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A proof of the Breuil-Schneider conjecture in the indecomposable case

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Abstract

This paper contains a proof of a conjecture of Breuil and Schneider on the existence of an invariant norm on any locally algebraic representation of $\mathrm{GL}(n)$, with integral central character, whose smooth part is given by a generalized Steinberg representation. In fact, we prove the analogue for any connected reductive group G. This is done by passing to a global setting, using the trace formula for an $\mathbb{R}$-anisotropic model of G. The ultimate norm comes from classical p-adic modular forms.

1. Introduction

The p-adic Langlands program is still in its initial stages, especially for groups of higher rank. For a p-adic field F, one anticipates a correspondence between certain Galois representations $\rho : \mathrm{Gal}(\bar{\mathbb{Q}}_p/F) \to \mathrm{GL}_n(\bar{\mathbb{Q}}_p)$ and certain representations $\hat{\pi}$ of $\mathrm{GL}_n(F)$ on p-adic Banach spaces. See Breuil's survey [Bre10] from the ICM 2010. This correspondence should somehow be compatible with reduction mod p, cohomology, and p-adic families. This is a (big) theorem for $\mathrm{GL}_2(\mathbb{Q}_p)$, due to the work of many people (Berger, Breuil, Colmez, Emerton, Paskunas, and others). However, beyond this example, not much is known, although the subject is rapidly developing. Even $\mathrm{GL}_2(F)$, for fields $F \neq \mathbb{Q}_p$, seems surprisingly hard to deal with. Let us return to $\mathrm{GL}_2(\mathbb{Q}_p)$ for a moment, and give more details. We start off with a potentially semistable Galois representation

$$\rho : \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathrm{GL}(V) \simeq \mathrm{GL}_2(E),$$

with coefficients in a finite extension $E/\mathbb{Q}_p$. We assume ρ is *regular*. That is, it has distinct Hodge-Tate weights $w_1 < w_2$. By a standard recipe of Fontaine, to be recalled below, one associates a Weil-Deligne representation $\mathrm{WD}(\rho)$. By the classical local Langlands correspondence, its Frobenius-semisimplification $\mathrm{WD}(\rho)^{F-ss}$ corresponds to an irreducible smooth representation π' of $\mathrm{GL}_2(\mathbb{Q}_p)$

over E. We let $\pi = \pi' \otimes |\det|^{-1/2}$ if π' is generic (that is, infinite-dimensional). If π' is nongeneric, we replace it by $\pi = \pi'' \otimes |\det|^{-1/2}$, where π'' is a certain parabolically induced representation with π' as its unique irreducible quotient. This is the *generic* local Langlands correspondence. Note that π may be reducible. Now, one attaches to ρ an admissible unitary Banach space representation $B(\rho)$ of $\mathrm{GL}_2(\mathbb{Q}_p)$ over E satisfying a list of desiderata [Br, p. 8]. Most important for us is that $B(\rho)$ is the completion, relative to a suitable invariant norm, of the locally algebraic representation (at least when ρ is irreducible)

$$B(\rho)^{alg} = \det{}^{w_1} \otimes_E \mathrm{Sym}^{w_1 - w_2 - 1}(E^2) \otimes_E \pi.$$

Moreover, $B(\cdot)$ is compatible with the mod p local Langlands correspondence.

The Breuil-Schneider conjecture mimics some of this for $\mathrm{GL}_n(F)$. Again, let

$$\rho : \mathrm{Gal}(\bar{\mathbb{Q}}_p/F) \to \mathrm{GL}(V) \simeq \mathrm{GL}_n(E)$$

be a potentially semistable Galois representation. With ρ, we associate a Weil-Deligne representation $\mathrm{WD}(\rho)$ and a multiset of integers $\mathrm{HT}(\rho)$ as follows. Pick a finite Galois extension F'/F such that $\rho|_{\mathrm{Gal}(\bar{\mathbb{Q}}_p/F')}$ is semistable. Then

$$D = (B_{st} \otimes_{\mathbb{Q}_p} V)^{\mathrm{Gal}(\bar{\mathbb{Q}}_p/F')}$$

is a free $F'_0 \otimes_{\mathbb{Q}_p} E$-module of rank n, where F'_0 is the maximal unramified subfield of F'. The module D comes equipped with a Frobenius ϕ, a monodromy operator N such that $N\phi = p\phi N$, and a commuting action of $\mathrm{Gal}(F'/F)$. Moreover, there is an admissible filtration of $D_{F'}$ by $\mathrm{Gal}(F'/F)$-invariant $F' \otimes_{\mathbb{Q}_p} E$-submodules, which allows one to recover ρ. Observe that one has a factorization

$$D_{F'} \simeq \prod_{\sigma : F \to E} D_{F',\sigma}, \quad D_{F',\sigma} = D_{F'} \otimes_{F' \otimes_{\mathbb{Q}_p} E} (F' \otimes_{F,\sigma} E).$$

Hence, for each σ, we are given a filtration $\mathrm{Fil}^i(D_{F',\sigma})$ by $\mathrm{Gal}(F'/F)$-invariant free $F' \otimes_{F,\sigma} E$-submodules. Admissibility means, intuitively, that the Hodge polygon lies beneath the Newton polygon. More formally, one introduces numbers $t_N(D)$ and $t_H(D_{F'})$ as in [BS07, p. 15]. The former is given purely in terms of ϕ, the latter in terms of the filtration. One requires that $t_H(D_{F'}) = t_N(D)$ and that $t_H(D'_{F'}) \leq t_N(D')$ for any subobject $D' \subset D$ (with the induced filtration).

Hodge-Tate numbers. For every embedding $\sigma : F \to E$, the n-element multiset $\mathrm{HT}_\sigma(\rho)$ contains $i \in \mathbb{Z}$ with multiplicity $\mathrm{rk}_{(F' \otimes_{F,\sigma} E)} \mathrm{gr}^i(D_{F',\sigma})$. We label these as

$$\mathrm{gr}^i(D_{F',\sigma}) \neq 0 \Leftrightarrow i \in \mathrm{HT}_\sigma(\rho) = \{i_{1,\sigma} \leq \cdots \leq i_{n,\sigma}\}.$$

We say ρ is regular (at σ) if all the Hodge-Tate numbers $i_{j,\sigma}$ are distinct.

Weil-Deligne representation. Forgetting the filtration, the (ϕ, N)-module D gives rise to $WD(\rho)$ as follows. Choose an embedding $F_0' \hookrightarrow E$, and consider $D_E = D \otimes_{F_0' \otimes_{\mathbb{Q}_p} E} E$ with the inherited monodromy operator N and W_F-action defined by the formula

$$r(w) = \phi^{-d(w)} \circ \bar{w}, \quad w \in W_F.$$

(Here $d(w)$ is the power of arithmetic Frobenius induced by w, its image in $\mathrm{Gal}(F'/F)$ is $\bar{w}$, and ϕ is the semilinear Frobenius on B_{st}.) Note that $r|_{W_{F'}}$ is unramified. This defines $\mathrm{WD}(\rho) = (r, N, D_E)$, a Weil-Deligne representation.

Conversely, suppose we are given a Frobenius-semisimple Weil-Deligne representation (r, N, D_E) of W_F over E, unramified when restricted to $W_{F'}$, and for each $\sigma : F \to E$ a set of n distinct integers,

$$i_{1,\sigma} < \cdots < i_{n,\sigma}.$$

When do these data arise from a potentially semistable ρ? By [BS07, p. 14] we know (r, N, D_E) corresponds to a $(\phi, N) \times \mathrm{Gal}(F'/F)$-module D. What we are asking for is an admissible filtration $\mathrm{Fil}^i(D_{F',\sigma})$ such that

$$\mathrm{gr}^i(D_{F',\sigma}) \neq 0 \Leftrightarrow i \in \{i_{1,\sigma} < \cdots < i_{n,\sigma}\}.$$

The Breuil-Schneider conjecture asserts that this is the case precisely when some locally algebraic representation $\xi \otimes_E \pi$ (constructed from the given data) carries an invariant norm. That is, a non-archimedean norm $\| \cdot \|$ such that $\mathrm{GL}_n(F)$ acts unitarily.

The algebraic representation ξ. This is constructed out of the tuples $i_{j,\sigma}$. Let

$$a_{j,\sigma} = -i_{n+1-j,\sigma} - (j-1), \quad a_{1,\sigma} \leq \cdots \leq a_{n,\sigma}.$$

That is, write $i_{j,\sigma}$ in the opposite order, change signs, subtract $(0, 1, \ldots, n-1)$. The sequence $a_{j,\sigma}$ is identified with a dominant weight for GL_n, relative to the lower triangular Borel. We let ξ_σ be the corresponding irreducible algebraic representation of GL_n, and $\xi = \otimes \xi_\sigma$, viewed as an irreducible algebraic representation of the restriction of scalars $\mathrm{Res}_{F/\mathbb{Q}_p}\mathrm{GL}_n$, over E.

The smooth representation π. This is constructed out of (r, N, D_E) via a modified local Langlands correspondence. Let π° be the smooth irreducible representation of $\mathrm{GL}_n(F)$ (over $\bar{\mathbb{Q}}_p$) associated with (r, N, D_E) by the usual unitary local Langlands correspondence (after fixing a square root of $q = \#\mathbb{F}_F$)

$$(r, N, D_E) \simeq \mathrm{rec}(\pi^\circ \otimes |\det|^{(1-n)/2}).$$

The twist $\pi^\circ(\frac{1-n}{2})$ does not depend on the choice of $q^{\frac{1}{2}}$, and it can be defined over E. By the Langlands classification (see [Kud94] for a useful survey), π°

is the unique irreducible quotient of a parabolically induced representation

$$\mathrm{Ind}_P^G(Q(\Delta_1)\otimes\cdots\otimes Q(\Delta_r)) \twoheadrightarrow Q(\Delta_1,\ldots,\Delta_r)\simeq \pi^\circ.$$

Here the induction is normalized. The $Q(\Delta_i)$ are generalized Steinberg representations, built from segments of supercuspidals, Δ_i, ordered in a suitable way. We define

$$\pi = \mathrm{Ind}_P^G(Q(\Delta_1)\otimes\cdots\otimes Q(\Delta_r))\otimes|\det|^{(1-n)/2}.$$

By [BS07, p. 16], this π can be defined over E. Note that π may be reducible, and it admits $\pi^\circ(\frac{1-n}{2})$ as its unique irreducible quotient. Moreover, $\pi\simeq\pi^\circ(\frac{1-n}{2})$ exactly when the representation π° is generic. Also, π is always generic [JS83]. This is the so-called generic local Langlands correspondence for GL_n.

We are now in a position to state the conjecture, announced in [BS07] and [Bre10]. These references build on [ST06], where the crystalline case was discussed in detail (but with somewhat inconvenient normalizations).

THE BREUIL-SCHNEIDER CONJECTURE. *Fix data (r,N,D_E) and $i_{j,\sigma}$ as above, and let π and ξ be the representations constructed therefrom. Then the following two conditions are equivalent*:

1. *The data arises from a potentially semistable Galois representation.*
2. *The representation $\xi\otimes_E\pi$ admits a $\mathrm{GL}_n(F)$-invariant norm $\|\cdot\|$.*

The implication (2) $\Rightarrow$ (1) is completely known. A few cases were worked out in [BS07], and Hu proved it in general in [Hu09]. In fact, Hu proves a lot more. He shows that (1) is *equivalent* to what he refers to as the Emerton condition, which is a purely group theoretic statement: With V denoting the space $\xi\otimes_E\pi$,

$$(3)\ V^{N_0,Z_M^+=\chi}\neq 0\Rightarrow|\delta_P^{-1}(z)\chi(z)|\leq 1$$

for all $z\in Z_M^+$. The implication (2) $\Rightarrow$ (3) is an easy exercise.

We are concerned with the converse, (1) $\Rightarrow$ (2). Our main result is

THEOREM A. *The conjecture holds when (r,N,D_E) is indecomposable.*

Recall that indecomposable Weil-Deligne representations are precisely those obtained as follows. Starting with an irreducible representation $\tilde{r}:W_F\to\mathrm{GL}(\tilde{D})$, with open kernel, and a positive integer $\in\mathbb{Z}_{>0}$, let

$$D=\tilde{D}^{\oplus s},\quad r=\tilde{r}\oplus\tilde{r}(1)\oplus\cdots\oplus\tilde{r}(s-1),\quad N:\tilde{r}(i-1)\xrightarrow{\sim}\tilde{r}(i).$$

Here $\tilde{r}(i)$ denotes twisting $\tilde{r}$ by the ith power of $|\cdot|$, the absolute value on W_F, transferred from F^* via the reciprocity map. Under the (classical) local Langlands correspondence, $\tilde{D}$ corresponds to a supercuspidal τ, and D corresponds to the generalized Steinberg representation $Q(\Delta)$, where Δ is the segment

$$\Delta = \tau \otimes \tau(1) \otimes \cdots \otimes \tau(s-1).$$

The Jacquet modules of $Q(\Delta)$ can be made explicit; see Lemma 3.1 in [Hu09], for example. They are irreducible if nonzero. From that, it is easy to see that condition (3) just amounts to saying $\xi \otimes_E \pi$ has integral central character. In fact, this was already observed in Proposition 5.3 in [BS07], where they also state the resulting conjecture explicitly (as Conjecture 5.5), which is what we prove. Our methods work for any connected reductive group G defined over $\mathbb{Q}_p$.

Theorem B. *Let G be a connected reductive group over $\mathbb{Q}_p$. Let ξ be any irreducible algebraic representation of $G_{\bar{\mathbb{Q}}_p}$, and let π be any essentially discrete series representation of G. Then $\xi \otimes \pi$ admits a G-invariant norm if and only if its central character is integral.*

Taking $G = \mathrm{Res}_{F/\mathbb{Q}_p}\mathrm{GL}(n)$ yields Conjecture 5.5 in [BS07]. Indeed, the generalized Steinberg representations coincide with the essentially discrete series representations for $\mathrm{GL}(n)$. This theorem, and its proof, is purely group-theoretical. There is no mention of Galois representations, and much of the previous discussion is meant to be motivation only.

The proof of Theorem B (which implies Theorem A) is by passing to a global setting and making use of algebraic modular forms. By some sort of averaging over finite (cohomology) groups, we first reduce to the case where G is simple and simply connected, in which case the condition on the central character is vacuous. For such G, a result of Borel and Harder allows us to find a global model $G_{/\mathbb{Q}}$ such that $G(\mathbb{R})$ is compact. If π is a discrete series, a trace formula argument (due to Clozel in greater generality) shows that $\xi \otimes \pi$ admits an automorphic extension. Fixing an isomorphism $\iota : \mathbb{C} \to \bar{\mathbb{Q}}_p$, we infer that π^K sits as a submodule of $\mathcal{A}_{G,\xi}^K$, a space of classical p-adic modular forms. Therefore, $\xi \otimes \pi$ contributes to the direct limit of all $\xi \otimes \mathcal{A}_{G,\xi}^K$, which in turn embeds in $\mathcal{C}_G$ the space of all continuous functions $G(\mathbb{Q})\backslash G(\mathbb{A}_f) \to \bar{\mathbb{Q}}_p$. This latter space carries a supremum-norm, which is obviously invariant under $G(\mathbb{A}_f)$.

In [Sor12], a sequel to this paper, we make progress in the *decomposable* case. For instance, we show that the Banach space representations $B_{\xi,\zeta}$ introduced in [ST06] are nonzero (as conjectured) when (ξ, ζ) is globally relevant. This is done by combining the ideas of this paper with local-global compatibility at $p = \ell$ in the book project context; see [BLGGT] and [Car].

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2. Modular forms on definite reductive groups

2.1. *The complex case.*

2.1.1. *Notation.* For now, we will study automorphic forms on an arbitrary connected reductive group G over $\mathbb{Q}$ such that $G^{\mathrm{der}}(\mathbb{R})$ is compact. Here G^{der} is the derived subgroup, which is then necessarily an $\mathbb{R}$-anisotropic semisimple group. As is standard, A_G denotes the maximal $\mathbb{Q}$-split central torus in G, and we choose any central torus Z_G (over $\mathbb{Q}$) containing A_G. We will often take it to be the whole identity component of the center. K_∞ is the maximal compact subgroup of $G(\mathbb{R})$, which is unique, and possibly bigger than $G^{\mathrm{der}}(\mathbb{R})$.

2.1.2. *Classical automorphic forms.* Let $\mathbb{A} = \mathbb{R} \times \mathbb{A}_f$ be the ring of rational adeles. Inside $G(\mathbb{A})$, we introduce the normal subgroup $G(\mathbb{A})^1$ cut out by all $|\chi|$, where χ ranges over the $\mathbb{Q}$-characters of G. It contains $G(\mathbb{Q})$ as a cocompact discrete subgroup, and one has a decomposition

$$G(\mathbb{A}) = A_G(\mathbb{R})^+ \times G(\mathbb{A})^1.$$

Automorphic forms are affiliated with a central character, which we fix throughout. That is, we pick an arbitrary continuous (possibly nonunitary) character

$$\omega : Z_G(\mathbb{Q})\backslash Z_G(\mathbb{A}) \to \mathbb{C}^*$$

and consider the Hilbert space $L^2_G(\omega)$ of all measurable ω-central functions

$$f : G(\mathbb{Q})\backslash G(\mathbb{A}) \to \mathbb{C}, \quad \int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} |f(x)|^2 dx < \infty.$$

The right regular representation of $G(\mathbb{A})$ is completely reducible, and $L^2_G(\omega)$ breaks up into (irreducible) automorphic representations $\pi = \pi_\infty \otimes \pi_f$, each occurring with finite multiplicity $m_G(\pi)$. The space of automorphic forms $\mathcal{A}_G(\omega)$ is the dense subspace of smooth functions f satisfying the usual finiteness properties under the action of K_∞ and the center of the universal enveloping algebra at infinity. We will restrict ourselves to *algebraic* π. That is, we will assume π_∞ is the restriction of an irreducible algebraic (finite-dimensional) representation

$$\xi : G_{\mathbb{C}} \to \mathrm{GL}(W),$$

which we fix throughout. Its isotypic component is $\xi \otimes \mathcal{A}_{G,\xi}(\omega)$, where we let

Definition 1. $\mathcal{A}_{G,\xi}(\omega) = \mathrm{Hom}_{G(\mathbb{R})}(\xi, \mathcal{A}_G(\omega)) = (\xi^\vee \otimes \mathcal{A}_G(\omega))^{G(\mathbb{R})}.$

This is an admissible smooth representation of $G(\mathbb{A}_f)$, which breaks up as a direct sum $\oplus_\pi m_G(\pi)\pi_f$, summing over automorphic π, of central character $\omega_\pi = \omega$, such that $\pi_\infty = \xi$. We view elements of $\mathcal{A}_{G,\xi}(\omega)$ as vector-valued functions.

LEMMA 1. *As a $G(\mathbb{A}_f)$-module, $\mathcal{A}_{G,\xi}(\omega)$ can be identified with the space of all ω_f-central smooth functions*:

$$f : G(\mathbb{A}_f) \to W^\vee, \quad f(\gamma_f x) = \xi^\vee(\gamma_\infty)f(x) \quad \forall \gamma \in G(\mathbb{Q}).$$

Proof. One introduces a third space, consisting of all smooth ω-central functions:

$$f : G(\mathbb{Q})\backslash G(\mathbb{A}) \to W^\vee, \quad f(xg) = \xi^\vee(g)^{-1}f(x) \quad \forall g \in G(\mathbb{R}).$$

Such a function f gives a $G(\mathbb{R})$-map $\xi \to \mathcal{A}_G(\omega)$ by sending a vector $w \in W$ to the automorphic form $g \mapsto \langle f(g), w\rangle$. On the other hand, restriction to $G(\mathbb{A}_f)$ identifies it with the space of functions in the lemma. $\square$

Remark. We always assume ξ and ω are compatible; that is, $\omega_\infty = \xi|_{Z_G(\mathbb{R})}$.

By smoothness, as K varies over all compact open subgroups of $G(\mathbb{A}_f)$, one has

$$\mathcal{A}_{G,\xi}(\omega) = \varinjlim_K \mathcal{A}_{G,\xi}(\omega)^K,$$

where $\mathcal{A}_{G,\xi}(\omega)^K$ is the subspace of K-invariants, a module for the Hecke algebra $\mathcal{H}_{G,K}$ of all K-biinvariant compactly supported $\mathbb{C}$-valued functions on $G(\mathbb{A}_f)$. Again, for this subspace to be nonzero, we need K and ω to be compatible in the sense that ω_f is trivial on $Z_G(\mathbb{A}_f) \cap K$.

Example. When $\xi = 1$, we are just looking at the space $\mathcal{A}_{G,1}(\omega)$ of all ω_f-central smooth $\mathbb{C}$-valued functions on the profinite (hence compact) set

$$\tilde{S} = G(\mathbb{Q})\backslash G(\mathbb{A}_f) = \varprojlim_K S_K, \quad S_K = G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K.$$

Moreover, $\mathcal{A}_{G,1}(\omega)^K$ is the space of ω_f-central functions on the finite set S_K.

2.2. *The p-adic case.*

2.2.1. *Notation.* We fix a prime number p, an algebraic closure $\bar{\mathbb{Q}}_p$, together with an (algebraic) isomorphism $\iota : \mathbb{C} \xrightarrow{\sim} \bar{\mathbb{Q}}_p$. We will occasionally make use of an algebraic closure $\bar{\mathbb{Q}}$, always assumed to be endowed with an embedding $\iota_\infty : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$. Correspondingly, $\iota_p = \iota \circ \iota_\infty$ is an embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$. Via ι, we base change ξ to an algebraic representation over $\bar{\mathbb{Q}}_p$:

$$\iota\xi : G_{\bar{\mathbb{Q}}_p} \to \mathrm{GL}(\iota W), \quad \iota W = W \otimes_{\mathbb{C},\iota} \bar{\mathbb{Q}}_p.$$

Our central character ω has a p-adic avatar, the continuous character

$$\omega_{f,p} : Z_G(\mathbb{Q})\backslash Z_G(\mathbb{A}_f) \to \bar{\mathbb{Q}}_p^*, \quad \omega_{f,p}(z) = \iota\omega_\xi(z_p) \cdot \iota\omega_f(z).$$

(Here ω_ξ denotes the central character of ξ; the restriction of its highest weight to the connected center Z_G.)

2.2.2. *Classical p-adic automorphic forms.* All constructions of the previous section can be transferred to $\bar{\mathbb{Q}}_p$ via ι. When we put an ι in front, we mean tensoring by $\bar{\mathbb{Q}}_p$, as in $\iota W = W \otimes_{\mathbb{C},\iota} \bar{\mathbb{Q}}_p$.

LEMMA 2. *As a $G(\mathbb{A}_f)$-module, $\iota\mathcal{A}_{G,\xi}(\omega)^K$ can be identified with the space of all $\omega_{f,p}$-central functions (smooth away from p)*

$$f : G(\mathbb{Q})\backslash G(\mathbb{A}_f) \to \iota W^\vee, \quad f(xk) = \iota\xi^\vee(k_p)^{-1} f(x) \quad \forall k \in K.$$

Proof. Given a complex form f, as in the previous lemma, one associates the function $x \mapsto \iota\xi^\vee(x_p)^{-1}\iota f(x)$. It is easy to check that one can recover f. □

Definition 2. $\mathcal{C}_G(\omega) = \{\text{continuous } \omega_{f,p}\text{-central } G(\mathbb{Q})\backslash G(\mathbb{A}_f) \xrightarrow{f} \bar{\mathbb{Q}}_p\}$.

Any function f, as in the lemma, yields a K-map $\iota\xi \to \mathcal{C}_G(\omega)$ by sending $w \in \iota W$ to the continuous (in fact, locally algebraic) function $g \mapsto \langle f(g), w\rangle$, and vice versa. Here K acts on $\iota\xi$ through the projection to $G(\mathbb{Q}_p)$. We have shown that

$$\iota\mathcal{A}_{G,\xi}(\omega)^K = \mathrm{Hom}_K(\iota\xi, \mathcal{C}_G(\omega)) = (\iota\xi^\vee \otimes \mathcal{C}_G(\omega))^K.$$

Note that the image of K in $G(\mathbb{Q}_p)$ is compact open, hence Zariski dense, so that $\iota\xi$ is an irreducible representation of K. Let us look at its isotypic subspace $\mathcal{C}_G(\omega)[\iota\xi]$; that is, the sum of all K-stable subspaces isomorphic to $\iota\xi$. This is a semisimple K-representation, and $\mathrm{Hom}_K(\iota\xi, \mathcal{C}_G(\omega))$ is its multiplicity space

$$\iota\xi \otimes \iota\mathcal{A}_{G,\xi}(\omega)^K \xrightarrow{\sim} \mathcal{C}_G(\omega)[\iota\xi] \subset \mathcal{C}_G(\omega).$$

As K varies, these identifications are compatible with inclusions among the spaces $\mathcal{A}_{G,\xi}(\omega)^K$. Taking the direct limit, we end up with the injection

$$\varinjlim_K \iota\xi \otimes \iota\mathcal{A}_{G,\xi}(\omega)^K \hookrightarrow \mathcal{C}_G(\omega).$$

It can be checked that this map is $G(\mathbb{A}_f)$-equivariant. The image is the subspace of locally ξ-algebraic functions. Altogether, we arrive at our key result.

THEOREM 1. *There is an injective $G(\mathbb{A}_f)$-map $\iota\xi \otimes \iota\mathcal{A}_{G,\xi}(\omega) \hookrightarrow \mathcal{C}_G(\omega)$.*

Remark. At least when Z_G is trivial (so that $\omega = 1$), this can be found in Section (3.2) of [Eme06]. It is part of a larger picture, which we learnt after writing this paper. For a general G, Emerton introduces a certain spectral sequence computing *completed cohomology* $\tilde{H}^n$. In our case, $\tilde{H}^0 = \mathcal{C}_G$, and the injection from Theorem 1 is (essentially) an edge map of that spectral sequence; that is, (0.3) on page 3 in [Eme06]. Now Theorem 1 is basically just Corollary 2.2.25, page 37 in [Eme06]. We adopt this point of view in [Sor12].

2.2.3. *Existence of invariant norms.* The space $\mathcal{C}_G(\omega)$, being a subspace of $\mathcal{C}(\tilde{S}, \bar{\mathbb{Q}}_p)$, has a natural sup-norm

$$\|f\| = \sup_{x \in G(\mathbb{A}_f)} |f(x)|_p = \max_{x \in G(\mathbb{A}_f)} |f(x)|_p,$$

which is obviously invariant under the $G(\mathbb{A}_f)$-action; that is, $\|g \cdot f\| = \|f\|$.

COROLLARY 1. *If $\pi = \xi \otimes \pi_f$ is an automorphic representation of $G(\mathbb{A})$, then $\iota\xi \otimes \iota\pi_f$ has a natural $G(\mathbb{A}_f)$-invariant norm. (Here $G(\mathbb{A}_f^p)$ acts through $\iota\pi_f^p$ and $G(\mathbb{Q}_p)$ acts diagonally.)*

Since $\iota\xi \otimes \iota\pi_f = (\iota\xi \otimes \iota\pi_p) \otimes \iota\pi_f^p$, we deduce

COROLLARY 2. *If π_p is an irreducible admissible representation of $G(\mathbb{Q}_p)$, which extends to an automorphic representation of $G(\mathbb{A})$ of weight ξ, then $\iota\xi \otimes \iota\pi_p$ has a $G(\mathbb{Q}_p)$-invariant norm.*

This norm is far from canonical. There may be many ways to extend π_p.

3. A Grunwald-Wang type theorem

3.1. *The Grunwald-Wang theorem for* GL(1). From [AT09, p. 103], we briefly recall the following result of Grunwald (as corrected by Wang). This goes to show that one has to be careful about the center when prescribing automorphic representations locally. In the subsequent section we will restrict to *simple* groups for that reason.

THEOREM 2. *Given a number field F, a finite set of places S, and for each $v \in S$ a character χ_v of F_v^* of finite order, there exists a finite order Hecke character χ of F extending $\chi_S = \otimes_{v \in S} \chi_v$.*

Furthermore, the order of χ can be taken to be the least common multiple of the orders of the χ_v, unless a special case occurs (where the order of χ becomes twice that). Given an arbitrary χ_S, we see that it can be extended to a Hecke character conditionally; precisely, when some twist $\chi_S|\cdot|_S^s$ is of finite order. This is a constraint among the $\{\chi_v\}_{v \in S}$ (as $s \in \mathbb{C}$ depends only on S).

(There is a variant for tori, which follows easily from the congruence subgroup problem, as affirmed by Chevalley for tori.)

3.2. *Clozel's argument on limit multiplicities.* We will use the trace formula in its absolute simplest form. Namely, we will assume, for a moment, that G is *semisimple*. We keep all other assumptions. In particular, $G(\mathbb{R})$ is compact. The trace formula for G is the following identity:

$$\mathrm{tr}(\phi : L^2_G) = \sum_{\pi} m_G(\pi)\mathrm{tr}\pi(\phi) = \sum_{\{\gamma\}} \mathrm{vol}(G_\gamma(\mathbb{Q})\backslash G_\gamma(\mathbb{A}))O_\gamma(\phi),$$

valid for any test function $\phi \in \mathcal{C}_c^\infty(G(\mathbb{A}))$. On the spectral side, we are summing over all automorphic representations π. On the geometric side, the sum ranges over $\gamma \in G(\mathbb{Q})$, up to conjugacy. We denote by G_γ its stabilizer and by O_γ the orbital integral. Measures are chosen compatibly.

We wish to quickly outline an argument of Clozel, giving an analogue of the Grunwald-Wang theorem for G. We start off with a finite set of places S of $\mathbb{Q}$, which we assume contains ∞. At each $v \in S$, we are given a discrete series representation π_v° of $G(\mathbb{Q}_v)$. (That is, its matrix coefficients are square-integrable.)

THEOREM 3. *There is a function $\phi_v^\circ \in \mathcal{C}_c^\infty(G(\mathbb{Q}_v))$ such that, for every tempered irreducible admisisble representation π_v,*

$$\mathrm{tr}\pi_v(\phi_v^\circ) = \begin{cases} 1, & \pi_v = \pi_v^\circ, \\ 0, & \pi_v \neq \pi_v^\circ. \end{cases}$$

(Such a ϕ_v° is called a pseudo-coefficient of π_v°.)

Proof. For $v = \infty$, this is in [CD85]. The case $v \neq \infty$ is in [Clo86, p. 278]. □

Note. There may be nontempered π_v for which $\mathrm{tr}\pi_v(\phi_v^\circ) \neq 0$, but only finitely many. See [Clo86, p. 269] and [Clo86, p. 280]. Let us introduce $\phi_S^\circ = \otimes_{v\in S}\phi_v^\circ$. Then $\mathrm{tr}\pi_S(\phi_S^\circ) \neq 0$ for only finitely many representations $\pi_S^\circ = \pi_{S,0}, \ldots, \pi_{S,r}$.

With this choice of ϕ_S°, the spectral side becomes

$$\sum_{\pi^S} m_G(\pi_S^\circ \otimes \pi^S)\mathrm{tr}\pi^S(\phi^S) + \sum_{i=1}^r \sum_{\pi^S} m_G(\pi_{S,i} \otimes \pi^S)\mathrm{tr}\pi_{S,i}(\phi_S^\circ)\mathrm{tr}\pi^S(\phi^S)$$

for all $\phi^S \in \mathcal{C}_c^\infty(G(\mathbb{A}^S))$. We will take this ϕ^S to be of the following form:

$$\phi^S = \mathrm{vol}(K^S)^{-1} \cdot \mathrm{char}_{K^S},$$

where $K^S \subset G(\mathbb{A}^S)$ is a compact open subgroup, which we will let shrink to the identity below. With this choice, the spectral side turns into

$$\dim \mathrm{Hom}_{G(\mathbb{Q}_S)}(\pi_S^\circ, (L^2_G)^{K^S}) + \sum_{i=1}^r \dim \mathrm{Hom}_{G(\mathbb{Q}_S)}(\pi_{S,i}, (L^2_G)^{K^S})\mathrm{tr}\pi_{S,i}(\phi_S^\circ).$$

In some sense, the key ingredient of Clozel's proof is the following limit multiplicity formula, based on a method of DeGeorge-Wallach.

LEMMA 3. $\lim_{K^S\to 1} \mathrm{vol}(K^S) \dim \mathrm{Hom}_{G(\mathbb{Q}_S)}(\pi_{S,i}, (L^2_G)^{K^S}) = 0$ *for* $i > 0$.

Proof. This is (a weak version of) Lemma 8, [Clo86, p. 274]. □

Now, let us focus on the geometric side,

$$\sum_{\{\gamma\}} \mathrm{vol}(G_\gamma(\mathbb{Q})\backslash G_\gamma(\mathbb{A}))O_{\gamma_S}(\phi_S^\circ)O_{\gamma^S}(\phi^S).$$

Here, by Lemma 5 in [Clo86, p. 271], for sufficiently small K^S, the factor $O_{\gamma^S}(\phi^S) = 0$ unless γ is unipotent. Since G is $\mathbb{Q}$-anisotropic, this means $\gamma = 1$. In the limit, as $K^S \to 1$, the geometric side reduces to just one term,

$$\mathrm{vol}(G(\mathbb{Q})\backslash G(\mathbb{A}))\phi_S^\circ(1)\mathrm{vol}(K^S)^{-1}.$$

Here $\phi_S^\circ(1) = d(\pi_S^\circ) > 0$ is the formal degree, by the Plancherel formula. See Lemmas 9 and 12 in [Clo86]. Putting all this together, we arrive at the following limit formula.

Theorem 4. *We have*

$$\mathrm{vol}(K^S)\dim \mathrm{Hom}_{G(\mathbb{Q}_S)}(\pi_S^\circ, (L_G^2)^{K^S}) \underset{K^S\to 1}{\longrightarrow} \mathrm{vol}(G(\mathbb{Q})\backslash G(\mathbb{A}))d(\pi_S^\circ).$$

This is a weak version of Theorems 1A and 1B in [Clo86], which control ramification away from just one prime. We will not need this. On the other hand, Clozel's theorems give lower bounds for $\liminf_{K^S\to 1}$, not exact limits.

The following extension theorem, in the vein of Grunwald-Wang, will be crucial for the applications we have in mind later on.

Corollary 3. *Let G be a semisimple anisotropic $\mathbb{Q}$-group. Given a discrete series representation π_S° of $G(\mathbb{Q}_S)$, where S is a finite set of places of $\mathbb{Q}$, there is an automorphic representation π of $G(\mathbb{A})$ such that $\pi_S = \pi_S^\circ$.*

Proof. This follows immediately from Theorem 4. Since π_S° is a discrete series, $d(\pi_S^\circ) > 0$, so that the limit is nonzero. Consequently, for small enough K^S, there is a $G(\mathbb{Q}_S)$-embedding of π_S° into $(L_G^2)^{K^S}$, which decomposes as a direct sum of π^{K^S} for automorphic π. □

4. **Invariant norms on discrete series**

4.1. *Forms of algebraic groups.* We will quote (and use) a result of Borel and Harder on locally prescribed forms of algebraic groups. Recall that if G is an algebraic group over a field F, an F-form of G is an F-group G' isomorphic to G over the algebraic closure $\bar{F}$. This gives rise to a cocycle $c : \mathrm{Gal}(\bar{F}/F) \to \mathrm{Aut}(G)$ in the obvious way and identifies the set of equivalence classes of forms with the non-abelian Galois cohomology set

$$H^1(F, \mathrm{Aut}(G)).$$

We will take F to be a number field. For each place v of F, there is an obvious restriction map

$$H^1(F, \mathrm{Aut}(G)) \to H^1(F_v, \mathrm{Aut}(G)),$$

which on forms corresponds to extending scalars $G' \rightsquigarrow G'_v = G' \otimes_F F_v$.

THEOREM 5. *Let F be a number field, S a finite set of places of F, and G an (absolutely) almost simple F-group that is either simply connected or of adjoint type. Then the canonical restriction map is surjective*:

$$H^1(F, \mathrm{Aut}(G)) \twoheadrightarrow \prod_{v \in S} H^1(F_v, \mathrm{Aut}(G)).$$

In other words, given an F_v-form G'_v for each $v \in S$, there is an F-form G' equivalent to G'_v at places in S.

Proof. This is Theorem B in [BH78]. □

If v is a real (infinite) place of F, there is always a unique compact form G'_v, up to equivalence. The corresponding cocycle c is essentially given by the Cartan involution. We immediately deduce the following existence result, which will be used in the next section.

COROLLARY 4. *Let G be an almost simple $\mathbb{Q}_p$-group that is either simply connected or of adjoint type. Then there is a model over $\mathbb{Q}$, still denoted by G, such that $G(\mathbb{R})$ is compact.*

Proof. The group $G_{\bar{\mathbb{Q}}_p} \simeq G_{\mathbb{C}}$ has a split model over $\mathbb{Q}$ (even over $\mathbb{Z}$, this is the theory of Chevalley groups), which we will denote by G^*. We apply the theorem to this group, with $S = \{\infty, p\}$. At ∞ we take the compact form of $G^*_{\mathbb{R}}$. At p, we take G. □

4.2. *The simple case.* The following result is at the heart of our method.

LEMMA 4. *Let G be an almost simple $\mathbb{Q}_p$-group that is either simply connected or of adjoint type. Let ξ be any irreducible algebraic representation of $G_{\bar{\mathbb{Q}}_p}$, and let π be any discrete series representation of $G(\mathbb{Q}_p)$ (both over $\bar{\mathbb{Q}}_p$). Then the locally algebraic representation $\xi \otimes \pi$ carries a norm, which is invariant under the $G(\mathbb{Q}_p)$-action.*

Proof. The key is to embed this in a global situation. Thus, as in the previous corollary, we first find a $\mathbb{Q}$-model G such that $G(\mathbb{R})$ is compact. With a choice of an isomorphism $\iota : \mathbb{C} \to \bar{\mathbb{Q}}_p$, we can confuse ξ and π with representations over $\mathbb{C}$ (of $G_{\mathbb{C}}$ and $G(\mathbb{Q}_p)$ respectively). We will change notation and denote the previous π by π_p°. Also, we let $\pi_\infty^\circ = \xi|_{G(\mathbb{R})}$. Both are discrete series, so by Corollary 3 there is an automorphic representation π of $G(\mathbb{A})$ such that $\pi_\infty = \xi$ and $\pi_p = \pi_p^\circ$. By Corollary 2, we see that $\iota\xi \otimes \iota\pi_p^\circ$ has an invariant norm. □

4.3. *The semisimple case.* From the simple case, we derive the semisimple case.

LEMMA 5. *Let G be a connected semisimple $\mathbb{Q}_p$-group. Let ξ be any irreducible algebraic representation of $G_{\bar{\mathbb{Q}}_p}$, and let π be any discrete series representation of $G(\mathbb{Q}_p)$ (both over $\bar{\mathbb{Q}}_p$). Then the locally algebraic representation $\xi \otimes \pi$ carries a norm, which is invariant under the $G(\mathbb{Q}_p)$-action.*

Proof. Now, suppose G is any connected semisimple $\mathbb{Q}_p$-group, and let $G^{\mathrm{sc}} \twoheadrightarrow G$ be its universal covering over $\mathbb{Q}_p$; see [PR94]. The kernel $\pi_1(G)$ is finite. Being simply connected, G^{sc} is an actual direct product $G_1 \times \cdots \times G_r$, of finitely many simply connected simple groups G_i. By the main theorem of [Sil79], the restriction of π to G^{sc} is a direct sum of finitely many irreducible admissible representations

$$\pi|_{G^{\mathrm{sc}}} \simeq \oplus_{j=1}^{s}(\tau_{1,j} \otimes \cdots \otimes \tau_{r,j}),$$

where $\tau_{i,j}$ is a discrete series representation of $G_i(\mathbb{Q}_p)$. The restriction $\xi|_{G^{\mathrm{sc}}}$ remains irreducible, and we continue to denote it simply by ξ. It factors as a tensor product $\xi_1 \otimes \cdots \otimes \xi_r$, where ξ_i is an irreducible algebraic representation of $G_{i,\bar{\mathbb{Q}}_p}$. According to Lemma 4, each $\xi_i \otimes \tau_{i,j}$ has a norm $\|\cdot\|_{i,j}$, invariant under the action of $G_i(\mathbb{Q}_p)$. On the tensor product, where j is fixed for now,

$$(\xi_1 \otimes \tau_{1,j}) \otimes \cdots \otimes (\xi_r \otimes \tau_{r,j}),$$

we put the tensor product norm; see [Sch02, p. 110] and Proposition 17.4 therein. It has the property that

$$\|v_1 \otimes \cdots \otimes v_r\|_j = \|v_1\|_{1,j} \cdots \|v_r\|_{r,j},$$

with $v_i \in \xi_i \otimes \tau_{i,j}$. It is defined, for sums of pure tensors, by the formula

$$\|v\|_j = \inf\{\max \|v_1\|_{1,j} \cdots \|v_r\|_{r,j} : v = \sum v_1 \otimes \cdots \otimes v_r\}.$$

Here the maximum is over the same index set as the summation. The infimum is over all possible expressions for v. This tensor product norm $\|\cdot\|_j$ is clearly invariant under $G^{\mathrm{sc}}(\mathbb{Q}_p)$. Taking the maximum of all these, over $j = 1, \ldots, s$, we have constructed a $G^{\mathrm{sc}}(\mathbb{Q}_p)$-invariant norm $\|\cdot\|$ on $\xi \otimes \pi$. Now, to make it invariant under $G(\mathbb{Q}_p)$, we note that

$$G(\mathbb{Q}_p)/\mathrm{im}(G^{\mathrm{sc}}(\mathbb{Q}_p) \to G(\mathbb{Q}_p)) \subset H^1(\mathbb{Q}_p, \pi_1(G))$$

is a finite abelian group. Pick a set of representatives R, and replace $\|\cdot\|$ with

$$\|v\|' = \max_{g \in R} \|g \cdot v\|.$$

By construction, this modification $\|\cdot\|'$ is a $G(\mathbb{Q}_p)$-invariant norm on $\xi \otimes \pi$. $\square$

4.4. *The reductive case.* From the semisimple case, we derive the general reductive case.

Definition 3. An irreducible admissible complex representation π of $G(\mathbb{Q}_p)$ is essentially discrete series if a twist $\pi \otimes \nu$ is (unitary) discrete series for some smooth character $\nu : G(\mathbb{Q}_p) \to \mathbb{C}^*$. The essentially discrete series representations over $\bar{\mathbb{Q}}_p$ are those of the form $\iota\pi$ for *some* isomorphism $\iota : \mathbb{C} \to \bar{\mathbb{Q}}_p$.

Remark. To put this definition (over $\bar{\mathbb{Q}}_p$) on more solid ground, we would like to know that we can in fact pick any ι; in other words, whether any $\mathrm{Aut}(\mathbb{C})$-conjugate of an essentially discrete series representation is again essentially discrete series.[1] This is predicted by the local Langlands conjecture. (The parameter does not map into a proper Levi.) If $\sigma \in \mathrm{Aut}(\mathbb{C})$, the matrix coefficients of $\sigma\pi$ are σ-conjugates of matrix coefficients of π. Hence, it is certainly true for supercuspidals, but square integrability seems to be a problem. We should mention that at least it is known to be true for $\mathrm{GL}(n)$. Indeed the work of Bernstein-Zelevinsky shows that the essentially discrete series representations for $\mathrm{GL}(n)$ coincide with the generalized Steinberg representations $Q(\Delta)$, built from a segment Δ of supercuspidals, and $\sigma Q(\Delta) = Q(\sigma\Delta)$ in a suitable (rational) normalization. See [Kud94] for a nice exposition of the Langlands classification.

THEOREM 6. *Let G be a connected reductive group over $\mathbb{Q}_p$. Let ξ be any irreducible algebraic representation of $G_{\bar{\mathbb{Q}}_p}$, and let π be any essentially discrete series representation of $G(\mathbb{Q}_p)$ (both over $\bar{\mathbb{Q}}_p$). Then the locally algebraic representation $\xi \otimes \pi$ admits a $G(\mathbb{Q}_p)$-invariant norm if and only if its central character $\omega_\xi \cdot \omega_\pi$ is integral (that is, maps into $\bar{\mathbb{Z}}_p^\times$).*

Proof. The only if part is obvious. We assume $\omega_\xi \cdot \omega_\pi$ is integral and seek a norm. The derived subgroup G^{der} is semisimple, $Z_G \cap G^{\mathrm{der}}$ is finite, and

$$1 \to Z_G \cap G^{\mathrm{der}} \to Z_G \times G^{\mathrm{der}} \to G \to 1$$

is exact. Here Z_G is the full identity component of the center. The restriction $\xi|_{G^{\mathrm{der}}}$ hence remains irreducible, and we will just write ξ. On the other hand, the restriction $\pi|_{G^{\mathrm{der}}(\mathbb{Q}_p)}$ may not be, but it breaks up as a direct sum

$$\pi|_{G^{\mathrm{der}}(\mathbb{Q}_p)} \simeq \tau_1 \oplus \cdots \oplus \tau_r$$

of discrete series representations τ_i of $G^{\mathrm{der}}(\mathbb{Q}_p)$. For example, see [Tad92, p. 381 and p. 385]. By Lemma 5, there is a norm $\|\cdot\|_i$ on $\xi \otimes \tau_i$, invariant

[1] Tadic informs me that, at least for classical groups, this is known for generic representations. Clozel has later informed me that he proved this in *general* long ago and that Vigneras [Vig96] has published a different proof.

under $G^{\mathrm{der}}(\mathbb{Q}_p)$. Their maximum defines a $G^{\mathrm{der}}(\mathbb{Q}_p)$-invariant norm $\|\cdot\|$ on $\xi\otimes\pi$, which is automatically $Z_G(\mathbb{Q}_p)$-invariant, by our assumption on the central character. We have that

$$G(\mathbb{Q}_p)/Z_G(\mathbb{Q}_p)G^{\mathrm{der}}(\mathbb{Q}_p)\subset H^1(\mathbb{Q}_p, Z_G\cap G^{\mathrm{der}})$$

is a finite abelian group. Pick representatives R, and replace $\|\cdot\|$ with

$$\|v\|'=\max_{g\in R}\|g\cdot v\|.$$

This is independent of R and defines a $G(\mathbb{Q}_p)$-invariant norm on $\xi\otimes\pi$. □

Taking $G=\mathrm{Res}_{F/\mathbb{Q}_p}\mathrm{GL}(n)$, for a finite extension $F/\mathbb{Q}_p$, yields

Corollary 5. *Conjecture* 5.5 *in* [BS07] *holds true.*

Proof. As already mentioned, by Bernstein-Zelevinsky, the essentially discrete series representations of $\mathrm{GL}(n)$ are precisely the generalized Steinberg representations. □

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