Annals of Mathematics **178** (2013), 1135–1171
http://dx.doi.org/10.4007/annals.2013.178.3.7

Characters of relative p'-degree over normal subgroups

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Abstract

Let Z be a normal subgroup of a finite group G, let $\lambda \in \mathrm{Irr}(Z)$ be an irreducible complex character of Z, and let p be a prime number. If p does not divide the integers $\chi(1)/\lambda(1)$ for all $\chi \in \mathrm{Irr}(G)$ lying over λ, then we prove that the Sylow p-subgroups of G/Z are abelian. This theorem, which generalizes the Gluck-Wolf Theorem to arbitrary finite groups, is one of the principal obstacles to proving the celebrated Brauer Height Zero Conjecture.

1. Introduction

Let G be a finite group, and let p be a prime number. One of the important theorems of the Representation Theory of Finite Groups in the 1980's was to prove that if G is p-solvable and $\lambda \in \mathrm{Irr}(Z)$ is an irreducible complex character of a normal subgroup $Z \triangleleft G$ such that $\chi(1)/\lambda(1)$ is not divisible by p for all $\chi \in \mathrm{Irr}(G)$ lying over λ, then G/Z has abelian Sylow p-subgroups. This theorem, established by D. Gluck and T. Wolf in [GW84b], [GW84a] (and to which the book [MW93] is mainly devoted), led to a proof of the Richard Brauer's Height Zero Conjecture for p-solvable groups.

Recall that if B is a Brauer p-block of G with defect group P, then the Brauer Height Zero Conjecture asserts that all irreducible complex characters in B have height zero if and only if P is abelian. The "if" direction of this conjecture, which is not difficult to prove for p-solvable groups, was reduced a long time ago to quasi-simple groups in [BK88]. Only now the knowledge of

The authors are grateful to the referee for careful reading of the paper and helpful comments that greatly improved the exposition of the paper. The research of the first author is partially supported by the Spanish Ministerio de Educación y Ciencia MTM2010-15296, and Prometeo/Generalitat Valenciana. The second author gratefully acknowledges the support of the NSF (grants DMS-0901241 and DMS-1201374).

blocks of this class of finite groups has become sufficient to complete a proof of it, and this has been very recently accomplished in [KM13].

What happens with the remaining, "only if," direction of the Brauer Height Zero Conjecture? In [NT12], using the recent results on the reduction of the McKay Conjecture in [IMN07], we were able to prove it for $p = 2$ and $P \in \mathrm{Syl}_p(G)$. Also, we pointed out the necessity of obtaining the Gluck-Wolf theorem for arbitrary finite groups (a purely character-theoretic statement independent of modular representation theory) before attacking the Height Zero Conjecture. This has been confirmed in [Mur12], where it has been proved that Dade's projective conjecture [Dad94] and a proof of the generalized Gluck-Wolf theorem for arbitrary finite groups reduces the Height Zero Conjecture to quasi-simple groups. Furthermore, in [NS], the same result is obtained if every non-abelian simple group satisfies a strong form (the so called *inductive form*) of the Alperin-McKay conjecture as formulated in [Spä13].

Our aim in this paper is to prove the Gluck-Wolf theorem for arbitrary finite groups.

THEOREM A. *Let Z be a normal subgroup of a finite group G, and let $\lambda \in \mathrm{Irr}(Z)$. Let p be a prime number, and let $P/Z \in \mathrm{Syl}_p(G/Z)$. If $\chi(1)/\lambda(1)$ is coprime to p for all $\chi \in \mathrm{Irr}(G)$ lying over λ, then P/Z is abelian.*

As a consequence, we can prove the following.

COROLLARY B. *Dade's Projective Conjecture implies Brauer's Height Zero Conjecture.*

Proof. It was already pointed out by Dade in [Dad94] that Dade's Projective Conjecture implies the "if" direction of Brauer's Height Zero Conjecture.

Assume now that Dade's Projective Conjecture holds for arbitrary finite groups. To establish the truth of the "only if" direction of Brauer's Height Zero Conjecture for all finite groups, by the main result of Murai [Mur12] and Theorem A, we only have to prove that this "only if" direction holds for quasisimple groups. The latter has been recently completed by R. Kessar and G. Malle in [KM]. $\square$

Also, we remark that the fact that the inductive form of the Alperin-McKay conjecture implies Brauer's Height Zero Conjecture, which constitutes the main result of [NS], assumes Theorem A of this paper.

The case $p = 2$ of Theorem A, which helped us prove the main result in [NT12], was obtained in [Mor06]. However, this case is significantly different from the p odd case: if $p = 2$ the hypotheses in Theorem A imply that the group G/Z is solvable, then the Gluck-Wolf theorem applies. Definitely, this is not true for p odd, and this makes the proof of Theorem A much more complicated.

As happened in the p-solvable case, in order to prove Theorem A one needs to classify the finite groups of order divisible by p that act faithfully and irreducibly on an $\mathbb{F}_p$-module V having all orbits of p'-size (*p-exceptional* linear groups in the language of [GLP$^+$]) and that have abelian Sylow p-subgroups. If this was a very complicated task for p-solvable groups, to obtain this classification for arbitrary finite groups has been even more difficult. In fact, it has required the participation of a team of researchers [GLP$^+$] in order to be achieved. This classification is also of importance in the primitive permutation group theory. We will state a partial result of this classification in Theorem 2.3, which will be enough for us in order to prove the main result of this paper, referring the reader to [GLP$^+$] for the full classification of the p-exceptional linear groups.

The paper and the proof of Theorem A are organized as follows. In Section 2 we state the two results on quasi-simple groups that we will use in the proof of Theorem A. One of them, Theorem 2.2, which essentially proves Theorem A for quasi-simple groups, is established in Section 4. The other result is Theorem 2.3, whose proof can be found in [GLP$^+$]. In Section 3, we prove Theorem A using the statements in Section 2 and leaving out three cases arising from the possibilities given in Theorem 2.3. These three cases are completed by Theorems 5.1 and 6.1, which are proved in Section 5 and Section 6, respectively.

Let us mention finally that even the case $Z = 1$ in Theorem A can only be proved by appealing to the celebrated Ito-Michler theorem on character degrees [Mic86]. Also, in our proof we will be using the original Gluck-Wolf theorem on p-solvable groups.

2. **Preliminaries**

First of all, we need the following structure theorem for finite groups with abelian Sylow p-subgroups, which follows from the Classification of Finite Simple Groups.

THEOREM 2.1. *Suppose that a finite group G has abelian Sylow p-subgroups. Let $N = \mathbf{O}_{p'}(G)$ and $U/N = \mathbf{O}^{p'}(G/N)$. Then $U/N = \mathbf{O}^p(U/N) \times \mathbf{O}_p(U/N)$, where $\mathbf{O}^p(U/N)$ is either trivial or a direct product of non-abelian simple groups with nontrivial abelian Sylow p-subgroups, and $\mathbf{O}_p(U/N)$ is abelian.*

Proof. By Theorem (2.1) of [KS95], we know that $U/N = (L/N)\times(M/N)$, where L/N is trivial, or a direct product of non-abelian simple groups with abelian Sylow p-subgroup, and M/N is an abelian p-subgroup. Since $\mathbf{O}^p(L/N) = L/N$, it follows that $L/N = \mathbf{O}^p(U/N)$. Since $\mathbf{O}_p(L/N) = N$, it follows that $M/N = \mathbf{O}_p(U/N)$. □

In order to prove Theorem A, it is necessary to classify the quasisimple groups satisfying the hypotheses of the theorem with $Z = \mathbf{Z}(G)$. We do this in Theorem 2.2 (whose proof we defer until Section 4), which is a particular case of Theorem A.

Our proof of Theorem A relies on recent advances on the proof of the McKay Conjecture that were obtained in [IMN07], as well as some results on complex representations of finite groups of Lie type (see §4.2) that may have independent interest. In fact, it is essential in our proof that the simple groups appearing in Theorem 2.2 are *McKay-good* in the sense of [IMN07]. (See Section 10 of [IMN07].) In order for a simple group S to be McKay-good for the prime p, for every perfect central p'-extension G of S one has to find a *correspondence subgroup* (see Section 11 of [IMN07]) that is a particular subgroup of G that contains a p-Sylow normalizer and that possesses certain properties. Ideally, this correspondence subgroup should be a Sylow normalizer, but in some cases, it is not.

If $Z \lhd G$ and $\lambda \in \mathrm{Irr}(Z)$, then we use $\mathrm{Irr}(G|\lambda)$ to denote the set of the irreducible characters $\chi \in \mathrm{Irr}(G)$ such that $[\chi_Z, \lambda] \neq 0$. In general, our notation for characters follows [Isa06].

THEOREM 2.2. *Suppose that G is a finite perfect group, p is a prime, $Z = \mathbf{Z}(G)$ is cyclic of order not divisible by p, and $S := G/Z$ is simple of order divisible by p. Let $\lambda \in \mathrm{Irr}(Z)$ be faithful. Suppose that p does not divide $\chi(1)$ for all $\chi \in \mathrm{Irr}(G|\lambda)$. Then Sylow p-subgroups of G are abelian and one of the following statements holds*:

(i) $G = \mathrm{SL}_2(p^a)$ *and p odd*; *or*
(ii) $(G, p) = (6\mathsf{A}_6, 5),\ (6\mathsf{A}_7, 5),\ (6\mathsf{A}_7, 7),\ (2\mathsf{A}_8, 5),\ (4_1 \cdot \mathrm{PSL}_3(4), 3),$ $(12_1 \cdot \mathrm{PSL}_3(4), 7),$ *or* $(12M_{22}, 11)$.

Furthermore, in each of these cases, S is McKay-good for the prime p and the normalizer of a p-Sylow subgroup of G is a correspondence subgroup in the Isaacs-Malle-Navarro bijection.

Finally, this is the classification theorem that we mentioned in the introduction.

THEOREM 2.3. *Let G be a finite group, and let $p > 2$ be a prime. Assume that $G = \mathbf{O}^{p'}(G) = \mathbf{O}^{p}(G)$ has abelian Sylow p-subgroups. Assume furthermore that every G-orbit on a finite-dimensional faithful irreducible $\mathbb{F}_pG$-module V has length coprime to p. Then one of the following statements holds*:

(i) $G = \mathrm{SL}_2(q)$ *and* $|V| = q^2$;
(ii) *G acts transitively on the n summands of a decomposition $V = \oplus_{i=1}^{n} V_i$, where $p < n < p^2$, $n \equiv -1 (\mathrm{mod}\, p)$. Furthermore,* $\mathrm{Stab}_G(V_1)$ *acts*

transitively on $V_1 \setminus \{0\}$, *and the action of* G *on* $\{V_1, ..., V_n\}$ *induces either* A_n *or the affine group* $2^3 : \mathrm{SL}_3(2)$ *for* $(n,p) = (8,3)$.

(iii) $(G, |V|) = (\mathrm{SL}_2(5), 3^4)$, $(2_-^{1+4} \cdot \mathsf{A}_5, 3^4)$, $(\mathrm{PSL}_2(11), 3^5)$, $(M_{11}, 3^5)$, $(\mathrm{SL}_2(13), 3^6)$.

Proof. This is a partial case (Corollary 5) of the main result of [GLP+]. □

3. **Proof of Theorem A**

In order to show how different parts of the paper come into play, in this section we prove Theorem A assuming Theorems 2.2, 2.3, 6.1, and 5.1, which we will prove in subsequent sections.

THEOREM 3.1. *Let* $\lambda \in \mathrm{Irr}(Z)$, *where* $Z \triangleleft G$. *Let* p *be a prime number, and let* $P/Z \in \mathrm{Syl}_p(G/Z)$. *If* $\chi(1)/\lambda(1)$ *is not divisible by* p *for all* $\chi \in \mathrm{Irr}(G|\lambda)$, *then* P/Z *is abelian.*

Proof. We argue by induction on $|G : Z|$. If $p = 2$, then Theorem 3.1 is proven in [Mor06]. So we assume that p is odd. If G is p-solvable, then this is the Gluck-Wolf theorem (Theorem A of [GW84a]). So we assume that G is not p-solvable. If $Z = 1$, then the theorem follows from the Ito-Michler theorem [Mic86]. Hence, we assume that $Z > 1$.

Step 1. We may assume that λ is G-invariant.

If T is the stabilizer of λ in G, then by the Clifford correspondence we know that induction provides a bijection $\mathrm{Irr}(T \,|\, \lambda) \to \mathrm{Irr}(G \,|\, \lambda)$. Hence we see that $|G : T|$ is not divisible by p. Therefore some G-conjugate of P/Z is contained in T/Z. Now, the hypotheses of Theorem 3.1 are satisfied in T/Z again by the Clifford correspondence, and if $T < G$, then we are done by induction.

Step 2. If $Z < K < G$ is a proper normal subgroup of G, then K/Z and G/K have abelian Sylow p-subgroups.

If $\tau \in \mathrm{Irr}(K|\lambda)$ and $\chi \in \mathrm{Irr}(G|\tau)$, then

$$\chi(1)/\lambda(1) = (\chi(1)/\tau(1))(\tau(1)/\lambda(1)),$$

and p does not divide $\tau(1)/\lambda(1)$ nor $\chi(1)/\tau(1)$. Then we may apply induction.

Step 3. We may assume that $Z = \mathbf{O}_{p'}(G) \leq \mathbf{Z}(G)$ and that λ is faithful. Also Z is cyclic and $Z \leq \Phi(G)$. Furthermore, $\mathbf{O}^{p'}(G) = G$.

By using the theory of character triples (Chapter 11 of [Isa06]), we may replace (G, Z, λ) by some other triple $(\tilde{G}, \tilde{Z}, \tilde{\lambda})$, where $\tilde{Z} \subseteq \mathbf{Z}(\tilde{G})$ and such that $G/Z \cong \tilde{G}/\tilde{Z}$. Hence, by working in $\tilde{G}$, there is no loss to assume that $Z \leq \mathbf{Z}(G)$. Also, it is clear that we may assume that λ is faithful, since $\mathrm{Ker}(\lambda)$ is a normal subgroup of G (using that λ is G-invariant) and working in the group $G/\mathrm{Ker}(\lambda)$. In particular, we may assume that Z is cyclic. Now, let $\chi \in \mathrm{Irr}(G|\lambda)$. Since p does not divide $\chi(1)$, it follows that χ_P contains some p'-degree irreducible constituent $\tau \in \mathrm{Irr}(P|\lambda)$. Since $\tau(1)$ divides $|P : Z|$ by Corollary (11.29) of [Isa06], we deduce that τ is linear. Hence $\tau_Z = \lambda$. In particular, λ_p (the p-part of λ in the group of linear characters, which is a power of λ) extends to P/Z. However λ_p extends to every Q/Z for $Q \in \mathrm{Syl}_q(G/Z)$ if $q \neq p$. We conclude that λ_p has some extension ρ to G by Corollary (11.31) of [Isa06]. Now, by Theorem (6.16) of [Isa06], we have that $\beta \mapsto \rho\beta$ is a bijection $\mathrm{Irr}(G|\lambda_{p'})$ to $\mathrm{Irr}(G|\lambda)$. Therefore we have that p does not divide $\beta(1)$ for all $\beta \in \mathrm{Irr}(G|\lambda_{p'})$. In particular, we may assume that $\lambda_p = 1$. In other words, we may assume that Z is a p'-group. If $Z < \mathbf{O}_{p'}(G)$, then we will have by Step 2 that $G/\mathbf{O}_{p'}(G)$ has abelian Sylow p-subgroups, and we will be done. Hence $Z = \mathbf{O}_{p'}(G)$. Also, $\mathbf{O}^{p'}(G/Z) = G/Z$, by Step 2 too.

Now, let M be any subgroup of G such that $MZ = G$ and such that $|M|$ is as small as possible. If $Z \cap M$ is not contained in $\Phi(M)$, then there would exist a proper subgroup U of M such that $U(Z \cap M) = M$, and therefore $UZ = G$. Hence, we see that $Z \cap M$ is contained in $\Phi(M)$. Now, since Z is central, it is well known that restriction defines a bijection $\mathrm{Irr}(G|\lambda) \to \mathrm{Irr}(M|\lambda_{M\cap Z})$. (Use, for instance, Corollary (4.2) of [Isa84].) Hence, we have that p does not divide $\gamma(1)$ for every $\gamma \in \mathrm{Irr}(M|\gamma_{M\cap Z})$. Also, G/Z and $M/M \cap Z$ are isomorphic and $\mathbf{O}_{p'}(M) = M \cap Z$. Hence, by working in M, it is no loss to assume that $Z \subseteq \Phi(G)$. Now, if $X = \mathbf{O}^{p'}(G)$, then $XZ = G$ by the previous paragraph, and therefore $X = G$.

Step 4. We have that G/Z has a unique proper minimal normal subgroup K/Z and that $\bar{G} = G/K$ has abelian Sylow p-subgroups. In particular, if $E/K = \mathbf{O}_{p'}(G/K)$, then we have that $G/E = Y/E \times L/E$, where $L/E = \mathbf{O}_p(G/E)$ is an abelian p-group and $Y/E = \mathbf{O}^p(G/E)$ is either trivial or a direct product of non-abelian simple groups with abelian Sylow p-subgroups.

Suppose that K_i/Z are two different minimal normal subgroups of G/Z. Hence $Z = K_1 \cap K_2$. Now, by Step 2 we know that G/K_i has abelian Sylow p-subgroups, and therefore so does G/Z. Hence, we may assume that G/Z has a unique minimal normal subgroup K/Z. Also, G/K has abelian Sylow p-subgroups by Step 2. Now we apply Theorem 2.1, using that $\mathbf{O}^{p'}(G/K) = G/K$ (by the previous Step) to get the desired structure of G/E. Finally, suppose that K/Z is not a proper subgroup of G/Z, whence G/Z is simple non-abelian. Since Z is a p'-group and G is not p-solvable, we have that $\mathbf{O}^p(G) = G$. Also, since $\mathbf{O}^{p'}(G) = G$ by Step 3, then we have that G is perfect. In this case, G has abelian Sylow p-subgroups by Theorem 2.2, and so we are done. Thus we may assume henceforth that K/Z is proper in G/Z.

Step 5. We have $K = Q \times Z$, where $1 \neq Q \in \mathrm{Syl}_p(K)$. Also, $L < G$.

Since $K < G$ by Step 4, we know that K has abelian Sylow p-subgroups by Step 2. If K/Z is not p-solvable, then we have that $K/Z = U_1/Z \times \cdots \times U_s/Z$ is a direct product of non-abelian isomorphic simple groups U_i/Z of order divisible by p. Set $U = U_1$. Now, let $\gamma \in \mathrm{Irr}(U|\lambda)$ and choose $\tau \in \mathrm{Irr}(K|\gamma)$. Since $U \triangleleft K$ and p does not divide $\tau(1)$, we conclude that p does not divide $\gamma(1)$. Since U/Z is non-abelian simple, we have that $U'Z = U$ and that U' is perfect. Also, if $Z_1 = U' \cap Z$ and $\lambda_1 = \lambda_{Z_1}$, then we have that restriction defines a bijection $\mathrm{Irr}(U \mid \lambda) \to \mathrm{Irr}(U'|\lambda_1)$. Therefore, if $\tau_1 \in \mathrm{Irr}(U' \mid \lambda_1)$, then p does not divide $\tau_1(1)$. We have then proved that the group U' satisfies the hypothesis of Theorem 2.2. Hence, we have that U'/Z_1 is one of the groups in the conclusion of that theorem. By Theorem 2.2, we have that the simple group U'/Z_1 is McKay-good with correspondence subgroup a Sylow normalizer. Now we are ready to apply Theorem (13.1) of [IMN07], and we notice that we can take the subgroup H in that theorem as $\mathbf{N}_K(Q)$ for some Sylow p-subgroup Q of K. (See the paragraph previous to Lemma (13.2) of [IMN07].) Set $H = \mathbf{N}_K(Q)$, and notice that $\mathbf{N}_G(H) = \mathbf{N}_G(Q) = N$. We wish to show that the group N satisfies the hypothesis of the theorem with respect to Z and λ. Let $\gamma \in \mathrm{Irr}(N|\lambda)$, and let $\rho \in \mathrm{Irr}(H|\lambda)$ under γ. Since H has a normal and abelian Sylow p-subgroup, we know that $\rho(1)$ is not divisible by p. By Theorem (13.1) of [IMN07] and our hypothesis, we conclude that there exists an equivariant character bijection $^* : \mathrm{Irr}(K|\lambda) \to \mathrm{Irr}(H|\lambda)$ satisfying certain properties. In particular, $\rho = \theta^*$ for a unique $\theta \in \mathrm{Irr}(K|\lambda)$. Now, let G_0 be the stabilizer of θ in G, so that $N_0 = G_0 \cap N$ is the stabilizer of θ^* in N. We now closely follow the conclusion of the proof of Theorem (13.1) in page 83 of [IMN07]. By the last paragraph in that proof, we have that there exists a bijection $^* : \mathrm{Irr}(G_0|\theta) \to \mathrm{Irr}(N_0 \mid \theta^*)$ such that if χ corresponds to χ^*, then $\chi(1)/\theta(1) = \chi^*(1)/\theta^*(1)$. Now, by hypothesis, we have that $|G : G_0| = |N : N_0|$ is not divisible by p and that every $\chi \in \mathrm{Irr}(G_0|\theta)$ has degree not divisible by p. Since by the Clifford correspondence, $\gamma = (\chi^*)^N$ for some $\chi \in \mathrm{Irr}(G_0|\theta)$, we deduce that $\gamma(1)$ has degree not divisible by p. Since $|N : Z| < |G : Z|$, then we conclude by induction that N has abelian Sylow p-subgroups. But N contains a Sylow p-subgroup of G, and therefore we are done in this case. Hence, we may assume that K/Z is p-solvable. Since $Z = \mathbf{O}_{p'}(G) = \mathbf{O}_{p'}(K)$, necessarily we conclude that K/Z is a p-group. Therefore $K = Q \times Z$, where $Q \in \mathrm{Syl}_p(K)$. Now, we conclude that L is p-solvable, and therefore $L < G$.

Step 6. G/E is a non-abelian simple group of order divisible by p, with abelian Sylow p-subgroups.

First we prove that if $E \leq M \leq Y$ is a proper normal subgroup of G, then $M = E$. Suppose that $E < M$. By Step 2, we know that M has abelian Sylow p-subgroups. Let $R/Z = \mathbf{O}^{p'}(M/Z)$. Since Y/E is a direct product of non-abelian simple groups of order divisible by p, then M/E is a also a direct product of non-abelian simple groups of order divisible by p. Hence $RE = M$. Since $K/Z \subseteq M/Z$ is a p-group, we have that $K/Z \subseteq R/Z$. Since $Z = \mathbf{O}_{p'}(R)$ (because $Z = \mathbf{O}_{p'}(G)$), $\mathbf{O}^{p'}(R/Z) = R/Z$, and R has abelian Sylow p-subgroups, then by Theorem 2.1 we have that R/Z is the direct product of two characteristic subgroups; one of them is $\mathbf{O}_p(R/Z)$, which contains K/Z. Since G/Z has a unique minimal normal subgroup $K/Z > 1$, then we conclude that R/Z is a p-group. Then M/E is a p-group, and this is not possible.

By applying this to $M = Y$, we conclude that $Y = G$ and that G/E is a non-abelian simple group.

Step 7. We have that G is perfect.

Suppose that $G' < G$. Then we have that G/G' is a p-group by Step 3. Since Z is a p'-group, then we have that $Z \subseteq G'$. Since G/E is simple non-abelian, then we have that $G'E = G$. If $G' \cap E = Z$, then $G/Z = G'/Z \times E/Z$, and this is impossible since G/Z has a unique minimal normal subgroup. So we may assume that $Z < G' \cap E$. But in this case $G' \cap E$ contains the unique minimal normal subgroup of G/Z which is K/Z. Since E/K is a p'-group, this is impossible.

Step 8. We have that $\mathbf{C}_G(Q) = K$.

Assume first that Q is central. We have that $K = Q \times Z$ by Step 5. Consider $\delta = \lambda \times \varepsilon$, where $1 \neq \varepsilon \in \mathrm{Irr}(Q)$. Let $\chi \in \mathrm{Irr}(G)$ be any character over δ. By hypothesis, we have that $\chi(1)$ is not divisible by p. Now, $\chi_Q = \chi(1)\varepsilon$. By taking determinants and using that G is perfect, we have that $\varepsilon^{\chi(1)} = 1$. But this is impossible.

Write $Z \subseteq C = \mathbf{C}_G(Q) < G$. By Step 2, we have that C has abelian Sylow p-subgroups and $Z = \mathbf{O}_{p'}(C)$. Let $W/Z = \mathbf{O}^{p'}(C/Z)$ which contains K/Z. Now, again by Theorem 2.1, we can write $W/Z = W_1/Z \times W_2/Z$, where $W_2/Z = \mathbf{O}_p(W/Z)$ contains K/Z and W_1/Z is characteristic in W/Z. Since G/Z has a unique minimal normal subgroup, we conclude that $W_1 = Z$, and hence W/Z is a p-group. Thus C is p-solvable. Since G/E is simple non-abelian of order divisible by p, then we conclude that $C \leq E$. Since $Q \in \mathrm{Syl}_p(E)$ is contained in C, then $C = Q \times R$, where $R = \mathbf{O}_{p'}(C)$. But $Z = \mathbf{O}_{p'}(G) = \mathbf{O}_{p'}(C)$, and it follows that $C = K$.

Step 9. Write $\bar{G} = G/K$. Then we have that $\mathbf{O}^{p'}(\bar{G}) = \bar{G} = \mathbf{O}^{p}(\bar{G})$ and that $\bar{G}/\mathbf{O}_{p'}(\bar{G})$ is a non-abelian simple group of order divisible by p with abelian Sylow p-subgroups. Also, $V = \mathrm{Irr}(Q)$ is a faithful irreducible $\bar{G}$-module such that for every $v \in V$, we have that $|\bar{G} : \mathbf{C}_{\bar{G}}(v)|$ is not divisible by p.

The first two parts are clear now. Since K/Z is a minimal normal subgroup of G/Z and $K = Q \times Z$, we have that Q is an irreducible and faithful $\bar{G}$-module. Hence it is well known that $V = \mathrm{Irr}(Q)$ is also an irreducible and faithful

$\bar{G}$-module. If $v \in V$, then the stabilizer T of $\lambda \times v$ is the stabilizer of v in G. Hence $T/K = \mathbf{C}_{\bar{G}}(v)$ has index not divisible by p, by the Clifford correspondence.

Step 10. Now we can apply Theorem 2.3 to classify the possible action of $\bar{G}$ on $\mathrm{Irr}(Q)$ and arrive at one of the three cases (i) and (ii), or (iii). In all these cases, we arrive at a contradiction by Theorems 6.1 and 5.1. $\square$

4. **Proof of Theorem 2.2**

This section is devoted to prove Theorem 2.2. Certainly, by the Ito-Michler theorem we may assume that $Z \neq 1$.

4.1. *Alternating and sporadic groups.*

LEMMA 4.1. *Theorem* 2.2 *holds if* $S = \mathsf{A}_n$ *with* $n \geq 8$.

Proof. As we noted above, we may assume that $G = 2\mathsf{A}_n$ and $n \geq p > 2$. Hence, for $(n, p) \neq (8, 5)$, it suffices to show that some faithful irreducible character of $H = 2\mathsf{S}_n$ has degree divisible by p. Such characters are labeled by partitions of n into distinct parts. In what follows, by $\langle\lambda\rangle$ we mean one of the (at most two) irreducible faithful characters of H corresponding to such a partition $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_m) \vdash n$; in particular,

$$\langle\lambda\rangle(1) = 2^{\lfloor \frac{n-m}{2} \rfloor} \cdot \frac{n!}{\prod_i \lambda_i!} \cdot \prod_{i<j} \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j}.$$

Assume that $p|n$ for $p \geq 5$, or $9|n$ for $p = 3$. Then

$$\langle(n-3, 2, 1)\rangle(1) = 2^{\lfloor \frac{n-5}{2} \rfloor} \cdot n(n-4)(n-5)/3$$

is divisible by p. Furthermore, if $p = 3|n$ and $n \geq 12$, then

$$\langle(n-5, 4, 1)\rangle(1) = 2^{\lfloor \frac{n-3}{2} \rfloor} \cdot \frac{3(n-6)(n-9)}{(n-1)(n-4)} \cdot \binom{n}{5} \equiv 0 (\mathrm{mod}\, p).$$

Next suppose that $n = mp + 2s$ with $m \geq 1$ and $1 \leq s \leq p-1$. Then p divides

$$\langle(mp + s, s)\rangle(1) = 2^{\lfloor \frac{n-2}{2} \rfloor} \cdot \frac{mp}{n} \cdot \binom{n}{s}.$$

In particular, we are done if $n = p + 2s$ with $s \geq 1$. Consider the case $p \nmid n$ and $n \geq 2p$. Then we write $n = kp + t$ with $1 \leq t \leq p-1$ and $k \geq 2$. Choose $(m, s) = (k, t/2)$ if t is even and $(m, s) = (k-1, (p+t)/2)$ if t is odd, so that $n = mp + 2s$ and we are done again.

Suppose $n = p + s$ with $1 \leq s < p - 2$ and s is odd. Then p divides

$$\langle(p-1, s+1)\rangle(1) = 2^{\lfloor \frac{n-2}{2} \rfloor} \cdot \frac{p(p-s-2)}{(s+1)(p+s)} \cdot \binom{n}{s}.$$

If $n = 2p-2$ and $p \geq 7$, then

$$\langle (p-1, p-3, 2)\rangle(1) = 2^{\lfloor \frac{n-5}{2} \rfloor} \cdot \frac{p(p-3)(p-5)}{(p-1)(p-2)} \cdot \binom{n}{p+1} \equiv 0 (\mathrm{mod}\, p).$$

It remains to consider the case $(n, p) = (8, 5)$. According to [CCN+85], this case can indeed occur, and certainly Sylow 5-subgroups of G are abelian. Other claims in Theorem 2.2 for this case are established in [Mal08b, Th. 3.1]. □

LEMMA 4.2. *Theorem* 2.2 *holds if* S *is a sporadic simple group.*

Proof. We need to consider only the cases where $\mathrm{Mult}(S) \neq 1$. Inspecting [CCN+85], we see that the only possibility is that $(G, p) = (12M_{22}, 11)$. In this case, the claims in Theorem 2.2 follow from Theorem 5.1 and Corollary 2.2 of [Mal08b]. □

LEMMA 4.3. *Theorem* 2.2 *holds if* S *is one of the following simple groups*: A_n *with* $5 \leq n \leq 7$, $\mathrm{PSL}_2(7)$, $\mathrm{PSL}_3(4)$, $\mathrm{SU}_4(2)$, $\mathrm{PSU}_4(3)$, $\mathrm{PSU}_6(2)$, $\mathrm{Sp}_6(2)$, $\Omega_7(3)$, $\Omega_8^+(2)$, ${}^2B_2(8)$, $G_2(3)$, $G_2(4)$, $F_4(2)$, ${}^2F_4(2)'$, ${}^2E_6(2)$.

Proof. (i) Inspecting character tables available in [CCN+85] (and [Lä] for the groups $3 \cdot {}^2E_6(2)$ and $6 \cdot {}^2E_6(2)$), we see that (G, p) is one of the following pairs: $(2\mathsf{A}_5 \cong \mathrm{SL}_2(5), 5)$, $(2\mathsf{A}_6 \cong \mathrm{SL}_2(9), 3)$, $(6\mathsf{A}_6, 5)$, $(6\mathsf{A}_7, 5 \text{ or } 7)$, $(\mathrm{SL}_2(7), 7)$, $(4_1 \cdot \mathrm{PSL}_3(4), 3)$, $(12_1 \cdot \mathrm{PSL}_3(4), 7)$. In these cases, the claims in Theorem 2.2 follow from [IMN07, Th. 15.3] and from Corollary 2.2 and Theorems 3.1, 4.1 of [Mal08b], unless $S \cong \mathrm{PSL}_3(4)$.

(ii) Suppose now that $S = \mathrm{PSL}_3(4)$; in particular, $\mathrm{Mult}(S) = C_4 \times C_{12}$ and $\mathrm{Out}(S) = D_{12}$ is dihedral of order 12. We need to verify that every perfect group L, where $L/\mathbf{Z}(L) \cong S$, $\mathbf{Z}(L)$ is a cyclic p'-group for $p \in \{3, 7\}$, satisfies the conditions described in [IMN07, §10] with a Sylow normalizer $\mathbf{N}_L(P)$ as a correspondence subgroup. Again, this claim in the cases where $|\mathbf{Z}(L)|$ is even has been verified in [Mal08b, Th. 4.1]. It therefore remains to consider the cases where either $(L, p) = (\mathrm{SL}_3(4), 7)$ or $L = S$.

Assume we are in the former case. Then $\mathbf{C}_{\mathrm{Aut}(S)}(\mathbf{Z}(L))/S \cong C_6$ is generated by the S-cosets of α and $\beta := \sigma\tau$, where α is (induced by) the conjugation by $\mathrm{diag}(a, 1, 1)$ with $a \in \mathbb{F}_4^\times$ of order 3, σ is induced by the field automorphism $x \mapsto x^2$, and τ is the inverse-transpose automorphism of S. To get the group G satisfying the conditions (5)–(7) of [IMN07, §10] we can just consider the semidirect product $\mathrm{GL}_3(4) \cdot \langle \beta \rangle$. Also, since $\mathbf{C}_{\mathrm{Aut}(S)}(\mathbf{Z}(L))/S \cong C_6$ is cyclic, the cohomology classes occurring in condition (8) of [IMN07, §10] are both trivial. Thus we need to produce a bijection $\mathrm{Irr}_{7'}(L) \to \mathrm{Irr}_{7'}(\mathbf{N}_L(P))$ satisfying condition (4) of [IMN07, §10]. Since $P \in \mathrm{Syl}_p(L)$ is cyclic, such a bijection exists by [Dad96, Th. 4.3].

In the latter case, let $A := \mathrm{Aut}(S)$ and $T := \mathbf{N}_S(P)$ for $P \in \mathrm{Syl}_p(S)$. Suppose first that $(L,p) = (S,3)$. Then the claim follows from Proposition 3.6, Lemma 3.7, and Proposition 3.8 of [Mal08a]. Next, let $(L,p) = (S,7)$. Then, according to [CCN+85] and [GAP],

$$\mathrm{Irr}_{7'}(S) = \{\chi_{1a}, \chi_{20a}, \chi_{64a}, \chi_{45a}, \chi_{45b}\},\ \mathrm{Irr}_{3'}(T) = \{\psi_{1a}, \psi_{1b}, \psi_{1c}, \psi_{3a}, \psi_{3b}\}.$$

Now the bijection * : $\mathrm{Irr}_{3'}(S) \to \mathrm{Irr}_{3'}(T)$ can be defined as follows. First, * sends χ_{1a} to ψ_{1a}, the principal character of T, and maps $\{\chi_{20a}, \chi_{64a}\}$ onto $\{\psi_{1b}, \psi_{1c}\}$. Using [GAP] one can check that all these six characters extend to A and $\mathbf{N}_A(P)$, respectively. Finally, * maps $\{\chi_{45a}, \chi_{45b}\}$ onto $\{\psi_{3a}, \psi_{3b}\}$; all these four characters have the same stabilizers in $\mathbf{N}_A(P)$. □

4.2. *Restrictions of Lusztig series.* We refer the reader to [Car85], [DM91], [Lus84], [Lus88] for basic facts on the Deligne-Lusztig theory of complex representations of finite groups of Lie type. By such a group we mean the fixed point subgroup $G := \mathcal{G}^F$, where $F : \mathcal{G} \to \mathcal{G}$ is a Frobenius endomorphism on a connected reductive algebraic group $\mathcal{G}$ over a field of characteristic $p > 0$. Let the pair $(\mathcal{G}^*, F^*)$ be in duality with $(\mathcal{G}, F)$, and let $G^* := (\mathcal{G}^*)^{F^*}$. Then $\mathrm{Irr}(G)$ is partitioned into (rational) Lusztig series $\mathcal{E}(G,(s))$, where (s) is a semisimple conjugacy class in G^*, and furthermore there is a bijection

$$\chi \in \mathcal{E}(G,(s)) \longleftrightarrow \psi \in \mathcal{E}(\mathbf{C}_{G^*}(s),(1))$$

such that $\chi(1) = [G^* : \mathbf{C}_{G^*}(s)]_{p'} \cdot \psi(1)$; cf. [Lus88, Prop. 5.1] (and also [DM91, Th. 13.23]). We use X' to denote the derived subgroup of any group X.

First we record the following known facts.

LEMMA 4.4. *Let $s \in G^*$ be a semisimple element.*

(i) *There is $\alpha \in \mathrm{Irr}(\mathbf{Z}(G))$ depending only on the conjugacy class (s) of s in G^* such that $\theta_{\mathbf{Z}(G)} = \alpha$ for all pairs $(\mathcal{T}, \theta)$ in the geometric conjugacy class determined by s and such that $\chi_{\mathbf{Z}(G)} = \chi(1)\alpha$ for all $\chi \in \mathcal{E}(G,(s))$.*

(ii) *Assume in addition that $s \in (G^*)'$ and $|\mathbf{Z}(G)| = |G^*/(G^*)'|$. Then all $\chi \in \mathcal{E}(G,(s))$ restrict trivially to $\mathbf{Z}(G)$.*

Proof. (i) is [Mal07, Lemma 2.2].

(ii) Suppose that $|\mathbf{Z}(G^*)| = |G/G'|$. Note that $\mathbf{Z}(\mathcal{G}^*)^F = \mathbf{Z}(G^*)$ by [Car85, Prop. 3.6.8]. Now we consider any $t \in \mathbf{Z}(G^*)$. Then the series $\mathcal{E}(G,(t))$ contains a character $\chi_t \in \mathrm{Irr}(G)$ labeled by the principal (unipotent) character of $\mathbf{C}_{G^*}(t) = G^*$, of degree 1. Thus χ_t is trivial at G'. Let $\mathcal{T}$ be any F-stable maximal torus of $\mathcal{G}$, and let $\mathcal{T}^*$ be the F^*-stable maximal torus in duality with $\mathcal{T}$. Then $t \in \mathbf{Z}(\mathcal{G}^*) \leq \mathcal{T}^*$, and if $(\mathcal{T}, \theta_t)$ is in the geometric conjugacy class determined by t, then θ_t is trivial at $\mathcal{T}^F \cap G'$ by (i). Clearly, if $t \neq t' \in \mathbf{Z}(G^*) \leq \mathbf{Z}(\mathcal{G}^*)$, then t and t' are not $\mathcal{G}^*$-conjugate, whence $(\mathcal{T}, \theta_t)$ and

$(\mathcal{T}, \theta_{t'})$ belong to different geometric conjugacy classes; in particular, $\theta_t \neq \theta_{t'}$. Next,

$$|\operatorname{Irr}(\mathcal{T}^F/\mathcal{T}^F \cap G')| = |\mathcal{T}^F/\mathcal{T}^F \cap G'| \leq |G/G'| = |\mathbf{Z}(G^*)|.$$

It follows that when t varies over $\mathbf{Z}(G^*)$, θ_t varies over $\operatorname{Irr}(\mathcal{T}^F/\mathcal{T}^F \cap G')$.

Recall (cf. [Car85, Prop. 4.4.1]) that there is a natural isomorphism $\Pi : (\mathcal{T}^*)^{F^*} \longleftrightarrow \operatorname{Irr}(\mathcal{T}^F)$ such that $(\mathcal{T}, \Pi(t))$ is in the geometric conjugacy class defined by $t \in (\mathcal{T}^*)^{F^*}$. We have therefore shown that Π yields a bijection between $\mathbf{Z}(G^*)$ and the set of irreducible characters $\theta \in \operatorname{Irr}(\mathcal{T}^F)$ that are trivial at $\mathcal{T}^F \cap G'$. Next, the pairing $(\beta, x) \mapsto \beta(x)$ gives isomorphisms between $\operatorname{Irr}(\mathcal{T}^F) \cong \mathcal{T}^F$ and $\operatorname{Irr}((\mathcal{T}^*)^{F^*}) \cong (\mathcal{T}^*)^{F^*}$. It follows that there is an isomorphism between the group of irreducible characters of $(\mathcal{T}^*)^{F^*}$ that are trivial at $\mathbf{Z}(G^*)$ and $\mathcal{T}^F \cap G'$, the common kernel of all $\theta \in \operatorname{Irr}(\mathcal{T}^F/\mathcal{T}^F \cap G')$.

Repeating the above argument but interchanging $\mathcal{G}$ and $\mathcal{G}^*$ (which is possible since $|\mathbf{Z}(G)| = |G^*/(G^*)'|$), we see that Π gives an isomorphism between $(\mathcal{T}^*)^{F^*} \cap (G^*)'$ and the group of irreducible characters of $\mathcal{T}^F$ that are trivial at $\mathbf{Z}(G)$.

Now our claim follows by applying this last statement to s and using (i). $\square$

The following extension of Lemma 4.4(ii) also seems to be known to the experts. For the reader's convenience, we give a proof of it.

Proposition 4.5. *Assume that* $\mathbf{Z}(G) \cong G^*/(G^*)'$ *is cyclic of order* m, $s \in G^*$ *is semisimple, and that* $[G^* : \langle s, (G^*)' \rangle] = d$ *for some* $d|m$.

(i) *If* $(\mathcal{T}, \theta)$ *is in the geometric conjugacy class determined by* s, *then* $\operatorname{Ker}(\theta) \cap \mathbf{Z}(G)$ *is cyclic of order* d.

(ii) *For any* $\chi \in \mathcal{E}(G, (s))$, $\operatorname{Ker}(\chi) \cap \mathbf{Z}(G)$ *is cyclic of order* d.

Proof. (i) Let $\mathbf{Z}(G) := \langle z \rangle$, and let $\mathcal{T}^*$ be an F^*-stable maximal torus in duality with $\mathcal{T}$. Recall we have shown in the proof of Lemma 4.4 that Π gives an isomorphism between $(\mathcal{T}^*)^{F^*} \cap (G^*)'$ and the group of irreducible characters of $\mathcal{T}^F$ that are trivial at $\mathbf{Z}(G)$. Our assumption implies that the coset $s(G^*)'$ has order m/d in $G^*/(G^*)'$. Hence $s^{m/d} \in (\mathcal{T}^*)^{F^*} \cap (G^*)'$, and so $\theta^{m/d} = \Pi(s^{m/d})$ is trivial at $\mathbf{Z}(G)$. Thus $\theta(z^{m/d}) = 1$ and $\operatorname{Ker}(\theta) \cap \mathbf{Z}(G)$ contains the subgroup $\langle z^{m/d} \rangle \cong C_d$.

Assume now that $\operatorname{Ker}(\theta) \cap \mathbf{Z}(G)$ has order dj for some $j|(m/d)$. Then $\operatorname{Ker}(\theta) \cap \mathbf{Z}(G) = \langle z^{m/dj} \rangle$, whence $\theta^{m/dj}(z) = \theta(z^{m/dj}) = 1$; i.e., $\Pi(s^{m/dj}) = \theta^{m/dj}$ is trivial at $\mathbf{Z}(G)$. Since $\Pi : (\mathcal{T}^*)^{F^*} \to \operatorname{Irr}(\mathcal{T}^F)$ is an isomorphism sending $(\mathcal{T}^*)^{F^*} \cap (G^*)'$ to $\operatorname{Irr}(\mathcal{T}^F/\mathbf{Z}(G))$, we must then have $s^{m/dj} \in (G^*)'$, yielding $j = 1$.

(ii) Now consider any generalized Deligne-Lusztig character $R^{\mathcal{G}}_{\mathcal{T},\theta}$ (with some irreducible constituent) belonging to $\mathcal{E}(G, (s))$. According to (i), $\theta(z^{m/d})$

$= 1$, whence $R^{\mathcal{G}}_{\mathcal{T},\theta}(z^{m/d}) = R^{\mathcal{G}}_{\mathcal{T},\theta}(1)$ by the character formula [Car85, Th. 7.2.8]. It follows essentially from the definition of $\mathcal{E}(G,(s))$ that any $\chi \in \mathcal{E}(G,(s))$ is a sum

$$\chi = \sum_{\mathcal{T},\theta} c_{\mathcal{T},\theta} R^{\mathcal{G}}_{\mathcal{T},\theta} + f \tag{1}$$

of a linear combination of those $R^{\mathcal{G}}_{\mathcal{T},\theta}$ belonging to $\mathcal{E}(G,(s))$ and a certain orthogonal function f that vanishes at all semisimple elements. It follows that $\chi(z^{m/d}) = \chi(1)$.

We have shown that $\mathrm{Ker}(\chi) \cap \mathbf{Z}(G)$ contains the subgroup $\langle z^{m/d}\rangle \cong C_d$ for any $\chi \in \mathcal{E}(G,(s))$. Assume now that $\mathrm{Ker}(\chi) \cap \mathbf{Z}(G)$ has order dk for some $1 < k|(m/d)$ and some $\chi \in \mathcal{E}(G,(s))$. Then $\chi(t) = \chi(1)$ for $t := z^{m/dk}$. Fix a primitive k^{th}-root of unity $\omega \in \mathbb{C}$, and let $J := \{j \mid 1 \le j \le k-1, \gcd(j,k) = 1\}$. Recall we have shown that $\mathrm{Ker}(\theta) \cap \mathbf{Z}(G) = \langle t^k\rangle$ for any $R^{\mathcal{G}}_{\mathcal{T},\theta}$ in (1). It follows that $\theta(t) = \omega^j$ for some $j \in J$. Now we break $\sum_{\mathcal{T},\theta}$ in (1) into sub-sums $\sum^j$, where $R^{\mathcal{G}}_{\mathcal{T},\theta}$ enters $\sum^j$ for $j \in J$ precisely when $\theta(t) = \omega^j$; that is,

$$\chi = \textstyle\sum_{j\in J} \sum^j c_{\mathcal{T},\theta} R^{\mathcal{G}}_{\mathcal{T},\theta} + f. \tag{2}$$

For any $0 \le i \le k-1$, $\chi(t^i) = \chi(1)$, since we are assuming that $\mathrm{Ker}(\chi) \cap \mathbf{Z}(G) = \langle t\rangle$. On the other hand, $\theta(t^i) = \theta(t)^i = \omega^{ij}\theta(1)$ if $R^{\mathcal{G}}_{\mathcal{T},\theta}$ enters $\sum^j$. Applying [Car85, Th. 7.2.8] again, we get $R^{\mathcal{G}}_{\mathcal{T},\theta}(t^i) = \omega^{ij} R^{\mathcal{G}}_{\mathcal{T},\theta}(1)$. Denoting $A_j := \sum^j c_{\mathcal{T},\theta} R^{\mathcal{G}}_{\mathcal{T},\theta}(1) \in \mathbb{C}$ for $j \in J$, (2) now implies that

$$\chi(1) = \chi(t^i) = \sum_{j\in J} A_j \omega^{ij}$$

for $0 \le i \le k-1$. In particular,

$$k\chi(1) = \sum_{i=0}^{k-1} \chi(t^i) = \sum_{i=0}^{k-1}\sum_{j\in J} A_j\omega^{ij} = \sum_{j\in J} A_j \left(\sum_{i=0}^{k-1} \omega^{ij}\right) = 0,$$

a contradiction. □

Remark 4.6. Note that $\mathbf{Z}(G) \cong G^*/(G^*)'$ if $\mathcal{G}$ is simple and moreover $G = \mathcal{G}^F$ is none of the groups of types A_1 or ${}^2G_2$ over $\mathbb{F}_3$, or of types A_1, ${}^2A_2$, B_2, G_2, or ${}^2F_4$ over $\mathbb{F}_2$. If in addition $\mathcal{G}$ is simply connected, then $\mathbf{Z}(G)$ is cyclic unless $\mathcal{G}^F \cong \mathrm{Spin}^+_{4n}(q)$.

Recall that a semisimple element $s \in \mathcal{G}$ is called *strongly regular* if $\mathbf{C}_{\mathcal{G}}(s)$ is a maximal torus.

LEMMA 4.7. *Let $\mathcal{T}$ be an F-stable maximal torus of $\mathcal{G}$, and let $\mathcal{T}^*$ be the F^*-stable maximal torus in duality with $\mathcal{T}$.*

(i) *Assume that $R^{\mathcal{G}}_{\mathcal{T},\theta^a}$ is irreducible up to sign for some $\theta \in \mathrm{Irr}(\mathcal{T}^F)$ and some $a \in \mathbb{Z}$. Then $R^{\mathcal{G}}_{\mathcal{T},\theta}$ is also irreducible up to sign.*

(ii) *Let r be an integer with the property that any element of order r in $(\mathcal{T}^*)^{F^*}$ is strongly regular. Then for any $\theta \in \mathrm{Irr}(\mathcal{T}^F)$ of order divisible by r, $R^{\mathcal{G}}_{\mathcal{T},\theta}$ is irreducible up to sign.*

Proof. (i) By [DM91, Cor. 11.15],

$$N(\alpha) := [R^{\mathcal{G}}_{\mathcal{T},\alpha}, R^{\mathcal{G}}_{\mathcal{T},\alpha}]_{\mathcal{G}^F} = |\{g \in \mathbf{N}_{\mathcal{G}^F}(\mathcal{T}) \mid \alpha^g = \alpha\}|/|\mathcal{T}^F|$$

for any $\alpha \in \mathrm{Irr}(\mathcal{T}^F)$. In particular, $1 \le N(\theta) \le N(\theta^a) = 1$, whence $N(\theta) = 1$; i.e., $R^{\mathcal{G}}_{\mathcal{T},\theta}$ is irreducible up to sign.

(ii) Suppose $\theta \in \mathrm{Irr}(\mathcal{T}^F)$ has order divisible by r. Then there is some $a \in \mathbb{Z}$ such that θ^a has order r. Again consider the isomorphism $\Pi : (\mathcal{T}^*)^{F^*} \longleftrightarrow \mathrm{Irr}(\mathcal{T}^F)$, and let $\Pi(s) = \theta^a$ for $s \in (\mathcal{T}^*)^{F^*}$. Then $|s| = |\theta^a| = r$, and so by our hypothesis, $\mathbf{C}_{\mathcal{G}^*}(s) = \mathcal{T}^*$. It follows by Proposition 2.3 and Remark 2.4 of [DM91] that the relative Weyl groups $W(s)$ and $W^\circ(s)$ are trivial. Hence $R^{\mathcal{G}}_{\mathcal{T},\theta^a}$ is irreducible up to sign by [DM91, Prop. 14.43]. Applying (i) we can conclude that $R^{\mathcal{G}}_{\mathcal{T},\theta}$ is also irreducible up to sign. $\square$

4.3. *Generic simple groups of Lie type.* Using the results of the previous subsection, we will now complete the proof of Theorem 2.2. By Lemmas 4.1–4.3, we may assume that S is a simple group of Lie type in characteristic ℓ, which is not isomorphic to any of the simple groups listed in Lemma 4.3. In particular, we can find a simple simply connected algebraic group $\mathcal{G}$ and a Frobenius endomorphism $F : \mathcal{G} \to \mathcal{G}$ such that $S = L/\mathbf{Z}(L)$ and $\mathrm{Mult}(S) = \mathbf{Z}(L)$ for $L := \mathcal{G}^F$. Let the pair $(\mathcal{G}^*, F^*)$ be dual to $(\mathcal{G}, F)$, and let $L^* := (\mathcal{G}^*)^{F^*}$. By Remark 4.6, Lemma 4.4, and Proposition 4.5 apply to L unless $L = \mathrm{Spin}^+_{4n}(q)$. We use the notation for various groups of Lie type as given in [Car85, §1.19]. Furthermore, SL^ε denotes SL when $\varepsilon = +$ and SU when $\varepsilon = -$, and similarly for GL^ε and PGL^ε. We also denote by E_6^ε the type E_6 when $\varepsilon = +$ and the type ${}^2E_6$ when $\varepsilon = -$.

Lemma 4.8. *Keep the above notation and the hypotheses of Theorem* 2.2. *To prove Theorem* 2.2 *in the case $S = L/\mathbf{Z}(L)$ is a simple group of Lie type, it suffices to show that if $L \not\cong \mathrm{SL}_2(q)$ and $\mathbf{Z}(L) \neq 1$, then for any proper subgroup $A < \mathbf{Z}(L)$ with $\mathbf{Z}(L)/A$ a cyclic p'-subgroup, there is an irreducible character $\chi \in \mathrm{Irr}(L)$ with $\mathrm{Ker}(\chi) \cap \mathbf{Z}(L) = A$ and $p|\chi(1)$.*

Proof. By the Ito-Michler theorem we may assume $Z \neq 1$, whence $\mathbf{Z}(L) \neq 1$. First we consider the case $S = \mathrm{PSL}_2(q)$ with $q \ge 11$ odd. In this case $G = L = \mathrm{SL}_2(q)$, $Z = C_2$, and since G has faithful irreducible characters of degree $q \pm 1$, we must have $q = p^a$. Certainly the Sylow p-subgroups of G are abelian. Furthermore, the fact that S is McKay-good for the prime p with a Sylow normalizer being a correspondence subgroup is proved in [IMN07, (15F)].

So we may assume that $L \not\cong \mathrm{SL}_2(q)$. We also have that $G = L/A$ where $A < \mathbf{Z}(L)$, and $Z = \mathbf{Z}(G) = \mathbf{Z}(L)/A$ is a cyclic p'-subgroup. By our assumption, there is an irreducible character $\chi \in \mathrm{Irr}(L)$ with $\mathrm{Ker}(\chi) \cap \mathbf{Z}(L) = A$ and $p|\chi(1)$. Then we can view χ as a faithful irreducible character of G. It is easy to see now that some Galois conjugate χ^σ of χ has degree divisible by p and lies above λ, contrary to the main hypothesis of Theorem 2.2. $\square$

Proposition 4.9. *Theorem* 2.2 *holds if S is a simple group of Lie type in characteristic $\ell = p$.*

Proof. (i) Recall that $\mathcal{G}^*$ is connected reductive. Hence, by Remark 2.4 and Lemma 13.14 of [DM91], for any semisimple element $s \in L^* = (\mathcal{G}^*)^{F^*}$ with centralizer $\mathcal{C} := \mathbf{C}_{\mathcal{G}^*}(s)$, $\mathcal{C}/\mathcal{C}^\circ$ can be embedded in $\mathbf{Z}(\mathcal{G})/\mathbf{Z}(\mathcal{G})^\circ$. In particular, $\mathcal{C}/\mathcal{C}^\circ$ is a finite abelian p'-group. Since $\mathcal{C}^\circ$ is connected and F^*-stable, $\mathcal{C}^{F^*}/(\mathcal{C}^\circ)^{F^*} \cong (\mathcal{C}/\mathcal{C}^\circ)^{F^*}$ is also a finite abelian p'-group. Hence, p divides $|\mathcal{C}^{F^*}|$ if and only if p divides $|(\mathcal{C}^\circ)^{F^*}|$. As in [DM91, p. 112], by a Steinberg character of the finite (possibly disconnected) reductive group $\mathcal{C}^{F^*}$ we mean any irreducible constituent of $\varphi^{\mathcal{C}^{F^*}}$, where φ is the Steinberg character of $(\mathcal{C}^\circ)^{F^*}$, of degree $\varphi(1) = |(\mathcal{C}^\circ)^{F^*}|_p$. Thus a Steinberg character of $\mathcal{C}^{F^*} = \mathbf{C}_{L^*}(s)$ has degree divisible by p precisely when p divides $|\mathbf{C}_{L^*}(s)|$. It follows that if $s \in L^*$ is not regular, then the character $\chi \in \mathcal{E}(L,(s))$ corresponding to a Steinberg character of $\mathbf{C}_{L^*}(s)$ has degree divisible by p.

(ii) Suppose now that $\mathbf{Z}(L)$ is cyclic. We will apply Lemma 4.8 and use the observation made in (i) to produce an irreducible character $\chi \in \mathcal{E}(L,(s))$ of degree divisible by p with $\mathrm{Ker}(\chi) \cap \mathbf{Z}(L) = A$. By Proposition 4.5(ii), it suffices to find a nonregular semisimple element $s \in L^*$ with $[L^* : \langle s, (L^*)' \rangle] = |A|$. It is straightforward to verify that such an element s exists when L is classical. If L is exceptional, then one can appeal to [Lü].

It remains to consider the case $\mathbf{Z}(L)$ is not cyclic, that is, $L = \mathrm{Spin}_{2n}^+(q)$ with $2|n \geq 4$ and $q = p^f$, so that $\mathrm{Mult}(S) = C_2^2$. Since $\mathbf{Z}(G) \neq 1$ is cyclic, we see that $G = H'$, where H is one of the three groups in the same isogeny class with L but neither of simply connected nor of adjoint type. Since $[H : G] = 2$ and $\mathbf{Z}(G) = \mathbf{Z}(H) \cong C_2$, it suffices to find faithful irreducible characters of H of degree divisibly by p. Now we can again apply Proposition 4.5(ii) to H and argue as above. $\square$

We note that Proposition 4.9 is also proved in [KM].

To deal with the cross characteristic case, i.e., where S is a simple group of Lie type in characteristic $\ell \neq p$, we need the following observation.

Lemma 4.10. *To prove Theorem* 2.2 *in the case $S = L/\mathbf{Z}(L)$ is a simple group of Lie type in characteristic $\ell \neq p$, it suffices to find two square-free* *integers r_i coprime to $|\mathbf{Z}(L)|$ and two F^*-stable maximal tori $\mathcal{T}_i^*$ of $\mathcal{G}^*$, $i = 1, 2$, such that*

(i) *r_i divides $|(\mathcal{T}_i^*)^{F^*}|$, and any element of order r_i in $(\mathcal{T}_i^*)^{F^*}$ is strongly regular*;

(ii) *$|(\mathcal{T}_1^*)^{F^*}| \cdot |(\mathcal{T}_2^*)^{F^*}|$ divides $|L|_{\ell'}$.*

Proof. We apply Lemma 4.8 and consider any proper subgroup $A < \mathbf{Z}(L)$ with $\mathbf{Z}(L)/A$ a cyclic p'-group. Then we can find $\lambda \in \mathrm{Irr}(\mathbf{Z}(L))$ with $\mathrm{Ker}(\lambda) = A$. Also consider the F-stable maximal torus $\mathcal{T}_i$ of $\mathcal{G}$ in duality with $\mathcal{T}_i^*$ for $i = 1, 2$, and decompose $(\mathcal{T}_i)^F = T_i^r \times T_i^s$ into the direct product of the π_i-part and the π_i'-part, where π_i is the set of prime divisors of r_i. Since r_i is coprime to $\mathbf{Z}(L)$, $T_i^s \geq \mathbf{Z}(L)$, and so λ extends to some $\beta_i \in \mathrm{Irr}(T_i^s)$. Since r_i divides $|(\mathcal{T}_i^*)^{F^*}| = |(\mathcal{T}_i)^F|$ and r_i is square-free, there is some $\alpha_i \in \mathrm{Irr}(T_i^r)$ of order r_i. Now the character $\theta_i := \alpha_i \times \beta_i$ is irreducible and has order divisible by r_i. By Lemma 4.7(ii), $\chi_i = \pm R^{\mathcal{G}}_{\mathcal{T}_i,\theta_i}$ is irreducible. Moreover, $\chi_i(z) = \theta_i(z)\chi_i(1)$ for any $z \in \mathbf{Z}(L)$ by [Car85, Th. 7.2.8]. It follows that

$$\mathrm{Ker}(\chi_i) \cap \mathbf{Z}(L) = \mathrm{Ker}(\beta_i) \cap \mathbf{Z}(L) = \mathrm{Ker}(\lambda) = A.$$

Furthermore,

$$\chi_1(1)\chi_2(1) = \frac{|L|_{\ell'}}{|(\mathcal{T}_1)^F|} \cdot \frac{|L|_{\ell'}}{|(\mathcal{T}_2)^F|}$$

is divisible by $|L|_{\ell'}$, and $p \neq \ell$. So p divides at least one of $\chi_i(1)$, as desired. $\square$

PROPOSITION 4.11. *Theorem* 2.2 *holds if S is a simple group of Lie type in characteristic $\ell \neq p$.*

Proof. Throughout this proof, $q = \ell^f$ is a power of ℓ.

(i) First we use Lemma 4.8 to rule out the cases $L = \mathrm{SL}_3^{\varepsilon}(q)$. Since $\mathbf{Z}(L) \neq 1$, we have that $3|(q - \varepsilon)$ (and $(q, \varepsilon) \neq (2, -)$). It is well known that there are faithful irreducible characters of L of degree $q^2 + \varepsilon q + 1$ and $(q - \varepsilon)(q^2 - 1)$. Since p divides $|L|_{\ell'}$, at least one of these degrees is divisible by p, and so we are done.

Similarly, if $L = \mathrm{Sp}_4(q)$ with q odd, then there exist faithful irreducible characters of degree $q^4 - 1$ of L, and so we are done again.

(ii) Next we deal with exceptional groups of Lie type. First suppose that $L = E_7(q)_{\mathrm{sc}}$ and q is odd. In this proof, we denote by $\mathrm{ppd}(m)$ any *primitive prime divisor* of $\ell^m - 1$, i.e., a prime divisor of $\ell^m - 1$ that does not divide $\prod_{i=1}^{m-1}(\ell^i - 1)$. According to [Zsi92], $\mathrm{ppd}(m)$ exists if $m > 2$ and $(\ell, m) \neq (2, 6)$. Furthermore, $\Phi_m(q)$ denotes the m^{th} cyclotomic polynomial in q. Now choose $r_i = \mathrm{ppd}(m_i f)$ with $(m_1, m_2) = (18, 14)$. By [MT08, Lemma 2.3] (and its proof), any element s_i of order r_i in $\mathcal{G}^*$ is strongly regular, with $|\mathbf{C}_{L^*}(s_i)| = \Phi_{m_i}(q)\Phi_2(q)$. Taking $\mathcal{T}_i^* = \mathbf{C}_{\mathcal{G}^*}(s_i)$ for $i = 1, 2$, we see that the primes r_i and the tori $\mathcal{T}_i^*$ satisfy all the conditions set up in Lemma 4.10. Hence we are done in this case.

Next suppose that $L = E_6^\varepsilon(q)_{\rm sc}$ with $3|(q-\varepsilon)$ (and $(q,\varepsilon) \neq (2,-)$). If $\varepsilon = +$, choose $r_i = \mathrm{ppd}(m_i f)$ with $(m_1, m_2) = (9, 12)$. By [MT08, Lemma 2.3], any $s_i \in \mathcal{G}^*$ of order r_i is strongly regular with $|\mathbf{C}_{L^*}(s_i)| = \Phi_9(q)$, respectively $\Phi_{12}(q)\Phi_3(q)$, for $i = 1, 2$. Now we can choose $\mathcal{T}_i^* = \mathbf{C}_{\mathcal{G}^*}(s_i)$ and apply Lemma 4.10. If $\varepsilon = -$, choose $r_i = \mathrm{ppd}(m_i f)$ with $(m_1, m_2) = (18, 12)$. By [MT08, Lemma 2.3], any $s_i \in \mathcal{G}^*$ of order r_i is strongly regular with $|\mathbf{C}_{L^*}(s_i)| = \Phi_{18}(q)$, respectively $\Phi_{12}(q)\Phi_6(q)$, for $i = 1, 2$. Now we are done by taking $\mathcal{T}_i^* = \mathbf{C}_{\mathcal{G}^*}(s_i)$.

From now on we may assume that L is a classical group with $\mathbf{Z}(L) \neq 1$ and not isomorphic to $\mathrm{SL}_2(q)$, $\mathrm{SL}_3^\pm(q)$, and $\mathrm{Sp}_4(q)$.

(iii) Consider the case $L = \mathrm{SL}_n(q)$ with $n \geq 4$. We may also assume that $q > 2$ and $(n,q) \neq (4,4)$ as $\mathbf{Z}(L) \neq 1$. The conditions on (n,q) ensure that $r_1 = \mathrm{ppd}(nf)$ and $r_2 = \mathrm{ppd}((n-1)f)$ exist. Also observe that $r_1 \geq nf + 1 \geq n + 1$ and $r_2 \geq (n-1)f + 1 \geq n$. Furthermore, if $r_2 = n$ then, since $r_2 \nmid (q-1)$, we have $\mathbf{Z}(L) = 1$, contrary to our assumption. In particular, both r_1 and r_2 are coprime to $|\mathbf{Z}(L)| = \gcd(n, q-1)$. By [MT08, Lemma 2.4] (and its proof), any element $s_i \in \mathcal{G}^*$ of order r_i is strongly regular, with $|\mathbf{C}_{L^*}(s_i)| = (q^n-1)/(q-1)$, respectively $q^{n-1}-1$, for $i = 1, 2$. Now we take $\mathcal{T}_i^* = \mathbf{C}_{\mathcal{G}^*}(s_i)$ and apply Lemma 4.10.

(iv) Next suppose that $L = \mathrm{SU}_n(q)$ with $n \geq 4$ and $|\mathbf{Z}(L)| = \gcd(n, q+1) > 1$. We may also assume that $(n,q) \neq (4,2), (4,3), (6,2)$ by Lemma 4.3. Assume first that n is even. Set $r_1 = \mathrm{ppd}(2(n-1)f)$, and $r_2 = \mathrm{ppd}(nf)$ if $4|n$, $r_2 = \mathrm{ppd}(nf) \cdot \mathrm{ppd}(nf/2)$ if $n \equiv 2 (\bmod 4)$. The conditions on (n,q) guarantee that the r_i exist and are coprime to n and to $|\mathbf{Z}(L)|$. Since $\mathbf{Z}(\mathcal{G}) \leq C_n$, it follows that $\mathbf{C}_{\mathcal{G}^*}(y)$ is connected for any element $y \in \mathcal{G}^*$ of order r_i. Moreover, by [MT08, Lemma 2.4], any element $s_1 \in \mathcal{G}^*$ of order r_1 is strongly regular, with $|\mathbf{C}_{L^*}(s_1)| = q^{n-1} + 1$. Arguing similarly, one can show that any element $s_2 \in \mathcal{G}^*$ of order r_2 is strongly regular, with $|\mathbf{C}_{L^*}(s_2)| = (q^n-1)/(q+1)$. Now we can take $\mathcal{T}_i^* = \mathbf{C}_{\mathcal{G}^*}(s_i)$ and apply Lemma 4.10.

Now let $n \geq 5$ be odd. Set $r_1 = \mathrm{ppd}(2nf)$, and $r_2 = \mathrm{ppd}((n-1)f)$ if $n \equiv 1 (\bmod 4)$, $r_2 = \mathrm{ppd}((n-1)f) \cdot \mathrm{ppd}((n-1)f/2)$ if $n \equiv 3 (\bmod 4)$. The conditions on (n,q) guarantee that the r_i exist; furthermore, $r_1 \geq 2n+1$, $r_{21} := \mathrm{ppd}((n-1)f) \geq (n-1)f + 1 \geq n$, and $r_{22} := \mathrm{ppd}((n-1)f/2) \geq (n-1)f/2 + 1 > n/2$. If $r_{2j}|n$ for some $j = 1, 2$, then $r_{2j} = n$, and since $r_{2j} \nmid (q+1)$, we have $\mathbf{Z}(L) = 1$, contrary to our assumption. Thus both r_1 and r_2 are coprime to n and to $|\mathbf{Z}(L)|$. Since $\mathbf{Z}(\mathcal{G}) \leq C_n$, it follows that $\mathbf{C}_{\mathcal{G}^*}(y)$ is connected for any element $y \in \mathcal{G}^*$ of order r_i. Moreover, by [MT08, Lemma 2.4], any element $s_1 \in \mathcal{G}^*$ of order r_1 is strongly regular, with $|\mathbf{C}_{L^*}(s_1)| = (q^n+1)/(q+1)$. One can show that any element $s_2 \in \mathcal{G}^*$ of order r_2 is strongly regular, with $|\mathbf{C}_{L^*}(s_2)| = q^{n-1} - 1$. Now we can take $\mathcal{T}_i^* = \mathbf{C}_{\mathcal{G}^*}(s_i)$ and apply Lemma 4.10.

(v) Now we consider the case $L = \mathrm{Sp}_{2n}(q)$ or $\mathrm{Spin}_{2n+1}(q)$ with $n \geq 3$ and q odd. We may also assume that $L \ncong \Omega_7(3)$ by Lemma 4.3. Since $\mathbf{Z}(L) = C_2$, it suffices to produce a faithful irreducible character of L of degree divisible by p. By [MT08, Lemma 2.4], any element $t \in \mathcal{G}^*$ of order $r := \mathrm{ppd}(2nf)$ is strongly regular, with $|\mathbf{C}_{L^*}(t)| = q^n + 1$. The proof of Lemma 4.10 yields a faithful irreducible character ϑ of L of degree $|L|_{\ell'}/(q^n + 1)$. Thus we are done if $p|\vartheta(1)$. In the remaining case, p divides $q^n + 1$ but not $\prod_{i=1}^{n-1}(q^{2i} - 1)$. Here, we consider the element $s \in L^*$ constructed in part (ii) of the proof of Proposition 4.9 and let $\varsigma \in \mathcal{E}(L, (s))$ labeled by the principal character of $\mathbf{C}_{L^*}(s)$. Then $\varsigma(1) = [L : \mathbf{C}_{L^*}(s)]_{\ell'}$, which is divisible by $(q^{2n} - 1)/(2(q \pm 1))$ if $L = \mathrm{Sp}_{2n}(q)$ and by $(\prod_{i=1}^{n}(q^i + 1))/2$ if $L = \mathrm{Spin}_{2n+1}(q)$.

(vi) Suppose that $L = \mathrm{Spin}^+_{2n}(q)$ with $n \geq 4$ and q odd. Set $r_1 = \mathrm{ppd}(2(n-1)f)$, and $r_2 = \mathrm{ppd}(nf)$ if $n \equiv 1(\mathrm{mod}\, 2)$, $r_2 = \mathrm{ppd}((n-1)f)$ if $2|n$. The conditions on (n, q) guarantee that the r_i exist and are coprime to $|\mathbf{Z}(L)|$. By [MT08, Lemma 2.4], any element $s_1 \in \mathcal{G}^*$ of order r_1 is strongly regular, with $|\mathbf{C}_{L^*}(s_1)| = (q^{n-1} + 1)(q + 1)$. Similarly, any element $s_2 \in \mathcal{G}^*$ of order r_2 is strongly regular, with $|\mathbf{C}_{L^*}(s_2)| = q^n - 1$ if n is odd and $|\mathbf{C}_{L^*}(s_2)| = (q^{n-1} - 1)(q - 1)$ if $2|n$. Now we can take $\mathcal{T}_i^* = \mathbf{C}_{\mathcal{G}^*}(s_i)$ and apply Lemma 4.10.

Finally we consider the case $L = \mathrm{Spin}^-_{2n}(q)$ with $n \geq 4$ and q odd. Set $r_1 = \mathrm{ppd}(2nf)$ and $r_2 = \mathrm{ppd}(2(n-1)f)$. The conditions on (n, q) guarantee that the r_i exist and are coprime to $|\mathbf{Z}(L)|$. By [MT08, Lemma 2.4], any element $s_i \in \mathcal{G}^*$ of order r_i is strongly regular, with $|\mathbf{C}_{L^*}(s_i)| = q^n + 1$, respectively $(q^{n-1} + 1)(q - 1)$ for $i = 1, 2$. Taking $\mathcal{T}_i^* = \mathbf{C}_{\mathcal{G}^*}(s_i)$, we are done by Lemma 4.10. $\square$

We have now completed the proof of Theorem 2.2.

5. **Small exceptions in Theorem 2.3**

THEOREM 5.1. *Let $p > 2$ be a prime, and let G be a finite perfect group with $G = \mathbf{O}^{p'}(G)$ and with normal subgroups $E \geq K > Z$ such that*

(i) *$K = Q \times Z$, $Q \in \mathrm{Syl}_p(K)$, $Z \leq \mathbf{Z}(G)$ is a p'-subgroup, and $\lambda \in \mathrm{Irr}(Z)$ is faithful*;

(ii) *$E/K = \mathbf{O}_{p'}(G/K)$, and $S := G/E$ is a non-abelian simple group of order divisible by p with abelian Sylow p-subgroups*;

(iii) *Q is elementary abelian, $\mathbf{C}_G(Q) = K$, and G/K acts irreducibly and p-exceptionally on $V := \mathrm{Irr}(Q)$ as described in cases* (i) *or* (iii) *of Theorem* 2.3.

Then there exists $\chi \in \mathrm{Irr}(G|\lambda)$ of degree divisible by p.

Proof. By the Ito-Michler theorem, we may assume that $Z \neq 1$. Consider the cases where $\bar{L} := G/K$ is quasisimple in Theorem 2.3(i) or (iii): $(G/K, |V|) = (\mathrm{SL}_2(q), q^2)$, $(\mathrm{SL}_2(5), 3^4)$, $(\mathrm{PSL}_2(11), 3^5)$, $(M_{11}, 3^5)$, or $(\mathrm{SL}_2(13), 3^6)$. The conditions $\mathbf{C}_G(Q) = K$ and G/K acts irreducibly on $\mathrm{Irr}(Q)$ imply that $Z = \mathbf{Z}(G)$. Since $G/Q = Z \cdot \bar{L}$ and G is perfect, we see that G/Q is a perfect central extension of $\bar{L}$. Furthermore, Z is a p'-group and the p'-part of the Schur multiplier of $\mathrm{SL}_2(q)$ is 1 when $p|q \geq 4$. We conclude $(G/Q, p) = (\mathrm{SL}_2(11), 3)$ and $Z = C_2$. Inspecting [CCN$^+$85], we see that there is some $\chi \in \mathrm{Irr}(G|\lambda \times 1_Q)$ of degree divisible by p.

It remains to consider the case $(G/K, |V|) = (2_-^{1+4} \cdot \mathsf{A}_5, 3^4)$. Then G/Q is a perfect, central extension of $H \cong 2_-^{1+4} \cdot \mathsf{A}_5$ with kernel $\geq Z$. Using [GAP], one can check that the universal cover L of H has order $2|H|$; in particular, $|Z| = 2$. Moreover, L has three central subgroups Z_i of order 2, and for each of them, there is an irreducible character $\chi \in \mathrm{Irr}(L)$ of degree divisible by 3 that lies above the unique nonprincipal character of Z_i. Hence, there is some $\chi \in \mathrm{Irr}(G|\lambda \times 1_Q)$ of degree divisible by p. □

6. The imprimitive case of Theorem 2.3

The main goal of this section is to prove the following theorem, which handles the imprimitive case of Theorem 2.3.

THEOREM 6.1. *Let $p > 2$ be a prime, and let G be a finite perfect group with $G = \mathbf{O}^{p'}(G)$ and with normal subgroups $E \geq K > Z$ such that*

(i) *$K = Q \times Z$, $Q \in \mathrm{Syl}_p(K)$, $Z \leq \mathbf{Z}(G)$ is a p'-subgroup, and $\lambda \in \mathrm{Irr}(Z)$ is faithful*;

(ii) *$E/K = \mathbf{O}_{p'}(G/K)$, and $S := G/E$ is a non-abelian simple group of order divisible by p with abelian Sylow p-subgroups*;

(iii) *Q is elementary abelian, $\mathbf{C}_G(Q) = K$, and G/K acts irreducibly, imprimitively, and p-exceptionally on $V := \mathrm{Irr}(Q)$ as described in Theorem 2.3(ii).*

Then there exists $\chi \in \mathrm{Irr}(G|\lambda)$ of degree divisible by p.

To this end, we will prove a slightly more general result.

THEOREM 6.2. *Let G be a finite group with normal subgroups $M \geq R \geq \mathbf{Z}(G)$, $p > 2$ a prime, and $\lambda \in \mathrm{Irr}(\mathbf{Z}(G))$ be such that*

(i) *$S := G/M$ is simple, and R/Z is a solvable p'-group. Moreover, if r_0 denotes the maximal rank of chief factors of G in $R/\mathbf{Z}(G)$, then one of the following holds*:

 (a) *$p > 3$, and $S = \mathsf{A}_m$ where $m = ap-2$ with $3 \leq a \leq p$ or $m = 2p-1$. Furthermore, $r_0 \leq 2m + 2$.*

(b) *$p = 3$, and $S \in \{\mathsf{A}_5, \mathsf{A}_7, \mathrm{SL}_3(2)\}$. Furthermore, if $S = \mathrm{SL}_3(2)$ and 7 divides $|R/\mathbf{Z}(G)|$, then $r_0 \leq 7$.*

(ii) *If $M > R$, then $p \in \{11, 19, 29, 59\}$, $R/\mathbf{Z}(G) = \mathbf{Z}(M/\mathbf{Z}(G))$, and all composition factors of M/R are isomorphic to A_5.*

Then there exists $\chi \in \mathrm{Irr}(G|\lambda)$ of degree divisible by p.

6.1. *The structure of certain linear groups.*

Lemma 6.3. *Let a finite group $G < \mathrm{GL}(V)$ act on the n summands of a decomposition $V = V_1 \oplus \cdots \oplus V_n$ of a finite-dimensional vector space V over $\mathbb{F}_p$, with kernel B. Suppose that, for all i, one can identify V_i with $\mathbb{F}_{p^d}$ such that $\mathrm{Stab}_G(V_i)$ acts on V_i as a subgroup of the group $\Gamma\mathrm{L}_1(p^d)$ of all the $\mathbb{F}_{p^d}$-semilinear transformations of $\mathbb{F}_{p^d}$. Then all chief factors of G in B are elementary abelian of rank $\leq n$. If in addition G fixes V_1, then all chief factors of G in B are elementary abelian of rank $\leq n-1$.*

Proof. Note that $H := \Gamma\mathrm{L}_1(p^d)$ is the semidirect product of the cyclic group $\mathrm{GL}_1(p^d)$ of order $p^d - 1$ by the cyclic group $\mathrm{Gal}(\mathbb{F}_{p^d}/\mathbb{F}_p) \cong C_d$. It is easy to see that all the elements of order $p^d - 1$ of H generate the cyclic subgroup $\mathrm{GL}_1(p^d)$ and so $\mathrm{GL}_1(p^d)$ is characteristic in H. Then we can refine the normal series $1 < \mathrm{GL}_1(p^d) < H$ to a series of characteristic subgroups $1 = H_0 < H_1 < \cdots < H_m = H$ such that all the quotients H_i/H_{i-1} are of prime order.

Suppose, for instance, that G is transitive on $\{V_1, \ldots, V_n\}$. For each i, fix $g_i \in G$ such that $g_i(V_1) = V_i$, $g_1 = 1$. Also fix a basis $(e^1_k \mid 1 \leq k \leq d)$ of V_1, and let $e^i_k = g_i(e^1_k)$, so that $(e^i_k \mid 1 \leq k \leq d)$ is a basis of V_i. Without loss we may assume that the action of $\mathrm{Stab}_G(V_i)$ written in this basis is contained in H. Using this basis $(e^i_k \mid 1 \leq i \leq n, 1 \leq k \leq d)$ of V, we can form the wreath product $L := H \wr \mathsf{S}_n$ in $\mathrm{GL}(V)$, where each $\pi \in \mathsf{S}_n$ acts via $\pi(e^i_k) = e^{\pi(i)}_k$. Observe that $G \leq L$. Indeed, consider any $g \in G$ and suppose g induces the permutation π on $\{V_1, \ldots, V_n\}$. If $g(V_i) = V_j$, then the element $g_i g_j^{-1} g \in G$ fixes V_i. Thus there is an element $h_i \in H$ that acts as $g_i g_j^{-1} g$ on V_i; in particular, $g|_{V_i} = g_j g_i^{-1} h_i$. Set $h = (h_1, \ldots, h_n) \in H^n$, and consider the canonical element $\pi \in L$ corresponding to the permutation π. Then one can check that $g = \pi h$ and so $g \in L$. Repeating this argument for each orbit of G on $\{V_1, \ldots, V_n\}$, we may now assume that G is contained in a subgroup $L = H \wr \mathsf{S}_n$ of $\mathrm{GL}(V)$. (In fact, it is contained in $X \wr \mathsf{S}_n$, where $X \leq \Gamma\mathrm{L}_1(p^d)$ is chosen to contain the subgroup induced by the action of $\mathrm{Stab}_G(V_i)$ on V_i for all i.)

Now the base subgroup $K := H^n$ of L has a chain of subgroups $1 = K_0 < K_1 < \cdots < K_m = K$ such that all the subgroups $K_i := H_i^n$ are normal in L and all the quotients K_i/K_{i-1} are elementary abelian of rank n. As mentioned above, $G \le L$; furthermore, $B = K \cap G$. Then $K_i \cap G \lhd G$, and

$$(K_i \cap G)/(K_{i-1} \cap G) \cong (K_i \cap G)K_{i-1}/K_{i-1} \le K_i/K_{i-1}$$

are elementary abelian of rank $\le n$.

Assume in addition that G fixes V_1. Then we can apply the above argument to the subgroup $M = \mathrm{Stab}_L(V_1) \cong K : \mathsf{S}_{n-1}$ with the following chain,

$$1 = K_0 < K_1 < \cdots < K_{2m} = K,$$

of normal subgroups in M, where

$$K_i := H_i \times 1 \times 1 \cdots \times 1,\ K_{m+i} := H \times H_i \times H_i \times \cdots \times H_i$$

for $0 \le i \le m$. $\square$

LEMMA 6.4. *Assume we are in situation* (ii) *of Theorem* 2.3. *Assume in addition that* $S := G/\mathbf{O}_{p'}(G)$ *is simple. Let* $H \le \mathrm{GL}(V_1)$ *be induced by the action of* $\mathrm{Stab}_G(V_1)$ *on* V_1, $d := \dim_{\mathbb{F}_p}(V_1)$, *and let* B *be the kernel of the action of* G *on* $\{V_1, \dots, V_n\}$. *Then the following statements hold*:

(i) $B \le \mathbf{O}_{p'}(G)$. *In fact, either* $B = \mathbf{O}_{p'}(G)$ *or* $(S, n, p) = (\mathrm{SL}_3(2), 8, 3)$ *and* $\mathbf{O}_{p'}(G)/B = 2^3$.

(ii) *If* $(S, p) = (\mathsf{A}_5, 3)$, *then* H *is solvable.*

(iii) *Assume* $(S, p) \ne (\mathsf{A}_5, 3)$. *Then* H *is a* p'-*group, and one of the following statements holds*:

  (a) $H \le \Gamma\mathrm{L}_1(p^d)$;

  (b) $p^d = 11^2,\ 19^2,\ 29^2,\ 59^2$, *and* $\mathrm{SL}_2(5) \lhd H \le C_{p-1} * \mathrm{SL}_2(5)$;

  (c) $p^d = 5^2,\ 7^2,\ 11^2,\ 23^2$, *and* $H \le N_{\mathrm{GL}_d(p)}(Q_8) \le (C_{p-1} * Q_8) \cdot \mathsf{S}_3$;

  (d) $p^d = 3^4$ *and* H *is a solvable subgroup of* $N_{\mathrm{GL}_4(3)}(2_-^{1+4})$;

  (e) $p^d = 3^2$ *and* $H = Q_8$.

Proof. By the description in Theorem 2.3(ii), we see that either $G/B \cong \mathsf{A}_n$ with $n \ge 2p - 1 \ge 5$ or $(n, p) = (8, 3)$ and $G/B \cong 2^3 : \mathrm{SL}_3(2)$. On the other hand, by assumption G has a unique non-abelian composition factor of order divisible by p, with multiplicity 1, which is $S = G/\mathbf{O}_{p'}(G)$. It follows that $B \le \mathbf{O}_{p'}(G)$ and, in fact, $\mathbf{O}_{p'}(G)/B$ is trivial in the former case and 2^3 in the latter case. Moreover, $S = \mathsf{A}_n$, resp. $S = \mathrm{SL}_3(2)$.

Recall that $H \le \mathrm{GL}(V_1)$ is transitive on $V_1 \setminus \{0\}$, hence Hering's Theorem (cf. [Lie87, App. 1]) applies to H. In particular, it follows that either H is solvable or H has a unique non-abelian composition factor X, where $X = \mathrm{PSL}_a(q)$ or $\mathrm{PSp}_a(q)$ with $p^d = q^a$, or $X = \mathsf{A}_5$, or $X = \mathrm{PSL}_2(13)$. In all case, the order of such an X is divisible by 3. Now if $(S, p) = (\mathsf{A}_5, 3)$, then $\mathrm{Stab}_G(V_1)$ is an extension of the $3'$-group B by A_4, and so H as a homomorphic image of $\mathrm{Stab}_G(V_1)$ is solvable.

From now on we assume $(S, p) \ne (\mathsf{A}_5, 3)$. Then, again $\mathrm{Stab}_G(V_1)$ is an extension of the p'-group B by the simple group $T = \mathsf{A}_{n-1}$ or $T = \mathrm{SL}_3(2)$.

Note that T and X cannot be isomorphic, but p divides $|T|$. Since H is a homomorphic image of $\mathrm{Stab}_G(V_1)$, we conclude that T must be a composition factor of the kernel of the action of $\mathrm{Stab}_G(V_1)$ on V_1. Hence H is a p'-group.

Now we apply Hering's Theorem as formulated in [Lie87, App. 1] to H again. If H belongs to case (A) listed there, then we arrive at (a). Case (B) leads to the possibilities (c)–(e), and case (C) leads to (b). (Note that for p not dividing $|\mathrm{SL}_2(5)|$, the 2-dimensional irreducible representations of $\mathrm{SL}_2(5)$ on $\mathbb{F}_p^2$ (if they exist) do not extend to $\mathrm{SL}_2(5)\cdot 2$ and so $N_{\mathrm{GL}_2(p)}(\mathrm{SL}_2(5)) = C_{p-1} * \mathrm{SL}_2(5)$.) □

Lemma 6.5. *Let a finite group $G < \mathrm{GL}(V)$ act transitively on the n summands of a decomposition $V = V_1 \oplus \cdots \oplus V_n$ of $V = \mathbb{F}_p^{2n}$ with kernel B, where $p = 11$, 19, 29, or 59. Suppose the subgroup $X \leq \mathrm{GL}(V_1)$ induced by the action of $\mathrm{Stab}_G(V_1)$ on V_1 is contained in the irreducible subgroup $H := C_{p-1} * \mathrm{SL}_2(5)$ of $\mathrm{GL}(V_1)$. Then there is a normal subgroup $C \lhd G$ inside B such that both of the following hold*:

(i) *All chief factors of G in C are elementary abelian of rank $\leq 2n$.*
(ii) *If $C < B$, then $C = \mathbf{Z}(B)$ and all composition factors of B/C are isomorphic to A_5.*

Proof. (a) Note that $\mathbf{Z}(H) = \mathbf{Z}(\mathrm{GL}(V_1)) = C_{p-1}$ and $H/\mathbf{Z}(H) \cong \mathsf{A}_5$. Certainly the action of B on V_1 induces a subgroup $Y \lhd X \leq H$, and by the transitivity of G on the V_i's, $B \leq Y^n$. As shown in the proof of Lemma 6.3, $G \leq X \wr \mathsf{S}_n$. Also, $Y \cap \mathbf{Z}(H)$ is cyclic. Hence there is a series of X-invariant subgroups $1 < Y_1 < \cdots < Y_a = Y \cap \mathbf{Z}(H)$, where each quotient Y_i/Y_{i-1} is of prime order.

Suppose in addition that Y is solvable. Then $\bar{Y} := Y/(Y \cap \mathbf{Z}(H))$ is a solvable subgroup of $H/\mathbf{Z}(H) \cong \mathsf{A}_5$. Hence $\bar{Y}$ has a series of characteristic subgroups $1 < \bar{Y}_{a+1} < \cdots < \bar{Y}_{a+b} = \bar{Y}$ where each quotient $\bar{Y}_{i+1}/\bar{Y}_i$ is elementary abelian of rank ≤ 2. Setting $\bar{Y}_j = Y_j/Y_a$ for $j > a$, we see that Y has a series of X-invariant subgroups $1 < Y_1 < \cdots < Y_{a+b} = Y$, where each quotient Y_{i+1}/Y_i is elementary abelian of rank ≤ 2. It follows that Y^n has a series of $X \wr \mathsf{S}_n$-invariant subgroups $1 < Y_1^n < \cdots < Y_{a+b}^n = Y^n$, where each quotient Y_i^n/Y_{i-1}^n is elementary abelian of rank $\leq 2n$. Taking $C = B$ and arguing as in the proof of Lemma 6.3, we are done in this case.

(b) Now assume that Y is nonsolvable; in particular, $Y/(Y \cap \mathbf{Z}(H)) \cong \mathsf{A}_5$. Let $\pi_i : B \to \mathrm{PGL}(V_i)$ be induced by the action of B on the projective space $\mathbb{P}V_i$, so that $\mathrm{Im}(\pi_i) \cong \mathsf{A}_5$. Then $C := \cap_{i=1}^n \mathrm{Ker}(\pi_i) \leq (Y \cap \mathbf{Z}(H))^n$ is centralized by B. Arguing as in the proof of Lemma 6.3, we see that each chief factor of G in C is elementary abelian of rank $\leq n$. It remains to show that all nontrivial composition factors of B/C are isomorphic to A_5. Consider the following chain of normal subgroups of B: $B_0 = B$, $B_{i+1} = B_i \cap \mathrm{Ker}(\pi_{i+1})$ for $0 \leq i \leq n-1$.

Then $B_n = C$. Furthermore, $B_i/B_{i+1} \lhd \mathrm{Im}(\pi_{i+1}) \cong \mathsf{A}_5$, so either $B_i = B_{i+1}$ or $B_i/B_{i+1} \cong \mathsf{A}_5$. $\square$

6.2. *Some reductions.* It is convenient to have the following easy reduction step.

Lemma 6.6. *Let $Z \leq \mathbf{Z}(G)$, p a prime, and $\lambda \in \mathrm{Irr}(Z)$ be such that $p \nmid \chi(1)$ for all $\chi \in \mathrm{Irr}(G|\lambda)$. Then there is a subgroup $H \lhd G$ such that*

(i) *$Z \leq H$, H/Z is perfect;*
(ii) *$p \nmid \alpha(1)$ for all $\alpha \in \mathrm{Irr}(H|\lambda)$;*
(iii) *if X is a finite group with only non-abelian composition factors and G maps surjectively onto X, then so does H.*

Proof. We choose $H := G^{(\infty)}Z$ to fulfill (i). Since $Z \leq H \lhd G$, (ii) is also satisfied. Suppose now that $K \lhd G$ and $G/K \cong X$. Then G/HK is a solvable quotient of G/K, so $HK = G$, and $H/(H \cap K) \cong G/K = X$, as required. $\square$

We will need the following character-theoretic fact.

Lemma 6.7. *Let $Z \leq \mathbf{Z}(G)$, $\lambda \in \mathrm{Irr}(Z)$ a faithful irreducible character, and let K/Z be a nontrivial abelian chief factor of G, of order r^m for a prime r. Then one of the following statements holds:*

(i) *There is a G-invariant $\vartheta \in \mathrm{Irr}(K|\lambda)$.*
(ii) *K is abelian, and $\mathrm{Irr}(K|\lambda)$ consists of exactly $|K/Z|$ distinct linear characters. Moreover, if $G/\mathbf{Z}(G)$ is perfect in addition, then there is an elementary abelian r-subgroup $L \lhd G$ such that $K = L * Z$ is a central product, $L \cap Z \cong C_r$, and the $\mathbb{F}_rG$-module L is indecomposable.*

Proof. Certainly, λ^K equals 0 on $K \setminus Z$ and $|K/Z| \cdot \lambda$ on Z. Hence $[\lambda^K, \lambda^K]_K = |K/Z| > 1$; i.e., λ^K is reducible. According to [Isa06, Ex. (6.12)], in this situation, either $\lambda^K = e\vartheta$ with $\vartheta \in \mathrm{Irr}(K)$ and $e^2 = |K/Z|$ or $\lambda^K = \sum_{i=1}^{t} \mu_i$, where the $\mu_i \in \mathrm{Irr}(K)$ are distinct and $t = |K/Z|$. In the former case, certainly ϑ is G-invariant, and so we arrive at (i). Assume we are in the latter case. Then all μ_i are linear. Furthermore, $\mathrm{Ker}(\lambda^K) = \mathrm{Ker}(\lambda) = 1$. Hence, any $\mathbb{C}K$-representation affording the character λ^K maps K injectively into a group of diagonal matrices, whence K is abelian.

Assume furthermore in the case of (ii) that $G/\mathbf{Z}(G)$ is perfect. Since K is abelian, $K = \mathbf{O}_r(K) \times \mathbf{O}_{r'}(K) = \mathbf{O}_r(K) \times \mathbf{O}_{r'}(Z)$. Observe that Z is cyclic since λ is faithful. Consider $L = \Omega_1(\mathbf{O}_r(K)) \lhd G$. Suppose $L \cong C_r$. Then $\mathbf{O}_r(K)$ is cyclic, and so K is cyclic. But $G/\mathbf{Z}(G)$ is perfect, so $G/\mathbf{Z}(G)$ acts trivially on K; i.e., $K \leq \mathbf{Z}(G)$. In this case, any $\mu_i \in \mathrm{Irr}(K|\lambda)$ is G-invariant, and we arrive at (i) again. So we may assume L is not cyclic. Now $L \cap Z$ has order 1 or r as Z is cyclic, so $1 \neq LZ/Z \leq K/Z$. But K/Z is a chief factor, hence $K = LZ = L * Z$. If $L \cap Z = 1$, then we can take $\vartheta := 1_K \times \lambda$ and return to (i). Similarly, if $L \cap Z = C_r$ and the $\mathbb{F}_r G$-module L is decomposable, then we can find a normal subgroup $L_1 \lhd G$ such that $L = L_1 \times (L \cap Z)$, in which case $K = L_1 \times Z$, and so $\vartheta := 1_{L_1} \times \lambda$ is again G-invariant. $\square$

In certain situations involving Lemma 6.7(ii), we will need the following counting argument.

LEMMA 6.8. *Let $Z \leq \mathbf{Z}(G)$ and let $1 \leq K/Z$ be an r-group that is a chief factor of G. Assume $p \neq r$ is a prime and $\lambda \in \mathrm{Irr}(Z)$ is such that $p \nmid \chi(1)$ for all $\chi \in \mathrm{Irr}(G|\lambda)$. Assume in addition that K is abelian. Then considering $W := \mathrm{Irr}(K/Z)$ as an $\mathbb{F}_r G$-module, we have that*

$$|W| \leq \sum_{\bar{P} \in \mathrm{Syl}_p(G/\mathbf{C}_G(K))} |\mathbf{C}_W(\bar{P})|.$$

Proof. Since K is abelian, $\mathrm{Irr}(K|\lambda)$ consists of exactly $|W|$ linear characters. Denote $C := \mathbf{C}_G(K)$. By assumption, the G-orbit of any $\mu \in \mathrm{Irr}(K|\lambda)$ is of p'-length, and C acts trivially on this set, hence μ is fixed by some $P/C \in \mathrm{Syl}_p(G/C)$. Now we can find $Q/K \in \mathrm{Syl}_p(P/K)$ such that $P/K = (C/K)(Q/K)$. Since $p \neq r$, Q/K acts coprimely on K/Z. Hence by [Isa06, Th. (13.31)], there is $\varrho \in \mathrm{Irr}(K|\lambda)$ that is Q-invariant. Clearly, ϱ is also C-invariant. Now $\mu \in \mathrm{Irr}(K|\lambda)$ is P-invariant precisely when $\mu = \varrho\alpha$ for some P/C-invariant $\alpha \in \mathrm{Irr}(K/Z) = W$. Thus the number of P/C-invariant characters in $\mathrm{Irr}(K|\lambda)$ equals $|\mathbf{C}_W(P/C)|$, and so the claim follows. $\square$

Note that, in the situation of Theorem 6.2, R is just $\mathrm{sol}(G)$, the solvable radical of G.

Now we prove a key reduction for Theorem 6.2.

PROPOSITION 6.9. *Let G be a counterexample to Theorem 6.2 such that $G/\mathbf{Z}(G)$ has smallest possible order and then with $|\mathbf{Z}(G)|$ minimal possible. Then the following statements hold*:

(i) *Assume that $\mathbf{Z}(G) < N \lhd G$ and that either $N \leq R$ or $R \leq N \leq M$. Then no $\varrho \in \mathrm{Irr}(N|\lambda)$ can be G-invariant. Furthermore, the G-orbit of any $\vartheta \in \mathrm{Irr}(N|\lambda)$ has length coprime to p.*

(ii) *$G/\mathbf{Z}(G)$ is perfect, and λ is faithful.*

(iii) *$M \not\leq \mathbf{Z}(G)$.*

(iv) *Let $1 \neq K/\mathbf{Z}(G)$ be a chief factor of G in $R/\mathbf{Z}(G)$. Then $K \leq \mathbf{Z}(M)$.*

Proof. (i) Assume the contrary: $\varrho \in \mathrm{Irr}(N|\lambda)$ is G-invariant. Since G is a counterexample, p is coprime to $\chi(1)$ and $\chi(1)/\varrho(1)$ for every $\chi \in \mathrm{Irr}(G|\varrho)$. Now (G, N, ϱ) is a character triple, and so it is isomorphic to a character triple (G^*, N^*, μ) with $N^* \leq \mathbf{Z}(G^*)$ by [Isa06, Th. (11.28)]. This isomorphism preserves ratios of character degrees (see [Isa06, Lemma (11.24)]), hence $\theta(1)$ is coprime to p for all $\theta \in \mathrm{Irr}(G^*|\mu)$. Recall that this isomorphism induces an isomorphism $\tau : G^*/N^* \to G/N$. For any $N \leq H \leq G$, let H^τ be the inverse image of $\tau(H/N)$ in G^*; in particular, $G^\tau = G^*$ and $N^\tau = N^*$. Set $R^* := R^\tau \mathbf{Z}(G^*)$ if $N \leq R$, and set $R^* := \mathbf{Z}(G^*)$ if $R \leq N \leq M$. Also set $M^* := M^\tau$. Then one readily checks that G^* has the structure prescribed in Theorem 6.2, with (G^*, M^*, R^*, μ) playing the role of (G, M, R, λ). Thus G^* is a counterexample to Theorem 6.2 with $|G^*/\mathbf{Z}(G^*)| \leq |G^*/N^*| = |G/N| < |G/\mathbf{Z}(G)|$, contrary to the choice of G.

The second claim follows by Clifford's Theorem from the assumption that $p \nmid \chi(1)$ for all $\chi \in \mathrm{Irr}(G|\lambda)$.

(ii) Let $Y := G/\mathrm{Ker}(\lambda)$. Since Y is also a counterexample to Theorem 6.2 (with $(Y, M/\mathrm{Ker}(\lambda), R/\mathrm{Ker}(\lambda), \lambda)$ playing the role of (G, M, R, λ)) and $|Y/\mathbf{Z}(Y)| \leq |G/\mathbf{Z}(G)|$, we must have that $|Y/\mathbf{Z}(Y)| = |G/\mathbf{Z}(G)|$, and so $\mathbf{Z}(Y) = \mathbf{Z}(G)/\mathrm{Ker}(\lambda)$. But then the minimality of G forces $\mathrm{Ker}(\lambda) = 1$. Also, by Lemma 6.6 and its proof, we see that $H := G^{(\infty)}\mathbf{Z}(G)$ is a counterexample to Theorem 6.2, with $(H, H \cap M, H \cap R, \nu)$ playing the role of (G, M, R, λ), where $\nu \in \mathrm{Irr}(\mathbf{Z}(H)|\lambda)$. But $|H/\mathbf{Z}(H)| \leq |G/\mathbf{Z}(G)|$, so by the choice of G, $G = H$, whence $G/\mathbf{Z}(G)$ is perfect.

(iii) Suppose $M = \mathbf{Z}(G)$, and let $E = G^{(\infty)}$. Then $G = E * \mathbf{Z}(G)$, and E is a quasisimple group. Furthermore, denoting $\alpha := \lambda_{\mathbf{Z}(E)}$, we see that all $\theta \in \mathrm{Irr}(E|\alpha)$ are of degree coprime to p. As λ is faithful, α is also faithful. Since E is perfect, $\det(\theta) = 1_E$, whence $\alpha^{\theta(1)} = 1_{\mathbf{Z}(E)}$, and so $|\mathbf{Z}(E)|$ is coprime to p. Now we can apply Theorem 2.2 to get the desired contradiction.

(iv) Let K/Z be an elementary abelian r-group of rank m. Since λ is faithful by (ii), we can apply Lemma 6.7 to K. By (i), no $\varrho \in \mathrm{Irr}(K|\lambda)$ can be G-invariant. Hence K is abelian, and $K = L * Z$, where $Z := \mathbf{Z}(G)$ and L is the elementary abelian r-group specified in Lemma 6.7(ii).

(a) Denote $D := \mathbf{C}_R(L/(Z \cap L))$ and $C := \mathbf{C}_R(L) \geq K$. First we show that $C = D$. Assume the contrary: $C < D$. View L as an $\mathbb{F}_r$-space with a basis $(e_0, e_1, \ldots, e_m)$, where $\langle e_0 \rangle = Z \cap L$, and write down the representation Φ of G/C on L in this basis,

$$\Phi(x) = \begin{pmatrix} 1 & f(x) \\ 0 & I_m \end{pmatrix}, \; \Phi(g) = \begin{pmatrix} 1 & f(g) \\ 0 & \Psi(g) \end{pmatrix},$$

where $x \in D/C$, $g \in G/C$, $f : G/C \to \mathbb{F}_r^m$, and $\Psi \in \mathrm{Hom}(G/C, \mathrm{GL}_m(r))$. Since $D/C \neq 1$ acts faithfully on L, $F := \{f(x) \mid x \in D/C\}$ is a nonzero subspace of $\mathbb{F}_r^m = L/(Z \cap L)$ fixed by the irreducible subgroup $\Psi(G/C)$ of $\mathrm{GL}_m(r)$. It follows that $F = \mathbb{F}_r^m$. Therefore, if $(e_0^*, e_1^*, \ldots, e_m^*)$ denotes the dual basis of $L^* := \mathrm{Hom}_{\mathbb{F}_r}(L, \mathbb{F}_r)$, then D/C acts transitively on the r^m representatives of any nonzero coset in L^*/W, where $W := \langle e_1^*, \ldots, e_m^* \rangle_{\mathbb{F}_r}$. Now we can identify L^* with $\mathrm{Irr}(L)$ and W with $\mathrm{Irr}(L/(Z \cap L))$. Then $\mathrm{Irr}(L|\lambda_{Z \cap L})$ is just one of these nonzero cosets, as λ is faithful by (ii). Thus D/C acts transitively on the set $\mathrm{Irr}(L|\lambda_{Z\cap L})$. Since $K = L * Z$, D/C also acts transitively on $\mathrm{Irr}(K|\lambda)$.

Fix some $\mu \in \mathrm{Irr}(K|\lambda)$, and let $J := \mathrm{Stab}_G(\mu) \geq K$. Since G/C acts on $\mathrm{Irr}(K|\lambda)$ of cardinality r^m and D/C acts transitively on it, we have that $G = DJ = RJ$ and $[G : J] = r^m > 1$. Next, $J/(R\cap J) \cong G/R$, so J also has the structure described in Theorem 6.2, with $(J, J \cap M, J \cap R)$ playing the role of (G, M, R). (Notice that $R \cap J \geq K$.) Furthermore, μ is J-invariant and, since G is a counterexample, by Clifford's Theorem every $\theta \in \mathrm{Irr}(J|\mu)$ has degree coprime to p. Replacing the character triple (J, K, μ) by an isomorphic triple, we may assume that $K \leq \mathbf{Z}(J)$. But then J is a counterexample to Theorem 6.2 with $|J/\mathbf{Z}(J)| \leq |J/K| < |G/Z|$, contradicting the minimality of G.

(b) Now we show that R centralizes L. Assume the contrary: $C = D < R$. Since $L/(Z \cap L) \cong K/Z$ is an irreducible $\mathbb{F}_rG$-module and R/D acts faithfully on it, $\mathbf{O}_r(R/D) = 1$. Since R/D is solvable, $X := \mathbf{O}_{r'}(R/D) \neq 1$. Now X acts coprimely on the abelian group L and trivially on $Z \cap L$, so by [KS98, 8.4.2] we can write $L = \mathbf{C}_L(X) \times [L, X]$ with $\mathbf{C}_L(X) \geq Z \cap L$. Again by faithfullness, $\mathbf{C}_L(X) < L$, and so by irreducibility, $\mathbf{C}_L(X) = Z \cap L$. Thus $[L, X]$ is a G-invariant complement for $Z \cap L$ in L; i.e., the $\mathbb{F}_rG$-module L is decomposable, contradicting Lemma 6.7(ii).

We have shown that R centralizes L and $K = LZ$; in particular, $K \leq \mathbf{Z}(M)$ if $M = R$. Suppose that $M > R$. Then, by the assumptions, the perfect group M/R acts trivially on K/Z and on Z. Hence M centralizes K by the Three Subgroups Lemma. □

6.3. *The case* $p > 3$. For any alternating group A_n with $n \geq 5$ and any prime r, consider the natural permutation module $\mathcal{N} = \langle e_1, \ldots, e_n \rangle_{\mathbb{F}_r}$ and its submodules $I := \langle \sum_{i=1}^n e_i \rangle_{\mathbb{F}_r}$, $\mathcal{N}^0 = \{\sum_{i=1}^n a_i e_i \in \mathcal{N} \mid \sum_{i=1}^n a_i = 0\}$. Then $\mathcal{D} := \mathcal{N}^0/(\mathcal{N}^0 \cap I)$ is an (absolutely) irreducible, self-dual, $\mathbb{F}_r\mathsf{A}_n$-module.

For the reader's convenience we recall the following presumably well-known fact.

Lemma 6.10. *For* $n \geq 5$,

$$H^1(\mathsf{A}_n, \mathcal{D}) \cong \begin{cases} 0 & \text{if } r \nmid n \text{ and } (r,n) \neq (3,5), \\ \mathbb{F}_r & \text{if } r|n \text{ and } (r,n) \neq (3,6), \\ \mathbb{F}_3 & \text{if } (r,n) = (3,5), \\ \mathbb{F}_3^2 & \text{if } (r,n) = (3,6). \end{cases}$$

Proof. By [KP93, Lemma 1], $H^1(\mathsf{A}_n, \mathcal{N}/I) = 0$ except for $r = 3$ and $n = 5, 6$, in which cases it is $\mathbb{F}_r$. Now if $r \nmid n$, then $\mathcal{D} \cong \mathcal{N}^0 \cong \mathcal{N}/I$, and the claim follows. Suppose $r|n$. Then the long exact cohomology sequence yields

$$0 \to \mathbb{F}_r \to H^1(\mathsf{A}_n, \mathcal{D}) \to H^1(\mathsf{A}_n, \mathcal{N}/I) \to H^1(\mathsf{A}_n, I) \to H^2(\mathsf{A}_n, \mathcal{D}) \to \cdots .$$

If $(r,n) \neq (3,6)$, then $H^1(\mathsf{A}_n, \mathcal{N}/I) = 0$, and so we obtain $H^1(\mathsf{A}_n, \mathcal{D}) \cong \mathbb{F}_r$. If $(r,n) = (3,6)$, then $H^1(\mathsf{A}_n, I) = 0$, yielding $H^1(\mathsf{A}_6, \mathcal{D}) \cong \mathbb{F}_3^2$. $\square$

When $r|n$, then $\mathcal{D}$ is a submodule of codimension 1 in $\mathcal{N}/I$. By a *nonzero $\mathcal{D}$-coset* in $\mathcal{N}/I$ we mean any coset different from $\mathcal{D}$.

Lemma 6.11. *Let $2 < p \leq n$ be a prime, and let r be a prime divisor of n. Assume in addition that $n \notin \{2p-2, ap-1 \mid a \geq 3\}$ for $r = 2$. Then any nonzero $\mathcal{D}$-coset in $\mathcal{N}/I$ contains a representative $\bar{v}$ such that the A_n-orbit of $\bar{v}$ has length divisible by p.*

Proof. Without loss we may assume that the coset in question is

$$\mathcal{C} := \left\{ \bar{x} = x + I \in \mathcal{N}/I \mid x = \sum_i x_i e_i,\ x_i \in \mathbb{F}_r,\ \sum_i x_i = 1 \right\}.$$

Denote $G := \mathsf{A}_n$ and $H := \mathsf{S}_n$. We will also write $\sum_i x_i e_i$ sometimes as $(x_1, \ldots, x_n)$.

(i) Consider any $\bar{x} = x + I \in \mathcal{C}$. For any $\alpha \in \mathbb{F}_r$, let $J_\alpha := \{i \mid x_i = \alpha\}$, and let $j := \max_{\alpha \in \mathbb{F}_r} |J_\alpha|$. Suppose that $|J_\alpha| = j$ for precisely $k \leq 2$ distinct values $\alpha_1, \ldots, \alpha_k \in \mathbb{F}_r$. Then we claim that p divides $|\bar{x}^G|$ if $p|N_k$, where $N_1 := \binom{n}{j}$ and $N_2 := \frac{n!}{(j!)^2(n-2j)!}$. Indeed, assume that $g(x) = y$ with $y := x + a\sum_i e_i$ for some $g \in G$ and $a \in \mathbb{F}_r$. Then $\alpha_1, \ldots, \alpha_k$ each occurs as a coordinate of x and $g(x)$ with largest possible multiplicity j, and similarly, $\alpha_1 + a, \ldots, \alpha_k + a$ each occurs as a coordinate of y with largest possible multiplicity j. Now if $k = 1$, then it forces $\alpha_1 + a = \alpha_1$, $a = 0$, $y = x$, and g preserves J_{α_1}. Thus $\mathrm{Stab}_G(\bar{x}) \leq \mathrm{Stab}_G(J_{\alpha_1})$. Since $[H : G] = 2 < p$ and $[H : \mathrm{Stab}_H(J_{\alpha_1})] = N_1$, the claim follows. Assume $k = 2$. Then $\{\alpha_1 + a, \alpha_2 + a\} = \{\alpha_1, \alpha_2\}$, and so either $a = 0$ or $r = 2$ and $\alpha_2 = \alpha_1 + a$. In the former case g stabilizes each of J_{α_1} and J_{α_2}, whereas in the latter case g interchanges the subsets J_{α_1} and J_{α_2}. Again, since $[H : G] = 2 < p$ and $[H : \mathrm{Stab}_H(J_{\alpha_1}, J_{\alpha_2})] = N_2$, the claim follows.

(ii) Consider the case $r \geq 3$, and fix some $a \in \mathbb{F}_r^\times$. First assume that $n \not\equiv -1 (\mathrm{mod}\, p)$. Since $r \geq 3$, there is some $b \in \mathbb{F}_r \setminus \{0, (2-p)a\}$. Choose $c = 1/(b + (p-2)a)$, and set

$$v = c(\underbrace{a, a, \ldots, a}_{p-2}, b, \underbrace{0, 0, \ldots, 0}_{n-p+1}).$$

Then $\bar{v} \in \mathcal{C}$. Furthermore, the parameters j and k defined in (i) for $\bar{v}$ are as follows:

$$(j,k)=\begin{cases}(n-p+1,1) & \text{if } n\geq 2p, \text{ or } n=2p-2 \text{ and } b\neq a,\\ (p-1,1) & \text{if } n\leq 2p-3 \text{ and } a=b,\\ (p-2,1) & \text{if } n\leq 2p-4 \text{ and } b\neq a,\\ (p-1,2) & \text{if } n=2p-2 \text{ and } a=b,\\ (p-2,2) & \text{if } n=2p-3 \text{ and } a\neq b.\end{cases}$$

In all cases, p divides N_k, and so we are done.

Assume that $n=dp-1$ with $d\geq 3$. Suppose in addition that there is some $b\in\mathbb{F}_r\setminus\{0,a,(p-n)a\}$. Choose $c=1/(b+(n-p)a)$, and set

$$v=c(\underbrace{a,a,\dots,a}_{n-p},b,\underbrace{0,0,\dots,0}_{p-1}).$$

Then $\bar{v}\in\mathcal{C}$, and $(j,k)=(n-p,1)$. Arguing as in (i), we see that any $g\in\mathrm{Stab}_G(\bar{v})$ must fix $\{1,2,\dots,n-p\}$ and $n-p+1$. Hence $2|\bar{v}^G|$ is divisible by $p\cdot\binom{n}{p}$, a multiple of p. Now suppose that such a b does not exist. Then $r=3$ and $p\equiv 2(\mathrm{mod}\,3)$; in particular, $p\geq 5$. In this case we choose

$$v=(\underbrace{1,1,\dots,1}_{n-p-1},2,2,\underbrace{0,0,\dots,0}_{p-1}).$$

Then $\bar{v}\in\mathcal{C}$ and $(j,k)=(n-p-1,1)$. Arguing as in (i) we see that any $g\in\mathrm{Stab}_G(\bar{v})$ must fix $\{1,2,\dots,n-p-1\}$ and $\{n-p,n-p+1\}$. Hence $4|\bar{v}^G|$ is divisible by $p(n-p)\cdot\binom{n}{p}$, a multiple of p.

Assume now that $n=2p-1$. Suppose there is some $b\in\mathbb{F}_r\setminus\{0,a,(1-p)a\}$. Choose $c=1/(b+(p-1)a)$, and set

$$v=c(\underbrace{a,a,\dots,a}_{p-1},b,\underbrace{0,0,\dots,0}_{p-1}).$$

Then $\bar{v}\in\mathcal{C}$, and $(j,k)=(p-1,2)$. Since $p|N_k$, we are done. Now suppose that such a b does not exist. Then again $r=3$ and $p\equiv 2(\mathrm{mod}\,3)$; in particular, $p\geq 5$. In this case we choose

$$v=(\underbrace{1,1,\dots,1}_{p-2},2,2,\underbrace{0,0,\dots,0}_{p-1}).$$

Then $\bar{v}\in\mathcal{C}$ and $(j,k)=(p-1,1)$. Arguing as in (i), we see that any $g\in\mathrm{Stab}_G(\bar{v})$ must fix $\{1,2,\dots,p-1\}$ and $\{p,p+1\}$. Hence $4|\bar{v}^G|$ is divisible by $p(p-1)\cdot\binom{n}{p}$, again a multiple of p.

(iii) Now we assume that $r=2$ and so $2|n$. First we consider the case $n=dp+s$ with $d\geq 2$ and $0\leq s\leq p-2$. Then choosing

$$v=(\underbrace{1,1,\dots,1}_{2p-1},\underbrace{0,0,\dots,0}_{(d-2)p+s+1}),$$

we have that $\bar{v} \in \mathcal{C}$ and $k \leq 2$. Since $((d-2)p+s+1)N_k = 2p \cdot \binom{n}{2p}$ and $s \leq p-2$, we are done.

It remains to consider the case $p+1 \leq n \leq 2p-4$; in particular, $p \geq 5$. Choosing

$$v = (\underbrace{1, 1, \dots, 1}_{p-2}, \underbrace{0, 0, \dots, 0}_{n-p+2}),$$

we have that $\bar{v} \in \mathcal{C}$, $k \leq 2$, and $N_k = \binom{n}{p-2}$ is divisible by p. □

PROPOSITION 6.12. *Let G be a minimal counterexample to Theorem* 6.2 (*as in Proposition* 6.9) *with $p > 3$. Then $R = \mathbf{Z}(G)$.*

Proof. (i) Assume the contrary: $R > Z := \mathbf{Z}(G)$. Hence we can find an abelian chief factor $K/Z \neq 1$ of G in R/Z and apply Lemma 6.7 and Proposition 6.9 to K/Z. In particular, G/Z is perfect, $K = LZ$ is abelian, M acts trivially on K, and the $\mathbb{F}_r(G/M)$-module L is indecomposable with two composition factors $L \cap Z \cong \mathbb{F}_r = I$ and $W := L/(L \cap Z)$. Recall that $G/M = \mathsf{A}_m$ is simple, where $m = 2p-1$ or $m = ap-2$ with $a \geq 3$; in particular, $m \geq 9$. Let $\mathbb{E} := \mathrm{End}_{G/M}(W)$. Then W is an absolutely irreducible $\mathbb{E}G$-module. If in addition $\dim_{\mathbb{E}} W = 1$, then W is trivial and the action of the perfect group G/M on L maps it into an abelian (unipotent) subgroup of $\mathrm{GL}(L)$; i.e., G centralizes L and $L \leq Z$, a contradiction. Thus $2 \leq \dim_{\mathbb{E}} W \leq 2m+2$ by condition (a) of Theorem 6.2.

(ii) Consider the case $W \cong \mathcal{D} = \mathcal{N}^0/(\mathcal{N}^0 \cap I)$, the heart of the natural permutation module of A_m. Recall that $\mathrm{Ext}^1_{\mathsf{A}_m}(I, \mathcal{D}) \cong H^1(\mathsf{A}_m, I^* \otimes D) = H^1(\mathsf{A}_m, \mathcal{D})$. Since L is indecomposable, by Lemma 6.10 we have $r|m$ and can identify L with $\mathcal{N}^0$. In this case, we can also identify $\mathrm{Irr}(L)$ with $\mathcal{N}/I$ as a G-module. By Proposition 6.9(ii), λ is faithful, hence $\alpha := \lambda_{L\cap Z} \neq 1_{L\cap Z}$. Thus $\mathrm{Irr}(L|\alpha)$ is just a nonzero $\mathcal{D}$-coset in $\mathcal{N}/I$. By Lemma 6.11, there is some $\mu \in \mathrm{Irr}(L|\alpha)$ such that the G-orbit of μ has length divisible by p. Since $K = LZ$, the character μ has a unique extension ϑ to K with $\vartheta_Z = \lambda$. Consequently, the G-orbit of $\vartheta \in \mathrm{Irr}(K|\lambda)$ has length divisible by p, contradicting Proposition 6.9(i).

(iii) We have shown that $W \not\cong \mathcal{D}$. Suppose in addition that $m \geq 17$. Then $2 \leq \dim_{\mathbb{E}} W \leq 2m+2 < m(m-5)/2$. By [GT05, Lemma 6.1], this implies that $W \cong \mathcal{D}$, a contradiction. The bounds on $\dim_{\mathbb{E}} W$ also imply $W \cong \mathcal{D}$ when $m = 13$ by [JLPW95]. Since $m \in \{2p-1, ap-2 \mid a \geq 3\}$ and $p \geq 5$, it follows that $m = 9$ and $p = 5$. If $r = 3$, then the bounds on $\dim_{\mathbb{E}} W$ again imply $W \cong \mathcal{D}$ by [JLPW95]. Hence $r \neq 3$. Now $G/M = \mathsf{A}_9$ acts on the set $\mathrm{Irr}(K|\lambda)$, of size $|K/Z|$ by Lemma 6.7. Since $|K/Z|$ is coprime to 3, there must be some $\varrho \in \mathrm{Irr}(K|\lambda)$ that is fixed by a Sylow 3-subgroup P of G/M. According to [CCN+85], $J := \mathrm{Stab}_{G/M}(\varrho)$ either equals to $G/M = \mathsf{A}_9$ or has index divisible by $p = 5$ in A_9. Thus the G-orbit of ϱ has length 1 or divisible by p, contradicting Proposition 6.9(i). $\square$

PROPOSITION 6.13. *Theorem* 6.2 *holds for* $p > 3$.

Proof. Let G be a minimal counterexample to Theorem 6.2 as in Proposition 6.9 for $p > 3$. By Proposition 6.12, $R = \mathbf{Z}(G) =: Z$, whence $M > R$ by Proposition 6.9(iii). Hence, by condition (ii) of Theorem 6.2, there is a chief factor $1 \neq K/Z = S_1 \times \cdots \times S_d$ of G in M/Z, with $S_i \cong \mathsf{A}_5$ and G permuting $S_1, \ldots, S_d$ transitively. For each i, let K_i be the full inverse image of S_i in K and let $L_i := K_i^{(\infty)}$. Then G permutes these d quasisimple subgroups $L_1, \ldots, L_d$ transitively; in particular, $L_1 \cong \ldots \cong L_d$. Furthermore, since $[L_i, L_j] \leq Z$ for $i \neq j$, by the Three Subgroups Lemma we have $[L_i, L_j] = 1$ for $i \neq j$. Now $L = L_1 * \cdots * L_d \lhd G$ and $K = L * Z$.

Note that $L_i \cong \mathsf{A}_5$ or $\mathrm{SL}_2(5)$. In the former case, $L = L_1 \times \cdots \times L_d$ and $L \cap Z = 1$, whence $1_L \times \lambda \in \mathrm{Irr}(G|\lambda)$ is G-invariant, contradicting Proposition 6.9(i). Suppose we are in the latter case. Then $1 \neq \mathbf{Z}(L) \leq Z$ and $\mathbf{Z}(L)$ is an elementary abelian 2-group (as L is a quotient of $\mathrm{SL}_2(5)^d$). On the other hand, Z is cyclic by Proposition 6.9(ii). Hence $\mathbf{Z}(L) = \mathbf{Z}(L_i) \cong C_2$ for all i. Now $\lambda_{\mathbf{Z}(L_i)}$ is faithful, and there is a unique $\gamma_i \in \mathrm{Irr}(L_i|\lambda_{\mathbf{Z}(L_i)})$ of degree 4. It follows that there is a unique $\gamma \in \mathrm{Irr}(L|\lambda_{\mathbf{Z}(L)})$ of degree 4^d. Since $K = LZ$, there is a unique $\varrho \in \mathrm{Irr}(K|\lambda)$ of degree 4^d. Consequently, ϱ is G-invariant, contradicting Proposition 6.9(i). $\square$

6.4. *The case* $p = 3$.

PROPOSITION 6.14. *Let G be a minimal counterexample to Theorem* 6.2 (*as in Proposition* 6.9) *with* $p = 3$. *Then G/M cannot be isomorphic to* A_5 *or* A_7.

Proof. (i) Assume the contrary: $G/M \cong \mathsf{A}_m$ with $m \in \{5, 7\}$. By Proposition 6.9(iii), $R = M \not\leq Z := \mathbf{Z}(G)$. Hence we can find an abelian chief factor $K/Z \neq 1$ of G in R/Z and apply Lemma 6.7 and Proposition 6.9 to K/Z. In particular, G/Z is perfect, $K = LZ$ is abelian, M acts trivially on K, and the $\mathbb{F}_r(G/M)$-module L is indecomposable with two composition factors $L \cap Z \cong \mathbb{F}_r = I$ and $W := L/(L \cap Z)$. Furthermore, $G/M = \mathsf{A}_m$ acts on the set $\mathrm{Irr}(K|\lambda)$ of size $|K/Z| = r^d$ for some prime r and some integer d. As in the proof of Proposition 6.12, we can see that $d > 1$, W is nontrivial, and that $W \not\cong \mathcal{D}$ if $(r, m) \neq (3, 5)$.

(ii) Consider the case $G/M = \mathsf{A}_5$. If $r = 5$, then since W is nontrivial and $W \not\cong \mathcal{D}$, we have $W \cong \mathbb{F}_5^5$, an irreducible module of 5-defect 0. But in this case L must be decomposable, a contradiction. Hence $r \neq 5$. Since $|\mathrm{Irr}(K|\lambda)|$ is coprime to 5, there must be some $\varrho \in \mathrm{Irr}(K|\lambda)$ that is fixed by a Sylow

5-subgroup P of G/M. Inspecting the list of maximal subgroups of A_5 given in [CCN+85], we see that $J := \mathrm{Stab}_{G/M}(\varrho)$ either equals to $G/M = \mathsf{A}_5$ or has index divisible by $p = 3$ in A_5. Thus the G-orbit of ϱ has length 1 or divisible by p, contradicting Proposition 6.9(i).

(iii) Now let $G/M = \mathsf{A}_7$. First suppose that $r \neq 7$. Since $|\,\mathrm{Irr}(K|\lambda)|$ is coprime to 7, there must be some $\varrho \in \mathrm{Irr}(K|\lambda)$ that is fixed by a Sylow 7-subgroup P of G/M. According to [CCN+85], $J := \mathrm{Stab}_{G/M}(\varrho)$ either equals to $G/M = \mathsf{A}_7$ or has index divisible by $p = 3$ in A_7. Thus the G-orbit of ϱ has length 1 or divisible by p, contradicting Proposition 6.9(i).

We have shown that $r = 7$. Note that all irreducible $\bar{\mathbb{F}}_7\mathsf{A}_7$-representations can be realized over $\mathbb{F}_7$; cf. [JLPW95]. Hence the indecomposability of L implies that W belongs to the principal 7-block of A_7; also, $W \not\cong \mathcal{D}$ as noted in (i). It follows that $W = \mathbb{F}_7^{10}$. If φ denotes the Brauer character of W, then $\varphi = \bar{\varphi}$ and $\varphi(x) = 1$ for all elements $x \in \mathsf{A}_7$ of order 3; see [JLPW95]. It follows that $|\mathbf{C}_W(P)| = 7^2$ for any $P \in \mathrm{Syl}_3(\mathsf{A}_7)$. Since we have 70 Sylow 3-subgroups in A_7, we arrive at a contradiction with Lemma 6.8. $\square$

To rule out the case $G/M = \mathrm{SL}_3(2)$ and $p = 3$, we will need the following statement, where we again denote by $\mathrm{sol}(X)$ the solvable radical of any finite group X.

Lemma 6.15. *Let X be a finite perfect group with $X/Y \cong \mathrm{SL}_3(2)$ for $Y := \mathrm{sol}(X)$.*

(i) *If X acts transitively on a set Ω, then $|\Omega| = 1$ or $|\Omega| \geq 7$.*
(ii) *Suppose X acts irreducibly on $W = \mathbb{F}_2^d$ with $d \leq 7$. Then $\mathrm{sol}(X)$ acts trivially on W.*

Proof. (i) is straightforward.

(ii) First we observe that X is primitive on W. (Indeed, suppose X acts transitively on the summands of some decomposition $W = W_1 \oplus \cdots \oplus W_k$ with $k > 1$. By (i), $k = 7$, and so $W_i = \{0, e_i\}$ for some $e_i \in W$. But then X permutes the set $\{e_1, \ldots, e_7\}$. In particular, it fixes $\sum_{i=1}^{7} e_i$, contrary to the irreducibility of W.) Thus for any $N \lhd X$, the N-module W_N is isotypic.

Let K be the kernel of the action of X on W. If $K \not\leq Y$, then $X = KY$, $X/K \cong Y/(Y \cap K)$ is both perfect and solvable, whence $K = X$, and so we are done. So we may assume that $K \leq Y$. Modding out by K, it suffices to show that $Y = \mathrm{sol}(X) = 1$ if X is an irreducible subgroup of $\mathrm{GL}(W)$.

Assume $Y \neq 1$. Then we can find a nontrivial minimal normal r-subgroup $P \lhd X$ inside Y for some prime r. Since X is irreducible on W, $r \neq 2$. As shown above, $W_P \cong eU$ is a direct sum of e copies of U, where U is an irreducible $\mathbb{F}_2P$-module. Since P is abelian, $\dim_{\mathbb{E}} U = 1$, where $\mathbb{E} := \mathrm{End}_P(U) \cong \mathbb{F}_{2^s}$ for some integer s. Thus P acts on U as a cyclic subgroup of $\mathrm{GL}_1(\mathbb{E})$. But $W_P \cong eU$, and P acts faithfully on W, so P is cyclic. Since X is perfect and $P \lhd X$, it follows that $P \leq \mathbf{Z}(X)$. Also, $es = d \leq 7$. As $P \neq 1$ and P is minimal, we can find a nontrivial generator $z \in P$ of prime order r that acts scalarly and nontrivially on the e-dimensional $\mathbb{E}$-space W; in particular, $s > 1$. We may also replace s by the smallest positive integer s such that $|z|$ divides $2^s - 1$. Then $X \leq \mathbf{C}_{\mathrm{GL}(W)}(z) = \mathrm{GL}_e(2^s)$. Since X is perfect, we obtain $X \leq \mathrm{SL}_2(4)$, $\mathrm{SL}_2(8)$, or $\mathrm{SL}_3(4)$. Recall that $\mathbf{O}_r(X) \geq P > 1$ and $X/Y \cong \mathrm{SL}_3(2)$. Inspecting the list of maximal subgroups of these quasisimple groups in [CCN⁺85], we get $X \leq 3 \cdot \mathrm{SL}_3(2)$. But X is perfect and $\mathrm{Mult}(\mathrm{SL}_3(2)) = C_2$, so $X = \mathrm{SL}_3(2)$ and $\mathbf{O}_r(X) = 1$, a contradiction. $\square$

Proposition 6.16. *Let G be a minimal counterexample to Theorem* 6.2 (*as in Proposition* 6.9) *with $p = 3$. Then G/M cannot be isomorphic to* $\mathrm{SL}_3(2)$.

Proof. Assume the contrary: $G/M \cong \mathrm{SL}_3(2)$. By Proposition 6.9(ii) and (iii), $R = M \not\leq Z := \mathbf{Z}(G)$ and G/Z is perfect; also, R/Z is a $3'$-group.

(i) Suppose that $\mathbf{O}_{2'}(R/Z) \neq 1$. Then we can find an abelian chief factor $K/Z \neq 1$ of G in R/Z, of order r^d for some odd prime r, and apply Lemma 6.7 and Proposition 6.9 to K/Z. In particular, $K = LZ$ is abelian, M acts trivially on K, and the $\mathbb{F}_r(G/M)$-module L is indecomposable with two composition factors $L \cap Z \cong \mathbb{F}_r = I$ and $W := L/(L \cap Z)$. As in the proof of Proposition 6.12, we can see that $d > 1$ and W is nontrivial.

Assume in addition that $r \neq 7$. Then $G/M = \mathrm{SL}_3(2)$ acts coprimely on the abelian group L and trivially on $L \cap Z$. Hence by [KS98, 8.4.2] we can write $L = \mathbf{C}_L(G) \times [G, L]$ with $\mathbf{C}_L(G) = L \cap Z$ by the irreducibility of $L/(L \cap Z)$. Thus $[G, L]$ is a G-invariant complement for $L \cap Z$ in L, contradicting the indecomposability of L.

We may now assume that $r = 7$. Note that all irreducible $\bar{\mathbb{F}}_7\,\mathrm{SL}_3(2)$-representations can be realized over $\mathbb{F}_7$; cf. [JLPW95]. Hence the indecomposability of L implies that W belongs to the principal 7-block of $\mathrm{SL}_3(2)$. It follows that $W = \mathbb{F}_7^d$ with $d = 3$, resp. 5. If φ denotes the Brauer character of W, then $\varphi = \bar{\varphi}$ and $\varphi(x) = 0$, resp. -1 for all elements $x \in \mathrm{SL}_3(2)$ of order 3; see [JLPW95]. It follows that $|\mathbf{C}_W(P)| = 7$ for any $P \in \mathrm{Syl}_3(\mathrm{SL}_3(2))$. Since we have 28 Sylow 3-subgroups in $\mathrm{SL}_3(2)$, we arrive at a contradiction with Lemma 6.8.

(ii) Here we consider the case R/Z is a $\{3, 7\}'$-group, and we let J/R be a (Frobenius) subgroup of order 21 of $G/R = \mathrm{SL}_3(2)$. Then J/R acts coprimely on R/Z. So by [Isa06, Th. (13.31)], there is some $\mu \in \mathrm{Irr}(R|\lambda)$ that is J-invariant. By Proposition 6.9(i), μ is not G-invariant. Since J is maximal in G, we conclude that $J = \mathrm{Stab}_G(\mu)$. Now all the Sylow subgroups of J/R are cyclic, so by Corollaries (11.22) and (11.31) of [Isa06], μ extends to a character $\vartheta \in \mathrm{Irr}(J|\mu)$. Since J/R is Frobenius of order 21, there is some $\alpha \in \mathrm{Irr}(J/R)$

of degree 3. By Gallagher's Theorem, $\vartheta\alpha \in \mathrm{Irr}(J|\mu)$ and $(\vartheta\alpha)(1) = 3\vartheta(1)$. Finally, by the Clifford correspondence, $\chi := (\vartheta\alpha)^G$ lies in $\mathrm{Irr}(G|\lambda)$ and it has degree divisible by 3, a contradiction.

(iii) We have shown that $\mathbf{O}_{2'}(R/Z) = 1$ and 7 divides $|R/Z|$. By the hypothesis (b) in Theorem 6.2, any chief factor of G in R/Z has rank ≤ 7. Also, since R/Z is solvable, we see that the Fitting subgroup $F(R/Z)$ is just $E/Z := \mathbf{O}_2(R/Z) \neq 1$. Consider any chief factor K/L of G in E/Z. Then K/L is an elementary abelian 2-group of rank at most 7, on which $X := G/K$ acts irreducibly. Furthermore, since $R \geq K \geq Z$, we see that X is perfect, $\mathrm{sol}(X) = R/K$, and $X/\mathrm{sol}(X) = G/R = \mathrm{SL}_3(2)$. Hence by Lemma 6.15, R acts trivially on K/L.

Thus if x is any $2'$-element in R/Z, then x acts trivially and coprimely on each chief factor K/L of G in E/Z. It follows that x centralizes E/Z. In this case, $x \in \mathbf{C}_{R/Z}(E/Z) \leq E/Z$ since $E/Z = F(R/Z)$, whence $x = 1$ as $E/Z = \mathbf{O}_2(R/Z)$. It follows that R/Z is a 2-group, which is a contradiction since 7 divides $|R/Z|$. $\square$

Combining Propositions 6.13, 6.14, and 6.16 together, we have completed the proof of Theorem 6.2. $\square$

6.5. *Proof of Theorem* 6.1. Let G be as in Theorem 6.1.

(i) Since G is perfect, $G = \mathbf{O}^p(G)$; also $G = \mathbf{O}^{p'}(G)$ by the hypotheses. Hence G/K (acting on V) satisfies the assumptions of Theorem 2.3, and the main hypothesis in Theorem 6.1 is that this action is described in Theorem 2.3(ii). Observe that $\mathbf{Z}(G) = Z$ since $\mathbf{C}_G(Q) = Z \times Q$ and G/K acts irreducibly and nontrivially on Q.

Let $H \leq \mathrm{GL}(V_1)$ be the subgroup induced by the action of $\mathrm{Stab}_G(V_1)$ on V_1. As mentioned in the proof of Lemma 6.3, $G/K \leq H \wr \mathsf{S}_n$, and the structure of H is determined by Lemma 6.4 applied to G/K. Also, the kernel B of the action of G on $\{V_1, \dots, V_n\}$ is contained in E, and $B/K \leq H^n$. Assume in addition that $(S, p) = (\mathsf{A}_5, 3)$. Then $B/K = E/K \leq H^n$ is a solvable $3'$-group by Lemma 6.4(i) and (ii). Setting $M = R = B$, we see that G/Q has the structure described in Theorem 6.2. Hence by Theorem 6.2 there exists $\chi \in \mathrm{Irr}(G|\lambda \times 1_Q)$ of degree divisible by p, and we are done in this case.

From now on we may assume that $(S, p) \neq (\mathsf{A}_5, 3)$; in particular, H and B/K are p'-groups.

(ii) Here we consider the case $S \cong \mathsf{A}_n$ with $n = 2p - 1$ and $p > 3$. Then $B = E$ by Lemma 6.4(i). We will again show that G/Q has the structure described in Theorem 6.2. It then follows by Theorem 6.2 that there is $\chi \in \mathrm{Irr}(G|\lambda \times 1_Q)$ of degree divisible by p, and so we are done in this case.

Suppose we are in case (iii)(a) of Lemma 6.4. Then $H \leq \Gamma\mathrm{L}_1(p^d)$ is a solvable p'-group, and so is $B/K \leq H^n$. Setting $M = R = B$, by Lemma 6.3 we see that G/Q has the structure described in Theorem 6.2.

Assume now that we are in case (iii)(b) of Lemma 6.4. Then we can apply Lemma 6.5 to G/K to get the subgroup C of G/K. Let $M := B$, and let R be the complete inverse image of C in G. By Lemma 6.5 we see that G/Q has the structure described in Theorem 6.2 as stated.

Lastly, suppose that we are in case (iii)(c) of Lemma 6.4. Then $p \in \{5, 7, 11, 23\}$ and H is a solvable p'-subgroup of $X := \mathrm{GL}_2(p)$; in fact, $H \leq (C_{p-1} * Q_8) \cdot \mathsf{S}_3$. It is well known that Sylow 2-subgroups of $\mathrm{PGL}_2(p)$ can never have elementary abelian sections of rank 3. It follows that no section of $H/(H \cap \mathbf{Z}(X)) < X/\mathbf{Z}(X) = \mathrm{PGL}_2(p)$ can be elementary abelian of 2-rank ≥ 3. Hence we can refine the series $1 \leq H \cap \mathbf{Z}(X) \leq H$ to a series $1 < H_1 < \cdots < H_a = H$ of normal subgroups in H, where all quotients H_{i+1}/H_i are elementary abelian of rank ≤ 2. Arguing as in the proof of Lemma 6.3, we see that the chief factors of G/K in B/K are elementary abelian of rank $\leq 2n$. Setting $M = R = B$, we see that G/Q has the structure described in Theorem 6.2, and so we are done.

(iii) Now we may assume that either $S = \mathsf{A}_n$, where $n = ap - 1 \geq 8$ with $p \geq a \geq 3$, or $(G/B, n, p) = (2^3 : \mathrm{SL}_3(2), 8, 3)$. By way of contradiction, assume that $p \nmid \chi(1)$ for all $\chi \in \mathrm{Irr}(G|\lambda)$. Fix a nonzero $v \in V_1$, and identify it with $\alpha \in \mathrm{Irr}(Q)$. Also set $I := \mathrm{Stab}_G(v) = \mathrm{Stab}_G(\alpha) = \mathrm{Stab}_G(\lambda \times \alpha)$, and let J be the kernel of the action of I on V_1. By our assumption and by the Clifford correspondence, every $\varrho \in \mathrm{Irr}(I|\lambda \times \alpha)$ has degree coprime to p. Since $I \rhd J \geq K$, every $\vartheta \in \mathrm{Irr}(J|\lambda \times \alpha)$ has degree coprime to p.

Certainly J/K acts faithfully on $W := V_2 \oplus \cdots \oplus V_n$. Now we observe that J acts transitively on $\{V_2, \ldots, V_n\}$, inducing A_{n-1} or $\mathrm{SL}_3(2)$ (and in the latter case $(n, p) = (8, 3)$), with kernel $B \cap J$. Indeed, J equals the kernel of the action of $G_1 := \mathrm{Stab}_G(V_1)$ on V_1 and, by Lemma 6.4, $G_1/J \cong H$ has at most one non-abelian composition factor, which, if exists, is isomorphic to A_5. On the other hand, G_1/B is A_{n-1} or $\mathrm{SL}_3(2)$ by Theorem 2.3(ii). It follows that $J \not\leq B$, whence $BJ = G_1$ and $J/(B \cap J) \cong G_1/B$.

We have shown that $J/(B \cap J)$ is simple and isomorphic to A_{n-1} or $\mathrm{SL}_3(2)$. Recall that B/K is a p'-group; in particular, $(B \cap J)/K = \mathbf{O}_{p'}(J/K)$. Also, $J/K \leq H \wr \mathsf{S}_{n-1}$. Hence we can apply Lemma 6.4 to the action of J/K on W. If in addition $(S, p) = (\mathrm{SL}_3(2), 3)$ and 7 divides $|H|$, then only the case (iii)(a) of Lemma 6.4 is applicable to H; i.e., $H \leq \Gamma\mathrm{L}_1(p^d)$. Now arguing as in part (ii) above, we see that $J/\mathrm{Ker}(\alpha)$ has the structure described in Theorem 6.2. But then by Theorem 6.2 applied to $J/\mathrm{Ker}(\alpha)$, some $\vartheta \in \mathrm{Irr}(J|\lambda \times \alpha)$ has degree divisible by p, the final contradiction. $\square$

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(Received: February 2, 2012)
(Revised: October 12, 2012)

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