

Groups of oscillating intermediate growth

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Abstract

We construct an uncountable family of finitely generated groups of intermediate growth, with *growth functions* of new type. These functions can have large oscillations between lower and upper bounds, both of which come from a wide class of functions. In particular, we can have growth oscillating between e^{n^α} and any prescribed function, growing as rapidly as desired. Our construction is built on top of any of the Grigorchuk groups of intermediate growth and is a variation on the limit of permutational wreath product.

1. Introduction

The growth of finitely generated groups is a beautiful subject rapidly developing in the past few decades. With a pioneer invention of groups of *intermediate growth* by Grigorchuk [Gri83] about thirty years ago, it became clear that there are groups whose growth is given by a difficult to analyze function. Even now, despite multiple improvements, much about their growth functions remains open (see [Gri05], [Gri11a]), with the sharp bounds constructed in 2011 in groups specifically designed for that purpose [BEb].

In the other direction, the problem of characterizing growth functions of groups remains a major open problem, with only partial results known. Part of the problem is a relative lack of constructions of intermediate growth groups, many of which are natural subgroups of $\text{Aut}(\mathbf{T}_k)$, similar to the original Grigorchuk groups in both the structure and analysis. In this paper we propose a new type of groups of intermediate growth, built by combining the action of Grigorchuk groups on \mathbf{T}_k and its action on a product of copies of certain finite groups H_i . By carefully controlling groups H_i , and by utilizing delicate expansion results, we ensure that the growth oscillates between two given functions. Here the smaller function is controlled by a Grigorchuk group and the larger function can be essentially *any* sufficiently rapidly growing function (up to some technical conditions). This is the first result of this type, as even

the simplest special cases could not be attained until now (see the corollaries below).

Our main result (Main [Theorem 2.3](#)) is somewhat technical and is postponed until the next section. Here in the introduction we give a rough outline of the theorem, state several corollaries, connections to other results, etc. For more on the history of the subject, general background and further references, see [Section 11](#).

For a group Γ with a generating set S , let $\gamma_\Gamma^S(n) = |B_{\Gamma,S}(n)|$, where $B_{\Gamma,S}(n)$ is the set of elements in Γ with word length $\leq n$. Suppose $f_1, f_2, g_1, g_2 : \mathbb{N} \rightarrow \mathbb{N}$ monotone increasing subexponential integer functions that satisfy

$$(*) \quad f_1 \succ f_2 \succ g_1 \succ g_2 = \gamma_{\mathbf{G}_\omega}^S, \text{ where } \mathbf{G}_\omega = \langle S \rangle \text{ is a Grigorchuk group [Gri85].}$$

Roughly, the Main Theorem states that under further technical conditions strengthening $(*)$, there exists a finitely generated group Γ and a generating set S , with growth function $h(n) = \gamma_\Gamma^S(n)$, such that $g_2(n) < h(n) < f_1(n)$ and $h(n)$ takes values in the intervals $[g_2(n), g_1(n)]$ and $[f_2(n), f_1(n)]$ infinitely often. We illustrate the theorem in [Figure 1](#).

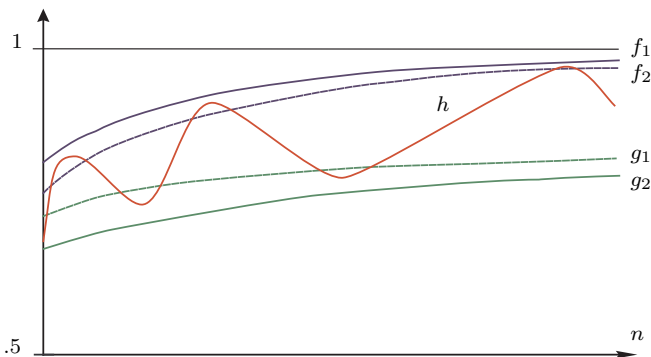


Figure 1. The graph of $\log_n \log$ of functions f_1, f_2, g_1, g_2 and h , as in the Main Theorem.

Stated differently, the Main Theorem implies that one can construct groups with growth that is oscillating within a certain range, between $\exp(n^{\alpha(n)})$, where $\alpha(n)$ is bounded from below by a constant, and a function that converges to $(1-)$ as rapidly as desired. Of course, it is conjectured that $\alpha(n) \geq 1/2$ for all groups of intermediate growth [[Gri05](#)] and n large enough (cf. [Section 11.8](#)).

To get some measure of the level of complexity of this result, let us state a corollary of independent interest. Here we take the *first Grigorchuk group* \mathbf{G} (cf. [Section 11.2](#)) and omit both f_1 and g_2 , taking g_1 to be slightly greater than the best known upper bound for the growth of \mathbf{G} .

COROLLARY 1.1 (Oscillating Growth Theorem). *For every increasing function $\mu : \mathbb{N} \rightarrow \mathbb{R}_+$, $\mu(n) = o(n)$, there exists a finitely generated group Γ and a generating set $\langle S \rangle = \Gamma$ such that*

$$\gamma_\Gamma^S(n) < \exp(n^{4/5}) \quad \text{for infinitely many } n \in \mathbb{N}$$

and

$$\gamma_\Gamma^S(n') < \exp(\mu(n')) \quad \text{for infinitely many } n' \in \mathbb{N}.$$

The corollary states that the growth of Γ is both *intermediate* (i.e., super polynomial and subexponential) and *oscillating* between two growth functions that may have different asymptotic behavior. In fact, $\mu(n)$ can be as close to the linear function as desired, so for example one can ensure that the ball sizes are $\leq \exp(n^\alpha)$ for some n and $\geq \exp(n/\log \log \log n)$ for other n , both possibilities occurring infinite often. Let us now state a stronger version of the upper bound in the same setting.

COROLLARY 1.2 (Oscillating growth with an upper bound). *For $\mu(n) = n^\alpha \log^\beta n$ with $5/6 < \alpha < 1$, or $\alpha = 1$ and $\beta < 0$, there exist a finitely generated group Γ and a generating set $\langle S \rangle = \Gamma$ such that*

$$\gamma_\Gamma^S(n) < \exp(n^{4/5}), \quad \gamma_\Gamma^S(n') < \exp(\mu(n')) \quad \text{for infinitely many } n, n' \in \mathbb{N}$$

and

$$\gamma_\Gamma^S(m) < \exp(\mu(9m)) \quad \text{for all sufficiently large } m \in \mathbb{N}.$$

In other words, $\gamma_\Gamma(n_i)$ has the same asymptotics as $e^{\mu(n_i)}$ for a certain infinite subsequence $\{n_i\}$. Note a mild restriction on α , which is a byproduct of our technique. (See other examples in the next section.)

Now, the Main Theorem and the corollaries are related to several other results. On the one hand, the growth of balls in the first Grigorchuk group \mathbf{G} is bounded from above and below by

$$\exp(n^\beta) < \gamma_{\mathbf{G}}^S(n) < \exp(n^\alpha) \quad \text{for all generating sets } S,$$

integers n large enough, and where $\alpha = 0.7675$ and $\beta = 0.5207$. Since this α is the smallest available upper bound for any known group of intermediate growth (see Section 11.2), this explain the lower bound in the corollaries. (In fact, the power $4/5$ there can be lowered to any $\alpha' > \alpha$.)

For the upper bound, a result by Erschler [Ers05] states that there is a group of intermediate growth such that $\gamma_\Gamma^S(n) > f(n)$ for all n large enough. This result does not specify exactly the asymptotic behavior of the growth function $\gamma_\Gamma^S(n)$, and it is the opposite extreme when compared to the Oscillating Growth Theorem, as here *both* the upper and lower bounds can be as close to the exponential function as desired. This also underscores the major difference

with our Main Theorem, as in this paper we emphasize the upper bounds on the growth, which can be essentially any subexponential function.

Combined with this Erschler’s result, the Main Theorem states that one can get an oscillating growth phenomenon as close to the exponential function as desired. For example, Erschler showed in [Ers04] that a certain group Grigorchuk group \mathbf{G}_ω has growth between $g_2 = \exp(n/\log^{2+\epsilon} n)$ and $g_1 = \exp(n^{1-\epsilon}/\log n)$ for any $\epsilon > 0$. The following result is a special case of the Main Theorem applied to this group \mathbf{G}_ω .

COROLLARY 1.3. *Fix $\epsilon > 0$. Define four functions: $g_2(n) = e^{n/\log^{2+\epsilon} n}$, $g_1(n) = e^{n/\log^{1-\epsilon} n}$, $f_2(n) = e^{n/\log \log n}$, and $f_1(n) = e^{n\sqrt{\log \log \log n}/\log \log n}$. Then there exists a finitely generated group Γ and a generating set $\langle S \rangle = \Gamma$, with growth function $h(n) = \gamma_\Gamma^S(n)$ such that*

$$g_2(n) < h(n) < f_1(n) \text{ for all } n \text{ large enough}$$

and

$$h(m) < g_1(m), \quad h(m') > f_2(m') \text{ for infinitely many } m, m' \in \mathbb{N}.$$

Of course, here the functions f_1, f_2 are chosen somewhat arbitrarily to illustrate the power of our Main Theorem.

Let us now say a few words about the *oscillation phenomenon*. In a recent paper [Bri], Briussel shows that there is a group Γ of intermediate growth such that

$$\liminf \log_n \log \gamma_\Gamma(n) = \alpha_- \quad \text{and} \quad \limsup \log_n \log \gamma_\Gamma(n) = \alpha_+$$

for any fixed $\alpha = 0.7675 \leq \alpha_- \leq \alpha_+ \leq 1$. This result is somewhat weaker than our Main Theorem when it comes to the range of asymptotics of upper limits, but it is stronger in a sense that the lower limits can be prescribed in advance, and $\alpha_+ - \alpha_-$ can be as small as desired (cf. Example 2.4). Since Briussel uses groups different from \mathbf{G}_ω , the Main Theorem cannot use them as an input. We postpone further discussion of this until Section 10 (see also Section 11.14).

A starting point for the construction in the proof of the Main Theorem (Theorem 2.3) is a sequence of finite groups G_i with generating sets S_i such that the growth of small balls in $X_i = \text{Cayley}(G_i, S_i)$ is roughly $\gamma_{\mathbf{G}_\omega}$, but the diameter of Cayley graphs X_i is close to logarithmic; i.e., the growth of large balls is almost exponential. These groups and generating sets can be combined into an infinite group Γ and generating set S such that for certain n , balls $\gamma_\Gamma^S(n)$ behave as small balls in X_i , while for other values of n , these balls behave as large balls in X_i , and thus have almost exponential size. This behavior implies that the size of these balls oscillates as in the theorem.

The rest of the paper is structured as follows. We begin with the statement of the Main Theorem in Section 2, where we also give examples and a very brief outline of the proof idea. We continue with basic definitions and notations in Section 3. We then explore graph and group limits in Section 4. We explore the Grigorchuk groups \mathbf{G}_ω in Section 5, giving some preliminary technical results, which we continue in Section 6. We then prove the Oscillating Growth Theorem (Corollary 1.1) in Section 7, as without functions g_1 and f_2 the result is technically easier to obtain. We then prove the Main Theorem 2.3 in Section 8. A key technical result (Main Lemma 8.1) is postponed until Section 9, while further generalizations are presented in Section 10. We conclude with final remarks and open problems in Section 11.

2. The Main Theorem

2.1. *The statement.* We begin with two technical definitions.

Definition 2.1. A function $f : \mathbb{N} \rightarrow \mathbb{R}$ is called *admissible* if $f(n)$ is increasing, subexponential, and the ratio

$$\frac{n}{\log f(n)} \text{ is increasing.}$$

Definition 2.2. If f is an admissible function, define a function $f^* : \mathbb{R} \rightarrow \mathbb{R}$ as the following inverse function:

$$f^*(z) = \Phi^{-1}(z), \quad \text{where } \Phi(x) = \frac{x}{\log f(x)}.$$

THEOREM 2.3 (Main Theorem). *Let $f_1, f_2, g_1, g_2 : \mathbb{N} \rightarrow \mathbb{N}$ be functions that satisfy the following conditions:*

- (i) f_1 and f_2 are admissible;
- (ii) $f_2(n)^3 = o(f_1(n))$;
- (iii) f_1/g_1 is increasing;
- (iv) $g_2(n) \leq \gamma_{\mathbf{G}_\omega}(n)$, where \mathbf{G}_ω is a Grigorchuk group of intermediate growth;
- (v) $\gamma_{\mathbf{G}_\omega}(n) = o(g_1(n))$;
- (vi) $\exp \left[\frac{\log g_1(n)}{Cn^2} f_1^* \left(\frac{n}{C \log g_1(n)} \right) \right] > \frac{C f_2^*(Cn)}{n^2}$ for all $C > 0$ and $n = n(C)$ sufficiently large.

Then there exist a finitely generated group Γ and a generating set $\langle S \rangle = \Gamma$, with growth function $h(n) = \gamma_\Gamma^S(n)$, such that

- (1) $h(n) < f_1(n)$ for all $n \in \mathbb{N}$ large enough,
- (2) $h(n) > f_2(n)$ for infinitely many $n \in \mathbb{N}$,
- (3) $h(n) < g_1(n)$ for infinitely many $n \in \mathbb{N}$,
- (4) $h(n) \geq g_2(n)$ for all $n \in \mathbb{N}$.

Although the conditions are technical, they are mostly mild in the sense that many natural functions satisfy them. For example, [condition \(ii\)](#) may seem strong, but notice that our functions are greater than $\exp(n^\alpha)$, in which case $(f_2)^3 \preceq f_2$. Similarly, [condition \(iv\)](#) may seem restrictive, but in fact, due to Erschler’s theorem [Ers05], the growth of such Grigorchuk groups can be as large as desired, even if we do not know anything else about these growth functions and with the currently available tools cannot yet control their growth.

2.2. Examples. [Condition \(vi\)](#) implies that the growth of f_1 is somewhat faster than that of g_1 . However, if the growth of g_1 is close to exponential, this condition also implies that f_1 is significantly larger than f_2 , since $f_1^*(\cdot)$ is small in that case. To clarify this condition, we list some examples below.

Example 2.4. Let $\log g_1(n) \sim n^\alpha$ and $\log f_1(n) \sim n^\beta$ for some $0 < \alpha < \beta \leq 1$. [Condition \(vi\)](#) in this case says that $\beta > 1/(2 - \alpha) > \alpha$. This implies that the interval (α, β) cannot be arbitrary and thus, in particular, the Main Theorem cannot imply Briussel’s theorem [Bri] (see below).

Example 2.5. Let $g_1(n) \sim n/\log^\alpha n$, $\log f_2(n) \sim n/\log^\nu n$, and $\log f_1(n) \sim n/\log^\beta n$, where $0 < \beta \leq \nu \leq \alpha$. [Condition \(vi\)](#) in this case says that $\nu > 1/(\alpha/\beta - 2)$. For example, $\alpha = 5$, $\nu = 3$ and $\beta = 2$ works, but in order to have $\nu = \beta = 2$, one needs $\alpha > 5$.

Example 2.6. Let $g_1(n) \sim n/\log^\alpha n$ be as before, but now $\log f_2(n) \sim n/(\log \log n)^\nu$, and $\log f_1(n) \sim n/(\log \log n)^\beta$, where $0 < \beta \leq \nu$ and $\alpha > 0$. [Condition \(vi\)](#) in this case says that $\nu > 1/(\alpha/\beta - 1)$. For example, $\alpha = 4$, $\nu = \beta = 2$ works fine. More generally, any $\nu = \beta > \alpha$ satisfy the condition, as well as $0 < \nu = \beta < \alpha - 1$.

2.3. A sketch of the group construction and the proof. The group Γ is constructed from a sequence of integers $\{m_i\}$ and a sequences of finite groups $\{H_i \subset \text{Sym}(k_i)\}$ generated by four involutions. The group Γ acts on a decorated binary tree $\widehat{\mathbf{T}}_2$ obtained from the (usual) infinite binary tree \mathbf{T}_2 as follows.¹ To each vertex on level m_i we attach k_i leaves, which are permuted by the group H_i . The group Γ , like the Grigorchuk group \mathbf{G}_ω , is generated by four involutions, whose faithful action is recursively defined. The definition is similar to the usual one; however, once we reach a vertex with leaves, one needs to specify the action on these leaves, which is given by a generator of the group H_i . We illustrate the action of Γ on $\widehat{\mathbf{T}}_2$ in [Figure 2](#), where the set of leaves of size H_i is decorating all vertices on m_i -th level. Now, the reasoning

¹The description of the group Γ is written in the language of [GP08], which is different from the rest of this paper.

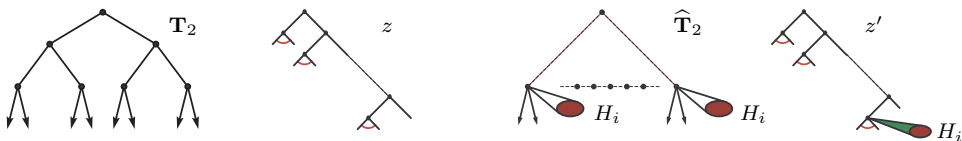


Figure 2. The binary tree \mathbf{T}_2 , generator $z \in \{a, b, c, d\}$ of \mathbf{G}_ω acting by transposing selected branches in \mathbf{T}_2 , the decorated binary tree $\widehat{\mathbf{T}}_2$, and the generator z' of Γ acting on $\widehat{\mathbf{T}}_2$.

behind the proof of the Main Theorem is the following. Roughly speaking, the balls of radius n does not depend on the group H_i for $m_i > \log n$. Therefore if the levels m_i grow sufficiently fast, the growth of Γ is similar to the growth of \mathbf{G}_ω in the last final interval before the i -th level is reached. However, once we reach an element in H_i , the growth of Γ is determined by the growth of H_i and can be much more rapid in this period. If the groups H_i have logarithmic diameter and their sizes increase sufficiently fast, we can ensure that the growth of Γ is as close to the exponential function as desired.

3. Basic definitions and notations

3.1. *Growth of groups.* Let $f, g : \mathbb{N} \rightarrow \mathbb{N}$ be two integer functions such that $f(n), g(n) \rightarrow \infty$ as $n \rightarrow \infty$. We write

$$\begin{aligned}
 f \ll g & \text{ if } f(n) < g(cn) \quad \text{for some } c > 0 \text{ and infinitely many } n \in \mathbb{N}, \\
 f \lesssim g & \text{ if } f(n) < g(cn) \quad \text{for some } c > 0 \text{ and all sufficiently large } n \in \mathbb{N}.
 \end{aligned}$$

For example, $n^{100} \lesssim 3^n \lesssim 2^n$ and $2^n \ll n^{n(n \bmod 2)} \ll n^2$. Note here that “ \ll ” is not transitive since 2^n is *not* $\ll n^2$. We write $f \sim g$ if $f \lesssim g$ and $g \lesssim f$.

Let Γ be a finitely generated group and $S = S^{-1}$ a symmetric generating set, $\Gamma = \langle S \rangle$. Denote by $B_{\Gamma,S}(n)$ the set of elements $g \in \Gamma$ such that $\ell_S(g) \leq n$, where ℓ_S is the *word length*, and let $\gamma_\Gamma^S(n) = |B_{\Gamma,S}(n)|$. For every other symmetric generating set $\langle S' \rangle = \Gamma$, we have $C_1 \ell_S(g) \leq \ell_{S'}(g) \leq C_2 \ell_S(g)$, which implies that $\gamma_\Gamma^S(n) \lesssim \gamma_\Gamma^{S'}(n) \lesssim \gamma_\Gamma^S(n)$. In other words, the asymptotics of $\gamma_\Gamma^S(n)$ are independent of the generating set S , so whenever possible we will write $\gamma_\Gamma(n)$ for simplicity.

The group Γ has *exponential growth* if $\gamma_\Gamma(n) \gtrsim \exp(n)$ and *polynomial growth* if $\gamma_\Gamma(n) \lesssim n^c$ for some $c > 0$. Similarly, Γ has *intermediate growth* if $\gamma_\Gamma(n) \gtrsim n^c$ for all $c > 0$ and $\gamma_\Gamma(n) \lesssim \exp f(n)$ for some $f(n)/n \rightarrow 0$, as $n \rightarrow \infty$.

Remark 3.1. Using this notation, the Main Theorem says that for any functions

$$f_1 \gtrsim f_2 \gtrsim g_1 \gtrsim g_2 = \gamma_{\mathbf{G}_\omega}^S,$$

which satisfy additional technical assumptions, and where \mathbf{G}_ω is a *Grigorchuk group* of intermediate growth, there exists a group Γ , whose growth function satisfies

$$f_1 \asymp \gamma_\Gamma \asymp g_2, \quad \gamma_\Gamma \gg f_2, \quad \text{and} \quad \gamma_\Gamma \ll g_1.$$

In the special case when $f_1 \sim f_2$ and $g_1 \sim g_2$, this means that

$$f_1 \asymp \gamma_\Gamma \asymp g_2 \quad \text{and} \quad f_1 \ll \gamma_\Gamma \ll g_2.$$

For a group of intermediate growth, define

$$\alpha(\Gamma) = \lim_{n \rightarrow \infty} \log_n \log \gamma_\Gamma(n) \quad \text{if this limit exists,}$$

$$\alpha_+(\Gamma) = \limsup_{n \rightarrow \infty} \log_n \log \gamma_\Gamma(n), \quad \text{and} \quad \alpha_-(\Gamma) = \liminf_{n \rightarrow \infty} \log_n \log \gamma_\Gamma(n).$$

Our Main Theorem (and the result in [Bri]) shows that $\alpha(\Gamma)$ does not necessarily exist, the first construction of this kind. The result of Briussel mentioned in the introduction implies that there exist a group Γ of intermediate growth with $\alpha(\Gamma) = \nu$ for any given $\alpha \leq \nu \leq 1$.

In addition to the growth function γ_G^S , we define a *normal growth function* $\tilde{\gamma}_{G,X}^S$ as the number of elements in the group G that can be expressed as words in the free group on length n , which also lie in the normal closure of elements $X \subset G$. In this paper we consider only the case $X = \{r\}$, which we denote $\tilde{\gamma}_{G,r}^S$.

3.2. Notation for groups and their products. To simplify the notation, we use \mathbb{Z}_m for $\mathbb{Z}/m\mathbb{Z}$ and trust this will not leave to any confusion. Let $\text{PSL}_2(N)$ denote the group $\text{PSL}_2(\mathbb{Z}/N\mathbb{Z})$ and \mathbf{F}_k denote the free group on k generators. By $G \rightarrow H$ we denote an *epimorphism* between the groups.

The group with presentation

$$\mathcal{G} = \langle a, b, c, d \mid a^2 = b^2 = c^2 = d^2 = bcd = 1 \rangle$$

will play a central role throughout the paper, as all our groups and also all Grigorchuk groups are homomorphic images of \mathcal{G} . We call it a *free Grigorchuk group*.

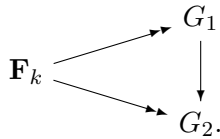
The *direct product* of groups G and H is denoted $G \oplus H$, rather than the more standard $G \times H$. This notation allows us to write infinite product as $\bigoplus G_i$, where all but finitely many terms are trivial, and we will typically omit the index of summation. We denote by $\prod G_i$ the (usually uncountable) group of sequences of group elements, without any finiteness conditions. Of course, the groups $\bigoplus G_i$ and $\prod G_i$ often are not finitely generated.

Finally, let $H \wr G = G \ltimes H^\ell$ denote the permutation wreath product of the groups, where $G \subset \Sigma_\ell$ is a permutation group.

3.3. *Marked groups and their homomorphisms.* All groups we will consider will have ordered finite generating sets of the same size k . Whenever we mention a group G , we will mean a pair (G, S) where $S = \{s_1, \dots, s_k\}$ is an ordered generating set of G of size k . Although typically described with the Cayley graph, the order on the generators is crucial for our results. We call these *marked groups*, and k will always denote the size of the generating set. By a slight abuse of notation, we will often drop S and refer to a *marked group* G , when S is either clear from the context or not very relevant.

Throughout the paper, the homomorphisms between marked groups will send one generating set to the other. Formally, let (G, S) and (G', S') be marked groups, where $S = \{s_1, \dots, s_k\}$ and $S' = \{s'_1, \dots, s'_k\}$. Then $\phi : (G, S) \rightarrow (G', S')$ is a *marked group homomorphism* if $\phi(s_j) = s'_j$, and this map on generators extends to the (usual) homomorphism between groups: $\phi : G \rightarrow G'$.

An equivalent way to think of marked groups is to consider epimorphisms $\mathbf{F}_k \twoheadrightarrow G_1, \mathbf{F}_k \twoheadrightarrow G_2$ so that the map between groups correspond to commutative diagram



3.4. *Direct sums of marked groups.* Let $\{G_i\}$ be a sequence of marked groups defined above, or more formally $\{(G_i, S_i)\}$. Denote by $G = \otimes G_i$ the subgroup of $\prod G_i$ generated by diagonally embedding the generating sets S_i , see [Definition 4.1](#). Of course, group G critically depends on the ordering of elements in S_i .

3.5. *Miscellanea.* With $\omega = (x_1, x_2, \dots)$ we denote an infinite word in $\{0, 1, 2\}$ which will be used to construct the Grigorchuk group \mathbf{G}_ω . Such a word is called stabilizing if all x_i are eventually the same. In this case the group \mathbf{G}_ω becomes virtually nilpotent.

We use $\log n$ to denote natural logarithms, but normally the base will be irrelevant. The radius of balls in the groups will be denoted with n . Finally, we use $\mathbb{N} = \{1, 2, \dots\}$.

4. Limits of groups

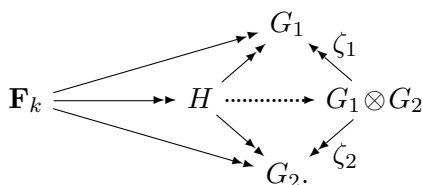
All marked groups we consider will have ordered finite generating sets of the same size k , and all maps between marked groups will send one generating set to the other.

Definition 4.1. Let $\{(G_i, S_i)\}, i \in I$ be a sequence of marked groups with generating sets $S_i = \{s_{i1}, \dots, s_{ik}\}$. Define the $(\Gamma, S) = (\otimes G_i, S)$ to be the subgroup of $\prod G_i$ generated by diagonally embedding the generating sets of

each G_i ; i.e., $\otimes G_i = \langle s_1, \dots, s_k \rangle$ where $s_j = \{s_{ij}\} \in \prod G_i$. Notice that Γ comes with canonical epimorphisms $\zeta_i : \Gamma \rightarrow G_i$. Often the generating sets will be clear from the context, and we will simply use $\Gamma = \otimes G_i$. When the index set contains only two elements we denote the product by $G_1 \otimes G_2$.

Remark 4.2. The notations $\otimes G_i$ and $G_1 \otimes G_2$ are slightly misleading since these products depend not only on the groups but also on the generating sets. In this paper all groups are marked and come with a fixed generations set, which justifies this abuse of the notation.

Remark 4.3. The group $G_1 \otimes G_2$ satisfies the following universal property. For any marked group H such that the left two triangles commute, there exists a homomorphism $H \rightarrow G_1 \otimes G_2$:



LEMMA 4.4. (i) If G_i is any sequence of marked groups, then growth function of $\Gamma = \otimes G_i$ is larger than the growth functions of each G_i ; i.e.,

$$\gamma_\Gamma(n) \geq \gamma_{G_i}(n) \quad \text{for all } i.$$

(ii) If G_1 and G_2 are two marked groups, then the growth function of $G_1 \otimes G_2$ is bounded by the product of the growth functions for G_i ,

$$\gamma_{G_1 \otimes G_2}(n) \leq \gamma_{G_1}(n) \cdot \gamma_{G_2}(n).$$

Proof. By definition of the product of marked groups, the map $\zeta_i : \Gamma \rightarrow G_i$ is not only surjective but also satisfies $\zeta_i(B_{\Gamma,S}(n)) = B_{G_i,S_i}(n)$, which implies the first part. The injectivity of the product of the projections ζ_1 and ζ_2 and the observation that

$$\zeta_1 \times \zeta_2 : B_{\Gamma,S}(n) \hookrightarrow B_{G_1,S_1}(n) \times B_{G_2,S_2}(n)$$

imply the second part. □

Definition 4.5. We say that the sequence of marked groups $\{(G_i, S_i)\}$ converge (in the the so-called *Chabauty topology*) to a group (G, S) if for any n , there exists $m = m(n)$ such that for any $i > m$, the ball of radius n in G_i is the same as the ball of radius n in G . We write $\lim G_i = G$.

Equivalently, this can be stated as follows. If $R_i = \ker(\mathbf{F}_k \rightarrow G_i)$ and $R = \ker(\mathbf{F}_k \rightarrow G)$, then

$$\lim_{i \rightarrow \infty} R_i \cap B_{\mathbf{F}_k}(n) = R \cap B_{\mathbf{F}_k}(n);$$

i.e., for a fixed n and sufficiently large i the sets $R_i \cap B_{\mathbf{F}_k}(n)$ and $R \cap B_{\mathbf{F}_k}(n)$ coincide.

LEMMA 4.6. *Let $\{G_i\}$ be a sequence of marked groups that converge to a marked group G . Define the $\Gamma = \otimes G_i$. Then there is an epimorphism $\pi : \Gamma \rightarrow G$. Moreover, the kernel of π is equal to the intersection $\Gamma \cap \bigoplus G_i$.*

Proof. There is an obvious map π that sends the generators of Γ to the generators of G . A word w represents the trivial element in Γ if and only if w is trivial in all G_i . Therefore, this word is trivial in infinitely many of G_i and is as well trivial in the limit G ; i.e., the map π extends to a group homomorphism.

The convergence of $\{G_i\} \rightarrow G$ implies that if a word $w \in \mathbf{F}$ of length n that evaluates to $\{g_i\} \in \prod G_i$ is in the kernel of π , then the components g_i have to be trivial for large i . (Otherwise, the ball of radius n in the Cayley graph of G_i will be different from the one in G .) Therefore, the word $w \in \bigoplus G_i$. The other inclusion is obvious. □

Lemma 4.6 allows us to think of G as the *group at infinity* for Γ . We will be interested in sequences of groups that satisfy the additional property that

$$\text{(splitting)} \quad \lim_{i \rightarrow \infty} G_i = G \quad \text{and} \quad \Gamma = \left[\otimes G_i \right] \cap \left[\bigoplus G_i \right] = \bigoplus N_i,$$

where N_i are necessary normal subgroups of G_i .

LEMMA 4.7. *If the groups G_i satisfy the condition (splitting), then there exists a group homomorphism $\pi_i : G \rightarrow G_i/N_i$ that makes the following diagram commute:*

$$\begin{array}{ccccccc} 1 & \longrightarrow & \ker \pi \subset & \longrightarrow & \Gamma & \xrightarrow{\pi} & G \longrightarrow 1 \\ & & \zeta_i \downarrow & & \zeta_i \downarrow & & \pi_i \downarrow \vdots \\ 1 & \longrightarrow & N_i \subset & \longrightarrow & G_i & \longrightarrow & G_i/N_i \longrightarrow 1. \end{array}$$

Proof. The existence and uniqueness of the homomorphism π follows from the exactness of the rows in the diagram above. □

Using these maps one can obtain estimates for the size of the ball in the groups Γ .

LEMMA 4.8. *If the groups $\{G_i\}$ satisfy the condition (splitting) and $B_{G_i}(n)$ is the same as $B_G(n)$ for $i > m$, then*

$$\gamma_\Gamma(n) \leq \gamma_G(n) \prod_{j \leq m} |N_j|.$$

Proof. If two elements $g, h \in B_\Gamma(n)$ are sent to the same elements in $B_G(n)$, then they are also the same in $B_{G_j}(n)$ for all $j > m$; i.e., their difference $g^{-1}h$ is inside

$$\Gamma \cap \bigoplus_{j \leq m} G_i = \bigoplus_{j \leq m} N_j.$$

Thus, the fibers of the restriction of π to $B_\Gamma(n)$ have size at most $\prod_{j \leq m} |N_j|$, which implies the inequality in the lemma. \square

5. The Grigorchuk group

5.1. *Basic results.* In this section we present variations of standard results on the Grigorchuk group \mathbf{G}_ω (cf. Section 11.1). Rather than give standard definitions as a subgroup of $\text{Aut}(\mathbf{T}_2)$, we define \mathbf{G} via its properties. We refer to [GP08], [Har00] for a more traditional introduction and most results in this subsection.

Definition 5.1. Let $\varphi : \mathcal{G} \rightarrow \mathcal{G}$ denote the automorphism of order 3 of the group G which fixes a , and cyclicly permutes the generators b, c and d ; i.e., $\varphi(a) = a, \varphi(b) = c, \varphi(c) = d, \varphi(d) = b$.

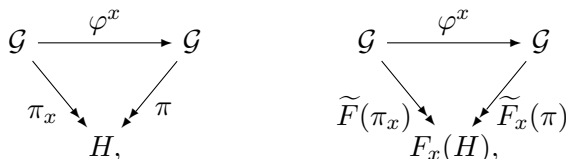
Definition 5.2. Let $\pi : \mathcal{G} \rightarrow H$ be an epimorphism; i.e., suppose the group H comes with a generating set consisting of four involutions $\{a, b, c, d\}$ that satisfy $bcd = 1$. By $F(H)$ we define the subgroups of $H \wr \mathbb{Z}_2 = \mathbb{Z}_2 \times (H \oplus H)$ generated by the elements A, B, C and D defined as

$$A = (\xi; 1, 1), \quad B = (1; a, b), \quad C = (1; a, c) \quad \text{and} \quad D = (1; 1, d),$$

where $\xi^2 = 1$ is the generator of \mathbb{Z}_2 . It is easy to verify that A, B, C and D are involutions that satisfy $BCD = 1$, which allows us to define an epimorphism $\tilde{F}(\pi) : \mathcal{G} \rightarrow F(H)$.

The construction can be twisted by the powers automorphism φ :

$$\tilde{F}_x(\pi) := \tilde{F}(\pi \circ \varphi^{-x}) \circ \varphi^x,$$



An equivalent way of defining the group $F_x(H)$ is as the subgroups generated by

$$\begin{aligned} A_0 &= (\xi; 1, 1), & B_0 &= (1; a, b), & C_0 &= (1; a, c), & D_0 &= (1; 1, d), \\ A_1 &= (\xi; 1, 1), & B_1 &= (1; a, b), & C_1 &= (1; 1, c), & D_1 &= (1; a, d), \\ A_2 &= (\xi; 1, 1), & B_2 &= (1; 1, b), & C_2 &= (1; a, c), & D_2 &= (1; a, d), \end{aligned}$$

Remark 5.3. Strictly speaking, the notation $F_i(H)$ is not precise since in order to define this group we need to specify a generating set; thus, the correct notation should be $\tilde{F}_i(\pi)$. However, since all groups H are marked, i.e., come with an epimorphism $\mathcal{G} \rightarrow H$, this allows us to slightly simplify the notation.

PROPOSITION 5.4. *Each F_x is an endofunctor of the category of homomorphic images of \mathcal{G} ; i.e., a group homomorphism $H_1 \rightarrow H_2$ that preserves the generators induces a group homomorphism $F_x(H_1) \rightarrow F_x(H_2)$:*



PROPOSITION 5.5. *The functor F_x commutes with the products of marked groups; i.e.,*

$$F_x \left(\bigotimes H_j \right) = \bigotimes F_x(H_j).$$

Proof. This is an immediate consequence of the functoriality of F_i and the universal property of the products of marked groups. Equivalently, one can check it directly from the definitions. □

Definition 5.6. One can define the functor F_ω for any finite word $\omega \in \{0, 1, 2\}^*$ as follows:

$$F_{x_1x_2\dots x_i}(H) := F_{x_1}(F_{x_2}(\dots F_{x_i}(H) \dots)).$$

If ω is an infinite word on the letters $\{0, 1, 2\}$ by F_ω^i , we will denote the functor F_{ω_i} , where ω_i is the prefix of ω of length i .

THEOREM 5.7 (cf. [Gri85]). *The Grigorchuk group \mathbf{G} is the unique group such that $\mathbf{G} = F_{012}(\mathbf{G})$.*

Remark 5.8. In [Gri85], Grigorchuk defined a group \mathbf{G}_ω for any infinite word ω . One way to define these groups is by $\mathbf{G}_{x\omega} = F_x(\mathbf{G}_\omega)$, where x is any letter in $\{0, 1, 2\}$. The first Grigorchuk group is denoted $\mathbf{G} = \mathbf{G}_{(012)^\infty}$, which corresponds to a periodic infinite word. If the word ω stabilizes, then the group \mathbf{G}_ω is virtually nilpotent and has polynomial growth.

Although, we will not use Theorem 5.7, the following construction gives the idea of the connection. Let $\mathbf{G}_{\omega,i} = F_\omega^i(\mathbf{1})$, where $\mathbf{1}$ denotes the trivial group with one element (with the trivial map $\mathcal{G} \rightarrow \mathbf{1}$).

PROPOSITION 5.9. *There is a canonical epimorphism $\mathbf{G}_\omega \rightarrow \mathbf{G}_{\omega,i}$. The groups $\mathbf{G}_{\omega,i}$ naturally act on the finite binary rooted tree of depth i , and this action comes from the standard action of the Grigorchuk group on the infinite binary tree \mathbf{T}_2 .* □

Remark 5.10. The group $F_\omega^i(H)$ is a subgroup of the permutational wreath product $H \wr_{X_i} \mathbf{G}_{\omega,i}$, where X_i is the set of leaves of the binary tree of depth i (cf. Section 2.3).

5.2. Contraction in Grigorchuk groups.

LEMMA 5.11. *Let $\pi : \mathcal{G} \rightarrow H$ be an epimorphism; i.e., the group H is generated by four nontrivial involutions that satisfy $bcd = 1$. If the word ω does not stabilize, then the balls of radius $n \leq \vartheta(m)$ in the groups $F_\omega^m(H)$ and \mathbf{G}_ω coincide, where $\vartheta(m) = 2^m - 1$ is the strictly increasing function $\vartheta : \mathbb{N} \rightarrow \mathbb{N}$.*

Proof. It is enough to show that the set of words of length $2\vartheta(m)$ that are trivial in $F_\omega^m(H)$ is the same as the ones that are trivial in \mathbf{G}_ω .

Observe that every word $w \in \mathcal{G}$ can be expanded to $(\xi^{a_w}; w', w'') \in F(\bar{H})$, where w' and w'' are words of length $\leq (|w| + 1)/2$. If $a_w \neq 0$, then w is not zero in $F(\bar{H})$ for any group \bar{H} . Iterating this m times shows that any word w of length $< 2\vartheta(m)$ is either nontrivial in both $F_\omega^m(H)$ and \mathbf{G}_ω , or evaluates to many words of length at most 1 acting on the copies of H . If one of these words is nontrivial, then w is nontrivial; otherwise it is trivial.

Here we are using that the nonstabilization on ω implies that the elementals a, b, c, d are nontrivial in $\mathbf{G}_{\bar{\omega}}$ for any suffix $\bar{\omega}$ of ω . □

Remark 5.12. One can show that a stronger result holds if ω does not contain $0^k, 1^k$ and 2^k as subwords. Indeed, then the balls in $F_\omega^m(H)$ and \mathbf{G}_ω of radius $\vartheta(m)$ are the same as the balls in $\mathbf{G}_{\omega, m+k+1}$. The last group is of the form $F_\omega^m(H')$, where $H' = F_{\omega'}(\mathbf{1})$ and ω' is a subword of ω of length $k+1$. The condition on ω implies that the generators a, b, c and d are nontrivial in H' .

Remark 5.13. Here we use that the length of each word w' and w'' is shorter than w . In many cases one can also show that the sum of the lengths (or some suitably defined norm) of these words is less than that of w . Such a contracting property is used to obtain upper bounds for the growth of \mathbf{G}_ω , see [Bar98], [BGŠ03], [Gri85], [MP01].

We conclude with an immediate corollary of Proposition 5.9 and Lemma 5.11, which can also be found in [Gri11a].

COROLLARY 5.14. *Let $\{\mathcal{G} \rightarrow H_i\}$ be any sequence of groups generated by $k = 4$ nontrivial involutions, and let $\{m_i\}$ be an increasing sequence. Then the sequence of groups $\{F_\omega^{m_i}(H_i)\}$ converges (in the Chabauty topology) to \mathbf{G}_ω .*

Remark 5.15. This can be used as an alternative definition of the groups \mathbf{G}_ω , which shows that there exists a canonical epimorphism $\mathcal{G} \rightarrow \mathbf{G}_\omega$.

5.3. Growth lemmas.

Let r denote the element $[c, [d, [b, (ad)^4]]] \in \mathcal{G}$, and let $r_x = \varphi^x(r)$ be its twists by the automorphism φ described in Definition 5.1.

LEMMA 5.16. *Let $\mathcal{G} \rightarrow H$ be a finite image of \mathcal{G} that is normally generated by element $r_{x_{k+1}}$ defined above. Then the kernel of the map $F_\omega^k(H) \rightarrow \mathbf{G}_{\omega, k}$*

induced by F_ω^k from the trivial homomorphism $H \rightarrow \mathbf{1}$, is isomorphic to $H^{\oplus 2^k}$. Moreover, there exists a word $\eta_{\omega,k} \in F$ of length $\leq K \cdot 2^k$ such that the image of $\eta_{\omega,k}$ in $F_\omega^k(H)$ normally generated this kernel and $\eta_{\omega,k}$ is trivial in $F_\omega^{k+1}(H')$ for every $\mathcal{G} \rightarrow H'$.

Proof. Consider the substitutions σ, τ (endomorphisms $\mathcal{G} \rightarrow \mathcal{G}$), defined as follows:

- $\sigma(a) = aca$ and $\sigma(s) = s$ for $s \in \{b, c, d\}$;
- $\tau(a) = c, \tau(b) = \tau(c) = a$ and $\tau(d) = 1$.

It is easy to see that for any word η , the evaluation of $\sigma(\eta)$ in $F(H)$ is equal to

$$(1; \tau(\eta), \eta) \in \{1\} \times H \times H \subset H \wr \mathbb{Z}_2.$$

Define words $\{w_i\}$ for $i = 0, \dots, k$ as follows: $w_0 = r_{x_{k+1}}$ and $w_{i+1} = \sigma_{x_{k-i}}(w_i)$, where $\sigma_{x_i} = \varphi^{x_i} \sigma \varphi^{-x_i}$ is the twist of the substitution σ . Notice that all these words have the form $[c, [d, [b, *]]]$ because σ_{x_i} fixes b, c and d . Therefore $\tau_x(w_i) = 1$

By construction, the word $\eta_{\omega,k} = w_k$ evaluates in $F_\omega^k(\bar{H})$ to $r_{x_{k+1}}$ in one of the copies of \bar{H} for any group \bar{H} . The expression $(ad)^4$ inside r ensures that $r_{x_{k+1}}$ is trivial if \bar{H} is of the form $F_{x_{k+1}}(\bar{H}')$, which proves the last claim.

The first claim follows from the transitivity of the action of \mathbf{G}_ω (and $F_\omega^k(H)$) on the m -th level of the binary tree and the assumption that H is normally generated by $r_{x_{k+1}}$. □

Remark 5.17. The lemma says that if H is normally generated by the element r , then the inclusion in [Remark 5.10](#) is an equality.

COROLLARY 5.18. *For H as in [Lemma 5.16](#) and every integer $n \geq 1$, we have*

$$\gamma_H(n) \leq \gamma_{F_\omega^k(H)}(c_k n), \quad \text{where } c_k = 2^{k+1} - 1.$$

Proof. Use that $\sigma_{k,\omega}(B_H(n)) \subset B_{F_\omega^k(S)}(c_k n)$ because the composition $\sigma_{k,\omega}$ of k substitutions σ_{x_j} increases the lengths of the words at most $2^{k+1} - 1$ times. □

COROLLARY 5.19. *For H as in [Lemma 5.16](#) and every integer $n \geq 1$ and any $t < 2^k$, we have*

$$\tilde{\gamma}_{H,r_{x_{k+1}}}(n)^t \leq \gamma_{F_\omega^k(H)}(c_k t n),$$

where $\tilde{\gamma}_H(n)$ is the normal growth function, i.e. the number of elements in H that can be expressed as words of length less than n in the normal subgroup $X = \langle r_{x_{k+1}} \rangle^{\mathbf{F}_4}$ of the free group $\mathbf{F}_4 = \langle a, b, c, d \rangle$.

Proof. As in [Corollary 5.18](#), but use the fact that there are many copies of H . □

6. Growth in $\mathrm{PSL}_2(\mathbb{Z}_N)$

For the proof of [Theorem 2.3](#), we need the following technical result.

LEMMA 6.1. *Let N be such that -1 is a square in \mathbb{Z}_N and $2 \nmid N$, i.e., the only prime factors that appear in the prime decomposition of N are of the form $p \equiv 1 \pmod{4}$. Then there exists a generating set $S_N = \{a, b, c, d\}$ of the group $H_N = \mathrm{PSL}_2(\mathbb{Z}_N)$ such that*

- (1) *there is an epimorphism of marked groups $\mathcal{G} \rightarrow H_N = \mathrm{PSL}_2(\mathbb{Z}_N)$;*
- (2) *the group H_N is normally generated by the image of element*

$$r = [c, [d, [b, (ad)^4]]];$$

- (3) *$\gamma_{H_N}(n) > \exp(n/K)$, for $n < K \log |H_N| < 3K \log N$, and $K > 0$ is an absolute constant;*
- (4) *$\tilde{\gamma}_{H_N, r}(n) > \exp(n/K)$, for $n < K' \log |H_N| < 3K' \log N$, and $K' > 0$ is an absolute constant.*

Here [property \(3\)](#) means that the size of balls in the Cayley graphs of $\mathrm{PSL}_2(\mathbb{Z}_N)$ grow exponentially. For the proof of [Corollary 1.1](#), we do not really need the exact form of these groups or [property \(4\)](#), only the fact their sizes go to infinity. However, the proof of [Theorem 2.3](#) uses that these groups are related to $\mathrm{PSL}_2(\mathbb{Z})$.

Proof. Consider the following matrices in $\mathrm{PSL}_2(\mathbb{Z}[\mathbf{i}, 1/2])$, where $\mathbf{i}^2 = -1$:

$$a = \begin{pmatrix} \mathbf{i} & \mathbf{i}/4 \\ 0 & -\mathbf{i} \end{pmatrix}, \quad b = \begin{pmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix}, \quad c = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad d = \begin{pmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{pmatrix}.$$

A direct computation shows that these elements are of order 2 and $bcd = 1$; i.e., there is a (nonsurjective²) homomorphism $\mathcal{G} \rightarrow \mathrm{PSL}_2(\mathbb{Z}[\mathbf{i}, 1/2])$. Moreover, we have

$$(ad)^4 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad [c, [d, [b, (ad)^4]]] = \begin{pmatrix} -1 & 2 \\ 2 & -5 \end{pmatrix}.$$

This implies that the image of $\{a, b, c, d\}$ in $\mathrm{PSL}_2(\mathbb{Z}_N)$ satisfies [properties \(1\)](#) and [\(2\)](#), because r is not contained in any proper finite index normal subgroup of the image.

[Property \(3\)](#) is satisfied because the standard expander generators of $\mathrm{PSL}_2(\mathbb{Z}_N)$ can be expressed as short words in the generators (see, e.g., [\[HLW06\]](#), [\[Lub94\]](#)):

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = (da)^4 \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = c(ad)^4c.$$

²The image contains $\mathrm{PSL}_2(\mathbb{Z}[1/2])$ as a subgroup of index 2.

Property (4) follows since r and r^a generate a nonsolvable subgroup of $\mathrm{PSL}_2(\mathbb{Z})$. Now the recent expansion results [BG08] imply that the Cayley graphs of $\mathrm{PSL}_2(\mathbb{Z}_N)$ with respect to the images of r and r^a are expanders. This completes the proof (cf. Section 11.12). \square

Remark 6.2. Twisting by the automorphism φ^x one sees that the lemma remains valid if r is replaced by r_x .

Remark 6.3. If $\{p_j\}$ is a finite sequence of different primes which are 3 mod 4, then the product of $\mathrm{PSL}_2(p_j)$ as marked groups (with respect to the generating sets constructed above) is

$$\bigotimes \mathrm{PSL}_2(p_j) = \mathrm{PSL}_2(\mathbb{Z}_N), \quad \text{where } N = \prod p_i.$$

The following is an immediate consequence of Corollaries 5.18 and 5.19.

COROLLARY 6.4. *Let N satisfy the conditions of Lemma 6.1. The size of a ball of radius $n < D_i \log |\mathrm{PSL}_2(\mathbb{Z}_N)|$ inside $F_\omega^i(\mathrm{PSL}_2(\mathbb{Z}_N))$ is more than e^{n/D_i} where $D_i = 2^{i+1}K$.*

COROLLARY 6.5. *Let N satisfy the conditions of Lemma 6.1. The size of the intersection of a ball of radius $n < 2^i D'_i \log |\mathrm{PSL}_2(\mathbb{Z}_N)|$ inside $F_\omega^i(\mathrm{PSL}_2(\mathbb{Z}_N))$, with the subgroup $\mathrm{PSL}_2(\mathbb{Z}_N)^{2^i} = \ker\{F_\omega^i(\mathrm{PSL}_2(\mathbb{Z}_N)) \rightarrow \mathbf{G}_{\omega,i}\}$ is more than e^{n/D'_i} where $D'_i = 2^{i+1}K'$.*

7. Proof of the Oscillating Growth Theorem

We are going to prove the following result, which implies the Oscillating Growth Theorem (see Corollary 1.1). It is a stepping stone to the proof of the Main Theorem.

THEOREM 7.1. *For every admissible integer function $f, g : \mathbb{N} \rightarrow \mathbb{N}$ such that $\lim g(n)/\gamma_{\mathbf{G}_\omega}(n) = \infty$ and any sequences of integers $\{a_i\} \rightarrow \infty$ and $\{b_i\} \rightarrow \infty$, there exists a finitely generated group Γ and a generating set $\langle S \rangle = \Gamma$ such that*

$$\gamma_\Gamma^S(n) < g(n) \quad \text{for infinitely many } n \in \{b_i\}$$

and

$$\gamma_\Gamma^S(n) > f(n) \quad \text{for infinitely many } n \in \{a_i\}.$$

Remark 7.2. The reason for including the subsequences a_i and b_i is to be able to ensure that for any $f \gg f'$ and $g \ll g'$, we have $\gamma_\Gamma \gg f'$ and $\gamma_\Gamma \ll g'$. In particular, we can guarantee that

$$\liminf_{n \rightarrow \infty} \log_n \log \gamma_\Gamma(n) = \liminf_{n \rightarrow \infty} \log_n \log \gamma_{\mathbf{G}_\omega}(n).$$

Proof of Theorem 7.1. The group Γ will be the product of marked groups $G_i = F_\omega^{m_i}(H_i)$, where $H_i = \mathrm{PSL}_2(p_i)$ and where $\{m_i\}$ and $\{p_i\}$ are sequences

which grow sufficiently fast; these sequences will be constructed later using the functions f and g .

Lemmas 5.11 and 5.16 imply that the sequence of groups G_i converges to \mathbf{G} and satisfies the **condition (splitting)** with $N_i = H_i^{\oplus 2^{m_i}}$.

By **Corollary 6.5**, the growth of γ_Γ is faster than the growth of the each γ_{G_i} . When p_i is sufficiently large one can find $n_i \in \{a_j\}$ such that

$$\text{(lower)} \quad \gamma_\Gamma(n_i) \geq \gamma_{G_i}(n_i) > f(n_i),$$

which guarantees that $\gamma_\Gamma \gg f$.

Also, if m_i grows sufficiently fast, then the G_i converge very quickly to \mathbf{G}_ω and by **Lemma 4.8**, there exists $n'_i \in \{b_j\}$ such that

$$\text{(upper)} \quad \gamma_\Gamma(n'_i) \leq \gamma_{\mathbf{G}}(n'_i) \prod_{j < i} |N_j| < g(n'_i).$$

This guarantees that $\gamma_\Gamma \ll g$.

The only thing left is to determine how fast the sequences $\{m_i\}$ and $\{p_i\}$ have to grow in order to ensure the above inequalities. Define the sequences m_i and p_i as follows:

- $m_1 = 1$ and $H_1 = \mathbf{1}$.
- Let $n_i \in \{b_j\}$ be in integer such that $\frac{g(n_i)}{\gamma_{\mathbf{G}_\omega}(n_i)} > \prod_{j < i} |H_j|^{2^{m_j}}$. (Such an integer exists since $\{b_s\} \rightarrow \infty$ and $\frac{g(n)}{\gamma_{\mathbf{G}_\omega}(n)} \rightarrow \infty$.) Define m_i such that $\vartheta(m_i) > n_i$ and $m_i > m_{i-1}$. This choice of m_i ensures that the **inequalities (upper)** are satisfied because the kernel N_i of the map $G_i \rightarrow \mathbf{G}_{\omega, m_i}$ is isomorphic to the direct sum of 2^{m_i} copies of H_i by **Lemma 5.16**.
- Let $n'_i \in \{a_j\}$ be an integer such that

$$\frac{n'_i}{\log f(n'_i)} \geq D_{m_i} = K \cdot 2^{m_i},$$

where D_i is the constant from **Corollary 6.4**. Such an integer exists since $\{a_i\} \rightarrow \infty$ and $n/\log f(n) \rightarrow \infty$. Define H_i to be a group together with the generating set (twisted by $\varphi^{x_{m_i+1}}$) from **Lemma 6.1** of size more than $e^{n'_i}$. Again this choice of n'_i ensures that the **inequalities (lower)** are satisfied.

These two (rather crude) estimates for the size of the balls in Γ show that **conditions (lower)** and **(upper)** are satisfied. Therefore the growth function of Γ is infinitely often larger than f and infinitely often smaller than g . □

8. Control of the upper bound

8.1. Roughly speaking, we obtain a very good control over the upper bound by using finite groups H_i of the carefully chosen size. We observe that

Lemma 6.1 gives us an “almost continuous” family of finite groups that can be plugged into the construction.

Unfortunately, the growth estimates we have so far are too crude for such results. If the sequence m_i grows sufficiently fast, then the growth of the group Γ (in a certain range) is very well approximated by the growth of the group $\Gamma_i = G_i \otimes \mathbf{G}_\omega$. This is because

$$\begin{aligned} \gamma_\Gamma(n) &\geq \gamma_{\Gamma_i}(n) \quad \text{for all } n, \\ \gamma_\Gamma(n) &< L_i \gamma_{\Gamma_i}(n) \quad \text{for all } n \leq \vartheta(m_{i+1}) \text{ and } L_i = \prod_{j < i} |N_j|. \end{aligned}$$

The first condition follows from the observation that there are maps from the marked group Γ to both G_i and \mathbf{G}_ω . By Remark 4.3 this gives a map onto their product Γ_i . Therefore the growth in the image is slower than the growth of Γ .

The second condition is a consequence of the fact that the ball of radius $\vartheta(m_{i+1})$ in Γ is the same as the ball in the product $\otimes_{j \leq i+1} G_j$ or in $\mathbf{G}_\omega \otimes [\otimes_{j \leq i} G_j]$ and that

$$\left| \ker \left(\mathbf{G}_\omega \otimes \left[\otimes_{j \leq i} G_j \right] \rightarrow \Gamma_i \right) \right| = L_i.$$

For m_{i+1} very large, it suffices to find H_i such that the growth of Γ_i is always below f_1 , but sometimes it is above f_2 .

8.2. Below we present much better bounds on the growth in the following marked group:

$$\Lambda_\omega^i(H) = F_\omega^i(H) \otimes \mathbf{G}_\omega \subset F_\omega^i(H) \oplus \mathbf{G}_\omega,$$

which is closely related to the group Γ_i mentioned above. The growth of the balls in Λ is in three different regimes depending on the scale. For small radius $n < t_i$, the balls are the same as the ball in G_ω and grow sub-exponentially.

For big radius $n > T_i = 2^i D_i \text{diam } |H|$, the finite group $F_\omega^i(H)$ has been exhausted and the size of the ball of radius n is very close to $|H|^{2^i}$ times the size of the ball in \mathbf{G}_ω and again is sub-exponential.

In the intermediate range $t_i < n < T_i$ the growth is more complicated — it is similar to the growth in the finite group H and therefore is “locally” very close to exponential. However, the proof of the next results requires us to obtain some bounds for this intermediate range. As usual in such situations, understanding the exact growth in the intermediate range is extremely difficult and our bounds are far from optimal. Improving these bounds will result in weakening the technical conditions (vi) in Theorem 2.3 and (v) in Lemma 8.1.

8.3. The following technical lemma ensures that we can find the group H such that the growth of the group $\Lambda_\omega^i(H)$ is between f_1 and f_2 . We postpone the proof until the next section.

LEMMA 8.1 (Main Lemma). *Let $f_1, f_2, g : \mathbb{N} \rightarrow \mathbb{N}$ be admissible functions that satisfy*

- (i) $f_1(n)/g(n)$ is an increasing function;
- (ii) $f_1(n) > g(n)^3$ for all sufficiently large n ;
- (iii) $f_1(n) > f_2(n)^3$ for all sufficiently large n ;
- (iv) $g(n) \geq \gamma_{\mathbf{G}_\omega}(n)$, where \mathbf{G}_ω is a Grigorchuk group of intermediate growth;
- (v) $\exp \left[\frac{\log g(n)}{Cn^2} f_1^* \left(\frac{n}{C \log g(n)} \right) \right] < \frac{C f_2^*(Cn)}{n^2}$ for any $C > 0$ and sufficiently large $n = n(C)$.

Then, for every $L > 0$ and all sufficiently large i , one can find a finite marked group H_i such that

- (1) H_i is normally generated by $r_{x_{i+1}}$;
- (2) there exists n such that $\gamma_\Delta(n) < f_2(n)$;
- (3) $f_1(n) < L \gamma_\Delta(n)$ for all $n > \vartheta(i)$,

where $\gamma_\Delta = \gamma_{\Lambda_\omega^i(H_i)}$.

Proof of Theorem 2.3. The proof is almost the same as the proof of Theorem 7.1, but one needs to pick the groups H_i of the correct size. First we pick m_1 such that for $n > \vartheta(m_1)$, we have

$$f_1(n) > f_2(n)^3 \quad \text{and} \quad f_1(n) > g_1(n) > \gamma_{\mathbf{G}_\omega}(n),$$

which is possible because the functions satisfy conditions (i)–(v).

When choosing the depths m_i one need to satisfy three conditions. The first one is that $m_i > m_{i-1}$ ensures that the groups grows. The second one, as in Theorem 7.1, is that $\frac{g_1(n_i)}{\mathbf{G}_\omega(n_i)} \geq L$ for some $n_i \leq \vartheta(m_i)$, where $L = \prod_{j < i} |H_j|^{2^{m_j}}$, which guarantees that the growth of Γ will be sometimes smaller than g_1 . The last one is that m_i is larger than the bound for i in Lemma 8.1, which depends on L .

If m_i is chosen as above, then we can apply the Lemma 8.1 (with g_1 instead of g) and obtain the group H_i . The second property of H_i implies that the growth of Γ is larger than f_2 for some $n > \vartheta(m_i)$, and the third implies that it is bellow f_1 for $\vartheta(m_i) < n < \vartheta(m_{i+1})$.

As a result we have that the growth of the group γ_Γ is between f_1 and g_2 for all sufficiently large $n > \vartheta(m_1)$ and is above f_2 and below g_1 at least once in each interval $\vartheta(m_i) < n < \vartheta(m_{i+1})$, which completes the proof. \square

9. Proof of Main Lemma 8.1

9.1. *Outline.* The following is a rough outline of the proof. We start with some estimates of the growth of Λ_ω^i in the intermediate range: the upper bound is coming from the submultiplicativity of the growth functions, and the lower is based on the growth inside H . It is clear that the lower bound is far from

being optimal, but we suspect that the upper one is relatively close to the optimal bound.

These bounds (Corollary 9.3) give that if the group H is small, then the growth of $\Lambda_\omega^i(H)$ is slower than f_1 , and using Corollary 6.5, this growth is faster than f_2 if the group H is big (Corollary 9.4). If the gap between f_1 and f_2 is sufficiently large, then these sets have a nontrivial intersection, which implies the existence of H satisfying the requirements of the lemma.

Unfortunately, this strategy only works if the gap between f_1 and f_2 is very big. The reason for this is that we are using very crude estimates for the sizes of balls, which does not allow us to obtain better estimates for the growth of the group $\Lambda_\omega^i(H)$. In order to obtain results where the functions f_1 and f_2 are not too far apart, we argue by contradiction. As a result, we only show the existence of the group H , but not an algorithm to construct it.

9.2. *Three classes of marked finite groups.* First we divide the finite groups H into three classes: $\mathcal{D}_-, \mathcal{D}_+$ and \mathcal{D}_\circ , depending how the growth of $\Lambda_\omega^M(H)$ compares with f_1 and f_2 . If one assumes that the class \mathcal{D}_\circ is empty (and the gap between f_1 and f_2 is not too small), then \mathcal{D}_- is closed under products of marked groups (Corollary 9.7), which allows us to construct a group in \mathcal{D}_- that is much larger than the bound in Corollary 9.3. Finally one obtains a contradiction if the size of this group is larger than the estimate from Corollary 6.5.

Definition 9.1. Given a marked group H , to simplify the notation denote by γ_Δ the growth function of $\Lambda_\omega^i(H)$. Let \mathcal{D}_-^i denote the set of marked groups H such that $f_2(n) > \gamma_\Delta(n)$ for all $n > \vartheta(i)$. Similarly, let \mathcal{D}_+^i denote the set of marked groups H such that $f_1(n)^{2/3} \leq \gamma_\Delta(n)$ for some $n > \vartheta(i)$. Finally, by \mathcal{D}_\circ^i denote the set of marked groups H such that $f_1(n)^{2/3} > \gamma_\Delta(n)$ for all $n > \vartheta(i)$, but $f_2(n) \leq \gamma_\Delta(n)$ for some n .

The conclusion of Lemma 8.1 is equivalent to saying that \mathcal{D}_\circ^i is not empty when i is sufficiently large, since we can guarantee that $f_1(n)^{1/3} > L$ for $n > \vartheta(i)$.

9.3. *Details: the large gap.*

LEMMA 9.2. *Fix the group H . The growth of the function γ_Δ is bounded above by the function Υ_T defined as follows:*

$$\Upsilon_T(n) = \begin{cases} g(n) & \text{for } n \leq \vartheta(i), \\ \exp(n/\phi_i) & \text{for } \vartheta(i) \leq n \leq T, \\ |H|^{2^i} g(n) & \text{for } n \geq T, \end{cases}$$

where $\phi_i = \min \left\{ \frac{n}{\log g(n)} \mid \frac{\vartheta(i)}{2} \leq n \leq \vartheta(i) \right\}$.

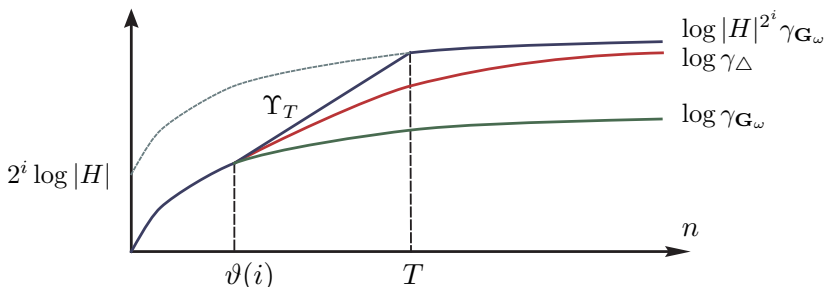


Figure 3. The graph of functions as in Lemma 9.2.

Proof. The ball of radius less than $\vartheta(i)$ in $\Lambda_\omega^i(H)$ is the same as in the group \mathbf{G}_ω , which gives the bound for small n . The kernel of $\Lambda_\omega^i(H) \rightarrow \mathbf{G}_\omega$ has size $|H|^{2^i}$, which easily implies the bound for large n .

In the middle range one uses submultiplicativity of growth functions $\gamma_\Delta(a+b) \leq \gamma_\Delta(a) \cdot \gamma_\Delta(b)$ for any a, b . This implies that if $n \geq m$, then

$$\frac{\log \gamma_\Delta(n)}{n} \leq \max_{m/2 \leq s \leq m} \frac{\log \gamma_\Delta(s)}{s}.$$

This inequality for $m = \vartheta(i)$ is equivalent to the bound in the middle range (see Figure 3). □

The lemma implies that if the size of H is very small, then the growth of $\Lambda_\omega^i(H)$ is smaller than f_1 .

COROLLARY 9.3.

- (i) Let f be an admissible function such that $f(n)/g(n)$ for some $g(n) > \gamma_{\mathbf{G}_\omega}(n)$ is increasing. If

$$|H| < \exp \left[\frac{f^*(\phi_i)}{2^i \phi_i} - \frac{\log g(f^*(\phi_i))}{2^i} \right],$$

then $f(n) > \gamma_\Delta(n)$ for all integer n .

- (ii) Moreover, if f also satisfies $f(n) \geq g(n)^3$ for $n \geq \vartheta(i)$, then for

$$|H| < U_i(f) := \exp \left[\frac{f^*(\frac{2}{3}\phi_i)}{2^{i+1} \phi_i} \right],$$

we have that $f(n)^{2/3} > \gamma_\Delta(n)$ for all integers $n \geq \vartheta(i)$.

Proof. For the first part, compute the point T where the graph of $(f_1)^{2/3}$ intersects with $\exp(n/\phi)$. By Lemma 9.2, if $|H|^{2^i} \leq \frac{f_1(T)}{g(T)}$, then the growth of $\Lambda_\omega^i(H)$ is slower than Υ_T , which is less than f . The second part uses the estimate $\frac{f_1(T)}{g(T)} \geq [f_1(T)]^{2/3}$. □

This following result is a strengthening of [Corollary 6.5](#).

COROLLARY 9.4. *If $H = \text{PSL}_2(\mathbb{Z}_N)$ with the generating set from [Lemma 6.1](#) and*

$$|H| < L_i(f) = \exp \left[\frac{f^*(D'_i)}{2^i D'_i} \right],$$

then $f(n) < \gamma_\Delta(n)$ for some n .

Proof. By [Corollary 6.5](#), $\gamma_\Delta(n) \geq \exp(n/D'_i)$ for $n \leq 2^i D'_i \log |H|$. For $n = f^*(D_i)$, the bound is the same as $f(n)$, but we can apply the estimate only if $|H| > \exp(n/2^i D'_i)$. □

Proof of [Lemma 8.1](#) for large gap. Let the functions f_1 and f_2 satisfy

(big gap)
$$\frac{\log g(n)}{n^2} f_1^* \left(\frac{n}{C \log g(n)} \right) > \frac{C}{n^2} f_2^*(Cn)$$

for any constant C and any sufficiently large n . Notice that, up to constants, both $\vartheta(i)$ and D'_i are equal to 2^i . Substituting $\varrho = 2^i$ one gets

$$L_i(f_2) \approx \exp \left[\frac{C_1}{\varrho^2} f_2^*(C_2 \varrho) \right],$$

where C_1 and C_2 are universal constants. Similarly,

$$U_i(f_1) \approx \exp \left[\frac{\log g(\varrho)}{C_3 \varrho^2} f_1^* \left(\frac{\varrho}{C_4 \log g(\varrho)} \right) \right].$$

If the functions f_1 and f_2 satisfy the [equation \(big gap\)](#), both $U_i(f_1)/L_i(f_2)$ and $U_i(f_1)$ tend to ∞ as i increases. Therefore, there exists i_0 , such that for $i > i_0$, we have

$$[f_1(\vartheta(i))]^{1/3} > L, \quad U_i(f_1)/L_i(f_2) > 10, \quad U_i(f_1) > 1000.$$

Under these conditions there exists a prime $p_i \equiv 1 \pmod{4}$ such that $U_i(f_1) > \text{PSL}_2(p_i) > L_i(f_2)$ because the above conditions translate to $A_i > p_i > B_i$, where $A_i/B_i > 2$ and $A_i > 13$, which allows us to apply Bertrand's postulate.

[Corollary 9.3](#) implies that the growth $\gamma_i = \gamma_{\Lambda_\omega^i(H_i)}$ of group $\Lambda_\omega^i(H_i)$, where $H_i = \text{PSL}_2(p_i)$, is slower than f_1 . Therefore,

$$L\gamma_i(n) < Lf_1(n)^{2/3} < f_1(n) \quad \text{for all } n \geq \vartheta(i).$$

Also by [Corollary 9.4](#) the growth of is not slower than $\Lambda_\omega^i(H_i)$; i.e.,

$$\gamma_i(n) < f_2(n) \quad \text{for some } n \geq \vartheta(i).$$

Therefore, the group $H_i = \text{PSL}_2(p_i)$ has all necessary properties. □

9.4. *Details: the small gap.* Unfortunately, if $f_1 \sim f_2$, then the gap between the functions f_1 and f_2 is not sufficiently big for the above argument to work. From this point on, we assume that the functions $f_1, f_2 : \mathbb{N} \rightarrow \mathbb{N}$ satisfy the following conditions:

- $f_1(n) > (f_2(n))^3$,
- $f_2(n) \geq \gamma_{\mathbf{G}_\omega}(n)$.

Remark 9.5. **Corollary 9.3** gives that (for a fixed i) if the size of H is small then $H \in \mathcal{D}_-^i$. In particular, by **Lemma 6.1** the class \mathcal{D}_-^i contains the groups $\text{PSL}_2(p)$ for $p^3/2 < U_i(f_2)$ and $p \equiv 3 \pmod 4$.

LEMMA 9.6. *If $H_1, H_2 \in \mathcal{D}_-^i$, then $H_1 \otimes H_2$ is not in \mathcal{D}_+^i .*

Proof. **Proposition 5.5** implies that

$$\Lambda_\omega^i(H_1 \otimes H_2) = \Lambda_\omega^i(H_1) \otimes \Lambda_\omega^i(H_2).$$

By **Lemma 4.6**,

$$\gamma_{\Lambda_\omega^i(H_1 \otimes H_2)}(n) \leq \gamma_{\Lambda_\omega^i(H_1)}(n) \cdot \gamma_{\Lambda_\omega^i(H_2)}(n) \leq f_2(n) \cdot f_2(n) < f_1(n);$$

i.e., the growth of $\Lambda_\omega^i(H_1 \otimes H_2)$ is slower than f_1 and the group $H_1 \otimes H_2$ is not in \mathcal{D}_+^i . □

COROLLARY 9.7. *If the set \mathcal{D}_\circ^i is empty, then \mathcal{D}_-^i is closed under \otimes .*

COROLLARY 9.8. *If the set \mathcal{D}_\circ^i is empty, then \mathcal{D}_-^i contain $\text{PSL}_2(\mathbb{Z}_N)$, where N is the product of all primes $p \equiv 3 \pmod 4$ such that $p^3/2 < U_i(f_1)$.*

Proof. **Corollary 9.3** says that $\text{PSL}_2(p)$ is not inside \mathcal{D}_+^i if $|\text{PSL}_2(p)| = p^3/2(1 + o(1)) < U_i(f_1)$. The previous corollary and **Remark 6.3** finish the proof. □

COROLLARY 9.9. *If the set \mathcal{D}_\circ^i is empty, then \mathcal{D}_-^i contain $\text{PSL}_2(\mathbb{Z}_N)$*

$$\log N \approx N_i(f_1) = \frac{1}{2} [2U_i(f_1)]^{1/3}.$$

Proof. This follows easily from the Dirichlet theorem on the distribution of primes (mod 4) and a calculation of the product of primes given in **[Rui97]**. We omit the (easy) details. □

Proof of Lemma 8.1. The idea is the same as in the case of the large gap between f_1 and f_2 . If one assumes that \mathcal{D}_\circ^i is empty, then one can use **Corollary 9.9** to construct groups in \mathcal{D}_-^i .

Again, up to a constants, both $\vartheta(i)$ and D'_i are equal to 2^i . Substituting $\varrho = 2^i$, one gets

$$L_i(f_2) \approx \exp \left[\frac{C_1}{\varrho^2} f_2^*(C_2 \varrho) \right],$$

where C_1 and C_2 are universal constants. Similarly,

$$U_i(f_1) \approx \exp \left[\frac{\log g(\varrho)}{C_3 \varrho^2} f^* \left(\frac{\varrho}{C_4 \log g(\varrho)} \right) \right]$$

and

$$N_i(f_1) \approx \exp \left(\exp \left[\frac{\log g(\varrho)}{C_5 \varrho^2} f^* \left(\frac{\varrho}{C_4 \log g(\varrho)} \right) \right] \right).$$

Condition (v) implies that

$$N_i(f_1)/L_i(f_2) \rightarrow \infty \quad \text{and} \quad U_i(f_1) \rightarrow \infty, \quad \text{as } i \rightarrow \infty.$$

Therefore, there exists i_0 such that for $i > i_0$, we have

$$[f_1(\vartheta(i))]^{1/3} > L, \quad N_i(f_1)/L_i(f_2) > 2, \quad U_i(f_1) > 1000.$$

However, Corollaries 9.4 and 9.9 imply that if $\mathcal{D}_\circ^i = \emptyset$, then the group $\text{PSL}_2(N)$ is neither in \mathcal{D}_-^i nor in \mathcal{D}_+^i , a contradiction. This implies that \mathcal{D}_\circ^i is not empty, and therefore there exists a group H with the desired properties. \square

10. Generalizations of the construction

10.1. Suppose G acts on a set X by $H \wr_X G$; we denote the restricted wreath product $G \times \bigoplus_{i \in X} H$. One easy modification of our construction is to use the permutation wreath product $P \wr_X \mathbf{G}_\omega$ as groups at infinity, where P is a finite group and X is an orbit in the action of \mathbf{G}_ω on the boundary of the binary tree \mathbf{T}_2 . The advantage of using these groups is that (unlike the groups \mathbf{G}_ω) their growth rate is known in some cases; see [BEb].

THEOREM 10.1. *The same as Theorem 2.3, but we use the growth of the group $\mathbb{Z}_2 \wr_X \mathbf{G}_\omega$ instead of the growth of the group \mathbf{G}_ω . Here X is the boundary of the binary tree \mathbf{T}_2 .*

Outline of the proof. The following is the list of changes we need in the construction:

- instead of the group \mathcal{G} , consider the free product $\mathcal{G} * \mathbb{Z}_2$;
- modify the functors F_i to include the extra generators by adding $G = (1; 1, g)$ for every $g \in \mathbb{Z}_2$;
- use \mathbb{Z}_2 in place of the trivial group $\mathbf{1}$,
- use that the limit of the groups $F_\omega^i(\mathbb{Z}_2)$ is $\mathbb{Z}_2 \wr_X \mathbf{G}_\omega$;
- change the groups H_i constructed in Lemma 6.1 to contain the group \mathbb{Z}_2 .

The rest of the proof follows verbatim. We omit the details. \square

10.2. It is easy to see that the growth types of the groups $P \wr_X G$ for fixed G and different P are the same if P is finite and nontrivial. Thus, in the theorem one can replace $\mathbb{Z}_2 \wr_X \mathbf{G}_\omega$ with $P \wr_X \mathbf{G}_\omega$ for any finite group P .

It seems possible to extend this result to wreath products of the forms $P \wr_X \mathbf{G}_\omega$ where the group P is not finite, but one needs a sofic approximation (a sequence of finite groups $\{P_i\}$ that converge to P) of the group P instead. The above outline needs to be modified by replacing \mathbb{Z}_2 with P_i .

10.3. Another possible generalization is to use the groups constructed in [Seg01] instead of the Grigorchuk groups \mathbf{G}_ω . However, this will make the words needed in Lemma 5.16 and 6.1 not so explicit, although it is clear that such words exist. In fact, as far as we are aware, the growth of these groups has not been studied and it is not clear if there any examples of this type that are of intermediate growth.

10.4. It would be interesting to analyze for which groups \mathbf{G} and subexponential functions f there exists a sequence of finite groups G_i , which converge to \mathbf{G} , and the growth of $\otimes G_i$ oscillates between $\gamma_{\mathbf{G}}$ and the function f . Theorem 2.3 shows that this is possible when \mathbf{G} is the Grigorchuk group \mathbf{G}_ω , and Theorem 10.1 shows it is possible when \mathbf{G} is a wreath product of \mathbf{G}_ω with a finite group. We believe that for any group \mathbf{G} of intermediate growth, such a sequence exists, provided the gap between the growth of \mathbf{G} and f is sufficiently large.

CONJECTURE 10.2. *For every group G of intermediate growth and a subexponential function $f(n)$ that grows sufficiently fast (depending on γ_G), there exists a sequence of finite groups $\{G_i\}$ such that*

- (1) $\lim G_i = G$, and
- (2) *the growth of $\Gamma = \otimes G_i$ oscillates between γ_G and f .*

11. Historical remarks and open problems

11.1. We refer to [GP08], [Har00] for the introduction to groups of intermediate growth and to [BGŠ03], [Gri05], [Gri11a], [GdlH97], [GNS00] for surveys on the subject and open problems.

Although \mathbf{G} is historically the first group of intermediate growth [Gri80], [Gri83], there is now a large number of constructions of intermediate growth *branch groups* (see [BGŠ03]). Groups \mathbf{G}_ω corresponding to infinite words $\omega \in \{0, 1, 2\}^\infty$ were introduced by Grigorchuk in [Gri85]. They form a continuum family of intermediate growth groups. In this setting, the Grigorchuk group $\mathbf{G} = \mathbf{G}_{(012)^\infty}$ corresponds to a periodic word sequence and is sometime called the *first Grigorchuk group* [BGŠ03].

11.2. Let us mention that Grigorchuk's original bounds for \mathbf{G} were $\alpha_-(\mathbf{G}) \geq 0.5$ and $\alpha_+(\mathbf{G}) \leq 0.991$. These bounds were successively improved, with the current records being

$$\alpha_-(\mathbf{G}) \geq 0.5207, \quad \alpha_+(\mathbf{G}) \leq 0.7675,$$

where both constants correspond to solutions of certain algebraic equations. The bound for α_- is in [Bri08] (see also [Bar01], [Leo00]), and the bound for $\alpha_+(\mathbf{G})$ is in [Bar98], [MP01]. Whether the limit $\alpha(\mathbf{G})$ exists remains an open

problem. However, Grigorchuk conjectures that $\alpha_-(\Gamma) \geq \alpha_-(\mathbf{G})$ for every group of intermediate growth [Gri05].

11.3. One of the few examples of groups of intermediate growth where the type of the growth function is known precisely are permutational wreath products $\Gamma = P \wr_X \mathbf{G}_\omega$ where $\omega = (012)^\infty$, where X is the boundary of the binary tree. If P is a finite group, then $\gamma_\Gamma \sim \exp(n^\alpha)$ for $\alpha = 0.7675$. If $P = \mathbb{Z}$, then the growth is $\gamma_\Gamma \sim \exp(n^\alpha \log n)$; see [BEb].

11.4. The free Grigorchuk group \mathcal{G} defined in Section 3 is clearly isomorphic to a free product $\mathbb{Z}_2^2 * \mathbb{Z}_2$, and thus it is *nonamenable*. It should not be confused with the *universal Grigorchuk group*

$$\bigotimes_\omega \mathbf{G}_\omega = \mathcal{G} / \bigcap_\omega \ker(\mathcal{G} \rightarrow \mathbf{G}_\omega),$$

which is known to have exponential growth and is conjectured to be amenable [Gri11b, §8].

11.5. It is well known and easy to see ([Har02]) that groups of exponential growth cannot have oscillations:

$$\liminf_{n \rightarrow \infty} \frac{\log \gamma_\Gamma^S(n)}{n} = \limsup_{n \rightarrow \infty} \frac{\log \gamma_\Gamma^S(n)}{n} \quad \text{for all } \langle S \rangle = \Gamma.$$

Denote this limit by $\varkappa(\Gamma, S) > 1$. It was recently discovered by Wilson [Wil04] that there exist groups with $\inf_S \varkappa(\Gamma, S) = 1$ (see also [Bar98]).

11.6. We conjecture that condition (vi) in the Main Theorem can be weakened to

$$(vi') \quad \frac{\log g_1(n)}{n^2} \cdot f_1^* \left(\frac{n}{\log g_1(n)} \right) \rightarrow \infty.$$

If true, this would significantly weaken the conditions on the growth of f_1 and f_2 , allowing further values of parameters in the examples from Section 2.2.

Heuristically, one expects that the growth of $\Lambda_\omega^i(\text{PSL}_2(\mathbb{Z}_N))$ behave reasonably with N . This implies that if \mathcal{D}_-^i contains enough groups, then \mathcal{D}_\circ^i is not empty. It is possible to prove such a statement for a fixed i using that the group $\Lambda_\omega^i(\text{PSL}_2(\mathbb{Z}))$ is reasonably close to a nice arithmetic group. However it is far from clear how to do this for all sufficiently large i .

11.7. In the context of Section 2.3, in order for this strategy to work, infinitely many groups H_i need to be nontrivial. However, one can show that taking the limits in Section 4 cannot possibly work if the group is not finitely presented. The following lemma clarifies our reasoning.

LEMMA 11.1. *In the context of Section 4, if the limit group G is finitely presented, then almost all groups N_i are trivial. Consequently, $\Gamma = G \times N$ for some finite group N .*

Proof. Suppose that G has a presentation where all relators have length at most k . If the ball of radius k in the group H coincides with the ball of radius k in G , then H is a homomorphic image of G since all defining relations of G are satisfied in H . Therefore, groups G_i are images of G for big i , which implies that N_i are trivial (again for big i). \square

Of course, in particular, the lemma shows that the Grigorchuk groups \mathbf{G}_ω of intermediate growth are *not* finitely presented, a well-known result in the field [Gri11a], [Gri11b]. Similarly, the lemma shows that Conjecture 10.2 implies that all groups of intermediate growth are not finitely presented, a classical open problem [Gri05], [Gri11a].

11.8. It follows from Shalom and Tao's recent extension [ST10] of the Gromov's theorem [Gro87] that every group of growth $n^{(\log \log n)^{o(1)}}$ must be virtually nilpotent and thus have polynomial growth. It is a major open problem whether this result can be extended to groups of growth $e^{o(\sqrt{n})}$. Only partial results have been obtained in this direction [Gri05] (see also [BGŠ03], [Gri86], [Gri11a]).

11.9. The growth of groups is in many ways parallel to the study of *subgroup growth* (see [Lub95], [LS03]). In this case, a celebrated construction of Segal [Seg01] (see also [Neu86]) showed that the group can have nearly polynomial growth without being virtually solvable of finite rank. In other words, the Shalom-Tao extension of Gromov theorem does not have a subgroup growth analogue. Interestingly, Segal's construction also uses the Grigorchuk type groups and takes the iterated permutational wreath product of permutation groups; it is one of the motivations behind our construction.

Let us mention here that Pyber completely resolved the "gap problem" by describing groups with subgroup growth given by any prescribed increasing function (within a certain range). His proof also relies on sequences of finite alternating groups of pairwise different degrees [Pyb04], generalizing a classical construction of B. H. Neumann [Neu37] (see also [Pyb03]).

On the other end of the spectrum, let us mention that for subgroup growth there is no strict lower bound; i.e., for any function $f(n)$, there is a finitely generated group where the number of subgroups of index n is less than $f(n)$ infinitely often [KN06], [Seg01].

11.10. Another variation on the group growth is the *representation growth*, defined via the number $r_n(G)$ of irreducible complex representations of dimension n , whose kernel has finite index. In this case there is again no upper bound for the growth of $r_n(G)$. (This follows from [KN06].) We refer to [LL08] for an

introduction to the subject and to [Cra10] for lower bound on representation growth. See also [JZ06] and [Vol10] for the zeta-function approach.

11.11. The growth of algebras (rather than group) is well understood, and much more flexibility is possible (see, e.g., [Ufn95]). The results in this paper and [Bri], it seems, suggest that the growth of groups can be much less rigid than previously believed.

11.12. In the proof of Lemma 6.1, using the length ≤ 10 of standard generators in $\{a, b, c, d\}$, one can get an explicit bound $K < 10 \cdot 2000$ (see [Lub94, §8] and [HLW06, §11.2]). Bounds in [BG08], giving K' , can also potentially be made explicit.

Most recently, it was shown that for primes $p = 1 \pmod 4$ of positive density, all Cayley graphs of $\mathrm{PSL}_2(p)$ have universal expansion, a result conjectured for all primes [BG10]. These most general bounds have yet to be made explicit, however.

11.13. A finitely generated group G is called *sofic* if it is a limit of some sequence of finite groups. The existence of finitely generated nonsocfic groups is a well-known open problem [Pes08, §3].

Let us also mention that convergence of Grigorchuk groups was also studied in [Gri85], and is related to the notion of *Benjamini-Schramm convergence* for graph sequences [BS01].

11.14. As a minor but potentially important difference, let us mention that the oscillating growth established by Brioussell [Bri] does not give explicit bounds on the “oscillation times,” while in this paper we compute them explicitly, up to some global constants. Since our results mostly do not overlap with those in [Bri], it would be useful to quantify the former.

Interestingly, both this paper and [Bri] have been obtained independently. Using different tools, they were both originally motivated by probabilistic applications (see [BKP13]) to the analysis of the *return probability* and the *rate of escape* of a random walk on groups. We refer to [Woe00] for a general introduction to the subject.

11.15. In a recent preprint, Bartholdi and Eschler constructed, for any subexponential function f satisfying some mild conditions, a finitely generated group G whose growth type is the same as f . The idea is based on the results in [BEa] — the group G is isomorphic to $\mathbb{Z}_2 \wr_X \mathbf{G}_\omega$ where X is the boundary of the binary tree \mathbf{T}_2 , where ω is carefully chosen infinite word. This result is neither stronger nor weaker than our main results; it implies that there exists a group Γ with

$$f_1 \asymp \gamma_\Gamma \asymp g_2 \quad \text{and} \quad f_1 \ll \gamma_\Gamma \ll g_2$$

but does not give pointwise inequalities like $\gamma_\Gamma(n) \leq f_2(n)$ for infinitely many n 's.

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