}^1, \widetilde{x}^2, 0)^\perp|^2\, d\|V\|(\widetilde{X}) + \int_F |\widetilde{x}^1|^2\, d\|V\|(\widetilde{X}) \\
&\qquad\qquad \leq C \int_{\mathbf{R} \times B_{15|x|/16}(x,y)} \operatorname{dist}^2(\widetilde{X}, \operatorname{spt}\|\mathbf{C}\|)\, d\|V\|(\widetilde{X}).
\end{aligned}$$

Combining the two preceding integral estimates, we conclude that for each $(x,y)\in\mathcal{U}$,

$$\int_{(\mathbf{R}\times B_{|x|/4}(x,y))\cap\mathcal{C}_{H_{(x,y)}}(\Sigma_{(x,y)})}\widetilde{r}^2\,d\|V\|(\widetilde{X}) + \int_{(\mathbf{R}\times B_{|x|/4}(x,y))\setminus\mathcal{C}_{H_{(x,y)}}(\Sigma_{(x,y)})}|(\widetilde{x}^1,\widetilde{x}^2,0)^{\perp}|^2\,d\|V\|(\widetilde{X}) \leq C\int_{\mathbf{R}\times B_{15|x|/16}(x,y)}\mathrm{dist}^2(\widetilde{X},\mathrm{spt}\,\|\mathbf{C}\|)\,d\|V\|(\widetilde{X}), \tag{10.17}$$

where $C=C(n,q,\alpha)\in(0,\infty)$. We claim that (10.17) also holds trivially (by taking $\Sigma_{(x,y)}$ to be equal to any component of $\mathrm{spt}\,\|\mathbf{C}\|\cap(\mathbf{R}\times B_{|x|/4}(x,y))$) whenever $(x,y)\in\mathcal{W}$. Indeed,

$$\begin{aligned}
\int_{\mathbf{R}\times B_{|x|/4}(x,y)}\widetilde{r}^2\,d\|V\|(\widetilde{X})
&= \int_{(\mathbf{R}\times B_{|x|/4}(x,y))\cap\{|\widetilde{x}^1|<|x|\}}\widetilde{r}^2\,d\|V\|(\widetilde{X})\\
&\quad + \int_{(\mathbf{R}\times B_{|x|/4}(x,y))\cap\{|\widetilde{x}^1|\geq|x|\}}\widetilde{r}^2\,d\|V\|(\widetilde{X})\\
&\leq \frac{81}{16}|x|^2\|V\|((\mathbf{R}\times B_{|x|/4}(x,y))\cap\{|\widetilde{x}^1|<|x|\})\\
&\quad + 50\int_{\mathbf{R}\times B_{|x|/4}(x,y)}\mathrm{dist}^2(\widetilde{X},\mathrm{spt}\,\|\mathbf{C}\|)\,d\|V\|(\widetilde{X})\\
&\leq C|x|^{n+2}+C\int_{\mathbf{R}\times B_{|x|/4}(x,y)}\mathrm{dist}^2(\widetilde{X},\mathrm{spt}\,\|\mathbf{C}\|)\,d\|V\|(\widetilde{X})\\
&\leq C\int_{\mathbf{R}\times B_{15|x|/16}(x,y)}\mathrm{dist}^2(\widetilde{X},\mathrm{spt}\,\|\mathbf{C}\|)\,d\|V\|(\widetilde{X})
\end{aligned}$$

whenever $(x,y)\in\mathcal{W}$, where $C=C(n,q,\alpha)\in(0,\infty)$. Thus (10.17) holds for each $(x,y)\in\mathcal{Y}$ and some $\mathcal{H}^n$-measurable subset $\Sigma_{(x,y)}\subset H_{(x,y)}\cap\mathrm{spt}\,\|\mathbf{C}\|\cap(\mathbf{R}\times B_{|x|/4}(x,y))$.

Now choose a countable collection $\mathcal{I}$ of points $(x,y)\in\mathcal{Y}$ such that $\mathcal{Y}\subset\bigcup_{(x,y)\in\mathcal{I}}B_{|x|/8}(x,y)$ and the collection $\{B_{15|x|/16}(x,y)\}_{(x,y)\in\mathcal{I}}$ can be decomposed into at most $N=N(n)$ pairwise disjoint sub-collections. This can be achieved, e.g., as follows: Use the "5-times covering lemma" [Sim83, Th. 3.3] to extract a countable collection $\mathcal{I}$ of points $(x,y)\in\mathcal{Y}$ such that the collection of closed balls $\{\overline{B}_{|x|/41}(x,y)\}_{(x,y)\in\mathcal{I}}$ is pairwise disjoint and $\mathcal{Y}\subset\bigcup_{(x,y)\in\mathcal{I}}B_{|x|/8}(x,y)$. Then the collection $\mathcal{B}=\{B_{15|x|/16}(x,y)\}_{(x,y)\in\mathcal{I}}$ automatically will have the property that for each $(x_0,y_0)\in\mathcal{I}$,

$$\mathrm{card}\,\{(x,y)\in\mathcal{I}:B_{15|x|/16}(x,y)\cap B_{15|x_0|/16}(x_0,y_0)\neq\emptyset\}\leq N \tag{\dagger}$$

for some fixed constant $N = N(n)$, from which it follows as required that $\cup\mathcal{B} = \bigcup_{j=1}^{N} \cup\mathcal{B}_j$ where $\mathcal{B}_1, \ldots, \mathcal{B}_N \subset \mathcal{B}$ and each $\mathcal{B}_j$ consists of pairwise disjoint balls. To see (†), note that $B_{15|x|/16}(x,y) \cap B_{15|x_0|/16}(x_0,y_0) \neq \emptyset \implies |(x,y)-(x_0,y_0)| \leq 15|x_0|/16 + 15|x|/16$, whence $|x| \leq 31|x_0| \leq 31 \times 31|x|$ and $|(x,y)-(x_0,y_0)| \leq c|x_0| - |x|/41$ where $c = 15/16 + (31\times 15)/16 + 31/41$, which say that $B_{|x_0|/(31\times 41)}(x,y) \subset B_{|x|/41}(x,y) \subset B_{c|x_0|}(x_0,y_0)$; since $B_{|x|/41}(x,y)$, $(x,y) \in \mathcal{I}$ are pairwise disjoint, the assertion (†) follows. Let

$$\mathcal{G} = \bigcup_{(x,y)\in\mathcal{I}} \left((\mathbf{R} \times B_{|x|/8}(x,y)) \setminus \mathcal{C}_{H_{(x,y)}}(\Sigma_{(x,y)}) \right).$$

We deduce from (10.17) that

$$\int_{(\mathbf{R}\times\mathcal{Y})\setminus\mathcal{G}} |\widetilde{r}|^2 \, d\|V\|(\widetilde{X}) + \int_{(\mathbf{R}\times\mathcal{Y})\cap\mathcal{G}} |(\widetilde{x}^1, \widetilde{x}^2, 0)^\perp|^2 \, d\|V\|(\widetilde{X}) \leq C \int_{\mathbf{R}\times B_1} \operatorname{dist}^2(\widetilde{X}, \operatorname{spt}\|\mathbf{C}\|) \, d\|V\|(\widetilde{X}), \tag{10.18}$$

where $C = C(n,q,\alpha) \in (0,\infty)$.

Now let $\mathcal{J}$ be a collection of $J = J(n)$ points $w \in B_{15/16} \setminus \{|\widetilde{x}^2| < 1/28\}$ such that $B_{15/16} \setminus \{|\widetilde{x}^2| < 1/28\} \subset \bigcup_{w\in\mathcal{J}} B_{1/64}(w)$. For $z \in \mathbf{R}^n$ and $\rho > 0$, let $T_\rho(z) = \{(\widetilde{x}\sin\theta, \widetilde{x}\cos\theta, \widetilde{y}) : (\widetilde{x}, \widetilde{y}) \in B_\rho(z), \quad \theta \in [0, 2\pi)\}$. Note that if $\varepsilon_0 = \varepsilon_0(n,q,\alpha,1/32) \in (0,1)$, $\gamma_0 = \gamma_0(n,q,\alpha,1/32) \in (0,1)$, $\beta_0 = \beta_0(n,q,\alpha,1/32) \in (0,1)$ are sufficiently small, then for each $(x,y) \in \mathcal{I}$,

$$\begin{aligned} &\left((\mathbf{R} \times B_{|x|/8}(x,y)) \setminus \mathcal{C}_{H_{(x,y)}}(\Sigma_{(x,y)}) \right) \cap \operatorname{spt}\|V\| \\ &\quad\subseteq \left(T^{\pm}_{9|x|/64}(x,y) \setminus \mathcal{C}_{H_{(x,y)}}(\Sigma_{(x,y)}) \right) \cap \operatorname{spt}\|V\| \\ &\quad\subseteq \left(T^{\pm}_{3|x|/16}(x,y) \setminus \mathcal{C}_{H_{(x,y)}}(\Sigma_{(x,y)}) \right) \cap \operatorname{spt}\|V\| \\ &\quad\subseteq \left((\mathbf{R} \times B_{|x|/4}(x,y)) \setminus \mathcal{C}_{H_{(x,y)}}(\Sigma_{(x,y)}) \right) \cap \operatorname{spt}\|V\| \end{aligned} \tag{10.19}$$

and for each $w \in \mathcal{J}$,

$$\mathbf{R} \times B_{1/64}(w) \cap \operatorname{spt}\|V\| \subseteq T^{\pm}_{9/512}(w) \cap \operatorname{spt}\|V\| \subseteq T^{\pm}_{3/128}(w) \cap \operatorname{spt}\|V\|, \tag{10.20}$$

where $T^+_\rho(z) = T_\rho(z) \cap \{|\widetilde{x}^1| < |\widetilde{x}^2|\} \cap \{\widetilde{x}^2 > 0\}$; $T^-_\rho(z) = T_\rho(z) \cap \{|\widetilde{x}^1| < |\widetilde{x}^2|\} \cap \{\widetilde{x}^2 < 0\}$; in (10.19) we choose the $+$ sign if $x > 0$ and the $-$ sign if $x < 0$; in (10.20) we choose the $+$ sign if $e_2 \cdot w > 0$ and the $-$ sign if $e_2 \cdot w < 0$.

Now, applying [Fed69, 3.1.13] with

$$\Phi = \{B_{3|x|/16}(x,y)\}_{(x,y)\in\mathcal{I}} \cup \{B_{3/128}(w)\}_{w\in\mathcal{J}}$$

and letting

$$h(p) = \frac{1}{20} \sup\{\inf\{1, \operatorname{dist}(p, \mathbf{R}^n \setminus B)\} : B \in \Phi\}$$

for $p \in \cup\Phi$, we obtain a smooth partition of unity $\{\varphi_s\}_{s\in\mathcal{S}}$ having the following properties:

(i) $\mathcal{S}$ is a countable subset of $\cup\Phi$ and $\varphi_s : \cup\Phi \to [0,1]\ \forall\ s \in \mathcal{S}$.
(ii) $\{B_{h(s)}(s)\}_{s\in\mathcal{S}}$ is pairwise disjoint and for each $s \in \mathcal{S}$, $B_{h(s)}(s) \subset \operatorname{spt}\varphi_s \subset B_{10h(s)}(s) \subset B$ for some $B \in \Phi$.
(iii) $\sum_{s\in\mathcal{S}} \varphi_s(p) = 1$ for each $p \in \cup\Phi$.
(iv) $|D\varphi_s(p)| \leq Ch(p)^{-1}$ for each $s \in \mathcal{S}$ and each $p \in \cup\Phi$, where $C = C(n) \in (0,\infty)$.

In particular, note that it follows from (iv) and the definition of $h(\cdot)$ that for each $s \in \mathcal{S}$,

$$|D\varphi_s(\widetilde{x},\widetilde{y})| \leq C\,|\widetilde{x}|^{-1} \tag{10.21}$$

whenever $(\widetilde{x},\widetilde{y}) \in \bigcup_{(x,y)\in\mathcal{I}} B_{5|x|/32}(x,y) \cup \bigcup_{w\in\mathcal{J}} B_{5/256}(w)$, where $C = C(n) \in (0,\infty)$. For each $s \in \mathcal{S}$, extend φ_s to $\mathbf{R}^n$ by setting $\varphi_s(x) = 0$ for $x \in \mathbf{R}^n \setminus \cup\Phi$, and let $\widetilde{\varphi}_s$ be the (smooth) extension of φ_s to $\{\widetilde{X} = (\widetilde{x}^1,\widetilde{x}^2,\widetilde{y}) \in \mathbf{R}^{n+1} : |\widetilde{x}^1| < |\widetilde{x}^2|\}$ defined by $\widetilde{\varphi}_s(\widetilde{x}^1,\widetilde{x}^2,\widetilde{y}) = \varphi_s(\pm\sqrt{|\widetilde{x}^1|^2+|\widetilde{x}^2|^2},\widetilde{y})$, where the $+$ sign is chosen if $\widetilde{x}^2 > 0$ and the $-$ sign if $\widetilde{x}^2 < 0$.

Let $\widetilde{\mathcal{G}} = \mathcal{G} \cup \left(\mathbf{R} \times (B_{15/16} \setminus \{|\widetilde{x}^2| < 1/28\})\right)$. We claim that there exists a fixed constant $M = M(n)$ such that for each $(x,y) \in \mathcal{I}$,

$$\operatorname{card}\{s \in \mathcal{S} : \operatorname{spt}\widetilde{\varphi}_s \subset T_{3|x|/16}(x,y) \ \text{ and } \ \operatorname{spt}\widetilde{\varphi}_s \cap \widetilde{\mathcal{G}} \cap \operatorname{spt}\|V\| \neq \emptyset\} \leq M \tag{10.22}$$

and for each $w \in \mathcal{J}$,

$$\operatorname{card}\{s \in \mathcal{S} : \operatorname{spt}\widetilde{\varphi}_s \subset T_{3/128}(w) \ \text{ and } \ \operatorname{spt}\widetilde{\varphi}_s \cap \widetilde{\mathcal{G}} \cap \operatorname{spt}\|V\| \neq \emptyset\} \leq M. \tag{10.23}$$

To see (10.22), fix $(x,y) \in \mathcal{I}$ and let

$$\mathcal{S}_{(x,y)} = \{s \in \mathcal{S} : \operatorname{spt}\widetilde{\varphi}_s \subset T_{3|x|/16}(x,y) \text{ and } \operatorname{spt}\widetilde{\varphi}_s \cap \widetilde{\mathcal{G}} \cap \operatorname{spt}\|V\| \neq \emptyset\}.$$

Note that $\operatorname{spt}\widetilde{\varphi}_s \subset T_{3|x|/16}(x,y) \iff \operatorname{spt}\varphi_s \subset B_{3|x|/16}(x,y)$, and since

$$\widetilde{\mathcal{G}} \cap \operatorname{spt}\|V\| \subset \bigcup_{(x,y)\in\mathcal{I}} \left((\mathbf{R} \times B_{|x|/8}(x,y)) \setminus \mathcal{C}_{H_{(x,y)}}(\Sigma_{(x,y)})\right)$$
$$\cup \bigcup_{w\in\mathcal{J}} (\mathbf{R} \times B_{1/64}(w)) \cap \operatorname{spt}\|V\|,$$

it follows from (10.19), (10.20) and (ii) above that if $s \in \mathcal{S}_{(x,y)}$, then either

($\star$) $B_{3|x|/16}(x,y) \cap B_{9|x'|/64}(x',y') \neq \emptyset$ and $B_{10h(s)}(s) \cap B_{9|x'|/64}(x',y') \neq \emptyset$ for some $(x',y') \in \mathcal{I}$, or
($\star\star$) $B_{10h(s)}(s) \cap B_{9/512}(w') \neq \emptyset$ for some $w' \in \mathcal{J}$.

If ($\star$) holds, then $|x - x'| < 3|x|/16 + 9|x'|/64$ whence $|x'| > 52|x|/73$, and $|s - (x',y')| < 10h(s) + 9|x'|/64$; so if $h(s) < |x'|/640$, then $s \in B_{5|x'|/32}(x',y')$ and hence, since $B_{3|x'|/16}(x',y') \in \Phi$, it follows from the definition of $h(s)$ that $h(s) \geq |x'|/640$ contrary to our assumption. Hence in case ($\star$) holds, we must have that $h(s) \geq 52|x|/(640 \times 73)$. In case ($\star\star$) holds, similar reasoning shows that $h(s) \geq 1/5120$. Thus for any fixed $(x,y) \in \mathcal{I}$, we have established that $s \in \mathcal{S}_{(x,y)} \implies h(s) \geq \min\{52|x|/(640\times73), 1/5120\}$ and (by (ii) above) $B_{h(s)}(s) \subset$

$B_{3|x|/16}(x,y)$. Since $B_{h(s)}(s)$, $s \in \mathcal{S}$ are pairwise disjoint, this establishes (10.22) for some fixed $M = M(n)$. Identical reasoning (using the fact that $|e_2 \cdot w| > 1/28$ for each $w \in \mathcal{J}$) establishes (10.23).

Noting, by (10.19) and the definition of $\Sigma_{(x,y)}$, that the set

$$\left(T^{\pm}_{3|x|/16}(x,y) \setminus \mathcal{C}_{H_{(x,y)}}(\Sigma_{(x,y)})\right) \cap \operatorname{spt} \|V\|,$$

if nonempty, can be written as the union of normal graphs of Lipschitz functions defined over subsets of a sub-collection of the half-hyperplanes $G_1, \ldots, G_q$ (if $x > 0$) or of the half-hyperplanes $H_1, \ldots, H_q$ (if $x < 0$), we see from the area formula and Remark (3) of Section 8 that for any given $(x,y) \in \mathcal{I}$ and any $s \in \mathcal{S}$ with $\operatorname{spt} \widetilde{\varphi}_s \subset T_{3|x|/16}(x,y)$,

(10.24)

$$\begin{aligned}
&\int_{\mathcal{G} \cup (\mathbf{R} \times (B_{15/16} \setminus \{|\widetilde{x}^2| < 1/28\}))} \widetilde{\varphi}_s(\widetilde{X}) \widetilde{r}^2 \widetilde{R}^{-1} \psi(\widetilde{R}) \psi'(\widetilde{R}) \, d\|V\|(\widetilde{X}) \\
&\quad = \sum_{k=1}^{\ell(x,y)} \sum_{i=1}^{q_k(x,y)} \int_{\Omega_k(x,y)} \varphi_s\left(\pm\sqrt{\widetilde{r}^2 + |u^i_k(\widetilde{X})|^2}, \widetilde{y}\right) \\
&\qquad\qquad\qquad \times \widetilde{r}^2_{u^i_k} \widetilde{R}^{-1}_{u^i_k} \psi(\widetilde{R}_{u^i_k}) \psi'(\widetilde{R}_{u^i_k}) \sqrt{1 + |\nabla u^i_k|^2} \, d\mathcal{H}^n(\widetilde{X}) \\
&\quad = \sum_{k=1}^{\ell(x,y)} q_k(x,y) \int_{\Omega_k(x,y)} \varphi_s(\pm \widetilde{r}, \widetilde{y}) \widetilde{r}^2 \widetilde{R}^{-1} \psi(\widetilde{R}) \psi'(\widetilde{R}) \, d\mathcal{H}^n(\widetilde{X}) \\
&\qquad + \sum_{k=1}^{\ell(x,y)} \sum_{i=1}^{q_k(x,y)} \int_{\Omega_k(x,y)} \left(\varphi_s\left(\pm\sqrt{\widetilde{r}^2 + |u^i_k(\widetilde{X})|^2}, \widetilde{y}\right) - \varphi_s(\pm\widetilde{r}, \widetilde{y})\right) \\
&\qquad\qquad\qquad \times \widetilde{r}^2 \widetilde{R}^{-1} \psi(\widetilde{R}) \psi'(\widetilde{R}) \, d\mathcal{H}^n(\widetilde{X}) + E,
\end{aligned}$$

where we choose the $+$ sign if $x > 0$ and the $-$ sign if $x < 0$; $\ell(x,y)$ is a positive integer $\le q$; $q_k(x,y)$ are positive integers with

$$\sum_{k=1}^{\ell(x,y)} q_k(x,y) \le q; \tag{10.25}$$

$\Omega_k(x,y)$ is, by (10.19) and (10.20), a measurable subset of

$$\left(\textstyle\bigcup_{(x',y') \in \mathcal{I}} T_{19|x'|/128}(x',y') \cup \bigcup_{w' \in \mathcal{J}} T_{19/1024}(w')\right) \cap (\mathbf{R} \times B_{|x|/4}(x,y)) \cap G_{j_k(x,y)}$$

(if $x > 0$) or of

$$\left(\textstyle\bigcup_{(x',y') \in \mathcal{I}} T_{19|x'|/128}(x',y') \cup \bigcup_{w' \in \mathcal{J}} T_{19/1024}(w')\right) \cap (\mathbf{R} \times B_{|x|/4}(x,y)) \cap H_{j_k(x,y)}$$

(if $x < 0$) for some integer $j_k(x,y) \in \{1, 2, \ldots, q\}$; u^i_k are the Lipschitz functions as in Remark (3) of Section 8 (applied with $\eta_{Z^{(x,y)}, 7|x|/8\,\#} V$, $\eta_{Z^{(x,y)}, 7|x|/8} \operatorname{spt} \|\mathbf{C}\|$

in place of V,$\mathbf{P}$); $\widetilde{r}_{u_k^i} = \sqrt{\widetilde{r}^2 + |u_k^i|^2}$; $\widetilde{R}_{u_k^i} = \sqrt{\widetilde{R}^2 + |u_k^i|^2}$ and, by the estimates of Remark (3) of Section 8,

$$|E| \leq C \int_{\mathbf{R}\times B_{15|x|/16}(x,y)} \mathrm{dist}^2(\widetilde{X}, \mathrm{spt}\,\|\mathbf{C}\|)\, d\|V\|(\widetilde{X})$$

for some constant $C = (n, q) \in (0, \infty)$. Still assuming $\mathrm{spt}\,\widetilde{\varphi}_s \subset T_{3|x|/16}(x, y)$, we also see in view of (10.25) that

(10.26)

$$\begin{aligned}
&\sum_{k=1}^{\ell(x,y)} q_k(x,y) \int_{\Omega_k(x,y)} \varphi_s(\pm\widetilde{r},\widetilde{y})\widetilde{r}^2\widetilde{R}^{-1}\psi(\widetilde{R})\psi'(\widetilde{R})\, d\mathcal{H}^n(\widetilde{X})\\
&\quad\geq \sum_{k=1}^{\ell(x,y)} q_k(x,y) \int_{P_k\cap T_{3|x|/16}(x,y)} \varphi_s(\pm\widetilde{r},\widetilde{y})\widetilde{r}^2\widetilde{R}^{-1}\psi(\widetilde{R})\psi'(\widetilde{R})\, d\mathcal{H}^n(\widetilde{X})\\
&\quad= \left(\sum_{k=1}^{\ell(x,y)} q_k(x,y)\right) \int_{B_{3|x|/16}(x,y)} \varphi_s(\pm\widetilde{r},\widetilde{y})\widetilde{r}^2\widetilde{R}^{-1}\psi(\widetilde{R})\psi'(\widetilde{R})\, d\mathcal{H}^n(\widetilde{X})\\
&\quad\geq q \int_{B_{3|x|/16}(x,y)} \varphi_s(\pm\widetilde{r},\widetilde{y})\widetilde{r}^2\widetilde{R}^{-1}\psi(\widetilde{R})\psi'(\widetilde{R})\, d\mathcal{H}^n(\widetilde{X})\\
&\quad= \int_{\mathbf{R}\times B_{15/16}} \varphi_s(\pm\widetilde{r},\widetilde{y})\widetilde{r}^2\widetilde{R}^{-1}\psi(\widetilde{R})\psi'(\widetilde{R})\, d\|\mathbf{C}\|(\widetilde{X}),
\end{aligned}$$

where $P_k = G_{j_k}$ if $x > 0$ and $P_k = H_{j_k}$ if $x < 0$. Since we may bound, using the sup estimate of Remark (3)(b) of Section 8 and (10.21) (keeping in mind that $\Omega_k(x,y) \subset \left(\bigcup_{(x',y')\in\mathcal{I}} T_{19|x'|/128}(x',y') \cup \bigcup_{w'\in\mathcal{J}} T_{19/1024}(w')\right) \cap \mathrm{spt}\,\|\mathbf{C}\|$), the absolute value of the middle term of the last line of (10.24) by a constant times $\int_{\mathbf{R}\times B_{15|x|/16}(x,y)} \mathrm{dist}^2(\widetilde{X}, \mathrm{spt}\,\|\mathbf{C}\|)\, d\|V\|(\widetilde{X})$, we conclude from (10.24) and (10.26) that for each $(x, y) \in \mathcal{I}$ and each $s \in \mathcal{S}$ with $\mathrm{spt}\,\widetilde{\varphi}_s \subset T_{3|x|/16}(x, y)$,

(10.27)
$$\begin{aligned}
&\int_{\mathcal{G}\cup(\mathbf{R}\times(B_{15/16}\setminus\{|\widetilde{x}^2|<1/28\}))} \widetilde{\varphi}_s(\widetilde{X})\widetilde{r}^2\widetilde{R}^{-1}\psi(\widetilde{R})\psi'(\widetilde{R})\, d\|V\|(\widetilde{X})\\
&\quad\geq \int_{\mathbf{R}\times B_{15/16}} \varphi_s(\pm\widetilde{r},\widetilde{y})\widetilde{r}^2\widetilde{R}^{-1}\psi(\widetilde{R})\psi'(\widetilde{R})\, d\|\mathbf{C}\|(\widetilde{X})\\
&\qquad - C\int_{\mathbf{R}\times B_{15|x|/16}(x,y)} \mathrm{dist}^2(\widetilde{X}, \mathrm{spt}\,\|\mathbf{C}\|)\, d\|V\|(\widetilde{X}),
\end{aligned}$$

where $C = C(n, q) \in (0, \infty)$; the $+$ sign is chosen if $x > 0$ and the $-$ sign if $x < 0$. By a similar argument, using part (a) and elliptic estimates, we also see that for each $w \in \mathcal{J}$ and each $s \in \mathcal{S}$ with $\mathrm{spt}\,\widetilde{\varphi}_s \subset T_{3/128}(w)$,

$$\begin{aligned}(10.28)\qquad &\int_{\mathcal{G}\cup(\mathbf{R}\times(B_{15/16}\setminus\{|\widetilde{x}^2|<1/28\}))}\widetilde{\varphi}_s(\widetilde{X})\widetilde{r}^2\widetilde{R}^{-1}\psi(\widetilde{R})\psi'(\widetilde{R})\,d\|V\|(\widetilde{X})\\ &\quad\geq\int_{\mathbf{R}\times B_{15/16}}\varphi_s(\pm\widetilde{r},\widetilde{y})\widetilde{r}^2\widetilde{R}^{-1}\psi(\widetilde{R})\psi'(\widetilde{R})\,d\|\mathbf{C}\|(\widetilde{X})\\ &\qquad-C\int_{\mathbf{R}\times B_{1/32}(w)}\operatorname{dist}^2(X,\operatorname{spt}\|\mathbf{C}\|)\,d\|V\|(\widetilde{X}),\end{aligned}$$

where $C=C(n,q)\in(0,\infty)$; the $+$ sign is chosen if $e_2\cdot w>0$ and the $-$ sign if $e_2\cdot w<0$.

Now choose enumerations $\mathcal{J}=\{w_j\}_{j=1}^J$ and $\mathcal{I}=\{(x_{J+j},y_{J+j})\}_{j=1}^\infty$, let

$$\begin{aligned}\mathcal{S}_j=\Big\{&s\in\mathcal{S}:\operatorname{spt}\widetilde{\varphi}_s\subset T_{3/128}(w_j),\\ &\operatorname{spt}\widetilde{\varphi}_s\cap\Big(\mathcal{G}\cup\Big(\mathbf{R}\times(B_{15/16}\setminus\{|\widetilde{x}^2|<1/28\})\Big)\Big)\cap\operatorname{spt}\|V\|\neq\emptyset\Big\}\end{aligned}$$

for $1\leq j\leq J$ and

$$\begin{aligned}\mathcal{S}_j=\{&s\in\mathcal{S}:\operatorname{spt}\widetilde{\varphi}_s\subset T_{3|x_j|/16}(x_j,y_j),\\ &\operatorname{spt}\widetilde{\varphi}_s\cap\Big(\mathcal{G}\cup\Big(\mathbf{R}\times(B_{15/16}\setminus\{|\widetilde{x}^2|<1/28\})\Big)\Big)\cap\operatorname{spt}\|V\|\neq\emptyset\}\end{aligned}$$

for $j\geq J+1$, write

$$\begin{aligned}\Big\{s\in\mathcal{S}:\operatorname{spt}\widetilde{\varphi}_s\cap\Big(\mathcal{G}\cup\Big(\mathbf{R}\times(B_{15/16}\setminus\{|\widetilde{x}^2|<1/28\})\Big)\Big)&\\ \cap\operatorname{spt}\|V\|\neq\emptyset\Big\}&=\textstyle\bigcup_{j=1}^\infty\mathcal{S}'_j,\end{aligned}$$

where $\mathcal{S}'_1=\mathcal{S}_1$ and $\mathcal{S}'_j=\mathcal{S}_j\setminus\bigcup_{i=1}^{j-1}\mathcal{S}'_i$ for $j\geq 2$, and note that $\mathcal{S}'_j$ are pairwise disjoint and, by (10.22), (10.23), that $\operatorname{card}(\mathcal{S}'_j)\leq M=M(n)$. Summing in (10.27), (10.28) first over $s\in\mathcal{S}'_j$ for fixed j, and then over j (where $j\in\{1,2,\ldots,J\}$ in (10.28) and $j\geq J+1$ in (10.27)) keeping in mind that the collection of balls $\{B_{15|x|/16}(x,y)\}_{(x,y)\in\mathcal{I}}=\{B_{15|x_j|/16}(x_j,y_j)\}_{j=J+1}^\infty$ can be subdivided into at most $N=N(n)$ sub-collections of pairwise disjoint balls, and adding the two resulting inequalities (and using the fact that $\sum_{s\in\mathcal{S}}\widetilde{\varphi}_s(\widetilde{X})=1$ for each point $\widetilde{X}\in\mathcal{G}\cup\Big(\mathbf{R}\times(B_{15/16}\setminus\{|\widetilde{x}^2|<1/28\})\Big)\cap\operatorname{spt}\|V\|$ and $\sum_{s\in\mathcal{S}}\varphi_s(\pm\widetilde{r},\widetilde{y})\leq 1$ for each point $\widetilde{X}=(\widetilde{x}^1,\widetilde{x}^2,\widetilde{y})\in\operatorname{spt}\|\mathbf{C}\|$), we conclude that

$$\begin{aligned}(10.29)\qquad &\int_{\mathcal{G}\cup(\mathbf{R}\times(B_{15/16}\setminus\{|\widetilde{x}^2|<1/28\}))}\widetilde{r}^2\widetilde{R}^{-1}\psi(\widetilde{R})\psi'(\widetilde{R})\,d\|V\|(\widetilde{X})\\ &\quad-\int_{\mathbf{R}\times B_{15/16}}\widetilde{r}^2\widetilde{R}^{-1}\psi(\widetilde{R})\psi'(\widetilde{R})\,d\|\mathbf{C}\|(\widetilde{X})\\ &\quad\geq-C\int_{\mathbf{R}\times B_1}\operatorname{dist}^2(\widetilde{X},\operatorname{spt}\|\mathbf{C}\|)\,d\|V\|(\widetilde{X}),\end{aligned}$$

where $C=C(n,q)\in(0,\infty)$. In view of (10.13), (10.14), (10.15), conclusions (b) and (c) now follow from the estimates (10.18), (10.29) and conclusion (a).

Conclusion (d) follows from conclusion (b) by exactly the same argument as for the corresponding estimate in Lemma 3.4 of [Sim93]. □

For the proof of Corollary 10.2 below and subsequently, we shall need the following elementary fact: If $\mathbf{C} \in \mathcal{C}_q$ is as in Hypothesis 10.1(2) and if $Z = (\zeta^1, \zeta^2, \eta) \in \mathbf{R} \times \mathbf{R} \times \mathbf{R}^{n-1} \equiv \mathbf{R}^{n+1}$, then for any $X \in \mathbf{R}^{n+1}$,

$$|\mathrm{dist}(X, \mathrm{spt}\,\|\mathbf{C}\|) - \mathrm{dist}(X, \mathrm{spt}\,\|T_{Z\,\#}\,\mathbf{C}\|)| \leq |\zeta^1| + \nu|\zeta^2|, \tag{10.30}$$

where $T_Z : \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ is the translation $X \mapsto X + Z$ and

$$\nu = \max\{|\lambda_1|, \ldots, |\lambda_q|, |\mu_1|, \ldots, |\mu_q|\}.$$

Indeed, by the triangle inequality

$$|\mathrm{dist}(X, \mathrm{spt}\,\|\mathbf{C}\|) - \mathrm{dist}(X, \mathrm{spt}\,\|T_{Z\,\#}\,\mathbf{C}\|)| \leq \mathrm{dist}_{\mathcal{H}}(\mathrm{spt}\,\|\mathbf{C}\|, \mathrm{spt}\,\|T_{Z\,\#}\,\mathbf{C}\|)$$

and by translation invariance of $\mathbf{C}$ along $\{0\} \times \mathbf{R}^{n-1}$,

$$\begin{aligned}
&\mathrm{dist}_{\mathcal{H}}(\mathrm{spt}\,\|\mathbf{C}\|, \mathrm{spt}\,\|T_{Z\,\#}\,\mathbf{C}\|)\\
&\qquad = \mathrm{dist}_{\mathcal{H}}(\mathrm{spt}\,\|\mathbf{C}\|, \mathrm{spt}\,\|T_{(\zeta^1,\zeta^2,0)\,\#}\,\mathbf{C}\|)\\
&\qquad \leq \mathrm{dist}_{\mathcal{H}}(\mathrm{spt}\,\|\mathbf{C}\|, \mathrm{spt}\,\|T_{(\zeta^1,0,0)\,\#}\,\mathbf{C}\|)\\
&\qquad\quad + \mathrm{dist}_{\mathcal{H}}(\mathrm{spt}\,\|T_{(\zeta^1,0,0)\,\#}\,\mathbf{C}\|, \mathrm{spt}\,\|T_{(\zeta^1,\zeta^2,0)\,\#}\,\mathbf{C}\|)\\
&\qquad = \mathrm{dist}_{\mathcal{H}}(\mathrm{spt}\,\|\mathbf{C}\|, \mathrm{spt}\,\|T_{(\zeta^1,0,0)\,\#}\,\mathbf{C}\|)\\
&\qquad\quad + \mathrm{dist}_{\mathcal{H}}(\mathrm{spt}\,\|\mathbf{C}\|, \mathrm{spt}\,\|T_{(0,\zeta^2,0)\,\#}\,\mathbf{C}\|) \leq |\zeta^1| + \nu|\zeta^2|.
\end{aligned}$$

COROLLARY 10.2. *Let q be an integer ≥ 2 and $\alpha \in (0,1)$. For each $\rho \in (0, 1/4]$, there exist numbers $\varepsilon = \varepsilon(n, q, \alpha, \rho) \in (0,1)$, $\gamma = \gamma(n, q, \alpha, \rho) \in (0,1)$ and $\beta = \beta(n, q, \alpha, \rho) \in (0,1)$ such that the following is true: If $V \in \mathcal{S}_\alpha$, $\mathbf{C} \in \mathcal{C}_q$ satisfy Hypotheses 10.1, Hypothesis $(\star)$ with $M = \frac{3}{2}M_0^3$ and Hypothesis $(\star\star)$, and if the induction hypotheses* (H1), (H2) *hold, then for each $Z = (\zeta^1, \zeta^2, \eta) \in \mathrm{spt}\,\|V\| \cap (\mathbf{R} \times B_{3/8})$ with $\Theta(\|V\|, Z) \geq q$ and each $\mu \in (0,1)$ we have the following*:

(a) $$|\zeta^1|^2 + \hat{E}_V^2|\zeta^2|^2 \leq C \int_{\mathbf{R}\times B_1} \mathrm{dist}^2(X, \mathrm{spt}\,\|\mathbf{C}\|)\, d\|V\|(X);$$

(b) $$\int_{B^{n+1}_{5\rho/8}(Z)} \frac{\mathrm{dist}^2(X, \mathrm{spt}\,\|T_{Z\,\#}\,\mathbf{C}\|)}{|X - Z|^{n+2-\mu}}\, d\|V\|(X) \leq \widetilde{C}\rho^{-n-2+\mu} \int_{\mathbf{R}\times B_\rho(\zeta^2,\eta)} \mathrm{dist}^2(X, \mathrm{spt}\,\|T_{Z\,\#}\,\mathbf{C}\|)\, d\|V\|(X).$$

Here $T_Z : \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ is the translation $X \mapsto X + Z$; $C = C(n, q, \alpha) \in (0, \infty)$ and $\widetilde{C} = \widetilde{C}(n, q, \alpha, \mu) \in (0, \infty)$. (In particular, C, $\widetilde{C}$ do not depend on ρ.)

Our proof of this corollary will be based on several preliminary results, given below as Lemma 10.3, Lemma 10.4, Proposition 10.5, Lemma 10.6 and Proposition 10.7.

LEMMA 10.3. *For any given $\delta \in (0,1)$, there exist $\varepsilon' = \varepsilon'(n,q,\alpha,\delta)$, $\gamma' = \gamma'(n,q,\alpha,\delta) \in (0,1)$ such that if Hypotheses* 10.1 *with ε', γ' in place of ε, γ are satisfied by $V \in \mathcal{S}_\alpha$ and $\mathbf{C} \in \mathcal{K}$, and also Hypothesis $(\star)$ with $M = \frac{3}{2}M_0^3$ are satisfied by V, then*

$$|\zeta^1|^2 + \hat{E}_V^2|\zeta^2|^2 < \delta \hat{E}_V^2$$

for each $Z = (\zeta^1,\zeta^2,\eta) \in \operatorname{spt}\|V\| \cap (\mathbf{R} \times B_{3/8})$ with $\Theta(\|V\|, Z) \geq q$.

Proof. The lemma follows by arguing by contradiction, using Remark (3) of Section 6, Theorem 3.5, the remark at the end of Section 8 and the bounds (10.1), (10.2). □

LEMMA 10.4. *Let q be an integer ≥ 3. For any given $\delta \in (0,1)$, there exist $\varepsilon = \varepsilon(n,q,\alpha,\delta)$, $\gamma = \gamma(n,q,\alpha,\delta)$ and $\beta = \beta(n,q,\alpha,\delta) \in (0,1)$ such that if*

(a) *$p' \in \{5,\ldots,2q\}$ and*
(b) *Hypotheses* 10.1, *Hypothesis $(\star)$ with $M = \frac{3}{2}M_0^3$ and Hypothesis $(\star\star)$ are satisfied with $V \in \mathcal{S}_\alpha$, $\mathbf{C} \in \mathcal{C}_q(p')$,*

then

$$|\zeta^1|^2 + \hat{E}_V^2|\zeta^2|^2 \leq \delta \left(Q_V^\star(p'-1)\right)^2$$

for each $Z = (\zeta^1,\zeta^2,\eta) \in \operatorname{spt}\|V\| \cap (\mathbf{R} \times B_{3/8})$ with $\Theta(\|V\|, Z) \geq q$.

We shall eventually prove this lemma by induction on p', but first we need to establish the following:

PROPOSITION 10.5. *Let q be an integer ≥ 2, $p \in \{4,\ldots,2q\}$, and suppose that either*

(i) *$p = 4$, or*
(ii) *$q \geq 3$, $p \geq 5$ and Lemma* 10.4 *holds whenever $p' \in \{5,\ldots,p\}$.*

Then Corollary 10.2 *holds whenever $\mathbf{C} \in \bigcup_{k=4}^p \mathcal{C}_q(k)$.*

Proof. Let ε_0, γ_0 and β_0 be the constants given by Theorem 10.1 taken with $\tau = 1/16$ (say). Suppose that the hypotheses of the proposition are satisfied. Let $\rho \in (0,1/4]$, and suppose that the hypotheses of Corollary 10.2, for suitably small $\varepsilon, \gamma, \beta$ to be determined depending only on n, q, α and ρ, are satisfied by a varifold $V \in \mathcal{S}_\alpha$ and a cone $\mathbf{C} \in \bigcup_{k=4}^p \mathcal{C}_q(k)$.

To show that the conclusions of Corollary 10.2 follow, we need to apply Theorem 10.1 with $\tau = 1/16$ and $\eta_{Z,\rho\,\#}\,V$ in place of V for any $Z = (\zeta^1,\zeta^2,\eta) \in \operatorname{spt}\|V\| \cap (\mathbf{R} \times B_{3/8})$ with $\Theta\left(\|V\|, Z\right) \geq q$. Thus we need to show that it is possible to choose ε, γ, β depending only on n, q, α, ρ such that Hypotheses 10.1, Hypothesis ($\star$) and Hypothesis ($\star\star$) are satisfied with the varifold $\widetilde{V} = \eta_{Z,\rho\,\#}\, V$ in place of V, with ε_0 , γ_0, β_0 in place of ε, γ, β respectively and with $M = \frac{3}{2}M_0^4$. If this is so, then part (b) of Corollary 10.2 follows as the result of a direct application of Theorem 10.1(d) with $\widetilde{V}$ in place of V, and part (a) of Corollary 10.2 follows from the argument of [Wic04, Lemma 6.21] (which in turn is a minor modification of the corresponding argument of [Sim93]), which also requires application of Theorem 10.1(d) with $\widetilde{V}$ in place of V.

Hypothesis 10.1(1) with $\widetilde{V}$ in place of V follows from Theorem 5.1; Hypothesis 10.1(3) with $\widetilde{V}$ in place of V and ε_0 in place of ε holds if $\varepsilon < \rho^{n+2}\varepsilon_0$. Hypothesis 10.1(4) with $\widetilde{V}$ in place of V is satisfied since by the remark at the end of Section 8, we may choose $\varepsilon = \varepsilon(n, q, \alpha, \rho)$, $\gamma = \gamma(n, q, \alpha, \rho)$ sufficiently small to ensure that $\{Z : \Theta(\|V\|, Z) \geq q\} \cap (\mathbf{R} \times B_{1/2}) \subset \{|x^2| < \rho/128\}$.

To verify that Hypotheses 10.1(5) is satisfied with $\widetilde{V}$ in place of V and γ_0 in place of γ, we proceed as follows: First, using Theorem 10.1(a) with $\tau = \rho/32$, we note that for $\varepsilon = \varepsilon(n, q, \alpha, \rho), \gamma = \gamma(n, q, \alpha, \rho), \beta = \beta(n, q, \alpha, \rho) \in (0, 1)$ sufficiently small,

$$
\begin{aligned}
(10.31)\quad & \rho^{-n-2}\int_{\mathbf{R}\times B_\rho(\zeta^2,\eta)} |x^1 - \zeta^1|^2\, d\|V\|(X) \\
&\geq \rho^{-n-2}\sum_{j=1}^{q}\Bigg(\int_{B_\rho(\zeta^2,\eta)\cap\{x^2<-\rho/16\}} |h_j + u_j - \zeta^1|^2 \\
&\qquad\qquad + \int_{B_\rho(\zeta^2,\eta)\cap\{x^2>\rho/16\}} |g_j + w_j - \zeta^1|^2\Bigg) \\
&\geq \frac{1}{2}\rho^{-n-2}\sum_{j=1}^{q}\left(\int_{B_{\rho/2}\cap\{x^2<-\rho/16\}} |h_j|^2 + \int_{B_{\rho/2}\cap\{x^2>\rho/16\}} |g_j|^2\right) \\
&\quad - \rho^{-n-2}\sum_{j=1}^{q}\left(\int_{B_\rho(\zeta^2,\eta)\cap\{x^2<-\rho/16\}} |u_j|^2 + \int_{B_\rho(\zeta^2,\eta)\cap\{x^2>\rho/16\}} |w_j|^2\right) \\
&\quad - C\rho^{-2}|\zeta^1|^2 \\
&\geq 2^{-n-3}\overline{C}_1\left(\sum_{j=1}^{q} |\lambda_j|^2 + |\mu_j|^2\right) - \rho^{-n-2}E_V^2 - C\rho^{-2}|\zeta^1|^2,
\end{aligned}
$$

where

$$
E_V^2 = \int_{\mathbf{R}\times B_1} \operatorname{dist}^2(X, \operatorname{spt}\|\mathbf{C}\|)\, d\|V\|(X),
$$

$$
\overline{C}_1 = \overline{C}_1(n) \equiv \int_{B_{1/2}\setminus\{x^2>1/16\}} |x^2|^2\, d\mathcal{H}^n(x^2, y), \qquad C = C(n, q) \in (0, 1)
$$

and the rest of the notation is as in Theorem 10.1(a). If $\varepsilon = \varepsilon(n,q,\alpha,\rho)$, $\gamma = \gamma(n,q,\alpha,\rho) \in (0,1)$ are sufficiently small, it follows from (10.31), (10.2) and Lemma 10.3 that

$$\hat{E}_{\widetilde{V}} \geq C\hat{E}_V, \tag{10.32}$$

where $C = C(n,q) \in (0,\infty)$. On the other hand, by (10.30) and (10.1), we have that

$$\begin{aligned}
&\int_{\mathbf{R}\times B_1} \operatorname{dist}^2(X, \operatorname{spt}\|\mathbf{C}\|)\, d\|\widetilde{V}\|(X) \\
&\qquad \leq 2\rho^{-n-2}\int_{\mathbf{R}\times B_1} \operatorname{dist}^2(X, \operatorname{spt}\|\mathbf{C}\|)\, d\|V\|(X) + C\rho^{-2}\left(|\zeta^1|^2 + \hat{E}_V^2|\zeta^2|^2\right),
\end{aligned}$$

where $C = C(n,q) \in (0,\infty)$ and, provided $\varepsilon = \varepsilon(n,q,\alpha,\rho)$, $\gamma = \gamma(n,q,\alpha,\rho)$, $\beta = \beta(n,q,\alpha,\rho)$ are sufficiently small,

$$\begin{aligned}
&\int_{\mathbf{R}\times\left(B_{1/2}\setminus\{|x^2|<1/16\}\right)} \operatorname{dist}^2(X, \operatorname{spt}\|\widetilde{V}\|)\, d\|\mathbf{C}\|(X) \\
&\quad = \rho^{-n-2}\int_{\mathbf{R}\times(B_\rho(Z)\setminus\{|x^2-\zeta^2|<\rho/16\})} \operatorname{dist}^2(X, \operatorname{spt}\|V\|)\, d\|T_{Z\,\#}\,\mathbf{C}\|(X) \\
&\quad \leq \rho^{-n-2}\int_{\mathbf{R}\times\left(B_{17\rho/16}(0,\eta)\setminus\{|x^2|<\rho/32\}\right)} \operatorname{dist}^2(X, \operatorname{spt}\|V\|)\, d\|T_{Z\,\#}\,\mathbf{C}\|(X) \\
&\quad \leq \rho^{-n-2}\int_{\mathbf{R}\times\left(B_{5/8}(0)\setminus\{|x^2|<\rho/32\}\right)} \operatorname{dist}^2(X, \operatorname{spt}\|V\|)\, d\|\mathbf{C}\|(X) \\
&\qquad + C\rho^{-2}\left(|\zeta^1|^2 + \hat{E}_V^2|\zeta^2|^2\right) \\
&\quad \leq C\rho^{-n-2}\int_{\mathbf{R}\times B_1} \operatorname{dist}^2(X, \operatorname{spt}\|\mathbf{C}\|)\, d\|V\|(X) + C\rho^{-2}\left(|\zeta^1|^2 + \hat{E}_V^2|\zeta^2|^2\right),
\end{aligned}$$

where $C = C(n,q) \in (0,\infty)$, the second inequality follows from the area formula and (10.1), and the last inequality follows from Theorem 10.1(a) applied with $\tau = \rho/64$. Thus

$$Q^2_{\widetilde{V}}(\mathbf{C}) \leq C\left(\rho^{-n-2}Q^2_V(\mathbf{C}) + \rho^{-2}(|\zeta^1|^2 + \hat{E}_V^2|\zeta^2|^2)\right) \tag{10.33}$$

which, in view of (10.32) and Lemma 10.3 applied with sufficiently small $\delta = \delta(n,q,\alpha,\rho) \in (0,1)$, implies that Hypothesis 10.1(5) is satisfied with $\widetilde{V}$ in place of V and γ_0 in place of γ.

To verify that Hypothesis ($\star$) is satisfied with $\widetilde{V}$ in place of V and $M = \frac{3}{2}M_0^4$, reasoning again as in (10.31), we see first that for any hyperplane P of the form $P = \{x^1 = \lambda x^2\}$ with $|\lambda| < 1$,

$$
\begin{aligned}
(10.34)\quad \rho^{-n-2}&\int_{\mathbf{R}\times B_\rho(\zeta^2,\eta)} \mathrm{dist}^2(X-Z,P)\,d\|V\|(X)\\
&\geq 2^{-n-4}\overline{C}_1\bigg(\sum_{j=1}^q |\lambda_j-\lambda|^2+|\mu_j-\lambda|^2\bigg)\\
&\quad -\frac{1}{2}\rho^{-n-2}E_V^2 - C\rho^{-2}|\zeta^1-\lambda\zeta^2|^2\\
&\geq 2^{-n-4}\overline{C}_1\mathrm{dist}^2_{\mathcal{H}}(\mathrm{spt}\,\|\mathbf{C}\|\cap(\mathbf{R}\times B_1), P\cap(\mathbf{R}\times B_1))\\
&\quad -\frac{1}{2}\rho^{-n-2}E_V^2 - C\rho^{-2}|\zeta^1-\lambda\zeta^2|^2\\
&\geq 2^{-n-4}\omega_n^{-1}(2q+1)^{-1}\overline{C}_1\int_{\mathbf{R}\times B_1}\mathrm{dist}^2(X,P)\,d\|V\|(X)\\
&\quad -\Big(2^{-n-3}\omega_n^{-1}(2q+1)^{-1}\overline{C}_1+2^{-1}\rho^{-n-2}\Big)E_V^2 - C\rho^{-2}|\zeta^1-\lambda\zeta^2|^2,
\end{aligned}
$$

where $C = C(n,q) \in (0,\infty)$ and we have used the triangle inequality in the last step. On the other hand, noting, by the Constancy Theorem, that $(\omega_n(2\rho)^n)^{-1}\,\|V\|\,(\mathbf{R}\times B_{2\rho}(0,\eta)) \leq q+1/2$ provided $\varepsilon=\varepsilon(n,q,\rho)\in(0,1)$ is sufficiently small, we see by Lemma 10.3 and the triangle inequality again that

$$
\begin{aligned}
(10.35)\quad \rho^{-n-2}&\int_{\mathbf{R}\times B_\rho(\zeta^2,\eta)} |x^1-\zeta^1|^2\,d\|V\|(X)\\
&\leq 2\rho^{-n-2}\|V\|(\mathbf{R}\times B_{2\rho}(0,\eta))\mathrm{dist}^2_{\mathcal{H}}\left(\mathrm{spt}\,\|\mathbf{C}\|\cap(\mathbf{R}\times B_{2\rho}),\{0\}\times B_{2\rho}\right)\\
&\quad + 2\rho^{-n-2}E_V^2 + C\rho^{-2}\delta\hat{E}_V^2\\
&\leq \Big(2^{n+2}\omega_n(2q+1)c_1^2+C\rho^{-2}\delta\Big)\hat{E}_V^2 + 2\rho^{-n-2}E_V^2,
\end{aligned}
$$

where $c_1 = c_1(n)$ is as in (10.1). Since

$$
\hat{E}_V^2 \leq \frac{3}{2}M_0^3 \inf_{\{P=\{x^1=\lambda x^2\}\}}\int_{\mathbf{R}\times B_1}\mathrm{dist}^2(X,P)\,d\|V\|(X)
$$

by hypothesis (of Corollary 10.2), in view of the fact that

$$
\begin{aligned}
\inf_{\{P=\{x^1=\lambda x^2\}\}}&\int_{\mathbf{R}\times B_1}\mathrm{dist}^2(X,P)\,d\|V\|(X)\\
&= \inf_{\{P=\{x^1=\lambda x^2\},\,|\lambda|<C\hat{E}_V\}}\int_{\mathbf{R}\times B_1}\mathrm{dist}^2(X,P)\,d\|V\|(X),
\end{aligned}
$$

where $C=C(n)\in(0,\infty)$, we deduce from Lemma 10.3, (10.34) and (10.35) that

$$
\begin{aligned}
\hat{E}^2_{\widetilde{V}} &\leq \frac{(2^{2n+6}\omega_n^2(2q+1)^2c_1^2+2^{n+4}\omega_n(2q+1)(2\rho^{-n-2}\gamma+C\rho^{-2}\delta))\,\frac{3}{2}M_0^3}{\overline{C}_1-2^{n+4}\omega_n(2q+1)\Big((2^{-1}\rho^{-n-2}+2^{-n-3}\omega_n^{-1}(2q+1)^{-1}\overline{C}_1)\gamma+C\rho^{-2}\delta\Big)\frac{3}{2}M_0^3}\\
&\quad\times \inf_{\{P=\{x^1=\lambda x^2\}\}}\int_{\mathbf{R}\times B_1}\mathrm{dist}^2(X,P)\,d\|\widetilde{V}\|(X)\\
&\leq \frac{3}{2}M_0^4\inf_{\{P=\{x^1=\lambda x^2\}\}}\int_{\mathbf{R}\times B_1}\mathrm{dist}^2(X,P)\,d\|\widetilde{V}\|(X)
\end{aligned}
$$

provided $\varepsilon=\varepsilon(n,q,\alpha,\rho)$, $\gamma=\gamma(n,q,\alpha,\rho)\in(0,1)$ are sufficiently small.

It only remains to verify that Hypothesis $(\star\star)$ with $\widetilde{V}$ in place of V and β_0 in place of β is satisfied whenever $\mathbf{C} \in \bigcup_{k=4}^{p} \mathcal{C}_q(k)$. If $p = 4$, then $\mathbf{C} \in \mathcal{C}_q(4)$ and there is nothing further to verify, so assume that $q \geq 3$ and $\mathbf{C} \in \mathcal{C}_q(p')$ for some $p' \in \{5, \ldots, p\}$. Then for any $\mathbf{C}' \in \bigcup_{k=4}^{p'-1} \mathcal{C}_q(k)$, we have by the definition of $Q^\star_V(p'-1)$, the triangle inequality and Hypothesis $(\star\star)$ (for V and $\mathbf{C}$ with sufficiently small β) that $\mathrm{dist}^2_{\mathcal{H}}(\mathrm{spt}\,\|\mathbf{C}'\| \cap (\mathbf{R} \times B_1), \mathrm{spt}\,\|\mathbf{C}\| \cap (\mathbf{R} \times B_1)) \geq C\left(Q^\star_V(p'-1)\right)^2$, where $C = C(n,q) \in (0, \infty)$, and hence by Theorem 10.1(a), for sufficiently small $\varepsilon = \varepsilon(n,q,\alpha,\rho)$, $\gamma = \gamma(n,q,\alpha,\rho)$, $\beta = \beta(n,q,\alpha,\rho) \in (0,1)$, that

(10.36)

$$\begin{aligned}
&\int_{\mathbf{R}\times B_1} \mathrm{dist}^2\left(X, \mathrm{spt}\,\|\mathbf{C}'\|\right) d\|\widetilde{V}\|(X)\\
&\geq \sum_{j=1}^{q} \rho^{-n-2} \int_{B_\rho(\zeta^2,\eta)\cap\{x^2<-\frac{\rho}{16}\}} \mathrm{dist}^2\left((h^j(X') + u^j(X'), X') - Z, \mathrm{spt}\,\|\mathbf{C}'\|\right) dX'\\
&\quad + \sum_{j=1}^{q} \rho^{-n-2} \int_{B_\rho(\zeta^2,\eta)\cap\{x^2>\frac{\rho}{16}\}} \mathrm{dist}^2\left((g^j(X') + w^j(X'), X') - Z, \mathrm{spt}\,\|\mathbf{C}'\|\right) dX'\\
&\geq \sum_{j=1}^{q} \rho^{-n-2} \int_{B_{\rho/2}(0,\eta)\cap\{x^2<-\frac{\rho}{16}\}} \mathrm{dist}^2\left((h^j(X') + u^j(X'), X'), \mathrm{spt}\,\|\mathbf{C}'\|\right) dX'\\
&\quad + \sum_{j=1}^{q} \rho^{-n-2} \int_{B_{\rho/2}(0,\eta)\cap\{x^2>\frac{\rho}{16}\}} \mathrm{dist}^2\left((g^j(X') + w^j(X'), X'), \mathrm{spt}\,\|\mathbf{C}'\|\right) dX'\\
&\quad - C'\rho^{-2}(|\zeta^1|^2 + \delta(\mathbf{C}')|\zeta^2|^2)\\
&\geq C\left(Q^\star_V(p-1)\right)^2 - \rho^{-n-2}E_V^2 - C'\rho^{-2}(|\zeta^1|^2 + \delta(\mathbf{C}')|\zeta^2|^2)\\
&\geq \frac{1}{2} C\left(Q^\star_V(p'-1)\right)^2 - C'\rho^{-2}(|\zeta^1|^2 + \delta(\mathbf{C}')|\zeta^2|^2)
\end{aligned}$$

where $C = C(n,q)$, $C' = C'(n,q) \in (0,\infty)$ and

$$\delta(\mathbf{C}') = \mathrm{dist}^2_{\mathcal{H}}\left(\mathrm{spt}\,\|\mathbf{C}'\| \cap (\mathbf{R} \times B_1), \{0\} \times B_1\right).$$

Since $\hat{E}^2_{\widetilde{V}} \leq C\rho^{-n-2}\hat{E}^2_V$ where $C = C(n,q) \in (0,\infty)$, we have that

$$Q^\star_{\widetilde{V}}(p'-1) = \inf_{\{\mathbf{C}' \in \bigcup_{k=4}^{p'-1} \mathcal{C}_q(k) : \delta(\mathbf{C}') < C\rho^{-n-2}\hat{E}^2_V\}} Q_{\widetilde{V}}(\mathbf{C}'),$$

so it follows from (10.36) that

$$\left(Q^\star_{\widetilde{V}}(p'-1)\right)^2 \geq C\left(Q^\star_V(p'-1)\right)^2 - C'\rho^{-n-4}(|\zeta^1|^2 + \hat{E}^2_V|\zeta^2|^2), \tag{10.37}$$

where $C = C(n,q)$, $C' = C'(n,q) \in (0,\infty)$. On the other hand, by (10.33),

$$\begin{aligned}
Q^2_{\widetilde{V}}(\mathbf{C}) &\leq C_1\left(\rho^{-n-2}Q^2_V(\mathbf{C}) + \rho^{-2}(|\zeta^1|^2 + \hat{E}^2_V|\zeta^2|^2)\right)\\
&\leq C_1\beta\rho^{-n-2}\left(Q^\star_V(p'-1)\right)^2 + C_1\rho^{-2}(|\zeta^1|^2 + \hat{E}^2_V|\zeta^2|^2),
\end{aligned} \tag{10.38}$$

where $C_1 = C_1(n,q) \in (0,\infty)$. Since by assumption Lemma 10.4 holds whenever $p' \in \{5,\ldots,p\}$, we may apply Lemma 10.4 with any $\delta = \delta(n,q,\alpha,\rho) \in (0,1)$ satisfying $\max\{C', \beta_0^{-1}C_1\}\rho^{-n-4}\delta < C/2$, where C, C', C_1 are as in (10.37) and (10.38), to conclude that Hypothesis $(\star\star)$ with $\widetilde{V}$, β_0 in place of V, β is satisfied. The proof of the proposition is thus complete. $\square$

LEMMA 10.6. *Let $q \geq 3$ and $\delta \in (0,1)$. There exist $\widetilde{\varepsilon}_1 = \widetilde{\varepsilon}_1(n,q,\alpha,\delta)$, $\widetilde{\beta}_1 = \widetilde{\beta}_1(n,q,\alpha,\delta)$, $\gamma_1 = \gamma_1(n,q,\alpha,\delta)$ and $\beta_1 = \beta_1(n,q,\alpha,\delta) \in (0,1)$ such that if*

(a) $p' \in \{5,\ldots,2q\}$;

(b) *Hypotheses* 10.1(1)–(4), *Hypothesis* $(\star)$ *and Hypothesis* $(\star\star)$ *are satisfied with $V \in \mathcal{S}_\alpha$, $\mathbf{C} \in \mathcal{C}_q(p')$, $M = \frac{3}{2}M_0^3$ and with $\widetilde{\varepsilon}_1, \widetilde{\beta}_1$ in place of ε, β respectively*;

(c) *either*

 (i) $(Q_V^\star(4))^2 \leq \gamma_1 \hat{E}_V^2$, *or*

 (ii) $p' \in \{6,\ldots,2q\}$, $(Q_V^\star(p'-j'))^2 \leq \beta_1 (Q_V^\star(p'-j'-1))^2$ *and* $(Q_V^\star(p'-j'))^2 \leq \gamma_1 \hat{E}_V^2$ *for some* $j' \in \{1,\ldots,p'-5\}$,

then for each $Z = (\zeta^1,\zeta^2,\eta) \in \operatorname{spt}\|V\| \cap (\mathbf{R}\times B_{3/8})$ with $\Theta(\|V\|,Z) \geq q$,

$$|\zeta_1|^2 + \hat{E}_V^2|\zeta_2|^2 < \delta\,(Q_V^\star(4))^2$$

in case (c)(i) *holds and*

$$|\zeta_1|^2 + \hat{E}_V^2|\zeta_2|^2 < \delta\,(Q_V^\star(p'-j'))^2$$

in case (c)(ii) *holds.*

This lemma will follow, in view of the following proposition, from our inductive proof of Lemma 10.4 given below.

PROPOSITION 10.7. *Let q be an integer ≥ 3, $p \in \{5,\ldots,2q\}$, and suppose that either*

 (i) $p = 5$, *or*

 (ii) $p \in \{6,\ldots,2q\}$ *and Lemma* 10.4 *holds whenever* $p' \in \{5,\ldots,p-1\}$.

Then Lemma 10.6 *holds whenever* $p' = p$.

Proof. We argue by contradiction. Fix $p \in \{5,\ldots,2q\}$, and suppose that the hypotheses of the proposition are satisfied.

Note that if Lemma 10.6 with $p' = p$ does not hold, then there exist a number $\delta \in (0,1)$, an integer $j' \in \{1,\ldots,p-5\}$ in case $p \in \{6,\ldots,2q\}$ and, for each $k = 1,2,\ldots$, a varifold $V_k \in \mathcal{S}_\alpha$, a point $Z_k = (\zeta_k^1, z_k^2, \eta_k) \in \operatorname{spt}\|V_k\| \cap (\mathbf{R}\times B_{3/8})$ with $\Theta(\|V_k\|,Z_k) \geq q$, a cone $\mathbf{C}_k \in \mathcal{C}_q(p)$ such that Hypotheses 10.1(1), 10.1(2), 10.1(4) and Hypothesis $(\star)$ are satisfied with V_k in place of V, $\mathbf{C}_k$ in place of $\mathbf{C}$, $M = \frac{3}{2}M_0^3$;

$$\hat{E}_k \to 0;$$

(10.39) $$\left(Q_k^\star(p-1)\right)^{-1} Q_{V_k}(\mathbf{C}_k) \to 0;$$

(10.40) $$\text{either } \hat{E}_k^{-1} Q_k^\star(4) \to 0 \text{ or}$$

(10.41)
$$p \in \{6,\ldots,2q\},\ \left(Q_k^\star(p-j'-1)\right)^{-1} Q_k^\star(p-j') \to 0 \text{ and } \hat{E}_k^{-1} Q_k^\star(p-j') \to 0$$

(or both) and yet

(10.42) $$|\zeta_k^1|^2 + \hat{E}_k^2 |\zeta_k^2|^2 \geq \delta \left(Q_k^\star(4)\right)^2 \text{ in case (10.40) holds and}$$

(10.43) $$|\zeta_k^1|^2 + \hat{E}_k^2 |\zeta_k^2|^2 \geq \delta \left(Q_k^\star(p-j')\right)^2 \text{ in case (10.41) holds,}$$

where we have used the notation $\hat{E}_k = \hat{E}_{V_k}$ and $Q_k^\star(\cdot) = Q_{V_k}^\star(\cdot)$.

For each $k = 1, 2, \ldots$, let $\overline{\mathbf{C}}_k \in \mathcal{C}_q$ be chosen as follows: in case (10.40) holds, $\overline{\mathbf{C}}_k \in \mathcal{C}_q(4)$ is such that $\left(Q_{V_k}(\overline{\mathbf{C}}_k)\right)^2 < \frac{3}{2}\left(Q_{V_k}^\star(4)\right)^2$; in case (10.41) holds, $\overline{\mathbf{C}}_k \in \mathcal{C}_q(p-j')$ is such that $\left(Q_{V_k}(\overline{\mathbf{C}}_k)\right)^2 < \frac{3}{2}\left(Q_{V_k}^\star(p-j')\right)^2$. Note that since the rest of our argument is the same for either case, we use the same notation $\overline{\mathbf{C}}_k$ for either case. Let $\tau_k \in (0, 1/8)$ be such that $\tau_k \searrow 0^+$. By passing to appropriate subsequences without changing notation, we have by Proposition 10.5 and Corollary 10.2 that for each $k = 1, 2, \ldots$,

(10.44) $$|\zeta_k^1|^2 + \hat{E}_k^2 |\zeta_k^2|^2 \leq C E_k^2,$$

where $C = C(n, q, \alpha) \in (0, \infty)$, and for each $\mu \in (0, 1)$,

(10.45)
$$\sum_{j=1}^q \int_{B_{1/8}(\zeta_k^2, \eta_k) \cap \{x^2 < -\tau_k/4\}} \frac{|u_j^k(X') - (\zeta_k^1 - \overline{\lambda}_j^k \zeta_k^2)|^2}{|(h_j^k(X') + u_j^k(X'), X') - (\zeta_k^1, \zeta_k^2, \eta_k)|^{n+2-\mu}}\, dX'$$
$$+ \sum_{j=1}^q \int_{B_{1/8}(\zeta_k^2, \eta_k) \cap \{x^2 > \tau_k/4\}} \frac{|w_j^k(X') - (\zeta_k^1 - \overline{\mu}_j^k \zeta^2)|^2}{|(g_j^k(X') + w_j^k(X'), X') - (\zeta_k^1, \zeta_k^2, \eta_k)|^{n+2-\mu}}\, dX'$$
$$\leq \widetilde{C} E_k^2,$$

where $\widetilde{C} = \widetilde{C}(n, q, \alpha, \mu) \in (0, \infty)$. Here,

$$E_k^2 = \int_{\mathbf{R} \times B_1} \operatorname{dist}^2(X, \operatorname{spt} \|\overline{\mathbf{C}}_k\|)\, d\|V_k\|(X)$$

for each $j \in \{1, 2, \ldots, q\}$ and $k = 1, 2, \ldots$, the functions u_j^k, w_j^k correspond to u_j, w_j of Theorem 10.1(a) when V_k, $\overline{\mathbf{C}}_k$ are taken in place of V, $\mathbf{C}$, and the numbers $\overline{\lambda}_j^k$, $\overline{\mu}_j^k$ correspond to λ_j, μ_j of Hypothesis 10.1(2) when $\overline{\mathbf{C}}_k$ is taken in place of $\mathbf{C}$. Note then that $\overline{\lambda}_1^k \geq \overline{\lambda}_2^k \geq \cdots \geq \overline{\lambda}_q^k$, $\overline{\mu}_1^k \leq \overline{\mu}_2^k \leq \cdots \leq \overline{\mu}_q^k$ and by (10.1) and (10.2),

(10.46) $$c\hat{E}_k \leq \max\{|\overline{\lambda}_1^k|, |\overline{\lambda}_q^k|\} \leq c_1 \hat{E}_k, \quad c\hat{E}_k \leq \max\{|\overline{\mu}_1^k|, |\overline{\mu}_q^k|\} \leq c_1 \hat{E}_k,$$
$$\min\{|\overline{\lambda}_1^k - \overline{\lambda}_q^k|, |\overline{\mu}_1^k - \overline{\mu}_q^k|\} \geq 2c\hat{E}_k,$$

where $c_1 = c_1(n), c = c(n, q) \in (0, \infty)$ are as in (10.1) and (10.2).

Writing $Q_k = Q_{V_k}(\overline{\mathbf{C}}_k)$, we see by Theorem 10.1(a) and elliptic estimates that for each $j \in \{1, 2, \ldots, q\}$, there exist harmonic functions $\varphi_j : B_{3/4} \cap \{x^2 < 0\} \to \mathbf{R}$, $\psi_j : B_{3/4} \cap \{x^2 > 0\} \to \mathbf{R}$ such that $Q_k^{-1} u_j^k \to \varphi_j$, $Q_k^{-1} w_j^k \to \psi_j$ where the convergence is in $C^2(K)$ for each compact subset K of the respective domains of φ_j, ψ_j. By (10.39), $Q_k^{-1} Q_{V_k}(\mathbf{C}_k) \to 0$, which implies that for each $j \in \{1, 2, \ldots, q\}$, there exist constants $\overline{\lambda}_j$, $\overline{\mu}_j$ such that $\varphi_j(x^2, y) = \overline{\lambda}_j x^2$ for $(x^2, y) \in B_{1/2} \cap \{x^2 < 0\}$ and $\psi_j(x^2, y) = \overline{\mu}_j x^2$ for $(x^2, y) \in B_{1/2} \cap \{x^2 > 0\}$. We find a point $\eta \in \{0\} \times \mathbf{R}^{n-1} \cap \overline{B_{3/8}(0)}$ and, by (10.44) and (10.46), numbers $\kappa_1, \kappa_2, \ell_1, \ldots, \ell_q, m_1, \ldots, m_q$ such that, passing to further subsequences without changing notation, $\eta_k \to \eta$, $Q_k^{-1} \zeta_k^1 \to \kappa_1$, $Q_k^{-1} \hat{E}_k \zeta_k^2 \to \kappa_2$, $\hat{E}_k^{-1} \lambda_j^k \to \ell_j$ and $\hat{E}_k^{-1} \mu_j^k \to m_j$. We deduce from (10.45) that

$$\sum_{j=1}^{q} \int_{B_{1/8}(0,\eta) \cap \{x^2 < 0\}} \frac{|\overline{\lambda}_j x^2 - (\kappa_1 - \ell_j \kappa_2)|^2}{(|x^2|^2 + |y - \eta|^2)^{\frac{n+2-\mu}{2}}} \, dx^2 dy + \sum_{j=1}^{q} \int_{B_{1/8}(0,\eta) \cap \{x^2 > 0\}} \frac{|\overline{\mu}_j x^2 - (\kappa_1 - m_j \kappa_2)|^2}{(|x^2|^2 + |y - \eta|^2)^{\frac{n+2-\mu}{2}}} \, dx^2 dy < \infty,$$

which readily implies that $\kappa_1 - \ell_j \kappa_2 = 0$ and $\kappa_1 - m_j \kappa_2 = 0$ for each $j = 1, 2, \ldots, q$. Since by (10.46) not all $\ell_1, \ldots, \ell_q$ are equal, we must have that $\kappa_1 = \kappa_2 = 0$. This contradicts (10.42) in case (10.40) holds and (10.43) in case (10.41) holds. The proposition is thus proved. $\square$

Proof of Lemma 10.4. We prove the lemma by induction on p'. Let $\delta \in (0, 1)$ and consider first the case $p' = 5$. Noting, in view of Proposition 10.7, the validity of Lemma 10.6 with $p' = 5$, let $\widetilde{\varepsilon}_1 = \widetilde{\varepsilon}_1(n, q, \alpha, \delta)$, $\widetilde{\beta}_1 = \widetilde{\beta}_1(n, q, \alpha, \delta)$, $\gamma_1 = \gamma_1(n, q, \alpha, \delta)$, $\beta_1 = \beta_1(n, q, \alpha, \delta)$ be as in Lemma 10.6 with $p' = 5$, and suppose that the hypotheses of Lemma 10.4 with $p' = 5$ are satisfied by some $V \in \mathcal{S}_\alpha$ and $\mathbf{C} \in \mathcal{C}_q(5)$, with $\varepsilon = \min\{\widetilde{\varepsilon}, \varepsilon'(n, q, \alpha, \delta\gamma_1)\}$, $\beta = \widetilde{\beta}_1$ and $\gamma = \min\{\gamma_1, \gamma'(n, q, \alpha, \delta\gamma_1)\}$, where $\varepsilon' = \varepsilon'(n, q, \alpha, \cdot)$, $\gamma' = \gamma'(n, q, \alpha, \cdot)$ are as in Lemma 10.3. Then hypotheses (a) and (b) of Lemma 10.6 with $p' = 5$ are satisfied by V and $\mathbf{C}$. If also $(Q_V^\star(4))^2 \leq \gamma_1 \hat{E}_V^2$, then by Lemma 10.6 we have the desired conclusion. If on the other hand $(Q_V^\star(4))^2 > \gamma_1 \hat{E}_V^2$, then applying Lemma 10.3 with $\delta\gamma_1$ in place of δ, we again have the desired conclusion. So Lemma 10.4 is established in case $p' = 5$.

Now fix $p \in \{6, \ldots, 2q\}$, and suppose by induction that Lemma 10.4 holds whenever $p' \in \{5, \ldots, p - 1\}$. Then by Proposition 10.7, Lemma 10.6 with $p' = p$ holds. Let $\delta \in (0, 1)$, and let $\widetilde{\varepsilon}_1(n, q, \alpha, \cdot)$, $\widetilde{\beta}_1(n, q, \alpha, \cdot)$, $\gamma_1(n, q, \alpha, \cdot)$, $\beta_1 = \beta_1(n, q, \alpha, \cdot)$ be as in Lemma 10.6 with $p' = p$. Set $\beta_1^{(0)} = 1$, and for $j = 1, 2, 3, \ldots p - 5$, set $\beta_1^{(j)} = \beta_1(n, q, \alpha, \delta \Pi_{k=0}^{j-1} \beta_1^{(k)})$, $\widetilde{\varepsilon}_1^{(j)} = \widetilde{\varepsilon}_1(n, q, \alpha, \delta \Pi_{k=1}^{j} \beta_1^{(k)})$, $\widetilde{\beta}_1^{(j)} = \widetilde{\beta}_1(n, q, \alpha, \delta \Pi_{k=1}^{j} \beta_1^{(k)})$ and $\gamma_1^{(j)} = \gamma_1(n, q, \alpha, \delta \Pi_{k=1}^{j} \beta_1^{(k)})$. Again let $\varepsilon' =$

$\varepsilon'(n,q,\alpha,\cdot)$, $\gamma' = \gamma'(n,q,\alpha,\cdot)$ be as in Lemma 10.3, let $\delta' = \Pi_{j=1}^{p-5}\gamma_1^{(j)}\beta_1^{(j)}\widetilde{\beta}_1^{(j)}$ and let $\overline{\varepsilon} = \varepsilon'(n,q,\alpha,\delta\delta')$, $\overline{\gamma} = \gamma'(n,q,\alpha,\delta\delta')$.

Let $\mathbf{C} \in \mathcal{C}_q(p)$, $V \in \mathcal{S}_\alpha$, and suppose that the hypotheses of Lemma 10.4 are satisfied with $\varepsilon = \min\{\overline{\varepsilon}, \widetilde{\varepsilon}_1^{(j)} : 1 \leq j \leq p-5\}$, $\beta = \min\{\widetilde{\beta}_1^{(j)} : 1 \leq j \leq p-5\}$ and $\gamma = \min\{\overline{\gamma}, \gamma_1^{(j)} : 1 \leq j \leq p-5\}$. Consider the following exhaustive list of alternatives:

(a) $(Q_V^\star(p-1))^2 \; > \; \delta' \hat{E}_V^2$.

(b$_1$) $(Q_V^\star(p-1))^2 \leq \delta'\hat{E}_V^2$ and $(Q_V^\star(p-1))^2 \leq \beta_1^{(1)}\,(Q_V^\star(p-2))^2$.

(b$_2$) $(Q_V^\star(p-1))^2 \leq \delta'\hat{E}_V^2$, $(Q_V^\star(p-1))^2 > \beta_1^{(1)}\,(Q_V^\star(p-2))^2$ and $(Q_V^\star(p-2))^2 \leq \beta_1^{(2)}\,(Q_V^\star(p-3))^2$.

(b$_3$) $(Q_V^\star(p-1))^2 \leq \delta'\hat{E}_V^2$, $(Q_V^\star(p-1))^2 > \beta_1^{(1)}\,(Q_V^\star(p-2))^2$, $(Q_V^\star(p-2))^2 > \beta_1^{(2)}\,(Q_V^\star(p-3))^2$ and $(Q_V^\star(p-3))^2 \; \leq \; \beta_1^{(3)}\,(Q_V^\star(p-4))^2$.

...

(b$_{p-5}$) $(Q_V^\star(p-1))^2 \leq \delta'\hat{E}_V^2$, $(Q_V^\star(p-1))^2 > \beta_1^{(1)}\,(Q_V^\star(p-2))^2$, $(Q_V^\star(p-2))^2 > \beta_1^{(2)}\,(Q_V^\star(p-3))^2$, $(Q_V^\star(p-3))^2 > \beta_1^{(3)}\,(Q_V^\star(p-4))^2$, $\ldots$, $(Q_V^\star(6))^2 > \beta_1^{(p-6)}\,(Q_V^\star(5))^2$ and $(Q_V^\star(5))^2 \leq \beta_1^{(p-5)}\,(Q_V^\star(4))^2$.

(c) $(Q_V^\star(p-1))^2 \leq \delta'\hat{E}_V^2$, $(Q_V^\star(p-1))^2 > \beta_1^{(1)}\,(Q_V^\star(p-2))^2$, $(Q_V^\star(p-2))^2 > \beta_1^{(2)}\,(Q_V^\star(p-3))^2$, $(Q_V^\star(p-3))^2 > \beta_1^{(3)}\,(Q_V^\star(p-4))^2$, $\ldots$, $(Q_V^\star(6))^2 > \beta_1^{(p-6)}\,(Q_V^\star(5))^2$ and $(Q_V^\star(5))^2 > \beta_1^{(p-5)}\,(Q_V^\star(4))^2$.

The conclusion of Lemma 10.4 in case of alternative (a) follows from Lemma 10.3 applied with $\delta\delta'$ in place of δ; the conclusion of Lemma 10.4 in case of alternative (b$_1$) follows from Lemma 10.6 applied with $p' = p$ and $j = 1$; the conclusion of Lemma 10.4 in case of alternative (b$_2$) follows from Lemma 10.6 applied with $p' = p$, $j = 2$ and $\delta\beta_1^{(1)}$ in place of δ; similarly, the conclusion of Lemma 10.4 in case of any of the alternatives (b$_3$)–(b$_{p-5}$) follows from an application of Lemma 10.6 with $p' = p$ and appropriate value of j and δ; the conclusion of Lemma 10.4 in case of alternative (c) follows from Lemma 10.6 applied with $p' = 5$ and $\delta\Pi_{k=1}^{p-5}\beta_1^{(k)}$ in place of δ. Thus the inductive poof of Lemma 10.4 is complete. □

Proof of Lemma 10.6. Since we have now established Lemma 10.4 for all values of $p' \in \{5, \ldots, 2q\}$, Lemma 10.6 follows from Proposition 10.7. □

Proof of Corollary 10.2. Again, since Lemma 10.4 holds for all values of $p' \in \{5, \ldots, 2q\}$, Corollary 10.2 follows from Proposition 10.5. □

Remark. Note that the proof of Corollary 10.2 establishes that corresponding to each $\varepsilon, \gamma, \beta \in (0,1/2)$ and $\rho \in (0,1/2)$, there exist $\widetilde{\varepsilon} = \widetilde{\varepsilon}(n,q,\alpha,\rho,\varepsilon) \in (0,1/2)$, $\widetilde{\gamma} = \widetilde{\gamma}(n,q,\alpha,\rho,\gamma) \in (0,1/2)$, $\widetilde{\beta} = \widetilde{\beta}(n,q,\alpha,\rho,\beta) \in (0,1/2)$ such that the following is true: Let $V \in \mathcal{S}_\alpha$ and $\mathbf{C} \in \mathcal{C}_q$. If Hypotheses 10.1 are satisfied with $\widetilde{\varepsilon}$, $\widetilde{\gamma}$ in place of ε, γ respectively, Hypothesis ($\star$) is satisfied with $M = \frac{3}{2}M_0^2$ and Hypothesis ($\star\star$) is satisfied with $\widetilde{\beta}$ in place of β, and if the induction hypotheses (H1), (H2) hold, then, for each $Z \in \operatorname{spt}\|V\| \cap (\mathbf{R} \times B_{3/8})$, Hypotheses 10.1, Hypothesis ($\star$) with $M = \frac{3}{2}M_0^3$ and Hypothesis ($\star\star$) are satisfied with $\eta_{Z,\rho\,\#}\, V$ in place of V.

Lemma 10.8. *Let q be an integer ≥ 2, $\alpha \in (0,1)$, $\delta \in (0,1/8)$ and $\mu \in (0,1)$. There exist numbers $\varepsilon_1 = \varepsilon_1(n,q,\alpha,\delta) \in (0,1)$, $\gamma_1 = \gamma_1(n,q,\alpha,\delta) \in (0,1)$ and $\beta_1 = \beta_1(n,q,\alpha) \in (0,1)$ such that the following is true: If $V \in \mathcal{S}_\alpha$, $\mathbf{C} \in \mathcal{C}_q$ satisfy Hypotheses* 10.1, *Hypothesis ($\star$) with ε_1, γ_1 in place of ε, γ respectively and with $M = \frac{3}{2}M_0^3$, and if the induction hypotheses* (H1), (H2) *hold, then*

(a) *$B_\delta^{n+1}(0,y) \cap \{Z : \Theta(\|V\|, Z) \geq q\} \neq \emptyset$ for each point $(0,y) \in \{0\} \times \mathbf{R}^{n-1} \cap B_{1/2}$.*

(b) *If additionally V, $\mathbf{C}$ satisfy Hypothesis ($\star\star$) with β_1 in place of β, then*

$$\int_{B_{1/2}^{n+1}(0)\cap\{|(x^1,x^2)|<\sigma\}} \operatorname{dist}^2(X, \operatorname{spt}\|\mathbf{C}\|)\, d\|V\|(X) \leq C_1\sigma^{1-\mu} \int_{\mathbf{R}\times B_1} \operatorname{dist}^2(X, \operatorname{spt}\|\mathbf{C}\|)\, d\|V\|(X)$$

for each $\sigma \in [\delta, 1/4)$, where $C_1 = C_1(n,q,\alpha,\mu) \in (0,\infty)$. (In particular, C_1 is independent of δ and σ.)

Proof. If part (a) were false, then there would exist a number $\delta \in (0,1/2)$ and a sequence of varifolds $\{V_k\} \subset \mathcal{S}_\alpha$; a sequence of cones $\mathbf{C}_k = \sum_{j=1}^q |H_j^k| + |G_j^k| \in \mathcal{C}_q$ where, for each k, $H_j^k = \{(x^1,x^2,y) \in \mathbf{R}^{n+1} : x^2 < 0 \text{ and } x^1 = \lambda_j^k x^2\}$, $G_j^k = \{(x^1,x^2,y) \in \mathbf{R}^{n+1} : x^2 > 0 \text{ and } x^1 = \mu_j^k x^2\}$, with $\lambda_1^k \geq \lambda_2^k \geq \cdots \geq \lambda_q^k$ and $\mu_1^k \leq \mu_2^k \leq \cdots \leq \mu_q^k$; and a sequence of points $(0,y_k) \in \{0\} \times \mathbf{R}^{n-1} \cap B_{1/2}$ with $B_\delta^{n+1}(0,y_k) \cap \{Z : \Theta(\|V_k\|, Z) \geq q\} = \emptyset$ such that Hypotheses 10.1 (1), (2), (4) are satisfied with V_k, $\mathbf{C}_k$ in place of V, $\mathbf{C}$; Hypothesis ($\star$) is satisfied with $M = \frac{3}{2}M_0^3$ and V_k in place of V; $\hat{E}_k = \hat{E}_{V_k} \equiv \sqrt{\int_{\mathbf{R}\times B_1} |x^1|^2 d\|V_k\|(X)} \to 0$ and

$$\hat{E}_k^{-2} \int_{\mathbf{R}\times(B_{1/2}\setminus\{|x^2|<1/16\})} \operatorname{dist}^2(X, \operatorname{spt}\|V_k\|)\, d\|\mathbf{C}_k\|(X) + \hat{E}_k^{-2} \int_{\mathbf{R}\times B_1} \operatorname{dist}^2(X, \operatorname{spt}\|\mathbf{C}_k\|)\, d\|V_k\|(X) \to 0. \tag{10.47}$$

After passing to a subsequence without changing notation, $(0,y_k) \to (0,y)$ for some point $(0,y) \in \{0\} \times \mathbf{R}^{n-1} \cap \overline{B}_{1/2}$, and hence

$$B_{\delta/2}^{n+1}(0,y) \cap \{Z : \Theta(\|V_k\|, Z) \geq q\} = \emptyset$$

for all sufficiently large k. This implies, by Remark 3 of Section 6, that for all sufficiently large k, $\mathcal{H}^{n-7+\gamma}(\operatorname{sing} V_k \llcorner (B_{\delta/2}^{n+1}(0,y)) = 0$ for each $\gamma > 0$ if $n \geq 7$, and $\operatorname{sing} V_k \llcorner (B^{n+1}_{\delta/2}(0,y)) = \emptyset$ if $2 \leq n \leq 6$, so by Theorem 3.5 and elliptic theory,

$$V_k \llcorner \left(\mathbf{R} \times B_{\delta/4}((0,y))\right) = \sum_{j=1}^{q} |\operatorname{graph} u_j^k| \tag{10.48}$$

for all sufficiently large k, where $v_j^k \in C^\infty\left(B_{\delta/4}(0,y)\right)$, $u_1^k \leq u_2^k \leq \cdots \leq u_q^k$ on $B_{\delta/4}((0,y))$ and u_j^k are solutions of the minimal surface equation on $B_{\delta/4}((0,y))$ satisfying, by standard elliptic estimates,

$$\sup_{B_{\delta/16}(0,y)} |D^\ell\, u_j^k| \leq C\hat{E}_k$$

for $\ell = 0,1,2,3$ and $j = 1,2,\ldots,q$, where $C = C(n,\delta)$. Passing to a further subsequence without changing notation, we deduce that for each $j = 1,2,\ldots,q$, $\hat{E}_k^{-1}u_j^k \to v_j$ in $C^2(B_{\delta/16}(0,y))$ where v_j are harmonic in $B_{\delta/16}(0,y)$ with $v_1 \leq v_2 \leq \ldots v_q$ on $B_{\delta/16}(0,y)$. By (10.47), we see that

$$v_j\Big|_{B_{\delta/16}(0,y)\cap\{x^2<0\}} = \widetilde{h}_j\Big|_{B_{\delta/16}(0,y)\cap\{x^2<0\}}$$

and

$$v_j\Big|_{B_{\delta/16}(0,y)\cap\{x^2>0\}} = \widetilde{g}_j\Big|_{B_{\delta/16}(0,y)\cap\{x^2>0\}},$$

where $\widetilde{h}_j$ and $\widetilde{g}_j$ are linear functions of the form $\widetilde{h}_j(x^2,y) = \widetilde{\lambda}_j x^2$, $\widetilde{g}_j(x^2,y) = \widetilde{\mu}_j x^2$, with $\widetilde{\lambda}_j, \widetilde{\mu}_j \in \mathbf{R}$, $\widetilde{\lambda}_1 \geq \widetilde{\lambda}_2 \geq \cdots \geq \widetilde{\lambda}_q$ and $\widetilde{\mu}_1 \leq \widetilde{\mu}_2 \leq \cdots \leq \widetilde{\mu}_q$. By the maximum principle, we conclude that $\widetilde{\lambda}_j = \widetilde{\mu}_j = \lambda$ for some $\lambda \in \mathbf{R}$ and all $j = 1,2,\ldots,q$. Therefore, by (10.47) again, we see that the coarse blow-up (in the sense of Section 5) of $\{V_k\}$ and that of $\{\mathbf{C}_k\}$ are both equal to the hyperplane $x^1 = \lambda x^2$. But this is impossible in view of (10.2), so the assertion of part (a) must hold.

To see part (b), argue as in [Sim93, Cor. 3.2(ii)] noting that by (10.30), (10.1) and Corollary 10.2(a), we have that for each $Z \in \operatorname{spt}\|V\| \cap (\mathbf{R} \times B_{3/8})$ with $\Theta(\|V\|, Z) \geq q$ and any $X \in \mathbf{R}^{n+1}$,

$$\begin{aligned}|\operatorname{dist}(X, \operatorname{spt}\|\mathbf{C}\|) - \operatorname{dist}(X, \operatorname{spt}\|T_{Z\,\#}\,\mathbf{C}\|)|^2 \\ \leq C\int_{\mathbf{R}\times B_1} \operatorname{dist}^2(\widetilde{X}, \operatorname{spt}\|\mathbf{C}\|)\, d\|V\|(\widetilde{X}),\end{aligned}$$

where $C = C(n,q,\alpha) \in (0,\infty)$. □

11. **Blowing up by fine excess**

Let $\{\varepsilon_k\}$,$\{\gamma_k\}$ and $\{\beta_k\}$ be sequences of positive numbers such that $\varepsilon_k, \gamma_k, \beta_k \to 0$. Consider sequences of varifolds $V_k \in \mathcal{S}_\alpha$ and cones $\mathbf{C}_k \in \mathcal{C}_q$ such that, for each $k = 1,2,\ldots$, with V_k, $\mathbf{C}_k$ in place of V, $\mathbf{C}$ respectively, Hypotheses 10.1 hold with ε_k, γ_k in place of ε, γ; Hypothesis $(\star)$ holds with $M = \frac{3}{2}M_0^3$ and

Hypothesis ($\star\star$) holds with β_k in place of β. Thus, for each $k = 1, 2, \ldots$, we suppose:

(1_k) $\Theta(\|V_k\|, 0) \geq q$, $(2\omega_n)^{-1}\|V_k\|(B_2^{n+1}(0)) < q + 1/2$, $\omega_n^{-1}\|V_k\|(\mathbf{R} \times B_1) < q + 1/2$.

(2_k) $\mathbf{C}_k = \sum_{j=1}^{q} |H_j^k| + |G_j^k|$ where for each $j \in \{1, 2, \ldots, q\}$, H_j^k is the half-space defined by $H_j^k = \{(x^1, x^2, y) \in \mathbf{R}^{n+1} : x^2 < 0 \text{ and } x^1 = \lambda_j^k x^2\}$, G_j^k the half-space defined by $G_j^k = \{(x^1, x^2, y) \in \mathbf{R}^{n+1} : x^2 > 0 \text{ and } x^1 = \mu_j^k x^2\}$, with λ_j^k, μ_j^k constants, $\lambda_1^k \geq \lambda_2^k \geq \cdots \geq \lambda_q^k$ and $\mu_1^k \leq \mu_2^k \leq \cdots \leq \mu_q^k$.

(3_k) $\hat{E}_k^2 = \hat{E}_{V_k}^2 \equiv \int_{\mathbf{R} \times B_1} |x^1|^2 d\|V_k\|(X) < \varepsilon_k$.

(4_k) $\{Z : \Theta(\|V_k\|, Z) \geq q\} \cap \big(\mathbf{R} \times (B_{1/2} \setminus \{|x^2| < 1/16\})\big) = \emptyset$.

(5_k) $\hat{E}_k^{-2} \left(Q_k(\mathbf{C}_k)\right)^2 < \gamma_k$, where

$$
\begin{aligned}
(Q_k(\mathbf{C}_k))^2 &= (Q_{V_k}(\mathbf{C}_k))^2 \\
&= \Bigg(\int_{\mathbf{R} \times (B_{1/2} \setminus \{|x^2| < 1/16\})} \mathrm{dist}^2(X, \mathrm{spt}\,\|V_k\|)\, d\|\mathbf{C}_k\|(X) \\
&\qquad + \int_{\mathbf{R} \times B_1} \mathrm{dist}^2(X, \mathrm{spt}\,\|\mathbf{C}_k\|)\, d\|V_k\|(X)\Bigg).
\end{aligned}
$$

(6_k)
$$\hat{E}_k^2 < \frac{3}{2} M_0^3 \inf_{\{P = \{x^1 = \lambda x^2\}\}} \int_{\mathbf{R} \times B_1} \mathrm{dist}^2(X, P)\, d\|V_k\|(X).$$

(7_k) Either (i) or (ii) below holds:

(i) $\mathbf{C}_k \in \mathcal{C}_q(4)$.

(ii) $q \geq 3$, $\mathbf{C}_k \in \mathcal{C}_q(p_k)$ for some $p_k \in \{5, 6, \ldots, 2q\}$ and $(Q_k^\star)^{-2} \left(Q_k(\mathbf{C}_k)\right)^2 < \beta_k$, where

$$
\begin{aligned}
(Q_k^\star)^2 &= \left(Q_{V_k}^\star(p_k - 1)\right)^2 \\
&= \inf_{\widetilde{\mathbf{C}} \in \bigcup_{j=4}^{p_k - 1} \mathcal{C}_q(j)} \Bigg(\int_{\mathbf{R} \times (B_{1/2} \setminus \{|x^2| < 1/16\})} \mathrm{dist}^2(X, \mathrm{spt}\,\|V_k\|)\, d\|\widetilde{\mathbf{C}}\|(X) \\
&\qquad + \int_{\mathbf{R} \times B_1} \mathrm{dist}^2(X, \mathrm{spt}\,\|\widetilde{\mathbf{C}}\|)\, d\|V_k\|(X)\Bigg).
\end{aligned}
$$

Let $E_k = \sqrt{\int_{\mathbf{R} \times B_1} \mathrm{dist}^2(X, \mathrm{spt}\,\|\mathbf{C}_k\|)\, d\|V_k\|(X)}$ so that by (5_k),

(11.1)
$$\hat{E}_k^{-1} E_k \to 0.$$

Note also that in case $\mathbf{C}_k \notin \mathcal{C}_q(4)$ except for finitely many k, we have by (3_k) and (5_k) that $Q_k^\star \to 0$.

Let $\{\delta_k\}, \{\tau_k\}$ be sequences of decreasing positive numbers converging to 0. By passing to appropriate subsequences of $\{V_k\}$, $\{\mathbf{C}_k\}$, and possibly replacing $\mathbf{C}_k$ with a cone $\mathbf{C}_k' \in \mathcal{C}_q$ with $\mathrm{spt}\,\|\mathbf{C}_k'\| = \mathrm{spt}\,\|\mathbf{C}_k\|$ without changing notation (see Remark (2) following the statement of Hypothesis ($\star\star$)), we deduce that, for each $k = 1, 2, \ldots$, assertions (A_k)–(D_k) below hold:

(A_k) By Lemma 10.8,

$$B^{n+1}_{\delta_k}(0,y) \cap \{Z : \Theta(\|V_k\|, Z) \geq q\} \neq \emptyset \tag{11.2}$$

for each point $(0,y) \in \{0\} \times \mathbf{R}^{n-1} \cap B_{1/2}$ and

$$\int_{B^{n+1}_{1/2}(0)\cap\{|(x^1,x^2)|<\sigma\}} \operatorname{dist}^2(X, \operatorname{spt}\|\mathbf{C}_k\|)\, d\|V_k\|(X) \leq C\sigma^{1/2}E_k^2 \tag{11.3}$$

for each $\sigma \in [\delta_k, 1/4)$, where $C = C(n,q,\alpha) \in (0,\infty)$.

(B_k) By Theorem 10.1(a),

$$V_k \llcorner (\mathbf{R}\times(B_{3/4}\setminus\{|x^2| < \tau_k\})) = \sum_{j=1}^{q} |\operatorname{graph}(h_j^k+u_j^k)|+|\operatorname{graph}(g_j^k+w_j^k)|, \tag{11.4}$$

where h_j^k, g_j^k are the linear functions on $\mathbf{R}^n$ given by $h_j^k(x^2,y) = \lambda_j^k x^2$, $g_j^k(x^2,y) = \mu_j^k x^2$, $u_j^k \in C^2(B_{3/4} \cap \{x^2 < -\tau_k\})$, $w_j^k \in C^2(B_{3/4} \cap \{x^2 > \tau_k\})$ with $h_j^k + u_j^k$ and $g_j^k + w_j^k$ solving the minimal surface equation on their respective domains and satisfying

$$h_1^k + u_1^k \leq h_2^k + u_2^k \leq \cdots \leq h_q^k + u_q^k,$$
$$g_1^k + w_1^k \leq g_2^k + w_2^k \leq \cdots \leq g_q^k + w_q^k,$$
$$\operatorname{dist}((h_j^k(x^2,y) + u_j^k(x^2,y), x^2, y), \operatorname{spt}\|\mathbf{C}_k\|) = (1+(\lambda_j^k)^2)^{-1/2}|u_j^k(x^2,y)|$$

for $(x^2,y) \in B_{3/4} \cap \{x^2 < -\tau_k\}$ and

$$\operatorname{dist}((g_j^k(x^2,y) + w_j^k(x^2,y), x^2, y), \operatorname{spt}\|\mathbf{C}_k\|) = (1+(\mu_j^k)^2)^{-1/2}|w_j^k(x^2,y)|$$

for $(x^2,y) \in B_{3/4} \cap \{x^2 > \tau_k\}$.

(C_k) For each point $Z = (\zeta^1, \zeta^2, \eta) \in \operatorname{spt}\|V_k\| \cap (\mathbf{R} \times B_{3/8})$ with $\Theta(\|V_k\|, Z) \geq q$, by Corollary 10.2(a) (taken with $\rho = 1/4$, say),

$$|\zeta^1|^2 + \hat{E}_k^2|\zeta^2|^2 \leq CE_k^2, \tag{11.5}$$

where $C = C(n,q,\alpha) \in (0,\infty)$.

(D_k) By (10.1) and (10.2),

$$c\hat{E}_k \leq \max\{|\lambda_1^k|, |\lambda_q^k|\} \leq c_1\hat{E}_k, \quad c\hat{E}_k \leq \max\{|\mu_1^k|, |\mu_q^k|\} \leq c_1\hat{E}_k, \tag{11.6}$$
$$\min\{|\lambda_1^k - \lambda_q^k|, |\mu_1^k - \mu_q^k|\} \geq 2c\hat{E}_k,$$

where $c_1 = c_1(n) \in (0,\infty)$ and $c = c(n,q) \in (0,\infty)$.

Furthermore, by Corollary 10.2(b), (11.4) and the area formula, there exists, for each $\rho \in (0, 1/4]$, an integer $K = K(\rho) \geq 1$ such that the following assertion holds for each $k \geq K$:

(E_k) For each point $Z = (\zeta^1, \zeta^2, \eta) \in \operatorname{spt} \|V_k\| \cap (\mathbf{R} \times B_{3/8})$ with $\Theta(\|V_k\|, Z) \geq q$ and each $\mu \in (0,1)$,

(11.7)
$$\sum_{j=1}^{q} \int_{B_{\rho/2}(\zeta^2,\eta)\cap\{x^2<-\tau_k\}} \frac{|u_j^k(x^2,y) - (\zeta^1 - \lambda_j^k \zeta^2)|^2}{|(h_j^k(x^2,y) + u_j^k(x^2,y), x^2, y) - (\zeta^1,\zeta^2,\eta)|^{n+2-\mu}}\, dx^2\, dy$$
$$+ \sum_{j=1}^{q} \int_{B_{\rho/2}(\zeta^2,\eta)\cap\{x^2>\tau_k\}} \frac{|w_j^k(x^2,y) - (\zeta^1 - \mu_j^k \zeta^2)|^2}{|(g_j^k(x^2,y) + w_j(x^2,y), x^2, y) - (\zeta^1,\zeta^2,\eta)|^{n+2-\mu}}\, dx^2 dy$$
$$\leq C_1 \rho^{-n-2+\mu} \int_{\mathbf{R}\times B_\rho(\zeta^2,\eta)} \operatorname{dist}^2(X, \operatorname{spt} \|T_{Z\,\#}\, \mathbf{C}_k\|)\, d\|V_k\|(X),$$

where $C_1 = C_1(n,q,\alpha,\mu) \in (0,\infty)$.

Extend u_j^k, w_j^k to all of $B_{3/4} \cap \{x^2 < 0\}$ and $B_{3/4} \cap \{x^2 > 0\}$ respectively by defining values to be zero in $B_{3/4} \cap \{0 > x^2 \geq -\tau_k\}$ and $B_{3/4} \cap \{0 < x^2 \leq \tau_k\}$ respectively.

By (11.6), there exist numbers ℓ_j, m_j for each $j = 1, 2, \ldots, q$ with

$$c \leq \max\{|\ell_1|, |\ell_q|\} \leq c_1, \quad c \leq \max\{|m_1|, |m_q|\} \leq c_1, \tag{11.8}$$
$$\min\{|\ell_1 - \ell_q|, |m_1 - m_q|\} \geq 2c$$

such that after passing to appropriate subsequences without changing notation,

$$\hat{E}_k^{-1} \lambda_j^k \to \ell_j \quad \text{and} \quad \hat{E}_k^{-1} \mu_j^k \to m_j \tag{11.9}$$

for each $j = 1, 2, \ldots, q$. By (11.4) and elliptic estimates, there exist harmonic functions $\varphi_j : B_{3/4} \cap \{x^2 < 0\} \to \mathbf{R}$ and $\psi_j : B_{3/4} \cap \{x^2 > 0\} \to \mathbf{R}$ such that

$$E_k^{-1} u_j^k \to \varphi_j \quad \text{and} \quad E_k^{-1} w_j^k \to \psi_j, \tag{11.10}$$

where the convergence is in $C^2(K)$ for each compact subset K of the respective domains. From (11.3), it follows that

$$\int_{B_{1/2}\cap\{0>x^2>-\sigma\}} |\varphi|^2 \leq C\sigma^{1/2}, \quad \int_{B_{1/2}\cap\{0<x^2<\sigma\}} |\psi|^2 \leq C\sigma^{1/2}$$

for each $\sigma \in (0, 1/4)$, where $C = C(n,q,\alpha) \in (0,\infty)$, and hence that the convergence in (11.10) is, respectively, also in $L^2(B_{1/2} \cap \{x^2 < 0\})$ and $L^2(B_{1/2} \cap \{x^2 > 0\})$.

Set $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_q)$ and $\psi = (\psi_1, \psi_2, \ldots, \psi_q)$.

Definitions. (1) *Fine blow-ups.* Let $\varphi : B_{3/4} \cap \{x^2 < 0\} \to \mathbf{R}^q$ and $\psi : B_{3/4} \cap \{x^2 > 0\} \to \mathbf{R}^q$ be a pair of functions arising, in the manner described above, corresponding to

(i) a sequence of varifolds $\{V_k\} \subset \mathcal{S}_\alpha$ and a sequence of cones $\{\mathbf{C}_k\} \subset \mathcal{C}_q$ satisfying the hypotheses (1_k)–(7_k) for some sequences of numbers $\{\varepsilon_k\}$, $\{\gamma_k\}$, $\{\beta_k\}$ with $\varepsilon_k, \gamma_k, \beta_k \to 0^+$, and

(ii) sequences $\{\delta_k\}$, $\{\tau_k\}$ of decreasing positive numbers converging to zero such that (11.2), (11.3), (11.4) and (11.7) hold. We call the pair (φ, ψ) a *fine blow-up* of the sequence $\{V_k\}$ relative to $\{\mathbf{C}_k\}$.

(2) *The Class* $\mathcal{B}^F$. Let $\mathcal{B}^F$ be the collection of all fine blow-ups (φ, ψ) such that the corresponding sequences of varifolds $V_k \in \mathcal{S}_\alpha$ satisfies condition (6_k) with M_0^2 in place of M_0^3; thus we assume the stronger condition

$$\hat{E}_{V_k}^2 < \frac{3}{2} M_0^2 \inf_{\{P=\{x^1=\lambda x^2\}\}} \int_{\mathbf{R}\times B_1} \operatorname{dist}^2(X,P)\, d\|V_k\|(X), \quad k=1,2,3,\ldots$$

in place of (6_k) for any sequence $\{V_k\} \subset \mathcal{S}_\alpha$ giving rise to a fine blow-up belonging to $\mathcal{B}^F$.

12. Continuity estimates for the fine blow-ups and their derivatives

Here we first use estimates (11.5) and (11.7) to prove a continuity estimate (Lemma 12.1 below) for any $(\varphi, \psi) \in \mathcal{B}^F$. We then use it to establish the main result of this section (Theorem 12.2), namely, the continuity estimate for the first derivatives of $(\varphi, \psi) \in \mathcal{B}^F$.

LEMMA 12.1. *If* $(\varphi, \psi) \in \mathcal{B}^F$, *then*

$$\varphi \in C^{0,\beta}(\overline{B_{5/16} \cap \{x^2 < 0\}}; \mathbf{R}^q), \quad \psi \in C^{0,\beta}(\overline{B_{5/16} \cap \{x^2 > 0\}}; \mathbf{R}^q)$$

for some $\beta = \beta(n,q,\alpha) \in (0,1)$ *and the following estimates hold*:

$$\sup_{\overline{B_{5/16}\cap\{x^2<0\}}} |\varphi|^2 + \sup_{x,z\in\overline{B_{5/16}\cap\{x^2<0\}}, x\neq z} \frac{|\varphi(x)-\varphi(z)|^2}{|x-z|^{2\beta}} \leq C\left(\int_{B_{1/2}\cap\{x^2<0\}} |\varphi|^2 + \int_{B_{1/2}\cap\{x^2>0\}} |\psi|^2\right),$$

$$\sup_{\overline{B_{5/16}\cap\{x^2>0\}}} |\psi|^2 + \sup_{x,z\in\overline{B_{5/16}\cap\{x^2>0\}}, x\neq z} \frac{|\psi(x)-\psi(z)|^2}{|x-z|^{2\beta}} \leq C\left(\int_{B_{1/2}\cap\{x^2<0\}} |\varphi|^2 + \int_{B_{1/2}\cap\{x^2>0\}} |\psi|^2\right).$$

Here $C = C(n,q,\alpha) \in (0,\infty)$.

Proof. By the definition of fine blow-up, there are sequences $\{V_k\} \subset \mathcal{S}_\alpha$, $\{\mathbf{C}_k\} \subset \mathcal{C}_q$ and sequences of decreasing positive numbers $\{\varepsilon_k\}$, $\{\gamma_k\}$, $\{\beta_k\}$, $\{\delta_k\}$, $\{\tau_k\}$ converging to zero for which all of the assertions of Section 11 hold, with M_0^2 in place of M_0^3 in (6_k).

Let $Y \in \{0\}\times\mathbf{R}^{n-1}\cap B_{5/16}$ be arbitrary. By (11.2), for each $k=1,2,3,\ldots$, there exist $Z_k = (\zeta_1^k, \zeta_2^k, \eta^k) \in \operatorname{spt}\|V_k\|$ with $\Theta(\|V_k\|, Z_k) \geq q$ such that $Z_k \to Y$.

Using (11.3), (11.5), (11.7) (with $\zeta_1^k, \zeta_2^k, \eta^k$ in place of ζ_1, ζ_2, η and $\mu = 1/2$) and (11.9), we deduce that for each $\rho \in (0, 1/8]$,

$$\tag{12.1} \begin{aligned} &\sum_{j=1}^q \int_{B_{\rho/2}(Y)\cap\{x^2<0\}} \frac{|\varphi_j(x) - (\kappa_1(Y) - \ell_j\kappa_2(Y))|^2}{|x-Y|^{n+3/2}}\,dx \\ &\quad + \sum_{j=1}^q \int_{B_{\rho/2}(Y)\cap\{x^2>0\}} \frac{|\psi_j(x) - (\kappa_1(Y) - m_j\kappa_2(Y))|^2}{|x-Y|^{n+3/2}}\,dx \\ &\leq C_1\rho^{-n-3/2}\sum_{j=1}^q \int_{B_{\rho}(Y)\cap\{x^2<0\}} |\varphi_j - (\kappa_1(Y) - \ell_j\kappa_2(Y))|^2 \\ &\quad + C_1\rho^{-n-3/2}\sum_{j=1}^q \int_{B_{\rho}(Y)\cap\{x^2>0\}} |\psi_j - (\kappa_1(Y) - m_j\kappa_2(Y))|^2, \end{aligned}$$

where $C_1 = C_1(n, q, \alpha) \in (0, \infty)$ and we have set

$$\tag{12.2} \kappa_1(Y) = \lim_{k\to\infty} E_k^{-1}\zeta_1^k, \;\; \kappa_2(Y) = \lim_{k\to\infty} E_k^{-1}\hat{E}_k\zeta_2^k,$$

both of which limits exist after passing to a subsequence of the original sequence $\{k\}$. Note that by (11.5),

$$\tag{12.3} |\kappa_1(Y)|, |\kappa_2(Y)| \leq C, \;\; C = C(n, q, \alpha) \in (0, \infty).$$

We remark also that our notation here is appropriate, and the limits in (12.2) indeed depend only on Y and are independent of the sequence of points Z_k converging to Y; this follows directly from the finiteness of the integrals on the left-hand side of (12.1) and the fact that, by Lemma 9.1, at least two of the ℓ_j's and two of the m_j's are distinct.

For $Y \in \{0\} \times \mathbf{R}^{n-1} \cap B_{5/16}$ and each $j = 1, 2, \ldots, q$, define

$$\tag{12.4} \varphi_j(Y) = \kappa_1(Y) - \ell_j\kappa_2(Y) \quad\text{and}\quad \psi_j(Y) = \kappa_1(Y) - m_j\kappa_2(Y).$$

Then by (12.1),

$$\tag{12.5} \begin{aligned} &\sigma^{-n}\left(\int_{B_\sigma(Y)\cap\{x^2<0\}} |\varphi(x) - \varphi(Y)|^2\,dx \;+\; \int_{B_\sigma(Y)\cap\{x^2>0\}} |\psi(x) - \psi(Y)|^2\,dx\right) \\ &\quad\leq C_1\left(\frac{\sigma}{\rho}\right)^{3/2}\rho^{-n}\left(\int_{B_\rho(Y)\cap\{x^2<0\}} |\varphi(x) - \varphi(Y)|^2\,dx\right. \\ &\qquad\qquad\left. + \int_{B_\rho(Y)\cap\{x^2>0\}} |\psi(x) - \psi(Y)|^2\,dx\right) \end{aligned}$$

for each $0 < \sigma \leq \rho/2 \leq 1/32$ and for the same constant $C_1 = C_1(n, q, \alpha) \in (0, \infty)$ as in (12.1).

To complete the proof of the lemma, we follow the argument of Lemma 4.3. Consider an arbitrary point $z^+ \in B_{5/16} \cap \{x^2 > 0\}$ and let $\rho \in (0, 1/16]$.

Denote by z^- the image of z^+ under reflection across $\{0\}\times\mathbf{R}^{n-1}$. Letting $Y\in\{0\}\times\mathbf{R}^{n-1}$ be the point such that $|z^--Y|=|z^+-Y|=\operatorname{dist}(z^+,\{0\}\times\mathbf{R}^{n-1})$, and with $\gamma=\gamma(n,q,\alpha)\in(0,1/16]$ to be chosen, if $\operatorname{dist}(z^+,\{0\}\times\mathbf{R}^{n-1})<\gamma\rho$, then

(12.6)
$$
\begin{aligned}
&(\gamma\rho)^{-n}\left(\int_{B_{\gamma\rho}(z^-)\cap\{x^2<0\}}|\varphi-\varphi(Y)|^2+\int_{B_{\gamma\rho}(z^+)\cap\{x^2>0\}}|\psi-\psi(Y)|^2\right)\\
&\quad\le 2^n(\gamma\rho+|z^--Y|)^{-n}\left(\int_{B_{\gamma\rho+|z^--Y|}(Y)\cap\{x^2<0\}}|\varphi-\varphi(Y)|^2\right.\\
&\qquad\qquad\left.+\int_{B_{\gamma\rho+|z^+-Y|}(Y)\cap\{x^2>0\}}|\psi-\psi(Y)|^2\right)\\
&\quad\le 2^nC_1\left(\frac{\gamma\rho+|z^--Y|}{\rho-|z^--Y|}\right)^{3/2}(\rho-|z^--Y|)^{-n}\left(\int_{B_{\rho-|z^--Y|}(Y)\cap\{x^2<0\}}\right.\\
&\qquad\qquad\left.\cdot|\varphi-\varphi(Y)|^2+\int_{B_{\rho-|z^+-Y|}(Y)\cap\{x^2>0\}}|\psi-\psi(Y)|^2\right)\\
&\quad\le 4^nC_1\left(\frac{2\gamma}{1-\gamma}\right)^{3/2}\rho^{-n}\left(\int_{B_\rho(z^-)\cap\{x^2<0\}}|\varphi-\varphi(Y)|^2\right.\\
&\qquad\qquad\left.+\int_{B_\rho(z^+)\cap\{x^2>0\}}|\psi-\psi(Y)|^2\right).
\end{aligned}
$$

Choosing $\gamma=\gamma(n,q,\alpha)\in(0,1/16]$ such that $4^nC_1\left(\frac{2\gamma}{1-\gamma}\right)^{3/2}<1/4$, we deduce that

(12.7)
$$
\begin{aligned}
&(\gamma\rho)^{-n}\left(\int_{B_{\gamma\rho}(z^-)\cap\{x^2<0\}}|\varphi-\varphi(Y)|^2+\int_{B_{\gamma\rho}(z^+)\cap\{x^2>0\}}|\psi-\psi(Y)|^2\right)\\
&\quad\le 4^{-1}\rho^{-n}\left(\int_{B_\rho(z^-)\cap\{x^2<0\}}|\varphi-\varphi(Y)|^2+\int_{B_\rho(z^+)\cap\{x^2>0\}}|\psi-\psi(Y)|^2\right)
\end{aligned}
$$

for any $z^+\in B_{5/16}\cap\{x^2>0\}$ and $\rho\in(0,1/16]$ provided $\gamma\rho>|z^+-Y|=|z^--Y|=\operatorname{dist}(z^+,\{0\}\times\mathbf{R}^{n-1})$. If on the other hand $\gamma\rho\le\operatorname{dist}(z^+,\{0\}\times\mathbf{R}^{n-1})$, since φ and ψ are harmonic in $B_{1/2}\cap\{x^2<0\}$ and $B_{1/2}\cap\{x^2>0\}$ respectively, we have for each $\sigma\in(0,1/2]$ and any constant vectors $b^+,b^-\in\mathbf{R}^q$,

$$
\begin{aligned}
(12.8)\qquad &(\sigma\gamma\rho)^{-n}\left(\textstyle\int_{B_{\sigma\gamma\rho}(z^-)}|\varphi-\varphi(z^-)|^2+\int_{B_{\sigma\gamma\rho}(z^+)}|\psi-\psi(z^+)|^2\right)\\
&\quad\le C\sigma^2(\gamma\rho)^{-n}\left(\textstyle\int_{B_{\gamma\rho}(z^-)}|\varphi-b^-|^2+\int_{B_{\gamma\rho}(z^+)}|\psi-b^+|^2\right),
\end{aligned}
$$

where $C=C(n)\in(0,\infty)$.

Given any $z^+\in B_{5/16}\cap\{x^2>0\}$, let $j_\star\in\{0,1,2,\dots\}$ be such that $\gamma^{j_\star+1}<\operatorname{dist}(z^+,\{0\}\times\mathbf{R}^{n-1})\le\gamma^{j_\star}$. Then, with $Y\in\{0\}\times\mathbf{R}^{n-1}$ such that

$|z^+ - Y| = \mathrm{dist}(z^+, \{0\} \times \mathbf{R}^{n-1})$, by (12.8),

(12.9)
$$(\sigma\gamma^{j_\star+1})^{-n}\left(\int_{B_{\sigma\gamma^{j_\star+1}}(z^-)} |\varphi - \varphi(z^-)|^2 + \int_{B_{\sigma\gamma^{j_\star+1}}(z^+)} |\psi - \psi(z^+)|^2\right)$$
$$\leq C\sigma^2(\gamma^{j_\star+1})^{-n}\left(\int_{B_{\gamma^{j_\star+1}}(z^-)} |\varphi - \varphi(Y)|^2 + \int_{B_{\gamma^{j_\star+1}}(z^+)} |\psi - \psi(Y)|^2\right)$$

for any $\sigma \in (0, 1/2]$, and if $j_\star \geq 1$, by (12.7),

(12.10)
$$(\gamma^j)^{-n}\left(\int_{B_{\gamma^j}(z^-)\cap\{x^2<0\}} |\varphi - \varphi(Y)|^2 + \int_{B_{\gamma^j}(z^+)\cap\{x^2>0\}} |\psi - \psi(Y)|^2\right)$$
$$\leq 4^{-1}(\gamma^{j-1})^{-n}\left(\int_{B_{\gamma^{j-1}}(z^-)\cap\{x^2<0\}} |\varphi - \varphi(Y)|^2 \right.$$
$$\left. + \int_{B_{\gamma^{j-1}}(z^+)\cap\{x^2>0\}} |\psi - \psi(Y)|^2\right)$$
$$\leq 4^{-(j-1)}\gamma^{-n}\left(\int_{B_\gamma(z^-)\cap\{x^2<0\}} |\varphi - \varphi(Y)|^2 + \int_{B_\gamma(z^+)\cap\{x^2>0\}} |\psi - \psi(Y)|^2\right)$$

for $j = 1, 2, \ldots, j_\star$. If $j_\star \geq 1$, taking $j = j_\star$ in (12.10) and $\sigma = 1/2$ in (12.9), we see by the triangle inequality that

$$|\varphi(z^-) - \varphi(Y)|^2 + |\psi(z^+) - \psi(Y)|^2$$
$$\leq C4^{-(j_\star-1)}\gamma^{-n}\left(\int_{B_\gamma(z^-)\cap\{x^2<0\}} |\varphi - \varphi(Y)|^2 + \int_{B_\gamma(z^+)\cap\{x^2>0\}} |\psi - \psi(Y)|^2\right),$$

where $C = C(n, q, \alpha) \in (0, \infty)$, and hence by (12.10) and the triangle inequality, we again see that

(12.11)
$$(\gamma^j)^{-n}\left(\int_{B_{\gamma^j}(z^-)\cap\{x^2<0\}} |\varphi - \varphi(z^-)|^2 + \int_{B_{\gamma^j}(z^+)\cap\{x^2>0\}} |\psi - \psi(z^+)|^2\right)$$
$$\leq C4^{-(j-1)}\gamma^{-n}\left(\int_{B_\gamma(z^-)\cap\{x^2<0\}} |\varphi - \varphi(Y)|^2 + \int_{B_\gamma(z^+)\cap\{x^2>0\}} |\psi - \psi(Y)|^2\right)$$

for $j = 1, 2, \ldots, j_\star$, where $C = C(n, q, \alpha) \in (0, \infty)$.

By applying (11.5) with $\widetilde{V}_k \equiv \eta_{0,1/2\,\#}\, V_k$ in place of V_k, and noting (e.g., by the argument establishing (10.32)) that $\hat{E}_{\widetilde{V}_k} \geq C\hat{E}_{V_k}$ where $C = C(n, q) \in (0, \infty)$, also using (11.3) and (11.8), we deduce that for each $Y \in \{0\} \times \mathbf{R}^{n-1} \cap B_{5/16}$,

$$|\varphi(Y)|^2 + |\psi(Y)|^2 \leq C\left(\int_{B_{1/2}\cap\{x^2<0\}} |\varphi|^2 + \int_{B_{1/2}\cap\{x^2>0\}} |\psi|^2\right), \tag{12.12}$$

where $C = C(n, q, \alpha) \in (0, \infty)$. With the help of (12.9), (12.10), (12.11) and (12.12), we deduce that for any given $z^+ \in B_{5/16} \cap \{x^2 > 0\}$,

$$\rho^{-n}\left(\int_{B_\rho(z^-)\cap\{x^2<0\}} |\varphi - \varphi(z^-)|^2 + \int_{B_\rho(z^+)\cap\{x^2>0\}} |\psi - \psi(z^+)|^2\right) \leq C\rho^{2\beta}\left(\int_{B_{1/2}\cap\{x^2<0\}} |\varphi|^2 + \int_{B_{1/2}\cap\{x^2>0\}} |\psi|^2\right) \tag{12.13}$$

for all $\rho \in (0, \gamma]$, where $C = C(n, q, \alpha) \in (0, \infty)$ and $\beta = \beta(n, q, \alpha) \in (0, 1)$, by considering, for any given $\rho \in (0, \gamma]$, the alternatives $2\rho \leq \gamma^{j_\star+1}$, in which case $\rho = \sigma\gamma^{j_\star+1}$ for some $\sigma \in (0, 1/2]$ and we use (12.9) and (12.10) with $j = j_\star$, or $\gamma^{j+1} < 2\rho \leq \gamma^j$ for some $j \in \{1, 2, \ldots, j_\star\}$, in which case we use (12.11). The conclusions of the lemma follow readily from (12.13). □

THEOREM 12.2. *If* $(\varphi, \psi) \in \mathcal{B}^F$, *then*

$$\varphi \in C^2(\overline{B_{1/4} \cap \{x^2 < 0\}}; \mathbf{R}^q), \quad \psi \in C^2(\overline{B_{1/4} \cap \{x^2 > 0\}}; \mathbf{R}^q)$$

and the following estimates hold:

$$\sup_{\overline{B_{1/4}\cap\{x^2<0\}}} |D\varphi|^2 + \sup_{x,z\in\overline{B_{1/4}\cap\{x^2<0\}},\, x\neq z} \frac{|D\varphi(x) - D\varphi(z)|^2}{|x-z|^2} \leq C\left(\int_{B_{1/2}\cap\{x^2<0\}} |\varphi|^2 + \int_{B_{1/2}\cap\{x^2>0\}} |\psi|^2\right),$$

$$\sup_{\overline{B_{1/4}\cap\{x^2<0\}}} |D\psi|^2 + \sup_{x,z\in\overline{B_{1/4}\cap\{x^2<0\}},\, x\neq z} \frac{|D\psi(x) - D\psi(z)|^2}{|x-z|^2} \leq C\left(\int_{B_{1/2}\cap\{x^2<0\}} |\varphi|^2 + \int_{B_{1/2}\cap\{x^2>0\}} |\psi|^2\right).$$

Here $C = C(n, q, \alpha) \in (0, \infty)$.

Proof. By the definition of $\mathcal{B}^F$, there are sequences $\{V_k\} \subset \mathcal{S}_\alpha$, $\{\mathbf{C}_k\} \subset \mathcal{C}_q$ and sequences of decreasing positive numbers $\{\varepsilon_k\}$, $\{\gamma_k\}$, $\{\beta_k\}$, $\{\delta_k\}$, $\{\tau_k\}$ for which all of the assertions of Section 11 hold, with M_0^2 in place of M_0^3 in condition (6_k).

By (3.1),

$$\int_{\mathbf{R}\times B_1} \nabla^{V_k} x^1 \cdot \nabla^{V_k} \widetilde{\zeta}\, d\|V_k\|(X) = 0 \tag{12.14}$$

for each $k = 1, 2, \ldots$ and any $\widetilde{\zeta} \in C^1_c(\mathbf{R} \times B_1)$. Let $\tau \in (0, 1/32)$ be arbitrary. Choose any $\zeta \in C^2_c(B_{3/8})$ with $\frac{\partial \zeta}{\partial x^2} \equiv 0$ in $\{|x^2| < 2\tau\}$, and set $\zeta_1(x^1, x^2, y) = \zeta(x^2, y)$ for $(x^1, x^2, y) \in \mathbf{R} \times B_{1/2}$. Let $\widetilde{\zeta} \in C^1_c(\mathbf{R} \times B_{3/8})$ be such that $\widetilde{\zeta} \equiv \zeta_1$ in a neighborhood of spt $\|V_k\| \cap (\mathbf{R} \times B_{3/8})$ for all $k = 1, 2, \ldots$. By (12.14) and (11.4), for all sufficiently large k,

(12.15)

$$\int_{\mathbf{R}\times(B_{3/8}\cap\{|x^2|<2\tau\})} \nabla^{V_k} x^1 \cdot \nabla^{V_k} \widetilde{\zeta}\, d\|V_k\|(X)$$
$$+\sum_{j=1}^{q}\int_{B_{3/8}\cap\{x^2\leq -2\tau\}} \left(1+|D(h_j^k+u_j^k)|^2\right)^{-1/2} D(h_j^k+u_j^k)\cdot D\zeta\, dx$$
$$+\sum_{j=1}^{q}\int_{B_{3/8}\cap\{x^2\geq 2\tau\}} \left(1+|D(g_j^k+w_j^k)|^2\right)^{-1/2} D(g_j^k+w_j^k)\cdot D\zeta\, dx = 0.$$

Since $\frac{\partial \widetilde{\zeta}}{\partial x^1} = 0$ in a neighborhood of spt $\|V_k\| \cap (\mathbf{R}\times B_{1/2})$ and $\frac{\partial \widetilde{\zeta}}{\partial x^2} = 0$ in $\{|x^2| < 2\tau\}$, it follows that

$$(12.16)\qquad \left|\int_{\mathbf{R}\times(B_{3/8}\cap\{|x^2|<2\tau\})} \nabla^{V_k} x^1 \cdot \nabla^{V_k} \widetilde{\zeta}\, d\|V_k\|(X)\right|$$
$$= \left|\int_{\mathbf{R}\times(B_{3/8}\cap\{|x^2|<2\tau\})} e_1 \cdot \nabla^{V_k} \widetilde{\zeta}\, d\|V_k\|(X)\right|$$
$$\leq \sup|D\zeta| \sum_{j=3}^{n+1}\int_{\mathbf{R}\times(B_{3/8}\cap\{|x^2|<2\tau\})} |e_j^{\perp_k}|\, d\|V_k\|(X)$$
$$\leq \sup|D\zeta|\, (\|V_k\|(\mathbf{R}\times(B_{3/8}\cap\{|x^2|<2\tau\})))^{1/2}$$
$$\cdot\left(\sum_{j=3}^{n+1}\int_{\mathbf{R}\times B_{3/8}} |e_j^{\perp_k}|^2\, d\|V_k\|(X)\right)^{1/2}$$
$$\leq C\sup|D\zeta|\sqrt{\tau}E_k,$$

where $C = C(n,q,\alpha) \in (0,\infty)$, and the last inequality is a consequence of Theorem 10.1(c) and the fact that $\|V_k\|(\mathbf{R}\times(B_{3/8}\cap\{|x^2| < 2\tau\})) \leq C\tau$, $C = C(n,q,\alpha) \in (0,\infty)$ for all sufficiently large k.

Since $h_j^k(x) = \lambda_j^k x^2$, we have for each $j = 1,2,\ldots,q$ and $k = 1,2,\ldots,$

(12.17)

$$\int_{B_{3/8}\cap\{x^2\leq -2\tau\}} \left(1+|D(h_j^k+u_j^k)|^2\right)^{-1/2} D(h_j^k+u_j^k)\cdot D\zeta$$
$$= \int_{B_{3/8}\cap\{x^2\leq -2\tau\}} \left(1+|D(h_j^k+u_j^k)|^2\right)^{-1/2} Du_j^k\cdot D\zeta$$
$$-\alpha_j^k\int_{B_{3/8}\cap\{x^2\leq -2\tau\}} \frac{\left(1+|D(h_j^k+u_j^k)|^2\right)^{-1/2} D(2h_j^k+u_j^k)\cdot Du_j^k}{\sqrt{1+|D(h_j^k+u_j^k)|^2}+\sqrt{1+(\lambda_j^k)^2}}\frac{\partial\zeta}{\partial x^2}$$
$$+\alpha_j^k\int_{B_{3/8}\cap\{x^2\leq -2\tau\}} \frac{\partial\zeta}{\partial x^2},$$

where $\alpha_j^k = \lambda_j^k\left(1+(\lambda_j^k)^2\right)^{-1/2}$.

By the Cauchy-Schwarz inequality and elliptic estimates,

$$\begin{aligned}(12.18)\quad &\left|\int_{B_{3/8}\cap\{x^2\le -2\tau\}} \frac{\left(1+|D(h_j^k+u_j^k)|^2\right)^{-1/2} D(2h_j^k+u_j^k)\cdot Du_j^k}{\sqrt{1+|D(h_j^k+u_j^k)|^2}+\sqrt{1+(\lambda_j^k)^2}}\frac{\partial\,\zeta}{\partial\,x^2}\,dx\right| \\ &\le C(\tau)\,\sup|D\zeta|\sqrt{|\lambda_j^k|^2+\int_{B_{1/2}\cap\{x^2\le-\tau\}}|u_j^k|^2\,dx}\sqrt{\int_{B_{1/2}\cap\{x^2\le-\tau\}}|u_j^k|^2\,dx} \\ &\le C(\tau)\sup|D\zeta|\sqrt{|\lambda_j^k|^2+E_k^2}\,E_k.\end{aligned}$$

If ζ also satisfies

$$(12.19)\qquad \int_{B_{3/8}\cap(\{0\}\times\mathbf{R}^{n-1})}\zeta\,dy=0,$$

then, since

$$\int_{B_{3/8}\cap\{x^2\le-2\tau\}}\frac{\partial\,\zeta}{\partial\,x^2}\,dx=\int_{B_{3/8}\cap\{x^2\le 0\}}\frac{\partial\,\zeta}{\partial\,x^2}\,dx=-\int_{B_{3/8}\cap(\{0\}\times\mathbf{R}^{n-1})}\zeta\,dy,$$

the last term on the right-hand side of (12.17) will be zero. Thus, for each fixed $\tau>0$ and each $\zeta\in C_c^1(B_{3/8})$ with $\frac{\partial\zeta}{\partial x^2}=0$ in $\{|x^2|<2\tau\}$ and satisfying (12.19), we have

$$\begin{aligned}(12.20)\quad &\sum_{j=1}^q\int_{B_{3/8}\cap\{x^2\le-2\tau\}}\left(1+|D(h_j^k+u_j^k)|^2\right)^{-1/2}D(h_j^k+u_j^k)\cdot D\zeta\,dx \\ &=\sum_{j=1}^q\int_{B_{3/8}\cap\{x^2\le-2\tau\}}\left(1+|D(h_j^k+u_j^k)|^2\right)^{-1/2}Du_j^k\cdot D\zeta\,dx+\varepsilon_k^-\end{aligned}$$

and, by a similar argument,

$$\begin{aligned}(12.21)\quad &\sum_{j=1}^q\int_{B_{3/8}\cap\{x^2\ge2\tau\}}\left(1+|D(g_j^k+w_j^k)|^2\right)^{-1/2}D(g_j^k+w_j^k)\cdot D\zeta\,dx \\ &=\sum_{j=1}^q\int_{B_{3/8}\cap\{x^2\ge2\tau\}}\left(1+|D(g_j^k+w_j^k)|^2\right)^{-1/2}Dw_j^k\cdot D\zeta\,dx+\varepsilon_k^+,\end{aligned}$$

where $\lim_{k\to\infty}E_k^{-1}|\varepsilon_k^-|=\lim_{k\to\infty}E_k^{-1}|\varepsilon_k^+|=0$. We may divide (12.15) by E_k and let $k\to\infty$ to deduce, by (12.16), (12.20), (12.21) and (11.10), that for each $\tau\in(0,1/16)$,

$$(12.22)\quad \sum_{j=1}^q\int_{B_{3/8}\cap\{x^2\le-2\tau\}}D\varphi_j\cdot D\zeta+\sum_{j=1}^q\int_{B_{3/8}\cap\{x^2\ge2\tau\}}D\psi_j\cdot D\zeta+\varepsilon(\tau)=0,$$

where $\varepsilon(\tau) \to 0$ as $\tau \to 0$. Upon integration by parts (in view of the fact that $\frac{\partial \zeta}{\partial x^2} = 0$ in $\{|x^2| < 2\tau\}$), this gives

$$\sum_{j=1}^{q} \int_{B_{3/8}\cap\{x^2\leq -2\tau\}} \varphi_j \Delta\,\zeta + \sum_{j=1}^{q} \int_{B_{3/8}\cap\{x^2\geq 2\tau\}} \psi_j \Delta\,\zeta - \varepsilon(\tau) = 0. \tag{12.23}$$

Since $\varphi_j \in L^1(B_{1/2} \cap \{x^2 \leq 0\})$ and $\psi_j \in L^1(B_{1/2} \cap \{x^2 \geq 0\})$ for each $j = 1, 2, \ldots, q$, we may let $\tau \to 0$ in (12.23) to conclude that

$$\sum_{j=1}^{q} \int_{B_{3/8}\cap\{x^2\leq 0\}} \varphi_j \Delta\,\zeta + \sum_{j=1}^{q} \int_{B_{3/8}\cap\{x^2\geq 0\}} \psi_j \Delta\,\zeta = 0 \tag{12.24}$$

for any $\zeta \in C_c^2(B_{3/8})$ with $\frac{\partial \zeta}{\partial x^2} = 0$ in a neighborhood of $\{x^2 = 0\}$ and satisfying (12.19).

Now for any $\ell \in \{1, 2, \ldots, n-1\}$, $h \in (-1/16, 1/16)$ and any $\zeta \in C_c^2(B_{5/16})$ with $\frac{\partial \zeta}{\partial x^2} = 0$ in a neighborhood of $\{x^2 = 0\}$, we have that $\delta_{\ell,h}\,\zeta \in C_c^1(B_{3/8})$, $\frac{\partial}{\partial x^2}\,\delta_{\ell,h}\,\zeta = 0$ in a neighborhood of $\{x^2 = 0\}$ and $\delta_{\ell,h}\,\zeta$ satisfies (12.19), where $\delta_{\ell,h}\,\zeta(x^2, y) = \zeta(x^2, y^1, \ldots, y^\ell + h, y^{\ell+1}, \ldots, y^{n-1}) - \zeta(x^2, y)$. Thus, by (12.24),

$$\sum_{j=1}^{q} \int_{B_{3/8}\cap\{x^2< 0\}} \varphi_j\,\Delta\,\delta_{\ell,h}\,\zeta + \sum_{j=1}^{q} \int_{B_{3/8}\cap\{x^2> 0\}} \psi_j\,\Delta\,\delta_{\ell,h}\,\zeta = 0$$

and consequently,

$$\sum_{j=1}^{q} \int_{B_{5/16}\cap\{x^2< 0\}} \delta_{\ell,h}\,\varphi_j\,\Delta\,\zeta + \sum_{j=1}^{q} \int_{B_{5/16}\cap\{x^2> 0\}} \delta_{\ell,h}\,\psi_j\,\Delta\,\zeta = 0 \tag{12.25}$$

for any $\zeta \in C_c^2(B_{5/16})$ with $\frac{\partial \zeta}{\partial x^2} = 0$ in a neighborhood of $\{x^2 = 0\}$, any $\ell \in \{1, 2, \ldots, n-1\}$ and $h \in (-1/16, 1/16)$. Since any $\zeta \in C_c^2(B_{5/16})$ that is even in the x^2 variable can be approximate in $C^2(B_{5/16})$ by a sequence $\zeta_i \in C_c^2(B_{5/16})$ satisfying, for each $i = 1, 2, 3, \ldots$, $\frac{\partial \zeta_i}{\partial x^2} = 0$ in a neighborhood of $\{x^2 = 0\}$, we see that (12.25) holds for any $\zeta \in C_c^2(B_{5/16})$ that is even in the x^2 variable and for each $\ell \in \{1, 2, \ldots, n-1\}$ and $h \in (-1/16, 1/16)$. Thus

$$\int_{B_{5/16}} \Phi_{\ell,h} \Delta\,\zeta = 0 \tag{12.26}$$

for any $\zeta \in C_c^2(B_{5/16})$ that is even in the x^2 variable, any $\ell \in \{1, 2, \ldots, n-1\}$ and $h \in (-1/16, 1/16)$, where $\Phi_{\ell,h} : B_{3/8} \to \mathbf{R}$ is the function defined by $\Phi_{\ell,h}(x^2, y) = \sum_{j=1}^q \delta_{\ell,h}\,\varphi_j(-x^2, y) + \delta_{\ell,h}\,\psi_j(x^2, y)$ if $x^2 \geq 0$ and $\Phi_{\ell,h}(x^2, y) = \sum_{j=1}^q \delta_{\ell,h}\,\varphi_j(x^2, y) + \delta_{\ell,h}\,\psi_j(-x^2, y)$ if $x^2 < 0$. Since Φ is even in the x^2 variable, (12.26) holds also for any ζ that is odd in the x^2 variable. Thus (12.26) holds for every $\zeta \in C_c^2(B_{5/16})$, and hence $\Phi_{\ell,h}$ is a smooth harmonic function in

$B_{5/16}$. Since we have directly from the definition of $\Phi_{\ell,h}$ and Lemma 12.1 that

$$\left|\int_{B_{5/16}} h^{-1}\Phi_{\ell,h}\right| \le C\left(\int_{B_{1/2}\cap\{x^2\le 0\}} |\varphi|^2 + \int_{B_{1/2}\cap\{x^2\ge 0\}} |\psi|^2\right)^{1/2} \tag{12.27}$$

for all $h \in (-1/16, 1/16) \setminus \{0\}$, where $C = C(n,q,\alpha) \in (0,\infty)$, it follows from standard estimates for harmonic functions that there exists a harmonic function $\Phi_\ell : B_{9/32} \to \mathbf{R}$ such that $h^{-1}\Phi_{\ell,h} \to \Phi_\ell$ in $C^2(B_{9/32})$ as $h \to 0$, and

(12.28)
$$\sup_{B_{9/32}} |\Phi_\ell|^2 + |D\Phi_\ell|^2 + |D^2\Phi_\ell|^2 \le C\left(\int_{B_{1/2}\cap\{x^2\le 0\}} |\varphi|^2 + \int_{B_{1/2}\cap\{x^2\ge 0\}} |\psi|^2\right).$$

Let $\Phi : B_{1/2} \to \mathbf{R}$ be the function defined by $\Phi(x^2,y) = \sum_{j=1}^q \varphi_j(x^2,y) + \psi_j(-x^2,y)$ if $x^2 < 0$ and $\Phi(x^2,y) = \sum_{j=1}^q \varphi_j(-x^2,y) + \psi_j(x^2,y)$ if $x^2 \ge 0$. Since $\Phi_\ell = \frac{\partial}{\partial y^\ell}\Phi$ on $B_{1/2} \setminus (\{0\}\times\mathbf{R}^{n-1})$, it follows that for $(x^2,y) \in B_{9/32} \setminus (\{0\}\times\mathbf{R}^{n-1})$,

$$\begin{aligned}\Phi(x^2,y) &= \Phi(x^2,y^1,\ldots,y^{\ell-1},0,y^{\ell+1},\ldots,y^{n-1})\\ &\quad + \int_0^{y^\ell} \Phi_\ell(x^2,y^1,\ldots,y^{\ell-1},t,y^{\ell+1},\ldots,y^{n-1})\,dt,\end{aligned}$$

so we may let $x^2 \to 0$ on both sides of this and use Lemma 12.1, (12.4) and the arbitrariness of the index $\ell \in \{1,2,\ldots,n-1\}$ to conclude that, with $Y = (0,y)$,

$$\Phi(Y) = 2q\kappa_1(Y) - \left(\sum_{j=1}^q (\ell_j + m_j)\right)\kappa_2(Y) \tag{12.29}$$

is a C^∞ function of $Y \in B_{9/32} \cap (\{0\}\times\mathbf{R}^{n-1})$ (with $\frac{\partial}{\partial y^\ell}\Phi(Y) = \Phi_\ell(Y)$, $\frac{\partial^2}{\partial y^m\,\partial y^\ell}\Phi(Y) = \frac{\partial}{\partial y^m}\Phi_\ell(Y)$, $\frac{\partial^3}{\partial y^k\,\partial y^m\,\partial y^\ell}\Phi(Y) = \frac{\partial^2}{\partial y^k\,\partial y^m}\Phi_\ell(Y)$ for each $\ell, m, k \in \{1,2,\ldots,n-1\}$) satisfying, by (12.28) and Lemma 12.1, the estimate

$$\begin{aligned}\sup_{B_{9/32}\cap(\{0\}\times\mathbf{R}^{n-1})} &|\Phi|^2 + |D_Y\,\Phi|^2 + |D_Y^2\,\Phi|^2 + |D_Y^3\,\Phi|^2\\ &\le C\left(\int_{B_{1/2}\cap\{x^2\le 0\}} |\varphi|^2 + \int_{B_{1/2}\cap\{x^2\ge 0\}} |\psi|^2\right),\end{aligned} \tag{12.30}$$

where $C = C(n,q,\alpha) \in (0,\infty)$.

Next we derive regularity estimates for a different linear combination of κ_1 and κ_2. For this, we note that by (3.1) again,

$$\int_{\mathbf{R}\times B_1} \nabla^{V_k} x^2 \cdot \nabla^{V_k}\,\widetilde{\zeta}\,d\|V_k\|(X) = 0 \tag{12.31}$$

for each $k = 1,2,\ldots$ and each $\widetilde{\zeta} \in C_c^1(\mathbf{R}\times B_1)$. Let $\tau \in (0,1/16)$, $\zeta \in C_c^2(B_{3/8})$ and $\widetilde{\zeta}$ be as before so that, in particular, $\frac{\partial\zeta}{\partial x^2} = 0$ in $\{|x^2| < 2\tau\}$.

Note that the unit normal ν_j^k to $(\mathcal{M}_j^k)^- \equiv \text{graph}(h_j^k + u_j^k)$ is given by $\nu_j^k = \left((1+|D(h_j^k+u_j^k)|^2\right)^{-1/2}\left(1, -\lambda_j^k - \frac{\partial u_j^k}{\partial x^2}, -D_y\, u_j^k\right)$ so that, on $(\mathcal{M}_j^k)^-$,

$$\begin{aligned}
\nabla^{V_k} x^2 \cdot \nabla^{V_k} \widetilde{\zeta} &= e_2 \cdot \left(D\widetilde{\zeta} - (D\widetilde{\zeta}\cdot \nu_j^k)\nu_j^k\right) \\
&= \frac{\partial\,\zeta}{\partial\,x^2} - \left(1+|D(h_j^k+u_j^k)|^2\right)^{-1}\left(\lambda_j^k + \frac{\partial\,u_j^k}{\partial\,x^2}\right) \\
&\qquad \cdot \left(\left(\lambda_j^k + \frac{\partial\,u_j^k}{\partial\,x^2}\right)\frac{\partial\,\zeta}{\partial\,x^2} + D_y u_j^k \cdot D_y \zeta\right) \\
&= \left(1+|D(h_j^k+u_j^k)|^2\right)^{-1}\left(\left(1+|D_y\,u_j^k|^2\right)\frac{\partial\,\zeta}{\partial\,x^2} - \left(\lambda_j^k + \frac{\partial\,u_j^k}{\partial\,x^2}\right) D_y\,u_j^k \cdot D_y\,\zeta\right).
\end{aligned}$$

Using this and the analogous expression for $\nabla^{V_k} x^2 \cdot \nabla^{V_k}\widetilde{\zeta}$ on

$$(\mathcal{M}_j^k)^+ \equiv \text{graph}(g_j^k + w_j^k),$$

we deduce from (12.31) and Theorem 10.1(a) that

(12.32)

$$\begin{aligned}
&\int_{\mathbf{R}\times(B_{3/8}\cap\{|x^2|<2\tau\})} \nabla^{V_k} x^2 \cdot \nabla^{V_k}\widetilde{\zeta}\, d\|V_k\|(X) \\
&\quad + \sum_{j=1}^{q} \int_{B_{3/8}\cap\{x^2\le -2\tau\}} \frac{(1+|D_y\,u_j^k|^2)\frac{\partial\,\zeta}{\partial\,x^2} - (\lambda_j^k + \frac{\partial\,u_j^k}{\partial\,x^2})D_y\,u_j^k\cdot D_y\,\zeta}{\sqrt{1+|D(h_j^k+u_j^k)|^2}}\,dx \\
&\quad + \sum_{j=1}^{q} \int_{B_{3/8}\cap\{x^2\ge 2\tau\}} \frac{(1+|D_y\,w_j^k|^2)\frac{\partial\,\zeta}{\partial\,x^2} - (\mu_j^k + \frac{\partial\,w_j^k}{\partial\,x^2})D_y\,w_j^k\cdot D_y\,\zeta}{\sqrt{1+|D(g_j^k+w_j^k)|^2}}\,dx = 0.
\end{aligned}$$

Since $\frac{\partial\,\widetilde{\zeta}}{\partial\,x^1} = 0$ in a neighborhood of $\text{spt}\,\|V_k\| \cap (\mathbf{R}\times B_{1/2})$ and $\frac{\partial\,\widetilde{\zeta}}{\partial\,x^2} = 0$ in $\{|x^2| < 2\tau\}$, it follows that

(12.33)
$$\begin{aligned}
&\left|\int_{\mathbf{R}\times(B_{3/8}\cap\{|x^2|<2\tau\})} \nabla^{V_k} x^2 \cdot \nabla^{V_k}\widetilde{\zeta}\, d\|V_k\|(X)\right| \\
&\quad = \left|\int_{\mathbf{R}\times(B_{3/8}\cap\{|x^2|<2\tau\})} e_2 \cdot \nabla^{V_k}\widetilde{\zeta}\, d\|V_k\|(X)\right| \\
&\quad \le \sup|D\zeta| \sum_{j=3}^{n+1} \int_{\mathbf{R}\times(B_{3/8}\cap\{|x^2|<2\tau\})} |e_2^{\perp_k}||e_j^{\perp_k}|\, d\|V_k\|(X) \\
&\quad \le \sup|D\zeta| \left(\int_{\mathbf{R}\times(B_{3/8}\cap\{|x^2|<2\tau\})} |e_2^{\perp_k}|^2\, d\|V_k\|(X)\right)^{1/2} \\
&\qquad \cdot \left(\sum_{j=3}^{n+1} \int_{\mathbf{R}\times B_{3/8}} |e_j^{\perp_k}|^2\, d\|V_k\|(X)\right)^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&\leq \sup |D\zeta| \left(\int_{\mathbf{R}\times(B_{3/8}\cap\{|x^2|<2\tau\})} 1-|e_1^{\perp_k}|^2\, d\|V_k\|(X)\right)^{1/2}\\
&\quad\cdot\left(\sum_{j=3}^{n+1}\int_{\mathbf{R}\times B_{3/8}} |e_j^{\perp_k}|^2\, d\|V_k\|(X)\right)^{1/2}\\
&\leq C\sup|D\zeta|\,\tau^{1/4}\hat{E}_k E_k
\end{aligned}
$$

for all sufficiently large k, where $C = C(n, q, \alpha) \in (0, \infty)$ and the last inequality follows from Theorem 10.1(c), Theorem 7.1(b) and Lemma 10.8. Since

(12.34)

$$
\begin{aligned}
&\int_{B_{3/8}\cap\{x^2\leq -2\tau\}} \frac{(1+|D_y\, u_j^k|^2)\frac{\partial\zeta}{\partial x^2} - (\lambda_j^k + \frac{\partial u_j^k}{\partial x^2}) D_y\, u_j^k\cdot D_y\,\zeta}{\sqrt{1+|D(h_j^k+u_j^k)|^2}}\\
&\quad = -\int_{B_{3/8}\cap\{x^2\leq -2\tau\}} \frac{\lambda_j^k D_y\, u_j^k\cdot D_y\,\zeta}{\sqrt{1+|D(h_j^k+u_j^k)|^2}}\\
&\qquad + \int_{B_{3/8}\cap\{x^2\leq -2\tau\}} \frac{|D_y\, u_j^k|^2\frac{\partial\zeta}{\partial x^2} - \frac{\partial u_j^k}{\partial x^2} D_y\, u_j^k\cdot D_y\,\zeta}{\sqrt{1+|D(h_j^k+u_j^k)|^2}}\\
&\qquad - \int_{B_{3/8}\cap\{x^2\leq -2\tau\}}\\
&\qquad\qquad \frac{(2\lambda_j^k\frac{\partial u_j^k}{\partial x^2} + |D\, u_j^k|^2)\frac{\partial\zeta}{\partial x^2}}{\sqrt{1+|\lambda_j^k|^2}\sqrt{1+|D(h_j^k+u_j^k)|^2}\left(\sqrt{1+|\lambda_j^k|^2}+\sqrt{1+|D(h_j^k+u_j^k)|^2}\right)}\\
&\qquad + \frac{1}{\sqrt{1+|\lambda_j^k|^2}}\int_{B_{3/8}\cap\{x^2\leq -2\tau\}} \frac{\partial\,\zeta}{\partial\, x^2}
\end{aligned}
$$

it follows that if ζ also satisfies (12.19), then

(12.35)

$$
\begin{aligned}
&\int_{B_{3/8}\cap\{x^2\leq -2\tau\}} \frac{(1+|D_y\, u_j^k|^2)\frac{\partial\zeta}{\partial x^2} - (\lambda_j^k + \frac{\partial u_j^k}{\partial x^2}) D_y\, u_j^k\cdot D_y\,\zeta}{\sqrt{1+|D(h_j^k+u_j^k)|^2}}\, dx\\
&\quad = -\int_{B_{3/8}\cap\{x^2\leq -2\tau\}}\\
&\qquad \cdot \frac{2\lambda_j^k\frac{\partial u_j^k}{\partial x^2}\frac{\partial\zeta}{\partial x^2}}{\sqrt{1+|\lambda_j^k|^2}\sqrt{1+|D(h_j^k+u_j^k)|^2}\left(\sqrt{1+|\lambda_j^k|^2}+\sqrt{1+|D(h_j^k+u_j^k)|^2}\right)}\, dx\\
&\qquad - \int_{B_{3/8}\cap\{x^2\leq -2\tau\}} \frac{\lambda_j^k D_y\, u_j^k\cdot D_y\,\zeta}{\sqrt{1+|D(h_j^k+u_j^k)|^2}}\, dx + \eta_k^-
\end{aligned}
$$

where, by elliptic estimates,

$$|\eta_k^-| \leq C \sup |D\zeta|\, E_k^2, \quad C = C(n, q, \tau) \in (0, \infty). \tag{12.36}$$

By the same argument,

$$
\begin{aligned}
&\int_{B_{3/8}\cap\{x^2\geq 2\tau\}} \frac{(1+|D_y\, w_j^k|^2)\frac{\partial \zeta}{\partial x^2} - (\mu_j^k + \frac{\partial w_j^k}{\partial x^2}) D_y\, w_j^k \cdot D_y\, \zeta}{\sqrt{1+|D(g_j^k + w_j^k)|^2}} \\
&\quad = -\int_{B_{3/8}\cap\{x^2\geq 2\tau\}} \\
&\qquad\quad \cdot \frac{2\mu_j^k \frac{\partial w_j^k}{\partial x^2}\frac{\partial \zeta}{\partial x^2}}{\sqrt{1+|\mu_j^k|^2}\sqrt{1+|Dg_j^k + w_j^k)|^2}\left(\sqrt{1+|\mu_j^k|^2} + \sqrt{1+|D(g_j^k+w_j^k)|^2}\right)} \\
&\qquad - \int_{B_{3/8}\cap\{x^2\geq 2\tau\}} \frac{\mu_j^k D_y\, w_j^k \cdot D_y\, \zeta}{\sqrt{1+|D(g_j^k + u_j^k)|^2}}\, dx + \eta_k^+
\end{aligned} \tag{12.37}
$$

where, again by elliptic estimates,

$$|\eta_k^+| \leq C \sup |D\zeta|\, E_k^2, \quad C = C(n, q, \tau) \in (0, \infty). \tag{12.38}$$

Dividing (12.32) by $\hat{E}_k E_k$ and letting $k \to \infty$, we conclude with the help of (12.33), (12.35), (12.36), (12.37), (12.38), (11.1) and (11.9) that

$$\sum_{j=1}^q \ell_j \int_{B_{3/8}\cap\{x^2\leq -2\tau\}} D\varphi_j \cdot D\zeta + \sum_{j=1}^q m_j \int_{B_{3/8}\cap\{x^2\geq 2\tau\}} D\psi_j \cdot D\zeta + \eta(\tau) = 0 \tag{12.39}$$

for any $\zeta \in C_c^2(B_{3/8})$ with $\frac{\partial \zeta}{\partial x^2} = 0$ in $\{|x^2| < 2\tau\}$ and satisfying (12.19), where $\eta(\tau) \to 0$ as $\tau \to 0$.

It follows from (12.39) in the same way that (12.30) follows from (12.22) that if we let, for $Y \in B_{3/8} \cap (\{0\} \times \mathbf{R}^{n-1})$,

$$\Psi(Y) = \left(\sum_{j=1}^q (\ell_j + m_j)\right)\kappa_1(Y) - \left(\sum_{j=1}^q (\ell_j^2 + m_j^2)\right)\kappa_2(Y), \tag{12.40}$$

then Ψ is a C^∞ function on $B_{9/32} \cap (\{0\} \times \mathbf{R}^{n-1})$ satisfying the estimate

$$
\begin{aligned}
\sup_{B_{9/32}\cap(\{0\}\times\mathbf{R}^{n-1})} &|\Psi|^2 + |D_Y\, \Psi|^2 + |D_Y^2\, \Psi|^2 + |D_Y^3\, \Psi|^2 \\
&\leq C\left(\int_{B_{1/2}\cap\{x^2\leq 0\}} |\varphi|^2 + \int_{B_{1/2}\cap\{x^2\geq 0\}} |\psi|^2\right),
\end{aligned} \tag{12.41}
$$

where $C = C(n, q, \alpha) \in (0, \infty)$.

Note that

$$\begin{aligned} J &\equiv 2q\sum_{j=1}^{q}(\ell_j^2+m_j^2)-\left(\sum_{j=1}^{q}(\ell_j+m_j)\right)^2 \\ &= \frac{1}{2}\sum_{i=1}^{q}\sum_{j=1}^{q}\left((m_i-m_j)^2+(\ell_i-\ell_j)^2+2(\ell_i-m_j)^2\right), \end{aligned}$$

and so it follows from (11.8) that $\widetilde{C} \geq J \geq C > 0$, where $\widetilde{C} = \widetilde{C}(n,q) \in (0,\infty)$ and $C = C(n,q) \in (0,\infty)$; thus, by (12.29) and (12.40), we may express each of κ_1 and κ_2 as a linear combination of Φ and Ψ with coefficients, in absolute value, $\leq C = C(n,q) \in (0,\infty)$. Consequently, κ_1, κ_2 are in $C^\infty(B_{9/32} \cap (\{0\} \times \mathbf{R}^{n-1}))$ and, by (12.30) and (12.41), satisfy the estimates

$$\sup_{B_{9/32}\cap(\{0\}\times\mathbf{R}^{n-1})} |\kappa_i|^2 + |D_y\,\kappa_i|^2 + |D_y^2\,\kappa_i|^2 + |D_y^3\,\kappa_i|^2 \leq C\left(\int_{B_{1/2}\cap\{x^2\leq 0\}}|\varphi|^2 + \int_{B_{1/2}\cap\{x^2\geq 0\}}|\psi|^2\right) \tag{12.42}$$

for $i = 1, 2$, where $C = C(n,q,\alpha) \in (0,\infty)$. This in turn implies that for each $j = 1, 2, \ldots, q$, the functions

$$\varphi_j|_{B_{9/32}\cap(\{0\}\times\mathbf{R}^{n-1})} \quad (= \kappa_1 - \ell_j\kappa_2)$$

and

$$\psi_j|_{B_{9/32}\cap(\{0\}\times\mathbf{R}^{n-1})} \quad (= \kappa_1 - m_j\kappa_2)$$

belong to $C^\infty\left(B_{9/32} \cap (\{0\} \times \mathbf{R}^{n-1})\right)$ and satisfy the estimates

$$\sup_{B_{9/32}\cap(\{0\}\times\mathbf{R}^{n-1})} |\varphi_j|^2 + |D_y\,\varphi_j|^2 + |D_y^2\,\varphi_j|^2 + |D_y^3\,\varphi_j|^2 \leq C\left(\int_{B_{1/2}\cap\{x^2\leq 0\}}|\varphi|^2 + \int_{B_{1/2}\cap\{x^2\geq 0\}}|\psi|^2\right), \tag{12.43}$$

$$\sup_{B_{9/32}\cap(\{0\}\times\mathbf{R}^{n-1})} |\psi_j|^2 + |D_y\,\psi_j|^2 + |D_y^2\,\psi_j|^2 + |D_y^3\,\psi_j|^2 \leq C\left(\int_{B_{1/2}\cap\{x^2\leq 0\}}|\varphi|^2 + \int_{B_{1/2}\cap\{x^2\geq 0\}}|\psi|^2\right), \tag{12.44}$$

where $C = C(n,q,\alpha) \in (0,\infty)$. By Lemma 12.1 and the standard $C^{2,\alpha}$ boundary regularity theory for harmonic functions ([Mor66]), the desired conclusions of the present lemma, in particular, follow. □

13. Improvement of fine excess

Let q be an integer ≥ 2, $\alpha \in (0,1)$, and suppose that the induction hypotheses (H1), (H2) hold. The main result of this section (Lemma 13.3 below) establishes that there are fixed constants $\varepsilon = \varepsilon(n,q,\alpha) \in (0,1)$, $\gamma = \gamma(n,q,\alpha) \in (0,1)$ such that whenever $V \in \mathcal{S}_\alpha$, $\mathbf{C} \in \mathcal{C}_q$ satisfy Hypotheses 10.1 and Hypothesis $(\star)$ (of Section 10) with a suitable constant M depending only on n and q, the fine excess of V relative to a new cone $\mathbf{C}' \in \mathcal{C}_q$ decays by a fixed factor at one of several fixed smaller scales.

Lemma 13.1. *Let q be an integer ≥ 2, $\alpha \in (0,1)$ and $\theta \in (0,1/4)$. There exist numbers $\overline{\varepsilon} = \overline{\varepsilon}(n,q,\alpha,\theta) \in (0,1/2)$, $\overline{\gamma} = \overline{\gamma}(n,q,\alpha,\theta) \in (0,1/2)$ and $\overline{\beta} = \overline{\beta}(n,q,\alpha,\theta) \in (0,1/2)$ such that the following is true: If $V \in \mathcal{S}_\alpha$, $\mathbf{C} \in \mathcal{C}_q$ satisfy Hypotheses* 10.1, *Hypothesis $(\star)$ and Hypothesis $(\star\star)$ with $\varepsilon = \overline{\varepsilon}$, $\gamma = \overline{\gamma}$, $M = \frac{3}{2}M_0$, $\beta = \overline{\beta}$, and if the induction hypotheses* (H1), (H2) *hold, then there exist an orthogonal rotation Γ of $\mathbf{R}^{n+1}$ and a cone $\mathbf{C}' \in \mathcal{C}_q$ such that, with*

$$\hat{E}_V^2 = \int_{\mathbf{R}\times B_1} |x^1|^2 \, d\|V\|(X)$$

and

$$E_V^2 = \int_{\mathbf{R}\times B_1} \operatorname{dist}^2(X, \operatorname{spt}\|\mathbf{C})\|) \, d\|V\|(X),$$

the following hold:

(a) $|e_1 - \Gamma(e_1)| \leq \overline{\kappa} E_V$ *and* $|e_j - \Gamma(e_j)| \leq \overline{\kappa}\hat{E}_V^{-1}E_V$ *for each* $j = 2,3,\ldots,n+1$;

(b) $\operatorname{dist}^2_{\mathcal{H}}(\operatorname{spt}\|\mathbf{C}'\| \cap (\mathbf{R}\times B_1), \operatorname{spt}\|\mathbf{C}\| \cap (\mathbf{R}\times B_1)) \leq \overline{C}_0 E_V^2$;

(c) $\theta^{-n-2}\int_{\Gamma\left(\mathbf{R}\times\left(B_{\theta/2}\setminus\{|x^2|\leq\theta/16\}\right)\right)} \operatorname{dist}^2(X, \operatorname{spt}\|V\|)\, d\|\Gamma_{\#}\,\mathbf{C}'\|(X)$
$+ \theta^{-n-2}\int_{\Gamma(\mathbf{R}\times B_\theta)} \operatorname{dist}^2(X, \operatorname{spt}\|\Gamma_{\#}\,\mathbf{C}'\|)\, d\|V\|(X) \leq \overline{\nu}\theta^2 E_V^2$;

(d) $\left(\theta^{-n-2}\int_{\mathbf{R}\times B_\theta} \operatorname{dist}^2(X,P)\, d\|\Gamma_{\#}^{-1}\,V\|(X)\right)^{1/2}$
$\geq 2^{-\frac{n+4}{2}}\sqrt{\overline{C}_1}\operatorname{dist}_{\mathcal{H}}(\operatorname{spt}\|\mathbf{C}\| \cap (\mathbf{R}\times B_1), P\cap(\mathbf{R}\times B_1)) - \overline{C}_2 E_V$ *for any* $P \in G_n$ *of the form* $P = \{x^1 = \lambda x^2\}$ *for some* $\lambda \in (-1,1)$;

(e) $\{Z : \Theta(\|\Gamma_{\#}^{-1}\,V\|, Z) \geq q\} \cap \left(\mathbf{R}\times(B_{\theta/2}\cap\{|x^2| < \theta/16\})\right) = \emptyset$;

(f) $(\omega_n\theta^n)^{-1}\,\|\Gamma_{\#}^{-1}\,V\|(\mathbf{R}\times B_\theta) < q + 1/2$.

Here the constants $\overline{\kappa}, \overline{C}_0, \overline{\nu}, \overline{C}_2 \in (0,\infty)$, each depends only on n, q, α, and $\overline{C}_1 = \overline{C}_1(n) = \int_{B_{1/2}\cap\{x^2>1/16\}} |x^2|^2\, d\mathcal{H}^n(x^2,y)$.

Proof. Consider any sequence of varifolds $\{V_k\} \subset \mathcal{S}_\alpha$ and any sequence of cones $\{\mathbf{C}_k\} \subset \mathcal{C}_q$ satisfying, for each $k = 1,2,\ldots$, hypotheses (1_k)–(7_k) of Section 11 for some sequences $\{\varepsilon_k\}$, $\{\gamma_k\}$, $\{\beta_k\}$ of numbers with $\varepsilon_k, \gamma_k, \beta_k \to 0^+$ and with M_0 in place of M_0^3 (in hypothesis (6_k)). The lemma will be established by showing that for each of infinitely many k, there exist an orthogonal rotation

Γ_k of $\mathbf{R}^{n+1}$ and a cone $\mathbf{C}'_k \in \mathcal{C}_q$ such that the conclusions of the lemma hold with V_k, $\mathbf{C}_k$, $\mathbf{C}'_k$ in place of V, $\mathbf{C}$, $\mathbf{C}'$ respectively for fixed constants $\overline{\kappa}, \overline{C}_0, \overline{\gamma}_0, \overline{\nu}, \overline{C}_1, \overline{C}_2 \in (0,\infty)$ depending only on n, q and α.

Let $\hat{E}_k = \hat{E}_{V_k}$ and $E_k = E_{V_k}$. For $i = 1, 2, \ldots, (n-1)$, let $Y_i = \frac{1}{2}\theta\, e_{2+i} \in \{0\} \times \mathbf{R}^{n-1}$. We infer from (11.2) that passing to a subsequence of $\{k\}$ without changing notation, for each $k = 1, 2, 3, \ldots$, there exist points

$$Z_{i,\,k} = (\zeta_1^{i,\,k}, \zeta_2^{i,\,k}, \eta_{i,\,k}) \in \operatorname{spt} \|V_k\| \cap (\mathbf{R} \times B_1),$$

$i = 1, 2, \ldots, (n-1)$, such that $\Theta(\|V_k\|, Z_{i,\,k}) \geq q$ and $|Z_{i,\,k} - Y_i| \to 0$ as $k \to \infty$; also, we may find orthogonal rotations Γ'_k of $\mathbf{R}^{n+1}$ such that

$$\Gamma'_k(\Sigma_k) = \{0\} \times \mathbf{R}^{n-1} \quad \text{and} \quad \Gamma'_k\left(\frac{Z_{i,k}}{|Z_{i,k}|}\right) \to e_{2+i} \quad \text{for each } i = 1, 2, \ldots, (n-1),$$

where Σ_k is the $(n-1)$-dimensional subspace spanned by $\{Z_{i,\,k} : i = 1, 2, \ldots, (n-1)\}$. Let Γ''_k be the orthogonal rotation of $\mathbf{R}^{n+1}$ such that $\Gamma''_k(Y) = Y$ for each $Y \in \{0\} \times \mathbf{R}^{n-1}$ and $\Gamma''_k\left(\frac{\pi_{12}\,\Gamma'_k(e_1)}{|\pi_{12}\,\Gamma'_k(e_1)|}\right) = e_1$, where $\pi_{12} : \mathbf{R}^{n+1} \to \mathbf{R}^2 \times \{0\}$ is the orthogonal projection onto the x^1x^2-plane, and let $\Gamma_k = \Gamma''_k \circ \Gamma'_k$ so that

(13.1)
$$\Gamma_k(\Sigma_k) = \{0\} \times \mathbf{R}^{n-1}, \quad \Gamma_k\left(\frac{Z_{i,k}}{|Z_{i,k}|}\right) \to e_{2+i} \quad \text{for each } i = 1, 2, \ldots, (n-1).$$

Let $(\varphi, \psi) \in \mathcal{B}^F$ be the fine blow-up of a subsequence of $\{V_k\}$ relative to the corresponding subsequence of $\{\mathbf{C}_k\}$. Since $\Theta(\|V_k\|, 0) \geq q$, it follows from (11.7) that $\varphi(0) = \psi(0) = 0$, and consequently, from (12.2) and (12.42) that after passing to further subsequences without changing notation,

$$|\zeta_1^{i,\,k}| + \hat{E}_k |\zeta_2^{i,\,k}| \leq C\theta E_k \tag{13.2}$$

for each $i = 1, 2, \ldots, n-1$ and $k = 1, 2, \ldots$, where $C = C(n, q, \alpha) \in (0, \infty)$. With the help of (13.2), the following can then be verified:

(13.3)
$$|e_1 - \Gamma_k(e_1)| \leq CE_k \quad \text{and} \quad |e_j - \Gamma_k(e_j)| \leq C\hat{E}_k^{-1} E_k, \quad j = 2, 3, \ldots, n+1,$$

where $C = C(n, q, \alpha) \in (0, \infty)$. In particular, note that C here is independent of θ. Consequently, letting $\widetilde{V}_k = \eta_{0,7/8\,\#}(\Gamma_{k\,\#}\, V_k)$ and passing to a further subsequence without changing notation, we have for each $k = 1, 2, 3, \ldots$ that

$$d_{\mathcal{H}}(\Gamma_k^{-1}(\{0\} \times \mathbf{R}^n) \cap (\mathbf{R} \times B_1), \{0\} \times B_1) \leq CE_k \tag{13.4}$$

and

$$E_{\widetilde{V}_k}^2 \equiv \int_{\mathbf{R} \times B_1} \operatorname{dist}^2(X, \operatorname{spt} \|\mathbf{C}_k\|)\, d\|\widetilde{V}_k\|(X) \leq CE_k^2, \tag{13.5}$$

where $C = C(n, q, \alpha) \in (0, \infty)$. Furthermore, we claim that

$$\widetilde{C}\hat{E}_k \leq \hat{E}_{\widetilde{V}_k} \leq C\hat{E}_k \tag{13.6}$$

for constants $\widetilde{C} = \widetilde{C}(n,q,\alpha) \in (0,\infty)$ and $C = C(n,q,\alpha) \in (0,\infty)$. The second of these inequalities follows directly from the definition of $\widetilde{V}_k$ and inequality (13.4); to see the first, note first that since the coarse blow-up $v_\star$ of $\{V_k\}$ (by the excess $\hat{E}_k$) is homogeneous of degree 1 (in fact its graph is a union of half-hyperplanes meeting along $\{0\} \times \mathbf{R}^{n-1}$) and satisfies, by (11.6), $\int_{B_1} |v_\star|^2 \geq \widetilde{c}$ where $\widetilde{c} = \widetilde{c}(n,q) \in (0,1)$, we have for each $\sigma \in (0,1)$ that $\sigma^{-n-2} \int_{B_\sigma} |v_\star|^2 = \int_{B_1} |v_\star|^2 \geq \widetilde{c}$ so that

$$\begin{aligned}\int_{\mathbf{R}\times B_\sigma} |x^1|^2 \, d\|V_k\|(X) &= \sum_{j=1}^q \int_{B_\sigma} \sqrt{1+|Du_k^j|^2}|u_k^j|^2 \\ &\quad - \sum_{j=1}^q \int_{\Sigma_k} \sqrt{1+|Du_k^j|^2}|u_k^j|^2 + \int_{\mathbf{R}\times\Sigma_k} |x^1|^2 \, d\|V_k\|(X) \\ &\geq \int_{B_\sigma} |u_k|^2 - 2C_\sigma \left(\sup_{B_\sigma} |u_k|^2\right) \hat{E}_k^2 \\ &\geq \left(\frac{1}{2}\widetilde{c}\,\sigma^{n+2} - 2C_\sigma \left(\sup_{B_\sigma} |u_k|^2\right)\right) \hat{E}_k^2\end{aligned}$$

for sufficiently large k, where u_k, Σ_k correspond to u, Σ of Theorem 5.1 taken with V_k in place of V and the constant C_σ is the same as the constant C of Theorem 5.1(a). Thus for sufficiently large k depending on σ,

$$\int_{\mathbf{R}\times B_\sigma} |x^1|^2 \, d\|V_k\|(X) \geq c\hat{E}_k^2, \tag{13.7}$$

where $c = c(n,q,\sigma) \in (0,1)$, which, taken with a suitable choice of $\sigma \in (0,1)$, readily implies the first of the inequalities of (13.6).

Using Theorem 5.1, (11.5) and inequalities (13.3)–(13.6), we can now verify that after passing to another subsequence without changing notation, for each $k = 1, 2, \ldots$, the hypotheses (1_k)–(7_k) of Section 11 are satisfied with $\widetilde{V}_k$ in place of V_k, suitable numbers $\widetilde{\varepsilon}_k$, $\widetilde{\gamma}_k$, $\widetilde{\beta}_k \to 0^+$ in place of ε_k, γ_k, β_k respectively and with M_0^2 in place of M_0^3 (in (6_k)). Of these, verification of (1_k)–(5_k) is straightforward; to verify that (6_k) is satisfied with $\widetilde{V}_k$ in place of V and M_0^2 in place of M_0^3, we proceed as follows: We note first that by (13.6),

$$\begin{aligned}&\inf_{P=\{x^1=\lambda x^2\}} \int_{\mathbf{R}\times B_1} \mathrm{dist}^2(X,P) \, d\|\widetilde{V}_k\|(X) \\ &\qquad = \inf_{P=\{x^1=\lambda x^2\};|\lambda|\leq C\hat{E}_k} \int_{\mathbf{R}\times B_1} \mathrm{dist}^2(X,P) \, d\|\widetilde{V}_k\|(X),\end{aligned}$$

where $C = C(n,q,\alpha) \in (0,\infty)$, and that for any hyperplane $P = \{x^1 = \lambda x^2\}$ with $|\lambda| < C\hat{E}_k$ and for sufficiently large k,

$$
\begin{aligned}
\int_{\mathbf{R}\times B_1} \operatorname{dist}^2(X,P)\, d\|\widetilde{V}_k\|(X) &\geq \left(\frac{8}{7}\right)^{n+2} \int_{\mathbf{R}\times B_{1/2}} \operatorname{dist}^2(X,\Gamma_k^{-1}(P))\, d\|V_k\|(X)\\
&\geq \frac{1}{2}\left(\frac{8}{7}\right)^{n+2} \int_{\mathbf{R}\times B_{1/2}} \operatorname{dist}^2(X,P)\, d\|V_k\|(X)\\
&\quad - C \operatorname{dist}^2_{\mathcal{H}}(\Gamma_k^{-1}(P)\cap(\mathbf{R}\times B_{1/2}), P\cap(\mathbf{R}\times B_{1/2}))\\
&\geq 7^{-n-2}2^{n-1}\omega_n^{-1}(2q+1)^{-1}\overline{C}_1 \int_{\mathbf{R}\times B_1} \operatorname{dist}^2(X,P)\, d\|V_k\|(X) - CE_k^2\\
&\geq 7^{-n-2}2^{n-1}\omega_n^{-1}(2q+1)^{-1}\overline{C}_1 \left(\frac{3}{2}M_0\right)^{-1} \int_{\mathbf{R}\times B_1} |x^1|^2\, d\|V_k\|(X) - CE_k^2,
\end{aligned}
$$

where $C = C(n,q) \in (0,\infty)$, the third inequality follows from (10.34) with $\rho = 1/2$ and $Z = 0$, and the last inequality holds by hypothesis of the present lemma. On the other hand,

$$
\begin{aligned}
\int_{\mathbf{R}\times B_1} |x^1|^2\, d\|\widetilde{V}_k\|(X) &\leq 2\left(\frac{8}{7}\right)^{n+2} \int_{\mathbf{R}\times B_1} |x^1|^2\, d\|V_k\|(X)\\
&\quad + \left(\frac{8}{7}\right)^{n+2} \omega_n(2q+1)\operatorname{dist}^2(\Gamma_k^{-1}(\{0\}\times\mathbf{R}^n)\cap(\mathbf{R}\times B_1), \{0\}\times B_1)\\
&\leq 2\left(\frac{8}{7}\right)^{n+2} \int_{\mathbf{R}\times B_1} |x^1|^2\, d\|V_k\|(X) + CE_k^2,
\end{aligned}
$$

where $C = C(n,q,\alpha) \in (0,\infty)$. Hence

$$
\hat{E}^2_{\widetilde{V}_k} \leq \frac{3M_0}{2\left(2^{-2n-7}\omega_n^{-1}(2q+1)^{-1}\overline{C}_1 - C\gamma_k\right)} \int_{\mathbf{R}\times B_1} \operatorname{dist}^2(X,P)\, d\|\widetilde{V}_k\|(X),
$$

where $C = C(n,q,\alpha) \in (0,\infty)$, and it follows from this that hypothesis (6_k) with $\widetilde{V}_k$ in place of V_k and M_0^2 in place of M_0^3 is satisfied for all sufficiently large k; hypothesis (7_k) with $\widetilde{V}_k$ in place of V_k can easily be verified using the estimate $Q^\star_{\widetilde{V}_k}(p_k-1) \geq CQ^\star_{V_k}(p_k-1)$, where $C = C(n,q) \in (0,\infty)$, which follows from (13.3) and the fact that, for any $\mathbf{C} \in \bigcup_{j=4}^{p_k-1}\mathcal{C}_q(j)$,

$$
\begin{aligned}
&\int_{\mathbf{R}\times\left(B_{7/16}\setminus\{|x^2|\leq 7/(8\cdot 16)\}\right)} \operatorname{dist}^2(X, \operatorname{spt}\|V_k\|)\, d\|\mathbf{C}\|(X)\\
&\qquad + \int_{\mathbf{R}\times B_{7/8}} \operatorname{dist}^2(X,\operatorname{spt}\|\mathbf{C}\|)\, d\|V_k\|(X) \geq \tilde{c}_1\left(Q^\star_{V_k}(p_k-1)\right)^2,
\end{aligned}
$$

where $\tilde{c}_1 = \tilde{c}_1(n,q) \in (0,1)$, the validity of which can be seen by reasoning as in the proof of (13.7) using the fact that the blow-up of $\{V_k\}$ by $Q^\star_{V_k}(p_k-1)$ is homogeneous of degree 1 (by hypothesis (7_k)) and has, by (10.3), $L^2(B_1)$ norm $\geq c$, $c = c(n,q) \in (0,1)$.

Thus, the fine blow-up $(\tilde{\varphi},\tilde{\psi})$ of $\{\widetilde{V}_k\}$ relative to $\{\mathbf{C}_k\}$ belongs to $\mathcal{B}^F$. Furthermore, it follows from (13.1) and (11.7) (applied with $\widetilde{V}_k$ in place of V_k and $\frac{8}{7}\Gamma_k\, Z_{i,k}$, $i = 1,2,\ldots,n-1$, in place of Z) that for each $i = 1,2,\ldots,(n-1)$,

$\widetilde{\varphi}(Y_i) = \widetilde{\psi}(Y_i) = 0$ and consequently, since $Y_i = \frac{1}{2}\theta\, e_{i+2}$, that there exist points $S_{j,i}, T_{j,i} \in B_{\theta/2} \cap (\{0\} \times \mathbf{R}^{n-1})$ such that

$$\frac{\partial\, \widetilde{\varphi}_j}{\partial\, y^i}(S_{j,i}) = 0 \quad \text{and} \quad \frac{\partial\, \widetilde{\psi}_j}{\partial\, y^i}(T_{j,i}) = 0$$

for each $i = 1, 2, \ldots, n-1$ and $j = 1, 2, \ldots, q$. By the estimate of Theorem 12.2, this readily implies that

$$|D_y\widetilde{\varphi}(0)|^2 + |D_y\widetilde{\psi}(0)|^2 \leq C\theta^2 \left(\int_{B_{1/2}\cap\{x^2<0\}} |\widetilde{\varphi}|^2 + \int_{B_{1/2}\cap\{x^2>0\}} |\widetilde{\psi}|^2\right), \tag{13.8}$$

where $C = C(n, q, \alpha) \in (0, \infty)$. For $j = 1, 2, \ldots, q$ and $x = (x^2, y) \in \mathbf{R}^n$, letting $L^j_{\widetilde{\varphi}}(x) = D\widetilde{\varphi}_j(0) \cdot x$, $L^j_{\widetilde{\psi}}(x) = D\widetilde{\psi}_j(0) \cdot x$, $P^j_{\widetilde{\varphi}}(x^2, y) = \frac{\partial\, \widetilde{\varphi}_j}{\partial\, x^2}(0)x^2$ and $P^j_{\widetilde{\psi}}(x) = \frac{\partial\, \widetilde{\psi}_j}{\partial\, x^2}(0)x^2$, it follows from (13.8) that for each $(x^2, y) \in \mathbf{R}^n$,

$$\begin{aligned} &|P^j_{\widetilde{\varphi}}(x^2, y) - L^j_{\widetilde{\varphi}}(x^2, y)|^2 + |P^j_{\widetilde{\psi}}(x^2, y) - L^j_{\widetilde{\psi}}(x^2, y)|^2 \\ &\qquad\qquad \leq C\theta^2|y|^2 \left(\int_{B_{1/2}\cap\{x^2<0\}} |\widetilde{\varphi}|^2 + \int_{B_{1/2}\cap\{x^2>0\}} |\widetilde{\psi}|^2\right) \end{aligned}$$

and consequently from Theorem 12.2 that

$$\theta^{-n-2} \left(\int_{B_{2\theta}\cap\{x^2\leq 0\}} |\widetilde{\varphi} - P_{\widetilde{\varphi}}|^2 + \int_{B_{2\theta}\cap\{x^2\geq 0\}} |\widetilde{\psi} - P_{\widetilde{\psi}}|^2\right) \leq C\theta^2, \tag{13.9}$$
$$C = C(n, q, \alpha) \in (0, \infty).$$

For $j = 1, 2, \ldots, q$ and $k = 1, 2, \ldots,$ let

$$\begin{aligned} \lambda'^k_j &= \lambda^k_j + E_{\widetilde{V}_k} \frac{\partial\, \widetilde{\varphi}_j}{\partial\, x^2}(0), \\ \mu'^k_j &= \mu^k_j + E_{\widetilde{V}_k} \frac{\partial\, \widetilde{\psi}_j}{\partial\, x^2}(0), \\ H'^k_j &= \{(x^1, x^2, y) : x^1 = \lambda'^k_j x^2, \ \ x^2 \leq 0\}, \\ G'^k_j &= \{(x^1, x^2, y) : x^1 = \mu'^k_j x^2, \ \ x^2 \geq 0\}, \\ \mathbf{C}'_k &= \sum_{j=1}^q |H'^k_j| + |G'^k_j|. \end{aligned} \tag{13.10}$$

With the help of (13.1), (11.3) and (13.5), it is straightforward to verify that

$$\theta^{-n-2} \int_{\Gamma_k^{-1}(\mathbf{R}\times B_\theta)} \mathrm{dist}^2(X, \mathrm{spt}\, \|(\Gamma_k^{-1})_\#\, \mathbf{C}'_k\|)\, d\|V_k\|(X) \leq C\theta^2 E_k^2 \tag{13.11}$$

for all sufficiently large k, where $\mathbf{C}'_k$ is as above and Γ_k is as in (13.1), and $C = C(n, q, \alpha) \in (0, \infty)$. Furthermore, it follows from (13.5), (13.10) and

Theorem 12.2 that

$$\operatorname{dist}^2_{\mathcal{H}}(\operatorname{spt}\|\mathbf{C}'_k\| \cap (\mathbf{R}\times B_1), \operatorname{spt}\|\mathbf{C}_k\| \cap (\mathbf{R}\times B_1)) \leq C E_k^2, \quad C = C(n,q,\alpha) \in (0,\infty). \tag{13.12}$$

From (11.4) (applied with $\widetilde{V}_k$ in place of V_k), it follows that

$$\begin{aligned}\theta^{-n-2}\int_{\Gamma_k^{-1}\left(\mathbf{R}\times\left(B_{\theta/2}\setminus\{|x^2|\leq\theta/16\}\right)\right)} &\operatorname{dist}^2(X, \operatorname{spt}\|V_k\|)\, d\|\Gamma^{-1}_{k\,\#}\,\mathbf{C}'_k\|(X) \\ &\leq C\theta^{-n-2}\int_{\Gamma_k^{-1}(\mathbf{R}\times B_\theta)} \operatorname{dist}^2(X, \operatorname{spt}\|\Gamma^{-1}_{k\,\#}\,\mathbf{C}'_k\|)\, d\|V_k\|(X), \\ &C = C(n,q,\alpha)\in(0,\infty).\end{aligned} \tag{13.13}$$

Again by (11.4) (applied with $\widetilde{V}_k$ in place of V_k), (13.5), (13.11) and (13.12), we have that for any hyperplane P of the form $P = \{x^1 = \lambda x^2\}$, $|\lambda| < 1$, writing $\widetilde{\theta} = \frac{8}{7}\theta$,

$$\begin{aligned}
&\widetilde{\theta}^{-n-2}\int_{\mathbf{R}\times B_{\widetilde{\theta}}} \operatorname{dist}^2(X,P)\, d\|\widetilde{V}_k\|(X) \\
&\geq \frac{1}{2}\widetilde{\theta}^{-n-2}\sum_{j=1}^{q}\left(\int_{B_{\widetilde{\theta}/2}\cap\{x^2<-\widetilde{\theta}/16\}} |h^k_j - \lambda x^2 + \widetilde{u}^k_j|^2 \right. \\
&\qquad\qquad \left. + \int_{B_{\widetilde{\theta}/2}\cap\{x^2>\widetilde{\theta}/16\}} |g^k_j - \lambda x^2 + \widetilde{w}^k_j|^2\right) \\
&\geq \frac{1}{4}\widetilde{\theta}^{-n-2}\sum_{j=1}^{q}\left(\int_{B_{\widetilde{\theta}/2}\cap\{x^2<-\widetilde{\theta}/16\}} |h^k_j - \lambda x^2|^2 + \int_{B_{\widetilde{\theta}/2}\cap\{x^2>\widetilde{\theta}/16\}} |g^k_j - \lambda x^2|^2\right) \\
&\quad - \frac{1}{2}\widetilde{\theta}^{-n-2}\sum_{j=1}^{q}\left(\int_{B_{\widetilde{\theta}/2}\cap\{x^2<-\widetilde{\theta}/16\}} |\widetilde{u}^k_j|^2 + \int_{B_{\widetilde{\theta}/2}\cap\{x^2>\widetilde{\theta}/16\}} |\widetilde{w}^k_j|^2\right) \\
&\geq 2^{-n-4}\overline{C}_1 \operatorname{dist}^2_{\mathcal{H}}(\operatorname{spt}\|\mathbf{C}_k\|\cap(\mathbf{R}\times B_1), P\cap(\mathbf{R}\times B_1)) \\
&\quad - \frac{1}{2}\widetilde{\theta}^{-n-2}\int_{\mathbf{R}\times B_{\widetilde{\theta}}} \operatorname{dist}^2(X, \operatorname{spt}\|\mathbf{C}_k\|)\, d\|\widetilde{V}_k\|(X) \\
&\geq 2^{-n-4}\overline{C}_1 \operatorname{dist}^2_{\mathcal{H}}(\operatorname{spt}\|\mathbf{C}_k\|\cap(\mathbf{R}\times B_1), P\cap(\mathbf{R}\times B_1)) \\
&\quad - \widetilde{\theta}^{-n-2}\int_{\mathbf{R}\times B_{\widetilde{\theta}}} \operatorname{dist}^2(X, \operatorname{spt}\|\mathbf{C}'_k\|)\, d\|\widetilde{V}_k\|(X) - CE_k^2 \\
&\geq 2^{-n-4}\overline{C}_1 \operatorname{dist}^2_{\mathcal{H}}(\operatorname{spt}\|\mathbf{C}_k\|\cap(\mathbf{R}\times B_1), P\cap(\mathbf{R}\times B_1)) - CE_k^2,
\end{aligned}$$

where $\overline{C}_1 = \int_{B_{1/2}\cap\{x^2>1/16\}} |x^2|^2\, d\mathcal{H}^n(x^2,y)$, $C_2 = C_2(n,q,\alpha)\in(0,\infty)$ and the notation is as in Theorem 10.1 taken with $\widetilde{V}_k$ in place of V (in particular, with $\widetilde{u}^j_k$, $\widetilde{w}^j_k$ corresponding to u^j, w^j). This readily implies that

$$(13.14)\quad \left(\theta^{-n-2}\int_{\mathbf{R}\times B_\theta}\mathrm{dist}^2(X,P)\,d\|\Gamma_{k\,\#}\,V_k\|(X)\right)^{1/2}$$
$$\geq\ 2^{-\frac{n+4}{2}}\sqrt{C_1}\,\mathrm{dist}_{\mathcal{H}}(\mathrm{spt}\,\|\mathbf{C}_k\|\cap(\mathbf{R}\times B_1),P\cap(\mathbf{R}\times\times B_1))-CE_k$$

for each hyperplane $P=\{x^1=\lambda x^2\}$ with $|\lambda|<1$ and all sufficiently large k, where $C=C(n,q,\alpha)\in(0,\infty)$.

The inequalities (13.3) and (13.11)–(13.14) say that the conclusions (a)–(d) of the lemma, with V_k, $\mathbf{C}_k$, $\mathbf{C}_k'$, Γ_k^{-1} in place of V, $\mathbf{C}$, $\mathbf{C}'$, Γ, hold for all sufficiently large k. Conclusion (e) with V_k in place of V and Γ_k^{-1} in place of Γ is clear, for all sufficiently large k, by (11.5) applied with $\widetilde{V}_k$ in place of V_k. Conclusion (f) with V_k in place of V and Γ_k^{-1} in place of Γ follows, for sufficiently large k, from the Constancy Theorem for stationary integral varifolds and the fact that $q\leq\Theta(\|\widetilde{V}_k\|,0)\leq(\omega_n2^n)^{-1}\,\|\widetilde{V}_k\|(B_2^{n+1}(0))<q+1/2$ for each k. $\square$

LEMMA 13.2. *Let $q\geq 2$ be an integer, $\alpha\in(0,1)$ and $p\in\{4,5,\ldots,2q\}$. For $j=1,2,\ldots,p-3$, let $\theta_j\in(0,1/4)$ be such that $\theta_1>8\theta_2>64\theta_3>\cdots>8^{p-4}\theta_{p-3}$. There exist numbers $\varepsilon^{(p)}=\varepsilon^{(p)}(n,q,\alpha,\theta_1,\theta_2,\ldots,\theta_{p-3})\in(0,1/2)$, $\gamma^{(p)}=\gamma^{(p)}(n,q,\alpha,\theta_1,\theta_2\ldots,\theta_{p-3})\in(0,1/2)$ such that if $V\in\mathcal{S}_\alpha$, $\mathbf{C}\in\mathcal{C}_q(p)$ satisfy Hypotheses* 10.1 *and Hypothesis* $(\star)$ *with $\varepsilon=\varepsilon^{(p)}$, $\gamma=\gamma^{(p)}$, $M=\frac{3}{2}M_0$, and if the induction hypotheses* (H1), (H2) *hold, then there exist an orthogonal rotation Γ of $\mathbf{R}^{n+1}$ and a cone $\mathbf{C}'\in\mathcal{C}_q$ such that, with*

$$\hat{E}_V^2=\int_{\mathbf{R}\times B_1}|x^1|^2\,d\|V\|(X)$$

and

$$Q_V^2(\mathbf{C})=\int_{\mathbf{R}\times(B_{1/2}\setminus\{|x^2|<1/16\})}\mathrm{dist}^2(X,\mathrm{spt}\,\|V\|)\,d\|\mathbf{C}\|(X)$$
$$+\int_{\mathbf{R}\times B_1}\mathrm{dist}^2(X,\mathrm{spt}\,\|\mathbf{C})\|)\,d\|V\|(X),$$

we have the following:

(a) $|e_1-\Gamma(e_1)|\leq\kappa^{(p)}Q_V(\mathbf{C})$ *and* $|e_j-\Gamma(e_j)|\leq\kappa^{(p)}\hat{E}_V^{-1}Q_V(\mathbf{C})$ *for each* $j=2,3,\ldots,n+1$;

(b) $\mathrm{dist}^2_{\mathcal{H}}(\mathrm{spt}\,\|\mathbf{C}'\|\cap(\mathbf{R}\times B_1),\mathrm{spt}\,\|\mathbf{C}\|\cap(\mathbf{R}\times B_1))\leq C_0^{(p)}Q_V^2(\mathbf{C})$;

and for some $j\in\{1,2,\ldots,p-3\}$,

(c) $\theta_j^{-n-2}\int_{\Gamma\left(\mathbf{R}\times\left(B_{\theta_j/2}\setminus\{|x^2|\leq\theta_j/16\}\right)\right)}\mathrm{dist}^2(X,\mathrm{spt}\,\|V\|)\,d\|\Gamma_{\#}\,\mathbf{C}'\|(X)$
$+\,\theta_j^{-n-2}\int_{\Gamma(\mathbf{R}\times B_{\theta_j})}\mathrm{dist}^2(X,\mathrm{spt}\,\|\Gamma_{\#}\,\mathbf{C}'\|)\,d\|V\|(X)\leq\nu_j^{(p)}\theta_j^2Q_V^2(\mathbf{C})$;

(d) *for any* $P \in G_n$ *of the form* $P = \{x^1 = \lambda x^2\}$ *for some* $\lambda \in (-1, 1)$,

$$\left(\theta_j^{-n-2} \int_{\mathbf{R}\times B_{\theta_j}} \mathrm{dist}^2\,(X, P)\, d\|\Gamma_{\#}^{-1} V\|(X)\right)^{1/2} \geq 2^{-\frac{n+4}{2}} \sqrt{\overline{C}_1}\, \mathrm{dist}_{\mathcal{H}}(\mathrm{spt}\,\|\mathbf{C}\| \cap (\mathbf{R}\times B_1), P \cap (\mathbf{R}\times B_1)) - C_2^{(p)} Q_V(\mathbf{C});$$

(e) $\{Z : \Theta(\|\Gamma_{\#}^{-1} V\|, Z) \geq q\} \cap \left(\mathbf{R} \times (B_{\theta_j/2} \cap \{|x^2| < \theta_j/16\})\right) = \emptyset$;

(f) $\left(\omega_n \theta_j^n\right)^{-1} \|\Gamma_{\#}^{-1} V\|(\mathbf{R}\times B_{\theta_j}) < q + 1/2$.

Here the dependence of the various constants on the parameters is as follows:

$$\kappa^{(p)} = \kappa^{(p)}(n, q, \alpha, \theta_1, \ldots, \theta_{p-4}), \quad C_0^{(p)} = C_0^{(p)}(n, q, \alpha, \theta_1, \ldots, \theta_{p-4}),$$
$$C_2^{(p)} = C_2^{(p)}(n, q, \alpha, \theta_1, \ldots, \theta_{p-4})$$

in case $q \geq 3$ *and* $p \in \{5, 6, \ldots, 2q\}$; $\kappa^{(4)} = \overline{\kappa}$, $C_0^{(4)} = \overline{C}_0$, $C_2^{(4)} = \overline{C}_2$, *where* $\overline{\kappa} = \overline{\kappa}(n, q, \alpha)$, $\overline{C}_0 = \overline{C}_0(n, q, \alpha)$, $\overline{C}_2 = \overline{C}_2(n, q, \alpha)$ *are as in Lemma* 13.1; $\nu_1^{(p)} = \overline{\nu}$, *where* $\overline{\nu} = \overline{\nu}(n, q, \alpha)$ *is as in Lemma* 13.1; *and, in case* $q \geq 3$, *for each* $j = 2, 3, \ldots, p-3$, $\nu_j^{(p)} = \nu_j^{(p)}(n, q, \alpha, \theta_1, \ldots, \theta_{j-1})$. *In particular,* $\nu_j^{(p)}$ *is independent of* $\theta_j, \theta_{j+1}, \ldots, \theta_{p-3}$ *for* $j = 1, 2, \ldots, p-3$.

Proof. If $p = 4$, then we may simply set $\varepsilon^{(4)}(n, q, \alpha, \theta_1) = \overline{\varepsilon}(n, q, \alpha, \theta_1)$ and $\gamma^{(4)}(n, q, \alpha, \theta_1) = \overline{\gamma}(n, q, \alpha, \theta_1)$, where $\overline{\varepsilon}$, $\overline{\gamma}$ are as in Lemma 13.1, and deduce from Lemma 13.1 with $\theta = \theta_1$ that there exist a cone $\mathbf{C}' \in \mathcal{C}_q$ and an orthogonal rotation Γ of $\mathbf{R}^{n+1}$ such that the conclusions of the lemma hold with $j = 1$ in (c)–(f); with $\kappa^{(4)} = \overline{\kappa}$, $C_0^{(4)} = \overline{C}_0$, $C_2^{(4)} = \overline{C}_2$ and $\nu_1^{(4)} = \overline{\nu}$, where $\overline{\kappa}_j$, $\overline{C}_0$, $\overline{C}_2$, $\overline{\nu}$ are as in Lemma 13.1. Thus the lemma holds if $p = 4$.

Else $q \geq 3$ and $p \in \{5, 6, \ldots, 2q\}$. Assume by induction the validity of the lemma with any $p' \in \{4, 5, \ldots, p-1\}$ in place of p. Let $\theta_j \in (0, 1/4)$, $j = 1, 2, \ldots, p-3$ be given such that $\theta_1 > 8\theta_2 > 64\theta_3 > \cdots > 8^{p-4}\theta_{p-3}$. To prove the lemma as stated, it suffices to show that for arbitrary sequences $\{V_k\} \subset \mathcal{S}_\alpha$, $\{\mathbf{C}_k\} \subset \mathcal{C}_q(p)$ that satisfy hypotheses (1_k)–(5_k) of Section 11 as well as hypothesis (6_k) of Section 11 with M_0 in place of M_0^2, there exist a subsequence $\{k'\}$ of $\{k\}$ and, for each k', a cone $\mathbf{C}'_{k'} \in \mathcal{C}_q$ and an orthogonal rotation $\Gamma_{k'}$ of $\mathbf{R}^{n+1}$ such that the conclusions of the lemma hold with $V_{k'}$, $\mathbf{C}_{k'}$, $\mathbf{C}'_{k'}$, Γ'_k in place of V, $\mathbf{C}$, $\mathbf{C}'$, Γ respectively and with suitable constants $\kappa^{(p)}$, $C_0^{(p)}$, $C_2^{(p)}$ and $\nu_1^{(p)}, \ldots, \nu_{p-3}^{(p)}$ depending only on the parameters as specified in the statement of the lemma. So suppose, for $k = 1, 2, \ldots$, that $V_k \in \mathcal{S}_\alpha$, $\mathbf{C}_k \in \mathcal{C}_q(p)$ satisfy hypotheses (1_k)–(6_k) of Section 11 with M_0 in place of M_0^2. For each k, choose a cone $\widetilde{\mathbf{C}}_k \in \bigcup_{j=4}^{p-1} \mathcal{C}_q(j)$ such that

$$(13.15) \qquad \left(\widetilde{Q}_k\right)^2 \equiv \left(\int_{\mathbf{R}\times(B_{1/2}\setminus\{|x^2|<1/16\})} \mathrm{dist}^2(X, \mathrm{spt}\,\|V_k\|)\, d\|\widetilde{\mathbf{C}}_k\|(X) + \int_{\mathbf{R}\times B_1} \mathrm{dist}^2(X, \mathrm{spt}\,\|\widetilde{\mathbf{C}}_k\|)\, d\|V_k\|(X)\right) \leq \frac{3}{2}\left(Q_k^\star\right)^2,$$

where

$$(Q_k^\star)^2 = \inf_{\widetilde{\mathbf{C}}\in\bigcup_{j=4}^{p-1}\mathcal{C}_q(j)} \left(\int_{\mathbf{R}\times(B_{1/2}\setminus\{|x^2|<1/16\})} \mathrm{dist}^2(X, \mathrm{spt}\,\|V_k\|)\, d\|\widetilde{\mathbf{C}}\|(X) + \int_{\mathbf{R}\times B_1} \mathrm{dist}^2(X, \mathrm{spt}\,\|\widetilde{\mathbf{C}}\|)\, d\|V_k\|(X)\right).$$

Let $\beta = \overline{\beta}(n, q, \alpha, \theta_1)$, where $\overline{\beta}$ is as in Lemma 13.1, and consider the following two alternatives:

(A) for infinitely many k,

$$\int_{\mathbf{R}\times(B_{1/2}\setminus\{|x^2|<1/16\})} \mathrm{dist}^2(X, \mathrm{spt}\,\|V_k\|)\, d\|\mathbf{C}_k\|(X) + \int_{\mathbf{R}\times B_1} \mathrm{dist}^2(X, \mathrm{spt}\,\|\mathbf{C}_k\|)\, d\|V_k\|(X) < \beta\,(Q_k^\star)^2$$

(B) for all sufficiently large k,

$$\int_{\mathbf{R}\times(B_{1/2}\setminus\{|x^2|<1/16\})} \mathrm{dist}^2(X, \mathrm{spt}\,\|V_k\|)\, d\|\mathbf{C}_k\|(X) + \int_{\mathbf{R}\times B_1} \mathrm{dist}^2(X, \mathrm{spt}\,\|\mathbf{C}_k\|)\, d\|V_k\|(X) \geq \beta\,(Q_k^\star)^2\,.$$

If alternative (A) holds, we deduce directly from Lemma 13.1, applied with $\theta = \theta_1$, that for infinitely many k, there exist a cone $\mathbf{C}_k' \in \mathcal{C}_q$ and an orthogonal rotation Γ_k of $\mathbf{R}^{n+1}$ such that the conclusions of the present lemma hold with V_k, $\mathbf{C}_k$, $\mathbf{C}_k'$, Γ_k in place of V, $\mathbf{C}$, $\mathbf{C}'$, Γ; with $j = 1$ in the conclusions (c)–(f); and with $\overline{\kappa}$, $\overline{C}_0$, $\overline{C}_2$, $\overline{\nu}$ (as in Lemma 13.1) in place of $\kappa^{(p)}$, $C_2^{(p)}$, $\nu_1^{(p)}$.

If alternative (B) holds, we have by hypothesis (5_k) and (13.15) that for all sufficiently large k,

$$\left(\int_{\mathbf{R}\times(B_{1/2}\setminus\{|x^2|<1/16\})} \mathrm{dist}^2(X, \mathrm{spt}\,\|V_k\|)\, d\|\widetilde{\mathbf{C}}_k\|(X) + \int_{\mathbf{R}\times B_1} \mathrm{dist}^2(X, \mathrm{spt}\,\|\widetilde{\mathbf{C}}_k\|)\, d\|V_k\|(X)\right) \leq \frac{3\gamma_k}{2\beta}\hat{E}_k^2. \tag{13.16}$$

Since $\widetilde{\mathbf{C}}_k \in \mathcal{C}_q(p')$ for some $p' \in \{4, 5, \ldots, p-1\}$ and infinitely many k, we may, by the induction hypothesis, apply the lemma with p' in place of p and $\theta_2, \theta_3, \ldots, \theta_{p'-2}$ in place of $\theta_1, \theta_2, \ldots, \theta_{p'-3}$ to deduce that for infinitely many k, there exist a cone $\mathbf{C}_k' \in \mathcal{C}_q$ and an orthogonal rotation Γ_k of $\mathbf{R}^{n+1}$ such that the conclusions (a)–(f) hold with V_k, $\widetilde{\mathbf{C}}_k$, $\mathbf{C}_k'$, Γ_k in place of V, $\mathbf{C}$, $\mathbf{C}'$, Γ—in particular, with $\widetilde{Q}_k$ in place of $Q_V(\mathbf{C})$—and such that

(i) in case $p' = 4$ (which must be the case if $p = 5$), with $\kappa^{(4)} = \overline{\kappa}$, $C_0^{(4)} = \overline{C}_0$, $C_2^{(4)} = \overline{C}_2$ and $\nu_1^{(4)} = \overline{\nu}$ (where $\overline{\kappa}$, $\overline{C}_0$, $\overline{C}_1$, $\overline{C}_2$, $\overline{\nu}$ are as in Lemma 13.1 and C is as in Theorem 10.2(a)); and

(ii) in case $p' \in \{5, 6, \ldots, p-1\}$ (possible, of course, only if $p \geq 6$) with

$$\kappa^{(p')} = \kappa^{(p')}(n, q, \alpha, \theta_2, \ldots, \theta_{p'-3}),\ \ C_0^{(p')} = C_0^{(p')}(n, q, \alpha, \theta_2, \ldots, \theta_{p'-3}),$$

$$C_2^{(p')} = C_2^{(p')}(n, q, \alpha, \theta_2, \ldots, \theta_{p'-3})$$

in place of $\kappa^{(p)}$, $C_0^{(p)}$, $C_2^{(p)}$ respectively, with $\nu_1^{(p')}(n, q, \alpha) = \overline{\nu}$ (where $\overline{\nu}$ is as in Lemma 13.1) in place of $\nu_1^{(p)}$, and with $\nu_{j-1}^{(p')}(n, q, \alpha, \theta_2, \ldots, \theta_{j-1})$ in place of $\nu_{j-1}^{(p)}$ for each $j = 3, \ldots, p'-2$.

Since by (13.15) and the defining requirement of alternative (B) we have that

$$\widetilde{Q}_k^2 \leq \frac{3}{2\beta}\left(\int_{\mathbf{R}\times(B_{1/2}\setminus\{|x^2|<1/16\})} \operatorname{dist}^2(X, \operatorname{spt}\|V_k\|)\, d\|\mathbf{C}_k\|(X)\right.$$
$$\left. + \int_{\mathbf{R}\times B_1} \operatorname{dist}^2(X, \operatorname{spt}\|\mathbf{C}_k\|)\, d\|V_k\|(X)\right),$$

and $\operatorname{dist}_{\mathcal{H}}^2(\operatorname{spt}\|\mathbf{C}_k\| \cap (\mathbf{R}\times B_1), \operatorname{spt}\|\widetilde{\mathbf{C}}_k\| \cap (\mathbf{R}\times B_1)) \leq \overline{C}(\widetilde{Q}_k^2 + Q_k^2)$, where $\overline{C} = \overline{C}(n, q) \in (0, \infty)$, setting

$$\kappa^{(5)}(n, q, \alpha, \theta_1) = \frac{3\overline{\kappa}}{2\overline{\beta}(n, q, \alpha, \theta_1)},\ \ C_0^{(5)}(n, q, \alpha, \theta_1) = 2\overline{C} + \frac{3(\overline{C} + \overline{C}_0)}{\overline{\beta}(n, q, \alpha, \theta_1)},$$

$$C_2^{(5)}(n, q, \alpha, \theta_1) = 2^{-\frac{n+4}{2}}\sqrt{\overline{C}_1\overline{C}} + \left(2^{-\frac{n+4}{2}}\sqrt{\overline{C}_1\overline{C}} + \overline{C}_2\right)\sqrt{\frac{3}{2\overline{\beta}(n, q, \alpha, \theta_1)}},$$

$$\nu_1^{(5)}(n, q, \alpha) = \overline{\nu},\quad \nu_2^{(5)}(n, q, \alpha, \theta_1) = \frac{3\overline{\nu}}{2\overline{\beta}(n, q, \alpha, \theta_1)},$$

and, for $p \geq 6$,

$$\kappa^{(p)}(n, q, \alpha, \theta_1, \ldots, \theta_{p-4})$$
$$= \max\left\{\overline{\kappa},\ \frac{3}{2\overline{\beta}(n, q, \alpha, \theta_1)}\kappa^{(p')}(n, q, \alpha, \theta_2, \ldots, \theta_{p'-3}) : p' = 5, \ldots, p-1\right\},$$

$$C_0^{(p)}(n, q, \alpha, \theta_1, \ldots, \theta_{p-4})$$
$$= \max\left\{\overline{C}_0,\ 2\overline{C} + \frac{3}{\overline{\beta}(n, q, \alpha, \theta_1)}\left(\overline{C} + C_0^{(p')}(n, q, \alpha, \theta_2, \ldots, \theta_{p'-3})\right) : p' = 5, \ldots, p-1\right\},$$

$$C_2^{(p)}(n, q, \alpha, \theta_1, \ldots, \theta_{p-4})$$
$$= \max\left\{\overline{C}_2,\ a + \sqrt{\frac{3}{2\overline{\beta}(n, q, \alpha, \theta_1)}}\left(a + C_2^{(p')}(n, q, \alpha, \theta_2, \ldots, \theta_{p'-3})\right) : p' = 5, \ldots, p-1\right\},$$

where $a = 2^{-\frac{n+4}{2}}\sqrt{\overline{C}_1\overline{C}}$,

$$\nu_1^{(p)}(n, q, \alpha) = \overline{\nu},\quad \nu_2^{(p)}(n, q, \alpha, \theta_1) = \frac{3\overline{\nu}}{2\overline{\beta}(n, q, \alpha, \theta_1)}$$

and, for $j = 3, \ldots, p-3$,

$$\nu_j^{(p)}(n, q, \alpha, \theta_1, \ldots, \theta_{j-1}) = \max\left\{\frac{3}{2\overline{\beta}(n,q,\alpha,\theta_1)}\nu_{j-1}^{(p')}(n, q, \alpha, \theta_2, \ldots, \theta_{j-1}) : p' = j+2, \ldots, p-1\right\},$$

we see that if alternative (B) holds, the conclusions (a)–(f) of the lemma follow with V_k, $\mathbf{C}_k$, $\mathbf{C}'_k$, Γ_k in place of V, $\mathbf{C}$, $\mathbf{C}'$, Γ; with constants $\kappa^{(p)}, C_0'^{(p)}, C_2'^{(p)}$ depending only on n, q, α, $\theta_1, \theta_2, \ldots, \theta_{p-4}$; with $\nu_1^{(p)}$ depending only on n, q, α and for each $j = 2, 3, \ldots, p-3$, with $\nu_j^{(p)}$ depending only on n, q, α and $\theta_1, \theta_2, \theta_3, \ldots, \theta_{j-1}$. Note that in checking that conclusion (d) holds with V_k, $\mathbf{C}_k$ in place of V, $\mathbf{C}$, we have used the fact that

$$\begin{aligned}
&\mathrm{dist}_{\mathcal{H}}(\mathrm{spt}\,\|\widetilde{\mathbf{C}}_k\| \cap (\mathbf{R} \times B_1), \{0\} \times B_1) \\
&\quad\geq \mathrm{dist}_{\mathcal{H}}(\mathrm{spt}\,\|\mathbf{C}_k\| \cap (\mathbf{R} \times B_1), \{0\} \times B_1) \\
&\qquad - \mathrm{dist}_{\mathcal{H}}(\mathrm{spt}\,\|\mathbf{C}_k\| \cap (\mathbf{R} \times B_1), \mathrm{spt}\,\|\widetilde{\mathbf{C}}_k\| \cap (\mathbf{R} \times B_1)) \\
&\quad\geq \mathrm{dist}_{\mathcal{H}}(\mathrm{spt}\,\|\mathbf{C}_k\| \cap (\mathbf{R} \times B_1), \{0\} \times B_1) - \sqrt{\overline{C}}(\widetilde{Q}_k + Q_k).
\end{aligned}$$

Similar reasoning applies in checking conclusion (b). This completes the proof. □

LEMMA 13.3. *Let $q \geq 2$ be an integer and $\alpha \in (0,1)$. For $j = 1, 2, \ldots, 2q-3$, let $\theta_j \in (0, 1/4)$ be such that $\theta_1 > 8\theta_2 > 64\theta_3 > \cdots > 8^{2q-4}\theta_{2q-3}$. There exist numbers*

$$\begin{aligned}
\varepsilon &= \varepsilon(n, q, \alpha, \theta_1, \theta_2, \ldots, \theta_{2q-3}) \in (0, 1/2), \\
\gamma &= \gamma(n, q, \alpha, \theta_1, \theta_2, \ldots, \theta_{2q-3}) \in (0, 1/2)
\end{aligned}$$

such that the following is true: If $V \in \mathcal{S}_\alpha$, $\mathbf{C} \in \mathcal{C}_q$ satisfy Hypotheses 10.1 *and Hypothesis* $(\star)$ *with $M = \frac{3}{2}M_0$ and if the induction hypotheses* (H1), (H2) *hold, then there exist an orthogonal rotation Γ of $\mathbf{R}^{n+1}$ and a cone $\mathbf{C}' \in \mathcal{C}_q$ such that, with $\hat{E}_V$ and $Q_V(\mathbf{C})$ as defined in Lemma* 13.2, *we have the following*:

(a) $|e_1 - \Gamma(e_1)| \leq \kappa Q_V(\mathbf{C})$*and* $|e_j - \Gamma(e_j)| \leq \kappa \hat{E}_V^{-1} Q_V(\mathbf{C})$ *for each* $j = 2, 3, \ldots, n+1$;

(b) $\mathrm{dist}^2_{\mathcal{H}}(\mathrm{spt}\,\|\mathbf{C}'\| \cap (\mathbf{R} \times B_1), \mathrm{spt}\,\|\mathbf{C}\| \cap (\mathbf{R} \times B_1)) \leq C_0 Q_V^2(\mathbf{C})$;

and for some $j \in \{1, 2, \ldots, 2q-3\}$,

(c) $$\theta_j^{-n-2}\int_{\Gamma\left(\mathbf{R}\times\left(B_{\theta_j/2}\setminus\{|x^2|\leq\theta_j/16\}\right)\right)} \mathrm{dist}^2(X, \mathrm{spt}\,\|V\|)\, d\|\Gamma_\#\,\mathbf{C}'\|(X) + \theta_j^{-n-2}\int_{\Gamma(\mathbf{R}\times B_{\theta_j})} \mathrm{dist}^2(X, \mathrm{spt}\,\|\Gamma_\#\,\mathbf{C}'\|)\, d\|V\|(X) \leq \nu_j \theta_j^2 Q_V^2(\mathbf{C});$$

(d) *for any $P \in G_n$ of the form $P = \{x^1 = \lambda x^2\}$ for some $\lambda \in (-1, 1)$,*

$$\left(\theta_j^{-n-2} \int_{\mathbf{R} \times B_{\theta_j}} \operatorname{dist}^2(X, P)\, d\|\Gamma_{\#}^{-1} V\|(X)\right)^{1/2}$$
$$\geq 2^{-\frac{n+4}{2}} \sqrt{C_1}\, \operatorname{dist}_{\mathcal{H}}(\operatorname{spt}\|\mathbf{C}\| \cap (\mathbf{R} \times B_1), P \cap (\mathbf{R} \times B_1)) - C_2 Q_V(\mathbf{C});$$

(e) $\{Z : \Theta(\|\Gamma_{\#}^{-1} V\|, Z) \geq q\} \cap \left(\mathbf{R} \times (B_{\theta_j/2} \cap \{|x^2| < \theta_j/16\})\right) = \emptyset$;

(f) $\left(\omega_n \theta_j^n\right)^{-1} \|\Gamma_{\#}^{-1} V\|(\mathbf{R} \times B_{\theta_j}) < q + 1/2$.

Here the constants $\kappa, C_0, C_2 \in (0, \infty)$ depend only on n, α in case $q = 2$ and only on n, q, α and $\theta_1, \theta_2, \ldots, \theta_{2q-4}$ in case $q \geq 3$; $\nu_1 = \nu_1(n, q, \alpha)$; and, in case $q \geq 3$, for each $j = 2, 3, \ldots, 2q-3$, $\nu_j = \nu_j(n, q, \alpha, \theta_1, \ldots, \theta_{j-1})$. (In particular, ν_j is independent of $\theta_j, \theta_{j+1}, \ldots, \theta_{2q-3}$ for each $j = 1, 2, \ldots, 2q-3$.)

Proof. Set $\varepsilon = \min\left\{\varepsilon^{(4)}, \varepsilon^{(5)}, \ldots, \varepsilon^{(2q)}\right\}$ and $\gamma = \min\left\{\gamma^{(4)}, \gamma^{(5)}, \ldots, \gamma^{(2q)}\right\}$, where

$$\varepsilon^{(p)} = \varepsilon^{(p)}(n, q, \alpha, \theta_1, \ldots, \theta_{p-3}), \quad \gamma^{(p)} = \gamma^{(p)}(n, q, \alpha, \theta_1, \ldots, \theta_{p-3}), \quad 4 \leq p \leq 2q$$

are as in Lemma 13.2. Set $\nu_1 = \overline{\nu}$, and for each $j = 2, \ldots, 2q-3$, set

$$\nu_j = \max\left\{\nu_j^{(j+3)}, \nu_j^{(j+4)}, \ldots, \nu_j^{(2q)}\right\} \ \left(= \nu_j^{(2q)}\right),$$

where $\overline{\nu}$ is as in Lemma 13.1 and for each $p \in \{5, \ldots, 2q\}$, the numbers $\nu_j^{(p)}$ are as in Lemma 13.2 taken with scales $\theta_1, \ldots, \theta_{p-3}$. Note that then, $\nu_1 = \nu_1(n, q, \alpha)$ and in case $q \geq 3$,

$$\nu_j = \nu_j(n, q, \alpha, \theta_1, \ldots, \theta_{j-1}) \ \text{ for } \ 2 \leq j \leq 2q-3.$$

Set

$$\kappa = \max\left\{\kappa^{(4)}, \kappa^{(5)}, \ldots, \kappa^{(2q)}\right\} \ \left(= \kappa^{(2q)}\right),$$
$$C_0 = \max\left\{C_0^{(4)}, C_0^{(5)}, \ldots, C_0^{(2q)}\right\} \ \left(= C_0^{(2q)}\right),$$
$$C_2 = \max\left\{C_2^{(4)}, C_2^{(5)}, \ldots, C_2^{(2q)}\right\} \ \left(= C_2^{(2q)}\right),$$

where for each $p \in \{4, 5, \ldots, 2q\}$, the numbers $\kappa^{(p)}$, $C_0^{(p)}$, $C_2^{(p)}$ are as in Lemma 13.2 taken with scales $\theta_1, \ldots, \theta_{p-3}$. Since $\mathbf{C} \in \mathcal{C}_q$ implies that $\mathbf{C} \in \mathcal{C}_q(p)$ for some $p \in \{4, 5, \ldots, 2q\}$, the conclusions of the present lemma follow directly from Lemma 13.2. □

14. Properties of coarse blow-ups: Part III

Subject to the induction hypotheses (H1), (H2), in this section we complete the proof that $\mathcal{B}_q$ is a proper blow-up class by showing that $\mathcal{B}_q$ satisfies property ($\mathcal{B}7$). Recall that in order to do this, it only remains to rule out the possibility that $\mathcal{B}_q$ contains an element whose graph is the union of q half-hyperplanes in the half-space $\{x^2 \leq 0\}$ and q half-hyperplanes in $\{x^2 \geq 0\}$, with all half-hyperplanes meeting along $\{0\} \times \mathbf{R}^{n-1}$ and with at least two of the half-hyperplanes distinct on each side . (This is Case 2 stated at the beginning of Section 9.)

LEMMA 14.1. *Let $q \geq 2$ be an integer and $\alpha \in (0,1)$. There exist constants $\varepsilon_1 = \varepsilon_1(n,q,\alpha) \in (0,1)$ and $\gamma_1 = \gamma_1(n,q,\alpha) \in (0,1)$ such that if*

- *the induction hypotheses* (H1), (H2) *hold*,
- $V \in \mathcal{S}_\alpha$,
- $\Theta(\|V\|, 0) \geq q$,
- $(\omega_n 2^n)^{-1} \|V\|(B_2^{n+1}(0)) < q + 1/2$,
- $\omega_n^{-1} \|V\|(\mathbf{R} \times B_1) < q + 1/2$,
- $\{Z : \Theta(\|V\|, Z) \geq q\} \cap \big(\mathbf{R} \times (B_{1/2} \setminus \{|x^2| < 1/16\})\big) = \emptyset$,
- $\hat{E}_V^2 \equiv \int_{\mathbf{R} \times B_1} |x^1|^2 d\|V\|(X) < \varepsilon_1$ *and*
- $\hat{E}_V^2 < \frac{3}{2} \inf_{\{P = \{x^1 = \lambda x^2\}\}} \int_{\mathbf{R} \times B_1} \mathrm{dist}^2(X, P)\, d\|V\|(X)$,

then

$$\int_{\mathbf{R} \times (B_{1/2} \setminus \{|x^2| < 1/16\})} \mathrm{dist}^2(X, \mathrm{spt}\, \|V\|)\, d\|\mathbf{C}\|(X) + \int_{\mathbf{R} \times B_1} \mathrm{dist}^2(X, \mathrm{spt}\, \|\mathbf{C}\|)\, d\|V\|(X) \geq \gamma_1 \hat{E}_V^2$$

for any cone $\mathbf{C} \in \mathcal{C}_q$.

Proof. For $j = 1, 2, \ldots, 2q-3$, choose numbers $\theta_j = \theta_j(n,q,\alpha) \in (0, 1/2)$ as follows: First choose $\theta_1 = \theta_1(n,q,\alpha) \in (0,1/2)$ such that $\nu_1 \theta_1^{2(1-\alpha)} < 1$, where $\nu_1 = \nu_1(n,q,\alpha)$ is as in Lemma 13.3. Having chosen $\theta_1, \theta_2, \ldots, \theta_j$, $1 \leq j \leq 2q-4$, choose $\theta_{j+1} = \theta_{j+1}(n,q,\alpha)$ such that $\theta_{j+1} < 8^{-1}\theta_j$ and $\nu_{j+1}\theta_{j+1}^{2(1-\alpha)} < 1$, where $\nu_{j+1} = \nu_{j+1}(n,q,\alpha,\theta_1,\theta_2,\ldots,\theta_j)$ is as in Lemma 13.3.

Let $\varepsilon_1 \in (0,\varepsilon)$, $\gamma_1 \in (0,\gamma)$ be constants to be eventually chosen depending only on n, q and α, where $\varepsilon = \varepsilon(n,q,\alpha,\theta_1,\ldots,\theta_{2q-3})$ and $\gamma = \gamma(n,q,\alpha,\theta_1,\ldots,\theta_{2q-3})$ are as in Lemma 13.3. Suppose that the hypotheses of the present lemma are satisfied with $V \in \mathcal{S}_\alpha$ but the conclusion fails; i.e., there exists $\mathbf{C} \in \mathcal{C}_q$ such that

$$\int_{\mathbf{R} \times (B_{1/2} \setminus \{|x^2| < 1/16\})} \mathrm{dist}^2(X, \mathrm{spt}\, \|V\|)\, d\|\mathbf{C}\|(X) + \int_{\mathbf{R} \times B_1} \mathrm{dist}^2(X, \mathrm{spt}\, \|\mathbf{C}\|)\, d\|V\|(X) < \gamma_1 \hat{E}_V^2. \tag{14.1}$$

In particular, V, $\mathbf{C}$ then satisfy the hypotheses of Lemma 13.3. In what follows, for $\mathbf{C}' \in \mathcal{C}_q$, Γ an orthogonal rotation of $\mathbf{R}^{n+1}$ and $\rho \in (0,1]$, we shall use the notation

$$Q_V(\mathbf{C}', \Gamma, \rho) = \left(\rho^{-n-2} \int_{\Gamma\left(\mathbf{R}\times(B_{\rho/2}\setminus\{|x^2|<\rho/16\})\right)} \operatorname{dist}^2(X, \operatorname{spt}\|V\|)\, d\|\Gamma_{\#}\, \mathbf{C}'\|(X) \right.$$
$$\left. + \rho^{-n-2} \int_{\Gamma(\mathbf{R}\times B_\rho)} \operatorname{dist}^2(X, \operatorname{spt}\|\Gamma_{\#}\, \mathbf{C}'\|)\, d\|V\|(X) \right)^{1/2}.$$

We claim that we may apply Lemma 13.3 iteratively to obtain, for each $k = 0, 1, 2, 3, \ldots$, an orthogonal rotation Γ_k of $\mathbf{R}^{n+1}$ with $\Gamma_0 = $ Identity, and a cone $\mathbf{C}_k \in \mathcal{C}_q$ with $\mathbf{C}_0 = \mathbf{C}$, satisfying, for $k \geq 1$,

$$|\Gamma_k(e_1) - \Gamma_{k-1}(e_1)|^2 \leq C\delta_k Q_V^2; \tag{14.2}$$

$$|\Gamma_k(e_j) - \Gamma_{k-1}(e_j)|^2 \leq C\delta_k \hat{E}_V^{-2} Q_V^2; \tag{14.3}$$

$$\operatorname{dist}^2_{\mathcal{H}}(\operatorname{spt}\|\mathbf{C}_k\| \cap (\mathbf{R}\times B_1), \operatorname{spt}\|\mathbf{C}_{k-1}\| \cap (\mathbf{R}\times B_1)) \leq C\delta_k Q_V^2; \tag{14.4}$$

$$Q_V^2(\mathbf{C}_k, \Gamma_k, \sigma_k) \leq \nu_{j_k}\theta_{j_k}^2 Q_V^2(\mathbf{C}_{k-1}, \Gamma_{k-1}, \sigma_{k-1}) \leq \cdots \leq \delta_k Q_V^2 \tag{14.5}$$

for some $j_k \in \{1, 2, \ldots, 2q-3\}$;

$$\left(\sigma_k^{-n-2} \int_{\mathbf{R}\times B_{\sigma_k}} \operatorname{dist}^2(X, P)\, d\|\Gamma_{k\,\#}\, V\|(X)\right)^{1/2} \tag{14.6}$$
$$\geq 2^{-\frac{n+4}{2}} \sqrt{C_1}\, \operatorname{dist}_{\mathcal{H}}(\operatorname{spt}\|\mathbf{C}_{k-1}\| \cap (\mathbf{R}\times B_1), P \cap (\mathbf{R}\times B_1))$$
$$- C_2\, Q_V(\mathbf{C}_{k-1}, \Gamma_{k-1}, \sigma_{k-1})$$

for each $P \in G_n$ of the form $P = \{x^1 = \lambda x^2\}$ for some $\lambda \in (-1, 1)$;

$$\{Z : \Theta(\|\Gamma_{k\,\#}\, V\|, Z) \geq q\} \cap \left(\mathbf{R}\times (B_{\sigma_k} \setminus \{|x^2| < \sigma_k/16\})\right) = \emptyset; \tag{14.7}$$

and

$$(\omega_n \sigma_k^n)^{-1}\, \|\Gamma^{-1}_{k\,\#}\, V\|(\mathbf{R}\times B_{\sigma_k}) < q + 1/2, \tag{14.8}$$

where $Q_V = Q_V(\mathbf{C}, \Gamma_0, 1)$, $C = C(n, q, \alpha) \in (0, \infty)$, $C_2 = C_2(n, q, \alpha) \in (0, \infty)$ and, for each $k = 1, 2, 3, \ldots$,

$$\sigma_k = \theta_{j_k}\sigma_{k-1}, \quad \delta_k = \nu_{j_k}\theta_{j_k}^2 \delta_{k-1}$$

for some $j_k \in \{1, 2, \ldots, 2q-3\}$, where $\sigma_0 = \delta_0 = 1$. Thus

$$\sigma_k = \prod_{j=1}^{2q-3} \theta_j^{k_j} \qquad \text{and} \qquad \delta_k = \prod_{j=1}^{2q-3} \left(\nu_j \theta_j^2\right)^{k_j}$$

for some nonnegative integers $k_1, k_2, \ldots, k_{2q-3}$ such that $\sum_{j=1}^{2q-3} k_j = k$. Note, in particular, that

$$\theta_{2q-3}^k \leq \sigma_k \leq \theta_1^k, \qquad \delta_k < \sigma_k^{2\alpha} < 4^{-k\alpha} \qquad \text{and} \qquad \sum_{j=k}^{\infty} \delta_j < c\delta_k$$

for $k = 1, 2, \ldots$, where $c = c(\alpha) \in (0, \infty)$.

To verify these assertions inductively, note that (14.4)–(14.7) with $k=1$ follow directly from Lemma 13.3. Suppose $k \geq 2$ and that (14.4)–(14.7) hold with $1, 2, 3, \ldots, k-1$ in place of k. We wish to apply Lemma 13.3 with $\eta_{\sigma_{k-1}\,\#}\, \Gamma^{-1}_{k-1\,\#}\, V$ in place of V and $\mathbf{C}_{k-1}$ in place of $\mathbf{C}$. Note first that by the triangle inequality and (14.8) with $k-1$ in place of k,

$$\begin{aligned}\hat{E}^2_{\eta_{\sigma_{k-1}\,\#}\,\Gamma^{-1}_{k-1\,\#}\,V} &= \sigma_{k-1}^{-n-2}\int_{\mathbf{R}\times B_{\sigma_{k-1}}} |x^1|^2 \, d\|\Gamma^{-1}_{k-1\,\#}\,V\|(X)\\ &\leq 2\,\sigma_{k-1}^{-n-2}\int_{\mathbf{R}\times B_{\sigma_{k-1}}} \operatorname{dist}^2(X, \operatorname{spt}\|\mathbf{C}_{k-1}\|)\, d\|\Gamma^{-1}_{k-1\,\#}\,V\|(X)\\ &\quad + \omega_n(2q+1)\operatorname{dist}^2_{\mathcal{H}}(\operatorname{spt}\|\mathbf{C}_{k-1}\| \cap (\mathbf{R}\times B_1), \{0\}\times B_1)\end{aligned}$$

and by applying (14.4) with $1, 2, \ldots, k-1$ in place of k, summing over k, and using the fact that $\sum_{k=1}^{\infty} \delta_k^{1/2} < 2^{-\alpha}(1-2^{-\alpha})^{-1}$,

$$\begin{aligned}&\operatorname{dist}_{\mathcal{H}}(\operatorname{spt}\|\mathbf{C}_{k-1}\| \cap (\mathbf{R}\times B_1), \{0\}\times B_1)\\ &\qquad \leq \operatorname{dist}_{\mathcal{H}}(\operatorname{spt}\|\mathbf{C}\| \cap (\mathbf{R}\times B_1), \{0\}\times B_1) + CQ_V,\ C = C(n,q,\alpha) \in (0,\infty);\end{aligned}$$

thus,

$$\hat{E}^2_{\eta_{\sigma_{k-1}\,\#}\,\Gamma^{-1}_{k-1\,\#}\,V} \leq 2\omega_n(2q+1)\operatorname{dist}^2_{\mathcal{H}}(\operatorname{spt}\|\mathbf{C}\| \cap (\mathbf{R}\times B_1), \{0\}\times B_1) + CQ_V^2,$$

$C = C(n,q,\alpha) \in (0,\infty)$, so that, by (10.1),

$$\hat{E}^2_{\eta_{\sigma_{k-1}\,\#}\,\Gamma^{-1}_{k-1\,\#}\,V} \leq 2(2q+1)\omega_n c_1^2 \hat{E}_V^2 + CQ_V^2,\ C = C(n,q,\alpha) \in (0,\infty), \tag{14.9}$$

where $c_1 = c_1(n) \in (0,\infty)$ is as in (10.1); in particular,

$$\hat{E}^2_{\eta_{\sigma_{k-1}\,\#}\,\Gamma^{-1}_{k-1\,\#}\,V} \leq C\hat{E}_V^2, \quad C = C(n,q,\alpha) \in (0,\infty). \tag{14.10}$$

Again by (14.4),

$$\begin{aligned}&\operatorname{dist}_{\mathcal{H}}(\operatorname{spt}\|\mathbf{C}_{k-2}\| \cap (\mathbf{R}\times B_1), \{0\}\times B_1)\\ &\qquad \geq \operatorname{dist}_{\mathcal{H}}(\operatorname{spt}\|\mathbf{C}\| \cap \mathbf{R}\times B_1, \{0\}\times B_1)\\ &\qquad\qquad - \sum_{j=1}^{k-2} \operatorname{dist}_{\mathcal{H}}(\operatorname{spt}\|\mathbf{C}_{j-1}\| \cap (\mathbf{R}\times B_1), \operatorname{spt}\|\mathbf{C}_j\| \cap (\mathbf{R}\times B_1))\\ &\qquad \geq \operatorname{dist}_{\mathcal{H}}(\operatorname{spt}\|\mathbf{C}\| \cap (\mathbf{R}\times B_1), \{0\}\times B_1) - CQ_V \sum_{j=1}^{k-2} \delta_j^{1/2},\end{aligned}$$

which implies by (14.6) and (14.5) that

$$\hat{E}_{\eta_{\sigma_{k-1}\,\#}\,\Gamma^{-1}_{k-1\,\#}\,V} \geq 2^{-\frac{n+4}{2}}\sqrt{\overline{C}_1}\operatorname{dist}_{\mathcal{H}}(\operatorname{spt}\|\mathbf{C}\| \cap (\mathbf{R}\times B_1), \{0\}\times B_1) - CQ_V,$$

where $C = C(n,q,\alpha) \in (0,\infty)$. Hence by (10.2) and (14.1), we see that

$$\hat{E}_{\eta_{\sigma_{k-1}\,\#}\,\Gamma^{-1}_{k-1\,\#}\,V} \geq (C_1 - C\gamma_1)\hat{E}_V, \tag{14.11}$$

where $C_1 = C_1(n,q)$, $C = C(n,q) \in (0,\infty)$. Thus if $2C\gamma_1 < C_1$, it follows from (14.1), (14.5) and (14.11) that

(14.12)
$$\int_{\mathbf{R}\times(B_{1/2}\setminus\{|x^2|<1/16\})} \mathrm{dist}^2(X, \mathrm{spt}\,\|\eta_{\sigma_{k-1}\,\#}\,\Gamma_{k-1\,\#}^{-1}\,V\|)\,d\|\mathbf{C}_{k-1}\|(X)$$
$$+\int_{\mathbf{R}\times B_1} \mathrm{dist}^{\,2}(X, \mathrm{spt}\,\|\mathbf{C}_{k-1}\|)\,d\|\eta_{\sigma_{k-1}\,\#}\,\Gamma_{k-1\,\#}^{-1}\,V\|(X) \le C\gamma_1 \hat{E}^2_{\eta_{\sigma_{k-1}\,\#}\,\Gamma_{k-1\,\#}^{-1}\,V}$$

and from (14.10) that
$$\hat{E}^2_{\eta_{\sigma_{k-1}\,\#}\,\Gamma_{k-1\,\#}^{-1}\,V} \le C\varepsilon_1,$$
where $C = C(n,q,\alpha) \in (0,\infty)$. By (14.6) again with $k-1$ in place of k and (14.4) with $1, 2, \ldots, k-1$ in place of k,

$$\left(\sigma_{k-1}^{-n-2}\int_{\mathbf{R}\times B_{\sigma_{k-1}}} \mathrm{dist}^2(X,P)\,d\|\Gamma_{k-1\,\#}\,V\|(X)\right)^{1/2}$$
$$\ge 2^{-\frac{n+4}{2}}\sqrt{\overline{C}_1}\,\mathrm{dist}_{\mathcal{H}}(\mathrm{spt}\,\|\mathbf{C}\|\cap(\mathbf{R}\times B_1), P\cap(\mathbf{R}\times B_1)) - CQ_V$$

so that
$$\begin{aligned}
&\int_{\mathbf{R}\times B_1} \mathrm{dist}^2(X,P)\,d\|\eta_{\sigma_{k-1}\,\#}\Gamma_{k-1\,\#}\,V\|(X)\\
&\quad\ge 2^{-n-5}\overline{C}_1\,\mathrm{dist}^2_{\mathcal{H}}(\mathrm{spt}\,\|\mathbf{C}\|\cap(\mathbf{R}\times B_1), P\cap(\mathbf{R}\times B_1)) - CQ_V^2\\
&\quad\ge 2^{-n-5}\overline{C}_1\omega_n^{-1}(2q+1)^{-1}\int_{\mathbf{R}\times B_1}\mathrm{dist}^2(X,P)\,d\|V\|(X) - CQ_V^2\\
&\quad\ge 2^{-n-5}\overline{C}_1\omega_n^{-1}(2q+1)^{-1}\left(\frac{3}{2}\right)^{-1}\hat{E}_V^2 - CQ_V^2\\
&\quad\ge 2^{-n-6}\overline{C}_1\omega_n^{-2}(2q+1)^{-2}c_1^{-2}\left(\frac{3}{2}\right)^{-1}\hat{E}^2_{\eta_{\sigma_{k-1}\,\#}\Gamma_{k-1\,\#}\,V} - CQ_V^2\\
&\quad\ge \left(2^{-n-6}\overline{C}_1\omega_n^{-2}(2q+1)^{-2}c_1^{-2}\left(\frac{3}{2}\right)^{-1} - C\gamma_1\right)\hat{E}^2_{\eta_{\sigma_{k-1}\,\#}\Gamma_{k-1\,\#}\,V},
\end{aligned}$$
where $C = C(n,q,\alpha) \in (0,\infty)$ and we have used our hypothesis that
$$\hat{E}_V^2 < \frac{3}{2}\inf_{P=\{x^1=\lambda x^2\}}\int_{\mathbf{R}\times B_1}\mathrm{dist}^2(X,P)\,d\|V\|(X).$$
This readily implies that if we choose $\gamma_1 = \gamma_1(n,q,\alpha) \in (0,1)$ sufficiently small, then
$$\hat{E}^2_{\eta_{\sigma_{k-1}\,\#}\Gamma_{k-1\,\#}\,V} \le \frac{3}{2}M_0\int_{\mathbf{R}\times B_1}\mathrm{dist}^2(X,P)\,d\|\eta_{\sigma_{k-1}\,\#}\Gamma_{k-1\,\#}\,V\|(X)$$
for any hyperplane P of the form $P = \{x^1 = \lambda x^2\}$. So if we choose $\gamma_1 = \gamma_1(n,q,\alpha)$ and $\varepsilon_1 = \varepsilon_1(n,q,\alpha)$ sufficiently small, we can apply Lemma 13.3 with $\eta_{\sigma_{k-1}\,\#}\,\Gamma_{k-1\,\#}^{-1}\,V$ in place of V and $\mathbf{C}_{k-1}$ in place of $\mathbf{C}$ to obtain an

orthogonal rotation Γ of $\mathbf{R}^{n+1}$ and a cone $\mathbf{C}_k \in \mathcal{C}_q$ such that, with $\Gamma_k = \Gamma_{k-1} \circ \Gamma$, (14.2)–(14.8) hold. This completes the inductive proof that (14.2)–(14.8) hold for all $k = 1, 2, 3, \ldots$. Writing

$$\mathbf{C}_k = \sum_{j=1}^{q} |H_j^k| + |G_j^k|,$$

where for each $j \in \{1, 2, \ldots, q\}$, H_j^k is the half-space defined by

$$H_j^k = \{(x^1, x^2, y) \in \mathbf{R}^{n+1} : x^2 < 0 \text{ and } x^1 = \lambda_j^k x^2\}$$

and G_j^k is the half-space defined by

$$G_j^k = \{(x^1, x^2, y) \in \mathbf{R}^{n+1} : x^2 > 0 \text{ and } x^1 = \mu_j^k x^2\},$$

with λ_j^k, μ_j^k constants, $\lambda_1^k \geq \lambda_2^k \geq \cdots \geq \lambda_q^k$ and $\mu_1^k \leq \mu_2^k \leq \cdots \leq \mu_q^k$, note that by (10.2) (applied with $\eta_{\sigma_k\,\#}\,\Gamma_{k\,\#}\,V$ in place of V and $\mathbf{C}_k$ in place of $\mathbf{C}$), (14.11) and (14.12), we also have that

$$(14.13)\quad |\lambda_1^k - \lambda_q^k| \geq C\hat{E}_V \quad \text{and} \quad |\mu_1^k - \mu_q^k| \geq C\hat{E}_V, \quad C = C(n, q, \alpha) \in (0, \infty)$$

for all $k = 1, 2, 3, \ldots$.

By (14.4), $\{\operatorname{spt}\|\mathbf{C}_k\| \cap (\mathbf{R} \times B_1)\}$ is a Cauchy sequence (in Hausdorff distance) and hence, since $\Theta(\|\mathbf{C}_k\|, 0) = q$ for each $k = 1, 2, \ldots$, there is a varifold $\mathbf{H} \in \mathcal{C}_q$ such that passing to a subsequence $\{k'\}$ of $\{k\}$, $\mathbf{C}_{k'} \llcorner B_2^{n+1}(0) \to \mathbf{H} \llcorner B_2^{n+1}(0)$ and

$$(14.14)\qquad \operatorname{dist}_{\mathcal{H}}^2(\operatorname{spt}\|\mathbf{H}\| \cap (\mathbf{R} \times B_1), \operatorname{spt}\|\mathbf{C}_{k'}\| \cap (\mathbf{R} \times B_1)) \leq C\delta_{k'} Q_V^2$$

for each k', where $C = C(n, q, \alpha) \in (0, \infty)$. By (14.13), $\operatorname{spt}\|\mathbf{H}\|$ is not a hyperplane. Since $\delta_k \leq \sigma_k^{2\alpha}$, it follows from (14.5), (14.8) and (14.14) that

$$(14.15)\qquad \int_{\mathbf{R}\times B_1} \operatorname{dist}^2(X, \operatorname{spt}\|\mathbf{H}\|)\, d\|\eta_{\sigma_{k'}\,\#}\Gamma_{k'\,\#}^{-1}\,V\|(X) \leq C\sigma_{k'}^{2\alpha} Q_V^2$$

and

(14.16)
$$\int_{\mathbf{R}\times\left(B_{1/2}\setminus\{|x^2|<1/16\}\right)} \operatorname{dist}^2(X, \operatorname{spt}\|\eta_{\sigma_{k'}\,\#}\Gamma_{k'\,\#}^{-1}\,V\|)\, d\|\mathbf{C}_{k'}\|(X) \leq C\sigma_{k'}^{2\alpha} Q_V^2$$

for all k', where $C = C(n, q, \alpha) \in (0, \infty)$. Now, since $q \leq \Theta(\|V\|, 0) \leq (\omega_n 2^n)^{-1}\|V\|(B_2^{n+1}(0)) < q + 1/2$, it follows from the monotonicity formula that

$$\begin{aligned} q \leq \Theta(\|\eta_{\sigma_{k'}\,\#}\Gamma_{k'\,\#}^{-1}\,V\|, 0) &\leq (\omega_n 2^n)^{-1}\|\eta_{\sigma_{k'}\,\#}\Gamma_{k'\,\#}^{-1}\,V\|(B_2^{n+1}(0)) \\ &\leq (\omega_n 2^n)^{-1}\|V\|(B_2^{n+1}(0)) < q + 1/2. \end{aligned}$$

Hence, there is a stationary integral varifold W on $B_2^{n+1}(0)$ with

$$q \leq \Theta(\|W\|, 0) \leq (\omega_n 2^n)^{-1}\|W\|(B_2^{n+1}(0)) < q + 1/2$$

such that, passing to a further subsequence without changing notation,

$$\eta_{\sigma_{k'}\,\#}\Gamma_{k'\,\#}^{-1}V \to W \tag{14.17}$$

as varifolds on $B_2^{n+1}(0)$. The estimate (14.15) implies that $\operatorname{spt}\|W\|\cap(\mathbf{R}\times B_1)\subseteq \operatorname{spt}\|\mathbf{H}\|\cap(\mathbf{R}\times B_1)$. Since $\operatorname{dist}_{\mathcal{H}}(\operatorname{spt}\|\eta_{\sigma_{k'}\,\#}\Gamma_{k'\,\#}V\|\cap(\mathbf{R}\times B_1), \operatorname{spt}\|W\|\cap(\mathbf{R}\times B_1))\to 0$, it follows from (14.16), the triangle inequality and the weak convergence $\|\mathbf{C}_{k'}\|\to\|\mathbf{H}\|$ on $\mathbf{R}\times B_{1/2}$ that $\operatorname{spt}\|\mathbf{H}\|\cap(\mathbf{R}\times(B_{1/2}\setminus\{|x^2|<1/16\}))\subseteq \operatorname{spt}\|W\|\cap(\mathbf{R}\times(B_{1/2}\setminus\{|x^2|<1/16\}))$. Hence $\operatorname{spt}\|W\|\cap(\mathbf{R}\times B_1)=\operatorname{spt}\|\mathbf{H}\|\cap(\mathbf{R}\times B_1)$, from which it also follows that $\Theta(\|W\|,0)=q$. Thus (14.17) contradicts case $\Theta(\|\mathbf{C}_0\|,0)=q$ of the induction hypothesis (H2), proving the lemma. □

COROLLARY 14.2. *Let q be an integer ≥ 2, and suppose that the induction hypotheses* (H1), (H2) *hold. Then the class $\mathcal{B}_q$ (defined in Section* 5) *satisfies property* ($\mathcal{B}$7) *of the definition of proper blow-up classes (given in Section* 4).

Proof. If not, in view of Lemma 9.1, there exists an element $v_\star\in\mathcal{B}_q$ such that, for $j=1,2,\ldots,q$, $v_\star^j(x^2,y)=L_1^j(x^2,y)$ if $x^2<0$ and $v_\star^j(x^2,y)=L_2^j(x^2,y)$ if $x^2\geq 0$ where $L_1^j, L_2^j:\mathbf{R}^n\to\mathbf{R}$ are linear functions with $L_1^j(0,y)=L_2^j(0,y)=0$ and

$$L_1^{j_1}\neq L_1^{j_1+1}\quad\text{and}\quad L_2^{j_2}\neq L_2^{j_2+1}\quad\text{for some}\quad j_1,j_2\in\{1,2,\ldots,q-1\}. \tag{14.18}$$

Since the average $(v_\star)_a=q^{-1}\sum_{j=1}^q v_\star^j$ is linear (by property ($\mathcal{B}$3)) and $\|v_\star-(v_\star)_a\|_{L^2(B_1)}^{-1}(v_\star-(v_\star)_a)\in\mathcal{B}_q$ (by property ($\mathcal{B}$5 I)), where $v_\star-(v_\star)_a=(v_\star^1-(v_\star)_a,\ldots,v_\star^q-(v_\star)_a)$, we may assume without loss of generality that $(v_\star)_a=0$ and that

$$\|v_\star\|_{L^2(B_1)}=1. \tag{14.19}$$

By the definition of $\mathcal{B}_q$, for each $k=1,2,3,\ldots$, there exists a stationary integral varifold V_k of $B_2^{n+1}(0)$ with

$$(\omega_n 2^n)^{-1}\,\|V_k\|(B_2^{n+1}(0))<q+1/2,\quad q-1/2\leq\omega_n^{-1}\|V_k\|(\mathbf{R}\times B_1)<q+1/2 \tag{14.20}$$

and

$$\hat{E}_k^2\equiv\int_{\mathbf{R}\times B_1}|x^1|^2d\|V_k\|(X)\to 0 \tag{14.21}$$

as $k\to\infty$ such that the following hold: If $\sigma\in(0,1)$, k is sufficiently large depending on σ, $\Sigma_k\subset B_\sigma$ is the measurable set corresponding to Σ and $v_k^j: B_\sigma\to\mathbf{R}$, $j=1,2,\ldots,q$, are the Lipschitz functions corresponding to u^j in Theorem 5.1 applied with V_k in place of V, then by Theorem 5.1, $v_k^1\leq v_k^2\leq\cdots\leq v_k^q$,

$$\operatorname{Lip} v_k^j\leq 1/2\quad\text{for each}\quad j\in\{1,2,\ldots,q\}, \tag{14.22}$$

$$\|V_k\|(\mathbf{R} \times \Sigma_k) + \mathcal{H}^n(\Sigma_k) \leq C_\sigma \hat{E}_k^2, \tag{14.23}$$

where $C_\sigma \in (0, \infty)$ is a constant depending only on n, q and σ,

$$\operatorname{spt}\|V_k\| \cap (\mathbf{R} \times (B_\sigma \setminus \Sigma_k)) = \textstyle\bigcup_{j=1}^q \operatorname{graph} v_k^j \cap (\mathbf{R} \times (B_\sigma \setminus \Sigma_k)), \tag{14.24}$$

and

$$\hat{E}_k^{-1} v_k^j \to v_\star^j, \tag{14.25}$$

where the convergence is in $L^2(B_\sigma)$ and weakly in $W^{1,2}(B_\sigma)$. Note that by (14.20), after passing to a subsequence without changing notation, there exists a stationary integral varifold V of $B_2^{n+1}(0)$ such that $V_k \to V$, and by (14.21), $\operatorname{spt}\|V \llcorner (\mathbf{R} \times B_1)\| \subset \{0\} \times \overline{B}_1$, so by (14.20) and the Constancy Theorem for stationary integral varifolds, $V \llcorner (\mathbf{R} \times B_1) = q|\{0\} \times B_1|$. Hence by replacing V_k with $\eta_{0,1/2\,\#} V_k$ and noting that by homogeneity of $v_\star$, the coarse blow-up of $\{\eta_{0,1/2\,\#} V_k\}$ is still $v_\star$, we may assume that for all sufficiently large k,

$$q - 1/4 \leq (\omega_n 2^n)^{-1} \|V_k\|(B_2^{n+1}(0)) < q + 1/4. \tag{14.26}$$

By using the argument justifying the assertion (9.7), we may pass to a subsequence without changing notation and find points $Z_k \in \operatorname{spt}\|V_k\| \cap B_1^{n+1}(0)$ with $\Theta(\|V_k\|, Z_k) \geq q$ and $Z_k \to 0$. Replacing V_k with $\eta_{Z_k, 1-|Z_k|\,\#} V_k$, we may thus assume that

$$\Theta(\|V_k\|, 0) \geq q \tag{14.27}$$

for each $k = 1, 2, 3, \ldots$, and in view of (14.26), the monotonicity formula implies that the new V_k satisfy (14.20). We now argue that for each sufficiently large k, we must have that

$$\int_{\mathbf{R} \times B_1} |x^1|^2 \, d\|V_k\|(X) < \frac{3}{2} \inf_{\{P = \{x^1 = \lambda x^2\}\}} \int_{\mathbf{R} \times B_1} \operatorname{dist}^2(X, P) \, d\|V_k\|(X). \tag{14.28}$$

If this is false, then there is a subsequence $\{k'\}$ of $\{k\}$ and corresponding to each k', there is a number $\lambda_{k'} \in \mathbf{R}$ such that, with $P_{k'} = \{x^1 = \lambda_{k'} x^2\}$, we have

$$\int_{\mathbf{R} \times B_1} \operatorname{dist}^2(X, P_{k'}) \, d\|V_{k'}\|(X) \leq \frac{5}{6} \hat{E}_{k'}^2$$

for all k'. In particular, for each $\sigma \in (1/2, 1)$ and sufficiently large k',

$$(1 + \lambda_{k'}^2)^{-1} \sum_{j=1}^q \int_{B_\sigma \setminus \Sigma_{k'}} (v_{k'}^j(x^2, y) - \lambda_{k'} x^2)^2 \, dx^2 dy \leq \frac{5}{6} \hat{E}_{k'}^2, \tag{14.29}$$

whence $(1+\lambda_{k'}^2)^{-1}\lambda_{k'}^2\int_{B_{1/2}\setminus\Sigma_{k'}}|x^2|^2\,dx^2dy \leq \frac{11}{3}\hat{E}_{k'}^2$. Thus, $|\lambda_{k'}| \leq C\hat{E}_{k'}$ for all sufficiently large k', where $C = C(n) \in (0,\infty)$, and hence, passing to a further subsequence without changing notation, $\hat{E}_{k'}^{-1}\lambda_{k'} \to \ell$ for some $\ell \in \mathbf{R}$. It follows from (14.29) and (14.23) that

$$\sum_{j=1}^{q}\int_{B_\sigma}(v_{k'}^j - \lambda_{k'}x^2)^2\,dx^2dy \leq \frac{5}{6}(1+\lambda_{k'}^2)\hat{E}_{k'}^2 + 2C_\sigma\sup_{B_\sigma}(|v_{k'}|^2 + \lambda_{k'}^2|x^2|^2)\,\hat{E}_{k'}^2.$$

First dividing this by $\hat{E}_{k'}^2$ and letting $k' \to \infty$, and then letting $\sigma \to 1$, we see that $\sum_{j=1}^q\int_{B_1}(v_\star^j - \ell x^2)^2 \leq 5/6$. Since $v_\star^j(x^2,y) = \ell_j x^2$ if $x^2 < 0$ and $v_\star^j(x^2,y) = m_j x^2$ if $x^2 > 0$ for some $\ell_j, m_j \in \mathbf{R}$, this implies that $\int_{B_1}|v_\star|^2 - 2\ell\sum_{j=1}^q(\ell_j + m_j)\int_{B_1\cap\{x^2>0\}}|x^2|^2 + \ell^2\int_{B_1}|x^2|^2 \leq 5/6$, which is a contradiction since $(v_\star)_a \equiv 0$ (so that $\sum_{j=1}^q \ell_j = \sum_{j=1}^q m_j = 0$) and $\int_{B_1}|v_\star|^2 = 1$. Thus (14.28) must hold for all sufficiently large k.

For $j = 1,2,3,\ldots,q$ and $k = 1,2,3,\ldots$, let $h_j^k = \hat{E}_k L_1^j$, $g_j^k = \hat{E}_k L_2^j$, $H_j^k = \operatorname{graph} h_j^k$, $G_j^k = \operatorname{graph} g_j^k$ and $\mathbf{C}_k = \sum_{j=1}^q |H_j^k| + |G_j^k|$. By (14.22), (14.23) and (14.24),

(14.30)
$$\begin{aligned}\int_{\mathbf{R}\times B_\sigma}&\operatorname{dist}^2(X,\operatorname{spt}\|\mathbf{C}_k\|)\,d\|V_k\|(X)\\ &\leq 2\int_{B_\sigma}|v_k - \hat{E}_k v_\star|^2 + C_\sigma\sup_{X\in\operatorname{spt}\|V_k\|\cap(\mathbf{R}\times B_\sigma)}\operatorname{dist}^2(X,\operatorname{spt}\|\mathbf{C}_k\|)\hat{E}_k^2.\end{aligned}$$

By (14.19) and homogeneity of $v_\star$, $\int_{B_\sigma}|v^\star|^2 = \sigma^{n+2}$, so by (14.25), for each $\theta \in (0,1/8)$ and $\sigma \in (0,1)$, $\int_{B_\sigma}|v_k|^2 \geq (1-\theta)\sigma^{n+2}\hat{E}_k^2$ for sufficiently large k. Since

$$\begin{aligned}\int_{\mathbf{R}\times B_\sigma}|x^1|^2\,d\|V_k\|(X) &= \sum_{j=1}^q\int_{B_\sigma}\sqrt{1+|Dv_k^j|^2}|v_k^j|^2\\ &\quad - \sum_{j=1}^q\int_{\Sigma_k}\sqrt{1+|Dv_k^j|^2}|v_k^j|^2 + \int_{\mathbf{R}\times\Sigma_k}|x^1|^2\,d\|V_k\|(X)\\ &\geq \int_{B_\sigma}|v_k|^2 - 2C_\sigma\left(\sup_{B_\sigma}|v_k|^2\right)\hat{E}_k^2,\end{aligned}$$

it follows that

(14.31)
$$\int_{\mathbf{R}\times(B_1\setminus B_\sigma)}|x^1|^2\,d\|V_k\|(X) \leq \left(1-(1-\theta)\sigma^{n+2} + 2C_\sigma\left(\sup_{B_\sigma}|v_k|^2\right)\right)\hat{E}_k^2$$

for all sufficiently large k. By the triangle inequality,

$$\begin{aligned}(14.32)\quad &\int_{\mathbf{R}\times(B_1\setminus B_\sigma)} \operatorname{dist}^2(X, \operatorname{spt}\|\mathbf{C}_k\|)\, d\|V_k\|(X)\\ &\leq 2\int_{\mathbf{R}\times(B_1\setminus B_\sigma)} |x^1|^2\, d\|V_k\|(X)\\ &\quad + 3\operatorname{dist}^2_{\mathcal{H}}(\operatorname{spt}\|\mathbf{C}_k\|\cap(\mathbf{R}\times B_1), \{0\}\times B_1)\|V_k\|(\mathbf{R}\times(B_1\setminus B_\sigma))\\ &\leq 2\int_{\mathbf{R}\times(B_1\setminus B_\sigma)} |x^1|^2\, d\|V_k\|(X) + C\mathcal{H}^n(B_1\setminus B_\sigma)\hat{E}_k^2\end{aligned}$$

for all sufficiently large k, where $C = C(n,q) \in (0,\infty)$. Here we have used the fact that $V_k \llcorner (\mathbf{R}\times B_1) \to q|\{0\}\times B_1|$. Thus, if $\gamma_1 = \gamma_1(n,q,\alpha) \in (0,1/2)$ is the constant as in Lemma 14.1, then we may fix $\theta = \theta(n,q,\alpha) \in (0,1/8)$ sufficiently small and $\sigma = \sigma(n,q,\alpha) \in (0,1)$ sufficiently close to 1 in order to conclude from (14.25), (14.30), (14.31) and (14.32) that for all sufficiently large k,

$$(14.33)\qquad \int_{\mathbf{R}\times B_1} \operatorname{dist}^2(X, \operatorname{spt}\|\mathbf{C}_k\|)\, d\|V_k\|(X) \leq \frac{\gamma_1}{4}\hat{E}_k^2.$$

In view of (14.18), we have by the argument leading to (9.5) that for all sufficiently large k, $\Sigma_k \subset B_\sigma \cap \{|x^2| < 1/64\}$ and that

(14.34)

$$V_k \llcorner ((\mathbf{R}\times B_\sigma)\cap\{x^2 \leq -1/64\}) = \sum_{j=1}^q |\operatorname{graph} u_k^j| \llcorner ((\mathbf{R}\times B_\sigma)\cap\{x^2 \leq -1/64\})$$

and

(14.35)

$$V_k \llcorner ((\mathbf{R}\times B_\sigma)\cap\{x^2 \geq 1/64\}) = \sum_{j=1}^q |\operatorname{graph} w_k^j| \llcorner ((\mathbf{R}\times B_\sigma)\cap\{x^2 \geq 1/64\}),$$

where $u_k^1 \leq u_k^2 \leq \ldots u_k^q$ and $w_k^1 \leq w_k^2 \leq \ldots w_k^q$ (thus, $v_k|_{B_\sigma\cap\{x^2\leq -1/64\}} \equiv u_k$ and $v_k|_{B_\sigma\cap\{x^2\geq 1/64\}} \equiv w_k$), u_k^j, w_k^j are C^2 functions on $B_\sigma\cap\{x^2 \leq -1/64\}$, $B_\sigma\cap\{x^2\geq 1/64\}$ respectively, solving the minimal surface equation there, and satisfying, by elliptic theory,

$$(14.36)\qquad \sup_{B_{\kappa\sigma}\cap\{x^2\leq -1/64\}} |D\, u_k^j|^2 + \sup_{B_{\kappa\sigma}\cap\{x^2\geq 1/64\}} |D\, w_k^j|^2 \leq C(\kappa,\sigma)\hat{E}_k^2$$

for each $\kappa \in (0,1)$, $j = 1,2,\ldots,q$ where $C(\kappa,\sigma) \in (0,\infty)$ is a constant depending only on n, κ and σ. We see from (14.34), (14.35), (14.36) and (14.25) that

(14.37)

$$\begin{aligned}&\int_{\mathbf{R}\times(B_{1/2}\setminus\{|x^2|<1/64\})} \operatorname{dist}^2(X, \operatorname{spt}\|V_k\|)\, d\|\mathbf{C}_k\|(X)\\ &\leq 2\sum_{j=1}^q\left(\int_{B_{1/2}\cap\{x^2<-1/64\}} |\hat{E}_k L_1^j - u_k^j|^2 + \int_{B_{1/2}\cap\{x^2>1/64\}} |\hat{E}_k L_2^j - w_k^j|^2\right) \leq \eta_k \hat{E}_k^2,\end{aligned}$$

where $\eta_k \to 0$. By (14.33) and (14.37),

$$\int_{\mathbf{R}\times(B_{1/2}\setminus\{|x^2|<1/64\})} \operatorname{dist}^2(X, \operatorname{spt}\|V_k\|)\, d\|\mathbf{C}_k\|(X) + \int_{\mathbf{R}\times B_1} \operatorname{dist}^2(X, \operatorname{spt}\|\mathbf{C}_k\|)\, d\|V_k\|(X) \leq \frac{\gamma_1}{2}\hat{E}_k^2$$

for sufficiently large k, which in view of (14.20), (14.21) and (14.27) contradicts Lemma 14.1. ☐

THEOREM 14.3. *Let q be an integer ≥ 2, $\alpha \in (0,1)$, and suppose that the induction hypotheses* (H1), (H2) *hold. Let $\mathcal{B}_q$ be the class of functions defined in Section* 5. (*Thus, each $v \in \mathcal{B}_q$ is a coarse blow-up, in the sense described in Section* 5, *of a sequences of varifolds in $\mathcal{S}_\alpha$ converging weakly, in $\mathbf{R}\times B_1$, to $q|\{0\}\times B_1|$.*) *If $v = (v^1, v^2, \ldots, v^q) \in \mathcal{B}_q$, then v^j is harmonic in B_1 for each $j = 1, 2, \ldots, q$. Furthermore, if $\{V_k\} \subset \mathcal{S}_\alpha$ is a sequence whose coarse blow-up is v, and if for each of infinitely many values of k, there is a point $Z_k \in \operatorname{spt}\|V_k\| \cap (B_{3/4}\times\mathbf{R})$ with $\Theta(\|V_k\|, Z_k) \geq q$, then $v^1 = v^2 = \cdots = v^q$.*

Proof. By the discussion of Section 8 and Corollary 14.2, $\mathcal{B}_q$ is a proper blow-up class for a constant $C = C(n,q) \in (0,\infty)$. The present theorem follows from Theorem 4.1 and the remark at the end of Section 8. ☐

15. **The Sheeting Theorem**

This section is devoted to the proof of the Sheeting Theorem (Theorem 3.3′) subject to the induction hypotheses (H1), (H2).

LEMMA 15.1. *Let q be an integer ≥ 2, $\alpha \in (0,1)$ and $\theta \in (0,1/4)$. Suppose that the induction hypotheses* (H1), (H2) *hold. There exists a number $\beta_0 = \beta_0(n,q,\alpha,\theta) \in (0,1/2)$ such that if $V \in \mathcal{S}_\alpha$, $(\omega_n 2^n)^{-1}\|V\|(B_2^{n+1}(0)) < q+1/2$, $q-1/2 \leq (\omega_n)^{-1}\|V\|(B_1\times\mathbf{R}) < q+1/2$, and $\int_{\mathbf{R}\times B_1} \operatorname{dist}^2(X,P)\, d\|V\|(X) < \beta_0$ for some affine hyperplane P of $\mathbf{R}^{n+1}$ with $\operatorname{dist}_{\mathcal{H}}^2(P\cap(B_1\times\mathbf{R}), B_1\times\{0\}) < \beta_0$, then the following hold*:

(a) *Either $V \llcorner (B_{1/2}\times\mathbf{R}) = \sum_{j=1}^q |\operatorname{graph} u_j|$ where $u_j \in C^2(B_{1/2};\mathbf{R})$ for $j = 1, 2, \ldots, q$; $u_1 \leq u_2 \leq \cdots \leq u_q$ on $B_{1/2}$; $u_{j_0} < u_{j_0+1}$ on $B_{1/2}$ for some $j_0 \in \{1, 2, \ldots, q-1\}$ and, for each $j \in \{1, 2, \ldots, q\}$,*

$$\sup_{B_{1/2}} |u_j - p|^2 + |D\, u_j - D\, p|^2 + |D^2\, u_j|^2 \leq C \int_{\mathbf{R}\times B_1} \operatorname{dist}^2(X,P)\, d\|V\|(X),$$

where $C = C(n,q) \in (0,\infty)$ and $p : \mathbf{R}^n \to \mathbf{R}$ is the affine function such that graph $p = P$; *or, there exists an affine hyperplane P' with*

$$\operatorname{dist}_{\mathcal{H}}^2(P'\cap(\mathbf{R}\times B_1), P\cap(\mathbf{R}\times B_1)) \leq C_1 \int_{\mathbf{R}\times B_1} \operatorname{dist}^2(X,P)\, d\|V\|(X)$$

and

$$\theta^{-n-2}\int_{\mathbf{R}\times B_\theta}\mathrm{dist}^2(X,P')\,d\|V\|(X)\le C_2\theta^2\int_{\mathbf{R}\times B_1}\mathrm{dist}^2\,(X,P)\,d\|V\|(X),$$

where $C_1 = C_1(n,q)\in(0,\infty)$ *and* $C_2 = C_2(n,q)\in(0,\infty)$.

(b) $(\omega_n(2\theta)^n)^{-1}\,\|V\|(B^{n+1}_{2\theta}(0)) < q+1/2$ *and*

$$q-1/2\le(\omega_n\theta^n)^{-1}\|V\|(\mathbf{R}\times B_\theta) < q+1/2.$$

Proof. For each $k=1,2,3,\dots$, let $V_k\in\mathcal{S}_\alpha$ be such that

(15.1)
$$(\omega_n 2^n)^{-1}\|V_k\|(B^{n+1}_2(0)) < q+1/2 \quad\text{and}\quad q-1/2\le(\omega_n)^{-1}\|V_k\|(\mathbf{R}\times B_1) < q+1/2,$$

and let P_k be an affine hyperplane of $\mathbf{R}^{n+1}$ such that

$$\mathrm{dist}_{\mathcal{H}}(P_k\cap(\mathbf{R}\times B_1),\{0\}\times B_1)\to 0 \tag{15.2}$$

and

$$\int_{\mathbf{R}\times B_1}\mathrm{dist}^2(X,P_k)\,d\|V_k\|(X)\to 0. \tag{15.3}$$

The lemma will be established by proving that for each of infinitely many k, the conclusions hold with V_k in place of V, P_k in place of P and with fixed constants $C=C(n,q)$, $C_1=C_1(n,q)$, $C_2=C_2(n,q)\in(0,\infty)$.

By (15.2), (15.3) and the triangle inequality, $\hat{E}_k\equiv\sqrt{\int_{\mathbf{R}\times B_1}|x^1|^2\,d\|V_k\|(X)}\to 0$. Hence, by (15.1) and the Constancy Theorem, $V_k\llcorner(\mathbf{R}\times B_1)\to q|\{0\}\times B_1|$ so that

$$q-1/2\le(\omega_n\theta^n)^{-1}\,\|V_k\|(\mathbf{R}\times B_\theta) < q+1/2$$

for sufficiently large k. Furthermore, by monotonicity of mass ratio,

$$(\omega_n(2\theta)^n)^{-1}\,\|V_k\|(B^{n+1}_{2\theta}(0))\le(\omega_n 2^n)^{-1}\,\|V_k\|(B^{n+1}_2(0)) < q+1/2.$$

Thus, conclusion (b) with V_k in place of V holds for sufficiently large k.

For each $k=1,2,3,\dots$, there exists, by (15.2), a rigid motion Γ_k of $\mathbf{R}^{n+1}$ with $\Gamma_k\to$ Identity such that $\Gamma_k(P_k)=\{0\}\times\mathbf{R}^n$. Let $\widetilde{V}_k=\eta_{9/10\,\#}\,\Gamma_{k\,\#}\,V_k$. Then by (15.1), $(\omega_n 2^n)^{-1}\,\|\widetilde{V}_k\|(B^{n+1}_2(0)) < q+1/2$, and by (15.3),

(15.4)
$$\int_{\mathbf{R}\times B_{19/18}}|x^1|^2\,d\|\widetilde{V}_k\|(X)\le\left(\frac{9}{10}\right)^{-n-2}\int_{\mathbf{R}\times B_1}\mathrm{dist}^2\,(X,P_k)\,d\|V_k\|(X)\to 0.$$

It follows again by the Constancy Theorem, for all sufficiently large k,

$$q-1/2\le(\omega_n)^{-1}\|\widetilde{V}_k\|(\mathbf{R}\times B_1) < q+1/2.$$

Let $\widetilde{v}=(\widetilde{v}^1,\widetilde{v}^2,\dots,\widetilde{v}^q)\in\mathcal{B}_q$ be the coarse blow-up of (a subsequence) of $\widetilde{V}_k$ by the coarse excess $\hat{E}_{\widetilde{V}_k}\equiv\sqrt{\int_{\mathbf{R}\times B_1}|x^1|^2\,d\|\widetilde{V}_k\|(X)}$. Suppose first that the $\widetilde{v}^j$'s are not all identical to one another. Then by Theorem 14.3, for all sufficiently large k,

$$Z \in \operatorname{spt} \|\widetilde{V}_k\| \cap (\mathbf{R} \times B_{3/4}) \quad \Longrightarrow \quad \Theta(\|\widetilde{V}_k\|, Z) < q.$$

Hence, by Remark 3 of Section 6, we may apply Theorem 3.5 followed by elliptic theory to $\widetilde{V}_k$ and conclude, after transforming by $\Gamma_k^{-1} \circ \eta_{9/10}^{-1}$, that

$$V_k \,\llcorner (\mathbf{R} \times B_{1/2}) = \sum_{j=1}^{q} |\operatorname{graph} u_k^j|$$

for all sufficiently large k, where $u_k^j \in C^2(B_{1/2}; \mathbf{R})$, $u_k^1 \le u_k^2 \le \cdots \le u_k^q$; $u_k^{j_0} < u_k^{j_0+1}$ on $B_{1/2}$ for some $j_0 \in \{1, 2, \ldots, q-1\}$ and, for each $j \in \{1, 2, \ldots, q\}$,

$$\sup\nolimits_{B_{1/2}} |u_k^j - p_k|^2 + |D\, u_k^j - D\, p_k|^2 + |D^2\, u_k^j|^2 \le C \int_{\mathbf{R} \times B_1} \operatorname{dist}^2(X, P_k)\, d\|V_k\|(X),$$

where $C = C(n,q) \in (0, \infty)$ and $p_k : \mathbf{R}^n \to \mathbf{R}$ is the affine function such that $\operatorname{graph} p_k = P_k$.

On the other hand, if $\widetilde{v}^1 = \widetilde{v}^2 = \cdots = \widetilde{v}^q$ $(= \widetilde{v}$, say) on B_1, then letting $\widetilde{p}(x) = \widetilde{v}(0) + D\widetilde{v}(0) \cdot x$ and $\widetilde{P}_k = \operatorname{graph} \hat{E}_{\widetilde{V}_k} \widetilde{p}$, it follows from Theorem 5.1 and the standard estimates for harmonic functions that

$$\operatorname{dist}_{\mathcal{H}}(\widetilde{P}_k \cap (\mathbf{R} \times B_1), \{0\} \times B_1) \le C \hat{E}_{\widetilde{V}_k} \tag{15.5}$$

and

$$\theta^{-n-2} \int_{\mathbf{R} \times B_{2\theta}} \operatorname{dist}^2(X, \widetilde{P}_k)\, d\|\widetilde{V}_k\|(X) \le C\theta^2 \hat{E}^2_{\widetilde{V}_k} \tag{15.6}$$

for all sufficiently large k, where $C = C(n,q) \in (0, \infty)$. Setting $P_k' = \eta_{9/10}^{-1} \Gamma_k^{-1}\, \widetilde{P}_k$, it follows readily from (15.5), (15.6) and (15.4) that

$$\operatorname{dist}_{\mathcal{H}}(P_k' \cap (\mathbf{R} \times B_1), P_k \cap (\mathbf{R} \times B_1)) \le C \int_{\mathbf{R} \times B_1} \operatorname{dist}^2(X, P_k)\, d\|V_k\|(X)$$

and

$$\theta^{-n-2} \int_{\mathbf{R} \times B_\theta} \operatorname{dist}^2(X, P_k')\, d\|V_k\|(X) \le C\theta^2 \int_{\mathbf{R} \times B_1} \operatorname{dist}^2(X, P_k)\, d\|V_k\|(X)$$

for all sufficiently large k, where $C = C(n,q) \in (0, \infty)$. Thus, conclusion (a) with V_k, P_k, P_k' in place of V, P, P' and with a fixed constant $C = C(n,q) \in (0, \infty)$ holds for infinitely many k. □

THEOREM 15.2. *Let q be an integer ≥ 2, $\alpha \in (0,1)$, $\gamma \in (0,1)$, and suppose that the induction hypotheses* (H1), (H2) *hold. There exists a number $\varepsilon = \varepsilon(n, q, \alpha, \gamma) \in (0,1)$ such that the following is true: If $V \in \mathcal{S}_\alpha$, $(\omega_n 2^n)^{-1} \|V\|(B_2^{n+1}(0)) < q + 1/2$, $q - 1/2 \le \omega_n^{-1} \|V\|(B_1 \times \mathbf{R}) < q + 1/2$ and $\hat{E}_V^2 \equiv \int_{\mathbf{R} \times B_1} |x^1|^2\, d\|V\|(X) < \varepsilon$, then*

$$V \,\llcorner (B_{\gamma/2} \times \mathbf{R}) = \sum_{j=1}^{q} |\operatorname{graph} u_j|,$$

where $u_j \in C^{1,\lambda}(B_{\gamma/2})$ for each $j = 1, 2, \ldots, q$, $u_1 \leq u_2 \leq \cdots \leq u_q$ and

$$\sup_{B_{\gamma/2}} (|u_j| + |Du_j|) + \sup_{Y_1, Y_2 \in B_{\gamma/2},\, Y_1 \neq Y_2} \frac{|Du_j(Y_1) - u_j(Y_2)|}{|Y_1 - Y_2|^{\lambda}} \leq C \left(\int_{\mathbf{R} \times B_1} |x^1|^2 \, d\|V\|(X) \right)^{1/2}.$$

Here $C = C(n, q, \alpha, \gamma) \in (0, \infty)$ and $\lambda = \lambda(n, q, \alpha, \gamma) \in (0, 1)$. Furthermore, we have in fact that $u_j \in C^{\infty}(B_{\gamma/2})$ and u_j solves the minimal surface equation on $B_{\gamma/2}$ for each $j = 1, 2, \ldots, q$.

Proof. Let $\widetilde{\gamma} = (1 - \gamma)/4$. Let $C = C(n, q)$, $C_1 = C_1(n, q)$ and $C_2 = C_2(n, q)$ be the constants as in the conclusion of Lemma 15.1. Choose $\theta = \theta(n, q) \in (0, 1/4)$ such that $C_2 \theta^2 < 1/4$ and $\varepsilon = \varepsilon(n, q, \alpha, \gamma) \in (0, 1)$ such that $\varepsilon < (1 + C_1)^{-1} \widetilde{\gamma}^{n+2} \beta_0 / 8$, where $\beta_0 = \beta_0(n, q, \alpha, \theta)$ is as in Lemma 15.1. Additional restrictions on ε will be imposed during the course of the proof, but we will choose ε depending only on n, q, α and γ. Suppose that $\hat{E}_V^2 \equiv \int_{\mathbf{R} \times B_1} |x^1|^2 \, d\|V\|(X) \leq \varepsilon$, and let

$$\begin{aligned} \beta = \min \Big\{ & 4^{-1} (1 + 2C_1)^{-1} \beta_0, \;\; 4^{-1} \widetilde{\gamma}^n \omega_n^{-1} (2q + 1)^{-1} \beta_0, \\ & 4^{-1} (2 + \omega_n (2q + 1) C_1)^{-1} \left(\frac{2\theta}{3} \right)^{n+2} \overline{\varepsilon}_0, \\ & 8^{-1} \omega_n 4^{-n} \theta^n \left(256 \theta^{-2} + (q + 1) \overline{C} \right)^{-1} (2 + \omega_n (2q + 1) C_1)^{-1} \Big\}. \end{aligned}$$

Here $\overline{\varepsilon}_0 = \varepsilon_0(n, q, 5/6)$, $\overline{C} = C(n, q, 5/6)$, where $\varepsilon_0 = \varepsilon_0(n, q, \cdot)$ is as in Theorem 5.1 and $C = C(n, q, \cdot)$ is as in Theorem 5.1(a). Note that β depends only on n, q, α and γ. Let P_0 be any affine hyperplane such that

$$\text{dist}_{\mathcal{H}}^2 (P_0 \cap (\mathbf{R} \times B_1), \{0\} \times B_1) < \beta. \tag{15.7}$$

Fix any point $Y \in B_{\gamma}(0)$, and let $\widetilde{V} = \eta_{Y, \widetilde{\gamma}\, \#} V$. Note then that

(15.8)

$$\begin{aligned} \hat{E}_{\widetilde{V}, P_0}^2 &\equiv \int_{\mathbf{R} \times B_1} \text{dist}^2 (X, P_0) \, d\|\widetilde{V}\|(X) \\ &= \widetilde{\gamma}^{-n-2} \int_{\mathbf{R} \times B_{\widetilde{\gamma}}(Y)} \text{dist}^2 (X, Y + \widetilde{\gamma} P_0) \, d\|V\|(X) \\ &\leq 2 \widetilde{\gamma}^{-n-2} \int_{\mathbf{R} \times B_{\widetilde{\gamma}}(Y)} |x^1|^2 \, d\|V\|(X) \\ &\quad + 2 \widetilde{\gamma}^{-n-2} \|V\| (\mathbf{R} \times B_{\widetilde{\gamma}}(Y)) \text{dist}_{\mathcal{H}}^2 (Y + \widetilde{\gamma} P_0 \cap (\mathbf{R} \times B_{\widetilde{\gamma}}(Y)), \{0\} \times B_{\widetilde{\gamma}}(Y)) \\ &\leq 2 \widetilde{\gamma}^{-n-2} \hat{E}_V^2 + \widetilde{\gamma}^{-n} \omega_n (2q + 1) \text{dist}_{\mathcal{H}}^2 \, (P_0 \cap (\mathbf{R} \times B_1), \{0\} \times B_1) \\ &\leq 2 \widetilde{\gamma}^{-n-2} \varepsilon + \beta_0 / 4 < \beta_0. \end{aligned}$$

Furthermore, assuming $\varepsilon < \varepsilon_0\left(n, q, \frac{3+\gamma}{4}\right)$, where ε_0 is as in Theorem 5.1 and applying Theorem 5.1 with $\sigma = (3+\gamma)/4$, we have that

(15.9)

$$\begin{aligned}
&\left|(\omega_n \widetilde{\gamma}^n)^{-1} \|V\|(\mathbf{R} \times B_{\widetilde{\gamma}}(Y)) - q\right| \\
&\quad = \Bigg|(\omega_n \widetilde{\gamma}^n)^{-1} \sum_{j=1}^{q} \int_{B_{\widetilde{\gamma}} \setminus \Sigma} \left(\sqrt{1+|Du^j|^2} - 1\right) dx \\
&\qquad\quad - (\omega_n \widetilde{\gamma}^n)^{-1} \left(q\mathcal{H}^n(B_{\widetilde{\gamma}} \cap \Sigma) - \|V\|(\mathbf{R} \times (B_{\widetilde{\gamma}} \cap \Sigma))\right)\Bigg| \\
&\quad \leq (\omega_n \widetilde{\gamma}^n)^{-1} \sum_{j=1}^{q} \int_{B_{\widetilde{\gamma}} \setminus \Sigma} \frac{|Du^j|^2}{\sqrt{1+|Du^j|^2}}\, dx + (\omega_n \widetilde{\gamma}^n)^{-1} (q+1)\widetilde{C}\hat{E}_V^2 \\
&\quad \leq (\omega_n \widetilde{\gamma}^n)^{-1} \int_{\mathbf{R} \times B_{\widetilde{\gamma}}(Y)} |\nabla^V x^1|^2 \, d\|V\|(X) + (\omega_n \widetilde{\gamma}^n)^{-1} (q+1)\widetilde{C}\hat{E}_V^2 \\
&\quad \leq 16\widetilde{\gamma}^{-2} (\omega_n \widetilde{\gamma}^n)^{-1} \int_{\mathbf{R} \times B_{2\widetilde{\gamma}}(Y)} |x^1|^2 \, d\|V\|(X) + (\omega_n \widetilde{\gamma}^n)^{-1} (q+1)\widetilde{C}\hat{E}_V^2 \\
&\quad \leq (\omega_n \widetilde{\gamma}^n)^{-1} (16\widetilde{\gamma}^{-2} + (q+1)\widetilde{C})\hat{E}_V^2.
\end{aligned}$$

Here $\widetilde{C} = C\left(n, q, \frac{3+\gamma}{4}\right)$, where $C = C(n, q, \cdot)$ is as in Theorem 5.1(a) and u^j, Σ are as in Theorem 5.1; we have also used the fact that

$$\int_{\mathbf{R} \times B_{\widetilde{\gamma}}(Y)} |\nabla^V x^1|^2 \, d\|V\|(X) \leq 16\widetilde{\gamma}^{-2} \int_{\mathbf{R} \times B_{\widetilde{\gamma}}(Y)} |x^1|^2 \, d\|V\|(X),$$

which follows from (5.1). Thus if $\varepsilon = \varepsilon(n, q, \alpha, \gamma) \in (0,1)$ is sufficiently small, this says that

$$q - 1/2 \leq (\omega_n)^{-1} \|\widetilde{V}\|(\mathbf{R} \times B_1) < q + 1/2. \tag{15.10}$$

Since

$$\begin{aligned}
(\omega_n 2^n)^{-1} \|\widetilde{V}\|(B_2^{n+1}(0)) &= (\omega_n (2\widetilde{\gamma})^n)^{-1} \|V\|(B_{2\widetilde{\gamma}}^{n+1}(Y)) \\
&\leq (\omega_n (2\widetilde{\gamma})^n)^{-1} \|V\|(\mathbf{R} \times B_{2\widetilde{\gamma}}(Y)),
\end{aligned}$$

the same estimate with $2\widetilde{\gamma}$ in place of $\widetilde{\gamma}$ shows that

$$(\omega_n 2^n)^{-1} \|\widetilde{V}\|(B_2^{n+1}(0)) < q + 1/2 \tag{15.11}$$

provided $\varepsilon = \varepsilon(n, q, \alpha, \gamma) \in (0,1)$ is sufficiently small.

We claim that either (I) or (II) below must hold:

(I) For each $k \in \{0, 1, 2, \dots\}$,

$$\begin{aligned}
\left(\omega_n (2\theta^k)^n\right)^{-1} \|\widetilde{V}\|(B_{2\theta^k}^{n+1}(0)) &< q + 1/2, \qquad (15.12)\\
q - 1/2 \leq (\omega_n (\theta^k)^n)^{-1} \|\widetilde{V}\|(\mathbf{R} \times B_{\theta^k}) &< q + 1/2,
\end{aligned}$$

there exists an affine hyperplane P_k such that, if $k \geq 1$,

$$\text{dist}^2_{\mathcal{H}}\left(P_k \cap (\mathbf{R} \times B_1), P_{k-1} \cap (\mathbf{R} \times B_1)\right) \leq C_1 4^{-k} \hat{E}^2_{\widetilde{V},P_0} \tag{15.13}$$

and

$$\begin{aligned} &(\theta^k)^{-n-2} \int_{\mathbf{R}\times B_{\theta^k}} \text{dist}^2(X, P_k)\, d\|\widetilde{V}\|(X) \\ &\quad \leq 4^{-1}(\theta^{k-1})^{-n-2} \int_{\mathbf{R}\times B_{\theta^{k-1}}} \text{dist}^2(X, P_{k-1})\, d\|\widetilde{V}\|(X) \leq \cdots \leq 4^{-k} \hat{E}^2_{\widetilde{V},P_0}. \end{aligned} \tag{15.14}$$

(II) There exists $\rho_0 \in (0,1)$ such that $\widetilde{V} \llcorner (\mathbf{R} \times B_{\rho_0}) = \sum_{j=1}^q |\text{graph}\, u_j|$ for functions $u_j \in C^2(B_{\rho_0}; \mathbf{R})$, $j = 1, 2, \ldots, q$, satisfying $u_1 \leq u_2 \leq \cdots \leq u_q$ on B_{ρ_0}; $u_{j_0} < u_{j_0+1}$ on B_{ρ_0} for some $j_0 \in \{1, 2, \ldots, q-1\}$ and

$$\begin{aligned} &\sup_{B_{\rho_0/2}} \rho_0^{-2}|u_j|^2 + |D\, u_j|^2 + \rho_0^2|D^2\, u_j|^2 \\ &\quad \leq (C + 2C_1)\hat{E}^2_{\widetilde{V},P_0} + 4\text{dist}^2_{\mathcal{H}}(P_0 \cap (\mathbf{R} \times B_1), \{0\} \times B_1) \end{aligned} \tag{15.15}$$

for each $j \in \{1, 2, \ldots, q\}$; moreover,

$$\begin{aligned} &\rho^{-n} \int_{\mathbf{R}\times B_\rho} |\nabla^V x^1|^2\, d\|V\|(X) \\ &\quad \leq C' \left(\hat{E}^2_{\widetilde{V},P_0} + \text{dist}^2_{\mathcal{H}}(P_0 \cap (\mathbf{R} \times B_1), \{0\} \times B_1)\right) \text{ for } \rho_0 < \rho < \theta, \end{aligned} \tag{15.16}$$

where $C' = C'(n,q) \in (0,\infty)$.

To see this, let k_0 be the smallest integer (≥ 1) such that alternative (I) fails to hold. If $k_0 = 1$, in view of (15.8) and (15.10), it follows directly from Lemma 15.1 applied with $\widetilde{V}$ in place of V and with $P = P_0$ that (II) must hold with $\rho_0 = 1/2$. Suppose $k_0 \geq 2$. Then by assumption, the inequalities (15.12), (15.13) and (15.14) hold for each $k = 1, 2, \ldots, k_0 - 1$ and consequently, by (15.7), (15.8), (15.13) and the triangle inequality,

$$\begin{aligned} &\text{dist}^2_{\mathcal{H}}(P_{k_0-1} \cap (\mathbf{R} \times B_1), \{0\} \times B_1) \\ &\quad \leq \left(\sqrt{C_1}\hat{E}_{\widetilde{V},P_0} + \text{dist}_{\mathcal{H}}(P_0 \cap (\mathbf{R} \times B_1), \{0\} \times B_1)\right)^2 \\ &\quad \leq 4C_1\widetilde{\gamma}^{-n-2}\varepsilon + \beta_0/2 < \beta_0. \end{aligned} \tag{15.17}$$

Applying Lemma 15.1 with $\eta_{\theta^{k_0-1}\,\#}\,\widetilde{V}$ in place of V and P_{k_0-1} in place of P, we see by the defining property of k_0 that $\widetilde{V} \llcorner (\mathbf{R} \times B_{\theta^{k_0-1}/2}) = \sum_{j=1}^q |\text{graph}\, u_j|$, where $u_j \in C^2\left(B_{\theta^{k_0-1}/2}; \mathbf{R}\right)$ for $j = 1, 2, \ldots, q$; $u_1 \leq u_2 \leq \cdots \leq u_q$ on $B_{\theta^{k_0-1}/2}$; $u_j < u_{j+1}$ on $B_{\theta^{k_0-1}/2}$ for some $j \in \{1, 2, \ldots, q-1\}$ and

(15.18)

$$\begin{aligned}&\sup_{B_{\theta^{k_0-1}/2}} (\theta^{k_0-1})^{-2}|u_j - p|^2 + |D\,u_j - D\,p|^2 + (\theta^{k_0-1})^2|D^2\,u_j|^2 \\ &\quad \leq C(\theta^{k_0-1})^{-n-2}\int_{\mathbf{R}\times B_{\theta^{k_0-1}}} \mathrm{dist}^2(X, P_{k_0-1})\,d\|\widetilde{V}\|(X) \leq C4^{-(k_0-1)}\hat{E}^2_{\widetilde{V},P_0}\end{aligned}$$

for each $j \in \{1, 2, \ldots, q\}$. Here $p : \mathbf{R}^n \to \mathbf{R}$ is the affine function such that $\mathrm{graph}\,p = P_{k_0-1}$. In view of (15.8) and (15.17), this evidently implies alternative (II) with $\rho_0 = \theta^{k_0-1}$. Thus either (I) or (II) holds as claimed.

Suppose that (I) holds for some P_0 satisfying (15.7). It is standard then that there exists a hyperplane $\widetilde{P}$ with

$$\mathrm{dist}_{\mathcal{H}}(\widetilde{P} \cap (\mathbf{R} \times B_1), P_0 \cap (\mathbf{R} \times B_1)) \leq C_1\hat{E}_{\widetilde{V},P_0} \tag{15.19}$$

such that

$$\rho^{-n-2}\int_{\mathbf{R}\times B_\rho} \mathrm{dist}^2(X, \widetilde{P})\,d\|\widetilde{V}\|(X) \leq C_3\rho^{2\mu}\hat{E}^2_{\widetilde{V},P_0} \tag{15.20}$$

for each $\rho \in (0, 1)$, where $C_3 = C_3(n, q) \in (0, \infty)$ and $\mu = \mu(n, q) \in (0, 1)$. Note that $\widetilde{P}$ does not depend on P_0, nor do the constants C_3 and μ. Moreover, in this case, we claim that we have for each $\rho \in (0, 1/4)$ that

$$\begin{aligned}(\omega_n(2\rho)^n)^{-1}\,\|\widetilde{V}\|(B^{n+1}_{2\rho}(0)) &< q + 1/2, \\ q - 1/2 \leq (\omega_n\rho^n)^{-1}\|\widetilde{V}\|(\mathbf{R} \times B_\rho) &< q + 1/2.\end{aligned} \tag{15.21}$$

To see this, given $\rho \in (0, 1/4)$, choose k such that $\theta^{k+1} \leq 4\rho < \theta^k$ and note by (15.12), (15.13), (15.14) and the triangle inequality that

(15.22)

$$\begin{aligned}&\left(\theta^k\right)^{-n-2}\int_{\mathbf{R}\times B_{\theta^k}} |x^1|^2\,d\|\widetilde{V}\|(X) \\ &\quad\leq\; 2(2 + \omega_n(2q + 1)C_1)(\widetilde{\gamma}^{-n-2}\hat{E}^2_V + \mathrm{dist}^2(P_0 \cap (\mathbf{R} \times B_1), \{0\} \times B_1)) \\ &\quad\leq 2(2 + \omega_n(2q + 1)C_1)(\widetilde{\gamma}^{-n-2}\varepsilon + \beta).\end{aligned}$$

Thus provided $\varepsilon = \varepsilon(n, q, \alpha, \gamma) \in (0, 1)$ is sufficiently small, we may, in view of (15.12), apply Theorem 5.1 with $\eta_{\theta^k\,\#}\,\widetilde{V}$ in place of V, $\sigma = 5/6$ and estimate exactly as in (15.9) (with $\eta_{\theta^k\,\#}\,\widetilde{V}$ in place of V, $Y = 0$ and $\theta^{-k}\rho$ in place of $\widetilde{\gamma}$) to deduce that

$$\begin{aligned}&\left|\left(\omega_n\left(\theta^{-k}\rho\right)^n\right)^{-1}\|\eta_{\theta^k\,\#}\,\widetilde{V}\|\left(\mathbf{R} \times B_{\theta^{-k}\rho}(Y)\right) - q\right| \\ &\quad\leq 2\left(\omega_n\left(\theta^{-k}\rho\right)^n\right)^{-1}\left(16\left(\theta^{-k}\rho\right)^{-2} + (q + 1)\overline{C}\right) \\ &\qquad\times (2 + \omega_n(2q + 1)C_1)\left(\widetilde{\gamma}^{-n-2}\varepsilon + \beta\right) \\ &\quad\leq 2\omega_n^{-1}4^n\theta^{-n}\left(256\theta^{-2} + (q + 1)\overline{C}\right)(2 + \omega_n(2q + 1)C_1)\left(\widetilde{\gamma}^{-n-2}\varepsilon + \beta\right).\end{aligned}$$

(Recall that $\overline{C} = C(n,q,5/6)$, where $C = C(n,q,\cdot)$ is as in Theorem 5.1(a).) From this, (15.11) and the monotonicity formula, we deduce that (15.21) holds provided $\varepsilon = \varepsilon(n,q,\alpha,\gamma) \in (0,1)$ is sufficiently small. It then follows, if alternative (I) holds for some P_0 satisfying (15.7), that $\operatorname{spt}\|\widetilde{V}\| \cap \pi^{-1}(0)$ consists of a single point $(= \widetilde{P} \cap \pi^{-1}(0))$; to see this, first note that $\operatorname{spt}\|\widetilde{V}\| \cap \pi^{-1}(0) \neq \emptyset$ by the second inequality in (15.21). Let $Z \in \operatorname{spt}\|\widetilde{V}\| \cap \pi^{-1}(0)$ and $\mathbf{C}_Z$ be a tangent cone to $\widetilde{V}$ at Z. Thus $\eta_{Z,\sigma_j\,\#}\,\widetilde{V} \to \mathbf{C}_Z \neq 0$ for some sequence of numbers $\sigma_j \to 0^+$, and by (15.20), $\operatorname{dist}_{\mathcal{H}}(\operatorname{spt}\|\eta_{Z,\sigma_j\,\#}\,\widetilde{V}\| \cap (\mathbf{R} \times B_{1/2}), \sigma_j^{-1}(\widetilde{P} - Z) \cap (\mathbf{R} \times B_{1/2})) \to 0$, which can only be true if $Z \in \widetilde{P}$. But by (15.19), $\widetilde{P} \cap \pi^{-1}(0)$ consists of a single point.

Since alternative (II) implies that $\operatorname{spt}\|\widetilde{V}\| \cap \pi^{-1}(0)$ has at least two distinct points, we see that if alternative (I) holds for some P_0 satisfying (15.7), then (I) must hold for all P_0 satisfying (15.7). Taking $P_0 = \mathbf{R}^n \times \{0\}$, we deduce from (15.19) and (15.20) that

$$\operatorname{dist}_{\mathcal{H}}(\widetilde{P} \cap (\mathbf{R} \times B_1), \{0\} \times B_1)) \leq C_1 \widetilde{\gamma}^{-n-2} \hat{E}_V^2 \leq C_1 \widetilde{\gamma}^{-n-2} \varepsilon \tag{15.23}$$

and

$$\rho^{-n-2} \int_{\mathbf{R} \times B_\rho} \operatorname{dist}^2(X, \widetilde{P})\, d\|\widetilde{V}\|(X) \leq C_3 \widetilde{\gamma}^{-n-2} \hat{E}_V^2 \leq C_3 \widetilde{\gamma}^{-n-2} \varepsilon \tag{15.24}$$

for $\rho \in (0,1)$. So if we choose $\varepsilon = \varepsilon(n,q,\alpha,\gamma)$ such that $C_1 \widetilde{\gamma}^{-n-2} \varepsilon < \beta$, then we may, in particular, take $P_0 = \widetilde{P}$ in (15.20).

Thus we have so far established the following: Given q, α, γ as in the theorem and that the induction hypotheses (H1), (H2) hold, there exists $\varepsilon = \varepsilon(n,q,\alpha,\gamma) \in (0,1)$ such that if $V \in \mathcal{S}_\alpha$ satisfies the hypotheses of the theorem, $Y \in B_\gamma$, $\widetilde{V} = \eta_{Y,\widetilde{\gamma}\,\#}\, V$ where $\widetilde{\gamma} = (1-\gamma)/4$, then either alternative (I) above holds for all affine hyperplanes P_0 satisfying (15.7) or alternative (II) above holds for all such P_0. Furthermore, if alternative (I) holds, then the bounds (15.21) are satisfied for each $\rho \in (0,1/4)$, the estimates (15.23), (15.24) are satisfied and

$$\rho^{-n-2} \int_{\mathbf{R} \times B_\rho} \operatorname{dist}^2(X, \widetilde{P})\, d\|\widetilde{V}\|(X) \leq C_3 \rho^{2\mu} \int_{\mathbf{R} \times B_1} \operatorname{dist}^2(X, \widetilde{P})\, d\|\widetilde{V}\|(X) \tag{15.25}$$

for $\rho \in (0,1)$, where $\widetilde{P}$ is a (uniquely determined) affine hyperplane, $C_3 = C_3(n,q) \in (0,\infty)$ and $\mu = \mu(n,q) \in (0,1)$.

Now suppose the hypotheses of the theorem are satisfied with a number $\varepsilon' = \varepsilon'(n,q,\alpha,\gamma) \in (0,\varepsilon)$ in place of ε and that alternative (I) (with $Y \in B_\gamma$, $\widetilde{V} = \eta_{Y,\widetilde{\gamma}\,\#}\, V$ as above) still holds. Then for any $\rho \in (0,1/4)$, we have by (15.23), (15.24), (15.21) and the triangle inequality that

$$\rho^{-n-2} \int_{\mathbf{R} \times B_\rho} |x^1|^2\, d\|\widetilde{V}\|(X) \leq (2C_3 + \omega_n(2q+1)C_1) \widetilde{\gamma}^{-n-2} \hat{E}_V^2 < 2(C_3 + C_1)\varepsilon'.$$

Thus if we choose $\varepsilon' = \varepsilon'(n,q,\alpha,\gamma)$ sufficiently small, we may, in view of this and (15.21), repeat the argument leading to (15.25) with $\eta_{\rho\,\#}\widetilde{V}$ (for which alternative (I) must hold) in place of $\widetilde{V}$. By applying (15.25) with $\eta_{\rho\,\#}\widetilde{V}$ in place of $\widetilde{V}$ and $\rho^{-1}\sigma$ in place of ρ, we deduce that if alternative (I) holds for some P_0 satisfying (15.7), then there exists a unique affine hyperplane $\widetilde{P}$ satisfying (15.23) and

$$\sigma^{-n-2}\int_{\mathbf{R}\times B_\sigma}\operatorname{dist}^2(X,\widetilde{P})\,d\|\widetilde{V}\|(X) \leq C_3\left(\frac{\sigma}{\rho}\right)^{2\mu}\rho^{-n-2}\int_{\mathbf{R}\times B_\rho}\operatorname{dist}^2(X,\widetilde{P})\,d\|\widetilde{V}\|(X) \tag{15.26}$$

for each $0<\sigma<\rho<1/4$. On the other hand, if (I) fails for some P_0 satisfying (15.7), then it fails with $P_0=\{0\}\times\mathbf{R}^n$, in which case (by (II)) there exists $\rho_0\in(0,1)$ such that $\widetilde{V}\llcorner(\mathbf{R}\times B_{\rho_0}) = \sum_{j=1}^q|\operatorname{graph}u_j|$ for functions $u_j\in C^2(B_{\rho_0};\mathbf{R})$, $j=1,2,\ldots,q$ satisfying $u_1\leq u_2\leq\cdots\leq u_q$ on B_{ρ_0}; $u_j<u_{j+1}$ on B_{ρ_0} for some $j\in\{1,2,\ldots,q-1\}$ and

$$\sup\nolimits_{B_{\rho_0/2}}\rho_0^{-2}|u_j|^2+|D\,u_j|^2+\rho_0^2|D^2\,u_j|\leq(C+2C_1)\widetilde{\gamma}^{-n-2}\hat{E}_V^2$$

for each $j\in\{1,2,\ldots,q\}$.

Thus we have shown that if the hypotheses of the theorem are satisfied with sufficiently small $\varepsilon'=\varepsilon'(n,q,\gamma)\in(0,1)$ in place of ε, then for each point $Y\in B_\gamma$, precisely one of the following alternatives (I_Y) and (II_Y) must hold:

(I_Y) there exists an affine hyperplane P_Y with

$$\operatorname{dist}^2_{\mathcal{H}}(P_Y\cap(\mathbf{R}\times B_1(Y)),\{0\}\times B_1(Y))\leq C_1\widetilde{\gamma}^{-n-2}\hat{E}_V^2 \tag{15.27}$$

such that

$$\sigma^{-n-2}\int_{\mathbf{R}\times B_\sigma(Y)}\operatorname{dist}^2(X,P_Y)\,d\|V\|(X) \leq C_3\left(\frac{\sigma}{\rho}\right)^{2\mu}\rho^{-n-2}\int_{\mathbf{R}\times B_\rho(Y)}\operatorname{dist}^2(X,P_Y)\,d\|V\|(X) \tag{15.28}$$

for each $0<\sigma<\rho<\widetilde{\gamma}/4$, where $C_3=C_3(n,q)\in(0,\infty)$ and $\mu=\mu(n,q)\in(0,1)$; or

(II_Y) there exists $\rho_Y\in(0,1/2]$ such that $V\llcorner(\mathbf{R}\times B_{\rho_Y}(Y))=\sum_{j=1}^q|\operatorname{graph}u_j^Y|$ for functions $u_j^Y\in C^2(B_{\rho_Y}(Y);\mathbf{R})$, $j=1,2,\ldots,q$, satisfying $u_1^Y\leq u_2^Y\leq\cdots\leq u_q^Y$ on $B_{\rho_Y}(Y)$; $u_{j_0}^Y<u_{j_0+1}^Y$ on $B_{\rho_Y}(Y)$ for some $j_0\in\{1,2,\ldots,q-1\}$ and

$$\sup\nolimits_{B_{\rho_Y}(Y)}\rho_Y^{-2}|u_j^Y|^2+|D\,u_j^Y|^2+\rho_Y^2|D^2\,u_j^Y|^2\leq(C+2C_1)\widetilde{\gamma}^{-n-2}\hat{E}_V^2 \tag{15.29}$$

for each $j\in\{1,2,\ldots,q\}$, where $C=C(n,q)\in(0,\infty)$.

Let $\Omega = \{Y \in B_\gamma \,:\, (\mathrm{I}_Y) \text{ fails}\}$. Since $u_j^Y < u_{j+1}^Y$ on $B_{\rho_Y}(Y)$ for some j whenever (II_Y) holds, it follows that Ω is an open set. Hence, since for every $Y \in \Omega$, each of the functions u_j^Y as in (II_Y) solves the minimal surface equation on $B_{\rho_Y}(Y)$, by unique continuation of solutions to the minimal surface equation, we see that

$$V \llcorner (\mathbf{R} \times \Omega) = \sum_{j=1}^{q} |\mathrm{graph}\, u_j| \tag{15.30}$$

for functions $u_j \in C^\infty(\Omega; \mathbf{R})$, solving the minimal surface equation on Ω and satisfying $u_1 \leq u_2 \leq \cdots \leq u_q$ on Ω; $u_j < u_{j+1}$ for some $j \in \{1, 2, \ldots, q-1\}$ in each connected component of Ω (by the maximum principle) and

$$\sup_\Omega |u_j|^2 + |D\, u_j|^2 \leq (C + 2C_1)\widetilde{\gamma}^{-n-2}\hat{E}_V^2 \leq (C + 2C_1)\widetilde{\gamma}^{-n-2}\varepsilon' \tag{15.31}$$

for each $j \in \{1, 2, \ldots, q\}$. This implies that for each affine function $p : \mathbf{R}^n \to \mathbf{R}$ with $\sup_{B_1} |p|^2 \leq C_1\widetilde{\gamma}^{-n-2}\varepsilon'$ and each $j = 1, 2, \ldots, q$, the function $w_j = u_j - p \in C^\infty(\Omega)$ solves on Ω a uniformly elliptic equation of the type $a_{\ell k} D_\ell D_\kappa\, w_j + b_\ell D_\ell\, w_j = 0$ with smooth coefficients $a_{\ell k}$, b_ℓ satisfying $\sup_\Omega |a_{\ell k}| + |b_\ell| \leq \kappa$, $\kappa = \kappa(n, q, \gamma) \in (0, \infty)$. By using the standard second derivative estimates for solutions to such equations, we conclude that for each $Y \in \Omega$, each $j = 1, 2, \ldots, q$ and each affine function $p_j : \mathbf{R}^n \to \mathbf{R}$ with $\sup_{B_1} |p_j|^2 \leq C_1\widetilde{\gamma}^{-n-2}\varepsilon'$,

$$\sigma^{-n-2}\int_{B_\sigma(Y)} |u_j - p_j^Y|^2 \leq C_4\left(\frac{\sigma}{\rho}\right)^2 \rho^{-n-2}\int_{B_\rho(Y)} |u_j - p_j|^2 \tag{15.32}$$

for $0 < \sigma \leq \rho/2 < \frac{1}{2}\mathrm{dist}(Y, B_\gamma \setminus \Omega)$, where $C_4 = C_4(n, q, \gamma) \in (0, \infty)$ and $p_j^Y(X) = u_j(Y) + Du_j(Y)\cdot(X - Y)$. Since for each $Y \in B_\gamma \setminus \Omega$, $\mathrm{spt}\, \|V\| \cap \pi^{-1}(Y)$ consists of a single point (z_Y, Y) $(= P_Y \cap \pi^{-1}(Y))$, for each $j = 1, 2, \ldots, q$, we may extend u_j to all of B_γ by setting $u_j(Y) = z_Y$ for $Y \in B_\gamma \setminus \Omega$. Then by (15.30),

$$\mathrm{spt}\, \|V\| \cap (\mathbf{R} \times B_\gamma) = \textstyle\bigcup_{j=1}^q \mathrm{graph}\, u_j. \tag{15.33}$$

Now let $\widetilde{\Sigma}_1$, $\widetilde{\Sigma}_2$, $\widetilde{\Sigma}_3$, Σ' be the sets as in Theorem 5.1 taken with $\sigma = \gamma$. We claim that these sets are all empty if $\varepsilon' = \varepsilon'(n, q, \alpha, \gamma)$ is sufficiently small. Indeed, it is clear from (15.30), (15.31) and the definitions of $\widetilde{\Sigma}_j$, Σ' that $\widetilde{\Sigma}_j \cap (\mathbf{R} \times \Omega) = \emptyset$ for $j = 1, 2, 3$ and that $\Sigma' \cap \Omega = \emptyset$. For each $Y \in B_\gamma \setminus \Omega$, by applying (5.1) with $\Gamma_{Y\,\#}\, V$ in place of V where Γ_Y is a rigid motion of $\mathbf{R}^{n+1}$ that takes $(z_Y, Y) \in P_Y$ to the origin and P_Y to $\{0\} \times \mathbf{R}^n$, and using the estimate (15.28), we see that $Y \notin \pi\, \widetilde{\Sigma}_1$ provided $\varepsilon' = \varepsilon'(n, q, \alpha, \gamma)$ is sufficiently small. Since (15.28) implies that for each $Y \in B_\gamma \setminus \Omega$, the varifold V has a unique tangent cone at (z_Y, Y) with support equal to $P_Y - (z_Y, Y)$, it follows from the constancy theorem that $\Theta(\|V\|, (z_Y, Y))$ is a positive integer and furthermore, from the fact that varifold convergence implies Hausdorff convergence of supports, that $\mathrm{Tan}(\mathrm{spt}\, \|V\|, Y) = P_Y - (z_Y, Y)$. Consequently, we see that $Y \notin \pi\,\widetilde{\Sigma}_2$ and by (15.27), we see that $Y \notin \pi\,\widetilde{\Sigma}_3$. Finally, we argue that $\Theta(\|V\|, (z_Y, Y)) \geq q$ for each $Y \in B_\gamma \setminus \Omega$, from which it follows that $\Sigma' \cap (B_\gamma \setminus \Omega) = \emptyset$. If $\Theta(\|V\|, (z_{Y_0}, Y_0)) < q$ for some $Y_0 \in B_\gamma \setminus \Omega$, there is, by upper semi-continuity of density, some $\sigma_0 > 0$ such that $\Theta(\|V\|, X) < q$ for each $X \in \operatorname{spt}\|V\| \cap (\mathbf{R} \times B_{\sigma_0}(Y_0))$. Hence, by Remark 3 of Section 6, the estimate (15.28) taken with $\sigma = \sigma_0$, $\rho = \widetilde{\gamma}/8$ and the estimate (15.21) taken with $\rho = \widetilde{\gamma}^{-1}\sigma_0$, we may, provided $\varepsilon' = \varepsilon'(n, q, \alpha, \gamma)$ is sufficiently small, apply Theorem 3.5 to conclude that

$$V \llcorner (\mathbf{R} \times B_{\sigma_0/2}) = \sum_{j=1}^{q} |\operatorname{graph} w^j|$$

for smooth functions $w_1 \leq w_2 \leq \cdots \leq w_q$ on $B_{\sigma_0/2}(Y_0)$ solving the minimal surface equation. Since $\pi^{-1}(Y_0) \cap \operatorname{spt}\|V\|$ consists of a single point, by the maximum principle we must have that $w_1 = w_2 = \cdots = w_q$ on $B_{\sigma_0/2}(Y_0)$, contrary to the assumption that $\Theta(\|V\|, (z_{Y_0}, Y_0)) < q$. This concludes the proof of the claim that the sets $\widetilde{\Sigma}_j, \Sigma'$ are all empty. Then by Theorem 5.1 and (15.33), for each $j = 1, 2, \ldots, q$, the function $u_j : B_\gamma \to \mathbf{R}$ is Lipschitz with Lipschitz constant $\leq 1/2$ so that by (15.28),(15.31) and the area formula, it follows that

$$\sigma^{-n-2} \int_{B_\sigma(Y)} |u_j - p^Y|^2 = 2C_3 \left(\frac{\sigma}{\rho}\right)^{2\alpha} \rho^{-n-2} \int_{B_\rho(Y)} |u^j - p^Y|^2 \tag{15.34}$$

for each $Y \in B_\gamma \setminus \Omega$ and each σ, ρ with $0 < \sigma < \rho < \widetilde{\gamma}/4$, where $p^Y : \mathbf{R}^n \to \mathbf{R}$ is the affine function such that $\operatorname{graph} p^Y = P_Y$.

In view of (15.32) and (15.34), we conclude from Lemma 4.3 that $u_j \in C^{1,\lambda}(B_{\gamma/2})$ with

$$\sup_{B_{\gamma/2}} |u_j|^2 + |Du_j|^2 + \sup_{Y_1, Y_2 \in B_{\gamma/2},\, Y_1 \neq Y_2} \frac{|Du_j(Y_1) - Du_j(Y_2)|^2}{|Y_1 - Y_2|^{2\lambda}} \leq C_5 \hat{E}_V^2$$

for each $j = 1, 2, \ldots, q$, where $C_5 = C_5(n, q, \gamma) \in (0, \infty)$ and $\lambda = \lambda(n, q, \gamma) \in (0, 1)$.

To show that for each $j = 1, 2, \ldots, q$, the function $u_j \in C^\infty(B_{\gamma/2})$ and solves the minimal surface equation on $B_{\gamma/2}$, we argue as follows: We know that on the open set $\Omega \subseteq B_\gamma$, each $u_j \in C^2$ and solves the minimal surface equation (and hence is smooth), and on $B_\gamma \setminus \Omega$, the functions u_j all agree, so if $B_{\gamma/2} \subseteq \Omega$ or $B_{\gamma/2} \cap \Omega = \emptyset$, there is nothing further to prove. Else, for any connected component Ω' of Ω such that $B_{\gamma/2} \cap \Omega' \neq \emptyset$, we must have that $B_{\gamma/2} \setminus \Omega' \neq \emptyset$ whence $\partial\,\Omega' \cap B_{\gamma/2} \neq \emptyset$. Fix any such Ω', and let $B \subset \Omega'$ be an open ball such that $\overline{B} \cap \partial\,\Omega' \cap B_{\gamma/2} \neq \emptyset$. (To find such B, pick any point $p \in \Omega'$ closer to $\partial\,\Omega'$ than to $\partial\,B_{\gamma/2}$ and let $B = B_R(p)$, where $R = \sup\{r : B_r(p) \subset \Omega'\}$.) Let $x_0 \in \partial\,B \cap \partial\,\Omega' \cap B_{\gamma/2}$. Pick any $j \in \{1, 2, \ldots, q-1\}$, and let $w_j = u_{j+1} - u_j$. Then w_j solves in B a uniformly elliptic equation with smooth coefficients.

Since $w_j \in C^1(B_{\gamma/2})$, $w_j \geq 0$ and $w_j(x_0) = 0$, it follows that $Dw_j(x_0) = 0$, and hence by the Hopf boundary point lemma, $w_j \equiv 0$ in B. This implies by unique continuation of solutions to the minimal surface equation that $w_j \equiv 0$ in Ω' whence all of the u_j's agree on Ω', which is impossible by the definition of Ω (see the line preceding (15.31)). Thus we must have either $B_{\gamma/2} \subseteq \Omega$ or $B_{\gamma/2} \cap \Omega = \emptyset$, and the proof of the theorem is complete. $\square$

16. **The Minimum Distance Theorem**

Let q be an integer ≥ 2, and let $\mathbf{C}_0$ be a stationary integral hypercone in $\mathbf{R}^{n+1}$ such that $\text{spt}\,\|\mathbf{C}_0\|$ consists of three or more distinct half-hyperplanes of $\mathbf{R}^{n+1}$ meeting along a common $(n-1)$-dimensional subspace $L_{\mathbf{C}_0}$ of $\mathbf{R}^{n+1}$. In this section we will use the multiplicity q case of the Sheeting Theorem (i.e., Theorem 15.2) to establish, subject to the induction hypotheses (H1), (H2), the validity of Theorem 3.4 whenever

$$\Theta(\|\mathbf{C}_0\|, 0) \in \{q + 1/2, q + 1\}. \tag{16.1}$$

Our argument will also establish Theorem 3.4 in case $\Theta(\|\mathbf{C}_0\|, 0) \in \{3/2, 2\}$; see the remark at the end of this section.

Suppose that $\mathbf{C}_0$ satisfies (16.1), and without loss of generality assume that $L_{\mathbf{C}_0} = \{0\} \times \mathbf{R}^{n-1}$. Thus, $\text{spt}\,\|\mathbf{C}_0\| = \text{spt}\,\|\Delta_0\| \times \mathbf{R}^{n-1}$, where Δ_0 is a 1-dimensional stationary cone in $\mathbf{R}^2$, whence $\Delta_0 = \sum_{j=1}^{m_0} q_j^{(0)} |R_j^{(0)}|$ and

$$\mathbf{C}_0 = \sum_{j=1}^{m_0} q_j^{(0)} |H_j^{(0)}|, \tag{16.2}$$

where m_0 is an integer ≥ 3, $q_j^{(0)}$ is a positive integer for each $j = 1, 2, \ldots, m_0$, $R_j^{(0)} = \{t\mathbf{w}_j^{(0)} : t > 0\}$ for some unit vector $\mathbf{w}_j^{(0)} \in \mathbf{S}^1 \subset \mathbf{R}^2$ with $\mathbf{w}_j^{(0)} \neq \mathbf{w}_k^{(0)}$ for $j \neq k$, and $H_j^{(0)} = R_j^{(0)} \times \mathbf{R}^{n-1}$. Stationarity of $\mathbf{C}_0$ is equivalent to the requirement

$$\sum_{j=1}^{m_0} q_j^{(0)} \mathbf{w}_j^{(0)} = 0. \tag{16.3}$$

Since, by (16.1),

$$\sum_{j=1}^{m_0} q_j^{(0)} \in \{2q + 1, 2q + 2\}, \tag{16.4}$$

we see readily from (16.3) that

$$q_j^{(0)} \leq q \quad \text{for each } j = 1, 2, \ldots, m_0. \tag{16.5}$$

The theorem we wish to prove is the following:

THEOREM 16.1. *Let q be an integer ≥ 2, $\alpha \in (0,1)$, and suppose that the induction hypotheses* (H1), (H2) *hold. Let $\mathbf{C}_0$ be the stationary cone as in* (16.2), *where $m_0 \geq 3$ and $H_j^{(0)} \neq H_k^{(0)}$ for $j \neq k$, and suppose that $\mathbf{C}_0$ satisfies* (16.1). *For each $\gamma \in (0,1/2)$, there exists a number $\varepsilon_0 = \varepsilon_0(n,q,\alpha,\gamma,\mathbf{C}_0) \in (0,1)$ such that if $V \in \mathcal{S}_\alpha$, $\Theta(\|V\|,0) \geq \Theta(\|\mathbf{C}_0\|,0)$ and $(\omega_n 2^n)^{-1}\|V\|(B_2^{n+1}(0)) < \Theta(\|\mathbf{C}_0\|,0)+\gamma$, then*

$$\operatorname{dist}_{\mathcal{H}}(\operatorname{spt}\|V\| \cap B_1^{n+1}(0), \operatorname{spt}\|\mathbf{C}_0\| \cap B_1^{n+1}(0)) \geq \varepsilon_0.$$

Notation. Let $\mathbf{C}_0$ be as in (16.2), with the associated unit vectors $\mathbf{w}_j^{(0)} \in \mathbf{R}^2$, $j = 1,2,\ldots,m_0$, as described above. We shall use the following notation in connection with $\mathbf{C}_0$:

$\sigma_0 = \max\{\mathbf{w}_j^{(0)} \cdot \mathbf{w}_k^{(0)} : j,k = 1,2,\ldots,m_0,\ j \neq k\}$.

$N(H_j^{(0)})$ is the conical neighborhood of $H_j^{(0)}$ defined by

$$N(H_j^{(0)}) = \left\{(x,y) \in \mathbf{R}^2 \times \mathbf{R}^{n-1} : x \cdot \mathbf{w}_j^{(0)} > \sqrt{\frac{1+\sigma_0}{2}}|x|\right\}.$$

Given $\mathbf{C}_0$ as above, $\mathcal{K}$ denotes the family of hypercones $\mathbf{C}$ of $\mathbf{R}^{n+1}$ of the form

$$\mathbf{C} = \sum_{j=1}^{m_0}\sum_{\ell=1}^{q_j^{(0)}} |H_{j,\ell}|, \tag{16.6}$$

where $H_{j,\ell}$ are half-hyperplanes of $\mathbf{R}^{n+1}$ meeting along $\{0\}\times\mathbf{R}^{n-1}$ with $H_{j,\ell} \in N(H_j^{(0)})$ for each $j \in \{1,2,\ldots,m_0\}$, $\ell \in \{1,2,\ldots,q_j^{(0)}\}$, and $H_{j,1}, H_{j,2}, \ldots, H_{j,q_j^{(0)}}$ not necessarily distinct for each $j \in \{1,2,\ldots,m_0\}$. Note that unless otherwise specified, we do *not* assume a cone $\mathbf{C} \in \mathcal{K}$ is stationary in $\mathbf{R}^{n+1}$.

For $p \in \{m_0, m_0+1, \ldots, 2q\}$, $\mathcal{K}(p)$ denotes the set of cones $\mathbf{C} \in \mathcal{K}$ as in (16.6) such that the number of *distinct* elements in the set $\{H_{j,\ell} : j = 1,2,\ldots,m_0,\ \ell = 1,2,\ldots,q_j^{(0)}\}$ is p. Thus

$$\mathcal{K} = \textstyle\bigcup_{p=m_0}^{2\Theta(\|\mathbf{C}_0\|,0)}\mathcal{K}(p).$$

Also, for $X \in \mathbf{R}^{n+1}$, let $r(X) = \operatorname{dist}(X, \{0\}\times\mathbf{R}^{n-1})$.

For the rest of this section, we shall fix $\mathbf{C}_0$ as above, with fixed labelling of the elements of the set $\{H_j^{(0)} : j = 1,\ldots,m_0\}$ of constituent half-hyperplanes of $\operatorname{spt}\|\mathbf{C}_0\|$ and with $q_j^{(0)}$, $1 \leq j \leq m_0$, denoting the multiplicity on $H_j^{(0)}$.

For $\alpha \in (0,1)$, $\gamma \in (0,1/2)$ and appropriate $\varepsilon \in (0,1/2)$, consider the following:

Hypotheses 16.2.

(1) $V \in \mathcal{S}_\alpha$, $0 \in \operatorname{spt}\|V\|$, $\Theta(\|V\|,0) \geq \Theta(\|\mathbf{C}_0\|,0)$, $(\omega_n 2^n)^{-1}\|V\|(B_2^{n+1}(0)) < \Theta(\|\mathbf{C}_0\|,0)+\gamma$.

(2) $\mathbf{C} = \sum_{j=1}^{m_0} \sum_{\ell=1}^{q_j^{(0)}} |H_{j,\ell}| \in \mathcal{K}$, where $H_{j,\ell}$ are half-hyperplanes of $\mathbf{R}^{n+1}$ meeting along $\{0\} \times \mathbf{R}^{n-1}$ with $H_{j,\ell} \in N(H_j^{(0)})$ for each $j \in \{1, 2, \ldots, m_0\}$ and $\ell \in \{1, 2, \ldots, q_j^{(0)}\}$.

(3) $\mathrm{dist}_{\mathcal{H}}(\mathrm{spt}\, \|\mathbf{C}\| \cap B_1^{n+1}(0), \mathrm{spt}\, \|\mathbf{C}_0\| \cap B_1^{n+1}(0)) < \varepsilon$.

(4) $\displaystyle\int_{B_1^{n+1}(0)} \mathrm{dist}^2(X, \mathrm{spt}\, \|\mathbf{C}\|)\, d\|V\|(X) < \varepsilon$.

(5) For each $j = 1, 2, \ldots, m_0$,

$$\|V\|((B_{1/2}^{n+1}(0) \setminus \{r(X) < 1/8\}) \cap N(H_j^{(0)})) \geq \left(q_j^{(0)} - \frac{1}{4}\right) \mathcal{H}^n((B_{1/2}^{n+1}(0) \setminus \{r(X) < 1/8\}) \cap H_j^{(0)}).$$

Fix a number $s = s(n, q) \in (0, 1/16)$ such that

(16.7)
$$\mathcal{H}^n\left(B_{\frac{1}{2}-s}^n(0) \setminus \{r(X) < 1/8 + s\}\right) \geq \left(1 - \frac{1}{4q}\right) \mathcal{H}^n\left(B_{1/2}^n(0) \setminus \{r(X) < 1/8\}\right)$$

and note that by (16.5),

(16.8)
$$q_j^{(0)} \mathcal{H}^n((B_{\frac{1}{2}-s}^{n+1}(0) \setminus \{r(X) < 1/8 + s\}) \cap H_j^{(0)}) \geq \left(q_j^{(0)} - \frac{1}{4}\right) \mathcal{H}^n((B_{1/2}^{n+1}(0) \setminus \{r(X) < 1/8\}) \cap H_j^{(0)})$$

for each $j = 1, 2, \ldots, m_0$.

Remarks. (1) For each $\gamma \in (0, 1/2)$ and $\tau \in (0, 1/8)$, there exists $\varepsilon = \varepsilon(n, q, \tau, \gamma, \mathbf{C}_0) \in (0, 1)$ such that if the induction hypotheses (H1), (H2) and Hypotheses 16.2 hold, then

(a) $\{Z \in \mathrm{spt}\, \|V\| \cap B_{15/16}^{n+1}(0) : \Theta(\|V\|, Z) \geq q + 1/2\} \subset \{X \in \mathbf{R}^{n+1} : r(X) < \tau/2\}$; and

(b) for each $j \in \{1, 2, \ldots, m_0\}$ and $\ell \in \{1, 2, \ldots, q_j^{(0)}\}$, there exists a function

$$\widetilde{u}_{j,\ell} \in C^2\left(\left(B_{15/16}^{n+1}(0) \cap H_j^{(0)} \setminus \{r(X) < \tau\}\right); \left(H_j^{(0)}\right)^{\perp}\right)$$

with small C^2 norm such that $\widetilde{u}_{j,\ell}$ solves the minimal surface equation on its domain and

$$V \llcorner \left(B_{15/16}^{n+1}(0) \setminus \{r(X) < \tau\}\right) = \sum_{j=1}^{m_0} \sum_{\ell=1}^{q_j^{(0)}} |\mathrm{graph}\, \widetilde{u}_{j,\ell}|.$$

To see this, argue by contradiction, using the Constancy Theorem, upper semi-continuity of the density function $\Theta(\cdot, \cdot)$, (16.5), induction hypothesis (H1) and Theorem 15.2.

(2) For each $\gamma \in (0,1/2)$ and $\tau \in (0,1/8)$, there exists $\varepsilon = \varepsilon(n,q,\tau,\gamma,\mathbf{C}_0) \in (0,1)$ such that if Hypotheses 16.2(1)–(4) hold and if (in place of Hypothesis 16.2(5))

$$\int_{B^{n+1}_{1/2}(0)\setminus\{r(X)<1/8\}} \mathrm{dist}^2(X, \mathrm{spt}\,\|V\|)\, d\|\mathbf{C}\|(X) < \varepsilon,$$

then

$$\{Z \in \mathrm{spt}\,\|V\| \cap B^{n+1}_{15/16}(0) : \Theta(\|V\|, Z) \geq q + 1/2\} \subset \{X \in \mathbf{R}^{n+1} : r(X) < \tau\}.$$

Again, this is easily seen by arguing by contradiction using the Constancy Theorem, upper semi-continuity of density and (16.5).

(3) Let q be an integer ≥ 2. If the induction hypotheses (H1), (H2) hold, $V \in \mathcal{S}_\alpha$, $\Omega \subseteq B^{n+1}_2(0)$ is open and $\Theta(\|V\|, Z) < q{+}1/2$ for each $Z \in \mathrm{spt}\,\|V\| {\cap} \Omega$, then $\mathcal{H}^{n-7+\gamma}(\mathrm{sing}\, V \llcorner \Omega) = 0$ for each $\gamma > 0$ if $n \geq 7$, $\mathrm{sing}\, V \llcorner \Omega$ is discrete if $n = 7$ and $\mathrm{sing}\, V \llcorner \Omega = \emptyset$ if $2 \leq n \leq 6$. This can be seen by reasoning exactly as in Remarks (2) and (3) of Section 6, with the additional help of Theorem 15.2.

(4) Let $\gamma \in (0,1/2)$, $\rho \in (0,1/2]$ and $\varepsilon' \in (0,1/2)$. There exists a number $\varepsilon = \varepsilon(\rho,\varepsilon',\alpha,\gamma,\mathbf{C}_0) \in (0,1/2)$ such that if Hypotheses 16.2 are satisfied, then for each $Z \in \mathrm{spt}\,\|V\| \cap B^{n+1}_{1/8}(0)$ with $\Theta(\|V\|, Z) \geq q + 1/2$, Hypotheses 16.2 are also satisfied with $\eta_{Z,\rho\,\#}\, V$ in place of V and ε' in place of ε.

Indeed, given any $\rho \in (0,1/2]$, if V, $\mathbf{C}$ are as in Hypotheses 16.2 with sufficiently small $\varepsilon = \varepsilon(\rho,\alpha,\mathbf{C}_0) \in (0,1/2)$, then it follows from Remark (1) applied with suitably small $\tau = \tau(\rho,\gamma) \in (0,1/16)$ and the fact that $\|V\|(B^{n+1}_1(0) \cap \{X : r(X) < \tau\}) \leq C\tau$ where $C = C(n,q) \in (0,\infty)$ that for any $Z \in \mathrm{spt}\,\|V\| \cap B^{n+1}_{1/8}(0)$ with $\Theta(\|V\|, Z) \geq q + 1/2$, Hypothesis 16.2(1) is satisfied with $\eta_{Z,\rho\,\#}\, V$ in place of V. Also, since by the triangle inequality

$$\begin{aligned}\int_{B^{n+1}_1(0)} &\mathrm{dist}^2(X, \mathrm{spt}\,\|\mathbf{C}\|) d\|\eta_{Z,\rho\,\#}\, V\|(X) \\ &\leq 2\rho^{-n-2} \int_{B^{n+1}_\rho(Z)} \mathrm{dist}^2(X, \mathrm{spt}\,\|\mathbf{C}\|)\, d\|V\|(X) \\ &\quad + C\rho^{-2}\mathrm{dist}^2(Z, \{0\} \times \mathbf{R}^{n-1}),\end{aligned}$$

where $C = C(n,q,\gamma) \in (0,\infty)$, it follows again by Remark (1) (taken with $\tau = \rho\sqrt{(2C)^{-1}\varepsilon'}$) that if $\varepsilon = \varepsilon(\rho,\varepsilon',\alpha,\gamma,\mathbf{C}_0)$ is sufficiently small, then Hypothesis 16.2(4) is satisfied with $\eta_{Z,\rho\,\#}\, V$ in place of V and ε' in place of ε. Finally, applying Remark (1) once again with $\tau = \rho s$, where $s = s(n,q)$ is as in (16.7), we deduce using (16.8), the area formula and the inclusion

$$\begin{aligned}&\mathrm{spt}\,\|V\| \cap \left(B^{n+1}_{\rho-\tau}(0,\eta) \setminus B^2_{\frac{\rho}{8}+\tau}(0) \times \mathbf{R}^{n-1}\right) \\ &\qquad\qquad \subset \mathrm{spt}\,\|V\| \cap \left(B^{n+1}_\rho(Z) \setminus B^2_\tau(0) \times \mathbf{R}^{n-1}\right),\end{aligned}$$

where $(0,\eta)$ is the orthogonal projection of Z onto $\{0\}\times\mathbf{R}^{n-1}$, that if $\varepsilon = \varepsilon(\rho,\alpha,\gamma,\mathbf{C}_0)$ is sufficiently small, then Hypothesis 16.2(5) is satisfied with $\eta_{Z,\rho\,\#}\,V$ in place of V.

With the notation as above, for $V\in\mathcal{S}_\alpha$, $\mathbf{C}\in\mathcal{K}$ as in Hypotheses 16.2 and appropriate $\beta\in(0,1/2)$, we will also need to consider the following:

Hypothesis (†). *Either* (i) *or* (ii) *below holds*:

(i) $\mathbf{C}\in\mathcal{K}(m_0)$.

(ii) $2\Theta(\|\mathbf{C}_0\|,0)\geq m_0+1$, $\mathbf{C}\in\mathcal{K}(p)$ *for some* $p\in\{m_0+1,m_0+2,\ldots,2\Theta(\|\mathbf{C}_0\|,0)\}$ *and*

$$\begin{aligned}&\int_{B_1^{n+1}(0)}\operatorname{dist}^2(X,\operatorname{spt}\|\mathbf{C}\|)\,d\|V\|(X)\\&\qquad+\int_{B_1^{n+1}(0)\setminus\{r(X)<1/16\}}\operatorname{dist}^2(X,\operatorname{spt}\|V\|)\,d\|\mathbf{C}\|(X)\\&\leq\beta\inf_{\widetilde{\mathbf{C}}\in\bigcup_{j=m_0}^{p-1}\mathcal{K}(j)}\left(\int_{B_1^{n+1}(0)}\operatorname{dist}^2(X,\operatorname{spt}\|\widetilde{\mathbf{C}}\|)\,d\|V\|(X)\right.\\&\qquad\qquad\left.+\int_{B_1^{n+1}(0)\setminus\{r(X)<1/16\}}\operatorname{dist}^2(X,\operatorname{spt}\|V\|)\,d\|\widetilde{\mathbf{C}}\|(X)\right).\end{aligned}$$

Remark. If Hypotheses 16.2 and Hypothesis (†) for some $\beta\in(0,1/2)$ are satisfied, and if $\mathbf{C}'\in\mathcal{K}$ is such that $\operatorname{spt}\|\mathbf{C}'\|=\operatorname{spt}\|\mathbf{C}\|$, then Hypotheses 16.2 and Hypothesis (†), taken with $\mathbf{C}'$ in place of $\mathbf{C}$ and $2q\beta$ in place of β, will be satisfied.

Case $\Theta(\|\mathbf{C}_0\|,0)=q+1/2$. From now on until we state otherwise, we shall assume that $\Theta(\|\mathbf{C}_0\|,0)=q+1/2$.

The basic L^2-estimates of [Sim93, Th. 3.1] hold under our assumptions, namely, the induction hypotheses (H1), (H2), Hypotheses 16.2 and Hypothesis (†), and are given in Theorem 16.2 and Corollary 16.3 below:

Theorem 16.2. *Let q be an integer ≥ 2, $\alpha\in(0,1)$, $\gamma\in(0,1/2)$, $\mu\in(0,1)$ and $\tau\in(0,1/8)$. Suppose that the induction hypotheses* (H1), (H2) *hold. Let $\mathbf{C}_0$ be a stationary cone as above, with $\Theta(\|\mathbf{C}_0\|,0)=q+1/2$. There exist numbers $\varepsilon_0=\varepsilon_0(n,q,\alpha,\gamma,\tau,\mathbf{C}_0)\in(0,1/2)$, $\beta_0=\beta_0(n,q,\alpha,\gamma,\tau,\mathbf{C}_0)\in(0,1/2)$ such that if $V\in\mathcal{S}_\alpha$, $\mathbf{C}\in\mathcal{K}$ satisfy Hypotheses* 16.2 *with ε_0 in place of ε and Hypothesis* (†) *with β_0 in place of β, then, after taking appropriate $\mathbf{C}'\in\mathcal{K}$ with $\operatorname{spt}\|\mathbf{C}'\|=\operatorname{spt}\|\mathbf{C}\|$ in place of $\mathbf{C}$, relabelling $\mathbf{C}'$ as $\mathbf{C}$* (*see the preceding remark*) *and writing $\mathbf{C}=\sum_{j=1}^{m_0}\sum_{\ell=1}^{q_j^{(0)}}|H_{j,\ell}|$ where $H_{j,\ell}$ are half-hyperplanes of $\mathbf{R}^{n+1}$ meeting along $\{0\}\times\mathbf{R}^{n-1}$ with $H_{j,\ell}\in N(H_j^{(0)})$ for each $j\in\{1,2,\ldots,m_0\}$ and $\ell\in\{1,2,\ldots,q_j^{(0)}\}$, the following hold*:

(a) $V \llcorner (B^{n+1}_{7/8}(0) \setminus \{r(X) < \tau\}) = \sum_{j=1}^{m_0} \sum_{\ell=1}^{q_j^{(0)}} |\mathrm{graph}\, u_{j,\ell} \cap B^{n+1}_{7/8}(0)|$ *where* $u_{j,\ell} \in C^2(B^{n+1}_{7/8}(0) \cap H_{j,\ell} \setminus \{r(X) < \tau\}; H_{j,\ell}^{\perp})$ *for* $1 \leq j \leq m_0$, $1 \leq \ell \leq q_j^{(0)}$, $u_{j,\ell}$ *solves the minimal surface equation on* $B^{n+1}_{7/8}(0) \cap H_{j,\ell} \setminus \{r(X) < \tau\}$, $\mathrm{dist}(X + u_{j,\ell}(X), \mathrm{spt}\, \|\mathbf{C}\|) = |u_{j,\ell}(X)|$ *for* $X \in B^{n+1}_{7/8}(0) \cap H_{j,\ell} \setminus \{r(X) < \tau\}$ *and for each* $j \in \{1, \ldots, m_0\}$ *and* $\ell_1, \ell_2 \in \{1, \ldots, q_j^{(0)}\}$, *either* $\mathrm{graph}\, u_{j,\ell_1} \cap B^{n+1}_{7/8}(0) \equiv \mathrm{graph}\, u_{j,\ell_2} \cap B^{n+1}_{7/8}(0)$ *or* $\mathrm{graph}\, u_{j,\ell_1} \cap \mathrm{graph}\, u_{j,\ell_2} \cap B^{n+1}_{7/8}(0) = \emptyset$;

(b) $\int_{B^{n+1}_{3/4}(0)} \frac{|X^{\perp}|^2}{|X|^{n+2}}\, d\|V\|(X) \leq C \int_{B^{n+1}_1(0)} \mathrm{dist}^2(X, \mathrm{spt}\, \|\mathbf{C}\|)\, d\|V\|(X)$;

(c) $\sum_{j=3}^{n+1} \int_{B^{n+1}_{3/4}(0)} |e_j^{\perp}|^2\, d\|V\|(X) \leq C \int_{B^{n+1}_1(0)} \mathrm{dist}^2(X, \mathrm{spt}\, \|\mathbf{C}\|)\, d\|V\|(X)$;

(d) $\int_{B^{n+1}_{3/4}(0)} \frac{\mathrm{dist}^2(X, \mathrm{spt}\, \|\mathbf{C}\|)}{|X|^{n+2-\mu}}\, d\|V\|(X) \leq C_1 \int_{B^{n+1}_1(0)} \mathrm{dist}^2(X, \mathrm{spt}\, \|\mathbf{C}\|)\, d\|V\|(X)$.

Here $C = C(n, \alpha, \gamma, \mathbf{C}_0) \in (0, \infty)$ *and* $C_1 = C_1(n, \alpha, \gamma, \mu, \mathbf{C}_0) \in (0, \infty)$. *In particular,* C *and* C_1 *do not depend on* τ.

Proof. Note first that by Remark (1) following Hypotheses 16.2, provided the hypotheses of the theorem are satisfied with $\varepsilon_0 = \varepsilon_0(n, q, \alpha, \tau, \mathbf{C}_0)$ sufficiently small, we have that $\Theta(\|V\|, Z) < q + 1/2$ for each $Z \in \mathrm{spt}\, \|V\| \cap B^{n+1}_{15/16}(0) \setminus \{r(X) < \tau/2\}$, and

$$V \llcorner \left(B^{n+1}_{7/8}(0) \setminus \{r(X) < \tau\}\right) = \sum_{j=1}^{m_0} \sum_{\ell=1}^{q_j^{(0)}} |\mathrm{graph}\, \widetilde{u}_{j,\ell}|,$$

where for each $j \in \{1, 2, \ldots, m_0\}$ and $\ell \in \{1, 2, \ldots, q_j^{(0)}\}$,

$$\widetilde{u}_{j,\ell} \in C^2\left(\left(B^{n+1}_{7/8}(0) \cap H_j^{(0)} \setminus \{r(X) < \tau\}\right); \left(H_j^{(0)}\right)^{\perp}\right)$$

and $\widetilde{u}_{j,\ell}$ are solutions to the minimal surface equation over

$$H_j^{(0)} \cap \left(B^{n+1}_{7/8}(0) \setminus \{r(X) < \tau\}\right)$$

with small C^2 norm. So if $\mathbf{C} \in \mathcal{K}(m_0)$, then the desired conclusions in part (a) readily follow because then $\mathbf{C} = \sum_{j=1}^{m_0} q_j^{(0)} |H_j'|$ for distinct half-hyperplanes H_j' meeting along $\{0\} \times \mathbf{R}^{n-1}$, which, by Hypotheses 16.2(3), satisfy $\mathrm{dist}_{\mathcal{H}}(H_j' \cap B^{n+1}_1(0), H_j^{(0)} \cap B^{n+1}_1(0)) < \varepsilon_0$ for each $j \in \{1, 2, \ldots, m_0\}$. Otherwise we must have that $2\Theta(\|\mathbf{C}_0\|, 0) \geq m_0 + 1$ and that $\mathbf{C} \in \mathcal{K}(p)$ for some $p \in \{m_0 + 1, m_0 + 2, \ldots, 2\Theta(\|\mathbf{C}_0\|, 0)\}$. For each $j \in \{1, 2, \ldots, m_0\}$, let $q_j' \in \{1, 2, \ldots, q_j^{(0)}\}$ be the number of distinct elements in the set $\{H_{j,1}, H_{j,2}, \ldots, H_{j, q_j^{(0)}}\}$ and label them $H'_{j,\ell'}$, $\ell' = 1, 2, \ldots, q_j'$. Let $\mathbf{w}'_{j,\ell'} \in \mathbf{R}^2$ be the unit vector such that $H'_{j,\ell'} = \{(t\mathbf{w}'_{j,\ell'}, y) : t > 0, \ \ y \in \mathbf{R}^{n-1}\}$. Provided that $\beta_0 = \beta_0(\alpha, \gamma, \mathbf{C}_0) \in (0, 1/2)$ is sufficiently small, it follows from the definition of $\mathcal{K}$, Hypotheses 16.2(3) and Hypothesis (†)(ii) that for each $j \in \{1, 2, \ldots, m_0\}$ and $\ell_1', \ell_2' \in \{1, 2, \ldots, q_j'\}$,

$$|\mathbf{w}'_{j,\ell_1'} - \mathbf{w}'_{j,\ell_2'}| \geq c\, \mathcal{Q}'_V$$

for some constant $c = c(\alpha, \gamma, \mathbf{C}_0) \in (0, \infty)$, where

$$\mathcal{Q}'_V = \inf_{\widetilde{\mathbf{C}} \in \bigcup_{j=m_0}^{p-1} \mathcal{K}(j)} \left(\int_{B_1^{n+1}(0)} \operatorname{dist}^2(X, \operatorname{spt} \|\widetilde{\mathbf{C}}\|)\, d\|V\|(X) \right.$$
$$\left. + \int_{B_1^{n+1}(0) \setminus \{r(X) < 1/16\}} \operatorname{dist}^2(X, \operatorname{spt} \|V\|)\, d\|\widetilde{\mathbf{C}}\|(X) \right).$$

By exactly the same inductive proof of Theorem 10.1(a), conclusion (a) now follows from this provided $\varepsilon_0 = \varepsilon_0(n, \alpha, \gamma, \tau, \mathbf{C}_0), \beta_0 = \beta_0(n, \alpha, \gamma, \tau, \mathbf{C}_0) \in (0, 1/2)$ are sufficiently small.

The rest of the theorem is proved by arguing exactly as in [Sim93, Th. 3.4]; the key point that enables us to use the argument of [Sim93, Th. 3.4] is having at our disposal the appropriate regularity theorem, namely, Theorem 15.2. Specifically, letting

$$T_{\rho,\kappa}(\zeta) = \{(x, y) \in \mathbf{R}^2 \times \mathbf{R}^{n-1} : (|x| - \rho)^2 + |y - \zeta|^2 < \kappa^2 \rho^2/64\}$$

for $\kappa \in (0, 1]$, $\rho \in (0, 1/2)$ and $\zeta \in \mathbf{R}^{n-1}$, we have the following for any given $\beta \in (0, 1)$:

Claim. *There exists a constant* $\delta = \delta(n, q, \alpha, \gamma, \beta, \mathbf{C}_0) \in (0, 1/2)$ *such that if* V, $\mathbf{C}$ *are as in the theorem,* $(\xi, \zeta) \in \operatorname{spt} \|\mathbf{C}\| \cap B_{13/16}^{n+1}(0) \cap \{r(X) < 1/16\}$ *where* $\zeta \in \mathbf{R}^{n-1}$,

$$\operatorname{spt} \|V\| \cap T_{|\xi|, 1/16}(\zeta) \neq \emptyset \tag{16.9}$$

and

$$|\xi|^{-n-2} \int_{T_{|\xi|,1}(\zeta)} \operatorname{dist}^2(X, \operatorname{spt} \|\mathbf{C}\|)\, d\|V\|(X) < \delta, \tag{16.10}$$

then there exist distinct integers $j_1, j_2, \ldots, j_p \in \{1, 2, \ldots, m_0\}$ *and, for each* $k \in \{1, 2, \ldots, p\}$, *functions* $u_{j_k, k_\ell}^{(|\xi|, \zeta)} \in C^2(T_{|\xi|, 3/4}(\zeta) \cap H_{j_k, k_\ell}; H_{j_k, k_\ell}^{\perp})$ *with* $\ell = 1, 2, \ldots, n_k$ *for some* $n_k \leq q$ *such that*

$$V \llcorner T_{|\xi|, 1/2}(\zeta) = \sum_{k=1}^{p} \sum_{\ell=1}^{n_k} |\operatorname{graph} u_{j_k, k_\ell}^{(|\xi|, \zeta)} \cap T_{|\xi|, 1/2}(\zeta)| \tag{16.11}$$

and for each $k \in \{1, \ldots, p\}$, $\ell \in \{1, \ldots, n_k\}$,

$$|\xi|^{-1} \sup_{T_{|\xi|, 3/4}(\zeta) \cap H_{j_k, k_\ell}} |u_{j_k, k_\ell}^{(|\xi|, \zeta)}| + \sup_{T_{|\xi|, 3/4}(\zeta) \cap H_{j_k, k_\ell}} |\nabla\, u_{j_k, k_\ell}^{(|\xi|, \zeta)}| \leq \beta/2.$$

To verify this claim, observe first that by using the monotonicity of mass ratio and a covering argument, we see that under the hypotheses of the theorem, $\|V\|(B_1^{n+1}(0) \cap \{r(X) < \tau\}) \leq C\tau$ for each $\tau \in (0, 1/4)$, where $C = C(n,q) \in (0,\infty)$. Using this with sufficiently small $\tau = \tau(n,q) \in (0,1/2)$ and conclusion (a), we deduce that

$$\omega_n^{-1}\rho^{-n}\|V\|(B_\rho^{n+1}(Z)) \leq \omega_n^{-1}(16)^n\|V\|(B_{1/16}^{n+1}(Z)) < q + 3/4$$

for each $Z \in B_{13/16}^{n+1}(0)$ and $\rho \in (0, 1/16)$ provided $\varepsilon_0 = \varepsilon_0(n,q,\alpha,\gamma,\mathbf{C}_0) \in (0,1)$ is sufficiently small. Since (16.10) for sufficiently small $\delta = \delta(n,q,\alpha,\gamma,\mathbf{C}_0) \in (0,1/2)$ implies that $V \llcorner T_{|\xi|,7/8}(\zeta) = \sum_{j=1}^{m_0} V_j$ where $\operatorname{spt}\|V_j\| \subset N(H_j^{(0)}) \cap T_{|\xi|,7/8}(\zeta)$ (allowing for the possibility that $V_j = 0$ for some values of j), we see by applying Theorem 15.2 and Remark (3) at the end of Section 8 that (16.11) follows from (16.9) and (16.10) as claimed.

Now, as in Lemma 2.6 of [Sim93], let U be the union of all $T_{|\xi|,1/2}(\zeta) \cap \operatorname{spt}\|\mathbf{C}\|$ over all $(\xi,\zeta) \in \operatorname{spt}\|\mathbf{C}\| \cap B_{7/8}^{n+1}(0)$ such that for each $j \in \{1,\ldots,m_0\}$ and each $\ell \in \{1,\ldots,q_j^{(0)}\}$, there exists $u_{j,\ell}^{(|\xi|,\zeta)} \in C^2(T_{|\xi|,3/4}(\zeta) \cap H_{j,\ell}; H_{j,\ell}^\perp)$ with

$$|\xi|^{-1} \sup_{T_{|\xi|,3/4}(\zeta)\cap H_{j,\ell}} |u_{j,\ell}^{(|\xi|,\zeta)}| + \sup_{T_{|\xi|,3/4}(\zeta)\cap H_{j,\ell}} |\nabla u_{j,\ell}^{(|\xi|,\zeta)}| \leq \beta/2,$$

$$\operatorname{dist}(X + u_{j,\ell}^{(|\xi|,\zeta)}(X), \operatorname{spt}\|\mathbf{C}\|) = |u_{j,\ell}^{(|\xi|,\zeta)}(X)| \text{ for each } X \in T_{|\xi|,1/2}(\zeta) \cap H_{j,\ell}$$

and

$$V \llcorner T_{|\xi|,1/2}(\zeta) = \sum_{j=1}^{m_0}\sum_{\ell=1}^{q_j^{(0)}} |\operatorname{graph} u_{j,\ell}^{(|\xi|,\zeta)} \cap T_{|\xi|,1/2}(\zeta)|.$$

For each $j \in \{1,\ldots,m_0\}$, $\ell \in \{1,\ldots,q_j^{(0)}\}$, define $u_{j,\ell} \in C^2(U \cap H_{j,\ell}; H_{j,\ell}^\perp)$ by $u_{j,\ell}|_{T_{|\xi|,1/2}(\zeta)\cap H_{j,\ell}} = u_{j,\ell}^{(|\xi|,\zeta)}$. With the help of the above claim, we may now verify the validity of Lemma 2.6 of [Sim93] (by following the same proof), with the conclusion that for each $j \in \{1,\ldots,m_0\}$ and $\ell \in \{1,\ldots,q_j^{(0)}\}$,

$$H_{j,\ell} \cap B_{7/8}^{n+1}(0) \setminus \{r(X) < \tau\} \subset U;$$

there exists $u_{j,\ell} \in C^2(U \cap H_{j,\ell}; H_{j,\ell}^\perp)$ such that

$$\sup_{U\cap H_{j,\ell}} r^{-1}|u_{j,\ell}| + |\nabla u_{j,\ell}| \leq \beta;$$

and

$$\int_{B_{7/8}^{n+1}(0)\setminus G} r^2(X)\,d\|V\|(X) + \sum_{j=1}^{m_0}\sum_{\ell=1}^{q_j^{(0)}} \int_{U\cap H_{j,\ell}} r^2(X)|\nabla u_{j,\ell}(X)|^2\,d\mathcal{H}^n(X) \leq C\int_{B_1^{n+1}(0)} \operatorname{dist}^2(X, \operatorname{spt}\|\mathbf{C}\|)\,d\|V\|(X),$$

where $G = \bigcup_{j=1}^{m_0} \bigcup_{\ell=1}^{q_j^{(0)}} \mathrm{graph}\, u_{j,\ell}$ and $C = C(n, \alpha, \gamma, \mathbf{C}_0) \in (0, \infty)$. Consequently, the argument of Lemma 3.4 of [Sim93] carries over to give conclusions (b)–(d) of the present theorem. $\square$

Corollary 16.3. *Let q be an integer ≥ 2, $\alpha \in (0,1)$, $\gamma \in (0, 1/2)$ and $\mu \in (0,1)$. Suppose that the induction hypotheses* (H1), (H2) *hold, and let $\mathbf{C}_0$ be the stationary cone as in* (16.2), *with $\Theta(\|\mathbf{C}_0\|, 0) = q + 1/2$. For each $\rho \in (0, 1/4]$, there exist numbers $\varepsilon = \varepsilon(n, \alpha, \gamma, \tau, \rho, \mathbf{C}_0) \in (0, 1/2)$, $\beta = \beta(n, \alpha, \gamma, \tau, \rho, \mathbf{C}_0) \in (0, 1/2)$ such that if $V \in \mathcal{S}_\alpha$, $\mathbf{C} \in \mathcal{K}$ satisfy Hypotheses* 16.2 *and Hypothesis* (†), *then for each $Z = (\zeta^1, \zeta^2, \eta) \in \mathrm{spt}\, \|V\| \cap (B^{n+1}_{3/8}(0))$ with $\Theta(\|V\|, Z) \geq \Theta(\|\mathbf{C}_0\|, 0)$, we have the following*:

(a) $|\zeta^1|^2 + |\zeta^2|^2 \leq C \int_{B_1^{n+1}(0)} \mathrm{dist}^2(X, \mathrm{spt}\, \|\mathbf{C}\|)\, d\|V\|(X)$.

(b) $\displaystyle\int_{B^{n+1}_{\rho/2}(Z)} \frac{\mathrm{dist}^2(X, \mathrm{spt}\, \|T_{Z\,\#}\, \mathbf{C}\|)}{|X - Z|^{n+2-\mu}}\, d\|V\|(X)$
$\leq C_1 \rho^{-n-2+\mu} \displaystyle\int_{B^{n+1}_{\rho}(Z)} \mathrm{dist}^2(X, \mathrm{spt}\, \|T_{Z\,\#} \mathbf{C}\|)\, d\|V\|(X)$, *where* $T_Z : \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ *is the translation* $X \mapsto X + Z$.

Here $C = C(n, \alpha, \gamma, \mathbf{C}_0) \in (0, \infty)$ and $C_1 = C_1(n, \alpha, \gamma, \mu, \mathbf{C}_0) \in (0, \infty)$. (*In particular, C, C_1 do not depend on ρ.*)

Proof. The proof requires application of Theorem 16.2 with $\eta_{Z,\rho\,\#}\, V$ in place of V, where $Z \in \mathrm{spt}\, \|V\| \cap B^{n+1}_{3/8}(0)$ is any point such that $\Theta(\|V\|, Z) \geq \Theta(\|\mathbf{C}_0\|, 0)$.

It follows from Remark (4) above that whenever Hypotheses 16.2 are satisfied with $\varepsilon = \varepsilon(n, q, \alpha, \tau, \rho, \mathbf{C}_0)$ sufficiently small, they are also satisfied with $\eta_{Z,\rho\,\#}\, V$ in place of V and ε_0 (as in Theorem 16.2) in place of ε.

To verify that Hypothesis (†) is satisfied with $\eta_{Z,\rho\,\#}\, V$ in place of V and β_0 (as in Theorem 16.2) in place of β, and complete at the same time the proof of the corollary inductively, we may follow the steps of the proof of Corollary 10.2 (i.e., Lemmas 10.3, 10.4, 10.6 and Propositions 10.5, 10.7) in conjunction with the argument of Lemma 3.9 of [Sim93] (with modifications as in [Wic04]). $\square$

We shall need the following easy consequence of the preceding corollary for the proof of Theorem 16.1 at the end of this section:

Corollary 16.4. *Let q be an integer ≥ 2, $\alpha \in (0,1)$, $\gamma \in (0,1/2)$, $\varepsilon' \in (0, 1/2)$, and suppose that the induction hypotheses* (H1), (H2) *hold. Let $\mathbf{C}_0$ be the stationary cone as in* 16.2, *with $\Theta(\|\mathbf{C}_0\|, 0) = q + 1/2$. There exists a number $\varepsilon_1 = \varepsilon_1(n, \alpha, \gamma, \varepsilon', \mathbf{C}_0) \in (0, 1/2)$ such that if $V \in \mathcal{S}_\alpha$, $\mathbf{C} \in \mathcal{K}$ satisfy Hypotheses* 16.2 *with ε_1 in place of ε, then*

(a)
$$\int_{B_1^{n+1}(0)} \operatorname{dist}^2(X, \operatorname{spt}\|\mathbf{C}\|)\, d\|V\|(X) + \int_{B_1^{n+1}(0)\setminus\{r(X)<1/16\}} \operatorname{dist}^2(X, \operatorname{spt}\|V\|)\, d\|\mathbf{C}\|(X) < \varepsilon',$$

and for each $Z = (\zeta^1, \zeta^2, \eta) \in \operatorname{spt}\|V\| \cap (B_{3/8}^{n+1}(0))$ *with* $\Theta(\|V\|, Z) \geq \Theta(\|\mathbf{C}_0\|, 0)$, *we have that*

(b)
$$|\zeta^1|^2 + |\zeta^2|^2 \leq C\left(\int_{B_1^{n+1}(0)} \operatorname{dist}^2(X, \operatorname{spt}\|\mathbf{C}\|)\, d\|V\|(X) + \int_{B_1^{n+1}(0)\setminus\{r(X)<1/16\}} \operatorname{dist}^2(X, \operatorname{spt}\|V\|)\, d\|\mathbf{C}\|(X)\right),$$

(c)
$$\begin{aligned}&\int_{B_1^{n+1}(0)} \operatorname{dist}^2(X, \operatorname{spt}\|\mathbf{C}\|)\, d\|V^Z\|(X) + \int_{B_1^{n+1}(0)\setminus\{r(X)<1/16\}} \operatorname{dist}^2(X, \operatorname{spt}\|V^Z\|)\, d\|\mathbf{C}\|(X)\\ &\quad\leq C\left(\int_{B_1^{n+1}(0)} \operatorname{dist}^2(X, \operatorname{spt}\|\mathbf{C}\|)\, d\|V\|(X) + \int_{B_1^{n+1}(0)\setminus\{r(X)<1/64\}} \operatorname{dist}^2(X, \operatorname{spt}\|V\|)\, d\|\mathbf{C}\|(X)\right),\end{aligned}$$

where $V^Z = \eta_{Z,1/2\,\#}V$ *and* $C = C(n, \alpha, \gamma, \mathbf{C}_0) \in (0, \infty)$.

Proof. Conclusion (a) is easily seen by arguing by contradiction using Allard's integral varifold compactness theorem ([All72]; also [Sim83, §42.8]). Conclusion (b) in case $\mathbf{C} \in \mathcal{K}(m_0)$ follows directly from Corollary 16.3. So suppose that $\mathbf{C} \notin \mathcal{K}(m_0)$. Noting in this case that $2\Theta(\|\mathbf{C}_0\|, 0) \geq m_0 + 1$, fix $p \in \{m_0+1, m_0+2, \ldots, 2\Theta(\|\mathbf{C}_0\|, 0)\}$ and assume by induction that conclusion (b) of the corollary (taken with $\varepsilon' = 1/4$, say) holds whenever $\mathbf{C} \in \bigcup_{j=m_0}^{p-1}\mathcal{K}(j)$, with $\overline{\varepsilon}$ denoting the required value of ε_1. Choose a cone $\widetilde{\mathbf{C}}_1 \in \bigcup_{j=m_0}^{p-1}\mathcal{K}(j)$ such that

$$\begin{aligned}&\int_{B_1^{n+1}(0)} \operatorname{dist}^2(X, \operatorname{spt}\|\widetilde{\mathbf{C}}_1\|)\, d\|V\|(X) + \int_{B_1^{n+1}(0)\setminus\{r(X)<1/16\}} \operatorname{dist}^2(X, \operatorname{spt}\|V\|)\, d\|\widetilde{\mathbf{C}}_1\|(X)\\ &\quad\leq \frac{3}{2}\inf_{\widetilde{\mathbf{C}}\in\bigcup_{j=m_0}^{p-1}\mathcal{K}(j)}\left(\int_{B_1^{n+1}(0)} \operatorname{dist}^2(X, \operatorname{spt}\|\widetilde{\mathbf{C}}\|)\, d\|V\|(X) + \int_{B_1^{n+1}(0)\setminus\{r(X)<1/16\}} \operatorname{dist}^2(X, \operatorname{spt}\|V\|)\, d\|\widetilde{\mathbf{C}}\|(X)\right),\end{aligned}$$

and let $\beta_1 = \frac{2}{3}\beta(n, \alpha, \gamma, 1/4, 1/4, \mathbf{C}_0)$ where β is as in Corollary 16.3. Suppose $\mathbf{C} \in \mathcal{K}(p)$ and that Hypotheses 16.2 hold with the value of ε equal to

$\varepsilon(n, \alpha, \gamma, 1/4, 1/4, \mathbf{C}_0)$ where $\varepsilon(n, \alpha, \gamma, \cdot, \cdot, \mathbf{C}_0)$ is as in Corollary 16.3. If

$$\begin{aligned}
&\int_{B_1^{n+1}(0)} \operatorname{dist}^2(X, \operatorname{spt} \|\mathbf{C}\|)\, d\|V\|(X) \\
&\qquad + \int_{B_1^{n+1}(0)\setminus\{r(X)<1/16\}} \operatorname{dist}^2(X, \operatorname{spt} \|V\|)\, d\|\mathbf{C}\|(X) \\
&\quad \leq \ \beta_1 \left(\int_{B_1^{n+1}(0)} \operatorname{dist}^2(X, \operatorname{spt} \|\widetilde{\mathbf{C}}_1\|)\, d\|V\|(X) \right. \\
&\qquad\qquad \left. + \int_{B_1^{n+1}(0)\setminus\{r(X)<1/16\}} \operatorname{dist}^2(X, \operatorname{spt} \|V\|)\, d\|\widetilde{\mathbf{C}}_1\|(X) \right),
\end{aligned}$$

then conclusion (b) follows directly from Corollary 16.3. On the other hand, if the reverse inequality holds, then by taking $\varepsilon' = \varepsilon'(n, \alpha, \gamma, \mathbf{C}_0)$ sufficiently small in conclusion (a), we can ensure that Hypotheses 16.2 are satisfied with $\widetilde{\mathbf{C}}_1$ in place of $\mathbf{C}$ and $\overline{\varepsilon}$ in place of ε, so conclusion (b) in this case follows by the induction hypothesis. Thus conclusion (b) holds whenever $\mathbf{C} \in \mathcal{K}(p)$, and since $\mathbf{C} \in \mathcal{K} \implies \mathbf{C} \in \mathcal{K}(j)$ for some $j \in \{m_0, \ldots, 2\Theta(\|\mathbf{C}_0\|, 0)\}$, the inductive proof of conclusion (b) is complete.

To see conclusion (c), note that

$$\begin{aligned}
&\int_{B_1^{n+1}(0)} \operatorname{dist}^2(X, \operatorname{spt} \|\mathbf{C}\|)\, d\|V^Z\|(X) \\
&\quad = 2^{n+2} \int_{B_{1/2}^{n+1}(Z)} \operatorname{dist}^2(X, T_Z \operatorname{spt} \|\mathbf{C}\|)\, d\|V\|(X) \\
&\quad \leq 2^{n+2} \int_{B_1^{n+1}(0)} \operatorname{dist}^2(X, \operatorname{spt} \|\mathbf{C}\|)\, d\|V\|(X) + C\left(|\zeta^1|^2 + |\zeta^2|^2\right)
\end{aligned}$$

and

$$\begin{aligned}
&\int_{B_1^{n+1}(0)\setminus\{r(X)<1/16\}} \operatorname{dist}^2(X, \operatorname{spt} \|V^Z\|)\, d\|\mathbf{C}\|(X) \\
&\quad = 2^{n+2} \int_{B_{1/2}^{n+1}(Z)\setminus\{r(X-Z)<1/32\}} \operatorname{dist}^2(X, \operatorname{spt} \|V\|)\, d\|T_{Z\,\#}\, \mathbf{C}\|(X) \\
&\quad \leq 2^{n+2} \int_{B_{1/2}^{n+1}(Z)\setminus\{r(X)<1/64\}} \operatorname{dist}^2(X, \operatorname{spt} \|V\|)\, d\|T_{Z\,\#}\, \mathbf{C}\|(X) \\
&\quad \leq 2^{n+2} \int_{B_1^{n+1}(0)\setminus\{r(X)<1/64\}} \operatorname{dist}^2(X, \operatorname{spt} \|V\|) d\|\mathbf{C}\|(X) + C\left(|\zeta^1|^2 + |\zeta^2|^2\right),
\end{aligned}$$

where $C = C(n, q) \in (0, \infty)$, $T_Z : \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ is the translation $X \mapsto X + Z$ and we have used the fact that $\mathbf{C}$ is translation invariant along $\{0\} \times \mathbf{R}^{n-1}$ and assumed that $\varepsilon = \varepsilon(n, \alpha, \gamma, \mathbf{C}_0)$ is sufficiently small to ensure that $\operatorname{dist}(Z, \{0\} \times \mathbf{R}^{n-1}) < 1/64$. In view of conclusion (b), the validity of conclusion (c) readily follows from these two inequalities. $\square$

LEMMA 16.5. *Let q be an integer ≥ 2, $\alpha \in (0,1)$, $\delta \in (0,1/8)$, $\gamma \in (0,1/2)$ and $\mathbf{C}_0$ be as above. Suppose that the induction hypotheses* (H1), (H2) *hold and that $\Theta(\|\mathbf{C}_0\|, 0) = q + 1/2$. There exist numbers $\varepsilon_1 = \varepsilon_1(n, \alpha, \gamma, \delta, \mathbf{C}_0) \in (0, 1/2)$ and $\beta_1 = \beta_1(n, \alpha, \gamma, \mathbf{C}_0) \in (0, 1/2)$ such that if $V \in \mathcal{S}_\alpha$, $\mathbf{C} \in \mathcal{K}$ satisfy Hypotheses* 16.2 *with ε_1 in place of ε, then*

(a) $B_\delta^{n+1}(0, y) \cap \{Z : \Theta(\|V\|, Z) \geq q + 1/2\} \neq \emptyset$ *for each point* $(0, y) \in \{0\} \times \mathbf{R}^{n-1} \cap B_{1/2}^{n+1}(0)$, *and*

(b) *if additionally Hypothesis* (†) *holds with β_1 in place of β and if $\mu \in (0, 1)$, then*

$$\int_{B_{1/2}^{n+1}(0) \cap \{r(X) < \sigma\}} \operatorname{dist}^2(X, \operatorname{spt} \|\mathbf{C}\|) \, d\|V\|(X) \leq C_1 \sigma^{1-\mu} \int_{B_1^{n+1}(0)} \operatorname{dist}^2(X, \operatorname{spt} \|\mathbf{C}\|) \, d\|V\|(X)$$

for each $\sigma \in [\delta, 1/4)$, where $C_1 = C_1(n, q, \alpha, \mu, \mathbf{C}_0) \in (0, \infty)$. (*In particular, C_1 is independent of δ and σ.*)

Proof. Suppose for some number $\delta \in (0, 1/8)$ and some point $(0, y) \in \{0\} \times \mathbf{R}^{n-1} \cap B_{1/2}^{n+1}(0)$ that $B_\delta^{n+1}(0, y) \cap \{Z : \Theta(\|V\|, Z) \geq q + 1/2\} = \emptyset$. Then by Remark (3) following Hypotheses 16.2, it follows that

$$\mathcal{H}^{n-7+\gamma}(\operatorname{sing} V \llcorner (B_\delta^{n+1}(0, y))) = 0 \quad \text{if } n \geq 7, \qquad \operatorname{sing} V \llcorner (B_\delta^{n+1}(0, y)) = \emptyset \quad \text{if } 2 \leq n \leq 6. \tag{16.4}$$

From this and hypothesis ($\mathcal{S}2$) we deduce (with the help of an elementary covering argument in case $n \geq 7$) that

$$\int_{\operatorname{spt} \|V\| \cap B_\delta^{n+1}(0, y)} |A|^2 \zeta^2 \, d\mathcal{H}^n \leq \int_{\operatorname{spt} \|V\| \cap B_\delta^{n+1}(0, y)} |\nabla^V \zeta|^2 \, d\mathcal{H}^n$$

for any $\zeta \in C_c^1(B_\delta^{n+1}(0, y))$, where A denotes the second fundamental form of $\operatorname{reg} V$. Choosing $\zeta \in C_c^1(B_\delta^{n+1}(0, y))$ such that $\zeta \equiv 1$ in $B_{\delta/2}^{n+1}(0, y)$ and $|D\zeta| \leq 4\delta^{-1}$, we conclude from the preceding inequality that

$$\int_{\operatorname{spt} \|V\| \cap B_{\delta/2}^{n+1}(0, y)} |A|^2 \, d\mathcal{H}^n \leq C \delta^{n-2}, \tag{16.12}$$

where $C = C(n, q) \in (0, \infty)$. Now let $\tau \in (0, \delta/4)$ be arbitrary for the moment and assume that $\varepsilon \in (0, \varepsilon_0)$, where $\varepsilon_0 = \varepsilon_0(\alpha, \beta, \tau, \mathbf{C}_0)$ is as in Theorem 16.2. Using Theorem 16.2(a), (16.12) and the argument leading to the inequality (6.12) of [SS81] (with $\sigma = \tau$), we deduce, provided $\varepsilon = \varepsilon(\alpha, \beta, \tau, \mathbf{C}_0)$ is sufficiently small and positive, that

$$C \leq \tau^{1/2} \delta^{-1/2},$$

where $C = C(\beta, \mathbf{C}_0) \in (0, \infty)$. This however is a contradiction if we choose, e.g., $\tau = \frac{\delta}{8}$ if $4C^2 \geq 1$ or $\tau = \frac{C^2\delta}{2}$ if $4C^2 < 1$. We conclude that part (a) must hold provided $\varepsilon = \varepsilon(\alpha, \beta, \delta, \mathbf{C}_0) \in (0, 1/2)$ is sufficiently small. To prove the estimate of part (b), first note that in view of Corollary 16.3(a) and (b) (with $\tau = 1/16$, say), it follows from the argument leading to the estimate (3) of [Sim93, p. 619] that for each $Z \in \operatorname{spt} \|V\| \cap B^{n+1}_{3/8}(0)$ with $\Theta(\|V\|, Z) \geq q$,

$$\int_{B^{n+1}_{1/4}(Z)} \frac{\operatorname{dist}^2(X, \operatorname{spt} \|\mathbf{C}\|)}{|X - Z|^{n-\alpha}} \, d\|V\|(X) \leq C \int_{B^{n+1}_1(0)} \operatorname{dist}^2(X, \operatorname{spt} \|\mathbf{C}\|) \, d\|V\|(X),$$

where $C = C(\beta, \alpha, \mathbf{C}_0) \in (0, \infty)$. By the argument of [Sim93, Cor. 3.2(ii)] (cf. proof of Lemma 10.8(b)), the required estimate follows from this and part (a). □

Remark. Note that Theorem 16.2, Corollary 16.3, Corollary 16.4 and Lemma 16.5 all continue to hold in case $\Theta(\|\mathbf{C}_0\|, 0) = q + 1$ provided that Theorem 15.2 holds with $q + 1$ in place of q.

Let $\gamma \in (0, 1/2)$, and consider a sequence of varifolds $\{V_k\} \subset \mathcal{S}_\alpha$ and a sequence of cones $\{\mathbf{C}_k\}$ satisfying, for each $k = 1, 2, \ldots$, Hypotheses 16.2 and Hypothesis (†) with V_k, $\mathbf{C}_k$ in place of V, $\mathbf{C}$ and ε_k, β_k in place of ε, β, where $\varepsilon_k, \beta_k \to 0^+$. Thus, for each $k = 1, 2, \ldots$, we suppose that

(1_k) $V_k \in \mathcal{S}_\alpha$, $0 \in \operatorname{spt} \|V_k\|$, $\Theta(\|V_k\|, 0) \geq q + 1/2$, $(\omega_n 2^n)^{-1} \|V_k\|(B^{n+1}_2(0)) < q + 1/2 + \gamma$;

(2_k) $\mathbf{C}_k = \sum_{j=1}^{m_0} \sum_{\ell=1}^{q_j^{(0)}} |H^k_{j,\ell}| \in \mathcal{K}$, where $H^k_{j,\ell}$ are half-hyperplanes of $\mathbf{R}^{n+1}$ meeting along $\{0\} \times \mathbf{R}^{n-1}$ with $H^k_{j,\ell} \in N(H_j^{(0)})$ for each $j \in \{1, 2, \ldots, m_0\}$ and $\ell \in \{1, 2, \ldots, q_j^{(0)}\}$;

(3_k) $\operatorname{dist}_{\mathcal{H}}(\operatorname{spt} \|\mathbf{C}_k\| \cap B^{n+1}_1(0), \operatorname{spt} \|\mathbf{C}_0\| \cap B^{n+1}_1(0)) < \varepsilon_k$;

(4_k) $\displaystyle\int_{B^{n+1}_1(0)} \operatorname{dist}^2(X, \operatorname{spt} \|\mathbf{C}_k\|) \, d\|V_k\|(X) < \varepsilon_k$;

(5_k) for each $j = 1, 2, \ldots, m_0$,

$$\begin{aligned} &\|V_k\|((B^{n+1}_{1/2}(0) \setminus \{r(X) < 1/8\}) \cap N(H_j^{(0})) \\ &\qquad \geq \left(q_j^{(0)} - \frac{1}{4}\right) \mathcal{H}^n((B^{n+1}_{1/2}(0) \setminus \{r(X) < 1/8\}) \cap H_j^{(0)}); \end{aligned}$$

(6_k) either (i) or (ii) below holds:
 (i) $\mathbf{C}_k \in \mathcal{K}(m_0)$;
 (ii) $2\Theta(\|\mathbf{C}_0\|, 0) \geq m_0 + 1$, $\mathbf{C}_k \in \mathcal{K}(p_k)$ for some $p_k \in \{m_0 + 1, m_0 + 2, \ldots, 2\Theta(\|\mathbf{C}_0\|, 0)\}$ and

$$\begin{aligned}&\int_{B_1^{n+1}(0)} \operatorname{dist}^2(X, \operatorname{spt}\|\mathbf{C}_k\|)\, d\|V_k\|(X)\\ &\qquad + \int_{B_1^{n+1}(0)\setminus\{r(X)<1/16\}} \operatorname{dist}^2(X, \operatorname{spt}\|V_k\|)\, d\|\mathbf{C}_k\|(X)\\ &\quad \leq \beta_k \inf_{\widetilde{\mathbf{C}}\in\bigcup_{j=m_0}^{p_k-1}\mathcal{K}(j)} \left(\int_{B_1^{n+1}(0)} \operatorname{dist}^2(X, \operatorname{spt}\|\widetilde{\mathbf{C}}\|)\, d\|V_k\|(X)\right.\\ &\qquad\qquad\qquad\left. + \int_{B_1^{n+1}(0)\setminus\{r(X)<1/16\}} \operatorname{dist}^2(X, \operatorname{spt}\|V_k\|)\, d\|\widetilde{\mathbf{C}}\|(X)\right).\end{aligned}$$

Note that it follows from (2_k) and (3_k) that $H_{j,\ell}^k \to H_j^{(0)}$ for each $j\in\{1,\ldots,m_0\}$ and $\ell \in \{1,\ldots,q_j^{(0)}\}$.

Let $\mathcal{E}_k = \sqrt{\int_{B_1^{n+1}(0)} \operatorname{dist}^2(X, \operatorname{spt}\|\mathbf{C}_k\|)\, d\|V_k\|(X)}$. Let $\{\delta_k\}, \{\tau_k\}$ be sequences of decreasing positive numbers converging to 0. By passing to appropriate subsequences of $\{V_k\}$, $\{\mathbf{C}_k\}$ without changing notation, we have the following:

(A_k) By Lemma 16.5,

$$B_{\delta_k}^{n+1}(0,y) \cap \{Z : \Theta(\|V_k\|, Z) \geq q + 1/2\} \neq \emptyset \tag{16.13}$$

for each point $(0,y) \in \{0\}\times\mathbf{R}^{n-1}\cap B_{1/2}^{n+1}(0)$ and

$$\int_{B_{1/2}^{n+1}(0)\cap\{r(X)<\sigma\}} \operatorname{dist}^2(X, \operatorname{spt}\|\mathbf{C}_k\|)\, d\|V_k\|(X) \leq C\sigma^{1/2}\mathcal{E}_k^2 \tag{16.14}$$

for each $\sigma \in [\delta_k, 1/4)$, where $C = C(n, q, \alpha, \gamma, \mathbf{C}_0) \in (0,\infty)$.

(B_k) By Theorem 16.2 (a),

$$V_k \llcorner (B_{7/8}^{n+1}(0)\setminus\{r(X)<\tau_k\}) = \sum_{j=1}^{m_0}\sum_{\ell=1}^{q_j^{(0)}} |\operatorname{graph} u_{j,\ell}^k| \tag{16.15}$$

where, for each $k = 1, 2, \ldots,$ $j \in \{1, 2, \ldots, m_0\}$ and $\ell \in \{1, 2, \ldots, q_j^{(0)}\}$, $u_{j,\ell}^k \in C^2(B_{7/8}^{n+1}(0)\cap H_{j,\ell}^k\setminus\{r(X)<\tau_k\}; (H_{j,\ell}^k)^\perp)$, $u_{j,\ell}^k$ solves the minimal surface equation on $B_{7/8}^{n+1}(0)\cap H_{j,\ell}^k\setminus\{r(X)<\tau_k\}$ and satisfies $\operatorname{dist}(X + u_{j,\ell}^k(X), \operatorname{spt}\|\mathbf{C}\|) = |u_{j,\ell}^k(X)|$ for $X \in B_{7/8}^{n+1}(0)\cap H_{j,\ell}^k\setminus\{r(X)<\tau_k\}$.

(C_k) For each point $Z = (\zeta^1, \zeta^2, \eta) \in \operatorname{spt}\|V_k\|\cap B_{3/8}^{n+1}(0)$ with $\Theta(\|V_k\|, Z) \geq q + 1/2$, by Corollary 16.3 (a),

$$|\zeta^1|^2 + |\zeta^2|^2 \leq C\mathcal{E}_k^2, \tag{16.16}$$

where $C = C(n, q, \alpha, \gamma, \mathbf{C}_0) \in (0,\infty)$.

(D$_k$) For each fixed $\mu \in (0,1)$, $\rho \in (0,1/4)$, each sufficiently large k and each point $Z = (\zeta^1, \zeta^2, \eta) \in \operatorname{spt}\|V_k\| \cap B^{n+1}_{3/8}(0)$ with $\Theta(\|V_k\|, Z) \geq q + 1/2$, by Corollary 16.3(b),

$$\sum_{j=1}^{m_0}\sum_{\ell=1}^{q_j^{(0)}} \int_{B^{n+1}_{1/4}(Z)\cap H^k_{j,\ell}\setminus\{r(X)<\tau_k\}} \frac{|u^k_{j,\ell}(X) - (\zeta^1,\zeta^2,0)^{\perp_{H^k_j}}|^2}{|X + u^k_{j,\ell}(X) - Z|^{n+2-\mu}}\, dX \tag{16.17}$$
$$\leq C_1\rho^{-n-2+\mu}\int_{B^{n+1}_\rho(Z)} \operatorname{dist}^2(X, \operatorname{spt}\|T_{Z\,\#}\,\mathbf{C}_k\|)\, d\|V_k\|(X),$$

where $C_1 = C_1(\alpha,\gamma,\mu,\mathbf{C}_0) \in (0,\infty)$.

Extend $u^k_{j,\ell}$ to all of $B^{n+1}_{7/8}(0)\cap H^k_{j,\ell}$ by defining values to be zero in $B^{n+1}_{7/8}(0)\cap H^k_{j,\ell}\cap\{r(X)<\tau_k\}$. For each $j \in \{1,2,\ldots,m_0\}$ and $\ell \in \{1,2,\ldots,q_j^{(0)}\}$, let $h_{j,\ell} : H_j^{(0)} \to (H_j^{(0)})^\perp$ be the linear functions such that $\{X + h_{j,\ell}(X) : X \in H_j^{(0)}\} = H_{j,\ell}$ and let $\tilde{u}^k_{j,\ell}(X) = u^k_{j,\ell}(X + h_{j,\ell}(X))$. By (16.15) and elliptic estimates, for each $j = 1,2,\ldots,m_0$ and $\ell = 1,2,\ldots,q_j^{(0)}$ (for any manner in which the labelling is chosen for the elements of the sets $\{u^k_{j,1}, u^k_{j,2},\ldots,u^k_{j,q_j^{(0)}}\}$, $k = 1,2,3,\ldots$), there exist harmonic functions $v_{j,\ell} : B_{3/4}\cap H_j^{(0)} \to (H_j^{(0)})^\perp$ such that, after passing to a subsequence,

$$\mathcal{E}_k^{-1}\tilde{u}^k_{j,\ell} \to v_{j,\ell}, \tag{16.18}$$

where the convergence is in $C^2(K)$ for each compact subset K of $B_{3/4}\cap H_j^{(0)}$. From (16.14), it follows that for each $\sigma \in (0,1/4)$,

$$\sum_{j=1}^{m_0}\sum_{\ell=1}^{q_j^{(0)}} \int_{B^{n+1}_{3/4}(0)\cap H_j^{(0)}\cap\{r(X)<\sigma\}} |v_{j,\ell}|^2 \leq C\sigma^{1/2}, \quad C = C(\alpha,\gamma,\mathbf{C}_0)$$

and hence that the convergence in (16.18) is also in $L^2\,(B_{3/4}\cap H_j^{(0)})$.

Lemma 16.6. *For each $j \in \{1,2,\ldots,m_0\}$ and $\ell \in \{1,2,\ldots,q_j^{(0)}\}$, we have that*

$$v_{j,\ell} \in C^{0,\mu}\left(\overline{B^{n+1}_{5/16}(0)\cap H_j^{(0)}};\left(H_j^{(0)}\right)^\perp\right)$$

with the estimate

$$\sup_{B^{n+1}_{5/16}(0)\cap H_j^{(0)}} |v_{j,\ell}|^2 + \sup_{X_1,X_2\in B^{n+1}_{5/16}(0)\cap H_j^{(0)},\, X_1\neq X_2} \frac{|v_{j,\ell}(X_1) - v_{j,\ell}(X_2)|^2}{|X_1 - X_2|^{2\mu}}$$
$$\leq C\sum_{j=1}^{m_0}\sum_{\ell=1}^{q_j^{(0)}}\int_{B^{n+1}_{3/4}(0)\cap H_j^{(0)}} |v_{j,\ell}|^2,$$

where $\mu = \mu(n,q,\alpha,\gamma,\mathbf{C}_0) \in (0,1)$ and $C = C(n,q,\alpha,\gamma,\mathbf{C}_0) \in (0,\infty)$.

Proof. Note first that for each given $Y \in B^{n+1}_{5/16}(0) \cap \{0\} \times \mathbf{R}^{n-1}$, there exists, by (16.13), a sequence of points $Z_k = (\zeta_1^k, \zeta_2^k, \eta_k) \in \operatorname{spt} \|V_k\| \cap B^{n+1}_{3/4}(0)$ with $\Theta(\|V_k\|, Z_k) \geq q$ such that $Z_k \to Y$. Passing to a subsequence without changing notation, the limits $\lim_{k\to\infty} \mathcal{E}_k^{-1}\zeta_1^k$ and $\lim_{k\to\infty} \mathcal{E}_k^{-1}\zeta_2^k$ exist by (16.16). Write

$$\kappa(Y) = \left(\lim_{k\to\infty} \mathcal{E}_k^{-1}\zeta_1^k, \lim_{k\to\infty} \mathcal{E}_k^{-1}\zeta_2^k, 0\right), \tag{16.19}$$

and note by (16.16) that

$$|\kappa(Y)| \leq C, \tag{16.20}$$

where $C = C(n, q, \alpha, \gamma, \mathbf{C}_0) \in (0, \infty)$. It follows from (16.14), (16.15) and (16.17) that for each $\mu \in (0, 1)$,

$$\begin{aligned}\sum_{j=1}^{m_0}\sum_{\ell=1}^{q_j^{(0)}} \int_{B^{n+1}_{1/4}(Y)\cap H_j^{(0)}} \frac{|v_{j,\ell}(X) - \kappa(Y)^{\perp_{H_j^{(0)}}}|^2}{|X - Y|^{n+2-\mu}}\, dX \\ \leq C_1 \rho^{-n-2+\mu} \sum_{j=1}^{m_0}\sum_{\ell=1}^{q_j^{(0)}} \int_{B^{n+1}_{\rho}(Y)\cap H_j^{(0)}} |v_{j,\ell} - \kappa(Y)^{\perp_{H_j^{(0)}}}|^2\end{aligned} \tag{16.21}$$

for $\rho \in (0, 1/8]$, where $C_1 = C_1(\alpha, \gamma, \mu, \mathbf{C}_0) \in (0, \infty)$. In view of (16.20), this in particular implies that for each $j = 1, 2, \ldots, m_0$, $\kappa(Y)^{\perp_{H_j^{(0)}}}$ is uniquely defined (depending only on Y and independent of the sequence $\{Z_k\}$ tending to Y), and hence, since the set of normal directions to $H_j^{(0)}$, $j = 1, 2, \ldots, m_0$, spans $\mathbf{R}^2 \times \{0\}$, the vector $\kappa(Y)$ is also uniquely defined. For $Y \in B^{n+1}_{1/4}(0) \cap \{0\} \times \mathbf{R}^{n-1}$, $j \in \{1, 2, \ldots, m_0\}$ and $\ell \in \{1, 2, \ldots, q_j^{(0)}\}$, define $v_{j,\ell}(Y) = \kappa(Y)^{\perp_{H_j^{(0)}}}$. The proof of the lemma can now be completed by modifying the proof of Lemma 12.1 in an obvious way. □

THEOREM 16.7. *For each $j \in \{1, 2, \ldots, m_0\}$, $\ell \in \{1, 2, \ldots, q_j^{(0)}\}$, we have that*

$$v_{j,\ell} \in C^2\left(\overline{B^{n+1}_{1/4}(0) \cap H_j^{(0)}}; \left(H_j^{(0)}\right)^{\perp}\right)$$

with the estimate

$$\begin{aligned}\sup_{B^{n+1}_{1/4}(0)\cap H_j^{(0)}} |Dv_{j,\ell}|^2 + \sup_{X_1, X_2 \in B^{n+1}_{1/4}(0)\cap H_j^{(0)},\, X_1 \neq X_2} \frac{|Dv_{j,\ell}(X_1) - Dv_{j,\ell}(X_2)|^2}{|X_1 - X_2|^2} \\ \leq C \sum_{j=1}^{m_0}\sum_{\ell=1}^{q_j^{(0)}} \int_{B^{n+1}_{3/4}(0)\cap H_j^{(0)}} |v_{j,\ell}|^2,\end{aligned}$$

where $C = C(n, q, \alpha, \gamma, \mathbf{C}_0) \in (0, \infty)$.

Proof. For $Y \in B^{n+1}_{1/2}(0) \cap \{0\} \times \mathbf{R}^{n-1}$, let $\widetilde{\kappa}(Y) = \sum_{j=1}^{m_0} \kappa(Y)^{\perp_{H_j^{(0)}}}$ where κ is the function defined by (16.19). By modifying the argument leading to the estimate (12.30) in obvious ways, it can be seen that $\widetilde{\kappa} \in C^\infty(B^{n+1}_{1/2}(0) \cap \{0\} \times \mathbf{R}^{n-1}; \mathbf{R}^{n+1})$ with

$$\sup_{B^{n+1}_{1/2}(0)\cap(\{0\}\times\mathbf{R}^{n-1})} |\widetilde{\kappa}|^2 + |D_Y\,\widetilde{\kappa}|^2 + |D_Y^2\,\widetilde{\kappa}|^2 + |D_Y^3\,\widetilde{\kappa}|^2$$

$$\leq C\sum_{j=1}^{m_0}\sum_{\ell=1}^{q_j^{(0)}}\int_{B^{n+1}_{3/4}(0)\cap H_j^{(0)}} |v_{j,\ell}|^2,$$

where $C = C(\alpha,\gamma,\mathbf{C}_0) \in (0,\infty)$. Since the set of normal directions to $H_j^{(0)}$, $j = 1, 2, \ldots, m_0$, span $\mathbf{R}^2 \times \{0\}$, it follows that for each $j = 1, 2, \ldots, m_0$, $\kappa^{\perp_{H_j^{(0)}}} \in C^\infty(B^{n+1}_{1/2}(0) \cap \{0\} \times \mathbf{R}^{n-1}; \mathbf{R}^{n+1})$ with

$$\sup_{B^{n+1}_{1/2}(0)\cap(\{0\}\times\mathbf{R}^{n-1})} |\kappa^{\perp_{H_j^{(0)}}}|^2 + |D_Y\,\kappa^{\perp_{H_j^{(0)}}}|^2 + |D_Y^2\,\kappa^{\perp_{H_j^{(0)}}}|^2 \tag{16.22}$$

$$+ |D_Y^3\,\kappa^{\perp_{H_j^{(0)}}}|^2 \leq C\sum_{j=1}^{m_0}\sum_{\ell=1}^{q_j^{(0)}}\int_{B^{n+1}_{3/4}(0)\cap H_j^{(0)}} |v_{j,\ell}|^2,$$

where $C = C(\alpha,\gamma,\mathbf{C}_0) \in (0,\infty)$. Since by Lemma 16.6, for each $j = 1, 2, \ldots, m_0$ and $\ell = 1, 2, \ldots, q_j^{(0)}$, $v_{j,\ell}$ is continuous in $\overline{B^{n+1}_{1/2}(0) \cap H_j^{(0)}}$ with boundary values $v_{j,\ell}|_{B^{n+1}_{1/2}(0)\cap\{0\}\times\mathbf{R}^{n-1}} \equiv \kappa^{\perp_{H_j^{(0)}}}$, and $v_{j,\ell}$ is harmonic in $B^{n+1}_{3/4}(0) \cap H_j^{(0)}$, the desired conclusions of the lemma follow, in view of the estimate (16.22), from the standard boundary regularity theory for harmonic functions. □

Lemma 16.8. *Let q be an integer ≥ 2, $\alpha \in (0,1)$, $\gamma \in (0,1/2)$ and $\theta \in (0,1/4)$. Let $\mathbf{C}_0$ be the stationary cone as in* (16.2), *with $\Theta(\|\mathbf{C}_0\|, 0) = q + 1/2$. There exist numbers $\overline{\varepsilon} = \overline{\varepsilon}(\alpha,\gamma,\theta,\mathbf{C}_0) \in (0,1/2)$ and $\overline{\beta} = \overline{\beta}(\alpha,\gamma,\theta,\mathbf{C}_0) \in (0,1/2)$ such that if $V \in \mathcal{S}_\alpha$, $\mathbf{C} \in \mathcal{K}$ satisfy Hypotheses* 16.2 *and Hypothesis* (†) *with $\varepsilon = \overline{\varepsilon}$ and $\beta = \overline{\beta}$ and if the induction hypotheses* (H1), (H2) *hold, then there exist an orthogonal rotation Γ of $\mathbf{R}^{n+1}$ and a cone $\mathbf{C}' \in \mathcal{K}$ such that, with*

$$\mathcal{E}_V^2 = \int_{B_1^{n+1}(0)} \operatorname{dist}^2(X, \operatorname{spt}\|\mathbf{C})\|)\, d\|V\|(X),$$

the following hold:

(a) $\quad |e_j - \Gamma(e_j)| \leq \overline{\kappa}\mathcal{E}_V, \quad$ for $j = 1, 2, 3, \ldots, n+1$;

(b) $\quad \operatorname{dist}^2_{\mathcal{H}}(\operatorname{spt}\|\mathbf{C}'\| \cap B_1^{n+1}(0), \operatorname{spt}\|\mathbf{C}\| \cap B_1^{n+1}(0)) \leq \overline{C}_0\mathcal{E}_V^2$;

(c) $$\theta^{-n-2}\int_{\Gamma\left(B_\theta^{n+1}(0)\setminus\{r(X)\le\theta/16\}\right)}\operatorname{dist}^2(X,\operatorname{spt}\|V\|)\,d\|\Gamma_\#\,\mathbf{C}'\|(X) + \theta^{-n-2}\int_{B_\theta^{n+1}(0)}\operatorname{dist}^2(X,\operatorname{spt}\|\Gamma_\#\,\mathbf{C}'\|)\,d\|V\|(X)\le\overline{\nu}\theta^2\mathcal{E}_V^2.$$

Here the constants $\overline{\kappa},\overline{C}_0,\overline{\nu}\in(0,\infty)$ *each depends only on* α, γ *and* $\mathbf{C}_0$.

Proof. If the lemma is false, there exist a sequence of varifolds $\{V_k\}\subset\mathcal{S}_\alpha$ and a sequence of cones $\{\mathbf{C}_k\}\subset\mathcal{K}$ satisfying, for each $k=1,2,3,\ldots$, the conditions (1_k)–(6_k) above (listed immediately after the proof of Lemma 16.5) but not satisfying, for any choice of orthogonal rotation Γ of $\mathbf{R}^{n+1}$ and $\mathbf{C}'\in\mathcal{K}$, the conclusion of the lemma taken with V_k, $\mathbf{C}_k$ in place of V, $\mathbf{C}$. Choose any two sequences of decreasing positive numbers $\{\delta_k\}$ and $\{\tau_k\}$ with $\delta_k\to 0$ and $\tau_k\to 0$ and corresponding subsequences of $\{V_k\}$, $\{\mathbf{C}_k\}$ for which the assertions (16.13)–(16.17) are valid, and let $\{v_{j,\ell}\}_{j=1,2,\ldots,m_0;\ \ell=1,2,\ldots,q_j^{(0)}}$ be the blow-up of $\{V_k\}$ relative to $\{\mathbf{C}_k\}$. Thus, for each $j=1,2,\ldots,m_0$ and $\ell=1,2,\ldots,q_j^{(0)}$,

$$v_{j,\ell}\in L^2\left(B_{3/4}^{n+1}(0)\cap H_j^{(0)};\left(H_j^{(0)}\right)^\perp\right)\cap C^2\left(B_{3/4}^{n+1}(0)\cap H_j^{(0)};\left(H_j^{(0)}\right)^\perp\right)$$

are the functions produced as in (16.18). Note then that Theorem 16.7 is applicable to the functions $v_{j,\ell}$. By exactly following the corresponding steps in the proof of Lemma 13.1 and by using Theorem 16.7 where the proof of Lemma 13.1 depended on Theorem 12.2, we see that corresponding to infinitely many k, there are orthogonal rotations Γ_k, cones $\mathbf{C}_k'\in\mathcal{K}$ such that the conclusions of the present lemma hold with V_k, $\mathbf{C}_k$, $\mathbf{C}_k'$ and Γ_k in place of V, $\mathbf{C}$, $\mathbf{C}'$ and Γ, and with constants $\overline{\kappa}$, $\overline{C}_0$, $\overline{\nu}$ depending only on α, γ and $\mathbf{C}_0$. This contradicts our assumption, establishing the lemma. □

LEMMA 16.9. *Let* $\alpha\in(0,1)$, q *be an integer* ≥ 2 *and* $\gamma\in(0,1/2)$. *Let* $\mathbf{C}_0=\sum_{j=1}^{m_0}\sum_{\ell=1}^{q_j^{(0)}}|H_{j,\ell}|$ *be the stationary cone as in* (16.2) *with* $\Theta(\|\mathbf{C}_0\|,0)=q+1/2$. *For* $j=1,2,\ldots,2q-m_0+1$, *let* $\theta_j\in(0,1/4)$ *be such that* $\theta_1>8\theta_2>64\theta_3>\cdots>8^{2q-m_0}\theta_{2q-m_0+1}$. *There exists a number* $\varepsilon_0=\varepsilon_0(\alpha,\gamma,\theta_1,\theta_2,\ldots,\theta_{2q-m_0+1},\mathbf{C}_0)\in(0,1/2)$ *such that the following is true*: *If* $V\in\mathcal{S}_\alpha$, $\mathbf{C}\in\mathcal{K}$ *satisfy Hypotheses* 16.2 *and if the induction hypotheses* (H1), (H2) *hold*, *then there exist orthogonal rotations* Γ,Δ *of* $\mathbf{R}^{n+1}$ *and cones* $\mathbf{C}',\mathbf{C}''\in\mathcal{K}$ *such that*, *with*

$$\mathcal{Q}_V^2(\mathbf{C})=\int_{B_1^{n+1}(0)\setminus\{r(X)<1/16\}}\operatorname{dist}^2(X,\operatorname{spt}\|V\|)\,d\|\mathbf{C}\|(X) + \int_{B_1^{n+1}(0)}\operatorname{dist}^2(X,\operatorname{spt}\|\mathbf{C})\|)\,d\|V\|(X)$$

and

$$\mathcal{R}_V^2(\mathbf{C}) = \int_{B_1^{n+1}(0)\setminus\{r(X)<1/64\}} \operatorname{dist}^2(X, \operatorname{spt}\|V\|)\, d\|\mathbf{C}\|(X) + \int_{B_1^{n+1}(0)} \operatorname{dist}^2(X, \operatorname{spt}\|\mathbf{C})\|)\, d\|V\|(X),$$

we have the following:

(a) $|e_j - \Gamma(e_j)| \leq \kappa \mathcal{Q}_V(\mathbf{C})$ *and* $|e_j - \Delta(e_j)| \leq \kappa \mathcal{R}_V(\mathbf{C})$ *for* $j = 1, 2, 3, \ldots, n+1$;

(b) $\operatorname{dist}_{\mathcal{H}}^2(\operatorname{spt}\|\mathbf{C}'\| \cap B_1^{n+1}(0), \operatorname{spt}\|\mathbf{C}\| \cap B_1^{n+1}(0)) \leq C_0 \mathcal{Q}_V^2(\mathbf{C})$ *and* $\operatorname{dist}_{\mathcal{H}}^2(\operatorname{spt}\|\mathbf{C}''\| \cap B_1^{n+1}(0), \operatorname{spt}\|\mathbf{C}\| \cap B_1^{n+1}(0)) \leq C_0 \mathcal{R}_V^2(\mathbf{C})$;

(c) *for some* $j' \in \{1, 2, \ldots, 2q - m_0 + 1\}$,

$$\theta_{j'}^{-n-2} \int_{\Gamma\left(B_{\theta_{j'}}^{n+1}(0)\setminus\{r(X)\leq\theta_{j'}/16\}\right)} \operatorname{dist}^2(X, \operatorname{spt}\|V\|)\, d\|\Gamma_{\#}\,\mathbf{C}'\|(X) + \theta_{j'}^{-n-2} \int_{B_{\theta_{j'}}^{n+1}(0)} \operatorname{dist}^2(X, \operatorname{spt}\|\Gamma_{\#}\,\mathbf{C}'\|)\, d\|V\|(X) \leq \nu_{j'} \theta_{j'}^2 \mathcal{Q}_V^2(\mathbf{C}),$$

and for some $j'' \in \{1, 2, \ldots, 2q - m_0 + 1\}$,

$$\theta_{j''}^{-n-2} \int_{\Delta\left(B_{\theta_{j''}}^{n+1}(0)\setminus\{r(X)\leq\theta_{j''}/64\}\right)} \operatorname{dist}^2(X, \operatorname{spt}\|V\|)\, d\|\Delta_{\#}\,\mathbf{C}''\|(X) + \theta_{j''}^{-n-2} \int_{B_{\theta_{j''}}^{n+1}(0)} \operatorname{dist}^2(X, \operatorname{spt}\|\Delta_{\#}\,\mathbf{C}''\|)\, d\|V\|(X) \leq \nu_{j''} \theta_{j''}^2 \mathcal{R}_V^2(\mathbf{C}).$$

Here κ *and* C_0 *depend only on* $\alpha, \gamma, \mathbf{C}_0$ *in case* $2q = m_0$ *and only on* $\alpha, \gamma, \theta_1, \ldots, \theta_{2q-m_0}$ *and* $\mathbf{C}_0$ *in case* $2q \geq m_0+1$; $\nu_1 = \nu_1(\alpha, \gamma, \mathbf{C}_0)$ *and, in case* $2q \geq m_0+1$, *for each* $j = 2, 3, \ldots, 2q - m_0 + 1$, $\nu_j = \nu_j(\alpha, \gamma, \theta_1, \ldots, \theta_{j-1}, \mathbf{C}_0)$. *In particular,* ν_j *is independent of* $\theta_j, \theta_{j+1}, \ldots, \theta_{2q-m_0+1}$ *for* $j = 1, 2, \ldots, 2q - m_0 + 1$.

Proof. First use Lemma 16.8 and the argument of Lemmas 13.2 and 13.3 to obtain each of those conclusions above in which $\mathcal{Q}_V(\mathbf{C})$ appears on the right-hand side, with a set of constants κ_1, $C_0^{(1)}$, $\nu_j^{(1)}$ in place of κ, C_0, ν_j, $j = 1, 2, \ldots, 2q - m_0 + 1$, depending only on the allowed parameters stated in the conclusion. Then repeat the entire argument leading to these conclusions but with $\mathcal{R}_V(\mathbf{C})$ in place of $\mathcal{Q}_V(\mathbf{C})$ (so, in particular, part (ii) of Hypothesis (†) reads $\mathcal{R}_V(\mathbf{C}) \leq \beta \inf_{\widetilde{\mathbf{C}} \in \bigcup_{j=m_0}^{p-1} \mathcal{K}(j)} \mathcal{R}_V(\widetilde{\mathbf{C}})$) to obtain those conclusions above where $\mathcal{R}_V(\mathbf{C})$ appears on the right-hand side, with a set of constants κ_2, $C_0^{(2)}$, $\nu_j^{(2)}$ in place of κ, C_0, ν_j, $j = 1, 2, \ldots, 2q - m_0 + 1$, depending again only on the allowed parameters. Set $\kappa = \max\{\kappa_1, \kappa_2\}$, $C_0 = \max\{C_0^{(1)}, C_0^{(2)}\}$ and $\nu_j = \max\{\nu_j^{(1)}, \nu_j^{(2)}\}$ for $j = 1, 2, \ldots, 2q - m_0 + 1$. $\square$

Proof of Theorem 16.1. *Case* 1: $\Theta(\|\mathbf{C}_0\|, 0) = q + 1/2$, $q \geq 2$. If the theorem is false in this case, then there exist a number $\gamma \in (0, 1/2)$ and, for each $\ell = 1, 2, 3, \ldots$, a varifold $V_\ell \in \mathcal{S}_\alpha$ with $\Theta(\|V_\ell\|, 0) \geq q + 1/2$ and $(\omega_n 2^n)^{-1}\|V_\ell\|(B_2^{n+1}(0)) \leq q + 1/2 + \gamma$ such that

$$\mathrm{dist}_{\mathcal{H}}(\mathrm{spt}\,\|V_\ell\| \cap B_1^{n+1}(0), \mathrm{spt}\,\|\mathbf{C}_0\| \cap B_1^{n+1}(0)) \to 0$$

as $\ell \to \infty$. By Allard's integral varifold compactness theorem ([All72]; see also [Sim83, §42.8]) and the constancy theorem ([Sim83, §41]), it follows, after passing to a subsequence without changing notation, that

$$V_\ell \llcorner B_1^{n+1}(0) \to \left(\sum_{j=1}^{m_0} q_j^{(0)} |H_j^{(0)}|\right) \llcorner B_1^{n+1}(0)$$

as varifolds, where $q_j^{(0)}$, $j = 1, 2, \ldots, m_0$, are positive integers with $\sum_{j=1}^{m_0} q_j^{(0)} = 2q + 1$. We may assume, by redefining the multiplicities of the original cone $\mathbf{C}_0$ if necessary, that $\mathbf{C}_0 = \sum_{j=1}^{m_0} q_j^{(0)} |H_j^{(0)}|$. Thus

$$V_\ell \llcorner B_1^{n+1}(0) \to \mathbf{C}_0 \llcorner B_1^{n+1}(0) \quad \text{as varifolds.} \tag{16.23}$$

For $j = 1, 2, \ldots, 2q - m_0 + 1$, choose numbers $\theta_j = \theta_j(\alpha, \gamma, \mathbf{C}_0) \in (0, 1/8)$ as follows: First choose θ_1 such that $\nu_1 \theta_1^{2(1-\alpha)} < 1/4$, where $\nu_1 = \nu_1(\alpha, \gamma, \mathbf{C}_0)$ is as in Lemma 16.9. Having chosen $\theta_1, \theta_2, \ldots, \theta_j$, $1 \leq j \leq 2q - m_0$, choose θ_{j+1} such that $\theta_{j+1} < 8^{-1}\theta_j$ and $\nu_{j+1}\theta_{j+1}^{2(1-\alpha)} < 1/4$, where $\nu_{j+1} = \nu_{j+1}(\alpha, \gamma, \theta_1, \ldots, \theta_j, \mathbf{C}_0)$ is as in Lemma 16.9.

Note that it is easily seen by arguing by contradiction that corresponding to any given $\varepsilon' \in (0, 1/2)$, there exists $\varepsilon = \varepsilon(\varepsilon', \alpha, \gamma, \mathbf{C}_0) \in (0, 1/2)$ such that if Hypotheses 16.2 are satisfied, then

$$\mathcal{Q}_V(\mathbf{C}) \leq \mathcal{R}_V(\mathbf{C}) < \varepsilon'$$

where $\mathcal{Q}_V(\mathbf{C})$, $\mathcal{R}_V(\mathbf{C})$ are defined as in Lemma 16.9. By Remark (4) following the statement of Hypotheses 16.2, it then follows that if Hypotheses 16.2 are satisfied with sufficiently small $\varepsilon = \varepsilon(\varepsilon', \alpha, \gamma, \mathbf{C}_0)$, then for each $Z \in \mathrm{spt}\,\|V\| \cap B_{1/8}^{n+1}(0)$ with $\Theta(\|V\|, Z) \geq q + 1/2$,

$$\mathcal{Q}_{V^Z}(\mathbf{C}) \leq \mathcal{R}_{V^Z}(\mathbf{C}) < \varepsilon', \tag{16.24}$$

where $V^Z = \eta_{Z,1/2\,\#}\,V$.

Now fix ℓ sufficiently large, let $V = V_\ell$ and let $Z \in \mathrm{spt}\,\|V\| \cap B_{1/8}^{n+1}(0)$ with $\Theta(\|V\|, Z) \geq q + 1/2$. We claim that we may apply Lemma 16.9 iteratively to obtain, for each $k = 0, 1, 2, 3, \ldots$, an orthogonal rotation Γ_k^Z of $\mathbf{R}^{n+1}$ with $\Gamma_0^Z = \text{Identity}$, and a cone $\mathbf{C}_k^Z \in \mathcal{K}$ with $\mathbf{C}_0^Z = \mathbf{C}_0$ satisfying, for $k \geq 1$,

$$|\Gamma_k^Z(e_j) - \Gamma_{k-1}^Z(e_j)| \leq \kappa \left(\sigma_k^Z\right)^\alpha \mathcal{Q}_{V^Z}(\mathbf{C}_0), \qquad j = 1, 2, \ldots, n+1, \tag{16.25}$$

(16.26)
$$\mathrm{dist}_{\mathcal{H}}(\mathrm{spt}\,\|\mathbf{C}_k^Z\| \cap B_1^{n+1}(0), \mathrm{spt}\,\|\mathbf{C}_{k-1}^Z\| \cap B_1^{n+1}(0)) \leq C_0 \left(\sigma_k^Z\right)^{\alpha} \mathcal{Q}_{V^Z}(\mathbf{C}_0),$$

(16.27)
$$\left(\sigma_k^Z\right)^{-n-2} \int_{B^{n+1}_{\sigma_k^Z}(0)} \mathrm{dist}^2(X, \mathrm{spt}\,\| \left(\Gamma_k^Z\right)_{\#} \mathbf{C}_k^Z\|)\, d\|V^Z\|(X) \leq \left(\sigma_k^Z\right)^{2\alpha} \mathcal{Q}^2_{V^Z}(\mathbf{C}_0),$$

and

(16.28)
$$\left(\sigma_k^Z\right)^{-n-2} \int_{\Gamma_k^Z(B^{n+1}_{\sigma_k^Z}(0)\setminus\{|r(X)|\leq \sigma_k^Z/16\})} \mathrm{dist}^2(X, \mathrm{spt}\,\|V^Z\|)\, d\| \left(\Gamma_k^Z\right)_{\#} \mathbf{C}_k^Z\|(X)$$
$$\leq \left(\sigma_k^Z\right)^{2\alpha} \mathcal{Q}^2_{V^Z}(\mathbf{C}_0),$$

where $\kappa = \kappa(\alpha,\gamma,\mathbf{C}_0)$, $C_0 = C_0(\alpha,\gamma,\mathbf{C}_0)$ are as in Lemma 16.9 and $\{\sigma_k^Z\}$ is a sequence of positive numbers such that $\sigma_0^Z = 1$ and for each $k = 1, 2, \ldots$, $\sigma_k^Z = \theta_{j_k^Z}\sigma_{k-1}^Z$ for some $j_k^Z \in \{1, 2, \ldots, 2q-m_0+1\}$. To see this, note first that it follows from Remark (4) following the statement of Hypotheses 16.2 that if $V = V_\ell$ with ℓ fixed sufficiently large, then for each $Z \in \mathrm{spt}\,\|V\| \cap B^{n+1}_{1/8}(0)$ with $\Theta(\|V\|, Z) \geq q + 1/2$, Hypotheses 16.2 are satisfied with V^Z in place of V, $\mathbf{C}_0$ in place of $\mathbf{C}$ and with $\varepsilon = \varepsilon_0(\alpha,\gamma,\mathbf{C}_0)$, where ε_0 is as in Lemma 16.9. Hence by applying Lemma 16.9 with V^Z in place of V and $\mathbf{C} = \mathbf{C}_0$, we deduce that (16.25)–(16.28) hold in case $k = 1$. So let $k \geq 2$, and suppose by induction that (16.25)–(16.28) are valid with $1, 2, \ldots, k-1$ in place of k. Then for any given $\varepsilon \in (0, 1/4)$, provided $V = V_\ell$ with ℓ sufficiently large, Hypotheses 16.2 are satisfied with $\left(\Gamma_{k-1}^Z\right)^{-1}_{\#} \eta_{0,\sigma_{k-1}^Z\,\#} V^Z$ in place of V and with $\mathbf{C} = \mathbf{C}_{k-1}^Z$. Here, the validity of Hypotheses 16.2(1)–(4) with $\left(\Gamma_{k-1}^Z\right)^{-1}_{\#} \eta_{0,\sigma_{k-1}^Z\,\#} V^Z$ in place of V and $\mathbf{C}_{k-1}^Z$ in place of $\mathbf{C}$ is clear, and in verifying Hypothesis 16.2(5) with $\left(\Gamma_{k-1}^Z\right)^{-1}_{\#} \eta_{0,\sigma_{k-1}^Z\,\#} V^Z$ in place of V, note first that by Remarks (1) and (4) following the statement of Hypotheses 16.2 (taken with $\rho = \sigma_1^Z$ and $\tau = \frac{1}{32}\min\{\theta_1, \theta_2 \ldots, \theta_{2q-m_0+1}\} = \frac{1}{32}\theta_{2q-m_0+1}$), we have that
(16.29)
$$\eta_{0,\sigma_1^Z\,\#} V^Z \llcorner \left(B_1^{n+1}(0) \setminus \left\{r(X) < \frac{1}{32}\theta_{2q-m_0+1}\right\}\right) = \sum_{j=1}^{m_0}\sum_{i=1}^{q_j^{(0)}} |\mathrm{graph}\, \widetilde{u}_{j,i}|,$$

where for each $j \in \{1, 2, \ldots, m_0\}$ and $i \in \{1, 2, \ldots, q_j^{(0)}\}$,

$$\widetilde{u}_{j,i} \in C^2\left(H_j^{(0)} \cap \left(B_1^{n+1}(0) \setminus \left\{r(X) < \frac{1}{32}\theta_{2q-m_0+1}\right\}\right); \left(H_j^{(0)}\right)^{\perp}\right)$$

and $\widetilde{u}_{j,i}$ are solutions to the minimal surface equation over

$$H_j^{(0)} \cap \left(B_1^{n+1}(0) \setminus \{r(X) < \frac{1}{32}\theta_{2q-m_0+1}\}\right)$$

with small C^2 norm. So in particular, in view of (16.25), Hypothesis 16.2(5) is satisfied with $\left(\Gamma_1^Z\right)^{-1}_{\#} \eta_{0,\sigma_1^Z\,\#} V^Z$ in place of V. On the other hand, by (16.26), (16.27) and (16.28), we may apply Remarks (2) and (3) following the statement of Hypotheses 16.2 with $\left(\Gamma_r^Z\right)^{-1}_{\#} \eta_{0,\sigma_r^Z\,\#} V^Z$ in place of V and $\tau = \frac{1}{2}\theta_{2q-m_0+1}$, followed by Theorem 3.5, to deduce that for each $r \in \{2, 3, \ldots, k-1\}$,
(16.30)

$$\left(\Gamma_r^Z\right)^{-1}_{\#} \eta_{0,\sigma_r^Z\,\#} V^Z \llcorner \left(B_{1/2}^{n+1}(0) \setminus \left\{r(X) < \frac{1}{2}\theta_{2q-m_0+1}\right\}\right) = \sum_{j=1}^{m_0}\sum_{i=1}^{p_j^{Z,r}} |\mathrm{graph}\, \widetilde{u}_{j,i}^{Z,r}|$$

for some integers $p_j^{Z,r} \geq 1$, where for each $j \in \{1, \ldots, m_0\}$ and $i \in \{1, \ldots, p_j^{Z,r}\}$,

$$\widetilde{u}_{j,i}^{Z,r} \in C^2\left(H_j^{(0)} \cap \left(B_{1/2}^{n+1}(0) \setminus \left\{r(X) < \frac{1}{2}\theta_{2q-m_0+1}\right\}\right); \left(H_j^{(0)}\right)^{\perp}\right)$$

and $\widetilde{u}_{j,i}^{Z,r}$ are solutions to the minimal surface equation over

$$H_j^{(0)} \cap \left(B_{1/2}^{n+1}(0) \setminus \{r(X) < \frac{1}{2}\theta_{2q-m_0+1}\}\right)$$

with small C^2 norm. Since $\sigma_r^Z \geq \theta_{2q-m_0+1}\sigma_{r-1}^Z$ for each $r \geq 1$, it follows from (16.29), (16.30) and unique continuation of solutions to the minimal surface equation that

$$p_j^{Z,r} = q_j^{(0)} \tag{16.31}$$

for each $r \in \{2, 3, \ldots, k-1\}$ and $j \in \{1, 2, \ldots, m_0\}$, whence, by (16.30), we see that Hypothesis 16.2(5) with $\left(\Gamma_{k-1}^Z\right)^{-1}_{\#} \eta_{0,\sigma_{k-1}^Z\,\#} V^Z$ in place of V is satisfied as claimed. Hence we may apply Lemma 16.9 with $\left(\Gamma_{k-1}^Z\right)^{-1}_{\#} \eta_{0,\sigma_{k-1}^Z\,\#} V^Z$ in place of V and $\mathbf{C}_{k-1}^Z$ in place of $\mathbf{C}$ to obtain an orthogonal rotation Γ_k^Z of $\mathbf{R}^{n+1}$ and a cone $\mathbf{C}_k^Z \in \mathcal{K}$ satisfying (16.25)–(16.28). This inductively establishes the validity of (16.25)–(16.28) for each $k = 1, 2, 3, \ldots$. Using (16.25)–(16.28) in a standard way, we reach the conclusion that if $V = V_\ell$ with ℓ fixed sufficiently large, then corresponding to each $Z \in \mathrm{spt}\,\|V\| \cap B_{1/8}^{n+1}(0)$ with $\Theta(\|V\|, Z) \geq q + 1/2$, there exist a cone $\mathbf{C}^Z \in \mathcal{K}$ with

$$\mathrm{dist}_{\mathcal{H}}(\mathrm{spt}\,\|\mathbf{C}^Z\| \cap B_1^{n+1}(0), \mathrm{spt}\,\|\mathbf{C}_0\| \cap B_1^{n+1}(0)) \leq C\mathcal{Q}_{V^Z}(\mathbf{C}_0) \tag{16.32}$$

and an orthogonal rotation Γ^Z of $\mathbf{R}^{n+1}$ satisfying, for each $k = 0, 1, 2, \ldots$,

$$|\Gamma^Z(e_j) - \Gamma_k^Z(e_j)| \leq C\left(\sigma_k^Z\right)^{\alpha} \mathcal{Q}_{V^Z}(\mathbf{C}_0), \quad j = 1, 2, \ldots, n+1 \tag{16.33}$$

such that
(16.34)
$$\left(\sigma_k^Z\right)^{-n-2}\int_{\Gamma_k^Z(B^{n+1}_{\sigma_k^Z/2}(0)\setminus\{|r(X)|\leq\sigma_k^Z/16\})}\mathrm{dist}^2(X,\mathrm{spt}\,\|V^Z\|)\,d\|\Gamma^Z_{\#}\,\mathbf{C}^Z\|(X)$$
$$\leq C\left(\sigma_k^Z\right)^{2\alpha}\mathcal{Q}^2_{V^Z}(\mathbf{C}_0)$$

for each $k = 0, 1, 2, \ldots$ and

$$\rho^{-n-2}\int_{B^{n+1}_\rho(0)}\mathrm{dist}^2(X,\mathrm{spt}\,\|\Gamma^Z_{\#}\,\mathbf{C}^Z\|)\,d\|V^Z\|(X)\leq C\rho^{2\alpha}\mathcal{Q}^2_{V^Z}(\mathbf{C}_0) \tag{16.35}$$

for all $\rho\in(0,1/4]$, where $C = C(\alpha,\gamma,\mathbf{C}_0)\in(0,\infty)$.

Let $T_V = \{Z\in\mathrm{spt}\,\|V\| : \Theta(\|V\|,Z)\geq q+1/2\}\cap B_1^{n+1}(0)$. We now use the estimates (16.32)–(16.35), Lemmas 16.5(a) and 16.9 and Corollary 16.4 to establish that $T_V\cap B^{n+1}_{1/32}(0)$ is an $(n-1)$-dimensional embedded $C^{1,\alpha}$ submanifold of $B^{n+1}_{1/32}(0)$ containing the origin. Indeed, note first that estimates (16.32), (16.34) and (16.35) imply that for any given $\varepsilon\in(0,1/4)$, if $V = V_\ell$ with fixed ℓ sufficiently large, then for each $Z\in T_V\cap B^{n+1}_{1/16}(0)$ and each $k\geq 1$, Hypotheses 16.2 are satisfied with $\left(\Gamma^Z\right)^{-1}_{\#}\ \eta_{Z,\frac{1}{2}\sigma_k^Z\,\#}\,V$ in place of V and $\mathbf{C} = \mathbf{C}^Z$. (In verifying Hypothesis 16.2(5) with $\left(\Gamma^Z\right)^{-1}_{\#}\ \eta_{Z,\frac{1}{2}\sigma_k^Z\,\#}\,V = \left(\Gamma^Z\right)^{-1}_{\#}\ \eta_{0,\sigma_k^Z\,\#}\,V^Z$ in place of V, we argue exactly as we did in verifying Hypothesis 16.2(5) with $\left(\Gamma^Z_{k-1}\right)^{-1}_{\#}\ \eta_{0,\sigma^Z_{k-1}\,\#}\,V^Z$ in place of V as part of the inductive step described above.)

Consequently, we see that for each point $(0,y)\in\{0\}\times\mathbf{R}^{n-1}\cap B^{n+1}_{1/16}(0)$,

$$T_V\cap\mathbf{R}^2\times\{(0,y)\}\neq\emptyset; \tag{16.36}$$

for if there is a point $(0,y)\in\{0\}\times\mathbf{R}^{n-1}\cap B^{n+1}_{1/16}(0)$ with $T_V\cap\left(\mathbf{R}^2\times\{(0,y)\}\right) = \emptyset$, then, since $T_V\cap B^{n+1}_{1/16}(0)$ is a relatively closed subset of $B^{n+1}_{1/16}(0)$ and $0\in T_V$, we can find $r\in(0,1/16)$ such that $T_V\cap\left(\mathbf{R}^2\times B_r^{n-1}(0,y)\right) = \emptyset$ but $T_V\cap\left(\mathbf{R}^2\times\partial\,B_r^{n-1}(0,y)\right)\neq\emptyset$, whence we may, in view of (16.33), (16.34) and (16.35), pick any point $Z\in T_V\cap\left(\mathbf{R}^2\times\partial\,B_r^{n-1}(0,y)\right)$, choose k such that $\sigma_k^Z < r/4$ and apply Lemma 16.5(a) with $\left(\Gamma^Z\right)^{-1}_{\#}\ \eta_{Z,\frac{1}{2}\sigma_k^Z\,\#}\,V$ in place of V, $\mathbf{C} = \mathbf{C}^Z$ and $\delta = 1/8$ to get a contradiction with the assumption $T_V\cap\left(\mathbf{R}^2\times B_r^{n-1}(0,y)\right) = \emptyset$.

For $Z\in T_V$, let $S_Z = Z + \Gamma^Z\left(\{0\}\times\mathbf{R}^{n-1}\right)$ and note that for each $Z\in T_V$ and each $\rho\in(0,1/4]$,

$$T_V\cap\left(B_\rho^{n+1}(Z)\setminus\left\{X\in\mathbf{R}^{n+1} : \mathrm{dist}(X,S_Z)<\frac{1}{8}\rho\right\}\right) = \emptyset. \tag{16.37}$$

This is easily seen by choosing, for given $Z\in T_V$ and $\rho\in(0,1/4]$, the unique integer k such that $\frac{15}{32}\sigma^Z_{k+1}<\rho\leq\frac{15}{32}\sigma_k^Z$, and applying Remark (2) following

Hypotheses 16.2 with $\tau = \frac{1}{16}\theta_{2q-m_0+1}$ and with $\left(\Gamma^Z\right)^{-1}_{\#} \eta_{Z,\frac{1}{2}\sigma_k^Z\,\#} V$ in place of V. This and (16.36) imply that for each $(0,y) \in \{0\} \times \mathbf{R}^{n-1} \cap B^{n+1}_{1/16}(0)$, the set $T_V \cap \mathbf{R}^2 \times \{(0,y)\}$ consists of a unique point, so that

$$T_V \cap B^{n+1}_{1/16}(0) = \operatorname{graph} \varphi \tag{16.38}$$

for a function $\varphi = (\varphi_1, \varphi_2) : B^{n-1}_{1/16}(0) \to \mathbf{R}^2$. Moreover, (16.37) and the estimates (16.24), (16.33) say that φ is Lipschitz with $\mathrm{Lip}(\varphi) \leq 1$ and, writing $\widetilde{\varphi}(Z) = (\varphi_1(Z), \varphi_2(Z), Z)$ for $Z \in B^{n-1}_{1/16}(0)$, that

$$D\widetilde{\varphi}(Z)\left(\{0\} \times \mathbf{R}^{n-1}\right) = \Gamma^Z(\{0\} \times \mathbf{R}^{n-1}) \tag{16.39}$$

for $\mathcal{H}^{n-1}$-a.e. $Z \in B^{n-1}_{1/16}(0)$.

We now argue that $\varphi|_{B^{n-1}_{1/32}(0)}$ must be of class $C^{1,\alpha}$. For this, first observe that by employing exactly the argument leading to (16.32)–(16.35) but using those conclusions of Lemma 16.9 involving $\mathcal{R}_V(\mathbf{C})$ (in place of those involving $\mathcal{Q}_V(\mathbf{C})$), we obtain for each $Z \in T_V$ orthogonal rotations Δ^Z, Δ^Z_k of $\mathbf{R}^{n+1}$ for $k = 1, 2, 3, \ldots$; a cone $\mathbf{W}^Z \in \mathcal{K}$ and numbers $\tau^Z_k \in (0, 1]$ for $k = 1, 2, 3, \ldots$, where for each k, $\tau^Z_k = \theta_{\ell^Z_k}\tau^Z_{k-1}$ for some $\ell^Z_k \in \{1, 2, \ldots, 2q - m_0 + 1\}$, such that

$$\mathrm{dist}_{\mathcal{H}}(\operatorname{spt}\|\mathbf{W}^Z\| \cap B^{n+1}_1(0), \operatorname{spt}\|\mathbf{C}_0\| \cap B^{n+1}_1(0)) \leq C\mathcal{R}_{V^Z}(\mathbf{C}_0); \tag{16.40}$$

$$|\Delta^Z(e_j) - \Delta^Z_k(e_j)| \leq C\left(\tau^Z_k\right)^{\alpha} \mathcal{R}_{V^Z}(\mathbf{C}_0), \quad j = 1, 2, \ldots, n+1; \tag{16.41}$$

(16.42)
$$\left(\tau^Z_k\right)^{-n-2} \int_{\Delta^Z_k(B^{n+1}_{\tau^Z_k/2}(0) \setminus \{|r(X)| \leq \tau^Z_k/64\})} \mathrm{dist}^2(X, \operatorname{spt}\|V^Z\|)\, d\|\Delta^Z_{\#}\,\mathbf{W}^Z\|(X) \leq C\left(\tau^Z_k\right)^{2\alpha} \mathcal{R}^2_{V^Z}(\mathbf{C}_0)$$

for each $k = 0, 1, 2, \ldots$; and

$$\rho^{-n-2}\int_{B^{n+1}_\rho(0)} \mathrm{dist}^2(X, \operatorname{spt}\|\Delta^Z_{\#}\,\mathbf{W}^Z\|)\, d\|V^Z\|(X) \leq C\rho^{2\alpha}\mathcal{R}^2_{V^Z}(\mathbf{C}_0) \tag{16.43}$$

for all $\rho \in (0, 1/4]$, where $C = C(\alpha, \gamma, \mathbf{C}_0) \in (0, \infty)$.

Since the sequence of varifolds $W_k = \eta_{Z,\sigma^Z_k\,\#} V$, $k = 1, 2, 3, \ldots$ has a subsequence $W_{k'}$ that converges to a cone $\mathbf{P}$ satisfying, by (16.34) and (16.35), $\operatorname{spt}\|\mathbf{P}\| = \operatorname{spt}\|\Gamma^Z_{\#}\,\mathbf{C}^Z\|$, it follows from (16.43) taken with $\rho = \sigma^Z_{k'}$ that $\operatorname{spt}\|\Gamma^Z_{\#}\,\mathbf{C}^Z\| \subseteq \operatorname{spt}\|\Delta^Z_{\#}\,\mathbf{W}^Z\|$. The same reasoning applied to the sequence $\eta_{Z,\tau^Z_k\,\#} V$ establishes the reverse inclusion, so we have that $\operatorname{spt}\|\Gamma^Z_{\#}\,\mathbf{C}^Z\| = \operatorname{spt}\|\Delta^Z_{\#}\,\mathbf{W}^Z\|$ whence, in particular, we have that

$$\Gamma^Z(\{0\} \times \mathbf{R}^{n-1}) = \Delta^Z(\{0\} \times \mathbf{R}^{n-1}). \tag{16.44}$$

Recall (cf. the paragraph preceding (16.36)) that given any $\varepsilon \in (0,1/2)$, if $V = V_\ell$ with ℓ fixed sufficiently large depending on ε, then for each $Z \in T_V \cap B^{n+1}_{1/16}(0)$ and $k \geq 1$, Hypotheses 16.2 are satisfied with $V_{k,Z} \equiv \left(\Delta^Z\right)^{-1}_{\#} \eta_{Z,\frac{1}{2}\tau_k^Z\,\#} V$ in place of V and $\mathbf{W}^Z$ in place of $\mathbf{C}$. Consequently, by Remark (4) following the statement of Hypotheses 16.2, we see that given any $\varepsilon \in (0,1/2)$, if $V = V_\ell$ for ℓ fixed sufficiently large, then for any $Z \in T_V$, $k \geq 1$ and $\widetilde{Z} \in T_{V_{k,Z}}$, Hypotheses 16.2 are satisfied with $\eta_{\widetilde{Z},1/2\,\#} V_{k,Z}$ in place of V and $\mathbf{W}^Z$ in place of $\mathbf{C}$. Now take any two distinct points $Z_1, Z_2 \in T_V \cap B^{n+1}_{1/32}(0)$, let m be the unique integer satisfying $\tau^{Z_1}_{m+1} < 2|Z_1 - Z_2| \leq \tau^{Z_1}_m$ and let $\widetilde{V} = V_{m,Z_1} = \left(\Delta^{Z_1}\right)^{-1}_{\#} \eta_{Z_1,\frac{1}{2}\tau_m^{Z_1}\,\#} V$. Letting $\widetilde{Z} = \left(\Delta^{Z_1}\right)^{-1}\left(\frac{2(Z_2-Z_1)}{\tau_m^{Z_1}}\right)$ and noting that $\widetilde{Z} \in \operatorname{spt}\|\widetilde{V}\| \cap B^{n+1}_{1/16}(0)$ with $\Theta(\|\widetilde{V}\|, \widetilde{Z}) \geq q + 1/2$, we may apply Lemma 16.9 iteratively (utilising its conclusions involving $\mathcal{Q}_{(\cdot)}(\cdot)$), starting with $\widetilde{V}^{\widetilde{Z}} = \eta_{\widetilde{Z},1/2\,\#}\widetilde{V}$ in place of V and $\mathbf{W}^{Z_1}$ in place of $\mathbf{C}$ (and with $\theta_1, \theta_2, \ldots \theta_{2q-m_0+1}$ equal to the same fixed constants as chosen at the beginning of the proof of the present theorem), in the manner exactly as in the argument leading to (16.32)–(16.35), to conclude that there exist a cone $\widetilde{\mathbf{C}} \in \mathcal{K}$ with

$$\text{(16.45)}\qquad \operatorname{dist}_{\mathcal{H}}(\operatorname{spt}\|\widetilde{\mathbf{C}}\| \cap B_1^{n+1}(0), \operatorname{spt}\|\mathbf{W}^{Z_1}\| \cap B_1^{n+1}(0)) \leq C\mathcal{Q}_{\widetilde{V}^{\widetilde{Z}}}(\mathbf{W}^{Z_1});$$

orthogonal rotations $\widetilde{\Gamma}, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1, \ldots$ of $\mathbf{R}^{n+1}$ with $\widetilde{\Gamma}_0 =$ Identity; and a sequence of positive numbers $\{\widetilde{\sigma}_k\}$ with $\widetilde{\sigma}_0 = 1$ and $\widetilde{\sigma}_k = \theta_{\widetilde{j}_k}\widetilde{\sigma}_{k-1}$ for some $\widetilde{j}_k \in \{1, 2, \ldots, 2q - m_0 + 1\}$ and each $k \geq 1$, satisfying, for each $k = 0, 1, 2, \ldots$,

$$\text{(16.46)}\qquad |\widetilde{\Gamma}(e_j) - \widetilde{\Gamma}_k(e_j)| \leq C\,(\widetilde{\sigma}_k)^\alpha\,\mathcal{Q}_{\widetilde{V}^{\widetilde{Z}}}(\mathbf{W}^{Z_1}), \quad j = 1, 2, \ldots, n+1;$$

$$\text{(16.47)}\qquad (\widetilde{\sigma}_k)^{-n-2}\int_{\widetilde{\Gamma}_k(B^{n+1}_{\widetilde{\sigma}_k/2}(0)\setminus\{|r(X)|\leq\widetilde{\sigma}_k/16\})} \operatorname{dist}^2(X, \operatorname{spt}\|\widetilde{V}^{\widetilde{Z}}\|)\,d\|\widetilde{\Gamma}_{\#}\,\widetilde{\mathbf{C}}\|(X) \leq C\,(\widetilde{\sigma}_k)^{2\alpha}\,\mathcal{Q}^2_{\widetilde{V}^{\widetilde{Z}}}(\mathbf{W}^{Z_1});$$

and for each $\rho \in (0, 1/4]$,

$$\text{(16.48)}\qquad \rho^{-n-2}\int_{B_\rho^{n+1}(0)} \operatorname{dist}^2(X, \operatorname{spt}\|\widetilde{\Gamma}_{\#}\,\widetilde{\mathbf{C}}\|)\,d\|\widetilde{V}^{\widetilde{Z}}\|(X) \leq C\rho^{2\alpha}\mathcal{Q}^2_{\widetilde{V}^{\widetilde{Z}}}(\mathbf{W}^{Z_1}),$$

where $C = C(\alpha, \gamma, \mathbf{C}_0) \in (0, \infty)$ is as in (16.32)–(16.35). Noting that $\widetilde{V}^{\widetilde{Z}} = \left(\Delta^{Z_1}\right)^{-1}_{\#} \eta_{Z_2,\frac{1}{2}\tau_m^{Z_1}\,\#} V$, we deduce from (16.47), (16.48) and inequalities (16.34), (16.35) taken with $Z = Z_2$, and reasoning exactly as for (16.44), that

$$\text{(16.49)}\qquad \Delta^{Z_1} \circ \widetilde{\Gamma}(\{0\} \times \mathbf{R}^{n-1}) = \Gamma^{Z_2}(\{0\} \times \mathbf{R}^{n-1}).$$

This together with (16.44) taken with $Z = Z_1$ and (16.46) taken with $k = 0$ implies that

$$\mathrm{dist}_{\mathcal{H}}\left(\Gamma^{Z_1}\left(\{0\} \times \mathbf{R}^{n-1}\right) \cap B_1^{n+1}(0), \Gamma^{Z_2}\left(\{0\} \times \mathbf{R}^{n-1}\right) \cap B_1^{n+1}(0)\right) \leq C\mathcal{Q}_{\widetilde{V}\widetilde{Z}}(\mathbf{W}^{Z_1}),$$

where $C = C(\alpha, \gamma, \mathbf{C}_0) \in (0, \infty)$. On the other hand, we see directly from Corollary 16.4(c) (taken with $\widetilde{V}$ in place of V and $\mathbf{W}^{Z_1}$ in place of $\mathbf{C}$) that $\mathcal{Q}_{\widetilde{V}\widetilde{Z}}(\mathbf{W}^{Z_1}) \leq C\mathcal{R}_{\widetilde{V}}(\mathbf{W}^{Z_1})$ and by (16.42), (16.43) and (16.24) that $\mathcal{R}_{\widetilde{V}}(\mathbf{W}^{Z_1}) \leq C(\tau_m^{Z_1})^{\alpha}\mathcal{R}_{V^{Z_1}}(\mathbf{C}_0) \leq C|Z_1 - Z_2|^{\alpha}$, where $C = C(n, \alpha, \gamma, \mathbf{C}_0) \in (0, \infty)$. We have thus established that for any pair of points $Z_1, Z_2 \in T_V \cap B_{1/32}^{n+1}(0)$,

$$\mathrm{dist}_{\mathcal{H}}\left(\Gamma^{Z_1}\left(\{0\} \times \mathbf{R}^{n-1}\right) \cap B_1^{n+1}(0), \Gamma^{Z_2}\left(\{0\} \times \mathbf{R}^{n-1}\right) \cap B_1^{n+1}(0)\right) \leq C|Z_1 - Z_2|^{\alpha},$$

which in view of (16.39) says that

$$\varphi|_{B_{1/32}^{n-1}(0)} \in C^{1,\alpha}(B_{1/32}^{n-1}(0)). \tag{16.50}$$

Now fix $j \in \{1, 2, \ldots, m_0\}$ and assume, for notational convenience and without loss of generality, that $H_j^{(0)} = \{(0, x^2, y) \in \mathbf{R}^{n+1} : x^2 > 0,\ y \in \mathbf{R}^{n-1}\}$. Let T_V' be the orthogonal projection of $T_V \cap B_{1/32}^{n+1}(0)$ onto the hyperplane $\{x^1 = 0\} \equiv \mathbf{R}^n$, so that $T_V' = \{(0, \varphi_2(y), y) : y \in B_{1/32}^{n-1}(0)\}$. Assuming that $V = V_\ell$ with ℓ sufficiently large, note then that $T_V' \subset \{|x^2| < 1/128\}$ and by (16.38) and (16.50) that $B_{1/64}^n(0) \setminus T_V'$ has exactly two components. Let Ω' be the component of $B_{1/64}^n(0) \setminus T_V'$ containing $B_{1/64}^n(0) \cap \{x^2 > 1/128\}$. Keeping in mind that (16.30) and (16.31) are valid for each $Z \in T_V \cap B_{1/32}^{n+1}(0)$ and each $r = 1, 2, 3, \ldots$, it follows from (16.30), (16.31) and unique continuation of solutions to the minimal surface equation that

$$V \llcorner ((\mathbf{R} \times \Omega') \cap N_j) = \sum_{i=1}^{q_j^0} |\mathrm{graph}\, u_i|,$$

where $N_j = \bigcup_{Z \in T_V \cap B_{1/32}^{n+1}(0)} \left(Z + \Gamma^Z(N(H_j^{(0)}))\right)$ and, for each $i = 1, 2, \ldots, q_j^{(0)}$, $u_i \in C^2(\Omega')$ with u_i solving the minimal surface equation on Ω', $|Du_i| < 1$, $u_1 \leq u_2 \leq \cdots \leq u_{q_j^{(0)}}$ and, by the maximum principle, either $u_i \equiv u_{i+1}$ or $u_i < u_{i+1}$ for each $i = 1, 2, \ldots, q_j^{(0)} - 1$. Since for each $i = 1, 2, \ldots, q_j^{(0)}$, u_i extends to $\overline{\Omega'} \cap B_{1/64}^n(0)$ as a Lipschitz function with boundary values given by $u_j|_{\partial\Omega' \cap B_{1/64}^n(0)}(0, \varphi_2(y), y) = \varphi_1(y)$ for each point $(0, \varphi_2(y), y) \in \partial\Omega' \cap B_{1/64}^n(0) = T_V' \cap B_{1/64}^n(0)$, it follows from (16.50) and standard $C^{1,\alpha}$ boundary regularity theory for uniformly elliptic equations ([Mor66]) that $u_i \in C^{1,\alpha}(\overline{\Omega'} \cap B_{1/64}^n(0))$.

We have thus established that $V \llcorner B^{n+1}_{1/64}(0) = \sum_{j=1}^{2q+1} |M_j|$ where, for each $j \in \{1, 2, \ldots, 2q+1\}$, M_j is an embedded $C^{1,\alpha}$ hypersurface-with-boundary with $\partial M_j = T_V \cap B^{n+1}_{1/64}(0)$ and, for each $j, k \in \{1, 2, \ldots, 2q+1\}$, either $M_j \cap M_k = T_V \cap B^{n+1}_{1/64}(0)$ or $M_j = M_k$. This directly contradicts hypothesis $(\mathcal{S}3)$ that V is assumed to satisfy, completing the proof of the theorem in case $\Theta(\|\mathbf{C}_0\|, 0) = q + 1/2$.

Case 2: $\Theta(\|\mathbf{C}_0\|, 0) = q+1$, $q \geq 2$. Note that the validity of Theorem 16.1 in case $\Theta(\|\mathbf{C}_0\|, 0) = q + 1/2$ enables us to repeat the entire proof of Theorem 15.2 with $q + 1$ in place of q, yielding Theorem 15.2 with $q + 1$ in place of q. Consequently, the assertion of Remark (3) following the statement of Hypotheses 16.2 holds with $q+1$ in place of $q+1/2$. Thus we may simply repeat (see the remark following the proof of Lemma 16.5) all of the steps of the above argument taking $q + 1$ in place of $q + 1/2$. This establishes Theorem 16.1 in case $\Theta(\|\mathbf{C}_0\|, 0) = q + 1$.

The proof of Theorem 16.1 is now complete. □

Remark. The case $q = 1$ of Theorem 3.3′ is a special case of Allard's Regularity Theorem (which is reproduced by taking $q = 1$ in our proofs of Lemma 15.1 and Theorem 15.2). The validity of the case $\Theta(\|\mathbf{C}_0\|, 0) = 3/2$ of Theorem 3.4 follows from the validity of the case $q = 1$ of Theorem 3.3′; indeed, in this case, the same argument as for Theorem 16.1 carries over (with obvious simplifications) provided the induction hypothesis (H1) is replaced by Theorem 3.3′, case $q = 1$. In fact, when $\Theta(\|\mathbf{C}_0\|, 0) = 3/2$, Theorem 3.4 is true without the stability hypotheses $(\mathcal{S}2)$ on V (so V only needs to be stationary and satisfy $(\mathcal{S}3)$); see [Sim93, Cors. 2 and 3]. This in turn enables us to prove Theorem 3.4 in case $\Theta(\|\mathbf{C}_0\|, 0) = 2$ by repeating the above proof of Theorem 16.1 (case $\Theta\,(\|\mathbf{C}_0\|, 0) = q + 1$), taking $q = 1$ and, in place of induction hypotheses (H1) and (H2), case $q = 1$ of Theorem 3.3′ and case $\Theta(\|\mathbf{C}_0\|, 0) = 3/2$ of Theorem 3.4 respectively.

Theorem 15.2 and Theorem 16.1 together with the above remark and the remark preceding the statement of Theorem 3.3′ complete the inductive proof of both Theorem 3.3 and Theorem 3.4.

17. The Regularity and Compactness Theorem

Proof of Theorem 3.1. Note first that if $V \in \mathcal{S}_\alpha$, then it follows from Theorem 3.3, Theorem 3.4 and Remark 3 of Section 6 that $\mathcal{H}^{n-7+\gamma}(\text{sing}\, V \cap (B^{n+1}_2(0)) = 0$ for each $\gamma > 0$ if $n \geq 7$ and $\text{sing}\, V \cap B^{n+1}_2(0) = \emptyset$ if $2 \leq n \leq 6$.

Suppose, for each $k = 1, 2, 3, \ldots$, that $V_k \in \mathcal{S}_\alpha$ and that

$$\Lambda = \limsup_{k\to\infty} \|V_k\|(B^{n+1}_2(0)) < \infty.$$

By Allard's integer varifold compactness theorem, there exists a stationary integral varifold V of $B_2^{n+1}(0)$, with $\|V\|(B_2^{n+1}(0)) < \Lambda + 1$, such that, after passing to a subsequence, $V_k \to V$ as varifolds in $B_2^{n+1}(0)$. Set $K = \operatorname{sing} V \cap B_2^{n+1}(0)$.

We argue that $V \in \mathcal{S}_\alpha$ as follows. By Theorem 3.3 and unique continuation of solutions to the minimal surface equation, if M is a connected component of $\operatorname{reg} V$ and $0 < \rho' < \rho < 2$, there exists a number $\varepsilon = \varepsilon(M, \rho, \rho') \in (0, 1/2)$ such that for all sufficiently large k,

$$\begin{aligned}\operatorname{spt} \|V_k\| &\cap \{X \in B_\rho^{n+1}(0) : \operatorname{dist}(X, M \cap B_\rho^{n+1}(0)) < \varepsilon\} \\ &\supset \textstyle\bigcup_{j=1}^q \operatorname{graph} u_j^k \supset \operatorname{spt} \|V_k\| \cap \{X \in B_{\rho'}^{n+1}(0) : \operatorname{dist}(X, M \cap B_{\rho'}^{n+1}(0)) < \varepsilon\}\end{aligned}$$

for some integer $q \geq 1$ and functions $u_j^k \in C^{1,\alpha}(M \cap B_\rho^{n+1}(0); M^\perp)$ solving the minimal surface equation on $M \cap B_\rho^{n+1}(0)$. It follows that $\int_{\operatorname{reg} V} |A|^2 \zeta^2 \leq \int_{\operatorname{reg} V} |\nabla \zeta|^2$ for each $\zeta \in C_c^1(\operatorname{reg} V)$, where A denotes the second fundamental form of $\operatorname{reg} V$. It is also clear, from Theorem 3.4, that V satisfies the structural property ($\mathcal{S}3$); for if not, there exists a point $Z \in \operatorname{spt} \|V\| \cap B_2^{n+1}(0)$ such that the (unique) tangent cone $\mathbf{C}_Z$ to V at Z is supported by the union of a finite number (≥ 3) of half-hyperplanes meeting along an $(n-1)$-dimensional subspace. By the definition of tangent cone and the fact that varifold convergence of stationary integral varifolds implies convergence in Hausdorff distance of the supports of the associated weight measures, for any given $\varepsilon_1 > 0$, there exists a number $\sigma \in (0, \operatorname{dist}(Z, \partial B_2^{n+1}(0)))$ such that for all sufficiently large k, $\operatorname{dist}(\operatorname{spt} \|\eta_{Z,\sigma\,\#} V_k\| \cap B_1^{n+1}(0), \operatorname{spt} \|\mathbf{C}_Z\| \cap B_1^{n+1}(0)) < \varepsilon_1$. This however contradicts Theorem 3.4 if we take $\varepsilon_1 = \varepsilon(1/2, \mathbf{C}_Z)$, where ε is as in Theorem 3.4. Thus $V \in \mathcal{S}_\alpha$, and hence $\mathcal{H}^{n-7+\gamma}(K) = 0$ for each $\gamma > 0$ if $n \geq 7$ and $K = \emptyset$ if $2 \leq n \leq 6$.

Finally, suppose $n = 7$ and consider any $V \in \mathcal{S}_\alpha$. To complete the proof of the theorem, it only remains to show that K is discrete. If this were false, there would exist points $Z, Z_j \in K$, $j = 1, 2, 3 \ldots$, such that $Z_j \neq Z$ for each $j = 1, 2, 3, \ldots$ and $Z_j \to Z$ as $j \to \infty$. Letting $\sigma_j = |Z - Z_j|$, we obtain, passing to a subsequence without changing notation, a tangent cone $\mathbf{C} = \lim_{j\to\infty} \eta_{Z,\sigma_j\,\#} V$. By the discussion above, $\mathbf{C} \in \mathcal{S}_\alpha$. Since $\sigma_j^{-1}(Z_j - Z) \in \mathbf{S}^{n-1} \cap \operatorname{sing} \eta_{Z,\sigma_j\,\#} V$, it follows, passing to a further subsequence, that $\sigma_j^{-1}(Z_j - Z) \to Z^\star \in \mathbf{S}^{n-1}$ and by Hausdorff convergence and Theorem 3.3, $Z^\star \in \operatorname{sing} \mathbf{C}$. Since $\mathbf{C}$ is a cone, it follows that $\{tZ^\star : t > 0\} \subset \operatorname{sing} \mathbf{C}$, which is impossible since $\mathbf{C} \in \mathcal{S}_\alpha$ and we have established that for $n = 7$, $\mathcal{H}^\gamma(K) = 0$ for each $\gamma > 0$ and any $V \in \mathcal{S}_\alpha$. □

18. Generalization to Riemannian manifolds

Let N be a smooth $(n+1)$-dimensional Riemannian manifold (without boundary) and for $X \in N$, let $\exp_X$ denote the exponential map at X. For each $X \in N$, let $R_X \in (0, \infty]$ be the injectivity radius at X.

Let $\widetilde{V}$ be a stationary integral n-varifold on N. Let $X_0 \in \operatorname{spt} \|\widetilde{V}\|$, $\mathcal{N}_{\rho_0}(X_0)$ be a normal coordinate ball of radius $\rho_0 \in (0, R_{X_0})$ around X_0. Then $V = \exp^{-1}_{X_0\,\#} \widetilde{V} \llcorner \mathcal{N}_{\rho_0}(X_0)$ is an integral n-varifold on $B^{n+1}_{\rho_0}(0) \subset T_{X_0} N \approx \mathbf{R}^{n+1}$, which is stationary with respect to the functional

$$\mathcal{F}_{X_0}(V) = \int_{B^{n+1}_{\rho_0}(0) \times G_n} |\Lambda_n DF(X) \circ S| \, dV(X, S), \tag{18.1}$$

where $F \equiv \exp_{X_0}$. Let $\psi \in C^1_c(B^{n+1}_{\rho_0}(0); \mathbf{R}^{n+1})$, and let φ_t, $t \in (-\varepsilon, \varepsilon)$ be the flow generated by ψ. By computing directly the first variation $\delta_{\mathcal{F}_{X_0}} V(\psi) \equiv \frac{d}{dt}\Big|_{t=0} \mathcal{F}_{X_0}(\varphi_{t\,\#} V)$ of V with respect to $\mathcal{F}_{X_0}$ and setting $\delta_{\mathcal{F}_{X_0}} V(\psi) = 0$, we see that the following bound holds (cf. [SS81, (1.7), (1.9), (1.11)]) for some constant μ depending only on the metric on N. (Such $\mu \in (0, \infty)$ exists by replacing N with a suitable open subset of N if necessary.)

($\mathcal{S}^\star 1$) For all $\psi \in C^1_c(B^{n+1}_{\rho_0}(0); \mathbf{R}^{n+1})$,

$$\left| \int_{B^{n+1}_{\rho_0}(0) \times G_n} \operatorname{div}_S \psi(X) \, dV(X, S) \right| \leq \mu \int_{B^{n+1}_{\rho_0}(0)} (|\psi(X)| + |X||\nabla \psi(X)|) \, d\|V\|(X).$$

Furthermore, for $\psi \in C^1_c(B^{n+1}_{\rho_0}(0) \setminus \operatorname{sing} V; \mathbf{R}^{n+1})$, the second variation

$$\delta^2_{\mathcal{F}_{X_0}} V(\psi) \equiv \frac{d^2}{dt^2}\bigg|_{t=0} \mathcal{F}_{X_0}(\varphi_{t\,\#} V)$$

of V with respect to $\mathcal{F}_{X_0}$ is given by (cf. [SS81, (1.8), (1.10), (1.12)])

$$\delta^2_{\mathcal{F}_{X_0}} V(\psi) = \int_{\operatorname{reg} V} \left(\sum_{i=1}^n |(D_{\tau_i} \psi)^\perp|^2 + (\operatorname{div}_{\operatorname{reg V}} \psi)^2 - \sum_{i,j=1}^n (\tau_i \cdot D_{\tau j} \psi) \cdot (\tau_j \cdot D_{\tau_i} \psi) \right) d\mathcal{H}^n + R(\psi),$$

where $\{\tau_1, \tau_2, \ldots, \tau_n\}$ is an orthonormal basis for the tangent space $T_X(\operatorname{reg} V)$ of $\operatorname{reg} V$ at X, $D_\tau \psi$ denotes the directional derivative of ψ in the direction τ and

$$|R(\psi)| \leq c\mu \int_{\operatorname{reg} V} \left(\widetilde{c}\mu |\psi|^2 + |\psi||\nabla \psi| + |X||\nabla \psi|^2 \right) d\mathcal{H}^n,$$

with c, $\widetilde{c}$ absolute constants. If $\operatorname{reg} V$ is orientable and ν is a continuous choice of unit normal to $\operatorname{reg} V$, we may, for any $\zeta \in C^1_c(\operatorname{reg} V)$, extend $\zeta\nu$ to a vector field in $C^1_c(B^{n+1}_{\rho_0}(0)\setminus \operatorname{sing} V;\mathbf{R}^{n+1})$ and take in the above $\psi=\zeta\nu$ to deduce that (cf. [SS81, (1.14), (1.15)])

$$\delta^2_{\mathcal{F}_{X_0}} V(\psi)=\int_{\operatorname{reg} V}\left(|\nabla\zeta|^2-|A|^2\zeta^2+H^2\zeta^2\right)d\mathcal{H}^n+R(\psi),$$

where A denotes the second fundamental form of $\operatorname{reg} V$, $|A|$ the length of A, H the mean curvature of $\operatorname{reg} V$ and

$$|R(\psi)|\le c\mu\int_{\operatorname{reg} V}\left(\widetilde{c}\mu|\zeta|^2+|\zeta||\nabla\zeta|+\zeta^2|A||X||\nabla\zeta|^2+|X|\zeta^2|A|^2\right)d\mathcal{H}^n.$$

If $\delta^2_{\mathcal{F}_{X_0}}(\psi)\ge 0$ for all $\psi=\zeta\nu$, $\zeta\in C^1_c(\operatorname{reg} V)$, then we have (cf. [SS81, (1.17)])

($\mathcal{S}^\star 2$) For all $\zeta\in C^1_c(\operatorname{reg} V)$ where c_1, c_2 are constants depending only on n,

$$\begin{aligned}\int_{\operatorname{reg} V\cap B^{n+1}_{\rho_0}(0)}|A|^2\zeta^2\,d\mathcal{H}^n&\le\int_{\operatorname{reg} V\cap B^{n+1}_{\rho_0}(0)}|\nabla\zeta|^2\,d\mathcal{H}^n\\&\quad+c_1\mu\int_{\operatorname{reg} V\cap B^{n+1}_{\rho_0}(0)}\Big(c_2\mu\zeta^2+\zeta|\nabla\zeta|+\zeta^2|A|+|X||\nabla\zeta|^2\\&\qquad\qquad+|X|\zeta^2|A|^2+c_2\mu|X|^2\zeta^2|A|^2\Big)\,d\mathcal{H}^n.\end{aligned}$$

For the rest of this discussion, we take μ, c_1, c_2 to be chosen as above and fixed.

Definitions. Let μ, c_1, c_2 be the positive numbers as above.

(1) By a *stable integral n-varifold $\widetilde{V}$ on N* we mean a stationary integral n-varifold $\widetilde{V}$ on N such that for each $X_0\in\operatorname{spt}\|\widetilde{V}\|$ and each normal ball $\mathcal{N}_{\rho_0}(X_0)\subset N$ around X_0, the integral n-varifold $V=(\exp^{-1}_{X_0})_\#\,\widetilde{V}\llcorner\mathcal{N}_{\rho_0}(X_0)$ on $B^{n+1}_{\rho_0}(0)\subset\mathbf{R}^{n+1}$ satisfies ($\mathcal{S}^\star 2$).

(2) For $\alpha\in(0,1)$, let $\widetilde{\mathcal{S}}_\alpha$ denote the collection of stable integral n-varifolds on N satisfying the structural condition ($\mathcal{S}3$) of Section 3 taken with normal ball $\mathcal{N}_\rho(Z)\subset N$ in place of $B^{n+1}_\rho(Z)$.

(3) For $\alpha\in(0,1)$, let $\mathcal{S}^\star_\alpha$ denote the collection of integral n-varifolds V on $B^{n+1}_1(0)\subset\mathbf{R}^{n+1}$ such that

$$V=\eta_{0,\rho\,\#}\exp^{-1}_{X\,\#}\widetilde{V}\llcorner\mathcal{N}_\rho(X)\tag{18.2}$$

for some $\widetilde{V}\in\widetilde{\mathcal{S}}_\alpha$, $X\in\operatorname{spt}\|\widetilde{V}\|$ and $\rho\in(0,R_X)$.

(4) For $\rho\in(0,\infty)$ and $\alpha\in(0,1)$, let $\mathcal{S}^\star_\alpha(\rho)$ be the set of integral n-varifolds $V\in\mathcal{S}^\star_\alpha$ such that (18.2) holds for some $\widetilde{V}\in\widetilde{\mathcal{S}}_\alpha$ and $X\in\operatorname{spt}\|\widetilde{V}\|$ with $R_X\ge\rho$.

Remark. Let $\rho\in(0,\infty)$, and suppose that $V\in\mathcal{S}^\star_\alpha(\rho)$. Then for each $Y\in\operatorname{spt}\|V\|\cap B^{n+1}_{1/2}(0)$,

$$\eta_{0,\rho/2\,\#}\,\tau_{Y\,\#}\,V\in\mathcal{S}^\star_\alpha(\rho/2),\tag{18.3}$$

where $\tau_Y=\exp^{-1}_{\exp_X(\rho Y)}\circ\exp_X\circ\,\eta_{0,\rho^{-1}}$. Note that $\tau_Y(Y)=0$.

We assert that the following direct analog of Theorem 3.1 holds:

THEOREM 18.1 (Regularity and Compactness Theorem—Manifold version). *Let N be a smooth $(n+1)$-dimensional Riemannian manifold, $X_0 \in N$ and $\alpha \in (0, 1/2)$. Let $\{\widetilde{V}_k\} \subset \widetilde{\mathcal{S}}_\alpha$ be a sequence with $X_0 \in \operatorname{spt} \|\widetilde{V}_k\|$ for each $k = 1, 2, \ldots$ and with*

$$\limsup_{k\to\infty} \|\widetilde{V}_k\|(N) < \infty.$$

Then there exist a subsequence $\{k'\}$ of $\{k\}$ and a varifold $\widetilde{V} \in \widetilde{\mathcal{S}}_\alpha$ with $X_0 \in \operatorname{spt} \|\widetilde{V}\|$ and with $\mathcal{H}^{n-7+\gamma}(\operatorname{sing} \widetilde{V} \cap N) = 0$ for each $\gamma > 0$ if $n \geq 7$, $\operatorname{sing} \widetilde{V} \cap N$ discrete if $n = 7$ and $\operatorname{sing} \widetilde{V} \cap N = \emptyset$ if $2 \leq n \leq 6$ such that $\widetilde{V}_{k'} \to \widetilde{V}$ as varifolds of N and smoothly (i.e., in the C^m topology for every m) locally in $N \setminus \operatorname{sing} \widetilde{V}$. In particular, if $\widetilde{W} \in \widetilde{\mathcal{S}}_\alpha$, then $\mathcal{H}^{n-7+\gamma}(\operatorname{sing} \widetilde{W} \cap N) = 0$ for each $\gamma > 0$ if $n \geq 7$, $\operatorname{sing} \widetilde{W} \cap N$ is discrete if $n = 7$ and $\operatorname{sing} \widetilde{W} \cap N = \emptyset$ if $2 \leq n \leq 6$.

By the preceding discussion, this theorem is equivalent to the assertion obtained from it by replacing N with $B_1^{n+1}(0) \subset \mathbf{R}^{n+1}$, X_0 with 0 and $\widetilde{\mathcal{S}}_\alpha$ with $\mathcal{S}_\alpha^\star$; the proof of the latter amounts to making minor modifications, as described below, to the proof of Theorem 3.1.

Step 1. Let V be an integral n-varifold of $B_1^{n+1}(0)$ such that (18.2) holds for some stationary integral n-varifold $\widetilde{V}$ of N, $X_0 \in \operatorname{spt} \|\widetilde{V}\|$ in place of X and $\rho_0 \in (0, R_{X_0})$ in place of ρ. By the discussion involving (5.3)–(5.9) of [SS81], we have, for each $0 < \sigma < \delta$, where $\delta = \delta(n, \mu\rho_0) \in (0, 1)$, the following facts:

$$\tau^{-n}\|V\|(B_\tau^{n+1}(0)) \leq (1 + 12n\mu\rho_0\sigma)\sigma^{-n}\|V\|(B_\sigma^{n+1}(0)) \tag{18.4}$$

for all τ with $0 < \tau \leq \sigma$; the density $\Theta(\|V\|, 0) = \lim_{\tau\to 0} \frac{\|V\|(B_\tau^{n+1}(0))}{\omega_n \tau^n}$ exists (and is finite); the function $\Theta(\|\cdot\|, 0)$ is upper semi-continuous;

$$\int_{B_\sigma^{n+1}(0)} \frac{|X^\perp|^2}{|X|^{n+2}} d\|V\|(X) \leq \frac{\|V\|(B_\sigma^{n+1}(0))}{\omega_n \sigma^n} - \Theta(\|V\|, 0) + C\sigma \frac{\|V\|(B_\sigma^{n+1}(0))}{\omega_n \sigma^n}, \tag{18.5}$$

where $C = C(n, \mu\rho_0) \in (0, \infty)$; tangent cones to V at $0 \in \operatorname{spt} \|V\|$ exist and are stationary integral hypercones of $\mathbf{R}^{n+1}$.

Let $\operatorname{VarTan}(V, 0)$ denote the set of tangent cones to V at 0. For $Y \in \operatorname{spt} \|V\| \cap B_{1/2}(0)$, let $\Theta(\|V\|, Y) = \Theta(\|\eta_{0,\rho_0/2\,\#}\, \tau_{Y\,\#} V\|, 0)$ (see 18.3) and $\operatorname{VarTan}(V, Y) = \operatorname{VarTan}(\eta_{0,\rho_0/2\,\#}\, \tau_{Y\,\#} V, 0)$. Recalling the well-known fact that if $\mathbf{C}$ is a stationary cone in a Euclidean space $\mathbf{R}^m$, then the set $\{Z \in \mathbf{R}^m : \Theta(\|\mathbf{C}\|, Z) = \Theta(\|\mathbf{C}\|, 0)\}$ is a linear subspace of $\mathbf{R}^m$, we deduce by the argument of Almgren's generalised stratification of stationary integral varifolds ([Alm00, Rem. 2.28]; see also [Sim96, §3.4]) the following:

Let V be an integral n-varifold of $B_1^{n+1}(0)$ such that (18.2) holds for some stationary integral n-varifold $\widetilde{V}$ of N, $X \in \operatorname{spt}\|\widetilde{V}\|$ and $\rho \in (0, R_X)$. For $k = 0, 1, 2, \ldots, n$, let $S_k = \{Y \in \operatorname{spt}\|V\| \cap B_{1/2}^{n+1}(0) : \dim\{Z \in \mathbf{R}^{n+1} : \Theta(\|\mathbf{C}\|, Z) = \Theta(\|\mathbf{C}\|, 0)\} \leq k \ \ \forall \mathbf{C} \in \operatorname{VarTan}(V, Y)\}$. Then $\dim_{\mathcal{H}}(S_k) \leq k$.

Step 2. We claim that the following analogs of Theorems 3.3 and 3.4 hold.

THEOREM 18.2 (Sheeting Theorem—Manifold Version). *Let $\alpha \in (0, 1/2)$, $\rho_0 \in (0, \infty)$ and q be any integer ≥ 1. Let $\alpha' = (2\alpha + 1)/4$. There exists a number $\varepsilon_0 = \varepsilon_0(n, q, \alpha, \mu\rho_0) \in (0, 1)$ such that if $V \in \mathcal{S}_\alpha^\star(\rho_0)$, $\omega_n^{-1}\|V\|(B_1^{n+1}(0)) < q + 1/2$, $\sigma \in (0, 1/2)$, $(q - 1/2) \leq (\omega_n\sigma^n)^{-1}\,\|V\|(B_\sigma^{n+1}(0)) < (q + 1/2)$ and $\sigma^{-1}\operatorname{dist}_{\mathcal{H}}(\operatorname{spt}\|V\| \cap (\mathbf{R} \times B_\sigma), \{0\} \times B_\sigma) + \sigma^{2\alpha'} < \varepsilon_0$, then*

$$V \llcorner (\mathbf{R} \times B_{\sigma/2}) = \sum_{j=1}^{q} |\operatorname{graph} u_j|,$$

where $u_j \in C^{1,\beta}(B_{\sigma/2})$ for each $j = 1, 2, \ldots, q$; $u_1 \leq u_2 \leq \cdots \leq u_q$ and

$$\sigma^{-1} \sup_{B_{\sigma/2}} |u_j| + \sup_{B_{\sigma/2}} |Du_j| + \sigma^\beta \sup_{X_1, X_2 \in B_{\sigma/2},\, X_1 \neq X_2} \frac{|Du_j(X_1) - Du_j(X_2)|}{|X_1 - X_2|^\beta}$$
$$\leq C\left(\sigma^{-n-2}\int_{\mathbf{R}\times B_\sigma} |x^1|^2\, d\|V\|(X) + \sigma^{2\alpha'}\right)^{1/2}.$$

Here $C = C(n, q, \alpha, \mu\rho_0) \in (0, \infty)$ and $\beta = \beta(n, q, \alpha, \mu\rho_0) \in (0, 1)$.

Remark. If the conclusions of Theorem 18.2 hold and V corresponds, as in (18.2), to some $\widetilde{V} \in \widetilde{\mathcal{S}}_\alpha$, $X = X_0 \in N \cap \operatorname{spt}\|\widetilde{V}\|$ and $\rho = \rho_0 \in (0, R_{X_0})$, then it follows that for each $j \in \{1, 2, \ldots, q\}$, $V_j \equiv |\operatorname{graph} \rho_0 u_j(\rho_0^{-1}(\cdot))|$ is stationary with respect to the functional $\mathcal{F}(\cdot) = \mathcal{F}_{X_0}\left((\cdot) \llcorner \mathbf{R} \times B_{\sigma/2}\right)$, where $\mathcal{F}_{X_0}$ is as in (18.1). Thus, by computing the associated Euler-Lagrange equation and applying elliptic regularity theory, we see that $u_j \in C^\infty(B_{\sigma/2})$ and satisfies an equation of the form

$$\sum_{k,\ell=1}^{n} a_{k\ell}^j D_k D_\ell u_j = f^j \tag{18.6}$$

on $B_{\sigma/2}$, with $|f^j(x)| \leq \mu\rho_0$ and $a_{k\ell}(x) = \delta_{k\ell} - \frac{D_k u_j(x) D_\ell u_j(x)}{\sqrt{1 + |Du_j(x)|^2}} + b_{k\ell}^j(x)$, where $|b_{k\ell}^j(x)| \leq \mu\rho_0\sigma$, for $x \in B_{\sigma/2}$.

THEOREM 18.3 (Minimum Distance Theorem—Manifold Version). *Let $\alpha \in (0, 1/2)$, $\rho_0 \in (0, \infty)$ and $\gamma \in (0, 1/2)$. Let $\alpha' = (2\alpha + 1)/4$. Suppose that $\mathbf{C}_0$ is an n-dimensional stationary cone in $\mathbf{R}^{n+1}$ such that* $\operatorname{spt}\|\mathbf{C}_0\|$ *is equal to a finite union of at least three distinct n-dimensional half-hyperplanes of $\mathbf{R}^{n+1}$ meeting along an $(n-1)$-dimensional subspace. Then there exists*

$\varepsilon = \varepsilon(\alpha, \gamma, \mu\rho_0, \mathbf{C}_0) \in (0,1)$ *such that if* $V \in \mathcal{S}^\star_\alpha(\rho_0)$, $\sigma \in (0, 1/2)$, $\Theta(\|V\|, 0) \geq \Theta(\|\mathbf{C}_0\|, 0)$ *and* $(\omega_n)^{-1}\|V\|(B_1^{n+1}(0)) \leq \Theta_{\mathbf{C}_0}(0) + \gamma$, *then*

$$\sigma^{\alpha'} + \sigma^{-1}\text{dist}_{\mathcal{H}}(\text{spt}\,\|V\| \cap B_\sigma^{n+1}(0), \text{spt}\,\|\mathbf{C}_0\| \cap B_\sigma^{n+1}(0)) \geq \varepsilon.$$

In particular, $\sigma^{-1}\text{dist}_{\mathcal{H}}(\text{spt}\,\|V\| \cap B_\sigma^{n+1}(0), \text{spt}\,\|\mathbf{C}_0\| \cap B_\sigma^{n+1}(0)) \geq \varepsilon/2$ *for sufficiently small* $\sigma > 0$.

The proof of Theorems 18.2 and 18.3 amounts to an easy modification of the induction argument given above for Theorems 3.3′ and 3.4, which is the "Euclidean case," viz. the case when $\mu = 0$ (which corresponds to the case when N is an open subset of $\mathbf{R}^{n+1}$ in Theorem 18.1). We outline the proof as follows:

(i) It follows from [SS81, Th. 1], that Theorem 18.2 holds if V, in place of the structural condition $(\mathcal{S}3)$, satisfies that

$$\dim_{\mathcal{H}}(\text{sing V}) \leq n - 7 \text{ in case } n \geq 7 \text{ and } \text{sing}\, V = \emptyset \text{ in case } n \leq 6,$$

together with all other hypotheses as in Theorem 18.2.

(ii) Let $\rho_0 \in (0, \infty)$, and let V be an integral n-varifold on $B_1^{n+1}(0)$ such that (18.2) holds with $\rho = \rho_0$ for some stationary integral n-varifold $\widetilde{V}$ on N and $X_0 \in \text{spt}\,\|\widetilde{V}\|$ with $R_{X_0} \geq \rho_0$. Let $\sigma \in (0,1)$, $\Lambda \in [1, \infty)$, and suppose that $(\omega_n\sigma^n)^{-1}\|V\|(B_\sigma^{n+1}(0)) \leq \Lambda$ and $\sigma^{-n-2}\int_{\mathbf{R}\times B_\sigma} |x^1|^2\, d\|V\|(X) + \sigma < 1$. By taking $\psi(X) = x^1\widetilde{\zeta}^2(X)e^1$ in $(\mathcal{S}^\star 1)$, where $\widetilde{\zeta} \in C_c^1(\mathbf{R} \times B_{3/4})$, we deduce that (18.7)

$$\int_{\mathbf{R}\times B_{3/4}} |\nabla\, x^1|^2\widetilde{\zeta}^2\, d\,\|\eta_{0,\sigma\,\#}\, V\|(X) \leq C\left(\int_{\mathbf{R}\times B_{3/4}} |x^1|^2 |\nabla\, \widetilde{\zeta}|^2\, d\,\|\eta_{0,\sigma\,\#}\, V\|(X) + \sigma\right)$$

for each $\widetilde{\zeta} \in C_c^1(\mathbf{R} \times B_{3/4})$, where $C = C(n, \Lambda, M, \mu\rho_0) \in (0, \infty)$ and $M = \sup_{\text{spt}\,\|\eta_{0,\sigma\,\#}\, V\| \cap (\mathbf{R}\times B_{3/4})} |\widetilde{\zeta}| + |D\widetilde{\zeta}|$. Choosing $\widetilde{\zeta}$ such that $\widetilde{\zeta}(x^1, x') = \zeta(x')$ in a neighborhood of $\text{spt}\,\|\eta_{0,\sigma\,\#}\, V\| \cap (\mathbf{R} \times B_{3/4})$, where $\zeta \in C_c^1(B_{3/4})$ is such that $\zeta \equiv 1$ on $B_{1/2}$, $0 \leq \zeta \leq 1$ and $|D\zeta| \leq 8$, we deduce from this that (18.8)

$$\int_{\mathbf{R}\times B_{1/2}} |\nabla\, x^1|^2\, d\,\|\eta_{0,\sigma\,\#}\, V\|(X) \leq C\left(\int_{\mathbf{R}\times B_{3/4}} |x^1|^2\, d\,\|\eta_{0,\sigma\,\#}\, V\|(X) + \sigma\right),$$

where $C = C(n, \Lambda, \mu\rho_0)$.

(iii) Let ρ_0, V be as in (ii), and let $\sigma \in (0, 3/4)$. With $\eta_{0,\sigma\,\#}\, V$ in place of V,

$$\sqrt{\sigma^{-n-2}\int_{\mathbf{R}\times B_\sigma} |x^1|^2\, d\|V\|(X) + \sigma}$$

in place of $\hat{E}_V$ and with the constants ε_0, C depending on n, q, $\mu\rho_0$, Theorem 5.1 holds; its proof amounts to modifying the argument of [Alm00, Th. 3.8] in obvious ways, making use of (18.4), (18.5) and (18.8).

(iv) Consequently, the case $q = 1$ of Theorem 18.2 follows by the excess improvement argument as in [All72, Chap. 8].

(v) From (iii) and the inequalities (18.5), (18.7), we deduce that for ρ_0, V as in (ii) and $\sigma \in (0, 3/4)$, Theorem 7.1 hold with $\eta_{0,\sigma\,\#}\, V$ in place of V and

$$\sqrt{\sigma^{-n-2} \int_{\mathbf{R}\times B_\sigma} |x^1|^2 \, d\|V\|(X) + \sigma}$$

in place of $\hat{E}_V$, again with the constants ε_1, C etc. depending also on $\mu\rho_0$.

(vi) For what follows, fix $\alpha \in (0, 1/2)$, $\rho_0 \in (0, \infty)$, and let $\alpha' = (2\alpha+1)/4$. For $V \in \mathcal{S}^\star_\alpha(\rho_0)$ and $\sigma \in (0, 3/4)$, let

$$\hat{E}^\star_V(\sigma) = \sqrt{\sigma^{-n-2} \int_{\mathbf{R}\times B_\sigma} |x^1|^2 \, d\|V\|(X) + \sigma^{2\alpha'}}.$$

Let q be an integer ≥ 2, and assume inductively the validity of Theorem 18.2 with $1, 2, \ldots, q-1$ in place of q and that of Theorem 18.3 if $\Theta\left(\|\mathbf{C}_0\|, 0\right) \in \{3/2, 2, 5/2, \ldots, q - 1/2, q\}$.

(vii) For each $k = 1, 2, 3, \ldots$, let $\sigma_k \in (0, 3/4)$, $V_k \in \mathcal{S}^\star_\alpha(\rho_0)$ be such that $\omega_n^{-1}\|V_k\|(B^{n+1}_1(0)) < q+1/2$, $\sigma_k \to 0$ and $(q-1/2) \leq (\omega_n\sigma_k^n)^{-1}\, \|V_k\|(B^{n+1}_{\sigma_k}(0)) < (q + 1/2)$. If $\hat{E}^\star_{V_k}(\sigma_k) \to 0$, then as in the discussion following Theorem 5.1, we may blow up the sequence $\{\eta_{0,\sigma_k\,\#}\, V_k \llcorner B^{n+1}_1(0)\}$ by $\hat{E}^\star_{V_k}(\sigma_k)$. We shall continue to call a function $v \in W^{1,2}_{\mathrm{loc}}(B_1; \mathbf{R}^q) \cap L^2(B_1; \mathbf{R}^q)$ produced this way a coarse blow-up.

(viii) By the reasoning of Remarks 2 and 3 of Section 6 and Step 1 above, we have the following:

> *Let q be an integer ≥ 2 and suppose that the induction hypotheses as in* (vi) *hold. If $V \in \mathcal{S}^\star_\alpha(\rho_0)$, $\Omega \subseteq B^{n+1}_1(0)$ is open and $\Theta(\|V\|, Z) < q$ for each $Z \in \operatorname{spt}\|V\| \cap \Omega$, then $\mathcal{H}^{n-7+\gamma}(\operatorname{sing} V \llcorner \Omega) = 0$ for each $\gamma > 0$ if $n \geq 7$ and $\operatorname{sing} V \llcorner \Omega = \emptyset$ if $2 \leq n \leq 6$.*

(ix) The collection $\mathcal{B}^\star_q$ of all coarse blow-ups v (as in (vii)) is a proper blow-up class, viz. $\mathcal{B}^\star_q$ satisfies properties ($\mathcal{B}1$)–($\mathcal{B}7$) of Section 4. Verification of properties ($\mathcal{B}1$)–($\mathcal{B}3$), ($\mathcal{B}5$) and ($\mathcal{B}6$) proceeds in the same way as for the Euclidean case described in Section 8 above. In view of (i), property ($\mathcal{B}4$) follows from the corresponding argument for the Euclidean case, also described in Section 8, with the inequality (18.5) taking the place of the monotonicity identity (7.1).

Property ($\mathcal{B}7$) is verified by separately establishing the same two cases as Cases 1 and 2 of Section 9. With regard to Case 1, note that by taking $\psi(X) = \widetilde{\zeta}(X)e^2$ in ($\mathcal{S}^\star1$), where $\widetilde{\zeta} \in C^1_c(\mathbf{R} \times B_{3/4})$, it follows that for each $k = 1, 2, \ldots,$

$$\begin{aligned}
&\left|\int_{\mathbf{R}\times B_{3/4}} \nabla x^2 \cdot \nabla \widetilde{\zeta}\, d\|\eta_{0,\sigma_k\,\#}\, V_k\|(X)\right| \\
&\qquad \leq C \sup_{\operatorname{spt}\|\eta_{0,\sigma_k\,\#}\, V_k\|\cap(\mathbf{R}\times B_{3/4})} \left(|\widetilde{\zeta}| + |D\widetilde{\zeta}|\right)\sigma_k \\
&\qquad \leq C \sup_{\operatorname{spt}\|\eta_{0,\sigma_k\,\#}\, V_k\|\cap(\mathbf{R}\times B_{3/4})} \left(|\widetilde{\zeta}| + |D\widetilde{\zeta}|\right)\sigma_k^{1-2\alpha'} \left(\hat{E}^{\star}_{V_k}(\sigma_k)\right)^2,
\end{aligned}$$

where $C = C(n,q,\mu\rho_0)$. Case 1 is established by taking this in place of (9.8) and (18.7) in place of (5.1) in the argument of Lemma 9.1. With regard to Case 2, we note that the following analogue of Lemma 13.1 holds. Here $\mathcal{C}_q$, $\mathcal{C}_q(p)$ are as defined in Section 10.

Lemma 18.4. *Let q be an integer ≥ 2, $\alpha \in (0,1/2)$, $\theta \in (0,1/4)$ and $\rho_0 \in (0,\infty)$. There exist numbers $\overline{\varepsilon} = \overline{\varepsilon}(n,q,\alpha,\theta,\mu\rho_0) \in (0,1/2)$, $\overline{\gamma} = \overline{\gamma}(n,q,\alpha,\theta,\mu\rho_0) \in (0,1/2)$ and $\overline{\beta} = \overline{\beta}(n,q,\alpha,\theta,\mu\rho_0) \in (0,1/2]$ such that the following is true: Let $\sigma \in (0,1)$, and suppose that the induction hypotheses as in* (vi) *and the following hold.*

(1) $V \in \mathcal{S}^{\star}_{\alpha}(\rho_0)$, $\quad \Theta(\|V\|,0) \geq q$, $\quad (\omega_n\sigma^n)^{-1}\|V\|(B^{n+1}_{\sigma}(0)) < q + 1/2$.

(2) $\mathbf{C} = \sum_{j=1}^{q} |H_j| + |G_j| \in \mathcal{C}_q$, *where for each $j \in \{1,2,\ldots,q\}$, H_j is the half-space defined by $H_j = \{(x^1,x^2,y) \in \mathbf{R}^{n+1} : x^2 < 0 \text{ and } x^1 = \lambda_j x^2\}$ and G_j is the half-space defined by $G_j = \{(x^1,x^2,y) \in \mathbf{R}^{n+1} : x^2 > 0 \text{ and } x^1 = \mu_j x^2\}$, with λ_j, μ_j constants, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_q$ and $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_q$.*

(3) $\left(\hat{E}^{\star}_V(\sigma)\right)^2 \equiv \int_{\mathbf{R}\times B_1} |x^1|^2 d\|\eta_{0,\sigma\,\#}\, V\|(X) + \sigma^{2\alpha'} < \varepsilon$, *where* $\alpha' = (2\alpha+1)/4$.

(4) $\{Z : \Theta(\|\eta_{0,\sigma\,\#}\, V\|, Z) \geq q\} \cap \left(\mathbf{R}\times(B_{1/2}\setminus\{|x^2| < 1/16\})\right) = \emptyset$.

(5) $\int_{\mathbf{R}\times(B_{1/2}\setminus\{|x^2|<1/16\})} \operatorname{dist}^2(X, \operatorname{spt}\|\eta_{0,\sigma\,\#}\, V\|)\, d\|\mathbf{C}\|(X)$
$\qquad + \int_{\mathbf{R}\times B_1} \operatorname{dist}^2(X, \operatorname{spt}\|\mathbf{C}\|)\, d\|\eta_{0,\sigma\,\#}\, V\|(X) \leq \gamma \left(\hat{E}^{\star}_V(\sigma)\right)^2$.

(6) $\left(\hat{E}^{\star}_V\right)^2 < \frac{3}{2} M_0 \inf_{\{P\in G_n : P\cap(\{0\}\times\mathbf{R}^n) = \{0\}\times\mathbf{R}^{n-1}\}} \int_{\mathbf{R}\times B_1} \operatorname{dist}^2(X,P), d\|V\|(X)$ $+\sigma^{2\alpha'}$, *where $M_0 = M_0(n,q) \in (1,\infty)$ is the constant defined in Section* 10.

(7) *Either*
 (i) $\mathbf{C} \in \mathcal{C}_q(4)$, *or*
 (ii) $q \geq 3$, $\mathbf{C} \in \mathcal{C}_q(p)$ *for some* $p \in \{5,\ldots,2q\}$ *and*

$$\begin{aligned}
&\int_{\mathbf{R}\times(B_{1/2}\setminus\{|x^2|<1/16\})} \operatorname{dist}^2(X, \operatorname{spt}\|\eta_{0,\sigma\,\#}\, V\|)\, d\|\mathbf{C}\|(X) \\
&\qquad + \int_{\mathbf{R}\times B_1} \operatorname{dist}^2(X, \operatorname{spt}\|\mathbf{C}\|)\, d\|\eta_{0,\sigma\,\#}\, V\|(X) \\
&\quad \leq \beta \inf_{\widetilde{\mathbf{C}}\in\bigcup_{k=4}^{p-1}\mathcal{C}_q(k)} \left(\int_{\mathbf{R}\times(B_{1/2}\setminus\{|x^2|<1/16\})} \operatorname{dist}^2(X, \operatorname{spt}\|\eta_{0,\sigma\,\#}\, V\|)\, d\|\widetilde{\mathbf{C}}\|(X)\right. \\
&\qquad\qquad\qquad \left. + \int_{\mathbf{R}\times B_1} \operatorname{dist}^2(X, \operatorname{spt}\|\widetilde{\mathbf{C}}\|)\, d\|\eta_{0,\sigma\,\#}\, V\|(X)\right).
\end{aligned}$$

Then there exist an orthogonal rotation Γ *of* $\mathbf{R}^{n+1}$ *and a cone* $\mathbf{C}' \in \mathcal{C}_q$ *such that the conclusions of Lemma* 13.1 *hold with* $\eta_{0,\sigma\,\#}\,V$ *in place of* V, $\hat{E}^\star_V(\sigma)$ *in place of* $\hat{E}_V$,

$$E^\star_V(\mathbf{C},\sigma) \equiv \sqrt{\int_{\mathbf{R}\times B_1} \mathrm{dist}^2(X, \mathrm{spt}\,\|\mathbf{C}\|)\, d\|\eta_{0,\sigma\,\#}\,V\|(X) + \sigma^{2\alpha'}}$$

in place of E_V *and with the constants* $\overline{\kappa}$, $\overline{C}_0$, $\overline{\gamma}_0$, $\overline{\nu}$, $\overline{C}_1$, $\overline{C}_2 \in (0,\infty)$ *depending only on* n, q, α *and* $\mu\rho_0$.

In proving this, note first that if $\sigma^{2\alpha'} > \int_{\mathbf{R}\times B_1} |x^1|^2\, d\|\eta_{0,\sigma\,\#}\,V\|(X)$, then, provided $\gamma < \theta^{n+4}/2$, we trivially have that

$$\begin{aligned}\theta^{-n-2}\int_{\mathbf{R}\times B_\theta} \mathrm{dist}^2(X, \mathrm{spt}\,\|\mathbf{C}\|)\, d\|\eta_{0,\sigma\,\#}\,V\|(X) &\leq \theta^{-n-2}\gamma\left(\hat{E}^\star_V(\sigma)\right)^2 \\ &\leq 2\theta^{-n-2}\gamma\sigma^{2\alpha'} \leq \theta^2\sigma^{2\alpha'} \leq \theta^2\left(E^\star_V(\mathbf{C},\sigma)\right)^2.\end{aligned}$$

Thus conclusions (a)–(d) hold with $\mathbf{C}' = \mathbf{C}$ and $\Gamma=$ Identity, and conclusions (e) and (f) can be checked as in the proof of Lemma 13.1. Hence we may assume without loss of generality that

$$\left(\hat{E}^\star_V(\sigma)\right)^2 \leq 2\int_{\mathbf{R}\times B_1} |x^1|^2\, d\|\eta_{0,\sigma\,\#}\,V\|(X).$$

With this additional assumption and with the help of inequality (18.5), the obvious analogues of Theorem 10.1 and Corollary 10.2 can be established; consequently, Lemma 18.4 can be proved by making obvious modifications to the entire argument leading to Lemma 13.1, as described in Sections 10–13.

The obvious analog of Lemma 13.3 then follows; note, in particular, that in the conclusions of this modified lemma we must take

$$\begin{aligned}Q^\star_V(\mathbf{C},\sigma) \equiv \Bigg(&\int_{\mathbf{R}\times(B_{1/2}\setminus\{|x^2|<1/16\})} \mathrm{dist}^2(X, \mathrm{spt}\,\|\eta_{0,\sigma\,\#}\,\eta_{0,\sigma\,\#}\,V\|)\, d\|\mathbf{C}\|(X) \\ &+ \int_{\mathbf{R}\times B_1} \mathrm{dist}^2(X, \mathrm{spt}\,\|\mathbf{C}\|)\, d\|\eta_{0,\sigma\,\#}\,V\|(X) + \sigma^{2\alpha'}\Bigg)^{1/2}\end{aligned}$$

in place of Q_V, and note that the modified lemma yields that for some $j \in \{1, 2, \ldots, 2q-3\}$, $\mathbf{C}' \in \mathcal{C}_q$ and some orthogonal rotation Γ of $\mathbf{R}^{n+1}$,

$$\begin{aligned}&\int_{\mathbf{R}\times(B_{1/2}\setminus\{|x^2|<1/16\})} \mathrm{dist}^2(X, \mathrm{spt}\,\|\eta_{0,\theta_j\sigma\,\#}\,\eta_{0,\sigma\,\#}\,V\|)\, d\|\Gamma_\#\,\mathbf{C}'\|(X) \\ &\quad + \int_{\mathbf{R}\times B_1} \mathrm{dist}^2(X, \mathrm{spt}\,\|\Gamma_\#\,\mathbf{C}'\|)\, d\|\eta_{0,\theta_j\sigma\,\#}\,V\|(X) \leq \nu_j\theta_j^2\left(Q^\star_V(\sigma)\right)^2,\end{aligned}$$

where the parameters $\theta_1, \ldots, \theta_{2q-3}$ and the constants $\nu_1, \ldots \nu_{2q-3}$ are analogous to the same quantities as in Lemma 13.3, with ν_1 depending only on n, q, α, $\mu\rho_0$ and for $j \in \{2, 3, \ldots, 2q-3\}$, ν_j depending only on n, q, α, $\theta_1, \ldots, \theta_{j-1}$,$\mu\rho_0$. By choosing $\theta_1, \theta_2, \ldots, \theta_{2q-3}$ in that order, depending only on n, q, α and $\mu\rho_0$, to ensure that $\nu_j \theta_j^2 < \frac{1}{2}\theta_j^{2\alpha}$ and $\theta_j^{2\alpha'} < \frac{1}{2}\theta_j^{2\alpha}$ for each $j = 1, 2, \ldots, 2q-3$, we deduce that under the hypotheses of the modified lemma,

$$\left(Q_V^\star(\Gamma_\#\, \mathbf{C}', \theta_j \sigma)\right)^2 \leq \theta_j^{2\alpha} \left(Q_V^\star(\mathbf{C}, \sigma)\right)^2$$

for some $j \in \{1, 2, \ldots, 2q-3\}$, $\mathbf{C}' \in \mathcal{C}_q$ and an orthogonal rotation Γ of $\mathbf{R}^{n+1}$. In view of the remark preceding Theorem 18.1, the iterative application of this as in Lemma 14.1 gives the analog of Lemma 14.1; arguing as in Corollary 14.2 then establishes Case 2, completing the proof that $\mathcal{B}_q^\star$ is a proper blow-up class.

(x) In view of (i) and (18.6), the argument of Section 15 carries over to yield Theorem 18.2 for $q \geq 2$, subject to the induction hypotheses as in (vi). First in case $\Theta(\|\mathbf{C}_0\|, 0) = q + 1/2$ and then in case $\Theta(\mathbf{C}_0\|, 0) = q + 1$, again subject to the induction hypotheses as in (vi), Theorem 18.3 follows from the argument, with obvious modifications, of Section 16. In particular, note that in view of the "monotonicity inequality" (18.5) needed in the proof, and the need to use directly the first variation inequality $(\mathcal{S}^\star 1)$ in establishing regularity of blow-ups as in Theorem 16.7, we must take

$$\mathcal{E}_V^\star(\mathbf{C}, \sigma) \equiv \left(\int_{B_1^{n+1}(0)} \operatorname{dist}^2(X, \operatorname{spt} \|\mathbf{C})\|)\, d\|\eta_{0,\sigma\,\#}\, V\|(X) + \sigma^{2\alpha'} \right)^{1/2}$$

in place of the excess $\mathcal{E}$ used in Section 16 (see Lemma 16.8). Same modification applies to the excess $\mathcal{Q}$ used in Lemma 16.9.

Step 3: In view of Step 1, Step 2 and the fact that Allard's integral varifold compactness theorem ([All72, Th. 6.4]) holds in Riemannian manifolds, Theorem 18.1 follows from the argument of Theorem 3.1 in Section 17.

19. **A sharp varifold maximum principle**

We conclude this paper by pointing out an immediate application of Theorem 18.1; namely, the following optimal strong maximum principle for codimension 1 stationary integral varifolds.

THEOREM 19.1. *Let N be a smooth $(n+1)$-dimensional Riemannian manifold.*

(a) *If V_1, V_2 are stationary integral n-varifolds on N such that*

$$\mathcal{H}^{n-1}(\operatorname{spt} \|V_1\| \cap \operatorname{spt} \|V_2\|) = 0,$$

then $\operatorname{spt} \|V_1\| \cap \operatorname{spt} \|V_2\| = \emptyset$.

(b) *Let Ω_1, Ω_2 be open subsets of N with $\Omega_1 \subset \Omega_2$ and $M_i = \partial\, \Omega_i$, $i = 1, 2$. If for $i = 1, 2$, M_i is connected, $\mathcal{H}^{n-1}(\operatorname{sing} M_i) = 0$ and $V_i \equiv |M_i|$ is stationary in N, then either* $\operatorname{spt} \|V_1\| = \operatorname{spt} \|V_2\|$ *or* $\operatorname{spt} \|V_1\| \cap \operatorname{spt} \|V_2\| = \emptyset$. *Here* $\operatorname{sing} M_i = M_i \setminus \operatorname{reg} M_i$, *where* $\operatorname{reg} M_i$ *is the set of points $X \in M_i$ with the property that there exists a number $\sigma = \sigma(X) > 0$ such that $M_i \cap B_\sigma^{n+1}(X)$ is a smooth, properly embedded hypersurface of $B_\sigma^{n+1}(X)$ with no boundary in $B_\sigma^{n+1}(X)$.*

Remark. These results were established by T. Ilmanen ([Ilm96]) under the stronger hypotheses that

$$\mathcal{H}^{n-2}(\operatorname{spt}\|V_1\| \cap \operatorname{spt}\|V_2\|) = 0$$

in part (a) and

$$\mathcal{H}^{n-2}(\operatorname{sing} M_i) = 0, \quad i = 1, 2,$$

in part (b). Obvious examples show that for any $\gamma > 0$, neither of these hypotheses can be weakened to $\mathcal{H}^{n-1+\gamma}(\cdot) = 0$.

Proof. The argument of [Ilm96] carries over, with (2) of [Ilm96] replaced by the hypothesis

$$\mathcal{H}^{n-1}(\operatorname{sing} M) = 0$$

and Theorems (8), (9) therein replaced by our Theorem 18.1. □

References

[All72] W. K. Allard, On the first variation of a varifold, *Ann. of Math.* **95** (1972), 417–491. MR 0307015. Zbl 0252.49028. http://dx.doi.org/10.2307/1970868.

[Alm66] F. J. Almgren, Jr., Some interior regularity theorems for minimal surfaces and an extension of Bernstein's theorem, *Ann. of Math.* **84** (1966), 277–292. MR 0200816. Zbl 0146.11905. http://dx.doi.org/10.2307/1970520.

[Alm00] F. J. Almgren, Jr., *Almgren's Big Regularity Paper*: *Q-Valued Functions Minimizing Dirichlet's Integral and the Regularity of Area Minimizing Rectifiable Currents up to Codimension Two*, *World Scientific Monograph Series in Math.* **1**, World Scientific Publ. Co., Inc., River Edge, NJ, 2000. MR 1777737. Zbl 0985.49001.

[DG61] E. De Giorgi, *Frontiere Orientate di Misura Minima*, *Seminario di Matematica della Scuola Normale Superiore di Pisa*, 1960-61, Editrice Tecnico Scientifica, Pisa, 1961. MR 0179651. Zbl 0296.49031.

[Fed69] H. Federer, *Geometric Measure Theory*, *Grundlehren Math. Wissen.* **153**, Springer-Verlag, New York, 1969. MR 0257325. Zbl 0176.00801.

[Fed70] H. Federer, The singular sets of area minimizing rectifiable currents with codimension one and of area minimizing flat chains modulo two with arbitrary codimension, *Bull. Amer. Math. Soc.* **76** (1970), 767–771. MR 0260981. Zbl 0194.35803. http://dx.doi.org/10.1090/S0002-9904-1970-12542-3.

[Fle62] W. H. Fleming, On the oriented Plateau problem, *Rend. Circ. Mat. Palermo* **11** (1962), 69–90. MR 0157263. Zbl 0107.31304. http://dx.doi.org/10.1007/BF02849427.

[HS79] R. Hardt and L. Simon, Boundary regularity and embedded solutions for the oriented Plateau problem, *Ann. of Math.* **110** (1979), 439–486. MR 0554379. Zbl 0457.49029. http://dx.doi.org/10.2307/1971233.

[Ilm96] T. Ilmanen, A strong maximum principle for singular minimal hypersurfaces, *Calc. Var. Partial Differential Equations* **4** (1996), 443–467. MR 1402732. Zbl 0863.49030. http://dx.doi.org/10.1007/BF01246151.

[Mor66] C. B. Morrey, Jr., *Multiple Integrals in the Calculus of Variations*, *Grundlehren Math. Wissen.* **130**, Springer-Verlag, New York, 1966. MR 0202511. Zbl 0142.38701. http://dx.doi.org/10.1007/978-3-540-69952-1.

[Mos77] M. P. Moschen, Principio di massimo forte per le frontiere di misura minima, *Ann. Univ. Ferrara Sez. VII* **23** (1977), 165–168 (1978). MR 0482508. Zbl 0384.49030.

[Rei60] E. R. Reifenberg, Solution of the Plateau Problem for m-dimensional surfaces of varying topological type, *Acta Math.* **104** (1960), 1–92. MR 0114145. Zbl 0099.08503. http://dx.doi.org/10.1007/BF02547186.

[Ros10] L. Rosales, The geometric structure of solutions to the two-valued minimal surface equation, *Calc. Var. Partial Differential Equations* **39** (2010), 59–84. MR 2659679. Zbl 1195.49051. http://dx.doi.org/10.1007/s00526-009-0301-y.

[SS81] R. Schoen and L. Simon, Regularity of stable minimal hypersurfaces, *Comm. Pure Appl. Math.* **34** (1981), 741–797. MR 0634285. Zbl 0497.49034. http://dx.doi.org/10.1002/cpa.3160340603.

[Sim83] L. Simon, *Lectures on Geometric Measure Theory*, *Proc. Centre Math. Anal. Austral. Nat. Univ.* **3**, Australian National University Centre for Mathematical Analysis, Canberra, 1983. MR 0756417. Zbl 0546.49019.

[Sim87] L. Simon, A strict maximum principle for area minimizing hypersurfaces, *J. Differential Geom.* **26** (1987), 327–335. MR 0906394. Zbl 0625.53052. Available at http://projecteuclid.org/euclid.jdg/1214441373.

[Sim93] L. Simon, Cylindrical tangent cones and the singular set of minimal submanifolds, *J. Differential Geom.* **38** (1993), 585–652. MR 1243788. Zbl 0819.53029. Available at http://projecteuclid.org/euclid.jdg/1214454484.

[Sim96] L. Simon, *Theorems on Regularity and Singularity of Energy Minimizing Maps*, *Lectures Math. ETH Zürich*, Birkhäuser, Basel, 1996. MR 1399562. Zbl 0864.58015. http://dx.doi.org/10.1007/978-3-0348-9193-6.

[SW07] L. Simon and N. Wickramasekera, Stable branched minimal immersions with prescribed boundary, *J. Differential Geom.* **75** (2007), 143–173. MR 2282727. Zbl 1109.53064. Available at http://projecteuclid.org/euclid.jdg/1175266256.

[Sim68] J. Simons, Minimal varieties in riemannian manifolds, *Ann. of Math.* **88** (1968), 62–105. MR 0233295. Zbl 0181.49702. http://dx.doi.org/10.2307/1970556.

[SW89] B. Solomon and B. White, A strong maximum principle for varifolds that are stationary with respect to even parametric elliptic functionals, *Indiana Univ. Math. J.* **38** (1989), 683–691. MR 1017330. Zbl 0711.49059. http://dx.doi.org/10.1512/iumj.1989.38.38032.

[Wic04] N. Wickramasekera, A rigidity theorem for stable minimal hypercones, *J. Differential Geom.* **68** (2004), 433–514. MR 2144538. Zbl 1085.53055. Available at http://projecteuclid.org/euclid.jdg/1115669592.

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