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The lens space realization problem

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Dedicated to the memory of Professor Michael Moody

Abstract

We determine the lens spaces that arise by integer Dehn surgery along a knot in the three-sphere. Specifically, if surgery along a knot produces a lens space, then there exists an equivalent surgery along a Berge knot with the same knot Floer homology groups. This leads to sharp information about the genus of such a knot. The arguments rely on tools from Floer homology and lattice theory. They are primarily combinatorial in nature.

1. **Introduction**

What are all the ways to produce the simplest closed 3-manifolds by the simplest 3-dimensional topological operation? From the cut-and-paste point of view, the simplest 3-manifolds are the *lens spaces* $L(p, q)$, these being the spaces (besides S^3 and $S^1 \times S^2$) that result from identifying two solid tori along their boundaries, and the simplest operation is *Dehn surgery* along a knot $K \subset S^3$. With these meanings in place, the opening question goes back forty years to Moser [35], and its definitive answer remains unknown.

By definition, a *lens space knot* is a knot $K \subset S^3$ that admits a lens space surgery. Moser observed that all torus knots are lens space knots and classified their lens space surgeries. Subsequently, Bailey-Rolfsen [1] and Fintushel-Stern [14] gave more examples of lens space knots. The production of examples culminated in an elegant construction due to Berge that at once subsumed all the previous ones and generated many more classes [6]. Berge's examples are the knots that lie on a Heegaard surface Σ of genus two for S^3 and that represent a primitive element in the fundamental group of each handlebody. For this reason, such knots are called *doubly primitive*. Berge observed that performing surgery along such a knot K, with (integer) framing specified by a push-off of K on Σ, produces a lens space. Furthermore, he enumerated several different types of doubly primitive knots. By definition, the *Berge knots* are

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the doubly primitive knots that Berge specifically enumerated in [6]. They are reproduced in Section 1.2. (More precisely, the *dual* Berge knots are reported there.)

The most prominent question concerning lens space surgeries is the *Berge conjecture*.

CONJECTURE 1.1 ([28, Prob. 1.78]). *If integer surgery along a knot $K \subset S^3$ produces a lens space, then it arises from Berge's construction.*

Complementing Conjecture 1.1 is the cyclic surgery theorem of Culler-Gordon-Luecke-Shalen [10], which implies that if a lens space knot K is not a torus knot, then the surgery coefficient is an integer. Therefore, an affirmative answer to the Berge conjecture would settle Moser's original question. We henceforth restrict attention to *integer* slope surgeries as a result. The *sign* of integral surgery along a knot K in an arbitrary 3-manifold Y is the sign of the intersection pairing on the associated 2-handle cobordism from $-Y$ to the surgered manifold. In the case at hand, we may assume that the slope is *positive* by reflecting the knot, if necessary. Thus, in what follows, we attach to every lens space knot K a positive integer p for which p-surgery along K produces a lens space, and we denote the surgered manifold by K_p.

Using monopole Floer homology, Kronheimer-Mrowka-Ozsváth-Szabó related the knot genus and the surgery slope via the inequality

$$2g(K) - 1 \leq p \tag{1}$$

[29, Cor. 8.5]. Their argument utilizes the fact that the Floer homology of a lens space is as simple as possible: rk $\widehat{HF}(Y) = |H_1(Y;\mathbb{Z})|$. A space with this property is called an L-*space*, and a knot with a positive L-space surgery is an L-*space knot*. Their proof adapts to the setting of Heegaard Floer homology as well [38], the framework in place for the remainder of this paper. Ozsváth-Szabó established a significant constraint on the knot Floer homology groups $\widehat{HFK}(K)$ and hence the Alexander polynomial Δ_K [39, Th. 1.2 and Cor. 1.3]. Utilizing this result, Ni proved that K is fibered [36, Cor. 1.3].

As indicated by Berge, it is often preferable to take the perspective of surgery along a knot in a lens space. Corresponding to a lens space knot $K \subset S^3$ is a dual knot $K' \subset K_p$, the core of the surgery solid torus. Reversing the surgery, it follows that K' has a *negative* integer surgery producing S^3. Following custom, we refer to the dual of a Berge knot as a Berge knot as well, and we stress the ambient manifold to prevent confusion. As demonstrated by Berge [6, Th. 2], the dual to a doubly primitive knot takes a particularly pleasant form: it is an example of a *simple knot*, of which there is a unique one in each homology class in $L(p, q)$. Thus, each Berge knot in a lens space is specified by its homology class, and this is what we report in Section 1.2.

This point of view is taken up by Baker-Grigsby-Hedden [4] and J. Rasmussen [40], who have proposed programs to settle Conjecture 1.1 by studying knots in lens spaces with simple knot Floer homology.

1.1. *Results.* A derivative of the Berge conjecture is the *realization problem*, which asks for those lens spaces that arise by integer surgery along a knot in S^3. Closely related is the question of whether the Berge knots account for all the doubly primitive knots. Furthermore, the Berge conjecture raises the issue of tightly bounding the knot genus $g(K)$ from above in terms of the surgery slope p. The present work answers these three questions.

THEOREM 1.2. *Suppose that negative integer surgery along a knot $K \subset L(p,q)$ produces S^3. Then K lies in the same homology class as a Berge knot $B \subset L(p,q)$.*

The resolution of the realization problem follows at once. As explained in Section 10, the same result holds with S^3 replaced by any L-space homology sphere with d-invariant 0. As a corollary, we obtain the following result, the last part of which has been independently obtained by Berge [7].

THEOREM 1.3. *Suppose that $K \subset S^3$, p is a positive integer, and K_p is a lens space. Then there exists a Berge knot $B \subset S^3$ such that $B_p \cong K_p$ and $\widehat{HFK}(B) \cong \widehat{HFK}(K)$. Furthermore, every doubly primitive knot in S^3 is a Berge knot.*

Based on well-known properties of the knot Floer homology groups, it follows that K and B have the same Alexander polynomial, genus, and four-ball genus. Furthermore, the argument used to establish Theorem 1.2 leads to a tight upper bound on the knot genus $g(K)$ in relation to the surgery slope.

THEOREM 1.4. *Suppose that $K \subset S^3$, p is a positive integer, and K_p is a lens space. Then*

$$2g(K) - 1 \leq p - 2\sqrt{(4p+1)/5}, \tag{2}$$

unless K is the right-hand trefoil and $p = 5$. Moreover, this bound is attained by the type VIII *Berge knots specified by the pairs $(p,k) = (5n^2+5n+1, 5n^2-1)$.*

Theorem 1.4 was announced without proof in [23, Th. 1.2] (cf. [43]). As indicated in [23], for $p \gg 0$, Theorem 1.4 significantly improves on the bound $2g(K)-1 \leq p-9$ conjectured by Goda-Teragaito [21] for a hyperbolic knot K, and it can be used to show that the conjectured bound holds for all but at most two values $p \in \{14, 19\}$. In addition, one step involved in both approaches to the Berge conjecture outlined in [4, 40] is to argue the nonexistence of a nontrivial knot K for which $K_{2g(K)-1}$ is a lens space. This fact follows immediately from Theorem 1.4.

1.2. *Berge knots in lens spaces.* J. Rasmussen concisely tabulated the Berge knots $B \subset L(p,q)$ [40, §6.2]. To describe those with a *negative* S^3 surgery, select a positive integer k and produce a positive integer p in terms of it according to the table below. The value $k \pmod p$ represents the homology class of B in $H_1(L(p,q)) \cong \mathbb{Z}/p\mathbb{Z}$, $q \equiv -k^2 \pmod p$, as described at the end of Section 2. We reproduce the tabulation here.

Berge Type I$_\pm$: $p = ik \pm 1, \quad \gcd(i,k) = 1;$

Berge Type II$_\pm$: $p = ik \pm 1, \quad \gcd(i,k) = 2,\ i,k \geq 4;$

Berge Type III: $\begin{cases} (a)_\pm & p \equiv \pm(2k-1)d \pmod{k^2}, \quad d \mid k+1,\ \frac{k+1}{d} \text{ odd}; \\ (b)_\pm & p \equiv \pm(2k+1)d \pmod{k^2}, \quad d \mid k-1,\ \frac{k-1}{d} \text{ odd}; \end{cases}$

Berge Type IV: $\begin{cases} (a)_\pm & p \equiv \pm(k-1)d \pmod{k^2}, \quad d \mid 2k+1; \\ (b)_\pm & p \equiv \pm(k+1)d \pmod{k^2}, \quad d \mid 2k-1; \end{cases}$

Berge Type V: $\begin{cases} (a)_\pm & p \equiv \pm(k+1)d \pmod{k^2}, \quad d \mid k+1, d \text{ odd}; \\ (b)_\pm & p \equiv \pm(k-1)d \pmod{k^2}, \quad d \mid k-1, d \text{ odd}; \end{cases}$

Berge Type VII: $k^2 + k + 1 \equiv 0 \pmod p;$

Berge Type VIII: $k^2 - k - 1 \equiv 0 \pmod p;$

Berge Type IX: $p = \frac{1}{11}(2k^2 + k + 1), k \equiv 2 \pmod{11};$

Berge Type X: $p = \frac{1}{11}(2k^2 + k + 1), k \equiv 3 \pmod{11}.$

As indicated by J. Rasmussen, type VI occurs as a special case of type V and types XI and XII result from allowing negative values for k in IX and X, respectively.

1.3. *Overview and organization.* We now provide a detailed overview of the general strategy we undertake to establish the main results. We hope that this account will satisfy the interests of most readers and clarify the intricate combinatorial arguments that occupy the main body of the text.

Our approach draws inspiration from a remarkable pair of papers by Lisca [31], [32], in which he classified the sums of lens spaces that bound a smooth, rational homology ball. Lisca began with the observation that the lens space $L(p,q)$ naturally bounds a smooth, negative definite plumbing 4-manifold $X(p,q)$ (Section 2). If $L(p,q)$ bounds a rational ball W, then the 4-manifold $Z := X(p,q) \cup -W$ is a smooth, closed, negative definite 4-manifold with $b_2(Z) = b_2(X) =: n$. According to Donaldson's celebrated "Theorem A," the intersection pairing on $H_2(Z;\mathbb{Z})$ is isomorphic to *minus* the standard Euclidean integer lattice $-\mathbb{Z}^n$ [12]. As a result, it follows that the intersection pairing on $X(p,q)$, which we henceforth denote by $-\Lambda(p,q)$, embeds as a full-rank sublattice of $-\mathbb{Z}^n$. Lisca solved the combinatorial problem of determining the pairs (p,q) for which there exists an embedding $\Lambda(p,q) \hookrightarrow \mathbb{Z}^n$, subject to a certain additional constraint on the pair (p, q). By consulting an earlier tabulation of Casson-Gordon [9, p. 188], he observed that the embedding exists if and only if $\pm L(p, q)$ belongs to a family of lens spaces already known to bound a special type of rational ball (cf. Section 1.6). The classification of lens spaces that bound rational balls follows at once. Pushing this technique further, Lisca obtained the classification result for sums of lens spaces as well.

In our situation, we seek the pairs (p, q) for which $L(p, q)$ arises as positive integer surgery along a knot $K \subset S^3$. Thus, suppose that $K_p \cong L(p, q)$, and form a smooth 4-manifold $W_p(K)$ by attaching a p-framed 2-handle to D^4 along $K \subset \partial D^4$. This space has boundary K_p, so we obtain a smooth, closed, negative definite 4-manifold by setting $Z = X(p, q) \cup -W_p(K)$, where $b_2(Z) = n + 1$. By Donaldson's theorem, it follows that $\Lambda(p, q)$ embeds as a codimension one sublattice of $\mathbb{Z}^{n+1}$. However, this restriction is too weak: it is easy to produce pairs (p, q) that fulfill this condition, while $L(p, q)$ does not arise as a positive integer knot surgery (for example, $L(10, 1)$ and $L(17, 15)$; cf. Section 1.6).

Thankfully, we have another tool to work with: the *correction terms* in Heegaard Floer homology ([23, §2]). Ozsváth-Szabó defined these invariants and subsequently used them to phrase a necessary condition on the pair (p, q) in order for $L(p, q)$ to arise as a positive integer surgery [38, Cor. 7.5]. Using a computer, they showed that this condition is actually sufficient for $p \leq 1500$: every pair that fulfills it appears on Berge's list [39, Prop. 1.13]. Later, J. Rasmussen extended this result to all $p \leq 100,000$ [40, end of Section 6]. Following their work, it stood to reason that the Ozsváth-Szabó condition is both necessary and sufficient for *all* (p, q). However, it remained unclear how to manipulate the correction terms effectively towards this end.

The key idea here is to use the correction terms in tandem with Donaldson's theorem, in the form of an obstruction already extracted in [23, Th. 3.3]. The result is an enhanced lattice embedding condition (cf. [22], [25]). In order to state it, we first require a combinatorial definition.

Definition 1.5. A vector $\sigma = (\sigma_0, \ldots, \sigma_n) \in \mathbb{Z}^{n+1}$ with $1 = \sigma_0 \leq \sigma_1 \leq \cdots \leq \sigma_n$ is a *changemaker* if, for all $0 \leq k \leq \sigma_0 + \cdots + \sigma_n$, there exists a subset $A \subset \{0, \ldots, n\}$ such that $\sum_{i \in A} \sigma_i = k$. Equivalently, $\sigma_i \leq \sigma_0 + \cdots + \sigma_{i-1} + 1$ for all $1 \leq i \leq n$.

The reader may find it amusing to establish the equivalence stated in Definition 1.5; its proof appears in both [8] and [23, Lemma 3.2]. If we imagine the σ_i as values of coins, then this equivalence asserts a necessary and sufficient condition under which one can make exact change from the coins in any amount up to their total value. Note that Definition 1.5 differs slightly from the one used in [23], since here we require that $\sigma_0 = 1$.

Our lattice embedding condition now reads as follows. Again, we phrase it from the perspective of surgery along a knot in a lens space.

THEOREM 1.6. *Suppose that negative integer surgery along a knot* $K \subset L(p,q)$ *produces* S^3. *Then* $\Lambda(p,q)$ *embeds as the orthogonal complement to a changemaker* $\sigma \in \mathbb{Z}^{n+1}$, $n = b_2(X)$.

Our strategy is now apparent: determine the list of pairs (p,q) that pass this refined embedding obstruction, and check that it coincides with Berge's list. Indeed, this is the case.

THEOREM 1.7. *At least one of the pairs* (p,q), (p,q'), *where* $qq' \equiv 1 \pmod p$, *appears on Berge's list if and only if* $\Lambda(p,q)$ *embeds as the orthogonal complement to a changemaker in* $\mathbb{Z}^{n+1}$.

Furthermore, when $\Lambda(p,q)$ embeds, we recover a value $k \pmod p$ that represents the homology class of a Berge knot $K \subset L(p,q)$ (Proposition 2.2). Theorem 1.2 follows easily from this result.

To give a sense of the proof of Theorem 1.7, we first reflect on the lattice embedding problem that Lisca solved. He made use of the fact that $\Lambda(p,q)$ admits a special basis; in our language, it is a *linear lattice* with a distinguished *vertex basis* (Section 3.3). He showed that any embedding of a linear lattice as a full-rank sublattice of $\mathbb{Z}^n$ (subject to the extra constraint he posited) can be built from one of a few small embeddings by repeatedly applying a basic operation called *expansion*. Following this result, the identification of the relevant pairs (p,q) follows from a manipulation of continued fractions. The precise details of Lisca's argument are involved, but ultimately elementary and combinatorial in nature.

One is tempted to carry out a similar approach to Theorem 1.7. Thus, one might first attempt to address the problem of embedding $\Lambda(p,q)$ as a codimension one sublattice of $\mathbb{Z}^n$ and then analyze which of these are complementary to a changemaker. However, getting started in this direction is difficult, since Lisca's techniques do not directly apply.

More profitable, it turns out, is to turn this approach on its head. Thus, we begin with a study of the lattices of the form $(\sigma)^\perp \subset \mathbb{Z}^n$ for some changemaker σ; by definition, these are the *changemaker lattices*. A changemaker lattice is best presented in terms of its *standard basis* (Section 3.4). The question then becomes: when is a changemaker lattice isomorphic to a linear lattice? That is, how do we recognize whether there exists a change of basis from its standard basis to a vertex basis?

The key notion in this regard is that of an *irreducible* element in a lattice L. By definition, an element $x \in L$ is *reducible* if $x = y + z$, where $y, z \in L$ are nonzero and $\langle y, z\rangle \geq 0$; it is irreducible otherwise. Here $\langle\ ,\ \rangle$ denotes the pairing on L. As we show, the standard basis elements of a changemaker lattice are irreducible (Lemma 3.13), as are the vertex basis elements of a linear lattice. Furthermore, the irreducible elements in a linear lattice take a very specific form (Corollary 3.5). This leads to a variety of useful lemmas, collected in Section 4.2. For example, if a changemaker lattice is isomorphic to a linear lattice, then its standard basis does not contain three elements, each of norm ≥ 3, such that any two pair together nontrivially (Lemma 4.10).

Thus, we proceed as follows. First, choose a standard basis $S \subset \mathbb{Z}^n$ for a changemaker lattice L and suppose that L is isomorphic to a linear lattice. Then apply the combinatorial criteria of Section 4.2 to deduce the specific form that S must take. Standard basis elements come in three distinct flavors – *gappy*, *tight*, and *just right* (Definition 3.11) – and our case analysis decomposes according to whether S contains no gappy or tight vectors (Section 6), a gappy vector but not a tight one (Section 7), or a tight vector (Section 8). In addition, Section 5 addresses the case in which L is isomorphic to a (direct) sum of linear lattices. This case turns out the easiest to address, and the subsequent Sections 6–8 rely on it, while increasing in order of complexity.

The net result of Sections 5–8 is a collection of several structural propositions that enumerate the possible standard bases for a changemaker lattice isomorphic to a linear lattice or a sum thereof. Section 9 takes up the problem of converting these standard bases into vertex bases, extracting the relevant pairs (p, q) for each family of linear lattices, as well as the value $k \pmod p$ of Proposition 2.2. Here, as in Lisca's work, we make some involved calculations with continued fractions. Table 1 gives an overview of the correspondence between the structural propositions and the Berge types. Lastly, Section 10 collects the results of the earlier sections to prove the theorems stated above.

The remaining introductory sections discuss various related topics.

1.4. *Related progress.* A number of authors have recently addressed both the realization problem and the classification of doubly primitive knots in S^3. S. Rasmussen established Theorem 1.2 under the constraint that $k^2 < p$ [41, Th. 1.0.3]. This condition is satisfied precisely by Berge types I–V. Tange established Theorem 1.2 under a different constraint relating the values k and p [45, Th. 6]. The two constraints are not complementary, however, so the full statement of Theorem 1.2 does not follow on combination of these results. Also, as we indicated before Theorem 1.3, Berge has shown by direct topological methods that all doubly primitive knots are Berge knots; equivalently, a simple knot in a lens space has an S^3 surgery if and only if it is a (dual) Berge knot [7].

1.5. *Comparing Berge's and Lisca's lists.* Lisca's list of lens spaces that bound rational balls bears a striking resemblance to the list of Berge knots of type I–V (Section 1.2). J. Rasmussen has explained this commonality by way of the knots K in the solid torus $S^1 \times D^2$ that possess integer $S^1 \times D^2$

surgeries. The classification of these knots is due to Berge and Gabai [5], [18]. Given such a knot, we obtain a knot $K' \subset S^1 \times S^2$ via the standard embedding $S^1 \times D^2 \subset S^1 \times S^2$. Performing the induced surgery along K' produces a lens space $L(p,q)$, which we can effect by attaching a 2-handle along $K' \subset \partial(S^1 \times D^3)$. The resulting 4-manifold is a rational ball built from a single 0-, 1-, and 2-handle, and it has boundary $L(p,q)$.

As observed by J. Rasmussen, every lens space on Lisca's list actually arises in this way.[1] On the other hand, we obtain a knot $K'' \subset S^3$ from K via the standard embedding $S^1 \times D^2 \subset S^3$. Performing the induced surgery along K'' produces a lens space $L(r,s)$ and a dual knot representing some homology class $k \pmod r$. The Berge knots of type I–V arise in this way. The pair (p,q) comes from setting $p = k^2$ and $q = r$; in this way, we reconstruct Lisca's list (but not his result!) from Berge's.

Analogous to the Berge conjecture, Lisca's theorem raises the following conjecture.

Conjecture 1.8. *If a knot in $S^1 \times S^2$ admits an integer lens space surgery, then it arises from a knot in $S^1 \times D^2$ with an integer $S^1 \times D^2$ surgery.*

Since this paper first appeared as a preprint, Baker produced counterexamples to Conjecture 1.8 [3]. The natural revision to Conjecture 1.8 reads as follows.

Conjecture 1.9 (Baker-Greene). *If a knot in $S^1 \times S^2$ admits an integer lens space surgery, then it is doubly-primitive with respect to the standard genus two Heegaard splitting of $S^1 \times S^2$. Equivalently, a knot in a lens space with an integer $S^1 \times S^2$ surgery is simple.*

Baker's examples indicate that although it is known, following Lisca and J. Rasmussen, which lens spaces contain a knot with an integer $S^1 \times S^2$ surgery, it remains unknown which homology classes in these spaces contain such knots. Thus, it remains unknown which simple knots in lens spaces admit an integer $S^1 \times S^2$ surgery, in contrast with the situation described by Theorem 1.2.

1.6. *4-manifolds with small b_2.* Which lens spaces bound a smooth 4-manifold W with $b_1(W) = 0$ and $b_2(W) = b_2^+(W) = 1$? What if we ask that $\pi_1(W) = 1$? What if we ask that W be built from a single 0- and 2-handle? These requirements are increasingly stringent, and indeed these questions have different answers, as the following examples illustrate. Note that Theorem 1.2 can be read as an answer to the third question, while the answers to the first two remain unknown.

[1]To be accurate, Lisca's list $\mathcal{R}$ should include the case $\gcd(m,k) = 2$ in type (1) of [31, Def. 1.1]. This oversight was already reported in another footnote [30, p. 247].

First we argue that $L(10,1)$ bounds a 4-manifold of the first type, but not the second. This space is surgery along a nine-component link $\mathbb{L}$ as in Figure 1, with all framings equal to $+2$. By attaching a $+2$-framed 2-handle along an unknot linking the third component of $\mathbb{L}$, we obtain a positive definite cobordism from $L(10,1)$ to the Brieskorn sphere $\Sigma(2,3,7)$. Fintushel-Stern showed that this space, in turn, bounds a rational homology ball [15], so its union with the 2-handle cobordism provides a smooth 4-manifold W with $b_1(W)=0$, $b_2(W)=b_2^+(W)=1$, and boundary $L(10,1)$. On the other hand, W cannot be chosen simply-connected, thanks to the following argument due to Fintushel-Stern [16]. If such a manifold W existed, then we could glue it along its boundary to the disk bundle of Euler number 10 over $S := S^2$. The result is a smooth, closed, simply-connected 4-manifold Z with $b_2(Z)=b_2^+(Z)=2$ that contains a sphere S of self-intersection 10. By Donaldson's theorem, the intersection pairing on Z is isomorphic to $\mathbb{Z}^2$, and using this it follows that $[S]=3e_1+e_2$ for a suitable choice of orthonormal basis $\{e_1,e_2\}$ for $H_2(Z;\mathbb{Z})$. Thus, S represents a characteristic element in $H_2(Z;\mathbb{Z})$. On the other hand, Kervaire-Milnor showed that if a characteristic element for a smooth, simply-connected 4-manifold Z is represented by a sphere S, then $\sigma(Z)\equiv[S]^2 \pmod{16}$ [27]. As $\sigma(Z)=2$ and $[S]^2=10$ in the present case, we obtain the desired contradiction.

Next we note that $L(17,15)$ bounds a manifold of the second type, but not the third. Indeed, Tange has exhibited many examples of lens spaces that arise by positive surgery along a knot in the boundary of a contractible 4-manifold, but not along any knot in S^3 [46]. For the case of $L(17,15)$, Tange exhibits a positive definite 2-handle cobordism to the Brieskorn sphere $\Sigma(2,3,11)$, which is known to bound a contractible 4-manifold. On the other hand, an application of Theorem 1.6 shows that $L(17,15)$ does not bound a 4-manifold of the third type.

We reiterate that Donaldson's theorem places a restriction on which lens spaces may bound either of the first two types of 4-manifolds, and the Kervaire-Milnor result places an additional restriction in the second case. We do not know whether these conditions are sufficient in either case.

By contrast, the situation in the topological category is much simpler: a lens space $L(p,q)$ bounds a topological 4-manifold W with $b_2(W)=b_2^+(W)=1$ if and only if W can be chosen simply-connected and if and only if $-q$ is a square $\pmod p$.

Similarly, we ask: which lens spaces bound a smooth rational homology ball W? What if we ask that W be built from a single 0-, 1-, and 2-handle? As addressed in Section 1.5, the answers to these two questions are the same and are settled. Furthermore, Lisca showed that a two-bridge knot is smoothly slice if and only if its branched double-cover (a lens space) bounds a smooth rational homology ball. Donald-Owens extended this result to the case of two-bridge links using their notion of χ-*sliceness* [11].

Which lens spaces bound a topological rational homology ball? The answer to this question remains unknown. For that matter, it remains unknown which two-bridge links L are topologically χ-slice. Note that if L is topologically χ-slice, then the lens space that arises as its branched double-cover bounds a topological rational homology ball [11]. However, the converse is unknown: is it the case that a lens space bounds a topological rational homology ball if and only if the corresponding two-bridge link is topologically χ-slice? Are the answers to these questions the same as in the smooth category?

1.7. *The Poincaré sphere.* Tange constructed several families of simple knots in lens spaces with integer surgeries producing the Poincaré sphere P^3 [44, §5]. J. Rasmussen verified that Tange's knots account for all such simple knots in $L(p, q)$ with $|p| \leq 100,000$ and $2g(K) - 1 < p$ [40, end of Section 6]. Furthermore, he observed that in the homology class of each type VII Berge knot, there exists a $(1, 1)$-knot T_L with $2g(T_L)-1 = p$ as constructed by Hedden [26, Fig. 3], and it admits an integer P^3-surgery for values $p \leq 39$ [40, end of §5]. Baker subsequently succeeded in showing that this is the case for all p [2]. Combining conjectures of Hedden [26, Conj. 1.7] and J. Rasmussen [40, Conj. 1], it would follow that Tange's knots and the knots T_L homologous to type VII Berge knots are precisely the knots in lens spaces with an integer P^3-surgery. Conjecture 1.10 is the analogue to the realization problem in this setting.

CONJECTURE 1.10. *Suppose that integer surgery along a knot $K \subset L(p, q)$ produces P^3.*[2] *Then either $2g(K) - 1 < p$, and K lies in the same homology class as a Tange knot, or else $2g(K) - 1 = p$, and K lies in the same homology class as a Berge knot of type* VII.

Tange has obtained partial progress on Conjecture 1.10 [45]. The methodology developed here to establish Theorem 1.2 suggests a similar approach to Conjecture 1.10, making use of an unpublished variant on Donaldson's theorem due to Frøyshov [17, Prop. 2 and the remark thereafter]. Lastly, we remark that the determination of *non*integral P^3-surgeries along knots in lens spaces seems tractable, although it falls outside the scope of the cyclic surgery theorem.

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[2]Or, more generally, any L-space homology sphere with d-invariant -2.

[32] and Dusa McDuff's lecture on her joint work with Felix Schlenk [33] were especially influential along the way. The bulk of this paper was written at the Mathematical Sciences Research Institute in Spring 2010. Thanks to everyone connected with that institution for providing an ideal working environment.

2. **Topological preliminaries**

Given relatively prime integers $p > q > 0$, the lens space $L(p,q)$ is the oriented manifold obtained from $-p/q$ Dehn surgery along the unknot. It bounds a plumbing manifold $X(p,q)$, which has the following familiar description. Expand p/q in a Hirzebruch-Jung continued fraction

$$p/q = [a_1, a_2, \ldots, a_n]^- = a_1 - \cfrac{1}{a_2 - \cfrac{1}{\ddots - \cfrac{1}{a_n}}},$$

with each a_i an integer ≥ 2. Form the disk bundle X_i of Euler number $-a_i$ over S^2, plumb together X_i and X_{i+1} for $i = 1, \ldots, n-1$, and let $X(p,q)$ denote the result. The manifold $X(p,q)$ is *sharp* [23, §2]. It also admits a Kirby diagram given by the framed chain link $\mathbb{L} = L_1 \cup \cdots \cup L_n \subset S^3$, in which each L_i is a planar unknot framed by coefficient $-a_i$, oriented so that consecutive components link once positively (Figure 1).

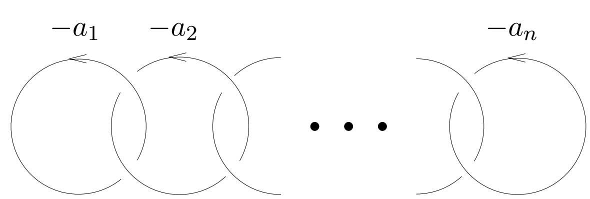

Figure 1. The oriented, framed link $\mathbb{L}$.

To describe the intersection pairing on $X(p,q)$, we make a definition.

Definition 2.1. The *linear lattice* $\Lambda(p,q)$ is the lattice freely generated by elements $x_1, \ldots, x_n$ with inner product given by

$$\langle x_i, x_j \rangle = \begin{cases} a_i, & \text{if } i = j; \\ -1, & \text{if } |i-j| = 1; \\ 0, & \text{if } |i-j| > 1. \end{cases}$$

A more detailed account about lattices (in particular, the justification for calling $\Lambda(p,q)$ a lattice) appears in Section 3. It follows at once that the inner product space $H_2(X(p,q), Q_X)$ equals *minus* $\Lambda(p,q)$; here and throughout, we take homology groups with integer coefficients. We note that $p/q' = [a_n, \ldots, a_1]^-$, where $0 < q' < p$ and $qq' \equiv 1 \pmod p$ (Lemma 9.4(4)). Thus, we obtain $\Lambda(p,q) \cong \Lambda(p,q')$ on the algebraic side and $L(p,q) \cong L(p,q')$ on the topological side (cf. Proposition 3.6).

Now suppose that negative integer surgery along a knot $K \subset L(p,q)$ produces S^3. Let W denote the associated 2-handle cobordism from $L(p,q)$ to S^3, capped off with a 4-handle. Orienting K produces a canonical generator $[\Sigma] \in H_2(-W)$ defined by the condition that $\langle [C], [\Sigma] \rangle = +1$, where C denotes the core of the 2-handle attachment. Form the closed, oriented, smooth, negative definite 4-manifold $Z = X(p,q) \cup -W$. By [23, Th. 3.3], it follows that $\Lambda(p,q)$ embeds in the orthogonal complement $(\sigma)^\perp \subset \mathbb{Z}^{n+1}$, where the changemaker σ corresponds to the class $[\Sigma]$. *A priori* σ could begin with a string of zeroes as in [23], but Theorem 1.6 rules this out and, moreover, shows that $\Lambda(p,q) \cong (\sigma)^\perp$. We establish Theorem 1.6 once we develop a bit more about lattices (cf. Section 3.4), and we make use of it in the remainder of this section.

We now focus on the issue of recovering the homology class $[K] \in H_1(L(p,q))$ from this embedding. Regard $\mathbb{L}$ as a surgery diagram for $L(p,q)$, and let $\mu_i, \lambda_i \subset \partial(nd(L_i))$ denote a meridian, Seifert-framed longitude pair for L_i, oriented so that $\mu_i \cdot \lambda_i = +1$. Let T_i denote the i^{th} surgery solid torus. The boundary of T_n is a Heegaard torus for $L(p,q)$; denote by a the core of T_n and by b the core of the complementary solid torus T_n'. We compute the self-linking number of b as $-q'/p \pmod 1$. (Cf. [40, §2], bearing in mind the opposite orientation convention in place there.) Thus, if $[K] = \pm k[b]$, then the self-linking number of K is $-k^2q'/p \pmod 1$. The condition that K has a negative integer homology sphere surgery amounts to the condition that $-k^2q' \equiv 1 \pmod p$ (ibid.), from which we derive $q \equiv -k^2 \pmod p$.

Define

$$x := \sum_{i=1}^{n} p_{i-1} x_i \in \Lambda(p,q), \tag{3}$$

where the values p_i are inductively defined by $p_{-1} = 0$, $p_0 = 1$, and $p_i = a_i p_{i-1} - p_{i-2}$ (cf. Definition 9.3 and Lemma 9.4(1)). We identify the elements x_i and x with their images under the embedding $\Lambda(p,q) \oplus (\sigma) \hookrightarrow \mathbb{Z}^{n+1}$. We denote by $\{e_0, \dots, e_n\}$ the orthonormal basis of $\mathbb{Z}^{n+1}$ with respect to which $\sigma = \sum_{i=0}^n \sigma_i e_i$.

Proposition 2.2. *Suppose that negative integer surgery along the oriented knot $K \subset L(p,q)$ produces S^3, let $\Lambda(p,q) \oplus (\sigma) \hookrightarrow \mathbb{Z}^{n+1}$ denote the corresponding embedding, and set $k = \langle e_0, x \rangle$. Then*

$$[K] = k\,[b] \in H_1(L(p,q)).$$

Proof. (I) We first express the homology class of a knot $\kappa \subset L(p,q)$ from the 3-dimensional point of view. To this end, we construct a compressing

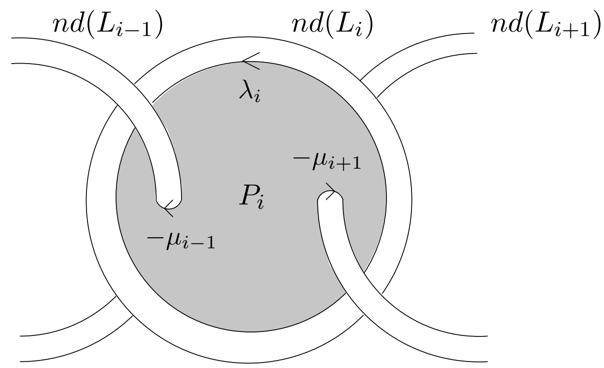

Figure 2. An oriented, twice-punctured disk P_i in $S^3 - nd(L)$, $0 < i < n$.

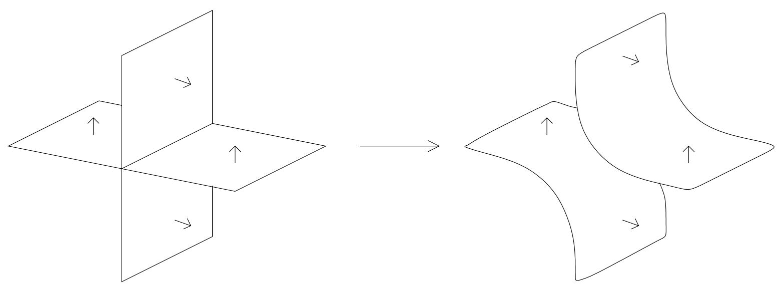

Figure 3. The oriented cut-and-paste of a pair of co-oriented, properly embedded surfaces nearby a transverse intersection.

disk $D \subset T'_n$ that is related to the class x. Let P_i denote an oriented, twice-punctured disk in $S^3 - nd(L)$ with $[\partial P_i] = [\lambda_i] - [\mu_{i-1}] - [\mu_{i+1}] \in H_1(\partial(nd(L_n)))$, as in Figure 2 (taking $\mu_0, \mu_{n+1} = \varnothing$). For $i = 0, \dots, n$, form p_{i-1} parallel, disjoint copies of P_i. We may arrange the surfaces so that any two in this collection intersect transversely and so that any three have empty intersection. Furthermore, each surface comes with a natural co-orientation induced by the given orientations of P_i and S^3. Thus, we may form the oriented cut-and-paste P of all these surfaces, as indicated by the local model in Figure 3.

We calculate

$$\begin{aligned}[\partial P] &= \sum_{i=1}^{n} p_{i-1}[\partial P_i] = \sum_{i=1}^{n} p_{i-1}([\lambda_i] - [\mu_{i-1}] - [\mu_{i+1}]) \\ &= \sum_{i=1}^{n-1} (p_{i-1}[\lambda_i] - (p_{i-2} + p_i)[\mu_i]) \ + \ (p_{n-1}[\lambda_n] - p_{n-2}[\mu_n]) \\ &= \sum_{i=1}^{n-1} p_{i-1}[\lambda_i - a_i \mu_i] \ + \ (p_{n-1}[\lambda_n] - p_{n-2}[\mu_n]).\end{aligned}$$

Let D_i denote a compressing disk for T_i. Since $[\partial D_i] = [\lambda_i - a_i\mu_i]$, it follows that we can form the union of P with p_{i-1} copies of $-D_i$ for $i = 1, \dots, n-1$ to produce a properly embedded, oriented surface $D \subset T'_n$. The boundary ∂D represents $p_{n-1}[\lambda_n] - p_{n-2}[\mu_n] \in H_1(\partial(nd(L_n)))$, and since $\gcd(p_{n-1}, p_{n-2}) = 1$, it follows that ∂D has a single component. Furthermore, a simple calculation shows that $\chi(D) = 1$. Therefore, D provides the desired compressing disk.

Since $b \cdot D = 1$, we calculate

$$[\kappa] = (\kappa \cdot D)[b] \in H_1(L(p,q)) \tag{4}$$

for an oriented knot $\kappa \subset L(p,q)$ supported in T'_n and transverse to D.

(II) Now we pull in the 4-dimensional point of view. Given $\alpha \in H_2(X, \partial X)$, represent the class $\partial_*\alpha$ by a knot $\kappa \subset L(p,q)$, isotop it into the complement of the surgery tori $T_1 \cup \cdots \cup T_n$, and regard it as knot in ∂D^4. Choose a Seifert surface for it, and push its interior slightly into D^4, producing a surface F.

Consider the Kirby diagram of X. We can represent the class x by the sphere $\mathcal{S}$ obtained by pushing the interior of P slightly into D^4, producing a surface P', and capping off $\partial P'$ with p_{i-1} copies of the core of handle attachment along L_i, for $i = 1, \dots, n$. It is clear that

$$\kappa \cdot D = \kappa \cdot P = F \cdot P' = F \cdot \mathcal{S} = \langle [F], x \rangle. \tag{5}$$

Since $\partial_*\alpha = \partial_*[F] = [\kappa]$, it follows that $\alpha - [F]$ represents an absolute class in $H_2(X)$. Since the pairing $H_2(X) \otimes H_2(X) \to \mathbb{Z}$ takes values in $|H_1(\partial X)| \cdot \mathbb{Z}$, it follows that $\langle \alpha, x \rangle \equiv \langle [F], x \rangle \pmod{p}$. Comparing with (4) and (5), we obtain

$$\partial_*\alpha = \langle \alpha, x \rangle [b]. \tag{6}$$

(III) At last we use the 2-handle cobordism W and the closed manifold Z. Given $\beta \in H_2(-W, -\partial W)$, write $\beta = n[C]$, where C denotes the core of the handle attachment along K. Since $\partial_*[C] = [K]$, it follows that

$$\partial_*\beta = \langle \beta, [\Sigma] \rangle [K]. \tag{7}$$

Finally, consider the commutative diagram

$$\begin{array}{ccccc} & & H_2(Z, -W) \xrightarrow[\text{exc.}]{\sim} H_2(X, \partial X) & & \\ & \nearrow & & \searrow{\scriptstyle \partial_*} & \\ H_2(Z) & & & & H_1(L(p,q)). \\ & \searrow & & \nearrow{\scriptstyle \partial_*} & \\ & & H_2(Z, X) \xrightarrow[\text{exc.}]{\sim} H_2(-W, -\partial W) & & \end{array}$$

Proceeding along the top, the image of a class $\gamma \in H_2(Z)$ in $H_1(L(p,q))$ is given by $\langle \gamma, x \rangle [b]$ according to (6). Similarly, proceeding along the bottom, its image in $H_1(L(p,q))$ is given by $\langle \gamma, \sigma \rangle [K]$ according to (7), switching to the use of σ for $[\Sigma]$. Thus, taking $\gamma = e_0$, we have

$$k[b] = \langle e_0, x \rangle [b] = \langle e_0, \sigma \rangle [K] = [K],$$

using the fact that $\sigma_0 = 1$. This completes the proof of the proposition. $\square$

Thus, for an *unoriented* knot $K \subset L(p,q)$, we obtain a pair of values $\pm k \pmod p$ that specify a pair of homology classes in $H_1(L(p,q))$, one for each orientation on K. Note that had we used the reversed basis $\{x_n, \ldots, x_1\}$, we would have expressed $[K]$ as a multiple $k'[a] \in H_1(L(p,q))$. Since $[a] = q[b]$, we obtain $k' \equiv kq' \equiv -k^{-1} \pmod p$, which is consistent with $q' \equiv -(k')^2 \pmod p$ and $L(p,q') \cong L(p,q)$. Thus, given a value $k \pmod p$, we represent equivalent (unoriented) knots by choosing any of the values $\{\pm k, \pm k^{-1}\} \pmod p$. For the latter Berge types listed in Section 1.2, we use a judicious choice of k. For example, Berge types IX and X involve a concise quadratic expression for p in terms of k, but there does not exist such a nice expression for it in terms of the least positive residue of $-k$ or $k^{-1} \pmod p$.

3. **Lattices**

3.1. *Generalities.* A *lattice* L consists of a finitely-generated free abelian group equipped with a positive-definite, symmetric bilinear pairing $\langle \; , \; \rangle : L \times L \to \mathbb{R}$. It is *integral* if the image of its pairing lies in $\mathbb{Z}$. In this case, its *dual lattice* is the lattice

$$L^* := \{x \in L \otimes \mathbb{R} \mid \langle x, y \rangle \in \mathbb{Z} \text{ for all } y \in L\},$$

and its *discriminant* $\mathrm{disc}(L)$ is the index $[L^* : L]$. All lattices will be assumed integral henceforth.

Given a vector $v \in L$, its *norm* is the value $|v| := \langle v, v \rangle$. It is *reducible* if $v = x + y$ for some nonzero $x, y \in L$ with $\langle x, y \rangle \geq 0$, and *irreducible* otherwise. It is *breakable* if $v = x + y$ for some $x, y \in L$ with $|x|, |y| \geq 3$ and $\langle x, y \rangle = -1$, and *unbreakable* otherwise. A lattice L is *decomposable* if it is an orthogonal direct sum $L = L_1 \oplus L_2$ with $L_1, L_2 \neq (0)$, and *indecomposable* otherwise.

Suppose that a lattice L has a basis $S = \{v_1, \ldots, v_n\}$ of irreducible vectors. We then define the *pairing graph* of S by

$$\widehat{G}(S) = (S, E), \quad E = \{(v_i, v_j) \mid i \neq j \text{ and } \langle v_i, v_j \rangle \neq 0\}.^{3}$$

Let G_k denote a connected component of $\widehat{G}(S)$ and $L_k \subset L$ the sublattice spanned by $V(G_k)$. If $L_k = L' \oplus L''$, then each vector in $V(G_k)$ must belong to one of L' or L'' by irreducibility. Since G_k is connected, it follows that they must all belong to the same summand, whence L_k is indecomposable.

[3] More on graph notation in Section 3.2.

A basic result, due to Eichler, asserts that this decomposition $L \cong \bigoplus_k L_k$ into indecomposable summands is unique up to reordering of its factors [34, Th. II.6.4].

3.2. *Graph lattices.* Let $G = (V, E)$ denote a finite, loopless, undirected graph. Write $v \sim w$ to denote $(v, w) \in E$. A *subgraph* of G takes the form $H = (V', E')$, where $V' \subset V$ and $E' \subset \{(v, w) \in E \mid v, w \in V'\}$; it is *induced* if "=" holds in place of "⊂," in which case we write $H = G|V'$. For a pair of disjoint subsets $T, T' \subset V$, write $E(T, T')$ for the set of edges between T and T', $e(T, T')$ for its cardinality, and set $d(T) = e(T, V - T)$. In particular, the *degree* of a vertex $v \in V$ is the value $d(v)$.

Form the abelian group $\overline{\Gamma}(G)$ freely generated by elements $[v]$, $v \in V$, and define a symmetric, bilinear pairing by

$$\langle [v], [w] \rangle = \begin{cases} d(v), & \text{if } v = w; \\ -e(v, w), & \text{if } v \neq w. \end{cases}$$

Let

$$[T] := \sum_{v \in T} [v],$$

and note that

$$\langle [T], [T'] \rangle = e(T \cap T', V - (T \cup T')) - e(T - T', T' - T).$$

In particular, $\langle [T], [T] \rangle = d(T)$, and $\langle [T], [T'] \rangle = -e(T, T')$ for disjoint T, T'.

Given $x \in \overline{\Gamma}(G)$, write $x = \sum_{v \in V} x_v [v]$. Observe that $|x| = \sum_{e \in E} (x_v - x_w)^2$, where v and w denote the endpoints of the edge e. It follows that $|x| \geq 0$, so the pairing on $\overline{\Gamma}(G)$ is positive semi-definite. Let $V_1, \dots, V_k$ denote the vertex sets of the connected components of G. It easy to see that $|x| = 0$ if and only if x belongs to the span of $[V_1], \dots, [V_k]$ and, moreover, that these elements generate $Z(G) := \{x \in \overline{\Gamma}(G) \mid \langle x, y \rangle = 0 \text{ for all } y \in \overline{\Gamma}(G)\}$. It follows that the quotient $\Gamma(G) := \overline{\Gamma}(G)/Z(G)$ is a lattice.

Definition 3.1. The *graph lattice* associated to G is the lattice $\Gamma(G)$.

Now assume that G is connected. For a choice of *root* $r \in V$, every element in $\overline{\Gamma}(G)$ is equivalent (mod $Z(G)$) to a unique element in the subspace of $\overline{\Gamma}(G)$ spanned by the set $\{[v] \mid v \in V - r\}$. In what follows, we keep a choice of root fixed and identify $\Gamma(G)$ with this subspace. We reserve the notation $[T]$ for $T \subset V - r$.

Definition 3.2. The set $\{[v] \mid v \in V - r\}$ constitutes a *vertex basis* for $\Gamma(G)$.

PROPOSITION 3.3. *The irreducible elements of* $\Gamma(G)$ *take the form* $\pm[T]$, *where* T *and* $V - T$ *induce connected subgraphs of* G.

Proof. Suppose that $0 \neq x = \sum_{v \in V-r} c_v[v] \in \Gamma(G)$ is irreducible. Replacing x by $-x$ if necessary, we may assume that $c := \max_v c_v \geq 1$. Let $T = \{v \mid c_v = c\}$; then

$$\begin{aligned}\langle [T], x-[T]\rangle &= \langle [T], (c-1)[T]\rangle + \langle [T], \sum_{v\in V-T} c_v[v]\rangle \\ &= (c-1)\cdot d(T) - \sum_{v\in V-T} c_v \cdot e(v,T) \\ &= \sum_{v\in V-T} (c-1-c_v)\cdot e(v,T) \geq 0.\end{aligned}$$

Since x is irreducible, it follows that $x = [T]$.

Next, we argue that $[T]$ is irreducible if and only if the induced subgraphs $G|T$ and $G|(V-T)$ are connected. Write $y = \sum_{v\in V} y_v[v] \in \overline{\Gamma}(G)$. Then

$$(8)\quad \langle y, y-[T]\rangle = \sum_C \sum_{(u,v)\in E(C)} (y_u - y_v)^2 + \sum_{(u,v)\in E(T,V-T)} (y_u-y_v)(y_u-y_v-1),$$

where C ranges over the connected components of $G|T$ and $G|(V-T)$. Each summand appearing in (8) is nonnegative. It follows that (8) vanishes identically if and only if (a) y_u is constant on each component C and (b) if a component $C_1 \subset G|T$ has an edge (u,v) to a component $C_2 \subset G|(V-T)$, then $y_u = y_v$ or $y_v + 1$. Now pass to the quotient $\Gamma(G)$. This has the effect of setting $y_r = 0$ in (8). If $G|(V-T)$ is disconnected, then we can choose a component C such that $r \notin V(C)$ and set $y_u = -1$ for all $u \in V(C)$ and 0 otherwise. Then y and $[T]-y$ are nonzero, orthogonal, and sum to $[T]$, so $[T]$ is reducible. Similarly, if $G|T$ is disconnected, then we can choose an arbitrary component C and set $y_u = 1$ if $u \in V(C)$ and 0 otherwise, and conclude once more that $[T]$ is reducible. Otherwise, both $G|T$ and $G|(V-T)$ are connected, and y vanishes on $G|(V-T)$ and equals 0 or 1 on $G|T$. Thus, $y = 0$ or $[T]$, and it follows that $[T]$ is irreducible. □

PROPOSITION 3.4. *Suppose that G does not contain a cut-edge, and suppose that $[T] = y + z$ with $\langle y, z\rangle = -1$. Then either*

(1) *$G|T$ contains a cut-edge e, $V(G|T - e) = T_1 \cup T_2$, and $\{y,z\} = \{[T_1],[T_2]\}$; or*
(2) *$G(V-T)$ contains a cut-edge e, $V(G|(V-T)-e) = T_1 \cup T_2$, $r \in T_2$, and $\{y,z\} = \{[T_1 \cup T], -[T_1]\}$.*

Proof. Reconsider (8). In the case at hand, the inner product is 1. Each term $(y_u - y_v)(y_u - y_v - 1)$ is either 0 or ≥ 2, so it must be the case that each such term vanishes and there exists a unique edge $e \in E(T) \cup E(V-T)$, $e = (u,v)$, for which $(y_u - y_v)^2 = 1$ and all other terms vanish. In particular, it follows that e is a cut-edge in either (a) $G|T$ or (b) $G|(V-T)$.

In case (a), write T_1 and T_2 for the vertex sets of the components of $G|T - e$. Then y is constant on T_1, T_2, and $V - T$; furthermore, it vanishes on $V - T$ and its values on T_1 and T_2 differ by one. Since e is not a cut-edge in G, it follows that $E(V - T, T_1), E(V - T, T_2) \neq \varnothing$, so the values on T_1 and T_2 differ from the value on $V - T$ by at most one. It follows that these values are 0 and 1 in some order. This results in (1).

In case (b), write T_1 and T_2 for the vertex sets of the components of $G|(V-T) - e$ with $r \in T_2$. Now y is constant on T_1, T_2, and T; furthermore, it vanishes on T_2 and its values on T_1 and T_2 differ by one. Hence the value on T_2 is 1 or -1. Since e is not a cut-edge in G, it follows that $E(T, T_1), E(T, T_2) \neq \varnothing$, so the value on T is 0 or 1 more than the values on T_1 and T_2. Thus, either the value on $T_1 \cup T$ is 1, or the value on T_1 is -1 and the value on T is 0. This results in (2). □

3.3. *Linear lattices.* Observe that a sum of linear lattices $L = \bigoplus_k L_k$ occurs as a special case of a graph lattice. Indeed, construct a graph G whose vertex set consists of one vertex for each generator x_i of L_k (Definition 2.1), as well as one additional vertex r. For a pair of generators x_i, x_j, declare $(x_i, x_j) \in E$ if and only if $\langle x_i, x_j \rangle = -1$, and define as many parallel edges between r and x_i as necessary so that $d(x_i) = a_i$. It is clear that $\Gamma(G) \cong L$, and this justifies the term linear *lattice*. Furthermore, the x_i comprise a vertex basis for L.

Given a linear lattice L and a subset of consecutive integers $\{i, \ldots, j\} \subset \{1, \ldots, n\}$, we obtain an *interval* $\{x_i, \ldots, x_j\}$. Two distinct intervals $T = \{x_i, \ldots, x_j\}$ and $T' = \{x_k, \ldots, x_l\}$ *share a common endpoint* if $i = k$ or $j = l$ and are *distant* if $k > j + 1$ or $i > l + 1$. If T and T' share a common endpoint and $T \subset T'$, then write $T \prec T'$. If $i = l + 1$ or $k = j + 1$, then T and T' are *consecutive* and write $T \dagger T'$. They *abut* if they are either consecutive or share a common endpoint. Write $T \pitchfork T'$ if $T \cap T' \neq \varnothing$ and T and T' do not share a common endpoint. Observe that if $T \pitchfork T'$, then the symmetric difference $(T - T') \cup (T' - T)$ is the union of a pair of distant intervals.

Corollary 3.5. *Let $L = \bigoplus_k L_k$ denote a sum of linear lattices.*

1. *The irreducible vectors in L take the form $\pm[T]$, where T is an interval in some L_k;*
2. *each L_k is indecomposable;*
3. *if $T \pitchfork T'$, then $[T - T'] \pm [T' - T]$ is reducible;*
4. *$[T]$ is unbreakable if and only if T contains at most one vertex of degree ≥ 3.*

Proof. (1) follows from Proposition 3.3, noting that for $T \subset V - \{r\}$, if T and $V - T$ induce connected subgraphs of the graph G corresponding to L, then T is an interval in some L_k. (2) follows since the elements of the vertex basis for L_k are irreducible and their pairing graph is connected. For item (3), write $(T - T') \cup (T' - T) = T_1 \cup T_2$ as a union of distant intervals. Then $[T - T'] \pm [T' - T] = \varepsilon_1[T_1] + \varepsilon_2[T_2]$ for suitable signs $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$, and $\langle \varepsilon_1[T_1], \varepsilon_2[T_2] \rangle = 0$. For (4), we establish the contrapositive in two steps.

($\Longrightarrow$) If an interval T contains a pair of vertices x_i, x_j of degree ≥ 3, then it breaks into consecutive intervals $T = T_i \cup T_j$ with $x_i \in T_i$ and $x_j \in T_j$. It follows that $[T]$ is breakable, since $[T] = [T_i] + [T_j]$ with $\langle [T_i], [T_j] \rangle = -1$ and $d(T_i), d(T_j) \geq 3$.

($\Longleftarrow$) If $[T] = y + z$ is breakable, then Proposition 3.4 applies. Observe that case (2) does not hold since breakability entails $|z| \geq 3$, while a cut-edge in $G(V - T)$ separates it into $T_1 \cup T_2$ with $d(T_1) = 2$ and $r \in T_2$. Thus, case (1) holds, and it follows that $d(T_1), d(T_2) \geq 3$, so both T_1 and T_2 contain a vertex of degree ≥ 3, which shows that T contains at least two such. □

Next we turn to the question of when two linear lattices are isomorphic. Let I denote the set of irreducible elements in L, and given $y \in I$, let

$$I(y) = \{z \in I \mid \langle y, z \rangle = -1, y + z \in I\}.$$

To unpack the meaning of this definition, suppose that $y \in I$, $|y| \geq 3$, and write $y = \varepsilon_y[T_y]$. If $z \in I(y)$ with $z = \varepsilon_z[T_z]$, then either $\varepsilon_y = \varepsilon_z$ and $T_y \dagger T_z$, or else $|z| = 2$, $\varepsilon_y = -\varepsilon_z$, and $T_z \prec T_y$. Now suppose that $z \in I(y)$ with $|z| = 3$. Choose elements $x_i \in T_y$ and $x_j \in T_z$ of norm ≥ 3 so that the open interval (x_i, x_j) contains no vertex of degree ≥ 3. If $w \in I(y) \cap (-I(z))$ with $w = \varepsilon_w[T_w]$, then $T_w \subset (x_i, x_j)$, and either $T_w \prec T_y$ and $\varepsilon_w = -\varepsilon_y$, or else $T_w \prec T_z$ and $\varepsilon_w = \varepsilon_z$. It follows that $|I(y) \cap (-I(z))| = |(x_i, x_j)| = |i - j| - 1$.

The following is the main result of [19]. An alternative proof follows from the results of [24] (notably Theorems 1.1, 1.2, and Proposition 4.6 there).

Proposition 3.6 (Gerstein). *If $\Lambda(p, q) \cong \Lambda(p', q')$, then $p = p'$, and $q = q'$ or $qq' \equiv 1 (\text{mod } p)$.*

Proof. Let L denote a linear lattice with standard basis $S = \{x_1, \ldots, x_n\}$. The proposition follows once we show that L uniquely determines the sequence of norms $\mathbf{x} = (|x_1|, \ldots, |x_n|)$ up to reversal, noting that if $p/q = [a_1, \ldots, a_n]^-$ and $p/q' = [a_n, \ldots, a_1]^-$, then $qq' \equiv 1 \pmod{p}$ (Lemma 9.4(4)).

Suppose that I contains an element of norm ≥ 3. In this case, select $y_1 \in I$ with minimal norm ≥ 3 subject to the condition that there does not exist a pair of orthogonal elements in $I(y_1)$. It follows that $y_1 = \varepsilon[T_1]$, where T_1 contains exactly one element $x_{j_1} \in S$ of norm ≥ 3, and j_1 is the smallest or largest index of an element in S with norm ≥ 3. Inductively select $y_i \in I(y_{i-1})$ with minimal norm ≥ 3 subject to the condition that $\langle y_i, y_j \rangle = 0$ for all $j < i$, until it is no longer possible to do so, terminating in some element y_k. It follows that $y_i = \varepsilon[T_i]$ for all i, where $\varepsilon \in \{\pm 1\}$ is independent of i; each T_i contains a unique $x_{j_i} \in S$ of norm ≥ 3; $T_i \dagger T_{i+1}$ for $i < k$; and each $x_j \in S$ of norm ≥ 3 occurs as some x_{j_i}. Therefore, up to reversal, the (possibly empty) sequence $(|y_1|, \ldots, |y_k|) = (|x_{j_1}|, \ldots, |x_{j_k}|)$ coincides with $\mathbf{x}$ with every occurrence of 2 omitted.

To recover $\mathbf{x}$ completely, assume for notational convenience that $j_1 < \cdots < j_k$. Set $n_i := |I(y_i) \cap (-I(y_{i+1}))|$ for $i = 1, \ldots, k-1$, so that $n_i = j_{i+1} - j_i - 1$. If $k \geq 2$, then set $n_0 = |I(y_1) - (-I(y_2))$, $n_k = |I(y_k) - (-I(y_{k-1}))|$, and observe that $n_0 = j_1 - 1$ and $n_k = n - j_k$. If $k = 1$, then decompose $I(y) = I_0 \cup I_1$, where $\langle z_i, z_i' \rangle \neq 0$ for all $z_i, z_i' \in I_i$, $i = 0, 1$. In this case, set $n_i = |I_i|$, and observe that $\{n_0, n_1\} = \{j_1 - 1, n - j_1\}$. Lastly, if $k = 0$, then set $n_0 = n$. Letting $2^{[t]}$ denote the sequence of 2's of length t, it follows that $\mathbf{x} = (2^{[n_0]}, |y_1|, 2^{[n_1]}, \ldots, 2^{[n_{k-1}]}, |y_k|, 2^{[n_k]})$.

Since the elements $y_1, \ldots, y_k$ and the values $n_0, \ldots, n_k$ depend solely on L for their definition, it follows that $\mathbf{x}$ is determined uniquely up to reversal, and the proposition follows. □

The following definition and lemma anticipate our discussion of the intersection graph in Section 4.2 (especially Lemma 4.11).

Definition 3.7. Given a collection of intervals $\mathcal{T} = \{T_1, \ldots, T_k\}$ whose classes are linearly independent, define a graph

$$G(\mathcal{T}) = (\mathcal{T}, \mathcal{E}), \quad \mathcal{E} = \{(T_i, T_j) \mid T_i \text{ abuts } T_j\}.$$

LEMMA 3.8. *Given a cycle $C \subset G(\mathcal{T})$, the intervals in $V(C)$ abut pairwise at a common end. That is, there exists an index j such that each $T_i \in V(C)$ has left endpoint x_{j+1} or right endpoint x_j. In particular, $V(C)$ induces a complete subgraph of $G(\mathcal{T})$.*

Proof. Relabeling as necessary, write $V(C) = \{T_1, \ldots, T_k\}$, where $(T_i, T_{i+1}) \in E(C)$ for $i = 1, \ldots, k$, subscripts (mod k). We proceed by induction on the number of edges $n \geq k$ in the subgraph induced on $V(C)$.

When $n = k$, C is an induced cycle. In this case, if three of the intervals abut at a common end, then they span a cycle, $k = 3$, and we are done. If not, then define a sign $\varepsilon_i = \pm 1$ by the rule that $\varepsilon_i = 1$ if and only if $T_i \dagger T_{i-1}$ and T_i lies to the right of T_{i-1}, or if T_i and T_{i-1} share a common left endpoint. Fix a vertex x_j, suppose that $x_j \in T_i$ for some i, and choose the next index l (mod k) for which $x_j \in T_l$. Observe, crucially, that $\varepsilon_i = -\varepsilon_l$. It follows that $\langle x_j, \sum_{i=1}^k \varepsilon_i [T_i] \rangle = 0$. As x_j was arbitrary, we obtain the linear dependence $\sum_{i=1}^k \varepsilon_i [T_i] = 0$, a contradiction. It follows that if $n = k$, then $k = 3$ and the three intervals abut at a common end.

Now suppose that $n > k$. Thus, there exists an edge $(T_i, T_j) \in E(C)$ for some pair of nonconsecutive indices i, j (mod k). Split C into two cycles C_1

and C_2 along (T_i, T_j). By induction, every interval in $V(C_1)$ and $V(C_2)$ abuts at the same end as T_i and T_j, so the same follows at once for $V(C)$. □

3.4. *Changemaker lattices.* Fix an orthonormal basis $\{e_0, \ldots, e_n\}$ for $\mathbb{Z}^{n+1}$.

LEMMA 3.9. *Suppose that* $L = (\sigma)^\perp \subset \mathbb{Z}^{n+1}$ *and* $\sigma_0 := \langle e_0, \sigma\rangle = 1$. *Then* $\mathrm{disc}(L) = |\sigma|$.

Proof. (cf. [34, proof of Lemma II.1.6]) Consider the map

$$\varphi : \mathbb{Z}^{n+1} \to \mathbb{Z}/|\sigma|\mathbb{Z}, \quad \varphi(x) = \langle x, \sigma\rangle \pmod{|\sigma|}.$$

As $\sigma_0 = 1$, the map φ is onto, so $K := \ker(\varphi)$ has discriminant $[\mathbb{Z}^{n+1} : K]^2 = |\sigma|^2$. On the other hand, $K = L \oplus (\sigma)$, so $\mathrm{disc}(L) = \mathrm{disc}(K)/|\sigma| = |\sigma|$. □

Proof of Theorem 1.6. We invoke [23, Th. 3.3] with $X = X(p, q)$. It follows that $\Lambda(p, q)$ embeds as a full-rank sublattice of $(\sigma)^\perp \subset \mathbb{Z}^{n+1}$, where $|\sigma| = p$ and σ is a changemaker according to the convention of [23, Lemma 3.2], which relaxes the condition that $\sigma_0 = 1$ to $\sigma_0 \geq 0$. It stands to verify that $\sigma_0 = 1$, so that σ is a changemaker according to the present convention, and furthermore that $\Lambda(p, q)$ actually equals $(\sigma)^\perp$ on the nose. First, if $\sigma_0 = 0$, then $\Lambda(p, q)$ would have a direct summand isomorphic to $(e_0) \cong \mathbb{Z}$, in contradiction to its indecomposability (Corollary 3.5(2)). Hence $\sigma_0 = 1$. Second, $\mathrm{disc}(\Lambda(p, q)) = p = |\sigma| = \mathrm{disc}((\sigma)^\perp)$, using Lemma 3.9 at the last step. Since $\mathrm{rk}\, \Lambda(p, q) = \mathrm{rk}\, (\sigma)^\perp$, the two lattices coincide. □

Definition 3.10. A *changemaker lattice* is any lattice isomorphic to $(\sigma)^\perp \subset \mathbb{Z}^{n+1}$ for some changemaker σ (Definition 1.5).

We construct a basis for a changemaker lattice L as follows. Fix an index $1 \leq j \leq n$, and suppose that $\sigma_j = 1 + \sum_{i=0}^{j-1} \sigma_i$. In this case, set $v_j = -e_j + 2e_0 + \sum_{i=1}^{j-1} e_i \in L$. Otherwise, $\sigma_j \leq \sum_{i=0}^{j-1} \sigma_i$. It follows that there exists a subset $A \subset \{0, \ldots, j-1\}$ such that $\sigma_j = \sum_{i \in A} \sigma_i$. Amongst all such subsets, choose the one maximal with respect to the total order $<$ on subsets of $\{0, 1, \ldots, n\}$ defined by declaring $A' < A$ if the largest element in $(A \cup A') \setminus (A \cap A')$ lies in A; equivalently, $\sum_{i \in A'} 2^i < \sum_{i \in A} 2^i$. Then set $v_j = -e_j + \sum_{i \in A} e_i \in L$. If $v = -e_j + \sum_{i \in A'} e_i$ for some $A' < A$, then write $v \ll v_j$.

The vectors $v_1, \ldots, v_n$ are clearly linearly independent. The fact that they span L is straightforward to verify, too: given $w \in L$, add suitable multiples of $v_n, \ldots, v_1$ to w in turn to produce a sequence of vectors with *support* decreasing to $\varnothing$. Recall that the support of a vector $v \in \mathbb{Z}^{n+1}$ is the set $\mathrm{supp}(v) = \{i \mid \langle v, e_i\rangle \neq 0\}$. For future reference, we also define

$$\mathrm{supp}^+(v) = \{i \mid \langle v, e_i\rangle > 0\} \quad \text{and} \quad \mathrm{supp}^-(v) = \{i \mid \langle v, e_i\rangle < 0\}.$$

Definition 3.11. The set $S = \{v_1, \dots, v_n\}$ constitutes the *standard basis* for L. A vector $v_j \in S$ is

- *tight*, if $v_j = -e_j + 2e_0 + \sum_{i=1}^{j-1} e_i$;
- *gappy*, if $v_j = -e_j + \sum_{i \in A} e_i$ and A does not consist of consecutive integers; and
- *just right*, if $v_j = -e_j + \sum_{i \in A} e_i$ and A consists of consecutive integers.

A *gappy index* for a gappy vector v_j is an index $k \in A$ such that $k+1 \notin A \cup \{j\}$.

Thus, every element of S belongs to exactly one of these three types.

We record a few basic observations before proceeding to some more substantial facts about changemaker lattices. Write $v_{jk} = \langle v_j, e_k \rangle$.

Lemma 3.12. *The following hold*:

(1) $v_{jj} = -1$ *for all* j, *and* $v_{j,j-1} = 1$ *unless* $j = 1$ *and* $v_{1,0} = 2$;
(2) *for any pair* v_i, v_j, *we have* $\langle v_i, v_j \rangle \geq -1$;
(3) *if* k *is a gappy index for some* v_j, *then* $|v_{k+1}| \geq 3$;
(4) *given* $z = \sum_{i=0}^{n} z_i e_i \in L$ *with* $|z| \geq 3$, $\mathrm{supp}^-(z) = \{j\}$, *and* $z_j = -1$, *it follows that* $j = \max(\mathrm{supp}(z))$.

Proof. (1) is clear, using the maximality of A for the second part. (2) is also clear. (3) follows from maximality, as otherwise $v_j \ll v_j - v_{k+1}$. For (4), suppose not, and select $k > j$ for which $z_k > 0$. We obtain the contradiction $0 = \langle z, \sigma \rangle > \sigma_k - \sigma_j \geq 0$, where the inequality is strict because $|z| \geq 3$. □

Lemma 3.13. *The standard basis elements of a changemaker lattice are irreducible.*

Proof. Choose a standard basis element $v_j \in S$, and suppose that $v_j = x + y$ for $x, y \in L$ with $\langle x, y \rangle \geq 0$. In order to prove that v_j is irreducible, it stands to show that one of x and y equals 0. Write $x = \sum_{i=0}^{n} x_i e_i$ and $y = \sum_{i=0}^{n} y_i e_i$.

Case 1. v_j *is not tight.* In this case, $|v_{ji}| \leq 1$ for all i. We claim that $x_i y_i = 0$ for all i. For suppose not. Since $\langle x, y \rangle \geq 0$, there exists an index i so that $x_i y_i > 0$. Then $|v_{ji}| = |x_i + y_i| \geq 2$, a contradiction. Since all but one coordinate of v_j is nonnegative, it follows that one of x and y has all its coordinates nonnegative. But the only such element in L is 0. It follows that v_j is irreducible. Notice that this same argument applies to any vector of the form $-e_j + \sum_{i \in A} e_i$.

Case 2. v_j *is tight.* We repeat the previous argument up to the point of locating an index i such that $x_i y_i > 0$. Now, however, we conclude that $x_0 = y_0 = 1$, and $x_i y_i \leq 0$ for all other indices i. In particular, there is at most one index k for which $x_k y_k = -1$, and $x_i y_i = 0$ for $i \neq 0, k$. If there is no such k, then we conclude as above that one of x and y has all its coordinates nonnegative, and v_j is irreducible as before. Otherwise, we may assume that $x_k = 1$. Then $\mathrm{supp}^-(y) = \{k\}$ and $\mathrm{supp}^-(x) = \{j\}$. Since $x_i + y_i = v_{ji} \neq 0$ for all $i \leq j$, it follows that $k > j$. But then $|x| \geq 3$ and $k = \max(\mathrm{supp}(x)) > j$, in contradiction to Lemma 3.12(4). Again it follows that v_j is irreducible. □

We collect a few more useful cases of irreducibility (cf. Lemma 4.2).

LEMMA 3.14. *Suppose that $v_t \in L$ is tight.*

(1) *If v_j is tight, $j \neq t$, then $v_j - v_t$ is irreducible.*
(2) *If $v_j = -e_j + e_{j-1} + e_t$, $j > t$, then $v_j + v_t$ is irreducible.*
(3) *If $v_{t+1} = -e_{t+1} + e_t + \cdots + e_0$, then $v_{t+1} - v_t$ is irreducible.*

Proof. In each case, we assume that the vector in question is expressed as a sum of nonzero vectors x and y with $\langle x, y\rangle \geq 0$. Recall that both x and y have entries of both signs.

(1) Assume without loss of generality that $j > t$. Thus, $v_j - v_t = -e_j + 2e_t + \sum_{i=t+1}^{j-1} e_i$, where the summation could be empty. As in the proof of Lemma 3.13, it quickly follows that $x_t = y_t = 1$, $x_k y_k = -1$ for some k, and otherwise $x_i y_i = 0$. Without loss of generality, $x_k = 1$. Thus, $\mathrm{supp}^-(x) = \{j\}$ and $\mathrm{supp}^-(y) = \{k\}$. By Lemma 3.12(4), it follows that $j = \max(\mathrm{supp}(x)) > k$. As $x_k + y_k = 0$, it follows that $k < t$. Now $0 = \langle y, \sigma\rangle \geq \sigma_t - \sigma_k > 0$, a contradiction. Therefore, $v_j - v_t$ is irreducible.

(2) It follows that $x_0 = y_0 = 1$, $x_k = -y_k = \pm 1$ for some value $k \geq t$, and otherwise $x_i y_i = 0$. Without loss of generality, say $x_k = 1$. Thus, $\mathrm{supp}^-(x) = \{j\}$ and $\mathrm{supp}^-(y) = \{k\}$. By Lemma 3.12(4), $j = \max(\mathrm{supp}(x))$. In particular, it follows that $k < j$. Another application of Lemma 3.12(4) implies that $k = \max(\mathrm{supp}(y))$. It follows that $y = -e_k + \sum_{i\in A} e_i$ for some $A \subset \{0, \ldots, t-1\}$. But then $0 = \langle y, \sigma\rangle = -\sigma_k + \sum_{i\in A} \sigma_i < -\sigma_k + 1 + \sum_{i=0}^{t-1} \sigma_i = -\sigma_k + \sigma_t \leq 0$, a contradiction. Therefore, $v_j + v_t$ is irreducible.

(3) We have $v_{t+1} - v_t = -e_{t+1} + 2e_t - e_0$. It follows that $x_t = y_t = 1$. If $x_i y_i = 0$ for every other index i, then $\{x, y\} = \{-e_{t+1} + e_t, e_t - e_0\}$, but $\langle e_t - e_0, \sigma\rangle = \sigma_t - \sigma_0 > 0$, a contradiction. It follows that there is a unique index k for which $x_k y_k = -1$, and otherwise $x_i y_i = 0$. Without loss of generality, say $x_k = 1$. Hence $y = e_t - \sum_{i\in A} e_i$ for some $A \subset \{0, k, t+1\}$. But A cannot contain an index $> t$, for then $0 = \langle y, \sigma\rangle \leq \sigma_t - \sigma_{t+1} < 0$, nor can it just contain indices $< t$, for then $0 = \langle y, \sigma\rangle \geq \sigma_t - \sum_{i=0}^{t-1} \sigma_i = 1$. Therefore, $v_{t+1} - v_t$ is irreducible. □

LEMMA 3.15. *If $v_j \in S$ is not tight, then it is unbreakable.*

Proof. Suppose that v_j is breakable and choose x and y accordingly. From the conditions that $\langle x, y\rangle = -1$ and $v_{j0} \neq 2$, it follows that $x_k y_k = -1$ for a single index k, and otherwise $x_i y_i = 0$. Without loss of generality, say $x_k = -1$. Then $\mathrm{supp}^-(x) = \{k\}$ and $\mathrm{supp}^-(y) = \{j\}$. By Lemma 3.12(4), it follows that $k = \max(\mathrm{supp}(x))$ and $j = \max(\mathrm{supp}(y))$. In particular, it follows that $k < j$, and that $y_i = v_{ji}$ for all $i > j$. On the other hand, $j \in \mathrm{supp}^+(y) - \mathrm{supp}^+(v_i)$. It follows that $v_j \ll y$, a contradiction. Hence v_i is unbreakable, as claimed. □

4. **Comparing linear lattices and changemaker lattices**

In this section we collect some preparatory results concerning when a changemaker lattice is isomorphic to a sum of one or more linear lattices. Thus, for the entirety of this section, let L denote a changemaker lattice with standard basis $S = \{v_1, \ldots, v_n\}$, and suppose that L is isomorphic to a linear lattice or a sum thereof. By Corollary 3.5 and Lemma 3.13, it follows that $v_i = \varepsilon_i[T_i]$ for some sign $\varepsilon_i = \pm 1$ and interval T_i. Let $\mathcal{T} = \{T_1, \ldots, T_n\}$. If v_i is not tight, then Corollary 3.5 and Lemma 3.15 imply that T_i contains at most one vertex of degree ≥ 3. If $[T_i]$ is unbreakable and $d(T_i) \geq 3$, then let z_i denote its unique vertex of degree ≥ 3.

4.1. *Standard basis elements and intervals.* Tight vectors, especially breakable ones, play an involved role in the analysis (Section 8). We begin with some basic observations about them.

LEMMA 4.1. *Suppose that v_t is tight, $j \neq t$, and $|v_j| \geq 3$. Then $\langle v_t, v_j \rangle$ equals*

(1) $|v_j| - 1$, *if and only if $T_j \prec T_t$;*
(2) $|v_j| - 2$, *if and only if $z_j \in T_t$ and $T_j \pitchfork T_t$, or $|v_j| = 3, T_j \dagger T_t$, and $\varepsilon_j \neq \varepsilon_t$;*
(3) $\varepsilon \in \{\pm 1\}$, *if and only if $T_j \dagger T_t$ and $\varepsilon_j \varepsilon_t \neq \varepsilon$, or $|v_j| = 3$, $z_j \in T_t$, $T_j \pitchfork T_t$, and $\varepsilon_j \varepsilon_t = \varepsilon$; or*
(4) 0, *if and only if $z_j \notin T_t$ and either T_j and T_t are distant or $T_j \pitchfork T_t$.*

If $|v_j| = 2$, then $|\langle v_t, v_j \rangle| \leq 1$, with equality if and only if T_t and T_j abut.

Proof sketch. Observe that $-1 \leq \langle v_i, v_j \rangle \leq |v_j| - 1$ for any pair of distinct i, j. Assuming that $|v_j| \geq 3$, the result follows by using the fact that T_j is unbreakable and by conditioning on how T_j meets T_t and whether or not $d(T_j) > 3$. □

LEMMA 4.2. *Suppose that $v_t \in S$ is tight.*

(1) *No other standard basis vector is tight.*
(2) *If $v_j = -e_j + e_{j-1} + e_t$, $j > t + 1$, then $T_t \dagger T_j$.*
(3) *If $v_{t+1} = -e_{t+1} + e_t + \cdots + e_0$, then $t = 1$ and $T_1 \dagger T_2$.*

Proof. We apply each case of Lemma 3.14 in turn.

(1) Suppose that there is another index j for which v_j is tight. Without loss of generality, we may assume that $j > t$. Then $\langle v_t, v_j\rangle = |v_t| - 2 \geq 3$. It follows that $\varepsilon_j = \varepsilon_t$ and $T_j \pitchfork T_t$. Thus, $[T_j - T_t] - [T_t - T_j]$ is reducible, but it also equals $\varepsilon_j(v_j - v_t)$, which is irreducible according to Lemma 3.14(1). This yields the desired contradiction.

(2) We have $\langle v_t, v_j\rangle = -1$ and $|v_j| = 3$, so either the desired conclusion holds, or else $z_j \in T_t$, $T_t \pitchfork T_j$, and $\varepsilon_t\varepsilon_j = -1$. If the latter possibility holds, then $[T_j - T_t] - [T_t - T_j]$ is reducible, but it also equals $\varepsilon_j v_j - \varepsilon_t v_t = \varepsilon_j(v_j + v_t)$, which is irreducible according to Lemma 3.14(2). It follows that $T_t \dagger T_j$.

(3) We have $\langle v_t, v_{t+1}\rangle = |v_{t+1}| - 2$, so either the desired conclusion holds, or else $z_{t+1} \in T_t$, $T_t \pitchfork T_{t+1}$, and $\varepsilon_t = \varepsilon_{t+1}$. If the latter possibility holds, then again $[T_{t+1} - T_t] - [T_t - T_{t+1}]$ is reducible, but it also equals $\varepsilon_t(v_{t+1} - v_t)$, which is irreducible according to Lemma 3.14(3). It follows that $t = 1$ and $T_1 \dagger T_2$. □

Lemmas 3.15 and 4.2(1) immediately imply the following result.

Corollary 4.3. *A standard basis S contains at most one breakable vector, and it is tight.* □

The following important lemma provides essential information about when two standard basis elements can pair nontrivially together; unless one is breakable or has norm 2, then they correspond to consecutive intervals.

Lemma 4.4. *Given a pair of unbreakable vectors $v_i, v_j \in S$ with $|v_i|, |v_j| \geq 3$, we have $|\langle v_i, v_j\rangle| \leq 1$, with equality if and only if $T_i \dagger T_j$ and $\varepsilon_i\varepsilon_j = -\langle v_i, v_j\rangle$.*

Proof. The lemma follows easily once we establish that $\langle [T_i], [T_j]\rangle \leq 0$. Thus, we assume that $\langle [T_i], [T_j]\rangle \geq 1$ and derive a contradiction. Since these classes are unbreakable and they pair positively, it follows that $z_i = z_j$. In particular, $d := |v_i| = d(T_i) = d(T_j) = |v_j|$. Now, either $T_i \pitchfork T_j$, in which case $\langle [T_i], [T_j]\rangle = d - 2$, or else T_i and T_j share a common endpoint, in which case $\langle [T_i], [T_j]\rangle = d - 1$.

Let us first treat the case in which $i = t$ and v_t is tight. By Corollary 4.3, it follows that v_j is not tight. Thus, $d = t + 4$, and $\mathrm{supp}(v_j)$ contains at least three values $> t$. If $v_{jt} = 1$, then $\langle v_t, v_j\rangle \leq d - 3$, while if $v_{jt} = 0$, then $\mathrm{supp}(v_j)$ contains at least four values $> t$, and again $\langle v_j, v_t\rangle \leq d - 3$. As $\langle v_t, v_j\rangle \geq -1$ and $d \geq 5$, we have $|\langle v_t, v_j\rangle| \leq d - 3$, whereas $|\langle [T_t], [T_j]\rangle| \geq d - 2$. This yields the desired contradiction to the assumption that $\langle [T_t], [T_j]\rangle \geq 1$ in this case.

Thus, we may assume that neither v_i nor v_j is tight, and without loss of generality that $j > i$. Suppose that $\varepsilon := \varepsilon_j = \varepsilon_i$. Thus $\langle v_i, v_j\rangle = \langle [T_i], [T_j]\rangle \geq 1$. If $v_{ji} = 1$, then $\langle v_j, v_i\rangle \leq d - 2$, with equality possible if and only if $v_{jk} = 1$

whenever $v_{ik} = 1$. But then $|v_j| > |v_i|$, a contradiction. Hence $v_{ji} = 0$. If $\langle v_j, v_i \rangle = d - 1$, then again $v_{jk} = 1$ whenever $v_{ik} = 1$. But then $v_j - v_i = -e_j + e_i \gg v_j$, a contradiction.

Still assuming that $\varepsilon_j = \varepsilon_i$, we are left to consider the case that $v_{ji} = 0$ and $\langle v_j, v_i \rangle = d-2$, and therefore $T_i \pitchfork T_j$. In this case, $\mathrm{supp}(v_j) - \mathrm{supp}(v_i) = \{j, k\}$ and $\mathrm{supp}(v_i) - \mathrm{supp}(v_j) = \{i, l\}$ for some indices k, l. If $\sigma_j = \sigma_k$, then $v_j = -e_j + e_k$, in contradiction to $|v_j| \geq 3$. If $\sigma_j = \sigma_i$, then either $i < k$, in which case we derive the contradiction $v_j = -e_j + e_k$ again, or else $k < i$, in which case we derive the contradiction $-e_j + e_i \gg v_j$. Therefore, $\sigma_j \neq \sigma_i, \sigma_k$. It easily follows that $v_j - v_i = -e_j + e_k + e_i - e_l$ is irreducible. On the other hand, $v_j - v_i = \varepsilon([T_j] - [T_i]) = \varepsilon[T_j - T_i] - \varepsilon[T_i - T_j]$ is reducible, a contradiction.

It follows that $\varepsilon := \varepsilon_j = -\varepsilon_i$. Hence $\langle [T_i], [T_j] \rangle = -\langle v_i, v_j \rangle \leq 1$. In case of equality, we have $d = 3$ and $T_i \pitchfork T_j$. So on the one hand, $v_j = -e_j + e_i + e_p$ and $v_i = -e_i + e_q + e_s$ for distinct indices i, j, p, q, s, whence $v_j + v_i = -e_j + e_p + e_q + e_s$ is irreducible (cf. the proof of Lemma 3.13, Case 1). On the other hand, it equals $\varepsilon([T_j] - [T_i]) = \varepsilon[T_j - T_i] - \varepsilon[T_i - T_j]$, which is reducible, a contradiction.

In total, $\langle [T_i], [T_j] \rangle \leq 0$ in every case, and the lemma follows. ☐

Corollary 4.5. *If T_i and T_j are distinct unbreakable intervals with $d(T_i)$, $d(T_j) \geq 3$, then $z_i \neq z_j$.*

4.2. *The intersection graph.* This subsection defines the key notion of the intersection graph and establishes the most important properties about it that are necessary to carry out the combinatorial analysis of Sections 5–8.

Recall that from the standard basis S, we obtain a collection of intervals $\mathcal{T}$. Let $\overline{S} \subset S$ denote the subset of unbreakable elements of S; thus, $\overline{S} = S - v_t$ if S contains a breakable element v_t, and $\overline{S} = S$ otherwise. For an index $1 \leq i \leq n$, let $S_i = \{v_1, \ldots, v_i\}$.

Definition 4.6 (Compare Definition 3.7). The *intersection graph* is the graph

$$G(S) = (S, E), \quad E = \{(v_i, v_j) \mid T_i \text{ abuts } T_j\}.$$

Write $G(S')$ to denote the subgraph induced by a subset $S' \subset S$. If $(v_i, v_j) \in E$ with $i < j$, then v_i is a *smaller neighbor* of v_j.

Observe that $G(S)$ is a subgraph of the pairing graph $\widehat{G}(S)$ (Section 3.1) and Lemma 4.4 implies that they coincide unless S contains a breakable element v_t. Furthermore, if $v_t \in S$ is breakable, then Lemma 4.1 implies that $(v_t, v_j) \in E$ if and only if $\langle v_t, v_j \rangle \in \{|v_j| - 1, 1, -1\}$, except in the special case that $|v_j| = 3$, $z_j \in T_t$, $T_j \pitchfork T_t$, and $\varepsilon_j \varepsilon_t = \langle v_t, v_j \rangle$. Therefore, $G(S)$ is determined by the pairings of vectors in S except in this special case, which fortunately arises just once in our analysis (Proposition 8.8).

We now collect several fundamental properties about the intersection graph $G(S)$.

Definition 4.7. The *claw* $(i;j,k,l)$ is the graph $Y = (V,E)$ with

$$V = \{i,j,k,l\} \quad \text{and} \quad E = \{(i,j),(i,k),(i,l)\}.$$

A graph G is *claw-free* if it does not contain an induced subgraph isomorphic to Y.

Equivalently, if three vertices in G neighbor a fourth, then some two of them neighbor.

LEMMA 4.8. *$G(S)$ is claw-free.*

Proof. If T_i abuts three intervals T_j, T_k, T_l, then it abuts some two at the same end, and then those two abut. □

Definition 4.9. A *heavy triple* (v_i, v_j, v_k) consists of distinct vectors of norm ≥ 3 contained in the same component of $G(\overline{S})$, none of which separates the other two in $G(\overline{S})$. In particular, if (v_i, v_j, v_k) spans a triangle, then it spans a *heavy triangle*.

LEMMA 4.10. *$G(\overline{S})$ does not contain a heavy triple.*

Proof. Since v_i, v_j, v_k belong to the same component of $G(\overline{S})$, the intervals T_i, T_j, T_k are subsets of some path $P \subset G - r$. Assume without loss of generality that z_i lies between z_j and z_k on P. Every unbreakable interval in P that avoids z_i lies to one side of it, and T_j and T_k lie to opposite sides by assumption. As each element in $\overline{S}$ is unbreakable and T_i is the unique interval containing z_i, it follows that v_j and v_k lie in separate components of $G(\overline{S}) - v_i$. □

LEMMA 4.11. *Every cycle in $G(\overline{S})$ has length 3 and contains a unique vector v_i of norm 2. Furthermore, if (v_i, v_j, v_k) is a cycle with $i < j < k$ and v_k is not gappy, then $|v_l| = 2$ for all $l \leq i$ if $\overline{S} = S$, or for all $t < l \leq i$ otherwise.*

Proof. Choose a cycle $C \subset G(\overline{S})$. By Lemma 3.8, it follows that $V(C)$ induces a complete subgraph. Thus, $V(C)$ cannot contain three vectors of norm ≥ 3, for then they would span a heavy triangle, in contradiction to Lemma 4.10. Note also that the vectors of norm 2 in S induce a union of paths in $G(S)$. Therefore, $V(C)$ cannot contain more than two such vectors. If it did contain two, then they must take the form $v_{i+1} = -e_{i+1} + e_i$ and $v_i = -e_i + e_{i-1}$. Choose any other $v_j \in V(C)$. Then either $v_{j,i} = 0$ and $v_{j,i\pm1} = 0$, or else $v_{j,i} = 0$ and $v_{j,i\pm1} = 1$. In the first case, $v_j \ll v_j - v_{i+1}$, and in the second, $v_j \ll v_j - v_i$, both of which entail contradictions. It follows that $V(C)$ contains at most two vectors of norm ≥ 3 and at most one vector of norm 2, hence exactly that many of each. This establishes the first part of the lemma.

For the second part, it follows at once that $\min(\mathrm{supp}(v_k)) = i$ and $|v_i| = 2$. If $|v_l| \geq 3$ for some largest value $l < i$ and $l \neq t$, then $(v_l, v_{l+1}, \ldots, v_i)$ induces a path in $G(\overline{S})$, and then (v_l, v_j, v_k) forms a heavy triple. The second part now follows as well. □

Corollary 4.12. *If $C \subset S$ spans a cycle in $G(S)$, then it induces a complete subgraph and $|V(C)| \leq 4$, with equality if and only if C contains a breakable vector v_t.*

Definition 4.13. If (v_i, v_j, v_k) spans a triangle in $G(S)$, then it is *positive* or *negative* according to the sign of $\langle v_i, v_j\rangle \cdot \langle v_j, v_k\rangle \cdot \langle v_k, v_i\rangle$.

Lemma 4.14. *If (v_i, v_j, v_k) spans a triangle in $G(S)$ and some pair of T_i, T_j, T_k are consecutive, then the triangle is positive.*

Proof. Observe that $\langle v_i, v_j\rangle \cdot \langle v_j, v_k\rangle \cdot \langle v_k, v_i\rangle = (\varepsilon_i \varepsilon_j \varepsilon_k)^2 \cdot \langle [T_i], [T_j]\rangle \cdot \langle [T_j], [T_k]\rangle \cdot \langle [T_k], [T_i]\rangle$. Two pairs of T_i, T_j, T_k are consecutive and the other pair shares a common endpoint, so the right-hand side of this equation is positive. □

Most of the case analysis to follow in Sections 5–8 involves arguing that elements of S must take a specific form, for otherwise we would obtain a contradiction to one of the preceding lemmas. In such cases, we typically just state something to the effect of "(v_i, v_j, v_k) forms a negative triple" without the obvious conclusion "a contradiction," to spare the use of this phrase several dozen times.

We conclude with one last basic observation.

Lemma 4.15. *Suppose that $v_s \in S$ has norm ≥ 3 with s chosen smallest and that it is not tight. Then v_s is just right, and $|v_s| \in \{s, s+1\}$.*

Proof. Recall that if v_g is gappy, then $|v_{k+1}| \geq 3$ for a gappy index k. By minimality of s, it follows that v_s is just right. If $|v_s| < s$, then $v_s = -e_s + e_{s-1} + \cdots + e_k$ for some $2 \leq k \leq s-2$, and $(v_k; v_{k-1}, v_{k+1}, v_s)$ induces a claw, a contradiction. □

5. **A decomposable lattice**

The goal of this section is to classify the changemaker lattices isomorphic to a sum of more than one linear lattice (Proposition 5.7). We begin with a basic result.

Lemma 5.1. *A changemaker lattice has at most two indecomposable summands. If it has two indecomposable summands, then there exists an index*

$s > 1$ *for which* $v_s = -e_s + \sum_{i=0}^{s-1} e_i$, $|v_i| = 2$ *for all* $1 \leq i < s$, *and* v_s *and* v_1 *belong to separate summands.*

Proof. For the first statement, it suffices to show that $\widehat{G}(S)$ has at most two connected components (cf. Section 3.1). Thus, suppose that $\widehat{G}(S)$ has more than one component, fix a component C that does not contain v_1, and choose $s > 1$ smallest such that $v_s \in V(C)$. Thus, $v_s \not\sim v_i$ for all $1 \leq i < s$. Let $k = \min(\mathrm{supp}(v_s))$. Then $k = 0$, since otherwise $v_s \sim v_k$. Furthermore, v_s is not gappy, for if l is a gappy index, then $v_s \sim v_{l+1}$. Therefore, $v_s = -e_s + \sum_{i=0}^{s-1} e_i$. If $|v_i| \geq 3$ for some $i < s$, then $v_s \sim v_i$. It follows that $|v_i| = 2$ for all $i < s$. As s is uniquely determined, it follows that C is as well, so $\widehat{G}(S)$ contains exactly two components. The statement of the lemma now follows. □

For the remainder of the section, suppose that L is a changemaker lattice isomorphic to a sum of two linear lattices.

LEMMA 5.2. *All elements of* S *are just right.*

Thus, $G(S) = \widehat{G}(S)$.

Proof. Suppose that $v_t \in S$ is tight. Then $\langle v_t, v_1 \rangle = 1$ and $\langle v_t, v_s \rangle \geq 1$ implies that v_1 and v_s belong to the same component of $\widehat{G}(S)$, a contradiction.

Next, suppose that $v_g \in S$ is gappy with g chosen minimal. Note that $|v_g| \geq 3$. Let k denote the minimal gappy index for v_g. Since v_{k+1} is not gappy, it follows that $v_{k+1,k-1} \neq 0$. Since $\langle v_g, v_{k+1} \rangle \leq 1$ by Lemma 4.4, it follows that $v_{g,k-1} = 0$, and now minimality of k implies that $k = \min(\mathrm{supp}(v_g))$. This implies that $v_g \sim v_k$. We cannot have $v_{k+1} \sim v_k$, since this forces $|v_k| \geq 3$, and then (v_k, v_{k+1}, v_j) forms a heavy triangle, in contradiction to Lemma 4.10. Hence $v_k \not\sim v_{k+1}$. As $G(S)$ does not contain a cycle of length > 3, it follows that v_k and v_{k+1} belong to separate components of $G(S_{g-1})$. (Recall from Section 4.2 that $S_i := \{v_1, \ldots, v_i\} \subset S$.) Since $G(S_{g-1})$ has at most two components, it follows that $G(S_g)$ is connected; hence $G(S)$ is as well, a contradiction. □

We define a few important families of basis sets.

Definition 5.3. Given positive integers $1 < s < m-1$, let $A_{s,m} = \{v_1, \ldots, v_m\}$, where

- $v_m = -e_m + e_{m-1} + \cdots + e_{s-1}$,
- $v_{s+1} = -e_{s+1} + e_s + e_{s-1}$,
- $v_s = -e_s + e_{s-1} + \cdots + e_0$, and
- $v_k = -e_k + e_{k-1}$ for all other $k < m$, $k \neq s, s+1$.

Given a positive integer $m \geq 3$, let $B_m = \{v_1, \ldots, v_m\}$, where

- $v_m = -e_m + e_{m-1} + \cdots + e_0$,

- $v_2 = -e_2 + e_1 + e_0$, and
- $v_k = -e_k + e_{k-1}$ for all other $k < m$, $k \neq 2$.

Given positive integers $1 < s < m$, let $C_{s,m} = \{v_1, \dots, v_m\}$, where

- $v_m = -e_m + e_{m-1} + \cdots + e_s$,
- $v_s = -e_s + e_{s-1} + \cdots + e_0$, and
- $v_k = -e_k + e_{k-1}$ for all other $k < m$, $k \neq s$.

LEMMA 5.4. *Suppose that v_m has multiple smaller neighbors. Then $m > s+1$ and $S_m = A_{s,m}$.*

Proof. Suppose that $v_m \sim v_i, v_j$ with $i < j < m$. As in the proof of Lemma 5.2, the vectors v_i and v_j cannot belong to separate components of $G(S_{m-1})$, for then $G(S)$ would be connected. Furthermore, $v_i \sim v_j$, since otherwise $G(S_m)$ would contain a cycle of length > 3. By Lemma 4.11, it follows that $|v_l| = 2$ for all $l \leq i$. Hence $s \geq i+1$. From $v_j \sim v_i$, it follows that $j \geq s+1$ and $v_j = -e_j + e_{j-1} + \cdots + e_i$. As $\langle v_j, v_m \rangle \leq 1$, it follows that $j = i+2$. Thus, $s = i+1$. As i and j are uniquely determined, it follows that $v_m \not\sim v_k$ for all $s+2 < k < m$, so $|v_k| = 2$ for all such k. Hence $S_m = A_{s,m}$, as claimed. □

The following definition is essential to describe the way in which we build families of standard bases. The terminology borrows from [31, Def. 3.4], although its meaning differs somewhat.

Definition 5.5. We call S_m an *expansion* of S_{m-1} if $v_m = -e_m + e_{m-1} + \cdots + e_k$ for some k, $|v_i| = 2$ for all $k+1 < i < m$, and $|v_{k+1}| \geq 3$ in case $m > k+1$.

LEMMA 5.6. *Suppose that v_m has a single smaller neighbor. Then S_m is either B_m, $C_{s,m}$, or an expansion of S_{m-1}.*

Proof. Suppose first that $v_{m0} = 1$. It follows by assumption on v_m that $|v_k| = 2$ for all $1 \leq k < n$ except for a single v_i, for which $|v_i| = 3$. On the other hand, $|v_s| = s+1$. It follows that $s = 2$ and $S_m = B_m$.

Thus, we may assume that $v_m = -e_m + e_{m-1} + \cdots + e_i$ for some $i > 0$. Now $v_m \sim v_i$, so $|v_k| = 2$ for all $i+1 < k < m$. Suppose that $i+1 < m$ and $|v_{i+1}| = 2$. If $v_i \sim v_l$ with $l < i$, then $(v_i; v_l, v_{i+1}, v_m)$ induces a claw, in contradiction to Lemma 4.8. It follows that $i = s$ and $S_m = C_{s,m}$.

The remaining cases that $m = i+1$, or that $m > i+1$ and $|v_{i+1}| > 2$, both imply that S_m is an expansion of S_{m-1}. □

Combining Lemmas 5.4 and 5.6 and induction, we obtain the following structural result.

PROPOSITION 5.7. *Suppose that a changemaker lattice is isomorphic to a sum of more than one linear lattice. Then its standard basis is built by a sequence of (possibly zero) expansions to $A_{s,m}$, B_m, $C_{s,m}$, or $\varnothing$ for some $m > s \geq 2$.*

In fact, we have established somewhat more: if a changemaker lattice is isomorphic to a linear lattice or a sum thereof, and $G(S_{n'})$ is disconnected for some $n' \leq n$, then $S_{n'}$ takes the form appearing in Proposition 5.7. We will utilize Proposition 5.7 in this stronger form on several occasions in Sections 6–8.

We obtain vertex bases for the families in Proposition 5.7 as follows:

$A_{s,m}$: $\{v_{m-1}, \dots, v_{s+1}, v_{s-1}, \dots, v_1, -(v_m + v_1 + \cdots + v_{s-1})\} \cup \{v_s\}$;
B_m: $\{v_{m-1} \dots, v_2, -v_m\} \cup \{v_1\}$;
$C_{s,m}$: $\{v_1, \dots, v_{s-1}\} \cup \{v_{m-1}, \dots, v_s, v_m\}$.

For expansion on $\varnothing$, the standard basis is, up to reordering, a vertex basis (cf. Proposition 9.2).

6. **All vectors just right**

Before proceeding further, we briefly comment on the purpose of this and the next two sections and establish some notation. Just as Proposition 5.7 describes the structure of the standard basis for a changemaker lattice isomorphic to a sum of more than one linear lattice, our goal in Sections 6–8 is to produce a comprehensive collection of *structural propositions* that do the same thing for the case of a single linear lattice. In each proposition we enumerate a specific family of standard bases, and in Section 9 we verify that each basis does, in fact, span a linear lattice by converting it into a vertex basis.

A posteriori, each standard basis $S = \{v_1, \dots, v_n\}$ contains at most one tight vector and two gappy vectors. We always denote the tight vector by v_t. We denote the gappy vector with the smaller index by v_g, which always takes the form $e_k + e_j + e_{j+1} + \cdots + e_{g-1} - e_g$ with $k < j + 1$. When there are two gappy vectors (8.6(1), 8.7(2), 8.8(1,2)), we specifically notate the one with the larger index. We write $s = \min\{i \mid |v_i| > 2\}$ when there is no tight vector and $s = \min\{i > t \mid |v_i| > 2\}$ when there is one. Otherwise, every standard basis element v_i is just right, so it is completely determined by i and its norm, which we report if and only if $|v_i| \geq 3$. In 7.5(2) and 8.6–8.8 we report some families of standard bases *up to truncation.* Thus in 7.5(2), we may truncate by taking $n = g$ and disregarding v_i for $i \geq g + 1$.

Example. The first structural Proposition 6.2(1) reports the family of standard bases parametrized by $s \geq 2$, where

- $v_i = e_{i-1} - e_i$ for $i = 1, \dots, s - 1$;
- $v_s = e_0 + \cdots + e_{s-1} - e_s$;
- $v_{s+1} = e_{s-1} + e_s - e_{s+1}$;

- $v_{s+2} = e_s + e_{s+1} - e_{s+2}$; and
- $v_n = v_{s+3} = e_{s-1} + e_s + e_{s+1} + e_{s+2} - e_{s+3}$.

For the remainder of this section, assume that L is a changemaker lattice isomorphic to a linear lattice and that every element of S is just right (hence also unbreakable).

6.1. $G(S)$ *contains a triangle.* A *sun* is a graph consisting of a triangle Δ on vertices $\{a_1, a_2, a_3\}$ together with three vertex-disjoint paths P_1, P_2, P_3 such that a_i is an endpoint of P_i, $i = 1, 2, 3$. The other endpoints of the P_i are the *extremal vertices* of the graph.

Lemma 6.1. *If $G(S)$ contains a triangle Δ, then $G(S)$ is a sun and $V(\Delta) = \{v_i, v_{i+2}, v_m\}$ for some $i + 2 < m$. Furthermore, $|v_l| = 2$ for all v_l along the path containing v_i.*

Proof. Choose a triangle $\Delta \subset G(S)$ with $V(\Delta) = \{v_i, v_j, v_m\}$, $i < j < m$. By Lemma 4.11, $|v_l| = 2$ for all $l \le i$ and $|v_j|, |v_m| \ge 3$. Since v_j and v_m are just right, we have $v_j = -e_j + \cdots + e_i$ and $v_m = -e_m + \cdots + e_i$. Since $\langle v_m, v_j \rangle \le 1$, it follows that $j = i + 2$.

Suppose by way of contradiction that $\Delta' \subset G(S)$ is another triangle with $V(\Delta') = \{v_{i'}, v_{j'}, v_{m'}\}$, $i' < j' < m'$. Then $|v_l| = 2$ for all $l \le i'$ and $j' = i' + 2$, so $i' \in \{i-1, i, i+1\}$. If $i' = i$, then $\langle v_{m'}, v_m \rangle \ge 2$, which cannot occur. If $i' = i+1$, then (v_{i+3}, v_{i+1}, v_i) is a path in $G(S)$, which implies that (v_{i+2}, v_{i+3}, v_m) forms a heavy triple, a contradiction. By symmetry, $i = i' + 1$ cannot occur either.

Consequently, $G(S)$ contains a unique triangle Δ. Furthermore, $G(S)$ is claw-free by Lemma 4.8. It follows that $G(S)$ is a sun. If some vector v_l on the path containing v_i had norm ≥ 3, then (v_l, v_k, v_m) forms a heavy triple, so this does not occur. ☐

Proposition 6.2. *Suppose that every element in S is just right and that $G(S)$ contains a triangle. Then either*

(1) $n = s + 3$, $|v_s| = s + 1$, $|v_{s+1}| = |v_{s+2}| = 3$, *and* $|v_{s+3}| = 5$;

(2) $n = s + 3$, $|v_s| = s + 1$, $|v_{s+1}| = 3$, *and* $|v_{s+2}| = |v_{s+3}| = 4$; *or*

(3) $|v_s| = s = 3$, $|v_m| = m$ *for some* $m > 3$, $|v_i| = 2$ *for all* $i < m$, $i \ne 3$, *and S is built from S_m by a sequence of expansions.*

Proof. We apply Lemma 6.1, keeping the notation therein.

(I) *Suppose that* $|v_{i+1}| > 2$. In this case, $s = i + 1$ and v_s has no smaller neighbor, so $|v_s| = s + 1$. Since $G(S)$ is connected, v_s has some neighbor $v_j = -e_j + \cdots + e_l$. Note that $v_s \not\sim v_{s+1}, v_m$, so $j \ne s + 1, m$. If $l < s$, then in fact $l < s - 1$ and (v_{s+1}, v_j, v_m) forms a heavy triangle, so it follows that $l = s$. Since $1 \ge \langle v_j, v_m \rangle = \min\{m, j\} - (s+1) \ge 1$, it follows that $\min\{m, j\} = s+2$.

(I.1) *Suppose that* $j = s + 2$. The subgraph H of $G(S)$ induced on

$\{v_1, \ldots, v_{s+2}, v_m\}$ is a sun with extremal vertices v_1, v_s, v_{s+1}. We claim that $G(S) = H$. For if not, then there exists some vector $v_l = -e_l + \cdots + e_k$ with a single edge to H, meeting it in an extremal vertex. This forces $k \leq s+1$, but then $\langle v_l, v_m \rangle \geq 1$, a contradiction. It follows that $n = m = s+3$, and case (1) results.

(I.2) *Suppose that* $m = s+2$. The subgraph H induced on $\{v_1, \ldots, v_{s+2}, v_j\}$ is a sun with extremal vertices v_1, v_s, v_{s+1} like before. The argument just given (with v_j in place of v_m) applies to show that $G(S) = H$. It follows that $n = j = s+3$, and case (2) results.

(II) *Suppose that* $|v_{i+1}| = 2$. In this case, $s = i+2 = 3$ and $|v_3| = 3$. If $|v_j| \geq 3$ for some $3 < j < m$ chosen smallest, then the subgraph induced on $V' = \{v_1, \ldots, v_{j-1}, v_m\}$ is a sun in which v_j has multiple neighbors, which cannot occur in the sun $G(S)$. Thus, $|v_j| = 2$ for all $3 < j < m$. Next, choose any $v_j = -e_j + \cdots + e_k$ with $j > m$. Then $G(S_{j-1})$ is a sun, so v_j has exactly one smaller neighbor. It easily follows that $k \geq 2$, $v_j \sim v_k$, and v_k has some smaller neighbor v_l. If $|v_j| \geq 3$, then $|v_{k+1}| \geq 3$, since otherwise $(v_k; v_l, v_{k+1}, v_j)$ induces a claw. It follows that $G(S_j)$ is an expansion of $G(S_{j-1})$. By induction, $G(S)$ is a sequence of expansions applied to $G(S_m)$, and case (3) results. □

6.2. *$G(S)$ does not contain a triangle.* In this case, $G(S)$ is a path.

6.2.1. *Some vertex has multiple smaller neighbors.* Suppose that $v_m \in S$ has multiple smaller neighbors. Since $G(S)$ is a path, it follows that $G(S_{m-1})$ consists of a union of two paths and that v_m is adjacent precisely to one endpoint of each. Therefore, m is the minimal index for which $G(S_m)$ is connected, which establishes that m is unique.

Lemma 6.3. *Suppose that every element in S is just right, $G(S)$ does not contain a triangle, $v_m \in S$ has multiple smaller neighbors, and v_{m-1} is not an endpoint of $G(S)$. Then $m = n$.*

Proof. Suppose by way of contradiction that $n > m$, and consider v_{m+1}. Its unique smaller neighbor v_j is an endpoint of the path $G(S_m)$, and $j < m-1$ by hypothesis. Therefore, $|v_{m+1}| \geq 4$, but this implies that $\langle v_{m+1}, v_m \rangle \geq 1$, a contradiction. □

Proposition 6.4. *Suppose that every element in S is just right, $G(S)$ does not contain a triangle, and some vector $v_m \in S$ has multiple smaller neighbors. Then either*

(1) $m = n = 4$, $|v_2| = |v_3| = 3$, *and* $|v_4| = 5$;
(2) $m = n = 4$, $|v_2| = 3$, *and* $|v_3| = |v_4| = 4$;
(3) $m = n = s+3$, $|v_s| = s+1$, $|v_{s+2}| = 3$, *and* $|v_{s+3}| = 5$;
(4) $m = n = s+3$, $|v_s| = s+1$, *and* $|v_{s+2}| = |v_{s+3}| = 4$; *or*
(5) $s = 3$, $|v_3| = 4$, $|v_m| = m > 3$, *and* $|v_{m+1}| = 3$ *in case* $n > m$.

Proof. Write $v_m = -e_m + \cdots + e_k$.

(I) *Suppose that* $k = 0$. Then $\langle v_m, v_s\rangle = s - 1 \geq 1$ forces $s = 2$. Further, v_m is not adjacent to v_1, but as v_m has a neighbor in the component of $G(S_{m-1})$ containing it, v_1 must have some neighbor $v_i = -e_i + \cdots + e_1$ with $i \geq 3$. Then $\langle v_m, v_i\rangle \geq i - 2 \geq 1$, so $i = 3$. Hence v_m neighbors v_2 and v_3 but no other v_j, $j < m$. It follows that $|v_j| = 2$ for $3 < j < m$. If $m > 4$, then $(v_3; v_1, v_4, v_m)$ induces a claw. Hence $m = 4$. By Lemma 6.3, it follows that $m = n$, and case (1) results.

(II) *Suppose that* $k > 0$. Hence $v_m \sim v_k, v_j$ for some $j \geq k + 2$.

(II.1) *Suppose that* $j > k+2$. In this case, $|v_j| = 3$ and $|v_{j-1}| = 2$, so v_{j-2} neighbors v_j and v_{j-1} and no other vector. It follows that $j - 2 \in \{1, s\}$. But $1 = j - 2 > k > 0$ cannot occur, so $j - 2 = s$. Since $0 = \langle v_m, v_s\rangle = s - k - 1$, it follows that $s = k + 1$. Moreover, since v_j is an endpoint of its path in $G(S_{m-1})$, it follows that $m = j + 1$. By Lemma 6.3, it follows that $m = n$, and case (3) results.

(II.2) *Suppose that* $j = k + 2$. Since v_j and v_k belong to different components of $G(S_{m-1})$, it follows that $|v_k| = 2$ and $|v_{k+2}| \geq 4$.

(II.2.i) *Suppose that* v_{k+2} *has no smaller neighbor*. In this case, $|v_{k+1}| = 2$ and $k + 2 = s$. As $v_k \sim v_{k+1}$ and v_k is an endpoint of its path in $G(S_{m-1})$, it follows that v_k has no smaller neighbor; hence $k = 1$. It follows that $s = 3$, $|v_3| = 4$, $|v_m| = m$, and $|v_i| = 2$ for all other $i < m$. If $n = m$, then we land in case (5). If $n > m$, then consider v_{m+1}. Its unique smaller neighbor v_j is an endpoint of $G(S_m)$. It follows that $j = m - 1$ and $|v_{m+1}| = 3$. By a similar argument, it follows that $|v_i| = 2$ for all $i > m+1$. Therefore, case (5) results.

(II.2.ii) *Suppose that* v_{k+2} *has a smaller neighbor*. Thus, it has no larger neighbor in $G(S_m)$ besides v_m, so $m = k + 3$.

(II.2.ii′) *Suppose that* $|v_{k+1}| \geq 3$. It follows that $v_{k+1} \sim v_{k+2}$, and v_{k+2} cannot have any other smaller neighbor. Hence $\min(\text{supp}(v_{k+2})) = 0$, $s = k+1$, and $1 \geq |\langle v_{k+1}, v_k\rangle| = k$, so $k = 1$. By Lemma 6.3, it follows that $m = n$, and case (2) results.

(II.2.ii″) *Suppose that* $|v_{k+1}| = 2$. In this case, $v_k \sim v_{k+1}$, so v_k has no smaller neighbor. Hence $k \in \{1, s\}$. However, if $k = 1$, then v_{k+2} would not have a smaller neighbor. Hence $k = s$, and as $\langle v_{k+2}, v_s\rangle = 0$ but v_{k+2} has a smaller neighbor, it follows that $|v_{k+2}| = 4$. By Lemma 6.3, it follows that $m = n$, and case (4) results. □

6.2.2. *No vector has multiple smaller neighbors.*

Proposition 6.5. *Suppose that every element in* S *is just right*, $G(S)$ *does not contain a triangle, and no vector in* S *has multiple smaller neighbors.*

Then either

(1) $|v_i| = 2$ *for all* i;

(2) $s = 3, |v_3| = 3, |v_4| = 5$; *or*

(3) $|v_s| = s$ *and* S *is built from* S_s *by a sequence of expansions.*

Proof. It must be the case that $G(S_j)$ is connected for all j, since $G(S)$ is connected, and if $G(S_{m-1})$ is disconnected for some $m > 0$, then v_m would have multiple smaller neighbors. Thus, unless case (1) occurs, it follows that $|v_s| = s \geq 3$.

(I) *Suppose that there exists an index* $m > s$ *for which* $\min(\text{supp}(v_m)) = 0$. In this case, $\langle v_m, v_s\rangle = s - 2 \geq 1$. It follows that $s = 3$ and v_3 is the unique smaller neighbor of v_m. If $m > 4$, then $(v_3; v_2, v_4, v_m)$ induces a claw. Hence $m = 4$. If $n > 4$, then choose j maximal for which $|v_5| = \cdots = |v_j| = 2$. Thus, $G(S_j)$ has endpoints v_j and v_2. If $n > j$, then v_{j+1} must neighbor one of v_j and v_2. However, $v_{j+1} \sim v_2$ implies that $\langle v_{j+1}, v_4\rangle \geq 1$, while $v_{j+1} \sim v_j$ implies that $|v_{j+1}| = 2$. Both result in contradictions, so it follows that $n = j$, and case (2) results.

(II) *Suppose that* $\min(\text{supp}(v_m)) > 0$ *for all* $m > s$. Consider $v_m = -e_m + \cdots + e_k$ with $m > s$ and $k > 0$. Thus, v_k is the unique smaller neighbor of v_m, so it is an endpoint of $G(S_{m-1})$, and $|v_i| = 2$ for all $k + 1 < i < m$. If $m > k + 1$ and $|v_{k+1}| = 2$, then $v_k \sim v_{k+1}$. As v_k is an endpoint of $G(S_{m-1})$, it follows that v_k has no smaller neighbor. But then $k = 1$ and $|v_i| = 2$ for all $i < m$, in contradiction to the assumption that $m > s$. It follows that $|v_{k+1}| \geq 3$, or else $m = k + 1$ and $|v_m| = 2$. Hence S_m is an expansion on S_{m-1}. By induction on m, it follows that case (3) results. □

7. A gappy vector, but no tight vector

In this section, assume that L is a changemaker lattice isomorphic to a linear lattice, S does not contain a tight vector, and it does contain a gappy vector v_g. We keep the notation introduced in the second paragraph of Section 6. Again, every vector in S is unbreakable, and $G(S) = \widehat{G}(S)$. For use in Section 8, the following lemma allows the possibility that S contains a tight, unbreakable vector.

LEMMA 7.1. *Suppose that* $v_g \in S$ *is gappy and that* S *contains no breakable vector. Then* v_g *is the unique gappy vector,* $v_g = -e_g + e_{g-1} + \cdots + e_j + e_k$ *for some* $k + 1 < j < g$, *and* v_k *and* v_{k+1} *belong to distinct components of* $G(S_{g-1})$.

Proof. Choose v_g with g minimal, and choose a minimal gappy index k for v_g. Then $|v_{k+1}| \geq 3$, and since v_{k+1} is not gappy, it follows that $v_{k+1,k-1} = 1$. Thus, $v_{g,k-1} = 0$, since otherwise $\langle v_g, v_{k+1}\rangle \geq 2$. It follows that $v_g \sim v_{k+1}, v_k$.

If $v_k \sim v_{k+1}$, then either $|v_k| \geq 3$, or else $k = 1$ and v_2 is tight. In the first case, the triangle (v_k, v_{k+1}, v_g) is heavy, and in the second case, it is negative. Hence $|v_k| = 2$ and $v_k \not\sim v_{k+1}$. If v_k and v_{k+1} are in the same component of $G(S_{g-1})$, then a shortest path between them, together with v_g, spans a cycle of length > 3 in $G(S)$. It follows that $G(S_{g-1})$ has two components, and v_k and v_{k+1} belong to separate components.

Suppose by way of contradiction that $l > k$ is another gappy index. Then $|v_{l+1}| \geq 3$, so $v_{k+1} \not\sim v_{l+1}$, since otherwise (v_{k+1}, v_{l+1}, v_g) forms a heavy triangle. Furthermore, $v_{l+1,k} = 0$, since otherwise $\langle v_g, v_{l+1}\rangle \geq 2$. It follows that $v_{l+1} \not\sim v_k$, too. But then $(v_g; v_k, v_{k+1}, v_{l+1})$ induces a claw. Hence, no other index l exists, and v_g takes the stated form.

Lastly, suppose by way of contradiction that v_h is another gappy vector, with $h > g$ chosen smallest. Note that $G(S_{h-1})$ is connected. It follows that v_h has at most two smaller neighbors and that they are adjacent, since otherwise there would exist a cycle of length > 3 in $G(S)$. Choose a minimal gappy index k' for v_h and let $l = \min(\mathrm{supp}(v_h))$. Then $|v_{k'+1}| \geq 3$, and since $\langle v_g, v_{k'+1}\rangle \leq 1$, it follows that $v_{k'+1,i} = 0$ for $i = l, \ldots, k'-1$. Thus, $v_{k'+1,i} = 1$ for some $i < l$, whence $l > 0$. Thus, $v_h \sim v_{k'+1}, v_l$, so $v_{k'+1} \sim v_l$. However, $|v_l| = 2$, so it follows that $v_{k'+1,l-1} = 1$; but then $v_{k'+1} \ll v_{k'+1} - v_l$, a contradiction. It follows that v_g is the unique gappy vector, as claimed. ☐

By Lemma 7.1, it follows that $G(S_{g-1})$ is disconnected, so S_{g-1} must take one of the forms described by Proposition 5.7. Lemmas 7.2 and 7.3 condition on these possible forms to determine the structure of S_g.

LEMMA 7.2. *Suppose that S_{g-1} is built from $\varnothing$ by a sequence of expansions. Then S_g takes one of the following forms*:

(1) $k = s-1$, $j = s+1$, $|v_s| = s+1$, *and* $|v_{s+2}| = 4$;
(2) $j = k+2$, $|v_i| = 2$ *for all* $i > k+1$, *and otherwise* S_{g-1} *is arbitrary*;
(3) $k = s$, $j = s+2$, $|v_s| = s+1$, $|v_{s+1}| = 3$, *and* $|v_{s+2}| = 3$;
(4) $k = 1$, $s = 2$, $|v_2| = 3$, $|v_j| = j$, *and* $|v_{j+1}| = 3$; *or*
(5) $k = 1$, $s = 2$, $|v_2| = 3$, *and* $|v_j| = j$.

Proof. (I) *Suppose that $v_g \sim v_j$.* Thus, $v_{jk} = 0$. If $v_j \not\sim v_{k+1}$, then $(v_g; v_k, v_{k+1}, v_j)$ induces a claw. Hence $v_j \sim v_{k+1}$. If $|v_j| \geq 3$, then (v_{k+1}, v_j, v_g) forms a heavy triangle. Hence $|v_j| = 2$ and $j = k+2$.

(I.1) *Suppose that $v_l \sim v_k$ with $k < l < g$.* Thus, $v_l = -e_l + \cdots + e_k$. Then $l > k+2$ and $\langle v_g, v_l\rangle \leq 1$, which implies that $l = k+3$ and $|v_{k+3}| = 4$. If $|v_i| \geq 3$ for some largest value $i \leq k$, then (v_i, v_{k+3}, v_g) forms a heavy triple. Hence $|v_i| = 2$ for all $i \leq k$, and $s = k+1$. If $v_l \sim v_{s+1}$ for some $s+1 < l < g$, then $(v_g; v_{s-1}, v_s, v_l)$ induces a claw. It follows that $|v_l| = 2$ for all $s+1 < l < g$, and case (1) results.

(I.2) *Suppose that* $v_l \not\sim v_k$ *for all* $k+1 < l < g$. It follows that $|v_l| = 2$ for all such l, and case (2) results.

(II) *Suppose that* $v_g \not\sim v_j$. Thus, $v_{jk} = 1$. Furthermore, the assumption on S_{g-1} implies that $k = \min(\mathrm{supp}(v_j))$ and that $|v_i| = 2$ for all $k+1 < i < j$. If v_k has a smaller neighbor v_l, then $(v_k; v_l, v_j, v_g)$ induces a claw. It follows that $k \in \{1, s\}$.

(II.1) *Suppose that* $k = s$. Since $|v_s| \geq 3$, it follows that $|v_{s+1}| = 3$. If $j > s+2$, then $|v_{s+2}| = 2$ and $(v_{s+1}; v_{s-1}, v_{s+2}, v_g)$ induces a claw. Hence $j = s+2$. If $v_{s+1} \sim v_l$ for some $s+2 < l < g$, then either $l = s+3$, in which case (v_s, v_l, v_g) forms a heavy triangle, or else $l > s+3$, in which case $\langle v_g, v_l \rangle \geq 2$. It follows that $|v_l| = 2$ for all $s+2 < l < g$, and case (3) results.

(II.2) *Suppose that* $k = 1$. It follows that $s = 2$.

(II.2′) *Suppose that* $v_l \sim v_{j-1}$ *for some* $j < l < g$. If $v_g \sim v_l$, then (v_{j-1}, v_l, v_g) forms a heavy triple. Hence $v_g \not\sim v_l$, so $l = j+1$ and $|v_{j+1}| = 3$. If $v_i \sim v_j$ for some $j < i < g$, then (v_j, v_i, v_g) forms a heavy triple. Hence $|v_i| = 2$ for all $j+1 < i < g$, and case (4) results.

(II.2″) *Suppose that* $v_l \not\sim v_{j-1}$ *for all* $j < l < g$. It follows that $|v_l| = 2$ for all $j < l < g$, and case (5) results. ☐

LEMMA 7.3. *Suppose that* S_{g-1} *is not built from* ∅ *by a sequence of expansions. Then* S_g *takes one of the following forms*:

(1) $k = s$, $j = s+2$, $|v_s| = s+1$, $|v_{s+1}| = 3$, *and* $|v_{s+2}| = 4$;
(2) $k = 1$, $j = 3$, $s = 2$, $|v_2| = 3$, *and* $|v_3| = 4$; *or*
(3) $k = s-1$, $j = s+1$, $|v_s| = s+1$, *and* $|v_{s+2}| = 3$.

Proof. Since S_{g-1} is not obtained from ∅ by a sequence of expansions, Proposition 5.7 implies that S_{g-1} is built by applying a sequence of expansions to $A_{s,m}, B_m$, or $C_{s,m}$, for some $m > s \geq 2$. We consider these three possibilities in turn.

(I) $S_m = A_{s,m}$. In this case, v_s is a singleton in $G(S_{g-1})$. It follows that $s \in \{k, k+1\}$. If $s = k+1$, then since (v_{s-1}, v_s, v_m) spans a triangle and $v_{s-1} \sim v_g$, it follows that (v_s, v_m, v_g) forms a heavy triple. Therefore, $s = k$. If $m \neq j$, then (v_{s+1}, v_m, v_g) forms a heavy triangle. Therefore, $m = j$. If $m > s+2$, then $|v_{s+1}| = 2$, and $(v_{s+1}; v_{s-1}, v_{s+2}, v_g)$ induces a claw. Therefore, $m = s+2$. It follows that S_{g-1} is built from $A_{s,s+2}$ by a sequence of expansions. If $v_l \sim v_{k+1}$ for some $s+2 < l \leq g-1$, then either $l = s+3$ and $(v_{s+1}, v_{s-1}, v_{s+3}, v_g)$ induces a claw, or else $l > s+3$ and (v_{s+1}, v_l, v_g) forms a heavy triple. Consequently, no such l exists, and therefore $|v_i| = 2$ for all $s+2 < i \leq g-1$. This results in case (1).

(II) $S_m = B_m$. In this case, v_1 is a singleton in $G(S_{g-1})$, so $k = 1$. If $m \neq j$, then (v_2, v_m, v_g) forms a heavy triangle. Therefore, $m = j$. If $m > 3$, then $(v_2; v_3, v_m, v_g)$ induces a claw. Hence $m = 3$. It follows that S_{g-1} is built from B_4 by a sequence of expansions. If $v_l \sim v_2$ for some $3 < l \le g-1$, then either (v_3, v_l, v_g) forms a heavy triangle, or $(v_3; v_4, v_l, v_g)$ induces a claw. It follows that $|v_i| = 2$ for all $4 < i \le g-1$. This results in case (2).

(III) $S_m = C_{s,m}$. In this case, $(v_1, \dots, v_{s-1})$ spans a component of $G(S_{g-1})$. It follows that $k = s-1$. If $v_m \sim v_g$, then (v_s, v_m, v_g) forms a heavy triangle. Hence $v_m \not\sim v_g$. If $j > s+1$, then $(v_s; v_{s+1}, v_m, v_g)$ induces a claw. Hence $j = s+1$. Since $v_m \not\sim v_g$, it follows that $m = s+2$. Therefore, S_{g-1} is built from $C_{s,s+2}$ by a sequence of expansions. If $v_l \sim v_{s+1}$ for some $s+2 < l \le g-1$, then $l = s+3$ since $\langle v_g, v_l \rangle \le 1$, and then $(v_g; v_{s-1}, v_s, v_{s+3})$ induces a claw. It follows that $|v_l| = 2$ for all $s+2 < l < g$. This results in case (3). ☐

Lemma 7.4. *Suppose that there exists $v_m \in S$ with multiple smaller neighbors and that $m > g$. Then $m = g+1$, $g = s+2$, $|v_s| = s+1$, $v_{s+2} = -e_{s+2} + e_{s+1} + e_{s-1}$, $|v_{s+3}| = 5$, and $|v_i| = 2$ for $i = 1, \dots, s-1, s+1$.*

Proof. Since $G(S_{m-1})$ is connected and $G(S_m)$ does not contain a cycle of length > 3, it follows that v_m has precisely two smaller neighbors v_a, v_b with $a < b$, and (v_a, v_b, v_m) spans a triangle. By Lemma 4.11, it follows that $v_m = -e_m + \cdots + e_a$, $|v_l| = 2$ for all $l \le a$, and $|v_b|, |v_m| \ge 3$. Furthermore, $|v_l| = 2$ for all $l < m$, $l \ne s, b$. As $|v_s|, |v_g| \ge 3$, it follows that $s = a+1$ and $b = g$; since $|v_{k+1}| \ge 3$, it follows that $k = s$; and since $\langle v_m, v_g \rangle \le 1$, it follows that $v_g = -e_g + e_{g-1} + e_{s-1}$ for some $g \ge s+2$. If $g < m-1$, then $(v_g; v_{g-1}, v_{g+1}, v_m)$ induces a claw, and if $g > s+2$, then $(v_g; v_s, v_{g-1}, v_m)$ induces a claw. It follows that $m = g+1$, $g = s+2$, and S_m takes the stated form. ☐

Proposition 7.5. *Suppose that S contains a gappy vector v_g but no tight vector. Then S takes one of the following forms*:

(1) *$n = g$ and S is as in Lemma* 7.2(2);
(2) *$n \ge g$, and up to truncation, $|v_{g+1}| = 3$, $|v_i| = 2$ for all $g+1 < i \le n$, and S_g is as in Lemmas* 7.2 *or* 7.3, *except for Lemma* 7.2(2);
(3) *S_{g+1} is as in Lemma* 7.4, *and $|v_i| = 2$ for all $g+1 < i \le n$.*

Proof. If $n = g$, then the result is immediate. Thus, suppose that $n > g$, and select any $g < m \le n$. If v_m has multiple smaller neighbors, then $m = g+1$ and S_{g+1} takes the form stated in Lemma 7.4. Assuming this is not the case, v_m has a unique smaller neighbor. If $l := \min(\mathrm{supp}(v_m)) = 0$, then $v_m \sim v_s, v_g$, a contradiction. Hence $l > 0$, and v_l is the unique smaller neighbor of v_m. Observe that $l \ne g$, since then $(v_g; v_k, v_{k+1}, v_m)$ induces a claw. It follows that v_g has no larger neighbor. If $|v_{l+1}| = 2$, then $s < g \le l$, so v_l has a smaller neighbor v_i, and then $(v_l; v_i, v_{l+1}, v_g)$ induces a claw. It follows that $|v_{l+1}| \ge 3$.

Consequently, if $m > g$ is chosen minimal with $|v_m| \ge 3$, then $m = g+1$ and either S_{g+1} is as in Lemma 7.4, or else $|v_{g+1}| = 3$. Furthermore, there does not exist any $m' > m$ with $|v_{m'}| \geq 3$, since then $\min(\text{supp}(v_{m'})) = g$ and $v_{m'} \sim v_g$, which does not occur. Therefore, $|v_i| = 2$ for all $g+1 < i \leq n$. Finally, S_g cannot take the form stated in Lemma 7.2(2), for then (v_{k+1}, v_g, v_{g+1}) forms a heavy triple. The statement of the proposition now follows. □

8. **A tight vector**

Suppose that S contains a tight vector v_t. By Lemma 4.2(1), the index t is unique. The arguments in this section reach slightly beyond the criteria laid out in Section 4.2 that sufficed to carry out the analysis in Sections 5–7. Nevertheless, the basic ideas are the same as before. Again, we keep the notation introduced in the second paragraph of Section 6.

8.1. *All vectors unbreakable.* Propositions 8.2 and 8.3 describe the structure of a standard basis that contains a tight, unbreakable element. However, we do not make any assumption on v_t just yet, as these results will apply in Section 8.2.

LEMMA 8.1. *S_{t-1} is built from $\varnothing$ by a sequence of expansions.*

Proof. If $|v_i| = 2$ for all $i < t$, then the result is immediate, so suppose that $|v_s| \geq 3$ with $s < t$ chosen smallest. Thus, $|v_s| = s$ or $s+1$. Let us rule out the first possibility. If $|v_s| = s$, then $s \geq 3$ and $\langle v_t, v_s \rangle = s - 2 \geq 1$. Hence either $T_s \pitchfork T_t$, or else $s = 3$ and $T_s \dagger T_t$. In the first case, $(v_1; v_2, v_s, v_t)$ induces claw, and in the second case, (v_1, v_s, v_t) forms a negative triangle. Therefore, $|v_s| = s+1$.

It follows that $\langle v_t, v_s \rangle = |v_s| - 1 \geq 2$, so that $T_s \prec T_t$. As $\langle v_1, v_s \rangle = 0$, it follows that T_1 and T_s abut T_t at opposite ends. We claim that v_1 and v_s belong to separate components of $G(S_{t-1})$. For suppose the contrary, and choose a shortest path between them. Together with v_t they span a cycle of length ≥ 4 in $G(S)$ that is missing the edge (v_1, v_s), contradicting Corollary 4.12.

Therefore, $G(S_{t-1})$ is disconnected. It follows by Proposition 5.7 that S_{t-1} is built from $A_{s,m}, B_m, C_{s,m}$, or $\varnothing$ by a sequence of expansions. Let us rule out the first three possibilities in turn.

(a) $A_{s,m}$. Since $|v_m| \geq 4$ and $\langle v_t, v_m \rangle = |v_m| - 2$, it follows that $T_m \pitchfork T_t$. Since $v_{s+1} \sim v_m$, it follows that $T_{s+1} \dagger T_m$, whence $T_{s+1} \not\dagger T_t$ since otherwise $T_m \dagger T_t$. Hence $T_{s+1} \pitchfork T_t$ as well. In particular, $z_{s+1}, z_m \in T_t$. On the other hand, $(v_1, \ldots, v_{s-1}, v_{s+1}, v_m)$ induces a sun, with $|v_{s+1}|, |v_m| \geq 3$. It follows that T_1 is contained in the open interval with endpoints z_{s+1} and z_m, so that T_1 and T_t do not abut, in contradiction to $v_1 \sim v_t$.

(b) B_m. In this case, T_2 and T_m both abut T_t, and at the opposite end as T_1. As $|v_2|, |v_m| \geq 3$, it follows that both $T_2, T_m \prec T_t$. Hence one of T_2, T_m contains the other, in contradiction to their unbreakability.

(c) $C_{s,m}$. Now $T_s \prec T_t$. If $T_m \pitchfork T_t$, then T_m and T_t abut T_s at opposite ends. However, T_{s+1} abuts T_s as well, but $v_s \not\sim v_t, v_{s+1}$. It follows that $m = s+2$ and $T_m \dagger T_t$. But then (v_s, v_m, v_t) forms a negative triangle. It follows that S_{t-1} is built from $\varnothing$ by a sequence of expansions, as desired. □

Proposition 8.2. *Suppose that $|v_i| \neq 2$ for some $i < t$. Then $S = S_t$.*

Proof. We proceed by way of contradiction. Thus, suppose that $S \neq S_t$, and consider v_{t+1}. Since $G(S_t)$ is a path, Lemma 7.1 implies that v_{t+1} is not gappy. Set $k := \min(\mathrm{supp}(v_{t+1}))$. By Lemma 4.2(3), it follows that $k > 0$. Hence $\langle v_t, v_{t+1}\rangle \leq |v_{t+1}| - 3$, so $|v_{t+1}| \in \{2,3,4\}$. If $|v_{t+1}| = 2$, then $(v_t; v_1, v_s, v_{t+1})$ induces a claw. Similarly, if $|v_{t+1}| = 4$, then $(v_t; v_1, v_s, v_{t+1})$ induces a claw unless $k \in \{1, s\}$; but if $k \in \{1, s\}$, then (v_k, v_t, v_{t+1}) forms a negative triangle.

It remains to consider the case that $|v_{t+1}| = 3$. In this case, v_{t-1} is the unique smaller neighbor of v_{t+1}, and $v_t \not\sim v_{t+1}$, so $z_{t+1} \notin T_t$. Let $P \subset G(S)$ denote the induced path with consecutive vertices $(v_{i_1}, \ldots, v_{i_l})$, where $i_1 = t$ and $i_l = t+1$. Thus, $i_2 \in \{1, s\}$ and $i_{l-1} = t-1$. Observe that if $m < t$ is maximal with the property that $|v_m| \geq 3$, then $v_m \in V(P)$; in fact, $i_j = m$, where $j + m = t+1$. Note that $z_{i_j} \notin T_{i_h}$ for all $h \neq 1, j$. Let x denote the endpoint of T_t at which $T_{i_1} = T_t$ and T_{i_2} abut. Without loss of generality, suppose that y is the left endpoint of T_t. Thus, $T_{i_{j-1}}$ abuts the left endpoint of T_m. It follows that $T_{i_{j+1}}$ abuts the right endpoint of T_{i_j}, since otherwise $(v_{i_{j-1}}, v_{i_j}, v_{i_{j+1}})$ induces a triangle, while P is a path. Hence z_m separates x from all T_{i_h} with $h > j$. In particular, z_m separates x from $z_{t+1} \in T_{t+1} = T_{i_l}$. As $z_{t+1} \notin T_t$, it follows that z_{t+1} lies to the right of T_t. Hence $T_t \subset T := \bigcup_{h=2}^{l} T_{i_h}$. However, $d(T_t) = t+4 > t+1 \geq d(T)$, a contradiction. It follows that v_{t+1} cannot exist, so $S = S_t$, as desired. □

Henceforth we assume that $|v_i| = 2$ for all $i < t$.

Proposition 8.3. *Suppose that $z_i \notin T_t$ for all $i \leq n' \leq n$ with $|v_i| \geq 3$. Then $S_{n'}$ takes one of the following forms:*

(1) *$t = 1$, $|v_s| = s+1$ for some $s > 1$, $|v_i| = 2$ for all $1 < i < s$, and $S_{n'}$ is built from S_s by a sequence of expansions;*
(2) *$t = 1$, $|v_s| = s$ for some $s > 1$, $|v_i| = 2$ for all $1 < i < s$, and $S_{n'}$ is built from S_s by a sequence of expansions; or*
(3) *$t > 1$, $|v_i| = 2$ for all $i < t$, and $S_{n'}$ is built from S_t by a sequence of expansions.*

Notice that Proposition 8.3(1) allows the possibility that $s = 2$, a slight divergence from our convention on the use of s stated at the outset of Section 6. Under the assumption that $n = n'$, Proposition 8.3 produces three broad families of examples. Assuming instead that $n > n'$, Propositions 8.6, 8.7, and 8.8 utilize this result to produce even more.

Proof. By Lemma 7.1, $S_{n'}$ does not contain a gappy vector. Choose any $m > t$, and suppose by way of contradiction that v_m has multiple smaller neighbors. Since $G(S_{m-1})$ is connected and $G(S_m)$ does not contain a cycle of length > 3, it follows that v_m has exactly two smaller neighbors v_k and v_j, $k < j$, and $v_k \sim v_j$. Therefore, $|v_i| = 2$ for all $k+1 < i < m$, $i \neq j$, and since $G(S_m)$ does not contain a heavy triple, it follows that $|v_i| = 2$ for all $i \leq k$. Hence $t \in \{k+1, j\}$. However, if $t = k+1$, then (v_t, v_j, v_m) forms a heavy triple, while if $t = j$, then $k = 1, t = 3$, and (v_k, v_t, v_m) forms a negative triangle. Therefore, v_m has exactly one smaller neighbor.

Set $k := \min(\mathrm{supp}(v_m))$, and suppose that $k = 0$. Then $\langle v_t, v_m\rangle = t$, so it follows that $t = 1$. Since v_m has no other smaller neighbor, it follows that $|v_i| = 2$ for all $1 < i < m$. Thus, S_m takes the form stated in (2) with $m = s$. Suppose instead that $|v_{k+1}| = 2$. Then v_k has no smaller neighbor v_i, since then $(v_k; v_i, v_{k+1}, v_m)$ induces a claw. As $t \leq k$, $G(S_k)$ is connected, so $k = t = 1$. Thus, S_m takes the form stated in (3) with $m = s$. If neither $k = 0$ nor $|v_{k+1}| = 2$, then it follows that S_m is an expansion on S_{m-1}. By induction, it follows that S takes one of the forms stated in the lemma. □

8.2. *A tight, breakable vector.* Now we treat the case that v_t is breakable. This is the final and most arduous step in the case analysis, resulting in Propositions 8.6, 8.7, and 8.8.

LEMMA 8.4. *Suppose that v_t is breakable, $g \neq t$, $|v_g| \geq 3$, and $z_g \in T_t$. Then $g > t+1$ and either $t > 1$, $v_g = -e_g + e_{g-1} + e_{t-1}$, and $T_g \pitchfork T_t$, or else $v_g = -e_g + e_{g-1} + e_{t-1} + \cdots + e_0$ and $T_g \prec T_t$.*

Note that we do not assume *a priori* that v_g is gappy.

Proof. (a) $g > t+1$. Otherwise, $\langle v_t, v_g\rangle \in \{|v_g| - 3, |v_g| - 2\}$, with the second possibility if and only if $\min(\mathrm{supp}(v_g)) = 0$. Lemma 4.1 rules out the first possibility and Lemma 4.2(3) the second.

It follows that $\mathrm{supp}(v_g)$ contains at least two values $> t$.

(b) $v_{gt} = 0$. Otherwise, $1 \leq \langle v_t, v_g\rangle \leq |v_g| - 3$. By Lemma 4.1, we must have $\langle v_t, v_g\rangle = 1$ and $|v_g| = 3$, so $v_g = -e_g + e_{g-1} + e_t$. Now Lemma 4.2(2) implies that $z_g \notin T_t$, a contradiction.

As $z_g \in T_t$, it follows that $\langle v_t, v_g\rangle > 0$, so v_g is gappy and there exists a gappy index $k < t$. Since $|v_{k+1}| \geq 3$, it follows that $k = t-1$, and $\mathrm{supp}(v_g) \cap \{0, \ldots, t-1\}$ consists of consecutive integers.

(c) $\mathrm{supp}(v_g)$ *contains exactly two values* $> t$. Otherwise, $0 \leq \langle v_t, v_g\rangle \leq |v_g| - 2$, where the latter inequality is attained precisely when $v_g = -e_g + e_{g-1} + e_m + e_{t-1} + \cdots + e_0$ for some $t < m < g-1$. Thus, $T_g \pitchfork T_t$, $\varepsilon_g = \varepsilon_t$, and $\varepsilon_g([T_g - T_t] - [T_t - T_g])$ is reducible. However, this equals $v_g - v_t = -e_g + e_{g-1} + e_m + e_t - e_0$. Since every nonzero entry in this vector is ± 1, a decomposition $v_g - v_t = x + y$ with $\langle x, y\rangle = 0$ satisfies $x_i y_i = 0$ for all i, and both x and y have a negative coordinate. Without loss of generality, $x_g = -1$ and $y_0 = -1$. Then $0 = \langle y, \sigma\rangle \geq -1 + \sigma_i$ for some $i \in \{t, m, g-1\}$; but $\sigma_i \geq \sigma_t = t + 1 > 1$, a contradiction.

It follows that $v_g = -e_g + e_{g-1} + e_{t-1} + \cdots + e_l$ for some $0 \leq l \leq t-1$. Suppose by way of contradiction that $0 < l < t-1$. Then $(v_l; v_i, v_{l+1}, v_g)$ induces a claw in $G(S)$, where $i = l-1$ if $l > 1$ and $i = t$ if $l = 1$. Therefore, $l \in \{0, t-1\}$, and the statement of the lemma follows on consideration of $\langle v_t, v_g\rangle$. □

Observe that if v_t is breakable, $z_i \notin T_t$ for all $i < t$, and g is chosen minimally as in Lemma 8.4, then S_{g-1} takes one of the forms stated in Proposition 8.3. We assume henceforth that this is the case, and $g > t + 1$ is chosen minimally with $z_g \in T_t$.

Lemma 8.5. *Suppose that T_t is breakable, $T_i \prec T_t$, and let $C = \{v_t\} \cup \{v_j \mid T_j \dagger T_i, T_t\}$. Then C separates v_i in $G(S)$ from every other v_l of norm ≥ 3 for which $z_l \notin T_t$.*

Proof. For suppose the contrary, and choose an induced path P in $G(S) - C$ with distinct endpoints v_i, v_l such that $|v_l| \geq 3$ and every vector interior to P has norm 2. Set $T = \bigcup_{v_k \in V(P)} T_k$. Since $V(P) \cap C = \varnothing$ and $z_l \notin T_t$, it follows that $T_t \subset T$ and $T_t - T_j$ contains no vertex of degree ≥ 3. But then T_t is unbreakable, a contradiction. □

Proposition 8.6. *Suppose that S_{g-1} is as in Proposition* 8.3(1). *Then $s = 2$, $n \geq g = 3$, and S takes one of the following forms* (*up to truncation*):

(1) $|v_m| = m - 1$ *for some* $m \geq 4$; *or*

(2) $v_4 = -e_4 + e_3 + e_0$ *and* $|v_m| = m - 1$ *for some* $m \geq 5$.

Proof. Lemma 8.4 implies that $v_g = -e_g + e_{g-1} + e_0$, $T_g \prec T_t$, and furthermore that $G(S) = \widehat{G}(S)$ in this case. (See the end of the paragraph following Definition 4.6.)

(a) $g = s + 1$, *and* $s = 2$. If $g > s + 1$, then $G(S_{g-1})$ is a path, and v_g neighbors v_1, v_s, and v_{g-1}, so $(v_1, \ldots, v_g)$ spans a cycle in $G(S)$ missing the edge (v_1, v_{g-1}), in contradiction to Lemma 3.8. If $s > 2$, then $(v_1; v_2, v_s, v_g)$ induces a claw.

Let h denote the maximum index of a vector v_h for which $|v_h| \geq 3$ and $z_h \in T_1$.

(b) $v_h = -e_h + e_{h-1} + e_0$ *and* $h \in \{3, 4\}$. The first statement follows from Lemma 8.4, which also implies that $\varepsilon_h = \varepsilon_1 = \varepsilon_g$. If $h > 4$, then $\langle v_g, v_h\rangle = 1$. But both v_g and v_h are unbreakable, so $\langle v_g, v_h\rangle = \varepsilon_g\varepsilon_h\langle [T_g], [T_h]\rangle = \langle [T_g], [T_h]\rangle \leq 0$, a contradiction.

(c) $v_{m0} = 1$ *for all* $m > h$. For suppose that $v_{m0} = 1$ for some $m > h$. Then $v_{m1} = 1$ by Lemma 8.4 and the definition of h, which implies that $\langle v_m, v_2\rangle \geq 1$. It follows that T_1, T_2, T_m abut in pairs. But this cannot occur, since $T_m \not\prec T_1$ by assumption, and T_2 and T_m are both unbreakable.

(d) $v_{m1} = 0$ *for all* $m > h$. For suppose that $v_{m1} = 1$ for some $m > h$. Thus, $T_m \dagger T_1$. If $v_{m2} = 0$, then (v_1, v_2, v_m) forms a negative triangle. If $v_{m2} = 1$, then T_m abuts T_1 at the same end as T_3, so $v_{m3} = 0$, and then (v_1, v_3, v_m) forms a negative triangle.

It follows that $\min(\mathrm{supp}(v_m)) \geq 2$ for all $m > h$. In particular, $\langle v_t, v_m\rangle = 0$.

(e) *There is no* $m > h$ *for which* v_m *is gappy.* Suppose by way of contradiction that v_m is gappy for some smallest $m > h$, and choose a minimal gappy index k for v_m. Take $i = 3$ in Lemma 8.5. Then $C = \{v_1\}$, $v_k \not\sim v_3$, and so $k > 2$. If $h = 4$, then take $i = 4$ in Lemma 8.5. Then $C = \{v_1, v_2\}$, $v_k \not\sim v_h$, and so $k > 3$. In any event, it follows that $k > h - 1$, so v_{k+1} is not gappy. It follows as in the proof of Lemma 7.1 that $v_m \sim v_k, v_{k+1}$. Now either $v_k \sim v_{k+1}$, in which case (v_k, v_{k+1}, v_m) forms a heavy triangle, or else $v_k \not\sim v_{k+1}$, and then the connectivity of $G(S_{m-1})$ implies that $G(S_m)$ contains an induced cycle of length > 3. Either case results in a contradiction.

Thus, if $|v_m| \geq 3$ for some $m > h$, then Lemma 8.5 implies that v_m does not lie in the same component of $G(S) - \{v_1, v_2\}$ as v_g or v_h. It quickly follows that there is at most one index $m > h$ for which $|v_m| \geq 3$, and if so, then $v_m \sim v_2$. It then follows that S takes one of the forms stated in the proposition. □

PROPOSITION 8.7. *Suppose that* S_{g-1} *is as in Proposition* 8.3(2). *Then* $n \geq g = s + 1$, *and* S *takes one of the following forms* (*up to truncation*):

(1) $s = 2$, $v_4 = -e_4 + e_3 + e_0$, *and* $|v_m| = m - 1$ *for some* $m \geq 5$;
(2) $|v_m| = m - g + 2$ *for some* $m \geq g$; *or*
(3) $|v_m| = m - g + 3$ *for some* $m \geq g$.

Proof. As in Proposition 8.6, Lemma 8.4 implies that $G(S) = \widehat{G}(S)$. In particular, if $\langle v_j, v_t\rangle = \pm 1$, then T_j abuts T_t. Observe that $v_1 \sim v_2, v_s$, but $v_2 \not\sim v_s$. It follows that $T_1 \dagger T_2, T_s$, and T_2 and T_s are distant. If $g > s + 1$, then $(v_1; v_2, v_s, v_g)$ induces a claw. Hence $g = s + 1$, $T_g \prec T_1$, and T_g abuts T_1 at the same end as T_s.

(I) *Suppose that* $v_{j0} = 1$ *for some* $j > g$.

(a) $v_{j1} = 0$. If $v_{j1} = 1$, then $T_j \dagger T_1$. As T_s and T_g abut T_1 at the same end, it follows that either $v_j \not\sim v_s, v_g$, or else $v_j \sim v_s, v_g$ and $s = |v_s| = 2$. Furthermore, $v_{ji} = 1$ for all $1 \leq i \leq s - 1$, since otherwise $v_j \ll v_j - v_i$ for any such i with $v_{ji} = 0$. Now, if $v_j \not\sim v_g$, then $v_{js} = 0$, which implies that $\langle v_s, v_j\rangle = s - 1 \geq 1$ and $v_j \sim v_s$, a contradiction. If instead $v_j \sim v_s$ and $s = 2$, then (v_1, v_2, v_j) forms a negative triangle. Therefore, $v_{j1} = 0$.

It follows that $T_j \prec T_1$, so $v_j = -e_j + e_{j-1} + e_0$ according to Lemma 8.4. Now, T_j and T_g abut T_1 at opposite ends, so $v_j \not\sim v_g$. It follows that $j = g+1$. Furthermore, $s = 2$, since otherwise T_2 abuts T_1 at the same end as T_j, but $v_j \not\sim v_2$. In summary, $s = 2$, $g = 3$, $j = 4$, $v_4 = -e_4 + e_3 + e_0$, $T_4 \prec T_1$, and T_4 abuts T_1 at the opposite end as do T_2 and T_3. In particular, $v_{m0} = 0$ for all $m > 4$.

Now suppose that there exists $m > 4$ with $|v_m| \geq 3$.

(b) $v_{m1} = 0$, *and* $v_{m2} = v_{m3} = v_{m4}$. For suppose that $v_{m1} = 1$. Then $v_{m2} = 1$ since $|v_2| = 2$. Thus, $v_m \sim v_1$ and $v_m \not\sim v_2$. It follows that T_m abuts T_1 at the same end as T_4. Thus, $v_m \not\sim v_3$, so $v_{m3} = 1$, and $v_m \sim v_4$, so $v_{m4} = 0$. But now (v_1, v_4, v_m) forms a negative triangle, a contradiction. Thus, $v_{m1} = 0$. It follows that $v_m \not\sim v_1$, so $v_m \not\sim v_3, v_4$. Thus, $v_{m4} = v_{m3} = v_{m2}$.

Let us further suppose that $m > 4$ is minimal subject to $|v_m| \geq 3$. If $k := \min(\text{supp}(v_m)) > 4$, then $(v_k; v_{k-1}, v_{k+1}, v_m)$ induces a claw. Hence $k = 2$. Since $|v_i| = 2$ for $4 < i < m$, it follows that v_m is not gappy, and $v_m = -e_m + e_{m-1} + \cdots + e_2$.

(c) *There does not exist* $m' > m$ *such that* $|v_{m'}| \geq 3$. For suppose otherwise, and choose m' minimal with this property. Thus, (b) implies that $v_{m'1} = 0$ and $v_{m'2} = v_{m'3} = v_{m'4}$. If these values all equal 1, then $\langle v_m, v_{m'} \rangle \geq 2$, a contradiction. Hence $k' := \min(\text{supp}(v_{m'})) > 4$. Now, Lemma 8.5 implies that $k' \geq m$, taking $i = 4$ and $l = m'$ therein. But now $|v_{k'+1}| = 2$ and $v_{k'}$ has a smaller neighbor v_i, so $(v_{k'}; v_i, v_{k'+1}, v_{m'})$ induces a claw. This is a contradiction.

In summary, (I) leads to case (1) of the proposition.

(II) *Suppose that* $v_{m0} = 0$ *for all* $m > g$.

(d) *If* $s > 2$, *then* $v_{m1} = 0$ *for all* $m > g$. Assume the contrary, and choose m accordingly. It follows that $v_m \sim v_1$, and moreover that $T_m \dagger T_1$. Since v_s is unbreakable, it follows that T_m abuts T_1 at the same end as T_2. Thus, $v_m \not\sim v_s, v_g$. From $v_m \not\sim v_s$ it follows that $v_{m2} = \cdots = v_{m,s-1} = 0$ and $v_{ms} = 1$, and from $v_m \not\sim v_g$ it subsequently follows that $v_{m0} = 0$ and $v_{mg} = 1$. But then (v_1, v_2, v_m) forms a negative triangle.

It follows that if $v_{m1} = 1$, then $s = 2$ and $T_m \dagger T_1$; otherwise $v_m \not\sim v_1$. We henceforth drop any assumption about s.

(e) v_m *is not gappy for any* $m > g$. For suppose some v_m is, choose m minimal with this property, and choose a minimal gappy index k for v_m. If $g = k+1$, then $v_m \sim v_g$. If $v_{m1} = 0$, then Lemma 8.5 implies a contradiction with $i = g$ and $l = m$, while if $v_{m1} = 1$, then $s = 2$, $\langle v_3, v_m \rangle \geq 0$ and $T_m \dagger T_1$ implies that (v_1, v_3, v_m) is a negative triangle. It follows in either case that $g \neq k+1$, and since m is chosen minimal, it follows that v_{k+1} is not gappy. It follows at once that $v_{m,k-1} = 0$, whence $v_m \sim v_k, v_{k+1}$. Furthermore, $k \neq 1$

since $|v_{k+1}| \geq 3$. Since $k \geq 2$, both v_k and v_{k+1} are unbreakable. Now, if $v_k \sim v_{k+1}$, then $|v_k| \geq 3$, and (v_k, v_{k+1}, v_m) forms a heavy triangle. Hence $v_k \not\sim v_{k+1}$; but then a shortest path between them in $G(S_{k+1})$, together with v_m, results in an induced cycle of length > 3, a contradiction. It follows that v_m is not gappy.

(f) *v_m does not have multiple smaller neighbors for any $m > g$.* For suppose that $v_m \sim v_j$ for some $j > k := \min(\mathrm{supp}(v_m)) \geq 1$. Note that $j \neq g$ because of the form v_g takes, so v_j is not gappy, and it follows that $j = k+2$ is uniquely determined. In particular, it follows that $k > 1$. Hence $v_j \sim v_k$, since otherwise $G(S)$ contains an induced cycle of length > 3. As $|v_j| \geq 3$, it follows that $|v_i| = 2$ for all $1 < i \leq k$, since otherwise (v_i, v_j, v_m) forms a heavy triple for some such i. Moreover, $|v_{k+1}| \geq 3$, since otherwise $(v_k; v_{k-1}, v_{k+1}, v_m)$ induces a claw. It follows that $k = s-1$, but then $j = g$ and $v_j \not\sim v_m$. Therefore, v_m does not have multiple smaller neighbors.

Thus, $v_m \sim v_k$ and v_m has no other smaller neighbor. Furthermore, $|v_{k+1}| \geq 3$, as argued in the last paragraph. If $|v_m| \geq 3$ for some smallest $m > g$, then $k \in \{s-1, s\}$, and v_{m-1} lies in the same component of $G(S) - \{v_1, v_s\}$ as v_g. Suppose by way of contradiction that there exists some smallest $m' > m$ for which $|v_{m'}| \geq 3$. It follows from the foregoing that $\min(\mathrm{supp}(v_{m'})) + 1 = m$. But then $v_{m'}$ lies in the same component of $G(S) - \{v_1, v_s\}$ as v_g, in contradiction to Lemma 8.5.

Therefore, $|v_m| \geq 3$ for at most one value $m > g$, and in this case, v_m is not gappy, and $\min(\mathrm{supp}(v_m)) \in \{s-1, s\}$. The two possibilities lead to cases (2) and (3), respectively. □

PROPOSITION 8.8. *Suppose that S_{g-1} is as in Proposition* 8.3(3). *Then $n \geq g = t+2$, $v_{t+2} = -e_{t+2} + e_{t+1} + e_{t-1} + \cdots + e_0$, and S takes one of the following forms* (*up to truncation*):

1. $|v_{t+1}| = 2$, $v_{t+3} = -e_{t+3} + e_{t+2} + e_{t-1}$, *and* $|v_m| = m - t$ *for some* $m \geq t+4$;
2. $|v_{t+1}| = 3$, $v_{t+3} = -e_{t+3} + e_{t+2} + e_{t-1}$, *and* $|v_m| = m - t$ *for some* $m \geq t+4$;
3. $|v_{t+1}| = 2$ *and* $|v_m| = m - t$ *for some* $m \geq t+3$.
4. $|v_{t+1}| = 3$ *and* $|v_m| = m - t$ *for some* $m \geq t+3$.

Proof. (a) $v_g = -e_g + e_{g-1} + e_{t-1} + \cdots + e_0$ *for some* $g \geq t+2$. By Lemma 8.4, it follows that $v_g = -e_g + e_{g-1} + e_{t-1}$ or $-e_g + e_{g-1} + e_{t-1} + \cdots + e_0$, and $g \geq t+2$. Let us rule out the first possibility. Thus, assume by way of contradiction that this is the case. It follows that $v_g \not\sim v_t$ in $G(S)$. Note that $G(S_{g-1})$ is a path and that $v_g \sim v_{t-1}$. Suppose that $v_g \sim v_{g-1}$. It follows that $v_{g-1} \sim v_{t-1}$, since otherwise $G(S)$ contains an induced cycle of length > 3. However, since S_{g-1} is built from S_t by a sequence of expansions, it follows that $t-1=\min(\mathrm{supp}(v_{g-1}))$. However, this implies that $v_g \not\sim v_{g-1}$, a contradiction. Hence $v_g \not\sim v_{g-1}$. But then $(v_{t-1}; v_i, v_{g-1}, v_g)$ induces a claw, where $i=t$ if $t=2$ and $i=t-2$ if $t>2$. This contradiction shows that $v_g = -e_g + e_{g-1} + e_{t-1} + \cdots + e_0$, as desired.

(b) $g=t+2$. Observe that $|v_{t+1}| \in \{2,3\}$, since S_{t+1} is an expansion on S_t. If $|v_{t+1}|=2$, then $v_{t+1} \sim v_g$, since otherwise $(v_t; v_1, v_{n+1}, v_g)$ induces a claw. If $|v_{t+1}|=3$, then $v_{t+1} \not\sim v_g$, since otherwise $(v_g, v_t, v_1, \cdots, v_{t-1})$ induces a cycle of length >3 in $G(S)$. It follows in either case that $g=t+2$, as desired.

(c) *If* $v_h = -e_h + e_{h-1} + e_{t-1} + \cdots + e_0$, *then* $h=g$. For if $h \neq g$, then $T_g, T_h \prec T_t$ and T_g and T_h are distant, and $(v_t; v_1, v_g, v_h)$ induces a claw.

(d) *If* $v_h = -e_h + e_{h-1} + e_{t-1}$, *then* $h = g+1 = t+3$. For if $h > g+1$, then $(v_h, v_g, v_t, v_1, \ldots, v_{t-1})$ spans a cycle in $G(S)$ that is missing the edge (v_h, v_t).

Henceforth we write $h=g+1$ if v_{g+1} takes the form in article (d), and $h=g$ otherwise. It follows from Lemma 8.4 and articles (c) and (d) that if $|v_m| \geq 3$ for some $m>h$, then $z_m \notin T_t$. In particular, $v_m \sim v_i$ if and only if T_m and T_i abut for all $m>h$.

(e) *If* $m>h$, *then* v_m *is not gappy.* For suppose that v_m is gappy for some minimal $m>h$, let $k=\min(\mathrm{supp}(v_m))$, and let j denote a minimal gappy index for v_m. Then $k>0$, since otherwise $|v_{j+1}| \geq 3$ implies that $\langle v_{j+1}, v_g \rangle \geq 2$, and then $t=j+1$ and $z_g \in T_t$, a contradiction. Now $\langle v_m, v_k \rangle = -1$ and $\langle v_m, v_{j+1} \rangle = 1$, so $v_m \sim v_k, v_{j+1}$, and since $G(S_{m-1})$ is connected, it follows that $v_k \sim v_{j+1}$. Furthermore, $T_g \dagger T_{j+1}$ since $|v_g|, |v_{j+1}| \geq 3$, and since (v_k, v_{l+1}, v_g) is a positive triangle, it follows that $\langle v_k, v_{l+1} \rangle = -1$. Thus, $v_{l+1,k}=1$, and since $\langle v_{l+1}, v_g \rangle \leq 1$, it follows that $l=k$. Now, $v_{k,k-1} \neq 0$, so it follows that $v_{k+1,k-1}=0$. Consequently, v_{k+1} is gappy. Since $m>h$ was chosen minimal, it follows that $k+1 \in \{g,h\}$. However, the only way that this can occur and satisfy $\langle v_k, v_{k+1} \rangle = -1$ is if $k+1=g=t+2$ and $|v_{t+1}|=2$. However, in this case, $(v_t, v_{t+1}, v_m, v_{t+2})$ spans a cycle that is missing the edge (v_t, v_m), a contradiction. It follows that no such m exists, as desired.

Thus, $z_m \notin T_t$.

(f) $\min(\mathrm{supp}(v_m)) = t+1$ *or* $\geq t+3$ *for all* $m>h$. Let $k=\min(\mathrm{supp}(v_m))$. Since $\langle v_g, v_m \rangle \leq 1$, it follows that $k \geq t-1$. Lemma 8.5 with $i=g$ and $l=m$ implies that $k \notin \{t-1, t+2\}$. Finally, $k \neq t$, since otherwise $(v_t; v_1, v_g, v_m)$ induces a claw.

Suppose that there exists a minimal $m>h$ such that $|v_m| \geq 3$.

(g) $k=t+1$. For if $k \geq t+3$, then v_k has a smaller neighbor v_i, and $(v_k; v_i, v_{k+1}, v_m)$ induces a claw.

(h) *There does not exist* $m'>m$ *for which* $|v_{m'}| \geq 3$. Suppose otherwise, and let $k'=\min(\mathrm{supp}(v_{m'}))$. If $k'=t+1$, then either $(v_{t+1}, v_m, v_{m'})$ or $(v_g, v_m, v_{m'})$ forms a heavy triple, depending on $|v_{t+1}| \in \{2,3\}$. From (f) it follows that $k' \geq t+3$. Thus, $k'+1 = m$, since otherwise $|v_{k'+1}| = 2$, $v_{k'}$ has a smaller neighbor v_i, and $(v_{k'}, v_i, v_{k'+1}, v_{m'})$ induces a claw. Thus, $(v_h, \ldots, v_{m-1}, v_{m'})$ induces a path. If $h = t+2$, then we obtain a contradiction to Lemma 8.5 with $i = g$ and $l = m'$. If $h = t+3$, then we obtain a similar contradiction with a bit more work. Specifically, considering the path $(v_t, v_1, \ldots, v_{t-1}, v_{t+3}, \ldots, v_{m-1}, v_{m'})$, it follows that the interval $T_{t+3} \cup \bigcup_{i=1}^{t-1} T_i$ contains one endpoint of T_t and the interval $T_{m'} \cup \bigcup_{i=t+3}^{m-1} T_i$ contains the other. As $z_{m'} \notin T_t$, it follows that z_{t+3} is the unique vertex of degree ≥ 3 in T_t, a contradiction, since $z_{t+2} \in T_t$ as well.

The four cases stated in the proposition now follow from the possibilities $|v_{t+1}| \in \{2, 3\}$ and $h \in \{t+2, t+3\}$. □

9. **Producing the Berge types**

The goal of this section is to show how the families of linear lattices enumerated in the structural propositions of Sections 6, 7, and 8 give rise to the homology classes of Berge knots tabulated in Section 1.2. Section 9.1 describes the methodology, and Table 1 collects the results. Section 9.2 contains the necessary background material about continued fractions, and Sections 9.3 and 9.4 carry out the details.

9.1. *Methodology.* Given a standard basis S expressed in one of the structural propositions, we show that the changemaker lattice it spans is isomorphic to a linear lattice $\Lambda(p,q)$ by converting S into a vertex basis $B = \{x_1, \ldots, x_n\}$ for it. Letting ν denote the sequence of norms $(|x_1|, \ldots, |x_n|)$, we recover p as the numerator $N[\nu]^- = N[|x_1|, \ldots, |x_n|]^-$ of the continued fraction. We recover the value k of Proposition 2.2, and hence $q \equiv -k^2 \pmod{p}$, in the following way. Let $B^\star$ denote the elements in B that pair nontrivially with e_0, let $\nu_i = (|x_1|, \ldots, |x_i|)$, and let $p_i = N[\nu_i]^-$. Then

$$k = \sum_{x_i \in B^\star} p_{i-1} \langle x_i, e_0 \rangle \tag{9}$$

according to (3), Proposition 2.2, and Lemma 9.4(1). In practice, $B^\star$ contains at most three elements, and each value $\langle x_i, e_0 \rangle$ is typically ± 1. (In case of Proposition 8.3, it can equal ± 2.)

Example. As an illustrative example, consider a standard basis S as in Proposition 6.2(1). By inspection, $\widehat{G}(S)$ is nearly a path, which suggests that S is not far off from a vertex basis. Indeed, a little manipulation shows that

$$B = \{-v_s^\star, -v_{s+2}, v_{s+3}, v_{s-1}, \ldots, v_1^\star, -(v_{s+1} + v_{s-1} + \cdots + v_1)^\star\}$$

is a vertex basis for the lattice spanned by S. The elements denoted by a star ($\star$) belong to $B^\star$. From B we obtain the sequence of norms

$$\nu = (s+1, 3, 5, 2^{[s-1]}, 3),$$

using $2^{[t]}$ as a shorthand for a sequence of t 2's. In order to determine p, we calculate

$$\begin{aligned} p &= N[s+1, 3, 5, 2^{[s-1]}, 3]^- = N[s+1, 3, 4, -s, 2]^- \\ &= N[s+1, -3, 4, s, -2]^+ = 22s^2 + 31s + 11, \end{aligned}$$

using Lemma 9.5(1) for the second equality and Mathematica [47] for the last one. In order to determine k, we consider the substrings

$$\nu_0 = \varnothing, \quad \nu_{s+1} = (s+1, 3, 5, 2^{[s-2]}), \quad \nu_{s+2} = (s+1, 3, 5, 2^{[s-1]}).$$

Weight the numerator of each $[\nu_{i-1}]^-$ by $\langle x_i, e_0\rangle$, which equals the sign ± 1 appearing on the leading term in the starred expression, and add them up to obtain the value k. Thus,

$$\begin{aligned} k &= -N[\varnothing]^- + N[s+1, 3, 5, 2^{[s-2]}]^- - N[s+1, 3, 5, 2^{[s-1]}]^- \\ &= -1 + N[s+1, 3, 4, -(s-1)]^- - N[s+1, 3, 4, -s]^- \\ &= -1 + N[s+1, -3, 4, s-1]^+ - N[s+1, -3, 4, s]^+ \\ &= -1 + (11s^2 - s - 5) - (11s^2 + 10s + 2) \\ &= 11(-s-1) + 3. \end{aligned}$$

Since $s \geq 2$ and $p = (2k^2+k+1)/11$, it follows that the standard bases of 6.2(1) correspond to Berge type X with $k \leq 11(-3) + 3$. The result of this example appears in Table 1 and as the first entry in Table 2.

In this manner we extract a linear lattice, described by the pair (p, k), from each standard basis expressed in the structural propositions. In the process, we show that these values account for precisely the pairs (p, k) tabulated in Section 1.2. Table 1 displays the results. Note that Proposition 7.5(2) gets reported in terms of its constituents, Lemmas 7.2(1,3,4,5) and 7.3.

Table 1. Structural Propositions sorted by Berge type.

I$_+$ 6.5(1,3)	**I**$_-$ 8.3(1,3)	**II**$_+$ 8.3(2)	**II**$_-$ 6.2(3)
III(a)$_+$ 8.7(1), 8.8(1)	(a)$_-$ 6.5(2), 7.3(1,2), 7.5(3)	(b)$_+$ 8.6(1), 8.8(4)	(b)$_-$ 7.3(3)
IV(a)$_+$ 8.6(2), 8.8(2)	(a)$_-$ 7.2(3)	(b)$_+$ 8.8(3)	(b)$_-$ 6.4(5), 7.2(1), 7.5(3)
V(a)$_+$ 8.7(2)	(a)$_-$ 6.4(5), 6.5(2), 7.2(4)	(b)$_+$ 8.7(3)	(b)$_-$ 7.2(5)
VII 7.5(1)	**VIII** 8.2	**IX** 6.2(2), 6.4(2,3)	**X** 6.2(1), 6.4(1,4)

We mention one caveat, which amounts to the overlap between different Berge types. For example, in type III(a)$_+$, 8.7(1) and 8.8(1) account for the cases that $d = 2$, $(k+1)/d \geq 5$ and $d = 3$, $(k+1)/d \geq 3$, respectively (Table 4). What happens when $(d, (k+1)/d) \in \{(2,3), (1,*), (*,1)\}$? For $(2,3)$, notice that we obtain the same family of examples by setting $d = 3, (k+1)/d = 2$ in type V(a)$_+$. This is covered by 8.7(2). Moreover, 8.7(2) fills out most of V(a)$_+$, while only this sliver of it applies to III(a)$_+$. For that reason, we only report 8.7(2) next to V(a)$_+$ in Table 1. Similarly, the cases of $(1,*)$ and $(*,1)$ correspond to II$_-$ with $i = 2$ and I$_-$ with $i = 1$, respectively. In general, it is not difficult to identify the overlaps of this sort and use Table 1 to obtain a complete correspondence between structural propositions and Berge types. In a few places the overlap is explicit: 6.4(5), 6.5(2), and 7.5(3) each appear twice in Table 1.

The correspondence between structural propositions and Berge types exhibits some interesting features. For example, amongst the "small" families (defined just below), and excluding the special cases of 6.4(5) and 6.5(1),

- all elements of S are just right if and only if L is an exceptional type (IX or X);
- S has a gappy vector but no tight one if and only if L is of $-$ type;
- S has a tight vector if and only if L is of $+$ type.

It would be interesting to examine the geometric significance of this correspondence.

In determining the values (p, k) from the structural propositions, it is useful to partition these families into two broad classes: *large families*, those that involve a sequence of expansions, and *small families*, those that do not. The large families (along with 6.5(1)) correspond to Berge types I, II, VII, and VIII in Table 1. Determining the relevant values (p, k) for these families occupies Section 9.3. The small families, while more numerous, are considerably simpler to address. We take them up in Section 9.4. Excluding 6.5(1), they correspond to Berge types III, IV, V, IX, and X in Table 1.

Lastly, we remark that the determination of the isomorphism types of the sums of linear lattices enumerated in Proposition 5.7 follows as well, and involves far fewer cases. As it turns out, they correspond precisely to the sums of lens spaces that arise by surgery along a torus knot or a cable thereof. For example, Proposition 9.2 enumerates the sums of linear lattices spanned by standard bases built from $\varnothing$ by a sequence of expansions. They correspond to the connected sums $-(L(p,q)\#L(q,p))$ that result from pq-surgery along the positive (p,q)-torus knots. In fact, [23, Th. 1.5] asserts a much stronger conclusion: if surgery along a knot produces a connected sum of lens spaces, then it is either a torus knot or a cable thereof. We refer to [23] for further details.

9.2. *Minding p's and q's.* Given a basis $C = \{v_1, \dots, v_n\}$ built from $\varnothing$ by a sequence of expansions, augment C by a vector $v'_{n+1} := \sum_{i=k}^{n} e_i$, where $k = 0$ if $|v_i| = 2$ for all $v_i \in C$; $k = n-1$ if $|v_n| \geq 3$; and k is the maximum index of a vector in C with norm ≥ 3 otherwise. Observe that $C' := C \cup \{v'_{n+1}\}$ spans a lattice isomorphic to a sum of two nonzero linear lattices for which C' is a vertex basis. More precisely, partition $C' = \{v_{i_1}, \dots, v_{i_l}\} \cup \{v_{j_1}, \dots, v_{j_m}\}$ into vertex bases for the two summands, where $i_1 > \cdots > i_l$ and $n+1 = j_1 > \cdots > j_m$, and write $(a_1, \dots, a_l) = (|v_{i_1}|, \dots, |v_{i_l}|)$ and $(b_1, \dots, b_m) = (|v_{j_1}|, \dots, |v_{j_m}|)$. Then $\langle C' \rangle \cong \Lambda(p,q) \oplus \Lambda(p',q')$, where $p/q = [a_1, \dots, a_l]^-$ and $p'/q' = [b_1, \dots, b_m]^-$. The following result sharpens this statement.

Lemma 9.1. *The lattice spanned by C' is isomorphic to $\Lambda(p,q) \oplus \Lambda(p,p-q)$ for some $p > q > 0$.*

Note that Lemma 9.1 implies a relationship between the Hirzebruch-Jung continued fraction expansions of p/q and $p/(p-q)$. This is nicely expressed by the Riemenschneider point rule. (See the German original, [42, pp. 222–223]; or [32, pp. 2158–2159].)

Proof. We proceed by induction on $n = |C|$. When $n = 1$, we have $C' = \{e_0 - e_1, e_0 + e_1\}$, and C' spans a lattice isomorphic to $\Lambda(2,1) \oplus \Lambda(2,1)$, from which the lemma follows with $p = 2$ and $q = 1$.

For $n > 1$, observe that C' is constructed from C'_{n-1} by either setting $v_n = v'_n - e_n$ and $v'_{n+1} = e_{n-1} + e_n$, or else $v_n = e_{n-1} - e_n$ and $v'_{n+1} = v'_n + e_n$. By induction, C'_{n-1} determines two strings of integers $(a_1, \dots, a_l)$ and $(b_1, \dots, b_m)$, and $\langle C'_{n-1} \rangle \cong \Lambda(p,q) \oplus \Lambda(p,p-q)$, where $p/q = [a_1, \dots, a_l]^-$ and $p/(p-q) = [b_1, \dots, b_m]^-$. Swapping the roles of q and $p-q$ if necessary, C' determines the strings $(2, a_1, \dots, a_l)$ and $(b_1 + 1, \dots, b_m)$, for which we calculate $[2, a_1, \dots, a_l]^- = 2 - 1/(p/(p-q)) = (p+q)/p$ and $[b_1 + 1, \dots, b_m]^- = 1 + p/(p-(p-q)) = (p+q)/p$. Therefore, $\langle C' \rangle = \Lambda(p+q,q) \oplus \Lambda(p+q,p)$, which takes the desired form and completes the induction step. ☐

Proposition 9.2. *Suppose that C is built from $\varnothing$ by a sequence of expansions. If $|v_i| = 2$ for all $v_i \in C$, then $L \cong \Lambda(n+1,n)$. Otherwise, $L \cong \Lambda(p,q) \oplus \Lambda(r,s)$ for some $p > q > 0$, where $r = p - q$ and s denotes the least positive residue of $-p \pmod r$.*

Proof. Augment C to C' as above, and write

$$\langle C' \rangle \cong \Lambda(p,q) \oplus \Lambda(p,p-q)$$

according to Lemma 9.1. Then C determines the strings $(a_1, \dots, a_l)$ and $(b_2, \dots, b_m)$, where the second string is empty in case $m = 1$. We have $b_1 = \lceil p/(p-q) \rceil$, from which it easily follows that $[b_2, \dots, b_m]^- = r/s$ when $m > 1$, with the values r and s as above. Thus, L takes the desired form in this case. Furthermore, when $m = 1$, it follows easily that $L \cong \Lambda(n+1, n)$. This establishes the proposition. $\square$

Definition 9.3. Given integers $a_1, \dots, a_l \geq 2$, write $p_j/q_j = [a_1, \dots, a_j]^-$, $r_j = p_j - q_j$, and $p_0 = 1$, and define integers $b_1, \dots, b_m \geq 2$ by $[b_1, \dots, b_m]^- = p_l/r_l$.

Thus, Definition 9.3 relates the strings $(a_1, \dots, a_l)$ and $(b_1, \dots, b_m)$ preceding Lemma 9.1.

LEMMA 9.4. *Given integers $a_1, \dots, a_n \geq 2$ and an indeterminate x, the following hold*:

(1) $p_j = p_{j-1}a_j - p_{j-2}$ *and* $q_j = q_{j-1}a_j - q_{j-2}$;
(2) $[a_1, \dots, a_n, x]^- = (p_n x - p_{n-1})/(q_n x - q_{n-1})$;
(3) $[a_j, \dots, a_1]^- = p_j/p_{j-1}$;
(4) p_{j-1} *is the least positive residue of* $q_j^{-1} \pmod{p_j}$;
(5) q_{j-1} *is the least positive residue of* $-p_j^{-1} \pmod{q_j}$;
(6) r_{j-1} *is the least positive residue of* $p_j^{-1} \equiv q_j^{-1} \pmod{r_j}$.

Proof sketch. Item (1) follows by induction on k, using the identity

$$[a_1, \dots, a_j, a_{j+1}]^- = [a_1, \dots, a_j - 1/a_{j+1}]^-.$$

Item (2) follows at once from (1). Item (3) follows from $[a_{j+1}, \dots, a_1]^- = a_{j+1} - q_j/p_j$ and (1). The identity

$$p_{j-1}q_j - p_j q_{j-1} = 1$$

follows from (1) and induction; the inequalities $0 < p_{j-1} < p_j$ and $0 < q_{j-1} < q_j$ follow from the fact that $a_j \geq 2$; and items (4) and (5) follow from these observations. From the preceding identity we obtain

$$p_j r_{j-1} - p_{j-1} r_j = 1 \quad \text{and} \quad q_j r_{j-1} - q_{j-1} r_j = 1,$$

and (1) implies that $0 < r_{j-1} < r_j$. Item (6) now follows as well. $\square$

We collect a few more useful facts whose straightforward proofs follow from Lemma 9.4. Following Lisca, we use the shorthand

$$(\dots, 2^{[t]}, \dots) := (\dots, \underbrace{2, \dots, 2}_{t}, \dots).$$

LEMMA 9.5. *The following identities hold*:

(1) $[\dots, b+1, 2^{[a-1]}, c+1, \dots]^- = [\dots, b, -a, c, \dots]^-$;
(2) $[2^{[a-1]}, b+1, \dots]^- = p/q \implies [-a, b, \dots]^- = -p/(p-q)$;
(3) $[\dots, b+1, 2^{[a-1]}]^- = [\dots, b, -a]^-$;
(4) $[b_m, \dots, b_2]^- = r_l/(r_l - r_{l-1})$;
(5) $[a_1, \dots, a_l, t+1, b_m, \dots, b_2]^- = (p_l r_l t + 1)/(q_l r_l t + 1)$;

(6) $[a_1, \dots, a_l + 1, 2^{[t-2]}, b_m + 1, \dots, b_2]^- = (p_l r_l t - 1)/(q_l r_l t - 1)$;
(7) $[a_1, \dots, a_l, b_m, \dots, b_1]^- = (p_l^2 - p_l p_{l-1} + p_{l-1}^2)/(p_l q_l - p_l q_{l-1} + p_{l-1} q_{l-1})$;
(8) $[a_1, \dots, a_l, b_m, \dots, b_2]^- = (p_l r_l - p_{l-1} r_l + p_{l-1} r_{l-1})/(q_l r_l - q_{l-1} r_l + q_{l-1} r_{l-1})$;
(9) $[a_1, \dots, a_l + b_m + 1, \dots, b_1]^- = (p_l^2 + p_l p_{l-1} - p_{l-1}^2)/(q_l p_l + q_{l-1} p_l - q_{l-1} p_{l-1} - 1)$;
(10) $[a_1, \dots, a_l + b_m + 1, \dots, b_2]^- = (p_l r_l + p_{l-1} r_l - p_{l-1} r_{l-1} - 1)/(q_l r_l + q_{l-1} r_l - q_{l-1} r_{l-1} - 1)$.

9.3. *Large families.* For each standard basis S occurring in a large family, we alter at most one $v_i \in S$ to another $\overline{v}_i$ such that $S - \{v_i\} \cup \{\overline{v}_i\}$ is a vertex basis, up to reordering and negating some elements. In each case, there exists a unique partition $\{1, \dots, n\} = \{i_1, \dots, i_\lambda\} \cup \{j_1, \dots, j_\mu\}$ with the following properties:

- $i_1 = n$, and in the case of Propositions 7.2(2) and 8.2, $j_1 = n - 1$;
- $i_1 > \cdots > i_\lambda$ and $j_1 > \cdots > j_\mu$;
- $\{i_1, j_1\} = \{1, \min\{j > 1 \mid |v_j| > 2\}\}$;
- the subgraphs of $\widehat{G}(S - \{v_i\} \cup \{\overline{v}_i\})$ induced on $\{v_{i_1}, \dots, v_{i_\lambda}\}$ and $\{v_{j_1}, \dots, v_{j_\mu}\}$ are paths with vertices appearing in consecutive order (replacing v_i by $\overline{v}_i$).

For the first five families below, we modify S to a related subset C built from $\varnothing$ by a sequence of expansions. We obtain a pair of strings $(a_1, \dots, a_l)$, $(b_1, \dots, b_m)$ from C', and we express the sequence of norms ν in terms of them. The values l and m are related to the values λ and μ. To determine p and k from ν, we apply Lemmas 9.4 and 9.5. Frequently it is easier to recover k' instead of k by reversing the order of the basis.

Proposition 6.5(3).

$B = \{v_{i_1}, \dots, v_{i_\lambda}, v_{j_\mu}, \dots, v_{j_1}\}$ and $B^\star = \{v_1\}$;
$C = S - \{v_1\} \subset \text{span}\langle e_1, \dots, e_n \rangle$;
$\nu = (a_1, \dots, a_l, 2, b_m, \dots, b_2)$;
$p = p_l r_l + 1$ by 9.5(5) with $t = 1$;
$k = p_l$ if $j_\mu = 1$; $k' = r_l$ if $i_\lambda = 1$.

Note that $p_l \geq 3$ and $r_l \geq 2$. With $\{i, k\} = \{p_l, r_l\}$, it follows that $p = ik + 1$ with $i, k \geq 2$ and $\gcd(i, k) = 1$. The values i and k are unconstrained besides these conditions. In summary, 6.5(3) accounts for Berge type I_+ with $i, k \geq 2$.

Proposition 8.3(1).

$B = \{-v_{i_1}, \dots, -v^\star_{i_\lambda}, v^\star_{j_\mu}, \dots, v_{j_1}\}$;
$C = S \cup \{v', v'_t\} - \{v_t\} \subset \text{span}\langle e', e_0, \dots, e_n \rangle$, where $v' = -e_0 + e'$ and $v'_t = v_t + v'$;
$\nu = (a_1, \dots, a_l + b_m, \dots, b_2)$;

$p = p_l r_l - 1$ by 9.5(5) with $t = -1$;
$k = p_l$ if $j_\mu = 1$, using $a_l = 2$ and 9.4(1); $k' = r_l$ if $i_\lambda = 1$ in the same way.

With $\{i, k\} = \{p_l, r_l\}$, it follows that $p = ik - 1$ with $i, k \geq 3$ and $\gcd(i, k) = 1$. The values i and k obey two further constraints coming from $\max\{a_l, b_m\} = 3$ and $\max\{a_{l-1}, \dots, a_1, b_{m-1}, \dots, b_2\} \geq s + 1 \geq 3$. In summary, 8.3(1) accounts for part of Berge type I_- with $i, k \geq 3$.

Proposition 8.3(3). The argument is identical to the case of 8.3(1), switching the conclusions in the case of $i_\lambda = 1$ and $j_\mu = 1$. Now we have the constraint that $\max\{a_l, b_m\} = t + 2 \geq 4$. In summary, 8.3(3) accounts for another part of Berge type I_- with $i, k \geq 3$.

Proposition 8.3(2).

$B = \{v_{i_1}, \dots, v_{i_\lambda}, v_{j_\mu}, \dots, v_{j_1}\}$ and $B^\star = \{v_1\}$;
$C = S - \{v_1\} \subset \operatorname{span}\langle e_1, \dots, e_n\rangle$;
$\nu = (a_1, \dots, a_l, 5, b_m, \dots, b_2)$;
$p = 4p_l r_l + 1$ by 9.5(5) with $t = 4$;
$k = 2p_l$ if $j_\mu = 1$; $k' = 2r_l$ if $i_\lambda = 1$.

With $\{i, k\} = \{2p_l, 2r_l\}$, it follows that $p = ik + 1$ with $i, k \geq 2$ and $\gcd(i, k) = 2$. The values i, k are unconstrained besides these conditions. However, the case $\min\{i, k\} = 2$ (which occurs when $s = 2$) accounts for Berge I_- with $\min\{i, k\} = 2$. In summary, 8.3(2) accounts for Berge type II_+ and this special case of Berge I_-.

Proposition 6.2(3).

$B = \{-v_{i_1}, \dots, -\overline{v}^\star_{i_{\lambda-2}}, v_{i_{\lambda-1}}, v^\star_{i_\lambda}, v_{j_\mu}, \dots, v_{j_1}\}$ if $i_\lambda = 1$, in which case $i_{\lambda-2} = m$, $\overline{v}_{i_{\lambda-2}} = v_m + v_1 + v_2$, $i_{\lambda-1} = 2$, and $j_\mu = 3$;
$C = S \cup \{v'_3, v'_m\} - \{v_1, v_2, v_3, v_m\} \subset \operatorname{span}\langle e_2, \dots, e_n\rangle$, where $v'_3 = v_3 - e_1$ and $v'_m = v_m - e_1$;
$\nu = (a_1, \dots, a_l + 1, 2, 2, b_m + 1, \dots, b_2)$;
$p = 4p_l r_l - 1$ by 9.5(6) with $t = 4$;
$k = 2p_l$ if $i_\lambda = 1$; $k' = 2r_l$ if $j_\mu = 1$.

With $\{i, k\} = \{2p_l, 2r_l\}$, it follows that $p = ik - 1$ where $i, k \geq 4$ and $\gcd(i, k) = 2$. A similar argument applies in case $j_\mu = 1$. In summary, 6.2(3) accounts for Berge type II_-.

For the two remaining large families, we modify S directly into a subset C' as in Section 9.2. Let $(a'_1, \dots, a'_l)$ and $(b'_1, \dots, b'_m)$ denote its corresponding strings, and let $(a_1, \dots, a_l)$ and $(b_1, \dots, b_m)$ denote their reversals ($a_i = a'_{l+1-i}$ and $b_j = b'_{m+1-j}$). This notational hiccup results in cleaner expressions for p and k. Note that these values are still related in the manner of Definition 9.3, so Lemma 9.5 applies.

Proposition 7.5(1). Here $n = g$.

$B = \{-v^\star_{i_\lambda}, \dots, -v_{i_1}, \overline{v}_{j_1}, \dots, v^\star_{j_\mu}\}$ if $i_\lambda = 1$, where $j_1 = n$ and $\overline{v}_n = \overline{v}_g = v_g + v_{g-1} + \cdots + v_{k+2}$;[4]

$C' = S \cup \{v'_n\} - \{v_n\} \subset \operatorname{span}\langle e_0, \dots, e_{n-1}\rangle$, where $v'_n = \overline{v}_n + e_n + e_{n-1}$;

$\nu = (a_1, \dots, a_l, b_m, \dots, b_1)$;

$p = p_l^2 - p_l p_{l-1} + p_{l-1}^2$ by 9.5(7);

$k = p_l r_l - p_{l-1} r_l + p_{l-1} r_{l-1} - 1$ by 9.5(8); also, observe that the difference between the numerator and denominator in $[\nu]^-$ is $D := r_l^2 - r_l r_{l-1} + r_{l-1}^2$.

Now we use the identity $(a^2 - ab + b^2)(c^2 - cd + d^2) = (e^2 - ef + f^2)$, where $e = ac - bc + bd$ and $f = ad - bc$. We apply this identity with $a = p_l, b = p_{l-1}, c = r_l$, and $d = r_{l-1}$, noting that $f = 1$. It follows that $p \cdot D = (k+1)^2 - (k+1) + 1 = k^2 + k + 1$. Up to renaming variables, the same argument applies in case $j_\mu = 1$. In summary, 7.2(2) accounts for Berge type VII.

Proposition 8.2. Here $n = t$.

$B' = \{v^\star_{i_\lambda}, \dots, v_{i_1}, \overline{v}_{j_1}, \dots, v^\star_{j_\mu}\}$ if $i_\lambda = 1$, where $j_1 = n$ and $\overline{v}_n = \overline{v}_t = v_t - (v_{t-1} + \cdots + v_1)$;

$C' = S \cup \{v'_n, v'_{n+1}\} - \{v_n\}$, where $v'_n = \overline{v}_n - e_{n-1}$ and $v'_{n+1} = e_{n-1} + e_n$;

$\nu = (a_1, \dots, a_l + b_m + 1, \dots, b_1)$;

$p = p_l^2 + p_l p_{l-1} - p_{l-1}^2$ by 9.5(9);

$k = p_l r_l + p_{l-1} r_l - p_{l-1} r_{l-1}$ by 9.5(10);

and the difference between the numerator and denominator in $[\nu]^-$ is $D = r_l^2 + r_l r_{l-1} - r_{l-1}^2$.

Now we use the identity $(a^2 + ab - b^2)(c^2 + cd - d^2) = (e^2 + ef - f^2)$, where $e = ac - bd + bc$ and $f = ad - bc$. As before, we apply it with $a = p_l, b = p_{l-1}, c = r_l$, and $d = r_{l-1}$, noting that $f = 1$. It follows that $p \cdot D = k^2 + k - 1$. Again, the same conclusion holds if, instead, $j_\mu = 1$. Replacing k by $-k$, it follows in summary that 8.2 accounts for Berge type VIII.

9.4. *Small families.* The 26 small families fall to a straightforward, though somewhat lengthy, analysis. In each case, converting the standard basis S into a vertex basis B usually involves altering just one element from S into a sum of several such, and then permuting these elements and replacing some of them by their negatives. In two cases (8.7(1) and 8.8(2)) there are two such alterations, and in a handful there are none.

From the sequence of norms ν, it is straightforward to obtain the values p and k as in the example of Section 9.1. Lemmas 9.5(1,2,3) help reduce the number of terms appearing in the continued fraction expansions under consideration; note that although Lemma 9.5(3) relates two different fractions, their numerators are opposite one another. In this way, we reduce each string to one

[4] Apology: this is the k of Lemma 7.2, not the homology class of K.

Table 2. Small families and their Berge types

Prop$^{\text{n}}$.	Berge type	B and $B^\star$; ν; k, p
6.2(1)	X, $k \le 11(-3)+3$	$\{-v_s^\star, -v_{s+2}, v_{s+3}, v_{s-1}, \dots, v_1^\star, -(v_{s+1}+v_{s-1}+\dots+v_1)^\star\}$
		$(a+1, 3, 5, 2^{[a-1]}, 3)$, $a = s \ge 2$
		$k = 11(-a-1)+3$, $p = (2k^2+k+1)/11$
6.2(2)	IX, $k \le 11(-3)+2$	$\{-v_s^\star, -v_{s+3}, v_{s+2}, v_{s-1}, \dots, v_1^\star, -(v_{s+1}+v_{s-1}+\dots+v_1)^\star\}$
		$(a+1, 4, 4, 2^{[a-1]}, 3)$, $a = s \ge 2$
		$k = 11(-a-1)+2$, $p = (2k^2+k+1)/11$
6.4(1)	X, $k = 11(-2)+3$	$\{-v_1^\star, -v_3, v_4^\star, -v_2^\star\}$
		$(2, 3, 5, 3)$
		$k = -19$, $p = (2k^2+k+1)/11 = 64$
6.4(2)	IX, $k = 11(-2)+2$	$\{-v_1^\star, -v_4, v_3^\star, -v_2^\star\}$
		$(2, 4, 4, 3)$
		$k = -20$, $p = (2k^2+k+1)/11 = 71$
6.4(3)	IX, $k \ge 11(2)+2$	$\{-v_1^\star, \dots, -v_{s-1}, -v_{s+3}, v_{s+2}, v_s^\star, v_{s+1}\}$
		$(2^{[a-1]}, 5, 3, a+1, 2), a = s \ge 2$
		$k = 11a+2$, $p = (2k^2+k+1)/11$
6.4(4)	X, $k \ge 11(2)+3$	$\{-v_1^\star, \dots, -v_{s-1}, -v_{s+2}, v_{s+3}, v_s^\star, v_{s+1}\}$
		$(2^{[a-1]}, 4, 4, a+1, 2)$, $a = s \ge 2$
		$k = 11a+3$, $p = (2k^2+k+1)/11$
6.4(5)	IV(b)$_-$, $d = 3$, $\frac{2k-1}{d} \ge 5$	$\{v_2, v_1^\star, v_m, -v_3^\star, \dots, -v_{m-1}, -v_{m+1}, \dots, -v_n\}$
	and	$(2, 2, a+3, 4, 2^{[a-1]}, 3, 2^{[b-1]})$, $a = m-3 \ge 1, b = n-m \ge 0$
	V(a)$_-$, $d = 3$, $\frac{k+1}{d} \ge 3$	$k = 3a+5$, $p = (b+1)k^2 - 3(k+1)$
6.5(1)	any type with $k = 1$	$\{v_1^\star, \dots, v_n\}$
		$(2^{[n]})$
		$k = 1$, $p = n+1$
6.5(2)	III(a)$_-$, $d = 2$, $\frac{k+1}{d} = 3$	$\{-v_2, -v_1^\star, v_3, v_4^\star, \dots, v_n\}$
	and	$(2, 2, 3, 5, 2^{[a-1]})$, $a \ge 1$
	V(a)$_-$, $d = 3$, $\frac{k+1}{d} = 2$	$k = 5$, $p = 25(a+1) - 18$

with at most three variables (a, b, c) and eight entries, which a computer algebra package or a tenacious person can evaluate. We used Mathematica [47] to perform these evaluations, relying on the command `FromContinuedFraction` and the conversion $[\dots, a_i, \dots]^- = [\dots, (-1)^{i+1} a_i, \dots]^+$. Tables 2 and 4 report the results. We use variables a, b, c (instead of g, m, n, s, t) to keep notation uniform across different families. As in Table 1, we report Lemmas 7.2(1,3,4,5) and 7.3 in place of Proposition 7.5(2).

Certain degenerations in our notation deserve mention. A string ending in $(\dots, x, y, 2^{[-1]})$ should be understood as $(\dots, x)$. Thus, in 6.4(5), taking $b = 0$, we obtain the string $(2, 2, a+3, 4, 2^{[a-1]})$. Furthermore, it follows that $n = m-1$ in this case, so the vertex basis truncates to $\{v_2, v_1, v_m, -v_3, \dots, -v_{m-1}\}$. In 8.8(1) and (2), there are two degenerations that can occur. In these cases, the degeneration $b = 0$ can occur only if $c = 0$. If $b = c = 0$, then we obtain the strings $(2^{[a-1]}, 4, a+2, 2, 2)$ and $(a+2, 4, 4, 2^{[a-1]}, 3)$, respectively.

Table 3. Small families (cont$^{\text{d}}$)

7.2(1)	IV(b)$_-$, $d \geq 5$, $\frac{2k-1}{d} \geq 5$	$\{v_s^\star, v_{s+1}, -(v_g + v_1 + \cdots + v_{s-1} + v_{s+1})^\star, v_1^\star, \ldots, v_{s-1}, v_{s+2}, \ldots, v_{g-1}, v_{g+1}, \ldots, v_n\}$ $(a+1, 2, b+3, 2^{[a-1]}, 4, 2^{[b-1]}, 3, 2^{[c-1]})$, $a = s \geq 2, b = g - s - 2 \geq 1, c = n - g \geq 0$ $k = 2ab + 3a + b + 2$, $p = (c+1)k^2 - (2a+1)(k+1)$
7.2(3)	IV(a)$_-$, $d \geq 5$, $\frac{2k+1}{d} \geq 5$	$\{-v_1^\star, \ldots, -v_{s-1}, -v_{s+1}, v_g, v_s^\star, v_{s+2}, \ldots, v_{g-1}, v_{g+1}, \ldots, v_n\}$ $(2^{[a-1]}, 3, b+2, a+1, 3, 2^{[b-1]}, 3, 2^{[c-1]})$, $a = s \geq 2, b = g - s - 2 \geq 1, c = n - g \geq 0$ $k = 2ab + 3a + b + 1$, $p = (c+1)k^2 - (2a+1)(k-1)$
7.2(4)	V(a)$_-$, $d \geq 5$, $\frac{k+1}{d} \geq 3$	$\{-v_j, -v_1^\star, -v_g, v_2^\star, \ldots, v_{j-1}, v_{j+1}, \ldots, v_{g-1}, v_{g+1}, \ldots, v_n\}$ $(a+2, 2, b+3, 3, 2^{[a-1]}, 3, 2^{[b-1]}, 3, 2^{[c-1]})$, $a = j - 2 \geq 1, b = g - j - 1 \geq 1, c = n - g \geq 0$ $k = 2ab + 4a + 3b + 5$, $p = (c+1)k^2 - (2a+3)(k+1)$
7.2(5)	V(b)$_-$, $d \geq 3$, $\frac{k-1}{d} \geq 2$	$\{-v_{j-1}, \ldots, -v_2^\star, v_g, v_1^\star, v_j, \ldots, v_{g-1}, v_{g+1}, \ldots, v_n\}$ $(2^{[a-1]}, 3, b+2, 2, a+2, 2^{[b-1]}, 3, 2^{[c-1]})$, $a = j - 2 \geq 1, b = g - j \geq 1, c = n - g \geq 0$ $k = 2ab + 2a + b + 2$, $p = (c+1)k^2 - (2a+1)(k-1)$
7.3(1)	III(a)$_-$, $d \geq 3$, $\frac{k+1}{d} \geq 5$	$\{v_s^\star, v_g, -v_{s+1}, -v_{s-1}, \ldots, -v_1^\star, (v_{s+2} + v_1 + \cdots + v_{s-1})^\star, \ldots, v_{g-1}, v_{g+1}, \ldots, v_n\}$ $(a+1, b+2, 3, 2^{[a-1]}, 4, 2^{[b-1]}, 3, 2^{[c-1]})$, $a = s \geq 2, b = g - s - 2 \geq 1, c = n - g \geq 0$ $k = 2ab + 3a + 2b + 2$, $p = (c+1)k^2 - (a+1)(2k-1)$
7.3(2)	III(a)$_-$, $d = 2$, $\frac{k+1}{d} \geq 5$	$\{v_1^\star, v_g, -v_2^\star, v_3^\star, \ldots, v_{g-1}, v_{g+1}, \ldots, v_n\}$ $(2, a+2, 3, 4, 2^{[a-1]}, 3, 2^{[b-1]})$, $a = g - 3 \geq 1, b = n - g \geq 0$ $k = 4a + 5$, $p = (b+1)k^2 - 2(2k-1)$
7.3(3)	III(b)$_-$, $d \geq 2$, $\frac{k-1}{d} \geq 5$	$\{-v_1^\star, \ldots, -v_{s-1}, -v_g, -v_{s+1}, (v_s + v_{s+1})^\star, v_{s+2}, \ldots, v_{g-1}, v_{g+1}, \ldots, -v_n\}$ $(2^{[a-1]}, b+3, 2, a+1, 3, 2^{[b-1]}, 3, 2^{[c-1]})$, $a = s \geq 2, b = g - s - 2 \geq 1, c = n - g \geq 0$ $k = 2ab + 3a + 1$, $p = (c+1)k^2 - a(2k+1)$
7.5(3)	III(a)$_-$, $d \geq 2$, $\frac{k+1}{d} = 3$ *and* IV(b)$_-$, $d \geq 3$, $\frac{2k-1}{d} = 3$	$\{-v_s^\star, -v_{s+1}, -(v_{s+2} - v_1 - \cdots - v_{s-1})^\star, v_1^\star, \ldots, v_{s-1}, v_{s+3}, \ldots, v_n\}$ $(a+1, 2, 3, 2^{[a-1]}, 5, 2^{[b-1]})$, $a = s \geq 2, b = n - s - 2 \geq 0$ $k = 3a + 2$, $p = (b+1)k^2 - (k+1)(2k-1)/3$

10. **Proofs of the main results**

Recall that we established Theorem 1.6 in Section 3.4 using [23, Th. 3.3].

Proof of Theorem 1.7. Suppose that (p, q) appears on Berge's list. Then $\Lambda(p, q)$ embeds as the orthogonal complement to a changemaker by Theorem 1.6. On the other hand, suppose that $\Lambda(p, q)$ is isomorphic to a changemaker lattice $L = (\sigma)^\perp \subset \mathbb{Z}^{n+1}$. Then L has a standard basis S appearing in

Table 4. Small families (cont[d])

8.6(1)	III(b)$_+$, $d=2$, $\frac{k-1}{d} \geq 3$	$\{-v_3^\star, \ldots, -v_{m-1}, -(v_1 - v_3 - \cdots - v_{m-1})^\star, v_2^\star, v_m, \ldots, v_n\}$ $(3, 2^{[a-1]}, 4, 3, a+2, 2^{[b-1]})$, $a = m-3 \geq 1, b = n-m+1 \geq 0$ $k = 4a+3,\ p = bk^2 + 2(2k+1)$
8.6(2)	IV(a)$_+$, $d=5$, $\frac{2k+1}{d} \geq 5$	$\{-v_3^\star, -(v_1 - v_3 - v_4 - \cdots - v_{m-1})^\star, -v_{m-1}, \ldots, -v_4^\star,$ $v_2^\star, v_m, \ldots, v_n\}$, $(3, 3, 2^{[a-1]}, 3, 3, a+3, 2^{[b-1]})$, $a = m-4 \geq 1, b = n-m+1 \geq 0$ $k = 5a+7,\ p = bk^2 + 5(k-1)$
8.7(1)	III(a)$_+$, $d=2$, $\frac{k+1}{d} \geq 5$	$\{v_4, \ldots, v_{m-1}, (v_1 - v_3 - v_4 - \cdots - v_{m-1}), (v_3 + v_2)^\star,$ $-v_2, v_m, \ldots, v_n\}$, $(3, 2^{[a-1]}, 3, 3, 2, a+3, 2^{[b-1]})$, $a = m-4 \geq 1, b = n-m+1 \geq 0$ $k = 4a+5,\ p = bk^2 + 2(2k-1)$
8.7(2)	V(a)$_+$, $d \geq 3$, $\frac{k+1}{d} \geq 2$	$\{v_s, \ldots, v_2, (v_1 - v_2 - \cdots - v_s - v_{s+2})^\star, v_{m-1}, \ldots, v_{s+2}^\star,$ $v_{s+1}, v_m, \ldots, v_n\}$, $(2^{[a-1]}, 4, 2^{[b-1]}, 3, a+1, b+2, 2^{[c-1]})$, $a = s \geq 1, b = m-s-2 \geq 1, c = n-m+1 \geq 0$ $k = 2ab + 2a + b,\ p = ck^2 + (2a+1)(k+1)$
8.7(3)	V(b)$_+$, $d \geq 3$, $\frac{k-1}{d} \geq 2$	$\{v_{s+1}, v_{s+2}^\star, \ldots, v_{m-1}, (v_1 - v_{s+2} - \cdots - v_{m-1})^\star, \ldots, v_s,$ $v_m, \ldots, v_n\}$, $(a+1, 3, 2^{[b-1]}, 4, 2^{[a-1]}, b+3, 2^{[c-1]})$, $a = s \geq 1, b = m-s-2 \geq 1, c = n-m+1 \geq 0$ $k = 2ab + 2a + b + 2,\ p = ck^2 + (2a+1)(k-1)$
8.8(1)	III(a)$_+$, $d \geq 3$, $\frac{k+1}{d} \geq 3$	$\{v_1^\star, \ldots, v_{t-1}, v_{t+3}, \ldots, v_{m-1}, v_t - v_1 - \cdots - v_{t-1} -$ $v_{t+2} - \cdots - v_{m-1}, v_{t+2}^\star, v_{t+1}, v_m, \ldots, v_n\}$ $(2^{[a-1]}, 3, 2^{[b-1]}, 3, a+2, 2, b+3, 2^{[c-1]})$, $a = t \geq 2, b = m-t-3 \geq 0, c = n-m+1 \geq 0$ $k = 2ab + 3a + 2b + 2,\ p = ck^2 + (a+1)(2k-1)$
8.8(2)	IV(a)$_+$, $d \geq 7$, $\frac{2k+1}{d} \geq 3$	$\{-v_{t+2}^\star, -v_t + v_1 + \cdots + v_{t-1} + v_{t+2} + \cdots + v_{m-1},$ $-v_{m-1}, \ldots, (-v_{t+3} + v_1 + \cdots + v_{t-1})^\star, v_1^\star, \ldots, v_{t-1}, v_{t+1},$ $v_m, \ldots, v_n\}$, $(a+2, 3, 2^{[b-1]}, 3, 2^{[a-1]}, 3, b+3, 2^{[c-1]})$, $a = t \geq 2, b = m-t-3 \geq 0, c = n-m+1 \geq 0$ $k = 2ab + 3a + 3b + 4,\ p = ck^2 + (2a+3)(k-1)$
8.8(3)	IV(b)$_+$, $d \geq 5$, $\frac{2k-1}{d} \geq 3$	$\{-v_{t-1}, \ldots, -v_1^\star, (-v_t + v_{t+2} + \cdots + v_{m-1})^\star,$ $v_{m-1}, \cdots, v_{t+2}^\star, v_{t+1}, v_m, \ldots, v_n\}$ $(2^{[a-1]}, 4, 2^{[b-1]}, a+2, 2, b+2, 2^{[c-1]})$, $a = t \geq 2, b = m-t-2 \geq 1, c = n-m+1 \geq 0$ $k = 2ab + a + b + 1,\ p = ck^2 + (2a+1)(k+1)$
8.8(4)	III(b)$_+$, $d \geq 3$, $\frac{k-1}{d} \geq 3$	$\{-v_{t+2}^\star, \ldots, -v_{m-1}, (-v_t + v_{t+2} + \cdots + v_{m-1})^\star,$ $v_1^\star, \ldots, v_{t-1}, v_{t+1}, v_m, \ldots, v_n\}$ $(a+2, 2^{[b-1]}, 4, 2^{[a-1]}, 3, b+2, 2^{[c-1]})$, $a = t \geq 2, b = m-t-2 \geq 1, c = n-m+1 \geq 0$ $k = 2ab + 2a + b + 2,\ p = ck^2 + (a+1)(2k+1)$

one of the structural propositions of Sections 6–8. Section 9 in turn exhibits an isomorphism $L \cong \Lambda(p', q')$, where the pair (p', q') appears on Berge's list. By Proposition 3.6, $p' = p$, and either $q' = q$ or $qq' \equiv 1 \pmod{p}$. Hence at least one of the pairs (p, q), (p, q') appears on Berge's list. □

Proof of Theorem 1.2. This follows from Theorems 1.6 and 1.7, making use of Proposition 2.2 and the analysis of Section 9 to pin down the homology class of the knot K. □

We note that the statement of Theorem 1.2 holds with S^3 replaced by an arbitrary L-space homology sphere Y with d-invariant 0. The only modification in the set-up is to use the 2-handle cobordism W from $L(p, q)$ to Y. The space $X(p, q) \cup W$ is negative-definite and has boundary Y. By [38, Cor. 9.7] and Elkies' Theorem [13], its intersection pairing is diagonalizable, so [23, Th. 3.3] and the remainder of the proof go through unchanged.

Proof of Theorem 1.3. Suppose that $K_p = L(p, q)$, and let $K' \subset L(p, q)$ denote the induced knot following the surgery. By Theorem 1.2, $[K'] = [B'] \in H_1(L(p, q); \mathbb{Z})$ for some Berge knot $B \subset S^3$. By [40, Th. 2], it follows that $\widehat{HFK}(K') \cong \widehat{HFK}(B')$. By [40, Prop. 3.1 and the remark thereafter], it follows that $\Delta_{K'} = \Delta_{B'}$, where Δ denotes the Alexander polynomial. Since Δ depends only on the knot complement, it follows that $\Delta_K = \Delta_B$. By [39, Th. 1.2], Δ_K and Δ_B determine $\widehat{HFK}(K)$ and $\widehat{HFK}(B)$; therefore, these groups are isomorphic.

Next, suppose that K is doubly primitive. As remarked in the introduction, both K' and B' are simple knots, and since they are homologous, they are isotopic. Thus, the same follows for K and B, whence every doubly primitive knot is a Berge knot. □

Proof sketch of Theorem 1.4. The main idea is to analyze the changemakers implicit in the structural propositions and apply Proposition 10.1, which restates the essential content of [23, Prop. 3.1]. The use of weight expansions draws inspiration from [33]. For a vector $x = (x_0, \ldots, x_n)$, we make use of its L^1 *norm* $|x|_1 := \sum_{i=0}^{n} |x_i|$.

Proposition 10.1. *Suppose that* $K_p = L(p, q)$, *and let* σ *denote the corresponding changemaker. Then*

$$2g(K) = p - |\sigma|_1.$$

Note that we obtain equality in Proposition 10.1 because the 4-manifold $X(p, q)$ is sharp.

Definition 10.2. A *weight expansion* is a vector of the form

$$w = (\underbrace{a_0, \dots, a_0}_{m_0}, \underbrace{a_1, \dots, a_1}_{m_1}, \dots, \underbrace{a_j, \dots, a_j}_{m_j}),$$

where each $m_i \geq 1$, $a_{-1} := 0, a_0 = 1$, and $a_i = m_{i-1}a_{i-1} + a_{i-2}$ for $i = 1, \dots, j$.

It is an amusing exercise to show that the entries of w form the sequence of side lengths of squares that tile an $a_j \times a_{j+1}$ rectangle (cf. Figure 4).

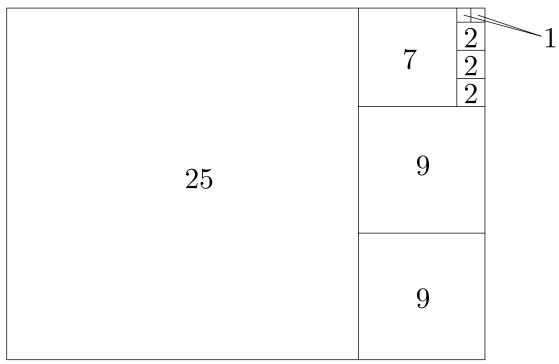

Figure 4. The tiling specified by the weight expansion $w = (1, 1, 2, 2, 2, 7, 9, 9, 25)$.

Another useful observation is $a_t = 1 + \sum_{i=0}^{t-1} m_i a_i - a_{t-1}$ for all $t \geq 1$. Thus,

$$|w| = a_j \cdot a_{j+1} \quad \text{and} \quad |w|_1 = a_j + a_{j+1} - 1, \tag{10}$$

which together with $a_{j+1} \geq a_j + 1$ leads to the bound

$$(|w|_1 + 1)^2 \geq 4|w| + 1. \tag{11}$$

Observe that a weight expansion is a special kind of changemaker. In fact, it is easy to check that a changemaker σ is a weight expansion if and only if the changemaker lattice $L = (\sigma)^\perp \subset \mathbb{Z}^{n+1}$ is built from $\varnothing$ by a sequence of expansions. Such lattices occur as one case of Proposition 5.7. Indeed, by inspection, for each changemaker lattice that appears in one of the structural propositions of Sections 5–8, the changemaker σ is just a slight variation on a weight expansion. For example, the changemakers implicit in Proposition 8.2 are obtained by augmenting a weight expansion by $a_j + a_{j+1}$, while those in Proposition 8.3(1,3) are obtained by deleting the first entry in a weight expansion with $m_0 \geq 2$.

Using Proposition 10.1, we obtain estimates on the genera of knots appearing in these families. For the changemakers σ specified by Proposition 8.2, (10) easily leads to the inequality

$$(|\sigma|_1 + 1)^2 \geq (4/5) \cdot (4|\sigma| + 1) \tag{12}$$

in the same way as (11). Furthermore, equality in (12) occurs precisely for changemakers σ of the form $(1,\ldots,1,n,2n+1)$, with 1 repeated n times. By Proposition 10.1, it follows that the bound (2) stated in Theorem 1.4 holds for type VIII knots, with equality attained precisely by knots K specified by the pairs $(p,k)=(5n^2+5n+1,5n^2-1)$.

Similarly, for the changemakers σ specified by Proposition 8.3(1,3), (10) easily leads to the bound $(|\sigma_1|+2)^2 \geq 4|\sigma|+5$. Furthermore, equality occurs precisely for changemakers σ of the form $(1,\ldots,1,n+1)$, with 1 repeated n times. By Proposition 10.1, it follows that the bound

$$2g(K)-1 \leq p+1-\sqrt{4p+5} \tag{13}$$

holds for type I_- knots, with equality attained precisely by knots K specified by the pairs $(p,k)=(n^2+3n+1,n+1)$. In fact, (12) holds for all the changemakers of Proposition 8.3(1,3) with the single exception of $(1,2)$. This corresponds to 5-surgery along a genus one L-space knot, which must be the right-hand trefoil by a theorem of Ghiggini [20]. Thus, the bound (2) holds for all the type I_- knots with the sole exception of 5-surgery along the right-hand trefoil.

The changemakers in the other structural propositions fall to the same basic analysis. Due to the abundance of cases, we omit the details, and instead happily report that the bound (2) is strict for the remaining lens space knots. This completes the proof sketch of Theorem 1.4. □

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