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Residual automorphic forms and spherical unitary representations of exceptional groups

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Abstract

Arthur has conjectured that the unitarity of a number of representations can be shown by finding appropriate automorphic realizations. This has been verified for classical groups by Mœglin and for the exceptional Chevalley group G_2 by Kim. In this paper we extend their results on spherical representations to the remaining exceptional groups E_6, E_7, E_8, and F_4. In particular, we prove Arthur's conjecture that the spherical constituent of an unramified principal series of a Chevalley group over any local field of characteristic zero is unitarizable if its Langlands parameter coincides with half the weighted marking of a coadjoint nilpotent orbit of the Langlands dual Lie algebra.

1. Introduction

The most trivial automorphic form — the constant function on the complex upper half plane — has a complicated construction as the residue of the usual nonholomorphic Eisenstein series $\sum_{(c,d)\in\mathbb{Z}^2-\{(0,0)\}} \frac{y^s}{|cz+d|^{2s}}$ at the polar point $s = 1$. The constant residue reflects the fact that the principal series representation associated to this Eisenstein series has a trivial quotient. This is the starting point for a fascinating mechanism of constructing automorphic realizations of certain "small" representations of real groups, meaning those with low Gelfand-Kirillov dimension (equivalently, those whose associated variety of the annihilator ideal is small). For example, the classical Jacobi θ-function is a residue of an Eisenstein series on the metaplectic double cover of $\mathrm{SL}(2,\mathbb{R})$. When the residues are square-integrable (which can be checked by a criterion of Langlands reviewed in Section 2), they lie in the discrete automorphic spectrum and hence automatically give unitary representations. This strategy was famously used by Speh [17] to construct new unitary representations of

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$\mathrm{SL}(4,\mathbb{R})$, representations which were difficult to approach until she added this arithmetic grasp.

As part of a broad set of conjectures, Arthur [2] has proposed that spherical constituents of principal series representations at certain points of reduction are unitary; moreover, their unitarity should, as above, be a consequence of an automorphic realization. In the case of unramified representations of split groups, one possibility is a precise realization of these representations as residues of Eisenstein series. This reduces Arthur's conjectures for these spherical constituents of principal series to verifying certain residual Eisenstein series are square-integrable. In this paper we prove the square-integrability and hence these conjectures, whose statement we now recall.

Let G denote a Chevalley group and $B = NA$ be a fixed Borel subgroup of G, with N a maximal unipotent subgroup and A a maximal torus. We shall assume, as we may, that N is generated by the one-parameter subgroups for each of the Chevalley basis positive root vectors and that the Lie algebra $\mathfrak{a}$ of A is spanned by the Chevalley basis coroot vectors. To any coadjoint nilpotent orbit $\mathcal{O}^\vee$ of $G^\vee(\mathbb{C})$ in the complexified Lie algebra $\mathfrak{g}^\vee \otimes \mathbb{C}$ of the Langlands dual group $G^\vee$, there is a "weighted marking" $2\lambda_0(\mathcal{O}^\vee) \in \mathfrak{a}^*$ specified by the condition that $\langle 2\lambda_0(\mathcal{O}^\vee), \alpha^\vee\rangle$ is equal to the integer 0,1, or 2 marking the node for the simple coroot $\alpha^\vee$ in the weighted Dynkin diagram of $\mathcal{O}^\vee$. For example, in the case of the "regular orbit" (i.e., the unique dense orbit) $\lambda_0(\mathcal{O}^\vee)$ is equal to ρ, half the sum of the positive roots. Suppose furthermore that $\mathcal{O}^\vee$ is *distinguished*, meaning it does not intersect any proper Levi subalgebra. Let F be a number field and $\mathbb{A}_F$ be its ring of adeles. Arthur conjectured that for any place v of F, the spherical constituent of the unramified principal series representation of $G(F_v)$ with Langlands parameter $\lambda_0(\mathcal{O}^\vee)$ is unitarizable and, moreover, occurs discretely in the automorphic spectrum $L^2(G(F)\backslash G(\mathbb{A}_F))$. The situation when $\mathcal{O}^\vee$ is not distinguished reduces to this; see Corollary 1.3 and the remarks following it.

The discrete spectrum includes cusp forms, which are very difficult to construct but can sometimes be counted using the trace formula (or more commonly, variants of the trace formula). The rest of the discrete spectrum consists of residual Eisenstein series at special, delicate points. (See [15] for a detailed general reference.) These are automorphic forms occurring as the leading coefficient in a multivariable Laurent series expansion of Eisenstein series, which are automorphic realizations of principal series representations. We now state the definition of the spherical, unramified Borel Eisenstein series, which are the relevant type in this paper. The adjoint action of $A(\mathbb{A}_F)$ on the Chevalley basis simple root vectors, composed with the global valuation on $\mathbb{A}_F$, gives rise to a character $a \mapsto a^\lambda$ of $A(F)\backslash A(\mathbb{A}_F)$ for each $\lambda \in \mathfrak{a}^* \otimes \mathbb{C}$, the complex span of the simple roots. Let $a(g)$ denote the Iwasawa A-factor of

$g \in G(\mathbb{A}_F)$. The unramified Borel Eisenstein series is defined as

$$E(\lambda, g) := \sum_{\gamma \in B(F)\backslash G(F)} a(\gamma g)^{\lambda+\rho}\,, \qquad \lambda \in \mathfrak{a}^* \otimes \mathbb{C}, \tag{1.1}$$

initially as an absolutely convergent sum when $\lambda - \rho$ lies in the interior of the positive Weyl chamber, and then for general λ by meromorphic continuation. At generic values of λ, $E(\lambda, g)$ and its right translates generate an automorphic realization of the local principal series representations $V_\lambda = \{f : G(F_v) \to \mathbb{C} \,|\, f(nag) = a^{\lambda+\rho} f(g), n \in N(F_v), a \in A(F_v)\}$ for each place v of F.

Arthur's conjectures suggest that the residues of $E(\lambda, g)$ at $\lambda = \lambda_0(\mathcal{O}^\vee)$ for distinguished orbits $\mathcal{O}^\vee$ lie in $L^2(G(F)\backslash G(\mathbb{A}_F))$ (though they would not be contradicted were this false, as there would remain the possibility of cuspidal realizations).[1] Such a function then generates an irreducible subrepresentation of $L^2(G(F)\backslash G(\mathbb{A}_F))$, whose unitarity comes from the L^2 inner product. Due to the agreement of infinitesimal characters, it must be the tensor product of the spherical constituent of each local principal series representation $V_{\lambda_0(\mathcal{O}^\vee)}$ over all completions F_v of F. Our main result is a proof of this L^2 property for the exceptional groups E_6, E_7, E_8, and F_4 (it is known for classical groups and G_2 [10], [14], [13], [11]). We summarize our findings as follows.

Theorem 1.2. *Let F be a number field, G be a Chevalley group of type E_6, E_7, E_8, or F_4, and $\mathfrak{g}$ its Lie algebra. Let $\lambda_0(\mathcal{O}^\vee) \in \mathfrak{a}^* \otimes \mathbb{C}$ and a coadjoint nilpotent orbit $\mathcal{O}$ in $\mathfrak{g}(\mathbb{C})$ be as defined by one of the following pairs, in which $\omega_1, \omega_2, \ldots$ refer to fundamental weights in the usual Bourbaki numbering and the orbit $\mathcal{O}$ is described by its Bala-Carter label* (*see* [6]):

G	$\lambda_0(\mathcal{O}^\vee)$	$\mathcal{O}$	G	$\lambda_0(\mathcal{O}^\vee)$	$\mathcal{O}$
E_6	$\omega_1+\omega_4+\omega_6$	A_2	E_8	ω_5	$E_8(a_7)$
E_6	$\rho-\omega_4$	A_1	E_8	$\omega_4+\omega_8$	$D_4(a_1)+A_2$
E_6	ρ	0	E_8	$\omega_4+\omega_7$	$D_4(a_1)+A_1$
E_7	$\omega_4+\omega_7$	$D_4(a_1)$	E_8	$\omega_4+\omega_7+\omega_8$	$D_4(a_1)$
E_7	$\omega_1+\omega_4+\omega_7$	A_2+2A_1	E_8	$\omega_1+\omega_4+\omega_7$	$2A_2$
E_7	$\omega_1+\omega_4+\omega_6+\omega_7$	A_2	E_8	$\omega_1+\omega_4+\omega_7+\omega_8$	A_2+2A_1
E_7	$\rho-\omega_4-\omega_6$	$2A_1$	E_8	$\omega_1+\omega_4+\omega_6+\omega_8$	A_2+A_1
E_7	$\rho-\omega_4$	A_1	E_8	$\rho-\omega_2-\omega_3-\omega_5$	A_2
E_7	ρ	0	E_8	$\rho-\omega_4-\omega_6$	$2A_1$
F_4	ω_3	$F_4(a_3)$	E_8	$\rho-\omega_4$	A_1
F_4	$\omega_1+\omega_3$	A_1+A_1s	E_8	ρ	0
F_4	$\rho-\omega_2$	A_1s			
F_4	ρ	0			

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[1] With the sole exception of the first E_8 case listed, the representations in Theorem 1.2 are not expected to occur cuspidally because the associated variety of their annihilator ideal $\overline{\mathcal{O}}$ is not distinguished.

Then the unramified Borel Eisenstein series (1.1) *has a square-integrable residue at* $\lambda = \lambda_0(\mathcal{O}^\vee)$. *Its local representation of* $G(F_v)$ *is unitary for each place* v *of* F *and, furthermore, has associated variety of the annihilator ideal equal to* $\overline{\mathcal{O}}$ *if* $F_v = \mathbb{R}$.

The case of $\lambda_0(\mathcal{O}^\vee) = \rho$ (which has a trivial residue) is of course well known. The cases with $\mathcal{O} = A_1$ for E_6, E_7, and E_8 are the automorphic realizations of the minimal representation constructed in [7], and the cases with $\mathcal{O} = 2A_1$ for E_7 and E_8 are likewise the automorphic realizations of the "next-to-minimal" representation constructed in [9]. Indeed, both the proof and immediate motivation of this paper arose out of the collaboration [9]. The appearance of small automorphic residual representations in certain string theory problems there and in [8] led us to develop the computational tools used here. The theorem's assertion about the associated variety of the annihilator ideal is a direct consequence of Theorem A.5 of [9, App. A], by Ciubotaru and Trapa.

Theorem 1.2 has well-known implications for the unitarity of spherical representations coming from nondistinguished orbits (which was shown by Barbasch-Moy [4] for nonarchimedean fields).

COROLLARY 1.3. *Let* $2\lambda_0(\mathcal{O}^\vee)$ *be the weighted marking of any coadjoint nilpotent orbit* $\mathcal{O}^\vee$ *of* $\mathfrak{g}^\vee \otimes \mathbb{C}$, *distinguished or not. Then the spherical constituent of* $V_{\lambda_0(\mathcal{O}^\vee)}$ *over any local field of characteristic zero is unitarizable.*

Proof. Every local field is a completion F_v of some number field F. By Theorem 1.2 we need only consider the case when $\mathcal{O}^\vee$ intersects the complexified Lie algebra $\mathfrak{l}(\mathbb{C})$ of the Levi component of a proper parabolic subgroup P, which we may assume to be minimal among such parabolics. We furthermore may assume $\mathfrak{l}$ is chosen compatibly with the Chevalley basis. Then $\mathcal{O}_L^\vee = \mathcal{O}^\vee \cap \mathfrak{l}(\mathbb{C})$ is distinguished in $\mathfrak{l}(\mathbb{C})$, and so by Theorem 1.2 and its known analog for classical groups it is associated to a spherical unitary representation of $L(F_v)$. Unitary induction of this representation from $L(F_v)$ to $G(F_v)$ gives a unitary representation that contains the spherical constituent of $V_{\lambda_0(\mathcal{O}^\vee)}$. Indeed, this last statement reduces to the compatibility of their respective infinitesimal characters; that, in turn, is equivalent to the agreement of the weighted marking of the nondistinguished orbit $\mathcal{O}^\vee$ of $\mathfrak{g}^\vee(\mathbb{C})$ with the weighted marking of the distinguished orbit $\mathcal{O}_L^\vee$ of $\mathfrak{l}(\mathbb{C})$. $\square$

Theorem A.5 of [9, App. A] also computes the associated variety of the annihilator ideal for these spherical representations when the ground field is $\mathbb{R}$ and $\mathcal{O}^\vee$ is even, i.e., $\lambda_0(\mathcal{O}^\vee)$ is an integral weight. The unitary induction in the proof has an automorphic analog, as Eisenstein series induced from automorphic forms on the Levi component of a proper parabolic subgroup. Thus the representations in the corollary also occur automorphically.

After these results were first announced, Jeffrey Adams informed the author that the algorithm of [1], as implemented in `Atlas 0.5.3` [3], has successfully verified the unitarity of all of the F_4 and E_7 representations in Theorem 1.2 over $\mathbb{R}$. This method is complementary to ours in the sense that it works best in the cases that are most difficult for us, and vice-versa.

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2. Langlands' constant term formula and L^2 condition

Let Δ^+ and Δ^- denote the positive and negative roots, respectively, of the Chevalley group G with respect to its fixed Borel subgroup B, and let Σ^+ denote its positive simple roots. We use the notation $\alpha^\vee$ for the coroot of $\alpha \in \Delta^+$. Langlands computed the constant term of the (unramified) Borel Eisenstein series (1.1) as

$$\int_{N(F)\backslash N(\mathbb{A}_F)} E(\lambda, ng)\, dn \;=\; \sum_{w\in W} M(w,\lambda)\, a(g)^{w\lambda+\rho}, \tag{2.1}$$

where W is the Weyl group,

$$M(w,\lambda) = \prod_{\substack{\alpha\in\Delta^+\\ w\alpha\in\Delta^-}} c(\langle \lambda, \alpha^\vee\rangle), \tag{2.2}$$

and $c(s)$ is a meromorphic function on $\mathbb{C}$ which can be expressed as an explicit ratio involving the Dedekind ζ-function of the number field F (see [16, §6] and [5, §3.7]). For example, $c(s) = \frac{\pi^{-s/2}\Gamma(s/2)\zeta(s)}{\pi^{-(s+1)/2}\Gamma((s+1)/2)\zeta(s+1)}$ when $F = \mathbb{Q}$. We shall not require a precise formula, but only its following direct consequences: $c(s)$ has a first order zero at $s = -1$, a first order pole at $s = 1$, no zeroes or poles in $\{\mathrm{Re}\ s < -1\}$, and satisfies $c(s)c(-s) = 1$, $c(0) = -1$. Langlands also showed the corresponding functional equation

$$E(\lambda, g) = M(w,\lambda)\, E(w\lambda, g) \tag{2.3}$$

of the meromorphic function $\lambda \mapsto E(\lambda, g)$, $\lambda \in \mathfrak{a}^* \otimes \mathbb{C}$.

Any coefficient in a (multivariable) Laurent expansion of an Eisenstein series in λ is an automorphic function in g. In particular, suppose that λ has the form $\lambda_1 + \varepsilon\lambda_2$ for $\varepsilon \in \mathbb{C}$ and fixed $\lambda_1, \lambda_2 \in \mathfrak{a}^*\otimes\mathbb{C}$, and that $E(\lambda, g)$ vanishes to order n at $\varepsilon = 0$. (By convention, $n < 0$ when there is a pole.) Grouping terms with similar powers together,

$$\begin{aligned}(2.4)\qquad &\int_{N(F)\backslash N(\mathbb{A}_F)} E(\lambda_1 + \varepsilon\lambda_2, ng)\, dn \\ &\quad = \sum_{\mu\in W\lambda_1} a(g)^{\mu+\rho} \sum_{w\in W(\lambda_1,\mu)} M(w, \lambda_1 + \varepsilon\lambda_2)\, a(g)^{\varepsilon w\lambda_2} \\ &\quad = \sum_{\mu\in W\lambda_1} a(g)^{\mu+\rho} \sum_{k\geq n} \varepsilon^k\, C(\mu, k, g),\end{aligned}$$

where $W\lambda_1$ is the Weyl orbit of λ_1, $W(\lambda_1, \mu) = \{w \in W | w\lambda_1 = \mu\}$, and the last expression represents the Laurent series expansion of the w-sum in ε. Thus the leading coefficient $\lim_{\varepsilon\to 0} \varepsilon^{-n} E(\lambda_1 + \varepsilon\lambda_2, g)$ in its Laurent expansion in ε is an automorphic form with constant term

$$(2.5)\qquad \int_{N(F)\backslash N(\mathbb{A}_F)} \lim_{\varepsilon\to 0} \varepsilon^{-n}\, E(\lambda_1 + \varepsilon\lambda_2, ng)\, dn = \sum_{\mu\in W\lambda_1} a(g)^{\mu+\rho}\, C(\mu, n, g).$$

The interchange of the limit and integration here is justified because the limit may be computed as an integral over a compact space using Cauchy's theorem. Note that certain μ in the Weyl orbit $W\lambda_1$ may have $C(\mu, n, g) \equiv 0$ as a function of g. Other terms may include logarithmic factors; however, the polynomial growth in g is always determined by the factor $a(g)^{\mu+\rho}$. Langlands [12, §5] gave the condition that because $\lim_{\varepsilon\to 0} \varepsilon^{-n} E(\lambda_1 + \varepsilon\lambda_2, g)$ is "concentrated along B," it is square-integrable if and only if the inequality

$$(2.6)\qquad \langle \mu, \omega_i\rangle < 0 \quad \text{for any fundamental weight } \omega_i$$

holds for each $\mu \in W\lambda_1$ in (2.5) such that $C(\mu, n, g) \not\equiv 0$.

3. Computational method

Explicit computations with (2.1) are frequently unwieldy with large Weyl groups W. However, a perturbation method was introduced in [8, §2] and [9, §7] that manageably reduces the size involved. In terms of the parametrization $\lambda = \lambda_1 + \varepsilon\lambda_2$ of the previous section, it corresponds to taking $\lambda_1 = 2s\omega_j - \rho$ and $\lambda_2 = \omega_j$ for some fixed $s \in \mathbb{C}$ and fundamental weight ω_j. Such a λ corresponds to a maximal parabolic Eisenstein series induced from the trivial representation. Of course few half-markings $\lambda_0(\mathcal{O}^\vee)$ satisfy these hypothesis for λ_1, but each from the statement of Theorem 1.2 turns out to have a Weyl translate that does. By the functional equation (2.3) the limiting value $\lim_{\varepsilon\to 0} \varepsilon^{-n} E(\lambda_1 + \varepsilon\lambda_2, g)$ corresponds to a nonzero multiple of a residue at $\lambda_0(\mathcal{O}^\vee)$. Hence there is no less of generality in studying the series expansion near $\lambda = \lambda_1$ instead of $\lambda = \lambda_0(\mathcal{O}^\vee)$.

With our choice of $\lambda = (2s+\varepsilon)\omega_j - \rho$, the inner products with simple coroots are

$$\langle \lambda, \alpha_i^\vee \rangle = \begin{cases} -1, & i \neq j \\ 2s+\varepsilon-1, & i = j. \end{cases} \tag{3.1}$$

If ε has sufficiently negative real part, then

$$c(\langle \lambda, \alpha^\vee \rangle) = 0 \quad \text{if} \quad \langle \lambda, \alpha^\vee \rangle = -1, \tag{3.2}$$

while no other factors $c(\langle \lambda, \alpha^\vee \rangle)$ in (2.2) have poles. Thus by analytic continuation in ε a term $M(w, \lambda) \equiv 0$ unless w lies in the set

$$W_{\text{rel}} := \{ w \in W \mid w\alpha_i > 0 \ \text{ for all } i \neq j \}. \tag{3.3}$$

As the tables in Section 4 show, W_{rel} is significantly smaller than W: it forms the Kostant coset representatives for W/W_M, where W_M is the Weyl group of the root system spanned by $\{\alpha_i | i \neq j\}$, and hence $\#W_{\text{rel}} = \#W/\#W_M$. Because it is so much smaller than W, it has fewer translates $W_{\text{rel}}\lambda_1$ of λ_1 than W has. The tables also indicate an overwhelming majority of the W_{rel}-translates satisfy (2.6). Square-integrability is therefore reduced to showing the existence of an integer m such that

(3.4)
- i) $C(\mu, m, g) \not\equiv 0$ for some $\mu \in W_{\text{rel}}\lambda_1$ satisfying (2.6), and
- ii) $C(\mu, m', g) \equiv 0$ for all $m' \leq m$ and all $\mu \in W_{\text{rel}}\lambda_1$ not satisfying (2.6).

Indeed, there then exists some $n \leq m$ such that $\lim_{\varepsilon \to 0} \varepsilon^{-n} E(\lambda_1 + \varepsilon\lambda_2, g)$ lies in $L^2(G(F)\backslash G(\mathbb{A}_F))$, n being the least integer such that $C(\mu, n, g) \not\equiv 0$ for some $\mu \in W_{\text{rel}}\lambda_1$ satisfying (2.6).

Thus the method boils down to the ability to efficiently compute certain coefficients $C(\mu, k, g)$ in the Laurent expansion (2.4). We used exact, symbolic calculation with integer arithmetic (which is rigorous), and we have made the programs available for download.[2] We used Mathematica v.8, though many other software packages would have sufficed. Mathematica is capable of exact calculations with the explicit formula $c(s) = \frac{\pi^{-s/2}\Gamma(s/2)\zeta(s)}{\pi^{-(s+1)/2}\Gamma((s+1)/2)\zeta(s+1)}$ when $F = \mathbb{Q}$, though they become cumbersome (and do not apply to general number fields); it also crashes both Windows and Unix operating systems for large symbolic computations such as ours, forcing us to rewrite them differently using some software tricks, which we now describe. We computed $c(\langle \lambda, \alpha^\vee \rangle) = c(\langle \lambda_1, \alpha^\vee \rangle + \varepsilon \langle \lambda_2, \alpha^\vee \rangle)$ differently depending on the value of $\langle \lambda_1, \alpha^\vee \rangle$ (which is always an integer in our examples): near $s \neq \pm 1$ we formally write $c(s) = -e^{c_\ell(s) - c_\ell(-s)}$ in order to satisfy the properties $c(s)c(-s) = 1$ and $c(0) = -1$,

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[2] Available at: http://annals.math.princeton.edu/supplemental/2013/177-3/p09/.

while at near $s = \pm 1$ we use the similar expressions

$$c(s) = \begin{cases} -(s-1)^{-1}\, e^{-c_{\ell,1}(1-s)} & \text{near } s = 1, \\ (s+1)\, e^{c_{\ell,1}(s+1)} & \text{near } s = -1, \end{cases} \tag{3.5}$$

which take into account the simple zero or pole. These formal expressions are valid in a neighborhood of the relevant integer, and thus can be rigorously used to derive formulas for Laurent series expansions in ε. As a result each $c(\langle \lambda, \alpha^\vee \rangle)$ can be written as a power of ε times the exponential of a formal function of $\langle \lambda_1, \alpha^\vee \rangle + \varepsilon \langle \lambda_2, \alpha^\vee \rangle$. By invariance properties we may take g in (2.4) to lie in $A(\mathbb{A}_F)$, so that $a(g)^{\varepsilon w \lambda_2}$ can be written as an exponential in terms of global valuation of the adjoint action of g on the simple root spaces. Hence $M(w, \lambda_1 + \varepsilon\lambda_2) a(g)^{\varepsilon w \lambda_2}$ can always be written as a power of ε times an exponential. We used the observation that this power of ε is constant over $W(\lambda_1, \mu)$ for a fixed μ in $W_{\mathrm{rel}}\lambda_1$ to further speed up the computations. These powers of ε come from w for which $\langle \lambda_1, \alpha^\vee \rangle$ is equal to -1 or 1 for some positive root α flipped by w. More generally, for any fixed integer n and μ in $W_{\mathrm{rel}}\lambda_1$, the quantity

$$\#\{\alpha \in \Delta^+ \cap w^{-1}\Delta^- | \langle \lambda_1, \alpha^\vee \rangle = n\} - \#\{\alpha \in \Delta^+ \cap w^{-1}\Delta^- | \langle \lambda_1, \alpha^\vee \rangle = -n\}$$

is constant over $w \in W(\lambda_1, \mu)$ (which also simplifies the computations).

At this point, for a fixed $\mu \in W_{\mathrm{rel}}\lambda_1$ each $M(w, \lambda_1 + \varepsilon\lambda_2) a(g)^{\varepsilon w \lambda_2}$, $w \in W(\lambda_1, \mu)$, is a fixed power of ε times an overall multiplicative constant (which comes from the terms with $\langle \lambda_1, \alpha^\vee \rangle \in \{-1, 0, 1\}$) and the exponential of a function of ε. To compute the Laurent series we compute the Taylor series development of that function in ε, and we then use the power series $e^x = \sum_{k \geq 0} x^k/k!$ to derive the Taylor series of the full exponential term in ε. We then multiply by the overall multiplicative constant and fixed power of ε, and then sum over $w \in W(\lambda_1, \mu)$. This computes the Laurent series of the inner sum in (2.4). In each case, the first nonvanishing $C(\mu, k, g)$ has a nonzero polynomial dependence in H with coefficients independent of the ground field F; this makes the nonvanishing independent of $c(\cdot)$ and hence also independent of F.

Ultimately, the calculation boils down to calculating averages of polynomials over highly symmetric, finite subsets of euclidean space — that is, a *design* computation. Its difficulty stems from the appearance of large symmetric subsets that cannot be distinguished from an equidistributed set until a high degree polynomial is taken. Such sets arise in these constant term calculations because of their similarity to the Weyl denominator formula. In the largest cases we took advantage of repeated occurrences of certain values of $M(w, \lambda_1 + \varepsilon\lambda_2)$ and its derivatives at $\varepsilon = 0$, by compressing them into new variables. This dramatically sped up symbolic calculations and reduced RAM requirements.

The computations were carried out on a Dell PowerEdge server, and required up to 45 GB of RAM. The lengthiest example (the first line in the $G = E_8$ table below) took about 3 days utilizing 8 CPUs, the vast majority of which was spent verifying that Mathematica had correctly manipulated a symbolic expression. Additionally, nearly all of these calculations have been reproduced with completely independent code using LiE and Sage, with full agreement in all cases that were practical to run within reasonable time constraints.[3]

4. **Results of the calculation**

Below we present some details of the calculation for Chevalley groups of type E_6, E_7, E_8, and F_4.

Legend for columns.

$2\lambda_0(\mathcal{O}^\vee)$: the weighted marking of a distinguished (and hence even) coadjoint nilpotent orbit $\mathcal{O}^\vee$. The integers $\langle 2\lambda_0(\mathcal{O}^\vee), \alpha_1^\vee\rangle, \langle 2\lambda_0(\mathcal{O}^\vee), \alpha_2^\vee\rangle, \ldots$ are strung together according to the standard Bourbaki numbering.

$\mathcal{O}$: the unique coadjoint nilpotent orbit whose closure is the associated variety of the annihilator ideal of the spherical constituent of $V_{\lambda_0(\mathcal{O}^\vee)}$ (for $F_v = \mathbb{R}$).

λ_1: a Weyl-equivalent point to $\lambda_0(\mathcal{O}^\vee)$ that is more useful for our computational purposes in Section 3, listed as $[\langle \lambda_1, \alpha_1^\vee\rangle, \langle \lambda_1, \alpha_2^\vee\rangle, \ldots]$.

W_{rel}: Weyl elements that give nonzero contributions to the constant term under the deformation $\lambda_1 + \varepsilon\lambda_2$ (defined in (3.3)).

$\# \cap -\mathcal{C}^\vee$: The first number indicates how many terms in $W_{\mathrm{rel}}\lambda_1$ lie in Langlands' region (2.6); the second number indicates how many do not satisfy (2.6); and the third indicates how many of these would instead not satisfy (2.6) if the condition's inequality was weakened from $\langle \lambda, \omega_i\rangle < 0$ to $\langle \lambda, \omega_i\rangle \le 0$ for each i.

ord: the order of vanishing of $E(\lambda_1 + \varepsilon\lambda_2, g)$ at $\varepsilon = 0$. This was called n in Section 2. The first entry for the E_8 table has an inequality; we did not determine n here but found a value of $m = 3$ (in the notation of (3.4)).

E_6

$2\lambda_0(\mathcal{O}^\vee)$	$\mathcal{O}$	λ_1	$\#W_{\mathrm{rel}}$	$\# \cap -\mathcal{C}^\vee$	ord
200202	A_2	[-1,4,-1,-1,-1,-1]	72	44/1/0	0
222022	A_1	[2,-1,-1,-1,-1,-1]	27	24/2/1	0
222222	0	[-1,-1,-1,-1,-1,-1]	1	1/0/0	0

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[3] This code is also available at: http://annals.math.princeton.edu/supplemental/2013/177-3/p09/.

E_7

$2\lambda_0(\mathcal{O}^\vee)$	$\mathcal{O}$	λ_1	$\#W_{\mathrm{rel}}$	$\# \cap -\mathcal{C}^\vee$	ord
0002002	$D_4(a_1)$	[-1,-1,4,-1,-1,-1,-1]	2016	638/27/2	1
2002002	A_2+2A_1	[-1,5,-1,-1,-1,-1,-1]	576	292/2/1	0
2002022	A_2	[7,-1,-1,-1,-1,-1,-1]	126	90/1/0	0
2220202	$2A_1$	[4,-1,-1,-1,-1,-1,-1]	126	115/3/1	0
2220222	A_1	[2,-1,-1,-1,-1,-1,-1]	126	97/28/0	0
2222222	0	[-1,-1,-1,-1,-1,-1,-1]	1	1/0/0	0

E_8

$2\lambda_0(\mathcal{O}^\vee)$	$\mathcal{O}$	λ_1	$\#W_{\mathrm{rel}}$	$\# \cap -\mathcal{C}^\vee$	ord
00002000	$E_8(a_7)$	[-1,-1,-1,-1,4,-1,-1,-1]	241920	18881/3897/1329	≤ 3
00020002	$D_4(a_1)+A_2$	[-1,7,-1,-1,-1,-1,-1,-1]	17280	3638/2/1	0
00020020	$D_4(a_1)+A_1$	[-1,6,-1,-1,-1,-1,-1,-1]	17280	8902/603/22	1
00020022	$D_4(a_1)$	[-1,-1,-1,-1,-1,-1,8,-1]	6720	3143/49/1	1
20020020	$2A_2$	[10,-1,-1,-1,-1,-1,-1,-1]	2160	1099/1/0	0
20020022	A_2+2A_1	[8,-1,-1,-1,-1,-1,-1,-1]	2160	1647/13/4	0
20020202	A_2+A_1	[7,-1,-1,-1,-1,-1,-1,-1]	2160	1763/157/26	0
20020222	A_2	[-1,-1,-1,-1,-1,-1,-1,13]	240	195/1/0	0
22202022	$2A_1$	[-1,-1,-1,-1,-1,-1,-1,8]	240	229/2/0	0
22202222	A_1	[-1,-1,-1,-1,-1,-1,-1,4]	240	224/15/0	0
22222222	0	[-1,-1,-1,-1,-1,-1,-1,-1]	1	1/0/0	0

F_4

$2\lambda_0(\mathcal{O}^\vee)$	$\mathcal{O}$	λ_1	$\#W_{\mathrm{rel}}$	$\# \cap -\mathcal{C}^\vee$	ord
0020	$F_4(a_3)$	[-1,1,-1,-1]	96	23/24/9	2
2020	A_1+A_1s	[2,-1,-1,-1]	24	15/2/1	0
2022	A_1s	[1,-1,-1,-1]	24	17/6/0	0
2222	0	[-1,-1,-1,-1]	1	1/0/0	0

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