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Limit theorems for translation flows

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Abstract

The aim of this paper is to obtain an asymptotic expansion for ergodic integrals of translation flows on flat surfaces of higher genus (Theorem 1) and to give a limit theorem for these flows (Theorem 2).

Contents

1. **Introduction**

1.1. *Outline of the main results.* A compact Riemann surface endowed with an abelian differential admits two natural flows, called, respectively, *horizontal* and *vertical*. One of the main objects of this paper is the space $\mathfrak{B}^+$ of Hölder cocycles over the vertical flow, invariant under the holonomy by the horizontal flow. Equivalently, cocycles in $\mathfrak{B}^+$ can be viewed, in the spirit of R. Kenyon [29] and F. Bonahon [8], [9], as finitely-additive transverse invariant measures for the horizontal foliation of our abelian differential. Cocycles in $\mathfrak{B}^+$ are closely connected to the invariant distributions for translation flows in the sense of G. Forni [21].

The space $\mathfrak{B}^+$ is finite-dimensional, and for a generic abelian differential, the dimension of $\mathfrak{B}^+$ is equal to the genus of the underlying surface. Theorem 1, which extends earlier work of A. Zorich [46] and G. Forni [21], states that the time integral of a Lipschitz function under the vertical flow can be uniformly approximated by a suitably chosen cocycle from $\mathfrak{B}^+$ up to an error that grows more slowly than any power of time. The renormalizing action of the Teichmüller flow on the space of Hölder cocycles now allows one to obtain limit theorems for translation flows on flat surfaces (Theorem 2).

The statement of Theorem 2 can be informally summarized as follows. Taking the leading term in the asymptotic expansion of Theorem 1, to a generic abelian differential one assigns a compactly supported probability measure on the space of continuous functions on the unit interval. The normalized distribution of the time integral of a Lipschitz function converges, with respect to weak topology, to the trajectory of the corresponding "asymptotic distribution" under the action of the Teichmüller flow. Convergence is exponential with respect to both the Lévy-Prohorov and the Kantorovich-Rubinstein metric.

1.2. *Hölder cocycles over translation flows.* Let $\rho \geq 2$ be an integer, let M be a compact orientable surface of genus ρ, and let ω be a holomorphic one-form on M. Denote by $\nu = i(\omega \wedge \overline{\omega})/2$ the area form induced by ω, and assume that $\nu(M) = 1$.

Let h_t^+ be the *vertical* flow on M (i.e., the flow corresponding to $\Re(\omega)$); let h_t^- be the *horizontal* flow on M (i.e., the flow corresponding to $\Im(\omega)$). The flows h_t^+, h_t^- preserve the area ν.

Take $x \in M$, $t_1, t_2 \in \mathbb{R}_+$, and assume that the closure of the set

$$\{h^+_{\tau_1} h^-_{\tau_2} x, 0 \leq \tau_1 < t_1, 0 \leq \tau_2 < t_2\} \tag{1}$$

does not contain zeros of the form ω. The set (1) is then called *an admissible rectangle* and denoted $\Pi(x, t_1, t_2)$. Let $\overline{\mathfrak{C}}$ be the semi-ring of admissible rectangles.

Consider the linear space $\mathfrak{B}^+$ of Hölder cocycles $\Phi^+(x,t)$ over the vertical flow h_t^+ that are invariant under horizontal holonomy. More precisely, a function $\Phi^+(x,t) : M \times \mathbb{R} \to \mathbb{R}$ belongs to the space $\mathfrak{B}^+$ if it satisfies

ASSUMPTION 1.1.
(1) $\Phi^+(x, t+s) = \Phi^+(x,t) + \Phi^+(h_t^+ x, s)$;
(2) *there exists* $t_0 > 0$, $\theta > 0$ *such that* $|\Phi^+(x,t)| \leq t^\theta$ *for all* $x \in M$ *and all* $t \in \mathbb{R}$ *satisfying* $|t| < t_0$;
(3) *if* $\Pi(x, t_1, t_2)$ *is an admissible rectangle*, *then* $\Phi^+(x, t_1) = \Phi^+(h^-_{t_2} x, t_1)$.

A cocycle $\Phi^+ \in \mathfrak{B}^+$ can equivalently be thought of as a finitely-additive Hölder measure defined on all arcs $\gamma = [x, h_t^+ x]$ of the vertical flow and invariant under the horizontal flow. It will often be convenient to identify the cocycle with the corresponding finitely-additive measure. For example, let ν^+ be the Lebesgue measure on leaves of the vertical foliation; the corresponding cocycle Φ_1^+ defined by $\Phi_1^+(x,t) = t$ of course belongs to $\mathfrak{B}^+$.

In the same way define the space $\mathfrak{B}^-$ of Hölder cocycles $\Phi^-(x,t)$ over the horizontal flow h_t^- that are invariant under vertical holonomy. A cocycle $\Phi^- \in \mathfrak{B}^-$ can equivalently be thought of as a finitely-additive Hölder measure defined on all arcs $\tilde{\gamma} = [x, h_t^- x]$ of the horizontal flow and invariant under the vertical flow. Let ν^- be the Lebesgue measure on leaves of the horizontal foliation. The corresponding cocycle Φ_1^- is defined by the formula $\Phi_1^-(x,t) = t$; of course, $\Phi_1^- \in \mathfrak{B}^-$.

Given $\Phi^+ \in \mathfrak{B}^+$, $\Phi^- \in \mathfrak{B}^-$, a finitely additive measure $\Phi^+ \times \Phi^-$ on the semi-ring $\overline{\mathfrak{C}}$ of admissible rectangles is introduced by the formula

$$\Phi^+ \times \Phi^-(\Pi(x,t_1,t_2)) = \Phi^+(x,t_1) \cdot \Phi^-(x,t_2). \tag{2}$$

In particular, for $\Phi^- \in \mathfrak{B}^-$, set $m_{\Phi^-} = \nu^+ \times \Phi^-$:

$$m_{\Phi^-}(\Pi(x,t_1,t_2)) = t_1 \Phi^-(x,t_2). \tag{3}$$

For any $\Phi^- \in \mathfrak{B}^-$, the measure m_{Φ^-} satisfies $(h_t^+)_* m_{\Phi^-} = m_{\Phi^-}$ and is an invariant distribution in the sense of G. Forni [20], [21]. For instance, $m_{\Phi_1^-} = \nu$.

An $\mathbb{R}$-linear pairing between $\mathfrak{B}^+$ and $\mathfrak{B}^-$ is given, for $\Phi^+ \in \mathfrak{B}^+$, $\Phi^- \in \mathfrak{B}^-$, by the formula

$$\langle \Phi^+, \Phi^- \rangle = \Phi^+ \times \Phi^-(M). \tag{4}$$

1.3. *Characterization of cocycles.* For an abelian differential $\mathbf{X} = (M, \omega)$, let $\mathfrak{B}_c^+(\mathbf{X})$ be the space of continuous holonomy-invariant cocycles. More precisely, a function $\Phi^+(x,t) : M \times \mathbb{R} \to \mathbb{R}$ belongs to the space $\mathfrak{B}_c^+(\mathbf{X})$ if it satisfies conditions 1 and 3 of Assumption 1.1, while condition 2 is replaced by the following weaker version: For any $\varepsilon > 0$, there exists $\delta > 0$ such that $|\Phi^+(x,t)| \le \varepsilon$ for all $x \in M$ and all $t \in \mathbb{R}$ satisfying $|t| < \delta$. Given an abelian differential $\mathbf{X} = (M, \omega)$, we now construct, following Katok [28], an explicit mapping of $\mathfrak{B}_c^+(M,\omega)$ to $H^1(M,\mathbb{R})$. A continuous closed curve γ on M is called *rectangular* if

$$\gamma = \gamma_1^+ \sqcup \cdots \sqcup \gamma_{k_1}^+ \sqcup \gamma_1^- \sqcup \cdots \sqcup \gamma_{k_2}^-,$$

where γ_i^+ are arcs of the flow h_t^+, γ_i^- are arcs of the flow h_t^-.

For $\Phi^+ \in \mathfrak{B}_c^+$, define

$$\Phi^+(\gamma) = \sum_{i=1}^{k_1} \Phi^+(\gamma_i^+);$$

similarly, for $\Phi^- \in \mathfrak{B}_c^-$, write

$$\Phi^-(\gamma) = \sum_{i=1}^{k_2} \Phi^-(\gamma_i^-).$$

Thus, a cocycle $\Phi^+ \in \mathfrak{B}_c$ assigns a number $\Phi^+(\gamma)$ to every closed rectangular curve γ. It is shown in Proposition 1.22 below that if γ is homologous to γ', then $\Phi^+(\gamma) = \Phi^+(\gamma')$. For an abelian differential $\mathbf{X} = (M, \omega)$, we thus obtain maps

$$\check{\mathcal{I}}_{\mathbf{X}}^+ : \mathfrak{B}_c^+(\mathbf{X}) \to H^1(M,\mathbb{R}), \ \check{\mathcal{I}}_{\mathbf{X}}^- : \mathfrak{B}_c^-(\mathbf{X}) \to H^1(M,\mathbb{R}). \tag{5}$$

For a generic abelian differential, the image of $\mathfrak{B}^+$ under the map $\check{\mathcal{I}}^+_{\mathbf{X}}$ is the strictly unstable space of the Kontsevich-Zorich cocycle over the Teichmüller flow. More precisely, let $\kappa = (\kappa_1, \dots, \kappa_\sigma)$ be a nonnegative integer vector such that $\kappa_1 + \dots + \kappa_\sigma = 2\rho - 2$. Denote by $\mathcal{M}_\kappa$ the moduli space of pairs (M, ω), where M is a Riemann surface of genus ρ and ω is a holomorphic differential of area 1 with singularities of orders $\kappa_1, \dots, \kappa_\sigma$. The space $\mathcal{M}_\kappa$ is often called the *stratum* in the moduli space of abelian differentials.

The Teichmüller flow $\mathbf{g}_s$ on $\mathcal{M}_\kappa$ sends the modulus of a pair (M, ω) to the modulus of the pair (M, ω'), where $\omega' = e^s \Re(\omega) + ie^{-s}\Im(\omega)$; the new complex structure on M is uniquely determined by the requirement that the form ω' be holomorphic. As shown by Veech, the space $\mathcal{M}_\kappa$ need not be connected; let $\mathcal{H}$ be a connected component of $\mathcal{M}_\kappa$.

Let $\mathbb{H}^1(\mathcal{H})$ be the fibre bundle over $\mathcal{H}$ whose fibre at a point (M, ω) is the cohomology group $H^1(M, \mathbb{R})$. The bundle $\mathbb{H}^1(\mathcal{H})$ carries the *Gauss-Manin connection*, which declares continuous integer-valued sections of our bundle to be flat and is uniquely defined by that requirement. Parallel transport with respect to the Gauss-Manin connection along the orbits of the Teichmüller flow yields a cocycle over the Teichmüller flow, called the *Kontsevich-Zorich cocycle* [31] and denoted $\mathbf{A} = \mathbf{A}_{KZ}$.

Let $\mathbb{P}$ be a $\mathbf{g}_s$-invariant ergodic probability measure on $\mathcal{H}$. For $\mathbf{X} \in \mathcal{H}$, $\mathbf{X} = (M, \omega)$, let $\mathfrak{B}^+_{\mathbf{X}}$, $\mathfrak{B}^-_{\mathbf{X}}$ be the corresponding spaces of Hölder cocycles.

Denote by $E^u_{\mathbf{X}} \subset H^1(M, \mathbb{R})$ the space spanned by vectors corresponding to the positive Lyapunov exponents of the Kontsevich-Zorich cocycle and by $E^s_{\mathbf{X}} \subset H^1(M, \mathbb{R})$ the space spanned by vectors corresponding to the negative exponents of the Kontsevich-Zorich cocycle.

PROPOSITION 1.2. *For $\mathbb{P}$-almost all $\mathbf{X} \in \mathcal{H}$, the map $\check{\mathcal{I}}^+_{\mathbf{X}}$ takes $\mathfrak{B}^+_{\mathbf{X}}$ isomorphically onto $E^u_{\mathbf{X}}$ and the map $\check{\mathcal{I}}^-_{\mathbf{X}}$ takes $\mathfrak{B}^-_{\mathbf{X}}$ isomorphically onto $E^s_{\mathbf{X}}$.*

The pairing $\langle , \rangle$ is nondegenerate and is taken by the isomorphisms $\mathcal{I}^+_{\mathbf{X}}$, $\mathcal{I}^-_{\mathbf{X}}$ to the cup-product in the cohomology $H^1(M, \mathbb{R})$.

Remark. In particular, if $\mathbb{P}$ is the Masur-Veech "smooth" measure [34], [37], then $\dim \mathfrak{B}^+_{\mathbf{X}} = \dim \mathfrak{B}^-_{\mathbf{X}} = \rho$.

Remark. The isomorphisms $\check{\mathcal{I}}^+_{\mathbf{X}}$, $\check{\mathcal{I}}^-_{\mathbf{X}}$ are analogues of G. Forni's isomorphism [21] between his space of invariant distributions and the unstable space of the Kontsevich-Zorich cocycle; cf. also the invariants of Aranson and Grines [2] in the fundamental group of the surface.

Now recall that to every cocycle $\Phi^- \in \mathfrak{B}^-_{\mathbf{X}}$ we have assigned a finitely-additive Hölder measure m_{Φ^-} invariant under the flow h^+_t. Considering these measures as distributions in the sense of Sobolev and Schwartz, we arrive at the following proposition.

Proposition 1.3. *Let $\mathbb{P}$ be an ergodic g_s-invariant probability measure on $\mathcal{H}$. Then for $\mathbb{P}$-almost every abelian differential (M, ω), the space $\{m_{\Phi^-}, \Phi^- \in \mathfrak{B}^-(M, \omega)\}$ coincides with the space of h_t^+-invariant distributions belonging to the Sobolev space H^{-1}.*

Proof. By definition, for any $\Phi^+ \in \mathfrak{B}^+$, the distribution m_{Φ^+} is h_t^--invariant and belongs to the Sobolev space H^{-1}. G. Forni has shown that for any g_s-invariant ergodic measure $\mathbb{P}$ and $\mathbb{P}$-almost every abelian differential (M, ω), the dimension of the space of h_t^--invariant distributions belonging to the Sobolev space H^{-1} *does not exceed* the dimension of the strictly expanding Oseledets subspace of the Kontsevich-Zorich cocycle. (Under mild additional assumption on the measure $\mathbb{P}$ G. Forni proved that these dimensions are in fact equal; see Theorem 8.3 and Corollary 8.3′ in [21]. Note, however, that the proof of the *upper* bound in Forni's Theorem only uses ergodicity of the measure.) Since the dimension of the space $\{m_{\Phi^-}, \Phi^- \in \mathfrak{B}^-\}$ equals that of the strictly expanding space for the Kontsevich-Zorich cocycle for $\mathbb{P}$-almost all (M, ω), the proposition is proved completely. □

Consider the inverse isomorphisms

$$\mathcal{I}_{\mathbf{X}}^+ = \left(\check{\mathcal{I}}_{\mathbf{X}}^+\right)^{-1}, \quad \mathcal{I}_{\mathbf{X}}^- = \left(\check{\mathcal{I}}_{\mathbf{X}}^-\right)^{-1}.$$

Let $1 = \theta_1 > \theta_2 > \cdots > \theta_l > 0$ be the distinct positive Lyapunov exponents of the Kontsevich-Zorich cocycle $\mathbf{A}_{KZ}$, and let

$$E_{\mathbf{X}}^u = \bigoplus_{i=1}^{l} E_{\mathbf{X},\theta_i}^u$$

be the corresponding Oseledets decomposition at $\mathbf{X}$.

Proposition 1.4. *Let $v \in E_{\mathbf{X},\theta_i}^u$, $v \neq 0$, and denote $\Phi^+ = \mathcal{I}_{\mathbf{X}}^+(v)$. Then for any $\varepsilon > 0$, the cocycle Φ^+ satisfies the Hölder condition with exponent $\theta_i - \varepsilon$ and for any $x \in M(\mathbf{X})$ such that $h_t^+ x$ is defined for all $t \in \mathbb{R}$, we have*

$$\limsup_{T\to\infty} \frac{\log|\Phi^+(x,T)|}{\log T} = \theta_i, \quad \limsup_{T\to 0} \frac{\log|\Phi^+(x,T)|}{\log T} = \theta_i.$$

Proposition 1.5. *If the Kontsevich-Zorich cocycle does not have zero Lyapunov exponent with respect to $\mathbb{P}$, then $\mathfrak{B}_c^+(\mathbf{X}) = \mathfrak{B}^+(\mathbf{X})$.*

Remark. The condition of the absence of zero Lyapunov exponents can be weakened: it suffices to require that the Kontsevich-Zorich cocycle act isometrically on the neutral Oseledets subspace corresponding to the Lyapunov exponent zero. Isometric action means here that there exists an inner product that depends measurably on the point in the stratum and that is invariant under the Kontsevich-Zorich cocycle. In all known examples (see, e.g., [24]) the action of the Kontsevich-Zorich cocycle on its neutral Lyapunov subspace is isometric; note, however, that the examples of [24] mainly concern measures invariant under the action of the whole group $SL(2,\mathbb{R})$.

Question. Does there exist a $\mathbf{g}_s$-invariant ergodic probability measure $\mathbb{P}'$ on $\mathcal{H}$ such that the inclusion $\mathfrak{B}^+ \subset \mathfrak{B}_c^+$ is proper almost surely with respect to $\mathbb{P}'$?

Remark. G. Forni has made the following remark. To a cocycle $\Phi^+ \in \mathfrak{B}^+$ assign a 1-current β_{Φ^+}, defined, for a smooth 1-form η on the surface M, by the formula

$$\beta_{\Phi^+}(\eta) = \int_M \Phi^+ \wedge \eta,$$

where the integral in the right-hand side is defined as the limit of Riemann sums. The resulting current β_{Φ^+} is a *basic current* for the horizontal foliation.

The mapping of Hölder cocycles into the cohomology $H^1(M,\mathbb{R})$ of the surface corresponds to G. Forni's map that to each basic current assigns its cohomology class. (The latter is well defined by the de Rham Theorem.) In particular, it follows that for any ergodic g_s-invariant probability measure $\mathbb{P}$ on $\mathcal{H}$ and $\mathbb{P}$-almost every abelian differential (M,ω), every basic current from the Sobolev space H^{-1} is induced by a Hölder cocycle $\Phi^+ \in \mathfrak{B}^+(M,\omega)$.

1.4. *Approximation of weakly Lipschitz functions.*

1.4.1. *The space of weakly Lipschitz functions.* The space of Lipschitz functions is not invariant under h_t^+, and a larger function space $\mathrm{Lip}_w^+(M,\omega)$ of weakly Lipschitz functions is introduced as follows. A bounded measurable function f belongs to $\mathrm{Lip}_w^+(M,\omega)$ if there exists a constant C, depending only on f, such that for any admissible rectangle $\Pi(x,t_1,t_2)$, we have

$$\left|\int_0^{t_1} f(h_t^+ x)dt - \int_0^{t_1} f(h_t^+(h_{t_2}^- x)dt\right| \le C. \tag{6}$$

Let C_f be the infimum of all C satisfying (6). We norm $\mathrm{Lip}_w^+(M,\omega)$ by setting

$$||f||_{\mathrm{Lip}_w^+} = \sup_M f + C_f.$$

By definition, the space $\mathrm{Lip}_w^+(M,\omega)$ contains all Lipschitz functions on M and is invariant under h_t^+. If Π is an admissible rectangle, then its characteristic function χ_Π is weakly Lipschitz. (I am grateful to C. Ulcigrai for this remark.)

We denote by $\mathrm{Lip}_{w,0}^+(M,\omega)$ the subspace of $\mathrm{Lip}_w^+(M,\omega)$ of functions whose integral with respect to ν is 0. For any $f \in \mathrm{Lip}_w^+(M,\omega)$ and any $\Phi^- \in \mathfrak{B}^-$, the integral $\int_M f dm_{\Phi^-}$ can be defined as the limit of Riemann sums.

1.4.2. *The cocycle corresponding to a weakly Lipschitz function.* If the pairing $\langle,\rangle$ induces an isomorphism between $\mathfrak{B}^+$ and the dual $(\mathfrak{B}^-)^*$, then one can assign to a function $f \in \mathrm{Lip}_w^+(M,\omega)$ the functional Φ_f^+ by the formula

$$\langle \Phi_f^+, \Phi^- \rangle = \int\limits_M f dm_{\Phi^-}, \Phi^- \in \mathfrak{B}^-. \tag{7}$$

By definition, $\Phi^+_{f\circ h_t^+} = \Phi_f^+$. We now proceed to the formulation of the first main result of this paper, the Approximation Theorem 1.

THEOREM 1. *Let $\mathbb{P}$ be an ergodic probability $\mathbf{g}_s$-invariant measure on $\mathcal{H}$. For any $\varepsilon > 0$, there exists a constant C_ε depending only on $\mathbb{P}$ such that for $\mathbb{P}$-almost every $\mathbf{X} \in \mathcal{H}$, any $f \in \mathrm{Lip}_w^+(\mathbf{X})$, any $x \in M$, and any $T > 0$, we have*

$$\left| \int_0^T f \circ h_t^+(x)dt - \Phi_f^+(x,T) \right| \le C_\varepsilon ||f||_{\mathrm{Lip}_w^+}(1+T^\varepsilon).$$

1.4.3. *Invariant measures with simple Lyapunov spectrum.* Consider the case in which the Lyapunov spectrum of the Kontsevich-Zorich cocycle is simple in restriction to the space E^u (as, by the Avila-Viana theorem [4], is the case with the Masur-Veech smooth measure). Let $l_0 = \mathrm{dim} E^u$, and let

$$1 = \theta_1 > \theta_2 > \cdots > \theta_{l_0} \tag{8}$$

be the corresponding simple expanding Lyapunov exponents.

Let Φ_1^+ be given by the formula $\Phi_1^+(x,t) = t$, and introduce a basis

$$\Phi_1^+, \Phi_2^+, \ldots, \Phi_{l_0}^+ \tag{9}$$

in $\mathfrak{B}^+_{\mathbf{X}}$ in such a way that $\check{\mathcal{I}}^+_{\mathbf{X}}(\Phi_i^+)$ lies in the Lyapunov subspace with exponent θ_i. By Proposition 1.4, for any $\varepsilon > 0$, the cocycle Φ_i^+ satisfies the Hölder condition with exponent $\theta_i - \varepsilon$, and for any $x \in M(\mathbf{X})$, we have

$$\limsup_{T\to\infty} \frac{\log|\Phi_i^+(x,T)|}{\log T} = \theta_i, \quad \limsup_{T\to 0} \frac{\log|\Phi_i^+(x,T)|}{\log T} = \theta_i.$$

Let $\Phi_1^-, \ldots, \Phi_{l_0}^-$ be the dual basis in $\mathfrak{B}^-_{\mathbf{X}}$. Clearly, $\Phi_1^-(x,t) = t$. By definition, we have

$$\Phi_f^+ = \sum_{i=1}^{l_0} m_{\Phi_i^-}(f)\Phi_i^+. \tag{10}$$

Noting that by definition we have

$$m_{\Phi_1^-} = \nu,$$

we derive from Theorem 1 the following corollary.

COROLLARY 1.6. *Let $\mathbb{P}$ be an invariant ergodic probability measure for the Teichmüller flow such that with respect to $\mathbb{P}$ the Lyapunov spectrum of the Kontsevich-Zorich cocycle is simple in restriction to its strictly expanding subspace. Then for any $\varepsilon > 0$, there exists a constant C_ε depending only on $\mathbb{P}$*

such that for $\mathbb{P}$-almost every $\mathbf{X} \in \mathcal{H}$, any $f \in \mathrm{Lip}_w^+(\mathbf{X})$, any $x \in \mathbf{X}$, and any $T > 0$, we have

$$\left| \int_0^T f \circ h_t^+(x)dt - T\Big(\int_M f d\nu \Big) - \sum_{i=2}^{l_0} m_{\Phi_i^-}(f)\Phi_i^+(x,T) \right| \le C_\varepsilon ||f||_{\mathrm{Lip}_w^+}(1 + T^\varepsilon).$$

For horocycle flows a related asymptotic expansion has been obtained by Flaminio and Forni [19].

Remark. If $\mathbb{P}$ is the Masur-Veech smooth measure on $\mathcal{H}$, then it follows from the work of G. Forni [20], [21], [23] and S. Marmi, P. Moussa, J.-C. Yoccoz [32] that the left-hand side is bounded for any $f \in C^{1+\varepsilon}(M)$ (in fact, for any f in the Sobolev space $H^{1+\varepsilon}$). In particular, if $f \in C^{1+\varepsilon}(M)$ and $\Phi_f^+ = 0$, then f is a coboundary.

1.5. *Holonomy invariant transverse finitely-additive measures for oriented measured foliations.* Holonomy-invariant cocycles assigned to an abelian differential can be interpreted as transverse invariant measures for its foliations in the spirit of Kenyon [29] and Bonahon [8], [9].

Let M be a compact oriented surface of genus at least two, and let $\mathcal{F}$ be a minimal oriented measured foliation on M. Denote by $\mathbf{m}_{\mathcal{F}}$ the transverse invariant measure of $\mathcal{F}$. If $\gamma = \gamma(t), t \in [0,T]$ is a smooth curve on M, and s_1, s_2 satisfy $0 \le s_1 < s_2 \le T$, then we denote by $\mathrm{res}_{[s_1,s_2]}\gamma$ the curve $\gamma(t), t \in [s_1, s_2]$.

Let $\mathfrak{B}_c(\mathcal{F})$ be the space of uniformly continuous finitely-additive transverse invariant measures for $\mathcal{F}$. In other words, a map Φ that to every smooth arc γ transverse to $\mathcal{F}$ assigns a real number $\Phi(\gamma)$ belongs to the space $\mathfrak{B}_c(\mathcal{F})$ if it satisfies the following

ASSUMPTION 1.7.

(1) (*Finite additivity*). *For $\gamma = \gamma(t), t \in [0,T]$ and any $s \in (0,T)$, we have*

$$\Phi(\gamma) = \Phi(\mathrm{res}_{[0,s]}\gamma) + \Phi(\mathrm{res}_{[s,T]}\gamma).$$

(2) (*Uniform continuity*). *For any $\varepsilon > 0$, there exists $\delta > 0$ such that for any transverse arc γ satisfying $\mathbf{m}_{\mathcal{F}}(\gamma) < \delta$, we have $|\Phi(\gamma)| < \varepsilon$.*

(3) (*Holonomy invariance*). *The value $\Phi(\gamma)$ does not change if γ is deformed in such a way that it stays transverse to $\mathcal{F}$ while the endpoints of γ stay on their respective leaves.*

A measure $\Phi \in \mathfrak{B}_c(\mathcal{F})$ is called Hölder with exponent θ if there exists $\varepsilon_0 > 0$ such that for any transverse arc γ satisfying $\mathbf{m}_{\mathcal{F}}(\gamma) < \varepsilon_0$, we have

$$|\Phi(\gamma)| \le (\mathbf{m}_{\mathcal{F}}(\gamma))^\theta .$$

Let $\mathfrak{B}(\mathcal{F}) \subset \mathfrak{B}_c(\mathcal{F})$ be the subspace of Hölder transverse measures. As before, we have a natural map

$$\mathcal{I}_{\mathcal{F}} : \mathfrak{B}_c(\mathcal{F}) \to H^1(M, \mathbb{R})$$

defined as follows. For a smooth closed curve γ on M and a measure $\Phi \in \mathfrak{B}_c(\mathcal{F})$, the integral $\int_\gamma d\Phi$ is well defined as the limit of Riemann sums; by holonomy-invariance and continuity of Φ, this operation descends to homology and assigns to Φ an element of $H^1(M, \mathbb{R})$.

Now take an abelian differential $\mathbf{X} = (M, \omega)$, and let $\mathcal{F}^-_{\mathbf{X}}$ be its horizontal foliation. We have a "tautological" isomorphism between $\mathfrak{B}_c(\mathcal{F}^-_{\mathbf{X}})$ and $\mathfrak{B}^+_c(\mathbf{X})$: every transverse measure for the horizontal foliation induces a cocycle for the vertical foliation and vice versa; to a Hölder measure corresponds a Hölder cocycle. For brevity, write $\mathcal{I}_{\mathbf{X}} = \mathcal{I}_{\mathcal{F}^-_{\mathbf{X}}}$. Denote by $E^u_{\mathbf{X}} \subset H^1(M, \mathbb{R})$ the unstable subspace of the Kontsevich-Zorich cocycle of the abelian differential $\mathbf{X} = (M, \omega)$.

Theorem 1 and Proposition 1.5 yield the following

COROLLARY 1.8. *Let $\mathbb{P}$ be a Borel probability measure on $\mathcal{H}$ invariant and ergodic under the action of the Teichmüller flow $\mathbf{g}_t$. Then for almost every abelian differential $\mathbf{X} \in \mathcal{H}$ the map $\mathcal{I}_{\mathbf{X}}$ takes $\mathfrak{B}(\mathcal{F}^-_{\mathbf{X}})$ isomorphically onto $E^u_{\mathbf{X}}$. If the Kontsevich-Zorich cocycle does not have zero Lyapunov exponents with respect to $\mathbb{P}$, then for almost all $\mathbf{X} \in \mathcal{H}$, we have $\mathfrak{B}_c(\mathcal{F}_{\mathbf{X}}) = \mathfrak{B}(\mathcal{F}_{\mathbf{X}})$.*

In other words, in the absence of zero Lyapunov exponents all continuous transverse finitely-additive invariant measures are in fact Hölder.

Remark. As before, the condition of the absence of zero Lyapunov exponents can be weakened: it suffices to require that the Kontsevich-Zorich cocycle act isometrically on the Oseledets subspace corresponding to the Lyapunov exponent zero.

By definition, the space $\mathfrak{B}(\mathcal{F}^-_{\mathbf{X}})$ only depends on the horizontal foliation of our abelian differential; so does $E^u_{\mathbf{X}}$.

1.6. *Finitely-additive invariant measures for interval exchange transformations.*

1.6.1. *The space of invariant continuous finitely-additive measures.* Let $m \in \mathbb{N}$. Let Δ_{m-1} be the standard unit simplex

$$\Delta_{m-1} = \left\{ \lambda \in \mathbb{R}^m_+,\ \lambda = (\lambda_1, \ldots, \lambda_m), \lambda_i > 0, \sum_{i=1}^m \lambda_i = 1 \right\}.$$

Let π be a permutation of $\{1, \ldots, m\}$ satisfying the *irreducibility* condition: we have $\pi\{1, \ldots, k\} = \{1, \ldots, k\}$ if and only if $k = m$.

On the half-open interval $I = [0, 1)$ consider the points

$$\beta_1 = 0, \quad \beta_i = \sum_{j<i} \lambda_j, \quad \beta_1^\pi = 0, \quad \beta_i^\pi = \sum_{j<i} \lambda_{\pi^{-1}j},$$

and denote $I_i = [\beta_i, \beta_{i+1})$, $I_i^\pi = [\beta_i^\pi, \beta_{i+1}^\pi)$. The length of I_i is λ_i, while the length of I_i^π is $\lambda_{\pi^{-1}i}$. Set

$$\mathbf{T}_{(\lambda,\pi)}(x) = x + \beta_{\pi i}^\pi - \beta_i \text{ for } x \in I_i.$$

The map $\mathbf{T}_{(\lambda,\pi)}$ is called *an interval exchange transformation* corresponding to (λ, π). By definition, the map $\mathbf{T}_{(\lambda,\pi)}$ is invertible and preserves the Lebesgue measure on I. By the theorem of Masur [34] and Veech [37], for any irreducible permutation π and for Lebesgue-almost all $\lambda \in \Delta_{m-1}$, the corresponding interval exchange transformation $\mathbf{T}_{(\lambda,\pi)}$ is uniquely ergodic: the Lebesgue measure is the only invariant probability measure for $\mathbf{T}_{(\lambda,\pi)}$.

Consider the space of complex-valued continuous finitely-additive invariant measures for $\mathbf{T}_{(\lambda,\pi)}$. More precisely, let $\mathfrak{B}_c(\mathbf{T}_{(\lambda,\pi)})$ be the space of all continuous functions $\Phi : [0, 1] \to \mathbb{R}$ satisfying

(1) $\Phi(0) = 0$;
(2) if $0 \le t_1 \le t_2 < 1$ and $\mathbf{T}_{(\lambda,\pi)}$ is continuous on $[t_1, t_2]$, then $\Phi(t_1) - \Phi(t_2) = \Phi(\mathbf{T}_{(\lambda,\pi)}(t_1)) - \Phi(\mathbf{T}_{(\lambda,\pi)}(t_2))$.

Each function Φ induces a finitely-additive measure on $[0, 1]$ defined on the semi-ring of subintervals. (For instance, the function $\Phi_1(t) = t$ yields the Lebesgue measure on $[0, 1]$.)

Let $\mathfrak{B}(\mathbf{T}_{(\lambda,\pi)})$ be the subspace of Hölder functions $\Phi \in \mathfrak{B}_c(\mathbf{T}_{(\lambda,\pi)})$. The classification of Hölder cocycles over translation flows and the asymptotic formula of Theorem 1 now yield the classification of the space $\mathfrak{B}(\mathbf{T}_{(\lambda,\pi)})$ and an asymptotic expansion for time averages of almost all interval exchange maps.

1.6.2. *The approximation of ergodic sums.* Let $\mathbf{X} = (M, \omega)$ be an abelian differential, and let $I \subset M$ be a closed interval lying on a leaf of a horizontal foliation. The vertical flow h_t^+ induces an interval exchange map $\mathbf{T}_I$ on I, namely, the Poincaré first return map of the flow. By definition, there is a natural tautological identification of the spaces $\mathfrak{B}_c(\mathbf{T}_I)$ and $\mathfrak{B}_c^-(\mathbf{X})$ as well as of the spaces $\mathfrak{B}(\mathbf{T}_I)$ and $\mathfrak{B}^-(\mathbf{X})$.

For $x \in M$, let $\tau_I(x) = \min\left\{t \geqslant 0 : h_{-t}^+ x \in I\right\}$. Note that the function τ_I is uniformly bounded on M. Now take a Lipschitz function f on I, and introduce a function $\widetilde{f}$ on M by the formula

$$\widetilde{f}(x) = \frac{f(h_{-\tau_I(x)}^+ x)}{\tau_I(x)}$$

(setting $\widetilde{f}(x) = 0$ for points at which τ_I is not defined).

By definition, the function $\widetilde{f}$ is weakly Lipschitz, and Theorem 1 is applicable to $\widetilde{f}$. The ergodic integrals of $\widetilde{f}$ under h_t^+ are of course closely related to ergodic sums of f under $\mathbf{T}_I$, and for any $N \in \mathbb{N}$, $x \in I$, there exists a time $t(x, N) \in \mathbb{R}$ such that

$$\int\limits_0^{t(x,N)} \widetilde{f} \circ h_s^+(x)\, ds = \sum_{k=0}^{N-1} f \circ \mathbf{T}_I^k(x).$$

By the Birkhoff–Khintchine Ergodic Theorem we have

$$\lim_{N\to\infty} \frac{t(x,N)}{N} = \frac{1}{\mathrm{Leb}(I)}\ ,$$

where $\mathrm{Leb}(I)$ stands for the length of I. Furthermore, Theorem 1 yields the existence of constants $C(I) > 0$, $\theta \in (0, 1)$, such that for all $x \in I$, $N \in \mathbb{N}$, we have

$$\left| t(x,N) - \frac{N}{\mathrm{Leb}(I)} \right| \leqslant C(I) \cdot N^{\theta}. \tag{11}$$

Indeed, the interval I induces a decomposition of our surface into weakly admissible rectangles $\Pi_1, \ldots, \Pi_m$; denote by h_i the height of the rectangle Π_i, and introduce a weakly Lipschitz function that assumes the constant value $\frac{1}{h_i}$ on each rectangle Π_i. Applying Theorem 1 to this function we arrive at the desired estimate.

In view of the estimate (11), Theorem 1 applied to the function $\widetilde{f}$ now yields the following Corollary.

Corollary 1.9. *Let $\mathbb{P}$ be a $\mathbf{g}_s$-invariant ergodic probability measure on $\mathcal{H}$. For any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ depending only on $\mathbb{P}$ such that the following holds: For almost every abelian differential $\mathbf{X} \in \mathcal{H}, \mathbf{X} = (M, \omega)$, any horizontal closed interval $I \subset M$, any Lipschitz function $f : I \to \mathbb{R}$, any $x \in I$, and all $n \in \mathbb{N}$, we have*

$$\left| \sum_{k=0}^{N-1} f \circ \mathbf{T}_I^k(x) - \Phi^+_{\widetilde{f}}(x, N) \right| \leqslant C_\varepsilon ||f||_{\mathrm{Lip}} N^{\varepsilon}.$$

Proof. Applying Theorem 1 to $\widetilde{f}$, using the estimate (11), and noting that the weakly Lipschitz norm of $\widetilde{f}$ is majorated by the Lipschitz norm of f, we obtain the desired corollary. □

Let $\theta_1 > \theta_2 > \cdots > \theta_{l_0} > 0$ be the distinct positive Lyapunov exponents of the measure $\mathbb{P}$, and let $d_1 = 1, d_2, \ldots, d_{l_0}$ be the dimensions of the corresponding subspaces. The tautological identification of $\mathfrak{B}(\mathbf{T}_I)$ and $\mathfrak{B}^-(\mathbf{X})$, together with the results of the previous corollary, now implies Zorich-type estimates for the growth of ergodic sums of $\mathbf{T}_I$. More precisely, we have the following

Corollary 1.10. *In the assumptions of the Corollary* 1.9, *the space* $\mathfrak{B}(\mathbf{T}_I)$ *admits a flag of subspaces*

$$0 = \mathfrak{B}_0 \subset \mathfrak{B}_1 = \mathbb{R}\,\mathrm{Leb}_I \subset \mathfrak{B}_2 \subset \cdots \subset \mathfrak{B}_{l_0} = \mathfrak{B}(\mathbf{T}_I)$$

such that for any finitely-additive measure $\Phi \in \mathfrak{B}_i$ *in Hölder with exponent* $\frac{\theta_i}{\theta_1} - \varepsilon$ *for any* $\varepsilon > 0$ *and that for any Lipschitz function* $f : I \to \mathbb{R}$ *and for any* $x \in I$, *we have*

$$\lim_{N\to\infty} \sup \frac{\log \left|\sum_{k=0}^{N-1} f\mathbf{T}_I^k(x)\right|}{\log N} = \frac{\theta_{i(f)}}{\theta_1},$$

where $i(f) = 1 + \max\{j : \int_I f d\Phi = 0$ *for all* $\Phi \in \mathfrak{B}_j\}$ *and by convention we set* $\theta_{l_0+1} = 0$. *If with respect to the measure* $\mathbb{P}$ *the Kontsevich-Zorich cocycle acts isometrically on its neutral subspaces*, *then we also have* $\mathfrak{B}_c(\mathbf{T}_I) = \mathfrak{B}(\mathbf{T}_I)$.

Remark. Corollaries 1.9 and 1.10 thus yield the asymptotic expansion in terms of Hölder cocycles as well as Zorich-type logarithmic estimates for almost all interval exchange transformations with respect to any conservative ergodic measure μ on the space of interval exchange transformations, invariant under the Rauzy-Veech induction map and such that the Kontsevich-Zorich cocycle is log-integrable with respect to μ. In particular, for the Lebesgue measure, if we let $\mathcal{R}$ be the Rauzy class of the permutation π, then, using the simplicity of the Lyapunov spectrum given by the Avila-Viana theorem [4], we obtain

Corollary 1.11. *For any irreducible permutation* π *and for Lebesgue-almost all* λ *all continuous finitely-additive measures are Hölder*, *we have*

$$\mathfrak{B}(\mathbf{T}_{(\lambda,\pi)}) = \mathfrak{B}_c(\mathbf{T}_{(\lambda,\pi)}).$$

For any irreducible permutation π, *there exists a natural number* $\rho = \rho(\mathcal{R})$ *depending only on the Rauzy class of* π *and such that*

(1) *for Lebesgue-almost all* λ, *we have* $\dim \mathfrak{B}(\lambda, \pi) = \rho$;
(2) *all the spaces* $\mathfrak{B}_i$ *are one-dimensional and* $l_0 = \rho$.

In the case of the Lebesgue measure on the space of interval exchange transformations, the second statement of Corollary 1.10 recovers the Zorich logarithmic asymptotics for ergodic sums [46], [47].

Remark. Objects related to finitely-additive measures for interval exchange transformations have been studied by X. Bressaud, P. Hubert and A. Maass in [10] and by S. Marmi, P. Moussa and J.-C. Yoccoz in [33]. In particular, the "limit shapes" of [33] can be viewed as graphs of the cocycles $\Phi^+(x,t)$ considered as functions in t.

1.7. *Limit theorems for translation flows.*

1.7.1. *Time integrals as random variables.* As before, (M, ω) is an abelian differential, and h_t^+, h_t^- are, respectively, its vertical and horizontal flows. Take $\tau \in [0,1]$, $s \in \mathbb{R}$, a real-valued $f \in \mathrm{Lip}_{w,0}^+(M, \omega)$, and introduce the function

$$\mathfrak{S}[f, s; \tau, x] = \int_0^{\tau \exp(s)} f \circ h_t^+(x) dt. \tag{12}$$

For fixed f, s, and x, the quantity $\mathfrak{S}[f, s; \tau, x]$ is a continuous function of $\tau \in [0,1]$; therefore, as x varies in the probability space (M, ν), we obtain a random element of $C[0,1]$. In other words, we have a random variable

$$\mathfrak{S}[f, s] : (M, \nu) \to C[0,1] \tag{13}$$

defined by the formula (12).

For any fixed $\tau \in [0,1]$, the formula (12) yields a real-valued random variable

$$\mathfrak{S}[f, s; \tau] : (M, \nu) \to \mathbb{R}, \tag{14}$$

whose expectation, by definition, is zero.

Our first aim is to estimate the growth of its variance as $s \to \infty$. Without losing generality, one may take $\tau = 1$.

1.7.2. *The growth rate of the variance in the case of a simple second Lyapunov exponent.* Let $\mathbb{P}$ be an invariant ergodic probability measure for the Teichmüller flow such that with respect to $\mathbb{P}$, the second Lyapunov exponent θ_2 of the Kontsevich-Zorich cocycle is positive and simple. (Recall that, as Veech and Forni showed, the first one, $\theta_1 = 1$, is always simple [41], [21] and that, by the Avila-Viana theorem [4], the second one is simple for the Masur-Veech smooth measure.)

For an abelian differential $\mathbf{X} = (M, \omega)$, denote by $E_{2,\mathbf{X}}^+$ the one-dimensional subspace in $H^1(M, \mathbb{R})$ corresponding to the second Lyapunov exponent θ_2, and let $\mathfrak{B}_{2,\mathbf{X}}^+ = \mathcal{I}_{\mathbf{X}}^+(E_{2,\mathbf{X}}^+)$. Similarly, denote by $E_{2,\mathbf{X}}^-$ the one-dimensional subspace in $H^1(M, \mathbb{R})$ corresponding to the Lyapunov exponent $-\theta_2$, and let $\mathfrak{B}_{2,\mathbf{X}}^- = \mathcal{I}_{\mathbf{X}}^-(E_{2,\mathbf{X}}^-)$.

Recall that the space $H^1(M, \mathbb{R})$ is endowed with the Hodge norm $|\cdot|_H$; the isomorphisms $\mathcal{I}_{\mathbf{X}}^{\pm}$ take the Hodge norm to a norm on $\mathfrak{B}_{\mathbf{X}}^{\pm}$; slightly abusing notation, we denote the latter norm by the same symbol.

Introduce a multiplicative cocycle $H_2(s, \mathbf{X})$ over the Teichmüller flow $\mathbf{g}_s$ by taking $v \in E_{2,\mathbf{X}}^+$, $v \neq 0$, and setting

$$H_2(s, \mathbf{X}) = \frac{|\mathbf{A}(s, \mathbf{X})v|_H}{|v|_H}. \tag{15}$$

The Hodge norm is chosen only for concreteness in (15); any other norm can be used instead.

By definition, we have

$$\lim_{s\to\infty} \frac{\log H_2(s,\mathbf{X})}{s} = \theta_2. \tag{16}$$

Now take $\Phi_2^+ \in \mathfrak{B}^+_{2,\mathbf{X}}$ $\Phi_2^- \in \mathfrak{B}^-_{2,\mathbf{X}}$ in such a way that $\langle \Phi_2^+, \Phi_2^- \rangle = 1$.

PROPOSITION 1.12. *There exists $\alpha > 0$ depending only on $\mathbb{P}$ and positive measurable functions*

$$C : \mathcal{H} \times \mathcal{H} \to \mathbb{R}_+,\ V : \mathcal{H} \to \mathbb{R}_+,\ s_0 : \mathcal{H} \to \mathbb{R}_+$$

such that the following is true for $\mathbb{P}$-almost all $\mathbf{X} \in \mathcal{H}$: If $f \in \mathrm{Lip}^+_{w,0}(\mathbf{X})$ satisfies $m_{\Phi_2^-}(f) \neq 0$, then for all $s \geq s_0(\mathbf{X})$, we have

$$\left| \frac{\mathrm{Var}_\nu \mathfrak{S}(f,x,e^s)}{V(\mathbf{g}_s\mathbf{X})(m_{\Phi_2^-}(f)|\Phi_2^+|H_2(s,\mathbf{X}))^2} - 1 \right| \leq C(\mathbf{X}, \mathbf{g}_s\mathbf{X}) \exp(-\alpha s). \tag{17}$$

Remark. Observe that the quantity $(m_{\Phi_2^-}(f)|\Phi_2^+|)^2$ does not depend on the specific choice of $\Phi_2^+ \in \mathfrak{B}^+_{2,\mathbf{X}}$, $\Phi_2^- \in \mathfrak{B}^-_{2,\mathbf{X}}$ such that $\langle \Phi_2^+, \Phi_2^- \rangle = 1$.

Remark. Note that by theorems of Egorov and Luzin, the estimate (17) holds *uniformly* on compact subsets of $\mathcal{H}$ of probability arbitrarily close to 1.

Proposition 1.12 is based on

PROPOSITION 1.13. *There exists a positive measurable function $V : \mathcal{H} \to \mathbb{R}_+$ such that for $\mathbb{P}$-almost all $\mathbf{X} \in \mathcal{H}$, we have*

$$\mathrm{Var}_{\nu(\mathbf{X})} \Phi_2^+(x, e^s) = V(\mathbf{g}_s\mathbf{X})|\Phi_2^+|^2 (H_2(s,\mathbf{X}))^2. \tag{18}$$

In particular, $\mathrm{Var}_\nu \Phi_2^+(x,e^s) \neq 0$ for any $s \in \mathbb{R}$. The function $V(\mathbf{X})$ is given by

$$V(\mathbf{X}) = \frac{\mathrm{Var}_{\nu(\mathbf{X})} \Phi_2^+(x,1)}{|\Phi_2^+|^2}.$$

Observe that the right-hand side does not depend on a particular choice of $\Phi_2^+ \in \mathfrak{B}^+_{2,\mathbf{X}}$, $\Phi_2^+ \neq 0$.

1.7.3. *The limit theorem in the case of a simple second Lyapunov exponent.* Go back to the $C[0,1]$-valued random variable $\mathfrak{S}[f,s]$, and denote by $\mathfrak{m}[f,s]$ the distribution of the normalized random variable

$$\frac{\mathfrak{S}[f,s]}{\sqrt{\mathrm{Var}_\nu \mathfrak{S}[f,s;1]}}. \tag{19}$$

The measure $\mathfrak{m}[f,s]$ is thus a probability distribution on the space $C[0,1]$ of continuous functions on the unit interval.

For $\tau \in \mathbb{R}$, $\tau \neq 0$, we also let $\mathfrak{m}[f, s; \tau]$ be the distribution of the $\mathbb{R}$-valued random variable

$$\frac{\mathfrak{S}[f, s; \tau]}{\sqrt{\mathrm{Var}_\nu \mathfrak{S}[f, s; \tau]}}. \tag{20}$$

If f has zero average then, by definition, $\mathfrak{m}[f, s; \tau]$ is a measure on $\mathbb{R}$ of expectation 0 and variance 1.

By definition, $\mathfrak{m}[f, s]$ is a Borel probability measure on $C[0, 1]$; furthermore, if $\xi = \xi(\tau) \in C[0, 1]$, then the following natural normalization requirements hold for ξ with respect to $\mathfrak{m}[f, s]$:

(1) $\xi(0) = 0$ almost surely with respect to $m[f, s]$;
(2) $\mathbb{E}_{\mathfrak{m}[f,s]}\xi(\tau) = 0$ for all $\tau \in [0, 1]$;
(3) $\mathrm{Var}_{\mathfrak{m}[f,s]}\xi(1) = 1$.

We are interested in the weak accumulation points of $\mathfrak{m}[f, s]$ as $s \to \infty$.

Consider the space $\mathcal{H}'$ given by the formula

$$\mathcal{H}' = \{\mathbf{X}' = (M, \omega, v), v \in E_2^+(M, \omega), |v|_H = 1\}.$$

By definition, the space $\mathcal{H}'$ is a $\mathbb{P}$-almost surely defined two-to-one cover of the space $\mathcal{H}$. The skew-product flow of the Kontsevich-Zorich cocycle over the Teichmüller flow yields a flow $\mathbf{g}'_s$ on $\mathcal{H}'$ given by the formula

$$\mathbf{g}'_s(\mathbf{X}, v) = \left(\mathbf{g}_s\mathbf{X}, \frac{\mathbf{A}(s, \mathbf{X})v}{|\mathbf{A}(s, \mathbf{X})v|_H}\right).$$

Given $\mathbf{X}' \in \mathcal{H}'$, set

$$\Phi^+_{2,\mathbf{X}'} = \mathcal{I}^+(v).$$

Take $\tilde{v} \in E_2^-(M, \omega)$ such that $\langle v, \tilde{v}\rangle = 1$, and set

$$\Phi^-_{2,\mathbf{X}'} = \mathcal{I}^-(v),\ m^-_{2,\mathbf{X}'} = m^-_{\Phi_{2,\mathbf{X}'}}.$$

Let $\mathfrak{M}$ be the space of all probability distributions on $C[0, 1]$, and introduce a $\mathbb{P}$-almost surely defined map $\mathcal{D}_2^+ : \mathcal{H}' \to \mathfrak{M}$ by setting $\mathcal{D}_2^+(\mathbf{X}')$ to be the distribution of the $C[0, 1]$-valued normalized random variable

$$\frac{\Phi^+_{2,\mathbf{X}'}(x, \tau)}{\sqrt{\mathrm{Var}_\nu \Phi^+_{2,\mathbf{X}'}(x, 1)}}, \quad \tau \in [0, 1].$$

By definition, $\mathcal{D}_2^+(\mathbf{X}')$ is a Borel probability measure on the space $C[0, 1]$; it is, besides, a compactly supported measure as its support consists of equibounded Hölder functions with exponent $\theta_2/\theta_1 - \varepsilon$.

Consider the set $\mathfrak{M}_1$ of probability measures $\mathfrak{m}$ on $C[0, 1]$ satisfying, for $\xi \in C[0, 1]$, $\xi = \xi(t)$, the following conditions:

(1) the equality $\xi(0) = 0$ holds $\mathfrak{m}$-almost surely;
(2) for all τ, we have $\mathbb{E}_\mathfrak{m}\xi(\tau) = 0$;
(3) we have $\mathrm{Var}_\mathfrak{m}\xi(1) = 1$ and for any $\tau \neq 0$, we have $\mathrm{Var}_\mathfrak{m}\xi(\tau) \neq 0$.

It will be proved in what follows that $\mathcal{D}_2^+(\mathcal{H}') \subset \mathfrak{M}_1$.

Consider a semi-flow J_s on the space $C[0,1]$ defined by the formula

$$J_s\xi(t) = \xi(e^{-s}t), \quad s \geq 0.$$

Introduce a semi-flow G_s on $\mathfrak{M}_1$ by the formula

$$G_s\mathfrak{m} = \frac{(J_s)_*\mathfrak{m}}{\mathrm{Var}_{\mathfrak{m}}(\xi(\exp(-s))}, \mathfrak{m} \in \mathfrak{M}_1. \tag{21}$$

By definition, the diagram

$$\begin{array}{ccc} \mathcal{H}' & \xrightarrow{\mathcal{D}_2^+} & \mathfrak{M}_1 \\ \Big\downarrow \mathbf{g}_s & & \Big\uparrow G_s \\ \mathcal{H}' & \xrightarrow{\mathcal{D}_2^+} & \mathfrak{M}_1 \end{array}$$

is commutative.

Let d_{LP} be the Lévy-Prohorov metric, and let d_{KR} be the Kantorovich-Rubinstein metric on the space of probability measures on $C[0,1]$ (see [6], [7] and the appendix).

We are now ready to formulate the main result of this subsection.

PROPOSITION 1.14. *Let $\mathbb{P}$ be a $\mathbf{g}_s$-invariant ergodic probability measure on $\mathcal{H}$ such that the second Lyapunov exponent of the Kontsevich-Zorich cocycle is positive and simple with respect to $\mathbb{P}$.*

There exist a positive measurable function $C : \mathcal{H} \times \mathcal{H} \to \mathbb{R}_+$ and a positive constant α depending only on $\mathbb{P}$ such that for $\mathbb{P}$-almost every $\mathbf{X}' \in \mathcal{H}'$, $\mathbf{X}' = (\mathbf{X}, v)$, and any $f \in \mathrm{Lip}^+_{w,0}(\mathbf{X})$ satisfying $m^-_{2,\mathbf{X}'}(f) > 0$, we have

$$d_{\mathrm{LP}}(\mathfrak{m}[f,s], \mathcal{D}_2^+(\mathbf{g}'_s\mathbf{X}')) \leq C(\mathbf{X}, \mathbf{g}_s\mathbf{X}) \exp(-\alpha s), \tag{22}$$

$$d_{\mathrm{KR}}(\mathfrak{m}[f,s], \mathcal{D}_2^+(\mathbf{g}'_s\mathbf{X}')) \leq C(\mathbf{X}, \mathbf{g}_s\mathbf{X}) \exp(-\alpha s). \tag{23}$$

Now fix $\tau \in \mathbb{R}$ and let $\mathfrak{m}_2(\mathbf{X}', \tau)$ be the distribution of the $\mathbb{R}$-valued random variable

$$\frac{\Phi^+_{2,\mathbf{X}'}(x,\tau)}{\sqrt{\mathrm{Var}_\nu \Phi^+_{2,\mathbf{X}'}(x,\tau)}}.$$

For brevity, write $\mathfrak{m}_2(\mathbf{X}', 1) = \mathfrak{m}_2(\mathbf{X}')$.

PROPOSITION 1.15. *For $\mathbb{P}$-almost any $\mathbf{X}' \in \mathcal{H}'$, the measure $\mathfrak{m}_2(\mathbf{X}', \tau)$ admits atoms for a dense set of times $\tau \in \mathbb{R}$.*

A more general proposition on the existence of atoms will be formulated in the following subsection.

Proposition 1.14 implies that the omega-limit set of the family $\mathfrak{m}[f,s]$ can generically assume at most two values. More precisely, the ergodic measure $\mathbb{P}$ on $\mathcal{H}$ is naturally lifted to its "double cover" on the space $\mathcal{H}'$: each point in the fibre is assigned equal weight; the resulting measure is denoted $\mathbb{P}'$. By definition, the measure $\mathbb{P}'$ has no more than two ergodic components. We therefore arrive at the following

COROLLARY 1.16. *Let $\mathbb{P}$ be a $\mathbf{g}_s$-invariant ergodic probability measure on $\mathcal{H}$ such that the second Lyapunov exponent of the Kontsevich-Zorich cocycle is positive and simple with respect to $\mathbb{P}$.*

There exist two closed sets $\mathfrak{N}_1, \mathfrak{N}_2 \subset \mathfrak{M}$ such that for $\mathbb{P}$-almost every $\mathbf{X} \in \mathcal{H}$ and any $f \in \mathrm{Lip}^+_{w,0}(\mathbf{X})$ satisfying $\Phi^+_f \neq 0$, the omega-limit set of the family $\mathfrak{m}[f,s]$ either coincides with $\mathfrak{N}_1$ or with $\mathfrak{N}_2$. If, additionally, the measure $\mathbb{P}'$ is ergodic, then $\mathfrak{N}_1 = \mathfrak{N}_2$.

Question. Do the sets $\mathfrak{N}_i$ contain measures with noncompact support?

For horocycle flows on compact surfaces of constant negative curvature, compactness of support for all weak accumulation points of ergodic integrals has been obtained by Flaminio and Forni [19].

Question. Is the measure $\mathbb{P}'$ ergodic when $\mathbb{P}$ is the Masur-Veech measure?

As we shall see in the next subsection, in general, the omega-limit sets of the distributions of the $\mathbb{R}$-valued random variables $\mathfrak{S}[f,s;1]$ contain the delta-measure at zero. As a consequence, it will develop that, under certain assumptions on the measure $\mathbb{P}$, which are satisfied, in particular, for the Masur-Veech smooth measure, for a generic abelian differential the random variables $\mathfrak{S}[f,s;1]$ *do not converge* in distribution, as $s \to \infty$, for any function $f \in \mathrm{Lip}_{w,0}$ such that $\Phi^+_f \neq 0$.

1.7.4. *The general case.* While, by the Avila-Viana Theorem [4], the Lyapunov spectrum of the Masur-Veech measure is simple, there are also natural examples of invariant measures with nonsimple positive second Lyapunov exponent due to Eskin-Kontsevich-Zorich [17], G. Forni [22], and C. Matheus (see [22, App. A.1]). A slightly more elaborate, but similar, construction is needed to obtain limit theorems in this general case.

Let $\mathbb{P}$ be an invariant ergodic probability measure for the Teichmüller flow, and let

$$\theta_1 = 1 > \theta_2 > \cdots > \theta_{l_0} > 0$$

be the distinct positive Lyapunov exponents of the Kontsevich-Zorich cocycle with respect to $\mathbb{P}$. We assume $l_0 \geq 2$.

As before, for $\mathbf{X} \in \mathcal{H}$ and $i = 2, \dots, l_0$, let $E^u_i(\mathbf{X})$ be the corresponding Oseledets subspaces, and let $\mathfrak{B}^+_i(\mathbf{X})$ be the corresponding spaces of cocycles. To make notation lighter, we omit the symbol $\mathbf{X}$ when the abelian differential is held fixed.

For $f \in \mathrm{Lip}_w^+(\mathbf{X})$, we now write

$$\Phi_f^+ = \Phi_{1,f}^+ + \Phi_{2,f}^+ + \cdots + \Phi_{l_0,f}^+,$$

with $\Phi_{i,f}^+ \in \mathfrak{B}_i^+$ and, of course, with

$$\Phi_{1,f}^+ = \Big(\int\limits_M f d\nu \Big) \cdot \nu^+,$$

where ν^+ is the Lebesgue measure on the vertical foliation.

For each $i = 2, \ldots, l_0$, introduce a measurable fibre bundle

$$\mathbf{S}^{(i)}\mathcal{H} = \{(\mathbf{X}, v) : \mathbf{X} \in \mathcal{H}, v \in E_i^+, |v| = 1\}.$$

The flow $\mathbf{g}_s$ is naturally lifted to the space $\mathbf{S}^{(i)}\mathcal{H}$ by the formula

$$\mathbf{g}_s^{\mathbf{S}^{(i)}}(\mathbf{X}, v) = \left(\mathbf{g}_s \mathbf{X}, \frac{\mathbf{A}(s, \mathbf{X})v}{|\mathbf{A}(s, \mathbf{X})v|} \right).$$

The growth of the norm of vectors $v \in E_i^+$ is controlled by the multiplicative cocycle H_i over the flow $\mathbf{g}_s^{\mathbf{S}^{(i)}}$ defined by the formula

$$H_i(s, (\mathbf{X}, v)) = \frac{\mathbf{A}(s, \mathbf{X})v}{|v|}.$$

For $\mathbf{X} \in \mathcal{H}$ and $f \in \mathrm{Lip}_{w,0}^+(\mathbf{X})$ satisfying $\Phi_f^+ \neq 0$, denote

$$i(f) = \min\{j : \Phi_{f,j}^+ \neq 0\}.$$

Define $v_f \in E_{i(f)}^u$ by the formula

$$\mathcal{I}_{\mathbf{X}}^+(v_f) = \frac{\Phi_{f,i(f)}^+}{|\Phi_{f,i(f)}^+|}.$$

The growth of the variance of the ergodic integral of a weakly Lipschitz function f is also, similarly to the case of the simple second Lyapunov exponent, described by the cocycle $H_{i(f)}$ in the following way.

PROPOSITION 1.17. *There exists $\alpha > 0$ depending only on $\mathbb{P}$ and, for any $i = 2, \ldots, l_0$, positive measurable functions*

$$V^{(i)} : \mathbf{S}^{(i)}\mathcal{H} \to \mathbb{R}_+, C^{(i)} : \mathcal{H} \times \mathcal{H} \to \mathbb{R}_+$$

such that for $\mathbb{P}$-almost every $\mathbf{X} \in \mathcal{H}$, the following holds. Let $f \in \mathrm{Lip}_{w,0}^+(\mathbf{X})$ satisfy $\Phi_f^+ \neq 0$. Then for all $s > 0$, we have

$$\left| \frac{\mathrm{Var}_\nu(\mathfrak{S}[f, s; 1])}{V^{(i(f))}(g_s^{\mathbf{S}^{(i(f))}}(\overline{\omega}, v_f))(H_{i(f)}(s, (\mathbf{X}, v_f)))^2} - 1 \right| \leqslant C^{(i(f))}(\mathbf{X}, \mathbf{g}_s\mathbf{X}) e^{-\alpha s}.$$

We proceed to the formulation and the proof of the limit theorem in the general case. For $i = 2, \dots, l_0$, introduce the map

$$\mathcal{D}_i^+ : \mathbf{S}^{(i)}\mathcal{H} \to \mathfrak{M}_1$$

by setting $\mathcal{D}_i^+(\mathbf{X}, v)$ to be the distribution of the $C[0,1]$-valued random variable

$$\frac{\Phi_v^+(x,\tau)}{\sqrt{\mathrm{Var}_{\nu(\overline{\omega})}(\Phi_v^+(x,1))}}, \quad \tau \in [0,1].$$

As before, we have a commutative diagram

$$\begin{array}{ccc} \mathbf{S}^{(i)}\mathcal{H} & \xrightarrow{\mathcal{D}_i^+} & \mathfrak{M}_1 \\ \Big\downarrow{\scriptstyle \mathbf{g}_s^{\mathbf{S}^{(i)}}} & & \Big\uparrow{\scriptstyle G_s} \\ \mathbf{S}^{(i)}\mathcal{H} & \xrightarrow{\mathcal{D}_i^+} & \mathfrak{M}_1. \end{array}$$

The measure $\mathfrak{m}[f,s]$, as before, stands for the distribution of the $C[0,1]$-valued random variable

$$\frac{\displaystyle\int_0^{\tau\exp(s)} f \circ h_t^+(x)dt}{\sqrt{\mathrm{Var}_\nu\Big(\displaystyle\int_0^{\exp(s)} f \circ h_t^+(x)dt\Big)}}, \quad \tau \in [0,1].$$

THEOREM 2. *Let $\mathbb{P}$ be an invariant ergodic probability measure for the Teichmüller flow such that the Kontsevich-Zorich cocycle admits at least two distinct positive Lyapunov exponents with respect to $\mathbb{P}$. There exists a constant $\alpha > 0$ depending only on $\mathbb{P}$ and a positive measurable map $C : \mathcal{H} \times \mathcal{H} \to \mathbb{R}_+$ such that for $\mathbb{P}$-almost every $\mathbf{X} \in \mathcal{H}$ and any $f \in \mathrm{Lip}_w^+(\mathbf{X})$, we have*

$$d_{\mathrm{LP}}(\mathfrak{m}[f,s], \mathcal{D}_{i(f)}^+(g_s^{\mathbf{S}^{(i(f))}}(\mathbf{X}, v_f))) \leqslant C(\mathbf{X}, \mathbf{g}_s\mathbf{X})e^{-\alpha s},$$

$$d_{\mathrm{KR}}(\mathfrak{m}[f,s], \mathcal{D}_{i(f)}^+(g_s^{\mathbf{S}^{(i(f))}}(\mathbf{X}, v_f))) \leqslant C(\mathbf{X}, \mathbf{g}_s\mathbf{X})e^{-\alpha s}.$$

1.7.5. *Atoms of limit distributions.* For $\Phi^+ \in \mathfrak{B}^+(\mathbf{X})$, let $\mathfrak{m}[\Phi^+, \tau]$ be the distribution of the $\mathbb{R}$-valued random variable

$$\frac{\Phi^+(x,\tau)}{\sqrt{\mathrm{Var}_\nu \Phi^+(x,\tau)}}\ .$$

PROPOSITION 1.18. *For $\mathbb{P}$-almost every $\mathbf{X} \in \mathcal{H}$, there exists a dense set $T_{\mathrm{atom}} \subset \mathbb{R}$ such that if $\tau \in T_{\mathrm{atom}}$, then for any $\Phi^+ \in \mathfrak{B}^+(\mathbf{X})$, $\Phi^+ \neq 0$, the measure $\mathfrak{m}(\Phi^+, \tau)$ admits atoms.*

1.7.6. *Nonconvergence in distribution of ergodic integrals.* Our next aim is to show that along certain subsequences of times the ergodic integrals of translation flows converge in distribution to the measure δ_0, the delta-mass at zero. Weak convergence of probability measures will be denoted by the symbol $\Rightarrow$.

We need the following additional assumption on the measure $\mathbb{P}$.

ASSUMPTION 1.19. *For any $\varepsilon > 0$, the set of abelian differentials $\mathbf{X} = (M, \omega)$ such that there exists an admissible rectangle $\Pi(x, t_1, t_2) \subset M$ with $t_1 > 1 - \varepsilon$, $t_2 > 1 - \varepsilon$ has positive measure with respect to $\mathbb{P}$.*

Of course, this assumption holds for the Masur-Veech smooth measure.

PROPOSITION 1.20. *Let $\mathbb{P}$ be an ergodic $\mathbf{g}_s$-invariant measure on $\mathcal{H}$ satisfying Assumption 1.19. Then for $\mathbb{P}$-almost every $\mathbf{X} \in \mathcal{H}$, there exists a sequence $\tau_n \in \mathbb{R}_+$ such that for any $\Phi^+ \in \mathfrak{B}^+(\mathbf{X})$, $\Phi^+ \neq 0$, we have*

$$\mathfrak{m}[\Phi^+, \tau_n] \Rightarrow \delta_0 \text{ in } \mathfrak{M}(\mathbb{R}) \text{ as } n \to \infty.$$

Theorem 2 now implies the following

COROLLARY 1.21. *Let $\mathbb{P}$ be an ergodic $\mathbf{g}_s$-invariant measure on $\mathcal{H}$ satisfying Assumption 1.19. Then for $\mathbb{P}$-almost every $\mathbf{X} \in \mathcal{H}$, there exists a sequence $s_n \in \mathbb{R}_+$ such that for any $f \in \mathrm{Lip}^+_{w,0}(\mathbf{X})$ satisfying $\Phi^+_f \neq 0$, we have*

$$\mathfrak{m}[f, s_n; 1] \Rightarrow \delta_0 \text{ in } \mathfrak{M}(\mathbb{R}) \text{ as } n \to \infty.$$

Consequently, if $f \in \mathrm{Lip}^+_{w,0}(\mathbf{X})$ satisfies $\Phi^+_f \neq 0$, then the family of measures $\mathfrak{m}[f, \tau; 1]$ does not converge in $\mathfrak{M}(\mathbb{R})$ and the family of measures $\mathfrak{m}[f, \tau]$ does not converge in $\mathfrak{M}(C[0,1])$ as $\tau \to \infty$.

1.8. *The mapping into cohomology.* In this subsection we show that for an arbitrary abelian differential $\mathbf{X} = (M, \omega)$, the map

$$\check{\mathcal{I}}_{\mathbf{X}} : \mathfrak{B}^+_c(M, \omega) \to H^1(M, \mathbb{R})$$

given by (5) is indeed well defined.

PROPOSITION 1.22. *Let γ_i, $i = 1, \ldots, k$, be rectangular closed curves such that the cycle $\sum_{i=1}^k \gamma_i$ is homologous to 0. Then for any $\Phi^+ \in \mathfrak{B}^+_c$, we have*

$$\sum_{i=1}^k \Phi^+(\gamma_i) = 0.$$

Informally, Proposition 1.22 states that the *relative* homology of the surface with respect to zeros of the form is not needed for the description of cocycles. Arguments of this type for invariant measures of translation flows go back to Katok's work [28].

We proceed to the formal proof. Take a fundamental polygon Π for M such that all its sides are simple closed rectangular curves on M. Let $\partial\Pi$ be the boundary of Π, oriented counterclockwise. By definition,

$$\Phi^+(\partial\Pi) = 0, \tag{24}$$

since each curve of the boundary enters $\partial\Pi$ twice and with opposite signs.

We now deform the curves γ_i to the boundary $\partial\Pi$ of our fundamental polygon.

Proposition 1.23. *Let $\gamma \subset \Pi$ be a simple rectangular closed curve. Then*

$$\Phi^+(\gamma) = 0.$$

Proof of Proposition 1.23. We may assume that γ is oriented counterclockwise and does not contain zeros of the form ω. By Jordan's theorem, γ is the boundary of a domain $N \subset \Pi$. Let $p_1, \dots, p_r$ be zeros of ω lying inside N; let κ_i be the order of p_i. Choose an arbitrary $\varepsilon > 0$, take $\delta > 0$ such that $|\Phi^+(\gamma)| \le \varepsilon$ as soon as the length of γ does not exceed δ, and consider a partition of N given by

$$N = \Pi_1^{(\varepsilon)} \sqcup \dots \sqcup \Pi_n^{(\varepsilon)} \sqcup \widetilde{\Pi}_1^{(\varepsilon)} \sqcup \dots \sqcup \widetilde{\Pi}_r^{(\varepsilon)}, \tag{25}$$

where all $\Pi_i^{(\varepsilon)}$ are admissible rectangles and $\widetilde{\Pi}_i^{(\varepsilon)}$ is a $4(\kappa_i + 1)$-gon containing p_i and no other zeros and satisfying the additional assumption that all its sides are no longer than δ. Let $\partial\Pi_i^{(\varepsilon)}$, $\partial\widetilde{\Pi}_i^{(\varepsilon)}$ stand for the boundaries of our polygons oriented counterclockwise.

We have

$$\Phi^+(\gamma) = \sum \Phi^+(\partial\Pi_i^{(\varepsilon)}) + \sum \Phi^+(\partial\widetilde{\Pi}_i^{(\varepsilon)}).$$

In the first sum, each term is equal to 0 by definition of Φ^+, whereas the second sum does not exceed, in absolute value, the quantity

$$C(\kappa_1, \dots, \kappa_r)\varepsilon,$$

where $C(\kappa_1, \dots, \kappa_r)$ is a positive constant depending only on $\kappa_1, \dots, \kappa_r$. Since ε may be chosen arbitrarily small, we have

$$\Phi^+(\gamma) = 0,$$

which is what we had to prove. □

For $A, B \in \partial\Pi$, let $\partial\Pi_A^B$ be the part of $\partial\Pi$ going counterclockwise from A to B.

Proposition 1.24. *Let $A, B \in \partial\Pi$, and let $\gamma \subset \Pi$ be an arbitrary rectangular curve going from A to B. Then*

$$\Phi^+(\partial\Pi_A^B) = \Phi^+(\gamma).$$

We may assume that γ is simple in Π since, by Proposition 1.23, self-intersections of γ (whose number is finite) do not change the value of $\Phi^+(\gamma)$. If γ is simple, then γ and $\Phi^+(\partial\Pi_B^A)$ together form a simple closed curve, and the proposition follows from Proposition 1.23.

COROLLARY 1.25. *If $\gamma \subset \Pi$ is a rectangular curve that yields a closed curve in M homologous to zero in M, then $\Phi^+(\gamma) = 0$.*

Indeed, by the previous proposition we need only consider the case when $\gamma \subset \partial\Pi$. Since γ is homologous to 0 by assumption, the cycle γ is in fact a multiple of the cycle $\partial\Pi$, for which the statement follows from (24).

1.9. *Markovian sequences of partitions.*

1.9.1. *The Markov property.* Let (M, ω) be an abelian differential. A rectangle $\Pi(x, t_1, t_2) = \{h^+_{\tau_1} h^-_{\tau_2} x, 0 \le \tau_1 < t_1, 0 \le \tau_2 < t_2\}$ is called *weakly admissible* if for all sufficiently small $\varepsilon > 0$ the rectangle $\Pi(h^+_\varepsilon h^-_\varepsilon x, t_1 - \varepsilon, t_2 - \varepsilon)$ is admissible. (In other words, the boundary of Π may contain zeros of ω but the interior does not.)

Assume we are given a natural number m and a sequence of partitions π_n

$$\pi_n : M = \Pi_1^{(n)} \sqcup \cdots \sqcup \Pi_m^{(n)}, \; n \in \mathbb{Z}, \tag{26}$$

where $\Pi_i^{(n)}$ are weakly admissible rectangles.

The sequence π_n of partitions of M into m weakly admissible rectangles will be called a *Markovian sequence* of partitions if for any $n_1, n_2 \in \mathbb{Z}$, $i_1, i_2 \in \{1, \ldots, m\}$, the rectangles $\Pi_{i_1}^{(n_1)}$ and $\Pi_{i_2}^{(n_2)}$ intersect in a Markov way in the following precise sense.

Take a weakly admissible rectangle $\Pi(x, t_1, t_2)$, and decompose its boundary into four parts:

$$\begin{aligned}
\partial_h^1(\Pi) &= \overline{\{h^+_{t_1} h^-_{\tau_2} x, 0 \le \tau_2 < t_2\}},\\
\partial_h^0(\Pi) &= \overline{\{h^-_{\tau_2} x, 0 \le \tau_2 < t_2\}},\\
\partial_v^1(\Pi) &= \overline{\{h^-_{t_2} h^+_{\tau_1} x, 0 \le \tau_1 < t_1\}},\\
\partial_v^0(\Pi) &= \overline{\{h^+_{\tau_1} x, 0 \le \tau_1 < t_1\}}.
\end{aligned}$$

The sequence of partitions π_n has the *Markov property* if for any $n \in \mathbb{Z}$ and $i \in \{1, \ldots, m\}$, there exist $i_1, i_2, i_3, i_4 \in \{1, \ldots, m\}$ such that

$$\begin{aligned}
\partial_h^1(\Pi_i^{(n)}) &\subset \partial_h^1 \Pi_{i_1}^{(n-1)},\\
\partial_h^0(\Pi_i^{(n)}) &\subset \partial_h^0 \Pi_{i_2}^{(n-1)},\\
\partial_v^1(\Pi_i^{(n)}) &\subset \partial_v^1 \Pi_{i_3}^{(n+1)},\\
\partial_v^0(\Pi_i^{(n)}) &\subset \partial_v^0 \Pi_{i_4}^{(n+1)}.
\end{aligned}$$

1.9.2. *Adjacency matrices.* To a Markovian sequence of partitions we assign the sequence of $m \times m$ *adjacency matrices* $A_n = A(\pi_n, \pi_{n+1})$ defined as follows: $(A_n)_{ij}$ is the number of connected components of the intersection

$$(\Pi_i^{(n)}) \cap \Pi_j^{(n+1)}.$$

A Markovian sequence of partitions π_n will be called an *exact* Markovian sequence of partitions if

$$\lim_{n\to\infty} \max_{i=1,\dots,m} \nu^+(\partial_v \Pi_i^{(n)}) = 0, \quad \lim_{n\to\infty} \max_{i=1,\dots,m} \nu^-(\partial_h \Pi_i^{(-n)}) = 0. \tag{27}$$

For an abelian differential both whose vertical and horizontal flows are minimal, there always exists $m \in \mathbb{N}$ and a sequence of partitions (26) having the Markov property and satisfying the exactness condition (27). A suitably chosen Markovian sequence of partitions will be essential for the construction of finitely-additive measures in the following section.

Remark. An exact Markovian sequence of partitions allows one to identify our surface M with the space of trajectories of a nonautonomous Markov chain or, in other words, a Markov compactum. The horizontal and vertical foliations then become the asymptotic foliations of the corresponding Markov compactum; the finitely-additive measures become finitely-additive measures on one of the asymptotic foliations invariant under holonomy with respect to the complementary foliation; the vertical and horizontal flow also admit a purely symbolic description as flows along the leaves of the asymptotic foliations according to an order induced by a Vershik's ordering (see, e.g., [43], [44], [45]) on the edges of the graphs forming the Markov compactum. The space of abelian differentials or, more precisely, the Veech space of zippered rectangles, is then represented as a subspace of the space of Markov compacta. The space of Markov compacta is a space of bi-infinite sequences of graphs and is therefore endowed with a natural shift transformation. Using Rauzy-Veech expansions of zippered rectangles, one represents the Teichmüller flow as a suspension flow over this shift. The Kontsevich-Zorich cocycle is then a particular case of the cocycle over the shift given by consecutive adjacency matrices of the graphs forming our Markov compactum. To an abelian differential, random with respect to a probability measure invariant under the Teichmüller flow, one can thus assign a Markov compactum corresponding to a sequence of graphs generated according to a stationary process. The relation between Markov compacta and abelian differentials is summarized in Table 1.

The main theorems of this paper, Theorems 1 and 2, are particular cases of general theorems on the asymptotic behaviour of ergodic averages of symbolic flows along asymptotic foliations of random Markov compacta; these generalizations, which will be published in the sequel to this paper, are proved in the preprint [14].

Markov compacta	**Abelian differentials**
Asymptotic foliations of a Markov compactum	Horizontal and vertical foliations of an abelian differential
Finitely-additive measures on asymptotic foliations of a Markov compactum	The spaces $\mathfrak{B}^+$ and $\mathfrak{B}^-$ of Hölder cocycles
Vershik's automorphisms	Interval exchange transformations
Suspension flows over Vershik's automorphisms	Translation flows on flat surfaces
The space of Markov compacta	The moduli space of abelian differentials
The shift on the space of Markov compacta	The Teichmüller flow
The cocycle of adjacency matrices	The Kontsevich-Zorich cocycle

Table 1.

For further results and background on Vershik's automorphisms, substitutions and symbolic dynamics for interval exchange transformations see, e.g., [1], [3], [12], [13], [18], [25], [35], [36], [38], [39], [40], [43], [44], [45].

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2. Construction of finitely-additive measures

2.1. *Equivariant sequences of vectors.* Let $\mathfrak{A}$ be a bi-invariant sequence of invertible $m \times m$-matrices with nonnegative entries

$$\mathfrak{A} = (A_n), \quad n \in \mathbb{Z}.$$

To a vector $v \in \mathbb{R}^n$ we assign the corresponding $\mathfrak{A}$-*equivariant* sequence $\mathbf{v} = (v^{(n)}),\ n \in \mathbb{Z}$, given by the formula

$$v^{(n)} = \begin{cases} A_{n-1} \cdots A_0 v, & n > 0, \\ v, & n = 0, \\ A_{-n}^{-1} \cdots A_{-1}^{-1} v, & n < 0. \end{cases}$$

We now consider subspaces in $\mathbb{R}^m$ consisting of vectors v such that the corresponding equivariant subsequence $\mathbf{v} = v^{(n)}$ decays as n tends to $-\infty$. More formally,we write

$$\begin{aligned} \mathfrak{B}_c^+(\mathfrak{A}) &= \{v \in \mathbb{R}^m : |v^{(-n)}| \to 0 \ as\ n \to +\infty\}, \\ \mathfrak{B}^+(\mathfrak{A}) &= \{v \in \mathbb{R}^m : \text{ there exists } \ C > 0, \alpha > 0 \\ &\qquad \text{such that } |v^{(-n)}| \leqslant Ce^{-\alpha n} \text{ for all } n \geqslant 0\}. \end{aligned}$$

It will sometimes be convenient to identify a vector with the corresponding equivariant sequence and, slightly abusing notation, we shall sometimes say that a given equivariant sequence belongs to the space $\mathfrak{B}_c^+(\mathfrak{A})$ or $\mathfrak{B}^+(\mathfrak{A})$.

2.2. *A canonical system of arcs corresponding to a Markovian sequence of partitions.* As before, let (M, ω) be an abelian differential. By an arc of the vertical flow we mean a set of the form $\{h_t^+ x, 0 \leqslant t \leqslant t_0\}$. Such a set will also sometimes be denoted $[x, x']$, where $x' = h_{t_0}^+ x$.

Let $\pi_n,\ n \in \mathbb{Z}$, be an exact Markovian sequence of partitions

$$\pi_n : M = \Pi_1^{(n)} \sqcup \cdots \sqcup \Pi_m^{(n)}, \quad n \in \mathbb{Z},$$

into weakly admissible rectangles. Take a rectangle $\Pi_i^{(n)}, n \in \mathbb{Z}, i \in \{1, \ldots, m\}$, and choose an arbitrary arc $\gamma_i^{(n)}$ of the vertical foliation going all the way from the lower boundary of $\Pi_i^{(n)}$ to the upper boundary. More formally, take $x \in \partial_h^{(0)} \Pi_i^{(n)}$, take t such that $h_t^+ x \in \partial_h^{(1)} \Pi_i^{(n)}$, and let $\gamma_i^{(n)}$ be the vertical arc $[x, h_t^+ x]$. An arc $\gamma_i^{(n)}$ of this form will be called a *Markovian* arc.

The family of arcs

$$\gamma_i^{(n)}, n \in \mathbb{Z}, i \in \{1, \ldots, m\},$$

will be called *a canonical system of arcs* assigned to the Markov sequence of partitions π_n. Of course, there is freedom in the choice of specific arcs $\gamma_i^{(n)}$, but our constructions will not depend on the specific choice of a canonical system.

Given a finitely-additive measure $\Phi^+ \in \mathfrak{B}_c^+(M, \omega)$, introduce a sequence of vectors $v^{(n)} \in \mathbb{R}^m, n \in \mathbb{Z}$, by setting

$$v_i^{(n)} = \Phi^+(\gamma_i^{(n)}). \tag{28}$$

Now let $\mathfrak{A} = (A_n), n \in \mathbb{Z}$, $A_n = A(\pi_n, \pi_{n+1})$ be the sequence of adjacency matrices of the sequence of partitions π_n, and assume all A_n to be invertible. By the horizontal holonomy invariance, the value $v_i^{(n)}$ does not depend on the specific choice of the arc $\gamma_i^{(n)}$ inside the rectangle $\Pi_i^{(n)}$. Finite additivity of the measure Φ^+ implies that the sequence $v^{(n)}, n \in \mathbb{Z}$, is $\mathfrak{A}$-equivariant. Exactness of the sequence of partitions $\pi_n, n \in \mathbb{Z}$, implies that that the equivariant sequence $v^{(n)}$ corresponding to a finitely-additive measure $\Phi^+ \in \mathfrak{B}_c^+(X, \omega)$ satisfies $v^{(0)} \in \mathfrak{B}_c^+(\mathfrak{A})$. We have therefore obtained a map

$$\mathrm{eval}_0^+ : \mathfrak{B}_c^+(X, \omega) \to \mathfrak{B}_c^+(\mathfrak{A}).$$

It will develop that under certain natural additional assumptions this map is indeed an isomorphism.

We now take an abelian differential (M, ω) whose vertical flow is uniquely ergodic and show that if the heights of the rectangles $\Pi_i^{(n)}$ decay exponentially as $n \to -\infty$, then the map eval_0^+ sends $\mathfrak{B}^+(X, \omega)$ to $\mathfrak{B}^+(\mathfrak{A})$. We proceed to precise formulations.

Introduce a sequence $h^{(n)} = (h_1^{(n)}, \ldots, h_m^{(n)})$ by setting $h_i^{(n)}$ to be the height of the rectangle $\Pi_i^{(n)}, i = 1, \ldots, m$. By the Markov property, the sequence $h^{(n)}$ is $\mathfrak{A}$-equivariant; unique ergodicity of the vertical flow and exactness of the sequence $\pi_n, n \in \mathbb{Z}$, imply that a positive $\mathfrak{A}$-equivariant sequence is unique up to scalar multiplication. By definition and, again, by exactness, we have

$$h^{(0)} \in \mathfrak{B}_c^+(\mathfrak{A}).$$

Proposition 2.1. *If* $h^{(0)} \in \mathfrak{B}^+(\mathfrak{A})$, *then*

$$\mathrm{eval}_0^+(\mathfrak{B}^+(X, \omega)) \subset \mathfrak{B}^+(\mathfrak{A}).$$

Proof. Let a canonical family of vertical arcs $\gamma_i^{(n)}$ corresponding to the Markovian sequence of partitions π_n be chosen as above. The condition $h^{(0)} \in \mathfrak{B}^+(\mathfrak{A})$ precisely means the existence of constants $C > 0, \alpha > 0$ such that for all $n \in \mathbb{N}, i = 1, \ldots, m$, we have $\nu^+(\gamma_i^{(n)}) \leqslant Ce^{-\alpha n}$.

Now if $\Phi^+ \in \mathfrak{B}^+(M,\omega)$, then there exists $\theta > 0$ such that for all sufficiently large n and all $i = 1, \dots, m$, we have

$$|\Phi^+(\gamma_i^{(n)})| \leqslant (\nu^+(\gamma_i^{(n)}))^\theta.$$

Consequently, $|\Phi^+(\gamma_i^{(n)})| \leqslant \widetilde{C} e^{-\tilde{\alpha} n}$ for some $\widetilde{C} > 0, \tilde{\alpha} > 0$ and all $n \in \mathbb{N}, i = 1, \dots, m$, which is what we had to prove. □

The scheme of the proof of the reverse inclusion can informally be summarized as follows. We start with an equivariant sequence $v^{(n)} \in \mathfrak{B}^+(\mathfrak{A})$, and we wish to recover a measure $\Phi^+ \in \mathfrak{B}^+(X,\omega)$. The equivariant sequence itself determines the values of the finitely-additive measure Φ^+ on all *Markovian* arcs, that is, arcs going from the lower horizontal to the upper horizontal boundary of a rectangle of one of the proposition π_n. To extend the measure Φ^+ from Markovian arcs to all vertical arcs, we approximate an arbitrary arc by Markovian ones. (Similar approximation lemmas were used by Forni [21] and Zorich [46].) The approximating series will be seen to converge because the number of terms at each stage of the approximation grows at most sub-exponentially, while the contribution of each term decays exponentially. For this argument to work, we assume that the norms of the matrices A_n grow sub-exponentially.

The measure Φ^+ is thus extended to all vertical arcs. To check the Hölder property for Φ^+, one needs additionally to assume that the heights of the Markovian rectangles $\Pi_i^{(n)}$ decay not faster than exponentially. More precise Oseledets-type assumptions on the sequence $\mathfrak{A}$ of adjacency matrices are used in order to obtain lower bounds on the Hölder exponent for Φ^+ and to derive the logarithmic asymptotics of the growth of Φ^+ at infinity. All our assumptions are verified for Markov sequences of partitions induced by Rauzy-Veech expansions of zippered rectangles as soon as one uses the Veech method of considering expansions corresponding to occurrences of a fixed renormalization matrix with positive entries.

2.3. *Strongly biregular sequences of matrices.* A sequence $\mathfrak{A} = (A_n), n \in \mathbb{Z}$, of $m \times m$-matrices will be called *balanced* if all entries of all matrices A_n are positive and, furthermore, there exists a positive constant C such that for any $n \in \mathbb{Z}$ and any $i_1, j_1, i_2, j_2 \in \{1, \dots, m\}$, we have

$$\frac{(A_n)_{i_1 j_1}}{(A_n)_{i_2 j_2}} < C.$$

A sequence $\mathfrak{A} = (A_n), n \in \mathbb{Z}$, of $m \times m$-matrices with nonnegative entries will be said to have *sub-exponential growth* if for any $\varepsilon > 0$ there exists C_ε such that for all $n \in \mathbb{Z}$, we have

$$\sum_{i,j=1}^{m} (A_n)_{ij} \leqslant C_\varepsilon e^{\varepsilon |n|}.$$

In order to formulate our next group of assumptions, we need to consider $\mathfrak{A}$-*reverse equivariant* sequences of vectors. To a vector $\tilde{\mathrm{v}} \in \mathbb{R}^m$ we assign the sequence $\tilde{\mathbf{v}} = (\tilde{v}^{(n)}), n \in \mathbb{Z}$, given by the formula

$$\tilde{v}^{(n)} = \begin{cases} (A_n^t)^{-1} \cdots (A_1^t)^{-1} \tilde{v}^{(n)}, & n > 0, \\ \tilde{v}^{(n)}, & n = 0, \\ A_{n+1}^t \dots A_{-1}^t A_0^t \tilde{v}^{(n)}, & n < 0. \end{cases}$$

By definition, if $\tilde{v}^{(n)}$ is an $\mathfrak{A}$-equivariant sequence, while $\tilde{v}^{(n)}$ is an $\mathfrak{A}$-reverse equivariant sequence, then the inner product

$$\left\langle v^{(n)}, \tilde{v}^{(n)} \right\rangle = \sum_{i=1}^{m} v_i^{(n)} \tilde{v}_i^{(n)}$$

does not depend on $n \in \mathbb{Z}$. In analogy to the spaces $\mathfrak{B}^+(\mathfrak{A})$ and $\mathfrak{B}_c^+(\mathfrak{A})$, we introduce the spaces

$$\begin{aligned} \mathfrak{B}_c^-(\mathfrak{A}) &= \{\tilde{v} : |\tilde{v}^{(n)}| \to 0 \ as \ n \to \infty\}, \\ \mathfrak{B}^-(\mathfrak{A}) &= \{\tilde{v} : \text{ there exists } C > 0,\ \alpha > 0 \\ &\qquad \text{such that } |\tilde{v}^{(n)}| \leqslant Ce^{-\alpha n} \text{ as } n \to \infty\}. \end{aligned}$$

The unique ergodicity of the vertical and the horizontal flow admits the following reformulation in terms of the spaces $\mathfrak{B}_c^+(\mathfrak{A})$, $\mathfrak{B}_c^-(\mathfrak{A})$.

ASSUMPTION 2.2. *The space $\mathfrak{B}_c^+(\mathfrak{A})$ contains an equivariant sequence $(h^{(n)}), n \in \mathbb{Z}$, such that $h_i^{(n)} > 0$ for all $n \in \mathbb{Z}$ and all $i \in \{1, \dots, m\}$. The space $\mathfrak{B}_c^-(\mathfrak{A})$ contains a reverse equivariant sequence $(\lambda^{(n)}), n \in \mathbb{Z}$, such that $\lambda_i^{(n)} > 0$ for all $n \in \mathbb{Z}$ and all $i \in \{1, \dots, m\}$. The sequences $(h^{(n)})$ and $(\lambda^{(n)})$ are unique up to scalar multiplication.*

A convenient normalization for us will be:

$$|\lambda^{(0)}| = 1, \langle \lambda^{(0)}, h^{(0)} \rangle = 1.$$

Our next assumption is the requirement of Lyapunov regularity for the sequence of matrices $\mathfrak{A} = (A_n), n \in \mathbb{Z}$. For renormalization matrices of Rauzy-Veech expansions this assumption will be seen to hold by the Oseledets Theorem applied to the Kontsevich-Zorich cocycle. In fact, we will assume the validity of all the statements of the Oseledets-Pesin Reduction Theorem ([5, Th. 3.3.5, p. 77]). We proceed to the precise formulation.

ASSUMPTION 2.3. *There exists $l_0 \in \mathbb{N}$, positive numbers*

$$\theta_1 > \theta_2 > \cdots > \theta_{l_0} > 0$$

and, for any $n \in \mathbb{Z}$, direct-sum decompositions

$$\mathbb{R}^m = E_n^1 \oplus \cdots \oplus E_n^{l_0} \oplus E_n^{cs},$$
$$\mathbb{R}^m = \widetilde{E}_n^1 \oplus \cdots \oplus \widetilde{E}_n^{l_0} \oplus \widetilde{E}_n^{cs}$$

such that the following holds:

(1) *For all $n \in \mathbb{Z}$, we have*

$$E_n^1 = \mathbb{R}h^{(n)}, \widetilde{E}_n^1 = \mathbb{R}\lambda^{(n)}.$$

(2) *For all $n \in \mathbb{Z}$ and all $i = 1, \dots, l_0$, we have*

$$A_n E_n^i = E_{n+1}^i, \quad A_n^t \widetilde{E}_{n+1}^i = \widetilde{E}_n^i.$$

(3) *For all $n \in \mathbb{Z}$, every $i = 1, \dots, l_0$, and any $v \in E_n^i \setminus \{0\}$, we have*

$$\lim_{k\to\infty} \frac{\log |A_{n+k-1} \cdots A_n v|}{k} = \lim_{k\to\infty} \frac{-\log |A_{n-k}^{-1} \cdots A_{n-1}^{-1} v|}{k} = \theta_i,$$

and the convergence is uniform on the sphere $\{v \in E_n^i : |v| = 1\}$.

(4) *For all $n \in \mathbb{Z}$, every $i = 1, \dots, l_0$, and any $v \in E_n^i \setminus \{0\}$, we have*

$$\lim_{k\to\infty} \frac{\log |A_{n-k}^t \cdots A_{n-1}^t v|}{k} = \lim_{k\to\infty} \frac{-\log |(A_{n+k-1}^t)^{-1} \cdots A_n^{t\,-1} v|}{k} = \theta_i,$$

and the convergence is uniform on the sphere $\{v \in \widetilde{E}_n^i : |v| = 1\}$.

(5) *For all $n \in \mathbb{Z}$, we have*

$$A_n E_n^{cs} = E_{n+1}^{cs}, \quad A_n^t E_{n+1}^{cs} = E_n^{cs}.$$

(6) *For any $\varepsilon > 0$, there exists C_ε such that for any $n \in \mathbb{Z}$ and $k \in \mathbb{N}$, we have*

$$||A_{n+k-1} \cdots A_n|_{E_n^{cs}}|| \leqslant C_\varepsilon e^{\varepsilon(k+|n|)},$$
$$||A_{n-k}^{-1} \cdots A_{n-1}^{-1}|_{E_n^{cs}}||^{-1} \leqslant C_\varepsilon e^{\varepsilon(k+|n|)},$$
$$||A_{n-k}^t \cdots A_{n-1}^t|_{\widetilde{E}_n^{cs}}|| \leqslant C_\varepsilon e^{\varepsilon(k+|n|)},$$
$$||(A_{n+k-1}^t)^{-1} \cdots (A_n^t)^{-1}|_{\widetilde{E}_n^{cs}}||^{-1} \leqslant C_\varepsilon e^{\varepsilon(k+|n|)}.$$

(7) *For all $n \in \mathbb{Z}$, we have $\dim E_n^{cs} = \dim \widetilde{E}_n^{cs}$ and, for any $i = 1, \dots, l_0$, we also have $\dim E_n^i = \dim \widetilde{E}_n^i$. If $v, \tilde{v} \in \mathbb{R}^m$ satisfy $\langle v, \tilde{v} \rangle \neq 0$, then $v \in E_n^i$ implies $\tilde{v} \in \widetilde{E}_n^i$, while $v \in E_n^{cs}$ implies $\tilde{v} \in \widetilde{E}_n^{cs}$, and vice versa.*

A balanced sequence $\mathfrak{A}$ of $m \times m$-matrices with positive entries, having sub-exponential growth and satisfying the unique ergodicity assumption as well as the Lyapunov regularity assumption, will be called a *strongly biregular* sequence or, for brevity, an SB-sequence. Using Markovian sequences of partitions induced by Rauzy-Veech expansions of zippered rectangles corresponding to consecutive occurrences of a fixed renormalization matrix with positive entries and applying the Oseledets Multiplicative Ergodic Theorem and the Oseledets-Pesin Reduction Theorem (see [5, Th. 3.5, p. 77]) to the Kontsevich-Zorich cocycle, we will establish in the next section the following simple

PROPOSITION 2.4. *Let $\mathbb{P}$ be an ergodic probability measure on a connected component $\mathcal{H}$ of the moduli space $\mathcal{M}_\kappa$ of abelian differentials. Then $\mathbb{P}$-almost every abelian differential $(M,\omega) \in \mathcal{H}$ admits an exact Markov sequence of partitions whose sequence of adjacency matrices belongs to the class* SB.

Let (M,ω) be an abelian differential whose horizontal and vertical foliations are both uniquely ergodic. Assume that (M,ω) is endowed with an exact Markov sequence $\pi_n, n \in \mathbb{Z}$, of partitions into weakly admissible rectangles such that the corresponding sequence $\mathfrak{A}$ of adjacency matrices belongs to the class SB.

Note that if $\mathfrak{A}$ is an SB-sequence, then

$$\mathfrak{B}^+(\mathfrak{A}) = E_0^1 \oplus \cdots \oplus E_0^{l_0},$$
$$\mathfrak{B}^-(\mathfrak{A}) = \widetilde{E}_0^1 \oplus \cdots \oplus \widetilde{E}_0^{l_0}.$$

Note also that there exists a constant $C > 0$ such that the positive equivariant sequence $h^{(n)}$ satisfies

$$\frac{h_i^{(n)}}{h_j^{(n)}} \leqslant C$$

for all $n \in \mathbb{Z}, i, j \in \{1, \ldots, n\}$. It follows that for any $\varepsilon > 0$, there exists a constant C_ε such that for all $n > 0$, we have

$$\min_i h_i^{(-n)} \geqslant C_\varepsilon e^{-(\theta_1+\varepsilon)n}.$$

2.4. *Characterization of finitely-additive measures.*

2.4.1. *The semi-rings of Markovian arcs.* Given a partition π of our surface M into weakly admissible rectangles $\Pi_1, \ldots, \Pi_m$, we consider the semi-ring $\mathfrak{C}^+(\pi)$ of arcs of the form $[x, x']$, where $x \in \partial_h^0 \Pi_i,\ x' \in \partial_h^1 \Pi_i$ for some i. (Recall here that $x \in \partial_h^0 \Pi_i$ stands for the lower horizontal boundary of $\Pi_i, \partial_h^1 \Pi_i$ for the upper horizontal boundary.)

Our Markov sequence π_n thus induces a sequence of semi-rings $\mathfrak{C}_n^+ = \mathfrak{C}^+(\pi_n)$; we write $\mathfrak{R}_n^+$ for the ring generated by the semi-ring $\mathfrak{C}_n^+$. Elements of $\mathfrak{R}_n^+$ are finite unions of arcs from $\mathfrak{C}_n^+$. For an arc γ of the vertical flow, let $\check{\gamma}_n$ be the largest by inclusion arc from the ring $\mathfrak{R}_{-n}^+$ contained in γ, and let $\hat{\gamma}_n$ be the smallest by inclusion arc from the ring $\mathfrak{R}_{-n}^+$ containing γ.

2.4.2. *Extension of finitely-additive measures.* The following lemma is immediate from the definitions. (Note that similar arc approximation lemmas were used by Forni in [21] and Zorich in [46].)

LEMMA 2.5. *Let π_n be an exact Markovian sequence of partitions such that the corresponding sequence $\mathfrak{A}$ of adjacency matrices has sub-exponential growth. Then for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that for any arc γ of the vertical flow and any $n \in \mathbb{N}$, the set $\hat{\gamma}_n \backslash \check{\gamma}_n$ consists of at most $C_\varepsilon e^{\varepsilon n}$ arcs from the semi-ring $\mathfrak{C}^+_{-n}$.*

Informally, Lemma 2.5 says that any arc of our symbolic flow is approximable by Markovian arcs with sub-exponential error; we illustrate this by Figure 1.

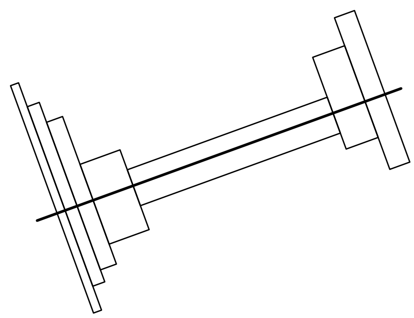

Figure 1. The number of small arcs grows at most subexponentially

We are now ready to identify $\mathfrak{B}^+(\mathfrak{A})$ and $\mathfrak{B}^+(X, \omega)$.

LEMMA 2.6. *Let $\pi_n, n \in \mathbb{Z}$, be a Markov sequence of partitions such that the corresponding sequence $\mathfrak{A}$ of adjacency matrices belongs to the class* SB. *Then for every equivariant sequence $v^{(n)} \in \mathfrak{B}^+(\mathfrak{A})$, there exists a unique finitely-additive measure $\Phi^+ \in \mathfrak{B}^+(X, \omega)$ such that*

$$\mathrm{eval}_0^+(\Phi^+) = v^{(0)}.$$

Proof. Indeed, the sequence $v^{(n)}$ itself prescribes the values of the Φ^+ on all Markovian arcs $\gamma \in \mathfrak{C}^+_n, n \in \mathbb{Z}$. For a general arc γ of the vertical flow, set

$$\Phi^+(\gamma) = \lim_{n\to\infty} \Phi^+(\hat{\gamma}_n) = \lim_{n\to\infty} \Phi^+(\check{\gamma}_n), \tag{29}$$

where the existence of both limits and the equality of their values immediately follow from Lemma 2.5. Finite-additivity of Φ^+ is again a corollary of Lemma 2.5. We have thus obtained a finitely-additive measure Φ^+ defined on all vertical arcs. The uniqueness of such a measure is clear by (29). The invariance of the resulting measure under horizontal holonomy is also clear by definition. To conclude the proof of Lemma 2.6, it remains to check that the obtained finitely-additive measure Φ^+ satisfies the Hölder property.

For Markovian arcs the Hölder upper bound is clear from the *upper* exponential bound

$$|v^{(-n)}| \leqslant Ce^{-\alpha n}$$

and the *lower* exponential bound

$$\min_i h_i^{(-n)} \geqslant C_1 e^{(-\alpha_1 n)}.$$

For general arcs, the Hölder property follows by Lemma 2.5. Lemma 2.6 is proved completely, and we have thus shown that under its assumptions the map

$$\mathrm{eval}_0^+ : \mathfrak{B}^+(X,\omega) \to \mathfrak{B}^+(\mathfrak{A})$$

is indeed an isomorphism. □

Remark. Under stronger assumptions of Lyapunov regularity we will also give a Hölder *lower* bound for the cocycles $\Phi^+ \in \mathfrak{B}^+$; see Proposition 2.9.

2.5. *Duality.* Let $v^{(n)} \in \mathfrak{B}^+(\mathfrak{A}), \tilde{v}^{(n)} \in \mathfrak{B}^-(\mathfrak{A})$. Let $\Phi^+ \in \mathfrak{B}^+(X,\omega), \Phi^- \in \mathfrak{B}^+(X,\omega)$ be the corresponding finitely-additive measures. The definitions directly imply

$$\int_M \Phi^+ \times \Phi^- = \sum_{i=1}^m v_i^{(0)} \tilde{v}_i^{(0)} = \langle v, \tilde{v} \rangle.$$

Duality between the spaces $\mathfrak{B}^+(X,\omega)$ and $\mathfrak{B}^-(X,\omega)$ follows now from the duality between the spaces $\mathfrak{B}^+(\mathfrak{A})$ and $\mathfrak{B}^-(\mathfrak{A})$, which holds by the Lyapunov regularity assumption for the sequence $\mathfrak{A}$.

2.6. *Proof of Theorem* 1.

2.6.1. *Approximation of almost equivariant sequences.* We start with a sequence of matrices $A_n, n \geqslant 0$ satisfying the following

ASSUMPTION 2.7. *There exists $\alpha > 0$ and, for every $n \geqslant 1$, a direct-sum decomposition*

$$\mathbb{R}^m = E_n^u \oplus E_{cs}^n$$

satisfying the following:

(1) $A_n E_n^u = E_{n+1}^u$ *and* $A_n|_{E_n^u}$ *is injective*;
(2) $A_n E_n^{cs} \subset E_{n+1}^{cs}$;
(3) *for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that for any $n \geqslant 1,\ k \geqslant 0$, we have*

$$\left|\left|(A_{n+k} \cdot \ldots \cdot A_n)^{-1}|_{E^u_{n+k+1}}\right|\right| \leqslant C_\varepsilon e^{\varepsilon n - \alpha k},$$
$$||A_{n+k} \cdot \ldots \cdot A_n|_{E_n^{cs}}|| \leqslant C_\varepsilon e^{\varepsilon(n+k)}.$$

LEMMA 2.8. *Let A_n be a sequence of matrices satisfying Assumption* 2.7. *Let $v_1, \dots, v_n, \dots$ be a sequence of vectors such that for any $\varepsilon > 0$, a constant C_ε can be chosen in such a way that for all n, we have*

$$|A_n v_n - v_{n+1}| \le C_\varepsilon \exp(\varepsilon n).$$

Then there exists a unique vector $v \in E_1^u$ such that

$$|A_n \cdots A_1 v - v_{n+1}| \le C'_\varepsilon \exp(\varepsilon n).$$

Proof. Denote $u_{n+1} = v_{n+1} - A_n v_n$, and decompose $u_{n+1} = u_{n+1}^+ + u_{n+1}^-$, where $u_{n+1}^+ \in E_{n+1}^u$, $u_{n+1}^- \in E_{n+1}^{cs}$. Let

$$v_{n+1}^+ = u_{n+1}^+ + A_n u_n^+ + A_n A_{n-1} u_{n-1}^+ + \cdots + A_n \cdots A_1 u_1^+,$$
$$v_{n+1}^- = u_{n+1}^- + A_n u_n^- + A_n A_{n-1} u_{n-1}^- + \cdots + A_n \cdots A_1 u_1^-.$$

We have $v_{n+1} \in E_{n+1}^u$, $v_{n+1}^- \in E_{n+1}^{cs}$, $v_{n+1} = v_{n+1}^+ + v_{n+1}^-$. Now introduce a vector

$$v = u_1^+ + A_1^{-1} u_1^+ + \cdots + (A_n \cdots A_1)^{-1} u_{n+1}^+ + \cdots .$$

By our assumptions, the series defining v converges exponentially fast and, moreover, we have

$$|A_n \cdots A_1 v - v_{n+1}^+| \le C'_\varepsilon \exp(\varepsilon n)$$

for some constant C'_ε.

Since by our assumptions we also have $|v_{n+1}^-| \le C_\varepsilon \exp(\varepsilon n)$, the lemma is proved completely. □

Uniqueness of the vector v follows from the fact that, by our assumptions, for any $\tilde{v} \ne 0$, $\tilde{v} \in E_0^u$ we have

$$|A_n \cdots A_1 \tilde{v}| \ge C'' \exp(\alpha n).$$

2.6.2. *Approximation of weakly Lipschitz functions.* Let $f : M \to \mathbb{R}$ be a weakly Lipschitz function and, as before, introduce a canonical family of Markovian curves $\gamma_i^n, n \in \mathbb{Z}, i = 1, \dots, m$, corresponding to the Markovian sequence of partitions $\pi_n, n \in \mathbb{Z}$.

Introduce a family of vectors $v(n) \in \mathbb{R}^m, n \in \mathbb{Z}$, by setting

$$(v(n))_i = \int_{\gamma_i^n} f d\nu^+, \ i = 1, \dots, m.$$

By definition of the adjacency matrices $A_n = A(\pi_n, \pi_{n+1})$, using the weak Lipschitz property of the function f and sub-exponential growth of the sequence $\mathfrak{A} = (A_n)$, we arrive for all $n \in \mathbb{N}$ at the estimate

$$|A_n v(n) - v(n+1)| \leqslant C_\varepsilon ||f||_{\mathrm{Lip}} \cdot e^{\varepsilon n}.$$

By Lemma 2.8, there exists a unique vector $v_f^+ \in \mathfrak{B}^+(\mathfrak{A})$ such that for all $n \in \mathbb{N}$ we have

$$\left| v(n) - A_{n-1} \cdot \ldots \cdot A_0 v_f^+ \right| \leqslant C'_\varepsilon ||f||_{\mathrm{Lip}} \cdot e^{\varepsilon n}.$$

We let $\Phi_f^+ \in \mathfrak{B}^+(X,\omega)$ be the finitely-additive measure corresponding to the vector v_f^+ or, in other words, the unique finitely additive measure in $\mathfrak{B}^+(X,\omega)$ satisfying

$$\mathrm{eval}_0^+(\Phi_f^+) = v_f^+.$$

The inequality

$$\left| \int_0^T f \circ h_t^+(x) dt - \Phi_f^+([x, h_T^+ x]) \right| \leqslant C''_\varepsilon ||f||_{\mathrm{Lip}} (1 + T^\varepsilon) \tag{30}$$

now holds for all $x \in M, T \in \mathbb{R}_+$.

Indeed, if $[x, h_t^+ x]$ is a Markovian arc, then (30) is clear by definition of the vector v_f^+ and the weak Lipschitz property of the function f, while for general arcs of the vertical flow, the inequality (30) follows by Lemma 2.5.

2.6.3. *Characterization of the cocycle* Φ_f^+. Our next step is to check that for every $\Phi^- \in \mathfrak{B}^-(X,\omega)$, we have

$$\langle \Phi_f^+ \times \Phi^- \rangle = \int_M f dm_{\phi^-},$$

where we recall that $m_\Phi^- = \nu^+ \times \Phi^-$.

As before, let γ_i^n be a canonical system of Markovian arcs of the vertical flow, corresponding to the Markov sequence of partitions $\pi_n, n \in \mathbb{Z}$, and let $\tilde\gamma_i^n$ be a canonical system of Markovian arcs of the horizontal flow corresponding to the Markov sequence of partitions $\pi_n, n \in \mathbb{Z}$. By definition, for any $n \in \mathbb{Z}$, we have

$$\int_M \Phi_f^+ \times \Phi^- = \sum_{i=1}^m \Phi_f^+(\gamma_i^n) \cdot \Phi^-(\tilde\gamma_i^n).$$

We now write the Riemann sum

$$S(n, f, \Phi^-) = \sum_{i=1}^m \int_{\gamma_i^n} f d\nu^+ \cdot \Phi^-(\tilde\gamma_i^n)$$

for the measure m_{Φ^-}, and let n tend to $+\infty$. By definition, we have

$$\lim_{n \to +\infty} S(n, f, \Phi^-) = \int_M f dm_{\Phi^-}.$$

Now for all $n \in \mathbb{N}, i = 1, \dots, m$, we have

$$\left| \int\limits_{\gamma_i^n} f d\nu^+ - \Phi_f^+(\gamma_i^n) \right| \leqslant C_\varepsilon e^{\varepsilon n}$$

while, by Lyapunov regularity, the quantity $\max\limits_{i=1,\dots,m} |\Phi^-(\tilde{\gamma}_i^n)|$ decays exponentially as $n \to \infty$. It follows that

$$\int\limits_M \Phi_f^+ \times \Phi^- = \int\limits_M f dm_{\Phi^-},$$

which is what we had to prove.

2.7. *The asymptotics at infinity for Hölder cocycles.* As was noted above, we identify a finitely-additive measure $\Phi^+ \in \mathfrak{B}_c^+(X, \omega)$ with a continuous cocycle over the vertical flow for which, slightly abusing notation, we keep the same symbol Φ^+; the identification is given by the formula

$$\Phi^+(x, t) = \Phi^+([x, h_t^+ x]).$$

The Hölder property of a finitely-additive measure is equivalent to the Hölder property of the cocycle, that is, to the requirement that the function $\Phi^+(x, t)$ be Hölder in t uniformly in x. Our next aim is to give Hölder lower bounds for the cocycles Φ^+ and to investigate the growth of $\Phi^+(x, T)$ as $T \to +\infty$.

Consider the direct-sum decomposition

$$\mathfrak{B}^+(\mathfrak{A}) = E_1^{(0)} \oplus \cdots \oplus E_{l_0}^{(0)}$$

and the corresponding direct-sum decomposition

$$\mathfrak{B}^+(X, \omega) = \mathfrak{B}_1^+(X, \omega) \oplus \mathfrak{B}_2^+(X, \omega) \oplus \cdots \oplus \mathfrak{B}_{l_0}^+(X, \omega)$$

with

$$\mathfrak{B}_i^+(X, \omega) = (\mathrm{eval}_0^+)^{-1}(E_i^{(0)}), \quad i = 1, \dots, l_0;$$

of course, we have

$$\mathfrak{B}_1^+(X, \omega) = \mathbb{R}\nu^+.$$

Take $\Phi^+ \in \mathfrak{B}^+(X, \omega)$, and write

$$\Phi^+ = \Phi_1^+ + \cdots + \Phi_{l_0}^+$$

with $\Phi_i^+ \in \mathfrak{B}^+(X, \omega)$. Take the smallest i such that $\Phi_i^+ \neq 0$; the exponent θ_i will then be called *the top Lyapunov exponent* of Φ^+; similarly, if j is the largest number such that $\Phi_j^+ \neq 0$, then θ_j will be called *the lower Lyapunov exponent* of Φ^+. We shall now see that the top Lyapunov exponent controls the growth of $\Phi^+(x, t)$ as $t \to \infty$, while the lower Lyapunov exponent describes the local Hölder behaviour of $\Phi^+(x, t)$.

PROPOSITION 2.9. *Let $r \in \{1, \dots, l_0\}$, let $\Phi^+ \in \mathfrak{B}_r^+, \Phi^+ \neq 0$, and let $x \in M$ be such that $h_t^+ x$ is defined for all $t \in \mathbb{R}$. Then*

$$\limsup_{|t| \to \infty} \frac{\log |\Phi^+(x,t)|}{\log |t|} = \limsup_{|t| \to 0} \frac{\log |\Phi^+(x,t)|}{\log |t|} = \theta_r.$$

Proof. We first let t tend to $+\infty$. Let $v^{(n)} \in \mathfrak{B}^+(\mathfrak{A})$ be the equivariant sequence corresponding to Φ^+; we have $v^{(n)} \in E_r^{(n)}$ and, consequently, for every $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ and, for every $n \in \mathbb{N}$, there exists $i(n) \in \{1, \dots, m\}$ such that

$$\left| v_{i(n)}^{(n)} \right| \geq C_\varepsilon e^{(\theta_r - \varepsilon) n}, \quad n \in \mathbb{N}. \tag{31}$$

Now let

$$t_n = \min\{t : t \geq h_{i(n)}^{(n)}, \quad h_{t_n}^+ x \in \partial_v^1 \Pi_{i(n)}^{(n)}\}.$$

Informally, t_n is the first such moment that the are $[x, h_{t_n}^+ x]$ contains a Markovian arc going all the way through the rectangle $\Pi_{i(n)}^{(n)}$. It is clear from the SB-property of the sequence $\mathfrak{A}$ that for any $\varepsilon > 0$, there exist constants $C'_\varepsilon, C''_\varepsilon > 0$ such that

$$C'_\varepsilon e^{(\theta_1 - \varepsilon) n} \leq t_n \leq C''_\varepsilon e^{(\theta_1 + \varepsilon) n}, \quad n \in \mathbb{N}. \tag{32}$$

Now denote

$$x'(n) = h^+_{t_n - h_{i(n)}^{(n)}} x, \quad x''(n) = h_{t_n}^+ x.$$

Since

$$\Phi^+([x, x''(n)]) = \Phi^+([x, x'(n)]) + \Phi^+([x'(n), x''(n)])$$

and

$$\Phi^+([x'(n), x''(n)]) = v_{i(n)}^{(n)},$$

it follows from (32) and (31) that we have

$$\limsup_{n \to +\infty} \frac{\max\{\log |\Phi^+([x, x'(n)])|, \log |\Phi^+([x, x''(n)])|\}}{n} = \frac{\theta_r}{\theta_1},$$

whence also

$$\limsup_{n \to +\infty} \max \left(\frac{\log |\Phi^+([x, x'(n)])|}{\log \nu^+([x, x'(n)])}, \frac{\log |\Phi^+([x'(n), x''(n)])|}{\log \nu^+([x, x''(n)])} \right) = \theta_r,$$

and finally

$$\limsup_{t \to +\infty} \frac{\log |\Phi^+(x,t)|}{\log t} = \theta_r.$$

The desired lower bound is established. We illustrate the argument in Figure 2.

The proof for $t \to -\infty$ is completely similar, while the case $t \to 0$ is obtained by taking $n \to -\infty$ and repeating the same argument. $\square$

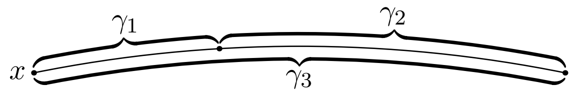

Figure 2. Proof of the lower bound in Proposition 2.9: $\gamma_1 = [x(n), x'(n)]$, $\gamma_2 = [x'(n), x''(n)]$. Either γ_1 or γ_3 satisfies the lower bound.

2.8. *Hyberbolic* SB-*sequences.* An SB-sequence $\mathfrak{A}$ will be called *hyperbolic* if $\mathfrak{B}_c^+(\mathfrak{A}) = \mathfrak{B}^+(\mathfrak{A})$. It is clear from the definitions that if an abelian differential (X, ω) admits an exact Markovian sequence of partitions such that the corresponding sequence $\mathfrak{A}$ is hyperbolic, then $\mathfrak{B}_c^+(X, \omega) = \mathfrak{B}^+(X, \omega)$.

In what follows, we shall check that if $\mathbb{P}$ is probability measure on $\mathcal{H}$, invariant under the Teichmüller flow and ergodic, and such that the Kontsevich-Zorich cocycle acts isometrically on its neutral Oseledets subspace, then $\mathbb{P}$-almost every abelian differential (X, ω) admits a Markovian sequence of partitions π_n such that the corresponding sequence $\mathfrak{A}$ of adjacency matrices is a hyperbolic SB-sequence. It will follow that for $\mathbb{P}$-almost all ω, we have $\mathfrak{B}^+(X, \omega) = \mathfrak{B}_c^+(X, \omega)$.

Remark. If $\mathfrak{B}_c^+(\mathfrak{A})$ is strictly larger than $\mathfrak{B}^+(\mathfrak{A})$, it does not follow that $\mathfrak{B}_c^+(X, \omega)$ is strictly larger than $\mathfrak{B}^+(X, \omega)$. Our constructions do not allow us to assign a finitely additive measure defined on all arcs of the vertical flow to a general equivariant sequence $v^{(n)} \in \mathfrak{B}^+(\mathfrak{A})$.

2.9. *Expectation and variance of Hölder cocycles.*

Proposition 2.10. *For any $\Phi^+ \in \mathfrak{B}^+$ and any $t_0 \in \mathbb{R}$, we have*

$$\mathbb{E}_\nu(\Phi^+(x, t_0)) = \langle \Phi^+, \nu^- \rangle \cdot t_0.$$

Proof. Since the proposition is clearly valid for $\Phi^+ = \nu^+$, it suffices to prove it in the case $\langle \Phi^+, \nu^- \rangle = 0$. But indeed, if $\mathbb{E}_\nu(\Phi^+(x, t)) \neq 0$, then the Ergodic Theorem implies

$$\limsup_{T \to \infty} \frac{\log |\Phi^+(x, T)|}{\log T} = 1,$$

and then $\langle \Phi^+, \nu^- \rangle \neq 0$.

Proposition 2.11. *For any $\Phi^+ \in \mathfrak{B}^+$ not proportional to ν^+ and any $t_0 \neq 0$, we have*

$$\mathrm{Var}_\nu \Phi^+(x, t_0) \neq 0.$$

Taking $\Phi^+ - \langle \Phi^+, \nu^- \rangle \cdot \nu^+$ instead of Φ^+, we may assume $\mathbb{E}_\nu(\Phi^+(x, t_0)) = 0$. If $\mathrm{Var}_\nu \Phi^+(x, t_0) = 0$, then $\Phi^+(x, t_0) = 0$ identically, but then

$$\limsup_{T \to \infty} \frac{\log |\Phi^+(x, T)|}{\log T} = 0,$$

whence $\Phi^+ = 0$, and the proposition is proved. $\square$

Remark. In the context of substitutions, cocycles related to the Hölder cocycles from $\mathfrak{B}^+$ have been studied by P. Dumont, T. Kamae and S. Takahashi in [16] as well as by T. Kamae in [27].

3. The Teichmüller flow on the Veech space of zippered rectangles

3.1. *Veech's space of zippered rectangles.*

3.1.1. *Rauzy-Veech induction.* The renormalization action of the Teichmüller flow on the spaces $\mathfrak{B}^+$ and $\mathfrak{B}^-$ of Hölder cocycles will play a main role in the proof of limit theorems for translation flows. We will use Veech's representation of abelian differentials by zippered rectangles, and in this section we recall Veech's construction using the notation of [11], [15]. For a different presentation of the Rauzy-Veech formalism, see Marmi-Moussa-Yoccoz [32].

We start by recalling the definition of the Rauzy-Veech induction. Let π be a permutation of m symbols, which will always be assumed irreducible in the sense that $\pi\{1, \ldots, k\} = \{1, \ldots, k\}$ implies $k = m$. The Rauzy operations a and b are defined by the formulas

$$a\pi(j) = \begin{cases} \pi j & \text{if } j \leq \pi^{-1}m, \\ \pi m & \text{if } j = \pi^{-1}m + 1, \\ \pi(j-1) & \text{if } \pi^{-1}m + 1 < j \leq m, \end{cases}$$

$$b\pi(j) = \begin{cases} \pi j & \text{if } \pi j \leq \pi m, \\ \pi j + 1 & \text{if } \pi m < \pi j < m, \\ \pi m + 1 & \text{if } \pi j = m. \end{cases}$$

These operations preserve irreducibility. The *Rauzy class* $\mathcal{R}(\pi)$ is defined as the set of all permutations that can be obtained from π by application of the transformation group generated by a and b. From now on we fix a Rauzy class $\mathcal{R}$ and assume that it consists of irreducible permutations.

For $i, j = 1, \ldots, m$, denote by E^{ij} the $m \times m$ matrix whose (i, j)-th entry is 1, while all others are zeros. Let E be the identity $m \times m$-matrix. Following

Veech [37], introduce the unimodular matrices

$$\mathcal{A}(a,\pi) = \sum_{i=1}^{\pi^{-1}m} E^{ii} + E^{m,\pi^{-1}m+1} + \sum_{i=\pi^{-1}m}^{m-1} E^{i,i+1}, \tag{33}$$

$$\mathcal{A}(b,\pi) = E + E^{m,\pi^{-1}m}. \tag{34}$$

For a vector $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m$, we write

$$|\lambda| = \sum_{i=1}^{m} \lambda_i.$$

Let

$$\Delta_{m-1} = \{\lambda \in \mathbb{R}^m : |\lambda| = 1,\ \lambda_i > 0 \text{ for } i = 1, \ldots, m\}.$$

One can identify each pair (λ, π), $\lambda \in \Delta_{m-1}$, with the *interval exchange map* of the interval $I := [0, 1)$ as follows. Divide I into the sub-intervals $I_k := [\beta_{k-1}, \beta_k)$, where $\beta_0 = 0$, $\beta_k = \sum_{i=1}^{k} \lambda_i$, $1 \leq k \leq m$, and then place the intervals I_k in I in the following order (from left to write): $I_{\pi^{-1}1}, \ldots, I_{\pi^{-1}m}$. We obtain a piecewise linear transformation of I that preserves the Lebesgue measure.

The space $\Delta(\mathcal{R})$ of interval exchange maps corresponding to $\mathcal{R}$ is defined by

$$\Delta(\mathcal{R}) = \Delta_{m-1} \times \mathcal{R}.$$

Denote

$$\Delta_\pi^+ = \{\lambda \in \Delta_{m-1} |\ \lambda_{\pi^{-1}m} > \lambda_m\}, \quad \Delta_\pi^- = \{\lambda \in \Delta_{m-1} |\ \lambda_m > \lambda_{\pi^{-1}m}\},$$

$$\Delta^+(\mathcal{R}) = \bigcup_{\pi \in \mathcal{R}} \{(\pi, \lambda) |\ \lambda \in \Delta_\pi^+\},$$

$$\Delta^-(\mathcal{R}) = \bigcup_{\pi \in \mathcal{R}} \{(\pi, \lambda) |\ \lambda \in \Delta_\pi^-\},$$

$$\Delta^\pm(\mathcal{R}) = \Delta^+(\mathcal{R}) \cup \Delta^-(\mathcal{R}).$$

The *Rauzy-Veech induction map* $\mathscr{T} : \Delta^\pm(\mathcal{R}) \to \Delta(\mathcal{R})$ is defined as follows:

$$\mathscr{T}(\lambda, \pi) = \begin{cases} \left(\frac{\mathcal{A}(a,\pi)^{-1}\lambda}{|\mathcal{A}(a,\pi)^{-1}\lambda|}, a\pi\right) & \text{if } \lambda \in \Delta_\pi^+, \\ \left(\frac{\mathcal{A}(b,\pi)^{-1}\lambda}{|\mathcal{A}(b,\pi)^{-1}\lambda|}, b\pi\right) & \text{if } \lambda \in \Delta_\pi^-. \end{cases} \tag{35}$$

One can check that $\mathscr{T}(\lambda, \pi)$ is the interval exchange map induced by (λ, π) on the interval $J = [0, 1-\gamma]$, where $\gamma = \min(\lambda_m, \lambda_{\pi^{-1}m})$; the interval J is then stretched to unit length.

Denote

$$\Delta^\infty(\mathcal{R}) = \bigcap_{n \geq 0} \mathscr{T}^{-n} \Delta^\pm(\mathcal{R}). \tag{36}$$

Every $\mathscr{T}$-invariant probability measure is concentrated on $\Delta^\infty(\mathcal{R})$. On the other hand, a natural Lebesgue measure defined on $\Delta(\mathcal{R})$, which is finite, but noninvariant, is also concentrated on $\Delta^\infty(\mathcal{R})$. Veech [37] showed that $\mathscr{T}$ has an absolutely continuous ergodic invariant measure on $\Delta(\mathcal{R})$, which is, however, infinite.

We have two matrix cocycles $\mathcal{A}^t$, $\mathcal{A}^{-1}$over $\mathscr{T}$ defined by

$$\mathcal{A}^t(n,(\lambda,\pi)) = \mathcal{A}^t(\mathscr{T}^n(\lambda,\pi)) \cdot \ldots \cdot \mathcal{A}^t(\lambda,\pi),$$
$$\mathcal{A}^{-1}(n,(\lambda,\pi)) = \mathcal{A}^{-1}(\mathscr{T}^n(\lambda,\pi)) \cdot \ldots \cdot \mathcal{A}^{-1}(\lambda,\pi).$$

We introduce the corresponding skew-product transformations $\mathscr{T}^{\mathcal{A}^t} : \Delta(\mathcal{R}) \times \mathbb{R}^m \to \Delta(\mathcal{R}) \times \mathbb{R}^m$, $\mathscr{T}^{\mathcal{A}^{-1}} : \Delta(\mathcal{R}) \times \mathbb{R}^m \to \Delta(\mathcal{R}) \times \mathbb{R}^m$,

$$\mathscr{T}^{\mathcal{A}^t}((\lambda,\pi),v) = (\mathscr{T}(\lambda,\pi), \mathcal{A}^t(\lambda,\pi)v),$$
$$\mathscr{T}^{\mathcal{A}^{-1}}((\lambda,\pi),v) = (\mathscr{T}(\lambda,\pi), \mathcal{A}^{-1}(\lambda,\pi)v).$$

3.1.2. *The construction of zippered rectangles.* Here we briefly recall the construction of the Veech space of zippered rectangles. We use the notation of [11].

Zippered rectangles associated to the Rauzy class $\mathcal{R}$ are triples (λ,π,δ), where $\lambda = (\lambda_1,\ldots,\lambda_m) \in \mathbb{R}^m$, $\lambda_i > 0$, $\pi \in \mathcal{R}$, $\delta = (\delta_1,\ldots,\delta_m) \in \mathbb{R}^m$, and the vector δ satisfies the following inequalities:

$$\delta_1 + \cdots + \delta_i \le 0, \quad i = 1,\ldots,m-1, \tag{37}$$

$$\delta_{\pi^{-1}1} + \cdots + \delta_{\pi^{-1}i} \ge 0, \quad i = 1,\ldots,m-1. \tag{38}$$

The set of all vectors δ satisfying (37) and (38) is a cone in $\mathbb{R}^m$; we denote it by $K(\pi)$.

For any $i = 1,\ldots,m$, set

$$a_j = a_j(\delta) = -\delta_1 - \cdots - \delta_j,\ h_j = h_j(\pi,\delta) = -\sum_{i=1}^{j-1}\delta_i + \sum_{l=1}^{\pi(j)-1}\delta_{\pi^{-1}l}. \tag{39}$$

3.1.3. *Zippered rectangles and abelian differentials.* Given a zippered rectangle $\mathscr{X} = (\lambda,\pi,\delta)$, Veech [37] takes m rectangles $\Pi_i = \Pi_i(\lambda,\pi,\delta)$ of girth λ_i and height h_i, $i = 1,\ldots,m$, and glues them together according to a rule determined by the permutation π. This procedure yields a Riemann surface M endowed with a holomorphic 1-form ω which, in restriction to each Π_i, is simply the form $dz = dx + idy$. The union of the bases of the rectangles is an interval $I^{(0)}(\lambda,\pi,\delta)$ of length $|\lambda|$ on M; the first return map of the vertical flow of the form ω is precisely the interval exchange $\mathbf{T}_{(\lambda,\pi)}$.

A zippered rectangle $\mathscr{X}$ by definition carries a partition $\pi_0 = \pi_0(\mathscr{X})$ of the underlying surface $M = M(\mathscr{X})$ into m weakly admissible rectangles Π_i:

$$\pi_0 : M = \Pi_1 \sqcup \cdots \sqcup \Pi_m.$$

The *area* of a zippered rectangle (λ, π, δ) is given by the expression

$$\operatorname{Area}(\lambda, \pi, \delta) := \sum_{r=1}^{m} \lambda_r h_r = \sum_{r=1}^{m} \lambda_r \left(-\sum_{i=1}^{r-1} \delta_i + \sum_{i=1}^{\pi r-1} \delta_{\pi^{-1} i} \right). \tag{40}$$

(Our convention is $\sum_{i=u}^{v} \cdots = 0$ when $u > v$.)

Furthermore, to each rectangle Π_i Veech [38] assigns a cycle $\gamma_i(\lambda, \pi, \delta)$ in the homology group $H_1(M, \mathbb{Z})$: namely, if P_i is the left bottom corner of Π_i and Q_i the left top corner, then the cycle is the union of the vertical interval $P_i Q_i$ and the horizontal subinterval of $I^{(0)}(\lambda, \pi, \delta)$ joining Q_i to P_i. It is clear that the cycles $\gamma_i(\lambda, \pi, \delta)$ span $H_1(M, \mathbb{Z})$.

3.1.4. *The space of zippered rectangles.* Denote by $\mathcal{V}(\mathcal{R})$ the space of all zippered rectangles corresponding to the Rauzy class $\mathcal{R}$, i.e.,

$$\mathcal{V}(\mathcal{R}) = \{(\lambda, \pi, \delta) : \lambda \in \mathbb{R}_+^m,\ \pi \in \mathcal{R},\ \delta \in K(\pi)\}.$$

Let also

$$\begin{aligned} \mathcal{V}^+(\mathcal{R}) &= \{(\lambda, \pi, \delta) \in \mathcal{V}(\mathcal{R}) : \lambda_{\pi^{-1}m} > \lambda_m\}, \\ \mathcal{V}^-(\mathcal{R}) &= \{(\lambda, \pi, \delta) \in \mathcal{V}(\mathcal{R}) : \lambda_{\pi^{-1}m} < \lambda_m\}, \\ \mathcal{V}^\pm(\mathcal{R}) &= \mathcal{V}^+(\mathcal{R}) \cup \mathcal{V}^-(\mathcal{R}). \end{aligned}$$

Veech [37] introduced the flow $\{P^s\}$ acting on $\mathcal{V}(\mathcal{R})$ by the formula

$$P^s(\lambda, \pi, \delta) = (e^s \lambda, \pi, e^{-s} \delta),$$

and the map $\mathcal{U} : \mathcal{V}^\pm(\mathcal{R}) \to \mathcal{V}(\mathcal{R})$, where

$$\mathcal{U}(\lambda, \pi, \delta) = \begin{cases} (\mathcal{A}(\pi, a)^{-1}\lambda, a\pi, \mathcal{A}(\pi, a)^{-1}\delta) & \text{if } \lambda_{\pi^{-1}m} > \lambda_m, \\ (\mathcal{A}(\pi, b)^{-1}\lambda, b\pi, \mathcal{A}(\pi, b)^{-1}\delta) & \text{if } \lambda_{\pi^{-1}m} < \lambda_m. \end{cases}$$

(The inclusion $\mathcal{U}\mathcal{V}^\pm(\mathcal{R}) \subset \mathcal{V}(\mathcal{R})$ is proved in [37].) The map $\mathcal{U}$ and the flow $\{P^s\}$ commute on $\mathcal{V}^\pm(\mathcal{R})$. They also preserve the area of a zippered rectangle (see (40)) and hence can be restricted to the set

$$\mathcal{V}^{1,\pm}(\mathcal{R}) := \{(\lambda, \pi, \delta) \in \mathcal{V}^\pm(\mathcal{R}) : \operatorname{Area}(\lambda, \pi, \delta) = 1\}.$$

For $(\lambda, \pi) \in \Delta(\mathcal{R})$, denote

$$\tau^0(\lambda, \pi) =: -\log(|\lambda| - \min(\lambda_m, \lambda_{\pi^{-1}m})). \tag{41}$$

From (33), (34) it follows that if $\lambda \in \Delta_\pi^+ \cup \Delta_\pi^-$, then

$$\tau^0(\lambda, \pi) = -\log |\mathcal{A}^{-1}(c, \pi)\lambda|, \tag{42}$$

where $c = a$ when $\lambda \in \Delta_\pi^+$ and $c = b$ when $\lambda \in \Delta_\pi^-$.

Next denote

$$\begin{aligned}\mathcal{Y}_1(\mathcal{R}) &:= \{x = (\lambda, \pi, \delta) \in \mathcal{V}(\mathcal{R}) : |\lambda| = 1,\ \mathrm{Area}(\lambda, \pi, \delta) = 1\}, \\ \tau(x) &:= \tau^0(\lambda, \pi) \text{ for } x = (\lambda, \pi, \delta) \in \mathcal{Y}_1(\mathcal{R}), \\ \mathcal{V}_{1,\tau}(\mathcal{R}) &:= \bigcup_{x \in \mathcal{Y}_1(\mathcal{R}),\ 0 \le s \le \tau(x)} P^s x.\end{aligned} \tag{43}$$

Let

$$\begin{aligned}\mathcal{V}^{1,\pm}_{\neq}(\mathcal{R}) &:= \{(\lambda, \pi, \delta) \in \mathcal{V}^{1,\pm}(\mathcal{R}) : a_m(\delta) \neq 0\}, \\ \mathcal{V}_\infty(\mathcal{R}) &:= \bigcap_{n \in \mathbb{Z}} \mathcal{U}^n \mathcal{V}^{1,\pm}_{\neq}(\mathcal{R}).\end{aligned}$$

Clearly $\mathcal{U}^n$ is well defined on $\mathcal{V}_\infty(\mathcal{R})$ for all $n \in \mathbb{Z}$.

We now set

$$\mathcal{Y}'(\mathcal{R}) := \mathcal{Y}_1(\mathcal{R}) \cap \mathcal{V}_\infty(\mathcal{R}), \quad \widetilde{\mathcal{V}}(\mathcal{R}) := \mathcal{V}_{1,\tau}(\mathcal{R}) \cap \mathcal{V}_\infty(\mathcal{R}).$$

The above identification enables us to define on $\widetilde{\mathcal{V}}(\mathcal{R})$ a natural flow, for which we retain the notation P^s. (Although the bounded positive function τ is not separated from zero, the flow P^s is well defined.)

Note that for any $s \in \mathbb{R}$, we have a natural "tautological" map

$$\mathfrak{t}_s : M(\mathscr{X}) \to M(P^s \mathscr{X}) \tag{44}$$

which on each rectangle Π_i is simply expansion by e^s in the horizontal direction and contraction by e^s in the vertical direction. By definition, the map $\mathfrak{t}_s$ sends the vertical and the horizontal foliations of $\mathscr{X}$ to those of $P^s \mathscr{X}$.

Introduce the space

$$\mathfrak{X}\widetilde{\mathcal{V}}(\mathcal{R}) = \{(\mathscr{X}, x) : \mathscr{X} \in \widetilde{\mathcal{V}}(\mathcal{R}),\ x \in M(\mathscr{X})\},$$

and endow the space $\mathfrak{X}\widetilde{\mathcal{V}}(\mathcal{R})$ with the flow $P^{s,\mathfrak{X}}$ given by the formula

$$P^{s,\mathfrak{X}}(\mathscr{X}, x) = (P^s \mathscr{X}, \mathfrak{t}_s x).$$

The flow P^s induces on the transversal $\mathcal{Y}_1(\mathcal{R})$ the first-return map $\overline{\mathscr{T}}$ given by the formula

$$\overline{\mathscr{T}}(\lambda, \pi, \delta) = \mathcal{U} P^{\tau^0(\lambda,\pi)}(\lambda, \pi, \delta). \tag{45}$$

Observe that, by definition, if $\overline{\mathscr{T}}(\lambda, \pi, \delta) = (\lambda', \pi', \delta')$, then $(\lambda', \pi') = \mathscr{T}(\lambda, \pi)$.

For $(\lambda, \pi, \delta) \in \widetilde{\mathcal{V}}(\mathcal{R})$, $s \in \mathbb{R}$, let $\tilde{n}(\lambda, \pi, \delta, s)$ be defined by the formula

$$\mathcal{U}^{\tilde{n}(\lambda,\pi,\delta,s)}(e^s \lambda, \pi, e^{-s}\delta) \in \mathcal{V}_{1,\tau}(\mathcal{R}).$$

Endow the space $\widetilde{\mathcal{V}}(\mathcal{R})$ with a matrix cocycle $\overline{\mathcal{A}}^t$ over the flow P^s given by the formula

$$\overline{\mathcal{A}}^t(s, (\lambda, \pi, \delta)) = \mathcal{A}^t(\tilde{n}(\lambda, \pi, \delta, s), (\lambda, \pi)),$$

and introduce the corresponding skew-product flow

$$P^{s,\overline{\mathcal{A}}^t} : \widetilde{\mathcal{V}}(\mathcal{R}) \times \mathbb{R}^m \to \widetilde{\mathcal{V}}(\mathcal{R}) \times \mathbb{R}^m$$

by the formula

$$P^{s,\overline{\mathcal{A}}^t}(\mathscr{X},v)=(P^s\mathscr{X},\overline{\mathcal{A}}^t(\mathscr{X},s)v).$$

We also have a natural cocycle $\overline{\mathcal{A}}$ over the inverse flow P^{-s} given by the formula

$$\overline{\mathcal{A}}(\mathscr{X},s)=(\overline{\mathcal{A}}^t(P^{-s}\mathscr{X},s))^t$$

and the natural skew-product flow

$$P^{-s,\overline{\mathcal{A}}}:\widetilde{\mathcal{V}}(\mathcal{R})\times\mathbb{R}^m\to\widetilde{\mathcal{V}}(\mathcal{R})\times\mathbb{R}^m$$

defined by the formula

$$P^{-s,\overline{\mathcal{A}}}(\mathscr{X},v)=(P^{-s}\mathscr{X},\overline{\mathcal{A}}(\mathscr{X},s)v).$$

The strongly unstable Oseledets bundle of the cocycle $\overline{\mathcal{A}}$ will be seen to describe all the measures $\Phi^-\in\mathfrak{B}^-$ in the same way in which the strongly unstable Oseledets bundle of the cocycle $\overline{\mathcal{A}}^t$ describes all the measures $\Phi^+\in\mathfrak{B}^+$.

Remark. The Kontsevich-Zorich cocycle is isomorphic to the inverse of its dual. (See, e.g., Statement 2 in Proposition 3.1 below.) This "self-duality" is, however, not used in the construction and characterization of finitely-additive invariant measures. The duality between the spaces $\mathfrak{B}^+$ and $\mathfrak{B}^-$ corresponds to the duality between the cocycle and its transpose, that is, in our notation, between $\overline{\mathcal{A}}^t$ and $\overline{\mathcal{A}}$. Such duality takes place for any invertible matrix-valued cocycle over any measure-preserving flow.

3.1.5. *The correspondence between cocycles.* To a connected component $\mathcal{H}$ of the space $\mathcal{M}_\kappa$ one can assign a Rauzy class $\mathcal{R}$ in such a way that the following is true [37], [30].

THEOREM 3 (Veech). *There exists a finite-to-one measurable map $\pi_{\mathcal{R}}$: $\widetilde{\mathcal{V}}(\mathcal{R})\to\mathcal{H}$ such that $\pi_{\mathcal{R}}\circ P^t=g_t\circ\pi_{\mathcal{R}}$. The image of $\pi_{\mathcal{R}}$ contains all abelian differentials whose vertical and horizontal foliations are both minimal.*

As before, let $\mathbb{H}^1(\mathcal{H})$ be the fibre bundle over $\mathcal{H}$ whose fibre at a point (M,ω) is the cohomology group $H^1(M,\mathbb{R})$. The Kontsevich-Zorich cocycle $\mathbf{A}_{KZ}$ induces a skew-product flow $g_s^{\mathbf{A}_{KZ}}$ on $\mathbb{H}^1(\mathcal{H})$ given by the formula

$$g_s^{\mathbf{A}_{KZ}}(\mathbf{X},v)=(\mathbf{g}_s\mathbf{X},\mathbf{A}_{KZ}v),\quad \mathbf{X}\in\mathcal{H},v\in H^1(M,\mathbb{R}).$$

Following Veech [37], we now explain the connection between the Kontsevich-Zorich cocycle $\mathbf{A}_{KZ}$ and the cocycle $\overline{\mathcal{A}}^t$. For any irreducible permutation π, Veech [38] defines an alternating matrix L^π by setting $L^\pi_{ij}=0$ if $i=j$ or if $i<j,\pi i<\pi j$, $L^\pi_{ij}=1$ if $i<j,\pi i>\pi j$, $L^\pi_{ij}=-1$ if $i>j,\pi i<\pi j$ and denotes by $N(\pi)$ the kernel of L^π and by $H(\pi)=L^\pi(\mathbb{R}^m)$ the image of L^π. The dimensions of $N(\pi)$ and $H(\pi)$ do not change as π varies in $\mathcal{R}$ and, furthermore, Veech [38] establishes the following properties of the spaces $N(\pi)$, $H(\pi)$.

PROPOSITION 3.1. *Let $c = a$ or b. Then*

(1) $H(c\pi) = \mathcal{A}^t(c,\pi)H(\pi)$, $N(c\pi) = \mathcal{A}^{-1}(c,\pi)N(\pi)$;

(2) *the diagram*

$$\begin{array}{ccc} \mathbb{R}^m/N(\pi) & \xrightarrow{L^\pi} & H(\pi) \\ \Big\downarrow{\scriptstyle \mathcal{A}^{-1}(\pi,c)} & & \Big\downarrow{\scriptstyle \mathcal{A}^{t}(\pi,c)} \\ \mathbb{R}^m/N(c\pi) & \xrightarrow{L^{c\pi}} & H(c\pi) \end{array}$$

is commutative and each arrow is an isomorphism;

(3) *for each π, there exists a basis $\mathbf{v}_\pi$ in $N(\pi)$ such that the map $\mathcal{A}^{-1}(\pi,c)$ sends every element of $\mathbf{v}_\pi$ to an element of $\mathbf{v}_{c\pi}$.*

Each space H^π is thus endowed with a natural anti-symmetric bilinear form $\mathcal{L}_\pi$ defined, for $v_1, v_2 \in H(\pi)$, by the formula

$$\mathcal{L}_\pi(v_1,v_2) = \langle v_1, (L^\pi)^{-1}v_2\rangle. \tag{46}$$

(The vector $(L^\pi)^{-1}v_2$ lies in $\mathbb{R}^m/N(\pi)$; since by definition we have $\langle v_1, v_2\rangle = 0$ for all $v_1 \in H(\pi)$, $v_2 \in N(\pi)$, the right-hand side is well defined.)

Consider the $\mathscr{T}^{\mathcal{A}^t}$-invariant subbundle $\mathscr{H}(\Delta(\mathcal{R})) \subset \Delta(\mathcal{R}) \times \mathbb{R}^m$ given by the formula

$$\mathscr{H}(\Delta(\mathcal{R})) = \{((\lambda,\pi),v),(\lambda,\pi) \in \Delta(\mathcal{R}), v \in H(\pi)\}$$

as well as a quotient bundle

$$\mathscr{N}(\Delta(\mathcal{R})) = \{((\lambda,\pi),v),(\lambda,\pi) \in \Delta(\mathcal{R}), v \in \mathbb{R}^m/N(\pi)\}.$$

The bundle map $\mathscr{L}_\mathcal{R} : \mathscr{H}(\Delta(\mathcal{R})) \to \mathscr{N}(\Delta(\mathcal{R}))$ given by $\mathscr{L}_\mathcal{R}((\lambda,\pi),v) = ((\lambda,\pi), L^\pi v)$ induces a bundle isomorphism between $\mathscr{H}(\Delta(\mathcal{R}))$ and $\mathscr{N}(\Delta(\mathcal{R}))$. Both bundles can be naturally lifted to bundles $\mathscr{H}(\widetilde{\mathcal{V}}(\mathcal{R}))$, $\mathscr{N}(\widetilde{\mathcal{V}}(\mathcal{R}))$ over the space $\widetilde{\mathcal{V}}(\mathcal{R})$ of zippered rectangles; they are naturally invariant under the corresponding skew-product flows $P^{s,\overline{\mathcal{A}}^t}$, $P^{-s,\overline{\mathcal{A}}}$, and the map $\mathscr{L}_\mathcal{R}$ lifts to a bundle isomorphism between $\mathscr{H}(\widetilde{\mathcal{V}}(\mathcal{R}))$ and $\mathscr{N}(\widetilde{\mathcal{V}}(\mathcal{R}))$.

Take $\mathscr{X} \in \widetilde{\mathcal{V}}(\mathcal{R})$, and write $\pi_\mathcal{R}(\mathscr{X}) = (M(\mathscr{X}), \omega(\mathscr{X}))$. Veech [39] has shown that the map $\pi_\mathcal{R}$ lifts to a bundle epimorphism $\tilde{\pi}_\mathcal{R}$ from $\mathscr{H}(\widetilde{\mathcal{V}}(\mathcal{R}))$ onto $\mathbb{H}^1(\mathcal{H})$ that intertwines the cocycle $\overline{\mathcal{A}}^t$ and the Kontsevich-Zorich cocycle $\mathbf{A}_{KZ}$:

PROPOSITION 3.2 (Veech). *For almost every $\mathscr{X} \in \widetilde{\mathcal{V}}(\mathcal{R})$, $\mathscr{X} = (\lambda,\pi,\delta)$, there exists an isomorphism $\mathcal{I}_\mathscr{X} : H(\pi) \to H^1(M(\mathscr{X}),\mathbb{R})$ such that*

(1) *the map $\tilde{\pi}_\mathcal{R} : \mathscr{H}(\widetilde{\mathcal{V}}(\mathcal{R})) \to \mathbb{H}^1(\mathcal{H})$ given by*

$$\tilde{\pi}_\mathcal{R}(\mathscr{X}, v) = (\pi_\mathcal{R}(\mathscr{X}), \mathcal{I}_X v)$$

induces a measurable bundle epimorphism from $\mathscr{H}(\widetilde{\mathcal{V}}(\mathcal{R}))$ onto $\mathbb{H}^1(\mathcal{H})$, which is an isomorphic on each fibre;

(2) *the diagram*

$$\begin{array}{ccc} \mathscr{H}(\widetilde{\mathcal{V}}(\mathcal{R})) & \xrightarrow{\tilde{\pi}_{\mathcal{R}}} & \mathbb{H}^1(\mathcal{H}) \\ \Big\downarrow{\scriptstyle P^{s,\overline{\mathcal{A}}^t}} & & \Big\downarrow{\scriptstyle g_s^{\mathbf{A}_{KZ}}} \\ \mathscr{H}(\widetilde{\mathcal{V}}(\mathcal{R})) & \xrightarrow{\tilde{\pi}_{\mathcal{R}}} & \mathbb{H}^1(\mathcal{H}) \end{array}$$

is commutative;

(3) *for* $\mathscr{X} = (\lambda, \pi, \delta)$, *the isomorphism* $\mathcal{I}_X$ *takes the bilinear form* $\mathcal{L}_\pi$ *on* $H(\pi)$, *defined by* (46), *to the cup-product on* $H^1(M(\mathscr{X}), \mathbb{R})$.

Proof. Recall that to each rectangle Π_i Veech [38] assigns a cycle $\gamma_i(\lambda, \pi, \delta)$ in the homology group $H_1(M, \mathbb{Z})$. If P_i is the left bottom corner of Π_i and Q_i the left top corner, then the cycle is the union of the vertical interval P_iQ_i and the horizontal subinterval of $I^{(0)}(\lambda, \pi, \delta)$ joining Q_i to P_i. It is clear that the cycles $\gamma_i(\lambda, \pi, \delta)$ span $H_1(M, \mathbb{Z})$; furthermore, Veech shows that the cycle $t_1\gamma_1 + \cdots + t_m\gamma_m$ is homologous to 0 if and only if $(t_1, \ldots, t_m) \in N(\pi)$. We thus obtain an identification of $\mathbb{R}^m/N(\pi)$ and $H_1(M, \mathbb{R})$. Similarly, the subspace of $\mathbb{R}^m$ spanned by the vectors $(f(\gamma_1), \ldots, f(\gamma_m))$, $f \in H^1(M, \mathbb{R})$, is precisely $H(\pi)$. The identification of the bilinear form $\mathcal{L}_\pi$ with the cup-product is established in [42, Prop. 4.19]. □

Let $\mathbb{P}_\mathcal{V}$ be an ergodic P^s-invariant probability measure for the flow P^s on $\mathcal{V}(\mathcal{R})$, and let $\mathbb{P}_\mathcal{H} = (\pi_\mathcal{R})_*\mathbb{P}_\mathcal{V}$ be the corresponding $\mathbf{g}_s$-invariant measure on $\mathcal{H}$. Let $\mathcal{E}^u_{\mathbb{P}_\mathcal{V}}(\widetilde{\mathcal{V}}(\mathcal{R}))$ be the strongly unstable bundle of the cocycle $\overline{\mathcal{A}}^t$. By Proposition 3.1, the bundle $\mathcal{E}^u_{\mathbb{P}_\mathcal{V}}(\widetilde{\mathcal{V}}(\mathcal{R}))$ is a subbundle of $\mathscr{H}(\widetilde{\mathcal{V}}(\mathcal{R}))$. It therefore follows from Proposition 3.2 that the map $\tilde{\pi}_\mathcal{R}$ isomorphically identifies the strongly unstable bundles of the cocycles $\overline{\mathcal{A}}^t$ and $\mathbf{A}_{KZ}$; this identification is equivariant with respect to the natural actions of the skew-product flows $P^{s,\overline{\mathcal{A}}^t}$ and $g_s^{\mathbf{A}_{KZ}}$ on the corresponding bundles.

3.1.6. *The correspondence between measures.*

Proposition 3.3. *Let* $\mathbb{P}$ *be an ergodic* $\mathbf{g}_s$*–invariant probability measure on* $\mathcal{H}$. *Then there exists an ergodic* P^s*-invariant probability measure* $\mathbb{P}_\mathcal{V}$ *on* $\mathcal{V}(\mathcal{R})$ *such that*

$$\mathbb{P} = (\pi_\mathcal{R})_*\mathbb{P}_\mathcal{V}.$$

This proposition is a corollary of the following general statement.

Proposition 3.4. *Let* Z_1, Z_2 *be standard Borel spaces, let* $g_s^1 : Z_1 \to Z_1$, $g_s^2 : Z_2 \to Z_2$ *be measurable flows, and let* $\pi_{12} : Z_1 \to Z_2$ *be a Borel measurable map such that*

(1) *for any* $z_2 \in Z_2$, *the preimage* $\{\pi_{12}^{-1}(z_2)\}$ *of* z_2 *is finite*;

(2) *the map* π_{12} *intertwines the flows* g_s^1, g_s^2 *in the sense that the diagram*

$$\begin{array}{ccc} Z_1 & \xrightarrow{\pi_{12}} & Z_2 \\ \big\downarrow{\scriptstyle g_s^1} & & \big\downarrow{\scriptstyle g_s^2} \\ Z_1 & \xrightarrow{\pi_{12}} & Z_2 \end{array}$$

is commutative.

Then for any Borel g_s^2*-invariant ergodic probability measure* $\mathbb{P}_2$ *on* Z_2*, there exists a Borel* g_s^1*-invariant ergodic probability measure* $\mathbb{P}_1$ *on* Z_1 *such that*

$$(\pi_{12})_*(\mathbb{P}_1) = \mathbb{P}_2.$$

The proof of Proposition 3.4 is routine. First, note that by ergodicity, for $\mathbb{P}_2$-almost every $z_z \in Z_2$, the cardinality of the preimage $\{\pi_{12}^{-1}(z_2)\}$ of z_2 is constant; now consider the normalized product $\widetilde{\mathbb{P}}_1$ of $\mathbb{P}_2$ and the counting measure in the preimage; the measure $\widetilde{\mathbb{P}}_1$ is by definition g_s^1-invariant, and for the measure $\mathbb{P}_1$ one may take an ergodic component (in fact, almost every ergodic component) of the measure $\widetilde{\mathbb{P}}_1$.

3.2. *A strongly biregular sequence of partitions corresponding to a zippered rectangle*. Given a zippered rectangle $\mathscr{X}$, we shall speak of its vertical and horizontal foliations, Hölder cocycles, and so on, meaning the corresponding objects for the underlying abelian differential, and we shall use the notation $\mathfrak{B}^+(\mathscr{X})$, $\mathfrak{B}^-(\mathscr{X})$, $\mathfrak{B}_c^+(\mathscr{X})$, $\mathfrak{B}_c^-(\mathscr{X})$ for the corresponding spaces of finitely-additive measures.

Recall that, by construction, a zippered rectangle carries the partition

$$\pi_0(\mathscr{X}) = \Pi_1^{(0)} \sqcup \cdots \sqcup \Pi_m^{(0)}$$

into weakly admissible rectangles. The Rauzy–Veech expansion of a zippered rectangle now yields a Markovian sequence of partitions π_n, $n \in \mathbb{Z}$. To construct it, first take $\mathscr{X} \in \mathcal{Y}(\mathcal{R})$ and recall that we have a natural "tautological" identification map

$$\mathfrak{t}_{\overline{\mathscr{T}}} : M(\mathscr{X}) \to M(\overline{\mathscr{T}}\mathscr{X}).$$

Now set

$$\pi_n(\mathscr{X}) = \left(\mathfrak{t}_{\overline{\mathscr{T}}}\right)^{-n} \pi_0\left(\overline{\mathscr{T}}^n \mathscr{X}\right). \tag{47}$$

By definition, the sequence $\pi_n(\mathscr{X})$, $n \in \mathbb{Z}$, is Markovian. Minimality of the horizontal and vertical flows implies exactness of the sequence $\pi_n(\mathscr{X})$. For a general zippered rectangle $\mathscr{X}' \in \mathcal{V}_{1,\tau}(\mathcal{R})$, write

$$\mathscr{X}' = P^{s_0}\,\mathscr{X}, \mathscr{X} \in \mathcal{Y}_1(\mathcal{R}), 0 \leqslant s_0 < \tau(\mathscr{X})$$

and set

$$\pi_n(\mathscr{X}) = \mathfrak{t}_{s_0}\left(\pi_n(\mathscr{X})\right).$$

(Informally, carry over sequence π_n from the "closest" zippered rectangle lying on the transversal $\mathcal{Y}_1(\mathcal{R})$.)

LEMMA 3.5. *For $\mathbb{P}_{\mathcal{V}}$-almost every zippered rectangle $\mathscr{X}$, there exists a sequence $n_k \in \mathbb{Z}$, $n_0 = 0$, $n_k < n_{k+1}$, $k \in \mathbb{Z}$, such that the sequence $\mathfrak{A} = \left(A\left(\pi_{n_k}, \pi_{n_{k+1}}\right)\right)$ of adjacency matrices of the exact Markovian subsequence $\pi_{n_k}(\mathscr{X})$, $k \in \mathbb{Z}$ satisfies the following*:

(1) $\mathfrak{A}$ *is an SB-sequence*,
(2) *the space $\mathfrak{B}^+(\mathfrak{A})$ coincides with the strongly unstable space of the cocycle $\overline{\mathcal{A}}^t$ at the point $\mathscr{X}$.*

The proof of the lemma is routine. One chooses a Rauzy-Veech matrix Q of the form

$$Q = Q_1\, Q_2,$$

where Q_1 and Q_2 are Rauzy-Veech matrices all whose entries are positive and such that $\mathbb{P}_{\mathcal{V}}$-almost all zippered rectangles $\mathscr{X}$ contain infinitely many occurrences of the matrix Q both in the past and in the future. The sequence n_k is then the sequence of consecutive occurrences of the matrix Q. Each adjacency matrix $A\left(\pi_{n_k}, \pi_{n_{k+1}}\right)$ now has the form $Q_2 \widetilde{A} Q_1$, where $\widetilde{A}$ is an integer matrix with nonnegative entries. It follows from the Oseledets Multiplicative Ergodic Theorem and the Oseledets-Pesin Reduction Theorem ([5, Th. 3.5.5, p. 77]) that $\mathfrak{A}$ is an SB-sequence and that $\mathfrak{B}^+(\mathfrak{A})$ coincides with the strongly unstable space of the cocycle $\overline{\mathcal{A}}^t$ at the zippered rectangle $\mathscr{X}$. The proof of the lemma is complete.

3.3. *The renormalization action of the Teichmüller flow on the space of finitely-additive measures.* We have the evaluation map

$$\mathrm{eval}^+_{\mathscr{X}} \colon \mathfrak{B}^+(\mathscr{X}) \to \mathbb{R}^m,$$

which to a finitely-additive measure $\Phi^+ \in \mathfrak{B}^+$ assigns the vector of its values on vertical arcs of the rectangles $\Pi_i^{(0)}$, $i = 1, \ldots, m$. We must now check that the map $\mathrm{eval}^+_{\mathscr{X}}$ is indeed an isomorphism between the space $\mathfrak{B}^+(\mathscr{X})$ and the strongly unstable space of the cocycle $\overline{\mathcal{A}}^t$.

Introduce a measurable fibre bundle $\mathfrak{B}^+\widetilde{\mathcal{V}}(\mathcal{R})$ over the Veech space $\mathcal{V}(\mathcal{R})$ by setting

$$\mathfrak{B}^+\widetilde{\mathcal{V}}(\mathcal{R}) = \left\{\left(\mathscr{X}, \Phi^+\right) \colon \mathscr{X} \in \widetilde{\mathcal{V}}(\mathcal{R}), \Phi^+ \in \mathfrak{B}^+(\mathscr{X})\right\}.$$

Extend the map $\mathrm{eval}^+_{\mathscr{X}}$ to a bundle morphism

$$\mathrm{eval}^+ \colon \mathfrak{B}^+\widetilde{\mathcal{V}}(\mathcal{R}) \to \widetilde{\mathcal{V}}(\mathcal{R}) \times \mathbb{R}^m,$$

given by the formula

$$\mathrm{eval}^+\left(\mathscr{X}, \Phi^+\right) = \left(\mathscr{X}, \mathrm{eval}^+_{\mathscr{X}}(\Phi^+)\right).$$

By definition, the map eval^+ intertwines the action of the flow $P^{s,\mathscr{X}}$ on the bundle $\mathfrak{B}^+\widetilde{\mathcal{V}}(\mathcal{R})$ with that of the flow $P^{s,\overline{\mathcal{A}}^t}$ on the trivial bundle $\widetilde{\mathcal{V}}(\mathcal{R})\times\mathbb{R}^m$.

Recall that for any $s\in\mathbb{R}$, we have a natural "tautological" map

$$\mathfrak{t}_s : M(\mathscr{X}) \to M(P^s\mathscr{X})$$

given by (44). The bundle $\mathfrak{B}^+\widetilde{\mathcal{V}}(\mathcal{R})$ is now endowed with a natural renormalization flow $P^{s,\mathfrak{B}^+}$ given by the formula

$$P^{s,\mathfrak{B}^+}(\mathscr{X},\Phi^+) = (P^s\mathscr{X},(\mathfrak{t}_s)_*\Phi^+).$$

We furthermore have a bundle morphism

$$\mathrm{eval}^+ : \mathfrak{B}^+\widetilde{\mathcal{V}}(\mathcal{R}) \to \widetilde{\mathcal{V}}(\mathcal{R})\times\mathbb{R}^m$$

given by the formula

$$\mathrm{eval}^+(\mathscr{X},\Phi^+) = (\mathscr{X},\mathrm{eval}^+_{\mathscr{X}}(\Phi^+).$$

The identification of cocycles now gives us the following

Proposition 3.6. *Let $\mathbb{P}_{\mathcal{V}}$ be an ergodic P^s-invariant probability measure for the flow P^s on $\mathcal{V}(\mathcal{R})$. We have a commutative diagram*

$$\begin{array}{ccc} \mathfrak{B}^+\widetilde{\mathcal{V}}(\mathcal{R}) & \xrightarrow{\mathrm{eval}^+} & \widetilde{\mathcal{V}}(\mathcal{R})\times\mathbb{R}^m \\ \Big\downarrow{\scriptstyle P^{s,\mathfrak{B}^+}} & & \Big\downarrow{\scriptstyle P^{s,\overline{\mathcal{A}}^t}} \\ \mathfrak{B}^+\widetilde{\mathcal{V}}(\mathcal{R}) & \xrightarrow{\mathrm{eval}^+} & \widetilde{\mathcal{V}}(\mathcal{R})\times\mathbb{R}^m. \end{array}$$

The map eval^+ *is injective in restriction each fibre. For $\mathbb{P}_{\mathcal{V}}$-almost every $\mathscr{X}\in\mathfrak{B}^+\widetilde{\mathcal{V}}(\mathcal{R})$, the map* eval^+ *induces an isomorphism between the space $\mathfrak{B}^+(\mathscr{X})$ and the strongly unstable Oseledets subspace of the cocycle $\overline{\mathcal{A}}^t$ at the point $\mathscr{X}$.*

Proof. Let π_{n_k} be the sequence of partitions given by Lemma 3.5, and let $\mathfrak{A}$ be the corresponding SB-sequence of matrices. Since $\mathfrak{A}$ is an SB-sequence, the map $\mathrm{eval}^+_{\mathscr{X}}$ induces an isomorphism between $\mathfrak{B}^+(\mathscr{X})$ and $\mathfrak{B}^+(\mathfrak{A})$. (Recall here that $n_0 = 0$.) Since $\mathfrak{B}^+(\mathfrak{A})$ coincides with the unstable space of the cocycle $\overline{\mathcal{A}}^t$, the proposition is proved completely. $\square$

Using Proposition 3.6, we will identify the action of $P^{s,\mathfrak{B}^+}$ on $\mathfrak{B}^+(\widetilde{\mathcal{V}}(\mathcal{R})$ with the action of $P^{s,\overline{\mathcal{A}}^t}$ on the strongly unstable Oseledets subbundle of $\widetilde{\mathcal{V}}(\mathcal{R})\times\mathbb{R}^m$ and speak of the action of the cocycle $\overline{\mathcal{A}}^t$ on the space of finitely-additive measures in this sense.

This renormalization action of the flow P^s on the space of finitely-additive measures will play a key role in the proof of the limit theorems in the next section. We close this section by giving a sufficient condition for the equality $\mathfrak{B}^+(X,\omega)=\mathfrak{B}^+_c(X,\omega)$.

3.4. *A sufficient condition for the equality* $\mathfrak{B}^+(X,\omega) = \mathfrak{B}_c^+(X,\omega)$. Let $(\mathcal{X},\mu)$ be a probability space endowed with a μ-preserving transformation T or flow g_s and an integrable linear cocycle A over g_s with values in $\mathrm{GL}(m,\mathbb{R})$. For $p \in \mathcal{X}$, let $E_{0,x}$ be the the *neutral* subspace of A at p, i.e., the Lyapunov subspace of the cocycle A corresponding to the Lyapunov exponent 0. We say that A *acts isometrically on its neutral subspaces* if for almost any p, there exists an inner product $\langle\cdot\rangle_p$ on $\mathbb{R}^m$ that depends on p measurably and satisfies

$$\langle A(1,p)v, A(1,p)v\rangle_{g_s p} = \langle v,v\rangle_p,\ v \in E_{0,p}$$

for all $s \in \mathbb{R}$. (Again, in the case of a transformation, g_s should be replaced by T in this formula.)

The third statement of Proposition 3.1 has the following immediate

Corollary 3.7. *Let $\mathbb{P}_{\mathcal{V}}$ be a Borel ergodic P^s-invariant probability measure on $\mathcal{V}(\mathcal{R})$, and let $\mathbb{P} = (\pi_{\mathcal{R}})_*\mathbb{P}_{\mathcal{V}}$ be the corresponding $\mathbf{g}_s$-invariant measure on $\mathcal{H}$. If the Kontsevich-Zorich cocycle acts isometrically on its neutral subspace with respect to $\mathbb{P}$, then the cocycle $\overline{\mathcal{A}}^t$ also acts isometrically on its neutral subspace with respect to $\mathbb{P}_{\mathcal{V}}$.*

Note that the hypothesis of Corollary 3.7 is satisfied, in particular, for the Masur-Veech smooth measure on the moduli space of abelian differentials.

The following proposition is clear from the definitions.

Proposition 3.8. *Let $\mathbb{P}_{\mathcal{V}}$ be a Borel ergodic P^s-invariant probability measure on $\mathcal{V}(\mathcal{R})$ such that the cocycle $\overline{\mathcal{A}}^t$ acts isometrically on its neutral subspace with respect to $\mathbb{P}_{\mathcal{V}}$. Let $\mathbb{P} = (\pi_{\mathcal{R}})_*\mathbb{P}_{\mathcal{V}}$ be the corresponding $\mathbf{g}_s$-invariant ergodic measure on $\mathcal{H}$. Then for $\mathbb{P}$-almost every abelian differential (M,ω), we have the equality*

$$\mathfrak{B}^+(M,\omega) = \mathfrak{B}_c^+(M,\omega).$$

In other words, if the cocycle $\overline{\mathcal{A}}^t$ acts isometrically on its neutral subspace with respect to $\mathbb{P}_{\mathcal{V}}$, then any continuous finitely-additive measure must in fact be Hölder. Note that the assumptions of the proposition are verified, in particular, for the Masur-Veech smooth measure on the moduli space of abelian differentials. To prove Proposition 3.8 we use Proposition 3.1, which implies that if the cocycle $\overline{\mathcal{A}}^t$ acts isometrically on its neutral subspace with respect to $\mathbb{P}_{\mathcal{V}}$, then $\mathbb{P}$-almost every abelian differential (M,ω) admits an exact Markovian sequence of partitions whose sequence of adjacency matrices is a hyperbolic SB-sequence which, in turn, is sufficient for the equality

$$\mathfrak{B}^+(M,\omega) = \mathfrak{B}_c^+(M,\omega).$$

4. **Proof of the limit theorems**

4.1. *Outline of the proof.* The main element in the proof of the limit theorem is the renormalization action of the Teichmüller flow P^s on the bundle $\mathfrak{B}^+\widetilde{\mathcal{V}}(\mathcal{R})$.

Start with the case when the second Lyapunov exponent of the cocycle $\overline{\mathcal{A}}^t$ is positive and simple with respect to a P^s-invariant ergodic probability measure $\mathbb{P}_{\mathcal{V}}$ on $\widetilde{\mathcal{V}}(\mathcal{R})$. Then, by Theorem 1, for $\mathbb{P}_{\mathcal{V}}$-almost every zippered rectangle $\mathscr{X}$ and a generic weakly Lipschitz function f of zero average, the ergodic integral

$$\int_0^{\exp(s)} f \circ h_t^+(x)dt$$

is approximated by an expression of the form

$$\mathrm{const} \cdot \Phi^+_{2,\mathscr{X}}(x, e^s),$$

where the constant depends on f and $\Phi^+_{2,\mathscr{X}} \in \mathfrak{B}^+(\mathscr{X})$ is a cocycle belonging to the second Lyapunov subspace of the cocycle $\overline{\mathcal{A}}^t$. Note that the cocycle $\Phi^+_{2,\mathscr{X}}$ is defined up to multiplication by a scalar; the double cover $\mathcal{H}'$ over the space $\mathcal{H}$ in the formulation of the limit theorem is considered precisely in order to distinguish between positive and negative scalars.

Now, Proposition 3.6 implies that the normalized distribution of the random variable $\Phi^+_{2,\mathscr{X}}(x, e^s)$ (considered as a function of x with fixed s) coincides with the normalized distribution of the random variable $\Phi^+_{2,P^s\mathscr{X}}(x, 1)$. Assigning to a zippered rectangle $\mathscr{X}$ the normalized distribution of the random variable $\Phi^+_{2,\mathscr{X}}(x, e^s)$ (considered as a function of x with fixed s) now yields the desired map $\mathcal{D}_2^+$ from the space of zippered rectangles (more precisely, from its double cover) to the space of distributions. The fact that the normalized distributions of the ergodic integrals are approximated by the image under the map $\mathcal{D}_2^+$ of the orbit of our zippered rectangle under the action of the Teichmüller flow P^s follows now from the asymptotic expansion of Theorem 1.

4.2. *The case of the simple second Lyapunov exponent.*

4.2.1. *The leading term in the asymptotic for the ergodic integral.* We fix a P^s-invariant ergodic probability measure $\mathbb{P}_{\mathcal{V}}$ on $\widetilde{\mathcal{V}}(\mathcal{R})$ and start with the case in which the second Lyapunov exponent of the cocycle $\overline{\mathcal{A}}^t$ is positive and simple with respect to the measure $\mathbb{P}_{\mathcal{V}}$. Consider the Oseledets subspace $E^u_{1,\mathscr{X}} = \mathbb{R}h_{\mathscr{X}}$ corresponding to the top Lyapunov exponent 1 and the one-dimensional Oseledets subspace $E^u_{2,\mathscr{X}}$ corresponding to the second Lyapunov exponent. Furthermore, let $E^u_{\geq 3,\mathscr{X}}$ be the subspace corresponding to the remaining Lyapunov exponents.

We have then the decomposition

$$E^u_{\mathscr{X}} = E^u_{1,\mathscr{X}} \oplus E^u_{2,\mathscr{X}} \oplus E^u_{\geq 3,\mathscr{X}}.$$

Denote by $\mathfrak{B}^+_{1,\mathscr{X}} = \mathbb{R}\nu^+$, $\mathfrak{B}^+_{2,\mathscr{X}}$, $\mathfrak{B}^+_{\geq 3,\mathscr{X}}$ the corresponding spaces of Hölder cocycles. A similar decomposition holds for the dual space $\widetilde{E}^u$, the strongly unstable space of the cocycle $\overline{\mathcal{A}}$:

$$\widetilde{E}^u_{\mathscr{X}} = \widetilde{E}^u_{1,\mathscr{X}} \oplus \widetilde{E}^u_{2,\mathscr{X}} \oplus \widetilde{E}^u_{\geq 3,\mathscr{X}}.$$

Again, denote by $\mathfrak{B}^-_{1,\mathscr{X}} = \mathbb{R}\nu^-$, $\mathfrak{B}^-_{2,\mathscr{X}}$, $\mathfrak{B}^-_{\geq 3,\mathscr{X}}$ the corresponding spaces of Hölder cocycles.

Choose $\Phi_2^+ \in \mathfrak{B}^+_{2,\mathscr{X}}$, $\Phi_2^- \in \mathfrak{B}^-_{2,\mathscr{X}}$ in such a way that

$$\langle \Phi_2^+, \Phi_2^- \rangle = 1.$$

Take $f \in \mathrm{Lip}^+_w(\mathscr{X})$, $x \in \mathscr{X}$, $T \in \mathbb{R}$, and observe that the expression

$$m_{\Phi_2^-}(f)\Phi_2^+(x,T) \tag{48}$$

does not depend on the precise choice of $\Phi_2^{\pm}$. (We have the freedom of multiplying Φ_2^+ by an arbitrary scalar, but then Φ_2^- is divided by the same scalar.)

Now for $f \in \mathrm{Lip}^+_w(\mathscr{X})$, write

$$\Phi_f^+(x,T) = \left(\int_{\mathscr{X}} f d\nu\right) \cdot T + m_{\Phi_2^-}(f)\Phi_2^+(x,T) + \Phi^+_{3,f}(x,T),$$

where $\Phi^+_{3,f} \in \mathcal{I}_{\mathscr{X}}(\widetilde{E}^u_{\geq 3,\mathscr{X}})$. In particular, there exist two positive constants C and α depending only on $\mathbb{P}$ such that for any function f satisfying

$$f \in \mathrm{Lip}^+_w(\mathscr{X}), \quad \int_{\mathscr{X}} f d\nu = 0,$$

we have the estimate

$$\left| \int_0^T f \circ h_t^+(x)dt - m_{\Phi_2^-}(f)\Phi_2^+(x,T) \right| \leq C||f||_{\mathrm{Lip}} T^{\overline{\theta}_2 - \alpha}. \tag{49}$$

4.2.2. *The growth of the variance.* In order to estimate the variance of the random variable $\int_0^T f \circ h_t^+(x)dt$, we start by studying the growth of the variance of the random variable $\Phi^+_{2,\mathscr{X}}(x,T)$ as $T \to \infty$.

Recall that $\mathbb{E}_\nu \Phi^+_{2,\mathscr{X}}(x,T) = 0$ for all T, while $\mathrm{Var}_\nu \Phi^+_{2,\mathscr{X}}(x,T) \neq 0$ for $T \neq 0$. Recall that for a cocycle $\Phi^+ \in \mathfrak{B}^+_{\mathscr{X}}$, $\Phi^+ = \mathcal{I}^+_{\mathscr{X}}(v)$, we have defined its norm $|\Phi^+|$ by the formula $|\Phi^+| = |v|$. Introduce a multiplicative cocycle $H_2(s,\mathscr{X})$ over the flow P^s by the formula

$$H_2(s,\mathscr{X}) = \frac{|\overline{\mathcal{A}}^t(s,\mathscr{X})v|}{|v|}, \quad v \in E^u_{2,\mathscr{X}},\ v \neq 0. \tag{50}$$

Observe that the right-hand side does not depend on the specific choice of $v \neq 0$.

By definition, we now have

$$\lim_{s\to\infty} \frac{\log H_2(s,\mathscr{X})}{s} = \overline{\theta}_2. \tag{51}$$

PROPOSITION 4.1. *There exists a positive measurable function $V : \widetilde{\mathcal{V}}(\mathcal{R}) \to \mathbb{R}_+$ such that the following equality holds for $\mathbb{P}_{\mathcal{V}}$-almost all $\mathscr{X} \in \widetilde{\mathcal{V}}(\mathcal{R})$:*

$$\mathrm{Var}_\nu \Phi_2^+(x,T) = V(P^s\mathscr{X})|\Phi_2^+|^2(H_2(s,\mathscr{X}))^2. \tag{52}$$

Indeed, the function $V(\mathscr{X})$ is given by

$$V(\mathscr{X}) = \frac{\mathrm{Var}_\nu \Phi_2^+(x,1)}{|\Phi_2^+|^2}, \tag{53}$$

and the proposition is an immediate corollary of Proposition 3.6. Observe that the right-hand side does not depend on a particular choice of $\Phi_2^+ \in \mathfrak{B}^+_{2,\mathscr{X}}$, $\Phi_2^+ \neq 0$.

Using (49), we now proceed to estimating the growth of the variance of the ergodic integral

$$\int_0^T f \circ h_t^+(x)dt.$$

We use the same notation as in the introduction. For $\tau \in [0,1]$, $s \in \mathbb{R}$, a real-valued $f \in \mathrm{Lip}^+_{w,0}(\mathscr{X})$, we write

$$\mathfrak{S}[f,s;\tau,x] = \int_0^{\tau \exp(s)} f \circ h_t^+(x)dt. \tag{54}$$

As before, let ν be the Lebesgue measure on the surface $M(\mathscr{X})$ corresponding to the zippered rectangle $\mathscr{X}$. As before, as x varies in the probability space $(M(\mathscr{X}),\nu)$, we obtain a random element of $C[0,1]$. In other words, we have a random variable

$$\mathfrak{S}[f,s] : (M(\mathscr{X}),\nu) \to C[0,1] \tag{55}$$

defined by the formula (54).

For any fixed $\tau \in [0,1]$, the formula (54) yields a real-valued random variable

$$\mathfrak{S}[f,s;\tau] : (M(\mathscr{X}),\nu) \to \mathbb{R}, \tag{56}$$

whose expectation, by definition, is zero.

PROPOSITION 4.2. *There exist $\alpha > 0$ depending only on $\mathbb{P}_{\mathcal{V}}$ and a positive measurable function $C : \widetilde{\mathcal{V}}(\mathcal{R}) \times \widetilde{\mathcal{V}}(\mathcal{R}) \to \mathbb{R}_+$ such that the following holds for $\mathbb{P}_{\mathcal{V}}$-almost all $\mathscr{X} \in \widetilde{\mathcal{V}}(\mathcal{R})$ and all $s > 0$. Let $\Phi^+_{2,\mathscr{X}} \in \mathfrak{B}^+_{2,\mathscr{X}}$, $\Phi^-_{2,\mathscr{X}} \in \mathfrak{B}^-_{2,\mathscr{X}}$ be chosen in such a way that $\langle \Phi^+_{2,\mathscr{X}}, \Phi^-_{2,\mathscr{X}} \rangle = 1$. Let $f \in \mathrm{Lip}^+_w(\mathscr{X})$ be such that*

$$\int_{M(\mathscr{X})} f d\nu = 0, \ m_{\Phi^-_{2,\mathscr{X}}}(f) \neq 0.$$

Then

$$\left| \frac{\mathrm{Var}_\nu \mathfrak{S}[f,s;1]}{V(P^s\mathscr{X})(m_{\Phi_2^-}(f)|\Phi_2^+|H_2(s,\mathscr{X}))^2} - 1 \right| \le C(\mathscr{X}, P^s\mathscr{X})\exp(-\alpha s). \tag{57}$$

Remark. Observe that the quantity $(m_{\Phi_2^-}(f)|\Phi_2^+|)^2$ does not depend on the specific choice of $\Phi_2^+ \in \mathfrak{B}_2^+$, $\Phi_2^- \in \mathfrak{B}_2^-$ such that $\langle \Phi_2^+, \Phi_2^- \rangle = 1$. Indeed, the proposition is immediate from Theorem 1, the inequality

$$|\mathbb{E}(\xi_1^2) - \mathbb{E}(\xi_2^2)| \le \sup|\xi_1 + \xi_2| \cdot \mathbb{E}|\xi_1 - \xi_2|,$$

which holds for any two bounded random variables ξ_1, ξ_2 on any probability space, and the following clear proposition which, again, is an immediate corollary of Theorem 1.

PROPOSITION 4.3. *There exist a constant $\alpha > 0$ depending only on $\mathbb{P}_{\mathcal{V}}$, a positive measurable function $C : \widetilde{\mathcal{V}}(\mathcal{R}) \times \widetilde{\mathcal{V}}(\mathcal{R}) \to \mathbb{R}_+$, and a positive measurable function $V' : \widetilde{\mathcal{V}}(\mathcal{R}) \to \mathbb{R}_+$ such that for all $s > 0$,we have*

$$\max_{x \in M} |\Phi_2^+(x, e^s)| = V'(P^s\mathscr{X})H_2(s,\mathscr{X}), \tag{58}$$

$$\left| \frac{\max\limits_{x \in M} \mathfrak{S}[f,s;1](x)}{V'(P^s\mathscr{X})(m_{\Phi_2^-}(f)|\Phi^+|H_2(s,\mathscr{X}))^2} - 1 \right| \le C(\mathscr{X}, P^s\mathscr{X})\exp(-\alpha s). \tag{59}$$

4.2.3. *Conclusion of the proof.* We now turn to the asymptotic behaviour of the distribution of the random variable $\mathfrak{S}[f,s]$ as $s \to \infty$. Again, we will use the notation $\mathfrak{m}[f,s]$ for the distribution of the normalized random variable

$$\frac{\mathfrak{S}[f,s]}{\sqrt{\mathrm{Var}_\nu \mathfrak{S}[f,s;1]}}. \tag{60}$$

The measure $\mathfrak{m}[f,s]$ is thus a probability distribution on the space $C[0,1]$ of continuous functions on the unit interval.

For $\tau \in \mathbb{R}$, $\tau \ne 0$, we again let $\mathfrak{m}[f,s;\tau]$ be the distribution of the $\mathbb{R}$-valued random variable

$$\frac{\mathfrak{S}[f,s;\tau]}{\sqrt{\mathrm{Var}_\nu \mathfrak{S}[f,s;\tau]}}. \tag{61}$$

If f has zero average then, by definition, $\mathfrak{m}[f,s;\tau]$ is a measure on $\mathbb{R}$ of expectation 0 and variance 1. Again, as in the introduction, we take the space $C[0,1]$ of continuous functions on the unit interval endowed with the Tchebyshev topology, and we let $\mathfrak{M}$ be the space of Borel probability measures on the space $C[0,1]$ endowed with the weak topology (see [7] or the appendix).

Consider the space $\widetilde{\mathcal{V}}(\mathcal{R})'$ given by the formula

$$\widetilde{\mathcal{V}}(\mathcal{R})' = \{\mathscr{X}' = (\mathscr{X}, v), v \in E^+_{2,\mathscr{X}}, |v| = 1\}.$$

The flow P^s is lifted to $\widetilde{\mathcal{V}}(\mathcal{R})'$ by the formula

$$P^{s,\prime}(\mathscr{X}, v) = \left(P^s \mathscr{X}, \frac{\overline{\mathcal{A}}^t(s, \mathscr{X})v}{|\overline{\mathcal{A}}^t(s, \mathscr{X})v|} \right).$$

Given $\mathscr{X}' \in \widetilde{\mathcal{V}}(\mathcal{R})'$, $\mathscr{X}' = (\mathscr{X}, v)$, write

$$\Phi^+_{2,\mathscr{X}'} = \mathcal{I}_{\mathscr{X}}(v).$$

As before, write

$$V(\mathscr{X}') = \mathrm{Var}_\nu \Phi^+_{2,\mathscr{X}'}(x, 1).$$

Now introduce the map

$$\mathcal{D}^+_2 : \widetilde{\mathcal{V}}(\mathcal{R})' \to \mathfrak{M}$$

by setting $\mathcal{D}^+_2(\mathscr{X}')$ to be the distribution of the $C[0,1]$-valued normalized random variable

$$\frac{\Phi^+_{2,\mathscr{X}'}(x, \tau)}{\sqrt{V(\mathscr{X}')}}, \ \tau \in [0,1].$$

Note here that by Proposition 2.11, for any $\tau_0 \neq 0$, we have $\mathrm{Var}_\nu \Phi^+_{2,\mathscr{X}}(x, \tau_0) \neq 0$ so, by definition, we have $\mathcal{D}^+_2(\mathscr{X}') \in \mathfrak{M}_1$.

Now, as before, we take a function $f \in \mathrm{Lip}^+_{w,\mathscr{X}}$ such that

$$\int_{M(\mathscr{X})} f d\nu = 0, \ m_{\Phi^-_{2,\mathscr{X}}}(f) \neq 0.$$

As before, d_{LP} stands for the Lévy-Prohorov metric on $\mathfrak{M}$, d_{KR} for the Kantorovich-Rubinstein metric on $\mathfrak{M}$.

Proposition 4.4. *Let $\mathbb{P}_{\mathcal{V}}$ be a P^s-invariant ergodic Borel probability measure on $\widetilde{\mathcal{V}}(\mathcal{R})$ such that the second Lyapunov exponent of the cocycle $\overline{\mathcal{A}}^t$ is positive and simple with respect to $\mathbb{P}_{\mathcal{V}}$. There exist a positive measurable function $C : \widetilde{\mathcal{V}}(\mathcal{R}) \times \widetilde{\mathcal{V}}(\mathcal{R}) \to \mathbb{R}_+$ and a positive constant α depending only on $\mathbb{P}_{\mathcal{V}}$ such that for $\mathbb{P}_{\mathcal{V}}$-almost every $\mathscr{X}' \in \widetilde{\mathcal{V}}(\mathcal{R})'$, $\mathscr{X}' = (\mathscr{X}, v)$, and any $f \in \mathrm{Lip}^+_{w,0}(\mathscr{X})$ satisfying $m^-_{2,\mathscr{X}'}(f) > 0$, we have*

$$d_{\mathrm{LP}}(\mathfrak{m}[f,s], \mathcal{D}^+_2(P^{s,\prime}\mathscr{X}')) \leq C(\mathscr{X}, P^s\mathscr{X}) \exp(-\alpha s), \tag{62}$$

$$d_{\mathrm{KR}}(\mathfrak{m}[f,s], \mathcal{D}^+_2(P^{s,\prime}\mathscr{X}')) \leq C(\mathscr{X}, P^s\mathscr{X}) \exp(-\alpha s). \tag{63}$$

Proof. We start with the simple inequality

$$\left| \frac{a}{b} - \frac{c}{d} \right| \leq |a| \cdot \left| \frac{b-d}{bd} \right| + \frac{|a-c|}{d}$$

valid for any real numbers a, b, c, d. For any pair of random variables ξ_1, ξ_2 taking values in an arbitrary Banach space and any positive real numbers

M_1, M_2, we consequently have

$$\sup\left|\frac{\xi_1}{M_1}-\frac{\xi_2}{M_2}\right| \le \sup|\xi_1|\cdot\left|\frac{M_1-M_2}{M_1M_2}\right| + \frac{\sup|\xi_1-\xi_2|}{M_2}. \tag{64}$$

We apply the inequality (64) to the $C[0,1]$-valued random variables

$$\xi_1 = \mathfrak{S}[f,s],\ \xi_2 = \Phi^+_{2,P^s\mathscr{X}}(x, \tau\cdot e^s),$$

letting M_1, M_2 be the corresponding normalizing variances: $M_1 = \mathrm{Var}_\nu \mathfrak{S}[f,s;1]$, $M_2 = \mathrm{Var}_\nu \mathfrak{m}[f,s;1]$.

Now take $\varepsilon > 0$, and let $\tilde{\xi}_1, \tilde{\xi}_2$ be two random variables on an arbitrary probability space $(\Omega, \mathbb{P})$ taking values in a complete metric space and such that the distance between their values does not exceed ε. In this case both the Lévy-Prohorov and the Kantorovich-Rubinstein distance between their distributions $(\tilde{\xi}_1)_*\mathbb{P}$, $(\tilde{\xi}_2)_*\mathbb{P}$ also does not exceed ε (see Lemma A.1). Proposition 4.4 is now immediate from equation (49) and Proposition 4.2.

It remains to derive Proposition 1.14 from Proposition 4.4. To do so, note that the map $\mathcal{D}_2^+$, originally defined on the double cover $\widetilde{\mathcal{V}}(\mathcal{R})'$ of the space of zippered rectangles, naturally descends to a map, for which we keep the same symbol $\mathcal{D}_2^+$, defined on the double cover $\mathcal{H}'$ of the connected component $\mathcal{H}$ of the moduli space of abelian differentials. Indeed, it is immediate from the definitions that the image $\mathcal{D}_2^+(\mathscr{X}')$ of an element $\mathscr{X}' \in \widetilde{\mathcal{V}}(\mathcal{R})'$, $\mathscr{X}' = (\mathscr{X}, v)$ only depends on the underlying element $(M(\mathscr{X}), \omega(\mathscr{X}), v)$ of the space $\mathcal{H}'$. Proposition 4.4 is now proved completely. □

4.3. *Proof of Corollary* 1.16. For $\mathscr{X}' \in \widetilde{\mathcal{V}}(\mathcal{R})'$, $\Phi^+ \in \mathfrak{B}^+_{\mathscr{X}}$, let $\mathfrak{m}[\Phi^+, \tau]$ be the distribution of the normalized $\mathbb{R}$-valued random variable

$$\frac{\Phi^+(x,\tau)}{\sqrt{\mathrm{Var}_\nu \Phi^+(x,\tau)}}.$$

PROPOSITION 4.5. *Let $\mathbb{P}_{\mathcal{V}}$ be a P^s-invariant ergodic Borel probability measure on $\widetilde{\mathcal{V}}(\mathcal{R})$. For $\mathbb{P}_{\mathcal{V}}$-almost every $\mathscr{X}$ and any $\Phi^+ \in \mathfrak{B}^+_{\mathscr{X}}$, $\Phi^+ \neq 0$, the correspondence*

$$\tau \to \mathfrak{m}[\Phi^+, \tau]$$

yields a continuous map from $\mathbb{R}\setminus 0$ to $\mathfrak{M}(\mathbb{R})$.

Proof. This is immediate from the Hölder property of the cocycle Φ^+ and the nonvanishing of the variance $\mathrm{Var}_\nu \Phi^+(x,\tau)$ for $\tau \neq 0$, which is guaranteed by Proposition 2.11. □

As usual, by the *omega-limit set* of a parametrized curve $p(s)$, $s \in \mathbb{R}$, taking values in a metric space, we mean the set of all accumulation points of our curve as $s \to \infty$.

We now use the following general statement.

Proposition 4.6. *Let* $(\Omega, \mathcal{B})$ *be a standard Borel space, and let* g_s *be a measurable flow on* Ω *preserving an ergodic Borel probability measure* μ. *Let* Z *be a separable metric space, and let* $\varphi : \Omega \to Z$ *be a measurable map such that for* μ*-almost every* $\omega \in \Omega$ *the curve* $\varphi(g_s\omega)$ *is continuous in* $s \in \mathbb{R}$. *Then there exists a closed set* $\mathfrak{N} \subset Z$, *such that for* μ*-almost every* $\omega \in \Omega$, *the set* $\mathfrak{N}$ *is the omega-limit set of the curve* $\varphi(g_s\omega)$, $s \in \mathbb{R}$.

The proof of Proposition 4.6 is routine. We choose a countable base $\mathscr{U} = \{U_n\}_{n\in\mathbb{N}}$ of open sets in Z. By ergodicity of g_s, continuity of the curves $\varphi(g_s\omega)$, and countability of the family $\mathscr{U}$, there exists a subset of full measure $\Omega' \subset \Omega$, $\mu(\Omega') = 1$, such that for any $U \in \mathscr{U}$ and any $\omega \in \Omega'$, the following conditions are satisfied:

(1) if $\mu(U) > 0$, then there exists an infinite sequence $s_n \to \infty$ such that $\varphi(g_{s_n}\omega) \in U$;

(2) if $\mu(U) = 0$, then there exists $s_0 > 0$ such that $\varphi(g_s\omega) \notin U$ for all $s > s_0$.

Now let $\mathfrak{N}$ be the set of all points $z \in Z$ such that $\mu(U) > 0$ for any open set $U \in \mathscr{U}$ containing the point z. By construction, for any $\omega \in \Omega'$, the set $\mathfrak{N}$ is precisely the omega-limit set of the curve $\varphi(g_s\omega)$. The proposition is proved.

Proposition 4.6 with $\Omega = \mathcal{H}'$, $\varphi = \mathcal{D}_2^+$ and μ an ergodic component of $\mathbb{P}'$ together with the Limit Theorem given by Propositions 1.14 and 4.4 immediately implies Corollary 1.16.

4.4. *The general case.*

4.4.1. *The fibre bundles* $\mathbf{S}^{(i)}\widetilde{\mathcal{V}}(\mathcal{R})$ *and the flows* $P^{s,\mathbf{S}^{(i)}}$ *corresponding to the strongly unstable Oseledets subspaces.* Let $\mathbb{P}_{\mathcal{V}}$ be an ergodic P^s-invariant probability measure on $\widetilde{\mathcal{V}}(\mathcal{R})$, and let

$$\theta_1 = 1 > \theta_2 > \cdots > \theta_{l_0} > 0$$

be the distinct positive Lyapunov exponents of $\overline{\mathcal{A}}^t$ with respect to $\mathbb{P}$. We assume $l_0 \geq 2$.

For $\mathscr{X} \in \widetilde{\mathcal{V}}(\mathcal{R})$, let

$$E^u_{\mathscr{X}} = \mathbb{R}h^{(0)}_{\mathscr{X}} + E_{2,\mathscr{X}} \oplus \cdots \oplus E_{l_0,\mathscr{X}}$$

be the corresponding direct-sum decomposition into Oseledets subspaces, and let

$$\mathfrak{B}^+_{\mathscr{X}} = \mathbb{R}\nu^+_{\mathscr{X}} \oplus \mathfrak{B}^+_{2,\mathscr{X}} \oplus \cdots \oplus \mathfrak{B}^+_{l_0,\mathscr{X}}$$

be the corresponding direct sum decomposition of the space $\mathfrak{B}^+_{\mathscr{X}}$.

For $f \in \mathrm{Lip}^+_w(\mathscr{X})$, we now write

$$\Phi^+_f = \Phi^+_{1,f} + \Phi^+_{2,f} + \cdots + \Phi^+_{l_0,f},$$

where $\Phi^+_{i,f} \in \mathfrak{B}^+_{i,\mathscr{X}}$ and, of course,

$$\Phi^+_{1,f} = \left(\int_{M(\mathscr{X})} f d\nu\right) \cdot \nu^+.$$

For each $i = 2, \ldots, l_0$, introduce a measurable fibre bundle

$$\mathbf{S}^{(i)}\widetilde{\mathcal{V}}(\mathcal{R}) = \{(\mathscr{X}, v) : \mathscr{X} \in \widetilde{\mathcal{V}}(\mathcal{R}), v \in E^+_{i,\mathscr{X}}, |v| = 1\}.$$

The flow P^s is naturally lifted to the space $\mathbf{S}^{(i)}\widetilde{\mathcal{V}}(\mathcal{R})$ by the formula

$$P^{s,\mathbf{S}^{(i)}}(\mathscr{X}, v) = \left(P^s\mathscr{X}, \frac{\overline{\mathcal{A}}^t(s, \mathscr{X})v}{|\overline{\mathcal{A}}^t(s, \mathscr{X})v|}\right).$$

4.4.2. *Growth of the variance.* The growth of the norm of vectors $v \in E^+_i$ is controlled by the multiplicative cocycle H_i over the flow $P^{s,\mathbf{S}^{(\mathbf{i})}}$ defined by the formula

$$H_i(s, (\mathscr{X}, v)) = \frac{|\overline{\mathcal{A}}^t(s, \mathscr{X})v|}{|v|}.$$

The growth of the variance of ergodic integrals is also, similarly to the previous case, described by the cocycle H_i.

For $\mathscr{X} \in \widetilde{\mathcal{V}}(\mathcal{R})$ and $f \in \mathrm{Lip}^+_{w,0}(\mathscr{X})$, we write

$$i(f) = \min\{j : \Phi^+_{f,j} \neq 0\}. \tag{65}$$

We now define a vector $v_f \in E^u_{i(f),\mathscr{X}}$ by the formula

$$\mathcal{I}^+_{\mathscr{X}}(v_f) = \frac{\Phi^+_{f,i(f)}}{|\Phi^+_{f,i(f)}|}. \tag{66}$$

Proposition 4.7. *There exists $\alpha > 0$ depending only on $\mathbb{P}_{\mathcal{V}}$ and, for any $i = 2, \ldots, l_0$, positive measurable functions*

$$V^{(i)} : \mathbf{S}^{(i)}\widetilde{\mathcal{V}}(\mathcal{R}) \to \mathbb{R}_+, C^{(i)} : \widetilde{\mathcal{V}}(\mathcal{R}) \times \widetilde{\mathcal{V}}(\mathcal{R}) \to \mathbb{R}_+$$

such that for $\mathbb{P}_{\mathcal{V}}$-almost every $\mathscr{X} \in \widetilde{\mathcal{V}}(\mathcal{R})$, any $f \in \mathrm{Lip}^+_{w,0}(\mathscr{X})$, and all $s > 0$, we have

$$\left|\frac{\mathrm{Var}_\nu(\mathfrak{S}[f, e^s; 1])}{V^{(i(f))}(P^{s,\mathbf{S}^{(i)}}(\mathscr{X}, v_f))(H_i(s, (\mathscr{X}, v_f)))^2} - 1\right| \leqslant C^{(i)}(\mathscr{X}, P^s\mathscr{X})e^{-\alpha s}.$$

Indeed, similarly to the case of a simple Lyapunov exponent, for $v \in E^i_{\mathscr{X}}$, we write $\Phi^+_v = \mathcal{I}^+_{\mathscr{X}}(v)$ and set

$$V^{(i)}(\mathscr{X}, v) = \mathrm{Var}_\nu \Phi^+_v(x, 1).$$

The proposition follows now in the same way as in the case of the simple second Lyapunov exponent. The pointwise approximation of the ergodic integral by the corresponding Hölder cocycle implies also that the variances of these random variables are exponentially close.

4.5. *Proof of Theorem* 2. For $i = 2, \dots, l_0$, introduce a map

$$\mathcal{D}_i^+ : \mathbf{S}^{(i)}\widetilde{\mathcal{V}}(\mathcal{R}) \to \mathfrak{M}$$

by setting $\mathcal{D}_i^+(\mathscr{X}, v)$ to be the distribution of the $C[0,1]$-valued random variable

$$\frac{\Phi_v^+(x,\tau)}{\sqrt{\mathrm{Var}_\nu(\Phi_v^+(x,1))}}, \quad \tau \in [0,1].$$

As before, by definition we have $\mathcal{D}_i^+(\mathscr{X}, v) \in \mathfrak{M}_1$. The measure $\mathfrak{m}[f,s] \in \mathfrak{M}$ is, as before, the distribution of the $C[0,1]$-valued random variable

$$\frac{\int_0^{\tau \exp(s)} f \circ h_t^+(x)dt}{\sqrt{\mathrm{Var}_\nu\left(\int_0^{\exp(s)} f \circ h_t^+(x)dt\right)}}, \quad \tau \in [0,1].$$

As before, let $l_0 = l_0(\mathbb{P}_{\mathcal{V}})$ be the number of distinct positive Lyapunov exponents of the measure $\mathbb{P}_{\mathcal{V}}$. For $f \in \mathrm{Lip}_{w,0}^+(\mathscr{X})$, we define the number $i(f)$ by (65) and the vector v_f by (66).

THEOREM 4. *Let $\mathbb{P}_{\mathcal{V}}$ be a Borel P^s-invariant ergodic probability measure on $\widetilde{\mathcal{V}}(\mathcal{R})$ satisfying $l_0(\mathbb{P}_{\mathcal{V}}) \geq 2$. There exist a constant $\alpha > 0$ depending only on $\mathbb{P}$ and a positive measurable map $C : \widetilde{\mathcal{V}}(\mathcal{R}) \times \widetilde{\mathcal{V}}(\mathcal{R}) \to \mathbb{R}_+$ such that for $\mathbb{P}_{\mathcal{V}}$-almost every $\mathscr{X} \in \widetilde{\mathcal{V}}(\mathcal{R})$ and any $f \in \mathrm{Lip}_{w,0}^+(\mathscr{X})$, we have*

$$d_{\mathrm{LP}}(\mathfrak{m}[f,s], D_{i(f)}^+(P^{s,\mathbf{S}^{(i(f))}}(\mathscr{X}, v_f))) \leqslant C(\mathscr{X}, P^s\mathscr{X})e^{-\alpha s},$$
$$d_{\mathrm{KR}}(\mathfrak{m}[f,s], D_{i(f)}^+(P^{s,\mathbf{S}^{(i(f))}}(\mathscr{X}, v_f))) \leqslant C(\mathscr{X}, P^s\mathscr{X})e^{-\alpha s}.$$

The proof is similar to the proof of Proposition 4.4. Again, the ergodic integral is uniformly approximated by the corresponding cocycle; the uniform bound on the difference yields the uniform bound on the difference and the ratio of variances of the ergodic integral and the cocycle considered as random variables; we proceed, as before, by using the inequality (64) with $\xi_1 = \mathfrak{m}[f,s]$, $\xi_2 = \Phi_{f,i(f)}^+(x,\tau)$, and M_1, M_2 the corresponding normalizing variances. We conclude, again, by noting that a uniform bound on the difference between two random variables implies the same bound on the Lévy-Prohorov or Kantorovich-Rubinstein distance between the distributions of the random variables (using Lemma A.1 in the appendix).

Theorem 4 now implies Theorem 2 in the same way in which Proposition 4.4 implies Proposition 1.14.

4.6. *Atoms of limit distributions.* Let $\mathscr{X}$ be a zippered rectangle, and let $(M,\omega) = (M(\mathscr{X}), \omega(\mathscr{X}))$ be the underlying abelian differential. For $x \in M(\mathscr{X})$, let $\gamma_\infty^+(x)$ stand for the leaf of the vertical foliation containing x, and let $\gamma_\infty^-(x)$ stand for the leaf of the horizontal foliation containing x. Our next aim is to show that atoms of limit distributions occur at all "homoclinic times," that is, moments of time t_0 such that there exists a point $\tilde{x} \in M$ satisfying $h^+_{t_0}(x) \in \gamma^-_\infty(\tilde{x})$.

PROPOSITION 4.8. *Let $\mathscr{X}$ be a zippered rectangle, and let*

$$(M, \omega) = (M(\mathscr{X}), \omega(\mathscr{X}))$$

be the underlying abelian differential. Let $\tilde{x} \in M$, and assume that $\tilde{x}$ does lies neither on a horizontal nor on a vertical leaf passing through a singularity of the abelian differential $\omega(\mathscr{X})$. Let $t_0 \in \mathbb{R}$ be such that $h^+_{t_0}\tilde{x} \in \gamma^-_\infty(\tilde{x})$. Then there exists a rectangle Π of positive area such that for any $x \in \Pi$ and any $\Phi^+ \in \mathfrak{B}^+(\mathscr{X})$, we have

$$\Phi^+(x, t_0) = \Phi^+(\tilde{x}, t_0). \tag{67}$$

Proof. Let $\hat{x} = h^+_{t_0}(\tilde{x})$, and write $\hat{x} = h^-_{t_1}(\tilde{x})$. Start with the case $t_0 > 0$, $t_1 > 0$. By our assumptions, for sufficiently small positive t_2, t_3, the rectangles

$$\Pi_1 = \Pi(\tilde{x}, t_2, t_1 + t_3), \quad \Pi_2 = \Pi(\tilde{x}, t_0 + t_2, t_3)$$

are both admissible.

The desired rectangle Π can now be taken of the form

$$\Pi = \Pi(\tilde{x}, t_2, t_3).$$

Indeed, take $x \in \Pi$. Our aim is to check the equality (67). Write $x = h^+_t x_1$, where $x_1 \in \partial^0_h(\Pi)$. We first check the equality

$$\Phi^+(x, t_0) = \Phi^+(x_1, t_0). \tag{68}$$

But indeed, $\Phi^+(x_1, t) = \Phi^+(h^+_{t_0 - t}x, t)$ since Π_1 is admissible, whence

$$\begin{aligned}\Phi^+(x, t_0) &= \Phi^+(x, t_0 - t) + \Phi^+(h^+_{t_0-t}x, t) \\ &= \Phi^+(x_1, t) + \Phi^+(h^+_t x_1, t_0 - t) = \Phi^+(x_1, t_0),\end{aligned}$$

as desired. The equality

$$\Phi^+(x_1, t_0) = \Phi^+(\tilde{x}, t_0) \tag{69}$$

is a direct corollary of admissibility of Π_2. Combining (68) with (69), we arrive at the desired equality (68), and Proposition 4.8 is proved. □

For a fixed zippered rectangle $\mathscr{X}$ both whose vertical and horizontal flows are minimal, the set of "homoclinic times" t_0 for which there exist $\tilde{x}, \hat{x} \in X$ satisfying $\tilde{x} \in \gamma^+_\infty(\hat{x})$, $\tilde{x} \in \gamma^-_\infty(\hat{x})$, $\hat{x} = h^+_{t_0}\tilde{x}$, is countable and dense in $\mathbb{R}$. Proposition 4.8 now implies the following

COROLLARY 4.9. *Let $\mathbb{P}_{\mathcal{V}}$ be a Borel P^s-invariant ergodic probability measure on $\widetilde{\mathcal{V}}(\mathcal{R})$. For $\mathbb{P}_{\mathcal{V}}$-almost every $\mathscr{X} \in \widetilde{\mathcal{V}}(\mathcal{R})$, there exists a dense set of times $t_0 \in \mathbb{R}$ such that for any $\Phi^+ \in \mathfrak{B}^+$, the distribution of the random variable $\Phi^+(x, t_0)$ has an atom.*

Our next step is to show that atoms of weight arbitrarily close to 1 occur for limit distributions of our Hölder cocycles. Informally, such atoms exist when one admissible rectangle occupies most of our surface. More precisely, we have the following

PROPOSITION 4.10. *Let* $\mathscr{X} \in \widetilde{\mathcal{V}}(\mathcal{R})$ *satisfy* $\lambda_1^{(0,\mathscr{X})} > 1/2$. *Then there exists a set* $\Pi \subset M(\mathscr{X})$ *such that*

(1) $\nu_{\mathscr{X}}(\Pi) \geq (2\lambda_1^{(0,\mathscr{X})} - 1)h_1^{(0,\mathscr{X})}$;

(2) *for any* $\Phi^+ \in \mathfrak{B}^+(\mathscr{X})$, *the function* $\Phi^+(x, h_1^{(0,\mathscr{X})})$ *is constant on* Π.

Proof. We consider $\mathscr{X}$ fixed and omit it from notation. Consider the partition

$$\pi_0(\mathscr{X}) = \Pi_1^{(0)} \sqcup \cdots \sqcup \Pi_m^{(0)}$$

of the zippered rectangle $\mathscr{X}$. Let I_k be the interval forming lower horizontal boundaries of the rectangles $\Pi_k^{(0)}$, $k = 1, \ldots, m$, and set

$$I = I_1 \sqcup \cdots \sqcup I_m.$$

The flow transversal I carries the Lebesgue measure ν_I invariant under the first-return map of the flow h_t^+ on I. We recall that the first return map is simply the interval exchange transformation (λ, π) of the zippered rectangle $\mathscr{X} = (\lambda, \pi, \delta)$. We recall that $\lambda_k^{(0)}$ is the length of I_k and that $h_k^{(0)}$ is the height of $\Pi_k^{(0)}$. For brevity, denote $t_1 = h_1^{(0)}$. By definition, $h_{t_1}^+ I_1 \subset I$ and we have

$$\nu_I \left(I_1 \bigcap h_{t_1}^+ I_1 \right) \geq 2\lambda_1^{(0)} - 1 > 0.$$

Introduce the set

$$\Pi = \{h_\tau^+ x, 0 < \tau < t_1, x \in I_1, h_{t_1} x \in I_1\}.$$

The first statement of the proposition is clear, and we proceed to the proof of the second. Note first that for any $\Phi^+ \in \mathfrak{B}^+(\mathscr{X})$ and any $\tau, 0 \leq \tau \leq t_1$, the quantity $\Phi^+(x, \tau)$ is constant as long as x varies in I_1.

Fix $\Phi^+ \in \mathfrak{B}^+(\mathscr{X})$, and take an arbitrary $\tilde{x} \in \Pi$. Write $\tilde{x} = h_{\tau_1}^+ x_1$, where $x_1 \in I_1$, $0 < \tau_1 < t_1$. We have $h_{t_1 - \tau_1}^+ \tilde{x} \in I_1$, whence

$$\Phi^+(h_{t_1-\tau_1}^+ \tilde{x}, \tau_1) = \Phi^+(x_1, \tau_1)$$

and

$$\begin{aligned} \Phi^+(\tilde{x}, t_1) &= \Phi^+(\tilde{x}, t_1 - \tau_1) + \Phi^+(h_{t_1-\tau_1}^+ \tilde{x}, \tau_1) \\ &= \Phi^+(h_{\tau_1}^+ x_1, t_1 - \tau_1) + \Phi^+(x_1, \tau_1) = \Phi^+(x_1, t_1), \end{aligned}$$

which concludes the proof of the proposition. We illustrate the proof by Figure 3. □

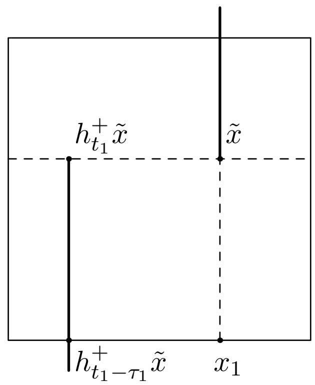

Figure 3. Atoms of limit distributions

4.7. *Accumulation at zero for limit distributions.* Recall that for $\mathscr{X}' \in \widetilde{\mathcal{V}}(\mathcal{R})'$, $\Phi^+ \in \mathfrak{B}^+_{\mathscr{X}}$, $\Phi^+ \neq 0$, and $\tau \in \mathbb{R}$, $\tau \neq 0$, the measure $\mathfrak{m}[\Phi^+, \tau]$ is the distribution of the normalized $\mathbb{R}$-valued random variable

$$\frac{\Phi^+(x,\tau)}{\sqrt{\mathrm{Var}_\nu \Phi^+(x,\tau)}}.$$

As before, let $\mathfrak{M}(\mathbb{R})$ be the space of probability measures on $\mathbb{R}$ endowed with the weak topology, and let $\delta_0 \in \mathfrak{M}(\mathbb{R})$ stand for the delta-measure at zero. Similarly to the introduction, we need the following additional assumption on our P^s-invariant ergodic probability measure $\mathbb{P}_{\mathcal{V}}$ on $\widetilde{\mathcal{V}}(\mathcal{R})$.

ASSUMPTION 4.11. *For any $\varepsilon > 0$, we have*

$$\mathbb{P}_{\mathcal{V}}(\{\mathscr{X} : \lambda_1^{(\mathscr{X})} > 1-\varepsilon, h_1^{(\mathscr{X})} > 1-\varepsilon\}) > 0.$$

By Proposition 4.10, in view of the ergodicity of $\mathbb{P}_{\mathcal{V}}$, for almost every $\mathscr{X} \in \widetilde{\mathcal{V}}(\mathcal{R})$ and every $\Phi^+ \in \mathfrak{B}^+_{\mathscr{X}}$, $\Phi^+ \neq 0$, the sequence of measures $\mathfrak{m}[\Phi^+, \tau]$ admits atoms of weight arbitrarily close to 1. The next simple proposition shows that the corresponding measures must then accumulate at *zero* (rather than at another point of the real line).

PROPOSITION 4.12. *Let μ_0 be a probability measure on $\mathbb{R}$ such that*

$$\int_{\mathbb{R}} x d\mu_0(x) = 0, \quad \int_{\mathbb{R}} x^2 d\mu_0(x) = 1.$$

Let $x_0 \in \mathbb{R}$, and assume that

$$\mu_0(\{x_0\}) = \beta.$$

Then

$$|x_0|^2 \leqslant \frac{1-\beta}{\beta^2}.$$

Proof. If $x_0 = 0$, then there is nothing to prove, so assume $x_0 > 0$. (The remaining case $x_0 < 0$ follows by symmetry.) We have

$$\int_0^{+\infty} x d\mu_0(x) \geqslant \beta x_0$$

and, consequently,

$$\int_{-\infty}^{0} x d\mu_0(x) \leqslant -\beta x_0.$$

Using the Cauchy-Bunyakovsky-Schwarz inequality, write

$$\frac{1}{\mu_0((-\infty,0))} \int_{-\infty}^{0} x^2 d\mu_0(x) \geqslant \left(\frac{1}{\mu_0((-\infty,0))} \int_{-\infty}^{0} x d\mu_0(x) \right)^2$$

whence, recalling that the variance of μ_0 is equal to 1, we obtain

$$\mu_0((-\infty,0)) \geqslant \left(\int_{-\infty}^{0} x d\mu_0(x) \right)^2$$

and, finally,

$$1 - \beta \geqslant \beta^2 x_0^2,$$

which is what we had to prove. □

As before, the symbol $\Rightarrow$ denotes weak convergence of probability measures.

PROPOSITION 4.13. *Let $\mathbb{P}_{\mathcal{V}}$ be a Borel ergodic P^s-invariant probability measure on $\widetilde{\mathcal{V}}(\mathcal{R})$ satisfying Assumption* 4.11. *Then for $\mathbb{P}_{\mathcal{V}}$-almost every $\mathscr{X} \in \widetilde{\mathcal{V}}(\mathcal{R})$ there exists a sequence $\tau_n \in \mathbb{R}_+$ such that for any $\Phi^+ \in \mathfrak{B}^+(\mathscr{X})$, we have*

$$\mathfrak{m}[\Phi^+, \tau_n] \Rightarrow \delta_0 \text{ as } n \to \infty.$$

This is immediate from Proposition 4.10 and Proposition 4.12.

COROLLARY 4.14. *Let $\mathbb{P}_{\mathcal{V}}$ be a Borel ergodic P^s-invariant probability measure on $\widetilde{\mathcal{V}}(\mathcal{R})$ satisfying Assumption* 4.11. *Then for $\mathbb{P}_{\mathcal{V}}$-almost every $\mathscr{X} \in \widetilde{\mathcal{V}}(\mathcal{R})$ there exists a sequence $s_n \in \mathbb{R}_+$ such that for any $f \in \mathrm{Lip}^+_{w,0}(\mathscr{X})$ satisfying $\Phi^+_f \neq 0$, we have*

$$\mathfrak{m}[f, s_n; 1] \Rightarrow \delta_0 \text{ as } n \to \infty.$$

Consequently, if $f \in \mathrm{Lip}^+_{w,0}(\mathscr{X})$ satisfies $\Phi^+_f \neq 0$, then the family of measures $\mathfrak{m}[f, s; 1]$ does not converge in the weak topology on $\mathfrak{M}(\mathbb{R})$ as $s \to \infty$ and the family of measures $\mathfrak{m}[f, s]$ does not converge in the weak topology on $\mathfrak{M}(C[0, 1])$ as $s \to \infty$.

Proof. The first claim is clear from Proposition 4.13 and the Limit Theorem 4. The second claim is obtained from the Limit Theorem 4 in the following way.

First note that the set

$$\{\mathfrak{m}[\Phi^+, 1], \Phi^+ \in \mathfrak{B}^+(\mathscr{X}), |\Phi^+| = 1\} \tag{70}$$

is compact in the weak topology. (Indeed, it is clear from the *uniform* convergence on spheres in the Oseledets Multiplicative Ergodic Theorem that the map $\Phi^+ \to \mathfrak{m}[\Phi^+, \tau]$ is continuous in restriction to the set $\{\Phi^+ : |\Phi^+| = 1\}$ whose image is therefore compact.) In particular, the set (70) is bounded away from δ_0, and the function

$$\kappa(\mathscr{X}) = \inf_{\Phi^+:|\Phi^+|=1} d_{\mathrm{LP}}(\mathfrak{m}[\Phi^+, 1], \delta_0)$$

is a positive measurable function on $\widetilde{\mathcal{V}}(\mathcal{R})$. Consequently, there exists $\kappa_0 > 0$ such that

$$\mathbb{P}_{\mathcal{V}}(\{\mathscr{X} : \kappa(\mathscr{X}) > \kappa_0\}) > 0.$$

From ergodicity of the measure $\mathbb{P}_{\mathcal{V}}$ and the Limit Theorem 4 it follows that the family $\mathfrak{m}[f, s; 1]$, $s \in \mathbb{R}$, does not converge to δ_0. On the other hand, as we have seen, the measure δ_0 is an accumulation point for the family. It follows that the measures $\mathfrak{m}[f, s; 1]$ do not converge in $\mathfrak{M}(\mathbb{R})$ as $s \to \infty$ and, *a fortiori*, that the measures $\mathfrak{m}[f, s]$ do not converge in $\mathfrak{M}(C[0, 1])$ as $s \to \infty$. Corollary 4.14 is proved completely. □

Appendix A. Metrics on the space of probability measures

A.1. *The weak topology.* In this appendix, we collect some standard facts about the weak topology on the space of probability measures. For a detailed treatment, see, e.g., [7].

Let (X, d) be a complete separable metric space, and let $\mathfrak{M}(X)$ be the space of Borel probability measures on X. The *weak topology* on $\mathfrak{M}(X)$ is defined as follows. Let $\varepsilon > 0$, $\nu_0 \in \mathfrak{M}(X)$, and let $f_1, \dots, f_k : X \to \mathbb{R}$ be bounded continuous functions. Introduce the set

$$U(\nu_0, \varepsilon, f_1, \dots, f_k) = \left\{ \nu \in \mathfrak{M}(X) : \left| \int_X f_i d\nu - \int_X f_i d\nu_0 \right| < \varepsilon, i = 1, \dots, k \right\}.$$

The basis of neighbourhoods for the weak topology is given precisely by sets of the form $U(\nu_0, \varepsilon, f_1, \dots, f_k)$ for all $\varepsilon > 0$, $\nu_0 \in \mathfrak{M}(X)$, $f_1, \dots, f_k$ continuous and bounded. The weak topology is metrizable and there are several natural metrics on $\mathfrak{M}(X)$ inducing the weak topology.

A.2. *The Kantorovich-Rubinstein metric.* Let

$$\mathrm{Lip}_1^1 = \left\{ f : X \to \mathbb{R} : \sup_X |f| \leqslant 1,\ |f(x_1) - f(x_2)| \leqslant d(x_1, x_2) \text{ for all } x_1, x_2 \in X \right\}.$$

The Kantorovich-Rubinstein metric is defined, for $\nu_1, \nu_2 \in \mathfrak{M}(X)$, by the formula

$$d_{\mathrm{KR}}(\nu_1, \nu_2) = \sup_{f \in \mathrm{Lip}_1^1(X)} \left| \int_X f d\nu_1 - \int_X f d\nu_2 \right|.$$

The Kantorovich-Rubinstein metric induces the weak topology on $\mathfrak{M}(X)$. By the Kantorovich-Rubinstein Theorem, the Kantorovich-Rubinstein metric admits the following equivalent dual description for bounded metric spaces. Given $\nu_1, \nu_2 \in \mathfrak{M}(X)$, let $\mathrm{Join}(\nu_1, \nu_2) \in \mathfrak{M}(X \times X)$ be the set of probability measures η on $X \times X$ such that projection of η on the first coordinate is equal to ν_1, the projection of η on the second coordinate is equal to ν_2. The Kantorovich-Rubinstein Theorem claims that

$$d_{\mathrm{KR}}(\nu_1, \nu_2) = \inf_{\eta \in \mathrm{Join}(\nu_1, \nu_2)} \int_{X \times X} d(x_1, x_2) d\eta.$$

A.3. *The Lévy-Prohorov metric.* Let $\mathcal{B}_X$ be the σ-algebra of Borel subsets of X. For $B \in \mathcal{B}_X, \varepsilon > 0$, set

$$B^\varepsilon = \{x \in X : \inf_{y \in B} d(x, y) \leqslant \varepsilon\}.$$

Given $\nu_1, \nu_2 \in \mathfrak{M}(X)$, introduce the Lévy-Prohorov distance between them by the formula

$$d_{\mathrm{LP}}(\nu_1, \nu_2) = \inf\{\varepsilon > 0 : \nu_1(B) \leqslant \nu_2(B^\varepsilon) + \varepsilon, \nu_2(B) \leqslant \nu_1(B^\varepsilon) + \varepsilon \text{ for any } B \in \mathcal{B}\}.$$

The Lévy-Prohorov metric also induces the weak topology on $\mathfrak{M}(X)$.

A.4. *An estimate on the distance between images of measures.* Consider a probability space $(\Omega, \mathfrak{B}_\Omega, \mathbb{P})$, and let $\xi_1, \xi_2 : \Omega \to X$ be two measurable maps. In the proof of the limit theorems, we use the following simple estimate on the Lévy-Prohorov and the Kantorovich-Rubinstein distance between the push-forwards $(\xi_1)_*\mathbb{P}, (\xi_2)_*\mathbb{P}$ of the measure $\mathbb{P}$ under the mappings ξ_1, ξ_2.

Lemma A.1. *Let $\varepsilon > 0$, and assume that for $\mathbb{P}$-almost all $\omega \in \Omega$, we have $d(\xi_1(\omega), \xi_2(\omega)) \leqslant \varepsilon$. Then we have*

$$d_{\mathrm{KR}}((\xi_1)_*\mathbb{P}, (\xi_2)_*\mathbb{P}) \leqslant \varepsilon,$$
$$d_{\mathrm{LP}}((\xi_1)_*\mathbb{P}, (\xi_2)_*\mathbb{P}) \leqslant \varepsilon.$$

The proof of Lemma A.1 is immediate from the definitions of the Kantorovich-Rubinstein and the Lévy-Prohorov metric.

References

[1] B. Adamczewski, Symbolic discrepancy and self-similar dynamics, *Ann. Inst. Fourier* (*Grenoble*) **54** (2004), 2201–2234 (2005). MR 2139693. Zbl 1066.11032. http://dx.doi.org/10.5802/aif.2079.

[2] S. H. Aranson and V. Z. Grines, Certain invariants of dynamical systems on two-dimensional manifolds (necessary and sufficient conditions for the topological equivalence of transitive systems), *Mat. Sb.* **90(132)** (1973), 372–402, 479. MR 0339275. Zbl 0252.54027. http://dx.doi.org/10.1070/SM1973v019n03ABEH001784.

[3] P. Arnoux and A. M. Fisher, Anosov families, renormalization and non-stationary subshifts, *Ergodic Theory Dynam. Systems* **25** (2005), 661–709. MR 2142941. Zbl 1140.37314. http://dx.doi.org/10.1017/S0143385704000641.

[4] A. Avila and M. Viana, Simplicity of Lyapunov spectra: proof of the Zorich-Kontsevich conjecture, *Acta Math.* **198** (2007), 1–56. MR 2316268. Zbl 1143.37001. http://dx.doi.org/10.1007/s11511-007-0012-1.

[5] L. Barreira and Y. Pesin, *Nonuniform Hyperbolicity. Dynamics of Systems with Nonzero Lyapunov Exponents*, *Encyclopedia Math. Appl.* **115**, Cambridge Univ. Press, Cambridge, 2007. MR 2348606. Zbl 1144.37002.

[6] P. Billingsley, *Convergence of Probability Measures*, second ed., *Wiley Series Prob. Stat.*, John Wiley & Sons, New York, 1999, A Wiley-Interscience Publication. MR 1700749. Zbl 0944.60003. http://dx.doi.org/10.1002/9780470316962.

[7] V. I. Bogachev, *Measure Theory. Vol. I*, *II*, Springer-Verlag, New York, 2007. MR 2267655. Zbl 1120.28001. http://dx.doi.org/10.1007/978-3-540-34514-5.

[8] F. Bonahon, Geodesic laminations with transverse Hölder distributions, *Ann. Sci. École Norm. Sup.* **30** (1997), 205–240. MR 1432054. Zbl 0887.57018. http://dx.doi.org/10.1016/S0012-9593(97)89919-3.

[9] F. Bonahon, Transverse Hölder distributions for geodesic laminations, *Topology* **36** (1997), 103–122. MR 1410466. Zbl 0871.57027. http://dx.doi.org/10.1016/0040-9383(96)00001-8.

[10] X. Bressaud, P. Hubert, and A. Maass, Persistence of wandering intervals in self-similar affine interval exchange transformations, *Ergodic Theory Dynam. Systems* **30** (2010), 665–686. MR 2643707. Zbl 1200.37002. http://dx.doi.org/10.1017/S0143385709000418.

[11] A. I. Bufetov, Decay of correlations for the Rauzy-Veech-Zorich induction map on the space of interval exchange transformations and the central limit theorem for the Teichmüller flow on the moduli space of abelian differentials, *J. Amer. Math. Soc.* **19** (2006), 579–623. MR 2220100. Zbl 1100.37002. http://dx.doi.org/10.1090/S0894-0347-06-00528-5.

[12] A. I. Bufetov, Suspension flows over Vershik's automorphisms. arXiv 0804.3970.

[13] A. I. Bufetov, Finitely-additive measures on the asymptotic foliations of a Markov compactum. arXiv 0902.3303.

[14] A. I. Bufetov, Limit theorems for translation flows. arXiv 0804.3970.

[15] A. I. Bufetov and B. M. Gurevich, Existence and uniqueness of a measure with maximal entropy for the Teichmüller flow on the moduli space of abelian differentials, *Mat. Sb.* **202** (2011), 3–42. MR 2857792. Zbl 1241.28011. http://dx.doi.org/10.1070/SM2011v202n07ABEH004172.

[16] J.-M. Dumont, T. Kamae, and S. Takahashi, Minimal cocycles with the scaling property and substitutions, *Israel J. Math.* **95** (1996), 393–410. MR 1418302. Zbl 0866.54033. http://dx.doi.org/10.1007/BF02761048.

[17] A. Eskin, M. Kontsevich, and A. Zorich, Lyapunov spectrum of square-tiled cyclic covers, *J. Mod. Dyn.* **5** (2011), 319–353. MR 2820564. Zbl 1254.32019. http://dx.doi.org/10.3934/jmd.2011.5.319.

[18] A. M. Fisher, Nonstationary mixing and the unique ergodicity of adic transformations, *Stoch. Dyn.* **9** (2009), 335–391. MR 2566907. Zbl 1231.28014. http://dx.doi.org/10.1142/S0219493709002701.

[19] L. Flaminio and G. Forni, Invariant distributions and time averages for horocycle flows, *Duke Math. J.* **119** (2003), 465–526. MR 2003124. Zbl 1044.37017. http://dx.doi.org/10.1215/S0012-7094-03-11932-8.

[20] G. Forni, Solutions of the cohomological equation for area-preserving flows on compact surfaces of higher genus, *Ann. of Math.* **146** (1997), 295–344. MR 1477760. Zbl 0893.58037. http://dx.doi.org/10.2307/2952464.

[21] G. Forni, Deviation of ergodic averages for area-preserving flows on surfaces of higher genus, *Ann. of Math.* **155** (2002), 1–103. MR 1888794. Zbl 1034.37003. http://dx.doi.org/10.2307/3062150.

[22] G. Forni, A geometric criterion for the nonuniform hyperbolicity of the Kontsevich-Zorich cocycle, *J. Mod. Dyn.* **5** (2011), 355–395. MR 2820565. Zbl 1238.30031. http://dx.doi.org/10.3934/jmd.2011.5.355.

[23] G. Forni, Sobolev regularity of solutions of the cohomological equation. arXiv 0707.0940.

[24] G. Forni, C. Matheus, and A. Zorich, Zero Lyapunov exponents of the Hodge bundle. arXiv 1201.6075.

[25] R. Gjerde and Ø. Johansen, Bratteli-Vershik models for Cantor minimal systems associated to interval exchange transformations, *Math. Scand.* **90** (2002), 87–100. MR 1887096. Zbl 1019.37006.

[26] S. Ito, A construction of transversal flows for maximal Markov automorphisms, *Tokyo J. Math.* **1** (1978), 305–324. MR 0519199. Zbl 0446.28017. http://dx.doi.org/10.3836/tjm/1270216501.

[27] T. Kamae, Linear expansions, strictly ergodic homogeneous cocycles and fractals, *Israel J. Math.* **106** (1998), 313–337. MR 1656897. Zbl 0914.28014. http://dx.doi.org/10.1007/BF02773474.

[28] A. B. Katok, Invariant measures of flows on orientable surfaces, *Dokl. Akad. Nauk SSSR* **211** (1973), 775–778. MR 0331438. Zbl 0298.28013.

[29] R. Kenyon, A group of paths in $\mathbf{R}^2$, *Trans. Amer. Math. Soc.* **348** (1996), 3155–3172. MR 1340179. Zbl 0896.52024. http://dx.doi.org/10.1090/S0002-9947-96-01562-0.

[30] M. Kontsevich and A. Zorich, Connected components of the moduli spaces of Abelian differentials with prescribed singularities, *Invent. Math.* **153** (2003), 631–678. MR 2000471. Zbl 1087.32010. http://dx.doi.org/10.1007/s00222-003-0303-x.

[31] M. Kontsevich and A. Zorich, Lyapunov exponents and Hodge theory, 2007. arXiv hep-th/9701164.

[32] S. Marmi, P. Moussa, and J.-C. Yoccoz, The cohomological equation for Roth-type interval exchange maps, *J. Amer. Math. Soc.* **18** (2005), 823–872. MR 2163864. Zbl 1112.37002. http://dx.doi.org/10.1090/S0894-0347-05-00490-X.

[33] S. Marmi, P. Moussa, and J.-C. Yoccoz, Affine interval exchange maps with a wandering interval, *Proc. Lond. Math. Soc.* **100** (2010), 639–669. MR 2640286. Zbl 1196.37041. http://dx.doi.org/10.1112/plms/pdp037.

[34] H. Masur, Interval exchange transformations and measured foliations, *Ann. of Math.* **115** (1982), 169–200. MR 0644018. Zbl 0497.28012. http://dx.doi.org/10.2307/1971341.

[35] Ya. G. Sinai and C. Ulcigrai, Weak mixing in interval exchange transformations of periodic type, *Lett. Math. Phys.* **74** (2005), 111–133. MR 2191950. Zbl 1105.37002. http://dx.doi.org/10.1007/s11005-005-0011-0.

[36] B. Solomyak, On the spectral theory of adic transformations, in *Representation Theory and Dynamical Systems*, *Adv. Soviet Math.* **9**, Amer. Math. Soc., Providence, RI, 1992, pp. 217–230. MR 1166205. Zbl 0770.28012.

[37] W. A. Veech, Gauss measures for transformations on the space of interval exchange maps, *Ann. of Math.* **115** (1982), 201–242. MR 0644019. Zbl 0486.28014. http://dx.doi.org/10.2307/1971391.

[38] W. A. Veech, The metric theory of interval exchange transformations. I. Generic spectral properties, *Amer. J. Math.* **106** (1984), 1331–1359. MR 0765582. Zbl 0631.28006. http://dx.doi.org/10.2307/2374396.

[39] W. A. Veech, The metric theory of interval exchange transformations. II. Approximation by primitive interval exchanges, *Amer. J. Math.* **106** (1984), 1361–1387. MR 0765583. Zbl 0631.28007. http://dx.doi.org/10.2307/2374397.

[40] W. A. Veech, The metric theory of interval exchange transformations. III. The Sah-Arnoux-Fathi invariant, *Amer. J. Math.* **106** (1984), 1389–1422. MR 0765584. Zbl 0631.28008. http://dx.doi.org/10.2307/2374398.

[41] W. A. Veech, The Teichmüller geodesic flow, *Ann. of Math.* **124** (1986), 441–530. MR 0866707. Zbl 0658.32016. http://dx.doi.org/10.2307/2007091.

[42] W. A. Veech, Moduli spaces of quadratic differentials, *J. Analyse Math.* **55** (1990), 117–171. MR 1094714. Zbl 0722.30032. http://dx.doi.org/10.1007/BF02789200.

[43] A. M. Vershik, A theorem on Markov periodic approximation in ergodic theory, *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov.* (*LOMI*) **115** (1982), 72–82, 306. MR 0660072. Zbl 0517.47005.

[44] A. M. Vershik, The adic realizations of the ergodic actions with the homeomorphisms of the Markov compact and the ordered Bratteli diagrams, *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov.* (*POMI*) **223** (1995), 120–126, 338. MR 1374314. Zbl 0884.28013. http://dx.doi.org/10.1007/BF02355799.

[45] A. M. Vershik and A. N. Livshits, Adic models of ergodic transformations, spectral theory, substitutions, and related topics, in *Representation Theory and*

Dynamical Systems, *Adv. Soviet Math.* **9**, Amer. Math. Soc., Providence, RI, 1992, pp. 185–204. MR 1166202. Zbl 0770.28013.

[46] A. Zorich, Deviation for interval exchange transformations, *Ergodic Theory Dynam. Systems* **17** (1997), 1477–1499. MR 1488330. Zbl 0958.37002. http://dx.doi.org/10.1017/S0143385797086215.

[47] A. Zorich, How do the leaves of a closed 1-form wind around a surface?, in *Pseudoperiodic Topology*, *Amer. Math. Soc. Transl. Ser.* 2 **197**, Amer. Math. Soc., Providence, RI, 1999, pp. 135–178. MR 1733872. Zbl 0976.37012.

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