Annals of Mathematics **177** (2013), 1029–1076
http://dx.doi.org/10.4007/annals.2013.177.3.5

Finite time singularities for Lagrangian mean curvature flow

By André Neves

Abstract

Given any embedded Lagrangian on a four-dimensional compact Calabi-Yau, we find another Lagrangian in the same Hamiltonian isotopy class that develops a finite time singularity under mean curvature flow. This contradicts a weaker version of the Thomas-Yau conjecture regarding long time existence and convergence of Lagrangian mean curvature flow.

1. **Introduction**

One of the hardest open problems regarding the geometry of Calabi-Yau manifolds consists in determining when a given Lagrangian admits a minimal Lagrangian (SLag) in its homology class or Hamiltonian isotopy class. If such SLag exists, then it is area-minimizing, and so one could approach this problem by trying to minimize area among all Lagrangians in a given class. Schoen and Wolfson [12] studied the minimization problem and showed that, when the real dimension is four, a Lagrangian minimizing area among all Lagrangians in a given class exists, is smooth everywhere except finitely many points, but is not necessarily a minimal surface. Later Wolfson [19] found a Lagrangian sphere with nontrivial homology on a given K3 surface such that the Lagrangian that minimizes area among all Lagrangians in this class is not an SLag and the surface that minimizes area among all surfaces in this class is not Lagrangian. This shows the subtle nature of the problem.

In another direction, Smoczyk [14] observed that when the ambient manifold is Kähler-Einstein, the Lagrangian condition is preserved by the gradient flow of the area functional (mean curvature flow), and so a natural question is whether one can produce SLag's using Lagrangian mean curvature flow. To that end, R. P. Thomas and S.-T. Yau [15, §7] considered this question and proposed a notion of "stability" for Lagrangians in a given Calabi-Yau, which we now describe.

Let $(M^{2n}, \omega, J, \Omega)$ be a compact Calabi-Yau with metric g, where Ω stands for the unit parallel section of the canonical bundle. Given $L \subseteq M$ Lagrangian, it is a simple exercise ([15, §2] for instance) to see that

$$\Omega_L = e^{i\theta}\mathrm{vol}_L,$$

where vol_L denotes the volume form of L and θ is a multivalued function defined on L called the *Lagrangian angle*. All the Lagrangians considered will be *zero-Maslov class*, meaning that θ can be lifted to a well-defined function on L. Moreover if L is zero-Maslov class with oscillation of Lagrangian angle less than π (called *almost-calibrated*), there is a natural choice for the phase of $\int_L \Omega$, which we denote by $\phi(L)$. Finally, given any two Lagrangians L_1, L_2, a connected sum operation $L_1 \# L_2$ is defined in [15, §3] (more involved than a simply topological connected sum). We refer the reader to [15, §3] for the details.

Definition 1.1 (Thomas-Yau Flow-Stability). Without loss of generality, suppose that the almost-calibrated Lagrangian L has $\phi(L) = 0$. Then L is flow-stable if either of the following two happen:

- L Hamiltonian isotopic to $L_1 \# L_2$, where L_1, L_2 are two almost-calibrated Lagrangians, implies that
$$[\phi(L_1), \phi(L_2)] \nsubseteq (\inf_L \theta, \sup_L \theta).$$
- L Hamiltonian isotopic to $L_1 \# L_2$, where L_1, L_2 are almost-calibrated Lagrangians, implies that
$$\mathrm{area}(L) \leq \int_{L_1} e^{-i\phi(L_1)}\Omega + \int_{L_2} e^{-i\phi(L_2)}\Omega.$$

Remark 1.2. The notion of flow-stability defined in [15, §7] applies to a larger class than almost-calibrated Lagrangians. For simplicity, but also because the author (unfortunately) does not fully understand that larger class, we chose to restrict the definition to almost calibrated.

In [15, §7] it is then conjectured

Conjecture (Thomas-Yau Conjecture). *Let L be a flow-stable Lagrangian in a Calabi-Yau manifold. Then the Lagrangian mean curvature flow will exist for all time and converge to the unique* SLag *in its Hamiltonian isotopy class.*

The intuitive idea is that if a singularity occurs, it is because the flow is trying to decompose the Lagrangian into "simpler" pieces and so, if we rule out this possibility, no finite time singularities should occur. Unfortunately, their stability condition is, in general, hard to check. For instance, the definition does not seem to be preserved by Hamiltonian isotopies, and so the existence of Lagrangians that are flow-stable and not SLag is a highly nontrivial problem. As a result, it becomes quite hard to disprove the conjecture because not many examples of flow-stable Lagrangians are known. For this reason there has been considerable interest in the following simplified version of the above conjecture (see [17, §1.4]).

Conjecture. *Let M be Calabi-Yau and Σ be a compact embedded Lagrangian submanifold with zero Maslov class. Then the mean curvature flow of Σ exists for all time and converges smoothly to a special Lagrangian submanifold in the Hamiltonian isotopy class of Σ.*

We remark that in [17] this conjecture is attributed to Thomas and Yau, but this is not correct because there is no mention of stability. For this reason, this conjecture, due to Mu-Tao Wang, is a weaker version of Thomas-Yau conjecture.

Schoen and Wolfson [13] constructed solutions to Lagrangian mean curvature flow that become singular in finite time and where the initial condition is homologous to a SLag Σ. On the other hand, we remark that the flow *does* distinguish between isotopy class and homology class. For instance, on a two-dimensional torus, a curve γ with a single self-intersection that is homologous to a simple closed geodesic will develop a finite time singularity under curve shortening flow, while if we make the more restrictive assumption that γ is isotopic to a simple closed geodesic, Grayson's Theorem [5] implies that the curve shortening flow will exist for all time and sequentially converge to a simple closed geodesic.

The purpose of his paper is to prove

Theorem 6.1. *Let M be a four real-dimensional Calabi-Yau and Σ an embedded Lagrangian. There is L Hamiltonian isotopic to Σ so that the Lagrangian mean curvature flow starting at L develops a finite time singularity.*

Remark 1.3. (1) If we take Σ to be a SLag, the theorem implies the second conjecture is false because L is then a zero-Maslov class Lagrangian.

(2) Theorem A provides the first examples of compact embedded Lagrangians that are not homologically trivial and for which mean curvature flow develops a finite time singularity. The main difficulty comes from the fact, due to the high codimension, that barrier arguments or maximum principle arguments do not seem to be as effective as in the codimension-one case and thus new ideas are needed.

(3) One way to picture L is to imagine a very small Whitney sphere N (a Lagrangian sphere with a single transverse self-intersection at p in Σ) and consider $L = \Sigma \# N$. (See the local picture in Figure 1.)

(4) If Σ is SLag, then for every ε we can make the oscillation for the Lagrangian angle of L lying in $[-\varepsilon, \pi + \varepsilon]$. Thus L is not almost-calibrated and so does not qualified to be flow-stable in the sense of Thomas-Yau.

(5) It is a challenging open question whether or not one can find L Hamiltonian isotopic to a SLag with arbitrarily small oscillation of the Lagrangian angle such that mean curvature flow develops finite time singularities. More generally, it is a fascinating problem to state a Thomas-Yau type conjecture that has an easier to check hypothesis on the initial condition and allows (or not) for the formation of a restricted type of singularities.

Acknowledgements. The author would like to thank Sigurd Angenent for his remarks regarding Section 7.2 and Richard Thomas and Dominic Joyce for their kindness in explaining the notion of stability for the flow. He would also like to express his gratitude to Felix Schulze for his comments on Lemma 5.13 and to one of the referees for the extensive comments and explanations that greatly improved the exposition of this paper.

2. **Preliminaries and sketch of proof**

In this section we describe the mains ideas that go into the proof of Theorem A, but first we have to introduce some notation.

2.1. *Preliminaries.* Fix (M, J, ω, Ω) a four-dimensional Calabi-Yau manifold with Ricci flat metric g, complex structure J, Kähler form ω, and calibration form Ω. For every R, set $g_R = R^2 g$, and consider G to be an isometric embedding of (M, g_R) into some Euclidean plane $\mathbb{R}^n$. L denotes a smooth Lagrangian surface contained in M and $(L_t)_{t\geq 0}$ a smooth solution to Lagrangian mean curvature flow with respect to one of the metrics g_R. (Different R simply change the time scale of the flow.) It is simple to recognize the existence of $F_t : L \longrightarrow \mathbb{R}^n$ so that the surfaces $L_t = F_t(L)$ solve the equation

$$\frac{dF_t}{dt}(x) = H(F_t(x)) = \bar{H}(F_t(x)) + E(F_t(x), T_{F_t(x)}L_t),$$

where $H(F_t(x))$ stands for the mean curvature with respect to g_R, $\bar{H}(F_t(x))$ stands for the mean curvature with respect to the Euclidean metric and E is some vector valued function defined on $\mathbb{R}^n \times G(2, n)$, with $G(2, n)$ being the set of 2-planes in $\mathbb{R}^n$. The term E can be made arbitrarily small by choosing R sufficiently large. In order to avoid introducing unnecessary notation, we will not be explicit whether we are regarding L_t being a submanifold of M or $\mathbb{R}^n$.

Given any (x_0, T) in $\mathbb{R}^n \times \mathbb{R}$, we consider the backwards heat kernel

$$\Phi(x_0, T)(x, t) = \frac{\exp\left(-\frac{|x-x_0|^2}{4(T-t)}\right)}{4\pi(T-t)}.$$

We need the following extension of Huisken's monotonicity [6] formula which follows trivially from [16, formula (5.3)].

LEMMA 2.1 (Huisken's monotonicity formula). *Let f_t be a smooth family of functions with compact support on L_t. Then*

$$\frac{d}{dt}\int_{L_t} f_t\Phi(x_0,T)d\mathcal{H}^2 = \int_{L_t}\left(\partial_t f_t - \Delta f_t\right)\Phi(x_0,T)d\mathcal{H}^2$$
$$-\int_{L_t}\left|\bar{H}+\frac{E}{2}+\frac{(\mathbf{x}-x_0)^\perp}{2(T-t_0)}\right|^2\Phi(x_0,T)d\mathcal{H}^2+\int_{L_t} f_t\Phi(x_0,T)\frac{|E|^2}{4}d\mathcal{H}^2.$$

We denote

$$A(r_1,r_2)=\{x\in\mathbb{R}^n \mid r_1<|x|<r_2\},\quad B_r = A(-1,r),$$

and define the $C^{2,\alpha}$ norm of a surface N at a point x_0 in $\mathbb{R}^n$ as in [18, §2.5]. This norm is scale invariant and, given an open set U, the $C^{2,\alpha}(U)$ norm of N denotes the supremum in U of the pointwise $C^{2,\alpha}$ norms. We say $\bar{N}$ is ν-close in $C^{2,\alpha}$ to N if there is an open set U and a function $u: N\cap U \longrightarrow \mathbb{R}^n$ so that $\bar{N}=u(N\cap U)$ and the $C^{2,\alpha}$ norm of u (with respect to the induced metric on N) is smaller than ν.

2.1.1. *Definition of* $N(\varepsilon,\underline{R})$. Let c_1, c_2, and c_3 be three half-lines in $\mathbb{C}$ so that c_1 is the positive real axis and c_2, c_3 are, respectively, the positive line segments spanned by $e^{i\theta_2}$ and $e^{i\theta_3}$, where $\pi/2<\theta_2<\theta_3<\pi$. These curves generate three Lagrangian planes in $\mathbb{R}^4$, which we denote by P_1, P_2, and P_3 respectively. Consider a curve $\gamma(\varepsilon):[0,+\infty)\longrightarrow\mathbb{C}$ such that (see Figure 1)

- $\gamma(\varepsilon)$ lies in the first and second quadrant and $\gamma(\varepsilon)^{-1}(0)=0$;
- $\gamma(\varepsilon)\cap A(3,\infty)=c_1^+\cap A(3,\infty)$ and $\gamma(\varepsilon)\cap A(\varepsilon,1)=(c_1^+\cup c_2\cup c_3)\cap A(\varepsilon,1)$;
- $\gamma(\varepsilon)\cap B_1$ has two connected components γ_1 and γ_2, where γ_1 connects c_2 to c_1^+ and γ_2 coincides with c_3;
- the Lagrangian angle of γ_1, $\arg\left(\gamma_1\frac{d\gamma_1}{ds}\right)$, has oscillation strictly smaller than $\pi/2$.

Set $\gamma(\varepsilon,\underline{R})=\underline{R}\gamma(\varepsilon/\underline{R})$. We define

$$N(\varepsilon,\underline{R})=\{(\gamma(\varepsilon,\underline{R})(s)\cos\alpha,\gamma(\varepsilon,\underline{R})(s)\sin\alpha)\mid s\geq 0,\alpha\in S^1\}. \tag{1}$$

We remark that one can make the oscillation for the Lagrangian angle of $N(\varepsilon,\underline{R})$ as close to π as desired by choosing θ_2 and θ_3 very close to $\pi/2$.

2.1.2. *Definition of a self-expander*. A surface $\Sigma\subseteq\mathbb{R}^4$ is called a *self-expander* if $H=\frac{x^\perp}{2}$, which is equivalent to saying that $\Sigma_t=\sqrt{t}\Sigma$ is a solution to mean curvature flow. We say that Σ is *asymptotic* to a varifold V if, when t tends to zero, Σ_t converges in the Radon measure sense to V. For instance, Anciaux [1, §5] showed there is a unique curve χ in $\mathbb{C}$ so that

$$\mathcal{S}=\{(\chi(s)\cos\alpha,\chi(s)\sin\alpha)\mid s\in\mathbb{R},\alpha\in S^1\} \tag{2}$$

is a self-expander for Lagrangian mean curvature flow asymptotic to P_1+P_2.

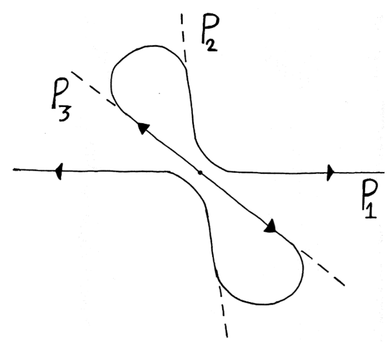

Figure 1. Curve $\gamma(\varepsilon) \cup -\gamma(\varepsilon)$.

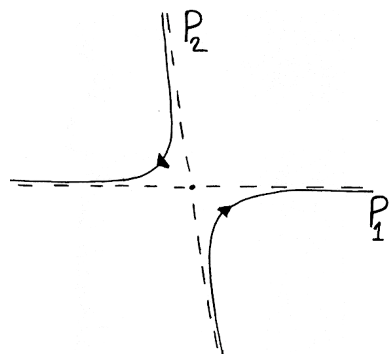

Figure 2. Curve $\chi \cup -\chi$.

2.2. *Sketch of Proof.*

THEOREM 6.1. *Let M be a four real-dimensional Calabi-Yau and Σ an embedded Lagrangian. There is L Hamiltonian isotopic to Σ so that the Lagrangian mean curvature flow starting at L develops a finite time singularity.*

Remark 2.2. The argument to prove Theorem 6.1 has two main ideas. The first is to construct L so that if the flow $(L_t)_{t\geq 0}$ exists smoothly, then L_1 and L will be in different Hamiltonian isotopy classes. Unfortunately this does not mean the flow must become singular because Lagrangian mean curvature flow is not an *ambient* Hamiltonian isotopy. This is explained below in the First Step and the Second Step.

The second main idea is to note that L_1 is very close to an SO(2)-invariant Lagrangian M_1 that has the following property. The flow $(M_t)_{t\geq 1}$ develops a singularity at some time T with the Lagrangian angle jumping by 2π at instant T. Because the solution $(L_t)_{t\geq 1}$ will be "nearby" $(M_t)_{t\geq 1}$, this jump will also occur on $(L_t)_{t\geq 1}$ around time T, which means that it must have a singularity as well.

Sketch of proof. It suffices to find a singular solution to Lagrangian mean curvature flow with respect to the metric $g_R = R^2 g$ for R sufficiently large. Pick Darboux coordinates defined on $B_{4\overline{R}}$ that send the origin into $p \in \Sigma$ so that $T_p\Sigma$ coincides with the real plane oriented positively and the pullback metric at the origin is Euclidean. (We can increase $\overline{R}$ by making R larger.) The basic approach is to remove $\Sigma \cap B_{2\overline{R}}$ and replace it with $N(\varepsilon, \underline{R}) \cap B_{2\overline{R}}$. Denote the resulting Lagrangian by L which, due to [4, Th. 1.1.A], we know to be Hamiltonian isotopic to Σ.

Assume that the Lagrangian mean curvature flow $(L_t)_{t\geq 0}$ exists for all time. The goal is to get a contradiction when $\overline{R}$, $\underline{R}$ are large enough and ε is small enough.

First Step. Because $L \cap A(1, 2\overline{R})$ consists of three planes that intersect transversely at the origin, we will use standard arguments based on White's Regularity Theorem [18] and obtain estimates for the flow in a smaller annular region. Hence, we will conclude the existence of R_1 uniform so that $L_t \cap A(R_1, \overline{R})$ is a small $C^{2,\alpha}$ perturbation of $L \cap A(R_1, \overline{R})$ for all $1 \leq t \leq 2$ and the decomposition of $L_t \cap B_{\underline{R}}$ into two connected components $Q_{1,t}$, $Q_{2,t}$ for all $0 \leq t \leq 2$, where $Q_{2,0} = P_3 \cap B_{\underline{R}}$. Moreover, we will also show that $Q_{2,t}$ is a small $C^{2,\alpha}$ perturbation of P_3 for all $1 \leq t \leq 2$. This is done in Section 3, and the arguments are well-known among the experts.

Second Step. In Section 4 we show that $Q_{1,1}$ must be close to $\mathcal{S}$, the smooth self-expander asymptotic to P_1 and P_2. (See (2) and Figure 2.) The geometric argument is that self-expanders act as attractors for the flow; i.e., because $Q_{1,0}$ is very close to $P_1 \cup P_2$ and $\sqrt{t}\mathcal{S}$ tends to $P_1 + P_2$ when t tends to zero, then $Q_{1,t}$ must be very close to $\sqrt{t}\mathcal{S}$ for all $1 \leq t \leq 2$. It is crucial for this part of the argument that $(Q_{1,t})_{0\leq t\leq 2}$ exists smoothly and that $P_1 + P_2$ is not area-minimizing. (See Theorem 4.2 and Remark 4.3 for more details.) This step is the first main idea of this paper.

From the first two steps it follows that L_1 is very close to a Lagrangian M_1 generated by a curve σ like the one in Figure 3. Because $Q_{1,0}$ is isotopic to $P_1 \# P_2$ but $Q_{1,1}$ is isotopic to $P_2 \# P_1$ (in the notation of [15]), we have that M_1 is not Hamiltonian isotopic to L. Thus it is not possible to connect the two by an ambient Hamiltonian isotopy. Nonetheless, as it was explained to the author by Paul Seidel, it is possible to connect them by smooth Lagrangian immersions that are neither rotationally symmetric nor embedded. Unfortunately it is not known whether Lagrangian mean curvature flow is a Hamiltonian isotopy (only infinitesimal Hamiltonian deformation is known), and so there is no topological obstruction to go from L to L_1 without singularities.

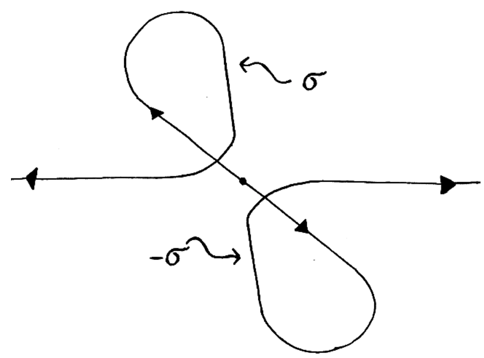

Figure 3. Curve $\sigma \cup -\sigma$.

Naturally we conjecture that does not occur and that $(L_t)_{0\leq t\leq 1}$ has a finite time singularity that corresponds to the flow developing a "neck-pinch" in order to get rid of the noncompact "Whitney Sphere" $N(\varepsilon, \underline{R})$ we glued to Σ. If the initial condition is simply $N(\varepsilon, \underline{R})$ instead of L, we showed in [9, §4] that this conjecture is true, but the arguments relied on the rotationally symmetric properties of $N(\varepsilon, \underline{R})$ and thus cannot be extended to arbitrarily small perturbations like L. If this conjecture were true, then the proof of Theorem 6.1 would finish here.

After several attempts, the author was unable to prove this conjecture, and this lead us to the second main idea of this paper described below. Again we stress that, conjecturally, this case will never occur without going through "earlier" singularities.

Third step. Denote by $(M_t)_{t\geq 1}$ the evolution by mean curvature flow of M_1, the Lagrangian that corresponds to the curve σ. In Theorem 5.3 we will show that M_t is SO(2)-invariant and can be described by curves σ_t that evolve the following way (see Figure 4). There is a singular time T so that for all $t < T$, the curves σ_t look like σ but with a smaller enclosed loop. When $t = T$, this enclosed loop collapses and we have a singularity for the flow. For $t > T$, the curves σ_t will become smooth and embedded.

We can now describe the second main idea of this paper. (See Remark 5.2 and Corollary 5.5 for more details.) Because σ_t "loses" a loop when t passes through the singular time, winding number considerations will show that the Lagrangian angle of M_t must suffer a discontinuity of 2π. Standard arguments will show that, because L_1 is very close to M_1, then L_t will be very close to M_t as well and so the Lagrangian angle of L_t should also suffer a discontinuity of approximately 2π when t passes through T. But this contradicts the fact that $(L_t)_{t\geq 0}$ exists smoothly. $\square$

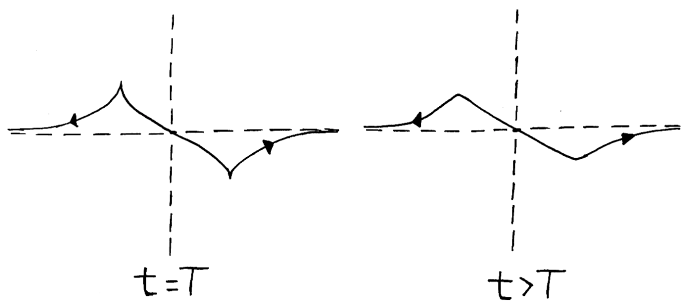

Figure 4. Evolution of σ_t.

2.3. *Organization.* The first step in the proof is done in Section 3. It consists mostly of standard but slightly technical results, all of which are well known. The second step is done in Section 4, and the third step is done in Section 5. Finally, in Section 6 the proof of Theorem 6.1 is made rigorous and in the appendix some basic results are collected.

Some parts of this paper are long and technical but can be skipped on a first reading. Section 3 can be skipped and consulted only when necessary. In Section 4 the reader can skip the proofs of Propositions 4.4 and 4.6 and read instead the outlines in Remarks 4.5 and 4.7. In Section 5 the reader can skip the proof of Theorem 5.3.

3. **First Step: General Results**

3.1. *Setup of Section* 3.

3.1.1. *Hypothesis on ambient space.* We assume the setting of Section 2.1 and the existence of a Darboux chart

$$\phi : B_{4\overline{R}} \longrightarrow M,$$

meaning $\phi^*\omega$ coincides with the standard symplectic form in $\mathbb{R}^4$ and ϕ^*J and $\phi^*\Omega$ coincide, respectively, with the standard complex structure and $dz_1 \wedge dz_2$ at the origin. Moreover, we assume that

- $\phi^* g_R$ is $1/\overline{R}$-close in C^3 to the Euclidean metric,
- $G \circ \phi$ is $1/\overline{R}$-close in C^3 to the map that sends x in $\mathbb{R}^4$ to $(x, 0)$ in $\mathbb{R}^n$,
- the $C^{0,\alpha}$ norm of E (defined in Section 2.1) is smaller than $1/\overline{R}$,
- and $G(M) \cap B_{4\overline{R}-1} \subseteq G \circ \phi(B_{4\overline{R}})$.

For the sake of simplicity, given any subset B of M, we freely identify B with $\phi^{-1}(B)$ in $B_{4\overline{R}}$ or $G(B)$ in $\mathbb{R}^n$.

3.1.2. *Hypothesis on Lagrangian L.* We assume that $L \subseteq M$ Lagrangian is such that

$$L \cap B_{2\overline{R}} = N(\varepsilon, \underline{R}) \cap B_{2\overline{R}} \text{ for some } \overline{R} \geq 4\underline{R}, \tag{3}$$

where $N(\varepsilon, \underline{R})$ was defined in (1). Thus $L \cap B_{2\underline{R}}$ consists of two connected components Q^1 and Q^2, where

$$Q^1 \setminus B_\varepsilon = (P_1 + P_2) \cap A(\varepsilon, 2\underline{R}) \text{ and } Q^2 = P_3 \cap B_{2\underline{R}}. \tag{4}$$

To be rigorous, one should use the notation $L_{\varepsilon,\underline{R}}$ for L. Nonetheless, for the sake of simplicity, we prefer the latter. Finally, we assume the existence of K_0 so that

- $\text{area}(L \cap B_r(x)) \leq K_0 r^2$ for every $x \in M$ and $r \geq 0$,
- the norm of second fundamental form of M in $\mathbb{R}^n$ is bounded by K_0,
- $\sup_{Q^1} |\theta| \leq \pi/2 - K_0^{-1}$ and we can find $\beta \in C^\infty(Q^1)$ such that $d\beta = \lambda$, and

$$|\beta(x)| \leq K_0(|x|^2 + 1) \text{ for all } x \in Q^1. \tag{5}$$

3.2. *Main results.* We start with two basic lemmas and then state the two main theorems.

LEMMA 3.1. *For all ε small, $\underline{R}$ large, and $T_1 > 0$, there is $D = D(T_1, K_0)$ so that*

$$\mathcal{H}^2(B_r(x) \cap L_t) \leq Dr^2 \textit{ for all } x \in \mathbb{R}^n,\ r > 0, \textit{ and } 0 \leq t \leq T_1.$$

Proof. Assuming a uniform bound on the second fundamental form of M in $\mathbb{R}^n$, it is a standard fact that uniform area bounds for L_t hold for all $0 \leq t \leq T_1$. (See, for instance, [9, Lemma A.3] if g is the Euclidean metric. A general proof could be given along the same lines provided we use the modification of monotonicity formula given in Lemma 2.1.) □

LEMMA 3.2. *For every δ small, $T_1 > 1$, and $\underline{R} > 0$, there is $R = R(T_1, \delta, \underline{R})$ so that, for every $1 \leq t \leq T_1$, L_t is δ-close in $C^{2,\alpha}$ to the plane P_1 in the annular region $A(R, \overline{R})$.*

Proof. Apply Lemma 3.9 with ν being δ given in this lemma, $S = 1$, and $\kappa = 1/T_1$. Because $L_0 \cap A(3\underline{R}, 2\overline{R}) = P_1 \cap A(3\underline{R}, 2\overline{R})$, it is simple to see that conditions (a), (b), and (c) of Lemma 3.9 are satisfied for all $x_0 \in L_0 \cap A(R, \overline{R})$ provided we choose R suitably large. Thus, the desired result follows from Lemma 3.9(ii). □

The next theorem is one of the main results of this section. The proof will be given at the end of Section 3 and can be skipped on a first reading.

THEOREM 3.3. *Fix ν. The constant Λ_0 mentioned below is universal.*

There are ε_1 and R_1, depending on the planes P_1, P_2, P_3, K_0, and ν, such that if $\varepsilon \le \varepsilon_1$ and $\underline{R} \ge 2R_1$ in (3), *then*

(i) *for every $0 \le t \le 2$, the $C^{2,\alpha}(A(R_1, \overline{R}))$ norm of L_t is bounded by $\Lambda_0 t^{-1/2}$ and*

$$F_t(x) \in A(R_1, \overline{R}) \implies |F_s(x) - x| < \Lambda_0 \sqrt{s} \text{ for all } 0 \le s \le 2;$$

(ii) *for every $1 \le t \le 2$, $L_t \cap A(R_1, \overline{R})$ is ν-close in $C^{2,\alpha}$ to L.*

Moreover, setting

$$Q_{1,t} = F_t(Q^1 \cap B_{\underline{R}}) \quad \text{and} \quad Q_{2,t} = F_t(Q^2 \cap B_{\underline{R}}),$$

we have that

(iii) *for every $0 \le t \le 2$,*

$$L_t \cap B_{\underline{R}-\Lambda_0} \subseteq Q_{1,t} \cup Q_{2,t} \subseteq B_{\underline{R}+\Lambda_0};$$

(iv) *for every $1 \le t \le 2$, $Q_{2,t}$ is ν-close in $C^{2,\alpha}(B_{R_1})$ to P_3.*

Remark 3.4. (1) We remark that Theorem 3.3(i) and (iii) have no ν dependence in their statements and so could have been stated independently of Theorem 3.3(ii) and (iv).

(2) The content of Theorem 3.3(i) and (ii) is that for all ε small and $\underline{R}$ large we have good control of L_t on an annular region $A(R_1, \overline{R})$ for all $t \le 2$. This is expected because, as we explain next, for all ε small and $\underline{R}$ sufficiently large, $L \cap A(1, 2\overline{R})$ has small $C^{2,\alpha}$ norm and area ratios close to one. In the region $A(1, \underline{R})$ this follows because, as defined in (3),

$$L \cap A(1, \underline{R}) = (P_1 \cup P_2 \cup P_3) \cap A(1, \underline{R}).$$

In the region $A(\underline{R}, 2\overline{R})$ this follows because the $C^{2,\alpha}$ norm and the area ratios of $L \cap A(\underline{R}, 2\overline{R})$ tend to zero as $\underline{R}$ tends to infinity.

(3) The content of Theorem 3.3(iii) is that $L_t \cap B_{\underline{R}}$ has two distinct connected components for all $0 \le t \le 2$, which we call $Q_{1,t}$ and $Q_{2,t}$. The idea is that initially $L \cap B_{\underline{R}}$ has two connected components and because we have control of the flow on the annulus $A(R_1, \underline{R})$ due to Theorem 3.3(i), then no connected component in $L_t \cap B_{\underline{R}}$ can be "lost" or "gained." Without the control on the annular region it is simple to construct examples where a solution to mean curvature flow in $B_1(0)$ consists initially of disjoint straight lines and at a later time is a single connected component.

(4) Theorem 3.3(iv) is also expected because $Q_{2,0}$ initially is just a disc and we have good control on $\partial Q_{2,t}$ for all $0 \le t \le 2$.

The next theorem collects some important properties of $Q_{1,t}$. The proof will be given at the end of Section 3 and, because it is largely standard, can be skipped on a first reading.

THEOREM 3.5. *There are D_1, R_2, and ε_2 depending only on K_0 so that if $\underline{R} \geq R_2$ and $\varepsilon \leq \varepsilon_2$ in* (3), *then for every $0 \leq t \leq 2$, the following properties hold*:

(i)
$$\sup_{0\leq t\leq 2}\sup_{Q_{1,t}}|\theta_t| \leq \pi/2 - 1/(2K_0).$$

(ii)
$$\mathcal{H}^2(\hat{B}_r(y)) \geq D_1 r^2,$$
where $\hat{B}_r(y)$ denotes the intrinsic ball of radius r in $Q_{1,t}$ centered at $y \in Q_{1,t}$ and $r < \operatorname{dist}(y, \partial Q_{1,t})$.

(iii) *All $Q_{1,t}$ are exact, and one can choose $\beta_t \in C^\infty(Q_{1,t})$ with*
$$d\beta_t = \lambda = \sum_{i=1}^{2} x_i dy_i - y_i dx_i$$
and
$$\frac{d}{dt}(\beta_t + 2t\theta_t) = \Delta(\beta_t + 2t\theta_t) + E_1,$$
where $E_1 = \sum_{i=1}^2 \nabla_{e_i}\lambda(e_i)$ and $\{e_1, e_2\}$ is an orthonormal basis for $Q_{1,t}$.

(iv)
$$|\beta_t|(x) \leq D_1(|x|^2 + 1) \quad \textit{for every } x \in Q_{1,t}.$$

(v) *If $\mu = x_1 y_2 - x_2 y_1$, then*
$$\frac{d\mu^2}{dt} \leq \Delta\mu^2 - 2|\nabla\mu|^2 + E_2,$$
where $E_2 = (|x|^3 + 1)O(1/\overline{R})$.

Remark 3.6. (1) We comment on Theorem 3.5(i). Recall that we are assuming $\sup_{Q^1}|\theta| \leq \pi/2 - K_0^{-1}$, where Q^1 is defined in (4). Because θ_t evolves by the heat equation, we have
$$\sup_{0\leq t\leq 2}\sup_{Q_{1,t}}|\theta_t| \leq \max\left\{\sup_{Q_{1,0}}|\theta|, \sup_{0\leq t\leq 2}\sup_{\partial Q_{1,t}}|\theta_t|\right\}.$$
Hence we need to control the Lagrangian angle along $\partial Q_{1,t}$ in order to obtain Theorem 3.5(i). The idea is to use the fact that Q^1 is very "flat" near $\partial Q_{1,0}$ to show that $F_t(Q^1)$ is a small C^1 perturbation of Q^1 near $\partial Q_{1,0}$, which means the Lagrangian angle along $\partial Q_{1,t}$ will not change much.

(2) Theorem 3.5(ii) is a consequence of the fact that $Q_{1,t}$ is almost-calibrated.

(3) Theorem 3.5(iii) and (v) are just derivations of evolution equations taking into account the error term one obtains from the metric g_R (defined in Section 2.1) not being Euclidean.

(4) Theorem 3.5(iv) gives the expected growth for β_t on $Q_{1,t}$, and its proof is a simple technical matter.

3.3. *Abstract results.* We derive some simple results that will be used to prove Theorems 3.3 and 3.5 as well as throughout the rest of the paper. They are presented in a fairly general setting in order to be used in various circumstances. The proofs are based on White's Regularity Theorem and Huisken's monotonicity formula.

Let E be a vector valued function defined on $\mathbb{R}^n \times G(2,n)$, Σ a smooth surface possibly with boundary, and $F_t : \Sigma \longrightarrow \mathbb{R}^n$ a smooth solution to

$$\frac{dF_t}{dt}(x) = H(F_t(x)) + E(F_t(x), T_{F_t(x)}M_t), \tag{6}$$

where $M_t = F_t(\Sigma)$ and F_0 is the identity map.

In what follows, Ω denotes a closed set of $\mathbb{R}^n$ and we use the notation

$$\Omega(s) = \{x \in \mathbb{R}^n \,|\, \mathrm{dist}(x, \Omega) < s\}.$$

We derive two lemmas that are well known among the experts. Denote $\bar{E} = \sup |E|_{0,\alpha}$, and let ε_0 be the constant given by White's Regularity Theorem [18, Th. 4.1].

Lemma 3.7. *Assume $T \leq 4$. There is $\Lambda = \Lambda(\bar{E}, n)$ so that for every $s \geq 0$, if*

(a) *for all $0 \leq t \leq 2T$, $y \in \Omega(s + 2\Lambda\sqrt{T})$, and $l \leq 2T$*

$$\int_{M_t} \Phi(y, l) d\mathcal{H}^2 \leq 1 + \varepsilon_0;$$

(b) *for all $0 \leq t \leq 2T$, $\partial M_t \cap \Omega(s + 2\Lambda\sqrt{T}) = \emptyset$;*

then for every $0 \leq t \leq T$, we have

(i) *the $C^{2,\alpha}$ norm of M_t on $\Omega(s + \Lambda\sqrt{T})$ is bounded by $\Lambda/\sqrt{t}$;*

(ii) $F_t'(x) \in \Omega(s) \implies |F_t(x) - x| < \Lambda\sqrt{t}$ *for all $0 \leq t \leq T$.*

Remark 3.8. The content of Lemma 3.9 is that if we know the Gaussian density ratios at a scale smaller than $2T$ in a region U are all close to one and ∂M_t lies outside U for all $t \leq 2T$, then we have good control of M_t for all $0 \leq t \leq T$ on a slightly smaller region. The proof is a simple consequence of White's Regularity Theorem.

Proof. Assume for all $0 \leq t \leq 2T$, $y \in \Omega(s + (\Lambda + 1)\sqrt{T})$, and $l \leq 2T$ that

$$\int_{M_t} \Phi(y, l) d\mathcal{H}^2 \leq 1 + \varepsilon_0,$$

where $\Lambda \geq 1$ is a constant to be chosen later. From White's Regularity Theorem [18, Th. 4.1] there is $K_1 = K_1(\bar{E}, n)$ so that the $C^{2,\alpha}$ norm of M_t on $\Omega(s + \Lambda\sqrt{T})$ is bounded by $K_1/\sqrt{t}$ and

$$\sup_{M_t \cap \Omega(s + \Lambda\sqrt{T})} |A|^2 \leq \frac{K_1}{t}$$

for every $t \leq T$. Thus from (6), we obtain

$$\left|\frac{dF_t}{dt}(x)\right| \leq \frac{K_1}{\sqrt{t}} + \bar{E}$$

whenever $F_t(x) \in \Omega(s + \Lambda\sqrt{T})$. Integrating the above inequality and using $T \leq 2$, we have the existence of $K_2 = K_2(\bar{E}, K_1)$ so that if

$$F_{t'}(x) \in \Omega(s + (\Lambda - K_2)\sqrt{T}) \text{ for some } 0 \leq t' \leq T,$$

then

$$F_t(x) \in \Omega(s + \Lambda\sqrt{T}) \quad \text{for every} \quad 0 \leq t \leq T$$

and

$$|F_t(x) - x| < K_2\sqrt{t} \quad \text{for every} \quad 0 \leq t \leq T.$$

Choose $\Lambda = \max\{K_1, K_2\}$. Then (i) and (ii) follow at once. □

LEMMA 3.9. *For every ν, S, and $0 < \kappa < 1$, there is δ, R so that if $x_0 \in M_0$ and*

(a) *the $C^{2,\alpha}$ norm of M_0 in $B_{R\sqrt{T}}(x_0)$ and the $C^{0,\alpha}(\mathbb{R}^n \times G(2, M))$ norm of E are smaller than $\delta/\sqrt{T}$;*
(b) *$\mathcal{H}^2(M_0 \cap B_r(x_0)) \leq 7\pi r^2$ for all $0 \leq r \leq R\sqrt{T}$;*
(c) *$\partial M_t \cap B_{R\sqrt{T}}(x_0) = \emptyset$ for all $0 \leq t \leq T$;*

then the following hold:

(i) *$\displaystyle\int_{M_t} \Phi(y, l) d\mathcal{H}^2 \leq 1 + \varepsilon_0$ for all $y \in B_{(S+1)\sqrt{T}}(x_0)$, $t \leq T$, and $l \leq 2T$;*
(ii) *For every $\kappa T \leq t \leq T$, there is a function*

$$u_t : T_{x_0}M_0 \cap B_{(S+1)\sqrt{T}}(x_0) \longrightarrow (T_{x_0}M_0)^\perp$$

with

$$\sup_{T_{x_0}M_0 \cap B_{(S+1)\sqrt{T}}} \left(|u_t|/\sqrt{T} + |\nabla u_t| + |\nabla^2 u_t|_{0,\alpha}\sqrt{T}\right) \leq \nu$$

and

$$M_t \cap B_{S\sqrt{T}}(x_0) \subseteq \{u_t(x) + x, \,|\, x \in T_{x_0}M_0 \cap B_{(S+1)\sqrt{T}}(x_0)\}.$$

Remark 3.10. This lemma, roughly speaking, says that for every S, there is R so that if the initial condition is very close to a disc in $B_{R\sqrt{T}}(x_0)$ (condition (a) and (b)) and ∂M_t lies outside $B_{R\sqrt{T}}(x_0)$ for all $0 \leq t \leq T$ (condition (c)), then we get good control of M_t inside $B_{S\sqrt{T}}(x_0)$.

Proof. It suffices to prove this for $T = 1$ and $x_0 = 0$. Consider a sequence of flows $(M_t^i)_{0 \leq t \leq 1}$ satisfying all the hypotheses with δ_i converging to zero and R_i tending to infinity. The sequence of flows $(M_t^i)_{t \geq 0}$ will converge weakly to $(\bar{M}_t)_{t \geq 0}$, a weak solution to mean curvature flow (see [7, §7.1]). The fact that the $C^{2,\alpha}_{\text{loc}}$ norm of M_0^i converges to zero implies that M_0^i converges in $C^{2,\alpha}_{\text{loc}}$ to a union of planes. From (b) we conclude that M_0^i converges to a multiplicity one plane P. Because ∂M_t^i lies outside B_{R_i} for all $0 \le t \le 1$ with R_i tending to infinity and

$$\lim_{i\to\infty}\int_{M_0^i}\Phi(y,l)d\mathcal{H}^2=\int_P\Phi(y,l)d\mathcal{H}^2=1 \text{ for every } y \text{ and } l,$$

we can still conclude from Huisken's monotonicity formula that for all i sufficiently large,

$$\int_{M_t^i}\Phi(y,l)d\mathcal{H}^2\le 1+\varepsilon_0 \quad \text{for all } y\in B_{S+1},\ t\le 1, \text{ and } l\le 2.$$

This proves (i). Moreover, the above inequality also implies, via White's Regularity Theorem, that M_t^i converges in $C^{2,\alpha}_{\mathrm{loc}}$ to P for all $\kappa \le t \le 1$ and so (ii) will also hold for all i sufficiently large. This implies the desired result. $\square$

3.4. *Proof of Theorems* 3.3 *and* 3.5.

Proof of Theorem 3.3. We first prove part (ii). Consider δ and R given by Lemma 3.9 when $\kappa = 1/2$, ν is the constant fixed in Theorem 3.3 and S is large to be chosen later. The same reasoning used in Remark 3.4(2) shows the existence of K_1 (depending on R and δ) so that for all ε small and $\underline{R}$ sufficiently large, the $C^{2,\alpha}$ norm of $L\cap A(K_1, 2\overline{R}-K_1)$ is smaller than $\delta/2$ and the area ratios with scale smaller than $2R$ are close to one. Thus, after relabelling K_1 to be $K_1-\sqrt{2}R$, we can apply Lemma 3.9(ii) (with $T=4$) to $M_0=L$ for all x_0 in $\Omega = L\cap A(K_1,2\overline{R}-K_1)$ and conclude Theorem 3.3(ii). Moreover, we also conclude from Lemma 3.9(i) that

$$\int_{L_t}\Phi(y,l)d\mathcal{H}^2\le 1+\varepsilon_0 \quad \text{for all } y\in \Omega(S),\ t\le 4, \text{ and } l\le 4, \tag{7}$$

where $\Omega(S)$ denotes the tubular neighbourhood of Ω in $\mathbb{R}^n$ with radius S.

We now prove part (i). From (7), we see that hypotheses (a) and (b) of Lemma 3.7 are satisfied with $T=2$, $s=0$, and $r=S-2^{3/2}\Lambda$ (which we assume to be positive). Hence Lemma 3.7(i) gives that the $C^{2,\alpha}$ norm of L_t in $\Omega(S-2^{1/2}\Lambda)$ is bounded by $\Lambda/\sqrt{t}$. Theorem 3.3(i) follows from this provided

$$L_t\cap A(K_1,2\overline{R}-K_1)\subset \Omega(S-2^{3/2}\Lambda). \tag{8}$$

This inclusion follows because, according to Brakke's Clearing Out Lemma [7, §12.2] (which can be easily extended to our setting assuming small $C^{0,\alpha}$ norm of $|E|$), there is a universal constant S_0 such that

$$L_t\cap A(K_1,2\overline{R}-K_1)\subset \Omega(S_0) \text{ for all } 0\le t\le 2.$$

Thus we simply need to require $S-2^{3/2}\Lambda > S_0$ in order to obtain (8). Furthermore, Lemma 3.7(ii) implies

$$F_t(x)\in\Omega(S-2^{3/2}\Lambda)\implies |F_s(x)-x|<\Lambda s^{1/2} \text{ for all } 0\le s\le 2,$$

which combined with (8) gives

$$F_t(x) \in A(K_1, 2\overline{R} - K_1) \implies |F_s(x) - x| < \Lambda s^{1/2} \text{ for all } 0 \le s \le 2, \tag{9}$$

and this proves the second statement of Theorem 3.3(i).

We now prove the first statement of Theorem 3.3(iii). Suppose

$$L_{t'} \cap B_{\underline{R}-\sqrt{2}\Lambda} \not\subseteq F_{t'}(L \cap B_{\underline{R}}) = Q_{1,t'} \cup Q_{2,t'},$$

meaning $F_{t'}(x) \in B_{\underline{R}-\sqrt{2}\Lambda}$ but $x \notin B_{\underline{R}}$. By continuity there is $0 \le t \le t'$ so that

$$F_t(x) \in A(K_1, \underline{R}),$$

and this implies from (8) that $F_t(x) \in \Omega(S - 2^{3/2}\Lambda)$, in which case we conclude from (9) that $|F_{t'}(x) - x| < \sqrt{2}\Lambda$, a contradiction. Similar reasoning shows the other inclusion in Theorem 3.3(iii).

Finally we show (iv). Apply Lemma 3.9, with $S = K_1/\sqrt{2}$, $\kappa = 1/2$, and ν the constant fixed in this theorem, to $M_0 = Q_{2,0} = P_3 \cap B_{\underline{R}}$ where $x_0 = 0$. Note that hypotheses (a) and (b) of Lemma 3.9 are satisfied with $T = 2$ if one assumes $\underline{R}$ sufficiently large. Moreover, hypothesis (c) is also satisfied because due to Theorem 3.3(i) we have $\partial Q_{2,t} \subset A(\underline{R} - 2\Lambda_0, \underline{R} + 2\Lambda_0)$. Thus $Q_{2,t}$ is ν-close in $C^{2,\alpha}(B_{K_1})$ to P_3 for every $1 \le t \le 2$. $\square$

Proof of Theorem 3.5. During this proof we will use Theorem 3.3(i) and (iii) with $\nu = 1$. Λ_0 is the constant given by that theorem.

From the maximum principle applied to θ_t we know that

$$\sup_{0\le t\le 2} \sup_{Q_{1,t}} |\theta_t| \le \max\left\{\sup_{Q_{1,0}} |\theta|, \sup_{0\le t\le 2} \sup_{\partial Q_{1,t}} |\theta_t|\right\}.$$

The goal now is to control the C^1 norm of $Q_{1,t}$ along $\partial Q_{1,t}$ so that we control $\sup_{\partial Q_{1,t}} |\theta_t|$.

Given η small, consider R and δ given by Lemma 3.9 when $\nu = \eta$, $S = 2\Lambda_0$, and $\kappa = 1/2$. We have

$$\partial Q_{1,0} = Q^1 \cap \{|x| = \underline{R}\},$$

where Q^1 is defined in (4). Thus, for all $\underline{R}$ sufficiently large and ε small, we have that $M_0 = Q^1$ satisfies hypotheses (a) and (b) of Lemma 3.9 for every $x_0 \in \partial Q_{1,0}$. Moreover $\partial(F_t(Q^1)) \cap B_{\overline{R}} = \emptyset$ by Theorem 3.3(i), and so hypothesis (c) is also satisfied because we are assuming $\overline{R} \ge 4\underline{R}$ (see (3)).

This means $F_t(Q^1) \cap B_{2\Lambda_0\sqrt{t}}(x_0)$ is graphical over $T_{x_0}Q^1$ with the C^1 norm being smaller than η for all $0 \le t \le 2$. Thus we can choose η small so that

$$\sup\{|\theta_t(y) - \theta_0(x_0)| : y \in F_t(Q^1) \cap B_{2\Lambda_0\sqrt{t}}(x_0)\} \le 1/(2K_0). \tag{10}$$

Using Theorem 3.3(i) we see that for every $y \in \partial Q_{1,t}$, there is $x_0 \in \partial Q_{1,0}$ so that $y \in B_{2\Lambda_0\sqrt{t}}(x_0)$. Thus from (10), we obtain

$$\sup_{\partial Q_{1,t}} |\theta_t| \leq \sup_{\partial Q_{1,0}} |\theta| + 1/(2K_0)$$

and this implies (i) because we are assuming $\sup_{Q^1} |\theta| \leq \pi/2 - K_0^{-1}$.

We now prove (ii). Assume for a moment that the metric g_R (defined in Section 2.1) is Euclidean in $B_{2\overline{R}}$. Because $Q_{1,t}$ is almost-calibrated, we have from [9, Lemma 7.1] the existence of a constant C depending only K_0 so that, for every open set B in $Q_{1,t}$ with rectifiable boundary,

$$\left(\mathcal{H}^2(B)\right)^{1/2} \leq C \,\mathrm{length}\,(\partial B).$$

It is easy to recognize the same is true (for some slightly larger C) if g_R is very close to the Euclidean metric. Set

$$\psi(r) = \mathcal{H}^2\left(\hat{B}_r(x)\right)$$

which has, for almost all $r < \mathrm{dist}(y, \partial Q_{1,t})$, the derivative given by

$$\psi'(r) = \mathrm{length}\,\left(\partial \hat{B}_r(x)\right) \geq C^{-1}(\psi(r))^{1/2}.$$

Hence, integration implies that for some other constant C, $\psi(r) \geq Cr^2$, and so (ii) is proven.

We now prove (iii). The Lie derivative of $\lambda_t = F_t^*(\lambda)$ is given by

$$\mathcal{L}_H \lambda_t = dF_t^*(H \lrcorner \lambda) + F_t^*(H \lrcorner 2\omega) = d(F_t^*(H \lrcorner \lambda) - 2\theta_t),$$

and so we can find $\beta_t \in C^\infty(Q_{1,t})$ with $d\beta_t = \lambda$ and

$$\frac{d\beta_t}{dt} = H \lrcorner \lambda - 2\theta_t. \tag{11}$$

A simple computation shows that $\Delta\beta_t = H \lrcorner \lambda + \sum_{i=1}^2 \nabla_{e_i}\lambda(e_i)$, which proves (iii).

We now prove (iv). Combining Theorem 3.3(i) and (11), we have that

$$\left|\frac{d\beta_t}{dt}(F_t(x))\right| \leq \frac{\Lambda_0}{\sqrt{t}}|F_t(x)| + \pi - K_0^{-1}$$

for every $x \in L \cap A(R_1 + 2\Lambda_0, \underline{R})$. Thus after integration in the t variable, assuming $t \leq 2$, and recalling (5), we obtain a constant $C = C(K_0, \Lambda_0)$ such that

$$|\beta_t(F_t(x))| \leq C(|F_t(x)| + |\beta(x))| + C \leq C(|F_t(x)|^2 + 1).$$

We are left to estimate β_t on $A_t = F_t(Q_{1,0} \cap B_{R_1+2\Lambda_0})$. From Theorem 3.3(i) we know that $A_t \subseteq B_{C_1}(0)$ for some $C_1 = C_1(K_0, \Lambda_0, R_1)$ and thus, provided we assume g_R to be sufficiently close to the Euclidean metric,

$$|\nabla\beta_t(x)| = |\lambda| \leq 2C_1 \quad \text{for every } x \in A_t.$$

Hence, if we fix x_1 in ∂A_t, we can find $C = C(K_0, \Lambda_0, R_1)$ so that for every y in A_t,

$$|\beta_t(y)| \le |\beta_t(x_1)| + C\mathrm{dist}_{A_t}(x_1, y) \le C(1 + \mathrm{dist}_{A_t}(x_1, y)),$$

where dist_{A_t} denotes the intrinsic distance in A_t. Property (ii) of this theorem, $A_t \subseteq B_{C_1}(0)$, and Lemma 3.1 are enough to bound uniformly the intrinsic diameter of A_t and thus bound β_t uniformly on A_t. Hence (iv) is proven.

We now prove (v). In what follows, E_2^j denotes any term with decay $(|x|^j + 1)O(1/\underline{R})$. Given a coordinate function $v = x_i$ or y_i, $i = 1, 2$, we have

$$\frac{dv}{dt} = \Delta v - \sum_{i=1}^{2} g(\nabla_{e_i} V, e_i) = \Delta v + E_2^0,$$

where V denotes the gradient of v with respect to g_R. Thus,

$$\frac{d\mu}{dt} = \Delta\mu + E_2^1 - 2g_R(X_1^\top, Y_2^\top) + 2g_R(Y_1^\top, X_2^\top),$$

where $X_i, Y_i, i = 1, 2$ denote the gradient of the coordinate functions with respect to g_R. If the ambient Calabi-Yau structure were Euclidean, then

$$\begin{aligned}\langle X_1^\top, Y_2^\top\rangle - \langle Y_1^\top, X_2^\top\rangle &= -\langle (JY_1)^\top, Y_2\rangle - \langle Y_1^\top, X_2\rangle \\ &= -\langle JY_1^\perp, Y_2\rangle - \langle Y_1^\top, X_2\rangle = -\langle Y_1^\perp + Y_1^\top, X_2\rangle = -\langle Y_1, X_2\rangle = 0.\end{aligned}$$

In general, it is easy to see that $g_R(X_1^\top, Y_2^\top) - g_R(Y_1^\top, X_2^\top) = E_2^0$ and so

$$\frac{d\mu^2}{dt} \le \Delta\mu^2 - 2|\nabla\mu|^2 + E_2^3. \qquad \square$$

4. **Second Step: Self-expanders**

The goal of this section is to prove the theorem below. For the reader's convenience, we recall that the planes P_1, P_2 are defined in Section 2.1.1, K_0 is defined at the beginning of Section 3, Q^1 is defined in (4), and $Q_{1,t}$ is defined in Theorem 3.3(iii). The self-expander equation is defined in Section 2.1.2, and the self-expander $\mathcal{S}$ is defined in (2) (see Figure 2).

THEOREM 4.1. *Fix S_0 and ν. There are ε_3 and R_3, depending on S_0, ν, and K_0, such that if $\underline{R} \ge R_3$, $\varepsilon \le \varepsilon_3$ in* (3), *and*

the flow $Q_{1,t}$ exists smoothly for all $0 \le t \le 2$,

then $t^{-1/2}Q_{1,t}$ is ν-close in $C^{2,\alpha}(B_{S_0})$ to $\mathcal{S}$ for every $1 \le t \le 2$.

As we will see shortly, this theorem follows from Theorem 4.2. Recall that, as seen in Theorem 3.5(iii), we can find β_t on $Q_{1,t}$ so that

$$d\beta_t = \lambda = \sum_{i=1}^{2} x_i dy_i - y_i dx_i.$$

THEOREM 4.2. *Fix* S_0 *and* ν. *There are* ε_4, R_4, *and* δ *depending on* S_0, ν, *and* K_0, *such that if* $\underline{R} \geq R_4$, $\varepsilon \leq \varepsilon_4$ *in* (3), *and*

- *the flow* $Q_{1,t}$ *exists smoothly for all* $0 \leq t \leq 2$;
-

$$\int_{Q^1 \cap B_{\underline{R}}} \beta^2 \exp(-|x|^2/8) d\mathcal{H}^2 \leq \delta; \tag{12}$$

then $t^{-1/2}Q_{1,t}$ *is* ν*-close in* $C^{2,\alpha}(B_{S_0})$ *to a smooth embedded self-expander asymptotic to* P_1 *and* P_2 *for every* $1 \leq t \leq 2$.

Remark 4.3. (1) If the ambient metric g_R (defined in Section 2.1) were Euclidean, then $|\nabla\beta(x)| = |x^\perp|$, and thus β would be constant exactly on cones. Hence, roughly speaking, the left-hand side of (12) measures how close $Q^1 \cap B_{\underline{R}}$ is to a cone.

(2) The content of the theorem is that given ν and S_0, there is δ so that if the initial condition is δ-close, in the sense of (12), to a non area-minimizing configuration of two planes $P_1 + P_2$ and the flow exists smoothly for all $0 \leq t \leq 2$, then the flow will be ν-close to a smooth self-expander in B_{S_0} for all $1 \leq t \leq 2$.

(3) The result is false if one removes the hypothesis that the flow exists smoothly for all $0 \leq t \leq 2$. For instance, there are known examples [9, Th. 4.1] where $Q_{1,0}$ is very close to $P_1 + P_2$ (see [9, Fig. 1]) and a finite-time singularity happens for a very short time T. In this case, $Q_{1,T}$ can be seen as a transverse intersection of small perturbations of P_1 and P_2 (see [9, Fig. 2]) and we could continue the flow past the singularity by flowing each component of $Q_{1,T}$ separately, in which case $Q_{1,1}$ would be very close to $P_1 + P_2$ and this is not a *smooth* self-expander. The fact the flow exists smoothly will be crucial to prove Lemma 4.10.

(4) The result is also false if $P_1 + P_2$ is area-minimizing. The reason is that in this case the self-expander asymptotic to $P_1 + P_2$ is simply $P_1 + P_2$, which is singular at the origin and thus not *smooth* as it is guaranteed by Theorem 4.2. The fact that $P_1 + P_2$ is not area-minimizing will be crucial to prove Lemma 4.10.

(5) The strategy to prove Theorem 4.2 is the following. The first step (Proposition 4.4) is to show that if the left-hand side of (12) is very small, then

$$\int_{Q_{1,1} \cap B_{\underline{R}/2}} (\beta_1 + 2\theta_1)^2 \Phi(0, 4-t) d\mathcal{H}^2 + \int_0^2 \int_{Q_{1,t}} |x^\perp - 2tH|^2 \Phi(0, 4-t) d\mathcal{H}^2 dt$$

is also very small. The second step (Proposition 4.6) in the proof will be to show that if

$$\int_{Q_{1,1}\cap B_{\underline{R}/2}}(\beta_1+2\theta_1)^2\Phi(0,4-t)d\mathcal{H}^2+\int_0^2\int_{Q_{1,t}}|x^\perp-2tH|^2\Phi(0,4-t)d\mathcal{H}^2dt$$

is very small, then $t^{-1/2}Q_{1,t}$ will be ν-close in $C^{2,\alpha}(B_{S_0})$ to a smooth self-expander. It is in this step that we use the fact that the flow exists smoothly and P_1+P_2 is not an area-minimizing configuration.

Proof of Theorem 4.1. The first step is to show that Theorem 4.2 can be applied, which amounts to showing that (12) holds if we choose ε sufficiently small and $\underline{R}$ sufficiently large. Thus, we obtain that $t^{-1/2}Q_{1,t}$ is ν-close in $C^{2,\alpha}(B_{S_0})$ to a smooth embedded self-expander asymptotic to P_1 and P_2 for every $1\le t\le 2$. The second step is to show that self-expander must be $\mathcal{S}$.

First Step. We note $Q^1\cap A(1,\underline{R})$ (defined in (4)) coincides with $(P_1\cup P_2)\cap A(1,\underline{R})$, and so the uniform control we have on β given by (5) implies that for all δ, there is r_1 large depending on K_0 and δ so that

$$\int_{Q^1\cap A(r_1,\underline{R})}\beta^2\exp(-|x|^2/8)d\mathcal{H}^2\le\frac{\delta}{2} \tag{13}$$

for all ε small and $\underline{R}$ large. Also, if we make ε tend to zero and $\underline{R}$ tend to infinity in (3), it is straightforward to see that Q^1 tends to $P_1\cup P_2$ smoothly on any compact set that does not contain the origin. Because β is constant on cones, we can choose β on Q^1 so that

$$\lim_{\varepsilon\to0,\underline{R}\to\infty}\int_{Q^1\cap B_{r_1}}\beta^2\exp(-|x|^2/8)d\mathcal{H}^2=0.$$

Combining this with (13) we obtain that for all ε small and $\underline{R}$ large,

$$\int_{Q^1\cap B_{\underline{R}}}\beta^2\exp(-|x|^2/8)d\mathcal{H}^2\le\delta.$$

Hence all the hypotheses of Theorem 4.2 hold.

Second Step. Let Q denote a smooth embedded Lagrangian self-expander asymptotic to P_1+P_2. Then $Q_t=\sqrt{t}Q$ and $\lim_{t\to0^+}Q_t=P_1+P_2$ as Radon measures. Thus, if we recall the function $\mu=x_1y_2-y_1x_2$ defined in Theorem 3.5(v), we have

$$\lim_{t\to0^+}\int_{Q_t}\mu^2\Phi(0,T-t)d\mathcal{H}^2=\int_{P_1+P_2}\mu^2\Phi(0,T)d\mathcal{H}^2=0. \tag{14}$$

Using the evolution equation for μ given in Theorem 3.5(v) (E_2 is identically zero) into Huisken's monotonicity formula (see Lemma 2.1), we have

$$\frac{d}{dt}\int_{Q_t}\mu^2\Phi(0,T-t)d\mathcal{H}^2\le0.$$

This inequality and (14) imply at once that

$$\int_{Q_t} \mu^2 \Phi(0,1) d\mathcal{H}^2 = 0 \text{ for all } t \geq 0$$

and so $Q \subset \mu^{-1}(0)$. A trivial modification of Lemma 7.1 implies the existence of γ asymptotic to χ (the curve defined in (2)) so that

$$Q = \{(\gamma(s)\cos\alpha, \gamma(s)\sin\alpha) \,|\, s \in \mathbb{R}, \alpha \in S^1\}.$$

From [1, §5] we know that $\chi = \gamma$, and so the result follows. □

4.1. *Proof of Theorem* 4.2. Throughout this proof we assume that $\underline{R}$ is sufficiently large and ε is sufficiently small so that Theorem 3.3 (with $\nu = 1$) and Theorem 3.5 apply. We also assume the flow $(Q_{1,t})_{0\leq t\leq 2}$ exists smoothly.

For simplicity, denote $Q_{1,t}$ simply by Q_t. We also recall that the constant K_0, which will appear multiple times during this proof, was defined at the beginning of Section 3.

Proposition 4.4. *Fix η. There are ε_5 and R_5 depending on η and K_0 so that if $\varepsilon \leq \varepsilon_5$ and $\underline{R} \geq R_5$ in* (3), *then*

$$\sup_{0\leq t\leq 2} \int_{Q_t \cap B_{\underline{R}/2}} (\beta_t + 2t\theta_t)^2 \Phi(0, 4-t) d\mathcal{H}^2 + \int_0^2 \int_{Q_t \cap B_{\underline{R}/2}} |x^\perp - 2tH|^2 \Phi(0, 4-t) d\mathcal{H}^2 dt \leq \frac{\eta}{2} + \int_{Q^1 \cap B_{\underline{R}}} \beta^2 \Phi(0,4) d\mathcal{H}^2.$$

Remark 4.5. The idea is to apply Huisken monotonicity formula for $(\beta_2 + 2t\theta_t)^2$. Some extra (technical) work has to be done because Q_t has boundary and the ambient metric g_R (defined in Section 3) is not Euclidean.

Proof. Let $\phi \in C^\infty(\mathbb{R}^4)$ such that $0 \leq \phi \leq 1$,

$$\phi = 1 \text{ on } B_{\underline{R}/2}, \quad \phi = 0 \text{ on } B_{2\underline{R}/3}, \quad |D\phi| + |D^2\phi| \leq \frac{\Lambda}{\underline{R}},$$

where Λ is some universal constant. By Theorem 3.3(i) we have that, provided we chose $\underline{R}$ large and ε small, $\partial Q_t \cap B_{\underline{R}/2} =$ and thus ϕ has compact support in Q_t.

Set $\gamma_t = \beta_t + 2t\theta_t$. Then on Q_t, we have from Theorem 3.5(iii) that

$$\frac{d(\gamma_t\phi)^2}{dt} = \Delta(\gamma_t\phi)^2 - 2|\nabla\gamma_t|^2\phi^2 + \langle H, D\phi^2\rangle\gamma_t^2 - 2\langle\nabla\gamma_t^2, D\phi^2\rangle - \gamma_t^2\Delta\phi^2 + \phi^2 E_1.$$

Thus, using Theorem 3.3(i) to estimate H and Theorem 3.5(iv), we have that for all $\underline{R}$ large and ε small,

$$\frac{d(\gamma_t\phi)^2}{dt} \leq \Delta(\gamma_t\phi)^2 - 2|\nabla\gamma_t|^2\phi^2 + \frac{C_1}{\sqrt{t}\underline{R}}(|x|^4+1)(1-\chi_{\underline{R}/2}) + |E_1|,$$

where $C_1 = C_1(K_0, \Lambda_0, D_1)$ and $\chi_{\underline{R}/2}$ denotes the characteristic function of $B_{\underline{R}/2}$. From Lemma 2.1, we conclude

$$\begin{aligned}&\frac{d}{dt}\int_{Q_t}(\gamma_t\phi)^2\Phi(0,4-t)d\mathcal{H}^2 + 2\int_{Q_t}|\nabla\gamma_t|^2\phi^2\Phi(0,4-t)d\mathcal{H}^2\\ &\leq \int_{Q_t}\left(\gamma_t^2\frac{|E|^2}{4}+|E_1|\right)\Phi(0,4-t)d\mathcal{H}^2 + \frac{C_1}{\sqrt{t}\underline{R}}\int_{Q_t\setminus B_{\underline{R}/2}}(|x|^4+1)\Phi(0,4-t)d\mathcal{H}^2.\end{aligned}$$

We now estimate the two terms on the right-hand side. If g_R (defined at the beginning of Section 3) were Euclidean, both terms $|E|^2$ and E_1 mentioned above would vanish. Otherwise it is easy to see that making $\underline{R}$ sufficiently large so that g_R becomes close to Euclidean, both terms can be made arbitrarily small. The growth of γ_t is quadratic (Theorem 3.5(i) and (iv)), and so choosing $\underline{R}$ sufficiently large and ε sufficiently small, we have

$$\int_{Q_t}\left(\gamma_t^2\frac{|E|^2}{4}+|E_1|\right)\Phi(0,4-t)d\mathcal{H}^2 \leq \frac{\eta}{8} \quad \text{for all } t\leq 2.$$

Using that $|x| \geq |x|^2/2 + \underline{R}^2/8$ outside $B_{\underline{R}/2}$, it is easy to see that

$$\Phi(0,4-t) \leq 2^{1/2}\Phi(0,2(4-t))\exp(-\underline{R}^2/(32(4-t))) \text{ on } \mathbb{R}^4\setminus B_{\underline{R}/2}.$$

Thus, for all $0\leq t\leq 2$, the uniform area bounds given in Lemma 3.1 imply

$$\begin{aligned}&\int_{Q_t\setminus B_{\underline{R}/2}}(|x|^4+1)\Phi(0,4-t)d\mathcal{H}^2\\ &\leq C_2\exp(-\underline{R}^2/C_2)\int_{Q_t\setminus B_{\underline{R}/2}}(|x|^4+1)\Phi(0,2(4-t))d\mathcal{H}^2 \leq C_3\exp(-\underline{R}^2/C_3),\end{aligned}$$

where C_2 and C_3 depend only on K_0. Therefore, we have

$$\begin{aligned}\frac{d}{dt}\int_{Q_t}(\gamma_t\phi)^2\Phi(0,4-t)d\mathcal{H}^2 &+ 2\int_{Q_t}|\nabla\gamma_t|^2\phi^2\Phi(0,4-t)d\mathcal{H}^2\\ &\leq \frac{C_4}{\sqrt{t}\underline{R}}\exp(-\underline{R}^2/C_4)+\frac{\eta}{8},\end{aligned}$$

where $C_4 = C_4(C_1, C_3)$. Integrating this inequality, for all $t\leq 2$, we obtain

$$\begin{aligned}(15)\quad &\int_{Q_t}\gamma_t^2\phi^2\Phi(0,4-t)d\mathcal{H}^2 + 2\int_0^t\int_{Q_s}|\nabla\gamma_t|^2\phi^2\Phi(0,4-s)d\mathcal{H}^2ds\\ &\leq \int_{Q^1\cap B_{\underline{R}}}\beta^2\Phi(0,4)d\mathcal{H}^2 + 2^{3/2}C_4\underline{R}^{-1}\exp(-\underline{R}^2/C_4)+\frac{\eta}{4}.\end{aligned}$$

If the metric g_R were Euclidean, then $|\nabla\gamma_t|^2 = |x^\perp - 2tH|^2$. Hence the result follows from (15) if we assume $\underline{R}$ is large enough so that $2|\nabla\gamma_t|^2 \leq |x^\perp - 2tH|^2$ and

$$2^{3/2}C_4\underline{R}^{-1}\exp(-\underline{R}^2/C_4) \leq \frac{\eta}{4}. \qquad \square$$

The next proposition is crucial to prove Theorem 4.2.

Proposition 4.6. *Fix ν and S_0. There are ε_6, R_6, and η depending on ν, K_0, and S_0, such that if $\underline{R} \geq R_5$, $\varepsilon \leq \varepsilon_5$ in* (3), *and*

$$(16) \quad \sup_{0\leq t\leq 2}\int_{Q_t\cap B_{\underline{R}/2}}(\beta_t + 2t\theta_t)^2\Phi(0,4-t)d\mathcal{H}^2 + \int_0^2\int_{Q_t\cap B_{\underline{R}/2}}|x^\perp - 2tH|^2\Phi(0,4-t)d\mathcal{H}^2dt \leq \eta,$$

then $t^{-1/2}Q_t$ is ν-close in $C^{2,\alpha}(B_{S_0})$ to a smooth embedded self-expander asymptotic to $P_1 + P_2$ for every $1 \leq t \leq 2$.

Remark 4.7. The strategy to prove this proposition is the following. We argue by contradiction, and standard arguments will give us a sequence of flows $(Q^i_t)_{0\leq t\leq 2}$ converging weakly to a Brakke flow $(\bar{Q}_t)_{0\leq t\leq 2}$, where in (3) we have $\underline{R}_i$ tending to infinity, ε_i tending to zero, and

$$(17) \quad \lim_{i\to\infty}\int_{Q^i_1\cap B_{\underline{R}_i/2}}(\beta^i_1 + 2\theta^i_1)^2\Phi(0,4-t)d\mathcal{H}^2 + \int_0^2\int_{Q^i_t\cap B_{\underline{R}_i/2}}|x^\perp - 2tH|^2\Phi(0,4-t)d\mathcal{H}^2dt = 0.$$

Standard arguments (Lemma 4.8) imply $\bar{Q}_t$ is a self-expander with

$$\lim_{t\to 0^+}\bar{Q}_t = P_1 + P_2.$$

The goal is to show that $\bar{Q}_1$ is smooth because we could have, for instance, $\bar{Q}_1 = P_1 + P_2$.

The first step (Lemma 4.10) is to show that $\bar{Q}_1$ is not stationary. The idea is the following. If $\bar{Q}_1$ were stationary, then $\bar{Q}_t = \bar{Q}_1$ for all t and so $\bar{Q}_1 = \lim_{t\to 0^+}\bar{Q}_t = P_1 + P_2$. On the other hand, from the control given in Theorem 3.3, we will be able to find $r_1 > 0$ large so that $Q^i_1 \cap B_{r_1}$ is connected. (If the flow had a singularity, this would not necessarily be true.) Furthermore, we will deduce from (17) that

$$\int_{Q^i_1\cap B_{r_1}}|\nabla\beta^i_1|^2d\mathcal{H}^2 = \lim_{i\to\infty}\int_{Q^i_1\cap B_{r_1}}|x^\perp|^2d\mathcal{H}^2 = 0.$$

Hence we can invoke [9, Prop. A.1] and conclude that β_1^i must tend to constant $\bar\beta$ in L^2. Combining this with (17), we have

$$\lim_{i\to\infty}\int_{Q_1^i\cap B_{r_1}}(\bar\beta+2\theta_1^i)^2 d\mathcal{H}^2=0,$$

and thus $\bar Q_1$ must be Special Lagrangian with Lagrangian angle $-\bar\beta/2$. This contradicts the choice of P_1 and P_2.

The second step (Lemma 4.11) is to show the existence of l_1 so that, for every $y\in\mathbb{C}^2$ and $l<l_1$, the Gaussian density ratios of $\bar Q_1$ centered at y with scale l defined by

$$\Theta(y,l)=\int_{\bar Q_1}\Phi(y,l)d\mathcal{H}^2$$

are very close to one. If true, then standard theory implies $\bar Q_1$ is smooth and embedded. The (rough) idea for the second step is the following. If this step fails for some $y\in\mathbb{C}^2$, then y should be in the singular set of $\bar Q_1$. Now $T_y\bar Q_1$ should be a union of (at least two) planes. Hence the Gaussian density ratios of $\bar Q_1$ at y for all small scales should not only be away from one but actually bigger than or equal to two. We know from Huisken's monotonicity formula that the Gaussian density ratios of $\bar Q_1$ at y and scale l are bounded from above by the Gaussian density ratios of $\bar Q_0=P_1+P_2$ at y and scale $l+1$. But these latter Gaussian ratios are never bigger than two (see Remark 4.12), which means equality must hold in Huisken's monotonicity formula and so $\bar Q_t$ must be a self-shrinker. Now $\bar Q_t$ is also a self-expander, and thus it must be stationary. This contradicts the first step.

Proof. Consider a sequence $(\underline{R}_i)$ converging to infinity and a sequence (ε_i) converging to zero in (3) that give rise to a sequence of smooth flows $(Q_t^i)_{0\le t\le 2}$ satisfying

$$\begin{aligned}(18)\quad \sup_{0\le t\le 2}\int_{Q_t^i\cap B_{\underline{R}^i/2}}&(\beta_t+2t\theta_t)^2\Phi(0,4-t)d\mathcal{H}^2\\ &+\int_0^2\int_{Q_t\cap B_{\underline{R}_i/2}}|x^\perp-2tH|^2\Phi(0,4-t)d\mathcal{H}^2dt\le\frac{1}{i}.\end{aligned}$$

We will show the existence of a smooth self-expander $\bar Q_1$ asymptotic to P_1 and P_2 so that, after passing to a subsequence, $t^{-1/2}Q_t^i$ converges in $C^{2,\alpha}(B_{S_0})$ to $\bar Q_1$ for every $1\le t\le 2$.

From compactness for integral Brakke motions [7, §7.1] we know that, after passing to a subsequence, $(Q_t^i)_{0\le t\le 2}$ converges to an integral Brakke motion $(\bar Q_t)_{0\le t\le 2}$, where Q_0^i converges in the varifold sense to the varifold P_1+P_2. Furthermore,

$$\lim_{i\to\infty}\int_0^2\int_{Q_t^i\cap B_{\underline{R}_i/2}}|x^\perp-2tH|^2\Phi(0,4-t)d\mathcal{H}^2dt=0,$$

which means

$$H = \frac{x^{\perp}}{2t} \text{ on } \bar{Q}_t \text{ for all } t > 0, \tag{19}$$

and so $\bar{Q}_t = \sqrt{t}\bar{Q}_1$ as varifolds for every $t > 0$. (See proof of [10, Th. 3.1] for this last fact.)

LEMMA 4.8. *As t tends to zero, $\bar{Q}_t$ converges, as Radon measures, to $P_1 + P_2$.*

Remark 4.9. This lemma is needed because the Brakke flow theory only assures that the support of the Radon measure obtained from $\lim_{t\to 0} \bar{Q}_t$ is contained in the support of $\lim_{i\to\infty} Q_0^i = P_1 + P_2$.

Proof. Set

$$\mu_t(\phi) = \int_{\bar{Q}_t} \phi \, d\mathcal{H}^2.$$

The Radon measure $\nu = \lim_{t\to 0^+} \mu_t$ is well defined by [7, Th. 7.2] and satisfies, for every $\phi \geq 0$ with compact support,

$$\nu(\phi) \leq \lim_{i\to\infty} \int_{Q_0^i} \phi \, d\mathcal{H}^2 = \int_{P_1+P_2} \phi \, d\mathcal{H}^2. \tag{20}$$

It is simple to recognize that ν must be either zero, P_1, P_2, or $P_1 + P_2$.

The measure ν is invariant under scaling, meaning that if we set $\phi_c(x) = \phi(cx)$, then

$$\begin{aligned}\nu(\phi_c) &= \lim_{t\to 0^+} \int_{\bar{Q}_t} \phi_c d\mathcal{H}^2 = c^{-2} \lim_{t\to 0^+} \int_{c\bar{Q}_t} \phi d\mathcal{H}^2 \\ &= c^{-2} \lim_{t\to 0^+} \int_{\bar{Q}_{c^2 t}} \phi d\mathcal{H}^2 = c^{-2} \lim_{t\to 0^+} \int_{\bar{Q}_t} \phi d\mathcal{H}^2 = c^{-2}\nu(\phi).\end{aligned}$$

From Theorem 3.3(i) and Theorem 3.5(ii) we have that the support of ν contains $(P_1 + P_2) \cap A(K_1, \infty)$ which, combined with the invariance of the measure we just mentioned, implies the support of ν coincides with $P_1 \cup P_2$. Thus $\nu = P_1 + P_2$, as we wanted to show. □

LEMMA 4.10. *$\bar{Q}_1$ is not stationary.*

Proof. If true, then $\bar{Q}_1$ needs to be a cone because $x^{\perp} = 2H = 0$ and so, because $\bar{Q}_t = \sqrt{t}\bar{Q}_1$, they are also cones for all $t > 0$. Hence we must have (from varifold convergence) that for every $r > 0$,

$$\lim_{i\to\infty} \int_0^2 \int_{Q_t^i \cap B_r} |x^{\perp}|^2 d\mathcal{H}^2 dt = 0,$$

which implies from (18) that

$$\lim_{i\to\infty} \int_0^2 \int_{Q_t^i \cap B_r} (t^2|H|^2 + |x^{\perp}|^2) d\mathcal{H}^2 dt = 0.$$

Therefore, we can assume without loss of generality that for every $r > 0$,

$$\lim_{i\to\infty}\int_{Q_1^i\cap B_r}(|H|^2+|x^\perp|^2)d\mathcal{H}^2 = 0 \tag{21}$$

and thus, by [9, Prop. 5.1], $\bar{Q}_1$ is a union of Lagrangian planes with possible multiplicities. We will argue that $\bar{Q}_1$ must be a Special Lagrangian, i.e., all the planes in $\bar{Q}_1$ must have the same Lagrangian angle. This gives us a contradiction for the following reason. On one hand, $\bar{Q}_t = \bar{Q}_1$ for all $t > 0$, which means $\lim_{t\to 0}\bar{Q}_t = \bar{Q}_1$. On the other hand, from Lemma 4.8, we have $\lim_{t\to 0}\bar{Q}_t = P_1 + P_2$, which means $\bar{Q}_1 = P_1 + P_2$ and therefore the Lagrangian angle of P_1 and P_2 must be the identical (or differ by a multiple of π). This contradicts how P_1 and P_2 were chosen.

From Theorem 3.3(ii) (which we apply with $\nu = 1$) we have that for all i sufficiently large, $Q_1^i \cap A(R_1, \underline{R}_i/2)$ is graphical over $(P_1 \cup P_2) \cap A(R_1, \underline{R}_i/2)$ with the $C^{2,\alpha}$ norm uniformly bounded. Hence we can find $r_1 \geq R_1$ so that if we set $N_i = Q_1^i \cap B_{3r_1}$, we have for all i sufficiently large that $N_i \cap B_{2r_1}$ connected. We note that if Q_t^i had a singularity for some $t < 1$ then N_i could be two discs intersecting transversally near the origin and thus $N_i \cap B_{2r_1}$ would not be connected.

Furthermore, we obtain from (21) that

$$\lim_{i\to\infty}\int_{N_i}|\nabla\beta^i|^2 d\mathcal{H}^2 = \lim_{i\to\infty}\int_{N_i}|x^\perp|^2 d\mathcal{H}^2 = 0$$

and so, because of Theorem 3.5(ii), we can apply [9, Prop. A.1] and conclude the existence of a constant $\bar{\beta}$ so that, after passing to a subsequence,

$$\lim_{i\to\infty}\int_{N_i\cap B_{r_1}}(\beta_1^i - \bar{\beta})^2 d\mathcal{H}^2 = 0. \tag{22}$$

Recall that from (18), we have

$$\lim_{i\to\infty}\int_{Q_1^i\cap B_{r_1}}(\beta_1^i + 2\theta_1^i)^2 d\mathcal{H}^2 = 0,$$

which combined with (22) implies

$$\lim_{i\to\infty}\int_{Q_1^i\cap B_{r_1}}(\bar{\beta} + 2\theta_1^i)^2 d\mathcal{H}^2 = 0.$$

Therefore $\bar{Q}_1$ must be a Special Lagrangian cone with Lagrangian angle $-\bar{\beta}/2$. □

In the next lemma, ε_0 denotes the constant given by White's Regularity Theorem [18].

LEMMA 4.11. *There is $l(t)$, a positive continuous function of $0 < t \leq 2$, so that*

$$\int_{\bar{Q}_t} \Phi(y,l)d\mathcal{H}^2 \leq 1 + \varepsilon_0/2 \quad \text{for every } l \leq l(t),\ y \in \mathbb{R}^4,\ \text{and } t > 0.$$

Remark 4.12. During the proof the following simple formula will be used constantly. Given $y \in \mathbb{C}^2$, let d_1, d_2 denote, respectively, the distance from y to P_1 and P_2. Then

$$\int_{P_1+P_2} \Phi(y,l)d\mathcal{H}^2 = \exp(-d_1^2/(4l)) + \exp(-d_2^2/(4l)) \leq 2. \tag{23}$$

Proof. It suffices to prove the lemma for $t = 1$ because, as we have seen, $\bar{Q}_t = \sqrt{t}\bar{Q}_1$ for all $t > 0$.

Claim. There is C_1 such that for every $l \leq 2$ and $y \in \mathbb{R}^4$,

$$\int_{\bar{Q}_1} \Phi(y,l)d\mathcal{H}^2 \leq 2 - C_1^{-1}. \tag{24}$$

From the monotonicity formula for Brakke flows [8, Lemma 7],

$$\begin{aligned}\int_{\bar{Q}_1} \Phi(y,l)d\mathcal{H}^2 + \int_0^1 \int_{\bar{Q}_t} \left| H + \frac{(x-y)^\perp}{2(l+1-t)} \right|^2 \Phi(y, l+1-t)d\mathcal{H}^2 dt \\ = \int_{P_1+P_2} \Phi(y, l+1)d\mathcal{H}^2 \leq 2.\end{aligned} \tag{25}$$

Suppose there is a sequence (y_i) and (l_i) with $0 \leq l_i \leq 2$ such that

$$\int_{\bar{Q}_1} \Phi(y_i, l_i)d\mathcal{H}^2 \geq 2 - \frac{1}{i}.$$

Then, from (25) we obtain

$$\lim_{i\to\infty} \int_{P_1+P_2} \Phi(y_i, l_i+1)d\mathcal{H}^2 = 2$$

and so, from (23), (y_i) must converge to zero. Assuming (l_i) converges to $\bar{l}$, we have again from (25) that

$$\begin{aligned}&\int_0^{1/2} \int_{\bar{Q}_t} \left| H + \frac{x^\perp}{2(\bar{l}+1-t)} \right|^2 \Phi(0, \bar{l}+1-t)d\mathcal{H}^2 dt \\ &\leq \lim_{i\to\infty} \int_0^1 \int_{\bar{Q}_t} \left| H + \frac{(x-y_i)^\perp}{2(l_i+1-t)} \right|^2 \Phi(y_i, l_i+1-t)d\mathcal{H}^2 dt \\ &\leq 2 - \lim_{i\to\infty} \int_{\bar{Q}_1} \Phi(y_i, l_i)d\mathcal{H}^2 = 0.\end{aligned}$$

As a result,

$$H + \frac{x^\perp}{2(\bar{l}+1-t)} = 0 \text{ on } \bar{Q}_t \text{ for all } 0 \leq t \leq 1/2,$$

and combining this with the fact that $H = \frac{x^\perp}{2t}$ on $\bar{Q}_t$, we obtain that $H = 0$ on $\bar{Q}_1 = t^{-1/2}\bar{Q}_t$, which contradicts Lemma 4.10. Thus, (24) must hold.

To finish the proof we argue again by contradiction and assume the lemma does not hold. Hence, there is a sequence $(y_j)_{j\in\mathbb{N}}$ of points in $\mathbb{R}^4$ and a sequence $(l_j)_{j\in\mathbb{N}}$ converging to zero for which

$$\int_{\bar{Q}_1} \Phi(y_j, l_j) d\mathcal{H}^2 \geq 1 + \frac{\varepsilon_0}{2}. \tag{26}$$

The first thing we do is to show (26) implies the existence of m so that $|y_j| \leq m$ for all j. The reason is that from (25) we obtain

$$\int_{P_1+P_2} \Phi(y_j, l_j + 1) d\mathcal{H}^2 \geq 1 + \frac{\varepsilon_0}{2}$$

and so, because (l_j) tends to zero, we obtain from (23) that the sequence (y_j) must be bounded.

The motivation for the rest of the argument is the following. The sequence (y_j) has a subsequence that converges to $\bar{y} \in \mathbb{C}^2$. From (26) we have that $\bar{y}$ must belong to the singular set of $\bar{Q}_1$. The tangent cone to $\bar{Q}_1$ at $\bar{y}$ is a union of (at least two) Lagrangian planes, and thus for all l very small, we must have

$$\int_{\bar{Q}_1} \Phi(\bar{y}, l) d\mathcal{H}^2 \geq 2 - \frac{1}{2C_1}.$$

This contradicts (24).

Recalling that the flow $(Q^i_t)_{0\leq t\leq 2}$ tends to $(\bar{Q}_t)_{0\leq t\leq 2}$, a standard diagonalization argument allows us to find a sequence of integers $(k_j)_{j\in\mathbb{N}}$ so that the blow-up sequence

$$\tilde{Q}^j_s = l_j^{-1/2}\left(Q^{k_j}_{1+sl_j} - y_j\right), \quad 0 \leq s \leq 1$$

has

$$-\frac{1}{j} \leq \int_{\tilde{Q}^j_0} \Phi(0,u) d\mathcal{H}^2 - \int_{l_j^{-1/2}(\bar{Q}_1 - y_j)} \Phi(0,u) d\mathcal{H}^2 \leq \frac{1}{j} \tag{27}$$

for every $1 \leq u \leq j$ and

$$\int_1^{1+l_j} \int_{Q^{k_j}_t \cap B_1(y_j)} \left| H - \frac{x^\perp}{2t} \right|^2 d\mathcal{H}^2 dt \leq l_j^2. \tag{28}$$

Thus, for every $r > 0$, we have from (28) and $|y_j| \leq m$ that

$$\int_0^1 \int_{\tilde{Q}^j_s \cap B_r(0)} |H|^2 d\mathcal{H}^2 ds = l_j^{-1} \int_1^{1+l_j} \int_{Q^{k_j}_t \cap B_{\sqrt{l_j} r}(y_j)} |H|^2 d\mathcal{H}^2 dt$$
$$\leq l_j^{-1} \int_1^{1+l_j} \int_{Q^{k_j}_t \cap B_{\sqrt{l_j} r}(y_j)} \left| H - \frac{x^\perp}{2t} \right|^2 + \left| \frac{x^\perp}{2t} \right|^2 d\mathcal{H}^2 dt \leq l_j + C_2 l_j,$$

where $C_2 = C_2(r, m, K_0)$. Therefore,

$$\lim_{j\to\infty}\int_0^1\int_{\tilde{Q}^j_s\cap B_r(0)}|H|^2 d\mathcal{H}^2 ds = 0$$

and so $(\tilde{Q}^j_s)_{0\leq s\leq 1}$ converges to an integral Brakke flow $(\tilde{Q}_s)_{0\leq s\leq 1}$ with $\tilde{Q}_s = \tilde{Q}$ for all s. From Proposition 5.1 in [9] we conclude that $\tilde{Q}$ is a union of Special Lagrangian currents. Note that

$$\int_{\tilde{Q}}\Phi(0,1)d\mathcal{H}^2 \geq 1+\varepsilon_0$$

and so $\tilde{Q}$ cannot be a plane with multiplicity one. The blow-down C of $\tilde{Q}$ is a union of Lagrangian planes (those are the only Special Lagrangian cones in $\mathbb{R}^4$), and so

$$\lim_{u\to\infty}\lim_{j\to\infty}\int_{\tilde{Q}^j_0}\Phi(0,u)d\mathcal{H}^2 = \lim_{u\to\infty}\int_{\tilde{Q}}\Phi(0,u)d\mathcal{H}^2 = \int_C \Phi(0,1)d\mathcal{H}^2 \geq 2. \tag{29}$$

From (29) and (27) one can find u_0 such that for every j sufficiently large, we have

$$\begin{aligned}2-\frac{1}{2C_1} \leq \int_{\tilde{Q}^j_0}\Phi(0,u_0)d\mathcal{H}^2 &\leq \int_{l_j^{-1/2}(\bar{Q}_1-y_j)}\Phi(0,u_0)d\mathcal{H}^2+\frac{1}{j}\\ &= \int_{\bar{Q}_1}\Phi(y_j,u_0 l_j)d\mathcal{H}^2+\frac{1}{j}.\end{aligned}$$

This contradicts (24) for all j large. □

The lemma we have just proven allows us to find l_0 so that for all $\hat{R}$ and all i sufficiently large,

$$\int_{Q^i_t}\Phi(y,l)d\mathcal{H}^2 \leq 1+\varepsilon_0 \quad \text{for all } y\in B_{\hat{R}},\ l\leq l_0, \text{ and } \frac{1}{2}\leq t\leq 2.$$

Thus, from White's Regularity Theorem [18] we have uniform bounds on the second fundamental form and all its derivatives on compact sets of Q^i_t for all $1\leq t\leq 2$. This implies $\bar{Q}_t$ is smooth and $t^{-1/2}Q^i_t$ converges in $C^{2,\alpha}_{\text{loc}}$ to $\bar{Q}_1$, a smooth self-expander asymptotic to P_1+P_2 by Lemma 4.8, which must be embedded due to Lemma 4.11. This finishes the proof of Proposition 4.6. □

Apply Proposition 4.6 with ν and S_0 given by Theorem 4.2, and then apply Proposition 4.4 with η being the one given by Theorem 4.2. Theorem 4.2 follows at once if we choose $\delta = \eta/2$, $\varepsilon_3 = \min\{\varepsilon_5, \varepsilon_6\}$, and $\underline{R}_3 = \max\{\underline{R}_5, \underline{R}_6\}$.

5. Third Step: Equivariant flow

5.1. *Setup of Section* 5. Consider a smooth curve $\sigma : [0,+\infty) \longrightarrow \mathbb{C}$ so that

- $\sigma^{-1}(0) = 0$ and $\sigma \cup -\sigma$ is smooth at the origin.
- σ has a unique self-intersection.
- Outside a large ball the curve σ can be written as the graph of a function u defined over part of the negative real axis with
$$\lim_{r\to-\infty} |u|_{C^{2,\alpha}((-\infty,r])} = 0.$$
- For some a small enough, we have

$$\sigma \subseteq C_a = \{r\exp(i\theta) \,|\, r \geq 0, \pi/2 + 2a < \theta < \pi + a\}. \tag{30}$$

The curve σ shown in Figure 3 has all these properties. Condition (30) is there for technical reasons that will be used during Lemma 5.6.

Denote by A_1 the area enclosed by the self-intersection of σ.

We assume that $L \subset M$ is a Lagrangian surface as defined in (3) and that ε, $\underline{R}$ are such that Theorem 3.3 (with $\nu = 1$) and Theorem 3.5 hold. We also assume that the solution to Lagrangian mean curvature flow $(L_t)_{t\geq 0}$ satisfies the following condition.

($\star$) There is a constant K_1, a disc D, and $F_t : D \longrightarrow \mathbb{C}^2$ a normal deformation defined for all $1 \leq t \leq 2$ so that
$$L_t \cap B_{\overline{R}/2} \subset F_t(D) \subset L_t \cap B_{\overline{R}}$$
and the $C^{2,\alpha}$ norm of F_t is bounded by K_1.

5.2. *Main result.*

THEOREM 5.1. *Assume condition* ($\star$) *holds. There are* η_0 *and* R_5, *depending on* K_1 *and* σ, *so that if* $\overline{R} \geq R_5$ *in* (3) *and* L_1 *is* η_0*-close in* $C^{2,\alpha}(B_{R_5})$ *to*
$$M_1 = \{(\sigma(s)\cos\alpha, \sigma(s)\sin\alpha) \,|\, s \in [0,+\infty), \alpha \in S^1\},$$
then $(L_t)_{t\geq 0}$ *must have a singularity before* $T_1 = 2A_1/\pi + 1$ (*with* A_1 *defined in Section* 5.1).

Remark 5.2. The content of the theorem is that if L_1 is very close to M_1 and $\overline{R}$ sufficiently large, then the flow $(L_t)_{1\leq t\leq T_1}$ must have a finite time singularity. The proof proceeds by contradiction, and we assume the existence of smooth flows $(L^i_t)_{0\leq t\leq T_1}$ with $\overline{R}^i$ tending to infinity and L^i_1 converging to M_1 in $C^{2,\alpha}_{\mathrm{loc}}$. Standard arguments show that $(L^i_t)_{1\leq t\leq T_1}$ converges to $(M_t)_{1\leq t\leq T_1}$ a (weak) solution to mean curvature flow starting at M_1. The rest of the argument will have two steps.

The first step, see Theorem 5.3(ii)–(iv), is to show the existence of a family of curves σ_t so that

$$M_t = \{(\sigma_t(s)\cos\alpha, \sigma_t(s)\sin\alpha) \mid s \in [0,+\infty), \alpha \in S^1\}$$

and show that $(\sigma_t)_{t\geq 1}$ behaves as depicted in Figures 3 and 4. More precisely, there is a singular time T_0 so that σ_t has a single self-intersection for all $1 \leq t < T_0$, σ_{T_0} is embedded with a singular point, and σ_t is an embedded smooth curve for $t > T_0$. Finally, and this will be important for the second step, we show in Theorem 5.3(i) that L_t^i converges in $C^{2,\alpha}$ to M_t in a small ball around the origin and outside a large ball for all $t \leq T_0 + 1$.

The second step (see details in Corollary 5.5) consists in considering the function

$$f(t) = \theta_t(\infty) - \theta_t(0),$$

where $\theta_t(0)$ is the Lagrangian angle of M_t at $0 \in M_t$ and $\theta_t(\infty)$ is the "asymptotic" Lagrangian angle of M_t, which makes sense because, due to Lemma 3.2, M_t is asymptotic to the plane P_1. On one hand, because the curve σ_t changes from a curve with a single self-intersection to a curve that is embedded as t crosses T_0, we will see that

$$\lim_{t\to T_0^-} f(t) = \lim_{t\to T_0^+} f(t) - 2\pi.$$

On the other hand, because L_t^i is smooth and converges to M_t in a small ball around the origin and outside a large ball for all $t \leq T_0 + 1$, we will see that the function $f(t)$ is continuous. This gives us a contradiction.

Proof of Theorem 5.1. We argue by contradiction and assume the theorem does not hold. In this case we can find $(L_t^i)_{0\leq t\leq T_1}$ a sequence of smooth flows that satisfies condition $(\star)$ with $\overline{R}^i$ tending to infinity and L_1^i converging to M_1 in $C^{2,\alpha}_{\text{loc}}$.

Compactness for integral Brakke motions [7, §7.1] implies that, after passing to a subsequence, $(L_t^i)_{0\leq t\leq T_1}$ converges to an integral Brakke motion $(M_t)_{0\leq t\leq T_1}$. The next theorem characterizes $(M_t)_{0\leq t\leq T_1}$.

THEOREM 5.3. *There is δ_0 small, r small, R large, $T_0 \in (1, T_1)$, and a continuous family of curves $\sigma_t : [0,+\infty) \longrightarrow \mathbb{C}$ with*

$$\sigma_1 = \sigma,\ \sigma_t^{-1}(0) = 0 \textit{ for all } 1 \leq t \leq T_0 + \delta_0,$$

and such that

(i) *For all $1 \leq t \leq T_0 + \delta_0$,*
- *M_t is smooth in $B_r \cup \mathbb{C}^2 \setminus B_R$ and*
- *L_t^i converges in $C^{2,\alpha}_{\text{loc}}$ to M_t in $B_r \cup \mathbb{C}^2 \setminus B_R$.*

(ii) *For all* $1 \leq t < T_0$, σ_t *is a smooth curve with a single self-intersection. Moreover,*

$$M_t = \{(\sigma_t(s)\cos\alpha, \sigma_t(s)\sin\alpha) \mid s \in [0, +\infty), \alpha \in S^1\} \tag{31}$$

and

$$\frac{dx}{dt} = \vec{k} - \frac{x^{\perp}}{|x|^2}. \tag{32}$$

Finally, for each $t < T_0$, L^i_t *converge in* $C^{2,\alpha}_{\text{loc}}$ *to* M_t.

(iii) *The curve* σ_{T_0} *has a singular point* Q *so that* $\sigma_{T_0} \setminus \{Q\}$ *consists of two disjoint smooth embedded arcs and, away from* Q, σ_t *converges to* σ_{T_0} *as* t *tends to* T_0.

(iv) *For all* $T_0 < t \leq T_0 + \delta_0$, σ_t *is a smooth embedded curve that satisfies* (31) *and* (32). *Moreover, for each* $T_0 < t \leq T_0 + \delta_0$, L^i_t *converge in* $C^{2,\alpha}_{\text{loc}}$ *to* M_t.

Remark 5.4. (1) The content of this theorem is to justify the behavior shown in Figures 3 and 4. More precisely, Theorem 5.3(ii) and (iii) say that the solution $(\sigma_t)_{t\geq 1}$ to (32) with $\sigma_1 = \sigma$ will have a singularity at time T_0 that corresponds to the loop enclosed by the self-intersection of σ_t collapsing. Theorem 5.3(iv) says that after T_0, the curves σ_t become smooth and embedded.

(2) The behavior described above follows essentially from Angenent's work [2], [3] on general one-dimensional curvature flows.

(3) We also remark that the fact M_t has the symmetries described in (31) up to the singular time T_0 is no surprise because that is equivalent to uniqueness of solutions with smooth controlled data. After the singular time T_0, there is no general principle justifying why M_t has the symmetries described in (31). The reason this occurs is because the function μ defined in Theorem 3.5(v) evolves by the linear heat equation and is zero if and only if M_t can be expressed as in (31). (See Claim 1 in the proof of Theorem 5.3 for details.)

(4) Theorem 5.3(i) is necessary so that we can control the flow in neighborhood of the origin because the right-hand side of (32) is singular at the origin. It is important for Corollary 5.5 that the convergence mentioned in Theorem 5.3(i) holds for all $t \leq T_0 + \delta_0$ including the singular time.

(5) The proof is mainly technical and will be given at the end of this section.

COROLLARY 5.5. *Assuming Theorem* 5.3 *we have that, for all* i *sufficiently large,* $(L^i_t)_{1\leq T_1}$ *must have a finite time singularity.*

In Remark 5.2 we sketched the idea behind the proof of this corollary.

Proof. From Theorem 5.3(i) we can find a small interval I containing T_0 (the singular time of σ_t) and pick $a_t \in \sigma_t \cap A(r/3, r/2)$, $b_t \in \sigma_t \cap A(2R, 3R)$ so that a_t, b_t are the endpoints of a segment $\bar{\sigma}_t \subseteq \sigma_t \cap A(r/3, 3R)$ and the paths $(a_t)_{t\in I}$, $(b_t)_{t\in I}$ are smooth. Consider the function

$$f(t) = \theta_t(b_t) - \theta_t(a_t).$$

We claim that

$$\lim_{t\to T_0^-} f(t) = \lim_{t\to T_0^+} f(t) - 2\pi. \tag{33}$$

Recall that the Lagrangian angle θ_t equals, up to a constant, the argument of the complex number $\sigma_t\sigma_t'$. Hence, for all $t \in I \setminus \{T_0\}$, we have

$$\theta_t(b_t) - \theta_t(a_t) = \int_{\bar{\sigma}_t} d\theta_t = \int_{\bar{\sigma}_t} \langle \vec{k}, \nu\rangle d\mathcal{H}^1 - \int_{\bar{\sigma}_t} \left\langle \frac{x}{|x|^2}, \nu \right\rangle d\mathcal{H}^1,$$

where ν is the normal obtained by rotating the tangent vector to $\bar{\sigma}_t$ counterclockwise and we are assuming that this segment is oriented from a_t to b_t. The curves $\bar{\sigma}_t$ are smooth near the endpoints by Theorem 5.3(i), have a single self-intersection for $t < T_0$ by Theorem 5.3(ii), and are embedded for $t > T_0$ by Theorem 5.3(ii) (see Figure 4). Thus, the rotation index of σ_t changes across T_0 and so

$$\lim_{t\to T_0^+} \int_{\bar{\sigma}_t} \langle \vec{k}, \nu\rangle d\mathcal{H}^1 = \lim_{t\to T_0^-} \int_{\bar{\sigma}_t} \langle \vec{k}, \nu\rangle d\mathcal{H}^1 + 2\pi. \tag{34}$$

The vector field $X = x|x|^{-2}$ is divergence free and so, because none of the segments $\bar{\sigma}_t$ wind around the origin, the Divergence Theorem implies

$$\lim_{t\to T_0^+} \int_{\bar{\sigma}_t} \left\langle \frac{x}{|x|^2}, \nu \right\rangle d\mathcal{H}^1 = \lim_{t\to T_0^-} \int_{\bar{\sigma}_t} \left\langle \frac{x}{|x|^2}, \nu \right\rangle d\mathcal{H}^1. \tag{35}$$

Claim (33) follows at once from (34) and (35).

From Theorem 5.3(i) we can choose a sequence of smooth paths $(a_t^i)_{t\in I}$, $(b_t^i)_{t\in I}$ converging to $(a_t)_{t\in I}$, $(b_t)_{t\in I}$ respectively, and such that $a_t^i, b_t^i \in L_t^i$. Consider the function

$$f^i(t) = \theta_t^i(b_t^i) - \theta_t^i(a_t^i).$$

For every $t \in I\setminus\{T_0\}$, we have from Theorem 5.3(ii) and (iv) that L_t^i converges in $C^{2,\alpha}_{\text{loc}}$ to M_t. As a result,

$$f_i(t) \text{ converges to } f(t) \text{ for all } t \in I \setminus \{T_0\}. \tag{36}$$

Because the flow $(L_t^i)_{t\in I}$ exists smoothly, the function $f^i(t)$ is smooth and

$$\frac{df_t^i(t)}{dt} = \Delta\theta_t^i(b_t^i) + \langle \nabla\theta_t^i, db_t^i/dt\rangle - \Delta\theta_t^i(a_t^i) - \langle \nabla\theta_t^i, da_t^i/dt\rangle.$$

Hence, Theorem 5.3(i) shows that $df^i(t)/dt$ is uniformly bounded (independently of i) for all $t \in I$. From (36) we obtain that the function f must be Lipschitz continuous, and this contradicts (33). $\square$

This corollary gives us the desired contradiction and finishes the proof of the theorem. $\square$

5.3. *Proof of Theorem* 5.3. Recall the function $\mu = x_1y_2 - y_1x_2$ defined in Theorem 3.5(v). We start by proving two claims.

Claim 1. $M_t \subseteq \mu^{-1}(0)$ and $|\nabla\mu| = 0$ for almost all $1 \le t \le T_1$.

From Lemma 2.1 and Theorem 3.5(v), we have

$$\begin{aligned}(37)\quad &\int_{L_t^i} \mu^2\Phi(0,1)d\mathcal{H}^2 + \int_1^t\int_{L_s^i}|\nabla\mu|^2\Phi(0,1+t-s)d\mathcal{H}^2ds \le \\ &\int_{L_1^i}\mu^2\Phi(0,1+t)d\mathcal{H}^2 + \int_1^t\int_{L_s^i}\left(\frac{|E|^2}{4}\mu^2 + E_2\right)\Phi(0,1+t-s)d\mathcal{H}^2ds.\end{aligned}$$

Because $M_1 \subseteq \mu^{-1}(0)$ and E, E_2 converge uniformly to zero when i goes to infinity, we obtain

$$\begin{aligned}\lim_{i\to\infty}\int_{L_1^i}\mu^2\Phi(0,1+t)d\mathcal{H}^2 + \int_1^t\int_{L_s^i}\left(\frac{|E|^2}{4}\mu^2 + E_2\right)\Phi(0,1+t-s)d\mathcal{H}^2ds \\ = \int_{M_1}\mu^2\Phi(0,1+t)d\mathcal{H}^2 = 0,\end{aligned}$$

which combined with (37) implies

$$\int_{M_t}\mu^2\Phi(0,1)d\mathcal{H}^2 + \int_1^t\int_{M_s}|\nabla\mu|^2\Phi(0,1+t-s)d\mathcal{H}^2ds = 0.$$

This proves the claim.

Claim 2. For every δ, there is $R = R(\delta, T_1)$ so that, in the annular region $A(R, \overline{R}_i)$, L_t^i is δ-close in $C^{2,\alpha}$ to the plane P_1 for all $1 \le t \le T_1$ and i sufficiently large.

According to Lemma 3.2 there is a constant $R = R(\delta, T_1, \underline{R}_i)$ so that, in the annular region $A(R, \overline{R}_i)$, L_t^i is δ-close in $C^{2,\alpha}$ to P_1 for all $1 \le t \le T_1$. Because L_1^i converges to M_1, we can deduce from Theorem 3.3(i) that $\underline{R}_i$ is bounded and thus the constant R depends only on δ and T_1 and not on the index i. This prove the claim.

Definition of "*singular time*" T_0: First we need to introduce some notation. Because condition $(\star)$ holds for the flow (L_t^i), there are a sequence of discs D_i of increasingly larger radius and normal deformations $F_t^i : D_i \longrightarrow \mathbb{C}^2$ so that, for all $1 \leq t \leq 2$, $F_t^i(D_i) \subseteq L_t^i$, and F_t^i converges in $C^{2,\alpha}_{\mathrm{loc}}$ to $F_t : \mathbb{R}^2 \longrightarrow \mathbb{C}^2$, where $M_t = F_t(\mathbb{R}^2)$.

Consider the following condition:

$$F_t^i \text{ converges in } C^{2,\alpha}_{\mathrm{loc}} \text{ to } F_t : \mathbb{R}^2 \longrightarrow \mathbb{C}^2, \text{ where } M_t = F_t(\mathbb{R}^2), \tag{38}$$

and set

$$T_0 = \sup\{l \mid F_t^i \text{ is defined and condition (38) holds for all } t \leq l\} \cap [1, T_1]. \tag{39}$$

Proof of Theorem 5.3(ii). By the way T_0 was chosen and Claim 1, we have that $M_t \subseteq \mu^{-1}(0)$ is a smooth surface diffeomorphic to $\mathbb{R}^2$. Thus Lemma 7.1 implies the existence of $(\sigma_t)_{1\leq t<T_0}$ so that (31) holds. Because $(M_t)_{1\leq t<T_0}$ is a smooth solution to mean curvature flow, it is immediate to conclude (32). From the definition of T_0 it is also straightforward to conclude that L_t^i converges in $C^{2,\alpha}_{\mathrm{loc}}$ to M_t if $t < T_0$. We are left to argue that σ_t has a single self-intersection for all $1 \leq t < T_0$. From Lemma 5.6 below we conclude that if σ_t develops a tangential self-intersection, it must be away from the origin. It is easy to see from the flow (32) that this cannot happen.

LEMMA 5.6. *There exists r so that $\sigma_t \cap B_r$ is embedded for all $1 \leq t < T_0$.*

Proof. Recall the definition of C_a in (30). We start by arguing that

$$\sigma_t \subseteq C_a \text{ for all } 1 \leq t < T_0. \tag{40}$$

The boundary of the cone C_a consists of two half-lines that are fixed points for the flow (32). From Claim 2 we see that M_t is asymptotic to P_1 and so σ_t does not intersect ∂C_a outside a large ball. Thus, because $\sigma_1 \subset C_a$, we conclude from Lemma 7.3 that $\sigma_t \subseteq C_a$ for all $1 \leq t < T_0$.

Denote by Γ a curve in $\mathbb{C}$ that is asymptotic at infinity to

$$\{r \exp(i(\pi + 3a/2)) \mid r \geq 0\} \cup \{r \exp(i(\pi/2 + 3a/2)) \mid r \geq 0\} \tag{41}$$

and generates, under the S^1 action described in (31), a Special Lagrangian asymptotic to two planes (Lawlor Neck). In particular, the curves $\Gamma_\delta = \delta\Gamma$ are fixed points for the flow (32) for all δ and, because of (40) and (41), σ_t does not intersect Γ_δ outside a large ball for all $1 \leq t < T_0$.

From the description of σ given at the beginning of Section 5, we find δ_0 so that for every $\delta < \delta_0$, the curve Γ_δ intersects σ only once. Hence, we can apply [3, Variation on Th. 1.3] and conclude that Γ_δ and σ_t intersect only once for all $1 \leq t < T_0$ and all $\delta < \delta_0$. It is simple to see that this implies the result we want to show provided we choose r small enough. □

Proof of Theorem 5.3(i). This follows from Claim 2 and the next lemma.

LEMMA 5.7. *There are r small and δ small so that $M_t \cap B_r$ is smooth, embedded, and L_t^i converges in $C^{2,\alpha}(B_r)$ to $M_t \cap B_r$ for all $1 \leq t \leq T_0 + \delta$.*

In particular, the curve $\sigma_t \cup -\sigma_t$ is smooth and embedded near the origin with bounds on its $C^{2,\alpha}$ norm for all $1 \le t \le T_0 + \delta$.

Remark 5.8. The key step to show Lemma 5.7 is to argue that $(M_t)_{t\geq 1}$ develops no singularity at the origin at time T_0. The idea is the following. First principles will show that a sequence of of blow-ups at the origin $(\sigma_t^j)_{s<0}$ of $(\sigma_t)_{t<T_0}$ converge in $C^{1,1/2}_{\text{loc}}(\mathbb{R}^2 - \{0\})$ to a union of half-lines. But Lemma (5.6) implies σ_t^j is embedded in B_1 for all j sufficiently large and so it must converge to a single half-line. White's Regularity Theorem implies no singularity occurs.

Proof. From the way T_0 was chosen (39) and Lemma 5.6 we know the existence of r so that $M_t \cap B_r$ is smooth, embedded, and L_t^i converges in $C^{2,\alpha}(B_r)$ to $M_t \cap B_r$ for all $1 \le t < T_0$. To extend this to hold up to $t = T_0$ (with possible smaller r) it suffices to show that $(M_t)_{1\le t<T_0}$ develops no singularity at the origin at time T_0.

Choose a sequence $(\lambda_j)_{j\in\mathbb{N}}$ tending to infinity, and set

$$M_t^j = \lambda_j M_{T_0 + t/\lambda_j^2}, \quad \text{for all } t < 0.$$

From [9, Lemma 5.4] we have the existence of a union of planes Q with support contained in $\mu^{-1}(0)$ such that, after passing to a subsequence and for almost all $t < 0$, M_t^j converges in the varifold sense to Q and

$$\lim_{j\to\infty} \int_{M_t^j} (|H|^2 + |x^\perp|^2) \exp(-|x|^2) d\mathcal{H}^2 = 0. \tag{42}$$

From (5.3) we can find curves σ_t^j so that

$$M_t^j = \{(\sigma_t^j(s)\cos\alpha, \sigma_t^j(s)\sin\alpha) \mid s \in [0, +\infty), \alpha \in S^1\}.$$

We obtain from (42) that for almost all t and every $0 < \eta < 1$,

$$\lim_{j\to\infty} \int_{\sigma_t^j \cap A(\eta,\eta^{-1})} |\vec{k}|^2 + |x^\perp|^2 d\mathcal{H}^1 = 0,$$

which implies that σ_t^j converges in $C^{1,1/2}_{\text{loc}}(\mathbb{R}^2 - \{0\})$ to a union of half-lines with endpoints at the origin. Lemma 5.6 implies that for all j sufficiently large σ_t^j is embedded inside the unit ball. Thus σ_t^j must converge to a single half-line and so Q is a multiplicity one plane. Thus there can be no singularity at time T_0 at the origin.

We now finish the proof of the lemma. So far we have proven that M_t is smooth and embedded near the origin for all $1 \le t \le T_0$ .Thus we can find l_0 small so that

$$\int_{M_t} \Phi(x,l) d\mathcal{H}^2 \le 1 + \varepsilon_0 \text{ for every } x \in B_{2l_0}, l \le 4l_0^2, \text{ and } 1 \le t \le T_0.$$

The monotonicity formula implies that

$$\int_{M_t} \Phi(x,l)d\mathcal{H}^2 \leq 1+\varepsilon_0 \text{ for every } x\in B_{l_0}, l\leq l_0^2, \text{ and } 1\leq t\leq T_0+l_0^2.$$

Because L_t^i converges to M_t as Radon measures, White's Regularity Theorem implies uniform $C^{2,\alpha}$ bounds in $B_{l_0/2}$ for L_t^i whenever i is sufficiently large and $t\leq T_0+l_0^2$. The lemma then follows straightforwardly. □

Proof of Theorem 5.3(iii). We need two lemmas first.

LEMMA 5.9. $T_0<T_1$.

Remark 5.10. The idea is to show that if $T_0\geq T_1$, then the loop of σ_t created by its self-intersection would have negative area.

Proof. Suppose $T_0=T_1$. Denote by q_t the single self-intersection of σ_t, by $c_t\subseteq\sigma_t$ the closed loop with endpoint q_t, by $\alpha_t\in[-\pi,\pi]$ the exterior angle that c_t has at the vertex q_t, by ν the interior unit normal, and by A_t the area enclosed by the loop. From the Gauss-Bonnet Theorem, we have

$$\int_{c_t}\langle\vec{k},\nu\rangle d\mathcal{H}^1+\alpha_t=2\pi \implies \int_{c_t}\langle\vec{k},\nu\rangle d\mathcal{H}^1\geq\pi.$$

A standard formula shows that

$$\frac{d}{dt}A_t=-\int_{c_t}\left\langle\vec{k}-\frac{x^\perp}{|x|^2},\nu\right\rangle d\mathcal{H}^1\leq-\pi+\int_{c_t}\left\langle\frac{x}{|x|^2},\nu\right\rangle d\mathcal{H}^1=-\pi,$$

where the last identity follows from the Divergence Theorem combined with the fact that c_t does not contain the origin in its interior. Hence $0\leq A_t\leq A_1-(t-1)\pi$. Making t tending to $T_1=2A_1/\pi+1$, we obtain a contradiction. □

LEMMA 5.11. *The curve σ_t must become singular when t tends to T_0.*

Remark 5.12. The flow $(M_t)_{t\geq0}$ is only a weak solution to mean curvature flow which means that, in principle, σ_{T_0} could be a smooth curve with a self-intersection and, right after, σ_t could split off the self-intersection and become instantaneously a disjoint union of a circle with a half-line. This lemma shows that, because M_t is a limit of smooth flows L_t^i, this phenomenon cannot happen. The proof is merely technical.

Proof. We are assuming F_t^i converges in $C^{2,\alpha}_{\text{loc}}$ to F_t for all $t<T_0$. Assuming σ_{T_0} is smooth, we have from parabolic regularity that $(\sigma_t)_{t\leq T_0}$ is a smooth flow. Thus, M_{T_0} is also smooth and the maps F_t converge smoothly to a map $F_{T_0}:\mathbb{R}^2\longrightarrow\mathbb{C}^2$. Therefore, there is a constant C that bounds the C^2 norm of F_t for all $T_0-1\leq t\leq T_0$. Hence, using Claim 2 to control the $C^{2,\alpha}$ norm of F_t^i outside a large ball, we obtain that for $\bar{t}<T_0$ and i sufficiently large, the C^2 norm of $F_{\bar{t}}^i$ is bounded by $2C$. Looking at the evolution equation of $|A|^2$

it is then a standard application of the maximum principle to find $\delta = \delta(C)$ such that the second fundamental form of the immersion F_t^i is bounded by $4C$ for all $\bar{t} \leq t \leq \bar{t} + \delta$. Therefore, choosing $\bar{t}$ such that $T_0 < \bar{t} + \delta$, parabolic regularity implies condition (38) holds for all t slightly larger than T_0 which, due to Lemma 5.9, contradicts the maximality of T_0. $\square$

Claim 2 and Lemma 5.7 give us control of the flow (32) outside an annulus. Hence we apply Theorem 7.2 and conclude the singular curve σ_{T_0} contains a point Q distinct from the origin such that $\sigma_{T_0} \setminus \{Q\}$ consists of two smooth disjoint arcs and, away from the singular point, the curves σ_t converge smoothly to σ_{T_0} (see Figure 4).

Proof of Theorem 5.3(iv). From Theorem 5.3(i) and Claim 1, we can apply Lemma 7.1 and conclude that M_t can be described by a one-dimensional varifold $\sigma_t \subset \mathbb{C}$ for almost all $T_0 < t < T_1$.

In [2, §8] Angenent constructed an embedded smooth solution $(\gamma_t)_{t>0}$ that tends to σ_{T_0} when t tends to zero and that looks like the solution described on Figure 4. The next lemma is the key to showing Theorem 5.3(iv).

LEMMA 5.13. *There is δ small so that $\gamma_t = \sigma_{T_0+t}$ for all $0 < t < \delta$.*

Remark 5.14. This lemma amounts to showing that there is a unique (weak) solution to the flow (32) that starts at σ_{T_0}.

The idea to prove this lemma, which we now sketch, is well known among the specialists. Consider γ_+^i, γ_-^i two sequences of smooth embedded curves with an endpoint at the origin and converging to σ_{T_0}, with γ_+^i, γ_-^i lying above and below σ_{T_0}, respectively. There is a region A_i that has $\sigma_{T_0} \subseteq A_i$ and $\partial A_i = \gamma_+^i \cup \gamma_-^i$. Denote the flows starting at γ_+^i and γ_-^i by $\gamma_{+,t}^i$ and $\gamma_{-,t}^i$ respectively, and use $A_i(t)$ to denote the region below $\gamma_{+,t}^i$ and above $\gamma_{-,t}^i$.

For the sake of the argument, we can assume that A_i is finite and tends to zero when i tends to infinity. A simple computation will show that area$(A_i(t))$ $\leq$ area(A_i) and so, like A_i, the area of $A_i(t)$ tends to zero when i tends to infinity. The avoidance principle for the flow implies that $\sigma_{T_0+t}, \gamma_t \subseteq A_i(t)$ for all i and t, and thus, making i tend to infinity, we obtain that $\sigma_{T_0+t} = \gamma_t$.

The proof requires some technical work to go around the fact that the curves γ_+^i, γ_-^i are noncompact and thus A_i could be infinity.

Proof. Let $\gamma_+^i, \gamma_-^i : [0, +\infty] \longrightarrow \mathbb{C}$ be two sequences of smooth embedded curves converging to σ_{T_0} with γ_+^i, γ_-^i lying above (below) σ_{T_0} and such that

$$(\gamma_\pm^i)^{-1}(0) = 0, \quad \gamma_\pm^i \cup -\gamma_\pm^i \text{ is smooth}, \quad \theta_+^i(0) < \theta_{T_0}(0) < \theta_-^i(0). \tag{43}$$

The convergence is assumed to be strong on compact sets not containing the cusp point of σ_{T_0}. Denote by $\gamma_{\pm,t}^i$ the solution to the equivariant flow (32) with initial condition $\gamma_\pm^i$. Short time existence was proven in [9, §4] provided we assume controlled behavior at infinity. The same arguments used to study σ_t (namely Lemma 5.7) show that embeddedness is preserved and no singularity of $\gamma^i_{\pm,t}$ can occur at the origin. Hence an immediate consequence of Theorem 7.2 is that the flow exists smoothly for all time.

From the last condition in (43) we know that γ^i_+ intersects γ^i_- transversely at the origin. Furthermore we can choose γ^i_+, γ^i_- to be not asymptotic to each other at infinity. Thus we can apply Lemma 7.3 and conclude that $\gamma^i_{+,t}$ and $\gamma^i_{-,t}$ intersect each other only at the origin. Hence there is an open region $A_i(t) \subset \mathbb{C}$ so that $\gamma^i_{+,t} \cup \gamma^i_{-,t} = \partial A_i(t)$.

From Claim 2 we know that γ_{T_0} is asymptotic to a straight line. Thus we can reason as in the proof of Theorem 3.5(i) and conclude the existence of R_i tending to infinity so that $\gamma^i_{\pm,t} \cap A(R_i/2, 2R_i)$ is graphical over the real axis with C^1 norm smaller than $1/i$ for all $0 \le t \le 1$.

Consider $B_i(t) = A_i(t) \cap \{(x, y) \,|\, x \ge -R_i\}$. This region has the origin as one of its "vertices" and is bounded by three smooth curves. The top curve is part of $\gamma^i_{+,t}$, the bottom curve is part of $\gamma^i_{-,t}$, and left-side curve is part of $\{x = -R_i\}$. Using the fact that

$$\operatorname{area}(B_i(t)) = \int_{\partial B_i(t)} \lambda,$$

differentiation shows that

$$\frac{d}{dt}\operatorname{area}(B_i(t)) = -\left(\theta^i_{+,t}(-R_i) - \theta^i_{-,t}(-R_i)\right) + \left(\theta^i_{+,t}(0) - \theta^i_{-,t}(0)\right),$$

where $\theta^i_{\pm,t}(-R_i)$ denote the Lagrangian angle of $\gamma^i_{\pm,t}$ at the intersection with $\{x = -R_i\}$ and $\theta^i_{\pm,t}(0)$ denotes the Lagrangian angle of $\gamma^i_{\pm,t}$ at the origin. Because $\gamma^i_{+,t}$ lies above $\gamma^i_{-,t}$ and they intersect at the origin, we have $\theta^i_{+,t}(0) \le \theta^i_{-,t}(0)$. Thus,

$$\frac{d}{dt}\operatorname{area}(B_i(t)) \le -\left(\theta^i_{+,t}(-R_i) - \theta^i_{-,t}(-R_i)\right).$$

Recalling that $\gamma^i_{\pm,t} \cap A(R_i/2, 2R_i)$ is graphical over the real axis with C^1 norm smaller than $1/i$ for all $t \le 1$, we have that the term on the right side of the above inequality tends to zero when i tends to infinity. Finally the curves can be chosen so that $\operatorname{area}(B_i(0)) \le 1/i$ and thus

$$\lim_{i\to\infty} \operatorname{area}(B_i(t)) \le \lim_{i\to\infty} \operatorname{area}(B_i(0)) = 0. \tag{44}$$

We now argue the existence of δ so that

$$\gamma_t, \sigma_{T_0+t} \subseteq A_i(t) \text{ for all } t \le \delta \text{ and all } i. \tag{45}$$

The inclusion for γ_t follows from Lemma 7.3. Next we want to deduce the inclusion for the varifolds σ_{T_0+t} (recall Lemma 7.1), which does not follow directly from Lemma 7.3 because σ_{T_0+t} might not be smooth. We remark that the right-hand side of (32) is the geodesic curvature with respect to the metric $h = (x_1^2 + y_1^2)(dx_1^2 + dy_1^2)$. Because $(M_t)_{t\geq 1}$ is a Brakke flow, it is not hard to deduce from Lemma 7.1 that $(\sigma_t)_{t\geq 1}$ is also a Brakke flow with respect to the metric h. This metric is singular at the origin and has unbounded curvature but fortunately, due to Claim 2 and Lemma 5.7, we already know that σ_t is smooth in a neighborhood of the origin and outside a compact set for all $t \leq T_0 + \delta$. Thus, the Inclusion Theorem proven in [7, 10.7 Inclusion Theorem] adapts straightforwardly to our setting, and this implies $\sigma_{T_0+t} \subseteq A_i(t)$ for all $t \leq \delta$.

Combining (44) with (45) we obtain that $\gamma_t = \sigma_{T_0+t}$ all $0 < t < \delta$. □

From Lemma (5.13) we obtain that M_t is smooth, embedded, and satisfies (31), (32) for all $T_0 < t < T_0 + \delta$. Finally, from the fact that M_t is embedded, it follows in a straightforward manner from White's Regularity Theorem that L_t^i converges in $C^{2,\alpha}_{\mathrm{loc}}$ to M_t. This completes the proof of Theorem 5.3.

6. **Main Theorem**

THEOREM 6.1. *For any embedded closed Lagrangian surface Σ in M, there is L Lagrangian in the same Hamiltonian isotopy class so that the Lagrangian mean curvature flow with initial condition L develops a finite time singularity.*

Proof. Setup. Given $\overline{R}$ large we can find a metric $g_R = R^2 g$ (see Section 2.1) so that the hypothesis on ambient space described in Section 3.1.1 are satisfied. Pick $p \in \Sigma$, and assume the Darboux chart ϕ sends the origin into $p \in \Sigma$ and $T_p\Sigma$ coincides with the real plane $\mathbb{R} \oplus i\mathbb{R} \subseteq \mathbb{C}^2$ oriented positively.

We can assume $\Sigma \cap B_{4\overline{R}}$ is given by the graph of the gradient of some function defined over the real plane, where the C^2 norm can be made arbitrarily small. It is simple to find $\overline{\Sigma}$ Hamiltonian isotopic to Σ that coincides with the real plane in $B_{3\overline{R}}$. Denote by L the Lagrangian that is obtained by replacing $\overline{\Sigma} \cap B_{3\overline{R}}$ with $N(\varepsilon, \underline{R})$ defined in (1). Using [4, Th. 1.1.A] we obtain at once that L is Hamiltonian isotopic to $\overline{\Sigma}$ and hence to Σ as well. Moreover, there is K_0 depending only on Σ so that the hypothesis on L described in Section 3.1.2 are satisfied for all $\overline{R}$ large.

We recall once more that L depends on $\varepsilon, \underline{R}, \overline{R}$ and that $\overline{R} \geq 4\underline{R}$. Assume the Lagrangian mean curvature flow $(L_t)_{t\geq 0}$ with initial condition L exists smoothly for all time.

First Step. Pick ν_0 small (to be fixed later), and choose ε, $\underline{R}$, and $\overline{R}$ so that Theorem 3.3 (with $\nu = \nu_0$) and Theorem 3.5 hold. Thus, there is $R_1 = R_1(\nu_0, K_0)$ so that (see Theorem 3.3(ii))

(A) for every $1 \leq t \leq 2$, $L_t \cap A(R_1, \overline{R})$ is ν_0-close in $C^{2,\alpha}$ to L.

Moreover, from Theorem 3.3(iii) and (iv), $L_t \cap B_{R_1}$ is contained in two connected components $Q_{1,t} \cup Q_{2,t}$ where

(B) for every $1 \leq t \leq 2$, $Q_{2,t}$ is ν_0-close in $C^{2,\alpha}(B_{R_1})$ to P_3.

Second Step. We need to control $Q_{1,t}$. Apply Theorem 4.1 with $S_0 = R_1$ and $\nu = \nu_0$. Thus for all ε small and $\underline{R}$ large, we have that

(C) for every $1 \leq t \leq 2$, $t^{-1/2}Q_{1,t}$ is ν_0-close in $C^{2,\alpha}(B_{R_1})$ to $\mathcal{S}$,

where $\mathcal{S}$ is the self-expander defined in (2) (see Figure 2). One immediate consequence of (A), (B), and (C) is the existence of K_1 so that for all ε small and $\underline{R}$ large, we have

($\star\star$) the existence of a disc D, and $F_t : D \longrightarrow \mathbb{C}^2$ a normal deformation defined for all $1 \leq t \leq 2$, so that

$$L_t \cap B_{\overline{R}/2} \subset F_t(D) \subset L_t \cap B_{\overline{R}}$$

and the $C^{2,\alpha}$ norm of F_t is bounded by K_1.

Third Step. Fix ε and $\underline{R}$ in the definition of L so that (A), (B), (C), and ($\star\star$) hold, but let $\overline{R}$ tend to infinity. We then obtain a sequence of smooth flows $(L_t^i)_{t\geq 0}$, where L_0^i converges strongly to $N(\varepsilon, \underline{R})$ defined in (1) (see Figure 1).

Lemma 6.2. *If ν_0 is chosen small enough, there is a curve $\sigma \subset \mathbb{C}$ with all the properties described in Section* 5.1 (*see Figure* 3) *and such that L_1^i tends in $C^{2,\alpha}_{\mathrm{loc}}$ to*

$$M_1 = \{(\sigma(s)\cos\alpha, \sigma(s)\sin\alpha) \mid \alpha \in S^1, s \in [0, +\infty)\}. \tag{46}$$

Assuming this lemma, we will show that $(L_t^i)_{t\geq 0}$ must have a singularity for all i sufficiently large, which finishes the proof of Theorem 6.1. Indeed, because the flow $(L_t^i)_{t\geq 0}$ has property ($\star\star$), we have at once that condition ($\star$) of Section 5.1 is satisfied. Hence Lemma 6.2 implies that L_1^i satisfies the hypothesis of Theorem 5.1 for all i sufficiently large, and thus Theorem 5.1 implies that $(L_t^i)_{t\geq 0}$ must have a finite time singularity.

Proof of Lemma 6.2. From condition ($\star\star$) we have that L_1^i converges in $C^{2,\alpha}_{\mathrm{loc}}$ to a smooth Lagrangian M_1 diffeomorphic to $\mathbb{R}^2$. Moreover, from (B) and (C) we see that we can choose ν_0 small so that M_1 is embedded in a small neighborhood the origin. We argue that $M_1 \subset \mu^{-1}(0)$, where the function $\mu = x_1y_2 - y_1x_2$ was defined in Theorem 3.5(v). From Lemma 2.1 and Theorem 3.5(v), we have

$$\begin{aligned}\int_{L_1^i} \mu^2\Phi(0,1)d\mathcal{H}^2 &+ \int_0^1\int_{L_t^i} |\nabla\mu|^2\Phi(0,2-t)d\mathcal{H}^2dt \\ &\leq \int_{L_0^i} \mu^2\Phi(0,2)d\mathcal{H}^2 + \int_0^1\int_{L_t^i}\left(\frac{|E|^2}{4}\mu^2 + E_2\right)\Phi(0,2-t)d\mathcal{H}^2dt.\end{aligned} \tag{47}$$

The terms E, E_2 converge uniformly to zero when i goes to infinity because the ambient metric converges to the Euclidean one. Moreover $N(\varepsilon, \underline{R}) \subset \mu^{-1}(0)$, and so we obtain from (47) that

$$\begin{aligned}\int_{M_1} \mu^2 \Phi(0,1) d\mathcal{H}^2 &= \lim_{i\to\infty} \int_{L_1^i} \mu^2 \Phi(0,1) d\mathcal{H}^2 \\ &\leq \lim_{i\to\infty} \int_{L_0^i} \mu^2 \Phi(0,2) d\mathcal{H}^2 + \int_0^1 \int_{L_t^i} \left(\frac{|E|^2}{4} \mu^2 + E_2 \right) \Phi(0, 2-t) d\mathcal{H}^2 dt \\ &= \int_{N(\varepsilon, \underline{R})} \mu^2 \Phi(0,2) d\mathcal{H}^2 = 0.\end{aligned}$$

Hence, $M_1 \subset \mu^{-1}(0)$ and we can apply Lemma 7.1 to conclude the existence of a curve σ so that (46) holds.

In order to check that σ has the properties described in Section 5.1, it suffices to see that σ has a single self-intersection and is contained in the cone C_a (defined in (30)) because the remaining properties follow from M_1 being diffeomorphic to $\mathbb{R}^2$, embedded near the origin, and asymptotic to the plane P_1 (Lemma 3.2).

Recall that $\gamma(\varepsilon, \underline{R})$, χ, and c_3 are the curves in $\mathbb{C}$ that define, respectively, the Lagrangian $N(\varepsilon, \underline{R})$, the self-expander $\mathcal{S}$, and the plane P_3. Now M_1, being the limit of L_1^i, also satisfies (A), (B), and (C). Hence we know that σ is ν_0-close in $C^{2,\alpha}$ to $\gamma(\varepsilon, \underline{R})$ in $\mathbb{C} \setminus B_{R_1}$ and that $\sigma \cap B_{R_1}$ has two connected components, one ν_0-close in $C^{2,\alpha}(B_{R_1})$ to c_3 and the other ν_0-close in $C^{2,\alpha}(B_{R_1})$ to χ. It is simple to see that if ν_0 is small, then indeed all the desired properties for σ follow. $\square$

7. **Appendix**

7.1. *Lagrangians with symmetries.* Recall that $\mu(x_1, y_1, x_2, y_2) = x_1 y_2 - x_2 y_1$, and consider two distinct conditions on M.

(C1) M is an integral Lagrangian varifold which is a smooth embedded surface in a neighborhood of the origin;

(C2) There is a smooth Lagrangian immersion $F : \mathbb{R}^2 \longrightarrow \mathbb{C}^2$ so that $M = F(\mathbb{R}^2)$ and M is a smooth embedded surface in a neighborhood of the origin.

LEMMA 7.1. *Assume $M \subseteq \mu^{-1}(0)$. If (C1) holds, then there is a one-dimensional integral varifold $\gamma \subset \mathbb{C}$ so that for every function ϕ with compact support,*

$$\int_M \phi d\mathcal{H}^2 = \int_\gamma |z| \int_0^{2\pi} \phi(z\cos\alpha, z\sin\alpha) d\alpha d\mathcal{H}^1. \tag{48}$$

If $(C2)$ *holds, then there is a smooth immersed curve* $\gamma : [0,\infty) \longrightarrow \mathbb{C}$ *with* $\gamma^{-1}(0) = 0$, *and*

$$M = \{(\gamma(s)\cos\alpha, \gamma(s)\sin\alpha) \,|\, s \in [0,+\infty), \theta \in S^1\}. \tag{49}$$

In both cases the curve (*or varifold*) $\gamma \cup -\gamma$ *is smooth near the origin.*

Proof. Consider the vector field

$$X = -JD\mu = (-x_2, -y_2, x_1, y_1).$$

A simple computation shows that for any Lagrangian plane P with orthonormal basis $\{e_1, e_2\}$, we have

$$\mathrm{div}_P X = \sum_{i=1}^{2} \langle D_{e_i} X, e_i \rangle = 0. \tag{50}$$

Finally, consider $(F_\alpha)_{\alpha \in S^1}$ to be the one-parameter family of diffeomorphisms in SU(2) such that

$$\frac{dF_\alpha}{d\alpha}(x) = X(F_\alpha(x))$$

Consider the functions $f_1(x) = \arctan(x_2/x_1)$ and $f_2(x) = \arctan(y_2/y_1)$, which are defined, respectively, in $U_1 = \{x_1 \neq 0\}$, $U_2 = \{y_1 \neq 0\}$. Assume that (C1) holds. We now make several remarks that will be important when one applies the co-area formula.

First, $F_\alpha(M) = M$; i.e.,

$$\int_M \phi \circ F_\alpha d\mathcal{H}^2 = \int_M \phi d\mathcal{H}^2 \text{ for all } \phi \text{ with compact support.}$$

Because $M \subseteq \mu^{-1}(0)$ and M is Lagrangian, we have that X is a tangent vector to M. Hence,

$$\frac{d}{d\alpha}\int_M \phi \circ F_\alpha d\mathcal{H}^2 = \int_M \langle D(\phi \circ F_\alpha), X \rangle d\mathcal{H}^2 = -\int_M (\phi \circ F_\alpha)\mathrm{div}_M X d\mathcal{H}^2 = 0,$$

where the last identity follows from (50).

Second, on M we have $|\nabla f_i|(x) = |x|^{-1}$. Indeed, for every $x \in U_i$ with $\mu(x) = 0$, it is a simple computation to see that $Df_i \in \mathrm{span}\{X(x), JX(x)\}$ and thus, because X is a tangent vector,

$$|\nabla f_i|(x) = |\langle Df_i(x), X(x)\rangle||x|^{-1} = |x|^{-1}.$$

Third, for almost all α and $i = 1, 2$, $f_i^{-1}(\alpha) \cap M$ is a one-dimensional varifold. Moreover, a simple computation shows $f_i \circ F_\alpha(x) = \alpha + f_i(x)$ for all $x \in U_i$ and all $\alpha \in (-\pi/2, \pi/2)$, and thus

$$F_\alpha(f_i^{-1}(0) \cap M) = f_i^{-1}(\alpha) \cap F_\alpha(M) = f_i^{-1}(\alpha) \cap M.$$

Fourth, the fact that $M \subseteq \mu^{-1}(0)$ implies that $f_i^{-1}(0) \cap M$ has support contained in $\{x_2 = y_2 = 0\} = \mathbb{C}$. Moreover, $f_1 = f_2$ on M, and so we set

$$\Gamma = f_1^{-1}(0) \cap M = f_2^{-1}(0) \cap M.$$

Fifth, one can check that F_π coincides with the antipodal map A. Thus

$$A(\Gamma) = A(f_i^{-1}(0)) \cap A(M) = f_i^{-1}(0) \cap M = \Gamma.$$

As a result, there is a one-dimensional varifold γ such that $\Gamma = A(\gamma) + \gamma$. (The choice of γ is not unique.)

Finally, we can apply the co-area formula and obtain for every ϕ with compact support in U_i,

$$\begin{aligned}
\int_M \phi d\mathcal{H}^2 &= \int_{-\pi/2}^{\pi/2} \int_{f_i^{-1}(\alpha)\cap M} \frac{\phi}{|\nabla f_i|} d\mathcal{H}^1 d\alpha = \int_{-\pi/2}^{\pi/2} \int_{f_i^{-1}(\alpha)\cap M} |x| \phi d\mathcal{H}^1 d\alpha \\
&= \int_{-\pi/2}^{\pi/2} \int_{f_i^{-1}(0)\cap M} |F_\alpha(x)| (\phi \circ F_\alpha) d\mathcal{H}^1 d\alpha \\
&= \int_\Gamma |x| \int_{-\pi/2}^{\pi/2} \phi(F_\alpha(x)) d\alpha d\mathcal{H}^1 \\
&= \int_\Gamma |z| \int_{-\pi/2}^{\pi/2} \phi(z\cos\alpha, z\sin\alpha) d\alpha d\mathcal{H}^1 \\
&= \int_\gamma |z| \int_0^{2\pi} \phi(z\cos\alpha, z\sin\alpha) d\alpha d\mathcal{H}^1.
\end{aligned}$$

This proves (48) for functions with support contained in U_i. Because M is smooth and embedded near the origin, it is straightforward to extend that formula to all functions with compact support.

Assume that (C2) holds. From what we have done it is straightforward to obtain the existence of a curve $\gamma : I \longrightarrow \mathbb{C}$, where I is a union of intervals, so that (49) holds. The fact that M is diffeomorphic to $\mathbb{R}^2$ implies that γ is connected and that $\gamma^{-1}(0)$ must be nonempty. The condition that M is embedded when restricted to a small neighborhood of the origin implies that $\gamma^{-1}(0)$ must have only one element, which we set to be zero. Finally, the fact that the map F is an immersion is equivalent to the curve $\gamma \cup -\gamma$ being smooth at the origin. ☐

7.2. *Regularity for equivariant flow.* In [2] and [3] Angenent developed the regularity theory for a large class of parabolic flows of curves in surfaces. We collect the necessary results, along with an improvement done in [11], which will be used in our setting.

Let $\gamma_t : [0, a] \longrightarrow \mathbb{C}$, $0 \leq t < T$, be a one-parameter family of smooth curves so that

(A1) There is $r > 0$ and $p \in \mathbb{C}$ so that for all $0 \leq t < T$, $\gamma_t(0) = 0$, $\gamma_t(a) \in B_r(p)$, γ_t has no self-intersections in $B_{2r}(0) \cup B_{2r}(p)$, and the curvature of γ_t along with $\frac{x^\perp}{|x|^2}$ and all its derivatives are bounded (independently of t) in $B_{2r}(0) \cup B_{2r}(p)$.

(A2) Away from the origin and for all $0 \leq t < T$, the curves γ_t solve the equation

$$\frac{dx}{dt} = \vec{k} - \frac{x^\perp}{|x|^2}.$$

A simple modification of [3, Th. 1.3] implies that, for $t > 0$, the self-intersections of γ_t are finite and nonincreasing with time.

THEOREM 7.2. *There is a continuous curve γ_T and a finite number of points $\{Q_1, \ldots, Q_m\} \subseteq \mathbb{C} \setminus B_{2r}(0) \cup B_{2r}(p)$ such that $\gamma_T \setminus \{Q_1, \ldots, Q_m\}$ consists of smooth arcs and away from the singular points the curves γ_t converge smoothly to γ_T. Any two smooth arcs intersect only in finitely many points.*

For each of the singular points Q_i and for each small ε, the number of self-intersections of γ_T in $B_\varepsilon(Q_i)$ is strictly less than the number of self-intersections of γ_{t_j} in $B_\varepsilon(Q_i)$ for some sequence $(t_j)_{j\in\mathbb{N}}$ converging to T.

Proof. Condition (A1) implies that the curves γ_t converge smoothly in $B_{2r}(0) \cup B_{2r}(p)$ as t tends to T. A slight modification of [2, Th. 4.1] shows that the quantity

$$\int_{\gamma_t} |\vec{k}| d\mathcal{H}^1$$

is uniformly bounded. Indeed the only change one has to make concerns the existence of boundary terms when integration by parts is performed. Fortunately, (A1) implies that the contribution from the boundary terms is uniformly bounded and so all the other arguments in [2, Th. 4.1] carry through.

The fact that the total curvature is uniformly bounded and that, on $\mathbb{C} \setminus B_{2r}(0)$, the deformation vector $\vec{k} - \frac{x^\perp}{|x|^2}$ satisfies conditions (V_1^*), (V_2), (V_3), (V_5^*), and (S) of [2], shows that we can apply [3, Th. 5.1] to conclude the existence of a continuous curve γ_T and a finite number of points $\{Q_1, \ldots, Q_m\} \subseteq \mathbb{C} \setminus (B_{2r}(0) \cup B_{2r}(p))$ such that $\gamma_T \setminus \{Q_1, \ldots, Q_m\}$ consists of smooth arcs and away from the singular points the curves γ_t converge smoothly to γ_T. We note that [3, Th. 5.1] is applied to close curves, but an inspection of the proof shows that all the arguments are local and so they apply with no modifications to γ_t provided hypothesis (A1) holds.

Oaks [11, Th. 6.1] showed that for each of the singular points Q_i and for each small ε, there is a sequence $(t_j)_{j\in\mathbb{N}}$ converging to T so that γ_{t_j} has self-intersections in $B_\varepsilon(Q_i)$ and either a closed loop of γ_{t_j} in $B_\varepsilon(Q_i)$ contracts as t_j tends to T or else there are two distinct arcs in the smooth part of γ_T that coincide in a neighborhood of Q_i (see [3, Fig. 6.2.]). Using the fact that the deformation vector is analytic in its arguments on $\mathbb{C} \setminus B_{2r}(0)$, we can argue as in [3, pp. 200–201] and conclude that the smooth part of γ_T must in fact be real analytic in $\mathbb{C} \setminus B_{2r}(0)$. Therefore, any two smooth arcs intersect only in finitely many points and this excludes the second possibility. $\square$

7.3. *Nonavoidance principle for equivariant flow.*

LEMMA 7.3. *For each $j = 1, 2$, consider smooth curves $\sigma_{j,t} : [-a, a] \longrightarrow \mathbb{C}$ defined for all $0 \le t \le T$ so that*

(i) *$\sigma_{j,t}(-s) = -\sigma_{j,t}(s)$ for all $0 \le t \le T$ and $s \in [-a, a]$.*

(ii) *The curves γ_t solve the equation*

$$\frac{dx}{dt} = \vec{k} - \frac{x^{\perp}}{|x|^2}.$$

(iii) *$\sigma_{1,0} \cap \sigma_{2,0} = \{0\}$ (nontangential intersection) and $(\partial\sigma_{1,t}) \cap \sigma_{2,t} = \sigma_{1,t} \cap (\partial\sigma_{2,t}) = \emptyset$ for all $0 \le t \le T$.*

For all $0 \le t \le T$, we have $\sigma_{1,t} \cap \sigma_{2,t} = \{0\}$.

Proof. Away from the origin, it is simple to see the maximum principle holds and so two disjoint solutions cannot intersect for the first time away from the origin. Thus it suffices to focus on what happens around the origin. Without loss of generality we assume that $\sigma_{j,t}(s) = (s, f_{j,t}(s))$ for all $s \in [-\delta, \delta]$ for all $t \le T_1$. The functions $\alpha_{j,t}(s) = s^{-1} f_{j,t}(s)$ are smooth by (i), and so we consider $u_t = \alpha_{1,t} - \alpha_{2,t}$ which, form (iii), we can assume to be initially positive and $u_t(\delta) = u_t(-\delta) > 0$ for all $t \le T_1$. It is enough to show that u_t is positive for all $t \le T_1$. We have at once that

$$\begin{aligned}
&\frac{df_{j,t}}{dt} = (\arctan(\alpha_{j,t}))' + \frac{f''_{j,t}}{1 + (f'_{j,t})^2} \\
&\quad\Longrightarrow\ \frac{d\alpha_{j,t}}{dt} = \frac{\alpha''_{j,t}}{1 + (s\alpha'_{j,t} + \alpha_{j,t})^2} + \frac{\alpha'_{j,t}}{s}\frac{1}{1 + \alpha^2_{j,t}} + \frac{\alpha'_{j,t}}{s}\frac{2}{1 + (s\alpha'_{j,t} + \alpha_{j,t})^2}.
\end{aligned}$$

The functions $s^{-1}\alpha'_{j,t}$ are smooth for all s, and so we obtain

$$\frac{du_t}{dt} = \frac{u''_t}{1 + C_1^2} + C_2 u'_t + C_3 u_t + \frac{u'_t}{s} C_4^2,$$

where C_k are smooth time dependent bounded functions for $k = 1, \dots, 4$.

Suppose T_1 is the first time at which u_t becomes zero, and consider $v_t = u_t e^{-Ct} + \varepsilon(t - T_1)$ with ε small and C large. The function v_t becomes zero for a first time $t \le T_1$ at some point s_0 for all small positive ε. At that time we have $u''_t(s_0) \ge 0$, $u'_t(s_0) = 0$, and thus, with an obvious abuse of notation,

$$0 \ge \frac{dv_t}{dt}(s_0) = \varepsilon + \frac{d}{dt}(u_t e^{-Ct})(s_0) \ge \varepsilon + \frac{u'_t(s_0)}{s_0} C_4^2 e^{-Ct}.$$

If s_0 is not zero, the last term on the right is zero. If s_0 is zero, then the last term on the right is $u_t''(0)C_4^2e^{-Ct}$, which is nonnegative. In any case we get a contradiction. □

References

[1] H. Anciaux, Construction of Lagrangian self-similar solutions to the mean curvature flow in $\mathbb{C}^n$, *Geom. Dedicata* **120** (2006), 37–48. MR 2252892. Zbl 1098.35074. http://dx.doi.org/10.1007/s10711-006-9082-z.

[2] S. Angenent, Parabolic equations for curves on surfaces. I. Curves with p-integrable curvature, *Ann. of Math.* **132** (1990), 451–483. MR 1078266. Zbl 0789.58070. http://dx.doi.org/10.2307/1971426.

[3] S. Angenent, Parabolic equations for curves on surfaces. II. Intersections, blow-up and generalized solutions, *Ann. of Math.* **133** (1991), 171–215. MR 1087347. Zbl 0749.58054. http://dx.doi.org/10.2307/2944327.

[4] Y. Eliashberg and L. Polterovich, Local Lagrangian 2-knots are trivial, *Ann. of Math.* **144** (1996), 61–76. MR 1405943. Zbl 0872.57030. http://dx.doi.org/10.2307/2118583.

[5] M. A. Grayson, Shortening embedded curves, *Ann. of Math.* **129** (1989), 71–111. MR 0979601. Zbl 0686.53036. http://dx.doi.org/10.2307/1971486.

[6] G. Huisken, Asymptotic behavior for singularities of the mean curvature flow, *J. Differential Geom.* **31** (1990), 285–299. MR 1030675. Zbl 0694.53005. Available at http://projecteuclid.org/euclid.jdg/1214444099.

[7] T. Ilmanen, Elliptic regularization and partial regularity for motion by mean curvature, *Mem. Amer. Math. Soc.* **108** (1994), x+90. MR 1196160. Zbl 0798.35066.

[8] T. Ilmanen, Singularities of mean curvature flow of surfaces, preprint.

[9] A. Neves, Singularities of Lagrangian mean curvature flow: zero-Maslov class case, *Invent. Math.* **168** (2007), 449–484. MR 2299559. Zbl 1119.53052. http://dx.doi.org/10.1007/s00222-007-0036-3.

[10] A. Neves and G. Tian, Translating solutions to Lagrangian mean curvature flow, preprint. arXiv 0711.4341.

[11] J. A. Oaks, Singularities and self intersections of curves evolving on surfaces, *Indiana Univ. Math. J.* **43** (1994), 959–981. MR 1305955. Zbl 0835.53048. http://dx.doi.org/10.1512/iumj.1994.43.43042.

[12] R. Schoen and J. Wolfson, Minimizing area among Lagrangian surfaces: the mapping problem, *J. Differential Geom.* **58** (2001), 1–86. MR 1895348. Zbl 1052.53056. Available at http://projecteuclid.org/euclid.jdg/1090348282.

[13] R. Schoen and J. Wolfson, Mean curvature flow and Lagrangian embeddings, preprint. Available at http://www.math.msu.edu/~wolfson/MCF_lagrangian.pdf.

[14] K. Smoczyk, A canonical way to deform a Lagrangian submanifold, preprint. arXiv dg-ga/9605005.

[15] R. P. Thomas and S.-T. Yau, Special Lagrangians, stable bundles and mean curvature flow, *Comm. Anal. Geom.* **10** (2002), 1075–1113. MR 1957663. Zbl 1115.53054.

[16] M.-T. Wang, Mean curvature flow of surfaces in Einstein four-manifolds, *J. Differential Geom.* **57** (2001), 301–338. MR 1879229. Zbl 1035.53094. Available at http://projecteuclid.org/euclid.jdg/1090348113.

[17] M.-T. Wang, Some recent developments in Lagrangian mean curvature flows, in *Surveys in Differential Geometry. Vol. XII. Geometric Flows*, *Surv. Differ. Geom.* **12**, Int. Press, Somerville, MA, 2008, pp. 333–347. MR 2488942. Zbl 1169.53052.

[18] B. White, A local regularity theorem for mean curvature flow, *Ann. of Math.* **161** (2005), 1487–1519. MR 2180405. Zbl 1091.53045. http://dx.doi.org/10.4007/annals.2005.161.1487.

[19] J. Wolfson, Lagrangian homology classes without regular minimizers, *J. Differential Geom.* **71** (2005), 307–313. MR 2197143. Zbl 1097.53052. Available at http://projecteuclid.org/euclid.jdg/1143651771.

(Received: September 8, 2010)
(Revised: March 30, 2012)

Imperial College London, London, UK
E-mail: aneves@imperial.ac.uk