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Stationary measures and invariant subsets of homogeneous spaces (III)

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Abstract

Let G be a real Lie group, Λ be a lattice in G and Γ be a compactly generated closed subgroup of G. If the Zariski closure of the group $\mathrm{Ad}(\Gamma)$ is semisimple with no compact factor, we prove that every Γ-orbit closure in G/Λ is a finite volume homogeneous space. We also establish related equidistribution properties.

Contents

1. **Introduction**

1.1. *Orbit closures, the real case.* Let G be a real Lie group with Lie algebra $\mathfrak{g}$. We let $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ denote the adjoint representation of G. In this article, by using the results of [4], we shall prove the following

THEOREM 1.1. *Let G be a real Lie group, Λ be a lattice in G and Γ be a compactly generated closed sub-semigroup of G. We assume that the Zariski closure of the semigroup $\mathrm{Ad}(\Gamma) \subset \mathrm{GL}(\mathfrak{g})$ is semisimple with no compact factor. Then, for every x in G/Λ, there exists a closed subgroup H of G with $\Gamma \subset H$ such that $\overline{\Gamma x} = Hx$ and Hx carries an H-invariant probability measure ν_x.*

Througout this article, by a semisimple algebraic group, we mean a *Zariski connected* semisimple algebraic group.

This result on orbit closures answers a question by Shah [25] and Margulis [16]. In case $\mathrm{Ad}\,\Gamma$ itself is a semisimple subgroup of $\mathrm{GL}(\mathfrak{g})$ with no compact factor, it follows from Ratner's Theorem [23]. Under the stronger assumption that G is simple and Γ is Zariski dense in G, Theorem 1.1 is the main result of [1].

Theorem 1.1 is already new in the following "concrete" cases:

– when $G = \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ and Γ is Zariski dense in G,

– when $G = \mathrm{SL}(3,\mathbb{R})$ and Γ is Zariski dense in the subgroup $\mathrm{SO}(2,1)$.

Let G, Λ and Γ be as above, and set $X = G/\Lambda$. For any x in X, we let ν_x be, as in Theorem 1.1, the unique probability measure on $\overline{\Gamma x}$ that is invariant under the stabilizer of this set in G. We shall say that a sequence of Borel probability measures (ν_n) on X converges toward a Borel probability measure ν if, for any continuous compactly supported function φ on X, $\int_X \varphi \,\mathrm{d}\nu_n$ converges toward $\int_X \varphi \,\mathrm{d}\nu$. Theorem 1.1 will follow from the following equidistribution result.

THEOREM 1.2. *Let G, Λ and Γ be as above, and let μ be a compactly supported Borel probability measure on Γ whose support spans a dense sub-semigroup of Γ. Then, for every x in G/Λ, one has*

$$\frac{1}{n}\sum_{k=0}^{n-1} \mu^{*k} * \delta_x \xrightarrow[n\to\infty]{} \nu_x.$$

Theorems 1.1 and 1.2 have been announced in [2]. Equidistribution in law as in Theorem 1.2 suffices to prove Theorem 1.1. However, we shall prove the following almost sure equidistribution property, which implies it and is also of independent interest.

THEOREM 1.3. *Let G, Λ, Γ and μ be as above. Let $g_1, \dots, g_n, \dots$ be a sequence of independent identically distributed random elements of Γ with law μ. Then, for every x in G/Λ, almost surely,*

$$\frac{1}{n}\sum_{k=0}^{n-1} \delta_{g_k \cdots g_1 x} \xrightarrow[n\to\infty]{} \nu_x.$$

In Theorem 1.3, "almost surely" means for $\mu^{\otimes \mathbb{N}^*}$-almost every choice of the sequence $g_1, \dots, g_n, \dots$.

This result may be seen as a random analogue of the equidistribution properties of unipotent flows on homogeneous spaces, due to Ratner [23] and Dani-Margulis [9].

1.2. *Orbit closures, the* S*-adic case.* Let p be a prime number. As in [24], a p-adic Lie group G is said to be weakly regular if any two of its one-parameter subgroups $\varphi_1, \varphi_2 : \mathbb{Q}_p \to G$ are equal as soon as their derivatives at e are equal. Any real Lie group is said to be weakly regular.

Fix a finite set S whose elements are prime numbers and ∞. In this paper, as in [4], by a weakly regular S-adic Lie group, we shall mean a topological group that is isomorphic to a closed subgroup of a product of weakly regular p-adic Lie groups, $p \in S$.

Let G be a weakly regular S-adic Lie group with Lie algebra $\mathfrak{g} = \bigoplus_{p\in S} \mathfrak{g}_p$ and Γ be a sub-semigroup of G. We let $\overline{\mathrm{Ad}\,\Gamma}^{\mathrm{Z}}$ denote the Zariski closure of the image of Γ under the adjoint representation of G, that is the product $\prod_{p\in S} \overline{\mathrm{Ad}_{\mathfrak{g}_p} \Gamma}^{\mathrm{Z}}$ of the Zariski closures of the images of Γ in $\mathrm{GL}(\mathfrak{g}_p)$, $p \in S$. We also let $\overline{\mathrm{Ad}\,\Gamma}^{\mathrm{Z,nc}}$ denote the smallest normal Zariski closed subgroup of $\overline{\mathrm{Ad}\,\Gamma}^{\mathrm{Z}}$ such that the image of Γ in $\overline{\mathrm{Ad}\,\Gamma}^{\mathrm{Z}}/\overline{\mathrm{Ad}\,\Gamma}^{\mathrm{Z,nc}}$ is bounded.

We get the following S-adic extension of Theorems 1.1, 1.2 and 1.3.

THEOREM 1.4. *Let G be a weakly regular S-adic Lie group, Λ be a lattice in G and Γ be a closed compactly generated sub-semigroup of G such that $\overline{\mathrm{Ad}\,\Gamma}^{\mathrm{Z}}$ is semisimple and equal to $\overline{\mathrm{Ad}\,\Gamma}^{\mathrm{Z,nc}}$.*

(a) *For every x in G/Λ, there exists a closed subgroup $H \supset \Gamma$ of G such that $\overline{\Gamma x} = Hx$ and Hx carries an H-invariant probability measure ν_x.*
(b) *If μ is a compactly supported Borel probability measure on Γ whose support spans a dense sub-semigroup of Γ, then*

$$\frac{1}{n}\sum_{k=0}^{n-1} \mu^{*k} * \delta_x \xrightarrow[n\to\infty]{} \nu_x.$$

(c) *More precisely, if* $g_1, \dots, g_n, \dots$ *is a sequence of independent identically distributed random elements of* Γ *with law* μ, *then, almost surely,*

$$\frac{1}{n}\sum_{k=0}^{n-1} \delta_{g_k \cdots g_1 x} \xrightarrow[n\to\infty]{} \nu_x.$$

Note that Theorem 1.4 is already new in the following "concrete" cases:

– when $G = \mathrm{SL}(2, \mathbb{Q}_p)$ and Γ is Zariski dense and unbounded in G,

– when $G = \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{Q}_p)$ and the projection of Γ on each factor is Zariski dense and unbounded.

1.3. *Equidistribution of invariant subsets.* Our methods also allow us to get some properties of the set of invariant homogeneous subsets.

Let G be a locally compact topological group, Λ be a discrete subgroup of G and $X = G/\Lambda$. We shall say that a closed subset Y of X is a *finite volume homogeneous subspace* if the stabilizer G_Y of Y in G acts transitively on Y and preserves a Borel probability measure ν_Y on Y. If Γ is a sub-semigroup of G_Y, we shall say that Y is Γ-*ergodic* if Γ acts ergodically on (Y, ν_Y).

Let Γ be a sub-semigroup of G. We set

$$\mathcal{S}_X(\Gamma) := \left\{ \begin{matrix} \Gamma\text{-invariant and } \Gamma\text{-ergodic finite volume} \\ \text{homogeneous subspaces } Y \text{ of } X \end{matrix} \right\}.$$

When G, Λ, X and Γ are as in Theorem 1.4, this set plays a key role in the proofs since, according to the main result of [4], every Γ-invariant Γ-ergodic probability measure on X is equal to ν_Y for some Y in $\mathcal{S}_X(\Gamma)$.

We may identify $\mathcal{S}_X(\Gamma)$ with a set of Borel probability measures on X through the map $Y \mapsto \nu_Y$. In particular, we endow $\mathcal{S}_X(\Gamma)$ with the topology of weak convergence, so that a sequence (Y_n) in $\mathcal{S}_X(\Gamma)$ converges toward $Y_\infty \in \mathcal{S}_X(\Gamma)$ if and only if ν_{Y_n} converges toward ν_{Y_∞}.

For every compact subset $K \subset X$, we set

$$\mathcal{S}_K(\Gamma) := \{Y \in \mathcal{S}_X(\Gamma) \mid Y \cap K \neq \emptyset\}.$$

As we will see in the corollaries, the following Theorem 1.5 is very efficient for computing the limit of a sequence in $\mathcal{S}_X(\Gamma)$.

THEOREM 1.5. *Let* G, Λ *and* Γ *be as in Theorem* 1.4, *and let* L *be the centralizer of* Γ *in* G.

(a) *For every compact subset* K *of* X, *the set* $\mathcal{S}_K(\Gamma)$ *is compact.*

(b) *If* $(Y_n) \subset \mathcal{S}_X(\Gamma)$ *converges to* $Y_\infty \in \mathcal{S}_X(\Gamma)$, *then there exists a sequence* $(\ell_n) \subset L$ *converging to* e *such that, for* n *large,* $Y_n \subset \ell_n Y_\infty$.

In particular, when Λ is cocompact, the set $\mathcal{S}_X(\Gamma)$ itself is compact.

We denote by $\overline{X} := X \cup \{\infty\}$ the one point compactification of X and by δ_∞ the Dirac mass at ∞. The set $\mathcal{S}_X(\Gamma) \cup \{\delta_\infty\}$ can be seen as a set of Borel probability measures on $\overline{X}$.

Corollary 1.6. *Let* G, Λ *and* Γ *be as in Theorem* 1.4. *Then the set* $\mathcal{S}_X(\Gamma) \cup \{\delta_\infty\}$ *is compact.*

Corollary 1.6 is an analogue of the main theorem of Mozes and Shah in [19] (see also [11]) which asserts, in case G is a real Lie group, if $\mathcal{E}$ is the space of finite volume homogeneous subsets of X that are invariant and ergodic under some Ad-unipotent one-parameter subgroup of G, then the set $\mathcal{E} \cup \{\delta_\infty\}$ is compact.

When Γ has discrete centralizer, the statement of Theorem 1.5 becomes simpler. A subset F of $X = G/\Lambda$ is said to be Γ-invariant if $gF \subset F$ for all g in Γ.

Theorem 1.7. *Let* G, Λ *and* Γ *be as in Theorem* 1.4. *Assume the centralizer* L *of* Γ *in* G *is discrete.*

(a) *The set* $\mathcal{S}_X(\Gamma)$ *is compact.*

(b) *If* $(Y_n) \subset \mathcal{S}_X(\Gamma)$ *converges to* $Y_\infty \in \mathcal{S}_X(\Gamma)$, *then, for* n *large, one has* $Y_n \subset Y_\infty$.

(c) *Every closed* Γ-*invariant subset of* X *is a finite union of elements of* $\mathcal{S}_X(\Gamma)$.

In particular, if (Y_n) is a sequence in $\mathcal{S}_X(\Gamma)$ such that, for any $Y \in \mathcal{S}_X(\Gamma)$ with $Y \neq X$, for all but finitely many n, one has $Y_n \not\subset Y$, then $\nu_{Y_n} \xrightarrow[n\to\infty]{} \nu_X$; that is, the orbits Y_n become equidistributed in X when n is large.

Let us state a particular case of this result. We will say that Γ is Ad-Zariski dense in G if $\overline{\mathrm{Ad}\,\Gamma}^{\mathrm{Z}} = \overline{\mathrm{Ad}\,G}^{\mathrm{Z}}$.

Corollary 1.8. *Let* G *be a connected semisimple real Lie group with no compact factor*, Λ *be an irreducible lattice in* G *and* Γ *be a* Ad-*Zariski dense subgroup of* G.

Every infinite Γ-*invariant subset of* X *is dense in* X; *any sequence* (Y_n) *of distinct finite* Γ-*orbits in* X *satisfies* $\nu_{Y_n} \xrightarrow[n\to\infty]{} \nu_X$.

Under the stronger assumption that G is simple, Corollary 1.8 is the main result of [1]. It also extends previous results by Clozel, Oh and Ullmo in [8] about equidistribution of Hecke orbits (see also [12]).

1.4. *Actions on tori and nilmanifolds.* We now specialize our results to automorphisms of tori and other nilmanifolds.

Let N be a connected simply connected nilpotent real Lie group, Λ be a lattice in N and X be the compact nilmanifold $X = N/\Lambda$. As in [4, §1.2], we say a closed subset of X is an *affine submanifold* if it is a translate of the image in X of some closed connected subgroup of N. By Mal'cev's rigidity theorem (see [22, II.2.11]), we may consider the group $\mathrm{Aut}(\Lambda)$ of automorphisms of Λ as a subgroup of the group $\mathrm{Aut}(N)$ of automorphisms of N. In particular, we may see X as a quotient of the group $\mathrm{Aut}(\Lambda) \ltimes N$ by its lattice $\mathrm{Aut}(\Lambda) \ltimes \Lambda$. Then, the action of $\mathrm{Aut}(\Lambda)$ on X reads as its natural action by automorphisms. Any homogeneous subspace of X, viewed as a homogeneous space of $\mathrm{Aut}(\Lambda) \ltimes N$, is a finite union of affine submanifolds.

Example 1.9. If $N = \mathbb{R}^d$ and $\Lambda = \mathbb{Z}^d$, we retrieve the standard action of $\mathrm{GL}(d, \mathbb{Z})$ on the torus $\mathbb{T}^d$. Any homogeneous subspace is a finite union of (parallel) subtori.

Theorems 1.4 and 1.5 and their corollaries now give the following partial answers to [16, Prob. 3].

COROLLARY 1.10. *Let $X = N/\Lambda$ be a compact nilmanifold and $\Gamma \subset \mathrm{Aut}(\Lambda)$ be a finitely generated sub-semigroup whose Zariski closure in $\mathrm{Aut}(N)$ is a Zariski connected semisimple subgroup with no compact factor. Let $L \subset N$ be the subgroup of Γ-invariant elements in N.*

(a) *Every Γ-orbit closure is a finite homogeneous union of affine submanifolds.*
(b) *Let μ be a finitely supported Borel probability measure on Γ whose support spans Γ and $g_1, \dots, g_n, \dots$ be a sequence of random independent identically distributed elements of $\mathrm{Aut}(\Lambda)$ with law μ. Then, almost surely, as n goes to ∞, $\frac{1}{n}\sum_{k=0}^{n-1} \delta_{g_k \cdots g_1 x}$ converges to the homogeneous probability measure of $\overline{\Gamma x}$.*
(c) *The set $\mathcal{S}_X(\Gamma)$ is compact. If $(Y_n) \subset \mathcal{S}_X(\Gamma)$ converges to $Y_\infty \in \mathcal{S}_X(\Gamma)$, then there exists a sequence $(\ell_n) \subset L$ converging to e such that, for n large, $Y_n \subset \ell_n Y_\infty$.*

COROLLARY 1.11. *Assume, moreover, that the centralizer L of Γ in N is trivial. Then every closed Γ-invariant subset F in X is a finite union of affine submanifolds.*

COROLLARY 1.12. *Assume X is a torus $\mathbb{T}^d$ and Γ acts strongly irreducibly on $\mathbb{Q}^d$. Then every infinite Γ-invariant subset of X is dense in X. Any sequence of distinct finite Γ-orbits in X equidistributes toward the Haar probability of $\mathbb{T}^d$.*

The statement about invariant closed subsets in Corollary 1.12 is due to Guivarc'h and Starkov [13] and Muchnik [20]. In case Γ is proximal and acts strongly irreducibly on $\mathbb{R}^d$, the one about equidistribution of finite orbits follows from the results of Bourgain, Furman, Lindenstrauss and Mozes [6].

To conclude, we point out that even the following special case of Corollary 1.10(b) seems to be new.

Corollary 1.13. *Let $g_1, \dots, g_n, \dots$ be a sequence of independent identically distributed random elements of* $\mathrm{SL}(2, \mathbb{Z})$ *whose law μ has finite support and generates a nonsolvable group. Then, starting from any irrational point x in the 2-torus $\mathbb{T}^2$, almost surely, the trajectory $g_n \cdots g_1 x$ equidistributes toward the Haar probability on $\mathbb{T}^2$.*

1.5. *Structure of the article.* The remainder of this paper is devoted to the proof of Theorems 1.4, 1.5 and 1.7. After replacing G by a compactly generated open subgroup that contains Γ and acts transitively on X, we may assume G is second countable.

In Section 2, we prove homogeneity of the orbit closures and convergence in law in Theorem 1.4. Knowing the recurrence results from [10] and [3] and the classification of stationary probability measures from [4], the main problem is to check that the centralizer L of Γ has only countably many orbits in $\mathcal{S}_X(\Gamma)$. We give the proof of this property for real Lie groups and postpone the general case to the appendix.

In Section 3, we establish preliminary results on Markov chains, mainly in order to prove the almost sure equidistribution statement in Theorem 1.4. Our starting point is Breiman's law of large numbers for Markov chains with a unique invariant measure [7]. In our case, as there may be several stationary measures, we have to prove that almost surely, starting from a point that does not belong to a given Y in $\mathcal{S}_X(\Gamma)$, the trajectory will spend most of the time far away from Y. This is done by exhibiting an exponentially recurrent subset in the complement of Y and by establishing large deviation properties for the return times in this subset. These techniques strengthen certain ideas from [17] and [4, §6].

In Section 4, we apply the results of Section 3 to the proofs of the almost sure equidistribution statement in Theorem 1.4 and of Theorem 1.5. We then deduce Theorem 1.7 from these.

In the appendix, we establish the countability of L-orbits in $\mathcal{S}_X(\Gamma)$ in the S-adic case. In Appendix A, we develop some tools to overcome the difficulty due to the fact that lattices in semiconnected S-adic Lie groups are not necessarily finitely generated, whereas Appendix B is devoted to the proof itself, whose structure mimics the real case.

2. Equidistribution in law

In this section, we establish the homogeneity of the orbit closures and the convergence in law in Theorem 1.4. We will first prove a countability statement for the set $\mathcal{S}_X(\Gamma)$ (Proposition 2.1). The claims will then follow from the recurrence result from [3] and the classification of stationary probability measures from [4].

2.1. *Countability of ergodic invariant homogeneous subsets.* Note that, if $\Gamma^{\pm}$ denotes the closed subgroup spanned by Γ, we have $\mathcal{S}_X(\Gamma) = \mathcal{S}_X(\Gamma^{\pm})$, so that we will assume in the next two sections Γ is a subgroup of G.

PROPOSITION 2.1. *Let G be a second countable weakly regular S-adic Lie group, Λ be a discrete subgroup of G, $X = G/\Lambda$, Γ be a compactly generated subgroup of G such that $\overline{\mathrm{Ad}\,\Gamma}^{\mathrm{Z}}$ is semisimple and equal to $\overline{\mathrm{Ad}\,\Gamma}^{\mathrm{Z,nc}}$, and L be the centralizer of Γ in G. Then, there exists a countable set $\mathcal{Y} \subset \mathcal{S}_X(\Gamma)$ such that*

$$\mathcal{S}_X(\Gamma) = \{\ell Y \mid \ell \in L \ , \ Y \in \mathcal{Y}\}.$$

In particular, when the centralizer of Γ in $\mathfrak{g}$ is null, the set $\mathcal{S}_X(\Gamma)$ is countable.

As the proof of Proposition 2.1 in the general S-adic case is highly technical, we will first give it in the real case. The general case is dealt with in the appendix. First, let us prove two elementary facts whose proofs are the same both in the real and S-adic cases and which will be of use below.

One is the following lemma which, in case H is discrete, follows from [4, Lemma 5.16].

LEMMA 2.2. *Let G be a second countable S-adic Lie group, H be a closed subgroup of G, Γ be a closed subgroup of G such that $\overline{\mathrm{Ad}\Gamma}^{\mathrm{Z}}$ is semisimple and L be the centralizer of Γ in G. When $S \neq \{\infty\}$, we assume that the group Γ is compactly generated. Then the set of fixed points of Γ in G/H is a countable union of L-orbits.*

Note that the semisimplicity assumption is crucial for this countability statement. Indeed, when $G = \mathrm{SL}(2, \mathbb{R})$ and $H = \Gamma$ is the subgroup of upper triangular unipotent matrices, the group L is equal to Γ and Γ has uncountably many fixed points in G/H.

Proof of Lemma 2.2. Set $X = G/H$, and let X^{Γ} be the set of fixed points of Γ in X. We shall prove that the orbits of L in X^{Γ} are open; that is, for any x in X^{Γ}, Lx contains a neighborhood of x in X^{Γ}. We may assume x is the base point of $X = G/H$. In particular, Γ is contained in H.

Let $\mathfrak{l}$ be the Lie algebra of L. If $S = \{\infty\}$, $\mathfrak{l}$ is necessarily the centralizer of Γ in $\mathfrak{g}$. If not, this is still the case by [4, Lemma 5.2], since we then assumed Γ to be compactly generated. As the linear span of $\mathrm{Ad}\Gamma$ in the space of endomorphisms of $\mathfrak{g}$ is finite dimensional, there exists $g_1, \ldots, g_r$ in Γ such that

$$\mathfrak{l} = \{v \in \mathfrak{g} \mid \forall 1 \leq i \leq r \ , \ g_i v = v\}.$$

Let $\mathfrak{h}$ be the Lie algebra of H. As $\overline{\mathrm{Ad}\Gamma}^{\mathrm{Z}}$ is semisimple, $\mathfrak{h}$ admits a Γ-invariant complementary subspace $\mathfrak{v}$. Now, there exists a standard open subset Ω of G (see [4, §5] or Section A.2 below), with exponential map $\exp_{\Omega}$:

$O \to \Omega$, such that the map $(O \cap \mathfrak{v}) \to X; v \mapsto \exp_\Omega(v)x$ is a diffeomorphism onto its image. We can assume $\exp_\Omega(O \cap \mathfrak{l}) \subset L$ and, by [4, Lemma 5.2], for any v in O and $1 \le i \le r$, if $g_i v \in O$, then $\exp_\Omega(g_i v) = g_i \exp_\Omega(v) g_i^{-1}$.

Set $U = O \cap \bigcap_{i=1}^r g_i^{-1} O \cap \mathfrak{v}$. Then $\exp_\Omega(U)x$ is a neighborhood of x and we shall prove that $\exp_\Omega(U)x \cap X^\Gamma \subset Lx$, which finishes the proof. Indeed, for $y = \exp_\Omega(v)x$ in X^Γ with v in U, we have, for any $1 \le i \le r$,

$$\exp_\Omega(g_i v)x = g_i \exp_\Omega(v) g_i^{-1} x = g_i \exp_\Omega(v)x = g_i y = y = \exp_\Omega(v)x;$$

hence, as both v and $g_i v$ belong to $O \cap \mathfrak{v}$, $g_i v = v$. This gives $v \in \mathfrak{l}$ and $y = \exp_\Omega(v)x \in Lx$, what should be proved. $\square$

We let $\mathbb{Q}_S$ denote the locally compact algebra $\prod_{p\in S} \mathbb{Q}_p$. By definition, a finite dimensional $\mathbb{Q}_S$-module is a product $V = \prod_{p \in S} V_p$ where, for any p in S, V_p is a finite dimensional $\mathbb{Q}_p$-vector space. We then let $\mathrm{GL}(V) = \prod_{p\in S} \mathrm{GL}(V_p)$ be the linear group of V and $\mathrm{Gr}(V) = \prod_{p \in S} \mathrm{Gr}(V_p)$ be its Grassmann variety.

One ingredient of the proof of Proposition 2.1 is the following more or less classical

Lemma 2.3. *Let V be a finitely generated $\mathbb{Q}_S$-module and Γ be a subgroup of $\mathrm{GL}(V)$. Assume that the Zariski closure $\overline{\Gamma}^{\mathrm{Z}}$ is semisimple and equal to $\overline{\Gamma}^{\mathrm{Z,nc}}$. Then every Γ-invariant probability measure η on $\mathrm{Gr}(V)$ is concentrated on the set of fixed points of Γ in $\mathrm{Gr}(V)$.*

Proof. By taking projections and replacing subspaces by exterior powers, it suffices to prove Lemma 2.3 when $V = V_p$ for some p in S, Γ acts irreducibly on V and η is a Γ-invariant probability measure on $\mathbb{P}(V)$. We will prove that the action of Γ on V is then trivial.

We first check that for any subspace $W \subsetneq V$, one has $\eta(\mathbb{P}(W)) = 0$. Indeed, let $\mathcal{W}$ be the set of subspaces W of V such that $\eta(\mathbb{P}(W)) > 0$ and the dimension of W is minimal among the subspaces satisfying this property. For any $W \neq W'$ in $\mathcal{W}$, we have

$$\eta(\mathbb{P}(W) \cup \mathbb{P}(W')) = \eta(\mathbb{P}(W)) + \eta(\mathbb{P}(W')).$$

Hence, $\mathcal{W}$ contains only finitely many elements $W_1, \ldots, W_r$ such that

$$\eta(\mathbb{P}(W_1)) = \cdots = \eta(\mathbb{P}(W_r)) = \max_{W \in \mathcal{W}} \eta(\mathbb{P}(W)).$$

Now, as η is Γ-invariant, the set $\{W_1, \ldots, W_r\}$ is Γ-invariant. As Γ is Zariski connected and acts irreducibly on V, we get $W_1 = \cdots = W_r = V$ and $\mathcal{W} = \{V\}$ as required.

Assume Γ acts nontrivially on V. By assumption, Γ is a nonrelatively compact subgroup of $\mathrm{SL}(V)$; hence the closure of $\mathbb{Q}_p\Gamma$ in the space of endomorphisms of V contains a nonzero singular map f. We have just shown $\eta(\mathbb{P}(\ker f)) = \eta(\mathbb{P}(\operatorname{im} f)) = 0$. We claim that $\eta(\mathbb{P}(\operatorname{im} f)) = 1$. Indeed write $f =$

$\lim_{n\to\infty} \lambda_n \gamma_n$ with λ_n in $\mathbb{Q}_p$ and γ_n in Γ, and note that, for all x in $\mathbb{P}(V) \smallsetminus \mathbb{P}(\ker f)$, the sequence $\gamma_n x$ accumulates towards $\mathbb{P}(\operatorname{im} f)$. Then, as η is Γ-invariant and as $\eta(\mathbb{P}(\ker f)) = 0$, for every continuous function φ with compact support on $\mathbb{P}(V) \smallsetminus \mathbb{P}(\operatorname{im} f)$, one has

$$\int \varphi(x)\, \mathrm{d}\eta(x) = \lim_{n\to\infty} \int \varphi(\gamma_n x)\, \mathrm{d}\eta(x) = 0.$$

This proves that $\eta(\mathbb{P}(V) \smallsetminus \mathbb{P}(\operatorname{im} f)) = 0$; hence $\eta(\mathbb{P}(\operatorname{im} f)) = 1$. This contradiction proves that Γ acts trivially on V. □

2.2. *Proof of countability in the real case.* Let G be a real Lie group, and let $\Delta \subset \Sigma$ be discrete subgroups of G.

Definition 2.4. We let $\mathcal{T}(G, \Delta, \Sigma)$ denote the set of closed subgroups H of G such that

(i) Σ is contained in H and Σ is a lattice in H;
(ii) one has $\Delta = \Sigma \cap H^\circ$, where H° is the connected component of H;
(iii) there exists a subgroup Γ of H such that $\overline{\operatorname{Ad}\Gamma}^{\mathrm{Z}}$ is semisimple with no compact factor and Γ acts ergodically on the H-invariant measure of H/Σ.

The core of the proof of Proposition 2.1 in the real case is the following

Lemma 2.5. *Let G be a second countable real Lie group and $\Delta \subset \Sigma$ be finitely generated discrete subgroups of G. The set $\mathcal{T}(G, \Delta, \Sigma)$ is countable.*

Note that this countability statement would not be true without the ergodicity condition (iii). Indeed, when $G = \mathrm{SL}(2, \mathbb{R})$, $\Delta = \Sigma = \{e\}$, G contains uncountably many compact subgroups H.

We shall need several preparatory lemmas. The semisimplicity assumption plays an essential role in the following

Lemma 2.6. *Let G be a real Lie group and $\Delta \subset \Sigma$ be discrete subgroups of G. Let H_1 and H_2 belong to $\mathcal{T}(G, \Delta, \Sigma)$. Then H_1 normalizes H_2°.*

A similar phenomenon appears in the proof of [24, Prop. 1.7].

Proof. It suffices to prove that H_1 normalizes the Lie algebra $\mathfrak{h}_2$ of H_2. Now, as Σ is contained in H_2, the map

$$H_1 \to \mathrm{Gr}(\mathfrak{g}); h_1 \mapsto \operatorname{Ad} h_1(\mathfrak{h}_2)$$

factors as a map $H_1/\Sigma \to \mathrm{Gr}(\mathfrak{g})$. Let η be the image of the H_1-invariant measure of H_1/Σ under this map. Let Γ be a subgroup of H_1 such that $\overline{\operatorname{Ad}\Gamma}^{\mathrm{Z}}$ is semisimple with no compact factor and that Γ acts ergodically on H_1/Σ. As η is Γ-ergodic, by Lemma 2.3, ν is a Dirac mass on a fixed point of Γ; that is, $\mathfrak{h}_2$ is normalized by H_1, what should be proved. □

We shall also need the two following elementary facts.

Lemma 2.7. *Let G be a second countable real Lie group. Then the set of normal compact subgroups of G is countable.*

Proof. Let K be a normal compact subgroup of G. The Lie algebra $\mathfrak{k}$ of K may be decomposed in a unique way as a direct sum $\mathfrak{k} = \mathfrak{s} \oplus \mathfrak{a}$, where $\mathfrak{s}$ is semisimple and $\mathfrak{a}$ is abelian. As $\mathfrak{k}$ is a G-invariant ideal of $\mathfrak{g}$, so are $\mathfrak{s}$ and $\mathfrak{a}$. As the Lie algebra $\mathfrak{g}$ of G contains only finitely many semisimple ideals, we may assume that $\mathfrak{s}$ is fixed. As the connected analytic subgroup S of G with Lie algebra $\mathfrak{s}$ is compact and normal in G, after replacing G by G/S, we may assume $\mathfrak{s} = \{0\}$.

In other terms, we have to prove that G contains countably many compact normal subgroups K whose Lie algebra $\mathfrak{k}$ is abelian. Since such a K is compact, $\mathfrak{k}$ admits a K-invariant complementary subspace $\mathfrak{v}$ in $\mathfrak{g}$. As $\mathfrak{v}$ is K-invariant and $\mathfrak{k}$ is an ideal of $\mathfrak{g}$, $\mathfrak{k}$ is central in $\mathfrak{g}$; hence K° is a central subgroup of G°. Now, the connected component of the center of G° is isomorphic to a product $\mathbb{R}^p \times \mathbb{T}^q$ and $\mathbb{T}^q$ admits countably many closed subgroups. Hence we may assume that K° is fixed. Thus, after replacing G by the group G/K°, we may assume that K is finite. Since K is normal, it then centralizes G° and Lemma 2.7 follows from Lemma 2.8 below. □

Lemma 2.8. *Let G be a second countable real Lie group. Then the set of compact subgroups of G which centralize G° is countable.*

Proof. By replacing G by the centralizer of G°, we may assume G° is central in G, and we have to prove that G contains countably many compact subgroups. Now, as above, G° being an abelian connected group, it admits countably many compact subgroups; hence we may fix the intersection of our compact subgroups with G°. As this intersection is central, up to replacing G by a quotient, we may assume it is trivial, and we therefore only have to prove that G contains countably many finite subgroups F such that $F \cap G^\circ = \{e\}$. For such a subgroup F, the group FG° is isomorphic to the product $F \times G^\circ$ and therefore contains only finitely many finite subgroups F' such that $F' \cap G^\circ = \{e\}$ and $F'G^\circ = FG^\circ$. As G/G° is countable, the result follows. □

We can now give the

Proof of Lemma 2.5. We can assume the set $\bigcup_{H \in \mathcal{T}(G,\Delta,\Sigma)} H$ spans a dense subgroup of G. Set $L = \bigcap_{H \in \mathcal{T}(G,\Delta,\Sigma)} H^\circ$. By Lemma 2.6, L is a normal subgroup of G. Since $L \cap \Sigma = \Delta$ is a lattice in L, the image of Σ in G/L is still discrete, so that, after replacing G by G/L, we may assume $L = \{e\}$. In particular, this gives $\Delta = \{e\}$ and thus, for any H in $\mathcal{T}(G,\Delta,\Sigma)$, H° is compact. As, still by Lemma 2.6, H° is normal in G, and, by Lemma 2.7, the set of normal compact subgroups of G is countable, we can suppose H° is fixed. Thus after replacing G by G/H°, we can assume H is discrete.

In other terms, we only have to prove that, setting $\mathcal{V}(G,\Sigma)$ to be the set of discrete subgroups H of G that contain Σ as a finite index subgroup and that admit a subgroup Γ such that $\overline{\mathrm{Ad}\Gamma}^{\mathrm{Z}}$ is semisimple with no compact factor and $H = \Gamma\Sigma$, then $\mathcal{V}(G,\Sigma)$ is countable. Now, if H belongs to $\mathcal{V}(G,\Sigma)$, H normalizes a finite index subgroup Θ of Σ. As Σ is finitely generated, the set of finite index subgroups of Σ is countable, and we can assume Θ is fixed. We set G' to be the closure of the subgroup of G spanned by all the H's in $\mathcal{V}(G,\Sigma)$ that normalize Θ and replace G by G'/Θ. Now, we just have to prove that, if Σ is finite, $\mathcal{V}(G,\Sigma)$ is countable. In this case, if H belongs to $\mathcal{V}(G,\Sigma)$, then H is finite and, if Γ is a subgroup of H such that $\overline{\mathrm{Ad}\Gamma}^{\mathrm{Z}}$ is semisimple with no compact factor and $H = \Gamma\Sigma$, then Γ is finite. Therefore $\mathrm{Ad}\,\Gamma$ is finite and hence trivial. In other terms, Γ is a finite subgroup of G that centralizes G°. By Lemma 2.8, the set of such subgroups is countable and we are done. □

We can now conclude the

Proof of Proposition 2.1 *in the real case.* Let Y be in $\mathcal{S}_X(\Gamma)$, and recall that G_Y denotes the stabilizer of Y in G. We pick g in G such that $g\Lambda$ belongs to Y, and we set $H = g^{-1}(\Gamma G_Y^\circ)g$. As H and H° are open in $g^{-1}G_Y g$ and $g\Lambda g^{-1} \cap G_Y$ is a lattice in G_Y, $\Lambda \cap H$ is a lattice in H and $\Lambda \cap H^\circ$ is lattice in H°. In particular, by [22, 6.18], $\Lambda \cap H^\circ$ is finitely generated. Since Γ is compactly generated, the real Lie group H is also compactly generated and its lattice $\Lambda \cap H$ is also finitely generated. In particular, as Λ admits countably many finitely generated subgroups, we can assume the groups

$$\Delta := \Lambda \cap H^\circ \ \text{ and } \ \Sigma := \Lambda \cap H$$

are fixed. Now, the group H belongs to $\mathcal{T}(G,\Delta,\Sigma)$ so that, by Lemma 2.5, we can also assume it to be fixed. The point $gH \in G/H$ is Γ-invariant. Hence, by Lemma 2.2, we can assume that the L-orbit LgH is fixed.

Now, if g_1 is an element of G such that $Lg_1H = LgH$, one can write $g_1 = \ell g h$ with ℓ in L and h in H. Hence one gets $g_1 H\Lambda = \ell Y$ and the result follows. □

2.3. *Proof of equidistribution in law.* We will need the following elementary lemma, which asserts that any two distinct elements in $\mathcal{S}_X(\Gamma)$ that are open in X are disjoint.

Lemma 2.9. *Let G be a second countable locally compact topological group, Λ be a discrete subgroup of G, $X = G/\Lambda$ and Γ be a closed sub-semigroup of G. Any two distinct elements $Y \neq Y'$ of $\mathcal{S}_X(\Gamma)$ that are open in X are disjoint.*

In particular, the set $\mathcal{S}_{\mathrm{op}}(\Gamma) := \{Y \in \mathcal{S}_X(\Gamma)\,, Y \text{ open in } X\}$ is countable.

Proof. Assume the intersection $Y'' = Y \cap Y'$ is not empty. Then, as Y'' is open in Y, one has $\nu_Y(Y'') > 0$ and, as Y'' is Γ-invariant and Y is Γ-ergodic, one has $\nu_Y(Y'') = 1$. Since Y'' is closed, we get $Y = Y''$. In the same way, $Y' = Y''$. $\square$

Using Proposition 2.1 and the results of [4] and [3], we get the

Proof of Theorem 1.4(a) *and* 1.4(b). We proceed by induction on the dimension of G. If this dimension equals 0, then the space G/Λ is finite and the result is evident.

Assume G has positive dimension, and fix x in X. If there exists a nonopen Y in $\mathcal{S}_X(\Gamma)$ such that x belongs to Y, we get the result by induction. Thus, we can assume this is not the case, and we will prove that there exists a unique Y_x in $\mathcal{S}_{\mathrm{op}}(\Gamma)$ containing x and that the sequence of probability measures $\nu_n = \frac{1}{n}\sum_{k=0}^{n-1}\mu^{*k} * \delta_x$ on X converges toward ν_{Y_x}. Now, let ν be a limit point of ν_n as n goes to ∞, so that ν is μ-stationary. By [3, Th. 7.2], ν is a probability measure.

By [4, Th. 2.5], ν being μ-stationary, it is Γ-invariant and every μ-ergodic component of ν is equal to some ν_Y for Y in $\mathcal{S}_X(\Gamma)$. Let L be the centralizer of Γ in G. By [4, Prop. 6.24 and Cor. 6.25], for every nonopen Y in $\mathcal{S}_X(\Gamma)$ and every compact subset K_L of L, one has $\nu(K_L Y) = 0$, hence $\nu(LY) = 0$. Since, by Proposition 2.1, $\mathcal{S}_X(\Gamma)$ is a countable union of L-orbits, almost every ergodic component of ν is equal to some ν_Y with Y in $\mathcal{S}_{\mathrm{op}}(\Gamma)$.

Since by Lemma 2.9, $\mathcal{S}_{\mathrm{op}}(\Gamma)$ is countable, there exists Y_x in $\mathcal{S}_{\mathrm{op}}(\Gamma)$ such that $\nu(Y_x) > 0$. But then, the point x belongs to Y_x, so that, by Lemma 2.9, Y_x is the unique element of $\mathcal{S}_{\mathrm{op}}(\Gamma)$ containing x. By construction, ν does not give mass to any other element of $\mathcal{S}_{\mathrm{op}}(\Gamma)$. Therefore, $\nu = \nu_{Y_x}$ and we are done. $\square$

3. **Markov operators**

We will now develop abstract probabilistic tools for proving the almost sure statement in Theorem 1.4. An important ingredient in our method comes from the proof of Breiman's law of large numbers for Markov chains [7], which, in the context of group actions, states as follows:

Let G be a locally compact group, X a compact metrizable G-space, and let μ be a Borel probability measure on G such that there exists a unique μ-stationary Borel probability measure ν on X. Let $g_1, \ldots, g_n, \ldots$ be a sequence of random elements of G that are independent and identically distributed with law μ. Then, for any x in X, almost surely, one has

$$\frac{1}{n}\sum_{k=0}^{n-1}\delta_{g_k\cdots g_1 x} \xrightarrow[n\to\infty]{} \nu.$$

In our situation, we have to update the strategy of [7] since X is not compact and there may be several μ-stationary probability measures on X. To do this, we shall need abstract general information on Markov chains.

3.1. *Markov measures and the law of large numbers.* Let $(X, \mathcal{X})$ be a standard Borel space. By a Markov operator on X, we mean a Borel map $x \mapsto P_x$ from X to the space of Borel probability measures on X. Given such an operator, for any bounded Borel function φ on X and any x in X, we set

$$P\varphi(x) = \int_X \varphi \, \mathrm{d}P_x.$$

Let us recall the construction of the Markov measures associated to P on the space of trajectories. We set $W = X^{\mathbb{N}}$, and we equip it with the product σ-algebra $\mathcal{X}^{\otimes \mathbb{N}}$. An element w in W will be written as a sequence $w = (w_0, w_1, w_2, \ldots)$. For any x in X, there exists a unique Borel probability measure ω_x on W such that, for any bounded Borel functions $\varphi_0, \ldots, \varphi_n$ on X, one has

$$\int_W \varphi_0(w_0) \cdots \varphi_n(w_n) \, \mathrm{d}\omega_x(w) = (\varphi_0 P(\cdots(\varphi_{n-1}P(\varphi_n))\cdots))(x).$$

In other terms, ω_x is implicitly defined by $\omega_x = \delta_x \otimes (\int_X \omega_y \, \mathrm{d}P_x(y))$. We say ω_x is the Markov measure associated to P and x.

Example 3.1. Let a locally compact topological group G act measurably on X. Fix a Borel probability measure μ on G and set, for any x in X, $P_x = \mu * \delta_x$. This defines a Markov operator P on X that represents formally the notion of a random walk on X with law μ. For any x in X, the associated Markov measure ω_x on W is the image of the measure $\mu^{\otimes \mathbb{N}}$ on $G^{\mathbb{N}}$ under the map $(g_k)_{k \in \mathbb{N}} \mapsto (g_{k-1} \cdots g_0 x)_{k \in \mathbb{N}}$.

Lemma 3.2. (Breiman [7]) *Let $(X, \mathcal{X})$ be a standard Borel space, P be a Markov operator and φ be a bounded Borel function on X. For every x in X, for ω_x-almost every w in W, one has*

$$\frac{1}{n} \sum_{k=0}^{n-1} \varphi(w_k) - \frac{1}{n} \sum_{k=0}^{n-1} P\varphi(w_k) \xrightarrow[n \to \infty]{} 0.$$

Proof. We first recall the following version of the classical law of large numbers:

> *Let $(Y, \mathcal{Y}, \eta)$ be a probability space and (ζ_n) be a bounded sequence of elements of $\mathrm{L}^2(Y, \mathcal{Y}, \eta)$ with, for any n, $\mathbb{E}(\zeta_n | \zeta_{n-1}, \ldots, \zeta_1) = 0$. Then $\frac{1}{n} \sum_{k=1}^{n} \zeta_k \xrightarrow[n \to \infty]{} 0$ almost everywhere.*

For any integer $n \geq 1$ set, for w in W,

$$\zeta_n(w) = \varphi(w_n) - P\varphi(w_{n-1}).$$

This sequence of functions on W is bounded by $2\sup_X |\varphi|$ and, as ζ_n only depends on $w_n, \dots, w_0$, one has, for any $n \geq 1$,

$$\mathbb{E}_{\omega_x}(\zeta_n | \zeta_{n-1}, \dots, \zeta_1) = \mathbb{E}_{\omega_x}(\mathbb{E}_{\omega_x}(\zeta_n | w_{n-1}, \dots, w_0) | \zeta_{n-1}, \dots, \zeta_1).$$

By construction, one has

$$\mathbb{E}_{\omega_x}(\zeta_n | w_{n-1}, \dots, w_0) = 0,$$

hence

$$\mathbb{E}_{\omega_x}(\zeta_n | \zeta_{n-1}, \dots, \zeta_1) = 0.$$

Therefore, we have, ω_x-almost everywhere, $\frac{1}{n}\sum_{k=1}^{n} \zeta_k \xrightarrow[n\to\infty]{} 0$. The result follows since φ is bounded. □

We say a Borel measure ν on X is P-invariant if, for any nonnegative Borel function φ on X, one has $\int_X P\varphi \, \mathrm{d}\nu = \int_X \varphi \, \mathrm{d}\nu$.

Recall that if X is a compact space, a Markov-Feller operator on X is a nonnegative operator P on the space of continuous functions on X such that $P\mathbf{1} = \mathbf{1}$. In other terms, a Markov-Feller operator is a Markov operator on X such that the map $x \mapsto P_x$ is continuous, when the space of Borel probability measures of X is equipped with the weak-$*$ topology.

From Lemma 3.2, we get

Corollary 3.3. *Let X be a compact metrizable topological space and P be a Markov-Feller operator on X. Then, for any x in X, for ω_x-almost any w in W, any weak-$*$ limit ν_∞ of $\nu_n := \frac{1}{n}\sum_{k=0}^{n-1} \delta_{w_k}$ is P-invariant.*

In particular, using the weak-$*$-compactness of the space of probability measures on X, we retrieve Breiman's law of large numbers in [7]:

> *If, moreover, there exists a unique P-invariant probability measure ν on X, then for any x in X, for ω_x-almost any w in W, one has $\frac{1}{n}\sum_{k=0}^{n-1} \delta_{w_k} \xrightarrow[n\to\infty]{} \nu$.*

Proof of Corollary 3.3. Let ν_{n_k} be a subsequence converging to ν_∞. For all continuous function φ on X, one has

$$\int_X \varphi \, \mathrm{d}\nu_\infty = \lim_{k\to\infty} \int_X \varphi \, \mathrm{d}\nu_{n_k}.$$

Since P is Feller, the function $P\varphi$ is also continuous and one also has

$$\int_X P\varphi \, \mathrm{d}\nu_\infty = \lim_{k\to\infty} \int_X P\varphi \, \mathrm{d}\nu_{n_k}.$$

Hence by Lemma 3.2, one has $\int_X \varphi \, \mathrm{d}\nu_\infty = \int_X P\varphi \, \mathrm{d}\nu_\infty$, and ν_∞ is P-invariant. □

3.2. *Recurrent subsets.* We need to understand weak limits as in Corollary 3.3 when the space X is not compact and, in particular, to get a manageable criterion for them to have total mass 1. To this aim, we study recurrent subsets.

If $Y \subset X$ is a Borel subset, we say Y is *P-recurrent* if, for any x in Y, one has $\omega_x(\{w \in W \mid \exists k \geq 1 \ w_k \in Y\}) = 1$; that is if, almost surely, the trajectories issued from Y turn back to Y. For any w in W, we let

$$\tau_Y(w) = \min\{k \geq 1 \mid w_k \in Y\} \in [1, \infty]$$

denote the first return time in Y. In the same way, for any w in W with $\sharp\{k \in \mathbb{N} \mid w_k \in Y\} = \infty$, we set $\tau_Y^1(w) = \tau_Y(w)$ and, for any $p \geq 2$,

$$\tau_Y^p(w) = \min\{k > \tau_Y^{p-1}(w) \mid w_k \in Y\}.$$

If $w_0 \in Y$, we also write $\tau_Y^0(w) = 0$. These are the successive return times of w in Y.

Assume Y is P-recurrent and let, for any x in Y, Q_y denote the probability measure on Y that is the image of ω_x under the map $w \mapsto w_{\tau_Y(w)}$, which is defined ω_x-almost everywhere on W. We say Q is the *Markov operator induced* by P on Y. One easily checks that, for any x in Y, the Markov measure associated to Q and x is the image of ω_x under the map

$$W \to Y^{\mathbb{N}}, w \mapsto (w_{\tau_Y^p(w)})_{p \in \mathbb{N}}.$$

We say a P-invariant Borel measure ν on X is ergodic if, for any Borel function φ on X with $P\varphi = \varphi$, φ is constant ν-almost everywhere.

LEMMA 3.4. *Let $(X, \mathcal{X})$ be a standard Borel space, P be a Markov operator on X, Y be a P-recurrent Borel subset of X and Q be the Markov operator induced by P on Y. Let ν be a P-invariant Borel measure on X. Then $\nu|_Y$ is Q-invariant. Moreover, if ν is P-ergodic, then $\nu|_Y$ is Q-ergodic.*

Proof. Let φ be a nonnegative Borel function on X, and let us prove $\int_Y Q\varphi \, \mathrm{d}\nu = \int_Y \varphi \, \mathrm{d}\nu$. We introduce the function $\psi : X \to [0, \infty)$ given by $\psi(x) = \varphi(x)$ if $x \in Y$ and $\psi(x) = \int_W \mathbf{1}_{\{\tau_Y(w) < \infty\}} \varphi(w_{\tau_Y(w)}) \, \mathrm{d}\omega_x(w)$ otherwise. By construction, one has

$$\begin{aligned} \psi &= \varphi \mathbf{1}_Y + \psi \mathbf{1}_{Y^c}, \\ P\psi &= (Q\varphi)\mathbf{1}_Y + \psi \mathbf{1}_{Y^c}; \end{aligned} \tag{3.1}$$

hence

$$\begin{aligned} \int_X \psi \, \mathrm{d}\nu &= \int_Y \varphi \, \mathrm{d}\nu + \int_{Y^c} \psi \, \mathrm{d}\nu, \\ \int_X P\psi \, \mathrm{d}\nu &= \int_Y Q\varphi \, \mathrm{d}\nu + \int_{Y^c} \psi \, \mathrm{d}\nu. \end{aligned}$$

Since ν is P-invariant, this gives $\int_Y Q\varphi \, \mathrm{d}\nu = \int_Y \varphi \, \mathrm{d}\nu$. Now, assume that ν is P-ergodic and $Q\varphi = \varphi$, and let ψ still be as above. From (3.1), we get $P\psi = \psi$; hence ψ is constant ν-almost everywhere and so is φ. $\square$

3.3. *Exponentially recurrent subsets.* Assuming the return times enjoy strong uniform moment properties, we will now get almost sure estimates on the asymptotic behaviour as p goes to ∞ of the p-th return time of a given trajectory.

Let still Y be P-recurrent. As in [4, §6], we say that Y is *exponentially P-recurrent* if there exists $0 < a < 1$ with

$$\sup_{x \in Y} \int_W a^{-\tau_Y} \, \mathrm{d}\omega_x < \infty.$$

The following lemma asserts that return times in exponentially recurrent subsets satisfy a large deviation principle.

LEMMA 3.5. *Let $(X, \mathcal{X})$ be a standard Borel space, P be a Markov operator on X and Y be an exponentially recurrent subset of X. We set*

$$\theta := \sup_{x \in Y} \int_W \tau_Y \, \mathrm{d}\omega_x < \infty.$$

Then, for any $\varepsilon > 0$, there exists $\alpha > 0$ such that, for any x in Y and p in $\mathbb{N}$, we have

$$\omega_x(\{w \in W \mid \tau_Y^p(w) \geq p(\theta + \varepsilon)\}) \leq e^{-p\alpha}.$$

In particular, for ω_x-almost any w in W, one has

$$\limsup_{p \to \infty} \frac{1}{p} \tau_Y^p(w) \leq \theta.$$

Proof. Let $\beta_0 > 0$ be such that

$$\sup_{x \in Y} \int_W e^{\beta_0 \tau_Y} \, \mathrm{d}\omega_x < \infty.$$

For any $0 < \beta \leq \beta_0$, for any x in Y, we get

$$\omega_x(\{w \in W \mid \tau_Y^p(w) \geq p(\theta + \varepsilon)\}) \leq e^{-p\beta(\theta+\varepsilon)} \int_W e^{\beta \tau_Y^p(w)} \, \mathrm{d}\omega_x(w).$$

Now, by the Markov property for the operator induced by P on Y, we have

$$\int_W e^{\beta \tau_Y^p} \, \mathrm{d}\omega_x \leq \left(\sup_{y \in Y} \int_W e^{\beta \tau_Y} \, \mathrm{d}\omega_y \right)^p.$$

Since for every $t \geq 0$ one has $e^t \leq 1 + t + t^2 e^t$, there exists $C > 0$ such that, for $0 < \beta \leq \beta_0$, one has

$$\sup_{y \in Y} \int_W e^{\beta \tau_Y} \, \mathrm{d}\omega_y \leq 1 + \beta\theta + C\beta^2.$$

Thus, if β is small enough, we get

$$e^{-\alpha} := e^{-\beta(\theta+\varepsilon)} \sup_{y\in Y} \int_W e^{\beta\tau_Y} \, \mathrm{d}\omega_y < 1$$

and the first part of the lemma is proved. The second part follows by the Borel-Cantelli lemma. □

Lemma 3.5 yields the following corollary, which we shall not use but which is of independent interest.

COROLLARY 3.6. *Let $(X, \mathcal{X})$ be a standard Borel space, $Y \supset Z$ be Borel subsets of X and P be a Markov operator on X. Assume Y is P-recurrent and let Q be the Markov operator induced by P on Y. If Y is exponentially P-recurrent and Z is exponentially Q-recurrent, then Z is exponentially P-recurrent.*

Proof. By Lemma 3.5, there exists $\gamma \geq 1$ and $\alpha > 0$ such that, for any x in Y and p in $\mathbb{N}$, one has

$$\omega_x(\{w \in W \mid \tau_Y^p(w) > \gamma p\}) \leq e^{-\alpha p}.$$

As Z is exponentially Q-recurrent, there exists $\beta > 0$ such that, for p large enough, for any x in Z, one has

$$\omega_x(\{w \in W \mid \tau_Z(w) > \tau_Y^p(w)\}) \leq e^{-\beta p}.$$

We get

$$\omega_x(\{w \in W \mid \tau_Z(w) > \gamma p\}) \leq e^{-\alpha p} + e^{-\beta p},$$

and the result follows. □

We now aim at proving that, on a given trajectory that starts from an exponentially recurrent subset Y, most of the time is spent at a close temporal distance from Y.

To be more precise, we introduce some notation. Let $Y \subset X$ still be a Borel subset. If $w \in W$ is such that $w_0 \in Y$ and $\sharp\{k \in \mathbb{N} \mid w_k \in Y\} = \infty$, we set, for any natural integers p, T,

$$\sigma_Y^p(w) := \tau_Y^{p+1}(w) - \tau_Y^p(w),$$

which are the successive excursion times of w out of Y, and

$$\sigma_{Y,T}^p(w) := \sigma_Y^p(w)\mathbf{1}_{\{\sigma_Y^p(w)\geq T\}} \text{ and } \tau_{Y,T}^p(w) := \sum_{0\leq q<p} \sigma_{Y,T}^p(w),$$

which is the total duration among the p first excursions outside Y of those of length $\geq T$.

LEMMA 3.7. *Let $(X,\mathcal{X})$ be a standard Borel space, P be a Markov operator on X and Y be an exponentially P-recurrent subset of X. For any $\varepsilon>0$, there exists T in $\mathbb{N}$ such that, for any x in Y, for ω_x-almost any w in W, one has*

$$\limsup_{p\to\infty}\frac{1}{p}\tau^p_{Y,T}(w)\leq\varepsilon.$$

Proof. Since Y is exponentially P-recurrent, there exists $\alpha_0>0$ such that $\sup_{x\in Y}\int_W e^{\alpha_0\tau_Y}\,\mathrm{d}\omega_x<\infty$. Hence, for T large enough, we have

$$\sup_{x\in Y}\int_W \tau^1_{Y,T}\,\mathrm{d}\omega_x\leq\varepsilon.$$

We can now conclude as in the proof of Lemma 3.5, using the fact that $\sup_{x\in Y}\int_W e^{\alpha_0\tau^1_{Y,T}}\,\mathrm{d}\omega_x<\infty$. □

3.4. *Excursions of trajectories.* Let $u:X\to[0,\infty]$ be a Borel function such that there exist $0\leq a<1$ and $C\geq 0$ with $Pu\leq au+C$. In [4, Prop. 6.3], we proved, if $M>0$ is large enough, the set $X_M=u^{-1}([0,M])$ is exponentially recurrent. In Proposition 3.9 below, we will see that, on the set $\{u<\infty\}$, almost surely, the trajectories spend most of the time in subsets of the form X_M with M large. This is a key step for proving Theorem 1.4(c).

We start with a much weaker result, whose conclusion serves as a motivation for the exact formulation of Proposition 3.9 and which will also be of use in the proof of Theorem 1.5.

LEMMA 3.8. *Let $(X,\mathcal{X})$ be a standard Borel space, P be a Markov operator on X and $u:X\to[0,\infty]$ be a Borel function such that there exist $0\leq a<1$ and $C\geq 0$ with $Pu\leq au+C$. Let ν be a P-invariant P-ergodic Borel probability measure on X such that $\nu(\{u<\infty\})>0$. Then one has $\int_X u\,\mathrm{d}\nu\leq\frac{C}{1-a}$.*

Proof. According to the Chacon-Ornstein ergodic theorem, for ν-almost every x in X, one has

$$\frac{1}{n}\sum_{k=0}^{n-1}P^k u(x)\xrightarrow[n\to\infty]{}\int_X u\,\mathrm{d}\nu.$$

Since $\nu(\{u<\infty\})>0$, we can choose such a x with $u(x)<\infty$. Now, by assumption, for every $k\geq 0$, one has

$$P^k u(x)\leq a^k u(x)+\frac{C}{1-a};$$

hence,

$$\frac{1}{n}\sum_{k=0}^{n-1}P^k u(x)\leq\frac{u(x)}{(1-a)n}+\frac{C}{1-a}.$$

Letting n go to infinity, we get $\int_X u\,\mathrm{d}\nu\leq\frac{C}{1-a}$. □

The main result of this section now states

Proposition 3.9. *Let $(X, \mathcal{X})$ be a standard Borel space, P be a Markov operator on X and $u : X \to [0, \infty]$ be a Borel function such that there exist $0 \le a < 1$ and $C \ge 0$ with $Pu \le au + C$. Then, for any x in X with $u(x) < \infty$, for ω_x-almost any w in W, for any $M > 0$, one has*

$$\limsup_{n\to\infty} \frac{1}{n} \sharp\{0 \le k < n \mid u(w_k) > M\} \le \frac{C}{(1-a)M}.$$

Note, if ν is a P-invariant Borel probability measure, by the Birkhoff theorem and Lemma 3.8 above, the conclusion of Proposition 3.9 holds for ν-almost any x with $u(x) < \infty$.

As a first step toward the proof, we establish a weaker result that would be sufficient for our purpose.

Lemma 3.10. *Let $(X, \mathcal{X})$ be a standard Borel space, P be a Markov operator on X and $u : X \to [0, \infty]$ be a Borel function such that there exist $0 \le a < 1$ and $C \ge 0$ with $Pu \le au + C$. Then, for any x in X with $u(x) < \infty$ and $\varepsilon > 0$, there exists $M > 0$ such that, for ω_x-almost any w in W, one has*

$$\limsup_{n\to\infty} \frac{1}{n} \sharp\{0 \le k < n \mid u(w_k) > M\} \le \varepsilon.$$

Proof. Choose $M_0 > 0$, and set $Y = u^{-1}([0, M_0])$. By [4, Prop. 6.3], if M_0 is large enough, the set Y is exponentially P-recurrent. More precisely, pick a_0 in $(a, 1]$ and assume $a_0 - a - C/M_0 > 0$. Reusing ideas of the proof of [4, Prop. 6.3], let us prove the following uniform bound for weighted Birkhoff sums of u up to the return time in Y: for any x in X,

$$\int_W \sum_{k=1}^{\tau_Y(w)} a_0^{-k} u(w_k)\, \mathrm{d}\omega_x(w) \le \frac{au(x) + C}{a_0 - a - C/M_0}. \tag{3.2}$$

In order to prove this bound we set, for x in X and $n \ge 1$,

$$U_n(x) := \int_W \sum_{k=1}^{\min(\tau_Y(w), n)} a_0^{-k} u(w_k)\, \mathrm{d}\omega_x(w),$$

which can be rewritten as

$$U_n(x) = \sum_{k=1}^{n} a_0^{-k} \int_{\{\tau_Y(w) \ge k\}} u(w_k)\, \mathrm{d}\omega_x(w).$$

In particular, one has $U_n(x) \le \sum_{k=1}^{n} a_0^{-k} P^k u(x) < \infty$. Besides, the function $\mathbf{1}_{\{\tau_Y \ge k\}}$ being a function of $w_1, \ldots, w_{k-1}$, by the Markov property, one gets

$$U_n(x) = \sum_{k=1}^{n} a_0^{-k} \int_{\{\tau_Y(w) \ge k\}} (Pu)(w_{k-1})\, \mathrm{d}\omega_x(w)$$

$$
\begin{aligned}
&\leq \sum_{k=1}^{n} a_0^{-k} \int_{\{\tau_Y(w)\geq k\}} (a\, u(w_{k-1}) + C)\, \mathrm{d}\omega_x(w) \\
&\leq \frac{au(x)+C}{a_0} + \sum_{k=2}^{n} a_0^{-k} \int_{\{\tau_Y(w)\geq k\}} (a + C/M_0) u(w_{k-1})\, \mathrm{d}\omega_x(w) \\
&\leq \frac{au(x)+C}{a_0} + \frac{a+C/M_0}{a_0} U_n(x);
\end{aligned}
$$

that is, since $a_0 - a - C/M_0 > 0$,

$$U_n(x) \leq \frac{au(x)+C}{a_0 - a - C/M_0}.$$

Letting n go to ∞, we get (3.2).

For any x in Y and p in $\mathbb{N}$, we set, for ω_x-almost any w in W,

$$v_p(w) = \max_{\tau_Y^p(w) \leq k < \tau_Y^{p+1}(w)} \log_+ u(w_k).$$

According to (3.2) with $a_0 = 1$, one has

$$\sup_{x\in Y} \int_W e^{v_0(w)}\, \mathrm{d}\omega_x < \infty.$$

Set $\rho = \sup_{x\in Y} \int_W v_0(w)\, \mathrm{d}\omega_x$. Reasoning again as in the proof of Lemma 3.5 yields, for any x in Y, for ω_x-almost any w in W,

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{p=0}^{n-1} v_p(w) \leq \rho.$$

Now, by Lemma 3.7, there exists T in $\mathbb{N}$ such that, for any x in Y, for ω_x-almost any w in W, one has

$$\limsup_{n\to\infty} \frac{1}{n} \tau_{Y,T}^n(w) \leq \frac{\varepsilon}{2}.$$

Since, for any n in $\mathbb{N}$, $\tau_Y^n \geq n$, we get, for any $M > 1$,

$$
\begin{aligned}
\sharp\{0 \leq k < n \mid u(w_k) > M\} &\leq \sum_{p=0}^{n-1} \sigma_Y^p(w)\, \mathbf{1}_{\{v_p(w) > \log M\}} \\
&\leq \tau_{Y,T}^n(w) + T\sharp\{0 \leq p < n \mid v_p(w) > \log M\} \\
&\leq \tau_{Y,T}^n(w) + \frac{T}{\log M} \sum_{p=0}^{n-1} v_p(w),
\end{aligned}
$$

thus, for ω_x-almost any w in W,

$$\limsup_{n\to\infty} \frac{1}{n} \sharp\{0 \leq k < n \mid u(w_k) > M\} \leq \frac{\varepsilon}{2} + \frac{T\rho}{\log M}.$$

The result follows, since M is arbitrarily large. $\square$

Proof of Proposition 3.9. For any $M > 0$, set $X_M = u^{-1}([0, M])$. Fix $\varepsilon > 0$ and $M > 0$. By Lemma 3.10, there exists $M' > 0$ such that, for ω_x-almost any w in W, for n large enough, one has

$$\frac{1}{n}\sum_{k=0}^{n-1} \mathbf{1}_{X^c_{M'}}(w_k) \leq \varepsilon.$$

Pick $m \geq 1$ such that $a^m \frac{M'}{M} \leq \varepsilon$. By Lemma 3.2, for ω_x-almost any w in W, for n large enough, we have

$$\begin{aligned}\frac{1}{n}\sum_{k=0}^{n-1} \mathbf{1}_{X^c_M}(w_k) &\leq \frac{1}{n}\sum_{k=0}^{n-1} P^m \mathbf{1}_{X^c_M}(w_k) + \varepsilon \\ &\leq \frac{1}{n}\sum_{k=0}^{n-1} \mathbf{1}_{X_{M'}}(w_k)\; P^m \mathbf{1}_{X^c_M}(w_k) + 2\varepsilon \leq \frac{C}{(1-a)M} + 3\varepsilon\end{aligned}$$

since, for any y in $X_{M'}$, one has

$$P^m \mathbf{1}_{X^c_M}(y) \leq \frac{1}{M} P^m u(y) \leq \frac{1}{M}\left(a^m u(y) + \frac{C}{1-a}\right) \leq \varepsilon + \frac{C}{(1-a)M}.$$

The proposition follows. □

4. Equidistribution through Markov chains

In this section, we apply the results on Markov chains established in Section 3 to the proof of Theorems 1.4(c), 1.5 and 1.7.

The key geometric input we will need from [4] and [3] is the following

LEMMA 4.1. *Let G, Λ, X, Γ and μ be as in Theorem* 1.4 *and L be the centralizer of Γ in G. We denote by P the Markov operator on X with transition probabilities $P_x = \mu * \delta_x$, $x \in X$.*

Let K be a compact subset of X. There exist a lower semicontinuous function $u : X \to [0, \infty]$ that is bounded on K and $0 \leq a < 1$ and $C > 0$ with $Pu \leq au + C$, such that, for any $M > 0$, the set $X_M = u^{-1}([0, M])$ is compact.

If X_M is P-recurrent, we let P_M denote the Markov operator induced by P on X_M.

Let Y be a closed Γ-invariant homogeneous subspace of X and K_L be a compact subset of L. If M is large enough, the set X_M is P-recurrent and there exist a lower semicontinuous function $v_M : X_M \to [0, \infty]$, a compact neighborhood K'_L of K_L in L and $0 \leq a_M < 1$ and $C_M > 0$ with

$$\begin{aligned} &P_M v_M \leq a_M v_M + C_M, \\ &\quad v_M < \infty \qquad\qquad \textit{on } (K'_L Y)^c \cap X_M, \\ &\quad v_M = \infty \qquad\qquad \textit{on } K_L Y \cap X_M. \end{aligned}$$

Proof. The existence of u follows from [3, Prop. 7.4]; the one of v_M follows from the proof of [4, Prop. 6.24]. Note that, by Proposition 3.9, the set X_M is P-recurrent as soon as $\frac{C}{(1-a)M} < 1$. $\square$

4.1. *Almost sure equidistribution of trajectories.*

Proof of Theorem 1.4(c). We set $X = G/\Lambda$ and P for the Markov operator on X with transition probabilities $P_x = \mu * \delta_x$, $x \in X$. Fix x in X. As in the first part of the proof of Theorem 1.4, we can assume x does not belong to any nonopen Y in $\mathcal{S}_X(\Gamma)$.

By Lemma 4.1, there exist a lower semicontinuous function $u : X \to [0, \infty]$ and $0 \leq a < 1$ and $C > 0$ with $Pu \leq au + C$, $u(x) < \infty$ and such that, for any $M > 0$, the set $X_M = u^{-1}([0, M])$ is compact. Therefore, by Proposition 3.9, for ω_x-almost any w in W, any weak limit point ν of $\frac{1}{n}\sum_{k=0}^{n-1}\delta_{w_k}$ is a probability measure. By Lemma 3.2, such a limit point is μ-stationary. By [4, Th. 2.5], ν is an average of probability measures ν_Y with Y in $\mathcal{S}_X(\Gamma)$. By Proposition 2.1 and Lemma 2.9, it suffices to prove that, for any nonopen Y in $\mathcal{S}_X(\Gamma)$ and any compact subset K_L of L, one has $\nu(K_L Y) = 0$ (where L is the centralizer of Γ in G).

Fix $0 < \varepsilon < 1$ and $M > 0$ large enough, so that $\frac{C}{(1-a)M} \leq \varepsilon$. By Proposition 3.9, the set $X_M = u^{-1}([0, M])$ is P-recurrent. We let P_M be the Markov operator induced by P on X_M and τ_M^p, $p \in \mathbb{N}$, be the successive return times in X_M. By Lemma 4.1, if M is large enough, there exist a lower semicontinuous function $v_M : X_M \to [0, \infty]$ and $0 \leq a_M < 1$ and $C_M > 0$ with $P_M v_M \leq a_M v_M + C_M$, $v_M(x) < \infty$ and such that $v_M = \infty$ on $K_L Y \cap X_M$. Therefore, by Proposition 3.9, applied to the Markov operator P_M, for ω_x-almost any w in W, any weak limit point of the sequence of probability measures $\frac{1}{p}\sum_{q=0}^{p-1}\delta_{w_{\tau_M^q(w)}}$ gives mass 0 to $K_L Y \cap X_M$. Hence, applying Proposition 3.9 to the operator P, we get $\nu(K_L Y \cap X_M) \leq \varepsilon$. The result follows since ε is arbitrarily small and M arbitrarily large. $\square$

4.2. *Equidistribution of invariant homogeneous subsets.* The proof of Theorem 1.5 essentially relies on the following

Lemma 4.2. *Let G, Λ, X and Γ be as in Theorem* 1.4 *and L be the centralizer of Γ in G. Fix a compact subset K of X, and let $(Y_n) \subset \mathcal{S}_K(\Gamma)$ and ν_∞ be a cluster point of ν_{Y_n} as $n \to \infty$.*

(a) *The measure ν_∞ has total mass* 1.
(b) *If $Y \in \mathcal{S}_X(\Gamma)$ is such that, for any compact subset K_L of L, for all but finitely many n, one has $Y_n \not\subset K_L Y$, then one has $\nu_\infty(LY) = 0$.*

Proof. The proof follows the same line as Theorem 1.4(c). For any n, we set $\nu_n = \nu_{Y_n}$. By replacing K by a compact neighborhood, we may assume one has $\nu_n(K) > 0$.

Let μ be a compactly supported Borel probability measure on Γ whose supports spans Γ, and let P be the Markov operator on X with transition probabilities $P_x = \mu * \delta_x$, $x \in X$. By Lemma 4.1, there exist a lower semicontinuous function $u : X \to [0,\infty]$ that is bounded on K and $0 \leq a < 1$ and $C > 0$ with $Pu \leq au + C$, such that, for any $M > 0$, the set $X_M = u^{-1}([0,M])$ is compact. Since for any n, ν_n is Γ-invariant, it is P-invariant. Hence, as $\nu_n(K) > 0$, by Lemma 3.8, one has $\int_X u \,\mathrm{d}\nu_n \leq \frac{C}{1-a}$ and therefore

$$\nu_n(X_M^c) \leq \frac{C}{(1-a)M}. \tag{4.1}$$

We get $\nu_\infty(X_M^c) \leq \frac{C}{(1-a)M}$ and ν_∞ is a probability measure; that is, (a) is proved.

Let Y be as in (b) and let us prove $\nu_\infty(LY) = 0$. Fix $0 < \varepsilon < 1$ and $M > 0$ large enough, so that $\frac{C}{(1-a)M} \leq \varepsilon$. By Proposition 3.9, the set X_M is P-recurrent. We let P_M be the induced Markov operator on X_M. Now, if M is large enough, the second part of Lemma 4.1 holds: there exist a lower semicontinuous function $v_M : X_M \to [0,\infty]$, a compact neighborhood K'_L of K_L in L and $0 \leq a_M < 1$ and $C_M > 0$ with $P_M v_M \leq a_M v_M + C_M$ such that $v_M < \infty$ on $(K'_L Y)^c \cap X_M$ and $v_M = \infty$ on $K_L Y \cap X_M$.

For any $M' > 0$, we set $X_{M,M'} = v_M^{-1}([0,M'])$. Let us dominate the measure $\nu_n(X_M \smallsetminus X_{M,M'})$. By Lemma 3.4, for all n, the restriction of ν_n to X_M is P_M-invariant and P_M-ergodic. By assumption, for all but finitely many n, we have $\nu_n(K'_L Y) = 0$. Hence, by Lemma 3.8, we get

$$\int_{X_M} v_M \,\mathrm{d}\nu_n \leq \nu_n(X_M)\frac{C_M}{(1-a_M)} \leq \frac{C_M}{(1-a_M)},$$

and $\nu_n(X_M \smallsetminus X_{M,M'}) \leq \frac{C_M}{(1-a_M)M'}$.

Since $X_{M,M'}$ is compact, using (4.1) and letting n go to ∞, this gives $\nu_\infty(X_{M,M'}) \geq 1-\varepsilon-\frac{C_M}{(1-a_M)M'}$ thus, since M' is arbitrary, $\nu_\infty(K_L Y \cap X_M) \leq \varepsilon$. Since ε is arbitrarily small and M arbitrarily large, this proves $\nu_\infty(LY) = 0$ as required. □

Proof of Theorem 1.5. Since (b) directly follows from Lemma 4.2, we only have to prove (a). Note that (b) implies $\mathcal{S}_K(\Gamma)$ is closed.

Let (Y_n) be a sequence in $\mathcal{S}_K(\Gamma)$, and let us construct a converging subsequence. First, we can assume the sequence (ν_{Y_n}) converges to a measure ν_∞. By Lemma 4.2(a), ν_∞ is a probability measure. Since ν_∞ is Γ-invariant, by [4, Th. 2.5], every Γ-ergodic component of ν_∞ is equal to ν_Y for some Y in $\mathcal{S}_X(\Gamma)$; hence, as by Proposition 2.1, $\mathcal{S}_X(\Gamma)$ is a countable union of L-orbits, there exists Y_∞ in $\mathcal{S}_X(\Gamma)$ such that $\nu_\infty(LY_\infty) > 0$. By Lemma 4.2.(b), there exists a compact subset K_L of L such that, for large n, one has $Y_n \subset K_L Y_\infty$.

Assume the dimension of Y_∞ is minimal, and let us prove $\nu_\infty = \nu_{\ell Y_\infty}$ for some ℓ in K_L.

Indeed, for n large, since Y_n is Γ-ergodic, there exists ℓ_n in K_L such that $Y_n \subset \ell_n Y_\infty$. After again extracting and replacing Y_∞ by a translate, we can assume $\ell_n \xrightarrow[n\to\infty]{} e$. Since $\nu_{Y_n} - \nu_{\ell_n^{-1}Y_n} \xrightarrow[n\to\infty]{} 0$, we can assume, for all n, one has $\ell_n = e$; that is, $Y_n \subset Y_\infty$. Now, still by [4, Th. 2.5], every Γ-ergodic component of ν_∞ is equal to ν_Y for some Y in $\mathcal{S}_X(\Gamma)$, $Y \subset Y_\infty$. But, by assumption and by Lemma 4.2.(b), if $Y \subsetneq Y_\infty$, one has $\nu_\infty(LY) = 0$. Hence, again by Proposition 2.1, we have $\nu_\infty = \nu_{Y_\infty}$, what should be proved. □

4.3. *Closed invariant subsets.*

Proof of Theorem 1.7(a) *and* (b). Note that (b) follows directly from Theorem 1.5.(b) and the fact that L is now assumed to be discrete.

Let us prove (a); that is, the set $\mathcal{S}_X(\Gamma)$ is compact. By Theorem 1.5(a), it suffices to construct a compact subset K of X such that, for every x in X, one has $\Gamma x \cap K \neq \emptyset$. Now, fix a compactly supported Borel probability measure μ on Γ whose support spans Γ. By [3, Th. 7.2.(b)] (see also [10] and [3, Th. 1.4] in the real case), there exists a compact subset K of X such that, for any x in X, for n large enough, one has $\mu^{*n} * \delta_x(K) \geq 1/2$ and we are done. □

To prove Theorem 1.7(c), we need the following complement to Lemma 2.9.

Lemma 4.3. *Let* G, Λ, X *and* Γ *be as in Theorem* 1.4. *If the centralizer of* Γ *in* G *is discrete, then the set* $\mathcal{S}_{\mathrm{op}}(\Gamma)$ *is a cover of* X *by finitely many disjoint open sets.*

Proof. We only have to check that the set $\mathcal{S}_{\mathrm{op}}(\Gamma)$ is finite and that one has $X = \bigcup_{Y\in\mathcal{S}_{\mathrm{op}}(\Gamma)} Y$. Assume there exists a sequence (Y_n) of distinct elements of $\mathcal{S}_{\mathrm{op}}(\Gamma)$. Since, by Theorem 1.7(a), $\mathcal{S}_X(\Gamma)$ is compact, we can assume (Y_n) converges to some Y in $\mathcal{S}_X(\Gamma)$. Now, by Theorem 1.7.(b), we get, for large n, $Y_n \subset Y$. Hence, since Y_n is open and Y is Γ-ergodic, $Y_n = Y$, which is a contradiction. Therefore, $\mathcal{S}_{\mathrm{op}}(\Gamma)$ is finite.

In particular, the set $X \smallsetminus \bigcup_{Y\in\mathcal{S}_{\mathrm{op}}(\Gamma)} Y$ is open in X. By Theorem 1.4, we have $X = \bigcup_{Y\in\mathcal{S}_X(\Gamma)} Y$. Since, by Proposition 2.1, the set $\mathcal{S}_X(\Gamma)$ is countable, we have

$$\nu_X(\textstyle\bigcup_{Y\notin\mathcal{S}_{\mathrm{op}}(\Gamma)} Y) = 0.$$

Thus, the set $X \smallsetminus \bigcup_{Y\in\mathcal{S}_{\mathrm{op}}(\Gamma)} Y$ is empty. □

Proof of Theorem 1.7(c). Let F be a closed Γ-invariant subset of X, and let us prove F is a finite union of elements of $\mathcal{S}_X(\Gamma)$. Using Lemma 4.3, we can assume X is Γ-ergodic. We proceed by induction on the dimension of X. If it is zero, there is nothing to prove. Assume it is positive.

If there exists $Y_1, \dots, Y_r$ in $\mathcal{S}_X(\Gamma) \smallsetminus \{X\}$ such that $F \subset Y_1 \cup \cdots \cup Y_r$, then we are done by induction since, for $1 \leq i \leq r$, $F \cap Y_i$ is a closed Γ-invariant subset of Y_i.

Assume this is not the case, and let us prove $F = X$. Let $(Y_n)_{n\geq 1}$ be the elements of $\mathcal{S}_X(\Gamma) \smallsetminus \{X\}$, which is countable by Proposition 2.1. For any $n \geq 1$, pick $x_n \in F \smallsetminus (Y_1 \cup \cdots \cup Y_n)$, and set $Z_n = \overline{\Gamma x_n}$. By Theorem 1.4, we have $Z_n \in \mathcal{S}_X(\Gamma)$. Since, by (a), $\mathcal{S}_X(\Gamma)$ is compact, (Z_n) admits a limit point Z_∞ in $\mathcal{S}_X(\Gamma)$. Now, by construction, for any $Y \neq X$ in $\mathcal{S}_X(\Gamma)$, for all but finitely many n, one has $Z_n \not\subset Y$; hence, by (b), $Z_\infty \not\subset Y$. We get $Z_\infty = X$, hence $F = X$, what should be proved. $\square$

Example 4.4. If L is not discrete, there may exist closed Γ-invariant subsets in X that are not finite unions of sets of the form $K_L Y$, where K_L is a compact subset of L and Y is in $\mathcal{S}_X(\Gamma)$. Here are two examples. We set $G = \mathrm{SL}(3, \mathbb{R})$, $\Lambda = \mathrm{SL}(3, \mathbb{Z})$ and $\Gamma \simeq \mathrm{GL}(2, \mathbb{Z})$ to be the stabilizer of the decomposition $\mathbb{Z}^3 = \mathbb{Z}^2 \times \mathbb{Z}$. Let $x_0 = \mathbb{Z}^3$ be the base point of X and $H \simeq \mathrm{SL}(2, \mathbb{R})$ be the semisimple part of the stabilizer of $\mathbb{R}^2 \times \{0\}$ in G, $L = \{\ell_t = \mathrm{diag}(t, t, t^{-2}) \mid t \neq 0\}$. We set $Y_\infty = Hx_0$, (Y_n) to be the sequence of all finite Γ-orbits in Y_∞ and $(t_n) \subset (1, \infty)$ a sequence converging toward $t_\infty = 1$. Then $F_1 = LY_\infty$ and $F_2 = Y_\infty \cup \bigcup_n \ell_{t_n} Y_n$ are closed Γ-invariant subsets.

Appendix A. Lattices in *S*-adic Lie groups

We now aim at proving Proposition 2.1 for general weakly regular S-adic Lie groups. In order to adapt the strategy we followed in the real case, we will use the notions introduced in [24] and [4, §5].

We will first study lattices in S-adic Lie groups and, in particular, give conditions for them to be finitely generated. The main results of this appendix are Proposition A.1 and Lemma A.9.

A.1. *Finite generation of lattices.* One of the main difficulties for extending the proof of Proposition 2.1 is the fact that some of the lattices we may encounter are not finitely generated. For example, if p is a prime number, the group $\mathbb{Z}[\frac{1}{p}]$ is not finitely generated, although it embeds as a lattice in $\mathbb{R} \times \mathbb{Q}_p$.

In this section, we precisely describe which S-adic Lie groups have finitely generated lattices.

Proposition A.1. *Let G be a compactly generated S-adic Lie group. Any lattice in G is finitely generated.*

In case G is S-algebraic and the lattice is arithmetic, this result is due to Kneser [14]. We follow the same strategy.

Note that, if a locally compact group admits a finitely generated lattice, it is compactly generated. Conversely, any cocompact lattice in a compactly generated locally compact group is finitely generated (see [15, IX.3] or [4, Prop. 5.22]), but this is not true for noncocompact lattices. Indeed, for any prime number power q, the group $\Lambda = \mathrm{SL}(2, \mathbb{F}_q[T^{-1}])$ embeds as a lattice in the compactly generated locally compact group $G = \mathrm{SL}(2, \mathbb{F}_q((T)))$, but Λ is not finitely generated.

Proof. We will show how to deduce this statement from the well-known case where G is a real Lie group. Let G° be the connected component of G. The closure Ω of the image of Λ in G/G° has finite covolume. Since G/G° is a compactly generated non-archimedean S-adic Lie group, by [5, Lemma 5.2], Ω is cocompact in G/G°, Hence it is compactly generated. (See, for instance, [4, Prop. 5.22].) We may thus assume that the group ΛG° is dense in G. We conclude thanks to Lemma A.2 below. □

LEMMA A.2. *Let G be a compactly generated locally compact group, G° its connected component and Λ be a lattice in G such that the group ΛG° is dense in G. Then the group Λ is finitely generated.*

Proof. By Montgomery-Zippin theorem in [18], there exist an open subgroup H of G and a compact normal subgroup K of H such that H/K is a connected real Lie group. (Note that if G is an S-adic Lie group, the existence of H and K does not rely on [18].) The group $\Lambda \cap H/\Lambda \cap K$ is a lattice in H/K. Hence, by [15, IX.3], $\Lambda \cap H$ is finitely generated.

Recall that if G' is a topological group, U' an open subset of G' that spans G' and Λ' a dense subgroup of G', then Λ' is spanned by $\Lambda' \cap U'$; indeed, the group spanned by $\Lambda' \cap U'$ is dense in Λ' for the induced topology.

Now, since the group $G' := G/G^\circ$ is compactly generated and since the image H' of H in G' is open, there exists a finite set F' in G' such that $F'H'$ generates G'. Since the image Λ' of Λ in the group G' is a dense subgroup, we may assume F' is contained in Λ'. Then, Λ' is generated by F' and $\Lambda' \cap H'$. Therefore, Λ' is finitely generated and so is Λ, since $\Lambda \cap G^\circ$ is contained in $\Lambda \cap H$. □

A.2. *Structure results*, *after Ratner.* We recall a few definitions and results from [24].

Let G be a weakly regular S-adic Lie group and $\mathfrak{g} = \bigoplus_{p \in S} \mathfrak{g}_p$ be the Lie algebra of G. We let G_∞ denote the connected component of G, which is also the analytic Lie subgroup of G whose Lie algebra is the archimedean part of $\mathfrak{g}$. We also let G_u denote the closure of the subgroup of G spanned by Ad-unipotent one-parameter subgroups of G and $G_{u,f}$ denote the closure of the subgroup of G spanned by Ad-unipotent one-parameter subgroups of G with derivative in $\mathfrak{g}_f = \bigoplus_{p \neq \infty} \mathfrak{g}_p$. The groups G_∞, G_u and $G_{u,f}$ are normal in G.

A difficulty in the study of S-adic Lie groups is that there is, in general, no normal open subgroup that plays the role of the connected component of real Lie groups. We may, however, define a weak analogue notion. A *standard* open subset Ω is a product of a small open neighborhood Ω_∞ of e in G_∞ and of a standard compact subgroup Ω_f of G with Lie algebra $\mathfrak{g}_f$ (See [4, §5.1] and note that the archimedean factor. Ω_∞ will play no role in the sequel.) For any closed subgroup H of G, we define the Ω-*semiconnected component of* H as its open subgroup $H_\Omega := (H \cap \Omega)H_{u,f}H_\infty$. The group H is said to be Ω-*semiconnected* if $H = H_\Omega$. The group H is said to be *semiconnected* if there exists a standard open subset Ω such that H is Ω-semiconnected. Note that semiconnected components are not necessarily normal subgroups as, for instance, if $G = \mathbb{Z}_p \ltimes (\oplus_{n\geq 1}\mathbb{Z}^{\mathbb{Z}/p^n\mathbb{Z}})$, where p is a prime number and the group $\mathbb{Z}_p$ of p-adic integers acts by translations on each $\mathbb{Z}/p^n\mathbb{Z}$.

Let us now focus on a particular class of S-adic Lie groups. For p in S, a p-adic Lie group N is said to be *algebraic unipotent* if it is isomorphic to the group of $\mathbb{Q}_p$-points of a unipotent $\mathbb{Q}_p$-group. These groups are extensively studied in [24, Sect.2] (where they are called *quasiconnected*).

If $p = \infty$, N is algebraic unipotent if and only if it is connected, simply connected and nilpotent. Then, the exponential map is a diffeomorphism $\mathfrak{n} \to N$, whose inverse is denoted by log.

If $p < \infty$, by [24, Prop. 2.1], N is algebraic unipotent if and only if it is weakly regular, spanned by Ad-unipotent one-parameter subgroups and has nilpotent Lie algebra. Then, for any g in N, the morphism $\mathbb{Z} \to N; n \mapsto g^n$ extends as a continuous morphism $\mathbb{Z}_p \to N$, whose derivative is denoted by $\log(g)$. The map $\log : N \to \mathfrak{n}$ is an analytic homeomorphism whose inverse is denoted by exp.

In any case, exp and log are $\mathrm{Aut}(N)$-equivariant and the map $\mathfrak{n} \times \mathfrak{n} \to \mathfrak{n}$; $(X, Y) \mapsto \log(\exp X \exp Y)$ is polynomial (and is therefore given by the Baker-Campbell-Hausdorff formula). In particular, if N' is a closed normal subgroup of N that is spanned by one-parameter subgroups, then N/N' is also an algebraic unipotent group.

An S-adic Lie group is said to be *algebraic unipotent* if it is a product of algebraic unipotent p-adic Lie groups, $p \in S$.

Let G be any weakly regular S-adic Lie group. Then, the group $G_{u,f}$ admits a *Levi decomposition*. Indeed this group $G_{u,f}$ is also Ad-*regular*, i.e., its center is equal to the Kernel of the adjoint representation, and Ratner has proven in [24, Cor. 2.1] that such a group always has a Levi decomposition. More precisely, let the *solvable radical* $R_{u,f}$ of $G_{u,f}$ be the closure of the subgroup of $G_{u,f}$ spanned by the Ad-unipotent one-parameter subgroups tangent to the solvable radical $\mathfrak{r}_{u,f}$ of $\mathfrak{g}_{u,f}$. We define a Levi subgroup $S_{u,f}$ of $G_{u,f}$ as being the closure of the subgroup of $G_{u,f}$ spanned by the Ad-unipotent one-parameter subgroups tangent to a given Levi subalgebra (i.e., a maximal semisimple Lie subalgebra) $\mathfrak{s}_{u,f}$ of $\mathfrak{g}_{u,f}$. By [21] and [24, Cor. 2.1], the group $S_{u,f}$ has Lie algebra $\mathfrak{s}_{u,f}$, the center of $S_{u,f}$ is finite, the group $R_{u,f}$ has Lie algebra $\mathfrak{r}_{u,f}$, is algebraic unipotent and one has

$$G_{u,f} = S_{u,f} R_{u,f}.$$

In particular, one has $S_{u,f} \cap R_{u,f} = \{e\}$ and the product map $S_{u,f} \times R_{u,f} \to G_{u,f}$ is a homeomorphism. It follows that every ad-nilpotent element in $\mathfrak{g}_{u,f}$ is tangent to an Ad-unipotent one-parameter subgroup in $G_{u,f}$.

A.3. *Unstable subgroups.* We will now give conditions for some weakly regular S-adic Lie groups to be compactly generated.

Let still G be a weakly regular S-adic Lie group with Lie algebra $\mathfrak{g}$, and let Γ be a closed subgroup of G. An element v of $\mathfrak{g}$ is said to be Γ-*unstable* if 0 belongs to the closure of the orbit $\operatorname{Ad}\Gamma(v)$. A one-parameter subgroup of G is said to be Γ-unstable if its derivative is a Γ-unstable vector. (Such a one-parameter subgroup is necessarily Ad-unipotent.) Let H be a closed subgroup of G with Lie algebra $\mathfrak{h}$ that is normalized by Γ. We will make repeated use of the following fact from [4, Lemma 5.12]:

(A.1) *if $\overline{\operatorname{Ad}\Gamma}^{\mathrm{Z}}$ is semisimple, any Γ-unstable vector in $\mathfrak{h}$ is tangent to a Γ-unstable one-parameter subgroup of H.*

LEMMA A.3. *Let G be a weakly regular S-adic Lie group, Γ be a closed subgroup of G such that $\overline{\operatorname{Ad}\Gamma}^{\mathrm{Z}}$ is semisimple and H be a closed subgroup of G that is normalized by Γ. The following are equivalent*:

(i) *The group $H_{u,f}$ is topologically spanned by Γ-unstable one-parameter subgroups.*
(ii) *The Lie algebra $\mathfrak{h}_{u,f}$ of $H_{u,f}$ is spanned by Γ-unstable vectors.*

Definition A.4. If either of the two properties of Lemma A.3 holds, we shall say H is Γ-unstable.

Note that if G is a quotient of the real Lie group $\widetilde{\mathrm{SL}(2,\mathbb{R})} \times \mathbb{T}$ by a discrete central subgroup whose projection on $\mathbb{T}$ is dense, then G is spanned topologically by G-unstable one-parameter subgroups, but its Lie algebra is not spanned by G-unstable vectors.

Proof of Lemma A.3. (ii)$\Rightarrow$(i) If $\mathfrak{h}_{u,f}$ is spanned by Γ-unstable vectors, by (A.1), the closure H' of the subgroup of $H_{u,f}$ spanned by Γ-unstable one-parameter subgroups has Lie algebra $\mathfrak{h}_{u,f}$. By [24, Cor. 2.1] (note that the S-adic group H' is also Ad-regular), every ad-nilpotent vector in $\mathfrak{h}_{u,f}$ is tangent to some Ad-unipotent one-parameter subgroup in H'; that is, $H' = H_{u,f}$.

(i)⇒(ii) Conversely, assume $H_{u,f}$ is topologically spanned by Γ-unstable one-parameter subgroups. Let $\mathfrak{r}_{u,f}$ be the radical of $\mathfrak{h}_{u,f}$ and set $\mathfrak{j} = \mathfrak{h}_{u,f}/\mathfrak{r}_{u,f}$. We will first prove that the Lie algebra $\mathfrak{j}$ is spanned by Γ-unstable vectors. Let $\mathfrak{l}$ be the subalgebra of those v in $\mathfrak{j}$ such that $\overline{\Gamma v}$ is a bounded subset of $\mathfrak{j}$ and $\mathfrak{j}_1$ be the subalgebra of $\mathfrak{j}$ spanned by Γ-unstable vectors, so that, since $\overline{\mathrm{Ad}\,\Gamma}^{\mathrm{Z}}$ is semisimple, we get $\mathfrak{j} = \mathfrak{l} + \mathfrak{j}_1$. As $[\mathfrak{l}, \mathfrak{j}_1] \subset \mathfrak{j}_1$, the subalgebra $\mathfrak{j}_1$ is an ideal of $\mathfrak{j}$. By construction, Γ has no unstable vector in $\mathfrak{j}/\mathfrak{j}_1$, hence, $H_{u,f}$ being topologically spanned by Γ-unstable one-parameter subgroups, the adjoint map $H_{u,f} \to \mathrm{Aut}(\mathfrak{j}/\mathfrak{j}_1)$ is trivial. Since $\mathfrak{j}/\mathfrak{j}_1$ is semisimple, we get $\mathfrak{j}_1 = \mathfrak{j}$; that is, $\mathfrak{j}$ is spanned as a Lie algebra by Γ-unstable vectors.

Now, as $\overline{\mathrm{Ad}\,\Gamma}^{\mathrm{Z}}$ is semisimple, the Lie algebra $\mathfrak{h}_{u,f}$ admits a Levi factor $\mathfrak{s}_{u,f}$ that is Γ-invariant and, since $\mathfrak{s}_{u,f}$ is Γ-isomorphic to $\mathfrak{j}$, it is spanned by Γ-unstable vectors. Let $\mathfrak{h}'$ be the subalgebra of $\mathfrak{h}_{u,f}$ spanned by Γ-unstable vectors and $\mathfrak{r}' = \mathfrak{h}' \cap \mathfrak{r}_{u,f}$. Since $\mathfrak{s}_{u,f} \subset \mathfrak{h}'$, we get $[\mathfrak{s}_{u,f}, \mathfrak{r}'] \subset \mathfrak{r}'$. Let $H_{u,f} = S_{u,f} R_{u,f}$ be the Levi decomposition of $H_{u,f}$ associated to the Levi decomposition $\mathfrak{h}_{u,f} = \mathfrak{s}_{u,f} \oplus \mathfrak{r}_{u,f}$ and $R' = \exp(\mathfrak{r}')$ be the unique algebraic unipotent subgroup of $R_{u,f}$ with Lie algebra $\mathfrak{r}'$. Then $S_{u,f}$ normalizes R' and hence $S_{u,f}R'$ is a closed subgroup in $H_{u,f}$. Now, by construction, every Γ-unstable one-parameter subgroup in $H_{u,f}$ is contained in $S_{u,f}R'$; that is, $H_{u,f} = S_{u,f}R'$, hence $\mathfrak{h}_{u,f} = \mathfrak{h}'$, what should be proved. □

We now get a criterion to ensure some of the groups we will encounter have finitely generated lattices.

LEMMA A.5. *Let G be a weakly regular S-adic Lie group and Γ be a closed compactly generated subgroup of G such that $\overline{\mathrm{Ad}\,\Gamma}^{\mathrm{Z}}$ is semisimple. Assume G admits a normal Γ-unstable semiconnected subgroup H such that $G = \Gamma H$. Then G is compactly generated. In particular, any lattice in G is finitely generated.*

Proof. Let v_i, $1 \le i \le \ell$, be Γ-unstable vectors that span the Lie algebra $\mathfrak{h}_{u,f}$. By (A.1), the lines they generate are tangent to one-parameter subgroups V_i of $H_{u,f}$. The group H' spanned by $V_1 \cup \cdots \cup V_\ell$. is open and hence closed in $H_{u,f}$. By [24, Cor. 2.1] (note that the S-adic group H' is also Ad-regular), every ad-nilpotent vector in $\mathfrak{h}_{u,f}$ is tangent to some Ad-unipotent one-parameter subgroup in H', that is $H' = H_{u,f}$. By construction, the group generated by Γ and $V_1, \ldots, V_\ell$ is compactly generated. Since this group is equal to $\Gamma H_{u,f}$, since H is a compact extension of $H_{u,f} H_\infty$, and since H_∞ is compactly generated, the group $G = \Gamma H$ is compactly generated. The property on lattices follows by Proposition A.1. □

A.4. *Cosolvable radicals.* We will construct large compactly generated subgroups in weakly regular semiconnected S-adic Lie groups.

Let $\mathfrak{g}$ be a Lie algebra. We shall say an ideal $\mathfrak{h}$ of $\mathfrak{g}$ is *cosolvable* if the algebra $\mathfrak{g}/\mathfrak{h}$ is solvable. As the intersection of any family of cosolvable ideals of $\mathfrak{g}$ still is a cosolvable ideal, $\mathfrak{g}$ admits a smallest cosolvable ideal $\mathfrak{c}$. We say $\mathfrak{c}$ is the *cosolvable radical* of $\mathfrak{g}$.

LEMMA A.6. *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{s}$ be a Levi subalgebra of $\mathfrak{g}$. Then the cosolvable radical of $\mathfrak{g}$ is the subalgebra of $\mathfrak{g}$ spanned by $[\mathfrak{s}, \mathfrak{g}]$.*

Proof. Let $\mathfrak{c}$ be the cosolvable radical of $\mathfrak{g}$, $\mathfrak{c}'$ be the subalgebra spanned by $[\mathfrak{s}, \mathfrak{g}]$ and $\mathfrak{l}$ be the centralizer of $\mathfrak{s}$. Since $\mathfrak{s} \subset \mathfrak{c}$, we have $\mathfrak{c}' \subset \mathfrak{c}$. Now, as $\mathfrak{s}$ is semisimple, we have $\mathfrak{g} = \mathfrak{l} \oplus [\mathfrak{s}, \mathfrak{g}]$. As $\mathfrak{l}$ normalizes $[\mathfrak{s}, \mathfrak{g}]$, it normalizes $\mathfrak{c}'$, so that $\mathfrak{c}'$ is an ideal. As $\mathfrak{s} \subset \mathfrak{c}'$, $\mathfrak{c}'$ is cosolvable, that is $\mathfrak{c} \subset \mathfrak{c}'$, and we are done. $\square$

For a weakly regular S-adic Lie group G, we define the cosolvable radical C_u of G_u as the closure of the subgroup of G_u spanned by the Ad-unipotent one-parameter subgroups tangent to the *cosolvable radical* $\mathfrak{c}_u$ of $\mathfrak{g}_u$.

LEMMA A.7. *Let G be a weakly regular S-adic Lie group with Lie algebra $\mathfrak{g}$, Ω be a standard open subset of G, $\mathfrak{c}_u$ be the cosolvable radical of $\mathfrak{g}_u$ and C_u be the cosolvable radical of G_u.*

(a) *The group $C_{u,f}$ has Lie algebra $\mathfrak{c}_{u,f}$, and the group $N_u := G_{u,f}/C_{u,f}$ is algebraic unipotent.*
(b) *The group $C_u G_\infty$ is compactly generated; it is the largest compactly generated closed subgroup H of G with $H = H_u H_\infty$.*
(c) *The group $\Omega C_u G_\infty$ is the largest compactly generated Ω-semiconnected subgroup of G.*

Proof. (a) Let $\mathfrak{r}_{u,f}$ be the solvable radical of $\mathfrak{g}_{u,f}$, $R_{u,f}$ be the solvable radical of $G_{u,f}$, $\mathfrak{s}_{u,f}$ be a Levi subalgebra of $\mathfrak{g}_{u,f}$ and $S_{u,f}$ be the associated Levi subgroup of $G_{u,f}$. By construction, the Lie algebra $\mathfrak{s}_{u,f}$ has no anisotropic factor and the group $\mathrm{Ad}(S_{u,f})$ has finite index in the connected algebraic subgroup of $\mathrm{GL}(\mathfrak{g}_{u,f})$ with Lie algebra $\mathfrak{s}_{u,f}$, so that $[\mathfrak{s}_{u,f}, \mathfrak{g}_{u,f}]$ is exactly the subspace spanned by $S_{u,f}$-unstable vectors in $\mathfrak{g}_{u,f}$ and, by Lemma A.6, $\mathfrak{c}_{u,f}$ is the subalgebra spanned by $S_{u,f}$-unstable vectors in $\mathfrak{g}_{u,f}$.

By (A.1), every such vector is tangent to a $S_{u,f}$-unstable one-parameter subgroup. Hence, by Lemma A.3, the group $C_{u,f}$ has Lie algebra $\mathfrak{c}_{u,f}$. In other terms, if $\mathfrak{r}_1 = \mathfrak{c}_{u,f} \cap \mathfrak{r}_{u,f}$, we have $C_{u,f} = S_{u,f} R_1$, where $R_1 = \exp \mathfrak{r}_1$ is the unique algebraic unipotent subgroup of $R_{u,f}$ with Lie algebra $\mathfrak{r}_1$. In particular, N_u is algebraic unipotent since it is isomorphic to $R_{u,f}/R_1$, and (a) is proved.

(b) There exists $v_1, \ldots, v_r$ in $\mathfrak{c}_{u,f}$ that span $\mathfrak{c}_{u,f}$ as a Lie algebra and are eigenvectors associated to eigenvalues with modulus < 1 of some $\gamma_1, \ldots, \gamma_r$ in $S_{u,f}$. Therefore, if Γ denotes the closure of the subgroup of $S_{u,f}$ spanned by $\gamma_1, \ldots, \gamma_r$, the group $C_{u,f}$ is Γ-unstable. By enlarging Γ, we can assume $\mathrm{Ad}(\Gamma)$

to be Zariski dense in $\mathrm{Ad}(S_{u,f})$. Thus, by Lemma A.5, $C_{u,f}G_\infty = C_uG_\infty$ is compactly generated.

Conversely, let H be a compactly generated closed subgroup of G with $H = H_uH_\infty$, so that H is contained in G_uG_∞. Since the closure of HC_uG_∞ is still compactly generated, we may assume $C_uG_\infty \subset H$.

We recall that in a non-archimedean algebraic unipotent group, every compactly generated subgroup is compact and every one-parameter subgroup has closed noncompact image (see [24, Prop. 2.1]). Now, since the quotient group G_uG_∞/C_uG_∞ is isomorphic to N_u, the image of H in N_u is trivial as required. We have proved (b).

(c) follows easily from (b). $\square$

We can now characterize the compactly generated weakly regular semiconnected S-adic Lie groups.

Corollary A.8. *Let G be weakly regular semiconnected S-adic Lie group and $\mathfrak{g}_{u,f}$ be the Lie algebra of the group $G_{u,f}$. The group G is compactly generated if and only if $\mathfrak{g}_{u,f} = [\mathfrak{g}_{u,f}, \mathfrak{g}_{u,f}]$.*

Proof. By Lemma A.7, one has the following equivalences: G is compactly generated $\iff$ $G_{u,f}$ equals its cosolvable radical $\iff$ the Lie algebra $\mathfrak{g}_{u,f}$ has no nonzero solvable quotient $\iff$ $\mathfrak{g}_{u,f} = [\mathfrak{g}_{u,f}, \mathfrak{g}_{u,f}]$. $\square$

A.5. *Finitely generated subgroups of lattices.* The proof of Proposition 2.1 for an S-adic Lie group G will rely on studying sets of subgroups of G that play the same role as the sets $\mathcal{T}(G, \Delta, \Sigma)$ in the real case. Thanks to Lemma A.5, we will know that the relevant subgroups Σ will be finitely generated and hence will vary in a countable set. But the groups Δ will be lattices in semiconnected groups and might not be finitely generated.

In this section we develop tools for overcoming this difficulty. Indeed, given Δ, we will exhibit a subgroup Δ' of Δ that is finitely generated and such that the group Δ is spanned by all the conjugates of Δ' under the elements of Σ. This subgroup Δ' will be constructed as the intersection of Δ with some large semiconnected compactly generated open subgroup as in Lemma A.7.

Lemma A.9. *Let H be a weakly regular S-adic Lie group, Ω be a standard open subset of H and Γ be a compactly generated subgroup of H such that $\overline{\mathrm{Ad}\,\Gamma}^{\mathrm{Z}}$ is semisimple and equal to $\overline{\mathrm{Ad}\,\Gamma}^{\mathrm{Z,nc}}$. Let Σ be a lattice in H such that Γ acts ergodically on H/Σ. Assume the Ω-semiconnected component H_Ω of H is a normal Γ-unstable subgroup and $H = \Gamma H_\Omega$. Set H' to be the largest compactly generated Ω-semiconnected subgroup of H, $\Delta = \Sigma \cap H_\Omega$ and $\Delta' = \Delta \cap H'$. Then Δ' is finitely generated and Δ is the smallest normal subgroup of Σ containing Δ'.*

The reader can keep in mind the example where $H = \Gamma \ltimes H_\Omega$ and $\Sigma = \Gamma \ltimes \Delta$ with $\Gamma = \mathrm{SL}(d, \mathbb{Z}[\frac{1}{p}])$ acting diagonally on the group $H_\Omega = \mathbb{Q}_p^d \times \mathbb{R}^d$ and on its lattice $\Delta = \mathbb{Z}[\frac{1}{p}]^d$ (for a prime number p). In this case, one can set $H' = \mathbb{Z}_p^d \times \mathbb{R}^d$ and $\Delta' = \mathbb{Z}^d$.

A key step for the proof is the following

LEMMA A.10. *Let G be a weakly regular S-adic Lie group and C_u be the cosolvable radical of G_u. Then, if Λ is a finite covolume subgroup of G, the group $\overline{C_u G_\infty \Lambda}$ contains G_u.*

Proof. By [4, Lemma 5.27] and [5, Lemma 5.2], this statement is true when the group G_u is nilpotent. We will reduce the general case to this one.

Indeed, after replacing G by a semiconnected component, we may assume G is semiconnected. By Lemma A.7, the group $N_u := G_{u,f}/C_{u,f}$ is algebraic unipotent. Since the quotient $N := G/C_u G_\infty$ is a compact extension of N_u, by [24, Prop. 1.2] every one-parameter subgroup of N is contained in N_u; hence N is weakly regular. After replacing G by N, we may assume G_u is algebraic unipotent and apply [4, Lemma 5.27]. $\square$

Lemma A.10 yields the following extension of [4, Lemma 5.28].

COROLLARY A.11. *Let G be a weakly regular S-adic Lie group and Ω be a standard open subset of G. Assume G is Ω-semiconnected, and let G' be the largest compactly generated Ω-semiconnected subgroup of G. Then, if Λ is a lattice in G, one has $G = G'\Lambda$.*

Proof. Let C_u be the cosolvable radical of G_u. By Lemma A.7, one has $G' = \Omega C_u G_\infty$. As G' is open, the set $G'\Lambda$ is closed. As it contains $\overline{C_u G_\infty \Lambda}$, by Lemma A.10, it contains G_u. The result follows since $G = G'G_u$. $\square$

We will also need the following

LEMMA A.12. *Let M be a closed subgroup of an algebraic unipotent p-adic Lie group N with $p < \infty$. Then the group M/M_u is compact.*

Proof. Replacing N with the quotient of the normalizer of M_u by M_u, we can assume that $M_u = \{e\}$, i.e., that M contains no one-parameter subgroups. We want to prove then that M is compact. This follows from the fact that the exponential map is an analytic isomorphism between $\mathfrak{n}$ and N ([24, Prop. 2.1]). $\square$

We can now give the

Proof of Lemma A.9. Let C_u be the cosolvable radical of H_u. By Lemma A.7, one has $H' = \Omega C_u H_\infty$. Since H' is compactly generated, by Proposition A.1, its lattice Δ' is finitely generated.

Let Δ'' be a subgroup of Δ that contains Δ' and is normal in Σ: we have to prove $\Delta'' = \Delta$. To do this, we will prove one has $H_u \subset \overline{C_u H_\infty \Delta''}$, as in

Lemma A.10, and infer, as in Corollary A.11, that this gives $H_\Omega = H'\Delta''$, which in turn yields $\Delta'' = \Delta$.

Let us therefore study the group $\overline{C_u H_\infty \Delta''}$. By Lemma A.7, the quotient group $N_u := H_u H_\infty / C_u H_\infty$ is algebraic unipotent. We denote by M the image of $\overline{C_u H_\infty \Delta''} \cap H_u H_\infty$ in N_u. We want to prove $M = N_u$.

First note that, by Lemma A.10, the group $\overline{C_u H_\infty \Delta}$ contains H_u. Thus, since H' is open in H and contains $C_u H_\infty$, the group $\overline{C_u H_\infty \Delta'}$ contains $H' \cap H_u$ and the group M is open in N_u. Let $\mathfrak{m}_u$ and $\mathfrak{n}_u$ respectively be the Lie algebras of M_u and N_u. As M is normalized by Σ, the map

$$H \to \mathrm{Gr}(\mathfrak{n}_u); h \mapsto \mathrm{Ad}\, h(\mathfrak{m}_u)$$

factors as an H-equivariant continuous map $H/\Sigma \to \mathrm{Gr}(\mathfrak{n}_u)$. As H/Σ is Γ-ergodic, by Lemma 2.3, this map is constant; that is, $\mathfrak{m}_u$ is an H-invariant ideal of $\mathfrak{n}_u$ and M_u is an H-invariant subgroup of N_u.

Since the S-adic Lie group N_u is algebraic unipotent and non-archimedean, by [24, Prop. 2.1], its closed subgroup M is a compact extension of M_u. Hence $P = \log(M/M_u)$ is a ΣH_∞-invariant compact open subset of $\mathfrak{n}_u/\mathfrak{m}_u$. Now, $\overline{\Sigma H_\infty}/H_\infty$ is a finite covolume subgroup in the non-archimedean S-adic Lie group H/H_∞; hence by [5, Prop. 5.1], the space $H/\overline{\Sigma H_\infty}$ is compact and the set $Q = \bigcup_{h\in H} hP$ is a compact open H-invariant subset of $\mathfrak{n}_u/\mathfrak{m}_u$. As Q is Γ-invariant, $\mathfrak{n}_u/\mathfrak{m}_u$ contains no Γ-unstable vector. On the other hand, since H_Ω is Γ-unstable, the Lie algebra $\mathfrak{n}_u$ is spanned by Γ-unstable vectors and so is $\mathfrak{n}_u/\mathfrak{m}_u$. This proves $\mathfrak{m}_u = \mathfrak{n}_u$; hence $M = N_u$ as required.

Since H' is open in H, this gives $H_u \subset H'\Delta''$; hence $H_\Omega = H'\Delta''$. As $\Delta'' \subset \Delta$ and $(\Delta \cap H') \subset \Delta''$, we get $\Delta = \Delta''$. □

Appendix B. Countability of invariant subspaces in the S-adic case

We now start the proof of Proposition 2.1. We will first prove an analogue of Lemma 2.5.

B.1. *Well-shaped compact open subgroups.* One of the difficulties we encounter is to extend Lemma 2.6. Indeed, if G is a weakly regular S-adic Lie group and H_1 and H_2 are subgroups of G, if H_1 normalizes the Lie algebra of H_2, there is no reason for H_1 to normalize a semiconnected component of H_2, as, for instance, when $G = H_1 = \mathrm{SL}(d, \mathbb{Q}_p)$ and $H_2 = \mathrm{SL}(d, \mathbb{Z}_p)$. In this section, we explain how to chose semiconnected components carefully in order to ensure properties of this kind to hold.

Let Γ be a subgroup of G such that $\overline{\mathrm{Ad}\,\Gamma}^{\mathrm{Z}}$ is semisimple and equal to $\overline{\mathrm{Ad}\,\Gamma}^{\mathrm{Z,nc}}$, and let $\mathfrak{l}$ be the centralizer of Γ in $\mathfrak{g}$. Let Ω be a standard open subset of G with exponential map $\exp_\Omega : O \to \Omega$.

As in [4, §5], we will say Ω is Γ-*good* if, for every v in O and γ in Γ with $\operatorname{Ad}\gamma(v)$ in O, one has

$$\exp_\Omega(\operatorname{Ad}\gamma(v)) = \gamma\exp_\Omega(v)\gamma^{-1}.$$

We will say Ω is Γ-*well shaped* if it is Γ-good and if, for every Lie subalgebra $\mathfrak{h}$ in $\mathfrak{g}$ that is normalized by Γ, setting $H_{\Gamma,u}$ to be the closure of the subgroup of G spanned by Γ-unstable one-parameter subgroups in $\mathfrak{h}$, one has

$$\exp_\Omega(\mathfrak{h}\cap O) \subset \exp_\Omega(\mathfrak{h}\cap\mathfrak{l}\cap O)H_{\Gamma,u}H_\infty.$$

Example B.1. There may exist Γ-good standard open subsets that are not Γ-well shaped. For instance, fix a prime number p, and set

$$G = \Gamma = \mathbb{Z}_p \times \mathrm{SL}(2,\mathbb{Q}_p),$$

$$\Omega = \{(t,g)\in\mathbb{Z}_p\times\mathrm{SL}(2,\mathbb{Z}_p) \mid g \equiv \begin{pmatrix} 1 & tp \\ 0 & 1\end{pmatrix} \bmod p^2\}.$$

One easily checks Ω is not contained in $\exp_\Omega(\mathfrak{l}\cap O)G_{\Gamma,u} = (p\mathbb{Z}_p)\times\mathrm{SL}(2,\mathbb{Q}_p)$.

Lemma B.2. *Let G be a weakly regular S-adic Lie group and Γ be a compactly generated subgroup of G such that $\overline{\operatorname{Ad}\Gamma}^{\mathrm{Z}}$ is semisimple and equal to $\overline{\operatorname{Ad}\Gamma}^{\mathrm{Z,nc}}$. Then G admits arbitrarily small Γ-well shaped standard open subsets.*

Proof. It suffices to deal with the case where $S=\{p\}$ for some $p<\infty$. By [4, Prop. 5.11], there exists a Γ-good standard open subset Ω. Let $\mathfrak{l}$ be the centralizer of Γ in $\mathfrak{g}$ and $\mathfrak{v}$ be the Γ-invariant complement of $\mathfrak{l}$. As the set of Γ-invariant subspaces of $\mathfrak{v}$ is compact, as each of these subspaces is spanned by Γ-unstable vectors and as two nearby such $\mathfrak{v}$'s are Γ-isomorphic, there exists an open subset $O'\subset O$ such that, for every Γ-invariant subspace $\mathfrak{w}$ of $\mathfrak{v}$, the sub-$\mathbb{Z}_p$-module spanned by the Γ-unstable vectors in $\mathfrak{w}\cap O$ contains $\mathfrak{w}\cap O'$. By shrinking O', we can assume it is arbitrarily small, it is a Lie sub-$\mathbb{Z}_p$-algebra and one has $O' = (\mathfrak{l}\cap O')\oplus(\mathfrak{v}\cap O')$. We set $\Omega' = \exp_\Omega(O')$. Let us prove that Ω' is a Γ-well shaped standard open subset of G.

Let $\mathfrak{h}$ be a Γ-invariant subalgebra of $\mathfrak{g}$, and set $\mathfrak{w} = \mathfrak{h}\cap\mathfrak{v}$. We have $\mathfrak{h}\cap O' = (\mathfrak{h}\cap\mathfrak{l}\cap O')\oplus(\mathfrak{w}\cap O')$; hence, if O' is small enough,

$$\exp_\Omega(\mathfrak{h}\cap O') = \exp_\Omega(\mathfrak{h}\cap\mathfrak{l}\cap O')\exp_\Omega(\mathfrak{w}\cap O'),$$

so that we just have to prove

$$\exp_\Omega(\mathfrak{w}\cap O')\subset H_{\Gamma,u},$$

where $H_{\Gamma,u}$ is defined as above. Indeed, let $P = \exp_\Omega^{-1}(H_{\Gamma,u}\cap\Omega)$. Then P is a sub-$\mathbb{Z}_p$-module of O and, since Ω is Γ-good, by (A.1), P contains every Γ-unstable element in $\mathfrak{w}\cap\Omega$. Hence, P contains $\mathfrak{w}\cap O'$, what should be proved. $\square$

B.2. *Subgroups with a given lattice.* Let us now introduce the set of subgroups of G that will play the same role in the general case as the one played by the set $\mathcal{T}(G,\Delta,\Sigma)$ in the real case.

Let $\Delta \subset \Sigma$ be discrete subgroups of G and Ω be a standard open subset of G with exponential map $\exp_\Omega : O \to \Omega$.

Definition B.3. We let $\mathcal{T}_\Omega(G,\Delta,\Sigma)$ denote the set of closed subgroups H of G satisfying the following properties:

(i) Σ is contained in H and Σ is a lattice in H;
(ii) the group $\exp_\Omega(\mathfrak{h} \cap O)$ is equal to $H \cap \Omega$, the Ω-semiconnected component H_Ω is a normal subgroup of H and one has $\Delta = \Sigma \cap H_\Omega$;
(iii) there exists a compactly generated subgroup Γ of H that acts ergodically on H/Σ, such that $\overline{\mathrm{Ad}\,\Gamma}^{\mathrm{Z}}$ is semisimple and equal to $\overline{\mathrm{Ad}\,\Gamma}^{\mathrm{Z,nc}}$, Ω is Γ-well shaped, H_Ω is Γ-unstable and $H = \Gamma H_\Omega$.

Lemma 2.5 admits the following weak analogue in the S-adic case.

Lemma B.4. *Let G be a second countable weakly regular S-adic Lie group, Ω be a standard open subset of G and $\Delta \subset \Sigma$ be discrete subgroups of G. Then, there exists a countable set $\mathcal{U}_\Omega(G,\Delta,\Sigma)$ of closed subgroups of G such that, for any H in $\mathcal{T}_\Omega(G,\Delta,\Sigma)$, there exists J in $\mathcal{U}_\Omega(G,\Delta,\Sigma)$ with $H_\infty \subset J \subset H$ such that H/J is compact and H virtually normalizes J.*

We say H virtually normalizes J if some finite index subgroup of H normalizes J.

The set $\mathcal{T}_\Omega(G,\Delta,\Sigma)$ might not be countable as, for instance when $G = \Omega = \mathbb{Z}_p^2$ and $\Delta = \Sigma = \{e\}$ since, in this case, H can be any subgroup of the form $G \cap D$, where D is a line in $\mathbb{Q}_p^2$.

We shall again need several preparatory lemmas. Thanks to our intricate definition, we have the following analogue of Lemma 2.6.

Lemma B.5. *Let G be a weakly regular S-adic Lie group, Ω be a standard open subset of G and $\Delta \subset \Sigma$ be discrete subgroups of G. Then, if H_1 and H_2 are in $\mathcal{T}_\Omega(G,\Delta,\Sigma)$, the group H_1 normalizes $H_{2,\Omega}$.*

Proof. We will first prove that $H_{1,\Omega}$ normalizes $H_{2,\Omega}$. As Σ normalizes $\mathfrak{h}_2$, we get an H_1-equivariant continuous map

$$H_1/\Sigma \to \mathrm{Gr}(\mathfrak{g});\ h_1\Sigma \mapsto \mathrm{Ad}\,h_1(\mathfrak{h}_2).$$

By Lemma 2.3, this map is constant; that is, H_1 normalizes $\mathfrak{h}_2$, hence H_1 normalizes $H_{2,\infty}$. In the same way, H_1 normalizes $\mathfrak{h}_{2,u}$, hence it normalizes $H_{2,u}$. Set $O = \log \Omega$. The normalizer H_1' of the group

$$H_{2,\Omega} = \exp_\Omega(\mathfrak{h}_2 \cap O) H_{2,u} H_{2,\infty}$$

in H_1 contains the groups $\exp_\Omega(\mathfrak{h}_1 \cap O)$, $H_{1,\infty}$ and Σ. In particular, H_1' is open in H_1 and, since Σ is a lattice in H_1, H_1' is a finite index subgroup of H_1. Therefore, since for any p in S, $\mathbb{Q}_p$ does not admit any proper finite index open subgroup, H_1' contains $H_{1,u}$ and the group

$$H_{1,\Omega} = \exp_\Omega(\mathfrak{h}_1 \cap O)H_{1,u}H_{1,\infty}$$

normalizes $H_{2,\Omega}$.

To conclude, let Γ be, as in the definition, a compactly generated subgroup of H_1 such that $\overline{\operatorname{Ad}\Gamma}^{\mathrm{Z}}$ is semisimple and equal to $\overline{\operatorname{Ad}\Gamma}^{\mathrm{Z,nc}}$, that Ω is Γ-well shaped and that $H_1 = H_{1,\Omega}\Gamma$, so that we only have to prove that Γ normalizes $H_{2,\Omega}$. Since H_1' has finite index in H_1, the group $H_{2,\Omega}$ is normalized by a finite index subgroup of Γ. Hence, setting $\mathfrak{l}$ to be the centralizer of Γ in $\mathfrak{g}$ and $\mathfrak{v}$ to be the Γ-invariant complementary subspace of $\mathfrak{l} \cap \mathfrak{h}_2$ in $\mathfrak{h}_2$, by (A.1), we have $\mathfrak{v} \subset \mathfrak{h}_{2,u}$. Thus, since Ω is Γ-well shaped, we have

$$\exp_\Omega(\mathfrak{h}_2 \cap O) \subset \exp_\Omega(\mathfrak{h}_2 \cap \mathfrak{l} \cap O)H_{2,u}H_{2,\infty}.$$

But then, since Γ commutes with $\exp_\Omega(\mathfrak{h}_2 \cap \mathfrak{l} \cap O)$ and normalizes $H_{2,u}H_{2,\infty}$, it also normalizes $H_{2,\Omega}$, what should be proved. $\square$

The proof of Lemma B.4 also uses an analogue of Lemma 2.7. Note that a second countable S-adic Lie group G may have uncountably many compact normal subgroups as, for instance, $G = \mathbb{Z}_p^2$.

Lemma B.6. *Let G be a second countable S-adic Lie group. Then the set of connected normal compact subgroups of G is countable.*

Proof. The proof mimics the real case. Let K be a connected normal compact subgroup of G, $\mathfrak{k}$ be the Lie algebra of K and K_∞ be the immersed real Lie subgroup of G with Lie algebra $\mathfrak{k}_\infty$. Since the group $\operatorname{Ad}_{\mathfrak{k}_\infty}(K_\infty)$ has compact closure (equal to $\operatorname{Ad}_{\mathfrak{k}_\infty}(K)$), the Lie algebra $\mathfrak{k}_\infty$ may be decomposed in a unique way as a direct sum of ideals $\mathfrak{k}_\infty = \mathfrak{s} \oplus \mathfrak{a}$, where $\mathfrak{s}$ is compact semisimple and $\mathfrak{a}$ is abelian. As $\mathfrak{k}_\infty$ is a G-invariant ideal of $\mathfrak{g}$, so are $\mathfrak{s}$ and $\mathfrak{a}$. As the Lie algebra $\mathfrak{g}_\infty$ contains only finitely many semisimple ideals, we may assume $\mathfrak{s}$ is fixed. As the connected real Lie subgroup S of G with Lie algebra $\mathfrak{s}$ is compact and normal in G, after replacing G by G/S, we may assume $\mathfrak{s} = \{0\}$.

Now, we have to prove G contains countably many abelian connected compact normal subgroups K. Since such a K is compact, $\mathfrak{k}_\infty$ admits a K-invariant complementary subspace $\mathfrak{v}$ in $\mathfrak{g}_\infty$. As $\mathfrak{v}$ is K-invariant and $\mathfrak{k}_\infty$ is an ideal of $\mathfrak{g}_\infty$, $\mathfrak{k}_\infty$ is contained in the center $\mathfrak{z}_\infty$ of $\mathfrak{g}_\infty$. Note that if $\mathfrak{a}$ and $\mathfrak{a}'$ are vector subspaces of $\mathfrak{z}_\infty$ and $\exp\mathfrak{a}$ and $\exp\mathfrak{a}'$ have compact closures in G, so has $\exp(\mathfrak{a}+\mathfrak{a}')$. Hence, if $\mathfrak{t}_\infty$ is the subspace of $\mathfrak{z}_\infty$ spanned by all such subspaces, then $\exp\mathfrak{t}_\infty$ has compact closure T in G. Hence we may assume that $G = T$

is a solenoid; i.e., G is isomorphic to a quotient of $\mathbb{R}^{d_\infty} \times \Omega_f$ by a cocompact lattice whose projection on Ω_f is dense, where $\Omega_f = \prod_{\substack{p \in S \\ p < \infty}} \mathbb{Z}_p^{d_p}$. Such a group admits only countably many closed subgroups. The result follows. □

We will need the following information on the normalizer of a totally discontinuous compact group.

LEMMA B.7. *Let G be an S-adic Lie group, H be a compact subgroup of G with $H_\infty = \{e\}$ and Γ be a subgroup of G normalizing H such that $\overline{\mathrm{Ad}\,\Gamma}^{\mathrm{Z}}$ is semisimple and equal to $\overline{\mathrm{Ad}\,\Gamma}^{\mathrm{Z,nc}}$. Then Γ centralizes the Lie algebra $\mathfrak{h}$ of H.*

Note that the analogue of this lemma is not true for real Lie groups as, for instance, when $G = \Gamma \ltimes H$ with $\Gamma = \mathrm{SL}(2, \mathbb{Z})$ acting on $H = \mathbb{T}^2$.

Proof. Assume by contradiction the action of Γ on $\mathfrak{h}$ is not trivial. In this case, by assumption, $\mathfrak{h}$ contains nonzero Γ-unstable vectors. Hence by (A.1), there exists a nontrivial one-parameter subgroup $\varphi : \mathbb{Q}_p \to H$. Since $H_\infty = \{e\}$, by [24, Prop. 1.2], φ is proper, which contradicts the fact that H is compact. The result follows. □

Finally, we shall also need

LEMMA B.8. *Let G be an S-adic Lie group, and let $\mathcal{K}$ be a set of normal compact subgroups of G.*

(i) *Assume $G_\infty = \{e\}$ and the set $\bigcup_{K \in \mathcal{K}} K$ spans a dense subgroup of G. Then, the Lie algebra of G is the linear span of the Lie algebras of the elements of $\mathcal{K}$.*

(ii) *Assume, for any K in $\mathcal{K}$, one has $\mathrm{Ad}\,K = \mathrm{Ad}\,G$ and the intersection of the Lie algebras of the elements of $\mathcal{K}$ is zero. Then the Lie algebra $\mathfrak{g}$ of G is abelian.*

Proof. (i) Let K be a compact normal subgroup of G that is generated by finitely many elements of $\mathcal{K}$ and whose Lie algebra is maximal. After having replaced G by G/K, we can assume the elements of $\mathcal{K}$ are finite, and we have to show that G is discrete.

Since $G_\infty = \{e\}$, there exists an open neighborhood U of e in G such that U does not contain any nontrivial torsion element. Then G contains a dense subgroup that meets U only at e; hence G is discrete.

(ii) Let $\mathfrak{z}$ be the center of $\mathfrak{g}$. By assumption, for any K in $\mathcal{K}$ with Lie algebra $\mathfrak{k}$, one has $\mathfrak{g} = \mathfrak{k} + \mathfrak{z}$, so that $\mathfrak{g}/\mathfrak{k}$ is abelian. Pick $K_1, \ldots, K_r$ in $\mathcal{K}$ whose Lie algebras have zero intersection. Then, the natural map $\mathfrak{g} \to \bigoplus_{i=1}^{r} \mathfrak{g}/\mathfrak{k}_i$ is injective. Hence $\mathfrak{g}$ is abelian. □

Another difficulty in the S-adic case is that the quotient G/H of a weakly regular p-adic Lie group G by a closed normal subgroup might not be weakly regular as, for instance, when $G = \mathbb{Q}_p^2$ and $H = \mathbb{Z}_p \times \{0\}$. This difficulty will weigh the following

Proof of Lemma B.4. We can assume that the set $\bigcup_{H \in \mathcal{T}_\Omega(G,\Delta,\Sigma)} H$ spans a dense subgroup of G. Then, in particular, by Lemma B.5, the group

$$H_0 = \bigcap_{H \in \mathcal{T}_\Omega(G,\Delta,\Sigma)} H_\Omega$$

is normal in G. We set

$$G' = G/H_0.$$

By [5, Lemmas 3.1 and 3.2], for any H in $\mathcal{T}_\Omega(G,\Delta,\Sigma)$, Δ is a lattice in both H_Ω and H_0. Hence the image H'_Ω of H_Ω in G' is a compact normal subgroup. Now since, by Lemma B.6, the set of connected normal compact subgroups of G' is countable, we can fix such a compact subgroup K and restrict our attention to the set $\mathcal{T}_\Omega^K(G,\Delta,\Sigma)$ of those H in $\mathcal{T}_\Omega(G,\Delta,\Sigma)$ such that $\overline{(H'_\Omega)_\infty} = K$. We set

$$K' = \bigcap_{H \in \mathcal{T}_\Omega^K(G,\Delta,\Sigma)} H'_\Omega$$

and

$$G'' = G'/K'.$$

For any H in $\mathcal{T}_\Omega^K(G,\Delta,\Sigma)$, we let H'' be the image of H in G'' and H''_Ω and Σ'' be the ones of H_Ω and Σ. As Σ is a lattice in H and Δ is a lattice in H_0, Σ'' is a lattice in H''. Let M be the closure of the subgroup of G'' spanned by the normal compact totally discontinuous subgroups H''_Ω as H varies in $\mathcal{T}_\Omega^K(G,\Delta,\Sigma)$.

We claim *the Lie algebra* $\mathfrak{m}$ *of* M *is abelian.* Indeed, first note that, since M is non-archimedean, by Lemma B.8(i), $\mathfrak{m}$ is the linear span of the Lie algebras of the normal compact subgroups H''_Ω, $H \in \mathcal{T}_\Omega^K(G,\Delta,\Sigma)$. Moreover, if $O = \log \Omega$, since, for any $H \in \mathcal{T}_\Omega^K(G,\Delta,\Sigma)$, one has $\exp_\Omega(\mathfrak{h} \cap O) \subset H_\Omega$, the Lie algebras of the H''_Ω, $H \in \mathcal{T}_\Omega^K(G,\Delta,\Sigma)$, have trivial intersection. Fix H in $\mathcal{T}_\Omega^K(G,\Delta,\Sigma)$, and let Γ be a compactly generated subgroup of H such that $\overline{\operatorname{Ad}\Gamma}^{\mathrm{Z}}$ is semisimple and equal to $\overline{\operatorname{Ad}\Gamma}^{\mathrm{Z,nc}}$, the action of Γ on H/Σ is ergodic and $H = \Gamma H_\Omega$. By Lemmas B.5 and B.7 applied to all the H''_Ω's, one has

$$\operatorname{Ad}_{\mathfrak{m}} \Gamma = \{e\},$$

and therefore $\operatorname{Ad}_{\mathfrak{m}} H = \operatorname{Ad}_{\mathfrak{m}} H_\Omega$. Besides, since the action of Γ on H/Σ is ergodic, there exists h in H such that $H = \overline{h\Gamma h^{-1}\Sigma}$, so that $\operatorname{Ad}_{\mathfrak{m}} H = \overline{\operatorname{Ad}_{\mathfrak{m}} \Sigma}$ and hence $\operatorname{Ad}_{\mathfrak{m}} H''_\Omega = \operatorname{Ad}_{\mathfrak{m}} M$. By Lemma B.8(ii), the Lie algebra $\mathfrak{m}$ is thus abelian.

Now, let us prove that *for any* H *in* $\mathcal{T}_\Omega^K(G,\Delta,\Sigma)$, H'' *virtually normalizes a finite index subgroup of* Σ''. Indeed, let Γ be as above. Since the adjoint action of Γ on the Lie algebra of H'' is trivial, there exists an open subgroup

U of H''_Ω such that Γ centralizes U. As the Lie algebra of H'' is abelian, we can assume U to be abelian. Since $H'' = \Gamma'' H''_\Omega$ and H''_Ω is compact, $\Gamma'' U$ has finite index in H''. Hence, the centralizer of U in H'' has finite index in H'' and the centralizer Σ''_U of U in Σ'' has finite index in Σ''. The group Σ''_U being a lattice in H'', the group $H''_U := \Sigma''_U U$ has finite index in H'' and normalizes Σ''_U as required.

Since, by Lemma A.5, Σ is finitely generated, the set of finite index subgroups of Σ is countable. Hence we can fix a finite index subgroup Θ of Σ and restrict our attention to the set of those H in $\mathcal{T}^K_\Omega(G, \Delta, \Sigma)$ such that some finite index open subgroup of H normalizes the image Θ'' of Θ in G'', and we set J to be the inverse image of Θ'' in G.

For such an H, this group J is included in H and contains Θ. Hence by [5, Lemma 3.1], the group J has finite covolume in H. Moreover J contains H_∞, hence by [5, Lem 5.2], the group J is cocompact in H. By construction, the group J is normalized by an open finite index subgroup in H. The result follows. $\square$

B.3. *Proof of countability in the* S-*adic case.*

Proof of Proposition 2.1 *in the general case.* We fix once for all a standard open subset Ω_0 in G, we set $O_0 = \log \Omega_0$ and we let $\mathfrak{l}$ denote the Lie algebra of L. Let Y be in $\mathcal{S}_X(\Gamma)$, fix g in G such that $x = g\Lambda$ belongs to Y and $\overline{\Gamma x} = Y$ and let $\mathfrak{h}$ denote the Lie algebra of $g^{-1} G_Y g$. By Lemma B.2, the group $g^{-1}\Gamma g$ admits a well-shaped standard open set $\Omega = \exp_{\Omega_0}(O) \subset \Omega_0$ such that $\exp_\Omega(\mathfrak{h} \cap O) \subset g G_Y g^{-1}$. As the compact group $O_{0,f}$ admits only countably many open subgroups, we can suppose Ω to be fixed.

We first construct an open compactly generated subgroup H of $g^{-1} G_Y g$ that contains $g^{-1}\Gamma g$. We let H_∞ denote the analytic subgroup of G with Lie algebra $\mathfrak{h}_\infty$, H_u denote the closed group spanned by the $g^{-1}\Gamma g$-unstable one-parameter subgroups in $g^{-1} G_Y g$, $\mathfrak{l}_g := \operatorname{Ad} g^{-1} \mathfrak{l}$ and

$$H := g^{-1}\Gamma g \exp_\Omega(\mathfrak{l}_g \cap \mathfrak{h} \cap O) H_u H_\infty.$$

This subgroup H of $g^{-1} G_Y g$ is open hence closed and, since Ω is $g^{-1}\Gamma g$-well shaped, the Ω-semiconnected component H_Ω of H satisfies

$$H_\Omega = \exp_\Omega(\mathfrak{l}_g \cap \mathfrak{h} \cap O) H_u H_\infty = \exp_\Omega(\mathfrak{h} \cap O) H_u H_\infty.$$

By construction, the subgroup H_Ω is normal in H and is $g^{-1}\Gamma g$-unstable. By Lemma A.5, H is compactly generated; hence the lattice $\Sigma := \Lambda \cap H$ is finitely generated. Since the countable group Λ admits countably many finitely generated subgroups, we can assume Σ to be fixed.

The subgroup $\Delta := \Lambda \cap H_\Omega$ is a lattice in H_Ω that might not be finitely generated. Therefore, let us introduce the largest Ω-semiconnected compactly generated subgroup H' of H, as in Lemma A.7. By Lemma A.9, the group

$\Delta' := \Delta \cap H'$ is finitely generated and the group Δ is the smallest normal subgroup of Σ containing Δ'. Hence we can assume Δ is fixed.

By construction, H belongs to $\mathcal{T}_\Omega(G,\Delta,\Sigma)$. Let $\mathcal{U}_\Omega(G,\Delta,\Sigma)$ be as in Lemma B.4. Then, there exists some J in $\mathcal{U}_\Omega(G,\Delta,\Sigma)$ such that H contains J, H virtually normalizes J and H/J is compact. As $\mathcal{U}_\Omega(G,\Delta,\Sigma)$ is countable, we can assume J to be fixed. Now, let Y_1 be another element of $\mathcal{S}_X(\Gamma)$. Proceed to the same construction, so that we get $x_1 = g_1\Lambda \in Y_1$ with $\overline{\Gamma x_1} = Y_1$ and a subgroup $H_1 \supset J$, virtually normalizing J, with H_1/J compact and $\Gamma \subset g_1 H_1 g_1^{-1} \subset G_{Y_1}$. By Lemma B.9 below, we can assume there exists ℓ in L with $g_1 J = \ell g J$. Thus, we have $\ell x \in Y_1$; hence $\ell Y = \overline{\Gamma \ell x} \subset Y_1$. In the same way, since $\ell^{-1}x_1$ belongs to Y, we get $\ell^{-1}Y_1 \subset Y$; hence $Y_1 = \ell Y$. The result follows. □

As the conclusion of Lemma B.4 is weaker than the one of Lemma 2.5, in the S-adic case we needed the following result, which is stronger than Lemma 2.2.

Lemma B.9. *Let G be a second countable S-adic Lie group, J be a closed subgroup of G, Γ be a closed compactly generated subgroup of G such that $\overline{\mathrm{Ad}\,\Gamma}^{\mathrm{Z}}$ is semisimple and equal to $\overline{\mathrm{Ad}\,\Gamma}^{\mathrm{Z,nc}}$ and L be the centralizer of Γ in G. Then the set*

$$Y = \{y = gJ \in G/J \mid \Gamma \text{ virtually normalizes } gJg^{-1} \text{ and } \overline{\Gamma y} \text{ is compact}\}$$

is a countable union of L-orbits.

Proof. Let N be the normalizer of J in G and, for $y = gJ$ in G/J, set $J_y = gJg^{-1}$ and $N_y = gNg^{-1}$. By assumption, for y in Y, the group $\Gamma_y = \Gamma \cap N_y$ is an open finite index subgroup of Γ. Since Γ is compactly generated, the set of such subgroups is countable, and since Γ and Γ_y have the same centralizer $\mathfrak{l}$ in $\mathfrak{g}$, the centralizer of Γ in G is open in the centralizer of Γ_y in G. Thus, after having replaced Γ by Γ_y, it suffices to prove that the set of y in G/J such that $\Gamma \subset N_y$ and $\overline{\Gamma y}$ is compact is a countable union of L-orbits.

For such a y, as the image of Γ in the group N_y/J_y is relatively compact and $\overline{\mathrm{Ad}\,\Gamma}^{\mathrm{Z}} = \overline{\mathrm{Ad}\,\Gamma}^{\mathrm{Z,nc}}$, the adjoint action of Γ on the Lie algebra $\mathfrak{n}_y/\mathfrak{j}_y$ of N_y/J_y is trivial. As $\overline{\mathrm{Ad}\,\Gamma}^{\mathrm{Z}}$ is semisimple, the action of Γ on $\mathfrak{n}_y$ is semisimple, and one has

$$\mathfrak{n}_y = (\mathfrak{n}_y \cap \mathfrak{l}) + \mathfrak{j}_y.$$

In other terms, since Γ is compactly generated, $(L \cap N_y)J_y$ is an open subgroup of N_y. Hence $N_y y$ is contained in a countable union of L-orbits in G/J. The result follows since, by Lemma 2.2, the set of fixed points of Γ in G/N is a countable union of L-orbits. □

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