Annals of Mathematics **179** (2014), 1–80
http://dx.doi.org/10.4007/annals.2014.179.1.1

Cannon-Thurston maps for surface groups

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Abstract

We prove the existence of Cannon-Thurston maps for simply and doubly degenerate surface Kleinian groups. As a consequence we prove that connected limit sets of finitely generated Kleinian groups are locally connected.

Contents

Research partially supported by DST Research grant DyNo. 100/IFD/8347/2008-2009.

1. **Introduction**

Let Γ be a finitely generated Kleinian group, i.e., a finitely generated discrete subgroup of $\mathrm{Isom}(\mathbf{H}^3)$ $(= \mathrm{PSL}_2(C))$, the isometry group of hyperbolic 3-space. Then Γ acts on the boundary Riemann sphere S^2 (of $\mathbf{H}^3$) by conformal automorphisms. The limit set of Γ, denoted by Λ_Γ, is the collection of accumulation points of any Γ-orbit in S^2. The limit set is independent of the Γ-orbit chosen. In particular, for any $z \in \Lambda_\Gamma$, the orbit $\Gamma.z$ is dense in Λ_Γ. The complement $S^2 \setminus \Lambda_\Gamma$ is called the domain of discontinuity of Γ and is denoted D_Γ. The action of Γ on D_Γ is properly discontinuous. Thus, the limit set Λ_Γ may be thought of as the locus of chaotic dynamics for the action of Γ on S^2 and it would be desirable to "tame" it.

Motivation and statement of results. Towards this, Thurston raises the following question (see [Thu82, Prob. 14]):

Question 1.1. Suppose Γ has the property that $(\mathbf{H}^3 \cup D_\Gamma)/\Gamma$ is compact. Then is it true that the limit set of any other Kleinian group Γ' isomorphic to Γ is the continuous image of the limit set of Γ by a continuous map taking the fixed points of an element γ to the fixed points of the corresponding element γ'?

Essentially the same question is raised by Cannon and Thurston in [CT85, §6], [CT07] in the specific context of surface Kleinian groups:

Question 1.2. Suppose that a surface group $\pi_1(S)$ acts freely and properly discontinuously on $\mathbb{H}^3$ by isometries such that the quotient manifold has no accidental parabolics. Does the inclusion $\tilde{i} : \widetilde{S} \to \mathbb{H}^3$ extend continuously to the boundary?

The authors of [CT85] point out that for a simply degenerate surface Kleinian group, this is equivalent, via the Carathéodory extension theorem, to asking if the limit set is locally connected. The most general question in this context is the following.

Question 1.3. Let Γ be a finitely generated Kleinian group such that the limit set Λ_Γ is connected. Is Λ_Γ locally connected?

It is a classical fact of general topology that a compact, connected, locally connected metric space X is homeomorphic to a Peano continuum; i.e., X is a continuous image of the closed interval $[0,1]$. Hence, asking if the limit set is locally connected is equivalent to asking if there is some parametrization by $[0,1]$. Question 1.1 makes this precise by asking for an explicit parametrization. For surface Kleinian groups, Question 1.2 asks for a parametrization of Λ_Γ by a circle. In this paper, we give a positive answer to Question 1.2.

Theorems 7.1 and 8.6. *Let ρ be a representation of a surface group $H(=\pi_1(S))$ into $\mathrm{PSL}_2(C)$ without accidental parabolics. Let M denote the (convex core of) $\mathbb{H}^3/\rho(H)$. Further suppose that $i: S \to M$, taking parabolic to parabolics, induces a homotopy equivalence. Then the inclusion $\tilde{i}: \widetilde{S} \to \widetilde{M}$ of universal covers extends continuously to a map $\hat{i}: \widehat{S} \to \widehat{M}$ between the compactifications of universal covers. Hence the limit set of $\rho(H)$ is locally connected.*

In [Mj10b] we extend the techniques of this paper to answer Question 1.1 affirmatively. The continuous boundary extensions above are called Cannon-Thurston maps.

Combining Theorems 7.1 and 8.6 with a theorem of Anderson and Maskit [AM96], we have the following affirmative answer to Question 1.3.

Theorem 8.9. *Let Γ be a finitely generated Kleinian group with connected limit set Λ. Then Λ is locally connected.*

Note that the limit set of a finitely generated Kleinian group Γ is connected if and only if the boundary of the *convex core* of $\mathbb{H}^3/\Gamma$ is incompressible away from cusps.

Relationship with the Ending Lamination Theorem. Seminal work of Minsky [Min10] and Brock-Canary-Minsky [BCM12], building on work of Masur-Minsky [MM99], [MM00], has resolved Thurston's Ending Lamination Conjecture. The Ending Lamination Theorem roughly says that for a simply or doubly degenerate surface Kleinian group Γ without accidental parabolics, the isometry type of the manifold $M = \mathbf{H}^3/\Gamma$ is determined by its end-invariants. For a doubly degenerate group, the end-invariants are two ending laminations, one each for the two geometrically infinite ends of M. For a simply degenerate group, the end-invariants are an ending lamination corresponding to the geometrically infinite end of M and a conformal structure corresponding to the geometrically finite end of M. The ending lamination corresponding to a geometrically infinite end may be regarded as a purely topological piece of data associated to the end. Thus, in the context of geometrically infinite Kleinian groups, the Ending Lamination Theorem roughly says that "*Topology implies Geometry*"; an analog of Mostow Rigidity for infinite covolume Kleinian groups.

Theorems 7.1 and 8.6 prove the existence of Cannon-Thurston maps for surface Kleinian groups but leave unanswered the question about the point preimages of these maps. In [Mj07], we relate the point preimages of Cannon-Thurston maps for simply and doubly degenerate surface Kleinian groups to ending laminations. In particular, the ending lamination corresponding to a degenerate end can be recovered from the Cannon-Thurston map. More generally, since topological conjugacies are compatible with Cannon-Thurston maps, a topological conjugacy of Γ-actions on limit sets comes from a bi-Lipschitz homeomorphism of quotient manifolds. Hence the Ending Lamination Theorem [Min10], [BCM12], in conjunction with Theorems 7.1 and 8.6 and the main result of [Mj07], shows that the geometry of M can be recovered from the action of Γ on the limit set Λ_Γ. This justifies the slogan "*dynamics on the limit set determines geometry in the interior.*"

History. Several authors have contributed to the theme of this paper. We shall give below a brief account of the history of the problem along with some further developments that use the results of this paper. Cannon and Thurston [CT07], Minsky [Min94], Alperin, Dicks and Porti [ADP99], Cannon and Dicks [CD02], [CD06], Klarreich [Kla99], McMullen [McM01], Bowditch [Bow13], [Bow07] and the author [Mit98b], [Mit98a], [Mj09], [Mj11], [Mj10a], [Mj05] have obtained partial positive answers to Questions 1.1 and 1.2. We describe some of this history in brief.

In [Abi76], Abikoff gave an approach to a negative answer to Question 1.3. However, around 1980, Thurston realized that this approach would not work. Then, in a foundational paper, Cannon and Thurston [CT85] gave the first examples furnishing a positive answer to Question 1.1 for geometrically infinite surface Kleinian groups; hence the term "Cannon-Thurston map." In approximate chronological order, the existence of Cannon-Thurston maps in the context of Kleinian groups was proven

(1) by Floyd [Flo80] for geometrically finite Kleinian groups;
(2) by Cannon and Thurston [CT85], [CT07] for fibers of closed hyperbolic 3-manifolds fibering over the circle and for simply degenerate groups with asymptotically periodic ends;
(3) by Minsky [Min94] for closed surface groups of bounded geometry (see also [Mit98b], [Mj10a]);

(4) by the author [Mit98b], and independently by Klarreich [Kla99] using different methods, for hyperbolic 3-manifolds of bounded geometry with an incompressible core and without parabolics;
(5) by Alperin-Dicks-Porti [ADP99] for fibers of the figure eight knot complement regarded as a fiber bundle over the circle;
(6) by McMullen [McM01] for punctured torus groups (see also [Mj11]).
(7) by Bowditch [Bow13], [Bow07] for punctured surface groups of bounded geometry (see also [Mj09]);
(8) by Miyachi [Miy02] for handlebody groups of bounded geometry (see also [Sou06]);
(9) by the author [Mj09] for hyperbolic 3-manifolds of bounded geometry with core incompressible away from cusps;
(10) by the author [Mj11], [Mj05] for special unbounded geometries.

Further developments. In [Mj07], we give an explicit parametrization of the limit set of a surface Kleinian group by describing the point pre-images of the Cannon-Thurston map and relating them to ending laminations. In a further follow-up paper [Mj10b], we answer Question 1.1 affirmatively and completely for all finitely generated Kleinian groups, using some preliminary work in [MD10]. The techniques of this paper can thus be strengthened to show that Cannon-Thurston maps exist in general for finitely generated Kleinian groups, thus answering a conjecture of McMullen [McM01].

Acknowledgments. I would like to thank Jeff Brock, Dick Canary and Yair Minsky for their help during the course of this work. In particular, Minsky and Canary brought a couple of critical gaps in previous versions of this paper to my notice. I would also like to thank Benson Farb for innumerable exciting conversations on relative hyperbolicity when we were graduate students. I would like to thank Caroline Series and Al Marden for a number of comments and corrections and Kingshook Biswas for comments that have improved the exposition. Special thanks are due to the referee for not unwelcome pressure to re-organize a couple of unwieldy manuscripts ([Mj05] and [Mj06]) into a relatively more streamlined version and also for painstakingly going through all the details and making several important and constructive suggestions. A major revision of this paper was done when the author was visiting Université Paris-Sud (Orsay) under the ARCUS Indo-French program.

Dedication. This paper is fondly dedicated to Gadai and Sarada for their support and indulgence.

1.1. *Broad scheme of proof.* Let M be a hyperbolic 3-manifold homotopy equivalent to a closed hyperbolic surface S. We think of S as an embedded incompressible surface in M. Let $\widetilde{S}$ and $\widetilde{M}(=\mathbf{H}^3)$ denote the universal covers of S, M respectively and $\tilde{i} : \widetilde{S} \to \widetilde{M}$ the inclusion of universal covers.

Given a hyperbolic geodesic segment λ in $\widetilde{S}$ lying outside a large ball about a fixed reference point $o \in \widetilde{S}$, our aim is to show that the geodesic in $\mathbf{H}^3$ joining the endpoints of $\widetilde{i}(\lambda)$ lies outside a large ball about $\widetilde{i}(o)$ in $\mathbf{H}^3$. This is sufficient to prove the existence of Cannon-Thurston maps (Lemma 1.8). Instead of proving this directly, our objective will be to construct a set $\mathcal{L}_\lambda$ (called a "ladder") containing $\widetilde{i}(\lambda)$ such that

(1) If λ lies outside a large ball in $\widetilde{S}$, then the ladder $\mathcal{L}_\lambda$ lies outside a a large ball in $\widetilde{M}$. It is much easier to show (and follows from an essentially elementary argument) that $\mathcal{L}_\lambda$ lies outside a a large ball than to find the exact (or even approximate) location of the geodesic in $\mathbf{H}^3$ joining the endpoints of $\widetilde{i}(\lambda)$. Hence this approach.
(2) $\mathcal{L}_\lambda$ is quasiconvex with respect to a modified (pseudo) metric d_G on $\widetilde{M}$, thus forcing the d_G geodesic joining the endpoints of λ to lie d_G-close to $\mathcal{L}_\lambda$.
(3) The pseudometric d_G is constructed in such a way that $\mathcal{L}_\lambda$ still controls (cf. Lemma 2.5) the location of the hyperbolic geodesic β^h in $\mathbf{H}^3$ joining the endpoints of $\widetilde{i}(\lambda)$, thus forcing β^h to lie outside a large ball in $\mathbf{H}^3$.

For ease of notation, we shall often identify any point or subset of $\widetilde{S}$ with its image under $\widetilde{i}$.

1.1.1. *The ladder.* One of the main steps in proving the sufficient condition of Lemma 1.8 (and hence concluding the existence of Cannon-Thurston maps) is to construct a quasiconvex "hyperbolic ladder" as in [Mit98b] and [Mit98a] containing λ. Suppose that a sequence $\{S_i\}$ of *disjoint*, *equispaced*, *embedded*, *bounded geometry* surfaces exiting an end E of M has been "judiciously" constructed. We shall describe a little later what "judicious" means. We think of $\{S_i\}$ as a sequence of surfaces exiting a vertical end E. Identify S with the base surface S_0.

Choose a basepoint in S and fix a lift p of the base-point in $\widetilde{S}$ as the origin. Let r be a quasigeodesic ray in M, starting at p, exiting E and making linear progress as it exits E. Suppose $\lambda = [a, b] \subset \widetilde{S}$ is a geodesic in the *intrinsic* metric on $\widetilde{S}$ joining two lifts $a(= a_0)$ and $b(= b_0)$ of p. Let r_a, r_b be the lifts of r starting at a, b respectively. Let a_i (resp. b_i) be the point at which r_a (resp. r_b) intersects $\widetilde{S_i}$. Let λ_i be the geodesic in the intrinsic metric on $\widetilde{S_i}$ joining a_i, b_i. The ladder associated to the sequence $\{S_i\}$ and the geodesic λ is $\mathcal{L}_\lambda = \bigcup_i \lambda_i$. To prove quasiconvexity of $\mathcal{L}_\lambda$ we construct a retraction Π_λ of $\bigcup_i \widetilde{S_i}$ onto $\mathcal{L}_\lambda = \bigcup_i \lambda_i$ by defining Π_λ on $\widetilde{S_i}$ as the nearest point retraction onto λ_i in the *intrinsic* metric on $\widetilde{S_i}$. We would like to ensure that Π_λ is coarsely Lipschitz. The construction of $\mathcal{L}_\lambda$ and Π_λ is detailed in Section 5.

The ladder $\mathcal{L}_\lambda$ has the following property that we want: If λ lies outside a large ball about the origin in $\widetilde{S}$, then $\mathcal{L}_\lambda$ lies outside a large ball about the origin in $\widetilde{M}$.

This construction works exactly for 3-manifolds of bounded geometry, where the S_i's may be chosen such that

(1) *Equispaced condition*: the regions between S_i and S_{i+1} are uniformly bi-Lipschitz to $S_i \times [0,1]$ (for all i).
(2) *Quasi-isometry condition*: The map from $\widetilde{S_i}$ to $\widetilde{S_{i+1}}$ that takes (x,i) to $(x,i+1)$ is a uniform quasi-isometry.

Both of these break down in general. In fact, quasiconvexity of $\mathcal{L}_\lambda$ is not in general true in the hyperbolic metric on $\widetilde{M}$ for the choice of the sequence $\{S_i\}$ we describe below.

The technical tool we shall use to address this issue in this paper is *electric geometry* and relative hyperbolicity (Section 2). Let $\mathcal{H} = \{H_i\}$ be a collection of quasiconvex subsets of $\mathbf{H}^3$. The electric (pseudo) metric obtained by electrocuting elements of $\mathcal{H}$ essentially allows one to travel for free within any H_i. However, this metric has the crucial feature that electric geodesics control hyperbolic geodesics (Lemma 2.5) and hence allows recovery of hyperbolic geodesics from electric geodesics. We emphasize that it is quasiconvexity of H_i's that allows this recovery.

1.1.2. *A motivational special case of split geometry.* We describe first a special case of the model geometry of a geometrically infinite unbounded geometry end E. This will be a particular case of what is referred to as "split geometry" later on in the paper and is representative in a sense to be explicated. (The model geometry described here was called "graph amalgamation geometry" in [Mj05].) Suppose we have the following situation:

(1) There exists a sequence $\{S_i\}$ of disjoint, embedded, bounded geometry surfaces exiting E. These are ordered in a natural way along E; i.e., $i < j$ implies that S_j is contained in the unbounded component of $E \setminus S_i$. The topological product region between S_i and S_{i+1} is denoted B_i.
(2) Corresponding to each such product region B_i, there exists a Margulis tube T_i such that $T_i \subset B_i$. Further, $T_i \cap S_i$ and $T_i \cap S_{i+1}$ are annuli on S_i and S_{i+1} respectively, with core curves homotopic to the core curve of T_i.

We think of the Margulis tube T_i as "splitting" the block B_i and hence the surfaces S_i and S_{i+1}; see Figure 1. The complementary components K_{ij} of $B_i \setminus T_i$ and their lifts $\widetilde{K_{ij}}$ to $\widetilde{E}$ will play a special role later.

Note that we have no control on the geometry of the complementary components K_{ij}. So the only thing we can do with them is to electrocute them and lose the geometry contained *within* any such component. Electrocuting K_{ij}'s forces S_i and S_{i+1} to be equispaced (about distance one apart from each

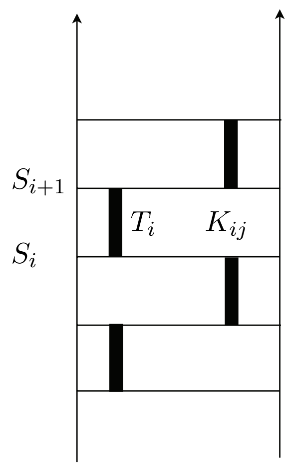

Figure 1. A special case of split geometry.

other). It is in this modified electric metric that the sequence $\{S_i\}$ satisfies both the equispaced condition and the quasi-isometry condition above and the ladder construction can go through.

It will turn out that the universal covers $\widetilde{K_{ij}} \subset \widetilde{E}$ are quasiconvex in a certain weak sense. Thus, we can electrocute such components and still hope to recover hyperbolic geodesics from electric geodesics using Lemma 2.5.

1.1.3. *Choice of the sequence of surfaces.* We shall first describe a couple of restrictive assumptions on a degenerate end that reduce it to the above model geometry. We shall then state (very briefly) how one needs to modify the above model to obtain a model geometry for a general degenerate end.

The special case. We give a brief sketch of the simplifying assumptions on a general degenerate end that leads us to a model geometry and a choice of a sequence $\{S_i\}$ as above. First one needs a linear order on incompressible (but not necessarily embedded) surfaces in E. It is at this stage that we need Minsky's model manifold from [Min10] and, more generally, the hierarchy machinery from [MM00]. The model manifold of [Min10] does not quite furnish a sequence of complete surfaces exiting E but rather a sequence of pants decompositions of S exiting E. A sequence $\{P_m\}$ of pants decompositions of S exiting E means the following. Fix an isometry type $\mathbb{P}$ of a pair of pants. Let τ_m denote the simple multicurve on S forming the boundary curves of the pants decomposition P_m. Then the complement of a thin (open) annular neighborhood of τ_m in S can be identified with P_m. We demand that this complementary region (identified with P_m) can be embedded in E such that P_m with the inherited metric is of uniformly bounded geometry; i.e., each pair of pants (component) in P_m is uniformly bi-Lipschitz to $\mathbb{P}$. We demand further that each such embedding of P_m can be extended to a topological embedding S_m of S and these topological embeddings $\{S_m\}$ exit the end E.

The sequence $\{P_m\}$ of (boundary curves of) pants decompositions exiting E is often referred to as a *resolution*. The sequence $\{P_m\}$ in [Min10] is chosen in such a way that the boundary curves of the pants decompositions $\{P_m\}$ occurring in the resolution have short geodesic realization in E.

Each pants decomposition gives a simplex in the curve complex $\mathrm{CC}(S)$ of the surface S. Hence the resolution furnishes a special kind of a path of simplices in $\mathrm{CC}(S)$. Associated to such a path is a *geodesic* of simplices in $\mathrm{CC}(S)$ called a *tight geodesic* [Min10]. A tight geodesic furnishes a "tight sequence" $\ldots, \tau_i, \tau_{i+1}, \ldots$ of multicurves on the surface S. (Note the difference between the suffixes i and m at this stage. This indicates that we are actually passing to a subsequence.) This material is detailed in Section 3.

Simplifying Assumption 1. Assume, for simplicity, that for all i, the length of exactly one curve in τ_i is sufficiently small, less than the Margulis constant in particular.

Call the short curve τ_i for convenience. The surface S_i (roughly) corresponds to the first occurrence of the vertex τ_i in the resolution. Since τ_i is short, the Margulis tube T_i corresponding to it *splits* both S_i and S_{i+1}.

Simplifying Assumption 2. Assume further that that the surfaces S_i have injectivity radius uniformly bounded below; i.e., the tube T_i is trapped entirely between S_i and S_{i+1}.

The product region B_i between S_i and S_{i+1} will be called a *split block* as it is split by T_i. This situation (an end E satisfying Simplifying Assumptions 1 and 2) gives us the model geometry (special case of split geometry) described above.

The general case. The construction of the sequence $\{S_i\}$ in general (without the simplifying assumptions of the special case) is described in detail in Section 4.1. Here we content ourselves by providing a couple of caveats.

Note first that $B_i \setminus T_i$ might be very far from a metric product. Thus electrocution is a necessity to make the S_i's equispaced.

We point out further that in general (when Simplifying Assumption 2 is no longer valid) the Margulis tube T_i may not be entirely contained in B_i but may extend into B_{i+1} or B_{i-1}. As a result, the surface S_i may have a thin part contained entirely in T_i, destroying the product structure of B_i.

To address this issue, we shall excise the interiors of Margulis tubes and "weld" the "vertical sides" (see Figure 1) of T_i together. The resulting manifold is called the *welded model manifold* M_{wel}. M_{wel} is thus a quotient space of M

homeomorphic to M itself. In the previous schematic figure, the thick dark vertical rectangle denotes a section of the Margulis tube T_i. The quotient map identifies the vertical sides of this vertical rectangle and collapses the horizontal I-direction to a point. (T_i should be thought of as a product of the dark vertical rectangle with a circle.) We shall also construct a new (pseudo) metric d_{tel} on B_i after welding the vertical sides (the "welded blocks"). This process is called *tube electrocution* and is carried out on the welded model manifold M_{wel} rather than the model manifold itself in Section 4.3. The pseudometric d_{tel} on the welded manifold M_{wel} roughly gives zero length to all horizontal circles of T_i and a uniformly bounded length to the vertical direction.

1.1.4. *Split geometry and graph-quasiconvexity.* Lifts $\widetilde{K}$ of components of $B_i \setminus T_i$ to the universal cover $\widetilde{M_{\text{wel}}}$ are called *split components.* We construct an auxiliary metric d_G called the graph (pseudo) metric on $\widetilde{M_{\text{wel}}}$ by *electrocuting* the family of split components in $\widetilde{M_{\text{wel}}}$. What this means is that for each split component $\widetilde{K} \subset \widetilde{M_{\text{wel}}}$, we attach a copy of $\widetilde{K} \times [0, \frac{1}{2}]$, identifying $\widetilde{K} \times \{0\}$ with $\widetilde{K} \subset \widetilde{M_{\text{wel}}}$ and equipping $\widetilde{K} \times \{\frac{1}{2}\}$ with the zero metric. (This is slightly different from Farb's coning construction [Far98].) A crucial fact we prove in Sections 4.4 and 4.6 is that the hyperbolic convex hull $\mathrm{CH}(\widetilde{K})$ has uniformly bounded diameter in the graph metric d_G. We describe this by saying that $\widetilde{K}$ is uniformly *graph-quasiconvex* as any hyperbolic geodesic joining points in $\widetilde{K}$ lies in a uniformly bounded neighborhood of $\widetilde{K}$ in the d_G-metric. It follows that $(\widetilde{M_{\text{wel}}}, d_G)$ is a (Gromov)-hyperbolic metric space. Equivalently, $\widetilde{M_{\text{wel}}}$ is *weakly hyperbolic* relative to the collection of split components. Note that we cannot in general use strong relative hyperbolicity as two adjacent split components $\widetilde{K_1}, \widetilde{K_2} \subset \widetilde{M_{\text{wel}}}$ intersect along a lift of a welded Margulis tube. This issue is responsible for much of the strife in the recovery step below (Sections 6.4 and 6.5).

Gromov-hyperbolicity of $(\widetilde{M_{\text{wel}}}, d_G)$ ensures quasiconvexity of the ladder $\mathcal{L}_\lambda$ in $(\widetilde{M_{\text{wel}}}, d_G)$ whose construction is described above. This is proven in Section 5.

1.1.5. *Recovery of hyperbolic geodesics.* There is a fair bit of technical difficulty at this stage. The graph metric is constructed on the welded model manifold. So we have to have a way of getting back to the model manifold from the welded model manifold. To do this, we note that the complement of the Margulis tubes in the model manifold and the complement of the welded tubes in the welded model manifold are the same. This allows us to construct a pseudometric quasi-isometric to d_G on the model manifold M itself. Abusing notation slightly, we call this pseudometric d_G also.

The split components of $\widetilde{M}$ are obtained from those of $\widetilde{M_{\text{wel}}}$ by adjoining certain Margulis tubes. Weak relative hyperbolicity of $\widetilde{M}$ relative to the collection of split components gives us control over hyperbolic geodesics in terms of geodesics in $(\widetilde{M_{\rm wel}}, d_G)$. The process of recovering a hyperbolic geodesic from a geodesic in $(\widetilde{M_{\rm wel}}, d_G)$ is detailed in Sections 6.4 and 6.5. A more detailed sketch of the scheme of recovery is given in Section 6.1.

1.1.6. *A flowchart of main ideas.* Here is a mnemonic flow-chart of the above scheme that may be useful:

- $\widetilde{M} \longrightarrow \widetilde{M_{\rm wel}}$ (welding),
- $\widetilde{M_{\rm wel}} \longrightarrow (\widetilde{M_{\rm wel}}, d_{\rm tel})$ (tube-electrocution),
- $(\widetilde{M_{\rm wel}}, d_{\rm tel}) \longrightarrow (\widetilde{M_{\rm wel}}, d_G)$ (split-component-electrocution),
- $(\widetilde{M_{\rm wel}}, d_G) \longrightarrow (\widetilde{M}, d_G) \longrightarrow \widetilde{M}$ (recovery).

The principal purpose behind carrying out each of these steps is given below in brief:

- Welding allows us to construct a sequence of *bounded geometry* surfaces exiting the end(s) of $M_{\rm wel}$, though such a sequence might not exist in M. The sequence of bounded geometry surfaces permits us to construct the ladder $\mathcal{L}_\lambda$ in $\widetilde{M_{\rm wel}}$.
- Tube electrocution and split-component-electrocution ensure both the equispaced condition and the quasi-isometry condition. In a certain sense therefore, the two electrocution steps allow us to reduce the problem to a model satisfying conditions (1) and (2) of the bounded geometry case. We can (as in the bounded geometry case) show that $\mathcal{L}_\lambda$ is quasiconvex in $(\widetilde{M_{\rm wel}}, d_G)$.
- Quasiconvexity of $\mathcal{L}_\lambda$ furnishes a d_G-quasigeodesic in $\widetilde{M_{\rm wel}}$ contained in $\mathcal{L}_\lambda$ joining the endpoints of λ.
- Finally, the recovery step allows us to come back from $(\widetilde{M_{\rm wel}}, d_G)$ to $\widetilde{M}$ via $(\widetilde{M}, d_G)$.

1.1.7. *Outline of the paper.* We recall the notions of relative hyperbolicity and electric geometry (cf. [Far98]) in Section 2 and derive some consequences that will be useful in this paper. In Section 3, we collect together features of the model manifold constructed by Minsky in [Min10] and proven to be a bi-Lipschitz model for simply and doubly degenerate manifolds by Brock-Canary-Minsky in [BCM12]. In Section 4, we select out a sequence of split surfaces from the split surfaces occurring in the model manifold and proceed to "fill" the intermediate spaces between successive split surfaces by special blocks homeomorphic to $S \times I$. This gives us a "split geometry" model for simply and doubly degenerate manifolds. We make crucial use of electric geometry and relative hyperbolicity at this stage. In Section 5, we construct a quasiconvex (Gromov) "hyperbolic ladder" in the (Gromov) hyperbolic electric space constructed in Section 4 and use it to construct a quasigeodesic in the electric metric joining the endpoints of λ. In Section 6, we recover information about the hyperbolic geodesic joining the endpoints of λ from the electric geodesic constructed in Section 5. In Section 7 we put all the ingredients together to prove the existence of Cannon-Thurston maps for closed surface Kleinian groups (Theorem 7.1). In Section 8 we describe the modifications necessary for punctured surfaces.

1.1.8. *Notation.* We shall in general use N (resp. N^h) to denote (the convex core of) a simply or doubly degenerate hyperbolic 3-manifold *without* (*resp. with*) *cusps*. For a manifold N^h with cusps, N will also denote N^h minus an open neighborhood of the cusps. M will denote the model manifold (Section 3).

Similarly, S (resp. S^h) shall denote a closed (resp. finite volume with cusps) hyperbolic surface. For a surface S^h with cusps, S will also denote S^h minus an open neighborhood of the cusps. We shall sometimes use S to denote a bi-Lipschitz homeomorphic image of a hyperbolic S. Thus M, N will both be homeomorphic to $S \times J$, where $J = [0, \infty)$ or $\mathbb{R}$ according as N is simply or doubly degenerate.

Since we shall not have specific need for manifolds with cusps until the last section of this paper, N will denote (the convex core of) a simply or doubly degenerate hyperbolic 3-manifold without cusps unless otherwise mentioned.

d will denote the hyperbolic (or bi-Lipschitz to hyperbolic) metric on S. d_M will denote the metric on the model manifold.

1.2. *Gromov hyperbolic metric spaces and Cannon-Thurston maps.* We start off with some preliminaries about hyperbolic metric spaces in the sense of Gromov [Gro87]. For details, see [CDP90], [GdlH90]. Let (X, d_X) be a (Gromov) hyperbolic metric space. The *Gromov boundary* of X, denoted by ∂X, is the collection of equivalence classes of geodesic rays $r : [0, \infty) \to X$ with $r(0) = x_0$ for some fixed $x_0 \in X$, where rays r_1 and r_2 are equivalent if $\sup\{d_X(r_1(t), r_2(t))\} < \infty$. Let $\widehat{X} = X \cup \partial X$ denote the natural compactification of X topologized the usual way (cf. [GdlH90, p. 124]).

We denote the k-neighborhood of a subset Z of (X, d_X) by $N_k(Z, d_X)$ or simply $N_k(Z)$ when d_X is understood.

Definition 1.4. A subset Z of (X, d_X) is said to be *k-quasiconvex* if any geodesic joining points of Z lies in a k-neighborhood $N_k(Z, d_X)$ of Z. A subset Z is quasiconvex if it is k-quasiconvex for some k.

For simply connected real hyperbolic manifolds, this is equivalent to saying that the convex hull $\mathrm{CH}(Z)$ of the set Z lies in a bounded neighborhood of Z. We shall have occasion to use this alternate characterization.

Definition 1.5. A map f from one metric space (Y, d_Y) into another metric space (Z, d_Z) is said to be a *(K, ε)-quasi-isometric embedding* if

$$\frac{1}{K}(d_Y(y_1, y_2)) - \varepsilon \leq d_Z(f(y_1), f(y_2)) \leq K d_Y(y_1, y_2) + \varepsilon.$$

If f is a quasi-isometric embedding and every point of Z lies at a uniformly bounded distance from some $f(y)$, then f is said to be a *quasi-isometry.*

A (K,ε)-quasi-isometric embedding that is a quasi-isometry will be called a (K,ε)-quasi-isometry.

A (K,ε)-*quasigeodesic* is a (K,ε)-quasi-isometric embedding of a closed interval in $\mathbb{R}$. A (K,K)-quasigeodesic will also be called a K-quasigeodesic. A (K,K)-quasigeodesic will simply be called a K-quasigeodesic

We shall say that two paths α,β in X *"C-track" each other* in $A\subset X$ if $\alpha\cap A$ and $\beta\cap A$ lie in a C-neighborhood of each other. The following Lemma says that quasigeodesics starting and ending close by track each other.

LEMMA 1.6 ([GdlH90]). *Let (X,d) be a δ-hyperbolic metric space. Then for any K,ε,D, there exists $C=C(\delta,K,\varepsilon,D)$ such that if α,β are two (K,ε)-quasi-geodesics whose starting points (as also ending points) are at most D apart, then $\alpha\subset N_C(\beta,d)$.*

The conclusion of Lemma 1.6 is also summarized by saying that α,β *C-fellow travel each other* and this property of quasigeodesics is called the C-fellow traveler property.

Let (X,d_X) be a (Gromov) hyperbolic metric space and Y be a subspace that is (Gromov) hyperbolic with the inherited path metric d_Y. By adjoining the Gromov boundaries ∂X and ∂Y to X and Y, one obtains their compactifications $\widehat{X}$ and $\widehat{Y}$ respectively. Let $i:Y\to X$ denote inclusion.

Definition 1.7. Let X and Y be (Gromov) hyperbolic metric spaces and $i:Y\to X$ be an embedding. A *Cannon-Thurston map* $\hat{i}$ from $\widehat{Y}$ to $\widehat{X}$ is a continuous extension of i.

The following lemma ([Mit98a, Lemma 2.1]) says a Cannon-Thurston map exists if for all $M>0$ and $y\in Y$, there exists $N>0$ such that if λ lies outside an N ball around y in Y, then any geodesic in X joining the endpoints of λ lies outside the M ball around $i(y)$ in X. For convenience of use later on, we state this somewhat differently and include the proof from [Mj10a] for completeness.

LEMMA 1.8. *A Cannon-Thurston map from $\widehat{Y}$ to $\widehat{X}$ exists if the following condition is satisfied. Given $y_0\in Y$, there exists a nonnegative function $M(N)$ such that $M(N)\to\infty$ as $N\to\infty$ and for all geodesic segments λ lying outside an N-ball around $y_0\in Y$, any geodesic segment in X joining the endpoints of $i(\lambda)$ lies outside the $M(N)$-ball around $i(y_0)\in X$.*

Proof. Suppose $i:Y\to X$ does not extend continuously. Since i is proper, there exist sequences x_m, $y_m\in Y$ and $p\in\partial Y$ such that $x_m\to p$ and $y_m\to p$ in $\widehat{Y}$, but $i(x_m)\to u$ and $i(y_m)\to v$ in $\widehat{X}$, where $u,v\in\partial X$ and $u\neq v$.

Since $x_m \to p$ and $y_m \to p$, any geodesic in Y joining x_m and y_m lies outside an N_m-ball $y_0 \in Y$, where $N_m \to \infty$ as $m \to \infty$. Any bi-infinite geodesic in X joining $u, v \in \partial X$ has to pass through some M-ball around $i(y_0)$ in X as $u \neq v$. There exist constants c and L such that for all $m > L$, any geodesic joining $i(x_m)$ and $i(y_m)$ in X passes through an $(M+c)$-neighborhood of $i(y_0)$. Since $(M+c)$ is a constant not depending on the index m, this proves the lemma. $\square$

The above result can be interpreted as saying that a Cannon-Thurston map exists if the space of geodesic segments in Y embeds properly in the space of geodesic segments in X.

2. **Relative hyperbolicity**

In this section, we shall recall first certain notions of relative hyperbolicity due to Farb [Far98], Klarreich [Kla99], Bowditch [Bow12] and the author [Mj11].

2.1. *Electric geometry.* We collect together certain facts about the electric metric that Farb proves in [Far98].

Definition 2.1. Given a metric space (X, d_X) and a collection $\mathcal{H}$ of subsets, let $\mathcal{E}(X,\mathcal{H}) = X \bigsqcup_{H \in \mathcal{H}} (H \times [0, \frac{1}{2}])$ be the identification space obtained by identifying $(h, 0) \in H \times [0, \frac{1}{2}]$ with $h \in X$. Each $\{h\} \times [0, \frac{1}{2}]$ is declared to be isometric to the interval $[0, \frac{1}{2}]$, and $H \times \{\frac{1}{2}\}$ is equipped with the zero metric.

A path $\sigma : I \to \mathcal{E}(X,\mathcal{H})$ is said to be *distinguished* if $\sigma(I) \cap \{h\} \times (0, \frac{1}{2})$ is either empty or all of $\{h\} \times (0, \frac{1}{2})$. The distance between two points in $\mathcal{E}(X,\mathcal{H})$ is defined to be the infimum of the lengths of distinguished paths between them.

The resulting pseudo-metric space $\mathcal{E}(X,\mathcal{H})$ is the *electric space* associated to X and the collection $\mathcal{H}$. We shall say that $\mathcal{E}(X,\mathcal{H})$ is constructed from X by *electrocuting* the collection $\mathcal{H}$, and the induced pseudo-metric d_e will be called the *electric metric.* If $\mathcal{E}(X,\mathcal{H})$ is (Gromov) hyperbolic, we say that X is *weakly hyperbolic* relative to $\mathcal{H}$.

The notion of electrocution above is slightly different from the coning construction introduced by Farb in [Far98], inasmuch as Farb [Far98]) collapses $H \times \{\frac{1}{2}\}$ to a point. Thus ours is a geometric (as opposed to topological) version of Farb's construction. *All paths in $\mathcal{E}(X,\mathcal{H})$ will henceforth be assumed to be distinguished.*

Let X be a geodesic metric space. Further suppose that each $H \in \mathcal{H}$ is closed. Then $(\mathcal{E}(X,\mathcal{H}), d_e)$ is a geodesic (pseudo) metric space. Geodesics and quasigeodesics in $(\mathcal{E}(X,\mathcal{H}), d_e)$ will be referred to as electric geodesics and electric quasigeodesics respectively.

Note that since $\mathcal{E}(X,\mathcal{H}) = X \bigsqcup_{H\in\mathcal{H}}(H\times[0,\frac{1}{2}])$, X can be naturally identified with a subspace of $\mathcal{E}(X,\mathcal{H})$. Paths in (X,d_X) (in particular, geodesics and quasigeodesics) can therefore be regarded as paths in $\mathcal{E}(X,\mathcal{H})$.

A collection $\mathcal{H}$ of subsets of (X,d_X) is said to be D-separated if $d_X(H_1,H_2) \geq D$ for all $H_1,H_2\in\mathcal{H}; H_1\neq H_2$. D-separatedness is only a technical restriction as the collection $\{H\times\{\frac{1}{2}\} : H\in\mathcal{H}\}$ is 1-separated in $\mathcal{E}(X,\mathcal{H})$.

Definition 2.2. Given a collection $\mathcal{H}$ of C-quasiconvex, D-separated sets in a (Gromov) hyperbolic metric space (X,d_X) and a number ε we shall say that a geodesic (resp. quasigeodesic) γ is a geodesic (resp. quasigeodesic) *without backtracking* with respect to ε-neighborhoods if γ does not return to $N_\varepsilon(H,d_X)$ after leaving it for any $H\in\mathcal{H}$. A geodesic (resp. quasigeodesic) γ is a geodesic (resp. quasigeodesic) without backtracking if it is a geodesic (resp. quasigeodesic) without backtracking with respect to ε-neighborhoods for $\varepsilon=0$.

Notation. For any pseudo metric space (Z,ρ) and $A\subset Z$, we shall use the notation $N_R(A,\rho)=\{x\in Z : \rho(x,A)\leq R\}$ as for metric spaces.

LEMMA 2.3 ([Far98, Lemma 4.5 and Prop. 4.6], [Kla99, Th. 5.3], [Bow12]). *Given δ, C, D, there exists Δ such that if (X,d_X) is a δ-hyperbolic metric space with a collection $\mathcal{H}$ of C-quasiconvex D-separated sets, then*

(1) Electric quasigeodesics electrically track (Gromov) hyperbolic geodesics: *For all $P>0$, there exists $K>0$ such that if β is any electric P-quasigeodesic from x to y and γ is a geodesic in (X,d_X) from x to y, then $\beta\subset N_K(\gamma,d_e)$.*
(2) $\gamma\subset N_K((N_0(\beta,d_e)),d_X)$.
(3) Relative Hyperbolicity: *X is weakly hyperbolic relative to $\mathcal{H}$. $\mathcal{E}(X,\mathcal{H})$ is Δ-hyperbolic.*

Let (X,d_X) be a δ-hyperbolic metric space and $\mathcal{H}$ a family of C-quasiconvex, D-separated, collection of subsets. Then X is weakly hyperbolic relative to $\mathcal{H}$ [Bow12]. Let $\alpha=[a,b]$ be a geodesic in (X,d_X) and β an electric quasigeodesic without backtracking (in $\mathcal{E}(X,\mathcal{H})$) joining a,b. Order from the left the collection of maximal subsegments of β contained entirely in some $H\times\frac{1}{2}$ for some $H\in\mathcal{H}$. Since β is a distinguished path (by our blanket assumption about paths in $\mathcal{E}(X,\mathcal{H})$), any such maximal subsegment can be extended by adjoining vertical subsegments at its endpoints to obtain a path of the form $\{p\}\times[0,\frac{1}{2}]\cup[p\times\frac{1}{2},q\times\frac{1}{2}]\cup\{q\}\times[0,\frac{1}{2}]$. We shall refer to these subpaths of β as extended maximal subsegments. Replace, as per the above ordering, extended maximal subsegment with endpoints p,q (say) by a geodesic $[p,q]$ in (X,d_X). (Note here that as per the definition of $\mathcal{E}(X,\mathcal{H})$, $(p,0)\in\mathcal{E}(X,\mathcal{H})$ is identified with $p\in X$; similarly for $(q,0)$ and q.) The resulting *connected* path β_q is called an *electro-ambient representative* of β in X. Also, if β is an electric P-quasigeodesic (resp. (K,ε)-quasigeodesic) without backtracking (in $\mathcal{E}(X,\mathcal{H})$), then β_q is called an *electro-ambient P-quasigeodesic* (*resp. electro-ambient* (K,ε)-*quasigeodesic*). If β is an electric geodesic (i.e., a $(1,0)$-quasigeodesic) without backtracking (in $\mathcal{E}(X,\mathcal{H})$), then β_q is simply called an *electro-ambient quasigeodesic.*

Remark 2.4. We emphasize a point about the terminology we use here. An electro-ambient quasigeodesic in our sense is the same as an electro-ambient $(1,0)$-quasigeodesic, *not* an electro-ambient (K,ε)-quasigeodesic for some K,ε.

Note that β_q *need not be a* (*Gromov*) *hyperbolic quasigeodesic*. However, the proof of Proposition 4.3 of Klarreich [Kla99] gives the following.

Lemma 2.5 (See [Kla99, Prop. 4.3] and [Mj11, Lemma 3.10]). *Given δ, C,P there exists C_3 such that the following holds. Let (X,d_X) be a δ-hyperbolic metric space and $\mathcal{H}$ a family of C-quasiconvex, collection of quasiconvex subsets. Let $(\mathcal{E}(X,\mathcal{H}),d_e)$ denote the electric space obtained by electrocuting elements of $\mathcal{H}$. Then, if α,β_q denote respectively a* (*Gromov*) *hyperbolic geodesic and an electro-ambient P-quasigeodesic with the same endpoints in X, then α lies in a* (*Gromov hyperbolic d_X-*) *C_3-neighborhood of β_q.*

For the convenience of the reader, we illustrate the content of Lemma 2.5 by Figure 2. The straight line indicates a hyperbolic geodesic, and the broken line built up of curves depicts an electro-ambient quasigeodesic.

Proof of Lemma 2.5. The proof closely follows Proposition 4.3 of Klarreich [Kla99]. Let $\alpha=[a,b](\subset X)$ be a geodesic, and let $\beta=\overline{ab}(\subset\mathcal{E}(X,\mathcal{H}))$ be an electric P-quasigeodesic with the same endpoints. Further, suppose that, for each $H\in\mathcal{H}$, $\beta\cap(H\times\{\frac{1}{2}\})$ is

(a) a maximal subsegment of β contained in $H\times\{\frac{1}{2}\}$,
(b) $\beta\cap(H\times\{\frac{1}{2}\})$ is a geodesic in $H\times\{\frac{1}{2}\}$ with respect to the intrinsic metric on $H(=H\times\{\frac{1}{2}\})$.

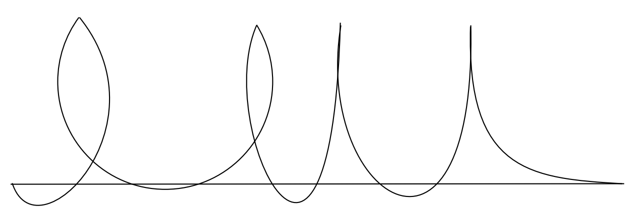

Figure 2. Hyperbolic geodesic lies in a neighborhood of an electro-ambient quasigeodesic.

For the purposes of this proof, we shall need to deal with two metrics (more precisely a metric and a pseudometric) on the *topological space* $\mathcal{E}(X, \mathcal{H})$:

(a) The first is the electric (pseudo) metric d_e described above.
(b) The other is the (genuine) metric on $X \bigcup_{H \in \mathcal{H}} H \times [0, \frac{1}{2}]$ obtained as a quotient space of X along with copies of $H \times [0, \frac{1}{2}]$. We call this metric d_q.

Thus d_e is obtained from d_q by redefining distance between points on $H \times \{\frac{1}{2}\}$ to be zero.

Recall that (by construction) the electro-ambient quasigeodesic β_q is obtained from β by "projecting" maximal subsegments of β to X. It therefore suffices to show that α lies in a (uniformly) bounded neighborhood of β in $(\mathcal{E}(X, \mathcal{H}), d_q)$.

Let $a = a_0, a_1, \ldots, a_n, a_{n+1} = b$ be a sequence of points on β such that for all i, $\overline{a_{2i}a_{2i+1}}(\subset \overline{ab})$ are maximal subsegments in $H_i \times \{\frac{1}{2}\}$ for some $H_i \in \mathcal{H}$. Also, assume that n is maximal; i.e., for all i, $\overline{a_{2i-1}a_{2i}}$ is a union of three segments:

(a) a vertical segment of the form $a_{2i-1} \times [0, \frac{1}{2}]$ traced from $a_{2i-1} \times \{\frac{1}{2}\}$ to $a_{2i-1} \times \{0\}$,
(b) a geodesic in (X, d_X) from a_{2i-1} (identified with $a_{2i-1} \times \{0\}$) to a_{2i} (identified with $a_{2i} \times \{0\}$),
(c) a vertical segment of the form $a_{2i} \times [0, \frac{1}{2}]$ traced from $a_{2i} \times \{0\}$ to $a_{2i} \times \{\frac{1}{2}\}$.

Note first that the collection $\{H \times \{\frac{1}{2}\}\}$, $H \in \mathcal{H}$ is automatically 1-separated. Hence $d_e(a_{2i-1}, a_{2i}) \geq 1$.

With this setup, the proof is a small reworking of Proposition 4.3 of [Kla99]. Choose an $R > 0$. Let $z \in [a, b]$ be a point for which no point of $\beta = \overline{ab}$ lies within R of z. Let (p, q) be a maximal subsegment of $[a, b]$ containing z such that no point of $\beta = \overline{ab}$ lies within R of (p, q).

Let $p_1 \in \overline{ab}$ and $q_1 \in \overline{ab}$ be points in $\overline{ab}$ closest to p, q respectively (with respect to the metric d_q). Let $\overline{p_1q_1}$ be the subpath of $\overline{ab}$ between p_1, q_1. Also, let $a_j, \ldots, a_{j+l}$ be the collection of vertices in $\overline{ab}$ between p_1, q_1. Then the proof of Proposition 4.3 of [Kla99] shows that there exists R_0 depending on δ, C, P such that for all $R \geq R_0$, $l(= (j + l) - j)$ is bounded in terms of R, δ, C, P. (This is essentially because l grows like $d_X(p, q)e^R$; cf. [Far98].) Let $l(R)$ be this bound for l.

Choosing $R = R_0$, we find that (p, q) is contained in a $(2R_0 + (l(R_0) + 4)\delta)$-neighborhood of $\beta = \overline{ab}$. This completes the proof. $\square$

Definition 2.6 ([Far98]). Two paths β, γ in (X, d_X) with the same endpoints are said to have *similar intersection patterns* with $\mathcal{H}$ if there exists $D > 0$, depending only on $(X, \mathcal{H})$, such that

- *Similar intersection patterns* 1: If precisely one of $\{\beta, \gamma\}$ meets some $H \in \mathcal{H}$, then the d_X-distance from the first entry point to the last exit point is at most D.
- *Similar intersection patterns* 2: If both $\{\beta, \gamma\}$ meet some $H \in \mathcal{H}$, then the distance from the first entry point of β to that of γ is at most D and similarly for the last exit points.

Definition 2.7 ([Far98]). Suppose that X is weakly hyperbolic relative to $\mathcal{H}$. Suppose that any two electric quasigeodesics without backtracking and with the same endpoints have similar intersection patterns with $\mathcal{H}$. Then $(X, \mathcal{H})$ is said to satisfy *bounded penetration* and X is said to be *strongly hyperbolic* relative to $\mathcal{H}$.

The next condition ensures that $(X, \mathcal{H})$ is *strongly hyperbolic* relative to $\mathcal{H}$.

Definition 2.8. A collection $\mathcal{H}$ of sets in a δ-hyperbolic metric space X is said to be *uniformly D-separated* if $d(H_i, H_j) \geq D$ for all $H_i, H_j \in \mathcal{H}; H_i \neq H_j$. A collection $\mathcal{H}$ of uniformly C-quasiconvex sets in a δ-hyperbolic metric space X is said to be *mutually D-cobounded* if for all $H_i, H_j \in \mathcal{H}$, $\pi_i(H_j)$ has diameter less than D, where π_i denotes a nearest-point projection of X onto H_i. A collection is *mutually cobounded* if it is mutually D-cobounded for some D.

Coboundedness ensures strong relative hyperbolicity.

LEMMA 2.9 ([Far98, Prop. 4.6], [Bow12]). *Let (X, d_X) be a (Gromov) hyperbolic metric space and $\mathcal{H}$ a collection of ε-neighborhoods of uniformly quasiconvex mutually cobounded uniformly separated subsets. Then X is strongly hyperbolic relative to the collection $\mathcal{H}$. Furthermore quasigeodesics without backtracking in (X, d_X) and $(\mathcal{E}(X, \mathcal{H}), d_e)$ have similar intersection patterns with elements of $\mathcal{H}$.*

Applications of Lemma 2.9 follow.

LEMMA 2.10. *Let M^h be a hyperbolic manifold. Let $\mathcal{T}$ and $\mathcal{H}$ denote respectively a collection of Margulis tubes and horoballs that are disjoint from one another. Then the elements of $\mathcal{T} \cup \mathcal{H}$ are mutually co-bounded. Hence $\widetilde{M^h}$ is strongly hyperbolic relative to the collection $\mathcal{T} \cup \mathcal{H}$.*

LEMMA 2.11. *Let S^h be a hyperbolic surface, with a finite collection of disjoint simple closed geodesics σ_i and cusps H_j. Let $\mathcal{S}$ denote the collection of lifts $\widetilde{\sigma_i}$ to $\mathbf{H}^2$, and let $\mathcal{H}$ denote the collection of lifts $\widetilde{H_j}$. Then the elements of $\mathcal{S} \cup \mathcal{H}$ are mutually co-bounded. Hence $\widetilde{S^h}$ is strongly hyperbolic relative to the collection $\mathcal{S} \cup \mathcal{H}$.*

A closely related theorem was proved by McMullen [McM01, Th. 8.1]. Let $\mathcal{H}$ be a locally finite collection of horoballs in a convex subset X of $\mathbb{H}^n$ (where the intersection of a horoball, which meets ∂X in a point, with X is called a horoball in X).

Definition 2.12. The ε-neighborhood of a bi-infinite geodesic in $\mathbb{H}^n$ will be called a *thickened geodesic*.

THEOREM 2.13 ([McM01]). *For $K, D \geq 1, \varepsilon \geq 0$, there exists $R \geq 0$ such that the following holds. Let X be a convex subset of $\mathbb{H}^n$, and let $\mathcal{H}$ denote a uniformly D-separated collection of horoballs and thickened geodesics. Let $Y = X \setminus \bigcup_{H\in\mathcal{H}} H$ and $\gamma : I \to Y$ be a (K, ε)-quasigeodesic in Y. Let η be the geodesic in X with the same endpoints as γ. Let $\mathcal{H}(\eta)$ be the union of all the horoballs and thickened geodesics in $\mathcal{H}$ meeting η. Then $\eta \cup \mathcal{H}(\eta)$ is R-quasiconvex and $\gamma(I) \subset B_R(\eta \cup \mathcal{H}(\eta))$. (The hyperbolic metric on $\mathbb{H}^n$ is understood.)*

2.2. *Electric geometry for surfaces.* We now specialize to surfaces. Let S be a hyperbolic surface with diameter bounded above by K. It follows that injectivity radius is bounded below by some $\varepsilon = \varepsilon(K)$. Let σ be a finite collection of disjoint simple closed geodesics on S. Component(s) of $S \setminus \sigma$ will be called the *amalgamation component*(*s*) of (S, σ). We shall denote an amalgamation components by S_A. Let $(S_{\text{Gel}}, d_{\text{Gel}}) = \mathcal{E}(S, S_A)$ be obtained from S by electrocuting S_A's, and let the universal cover of $(S_{\text{Gel}}, d_{\text{Gel}})$ with the lifted pseudometric be denoted $(\widetilde{S_{\text{Gel}}}, d_{\text{Gel}})$. A slightly different path pseudometric may be constructed on $\widetilde{S}$ by declaring that

(1) the length of any path that lies in the interior of an amalgamation component is zero,
(2) the length of any path that crosses σ once has length one,
(3) the length of any other path is the sum of lengths of pieces of the above two kinds.

This pseudometric differs from $(\widetilde{S_{\text{Gel}}}, d_{\text{Gel}})$ by at most one (due to the initial and final segments of length half). *We shall ignore this difference* (*cf. Lemma* 2.23).

The fundamental group $\pi_1(S)$ may be regarded as a graph of groups with vertex group(s) the subgroup(s) $\pi_1(S_A)$ corresponding to amalgamation component(s) and cyclic edge groups $\mathbb{Z}$ corresponding to σ. Then $(\widetilde{S_{\text{Gel}}}, d_{\text{Gel}})$ is quasi-isometric to the Bass-Serre tree of the splitting.

Continuous paths in S_{Gel} and $\widetilde{S_{\text{Gel}}}$ will be called *electric paths.* Continuous geodesics and quasigeodesics in the electric metric will be called electric geodesics and electric quasigeodesics respectively. We specialize Definition 2.2 to the present context, where it is slightly more restrictive.

Definition 2.14. An electric path $\gamma \subset \widetilde{S_{\text{Gel}}}$ is said to be an electric K-quasigeodesic in $(\widetilde{S_{\text{Gel}}}, d_{\text{Gel}})$ *without backtracking* if γ is a K-quasigeodesic in

$(\widetilde{S_{\mathrm{Gel}}}, d_{\mathrm{Gel}})$ and γ does not return to any lift $\widetilde{S_A}(\subset \widetilde{S_{\mathrm{Gel}}})$ of an amalgamation component $S_A \subset S$ after leaving it.

We now specialize the notion of an electro-ambient quasigeodesic to the context of surfaces.

Definition 2.15. An electric geodesic λ_e without backtracking in $(\widetilde{S_{\mathrm{Gel}}}, d_{\mathrm{Gel}})$ is called an *electro-ambient quasigeodesic* if

(a) each segment of λ_e lying inside a single lift $\widetilde{S_A}$ meets the boundary $\partial\widetilde{S_A}$ at most twice and is perpendicular to $\partial\widetilde{S_A}$ whenever they meet. We shall refer to these segments of λ_e as *amalgamation segments*.

(b) If a, b be the points of intersection of two distinct amalgamation segments of λ_e with a lift $\widetilde{\sigma}$ of σ, then $\lambda_e \cap \widetilde{\sigma}$ is equal to $[a, b]$, the geodesic segment in $\widetilde{\sigma}$ joining a, b. Such pieces $[a, b]$ shall be referred to as *interpolating segments*.

The underlying path of an electro-ambient quasigeodesic of the electro-ambient quasigeodesic in the *hyperbolic metric* on $\widetilde{S}$ shall be called the *electro-ambient representative* λ_q of λ_e.

See Figure 3, where the bold line indicates the electro-ambient quasigeodesic and the thin lines the geodesics $\widetilde{\sigma}$.

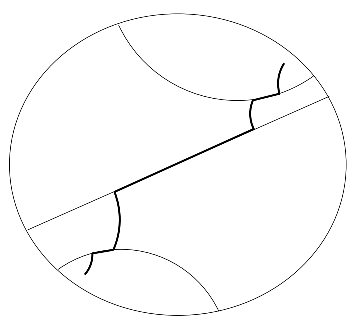

Figure 3. Electro-ambient quasigeodesic.

The next lemma justifies the terminology.

Lemma 2.16 (See [Mj11, Lemma 3.7]). *There exists (K, ε) such that each electro-ambient representative λ_q of an electric geodesic in $(\widetilde{S_{\mathrm{Gel}}}, d_{\mathrm{Gel}})$ is a (K, ε) hyperbolic quasigeodesic.*

Proof. Let $(S_{\mathrm{el}}, d_{\mathrm{el}})$ denote the surface S with the (collection of) geodesics σ electrocuted. Note that the electro-ambient quasigeodesics in $(\widetilde{S_{\mathrm{Gel}}}, d_{\mathrm{Gel}})$ coincide with those in the universal cover $(\widetilde{S_{\mathrm{el}}}, d_{\mathrm{el}})$. Hence it suffices to show that electro-ambient quasigeodesics in $(\widetilde{S_{\mathrm{el}}}, d_{\mathrm{el}})$ are uniform hyperbolic quasigeodesics.

Let λ_h denote the hyperbolic geodesic joining the endpoints of λ_e. By Lemmas 2.9 and 2.11, λ_h and λ_e, and hence λ_h and λ_q, have similar intersection patterns with $N_\varepsilon(\widetilde{\sigma})$ for some small $\varepsilon > 0$ and any lift $\widetilde{\sigma}$ of (an element of) σ. Also, λ_h and λ_q track each other off the collection $N_\varepsilon(\widetilde{\sigma})$. Further, each *interpolating segment* of λ_q being a *hyperbolic* geodesic, it follows (from the "K-fellow-traveler" property of hyperbolic geodesics starting and ending near each other; Lemma 1.6) that each interpolating segment of λ_q lies within a $(K + 2\varepsilon)$ neighborhood of λ_h for some fixed $K > 0$. Again, since each segment of λ_q that does not meet an electrocuted geodesic that λ_h meets is of uniformly bounded length (bounded by C say), we have finally that λ_q lies within a $(K + C + 2\varepsilon)$-neighborhood of λ_h. Finally, since λ_q is an electro-ambient representative, it does not backtrack. Hence the lemma. □

2.3. *Electric isometries.*

Definition 2.17. Let S be any hyperbolic surface and σ a collection of disjoint simple closed geodesics on S. A diffeomorphism ϕ of S will be called a *component preserving diffeomorphism* if it fixes σ pointwise and preserves each amalgamation component as a set; i.e., ϕ sends each amalgamation component of (S, σ) to itself.

Lemma 2.18. *Let ϕ denote a component preserving diffeomorphism of S_G. Then ϕ induces an isometry of $(S_{\mathrm{Gel}}, d_{\mathrm{Gel}})$.*

Proof. In the electrocuted surface $(S_{\mathrm{Gel}}, d_{\mathrm{Gel}})$, any electric geodesic λ_e has length equal to the number of times it crosses σ. Any component preserving diffeomorphism ϕ preserves the intersection pattern of λ_e with amalgamation components. Hence ϕ is an isometry of $(S_{\mathrm{Gel}}, d_{\mathrm{Gel}})$. □

The proof of Lemma 2.18 goes through verbatim after lifting to the universal cover $(\widetilde{S_{\mathrm{Gel}}}, d_{\mathrm{Gel}})$. We let $\widetilde{\phi}$ denote the lift of ϕ to $(\widetilde{S_{\mathrm{Gel}}}, d_{\mathrm{Gel}})$. This gives

Lemma 2.19. *Let $\widetilde{\phi}$ denote a lift of a component preserving diffeomorphism ϕ to $(\widetilde{S_{\mathrm{Gel}}}, d_{\mathrm{Gel}})$. Then $\widetilde{\phi}$ induces an isometry of $(\widetilde{S_{\mathrm{Gel}}}, d_{\mathrm{Gel}})$.*

2.4. *Nearest-point projections.* The next lemma says nearest-point projections in a δ-hyperbolic metric space do not increase distances much. This is a standard fact (cf. Lemma 3.1 of [Mit98b]).

Lemma 2.20. *Let (Y, d_Y) be a δ-hyperbolic metric space and let $\mu \subset Y$ be a C-quasiconvex subset; e.g., a geodesic segment. Let $\pi : Y \to \mu$ map $y \in Y$ to a point on μ nearest to y. Then $d_Y(\pi(x), \pi(y)) \leq C_3 d_Y(x, y)$ for all $x, y \in Y$ where C_3 depends only on δ, C.*

The next lemma (from [Mit98b]) says that quasi-isometries and nearest-point projections on (Gromov) hyperbolic metric spaces "almost commute."

Lemma 2.21 ([Mit98b, Lemma 3.5]). *Suppose (Y_1, d_1) and (Y_2, d_2) are δ-hyperbolic. Let μ_1 be some geodesic segment in Y_1 joining a, b and let p be any point of Y_1. Also let q be a point on μ_1 such that $d_1(p,q) \leq d_2(p,x)$ for all $x \in \mu_1$. Let ϕ be a (K,ε)-quasiisometric embedding from Y_1 to Y_2. Let μ_2 be a geodesic segment in Y_2 joining $\phi(a)$ to $\phi(b)$. Let r be a point on μ_2 such that $d_2(\phi(p), r) \leq d_2(\phi(p), x)$ for $x \in \mu_2$. Then $d_2(r, \phi(q)) \leq C_4$ for some constant C_4 depending only on K, ε and δ.*

We shall need the above lemma for quasi-isometries from $\widetilde{S_a}$ to $\widetilde{S_b}$ for two different bi-Lipschitz metrics on the same surface. We shall also need it for electrocuted surfaces.

Another property that we shall require for nearest point projections is that nearest-point projections in the hyperbolic metric on $\widetilde{S}$ and that in the electric metric $(\widetilde{S_{\rm Gel}}, d_{\rm Gel})$ almost agree. To make this precise, we make the following definition. The hyperbolic metric on S as well as $\widetilde{S}$ will be denoted by d.

Definition 2.22. Let $y \in (\widetilde{S}, d)$, and let μ_q be an electro-ambient representative of an electric geodesic $\mu_{\rm Gel}$ in $(\widetilde{S_{\rm Gel}}, d_{\rm Gel})$. Then $\pi_e(y) = z \in \mu_q$ if the ordered pair $\{d_{\rm Gel}(y, \pi_e(y)), d(y, \pi_e(y))\}$ is minimized at z in the lexicographical order on $(\mathbb{R}_+ \cup \{0\}) \times (\mathbb{R}_+ \cup \{0\})$.

The proof of the following lemma shows that this gives us a definition of π_e, which is ambiguous up to a finite amount of discrepancy not only in the electric metric but also in the hyperbolic metric.

Lemma 2.23. *Fix a hyperbolic surface S. For all $\varepsilon > 0$, there exists $C > 0$ such that if σ is a finite collection of disjoint simple closed geodesics such that $d(\sigma_i, \sigma_j) \geq \varepsilon$ for all $\sigma_i \neq \sigma_j \in \sigma$, then the following holds. Let μ be a hyperbolic geodesic in $(\widetilde{S}, d)$ joining $u, v \in \widetilde{S}$. Let $(S_{\rm Gel}, d_{\rm Gel})$ be the electric space obtained from S by electrocuting the amalgamation components of (S, σ). Let $\mu_{\rm Gel}$ be an electric geodesic in $(\widetilde{S_{\rm Gel}}, d_{\rm Gel})$ joining u, v, and let μ_q be its electro-ambient representative. Let π_h denote the nearest point projection of $(\widetilde{S}, d)$ onto μ. Then $d(\pi_h(y), \pi_e(y)) \leq C$.*

Proof. Let $[u,v]$ and $[u,v]_q$ denote respectively the hyperbolic geodesic and the electro-ambient quasigeodesic joining $u, v \in \widetilde{S}$. Since $[u,v]_q$ is a hyperbolic quasigeodesic by Lemma 2.16, the nearest-point projection of $y \in (\widetilde{S}, d)$ onto $[u,v]$ and $[u,v]_q$ almost agrees in the hyperbolic metric d. Thus, abusing notation slightly let π_h denote nearest-point projection of $(\widetilde{S}, d)$ onto $[u,v]_q$. Hence it suffices to show that for any $y \in \widetilde{S}$, its hyperbolic and electric projections $\pi_h(y), \pi_e(y)$ onto $[u,v]_q$ almost agree. See Figure 4, where we denote $\pi_h(y), \pi_e(y)$ by p, q respectively.

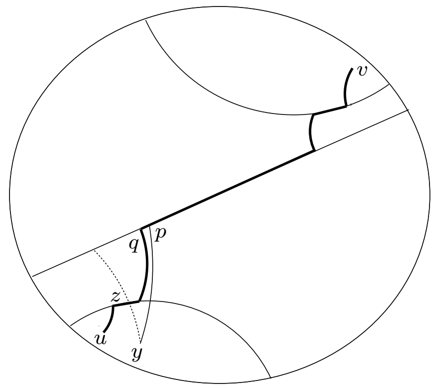

Figure 4. Electric and hyperbolic projections.

First note that any hyperbolic geodesic η in $\widetilde{S}$ is also an electric geodesic in $(\widetilde{S_{\mathrm{Gel}}}, d_{\mathrm{Gel}})$. This follows from the fact that $(\widetilde{S_{\mathrm{Gel}}}, d_{\mathrm{Gel}})$ maps to the Bass-Serre tree T of the splitting of S along σ such that the pre-image of every vertex is a set of diameter zero in the pseudometric d_{Gel}. If a path in $(\widetilde{S_{\mathrm{Gel}}}, d_{\mathrm{Gel}})$ projects to a path in T that is not a geodesic, then it must backtrack. Hence, it must leave an amalgamating component and return to it. Such a path can clearly not be a hyperbolic geodesic in $(\widetilde{S_{\mathrm{Gel}}}, d_{\mathrm{Gel}})$ since each amalgamating component is convex.

Next, it follows that hyperbolic projections automatically minimize electric distances. Otherwise, as in the preceding paragraph, $[y, \pi_h(y)]$ would have to cut a lift $\widetilde{\sigma_1}$ of σ that separates $[u,v]_q$. Further, $[y, \pi_h(y)]$ cannot return to $\widetilde{\sigma_1}$ after leaving it. Let z be the first point at which $[y, \pi_h(y)]$ meets $\widetilde{\sigma_1}$ (the intersection point of the dotted line with $\widetilde{\sigma_1}$ in Figure 4). Also let w be the point on $[u,v]_q \cap \widetilde{\sigma_1}$ that is nearest to z. Since amalgamation segments of $[u,v]_q$ meeting $\widetilde{\sigma_1}$ are perpendicular to the latter, it follows that $d(w,z) < d(w, \pi_h(y))$ and therefore $d(y,z) < d(y, \pi_h(y))$ contradicting the definition of $\pi_h(y)$. Hence hyperbolic projections automatically minimize electric distances.

Further, it follows by repeating the argument in the first paragraph that $[y, \pi_h(y)]$ and $[y, \pi_e(y)]$ pass through the same set of amalgamation components in the same order; in particular, they cut across the same set of lifts of $\widetilde{\sigma}$. Let $\widetilde{\sigma_2}$ be the last such lift. Then $\widetilde{\sigma_2}$ forms the boundary of an amalgamation component $\widetilde{S_A}$ whose intersection with $[u,v]_q$ is of the form $[a,b] \cup [b,c] \cup [c,d]$, where $[a,b] \subset \widetilde{\sigma_3}$ and $[c,d] \subset \widetilde{\sigma_4}$ are subsegments of two lifts of σ and $[b,c]$ is perpendicular to these two. Then the nearest-point projection of $\widetilde{\sigma_2}$ onto each of $[a,b], [b,c], [c,d]$ has uniformly bounded diameter. Hence the nearest-point projection of $\widetilde{\sigma_2}$ onto the hyperbolic geodesic $[a,d] \subset \widetilde{S_A}$ has uniformly bounded diameter. The result follows. $\square$

3. **The Minsky model**

In this section we summarize the notions and facts from [Min10], [BCM12] and [MM00] that we shall need. Let $\mathcal{C}(S)$ and $\mathcal{P}(S)$ denote respectively the curve complex and pants complex of a compact surface S, possibly with boundary, with the usual modifications for surfaces of small complexity. (See [MM00] for details.)

Split level surfaces. For our purposes, a pants decomposition of S will be a disjoint collection of 3-holed spheres $P_1, \dots, P_n$ embedded in S such that $S \setminus \bigcup_i P_i$ is a disjoint collection of nonperipheral annuli in S, no two of which are homotopic. We shall conflate a pants decomposition of S with the collection of (isotopy classes of) *nonperipheral* boundary curves of $P_1, \dots, P_n$. Thus when we refer to a pair of pants in a pants decomposition $P_1, \dots, P_n$ of S, we are referring to one of the P_i's, and when we refer to a curve in a pants decomposition of S, we are referring to one of the nonperipheral boundary curves of one of the P_i's.

Let N be the convex core of a simply or doubly degenerate hyperbolic 3-manifold minus an open neighborhood of the cusp(s). N is homeomorphic to $S \times [0, \infty)$ or $S \times \mathbb{R}$ according as N is simply or doubly degenerate, where S is a compact surface, possibly with boundary.

Let θ, ω be positive real numbers. A neighborhood $N_\varepsilon(\gamma)$ of a closed geodesic $\gamma (\subset N)$ is called a (θ, ω)-thin tube if the length of γ is less than θ and the length of the shortest geodesic on $\partial N_\varepsilon(\gamma)$ is greater than ω.

Let $\mathcal{T}$ denote a collection of disjoint, uniformly separated (θ, ω)-thin tubes in N such that all Margulis tubes in N belong to $\mathcal{T}$; in particular, θ is greater than the Margulis constant. Let M be a 3-manifold bi-Lipschitz homeomorphic to N and let $M(0)$ be the image of $N \setminus \bigcup_{T \in \mathcal{T}} \mathrm{Int}(T)$ in M under the bi-Lipschitz homeomorphism F. Let $\partial M(0)$ (resp. ∂M) denote the boundary of $M(0)$ (resp. M).

Let $(Q, \partial Q)$ be the unique hyperbolic pair of pants such that each component of ∂Q has length one. Q will be called the *standard* pair of pants. An isometrically embedded copy of $(Q, \partial Q)$ in $(M(0), \partial M(0))$ will be said to be *flat*.

Definition 3.1. A *split level surface* associated to a pants decomposition $\{Q_1, \dots, Q_n\}$ of S in $M(0) \subset M$ is an embedding $f : \cup_i (Q_i, \partial Q_i) \to (M(0), \partial M(0))$ such that

(1) each $f(Q_i, \partial Q_i)$ is flat,
(2) f extends to an embedding (also denoted f) of S into M such that the interior of each annulus component of $f(S \setminus \bigcup_i Q_i)$ lies entirely in $F(\bigcup_{T \in \mathcal{T}} \mathrm{Int}(T))$.

The class of *all* topological embeddings from S to M that agree with a split level surface f associated to a pants decomposition $\{Q_1, \ldots, Q_n\}$ on $Q_1 \cup \cdots \cup Q_n$ will be denoted by $[f]$.

We define a partial order $\leq_E$ on the collection of split level surfaces in an end E of M as follows: $f_1 \leq_E f_2$ if there exist $g_i \in [f_i]$, $i = 1, 2$ such that $g_2(S)$ lies in the unbounded component of $E \setminus g_1(S)$.

Tight geodesics. The *complexity* of a compact surface $S = S_{g,b}$ of genus g and b boundary components is defined to be $\xi(S_{g,b}) = 3g + b$.

For any simplex $\alpha \in \mathcal{C}(Y)$, γ_α will denote a collection of disjoint simple closed curves on S representing the (homotopy classes) of vertices of α. A pair of simplices α, β in $\mathcal{C}(Y)$ are said to *fill* an essential subsurface Y of S if all nontrivial nonperipheral curves in Y have essential intersection with at least one of γ_α or γ_β, where we assume that representatives γ_α and γ_β have been chosen to intersect each other minimally.

Given arbitrary simplices α, β in $\mathrm{CC}(S)$, form a regular neighborhood of $\gamma_\alpha \cup \gamma_\beta$ and fill in all disks and one-holed disks to obtain Y, which is said to be *filled* by α, β.

For a subsurface $X \subseteq Z$, let $\partial_Z(X)$ denote the *relative boundary* of X in Z, i.e., those boundary components of X that are nonperipheral in Z.

Definition 3.2. Let Y be an essential subsurface in S. If $\xi(Y) > 4$, a sequence of simplices $\{v_i\}_{i \in \mathcal{I}} \subset \mathcal{C}(Y)$ (where $\mathcal{I}$ is a finite or infinite interval in $\mathbb{Z}$) is called *tight* if

(1) for any vertices w_i of v_i and w_j of v_j where $i \neq j$, $d_{\mathcal{C}_1(Y)}(w_i, w_j) = |i-j|$;
(2) whenever $\{i-1, i, i+1\} \subset \mathcal{I}$, v_i represents the relative boundary $\partial_Y F(v_{i-1}, v_{i+1})$.

If $\xi(Y) = 4$, then a tight sequence is the vertex sequence of a geodesic in $\mathcal{C}(Y)$. A *tight geodesic* g in $\mathcal{C}(Y)$ consists of a tight sequence $v_0, \ldots, v_n$, and two simplices in $\mathcal{C}(Y)$, $\mathbf{I} = \mathbf{I}(g)$ and $\mathbf{T} = \mathbf{T}(g)$, called its initial and terminal markings such that v_0 (resp. v_n) is a sub-simplex of $\mathbf{I}$ (resp. $\mathbf{T}$). The length of g is n. v_i is called a simplex of g. Y is called the *domain or support* of g and is denoted as $Y = D(g)$. g is said to be supported in $D(g)$.

We denote the obvious linear order in g as $v_i < v_j$ whenever $i < j$. A geodesic supported in Y with $\xi(Y) = 4$ is called a 4-geodesic.

Given a surface W with $\xi(W) \geq 4$ and a simplex v in $\mathcal{C}(W)$ we say that Y is a *component domain of* (W, v) if Y is a component of $W \setminus \mathbf{collar}(v)$, where $\mathbf{collar}(v)$ denotes a thin collar neighborhood of the simple closed curves.

If g is a tight geodesic with domain $D(g)$, we call $Y \subset S$ a *component domain of* g if for some simplex v_j of g, Y is a component domain of $(D(g), v_j)$.

Hierarchies. The next definition is based on [MM00], describing certain special paths in $\mathcal{P}(S)$ and component domains associated to them. Paths in $\mathcal{P}(S)$ will be maps h from intervals I in $\mathbb{Z}$ into $\mathcal{P}(S)$ such that $h(i), h(i+1)$ are adjacent vertices of $\mathcal{P}(S)$ for all $i, i+1 \in I$. We reverse the logic of the exposition in [MM00] slightly here by defining a hierarchy path in $\mathcal{P}(S)$ first and then associating a hierarchy of tight geodesics to it.

Definition 3.3. A *hierarchy path* in $\mathcal{P}(S)$ joining pants decompositions P_1 and P_2 is a path $\rho : [0, n] \to P(S)$ joining $\rho(0) = P_1$ to $\rho(n) = P_2$ such that

(1) There is a collection $\{Y\}$ of essential, nonannular subsurfaces of S, called component domains for ρ such that for each component domain Y, there is a connected interval $J_Y \subset [0, n]$ with $\partial Y \subset \rho(j)$ for each $j \in J_Y$.
(2) For a component domain Y, there exists a tight geodesic g_Y supported in Y such that for each $j \in J_Y$, there is an $\alpha \in g_Y$ with $\alpha \in \rho(j)$.

A *hierarchy path* in $\mathcal{P}(S)$ is a sequence $\{P_n\}_n$ of pants decompositions of S such that for any $P_i, P_j \in \{P_n\}_n$, $i \leq j$, the finite sequence $P_i, P_{i+1}, \ldots, P_{j-1}, P_j$ is a hierarchy path joining pants decompositions P_i and P_j. The collection H of tight geodesics g_Y supported in component domains Y of ρ will be called the hierarchy of tight geodesics associated to ρ.

The notion of hierarchy in Definition 3.3 is a special case of "hierarchies without annuli" described in [MM00]. The next definition allows us to associate the extra piece of data coming from tight geodesics supported in component domains of a hierarchy path ρ to the hierarchy path ρ.

Definition 3.4. A *slice* of a hierarchy H associated to a hierarchy path ρ is a set τ of pairs (h, v), where $h \in H$ and v is a simplex of h, satisfying the following properties:

(1) A geodesic h appears in at most one pair in τ.
(2) There is a distinguished pair (h_τ, v_τ) in τ, called the bottom pair of τ. We call h_τ the bottom geodesic.
(3) For every $(k, w) \in \tau$ other than the bottom pair, $D(k)$ is a component domain of $(D(h), v)$ for some $(h, v) \in \tau$.

A resolution of a hierarchy H associated to a hierarchy path $\rho : I \to \mathcal{P}(S)$ is a sequence of slices $\tau_i = \{(h_{i1}, v_{i1}), (h_{i2}, v_{i2}), \ldots, (h_{in_i}, v_{in_i})\}$ (for $i \in I$, the same indexing set) such that the set of vertices of the simplices $\{v_{i1}, v_{i2}, \ldots, v_{in_i}\}$ is the same as the set of the nonperipheral boundary curves of the pairs of pants in $\rho(i) \in \mathcal{P}(S)$.

Minsky Blocks ([Min10, §8.1]). A tight geodesic in H supported in a component domain of complexity 4 is called a 4-geodesic, and an edge of a 4-geodesic in H is called a 4-edge.

Given a 4-edge e in H, let g be the 4-geodesic containing it, and let $D(e)$ be the domain $D(g)$. Let e^- and e^+ denote the initial and terminal vertices of e. As usual, let $\mathbf{collar}(v)$ denote a small collar neighborhood of v in $D(e)$.

To each e, a Minsky block $B(e)$ is assigned as as follows:

$$B(e) = (D(e) \times [-1,1]) \setminus (\mathbf{collar}(e^-) \times [-1,-1/2) \cup \mathbf{collar}(e^+) \times (1/2,1]).$$

That is, $B(e)$ is the product $D(e) \times [-1,1]$, with solid-torus trenches dug out of its top and bottom boundaries, corresponding to the two vertices e^- and e^+ of e.

The *gluing boundary* of $B(e)$ is

$$\partial_\pm B(e) \equiv (D(e) \setminus \mathbf{collar}(e^\pm)) \times \{\pm 1\}.$$

The gluing boundary is a union of 3-holed spheres. The rest of the boundary is a union of annuli. The top (resp. bottom) gluing boundaries of $B(e)$ are $(D(e) \setminus \mathbf{collar}(e^+)) \times \{1\}$ (resp. $(D(e) \setminus \mathbf{collar}(e^-)) \times \{-1\}$.

The model and the bi-Lipschitz Model Theorem. The following theorem summarizes and paraphrases what we need in this paper from the bi-Lipschitz Model Theorem of Minsky [Min10] and Brock-Canary-Minsky [BCM12]. (In particular, see [Min10, Th. 8.1].)

THEOREM 3.5 ([Min10], [BCM12]). *Let N be the convex core of a simply or doubly degenerate hyperbolic 3-manifold minus an open neighborhood of the cusp (s). Let S be a compact surface, possibly with boundary, such that N is homeomorphic to $S \times [0,\infty)$ or $S \times \mathbb{R}$ according as N is simply or doubly degenerate. There exist $L \geq 1$, $\theta, \omega, \varepsilon, \varepsilon_1 > 0$, a collection $\mathcal{T}$ of (θ,ω)-thin tubes containing all Margulis tubes in N, a 3-manifold M and an L-bi-Lipschitz homeomorphism $F : N \to M$ such that the following holds.*

Let $M(0) = F(N \setminus \bigcup_{T\in\mathcal{T}} \mathrm{Int}(T))$, and let $F(\mathcal{T})$ denote the image of the collection $\mathcal{T}$ under F. Let $\leq_E$ denote the partial order on the collection of split level surfaces in an end E of M. Then there exists a sequence S_i of split level surfaces associated to pants decompositions P_i exiting E such that

(1) *$S_i \leq_E S_j$ if $i \leq j$.*
(2) *The sequence $\{P_i\}$ is a hierarchy path in $\mathcal{P}(S)$.*
(3) *If $P_i \cap P_j = \{Q_1, \cdots Q_l\}$, then $f_i(Q_k) = f_j(Q_k)$ for $k = 1 \cdots l$, where f_i, f_j are the embeddings defining the split level surfaces S_i, S_j respectively.*
(4) *For all i, $P_i \cap P_{i+1} = \{Q_{i,1}, \ldots, Q_{i,l}\}$ consists of a collection of l pairs of pants such that $S \setminus (Q_{i,1} \cup \cdots \cup Q_{i,l})$ has a single nonannular component of complexity 4. Further, there exists a Minsky block W_i and an isometric*

map G_i of W_i into $M(0)$ such that $f_i(S \setminus (Q_{i,1} \cup \cdots \cup Q_{i,l})$ (resp. $f_{i+1}(S \setminus (Q_{i,1} \cup \cdots \cup Q_{i,l}))$ is contained in the bottom (resp. top) gluing boundary of W_i.

(5) *For each flat pair of pants Q in a split level surface S_i, there exists an isometric embedding of $Q \times [-\varepsilon, \varepsilon]$ into $M(0)$ such that the embedding restricted to $Q \times \{0\}$ agrees with f_i restricted to Q.*

(6) *For each $T \in \mathcal{T}$, there exists a split level surface S_i associated to pants decompositions P_i such that the core curve of T is isotopic to a nonperipheral boundary curve of P_i. The boundary $F(\partial T)$ of $F(T)$ with the induced metric d_T from $M(0)$ is a Euclidean torus equipped with a product structure $S^1 \times S^1_v$, where any circle of the form $S^1 \times \{t\} \subset S^1 \times S^1_v$ is a round circle of unit length and is called a horizontal circle; and any circle of the form $\{t\} \times S^1_v$ is a round circle of length l_v and is called a vertical circle.*

(7) *Let g be a tight geodesic other than the bottom geodesic in the hierarchy H associated to the hierarchy path $\{P_i\}$, let $D(g)$ be the support of g and let v be a boundary curve of $D(g)$. Let T_v be the tube in $\mathcal{T}$ such that the core curve of T_v is isotopic to v. If a vertical circle of $(F(\partial T_v), d_{T_v})$ has length l_v less than $n\varepsilon_1$, then the length of g is less than n.*

Since the above statement is culled out of a large amount of material, particularly from [Min10], we give specific references here. $M(0)$ (resp. M) above is denoted by $M_\nu(0)$ (resp. M_ν) in Section 8 of [Min10].

The collection $F(\mathcal{T})$ is denoted by $\mathcal{U}$ in [Min10] and is called the set of tubes in M_ν. The hierarchy H in item (7) of Theorem 3.5 is constructed in Lemma 5.13 of [Min10] (see also [MM00, Th. 4.6]) and the hierarchy path of item (2) is obtained from it by constructing a *resolution* sweeping through it in Lemma 5.8 of [Min10]. (We have thus reversed the logical order of hierarchies and hierarchy paths in our treatment.)

The estimate on the length of g in item (7) of Theorem 3.5 comes from equation 9.6 of [Min10], which gives estimates on meridian coefficients.

The Euclidean structure of $F(T)$ for $T \in \mathcal{T}$ in item (6) comes from gluing together the internal blocks (as well as boundary blocks) described in Section 8.1 and in Theorem 8.1 of [Min10].

Theorem 8.1 of [Min10] further describes the construction of *split level surfaces* and items (1), (3) and (4) follow from it. Item (5) simply ensures the existence of uniform product neighborhoods and follows from the fact that Minsky blocks are glued by isometries on their 3-holed sphere boundary components. In fact, $\varepsilon = \frac{1}{4}$ suffices. Finally [BCM12] ensures that the model constructed in [Min10] is indeed bi-Lipschitz homeomorphic to N.

We use the notation of Theorem 3.5 in the rest of this subsection, fixing N, M.

LEMMA 3.6. *Given $l > 0$, there exists $n \in \mathbb{N}$ such that the following holds. Let v be a vertex in the hierarchy H such that the length of the core curve of the Margulis tube T_v corresponding to v is greater than l. Next suppose $(h, v) \in \tau_i$ for some slice τ_i of the hierarchy H such that h is supported on Y, and suppose D is a component of $Y \setminus$ **collar**(v). Also suppose that $h_1 \in H$ such that D is the support of h_1. Then the length of h_1 is at most n.*

Proof. Let α be a meridian curve on $F(\partial T_v)$ such that $F^{-1}(\alpha)$ bounds a totally geodesic disk in T_v. By item (6) of Theorem 3.5, $F(\partial T_v)$ is a metric product $S^1 \times S^1_v$. Choose horizontal and vertical curves α_h, α_v on $F(\partial T_v)$. Then α is homologous to $(n\alpha_h + \alpha_v)$ for some integer n. Hence $l(\alpha) \geq l_v$, where $l(\alpha)$ is the length of α and l_v denotes the length of the vertical circle. Since F is an L-bi-Lipschitz homeomorphism by Theorem 3.5, it follows that $l(F^{-1}(\alpha)) \geq \frac{l_v}{L}$. Let Δ be the totally geodesic disk bounded by $F^{-1}(\alpha)$. Then the radius r_v of Δ is bounded below by $\sin h^{-1}(\frac{l_v}{L})$. Let l_c denote the length of the core curve c_v of T_v. Then any geodesic on ∂T_v homotopic to c_v in T_v has length bounded below by $\frac{l_v}{L} l_c$.

Also $l(F^{-1}(\alpha_h)) \leq L$ and $F^{-1}(\alpha_h)$ is homotopic to the core curve c_v. Hence $\frac{l_v}{L} l_c \leq L$. It follows that $l_v \leq \frac{L^2}{l_c} \leq \frac{L^2}{l}$. The lemma now follows from item (7) of Theorem 3.5. ☐

One last fallout of the Minsky model ([Min10, Th. 8.1] again) that we shall need is the following.

LEMMA 3.7. *Given $l > 0$ and $n \in \mathbb{N}$, there exists $L_2 \geq 1$ such that the following holds. Let S_i, S_j $(i < j)$ be split level surfaces associated to pants decompositions P_i, P_j such that*

(a) $(j - i) \leq n$;
(b) *$P_i \cap P_j$ is a (possibly empty) pants decomposition of $S \setminus W$, where W is an essential (possibly disconnected) subsurface of S such that each component W_k of W has complexity $\xi(W_k) \geq 4$;*
(c) *for any k with $i < k < j$, and $(g_D, v) \in \tau_k$ for $D \subset W_i$ for some i, no curve in v has a geodesic realization in N of length less than l.*

Then there exists an L_2-bi-Lipschitz embedding $G : W \times [-1, 1] \to M$ such that

(1) *W admits a hyperbolic metric given by $W = Q_1 \cup \cdots \cup Q_m$, where each Q_i is a flat pair of pants.*
(2) *$W \times [-1, 1]$ is given the product metric.*
(3) *$f_i(P_i \setminus P_i \cap P_j) \subset W \times \{-1\}$ and $f_j(P_j \setminus P_i \cap P_i) \subset W \times \{1\}$.*

Idea of proof. What Lemma 3.7 says is that if a "thick" piece of the manifold N is trapped between split level surfaces S_i, S_j, then it is bi-Lipschitz to a product region on the support of the hierarchy path between S_i, S_j. This follows from the construction of the model in Section 8.2 of [Min10] along with equation 9.6 of [Min10]. The lower bound on lengths of hierarchy curves in hypothesis (c) ensures an *upper bound* on the twist coefficient ($[h_v]$ in [Min10, eq. 9.6]) exactly as in the proof of Lemma 3.6. Hence the "full hierarchy" path (including annuli in the sense of [MM00]) between S_i, S_j equipped with markings is of length bounded in terms of l, n. This guarantees the existence of a bi-Lipschitz product region as required. Since this is the only place where we shall require full hierarchies and twist coefficients in this paper, and since the rest of the proof of Lemma 3.7 follows Lemma 3.6, we omit the details, referring the interested reader to Section 8.2 of [Min10]. (See also [MM00], where a quasi-isometry is constructed between the mapping class group and the full marking complex. Interpreted in these latter terms there is a bounded length element in the mapping class group $MCG(W)$ taking the marking on $S_i \cap W$ to the marking on $S_j \cap W$.) □

4. **Split geometry**

4.1. *Constructing split level surfaces.* The aim of this subsection is to extract a special sequence of split level surfaces from the sequence of split level surfaces constructed in Theorem 3.5. The main point is to ensure that successive split level surfaces are separated by a definite amount in $M(0)$. We continue with the notation of Theorem 3.5 in this subsection.

Fix an $l > 0$. The precise value of l will be less than the Margulis constant for hyperbolic 3-manifolds and will be determined by the Drilling Theorem 4.21 to be used in the next subsection. We shall henceforth refer to Margulis tubes that have core curve of length $\leq l$ as *thin Margulis tubes* and the corresponding vertex v as a *thin vertex*.

For convenience, start with a doubly degenerate surface group. Let $\rho(i) = \{P_i\}$ be a hierarchy path provided by item (2) of Theorem 3.5. Let H be the hierarchy of tight geodesics associated to $\{P_i\}$ and $\ldots, \tau_{i-1}, \tau_i, \tau_{i+1}, \ldots$ be a resolution. Let S_i be the split level surface corresponding to P_i, and let τ_i be the slice whose vertices comprise the curves in P_i. Let S_i^s denote the collection of flat pairs of pants occurring in the image of S_i in $M(0)$. The metric on the model manifold and the induced path metric on $M(0)$ will be denoted by d_M and will be called the *model metric*. Thus S_i is an embedding and S_i^s is the image in $M(0)$ of a collection of pairs of pants.

Definition 4.1. A curve v in H is *l-thin* if the core curve of the Margulis tube T_v has length less than or equal to l. A curve v is said to split a pair of split level surfaces S_i and S_j $(i < j)$ if v occurs as a vertex in both τ_i and τ_{j-1}. A pair of split level surfaces S_i and S_j $(i < j)$ is said to be an *l-thin pair* if there exists an l-thin curve v such that v occurs as a vertex in both τ_i and

τ_{j-1}. A pair of split level surfaces S_i and S_j $(i < j)$ is said to be an *l-thin pair on a component domain D* if

(a) $P_i \cap P_j$ is a pants decomposition of $S \setminus D$, none of whose curves are l-thin.
(b) There exists a tight geodesic $g_D \in H$ supported on D such that $(g_D, u) \in \tau_k$ for all $i < k < j$, where the multicurve u contains an l-thin curve. (Here D could be S itself.) Further we demand that the initial and final vertices of g_D consist of curves contained in (the boundary curves of) P_i, P_j respectively.

A pair of split level surfaces S_i and S_j $(i < j)$ is said to be an *l-thick pair* (or an l-thick pair on S) if no curve $v \in \tau_k$ is l-thin for $i < k < j$.

In fact in criterion (b) of the definition of an l-thin pair on a component domain D, we might as well have assumed that the initial and final markings of g_D, $I(g_D)$ and $T(g_D)$ respectively, are precisely P_i, P_j. This is the case when the markings are *complete* in the sense of [Min10].

By Definition 3.3(1), the set $J(v) = \{i : v \in \rho(i)\}$ is an interval. Consider the family of intervals $\{J(v) : v \in g_H\}$, where g_H is the distinguished main geodesic (bottom geodesic) for the hierarchy H. Then $\bigcup_v \{J(v) : v \in g_H\} = \mathbb{Z}$. This follows from the fact that each τ_i has a simple closed curve corresponding to some vertex in g_H.

Any pair v_i, v_{i+1} of simplices (multicurves) that form successive vertices of the base geodesic g_H are at a distance of 1 from each other by tightness of g_H.

Selecting split level surfaces. We shall now construct a subset $\mathbb{I}$ of $\mathbb{Z}$ by selecting a subsequence of the elements $\{P_i\}$ of the hierarchy path. Let τ_{m_i} be the first slice in the resolution such that $(g_H, v_i) \in \tau_{m_i}$. Let $\mathbb{I}_1 = \{m_i : i \in \mathbb{Z}\}$. We shall now expand the set $\mathbb{I}_1$, if necessary, as follows.

If some curve in v_i is l-thin, then we declare that $[m_i, m_{i+1}] \cap \mathbb{I} = \{m_i, m_{i+1}\}$; i.e., no integer strictly between m_i, m_{i+1} is added to $\mathbb{I}_1$. More generally, for any $j \in \mathbb{Z} \setminus \mathbb{I}_1$, choose i such that $m_i < j < m_{i+1}$. Then $j \in \mathbb{I}_2$ if and only if there exists k

(a) either with $j < k \leq m_{i+1}$ such that S_j, S_k form an l-thin pair on some component domain D,
(b) or with $k < j \leq m_{i+1}$ such that S_k, S_j form an l-thin pair on some component domain D.

Finally, set $\mathbb{I} = \mathbb{I}_1 \cup \mathbb{I}_2$. Then $\mathbb{I} = \{\cdots, n_{i-1}, n_i, n_{i+1}, \cdots\}$ inherits a linear order from $\mathbb{Z}$ such that $j < k$ implies $n_j < n_k$. Note that the same construction works for simply degenerate groups if we replace $\mathbb{Z}$ by $\mathbb{N}$.

The next few propositions identify some of the features of the selection $\mathbb{I}$. The main point is to show that the sequence of split level surfaces S_{n_i} makes definite progress out an end.

Proposition 4.2. *Let* $\mathbb{I} = \{\cdots, n_{i-1}, n_i, n_{i+1}, \cdots\}$ *be as above. There exists a positive integer* N_0 *such that for all* i,

(a) *either* $(S_{n_i}, S_{n_{i+1}})$ *is an* l*-thin pair on some component domain* D,
(b) *or* $(S_{n_i}, S_{n_{i+1}})$ *is an* l*-thick pair and* $n_{i+1} - n_i \leq N_0$.

Proof. Suppose that $(S_{n_i}, S_{n_{i+1}})$ is not an l-thin pair on some component domain. Then, by the construction of $\mathbb{I}$ and Lemma 3.6, there exists $N_1(= N_1(l))$ such that for all k with $n_i < k < n_{i+1}$, if $\tau_k = \{(h_{k1}, v_{k1}), (h_{k2}, v_{k2}), \ldots, (h_{km_k}, v_{km_k})\}$, then the length of the tight geodesic h_{ki} satisfies $l(h_{ki}) \leq N_1$. Further, none of the curves in v_{ki} are l-thin, ensuring l-thickness of the pair $(S_{n_i}, S_{n_{i+1}})$.

Note that $m_k \leq \xi(S)$, where $\xi(S)$ is the complexity of S. Also the number of component domains in $W \setminus \mathbf{collar}(v)$ for $W = D(h_{ki})$ is certainly bounded above by the number of pairs of pants in a pants decomposition of S and hence by $\xi(S)$. Therefore $(n_{i+1} - n_i)$ is bounded above by $N_1^{\xi(S)} \times \cdots \times N_1^{\xi(S)}$ ($\xi(S)$ times). Choosing $N_0 = N_1^{\xi(S)^2}$, we are done. $\square$

The next proposition asserts that between two successive split level surfaces S_{m_i} and $S_{m_{i+1}}$ selected from the base geodesic, our selection process "interpolates" a uniformly bounded number of new split level surfaces. Equivalently, the cardinality of the set $(\mathbb{I}_2 \cap [m_i, m_{i+1}])$ is uniformly bounded.

Proposition 4.3. *Let*

$$\mathbb{I} = \{\cdots, n_{i-1}, n_i, n_{i+1}, \cdots\}$$

and

$$\mathbb{I}_1 = \{\cdots, m_{i-1}, m_i, m_{i+1}, \cdots\}$$

be as above. There exists a positive integer N_2 *such that for all* i, *if* $n_j = m_i$ *and* $n_k = m_{i+1}$, *then* $k - j \leq N_2$.

Proof. Let $k \in \mathbb{I}$. So S_k is a split level surface interpolated between S_{m_i} and $S_{m_{i+1}}$ for some k with $m_i < k < m_{i+1}$. Let the corresponding slice $\tau_k = \{(h_{k1}, v_{k1}), (h_{k2}, v_{k2}), \ldots, (h_{km_k}, v_{km_k})\}$. Then there exists a unique "subslice" $\tau_k^0 = \{(h_{k1}, v_{k1}), (h_{k2}, v_{k2}), \ldots, (h_{kr_k}, v_{kr_k})\}$, with $r_k \leq m_k$ such that the length of the tight geodesic h_{ki} satisfies $l(h_{ki}) \leq N_1$ for all $i \leq r_k$ and S_k is a split level surface.

Since the total number of such choices is bounded above by N_0 by the proof of Proposition 4.2, and for each such choice at most two (by the construction of $\mathbb{I}_2$ above) split level surfaces are introduced, it follows that the total number of l-thin split level surfaces S_k with $m_i < k < m_{i+1}$ is bounded above by $2N_0 = 2N_1^{\xi(S)^2}$. Choosing $N_2 = 2N_0$, we are done. $\square$

LEMMA 4.4. *There exists n such that each thin curve splits at most n split level surfaces in the sequence* $\{S_{n_i} : i \in \mathbb{I}\}$.

Proof. Since for any i, the number of split level surfaces S_{n_i} between S_{m_i} and $S_{m_{i+1}}$ ($m_i, m_{i+1} \in \mathbb{I}_1$) is at most N_2 by Proposition 4.3, it suffices to prove that any thin curve splits a uniformly bounded number of S_{m_i}'s. If a curve v splits both S_{m_i} and S_{m_j}, then v belongs to both the pants decomposition P_{m_i} and $P_{m_j - 1}$.

Suppose $(g_S, v_{m_k}) \in \tau_{m_k}$ for $k = i, j$, where g_S denotes the bottom geodesic of the hierarchy H. Then the distance between v_{m_i} and v_{m_j} in $\mathcal{C}(S)$ is at most 3 by tightness; i.e., $|i - j| \leq 3$. Taking $n = 3N_2$, we are through. $\square$

Pushing split level surfaces apart. We shall now use item (5) of Theorem 3.5 to "thicken" each of the S_{n_i}'s if necessary, so that successive split level surfaces can be arranged to be uniformly separated. Recall that S_i^s is the collection of flat embedded pairs of pants in $M(0)$ corresponding to the split level surface S_i.

Definition 4.5. A pair of split level surfaces S_i and S_j ($i < j$) is said to be *k-separated* if

(a) for all $x \in S_i^s$, $d(x, S_j^s) \geq k$;
(b) similarly, for all $x \in S_j^s$, $d(x, S_i^s) \geq k$.

LEMMA 4.6. *Let* $\mathbb{I} = \{\cdots, n_{i-1}, n_i, n_{i+1}, \cdots\}$ *be as above. There exist* $k_0 > 0$ *and a sequence of split level surfaces* Σ_i *and a positive integer* N_0 *such that for all* i, (Σ_i, Σ_{i+1}) *is* k_0*-separated and*

(a) *either* (Σ_i, Σ_{i+1}) *is an l-thin pair on some component domain* D,
(b) *or* (Σ_i, Σ_{i+1}) *is an l-thick pair and* $n_{i+1} - n_i \leq N_0$.

Proof. By Proposition 4.2, the sequence $\{S_{n_i}\}_i$ satisfies one of the alternatives (a) or (b). It remains to modify $\{S_{n_i}\}_i$ such that $(S_{n_i}, S_{n_{i+1}})$ are k_0-separated for some $k_0 > 0$ and all i.

By item (5) of Theorem 3.5, there exists $\varepsilon > 0$ such that for all flat pairs of pants Q_i in $S_{n_i}^s$, there exists an isometric embedding $H_i : Q_i \times [-\varepsilon, \varepsilon]$ into $M(0)$. Also, by Lemma 4.4, there exists $n \in \mathbb{N}$ such that if $Q_k \in P_{n_i} \cap P_{n_j}$, then $|i - j| \leq n$. Further, the collection $\{i \in \mathbb{I} : Q_k \in P_{n_i}\} = \mathbb{I}_{Q_k}$ is an interval in $\mathbb{Z}$. Let $\varepsilon_2 = \frac{\varepsilon}{n}$. If $\mathbb{I}_{Q_k} = [a_k, b_k] \subset \mathbb{Z}$, define $Q_{ks} = H_k|_{Q_k \times s\varepsilon_2}$. Note that $(b_k - a_k) \leq n$.

For each embedding f_{n_i} defining the split level surface S_{n_i}, and $Q_k \in P_{n_i} \cap P_{n_j}$ for some $j \neq i$, let $\mathbb{I}_{Q_k} = [a_k, b_k]$ and $s = (n_i - a_k)$. Then define $f'_{n_i}|_{Q_k} = H_k|_{Q_k \times s\varepsilon_2}$.

Now, let Σ_i be the split level surface defined by $f'_{n_i}|_{Q_k}$, whenever $Q_k \in P_{n_i}$. Choosing $k_0 = \varepsilon_2$, it follows that successive split level surfaces are k_0-separated. $\square$

Let $\mathcal{T}_l$ denote the collection of tubes in $\mathcal{T}$ whose core curves have length less than l. Also let $M(l) = M(0) \bigcup_{T \in \mathcal{T} \setminus \mathcal{T}_l} F(T)$ denote the union of $M(0)$ and all l-thick tubes.

Definition 4.7. An L-bi-Lipschitz *split surface* in $M(l)$ associated to a pants decomposition $\{Q_1, \ldots, Q_n\}$ of S and a collection $\{A_1, \ldots, A_m\}$ of complementary annuli in S is an embedding $f : \cup_i Q_i \bigcup \cup_i A_i \to M(l)$ such that

(1) the restriction $f : \cup_i (Q_i, \partial Q_i) \to (M(0), \partial M(0))$ is a split level surface,
(2) the restriction $f : A_i \to M(l)$ is an L-bi-Lipschitz embedding,
(3) f extends to an embedding (also denoted f) of S into M such that the interior of each annulus component of $f(S \setminus (\cup_i Q_i \bigcup \cup_i A_i))$ lies entirely in $F(\bigcup_{T \in \mathcal{T}_l} \mathrm{Int}(T))$.

Note. The difference between a split level surface and a split surface is that the latter may contain bi-Lipschitz annuli in addition to flat pairs of pants.

Let Σ_i^s denote the union of the collection of flat pairs of pants and bi-Lipschitz annuli in the image of the embedding Σ_i.

THEOREM 4.8. *Let $N, M, M(0), S, F$ be as in Theorem* 3.5 *and E an end of M. For any l less than the Margulis constant, let $M(l) = \{F(x) : \mathrm{injrad_x}(N) \geq l\}$. Fix a hyperbolic metric on S such that each component of ∂S is totally geodesic of length one.* (*This is a normalization condition.*) *There exist $L_1 \geq 1$, $\varepsilon_1 > 0$, $n \in \mathbb{N}$, and a sequence Σ_i of L_1-bi-Lipschitz, ε_1-separated split surfaces exiting the end E of M such that for all i, one of the following occurs*:

(1) *An l-thin curve splits the pair (Σ_i, Σ_{i+1}); i.e., the associated split level surfaces form an l-thin pair.*
(2) *There exists an L_1-bi-Lipschitz embedding*

$$G_i : (S \times [0,1], (\partial S) \times [0,1]) \to (M, \partial M)$$

such that $\Sigma_i^s = G_i(S \times \{0\})$ and $\Sigma_{i+1}^s = G_i(S \times \{1\})$.

Finally, each l-thin curve in S splits at most n split level surfaces in the sequence $\{\Sigma_i\}$.

Proof. By Lemma 4.6, there exist $k > 0$, a positive integer N_0 and a sequence of *split level surfaces* Σ_i^0 such that for all i, $(\Sigma_i^0, \Sigma_{i+1}^0)$ is k-separated and

(a) either $(\Sigma_i^0, \Sigma_{i+1}^0)$ is an l-thin pair on some component domain D,
(b) or $(\Sigma_i^0, \Sigma_{i+1}^0)$ is an l-thick pair and $n_{i+1} - n_i \leq N_0$.

(We add the superscript 0 to indicate that we are still dealing with split level surfaces and not split surfaces.)

In case (a), there exists an l-thin curve splitting the pair of split level surfaces $(\Sigma_i^0, \Sigma_{i+1}^0)$. In case (b), let $P_{n_i}, P_{n_{i+1}}$ be the pants decompositions associated to $\Sigma_i^0, \Sigma_{i+1}^0$ and let $P_{n_i} \cap P_{n_{i+1}}$ be a (possibly empty) pants decomposition of $S \setminus W$, where W is an essential (possibly disconnected) subsurface of S such that each component W_k of W has complexity $\xi(W_k) \geq 4$. Hence by Lemma 3.7, there exist $L_2 \geq 1$ and an L_2-bi-Lipschitz embedding $G : W \times [-1, 1] \to M$ such that

(1) W admits a hyperbolic metric given by $W = Q_1 \cup \cdots \cup Q_m$, where each Q_i is a flat pair of pants;
(2) $W \times [-1, 1]$ is given the product metric;
(3) $f_{n_i}(P_{n_i} \setminus P_{n_i} \cap P_{n_{i+1}}) \subset W \times \{-1\}$ and $f_{n_{i+1}}(P_{n_{i+1}} \setminus P_{n_i} \cap P_i) \subset W \times \{1\}$.

Also, from the proof of Lemma 4.6, there exists $\varepsilon > 0$ such that for all i, there exists an isometric embedding $H_{n_i} : (P_{n_i} \cap P_{n_{i+1}}) \times [0, \varepsilon] \to M$ such that $H_{n_i}(P_{n_i} \cap P_{n_{i+1}}) \times \{0\} \subset f_{n_i}(P_{n_i})$ and $H_{n_i}(P_{n_{i+1}} \cap P_{n_{i+1}}) \times \{\varepsilon\} \subset f_{n_{i+1}}(P_{n_{i+1}})$.

Finally since $(\Sigma_i^0, \Sigma_{i+1}^0)$ is an l-thick pair, there exist standard annuli $A_1, \ldots, A_p$, $L_3 = L_3(l) \geq 1$, $\varepsilon_1 > 0$ and L_3-bi-Lipschitz embeddings $\Gamma_j : A_j \times [-1, 1] \to \bigcup_{T \in \mathcal{T} \setminus \mathcal{T}_l} F(\mathrm{Int}(T))$ such that

(a) $S = \cup_k P_k \bigcup \cup_j A_j$ is the union of the pairs of pants above along with the annuli A_j,
(b) f_{n_i} restricted to A_j agrees with Γ_j restricted to $A_j \times \{-1\}$,
(c) $f_{n_{i+1}}$ restricted to A_j agrees with Γ_j restricted to $A_j \times \{1\}$.

Pasting these maps,

(i) $G : W \times [-1, 1] \to M$,
(ii) $f_{n_i}(P_{n_i} \setminus P_{n_i} \cap P_{n_{i+1}}) \subset W \times \{-1\}$,
(iii) $H_{n_i} : (P_{n_i} \cap P_{n_{i+1}}) \times [0, \varepsilon] \to M$, and
(iv) $\Gamma_j : A_j \times [-1, 1] \to \bigcup_{T \in \mathcal{T} \setminus \mathcal{T}_l} F(\mathrm{Int}(T))$ along the common boundaries,

we obtain an L_1-bi-Lipschitz embedding

$$G_i : (S \times [0, 1], (\partial S) \times [0, 1]) \to (M, \partial M)$$

such that the *split surfaces* $\Sigma_i^s = G_i(S \times \{0\})$ and $\Sigma_{i+1}^s = G_i(S \times \{1\})$. Lemma 4.4 now proves the final assertion. $\square$

Pairs of split surfaces satisfying alternative (1) of Theorem 4.8 will be called an l-thin pair of split surfaces (or simply a thin pair if l is understood). Similarly, pairs of split surfaces satisfying alternative (2) of Theorem 4.8 will be called an l-thick pair (or simply a thick pair) of split surfaces.

Remark 4.9. The notion of split surface could be made a bit more general. We might as well require a split surface to be a (uniformly) bi-Lipschitz embedding of a bounded geometry subsurface of S containing a pants decomposition. Theorem 4.8 then summarizes the consequences of the Minsky model that we shall need in this paper. We have thus constructed the following from the Minsky model:

(1) A sequence of split surfaces S^s_i exiting the end(s) of M, where M is marked with a homeomorphism to $S \times J$. (J is $\mathbb{R}$ or $[0, \infty)$ according as M is totally or simply degenerate.) $S^s_i \subset S \times \{i\}$.
(2) A collection of Margulis tubes $\mathcal{T}$ in N with image $F(\mathcal{T})$ in M (under the bi-Lipschitz homeomorphism between N and M). We refer to the elements of $F(\mathcal{T})$ also as Margulis tubes.
(3) For each complementary annulus of S^s_i with core σ, there is a Margulis tube $T \in \mathcal{T}$ whose core is freely homotopic to σ such that $F(T)$ intersects S^s_i at the boundary. (What this roughly means is that there is an $F(T)$ that contains the complementary annulus.) We say that $F(T)$ splits S^s_i.
(4) There exist constants $\varepsilon_0 > 0, K_0 > 1$ such that for all i, either there exists a Margulis tube splitting both S^s_i and S^s_{i+1}, or else $S_i (= S^s_i)$ and $S_{i+1} (= S^s_{i+1})$ have injectivity radius bounded below by ε_0 and bound a *thick block* B_i, where a thick block is defined to be a K_0-bi-Lipschitz homeomorphic image of $S \times I$.
(5) $F(T) \cap S^s_i$ is either empty or consists of a pair of boundary components of S^s_i that are parallel in S_i.
(6) There is a uniform upper bound $n = n(M)$ on the number of surfaces that $F(T)$ splits.

For easy reference later on, a model manifold satisfying conditions (1)–(6) above is said to have *weak split geometry.*

We have isolated the features of weak split geometry in Remark 4.9 so as to emphasize the point that it is possible to make a definition independent of the Minsky model and the hierarchy machinery.[1] This will be useful for easy referencing in [Mj07] and [MS13]. In fact, a strengthening of weak split geometry will be enough to guarantee the existence of Cannon-Thurston maps, as we shall see below.

4.2. *Split blocks.*

Definition 4.10. Let $(\Sigma^s_i, \Sigma^s_{i+1})$ be a thick pair of split surfaces in M. The closure of the bounded component of $M \setminus (\Sigma^s_i \cup \Sigma^s_{i+1})$ will be called a thick block.

[1]This was our original approach to the main theorem of this paper: Prove it for more and more general model geometries; e.g., bounded geometry [Mj10a], i-bounded geometry [Mj11], amalgamation geometry and split geometry [Mj05]. Finally, prove that the Minsky model satisfies split geometry [Mj06].

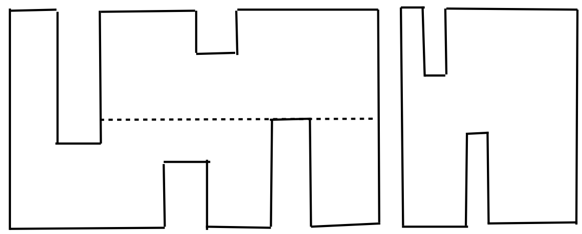

Figure 5. Split components of split block with hanging tubes.

Note that a thick block is uniformly bi-Lipschitz to the product $S \times [0,1]$ and that its boundary components are $\Sigma_i^s, \Sigma_{i+1}^s$.

Definition 4.11. Let $(\Sigma_i^s, \Sigma_{i+1}^s)$ be an l-thin pair of split surfaces in M and $F(\mathcal{T}_i)$ be the collection of l-thin Margulis tubes that split both $\Sigma_i^s, \Sigma_{i+1}^s$. The closure of the union of the bounded components of $M \setminus ((\Sigma_i^s \cup \Sigma_{i+1}^s) \bigcup_{T \in \mathcal{T}_i} F(T))$ will be called a split block. Equivalently, the closure of the union of the bounded components of $M(l) \setminus (\Sigma_i^s \cup \Sigma_{i+1}^s)$ is a split block.

Topologically, a split block B^s is a topological product $S^s \times I$ for some *not necessarily connected* S^s. However, its upper and lower boundaries need not be $S^s \times 1$ and $S^s \times 0$. We only require that the upper and lower boundaries be *split subsurfaces* of S^s. This is to allow for Margulis tubes starting (or ending) within the split block. Such tubes would split one of the horizontal boundaries but not both. We shall call such tubes *hanging tubes*. Connected components of split blocks are called *split components*. By l-thinness, there is a *nonempty* collection of l-thin Margulis tubes, called *splitting tubes*, splitting a split block. For each splitting tube $F(T)$ of a split block B^s, the intersection $(B^s \cap F(T)) \subset M$ is called the *vertical boundary* of the splitting tube. Note that the vertical boundary of a splitting tube is the union of two disjoint annuli.

See Figure 5, where the left split component has four hanging tubes and the right split component has two hanging tubes. The vertical space between the components is the place where an l-thin Margulis tube splits the split block into two split components.

Observe further that for each hanging tube $F(T)$, there exists a split surface S^s (marked with a dotted line in the figure) that intersects the boundary $F(\partial T)$ nontrivially and such that S^s contains an annulus whose core-curve is homotopic (in M) to the core curve of $F(T)$. Also, the closure of $(F(\partial T) \setminus S^s)$ consists of precisely two annuli called the *vertical boundary* of the hanging tube. We can assume further that

(a) $S^s \cap F(\partial T)$ is a bi-Lipschitz annulus called the *horizontal boundary* of $F(T)$ in the split block B^s,

(b) the union of the vertical and horizontal boundaries of an l-thin hanging tube $F(T)$ in B^s is precisely equal to $F(T) \cap B^s$.

Note that the whole manifold M is the union of

(a) thick blocks (bi-Lipschitz homeomorphic to $S \times I$),
(b) split blocks (homeomorphic to $S^s \times I$ for some split surfaces),
(c) l-thin Margulis tubes.

Also note that the union of thick and split blocks is $M(l)$, which is the complement (in M) of the union of l-thin Margulis tubes. Each of these Margulis tubes splits a uniformly bounded number of split blocks and might end in a hanging tube.

4.3. *Electrocutions.* For any hanging tube or splitting tube $F(T_j)$ in a split block B^s (with top and bottom split surfaces $\Sigma_k^s, \Sigma_{k+1}^s$, say), let $A_{ji} = S^1 \times [0, l_{ji}]$, $(i = 1, 2)$, be the vertical boundaries ($i = 1, 2$ correspond to the left and right vertical annuli in Figure 5). Let the metric product $S^1 \times [0, 1]$ be called the *standard annulus* for splitting tubes. For hanging tubes, the standard annulus will be $S^1 \times [0, 1/2]$.

We shall define a *welded split block* B_{wel} (homeomorphic to $S \times [0, 1]$) to be a split block with identifications on vertical boundaries of splitting tubes and hanging tubes. Let $H_{ji}^0 : [0, l_{ji}] \to [0, x]$ be the unique linear surjective map (scaling) taking 0 to 0 and l_{ji} to x, where x is 1 or 1/2 according as $F(T_j)$ is a splitting tube or a hanging tube. Now define $H_{ji} : S^1 \times [0, l_{ji}] \to S^1 \times [0, x]$ by $H_{ji}(y, z) = (y, H_{ji}^0(z))$. Finally, extend $H_{j1} \cup H_{j2}$ continuously to the horizontal boundaries $S^1 \times [-\varepsilon, \varepsilon]$ of hanging tubes $F(T_j)$ as Lipschitz maps to $S^1 \times \{p\}$ by $H_j(y, z) = (y, p)$, where p is either 0 or 1/2 according as the horizontal boundary of $F(T_j)$ lies at the bottom or the top of the hanging tube. (For instance, in Figure 5 the horizontal boundary marked with a dotted line lies at the top of a hanging tube.) Now glue the mapping cylinders of $H_{j1} \cup H_{j2} \cup H_j$ (for hanging tubes) and $H_{j1} \cup H_{j2}$ (for splitting tubes) to $F(\partial T_j) \cap B^s$ to obtain the *welded split block* B_{wel}. Note that B_{wel} is homeomorphic to $S \times [0, 1]$. The images of the standard annuli in B_{wel} after the identification shall simply be called *standard annuli in* B_{wel}.

For each hanging tube, there exists one distinguished curve on either Σ_k^s or Σ_{k+1}^s. When the hanging tube intersects Σ_{k+1}^s, this is the image of $S^1 \times \{1/2\}$ contained in the standard annulus after identification. Similarly, when the hanging tube intersects Σ_k^s, this is the image of $S^1 \times \{0\}$ contained in the standard annulus after identification. Again, for each splitting tube, there exist two distinguished curves, one each on Σ_k^s and Σ_{k+1}^s — the images of $S^1 \times \{0, 1\}$ contained in the standard annulus after identification. Such simple closed curves shall be called *weld curves.* The resulting metric on B_{wel} will be denoted by d_{wel}.

We shall equip B_{wel} with a new pseudometric. Equip the standard annulus $S^1 \times [0,x]$ (where x is 1 or 1/2) with the product of the zero metric on the S^1-factor and the Euclidean metric on the $[0,x]$ factor. Let $(S^1 \times [0,x], d_0)$ denote the resulting pseudometric. The *tube-electrocuted metric* d_{tel} is defined to be the pseudometric metric that agrees with d_{wel} away from the standard annuli in B_{wel} and with d_0 on the standard annuli in B_{wel}. To distinguish it from $(B_{\mathrm{wel}}, d_{\mathrm{wel}})$ we shall represent the new space and the pseudometric on it by $(B_{\mathrm{tel}}, d_{\mathrm{tel}})$. Note that the underlying topological spaces B_{wel} and B_{tel} are the same and homeomorphic to $S \times [0,1]$.

Recall that in defining thick blocks, S was equipped with a fixed hyperbolic metric. If Σ_k^s is the bottom split surface of the split block B_k^s and also the top split surface of a (thick or split) block block B_{k-1}^s, then the common split surface Σ_k^s can be easily extended over complementary annuli to a common uniformly bi-Lipschitz embedding of S into welded blocks $B_{\mathrm{wel},k}$ and $B_{\mathrm{wel},k-1}$, where we define $B_{\mathrm{wel},m} = B_m$ for thick blocks. When B_{k-1} is thick, this follows from the fact that the complementary annuli are uniformly bi-Lipschitz embeddings of $S^1 \times [0,1]$. When B_{k-1}^s is split, the mapping cylinder construction above restricted to Σ_k^s is the same whether Σ_k^s is regarded as the bottom split surface of B_k^s or the top split surface of B_{k-1}^s. We shall continue to denote the extended split surface by Σ_k^s and call it a split surface in $B_{\mathrm{wel},k}$. From now on, we shall drop the suffix wel from (thick or split) blocks $B_{\mathrm{wel},k}$ and denote them simply as B_k. Note that *all such extended split surfaces* are homeomorphic to S via uniformly bi-Lipschitz homeomorphisms.

The welded model manifold. Gluing successive welded blocks along common split surfaces, we obtain the *welded model manifold* $(M_{\mathrm{wel}}, d_{\mathrm{wel}})$ homeomorphic to $S \times J$, where $J = \mathbb{R}$ or $[0,\infty)$ according as the original manifold N is doubly or simply degenerate.

It remains to construct the tube-electrocuted pseudometric d_{tel} on M_{wel}. The tube electrocuted metrics on successive welded split blocks coincide on the common split surface. The same is clearly true if the successive blocks are thick.

If a weld curve lies in $\Sigma_k^s = B_{\mathrm{wel},k} \cap B_{\mathrm{wel},k-1}$ and precisely one of $B_{\mathrm{wel},k}$, $B_{\mathrm{wel},k-1}$ is a thick block, then we fix the convention that for the tube electrocuted metric d_{tel} on M_{wel}: *All weld curves have length zero.*

Gluing successive tube electrocuted blocks using the convention above, we obtain the *tube electrocuted manifold* $(M_{\mathrm{tel}}, d_{\mathrm{tel}})$. Observe that the underlying topological manifolds M_{wel} and M_{tel} are the same. (The notation $(M_{\mathrm{wel}}, d_{\mathrm{wel}})$ and $(M_{\mathrm{tel}}, d_{\mathrm{tel}})$ is used to distinguish the metrics.)

The union of the images of the contiguous mapping cylinders of maps $H_{j1} \cup H_{j2} \cup H_j$ (or $H_{j1} \cup H_{j2}$) in $(M_{\mathrm{tel}}, d_{\mathrm{tel}})$ associated to a particular l-thin Margulis tube T (and hence $F(T)$) is topologically a solid torus T^t. Equipped with the tube electrocuted metric, (T^t, d_{tel}) is of diameter at most n by Theorem 4.8. The collection of all T^t's in $(M_{\mathrm{tel}}, d_{\mathrm{tel}})$ is denoted $\mathcal{T}^t$. (We shall continue to use the same notation $\mathcal{T}^t$ for the collection of T^t's in (M_{wel}, d_G) to be defined below.)

The images of split components K of B^s in B_{tel} will continue to be called split components of B_{tel}. A lift $\widetilde{K}$ of a split component K of $(B_{\mathrm{tel}}, d_{\mathrm{tel}})$ to the universal cover $(\widetilde{B_{\mathrm{tel}}}, d_{\mathrm{tel}})$ shall be termed a *split component* of $\widetilde{B_{\mathrm{tel}}}$.

Let d_G be the (pseudo)-metric obtained by electrocuting the collection $\mathcal{K}$ of split components $\widetilde{K}$ in $(\widetilde{B_{\mathrm{tel}}}, d_{\mathrm{tel}}) \subset (\widetilde{M_{\mathrm{tel}}}, d_{\mathrm{tel}})$ as $(B_{\mathrm{tel}}, d_{\mathrm{tel}})$ ranges over all split blocks. d_G will be called the *graph metric* on $\widetilde{M_{\mathrm{tel}}}(= \widetilde{M_{\mathrm{wel}}})$. Thus $(\widetilde{M_{\mathrm{wel}}}, d_G)$ is isometric to $\mathcal{E}(\widetilde{M_{\mathrm{wel}}}, \mathcal{K})$ with the electric metric.

Remark 4.12 (alternate description). There is an alternate description of a pseudometric on $\widetilde{M}$ that makes it quasi-isometric to $(\widetilde{M_{\mathrm{wel}}}, d_G)$. For each lift $\widetilde{K} \subset \widetilde{M}$ of a split component K of a split block of $M(l) \subset M$, there are lifts of l-thin Margulis tubes that share the boundary of $\widetilde{K}$ in $\widetilde{M}$. Adjoining these lifts to $\widetilde{K}$ we obtain *extended split components*. Let $\mathcal{K}'$ denote the collection of extended split components in $\widetilde{M}$. We continue to denote the collection of split components in $\widetilde{M(l)} \subset \widetilde{M}$ by $\mathcal{K}$. Let $\widetilde{M(l)}$ denote the lift of $M(l)$ to $\widetilde{M}$. Then the inclusion of $\widetilde{M(l)}$ into $\widetilde{M}$ gives a quasi-isometry between $\mathcal{E}(\widetilde{M(l)}, \mathcal{K})$ and $\mathcal{E}(\widetilde{M}, \mathcal{K}')$ equipped with the respective electric metrics. This follows from the last assertion of Theorem 4.8.

Again, by the last assertion of Theorem 4.8, the inclusion of $\widetilde{M(l)}$ into $\widetilde{M_{\mathrm{wel}}}$ gives a quasi-isometry between $\mathcal{E}(\widetilde{M(l)}, \mathcal{K})$ and $\mathcal{E}(\widetilde{M_{\mathrm{wel}}}, \mathcal{K})(= (\widetilde{M_{\mathrm{wel}}}, d_G))$. Therefore $(\widetilde{M_{\mathrm{wel}}}, d_G)$ is quasi-isometric to $\mathcal{E}(\widetilde{M}, \mathcal{K}')$. We shall henceforth identify $\mathcal{E}(\widetilde{M}, \mathcal{K}')$ with $(\widetilde{M_{\mathrm{wel}}}, d_G)$ via this quasi-isometry without explicitly mentioning the quasi-isometry. The electric metric on $\mathcal{E}(\widetilde{M}, \mathcal{K}')$ shall therefore be denoted by d_G also. We shall find it easier to use $\mathcal{E}(\widetilde{M}, \mathcal{K}')$ when dealing with all of $\widetilde{M}$, whereas $(\widetilde{M_{\mathrm{wel}}}, d_G)$ will be more useful when dealing with the block structure of $\widetilde{M_{\mathrm{wel}}}$.

Remark 4.13. Here is the raison d'etre for the two closely related but different electric spaces. In the ladder construction of Section 5 below, it is important that a split surface goes "all the way across," i.e., is an embedded copy of S. There is no canonical way to do this in the model manifold M. In fact for the ladder construction of Section 5 to work, it is important that a split surface in $(M_{\mathrm{wel}}, d_{\mathrm{wel}})$ is an embedded copy of S having uniformly bounded geometry. This is simply not possible in M as Margulis tubes may be arbitrarily thin. On the other hand, we finally need to control hyperbolic geodesics in $\widetilde{N}$ by means of the ladder. Since M is bi-Lipschitz to N, we can equivalently control them in $\widetilde{M}$. The alternate description above establishes a way of transferring this control from $(\widetilde{M_{\mathrm{wel}}}, d_G)$ to $\widetilde{M}$, which is where we really want the control on geodesics.

The following definition illustrates this passing back and forth between these two quasi-isometric electric spaces.

Definition 4.14. Let $Y \subset \widetilde{N}$ and $X = F(Y)$. $X \subset \widetilde{M}$ is said to be Δ-graph quasiconvex if for any hyperbolic geodesic μ joining $a, b \in Y$, $F(\mu)$ lies inside $N_\Delta(X, d_G) \subset \mathcal{E}(\widetilde{M}, \mathcal{K}')$.

For X a split component, define $\mathrm{CH}(X) = F(\mathrm{CH}(Y))$, where $\mathrm{CH}(Y)$ is the convex hull of Y in $\widetilde{N}$. Then Δ-graph-quasiconvexity of X is equivalent to the condition that $\mathrm{dia}_G(\mathrm{CH}(X))$ is bounded by $\Delta' = \Delta'(\Delta)$ as any split component has diameter one in $(\widetilde{M_{\mathrm{tel}}}, d_G)$.

4.4. *Quasiconvexity of split components.* We now proceed to show further that split components are quasiconvex (not necessarily uniformly) in the hyperbolic metric, and uniformly quasiconvex in the graph metric; i.e., we need to show *hyperbolic quasiconvexity* and *uniform graph quasiconvexity* of split components.

Hyperbolic quasiconvexity. Let $N = \mathbf{H}^3/\Gamma$ be a complete hyperbolic 3-manifold. Then [CEG87] there exist a geometrically finite hyperbolic manifold with compact convex core N_{gf} and a strictly type-preserving embedding i of N_{gf} into N, which is a homotopy equivalence. Then for any boundary component S^h of N_{gf}, $i_*(\pi_1(S^h)) \subset \pi_1(N)$ is called a peripheral subgroup. In the theorem below, $\pi_1(N)$ will be identified with a Kleinian group Γ and the peripheral subgroup $i_*(\pi_1(S^h))$ with a Kleinian subgroup of Γ.

Covering Theorem 4.15 ([CEG87], [Can96]). *Let $N = \mathbf{H}^3/\Gamma$ be a complete hyperbolic 3-manifold. A finitely generated subgroup Γ' is geometrically infinite if and only if it contains a finite index subgroup of a geometrically infinite peripheral subgroup.*

We shall now specialize the Thurston-Canary Covering Theorem 4.15 to the case under consideration; namely, infinite index free subgroups of surface Kleinian groups.

Lemma 4.16. *Let N be a simply or doubly degenerate hyperbolic 3-manifold homotopy equivalent to a surface equipped with a weak split geometry model M. For K a split component, let $\tilde{K}$ be a lift to $\widetilde{N}$. Then there exists $C_0 = C_0(K)$ such that the convex hull of $\tilde{K}$ minus cusps lies in a C_0-neighborhood of $\tilde{K}$ in $\widetilde{N}$.*

Proof. Let $\Gamma = \pi_1(N)$ and $\Gamma' = i_*(\pi_1(K))(\subset \Gamma)$. Then Γ itself is the unique peripheral subgroup. Since Γ' has infinite index in Γ, it follows from

Theorem 4.15 that Γ' is geometrically finite. The result follows. (Cusps need to be excised because the model manifold is bi-Lipschitz homeomorphic to N minus cusps.) $\square$

Graph quasiconvexity. Next, we shall prove that each split component is uniformly graph quasiconvex. We begin with the following lemma. Recall that we are dealing with simply or totally degenerate groups without accidental parabolics.

LEMMA 4.17. *Let Σ be a component of a proper extended split subsurface S_i^s of S. Any (nonperipheral) simple closed curve in S appearing in the hierarchy whose free homotopy class has a representative lying in Σ must have a geodesic representative in M lying within a uniformly bounded distance of S_i^s in the graph metric d_G.*

Proof. Suppose a curve v in the hierarchy is homotopic into Σ. Then v is at a distance of at most 1 in the curve complex from each of the boundary components of Σ. Since Σ is a proper subsurface of S, the relative boundary $\partial_S(\Sigma) \neq \emptyset$. Let α be such a boundary component. Next, suppose that the geodesic realization of v in N intersects some block B_j^s (via the correspondence in Alternate Description 4.12). Then v must be at a distance of at most one from some curve σ in the base geodesic g_H forming an element of the pants decomposition of the split surface S_j^s.

By tightness, the distance from α to σ in the curve complex is at most 2. Hence the distance ($\leq |i-j|$) of S_j^s from S_i^s (in the d_G metric) is $\leq 2n$ from Lemma 4.4. Therefore v is realized within a distance $2n$ of S_i^s in the graph metric d_G. $\square$

Recall ([CEG87, Def. 8.8.1]) that a *pleated surface* in a hyperbolic 3-manifold N is a complete hyperbolic surface S of finite area, together with an isometric map $h : S \to N$ such that every $x \in S$ is in the interior of some geodesic segment in S that is mapped by h to a geodesic segment in N. Also, h maps cusps to cusps. We refer the reader to Section 8.8 of [CEG87] for further details. A pleated surface is said to be incompressible if $h_* : \pi_1(S) \to \pi_1(N)$ is injective. A standard fact about hyperbolic surfaces and pleated surfaces is Lemma 4.18 below. (See the proof of Proposition 8.8.5 of [CEG87] for instance.)

An *l-thin annulus* on a hyperbolic surface S^h is a maximal connected component of the set $\{x \in S^h : \mathrm{injrad}_x(S^h) < \frac{l}{2}\}$. This is the 2-dimensional analogue of an l-thin Margulis tube. Note that an l-thin annulus may also be a neighborhood of a cusp in S^h.

LEMMA 4.18 ([CEG87], [Bon86]). *For all $l > 0$ and $g, n \in \mathbb{N}$, there exists $\Delta = \Delta(l, g, n) > 0$ such that the following holds. Let S^h be any hyperbolic* *surface of genus g and n boundary components and/or cusps. Let $\mathcal{A}_l$ be the collection of l-thin annuli. Then $\mathcal{E}(S^h, \mathcal{A}_l)$ has diameter less than Δ in the electric metric. Again, let N be a hyperbolic 3-manifold, and let $\mathcal{T}_l$ be the collection of l-thin Margulis tubes and cusps in it. Let $h : S \to N$ be an incompressible pleated surface. Then $h(S)$ has diameter less than Δ in the electric metric on $\mathcal{E}(N, \mathcal{T}_l)$.*

Next, we show that any (nonperipheral) simple closed curve v_i in S^s_i (not just hierarchy curves as in Lemma 4.17) must be realized within a uniformly bounded distance of S^s_i in the graph metric. In fact we shall show further that any pleated surface that contains at least one boundary geodesic of Σ in its pleating locus lies within a uniformly bounded distance of S^s_i in the graph metric.

LEMMA 4.19. *There exists $B > 0$ such that the following holds. Let Σ be a proper split subsurface of S^s_i. Then any pleated surface with at least one boundary component coinciding with a geodesic representative of a nonperipheral component of $\partial\Sigma$ must lie within a B-neighborhood of S^s_i in $(M, d_G) = (\mathcal{E}(\widetilde{M}, \mathcal{K}'))/\Gamma$, where $\mathcal{K}'$ denotes the collection of extended split components in $\widetilde{M}$ and Γ is the fundamental group of M regarded as the group of deck transformations of $\widetilde{M}$. In particular, every simple closed curve in S homotopic into Σ has a geodesic representative within a B-neighborhood of S^s_i in (M, d_G).*

Proof. Choose a curve v_i homotopic to a simple closed curve on Σ. Let α denote its geodesic realization in N. Let Σ_p be any pleated (sub)surface whose boundary coincides with the geodesics representing the boundary components of Σ. (See [CEG87] for the construction of such pleated surfaces.) In particular, we may choose Σ_p such that its pleating locus contains v_i. Since S^s_i is a split surface in M, the topological type of Σ_p has finitely many possibilities. By Lemma 4.18 the diameter of Σ_p is bounded by $\Delta = \Delta(l)$ in the electric metric on $\mathcal{E}(N, \mathcal{T}_l)$. Since the l-thin components of the boundary of Σ_p are contained in l-thin tubes bounding S^s_i, it follows that Σ_p (and α in particular) lies in a Δ-neighborhood of S^s_i in the electric metric on $\mathcal{E}(N, \mathcal{T}_l)$. Since each $T \in \mathcal{T}_l$ is contained in the image of some $K \in \mathcal{K}'$ under the quotient map $(\mathcal{E}(\widetilde{M}, \mathcal{K}')) \to (\mathcal{E}(\widetilde{M}, \mathcal{K}'))/\Gamma = (M, d_G)$, the result follows. □

Remark 4.20. In [Bow05], Bowditch indicates a method to obtain a related (stronger) result that given $B_1 > 0$, there exists $B_2 > 0$ such that any two simple closed curves realized within a Hausdorff distance B_1 of each other in M are within a distance B_2 of each other in the curve complex.

4.5. *Drilling and filling.* In this subsection we summarize some material that will be needed in Section 4.6 to prove uniform graph-quasiconvexity of split components.

The *Drilling Theorem* of Brock and Bromberg [BB04], which built on work of Hodgson and Kerckhoff [HK98] [HK05], is given below. We shall invoke a version of this theorem that is closely related to one used by Brock and Souto in [BS06].

Theorem 4.21 ([BB04]). *For each $L > 1$ and n a positive integer, there is an $\ell > 0$ so that if N_{gf} is a geometrically finite hyperbolic 3-manifold and $c_1, \ldots, c_n$ are geodesics in N_{gf} with length $\ell_{N_{gf}}(c_i) < \ell$ for all c_i, then there is an L-bi-Lipschitz diffeomorphism of pairs*

$$h\colon (N_{gf} \setminus \cup_i \mathbb{T}(c_i), \cup_i \partial\mathbb{T}(c)) \to (N^0_{gf} \setminus \cup_i \mathbb{P}(c_i), \cup_i \partial\mathbb{P}(c_i)),$$

where $N_{gf} \setminus \cup_i \mathbb{T}(c_i)$ denotes the complement of a standard tubular neighborhood of $\cup_i c_i$ in N_{gf}, N^0_{gf} denotes the complete hyperbolic structure on $N_{gf} \setminus \cup_i c_i$, and $\mathbb{P}(c_i)$ denotes a standard rank-2 cusp corresponding to c_i.

N^0_{gf} is said to be obtained from N_{gf} by *drilling*. We remark here (following [BB04]) that the drilled manifold is the unique hyperbolic manifold that has the same conformal structure on its domain of discontinuity but has core curves of $\mathbf{T}$ drilled out to give rank-2 parabolics.

The Filling Theorem of Thurston [CEG87] (generalized by Canary [Can96]) we shall require is stated below.

Theorem 4.22 ([CEG87], [Can96]). *Given any quasifuchsian surface group Γ and $N = \mathbf{H}^3/\Gamma$, there exists $\delta > 0$ depending only on the Euler characteristic of the surface such that for all $x \in \mathrm{CC}(N)$, the convex core of N, there exists a pleated surface Σ such that $d(x, \Sigma) \leq \delta$.*

4.6. *Proof of uniform graph-quasiconvexity.* We need to prove the uniform graph-quasiconvexity of split components. Let B^s be a split block with a splitting l-thin Margulis tube T. We aim at showing

Proposition 4.23 (Uniform graph-quasiconvexity of split components). *Each (extended) split component $\widetilde{K}$ is uniformly graph-quasiconvex in $(\widetilde{M}, d_G)$.*

The proof of Proposition 4.23 will occupy this entire subsection. Let $B^s \subset B = S \times I$ be a split block with horizontal boundary consisting of split surfaces S^s_j, S^s_{j+1}. Let $\bigcup_i T_i$ be the union of l-thin Margulis tubes splitting B^s. (We suppress the dependence on the index j for the time being.) Let K be a split component. Then $K = (S_1 \times I)$ topologically for a subsurface S_1 of S. Also, let $\partial_s K = \partial S_1 \times I$ denote the collection of boundary annuli of K that abut the splitting tubes. Let $\partial S_1 = \bigcup_i \sigma_i = \sigma$ be the finite collection of boundary curves. σ is thus a *multicurve*. Each σ_i is homotopic to the core curve of an l-thin splitting Margulis tube T_i. Let $\mathbf{T} = \bigcup_i T_i$. $\mathbf{T}$ will be referred to as a *multi-Margulis tube*.

We have already shown in Lemma 4.16 that $\pi_1(S_1) \subset \pi_1(S)$ includes into $\pi_1(N)$ as a geometrically finite subgroup of $\mathrm{PSL}_2(C)$. Let N_1 be the cover of N corresponding to $\pi_1(K) = \pi_1(S_1)$. Then N_1 is geometrically finite. Let $\mathbf{T}^1$ be the multi-Margulis tube in N_1 that consists of tubes that are (individually) isometric to individual components of the multi-Margulis tube $\mathbf{T}$.

Let N_{1d} be the hyperbolic manifold obtained from N_1 by drilling out the core curves of $\mathbf{T}^1$. Since N_1 is geometrically finite, so is N_{1d}.

We first observe that the boundary of the augmented Scott core X of N_{1d} is incompressible away from cusps. To see this, note that X is double covered by a copy of $D \times I$ with solid tori drilled out of it, where D is the *double* of S_1 (obtained by doubling S_1 along its boundary circles).

Identify X with the convex core $\mathrm{CC}(N_{1d})$ of N_{1d}. We also identify D with the convex core boundary. Since D is incompressible away from cusps, we have the following.

Lemma 4.24 ([CEG87, Ch. 8]). *D is a pleated surface.*

Since N_1 is the cover of N corresponding to $\pi_1(K) \subset \pi_1(N)$, K lifts to an embedding into N_1. Adjoin the multi-Margulis tube $\mathbf{T}^1$ to (the lifted) K to get an augmented split component K_1. Let $K_{1d} \subset N_{1d}$ denote K_1 with the components of $\mathbf{T}^1$ drilled. We want to show that D lies within a uniformly bounded distance of K_{1d} in the lifted graph metric on N_{1d}. This would be enough to prove a version of Proposition 4.23 for the drilled manifold N_{1d} as the split geometry structure gives rise to a graph metric on N, hence a graph metric on N_1 and hence again, a graph metric on N_{1d}. Finally, we shall use the Theorem 4.21 to complete the proof of Proposition 4.23.

Lemma 4.25. *There exists C_1 such that for any split component K, D lies within a uniformly bounded neighborhood of K_{1d} in N_{1d}.*

Proof. Case 1: $D \cap K_{1d} \neq \emptyset$. If D intersects K_{1d}, then the lemma follows directly from Lemma 4.18: Incompressible pleated surfaces have bounded diameter in the graph metric d_G.

Case 2: $D \cap K_{1d} = \emptyset$. This is the more difficult case because *a priori* D might lie far from K_{1d}. Recall that $F : N \to M$ is a bi-Lipschitz homeomorphism between the hyperbolic manifold and the model manifold. Let $M_1 = F(N_1)$. Let B denote the block (split or thick) in the *model manifold* M containing $F(K)$. Let $B_1 \subset M_1$ denote its lift to M_1. Let B_{1d} denote B_1 with $\mathbf{T}^1$ drilled.

Then $B_{1d} - F(K_{1d})$ is topologically a disjoint union of "vertically thickened flaring annuli" $F(A_i)$, say. Each $A_i \subset F^{-1}(B_1)$ is of the form $S^1 \times [0, \infty)$ where $S^1 \times \{0\}$ lies on T_i.

More elaborately, what this means is the following. Identifying B with $S \times I$, we may identify B_1 with $S_1^a \times I$, where S_1^a is the cover of S corresponding to the subgroup $\pi_1(S_1) \subset \pi_1(S)$. Then S_1^a may be regarded as S_1 union a finite collection of flaring annuli $F(A_i)$ (one for each boundary component of S_1). Thus B_1 is the union of a core $F(K_1)$ and a collection of *vertically thickened flaring annuli* of the form $F(A_i) \times I$. Hence B_{1d} is the union of a core $F(K_{1d})$ and the collection of *vertically thickened flaring annuli* $F(A_i) \times I$. Also the boundary $\partial A_i = A_i \cap T_i$ is a curve of fixed length ε_0. Let us fix one such annulus A_1. Refer to Figure 6 (where we have removed subscripts for convenience).

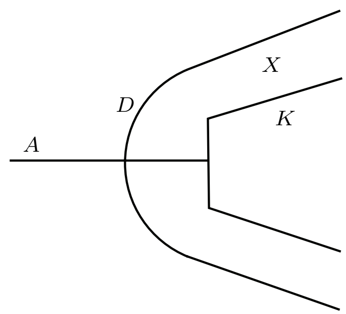

Figure 6. Graph quasiconvexity.

Recall that D bounds X and X contains K_{1d}. Thicken the convex core slightly to $N_\varepsilon(X)$ such that its boundary, D_ε is a smooth surface.

Let $\widetilde{M_1}$ denote the cover of M_1 corresponding to $i_*\pi_1(A_1)$, where i denotes the inclusion map. Let $\widetilde{D_\varepsilon}$ denote the lift of D_ε to $\widetilde{M_1}$. Then each lift $A_1 \times \{t\}$ separates $\widetilde{D_\varepsilon}$ since $i_*(\pi_1(A_1)) \subset \pi_1(D)$ is a subgroup such that the cover $\widetilde{D}$ has two ends. Hence, by a small homotopy of A_1, we can assume that

(a) $F^{-1}(F(A_1) \times I)$ is a smooth manifold (with boundary) bi-Lipschitz homeomorphic to $(F(A_1) \times I)$.
(b) $F(D_\varepsilon)$ is transverse to each $(F(A_1) \times \{t\})$ for t belonging to an interval I_1 of some fixed length $h_0 > 0$ (equal to the uniform lower bound on the height of split blocks B^s) contained in I.

Since $A_1 \times \{t\}$ separates $\widetilde{D_\varepsilon}$ in $\widetilde{N_1}$, it follows that for each $t \in I_1$, $F(D_\varepsilon)$ intersects $(F(A_1) \times \{t\})$ in an essential loop α_t parallel to ∂A_1. Hence D_ε must contain an annulus of the form $\alpha \times I_1 \subset F(A_1) \times I_1$. Also the length of I_1 is at least $\frac{h_0}{L}$, where L is the bi-Lipschitz constant for F. Since this is true for all ε, it is also true for the pleated surface D. Hence for at least some $t \in I$, the length of $\alpha \times \{t\}$ is uniformly bounded (by $\frac{2L\pi(4g-4)}{h_0}$) by the Gauss-Bonnet Theorem applied to D. Much more is true in fact, but this is enough for our purposes.

Since $\alpha\times\{t\}\subset A_1\times\{t\}$ and the latter is an exponentially flaring annulus, it follows that there exist uniform constants $C_0>0, \eta>1$ such that if $d(\alpha\times\{t\}, \partial(A_1\times\{t\})\geq d_0$, then the length of $\alpha\times\{t\}$ is bounded below by $C_0\eta^{d_0}$.

These two estimates imply that there is some point $p\in\alpha\times\{t\}\subset D$ such that $d(p,T_1)$ is uniformly bounded (in terms of the genus of S and the minimal height of split blocks h_0), where T_1 is the drilled Margulis tube intersecting $A_1\times\{t\}$ nontrivially.

By Lemma 4.18 and using the bi-Lipschitz homeomorphism F between N_{1d} and M_{1d}, the diameter of D is uniformly bounded in the graph metric lifted to M_{1d}. Hence, by the triangle inequality, D lies in a uniformly bounded neighborhood of K in the graph metric (using either of the descriptions of the graph metric in Remark 4.12). □

An alternate proof of Lemma 4.25. A simpler proof of the fact that D lies in a uniformly bounded neighborhood of K_1 in the graph metric may alternately be obtained directly as follows. First observe that M_1 is geometrically finite by the Covering Theorem of Thurston [CEG87] and Canary [Can96]. (See Lemma 4.16.) Next, by a theorem of Canary and Minsky [CM96], it follows that the convex hull boundary D of M_1 can be approximated by simplicial hyperbolic surfaces (see [CM96] for details) homotopic to D with short tracks. Thus any simplicial hyperbolic approximant D_a would have to have bounded area and hence bounded diameter modulo Margulis tubes (as in Lemma 4.19). Thus so would D. Now, we repeat the argument in the proof of Lemma 4.26 to conclude that D and hence the convex core $CC(M_1)$ of M_1 lies in a uniformly bounded neighborhood of K_1 in the graph metric. □

This approach would circumvent the use of the Drilling Theorem at this stage. However, since we shall again need it below, we retain our approach here.

Since D bounds X, we would like to claim that the conclusion of Lemma 4.25 follows with X in place of D. Though this does not *a priori* follow in the hyperbolic metric, it does follow for the graph metric. This is because the double cover of X is a "drilled quasifuchsian" manifold; i.e., it is essentially $(D\times I)$ with some short curves drilled. Further, any point in the convex core of a quasifuchsian $(D\times I)$ is close to a pleated surface by Theorem 4.22. Essentially the same argument as in Lemma 4.25 applies now. Details will be given below.

Lemma 4.26. *There exists C_1 such that for any split component K, K_{1d} is uniformly graph-quasiconvex in M_{1d}.*

Proof. X is double covered by $D\times I$ with cores of some Margulis tubes drilled. Let X_1 denote this double cover. Note that X_1 is convex, being a double cover of the convex compact X. By Theorem 4.21, there exists $l>0$

such that the drilled and undrilled manifolds are 2-bi-Lipschitz homeomorphic away from Margulis tubes and cusps provided the Margulis tubes are l-thin.

Perform $(1, m)$ Dehn filling on X_1 with sufficiently large $m = m(l)$ to ensure that the resulting Margulis tube is l-thin. Let X_{1f} be the resulting Dehn-filled manifold. By Theorem 4.21, X_{1f} is uniformly quasiconvex in $M_{1f} = \mathbf{H}^3/\Gamma$ where Γ is a quasi-Fuchsian surface group obtained by the above Dehn filling. (Theorem 4.21 gives a uniform bi-Lipschitz map outside Margulis tubes.)

Next, by Theorem 4.22, for all $x \in X_{1f}$, there exists a pleated surface $\Sigma \subset X_{1f}$ such that $d(x, \Sigma) \leq \delta$, where δ depends only on the genus of D.

Returning to X_1 via the Drilling Theorem 4.21, we see that for all $x \in X_1$,

(1) Either there exists a uniformly bi-Lipschitz image of a hyperbolic surface $\Sigma_1 \subset X_1$ such that $d(x, \Sigma_1) \leq \delta$. This is the case that the pleated surface Σ misses all filled Margulis tubes.
(2) Or, there exists a uniformly bi-Lipschitz image of a subsurface Σ_1 of a hyperbolic surface such that $d(x, \Sigma_1) \leq \delta$ and such that the boundary of Σ_1 lies in a Margulis tube. This is the case that the pleated surface Σ meets some filled Margulis tubes. Here, we can take Σ_1 to be the image of the component of (Σ *minus Margulis tubes*) that lies near x.

Again, passing down to X under the double cover (from X_1 to X), for all $x \in X$, we have

(1) Either there exists a uniformly bi-Lipschitz image of a hyperbolic surface $\Sigma_1 \subset X$ parallel to D such that $d(x, \Sigma_1) \leq \delta$.
(2) Or, there exists a uniformly bi-Lipschitz image of a subsurface Σ_1 of a hyperbolic surface such that $d(x, \Sigma_1) \leq \delta$ and such that the boundary of Σ_1 lies on a Margulis tube. Further, Σ_1 is incompressible in the complement of l-thin Margulis tubes.

In either case, the argument for Lemma 4.25 now shows that for all $x \in X$ the distance $d_G(x, K_{1d})$ is uniformly bounded in the graph-metric d_G. Thus, we have shown that K_{1d} is uniformly graph-quasiconvex in M_{1d}. $\square$

To complete the proof of Proposition 4.23 it is necessary to translate the content of Lemma 4.26 to the "undrilled" manifold N_1. We shall need to invoke the Drilling Theorem 4.21 again.

Concluding the proof of Proposition 4.23. While recovering data about N_1, it is slightly easier to handle the case where $D \cap K_{1d} = \emptyset$. Since we shall use the convex core boundary for both the drilled as well as the undrilled manifolds in the rest of the proof, we change notation slightly and use

(a) D_d for the convex core boundary of the drilled manifold N_d,
(b) D for the convex core boundary of the undrilled manifold N.

Case 1: $D_d \cap K_{1d} = \emptyset$. Filling N_{1d} along the (drilled) $\mathbf{T}^1$, we get back N_1. Since D_d misses K_{1d}, the filled image of X in N_1 is C_1-quasiconvex for some C_1, depending on the bi-Lipschitz constant of Theorem 4.21 above. (One can see this easily, for instance, from the fact that there is a uniform Lipschitz retract of $N_{1d} - X$ onto D_d.)

Case 2: $D_d \cap K_{1d} \neq \emptyset$. If D_d meets some Margulis tubes $\mathbf{T}^1$, we enlarge D to D' in X_1 by letting D' be the boundary of $X_1 = X \cup \mathbf{T}^1$. The annular intersections of D with Margulis tubes are replaced by boundary annuli contained in the boundary of $\mathbf{T}^1$.

It is easy enough to check that X_1 is uniformly quasiconvex in the hyperbolic metric: look at a universal cover $\widetilde{X_1}$ of X_1 in $\widetilde{N_1}$. Then $\widetilde{X_1}$ is a union of $\widetilde{X}$ and the lifts of T that intersect it. All these lifts of T are disjoint. Hence $\widetilde{X_1}$ is a "star" of convex sets, all of which intersect the convex set $\widetilde{X}$. By (Gromov) δ-hyperbolicity, such a set is uniformly quasiconvex. Then, as before, there is a uniform Lipschitz retract of $N_{1d} - X_1$ onto D'. But now D' misses the interior of K_{1d} and we can apply the previous argument.

By Theorem 4.21 above, the diameter of the convex core boundary D (or D' if D intersects some Margulis tubes) in N_1 is bounded in terms of the diameter of the convex core boundary D_d in N_{1d} and the *uniform* bi-Lipschitz constant L obtained from Theorem 4.21 above. Further, the distance of D from $K_1 \cup \mathbf{T}^1$ in N_1 is bounded in terms of the distance of D_d from $K_{1d} \cup \partial\mathbf{T}^1$ in N_{1d} and the bi-Lipschitz constant L.

Hence we can translate the content of Lemma 4.26 to the "undrilled" manifold N_1. This concludes the proof of Proposition 4.23: *Split components are uniformly graph-quasiconvex.* $\square$

Remark 4.27. Our proof above uses the fact that the convex core X of N_{1d} is a rather well-understood object, namely, a manifold double covered by a drilled convex hull of a quasi-Fuchsian group. Hence, it follows that the convex core X is uniformly congested; i.e., it has a uniform upper bound on its injectivity radius. This is an approach to a conjecture of McMullen [Bie]. (See also Fan [Fan97], [Fan99].)

A further point to be noted is that we have implicitly used here the idea of drilling *disk-busting curves* introduced by Canary in [Can93] and used again by Agol in his resolution of the tameness conjecture [Ago04].

Remark 4.28. Recall that *extended split components* were defined in $\widetilde{N}$ by adjoining Margulis tubes abutting lifts of split components to $\widetilde{N}$. The proof of Proposition 4.23 establishes also the uniform graph-quasiconvexity of *extended split components* in $\widetilde{N}$. The metric obtained by electrocuting the family of convex hulls of extended split components in $\widetilde{N}$ will be denoted as d_{CH}.

4.7. *Hyperbolicity in the graph metric.* First a word about the modifications necessary for simply degenerate groups.

Simply degenerate groups. We have so far mostly assumed, for simplicity, that we are dealing with totally degenerate groups. In a simply degenerate N, the Minsky model is *uniformly* bi-Lipschitz to N only in a neighborhood E of the end. In this case $(N \setminus E)$ is homeomorphic to $S \times I$. We declare $(N \setminus E)$ to be the first block — a "thick block" in the split geometry model. Thus the boundary blocks of Minsky are put together to form one initial thick block. This changes the bi-Lipschitz constant, but the rest of the discussion, including Proposition 4.23, go through as before.

Construct a second auxiliary metric $\widetilde{N}_2 = (\widetilde{N}, d_{\mathrm{CH}})$ by electrocuting the elements $\mathrm{CH}(\widetilde{K})$ of convex hulls of *extended split components*. We show that the spaces $\widetilde{N}_1 = (\widetilde{N}, d_G)$ and $\tilde{N}_2 = (\widetilde{N}, d_{\mathrm{CH}})$ are quasi-isometric. In fact we show that the identity map from $\widetilde{N}$ to itself induces this quasi-isometry after the two different electrocutions.

Lemma 4.29. *The identity map from $\widetilde{N}$ to itself induces a quasi-isometry of $\widetilde{N}_1$ and $\widetilde{N}_2$.*

Proof. We use d_1, d_2 as shorthand for the electric metrics d_G and d_{CH} on $\widetilde{N}_1$ and $\widetilde{N}_2$. Since $\widetilde{K} \subset \mathrm{CH}(\widetilde{K})$ for every split component, we have straightaway

$$d_1(x, y) \leq d_2(x, y) \text{ for all } x, y \in \widetilde{M}.$$

To prove a reverse inequality with appropriate constants, it is enough to show that each set $\mathrm{CH}(\widetilde{K})$ (of diameter one in $\widetilde{N}_2$) has uniformly bounded diameter in $\widetilde{N}_1$. To see this, note that by definition of graph-quasiconvexity, there exists n such that for all $\widetilde{K}$ and each point a in $\mathrm{CH}(\widetilde{K})$, there exists a point $b \in \widetilde{K}$ with $d_1(x, y) \leq n$. Hence by the triangle inequality,

$$d_2(x, y) \leq 2n + 1 \text{ for all } x, y \in \mathrm{CH}(\widetilde{K}).$$

Therefore,

$$d_2(x, y) \leq (2n + 1)(d_1(x, y) + 1) \text{ for all } x, y \in \widetilde{N}.$$

This proves the lemma. □

Corollary 4.30. $\widetilde{N}_1 = (\widetilde{N}, d_G)$ *is Gromov-hyperbolic.*

Proof. By Lemma 2.3, $\widetilde{N}_2 = (\widetilde{N}, d_{\mathrm{CH}})$ is a δ-hyperbolic metric space for some $\delta \geq 0$. By quasi-isometry invariance of Gromov hyperbolicity, so is $\widetilde{N}_1 = (\widetilde{N}, d_G)$. □

We have thus constructed a sequence of split surfaces that satisfy the following two conditions in addition to Conditions (1)–(6) of Remark 4.9 for the Minsky model of a simply or totally degenerate surface group.

Definition 4.31. A model manifold of weak split geometry is said to be of *split geometry* if

(7) Each split component $\widetilde{K}$ is quasiconvex (not necessarily uniformly) in the hyperbolic metric on $\widetilde{N}$.
(8) Equip $\widetilde{N}$ with the *graph-metric* d_G obtained by electrocuting (extended) split components $\widetilde{K}$. Then the convex hull $\mathrm{CH}(\widetilde{K})$ of any split component $\widetilde{K}$ has uniformly bounded diameter in the metric d_G.

Hence by Lemma 4.16 and Proposition 4.23, we have the following.

THEOREM 4.32. *Any simply or doubly degenerate surface group without accidental parabolics is bi-Lipschitz homeomorphic to a model of split geometry.*

5. Constructing quasiconvex ladders and quasigeodesics

To avoid confusion we summarize the various metrics on $\widetilde{M}, \widetilde{N}$ and related models that will be used:

(1) The hyperbolic metric d on $\widetilde{N}$.
(2) The weld-metric d_{wel} obtained after welding the boundaries of Margulis tubes of $\widetilde{M}$ to standard annuli (and before tube electrocution) where each horizontal circle of a Margulis tube T has a fixed nonzero length. This gives the welded model manifold $(M_{\mathrm{wel}}, d_{\mathrm{wel}})$.
(3) The tube-electrocuted metric $(M_{\mathrm{tel}}, d_{\mathrm{tel}})$. We remind the reader that the underlying manifolds $M_{\mathrm{wel}}, M_{\mathrm{tel}}$ are the same.
(4) The graph metric d_G. This is the notation for the electric metric on $\mathcal{E}(\widetilde{M_{\mathrm{wel}}}, \mathcal{K})$, where $\mathcal{K}$ denotes the collection of split components. We shall also use it for the electric metric on $\mathcal{E}(\widetilde{N}, \mathcal{K}')$, where $\mathcal{K}'$ denotes the collection of extended split components in $\widetilde{N}$. The two electric metrics are quasi-isometric by Remark 4.12.

There will be two (families of) metrics on the universal cover $\widetilde{S}$ of S:

(1) The graph-electrocuted metric d_{Gel} obtained by electrocuting the amalgamation components of $\widetilde{S}$ that the lift of a weld-curve cuts $\widetilde{S}$ into.
(2) The (Gromov) δ-hyperbolic metric d on $\widetilde{S}$ obtained by lifting the metric on the *welded* surface. Recall that the metric d on $\widetilde{S}$ is the lift to the universal cover of a metric on S obtained by cutting out thin annuli and then welding the boundaries of the resulting extended split surface together. The latter is uniformly bi-Lipschitz to a fixed hyperbolic structure on S. Hence we shall use d to denote both the hyperbolic metric as well as those uniformly bi-Lipschitz to it.

Note that the path metric induced on $\widetilde{S} \subset \widetilde{B}$ by the graph metric d_G on $\mathcal{E}(\widetilde{M_{\mathrm{wel}}}, \mathcal{K})$ is precisely d_{Gel}.

5.1. *Construction of quasiconvex sets for building blocks.* In this subsection, we describe the construction of a *hyperbolic ladder* $\mathcal{L}_\lambda$ restricted to building blocks B. Putting these together we will show later that $\mathcal{L}_\lambda$ is quasiconvex in $(\widetilde{M_{\mathrm{wel}}}, d_G)$.

Construction of $\mathcal{L}_\lambda(B)$: *thick block.* Let B be a thick block. By definition B is a uniformly bi-Lipschitz homeomorphic image of $S \times I$. Let $F_B : S \times I \to B$ denote the bi-Lipschitz homeomorphism.

Let $\lambda = [a, b]$ be a geodesic segment in $\widetilde{S}$. Let λ_{Bi} denote $F_B(\lambda \times \{i\})$ for $i = 0, 1$. Equivalently, let $\phi : F_B(\widetilde{S} \times \{0\}) \to F_B(\widetilde{S} \times \{1\})$ be given by $\phi(F_B(x, 0)) = F_B(x, 1)$. The induced map on geodesics will be denote by Φ, which can be described as follows. Let λ be a geodesic joining $a, b \in F_B(\widetilde{S} \times \{0\})$, and let $\Phi(\lambda)$ denote the a geodesic joining $\phi(a), \phi(b)$. Let λ_{B1} denote $\Phi(\lambda) \times \{1\}$.

For the universal cover $\widetilde{B}$ of the thick block B, define

$$\mathcal{L}_\lambda(B) = \bigcup_{i=0,1} \lambda_{Bi}.$$

Definition 5.1. Each $\widetilde{S} \times i$ for $i = 0, 1$ will be called a *horizontal sheet* of $\widetilde{B}$ when B is a thick block.

Construction of $\mathcal{L}_\lambda(B)$: *split block.* As above, let $\lambda = [a, b]$ be a geodesic segment in $\widetilde{S}$, where S is regarded as the base surface of B in the tube electrocuted model. Let λ_{B0} denote $\lambda \times \{0\}$. Then for each split component K, $K \cap (S \times i)$ $(i = 0, 1)$ is an *amalgamation component* of $\widetilde{S}$. Also, $S \times i$ $(i = 0, 1)$, are the boundary *welded split surfaces* forming the horizontal boundary of B, uniformly bi-Lipschitz to S with a fixed hyperbolic metric. Note further that the induced path metric d_{Gel} on $\widetilde{S} \times i$ $(i = 0, 1)$ is the electric pseudo-metric on $\widetilde{S}$ obtained by electrocuting amalgamation components of $\widetilde{S}$.

Let λ_{Gel} denote the electro-ambient quasigeodesic (Lemma 2.16) joining a, b in $(\widetilde{S}, d_{\mathrm{Gel}})$. Let λ_{B0} denote $\lambda_{\mathrm{Gel}} \times \{0\}$. Then the map $\phi : S \times \{0\} \to S \times \{1\}$ taking $(x, 0)$ to $(x, 1)$ is a component preserving diffeomorphism. Let $\widetilde{\phi}$ be the lift of ϕ to $\widetilde{S}$ equipped with the electric metric d_{Gel}. Then $\widetilde{\phi}$ is an isometry by Lemma 2.19. Let Φ denote the induced map on electro-ambient quasigeodesics; i.e., if $\mu = [x, y] \subset (\widetilde{S}, d_{\mathrm{Gel}})$, then $\Phi(\mu) = [\phi(x), \phi(y)]$ is the electro-ambient quasigeodesic joining $\phi(x), \phi(y)$. Let λ_{B1} denote $\Phi(\lambda_{\mathrm{Gel}}) \times \{1\}$.

For the universal cover $\widetilde{B}$ of the split block B, define

$$\mathcal{L}_\lambda(B) = \bigcup_{i=0,1} \lambda_{Bi}.$$

Definition 5.2. Each $\widetilde{S} \times i$ for $i = 0, 1$ will be called a *horizontal sheet* of $\widetilde{B}$ when B is a split block.

Construction of $\Pi_{\lambda,B}$: *thick block.* For $i = 0, 1$, let Π_{Bi} denote nearest point projection of $\widetilde{S} \times \{i\}$ onto λ_{Bi} in the path metric on $\widetilde{S} \times \{i\}$. For the universal cover $\widetilde{B}$ of the thick block B, define

$$\Pi_{\lambda,B}(x) = \Pi_{Bi}(x), x \in \widetilde{S} \times \{i\}, i = 0, 1.$$

Construction of $\Pi_{\lambda,B}$: *split block.* For $i = 0, 1$, let Π_{Bi} denote nearest-point projection of $\widetilde{S} \times \{i\}$ onto λ_{Bi}. Here the nearest-point projection is taken in the sense of the definition preceding Lemma 2.23, i.e., minimizing the ordered pair (d_{Gel}, d) in the lexicographic order on $\mathbb{R} \times \mathbb{R}$ (where d_{Gel}, d refer to electric and (bi-Lipschitz)-hyperbolic metrics respectively).

For the universal cover $\widetilde{B}$ of the split block B, define

$$\Pi_{\lambda,B}(x) = \Pi_{Bi}(x), x \in \widetilde{S} \times \{i\}, i = 0, 1.$$

$\Pi_{\lambda,B}$ *is a coarse Lipschitz retract*: *thick block.* The proof for a thick block is exactly as in [Mit98b] and [Mj10a]. We omit it here.

Lemma 5.3 ([Mj10a, Th. 3.1]). *There exists $C > 0$ such that the following holds. Let $x, y \in \widetilde{S} \times \{0, 1\} \subset \widetilde{B}$ for some thick block B. Then $d(\Pi_{\lambda,B}(x), \Pi_{\lambda,B}(y)) \le Cd(x, y)$.*

$\Pi_{\lambda,B}$ *is a retract*: *split block.*

Lemma 5.4. *There exists $C > 0$ such that the following holds. Let $x, y \in \widetilde{S} \times \{0, 1\} \subset \widetilde{B}$ for some split block B. Then $d_G(\Pi_{\lambda,B}(x), \Pi_{\lambda,B}(y)) \le Cd_G(x, y)$.*

Proof. It is enough to show this for the following cases.

Case 1: *$x, y \in \widetilde{S} \times \{0\}$ or $x, y \in \widetilde{S} \times \{1\}$.* This follows directly from Lemma 2.20.

Case 2: *$x = (p, 0)$ and $y = (p, 1)$ for some $p \in \widetilde{S}$.* First note that $(\widetilde{S}, d_{\mathrm{Gel}})$ is uniformly δ-hyperbolic as a metric space (in fact uniformly quasi-isometric to a tree) and $\widetilde{\phi} : \widetilde{S} \times \{0\} \to \widetilde{S} \times \{1\}$ induces an isometry of the d_{Gel} metric by Lemma 2.19 as ϕ is a component preserving diffeomorphism. Case 2 now follows from the fact that quasi-isometries and nearest-point projections almost commute (Lemma 2.21). □

In the next section, we shall come across the situation where one horizontal surface $\widetilde{S} \times \{i\}$ can occur as the bottom surface of a split block B_2 and as the top surface of a thick block B_1, or vice versa. Alternately it could occur as the bottom surface of a split block and as the top surface of a different split block where the collection of splitting tubes differ. In either situation we shall denote the bottom block by B_1 and the top block by B_2. In this case, the nearest-point projection could be in any of the following senses:

(a) projection onto a (bi-Lipschitz)-hyperbolic geodesic $[a, b]$ in the (bi-Lipschitz)-hyperbolic metric d on $\widetilde{S}$;
(b) projection onto an electro-ambient quasigeodesic $[a, b]_{\mathrm{ea}}$ minimizing the ordered pair (d_{Gel1}, d), where d_{Gel1} denotes the electric metric on S induced by the split block B_1;
(c) projection onto an electro-ambient quasigeodesic $[a, b]_{\mathrm{ea}}$ minimizing the ordered pair (d_{Gel2}, d), where d_{Gel2} denotes the electric metric on S induced by the split block B_2.

LEMMA 5.5. *$\Pi_{\lambda,B}$ is coarsely well defined. There exists $C_0 > 0$ such that the following holds. Suppose that $\Pi^1_{\lambda,B}$ and $\Pi^2_{\lambda,B}$ are projections defined in any two of the above senses. Then*

$$d(\Pi^1_{\lambda,B}(p), \Pi^2_{\lambda,B}(p)) \leq C_0$$

for all $p \in \widetilde{S}$.

Proof. By Lemma 2.23, hyperbolic and electric projections of p onto the (Gromov) δ-hyperbolic geodesic $[a, b]$ and the electro-ambient geodesic $[a, b]_{\mathrm{ea}}$ respectively "almost agree." If π_h and π_e denote the hyperbolic and electric projections, then there exists (uniform) $C_1 > 0$ such that $d(\pi_h(p), \pi_e(p)) \leq C_1$. The lemma follows if one of the blocks are thick.

If both blocks are split blocks, then $d(\pi_h(p), \Pi^i_{\lambda,B}(p)) \leq C_1$ for $i = 1, 2$ by the above argument. Taking $C_0 = 2C_1$, we are through. □

5.2. *Construction of $\mathcal{L}_\lambda$ and Π_λ.* A subset $Z \subset (X, d)$ shall be called a coarse k-net in X if $X = N_k(Z, d)$. A subset $Z \subset (X, d)$ shall be called a coarse net if it is a coarse k-net in X for some k.

Given a manifold M of split geometry, we know that M is homeomorphic to $S \times J$ for $J = [0, \infty)$ or $(-\infty, \infty)$. By definition of split geometry, there exists a sequence of blocks B_i (thick or split) such that $M_{\mathrm{wel}} = \cup_i B_i$. Denote

- $\mathcal{L}_{\mu,B_i} = \mathcal{L}_{i\mu}$,
- $\Pi_{\mu,B_i} = \Pi_{i\mu}$.

Now for a block $B = S \times I$ (thick or amalgamated), a natural map Φ_B may be defined taking $\mu = \mathcal{L}_\mu(B) \cap F_B(\widetilde{S} \times \{0\})$ to a geodesic $\mathcal{L}_\mu(B) \cap F_B(\widetilde{S} \times \{1\}) = \Phi_B(\mu)$. Similarly Φ_B^{-1} may be defined taking $\mu = \mathcal{L}_\mu(B) \cap F_B(\widetilde{S} \times \{1\})$ to $\mathcal{L}_\mu(B) \cap F_B(\widetilde{S} \times \{0\}) = \Phi_B^{-1}(\mu)$. Let the map Φ_{B_i} (resp. $\Phi^{-1}_{B_i}$) be denoted as Φ_i (resp. Φ_i^{-1}).

We start with a reference block B_0 and a reference geodesic segment $\lambda = \lambda_0$ on the "lower surface" $\widetilde{S} \times \{0\}$. Now inductively define

- $\lambda_{i+1} = \Phi_i(\lambda_i)$ for $i \geq 0$,
- $\lambda_{i-1} = \Phi_i^{-1}(\lambda_i)$ for $i \leq 0$.

Finally, define

$$\mathcal{L}_\lambda = \bigcup_i \lambda_i.$$

$\mathcal{L}_\lambda$ is the *hyperbolic ladder* promised.

Recall that each $\widetilde{S} \times i$ for $i = 0, 1$ is called a *horizontal sheet* of $\widetilde{B}$. We will restrict our attention to the union of the horizontal sheets $\widetilde{M_H} \subset \widetilde{M_{\text{wel}}}$ with the metric induced from the graph model. Since $\widetilde{M_H}$ is a *coarse* 1*-net* in $(\widetilde{M_{\text{wel}}}, d_G)$, we will be able to get all the coarse information we need by restricting ourselves to $\widetilde{M_H}$.

Clearly, $\mathcal{L}_\lambda \subset \widetilde{M_H} \subset \widetilde{M_{\text{wel}}}$. Let the bottom horizontal sheet of $\widetilde{B_i}$ be denoted as $\widetilde{S_i}$. $\Pi_{i\lambda}$ is defined to be the nearest-point projection of $\widetilde{S_i}$ onto λ_i.

Remark 5.6. As noted earlier, the nearest-point projection $\Pi_{i\lambda}$ could be in any of the following senses:

(a) projection onto a (bi-Lipschitz)-hyperbolic geodesic $[a, b]$ in the (bi-Lipschitz)-hyperbolic metric d on $\widetilde{S}$;
(b) projection onto an electro-ambient quasigeodesic $[a, b]_{\text{ea}}$ minimizing the ordered pair (d_{Gel1}, d), where d_{Gel1} denotes the electric metric on S induced by the split block B_1 whose top boundary is S;
(c) projection onto an electro-ambient quasigeodesic $[a, b]_{\text{ea}}$ minimizing the ordered pair (d_{Gel2}, d), where d_{Gel2} denotes the electric metric on S induced by the split block B_2 whose bottom boundary is S.

By Lemma 5.5, $\Pi_{i\lambda}$ *is coarsely well defined*; *i.e.*, *any two choices are a uniformly bounded d-distance apart.*

Hence we define the projection

$$\Pi_\lambda = \bigcup_i \Pi_{i\lambda}.$$

Π_λ is defined from $\widetilde{M_H}$ to $\mathcal{L}_\lambda$.

THEOREM 5.7. *There exists $C > 0$ such that for any geodesic $\lambda = \lambda_0 \subset \widetilde{S} \times \{0\} \subset \widetilde{B_0}$, the retraction $\Pi_\lambda : \widetilde{M_H} \to \mathcal{L}_\lambda$ satisfies*

$$d_G(\Pi_\lambda(x), \Pi_\lambda(y)) \leq C d_G(x, y) + C.$$

Proof. This is now a direct consequence of Lemmas 5.3 and 5.4 and Remark 5.6. □

For Theorem 5.7, note that all that we really require is that the universal cover $\widetilde{S}$ is a Gromov-hyperbolic metric space. There is *no restriction on* $\widetilde{M_H}$. In fact, Theorem 5.7 would hold for general stacks of (Gromov) hyperbolic metric spaces with blocks of split geometry. However, in the present situation we have more

COROLLARY 5.8. $\mathcal{L}_\lambda$ *is quasiconvex in* $(\widetilde{M_{\text{wel}}}, d_G)$.

Proof. By Corollary 4.30, $(\widetilde{M_{\mathrm{wel}}}, d_G)$ is (Gromov)-hyperbolic. Hence $\mathcal{L}_\lambda$ is a coarse Lipschitz retract in a (Gromov)-hyperbolic space by Theorem 5.7. Therefore $\mathcal{L}_\lambda$ is quasiconvex in $(\widetilde{M_{\mathrm{wel}}}, d_G)$. $\square$

5.3. *Heights of blocks.* Recall that each thick or split block B_i is identified with $S \times I$ where each fiber $\{x\} \times I$ has length $\leq l_i$ for some l_i, called the *thickness* of the block B_i.

Observation. $\widetilde{M_H}$ is a "coarse net" in $(\widetilde{M_{\mathrm{wel}}}, d_G)$ in the graph metric, but not in the weld metric d_{wel}, the tube-electrocuted metric d_{tel}, nor the model metric d_M (cf. Remark 4.12 for d_M). In the graph model, any point can be connected by a vertical segment of length ≤ 1 to one of the boundary horizontal sheets.

However, there are points within split components that are at a d_{wel}-distance of the order of l_i from the boundary horizontal sheets. Since l_i could be arbitrary, $\widetilde{M_H}$ is no longer necessarily a "coarse net" in $(\widetilde{M}, d_{\mathrm{wel}})$ or $(\widetilde{M}, d_{\mathrm{tel}})$.

Lemma 5.9. *There exists a function $g : \mathbb{Z} \to \mathbb{N}$ such that for any block B_i (resp. B_{i-1}) and $x \in \lambda_i$, there exists $x' \in \lambda_{i+1}$ (resp. λ_{i-1}) for $i \geq 0$ (resp. $i \leq 0$), satisfying*

$$d_{\mathrm{wel}}(x, x') \leq g(i), \quad d_M(x, x') \leq g(i).$$

Proof. Let $\mu \subset \widetilde{S} \times \{0\} \subset \widetilde{B_i}$ be a geodesic in a (thick or split) block. Then from the product structure on the block B_i, there exists a (K_i, ε_i)-quasi-isometry ψ_i from $\widetilde{S} \times \{0\}$ to $\widetilde{S} \times \{1\}$ and Ψ_i is the induced map on geodesics. Hence, for any $x \in \mu$, $\psi_i(x)$ lies within some bounded distance C_i of $\Psi_i(\mu)$. But x is connected to $\psi_i(x)$ by

Case 1: *thick blocks.* A vertical segment of uniformly bounded length ($\leq C$ say).

Case 2: *split blocks.* Thus x can be connected to a point $x' \in \Psi_i(\mu)$ by a path of length less than $g(i) = l_i + C_i + C$. Recall that λ_i is the geodesic on the lower horizontal surface of the block $\widetilde{B_i}$. The same can be done for blocks $\widetilde{B_{i-1}}$ and *going down* from λ_i to λ_{i-1}.

By Remark 4.12, the same argument works for the model manifold $(\widetilde{M}, d_M)$. $\square$

6. **Recovery**

The previous section was devoted to constructing a quasiconvex ladder in the graph metric that is an electric metric. In this section we shall be concerned with recovering information about hyperbolic geodesics from electric ones. Since a host of metrics will make their appearance in this section, we shall refer to (quasi)geodesics in $(\widetilde{M_{\mathrm{wel}}}, d_G)$, $(\widetilde{M_{\mathrm{wel}}}, d_{\mathrm{wel}})$, $(\widetilde{M_{\mathrm{tel}}}, d_{\mathrm{tel}})$ and $(\widetilde{M}, d_{\mathrm{CH}})$ as d_G-(quasi)geodesics, d_{wel}-(quasi)geodesics, d_{tel}-(quasi)geodesics and d_{CH}-(quasi)geodesics respectively. Recall that the union of the horizontal sheets $\widetilde{S_i} \subset \widetilde{M_{\mathrm{wel}}}$ is denoted as $\widetilde{M_H}$ and that the projection Π_λ occurring in Theorem 5.7 is defined only on $\widetilde{M_H}$ and *not* all of $\widetilde{M_{\mathrm{wel}}}$.

6.1. *Scheme of recovery.* The recovery is in several stages. We sketch the scheme of recovery in some detail in this subsection for the convenience of the reader. A first problem in recovering data about hyperbolic geodesics from d_G-geodesics is the absence of canonical representatives in $(\widetilde{M_{\mathrm{wel}}}, d_{\mathrm{wel}})$ of d_G-geodesics. In Section 6.2, we address this problem by making a choice of paths in $(\widetilde{M_{\mathrm{wel}}}, d_{\mathrm{wel}})$ representing d_G-geodesics. We call these *admissible paths.* Roughly speaking, admissible paths are built up of

(a) vertical segments of the form $\{x\} \times [0,1] \subset \widetilde{B} = \widetilde{S} \times [0,1]$, where B is a block (thick or split) and $x \in \widetilde{S}$;
(b) horizontal segments consisting of geodesics in the horizontal sheets of $\widetilde{M_H}$.

Let $\lambda \subset \widetilde{S}(\subset \widetilde{M_{\mathrm{wel}}})$ be a geodesic in the intrinsic metric on $\widetilde{S}$, where S is identified with the base surface $S \times \{0\}$ of the first block in $\widetilde{M_H}$. Let β_e denote an admissible path representing a d_G-geodesic joining the endpoints of λ in $(\widetilde{M_{\mathrm{wel}}}, d_G)$.

We would like to project β_e using Π_λ onto the ladder $\mathcal{L}_\lambda$ to obtain a quasigeodesic contained in $\mathcal{L}_\lambda$. Unfortunately, Π_λ is defined only on $\widetilde{M_H}$ and there is no natural way to extend it to all of $\widetilde{M_{\mathrm{wel}}}$. To circumvent this problem we first define in Section 6.2 a subcollection of the family of admissible paths, called $\mathcal{L}_\lambda$-admissible paths. Roughly speaking, $\mathcal{L}_\lambda$-admissible paths are those admissible paths whose horizontal segments lie on or near $\mathcal{L}_\lambda$.

Then in Section 6.3 we project $\beta_e \cap \widetilde{M_H}$ using Π_λ onto the ladder $\mathcal{L}_\lambda$. Since β_e is itself an admissible path, there is a sequence of points $a_1, b_1, a_2, b_2, \dots, a_k, b_k$ such that the piece of β_e joining a_i to b_i is horizontal, whereas the piece of β_e joining b_i to a_{i+1} is vertical. In particular, b_i and a_{i+1} must lie in the same split component if they lie in (the universal cover of) a split block. In this case $\Pi_\lambda(b_i)$ and $\Pi_\lambda(a_{i+1})$ must also lie in the same split component. This allows us to join the sequence of points

$$\Pi_\lambda(a_1), \Pi_\lambda(b_1), \Pi_\lambda(a_2), \Pi_\lambda(b_2), \dots, \Pi_\lambda(a_k), \Pi_\lambda(b_k)$$

by alternating horizontal and vertical segments to obtain an $\mathcal{L}_\lambda$-admissible path β_{adm} representing a (uniform) d_G-quasigeodesic joining the endpoints of λ in $(\widetilde{M_{\mathrm{wel}}}, d_G)$. Lemma 6.5 now establishes that if λ lies outside a large ball about a reference point in $\widetilde{S}$, then β_{adm} also lies outside a large ball about a reference point in $(\widetilde{M_{\mathrm{wel}}}, d_{\mathrm{wel}})$.

From the $\mathcal{L}_\lambda$-admissible paths constructed in Section 6.3, we construct in Section 6.4 an electro-ambient quasigeodesic $\beta_{\rm ea}$ in $(\widetilde{M_{\rm wel}}, d_G)$. The idea is simple. Denote by $\widetilde{K_{ij}} \subset \widetilde{B_i}$ the split components in the universal cover of a split block B_i. Replace the intersection $\beta_{\rm adm} \cap \widetilde{K_{ij}}$ (of $\beta_{\rm adm}$ with any such split component $\widetilde{K_{ij}}$) by a geodesic in $\widetilde{K_{ij}}$ joining the endpoints of $\beta_{\rm adm} \cap \widetilde{K_{ij}}$. Then $\beta_{\rm ea}$ continues to satisfy the conclusions of Lemma 6.5; i.e., if λ lies outside a large ball in $\widetilde{S}$, then $\beta_{\rm ea}$ lies outside a large ball in $(\widetilde{M_{\rm wel}}, d_{\rm wel})$. What is crucial at this stage of the recovery is the quasiconvexity of $\widetilde{S_i}$ and $\widetilde{S_{i+1}}$ in $\widetilde{B_i}$, where the quasiconvexity constant depends only on i.

Finally, in Section 6.5 we construct an electro-ambient quasigeodesic $\beta_{\rm ea2}$ in $(\widetilde{M}, d_{\rm CH})$ from $\beta_{\rm ea}$. To do this, we first replace $\beta_{\rm ea}$ by a path $\beta_{\rm ea1}$ in $\widetilde{M}$ such that $\beta_{\rm ea1}$ coincides with $\beta_{\rm ea}$ outside Margulis tubes and consists of hyperbolic geodesic segments within Margulis tubes. Then as above, we replace the intersection $\beta_{\rm ea1} \cap {\rm CH}(\widetilde{K_{ij}})$ (of $\beta_{\rm ea1}$ with the convex hull ${\rm CH}(\widetilde{K_{ij}})$ of a split component $\widetilde{K_{ij}}$) by a geodesic in ${\rm CH}(\widetilde{K_{ij}})$ joining the endpoints of $\beta_{\rm ea1} \cap {\rm CH}(\widetilde{K_{ij}})$. This gives us the required electro-ambient quasigeodesic $\beta_{\rm ea2}$ in $(\widetilde{M}, d_{\rm CH})$. Again, $\beta_{\rm ea2}$ continues to satisfy the conclusions of Lemma 6.5; i.e., if λ lies outside a large ball in $\widetilde{S}$, then $\beta_{\rm ea2}$ lies outside a large ball in $\widetilde{M}$ (where the latter is equipped with the model metric). The last statement follows from the uniform graph quasiconvexity of split components (Proposition 4.23).

It is a small step from here to the main Theorem 7.1 in Section 7, so we mention it here. Lemma 2.5 ensures that the geodesic β^h in $\widetilde{M}$ joining the endpoints of $\beta_{\rm ea2}$ lies in a uniformly bounded neighborhood of $\beta_{\rm ea2}$ (see Figure 2). Note that it is at this stage that we use explicitly the *weak relative hyperbolicity* of $\widetilde{M}$ relative to the collection of convex hulls of split components. Though $\beta_{\rm ea2}$ could be very far from a hyperbolic geodesic, Lemma 2.5 forces β^h to lie in a bounded neighborhood of it. Hence if λ lies outside a large ball in $\widetilde{S}$, then β^h lies outside a large ball in $\widetilde{M}$. Lemma 1.8 now furnishes the Cannon-Thurston map we want.

6.2. *Admissible paths.* We want to first define a collection of paths lying in a bounded neighborhood of $\mathcal{L}_\lambda$ in $(\widetilde{M_{\rm wel}}, d_G)$. Since $\mathcal{L}_\lambda$ is not connected, it does not make sense to speak of the path metric on $\mathcal{L}_\lambda$. To remedy this, in this subsection we shall introduce the class of $\mathcal{L}_\lambda$-*elementary admissible paths* whose horizontal pieces are contained in a neighborhood of $\mathcal{L}_\lambda$ in $\widetilde{M_H}$. Further the distance of $\mathcal{L}_\lambda$-elementary admissible paths from $\mathcal{L}_\lambda$ will be controlled. An $\mathcal{L}_\lambda$-*admissible path* will be a composition of $\mathcal{L}_\lambda$-elementary admissible paths.

We first define admissible paths in general. Let B be a thick or split block in $M_{\rm wel}$. We shall identify B with a product $S \times I$ as usual. In particular, for B a split block and any $x \in \widetilde{S}$, a vertical segment of the form $x \times [0, 1]$ will be assumed to be contained in some split component $\widetilde{K} \subset \widetilde{B}$.

Definition 6.1. An *admissible path in* $\widetilde{B}$ ($\subset \widetilde{M_{\mathrm{wel}}}$) is a path that can be decomposed into subpaths of the following two types:

(1) horizontal segments along some $\widetilde{S} \times \{i\}$ for $i = \{0, 1\}$,
(2) vertical segments of the form $x \times [0, 1]$ where $x \in \widetilde{S}$.

An *admissible path* σ in $\widetilde{M_{\mathrm{wel}}}$ is a path such that for every (thick or split) block B, any connected component of $\sigma \cap \widetilde{B}$ is an admissible path in $\widetilde{B}$. An *admissible* K*- quasigeodesic* is an admissible path that is a K-quasigeodesic in $(\widetilde{M_{\mathrm{wel}}}, d_G)$.

Lemma 6.2. *Given* $K \geq 1$, *there exists* $K_1 \geq 1$ *such that the following holds. Let* β_e *be a* (d_G-) K*-quasigeodesic in* $(\widetilde{M_{\mathrm{wel}}}, d_G)$. *Then there exists an admissible* K_1*-quasigeodesic* β'_e *joining the endpoints of* β_e.

Proof. Without loss of generality, we can assume that β_e does not back-track relative to the collection of split components, as any back-tracking can be removed without increasing the d_G-length of β_e (see [Far98] for instance). We shall now convert β_e into an admissible electric quasigeodesic without back-tracking, joining the same pair of points as β_e. To do this we shall look at connected components of $\beta_e \cap \widetilde{B}$ for any block B and replace them with admissible paths. We identify B with $S \times [0, 1]$, and we call $\widetilde{S} \times \{0\}$ and $\widetilde{S} \times \{1\}$ the lower and upper boundary components of $\widetilde{B}$. Also let P_0, P_1 denote the natural projections from $\widetilde{S} \times [0, 1]$ to $\widetilde{S} \times \{0\}$ and $\widetilde{S} \times \{1\}$ respectively given by $P_0(x, t) = (x, 0)$ and $P_1(x, t) = (x, 1)$.

Now let B be a block (thick or split), and let $\beta_e \cap \widetilde{B} \neq \emptyset$. Let β_1 be a connected component of $\beta_e \cap \widetilde{B}$. Let b_1, b_2 be the endpoints of β_1. Two cases arise.

- If both b_1, b_2 belong to the same boundary component, then we replace β_1 by $\beta'_1 = P_i(\beta_1)$, where $i = 0$ or 1 according as $b_1, b_2 \in \widetilde{S} \times \{0\}$ or $\widetilde{S} \times \{1\}$.
- If b_1, b_2 belong to different boundary components, then assume without loss of generality that $b_1 \in \widetilde{S} \times \{0\}$ and let $b_2 = (z, 1) \in \widetilde{S} \times \{1\}$. Then replace β_1 by $\beta'_1 = P_0(\beta_1) \cup \{z\} \times [0, 1]$.

Performing this replacement for every block B and every connected component of $\beta_e \cap \widetilde{B}$ we obtain the required admissible quasigeodesic β'_e joining the endpoints of β_e.

It remains to show that if β_e is a (d_G-quasi) K-quasiquasigeodesic, then β'_e is indeed an admissible K_1-quasiquasigeodesic, where K_1 depends only on K.

For B a split block, the d_G-length of any $\beta_1 \subset \widetilde{B}$ is the same as the d_G-length of the corresponding β'_1 constructed to replace it as above. This is because the d_G-length of β_1 is equal to the number of split blocks that β_1 cuts.

For B a thick block, the inclusion of $\widetilde{S} \times \{0\}$ (or $\widetilde{S} \times \{1\}$) into $\widetilde{B}$ is a uniform quasi-isometry as the thickness of thick blocks is uniformly bounded.

Hence β_1' is a K_1-quasiquasigeodesic where K_1 depends only on K. The lemma follows. $\square$

We shall now choose a subclass of these admissible paths to define $\mathcal{L}_\lambda$-elementary admissible paths. The constants $C, C(B), K(B)$, etc. below will be independent of the geodesic λ, the initial geodesic in the ladder $\mathcal{L}_\lambda$.

$\mathcal{L}_\lambda$-elementary admissible paths in the thick block. Let $B = S \times [i, i+1]$ be a thick block, where each (x, i) is connected by a vertical segment to $(x, i+1)$. Let ϕ be the map that takes (x, i) to $(x, i+1)$. Also let Φ be the map on geodesics induced by ϕ. Let $\mathcal{L}_\lambda \cap \widetilde{B} = \lambda_i \cup \lambda_{i+1}$, where λ_i lies on $\widetilde{S} \times \{i\}$ and λ_{i+1} lies on $\widetilde{S} \times \{i+1\}$. Let π_j for $j = i, i+1$ denote nearest-point projections of $\widetilde{S} \times \{j\}$ onto λ_j. Since ϕ is a quasi-isometry, there exists $C > 0$ such that

(a) for all $(x, i) \in \lambda_i$, $(x, i+1)$ lies in a C-neighborhood of $\Phi(\lambda_i) = \lambda_{i+1}$;
(b) for all $z \in \widetilde{S}$, $d_{\mathrm{wel}}(\pi_i(z, i), \pi_{i+1}(z, i+1)) \le C$ (by Lemma 5.3 or Theorem 5.7).

We emphasize here that C is independent of both the thick block B and the geodesic λ (and hence the ladder $\mathcal{L}_\lambda$). It depends only on the model manifold M.

The same conclusions hold for ϕ^{-1} and points in λ_{i+1}, where ϕ^{-1} denotes the *quasi-isometric inverse* of ϕ from $\widetilde{S} \times \{i+1\}$ to $\widetilde{S} \times \{i\}$. The *$\mathcal{L}_\lambda$-elementary admissible paths* in $\widetilde{B}$ are defined to be paths consisting of the following:

(1) Horizontal geodesic subsegments of λ_j, $j = \{i, i+1\}$.
(2) Vertical segments of d_G-length 1 joining $x \times \{0\}$ to $x \times \{1\}$. Note that for thick blocks, $d_G = d_{\mathrm{wel}}$.
(3) Horizontal geodesic segments lying in a C-neighborhood of λ_j, $j = i, i+1$.

$\mathcal{L}_\lambda$-elementary admissible paths in the split block. Let $B = S \times [i, i+1]$ be a split block, where each (x, i) is connected by a segment of d_G-length one and d_{wel}-length $\le C(B)$ (due to bounded thickness of B, Lemma 5.9) to $(x, i+1)$. As before we regard ϕ as the map from $\widetilde{S} \times \{i\}$ to $\widetilde{S} \times \{i+1\}$ that is the identity on the first component. Also let Φ be the map on electro-ambient quasigeodesics induced by ϕ. Let $\mathcal{L}_\lambda \cap \widetilde{B} = \bigcup_{j=i,i+1} \lambda_j$ where λ_j lies on $\widetilde{S} \times \{j\}$. π_j denotes nearest-point projection of $\widetilde{S} \times \{j\}$ onto λ_j (in the appropriate sense — minimizing the ordered pair of electric and hyperbolic distances). Since ϕ is an electric isometry, but a hyperbolic quasi-isometry, there exist $C > 0$ (uniform constant) and $K = K(B)$ such that

(a) for all $x \in \lambda_i$, $\phi(x)$ lies in a (d_G-quasi) C-neighborhood and a d_{wel}-quasi K-neighborhood of $\Phi(\lambda_i) = \lambda_{i+1}$;
(b) for all $z \in \widetilde{S}$, $d_G(\pi_i(z, i), \pi_{i+1}(z, i+1)) \le C$ (by Lemma 5.4 or Theorem 5.7) and $d_{\mathrm{wel}}(\pi_i(z, i), \pi_{i+1}(z, i+1)) \le K$ (by Lemma 5.3).

The last statement follows from the fact that the block B is topologically a product and hence the map ϕ is a quasi-isometry, with quasi-isometry constants depending on B.

We re-emphasize here that C is independent of both the split block B and the geodesic λ (and hence the ladder $\mathcal{L}_\lambda$), whereas $K = K(B)$ depends on the split block B but is independent of the geodesic λ.

The same holds for ϕ^{-1} and points in λ_{i+1}, where ϕ^{-1} denotes the *quasi-isometric inverse* of ϕ from $\widetilde{S} \times \{i+1\}$ to $\widetilde{S} \times \{i\}$. It is worth pointing out here that Remark 4.12 will be used later to pull back information from the graph metric in (M_{wel}, d_G) to the model manifold $(\widetilde{M}, d_M)$ and hence via the bi-Lipschitz homeomorphism F^{-1} to $\widetilde{N}$ to give information in the hyperbolic metric.

Again, since λ_i and λ_{i+1} are electro-ambient quasigeodesics, we further note that for all $(x,i) \in \lambda_i$, $(x, i+1) \in N_K(\lambda_{i+1}, d)$, where d is the (bi-Lipschitz) hyperbolic metric on $\widetilde{S}$.

The $\mathcal{L}_\lambda$-*elementary admissible paths* in $\widetilde{B}$ consist of the following:

(1) Horizontal subsegments of λ_j, $j = \{i, i+1\}$.
(2) Vertical segments joining $x \times \{i\}$ to $x \times \{i+1\}$. These have d_{wel}-quasi-“thickness” $l = l(B)$ and d_G-thickness one, by Lemma 5.9.
(3) Horizontal geodesic segments lying in a (*bi-Lipschitz*) *hyperbolic* $K(= K(B))$-neighborhood of λ_j, $j = i, i+1$.
(4) Horizontal (bi-Lipschitz) hyperbolic segments of *electric length* $\leq C$ and (*bi-Lipschitz*) *hyperbolic length* $\leq K(B)$ joining points of the form $(\phi(x), i+1)$ to a point on λ_{i+1} for $x \in \lambda_i$.
(5) Horizontal (bi-Lipschitz) hyperbolic segments of *electric length* $\leq C$ and (*bi-Lipschitz*) *hyperbolic length* $\leq K(B)$ joining points of the form $(\phi^{-1}(x), i)$ to a point on λ_i for $x \in \lambda_{i+1}$.

Definition. An $\mathcal{L}_\lambda$-admissible path is a continuous path that can be decomposed as a union of a sequence of $\mathcal{L}_\lambda$-elementary admissible paths with disjoint interiors.

The next lemma follows from the above definition and Lemma 5.9.

LEMMA 6.3. *There exists a function $g : \mathbb{Z} \to \mathbb{N}$ such that for any block B_i, and x lying on an $\mathcal{L}_\lambda$-admissible path in $\widetilde{B_i}$, there exist $y \in \lambda_i$ and $z \in \lambda_{i+1}$ such that*

$$\begin{aligned} d_{\mathrm{wel}}(x,y) &\leq g(i), \\ d_{\mathrm{wel}}(x,z) &\leq g(i), \\ d_M(x,y) &\leq g(i), \\ d_M(x,z) &\leq g(i). \end{aligned}$$

The following is an easy corollary of Lemma 6.3.

COROLLARY 6.4. *There exists a function $h : \mathbb{Z} \to \mathbb{N}$ such that for any block B_i, and x lying on a $\mathcal{L}_\lambda$-admissible path in $\widetilde{B_i}$, there exist $y \in \lambda_0 = \lambda$ such that*

$$d_{\mathrm{wel}}(x,y) \leq h(i),$$
$$d_M(x,y) \leq h(i).$$

Proof. Let $h(i) = \Sigma_{j=0\cdots i} g(j)$ be the sum of the values of $g(j)$ as j ranges from 0 to i (with the assumption that increments are by $+1$ for $i \geq 0$ and by -1 for $i \leq 0$). $\square$

Note. In Lemma 6.3 and Corollary 6.4, it is important to note that the distance d_{wel} (resp. d_M) is the weld (resp. model) metric, not the graph metric. This is because the lengths occurring in $\mathcal{L}_\lambda$-elementary admissible paths of types (4) and (5) above are (bi-Lipschitz) hyperbolic lengths depending only on i (in B_i).

LEMMA 6.5. *There exists a function $M(N) : \mathbb{N} \to \mathbb{N}$ such that $M(N) \to \infty$ as $N \to \infty$, for which the following holds. For any geodesic $\lambda \subset \widetilde{S} \times \{0\} \subset \widetilde{B_0}$, a fixed reference point $p \in \widetilde{S} \times \{0\} \subset \widetilde{B_0}$ and any x on an $\mathcal{L}_\lambda$-admissible path,*

$$d(\lambda, p) \geq N \Rightarrow d_{\mathrm{wel}}(x,p) \geq M(N) \text{ and } d_M(x,p) \geq M(N).$$

Proof. Suppose that λ lies outside $B_N(p)$, the N-ball about a fixed reference point p on the boundary horizontal surface $\widetilde{S} \times \{0\} \subset \widetilde{B_0}$. Then by Corollary 6.4, any x lying on an $\mathcal{L}_\lambda$-admissible path in $\widetilde{B_i}$ satisfies

$$d_{\mathrm{wel}}(x,p) \geq N - h(i).$$

Also, since the electric, and d_{wel}-"thickness" (the shortest distance between its boundary horizontal sheets) is $\geq k_0$ (by uniform k_0-separatedness of horizontal sheets), we get

$$d_{\mathrm{wel}}(x,p) \geq |i| k_0.$$

Assume for convenience that $i \geq 0$. (A similar argument works, reversing signs for $i < 0$.) Then,

$$d_{\mathrm{wel}}(x,p) \geq \min_i, \quad \max\{ik_0, N - h(i)\}.$$

Let $h_1(i) = h(i) + ik_0$. Then h_1 is a monotonically increasing function on the integers. If $M(N) = h_1^{-1}(N)$ denote the largest positive integer n such that $h_1(n) \leq N$, then clearly $M(N) \to \infty$ as $N \to \infty$. Also, $d_{\mathrm{wel}}(x,p) \geq k_0 M(N)$, and the first conclusion of the lemma follows. The same arguments work for $(\widetilde{M}, d_M)$. $\square$

6.3. *Projecting to $\mathcal{L}_\lambda$ and joining the dots.*

Definition 6.6. An *$\mathcal{L}_\lambda$-admissible* (d_G) *K-quasigeodesic* is an $\mathcal{L}_\lambda$-admissible path that is a K-quasigeodesic in $(\widetilde{M_{\rm wel}}, d_G)$.

Our strategy in this subsection is to project the intersection of an admissible quasigeodesic (Lemma 6.2) with the horizontal sheets $\widetilde{M_H}$ onto $\mathcal{L}_\lambda$ and then obtain a *connected* $\mathcal{L}_\lambda$-admissible quasigeodesic from it by interpolating $\mathcal{L}_\lambda$-admissible paths. We think of this last step as "*joining the dots.*" The end product is thus a *connected* d_G-quasigeodesic built up of $\mathcal{L}_\lambda$ admissible paths.

Lemma 6.7. *There exist $K \geq 1$ and a function $M(N) : \mathbb{N} \to \mathbb{N}$ with $M(N) \to \infty$ as $N \to \infty$ such that the following holds. Let B_0 denote the first block* (*thick or split*) *in $M_{\rm wel}$, and let $S \times \{0\}$ denote its lower boundary. For a fixed reference point $p \in \widetilde{S} \times \{0\} \subset \widetilde{B_0}$ and any geodesic $\lambda \subset \widetilde{S} \times \{0\} \subset \widetilde{B_0}$, there exists an $\mathcal{L}_\lambda$-admissible* (d_G) *K-quasigeodesic $\beta_{\rm adm} \subset \widetilde{M_{\rm wel}}$ without backtracking such that*

(1) *$\beta_{\rm adm}$ joins the endpoints of λ.*
(2) *$d(\lambda, p) \geq N \Rightarrow d_{\rm wel}(\beta_{\rm adm}, p) \geq M(N)$.*

Proof. Let a, b denote the endpoints of λ. First, by Lemma 6.2 there exists an admissible d_G-geodesic $\beta_e \subset \widetilde{M_{\rm wel}}$ joining a, b. We now look at $\Pi_\lambda(\beta_e \cap \widetilde{M_H})$ obtained by acting on $\beta_e \cap \widetilde{M_H}$ by Π_λ. From Theorem 5.7, we shall conclude that the image $\Pi_\lambda(\beta_e \cap \widetilde{M_H})$ is a d_G quasigeodesic carried by $\mathcal{L}_\lambda$ in an appropriate sense as explicated below.

Since β_e is itself an admissible path, there is a sequence of points $a = a_1, b_1, a_2, b_2, \ldots, a_k, b_k = b$ such that the piece of β_e joining a_i to b_i is horizontal, whereas the piece of β_e joining b_i to a_{i+1} is vertical. In particular, b_i and a_{i+1} must lie in the same split component if they lie in (the universal cover of) a split block. In this case, $\Pi_\lambda(b_i)$ and $\Pi_\lambda(a_{i+1})$ must also lie in the same split component. We shall now join the sequence of points $\Pi_\lambda(a_1), \Pi_\lambda(b_1), \Pi_\lambda(a_2), \Pi_\lambda(b_2), \ldots, \Pi_\lambda(a_k), \Pi_\lambda(b_k)$ by horizontal and vertical segments to obtain an $\mathcal{L}_\lambda$-admissible path $\beta_{\rm adm}$ as follows.

For all i, $[\Pi_\lambda(a_i), \Pi_\lambda(b_i)]$ will be a geodesic in the horizontal sheet $\widetilde{S_i}$, joining $\Pi_\lambda(a_i), \Pi_\lambda(b_i)$. The $\mathcal{L}_\lambda$-admissible path joining $\Pi_\lambda(b_i), \Pi_\lambda(a_{i+1})$ requires more care to define. For notational simplicity, let $b_i = p$ and $a_{i+1} = q$.

(1) Let $[p, q]$ be a vertical segment in a thick block joining p, q. Then $\Pi_\lambda(p), \Pi_\lambda(q)$ are a uniformly bounded $d_{\rm wel}$-distance apart by Theorem 5.7. Hence, by Lemma 5.3, we can join $\Pi_\lambda(p), \Pi_\lambda(q)$ by an $\mathcal{L}_\lambda$-admissible path of length bounded by some C_0 (independent of B, λ).

For a thick block, we define the $\mathcal{L}_\lambda$-admissible path joining $\Pi_\lambda(p), \Pi_\lambda(q)$ to be any such $\mathcal{L}_\lambda$-admissible path of uniformly bounded $d_{\rm wel}$-length.

(2) Let $[p, q]$ be a vertical segment in a split block $\widetilde{B_i}$ of d_G-length one and d_{wel}-length $\leq l_i$ joining p, q, where $p \in \widetilde{S_i}$, the lower horizontal boundary of $\widetilde{B_i}$ and $q \in \widetilde{S_{i+1}}$, the upper horizontal boundary of $\widetilde{B_i}$. Since p, q lie within a split component, $d_G(\Pi_\lambda(p), \Pi_\lambda(q)) = 1$; that is to say, $\Pi_\lambda(p), \Pi_\lambda(q)$ also lie within a split component. This is because the projection of a split component lies within a single split component. Hence there exists an admissible path $[\Pi_\lambda(p), \Pi_\lambda(q)]$ of d_G-length one joining $\Pi_\lambda(p), \Pi_\lambda(q)$. Further, by Lemma 5.3 again, we can join $\Pi_\lambda(p), \Pi_\lambda(q)$ by an $\mathcal{L}_\lambda$-admissible path of d_{wel}-length bounded by some C_i (dependent on B_i but independent of λ). Note that since C_i depends on B_i, it depends on l_i in particular.

(3) By Remark 5.6, the two images under nearest projection of a point in $\widetilde{S_i}$ onto respectively a hyperbolic geodesic and an electro-ambient quasigeodesic in $\widetilde{S_i}$ (joining any pair of points) are a uniformly bounded (bi-Lipschitz)-hyperbolic distance apart. Hence, by Lemma 5.5, we can join them by an $\mathcal{L}_\lambda$-admissible path of length bounded by some uniform C_1 (independent of B_i, λ).

A clarificatory remark as to why segments of type (3) *are necessary.* In defining $\mathcal{L}_\lambda$, we have had to make a choice. Suppose $\lambda_i \subset \widetilde{S}_i$. Then S_i is the common boundary of two blocks. In case both are split blocks then there is a choice of λ_i out of two electro-ambient quasigeodesics involved. If one is a split block and the other a thick block, then there is a choice of λ_i involved out of an electro-ambient quasigeodesic and a geodesic. The different nearest-point projections corresponding to the different choices of λ_i differ by a uniformly bounded amount (Remark 5.6). Segments of type (3) take care of this bounded discrepancy.

For a split block, we define the $\mathcal{L}_\lambda$-admissible path joining $\Pi_\lambda(p), \Pi_\lambda(q)$ to consist of one $\mathcal{L}_\lambda$-admissible path constructed in step (2) above and (at most) two segments of uniformly bounded d_{wel}-length as in step (3). Thus an $\mathcal{L}_\lambda$-admissible path joining $\Pi_\lambda(p), \Pi_\lambda(q)$ contains one vertical segment of type (2) typically sandwiched between two segments of type (3).

Joining $\Pi_\lambda(a_i), \Pi_\lambda(b_i)$ by $[\Pi_\lambda(a_i), \Pi_\lambda(b_i)]$ and $\Pi_\lambda(b_i), \Pi_\lambda(a_{i+1})$ by $\mathcal{L}_\lambda$-admissible paths as above, we obtain the required $\mathcal{L}_\lambda$-admissible (d_G) K-quasigeodesic $\beta_{\mathrm{adm}} \subset \widetilde{M_{\mathrm{wel}}}$.

By Theorem 5.7, there exists $K \geq 1$ such that β_{adm} represents a $(d_G){-}K$-quasigeodesic. This proves statement (1) of the lemma.

After "joining the dots" by $\mathcal{L}_\lambda$-admissible paths as above, we can assume further that the $\mathcal{L}_\lambda$-admissible quasigeodesic β_{adm} thus obtained does not backtrack relative to split components. Conclusion (2) of the lemma now follows from Lemma 6.5 since we have obtained an admissible quasigeodesic built up out of $\mathcal{L}_\lambda$-admissible paths. $\square$

6.4. *Recovering electro-ambient quasigeodesics* I. This subsection is devoted to extracting an electro-ambient quasigeodesic β_{ea} in $(\widetilde{M_{\mathrm{wel}}}, d_G)$ from an $\mathcal{L}_\lambda$-admissible quasigeodesic β_{adm}. β_{ea} shall satisfy the property indicated by Lemma 6.7.

LEMMA 6.8. *There exist $\kappa \geq 1$ and a function $M'(N) : \mathbb{N} \to \mathbb{N}$ with $M'(N) \to \infty$ as $N \to \infty$ such that the following holds. Let B_0 denote the first block (thick or split) in M_{wel}, and let $S \times \{0\}$ denote its lower boundary. For a fixed reference point $p \in \widetilde{S} \times \{0\} \subset \widetilde{B_0}$ and any geodesic $\lambda \subset \widetilde{S} \times \{0\} \subset \widetilde{B_0}$, there exists an electro-ambient κ-quasigeodesic β_{ea} without backtracking in $(\widetilde{M_{\mathrm{wel}}}, d_G)$ such that*

- *β_{ea} joins the endpoints of λ,*
- *$d(\lambda, p) \geq N \Rightarrow d_{\mathrm{wel}}(\beta_{\mathrm{ea}}, p) \geq M'(N)$.*

Proof. From Lemma 6.7, we have an $\mathcal{L}_\lambda$-admissible κ-quasigeodesic β_{adm} without backtracking (with respect to the collection $\mathcal{K}$ of split components $\widetilde{K}$) and a function $M(N)$ satisfying the conclusions of the lemma. Since β_{adm} does not backtrack, we can decompose it as a union of nonoverlapping segments $\beta_1, \ldots, \beta_k$ such that only successive β_i's intersect at one common endpoint and each β_i is

(a) either an $\mathcal{L}_\lambda$- admissible quasigeodesic lying outside split components,
(b) or an $\mathcal{L}_\lambda$-admissible quasigeodesic lying entirely within some split component $\widetilde{K}_{n(i)}$.

Further, since β_{adm} does not backtrack relative to split components, we can assume that all $\widetilde{K}_{n(i)}$'s are distinct; i.e., $i \neq j \Rightarrow \widetilde{K}_{n(i)} \neq \widetilde{K}_{n(j)}$.

We modify β_{adm} to an electro-ambient quasigeodesic β_{ea} in $(\widetilde{M_{\mathrm{wel}}}, d_G)$ as per the following recipe:

(1) β_{ea} coincides with β_{adm} outside split components.
(2) If some β_i lies within a split component $\widetilde{K}_{n(i)}$, then we replace it by a geodesic β_i^{ea} *in the intrinsic metric on* $\widetilde{K}_{n(i)}$ joining the endpoints of β_i. Of course β_i^{ea} lies within $\widetilde{K}_{n(i)}$.

Since β_{ea} coincides with β_{adm} outside split components and since β_{adm} is a (d_G) κ-quasigeodesic, thus β_{ea} represents a (d_G) κ-quasigeodesic. Hence, the resultant path β_{ea} is an electro-ambient κ-quasigeodesic without backtracking.

Next, since any amalgamation component of $\widetilde{S}$ is quasiconvex in the split component $\widetilde{K}$ containing it, each segment β_i^{ea} lies in a C_i-neighborhood of β_i. Here C_i depends on the quasiconvexity constants of the amalgamation components in split components and hence only on the thickness l_i of the split component $K_{n(i)}$.

We let $C(m)$ denote the maximum of the (finitely many) values of C_i for the split components of $\widetilde{B_m}$, where we take $C(m) = 0$ if B_m is thick. (This makes sense as β_{ea} coincides with β_{adm} outside split components.) Then, as in the proof of Lemma 6.5, for any $z \in \beta_{\mathrm{ea}} \cap \widetilde{B_m}$, we have

$$d(z, p) \geq \max(mk_0, M(N) - C(m)).$$

Again, as in Lemma 6.5, this gives us a (new) function $M'(N) : \mathbb{N} \to \mathbb{N}$ such that $M'(N) \to \infty$ as $N \to \infty$, for which

$$d(\lambda, p) \geq N \Rightarrow d_{\mathrm{wel}}(\beta_{\mathrm{ea}}, p) \geq M'(N).$$

This proves the lemma. $\square$

6.5. *Recovering electro-ambient quasigeodesics* II. This subsection is devoted to extracting an electro-ambient quasigeodesic β_{ea2} in $\widetilde{M}_2 = (\widetilde{M}, d_{\mathrm{CH}})$ from an electro-ambient quasigeodesic β_{ea} in $\widetilde{M}_1 = (\widetilde{M_{\mathrm{wel}}}, d_G)$. β_{ea2} shall satisfy the property indicated by Lemmas 6.7 and 6.8.

Recall that $\widetilde{M}_2 = (\widetilde{M}, d_{\mathrm{CH}})$ denotes $\widetilde{M}$ with the electric metric obtained by electrocuting the convex hulls $\mathrm{CH}(\widetilde{K})$ of extended split components $\widetilde{K}$. Also, recall that an electro-ambient k-quasigeodesic γ in $(\widetilde{M}, d_{\mathrm{CH}})$ is a k-quasigeodesic in $(\widetilde{M}, d_{\mathrm{CH}})$ such that in an ordering (from the left) of the convex hulls of split components that γ meets, each $\gamma \cap \mathrm{CH}(\widetilde{K})$ is a geodesic in the *intrinsic* metric on $\mathrm{CH}(\widetilde{K})$ (which in turn is uniformly bi-Lipschitz to the hyperbolic metric on $\mathrm{CH}(\widetilde{K})$ under the bi-Lipschitz homeomorphism between the model manifold M and the hyperbolic manifold N).

The underlying sets $\widetilde{M_{\mathrm{wel}}}$ (for $\widetilde{M}_1$) and $\widetilde{M}$ (for $\widetilde{M}_2$) are homeomorphic as topological spaces. Also, $\widetilde{M}_1$ is obtained by electrocuting the welded metric, i.e., $(\widetilde{M_{\mathrm{wel}}}, d_{\mathrm{wel}})$, whereas $\widetilde{M}_2$ is obtained by electrocuting the model metric, i.e., $(\widetilde{M}, d_M)$. Note further that the metrics $(\widetilde{M}, d_{\mathrm{wel}})$ and $(\widetilde{M}, d_M)$ coincide off Margulis tubes.

Now we need to set up a correspondence between paths in $(\widetilde{M_{\mathrm{wel}}}, d_{\mathrm{wel}})$ and $(\widetilde{M}, d_M)$, and hence between $\widetilde{M}_1 = (\widetilde{M}, d_G)$ and $\widetilde{M}_2 = (\widetilde{M}, d_{\mathrm{CH}})$.

Remark 6.9. Paths $\alpha_i \subset \widetilde{M}_i$ $(i = 1, 2)$ are said to *correspond* if

(1) they coincide off Margulis tubes,
(2) each piece of α_2 inside a (closed) Margulis tube is a geodesic in the model metric d_M.

It follows that any path $\alpha_1 \subset \widetilde{M}_1$ corresponds to a unique $\alpha_2 \subset \widetilde{M}_2$.

LEMMA 6.10. *There exist $\kappa \geq 1$ and a function $M'(N) : \mathbb{N} \to \mathbb{N}$ such that $M'(N) \to \infty$ as $N \to \infty$ for which the following holds. Let B_0 denote the first block* (*thick or split*) *in M_{wel}, and let $S \times \{0\}$ denote its lower boundary. For a fixed reference point $p \in \widetilde{S} \times \{0\} \subset \widetilde{B_0}$ and any geodesic $\lambda \subset \widetilde{S} \times \{0\} \subset \widetilde{B_0}$, there exists an electro-ambient κ-quasigeodesic β_{ea} without backtracking in $(\widetilde{M_{\mathrm{wel}}}, d_G)$ and a path β_{ea1} corresponding to β_{ea} in $(\widetilde{M}, d_{\mathrm{CH}})$ such that*

(1) *β_{ea1} joins the endpoints of λ,*
(2) *$d(\lambda, p) \geq N \Rightarrow d(\beta_{\mathrm{ea1}}, p) \geq M'(N)$.*

Proof. By Lemma 6.8, there exists an electro-ambient κ_0-quasigeodesic β_{ea} in $\widetilde{M}_1 = (\widetilde{M_{\mathrm{wel}}}, d_G)$ joining the endpoints of λ (where κ_0 is independent of λ). By Remark 6.9, β_{ea} *corresponds* to a unique path, which we call β_{ea1}, in $\widetilde{M}_2$. β_{ea1} is obtained by replacing intersections of β_{ea} with tube-electrocuted Margulis tubes by hyperbolic geodesics lying in the corresponding Margulis tubes as per Remark 6.9. From Lemma 4.29, $(\widetilde{M}, d_{\mathrm{CH}})(= \widetilde{M}_2)$ is quasi-isometric to $(\widetilde{M}, d_G)$. Hence there exists $\kappa \geq 1$ such that for any λ, the path β_{ea1} is a κ-quasigeodesic in $\widetilde{M}_2$.

Also by Lemma 6.8, there exists a function $M(N) : \mathbb{N} \to \mathbb{N}$ such that $M(N) \to \infty$ as $N \to \infty$ for which the following holds. If $d(\lambda, p) \geq N$, then β_{ea} lies outside a large $M(N)$-ball about p in $(\widetilde{M}_{\mathrm{wel}}, d_{\mathrm{wel}})$.

It follows that the intersection of β_{ea} with the boundary $\partial\widetilde{T}$ of the lift $\widetilde{T}$ of any Margulis tube T lies outside an $M(N)$-ball about p. Each point $x \in \beta_{\mathrm{ea}} \cap \partial\widetilde{T}$ lies on a unique totally geodesic hyperbolic disk $D_x \subset \widetilde{T}$. Also, $\beta_{\mathrm{ea1}} \cap \widetilde{T} \subset \bigcup_{x \in \beta_{\mathrm{ea}} \cap \partial\widetilde{T}} D_x$ by the convexity of $\bigcup_{x \in \beta_{\mathrm{ea}} \cap \partial\widetilde{T}} D_x$. Let the maximum diameter of Margulis tubes intersecting the ith block in $\widetilde{M}$ be t_i. Then $d_M(\beta_{\mathrm{ea1}} \cap \widetilde{B}_i, p) \geq d_{\mathrm{wel}}(\beta_{\mathrm{ea}} \cap \widetilde{B}_i, p) - t_i \geq M(N) - t_i$. Now, a reprise of the argument in Lemma 6.5 shows that β_{ea1} lies outside a large $M'(N)$ ball about p, where $M'(N) \to \infty$ as $N \to \infty$. $\square$

To obtain an electro-ambient quasigeodesic β_{ea2} in $(\widetilde{M}, d_{\mathrm{CH}})$ from β_{ea1}, first observe that there exists D_0 such that the diameter in the d_G metric $\mathrm{dia}_G(\beta_{\mathrm{ea1}} \cap \mathrm{CH}(\widetilde{K})) \leq D_0$ for any $\mathrm{CH}(\widetilde{K})$. This follows from the fact that β_{ea1} is a κ-quasigeodesic in $(\widetilde{M}, d_G)$ and from Lemma 4.29, which says that $(\widetilde{M}, d_{\mathrm{CH}})$ and $(\widetilde{M_{\mathrm{wel}}}, d_G)$ are quasi-isometric.

LEMMA 6.11. *For every $D_0 \geq 0$ and split component $\widetilde{K} \subset \widetilde{M_{\mathrm{wel}}}$, there exists $D_1 \geq 0$ such that the following holds. Let $\alpha \subset \mathrm{CH}(\widetilde{K}) \subset \widetilde{M}$ be a path such that the path η in $\widetilde{M_{\mathrm{wel}}}$ corresponding to it is of length at most D_0 in the d_G metric. Further suppose that*

(a) *$\alpha \cap \widetilde{C}(\subset \widetilde{M})$ for any split component $\widetilde{C}$ is a geodesic in the intrinsic metric on $\widetilde{C}$,*
(b) *$\alpha \cap \mathbb{T}$ is a hyperbolic geodesic for any lift $\mathbb{T}$ of a Margulis tube.*

Let $\gamma = [a, b]$ be the (model) hyperbolic geodesic in $(\widetilde{M}, d_M)$ joining the endpoints a, b of α. Then γ lies in a (d_M-) D_1-neighborhood of α.

Proof. Note first that the complement in $\widetilde{M}$ of the union of split components is the union of the universal covers of thick blocks and Margulis tubes. Hence by the hypotheses α can be described as the union of at most $3D_0$ pieces $\alpha_1, \ldots, \alpha_j (j \leq 3D_0)$ such that each α_i is either a geodesic in the intrinsic metric on $\widetilde{C}$ for some split component $\widetilde{C}$ or a geodesic in $(\widetilde{M}, d_M)$.

Let β_i be the geodesic in $(\widetilde{M}, d_M)$ joining the endpoints of α_i. Then $d(\gamma, \cup_i \beta_i) \leq j\delta_0 \leq 3D_0\delta_0$, where δ_0 is the (Gromov) hyperbolicity constant of $\widetilde{M}$.

Since α meets a bounded number of split components, there exists $C_1 \geq 0$ such that each split component $\widetilde{C}$ that α meets is C_1-quasiconvex. Note that C_1 depends only on the convex hull $\mathrm{CH}(\widetilde{K})$ and the fact that any $\mathrm{CH}(\widetilde{K})$ meets the lifts of only a uniformly bounded number of split components by graph-quasiconvexity (Theorem 4.31). Hence for any $\alpha_i \subset \widetilde{C}$, $d_M(\alpha_i, \beta_i) \leq C_1$. Choosing $D_1 = C_1 + 3D_0\delta_0$, we are through. $\square$

We are now in a position to obtain the last "recovery" lemma of this section. The main part of the argument is again a reprise of a similar argument in Lemma 6.5. We shall recount it briefly for completeness.

LEMMA 6.12. *There exist $\kappa \geq 1$ and a function $M_0(N) : \mathbb{N} \to \mathbb{N}$ such that $M_0(N) \to \infty$ as $N \to \infty$, for which the following holds. Let B_0 denote the first block (thick or split) in M_{wel}, and let $S \times \{0\}$ denote its lower boundary. For a fixed reference point $p \in \widetilde{S} \times \{0\} \subset \widetilde{B_0}$ and any geodesic $\lambda \subset \widetilde{S} \times \{0\} \subset \widetilde{B_0}$, there exists an electro-ambient κ-quasigeodesic β_{ea2} without backtracking in $(\widetilde{M}, d_{\mathrm{CH}})$ such that*

(1) *β_{ea2} joins the endpoints of λ,*
(2) *$d(\lambda, p) \geq N \Rightarrow d_M(\beta_{\mathrm{ea2}}, p) \geq M_0(N)$.*

Proof. By Lemma 6.10, there exist κ_0 and a function $M'(N) : \mathbb{N} \to \mathbb{N}$ such that for any geodesic $\lambda \subset \widetilde{S} \times \{0\} \subset \widetilde{B_0}$ with $d(\lambda, p) \geq N$, there exists a path α in $(\widetilde{M}, d_{\mathrm{CH}})$ *corresponding* (as per Remark 6.9) to an electro-ambient quasigeodesic in $(\widetilde{M}, d_G)$ satisfying the following:

(a) α joins the endpoints of λ,
(b) $d_M(\alpha, p) \geq M'(N)$,
(c) $N \to \infty \Rightarrow M'(N) \to \infty$.

Let β_{ea2} be an electro-ambient quasigeodesic in $(\widetilde{M}, d_{\mathrm{CH}})$ joining the endpoints of α. Let $\mathcal{CH}(\widetilde{\mathcal{K}})$ be the collection of (images under the bi-Lipschitz homeomorphism F of) convex hulls of extended split components. Recall that β_{ea2} is obtained by looking at the intervals of intersection of α with $\mathrm{CH}(\widetilde{K}) \in \mathcal{CH}(\widetilde{\mathcal{K}})$, ordered from the left, and replacing maximal intersections with (model) hyperbolic geodesics in $\mathrm{CH}(\widetilde{K})$.

Let $x \in \beta_{\mathrm{ea2}} \cap \mathrm{CH}(\widetilde{K})$ for an extended split component $\widetilde{K}$. Then by construction of the electro-ambient quasigeodesic β_{ea2} from α and Lemma 6.11, there exists $y \in \alpha \cap \mathrm{CH}(\widetilde{K})$ and $D_1 = D_1(K)$ such that $d(x, y) \leq D_1$.

By uniform graph-quasiconvexity (Theorem 4.31), for each i, there exist finitely many extended split components K such that $\widetilde{B_i} \cap \mathrm{CH}(\widetilde{K}) \neq \emptyset$. Let D_i be the maximum value of the $D_1(K)$'s for these split components. Hence $x \in$

$\beta_{\mathrm{ea2}} \cap \widetilde{B_i}$ implies that $d(x,p) \geq M'(N) - D_i$. Also, by uniform k_0-separatedness of split surfaces, $x \in \widetilde{B_i}$ implies that $d(x,p) \geq ik_0$. Therefore,

$$d(\beta_{\mathrm{ea2}}, p) \geq \min_i \max\left(ik_0, M'(N) - \sum_{j \leq i} D_j\right).$$

Defining $M_0(N)$ to be $M_0(N) = \min_i \max(ik_0, M'(N) - \sum_{j \leq i} D_j)$ and observing that $M_0(N) \to \infty$ as $N \to \infty$ (by the same argument as in Lemma 6.5), we are through. $\square$

6.6. *Application to sequences of surface groups.* The main proposition of this subsection will be used in [MS13]. The proof of Lemma 6.12 gives the following.

Corollary 6.13. *Let D be a positive integer. Let*

$$B_{-D}, \ldots, B_0, \ldots, B_n, \ldots, B_{n+D}$$

be a collection of split blocks, and let $\mathcal{B}_n^1$ be the union of these blocks glued along the common boundary split surfaces (i.e., B_{i-1} is glued to B_i along S_i). We assume that this gluing can be done consistently (i.e., the Margulis tubes are compatible). Let $\mathcal{B}_n = \bigcup_1^n B_i \subset \mathcal{B}_n^1$. Let M be a manifold of split geometry (not necessarily simply or doubly degenerate; i.e., we allow M to have finitely many split blocks) such that each split component is D-graph quasiconvex and $\mathcal{B}_n^1 \subset M$. Then for all $L \geq 0$, there exists $N \geq 0$ such that the following holds. For all geodesic segments λ lying outside an N-ball around $o \in \tilde{S}_0$ and any electro-ambient quasigeodesic β_{ea2}^n without backtracking in $\widetilde{M}$ joining the endpoints of λ, $\beta_{\mathrm{ea2}}^n \cap \widetilde{\mathcal{B}_n}$ lies outside the L-ball around $o \in \tilde{M}$.

Corollary 6.13 will be used to prove the convergence of Cannon-Thurston maps for quasi-Fuchsian groups converging strongly to a simply degenerate group.

Remark 6.14. In Corollary 6.13, we could replace $\mathcal{B}_n^1$ by

$$\mathcal{B}_n^2 = B_{-n-D}, \ldots, B_0, \ldots, B_n, \ldots, B_{n+D},$$

and the same conclusions follow. This will be used to prove the convergence of Cannon-Thurston maps for quasi-Fuchsian groups converging strongly to a doubly degenerate group.

7. Cannon-Thurston maps for surfaces without punctures

We note the following properties of $(\widetilde{M}, d_G)$ and $\mathcal{K}$ where $(\widetilde{M}, d_G)$ is the graph model of $\widetilde{M}$ and $\mathcal{K}$ consists of the split components. There exist C, D, Δ such that

(1) Each split component is C-graph quasiconvex by Theorem 4.31.
(2) $(\widetilde{M}, d_G)$ is Δ-hyperbolic.

(3) Given K, ε, there exists D_0 such that if γ is a (K,ε) quasigeodesic in $(\widetilde{M}, d_M)$ joining a, b and if β is a (K,ε) electro-ambient quasigeodesic in $(\widetilde{M}, d_G)$ joining a, b, then γ lies in a D_0-neighborhood of β in $(\widetilde{M}, d_M)$. This follows from Lemma 2.5.

We shall now assemble the proof of the main theorem.

Theorem 7.1. *Let M be a simply or doubly degenerate hyperbolic 3-manifold without parabolics, homeomorphic to $S \times J$ (for $J = [0,\infty)$ or $(-\infty,\infty)$ respectively). Fix a base surface $S_0 = S \times \{0\}$. Then the inclusion $i : \widetilde{S_0} \to \widetilde{M}$ extends continuously to a map between the compactifications $\hat{i} : \widehat{S_0} \to \widehat{M}$. Hence the limit set of $\widetilde{S_0}$ is locally connected.*

Proof. By Theorem 4.31, M has split geometry, and we may assume that $S_0 \subset B_0$, the first block. Let $(\widetilde{M}, d_{\mathrm{CH}})$ and $(\widetilde{M}, d_G)$ be as above, and let d_M be the model metric on $\widetilde{M}$. Suppose $\lambda \subset \widetilde{S_0}$ lies outside a large N-ball about p in the (bi-Lipschitz) hyperbolic metric on $\widetilde{S_0}$. By Lemma 6.12 we obtain an electro-ambient quasigeodesic without backtracking β_{ea2} joining the endpoints of λ and lying outside an $M_0(N)$-ball about p in $(\widetilde{M}, d_M)$, where $M_0(N) \to \infty$ as $N \to \infty$.

Suppose that β_{ea2} is a (κ,ε) electro-ambient quasigeodesic. Note that κ, ε depend on "the coarse Lipschitz constant" of Π_λ and hence only on $\widetilde{S_0}$ and $\widetilde{M}$.

From Lemma 2.5 we know that if β^h denotes the (model) hyperbolic geodesic in $\widetilde{M}$ joining the endpoints of λ, then β^h lies in a (uniform) C'-neighborhood of β_{ea2}.

Let $M_1(N) = M_0(N) - C'$. Then $M_1(N) \to \infty$ as $N \to \infty$. Further, the (model) hyperbolic geodesic β^h lies outside an $M_1(N)$-ball around p. Hence, by Lemma 1.8, the inclusion $i : \widetilde{S_0} \to \widetilde{M}$ extends continuously to a map $\hat{i} : \widehat{S_0} \to \widehat{M}$.

Since the continuous image of a compact locally connected set is locally connected [HY61] and the (intrinsic) boundary of $\widetilde{S_0}$ is a circle, we conclude that the limit set of $\widetilde{S_0}$ is locally connected. This proves the theorem. $\square$

8. Modifications for surfaces with punctures

In this section, we shall describe the modifications necessary to prove Theorem 7.1 for surfaces with punctures.

8.1. *Partial electrocution.* Two general references for this subsection are [MR08], [MP11], where much of what follows is done in a considerably more general setting. Let M be a convex hyperbolic 3-manifold with a neighborhood of the cusps excised. Then each component of the boundary of M is of the form $\sigma \times P$, where P is either an interval or a circle, and σ is a horocycle of some fixed length e_0. Each component of the boundary of the universal cover $\widetilde{M}$ is a flat horosphere of the form $\widetilde{\sigma} \times \tilde{P}$. Note that $\tilde{P} = P$ if P is an interval and $\mathbb{R}$ if P is a circle (the case for a $(Z+Z)$-cusp).

The construction of *partially electrocuted* horospheres below is half way between the spirit of Farb's construction (in Lemmas 2.3 and 2.9, where the entire horosphere is coned off) and McMullen's Theorem 2.13 (where nothing is coned off and properties of *ambient quasigeodesics* are investigated).

Partial electrocution of horospheres. Let Y be a convex simply connected hyperbolic 3-manifold. Let $\mathcal{B}$ denote a collection of horoballs. Let X denote Y minus the interior of the horoballs in $\mathcal{B}$. Let $\mathcal{H}$ denote the collection of boundary horospheres. Then each $H_\alpha \in \mathcal{H}$ with the induced metric is isometric to a Euclidean product $E^1 \times L_\alpha$ for an interval $L_\alpha \subset \mathbb{R}$. Here E^1 denotes Euclidean 1-space.

"Partially electrocute" each H_α by giving it the product of the zero metric with the Euclidean metric; i.e., on E^1 put the zero metric and on L_α put the Euclidean metric. Thus we are in the following situation:

(1) X is (strongly) hyperbolic relative to a collection $\mathcal{H}$ of horospheres.
(2) Each horosphere H_α is equipped with a pseudometric making it isometric to a Euclidean product $E^1 \times L_\alpha$ for an interval $L_\alpha \subset \mathbb{R}$. We shall denote the collection of L_α's by $\mathcal{L}$.

The resulting pseudometric space is denoted (X, d_{pel}) and is called the *partially electrocuted space* associated to the pair $(X, \mathcal{H})$.

Its worth pointing out here that (X, d_{pel}) is essentially what one would get (in the spirit of [Far98]) by gluing to each H_α the mapping cylinder of the projection of H_α onto the L_α-factor. Let $\mathcal{G}$ denote the collection of these projections $g_\alpha : H_\alpha \to L_\alpha$. Thus, instead of coning all of a horosphere down to a point, we cone only horocyclic leaves of a foliation of the horosphere. Effectively, therefore, we have a cone-line rather than a cone-point. We shall denote the union of X and all the mapping cylinders of g_α by $\mathcal{E}(X, \mathcal{H}, \mathcal{L}, \mathcal{G})$ in the spirit of the notation we have used for electric spaces. As pointed out above, $\mathcal{E}(X, \mathcal{H}, \mathcal{L}, \mathcal{G})$ and (X, d_{pel}) are quasi-isometric and both contain naturally embedded copies of X as a subset (though not as a metric subspace). We shall therefore conflate $\mathcal{E}(X, \mathcal{H}, \mathcal{L}, \mathcal{G})$ and (X, d_{pel}) in this subsection. Geodesics and quasigeodesics in the partially electrocuted space will be referred to as partially electrocuted geodesics and quasigeodesics respectively. In this situation, we conclude as in Lemma 2.3.

Lemma 8.1 ([MP11, Lemma 1.20]). *Let $(X, \mathcal{H}, \mathcal{L}, \mathcal{G})$ be a 4-tuple as above. Then the spaces $\mathcal{E}(X, \mathcal{H}, \mathcal{L}, \mathcal{G})$ as well as (X, d_{pel}) are hyperbolic metric spaces. Further, the subsets L_α and H_α are quasiconvex in $\mathcal{E}(X, \mathcal{H}, \mathcal{L}, \mathcal{G})$ and (X, d_{pel}) respectively.*

Recall that X is obtained from a simply connected convex hyperbolic manifold Y by excising a family of uniformly separated (open) horoballs.

Lemma 8.2 ([MP11, Lemma 1.21]). *Let* $(X, \mathcal{H}, \mathcal{L}, \mathcal{G})$ *be a* 4*-tuple as above. Given* $K, \varepsilon \geq 0$, *there exists* $C > 0$ *such that the following holds. Let* γ_{pel} *and* γ *denote respectively a* (K, ε) *partially electrocuted quasigeodesic in* $\mathcal{E}(X, \mathcal{H}, \mathcal{L}, \mathcal{G})$ *and a* (K, ε) *hyperbolic quasigeodesic in* Y *joining* a, b. *Then* $\gamma \setminus \bigcup_{H_\alpha \in \mathcal{H}} H_\alpha$ *lies in a* C*-neighborhood of* (*any representative of*) γ_{pel} *in* (X, d). *Further, outside of the horoballs that* γ *meets,* γ *and* γ_{pel} *track each other*; *i.e., they lie in a* C*-neighborhood of each other.*

Note. $\mathcal{E}(X, \mathcal{H}, \mathcal{L}, \mathcal{G})$ is strongly hyperbolic relative to the sets $\{L_\alpha\}$. In fact the space obtained by electrocuting the sets L_α in $\mathcal{E}(X, \mathcal{H}, \mathcal{L}, \mathcal{G})$ is just the space $\mathcal{E}(X, \mathcal{H})$ obtained by electrocuting the sets $\{H_\alpha\}$ in X.

Next, we show that partial electrocution preserves quasiconvexity.

Lemma 8.3. *Given* C, *there exists* C_1 *such that if* A *and* $A \cap B$ (*for any horoball* $B \in \mathcal{B}$) *are* C*-quasiconvex in* Y, *then* $(A \cap X, d_{\mathrm{pel}})$ *is* C_1*-quasiconvex in* (X, d_{pel}).

Proof. It is given that $A (\subset Y)$ as also $A \cap B$ for all $B \in \mathcal{B}$ are C-quasiconvex. Then given $a, b \in A \cap X$, the hyperbolic geodesic λ in Y joining a, b lies in a C-neighborhood of A. Since horoballs are convex, λ cannot backtrack. We let $H = \partial B$ be the boundary horosphere of the horoball B, and we let L be the element of $\mathcal{L}$ corresponding to H.

Let λ_{pel} be the partially electrocuted geodesic joining $a, b \in (X, d_{\mathrm{pel}})$. Clearly, λ_{pel} does not backtrack. Then by Lemma 8.2 above, we conclude that for all $H \in \mathcal{H}$ that λ intersects, there exist points a_H, b_H of λ_{pel} close (in Y) to the entry and exit points of λ with respect to H. The points a_H, b_H therefore lie close to $A \cap H$. Further, the corresponding L (resp. H) is quasiconvex in $\mathcal{E}(X, \mathcal{H}, \mathcal{L}, \mathcal{G})$ (resp. (X, d_{pel})) by Lemma 8.1. It follows that $\lambda_{\mathrm{pel}} \cap L$ (resp. $\lambda_{\mathrm{pel}} \cap H$) lies within a uniformly bounded distance of $A \cap H$ in $\mathcal{E}(X, \mathcal{H}, \mathcal{L}, \mathcal{G})$ (resp. (X, d_{pel})). The conclusion now follows from Lemma 8.2. $\square$

8.2. *Split geometry for surfaces with punctures.* Recall that N^h denotes (the convex core of) a simply or doubly degenerate hyperbolic 3-manifold *with cusps*. N will denote N^h minus an open neighborhood of the cusps. M will denote the model manifold (Section 3) bi-Lipschitz homeomorphic to N. Since the proof in the case of surfaces with punctures is only a small modification of the case of surfaces without punctures modulo known results (cf. [MP11, Mj09]), we shall only sketch the proof, indicating the necessary changes.

It is worth noting here that the purpose of the partial electrocution operation in the previous subsection is to ensure that successive split surfaces *with boundary* are uniformly separated so as to ensure a model of weak split geometry as defined in Remark 4.9. We shall proceed to construct a split geometry structure on M outlined in the steps below. In Steps (1)–(4) below we set up the model manifold of split geometry for S with boundary.

Step 1: *Preliminary.* For a hyperbolic surface S^h (possibly) with punctures, we fix a (small) e_0 and excise the cusps leaving horocyclic boundary components of (ordinary or Euclidean) length e_0. We then take the induced *path metric* on S^h minus cusps and call the resulting surface S. This induced path metric will still be referred to as the hyperbolic metric on S (with the understanding that now S possibly has boundary). Note that the horocycle boundary components are now totally geodesic in S.

Step 2: *Definition of thick and split blocks and hyperbolic quasiconvexity of split components.* A thick block in M is uniformly bi-Lipschitz to $S \times I$ as before.

The definitions and constructions of *split building blocks* and *split components* now go through with very little change. The only difference is that S now might have boundary curves of length e_0.

There is one subtle point about hyperbolic quasiconvexity (in $\widetilde{M}$) of split components. Hyperbolic quasiconvexity (cf. Lemma 4.16) does not hold in the metric obtained by merely excising the cusps and equipping the resulting horospheres with the Euclidean metric. What we demand is that each split component along with the parts of the horoballs that abut it be quasiconvex in $\tilde{N^h}$. Note that the intersection of split components in $\widetilde{M}$ with horoballs that abut it are (metric) products of horocycles with closed intervals. Lemma 4.16 furnishes the required quasiconvexity in this case.

When we excise horoballs from N^h to obtain N and then partially electrocute horospheres in N (or its bi-Lipschitz model M) in Step 3 below and we consider quasiconvexity in the resulting partially electrocuted space, split components will remain quasiconvex by Lemma 8.3.

Step 3: *Partially electrocuting horospherical boundaries in M.* Next, we modify the metric on M by partially electrocuting its boundary horospherical components so that the metric on the horospherical boundary components of any (thick or split) block $S \times I$ is the product of the zero metric on the horocycles of fixed (Euclidean) length e_0 and the Euclidean metric on the I-factor. The resulting blocks will be called *partially electrocuted blocks.* Note that M_{pel} may also be constructed directly from M by excising a neighborhood of the cusps and partially electrocuting the resulting horospheres. By Lemma 8.1, $\widetilde{M}_{\mathrm{pel}}$ is a hyperbolic metric space, and by Lemma 8.3, partially electrocuted split components are quasiconvex in $\widetilde{M}_{\mathrm{pel}}$.

Step 4: *Split blocks in $\widetilde{M}_{\mathrm{pel}}$ and graph-quasiconvexity.* Again, the definitions and constructions of *split blocks* and *split components* go through *mutatis*

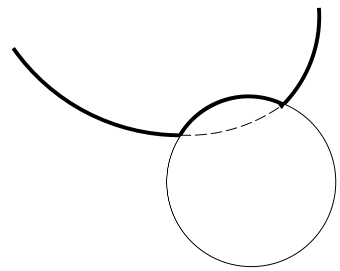

Figure 7. Horo-ambient quasigeodesic.

mutandis for the partially electrocuted manifold $\widetilde{M}_{\mathrm{pel}}$. By Lemma 8.3, quasiconvexity of split components as well as quasiconvexity of lifts of Margulis tubes are preserved by partial electrocution. Hence in the model M_{pel} obtained by gluing together partially electrocuted blocks, the split components are uniformly graph-quasiconvex.

In Steps (5)–(7) we indicate the modifications in the construction and use of the ladder $\mathcal{L}_\lambda$ and the retract Π_λ.

Step 5: *Horo-ambient quasigeodesics.* Let λ^h be a hyperbolic geodesic in $\tilde{S^h}$. We replace pieces of λ^h that lie within horodisks by shortest horocyclic segments joining its entry and exit points (into the corresponding horodisk). Such a path is called a *horo-ambient quasigeodesic*; cf. [Mj09]. See Figure 7.

A small modification might be introduced if we electrocute horocycles. Geodesics and quasigeodesics without backtracking then travel for free along the zero metric horocycles. This does not change matters much as the geodesics and quasigeodesics in the two constructions track each other by Lemma 2.9. Thus, our starting point for the construction of the hyperbolic ladder $\mathcal{L}_\lambda$ is not a hyperbolic geodesic λ^h but a horoambient quasigeodesic λ.

Step 6: *Construction of the ladder* $\mathcal{L}_\lambda$. The construction of $\mathcal{L}_\lambda, \Pi_\lambda$ and their properties go through *mutatis mutandis* and we conclude that $\mathcal{L}_\lambda$ is quasiconvex in the *graph metric* $(\widetilde{M}_{\mathrm{pel}}, d_G)$ on the partially electrocuted space $\widetilde{M}_{\mathrm{pel}}$. As before, $\widetilde{M_{H\mathrm{pel}}}$ will denote the collection of horizontal sheets.

The modification of Theorem 5.7 in this context is given below.

THEOREM 8.4. *There exists $C > 0$ such that for any horo-ambient geodesic $\lambda = \lambda_0 \subset \widetilde{S} \times \{0\} \subset \widetilde{B_0}$, the retraction $\Pi_\lambda : \widetilde{M_{H\mathrm{pel}}} \to \mathcal{L}_\lambda$ satisfies*

$$d_G(\Pi_\lambda(x), \Pi_\lambda(y)) \leq C d_G(x, y) + C.$$

Step 7: *Decomposing the ladder $\mathcal{L}_\lambda$ into $\mathcal{L}^c_\lambda$ and $\mathcal{L}^b_\lambda$.* From this step on, the modifications for punctured surfaces follow [Mj09]. As in [Mj09], we decompose λ into parts λ^c and λ^b consisting of (closures of) maximal segments that lie along horocycles and complementary pieces that do not intersect horocycles.

Accordingly, we decompose $\mathcal{L}_\lambda$ into two parts $\mathcal{L}_\lambda^c$ and $\mathcal{L}_\lambda^b$ consisting of parts that lie along horocycles and those that do not. As in Lemma 6.5, we get

LEMMA 8.5. *There exists a function $M(N) : \mathbb{N} \to \mathbb{N}$ such that $M(N) \to \infty$ as $N \to \infty$ for which the following holds. For any horo-ambient quasigeodesic $\lambda \subset \widetilde{S} \times \{0\} \subset \widetilde{B_0}$, a fixed reference point $p \in \widetilde{S} \times \{0\} \subset \widetilde{B_0}$ and any x on $\mathcal{L}_\lambda^b$,*

$$d(\lambda^b, p) \geq N \Rightarrow d_{\text{wel}}(x, p) \geq M(N).$$

In Steps (8)–(10) we indicate the process of recovering a hyperbolic geodesic.

Step 8: *Projecting and joining the dots.* Admissible paths are constructed as in Section 6.2. Now if $\lambda \subset \widetilde{S} \times \{0\} \subset \widetilde{B_0}$ is a horo-ambient geodesic joining a, b, let β be an admissible path representing a d_G geodesic in $\widetilde{M_{\text{pel}}}$. Project $\beta \cap \widetilde{M_H}_{\text{pel}}$ onto $\mathcal{L}_\lambda$ by Π_λ and "join the dots" as in Section 6.3 to get a connected ambient electric quasigeodesic β_{amb}.

Step 9: *Recovery.* As in Sections 6.4 and 6.5, construct from $\beta_{\text{amb}} \subset \widetilde{M}$ a partially electrocuted quasigeodesic γ in $(\widetilde{M}_{\text{pel}}, d_{\text{pel}})$. Observe that the parts of γ that do not lie along partially electrocuted horospheres lie close to $\mathcal{L}_\lambda^b$. Hence by Lemma 8.5, if λ^h lies outside large balls in S^h, then each point of $\gamma \setminus \bigcup_{H_\alpha \in \mathcal{H}} H_\alpha$ also lies outside large balls in $\widetilde{M}$.

At this stage we transfer the information to $\widetilde{N}$ ($=\widetilde{N^h}$ minus horoballs). Let $F : M \to N$ be the bi-Lipschitz homeomorphism between M and N, and let $\widetilde{F}$ denote its lift between universal covers. We thus conclude that if λ^h lies outside large balls in S^h, then each point of $\widetilde{F}(\gamma \setminus \bigcup_{H_\alpha \in \mathcal{H}} H_\alpha)$ also lies outside large balls in $\widetilde{N}$.

Note that in the case of surfaces without punctures, γ itself was a (bi-Lipschitz) hyperbolic geodesic in $\widetilde{M}$. However in the present situation of surfaces with punctures, one more step of recovery is necessary.

Step 10: *Conclusion.* Let γ^h denote the hyperbolic geodesic in $\widetilde{N}^h$ joining the endpoints of $\widetilde{F}(\gamma)$. By Lemma 8.2, $\widetilde{F}(\gamma)$ and γ^h track each other away from horoballs. Then, every point of $\gamma^h \cap \widetilde{N}$ must lie close to some point of $\widetilde{F}(\gamma)$ lying outside partially electrocuted horospheres. Hence from step (9), if λ^h lies outside a large ball about p in S^h, then $\gamma^h \cap \widetilde{N}$ also lies outside a large ball about p in $\widetilde{N}$. In particular, γ^h enters and leaves horoballs at large distances from p. From this it follows. (See [Mj09, Th. 5.9] for instance) that γ^h itself lies outside a large ball about p. Hence by Lemma 1.8, there exists a Cannon-Thurston map and the limit set is locally connected.

We now summarize the conclusion.

Theorem 8.6. *Let N^h be a simply or doubly degenerate 3-manifold homeomorphic to $S^h \times J$ (for $J = [0, \infty)$ or $(-\infty, \infty)$ respectively) for S^h a finite volume hyperbolic surface such that $i : S^h \to M^h$ is a proper map inducing a homotopy equivalence. Then the inclusion $i : \widetilde{S}^h \to \widetilde{N}^h$ extends continuously to a map $\hat{i} : \widehat{S}^h \to \widehat{N}^h$. Hence the limit set of $\widetilde{S}^h$ is locally connected.*

A part of the argument in Lemmas 6.8 and 6.10 and Step 9 does not use the full strength of the hypothesis that M is a model for a surface group. If we only assume that each end E of a manifold M is equipped with a split geometry structure where each split component is incompressible, then the same arguments furnish the following.

Lemma 8.7. *Let N be the convex core of a complete hyperbolic 3-manifold N^h minus a neighborhood of the cusps. Equip each degenerate end with a split geometry structure such that each split component is incompressible. Let M be the resulting model of split geometry and $F : N \to M$ be the bi-Lipschitz homeomorphism between the two. Let $\widetilde{F}$ be a lift of F to the universal covers. Then for all $C_0 > 0$ and $o \in \widetilde{N}$, there exists a function $\Theta : \mathbb{N} \to \mathbb{N}$ satisfying $\Theta(n) \to \infty$ as $n \to \infty$ such that the following holds. For any $a, b \in \widetilde{N} \subset \widetilde{N^h}$, let λ^h be the hyperbolic geodesic in $\widetilde{N^h}$ joining them and let $\lambda^h_{\mathrm{thick}} = \lambda^h \cap \widetilde{N}$. Similarly let β^h_{ea} be an electro-ambient C_0-quasigeodesic without backtracking in $\widetilde{M} \subset \mathcal{E}(\widetilde{M}, \mathcal{K}')$ joining $\widetilde{F}(a)$, $\widetilde{F}(b)$. Let $\beta_{\mathrm{ea}} = \beta^h_{\mathrm{ea}} \setminus \partial\widetilde{M}$ be the part of β^h_{ea} lying away from the (bi-Lipschitz) horospherical boundary of $\widetilde{M}$. Then $d_M(\beta_{\mathrm{ea}}, \widetilde{F}(o)) \geq n$ implies that $d_{\mathbf{H}^3}(\lambda^h_{\mathrm{thick}}, o) \geq \Theta(n)$.*

This will be useful in [Mj10b]

8.3. *Local connectivity of connected limit sets.* Here we shall use a theorem of Anderson and Maskit [AM96] along with Theorems 7.1 and 8.6 to prove that connected limit sets are locally connected. The connection between Theorems 7.1 and 8.6 and Theorem 8.9 via Theorem 8.8 is similar to one discussed by Bowditch in [Bow07].

Theorem 8.8 (Anderson-Maskit [AM96]). *Let Γ be an analytically finite Kleinian group with connected limit set. Then the limit set $\Lambda(\Gamma)$ is locally connected if and only if every simply degenerate surface subgroup of Γ without accidental parabolics has locally connected limit set.*

Combining Theorems 7.1 and 8.6 with Theorem 8.8, we have the following affirmative answer to Question 1.3.

Theorem 8.9. *Let Γ be a finitely generated Kleinian group with connected limit set Λ. Then Λ is locally connected.*

Note that Λ is connected if and only if the convex core of $\mathbb{H}^3/\Gamma$ is incompressible away from cusps. In [Mj07] we prove that for surface groups without accidental parabolics, the point pre-images of the Cannon-Thurston map for points having multiple pre-images are precisely the endpoints of leaves of the ending lamination. In [Mj10b] we shall use the techniques developed in this paper to answer Question 1.1 affirmatively.

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(Received: August 18, 2006)
(Revised: March 14, 2013)

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