

Kloosterman sheaves for reductive groups

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Abstract

Deligne constructed a remarkable local system on $\mathbb{P}^1 - \{0, \infty\}$ attached to a family of Kloosterman sums. Katz calculated its monodromy and asked whether there are Kloosterman sheaves for general reductive groups and which automorphic forms should be attached to these local systems under the Langlands correspondence.

Motivated by work of Gross and Frenkel-Gross we find an explicit family of such automorphic forms and even a simple family of automorphic sheaves in the framework of the geometric Langlands program. We use these automorphic sheaves to construct ℓ -adic Kloosterman sheaves for any reductive group in a uniform way, and describe the local and global monodromy of these Kloosterman sheaves. In particular, they give motivic Galois representations with exceptional monodromy groups G_2, F_4, E_7 and E_8 . This also gives an example of the geometric Langlands correspondence with wild ramification for any reductive group.

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Introduction

0.1. *Review of classical Kloosterman sums and Kloosterman sheaves.* Let n be a positive integer and p be a prime number. For every finite extension \mathbb{F}_q of \mathbb{F}_p and $a \in \mathbb{F}_q^\times$, the Kloosterman sum in n -variables is defined as the exponential sum

$$\mathrm{Kl}_n(a; q) := (-1)^{n-1} \sum_{x_1 x_2 \cdots x_n = a, x_i \in \mathbb{F}_q} \exp\left(\frac{2\pi i}{p} \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(x_1 + x_2 + \cdots + x_n)\right).$$

Kloosterman sums occur in the Fourier coefficients of modular forms. They satisfy the Weil bound

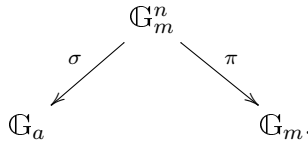
$$(0.1) \quad |\text{Kl}_n(a; q)| \leq nq^{(n-1)/2}.$$

When $n = 2$, for any $a \in \overline{\mathbb{F}}_p^\times$, define the angle $\theta(a) \in [0, \pi]$ to be such that

$$2p^{\deg(a)/2} \cos(\theta(a)) = \text{Kl}_2(a; p^{\deg(a)}),$$

where $\deg(a)$ is the degree of a over \mathbb{F}_p . Then the angles $\{\theta(a) | a \in \overline{\mathbb{F}}_p^\times\}$ are equidistributed according to the Sato-Tate measure $\frac{2}{\pi} \sin^2 \theta d\theta$ on $[0, \pi]$ as $\deg(a)$ tends to ∞ . In general, a similar equidistribution theorem for $\text{Kl}_n(a; q)$ was proved by Katz [26], using Deligne’s results in [11].

The above properties of Kloosterman sums were proved using a sheaf-theoretic incarnation. Let us recall this construction. In [10], Deligne considered the diagram



Here \mathbb{G}_m is the multiplicative group, $\mathbb{G}_a \cong \mathbb{A}^1$ is the additive group and σ (resp. π) is the map of taking the sum (resp. product) of the n -coordinates of \mathbb{G}_m^n . Let $\psi : \mathbb{F}_p \rightarrow \mathbb{Q}_\ell(\mu_p)^\times$ be a nontrivial character (here μ_p is the set of p -th roots of unity), and let AS_ψ be the associated Artin-Schreier local system on the additive group \mathbb{G}_a over \mathbb{F}_p . Deligne then defined the Kloosterman sheaf as the following complex of sheaves with $\mathbb{Q}_\ell(\mu_p)$ -coefficients on \mathbb{G}_m over \mathbb{F}_p :

$$\text{Kl}_n := R\pi_! \sigma^* \text{AS}_\psi[n - 1].$$

Fix an embedding $\iota : \mathbb{Q}_\ell(\mu_p) \hookrightarrow \mathbb{C}$ such that $\iota\psi(x) = \exp(2\pi i x/p)$ for $x \in \mathbb{F}_p$. For any $a \in \mathbb{G}_m(\mathbb{F}_q) = \mathbb{F}_q^\times$, denote by Frob_a the geometric Frobenius at a , acting on the geometric stalk $(\text{Kl}_n)_{\bar{a}}$. Then, by the Grothendieck-Lefschetz trace formula, we have

$$\text{Kl}_n(a; q) = \iota \text{Tr}(\text{Frob}_a, (\text{Kl}_n)_{\bar{a}}).$$

In this sense, Kl_n is a sheaf-theoretic incarnation of the Kloosterman sums $\{\text{Kl}_n(a; q)\}_{a \in \mathbb{F}_q^\times}$.

In [10, Th. 7.4, 7.8], Deligne proved

- (1) Kl_n is concentrated in degree 0 and is a local system of rank n .
- (2) Kl_n is tamely ramified around $\{0\}$, and the monodromy is unipotent with a single Jordan block.
- (3) Kl_n is totally wildly ramified around $\{\infty\}$ (i.e., the wild inertia at ∞ has no nonzero fixed vector on the stalk of Kl_n), and the Swan conductor $\text{Swan}_\infty(\text{Kl}_n) = 1$.
- (4) Kl_n is pure of weight $n - 1$ (which implies the estimate (0.1)).

In [26, §8.7], Katz proved the unicity of Kloosterman sheaves: If a rank- n local system L on \mathbb{G}_m satisfies [properties \(2\)](#) and [\(3\)](#) listed above, then L is isomorphic to Kl_n on $\mathbb{G}_m \otimes \overline{\mathbb{F}}_p$ up to a translation action on \mathbb{G}_m . Moreover, in [26, §11], Katz further studied the global geometric monodromy of Kl_n . Fix a geometric point $\bar{\eta}$ of \mathbb{G}_m , and denote by

$$\varphi : \pi_1(\mathbb{G}_m, \bar{\eta}) \rightarrow \mathrm{GL}_n(\mathbb{Q}_\ell(\mu_p))$$

the monodromy representation associated to the Kloosterman sheaf Kl_n . Let φ^{geo} be the restriction of φ to $\pi_1(\mathbb{G}_m \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q, \bar{\eta})$. Katz determined the Zariski closure \check{G}_{geo} of the image of φ^{geo} to be

$$(0.2) \quad \check{G}_{\mathrm{geo}} = \begin{cases} \mathrm{Sp}_n & n \text{ even,} \\ \mathrm{SL}_n & n \text{ odd, } p \text{ odd,} \\ \mathrm{SO}_n & n \text{ odd, } n \neq 7, p = 2, \\ G_2 & n = 7, p = 2. \end{cases}$$

Using Deligne’s equidistribution theorem [11, Th. 3.5.3], Katz showed that the conjugacy classes for the Frobenius elements Frob_a are equidistributed according to a “Sato-Tate” measure.

0.2. Motivation and goal of the paper. In view of the mysterious appearance of the exceptional group G_2 as the global geometric monodromy, Katz asked [27, p. I-5] whether all semisimple groups appear as geometric monodromy of local systems on \mathbb{G}_m . Alternatively, are there exponential sums whose equidistribution laws are governed by arbitrary simple groups, and especially by exceptional groups?

In this paper we find a uniform construction of such local systems. For any split reductive group \check{G} , we will construct a \check{G} -local system $\mathrm{Kl}_{\check{G}}$ on $\mathbb{G}_m = \mathbb{P}^1_{\setminus \{0, \infty\}}$ with similar local ramifications as Kl_n , and when $\check{G} = \mathrm{GL}_n$, we recover the Kloosterman sheaf Kl_n of Deligne. We will determine the Zariski closure of its global geometric monodromy (which turns out to be “large”), prove purity of the sheaf and deduce equidistribution laws. Finally, we give a conjecture about the unicity of such Galois representations (or local systems).

For the purpose of the introduction, let us restrict to the following cases. Assume G is either a split, almost simple and simply-connected group over $k = \mathbb{F}_q$, or $G = \mathrm{GL}_n$ over k . Let \check{G} be its Langlands dual group over $\mathbb{Q}_\ell(\mu_p)$, which is either a split, simple and adjoint group, or GL_n .

The motivation of our construction comes from the Langlands correspondence for the rational function field $K = k(t)$. Already in his study of Kloosterman sheaves, Katz [26] suggested the following.

“... It would be interesting to compare this result with the conjectural description of such sheaves, provided by the Langlands philosophy, in terms of automorphic forms.”

In a series of work [22], [21], [20], [15], Gross, partly joint with Reeder and with Frenkel, proposed a candidate automorphic representation π of $G(\mathbb{A}_K)$ which, in the case of $G = \mathrm{GL}_n$, should give the Kloosterman sheaf Kl_n . Let us briefly review their work.

In [22], Gross and Reeder gave the following construction of a representation V_ϕ of $G(k((s)))$. Fix a Borel subgroup $B \in G$, and denote by U the unipotent radical of B . Denote by $I(0) := \{g \in G(k[[s]]) \mid g \bmod s \in B\}$ the Iwahori subgroup of $G(k[[s]])$ and by $I(1) := \{g \in G(k[[s]]) \mid g \bmod s \in U\}$ the unipotent radical of $I(0)$. For any affine generic character $\phi: I(1) \rightarrow \mathbb{Q}_\ell(\mu_p)$ (see §1.3), let $V_\phi := \mathrm{c}\text{-Ind}_{I(1) \times Z(G)}^{G(k((s)))} (\phi \otimes 1)$ be the compactly induced representation of $G(k((s)))$. Gross and Reeder show that this representation is irreducible and supercuspidal.

For any global field F , Gross [21] managed to use the trace formula to obtain an expression for the multiplicities of automorphic representations π of $G(\mathbb{A}_F)$ whose ramified local components are either the Steinberg representation or the representation V_ϕ . In particular, when the global field is $K = k(t)$, this formula implies that, for any semisimple simply connected group G , there is unique cuspidal automorphic representation $\pi = \pi(\phi)$ of $G(\mathbb{A}_K)$ such that

- (1) π is unramified outside $\{0, \infty\}$;
- (2) π_0 is the Steinberg representation of $G(k((t)))$;
- (3) π_∞ is the simple supercuspidal representation V_ϕ of $G(k((s)))$, $s = t^{-1}$.

Motivated by the Langlands philosophy, Gross [20] raised the following conjecture. The automorphic representation $\pi = \pi(\phi)$ should correspond to a \check{G} -local system $\mathrm{Kl}_{\check{G}}(\phi)$ on $\mathbb{G}_m = \mathbb{P}^1_{\setminus \{0, \infty\}}$, which we will call the *Kloosterman sheaf* associated to \check{G} and ϕ , with the following properties parallel to that of π :

- (1) $\mathrm{Kl}_{\check{G}}(\phi)$ is a \check{G} -local system on $\mathbb{G}_m = \mathbb{P}^1_{\setminus \{0, \infty\}}$.
- (2) $\mathrm{Kl}_{\check{G}}(\phi)$ is tamely ramified around $\{0\}$, and the monodromy is a regular unipotent element in \check{G} .
- (3) The local system $\mathrm{Kl}_{\check{G}}^{\mathrm{Ad}}(\phi)$ associated to the adjoint representation of \check{G} (which is a local system on $\mathbb{P}^1_{\setminus \{0, \infty\}}$ of rank $\dim \check{G}$) is totally ramified around $\{\infty\}$ (i.e., the inert group at ∞ has no nonzero fixed vector), and $\mathrm{Swan}_\infty(\mathrm{Kl}_{\check{G}}^{\mathrm{Ad}}(\phi)) = r(\check{G})$, the rank of \check{G} .
- (4) For any irreducible representation V of \check{G} , the associated local system $\mathrm{Kl}_{\check{G}}^V(\phi)$ is pure.

For $\check{G} = \mathrm{GL}_n$, the sheaf Kl_n has the above properties. For \check{G} of type A, B, C and G_2 , \check{G} -local systems with the above properties were constructed

earlier by Katz [27], using a case-by-case construction and using Kl_n as building blocks.

In [15], Gross and Frenkel constructed an analog of such local systems over the complex numbers. Namely, they defined a $\check{G}_{\mathbb{C}}$ -connection $\nabla_{\check{G}}(\check{X})$ on $\mathbb{P}_{\mathbb{C}}^1 \setminus \{0, \infty\}$ (depending on the choice \check{X} of nonzero vectors in the affine simple root spaces of $\check{\mathfrak{g}}$). This connection has regular singularity at $\{0\}$ and irregular singularities at ∞ , parallel to [properties \(2\) and \(3\)](#) above. They furthermore show that $\nabla_{\check{G}}(\check{X})$ is cohomologically rigid [15, Th. 1] and compute the differential Galois group of these connections. For \check{G} of type A, B, C and G_2 , they verify that $\nabla_{\check{G}}(\check{X})$ coincide with connections constructed by Katz [27], which are the analogs of Katz's ℓ -adic \check{G} -local systems mentioned above. All these give strong evidence that $\nabla_{\check{G}}(\check{X})$ should be the correct de Rham analog of the conjectural local system $\text{Kl}_{\check{G}}(\phi)$.

The predictions about the conjectural local system $\text{Kl}_{\check{G}}(\phi)$ made in [15] served as a guideline for our work.

0.3. Method of construction. Our construction of the Kloosterman sheaves can be summarized as follows. We start with the automorphic representation π mentioned above. The key observation is that π contains a Hecke eigenfunction f_{ϕ} , which can be written down explicitly using the combinatorics of the double coset

$$G(K) \backslash G(\mathbb{A}_K) / \left(I_0 \times I_{\infty}(2) \times \prod_{x \neq 0, \infty} G(\mathcal{O}_x) \right)$$

(see §2.1).

The points of this double coset are the rational points of a moduli stack $\text{Bun}_{G(0,2)}$ of G -bundles on \mathbb{P}^1 with a particular level structure at 0 and ∞ , and we can upgrade the function f_{ϕ} to a sheaf A_{ϕ} on this stack. We then prove that this sheaf is indeed a Hecke eigensheaf. In particular, we show that the eigenvalues of the geometric Hecke operators applied to A_{ϕ} define a \check{G} -local system on $\mathbb{P}_{\mathbb{C}}^1 \setminus \{0, \infty\}$ — our Kloosterman sheaf $\text{Kl}_{\check{G}}(\phi)$. Technically, the proof of this result relies on the fact that the sheaf A_{ϕ} is a clean extension of a local system with affine support.

In the case $G = \text{GL}_n$, we compute the local system $\text{Kl}_{\check{G}}(\phi)$ explicitly. For a particular ϕ , the rank- n local system associated to $\text{Kl}_{\check{G}}(\phi)$ and the standard representation of $\check{G} = \text{GL}_n$ turns out to be the Kloosterman sheaf Kl_n of Deligne. Here Deligne's diagram will reappear as a part of a geometric Hecke transformation.

Our construction also works when we replace G by a certain class of quasi-split group schemes \mathcal{G} over $\mathbb{P}_{\mathbb{C}}^1 \setminus \{0, \infty\}$, and there is also a variant depending on an additional Kummer character. These generalizations seem to be the natural setup needed in order to compare Kloosterman sheaves for different groups.

Since the generalization does not require additional arguments, we will give the construction in this more general setup.

0.4. *Properties of Kloosterman sheaves.* The longest part of this article (§§4–6) is then devoted to the study of local and global properties of Kloosterman sheaves $\text{Kl}_{\check{G}}(\phi)$ for simple adjoint groups \check{G} . First, using [11], the expected purity property (4) is a corollary to our construction. When \check{G} is adjoint, we can moreover normalize $\text{Kl}_{\check{G}}^V(\phi)$ to be pure of weight 0. Thus, fixing an embedding $\iota : \mathbb{Q}_\ell(\mu_p) \hookrightarrow \mathbb{C}$ as before, the exponential sums

$$\text{Kl}_{\check{G}}^V(\phi; a; q) = \iota \text{Tr}(\text{Frob}_a, (\text{Kl}_{\check{G}}^V(\phi))_{\bar{a}}), \text{ for } a \in \mathbb{F}_q^\times, V \in \text{Rep}(\check{G})$$

satisfy the Weil bound

$$|\text{Kl}_{\check{G}}^V(\phi; a; q)| \leq \dim V.$$

Property (2) about the monodromy at 0 (see Theorem 1(2)) will be proved in Section 4.3. Property (3) about the monodromy at ∞ (see Theorem 2) will be proved in Section 5. The calculation of the Swan conductors involves a detailed study of the geometry of certain Schubert varieties in the affine Grassmannian, which occupies a large part of Section 5. Using a result of Gross-Reeder [22], one can give an explicit description of the monodromy at ∞ (Corollary 2.15). Together, these results show that the local system $\text{Kl}_{\check{G}}(\phi)$ satisfies properties (1)–(4) expected by Gross.

Again, let

$$\varphi : \pi_1(\mathbb{P}^1_{\setminus\{0,\infty\}}, \bar{\eta}) \rightarrow \check{G}(\mathbb{Q}_\ell(\mu_p))$$

be the monodromy representation associated to the Kloosterman sheaf $\text{Kl}_{\check{G}}(\phi)$, and let φ^{geo} be its restriction to $\pi_1(\mathbb{P}^1_{\setminus\{0,\infty\}} \otimes_k \bar{k}, \bar{\eta})$. We find (Theorem 3) that for $p > 2$ (and $p > 3$ in the case $\check{G} = B_3$), the Zariski closure \check{G}_{geo} of the image of φ^{geo} coincides with the differential Galois group \check{G}_{∇} of $\nabla_{\check{G}}(\check{X})$ calculated by Frenkel-Gross in [15, Cor. 9,10], which we list in Table 1.

\check{G}	\check{G}_{geo}
A_{2n}	A_{2n}
A_{2n-1}, C_n	C_n
$B_n, D_{n+1} \ (n \geq 4)$	B_n
E_7	E_7
E_8	E_8
E_6, F_4	F_4
B_3, D_4, G_2	G_2

Table 1. Global geometric monodromy of $\text{Kl}_{\check{G}}(\phi)$

Deligne’s equidistribution theorem [11, Th. 3.5.3] then implies the following equidistribution law for the image of the Frobenius elements under φ .

COROLLARY. *Suppose \check{G} is simple and adjoint. (In this case the whole image of φ lies in \check{G}_{geo} .) Let $\check{G}_{\text{geo},c} \subset \check{G}_{\text{geo}}(\mathbb{C})$ be a compact real form (using the embedding $\iota : \mathbb{Q}_\ell(\mu_p) \hookrightarrow \mathbb{C}$ to make sense of $\check{G}_{\text{geo}}(\mathbb{C})$). Let $\check{G}_{\text{geo},c}^{\text{h}}$ be the set of conjugacy classes of $\check{G}_{\text{geo},c}$.*

For $a \in \bar{k}^\times$, the conjugacy class $\varphi(\text{Frob}_a)$ of $\check{G}_{\text{geo}}(\mathbb{C})$ in fact belongs to $\check{G}_{\text{geo},c}^{\text{h}}$. As $\deg(a)$ (the degree of the field extension $k(a)/k$) tends to ∞ , the conjugacy classes

$$\{\varphi(\text{Frob}_a) \mid a \in \bar{k}^\times\} \subset \check{G}_{\text{geo},c}^{\text{h}}$$

become equidistributed according to the push-forward of the Haar measure from $\check{G}_{\text{geo},c}$ to $\check{G}_{\text{geo},c}^{\text{h}}$.

0.5. *Open problems.* Compared to the known results about Kl_n , the only missing piece for $\text{Kl}_{\check{G}}(\phi)$ is the unicity. We prove that $\text{Kl}_{\check{G}}(\phi)$ is *cohomologically rigid*; i.e.,

$$H^1(\mathbb{P}^1, j_{!*} \text{Kl}_{\check{G}}^{\text{Ad}}(\phi)) = 0.$$

This gives evidence for the *physical rigidity* of $\text{Kl}_{\check{G}}(\phi)$. Any other local system L on $\mathbb{P}^1_{\setminus\{0,\infty\}}$ satisfying [properties \(2\)](#) and [\(3\)](#) should be isomorphic to $\text{Kl}_{\check{G}}(\phi)$ over $\mathbb{P}^1_{\setminus\{0,\infty\}} \otimes_k \bar{k}$. We will state the precise unicity conjecture in [Conjectures 7.1](#) and [7.2](#).

[Table 1](#) also suggests that the Kloosterman sheaves for different groups \check{G} appearing in the same line of the table should be essentially the same (see [Conjecture 7.3](#)). This can be viewed as a functoriality statement for Kloosterman sheaves.

Also, the monodromy of Kloosterman sheaves is studied in detail only for split groups. In [Section 7.2](#), we state our predictions on the local and global monodromy of $\text{Kl}_{L_{\mathcal{G}}}(\phi)$ for \mathcal{G} a quasi-split, simple and simply-connected group scheme over \mathbb{P}^1 with good reduction at all places outside $\{0, \infty\}$, which is split by a tame Kummer cover $[N] : \mathbb{P}^1_{\setminus\{0,\infty\}} \rightarrow \mathbb{P}^1_{\setminus\{0,\infty\}}$ ($z \mapsto z^N$).

0.6. *Organization of the paper.* In [Section 1](#), we define the various group schemes $\mathcal{G}(m_0, m_\infty)$ over \mathbb{P}^1 encoding the level structures of the moduli stacks of G -bundles on which the automorphic sheaves will be defined. We also give a description of the geometry of these moduli stacks. In [Section 2](#), we construct the automorphic sheaf A_ϕ and state the main results of the paper. The proofs are given in [Sections 4–6](#). [Section 3](#) is devoted to the case $G = \text{GL}_n$, where we recover classical Kloosterman sheaves of Deligne. In [Section 7](#), we state our conjectures about rigidity of Kloosterman sheaves and our expectations about Kloosterman sheaves for quasi-split groups.

The paper contains four appendices. In [Appendix A](#), we prove the structure theorem for the moduli spaces $\text{Bun}_{\mathcal{G}}$ in the generality of quasi-split groups. In [Appendix B](#), we prove compatibility between outer automorphisms and the

geometric Satake equivalence. This is needed to describe the global monodromy image of Kloosterman sheaves and we couldn't find a reference for the result. In [Appendix C](#), we collect some facts about quasi-minuscule representations of \check{G} . In [Appendix D](#), we analyze the geometry of the adjoint Schubert variety for G_2 , which does not fit into the uniform treatment for groups of other types.

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1. Structural groups

We will work over a fixed finite field k of characteristic p . We fix a coordinate t of our base curve \mathbb{P}^1 , so that $\mathbb{A}^1 = \text{Spec}(k[t]) \subset \mathbb{P}^1$. We write $s := t^{-1}$ for the coordinate around $\infty \in \mathbb{P}^1$. For any closed point $x \in \mathbb{P}^1$, we will denote by \mathcal{O}_x the completed local ring at x and by K_x the fraction field of \mathcal{O}_x .

Since we are interested in the geometric Langlands correspondence for (wildly) ramified local systems, we will need to consider principal bundles with various level structures. It will be convenient to view these as torsors under group schemes \mathcal{G} over \mathbb{P}^1 . Moreover, in order to formulate the conjectured functoriality, it will be useful to allow quasi-split group schemes. We will introduce these group schemes in several steps.

1.1. *Quasi-split group schemes over $\mathbb{P}^1_{\setminus\{0,\infty\}}$.* We will assume $\mathcal{G}|_{\mathbb{P}^1_{\setminus\{0,\infty\}}}$ is a quasi-split reductive group. Moreover, we will assume that there is a finite extension k'/k and an integer N with $(N, p) = 1$ such that \mathcal{G} splits over the tame extension $[N]: \mathbb{G}_{m,k'} \rightarrow \mathbb{G}_{m,k'}$ defined by $t \mapsto t^N$. We may and will assume that k' contains all N -th roots of unity. We write μ_N for the group of N -th roots of unity.

We will fix subgroups $S \subset \mathcal{T}|_{\mathbb{P}^1_{\setminus\{0,\infty\}}} \subset \mathcal{B}|_{\mathbb{P}^1_{\setminus\{0,\infty\}}} \subset \mathcal{G}|_{\mathbb{P}^1_{\setminus\{0,\infty\}}}$, where S is a maximal split torus, $\mathcal{T}|_{\mathbb{P}^1_{\setminus\{0,\infty\}}}$ is a maximal torus and $\mathcal{B}|_{\mathbb{P}^1_{\setminus\{0,\infty\}}}$ is a Borel subgroup. We also fix a quasi-pinning of $\mathcal{G}|_{\mathbb{P}^1_{\setminus\{0,\infty\}}}$.

In order to describe our group schemes we will, for simplicity, assume that $k = k'$. In general, the construction we give will be invariant under $\text{Gal}(k'/k)$ so that Galois-descent will then give the general case.

By [13, Exp. XXIV, Th. 3.11], these groups can be described explicitly as follows. By assumption, there is a split reductive group G over k such that $[N]^*\mathcal{G}|\mathbb{G}_m = G \times \mathbb{G}_m$. The automorphism group of the covering $[N]$ is μ_N . We fix $T \subset B \subset G$ to be a split maximal torus and a Borel subgroup as well as a pinning \dagger of G .

This data defines a morphism $\sigma: \mu_N \rightarrow \text{Aut}^\dagger(G)$, where $\text{Aut}^\dagger(G)$ is the automorphisms of G respecting the pinning. Let us denote by $[N]_*(G \times \mathbb{G}_m)$ the Weil restriction of $G \times \mathbb{G}_m$ for the covering $[N]$. Then μ_N acts on this Weil restriction by the action of σ on G and by the action on the covering $[N]$. We will denote the diagonal action again by σ . Then descent implies

$$\mathcal{G}|\mathbb{P}_{\setminus\{0,\infty\}}^1 \cong ([N]_*(G \times \mathbb{P}_{\setminus\{0,\infty\}}^1))^\sigma = (G \times \mathbb{G}_m)/(\mu_N).$$

1.2. *Level structures at 0 and ∞ .* Next we need to describe several extensions of $\mathcal{G}|\mathbb{P}_{\setminus\{0,\infty\}}^1$ to \mathbb{P}^1 . We will again denote by $[N]: \mathbb{P}^1 \rightarrow \mathbb{P}^1$ the map given by $t \mapsto t^N$. The group $\mathcal{G}|\mathbb{P}_{\setminus\{0,\infty\}}^1$ will be extended as the (special) Bruhat-Tits group \mathcal{G} defined by the convex function, which is 0 on all roots. Since we assumed that $\mathcal{G}|\mathbb{P}_{\setminus\{0,\infty\}}^1$ splits over a tame extension, this group can be described as the connected component of the group $([N]_*(G \times \mathbb{P}^1))^\sigma$. The subgroups $S \subset \mathcal{T}|\mathbb{P}_{\setminus\{0,\infty\}}^1 \subset \mathcal{B}|\mathbb{P}_{\setminus\{0,\infty\}}^1 \subset \mathcal{G}|\mathbb{P}_{\setminus\{0,\infty\}}^1$ define closed subgroups $S \subset \mathcal{T} \subset \mathcal{B} \subset \mathcal{G}$.

We will need to introduce level structures at 0 and ∞ . These will correspond to the first steps of the Moy-Prasad filtration of \mathcal{G} . A self-contained exposition of the construction of these group schemes can be found in the preprint of Yu [37].

We first consider tori. For a split torus $T = \mathbb{G}_m^r \times \mathbb{P}^1$, we will denote by $T(m_0, m_\infty)$ the smooth group scheme over \mathbb{P}^1 such that

$$\begin{aligned} T(m_0, m_\infty)|\mathbb{P}_{\setminus\{0,\infty\}}^1 &= \mathbb{G}_m^r \times \mathbb{P}_{\setminus\{0,\infty\}}^1, \\ T(m_0, m_\infty)(\mathcal{O}_{\mathbb{P}^1,0}) &= \{g \in T(\mathcal{O}_{\mathbb{P}^1,0}) | g \equiv 1 \pmod{t^{m_0}}\}, \\ T(m_0, m_\infty)(\mathcal{O}_{\mathbb{P}^1,\infty}) &= \{g \in T(\mathcal{O}_{\mathbb{P}^1,\infty}) | g \equiv 1 \pmod{s^{m_\infty}}\}. \end{aligned}$$

This also defines a filtration for induced tori. For an arbitrary torus \mathcal{T} , pick an embedding into an induced torus $\mathcal{T} \hookrightarrow \mathcal{I}$ and define the filtration by pulling back the filtration on \mathcal{I} . By [37, §4], this definition is independent of the chosen embedding, and since we assumed that \mathcal{T} splits over a tame extension, the subgroups define connected groups.

For reductive groups $\mathcal{G}|\mathbb{P}_{\setminus\{0,\infty\}}^1$, we want to define models $\mathcal{G}(m_0, m_\infty)$ for $m_0, m_\infty \in \{0, 1, 2\}$, corresponding to the m_0 -th (resp. m_∞ -th) step in the Moy-Prasad filtration of the Iwahori subgroup at 0 (resp. ∞).

First assume that $\mathcal{G} = G \times \mathbb{P}^1$ is a split, semisimple group. As before, we fix $S = T \subset B \subset G$ to be a split maximal torus, a Borel subgroup and a pinning of G . We denote by $U \subset B$ the unipotent subgroup. Denote by $\Phi = \Phi(G, S)$ the set of roots and by $\Phi^\pm \subset \Phi$ the set of positive and negative roots with respect to the chosen Borel subgroup. Let $\alpha_1, \dots, \alpha_r$ denote the positive simple roots of G , and let $\{\theta_j\}$ denote the highest roots of G (one for each simple factor of G). Finally, for each root α , let U_α denote the corresponding root subgroup of G .

We will consider the following bounded subgroups of $G(k[[s]])$:

$$I(0) := \{g \in G(k[[s]]) \mid g \pmod s \in B\} \text{ is the Iwahori subgroup,}$$

$$I(1) := \{g \in G(k[[s]]) \mid g \pmod s \in U\} \text{ is the unipotent radical of } I(0).$$

To describe $I(2)$, let $f: \Phi \cup \{0\} \rightarrow \mathbb{N}$ be the concave function defined by $f(0) = 1$ and

$$f(\alpha) = \begin{cases} 0 & \text{if } \alpha \in \Phi^+ \setminus \{\alpha_i\} \text{ is positive, but not simple,} \\ 1 & \text{if } \alpha = \alpha_i \text{ is a positive simple root,} \\ 1 & \text{if } \alpha \in \Phi^- \setminus \{-\theta_j\}, \\ 2 & \text{if } \alpha = -\theta_j \text{ is the negative of a highest root.} \end{cases}$$

$I(2) \subset G(k[[s]])$ denotes the bounded subgroup defined by the concave function f , i.e., the subgroup generated by $\{u \in U_\alpha \mid u \equiv 1 \pmod{s^{f(i)}}\}$ and $\{g \in T(k[[s]]) \mid g \equiv 1 \pmod s\}$.

Note that by definition, $I(1)/I(2) \cong \bigoplus_{\alpha \text{ simple affine}} \mathbb{G}_a$ is isomorphic to the sum of the root subgroups $U_\alpha \subset G(k[[s]])$ for which α is a simple affine root. Similarly, we define $I(i)^{\text{opp}} \subset G(k[[t]])$ to be the analogous groups obtained by using the opposite Borel subgroup B^{opp} in the above definition.

Example. For $G = \text{SL}_n$, we choose $S = T$ to be the diagonal matrices and B the upper triangular matrices. Then the subgroup $I(0) \subset \text{SL}_n(k[[s]])$ is the subgroup of matrices such that the lower diagonal entries are divisible by s . $I(1) \subset I(0)$ is the subgroup such that the diagonal entries are elements of $1 + sk[[s]]$.

The root subgroups of SL_n for the simple roots are given by the above-diagonal entries $a_{i,i+1}$, and the $U_{-\tilde{\alpha}}$ of the negative of the longest root is given by the entry in the lower left corner. So $I(2)$ consists of matrices of the form

$$\begin{pmatrix} 1 + sk[[s]] & sk[[s]] & k[[s]] & k[[s]] \\ sk[[s]] & 1 + sk[[s]] & sk[[s]] & k[[s]] \\ sk[[s]] & sk[[s]] & 1 + sk[[s]] & sk[[s]] \\ s^2k[[s]] & sk[[s]] & sk[[s]] & 1 + sk[[s]] \end{pmatrix}.$$

We obtain an isomorphism $I(1)/I(2) \cong \mathbb{A}^n$ by mapping a matrix (a_{ij}) to the leading coefficients of the entries $a_{i,i+1}$ and $a_{n,1}$.

For a general, split reductive group $\mathcal{G} = G \times \mathbb{P}^1$, we consider the derived group G_{der} and the connected component of the center $Z(G)^\circ$, which is a torus. We will temporarily denote by $I_{G_{\text{der}}}(i), I_{G_{\text{der}}}^{\text{opp}}(i)$ the groups defined above for the semisimple group G_{der} and define

$$I(i) := Z(G)^\circ(i)(k[[s]]) \cdot I_{G_{\text{der}}}(i) \subset \mathcal{G}(k[[s]])$$

and similarly $I(i)^{\text{opp}} := Z(G)^\circ(i, m_\infty)(k[[t]])I_{G_{\text{der}}}^{\text{opp}}(i)$.

For split, reductive groups $\mathcal{G} = G \times \mathbb{P}^1$, the group $G(m_0, m_\infty)$ denotes the Bruhat Tits group scheme such that

$$\begin{aligned} G(m_0, m_\infty)|_{\mathbb{P}^1 \setminus \{0, \infty\}} &= G \times \mathbb{G}_m, \\ G(m_0, m_\infty)(\mathcal{O}_\infty) &= I(m_\infty), \text{ and} \\ G(m_0, m_\infty)(\mathcal{O}_0) &= I(m_0)^{\text{opp}}. \end{aligned}$$

Finally, for a general quasi-split group, we define

$$\mathcal{G}(m_0, m_\infty) := \left(([N]_* G(m_0, m_\infty))^\sigma \right)^\circ.$$

Note that for tori, this does not give a new definition.

We will abbreviate:

$$\begin{aligned} I(m_\infty) &:= \mathcal{G}(m_0, m_\infty)(\mathcal{O}_\infty), \\ I^-(m_0) &:= \mathcal{G}(m_0, m_\infty)(\mathbb{P}^1 - \{\infty\}). \end{aligned}$$

Recall from [33] that the groups $I(m)$ have a natural structure as (infinite dimensional) group schemes over k .

1.3. *Affine generic characters.* As indicated before, our construction depends on the choice of a character of $I(1)$. We call a linear function $\phi: I(1)/I(2) \rightarrow \mathbb{A}^1$ *generic* if for any simple affine root α , the restriction of ϕ to U_α is non-trivial. Throughout we will fix such a generic ϕ .

We will fix a nontrivial additive character $\psi: \mathbb{F}_p \rightarrow \overline{\mathbb{Q}}_\ell$ and denote the character $k \rightarrow \overline{\mathbb{Q}}_\ell$ defined as $\psi \circ \text{Tr}_{k/\mathbb{F}_p}$ again by ψ . With this notation, the character $\psi \circ \phi$ is called an *affine generic character* of $I(1)$.

1.4. *Principal bundles.* Having defined our groups, we need to collect some basic results on the geometry of the moduli stacks of \mathcal{G} -bundles $\text{Bun}_{\mathcal{G}}$. All of these are well known for constant groups. (See, e.g., [24], or [14] for a recent account.) In order to generalize these results to our setup we rely on [33] and [23], where the corresponding results on twisted loop groups are explained. Let us point out that, except for the computation of the connected components

of $\text{Bun}_{\mathcal{G}}$, all results are particular to group schemes over \mathbb{P}^1 that split over a tamely ramified covering $[N] : \mathbb{P}^1 \rightarrow \mathbb{P}^1$.

First, we recall some results and notations from [23]. We denote by \mathcal{N} the normalizer of \mathcal{T} and by $\widetilde{W} := \mathcal{N}(k((s)))/\mathcal{T}(k[[s]])$ the Iwahori-Weyl group. Furthermore, denote by $W_0 := \mathcal{N}(k((s)))/\mathcal{T}(k((s)))$ the relative Weyl group. This is isomorphic to the Weyl group of the reductive quotient of the special fiber $\mathcal{G}_{\infty, \text{red}}$ (loc. cit., Prop. 13) and $\widetilde{W} \cong \mathbb{X}_*(\mathcal{T})_{\pi_1(\mathbb{P}^1 \setminus \{0, \infty\})} \rtimes W_0$.

Denote by W_a the affine Weyl group of the root system of $\mathcal{G}(k((s)))$, which can be identified with the Iwahori-Weyl group of the simply connected cover of the derived group of \mathcal{G} ([23, p. 196]). This is a Coxeter group. Then $\widetilde{W} \cong W_a \rtimes \Omega$, where $\Omega \cong \mathbb{X}^*(Z(\check{G})^{\pi_1(\mathbb{G}_m)})$ is the stabilizer of an alcove.

Finally, for $x \in \mathbb{P}^1 - \{\infty\}$, we denote by $\text{Gr}_{\mathcal{G}, x}$ the affine Grassmannian (see [33]) so that $\text{Gr}_{\mathcal{G}, x}(k) = \mathcal{G}(K_x)/\mathcal{G}(\mathcal{O}_x)$. It can also be defined as the ind-scheme parametrizing \mathcal{G} -bundles \mathcal{P} on \mathbb{P}^1 together with a trivialization $\varphi : \mathcal{P}|_{\mathbb{P}^1 \setminus \{x\}} \xrightarrow{\cong} \mathcal{G}|_{\mathbb{P}^1 \setminus \{x\}}$ (see, e.g., [25]).

PROPOSITION 1.1. *Assume that the ground field k is either finite or algebraically closed. Let $\mathcal{G} = \mathcal{G}(0, 0)$ over \mathbb{P}^1 be a generically quasi-split group scheme, constructed as in Section 1.2. Then the following hold:*

- (1) *For any $x \in \mathbb{P}^1$, the canonical map $\text{Gr}_{\mathcal{G}, x} \rightarrow \text{Bun}_{\mathcal{G}}$ has sections, locally in the smooth topology on $\text{Bun}_{\mathcal{G}}$. Moreover, this map is essentially surjective on k points; i.e., for any \mathcal{G} -bundle over \mathbb{P}^1 , its restriction to $\mathbb{P}^1 \setminus \{x\}$ is trivial.*
- (2) $\pi_0(\text{Bun}_{\mathcal{G}}) \cong \pi_1([N]^*\mathcal{G}|_{\mathbb{G}_m})_{\pi_1(\mathbb{G}_m)} \cong \Omega$.
- (3) *Every $\mathcal{G}(0, 0)$ -bundle on \mathbb{P}^1 admits a reduction to \mathcal{T} .*
- (4) *(Birkhoff-Grothendieck decomposition)*

$$\mathcal{G}(0, 0)(k((s))) = \prod_{w \in \widetilde{W}} I^-(0) \cdot \widetilde{W} \cdot I(0).$$

- (5) *For $n \in \{0, 1, 2\}, m \in \{0, 1\}$, the map $\text{Bun}_{\mathcal{G}(m, n)} \rightarrow \text{Bun}_{\mathcal{G}(0, 0)}$ is an $I^-(0)/I^-(m) \times I(0)/I(n)$ -torsor. In particular, $\pi_0(\text{Bun}_{\mathcal{G}(m, n)}) \cong \pi_0(\text{Bun}_{\mathcal{G}})$.*

If $\mathcal{G} = G \times \mathbb{P}^1$ is a split group, a proof of this result can be found in [14] or [24]. Since the case of split groups was our starting point, we will postpone the proof of the general case to Appendix A. For $\gamma \in \Omega$, we will denote by $\text{Bun}_{\mathcal{G}(m, n)}^\gamma$ the corresponding connected component of $\text{Bun}_{\mathcal{G}(m, n)}$.

Let us collect some consequences of this result.

COROLLARY 1.2. *Let $n \in \{0, 1, 2\}, m \in \{0, 1\}$. Any $\gamma \in \Omega$ defines an isomorphism $\text{Hk}_\gamma : \text{Bun}_{\mathcal{G}(m, n)} \rightarrow \text{Bun}_{\mathcal{G}(m, n)}$. This induces an isomorphism of the connected components $\text{Bun}_{\mathcal{G}(m, n)}^0 \rightarrow \text{Bun}_{\mathcal{G}(m, n)}^\gamma$.*

In particular, all connected components of $\text{Bun}_{\mathcal{G}(m, n)}$ are isomorphic.

Proof. By [Proposition 1.1\(2\)](#), the connected components of $\text{Bun}_{\mathcal{G}}(m, n)$ are indexed by $\Omega \subset \widetilde{W}$. Since Ω is the stabilizer of the alcove defining $I(0)^{\text{opp}}$, any $\gamma \in \Omega$ normalizes $I(0)^{\text{opp}}$ and its unipotent radical $I(1)^{\text{opp}}$. Thus Ω acts by right multiplication on $\text{Gr}_{\mathcal{G}(m,n),0} = \mathcal{G}(k((t)))/I(m)^{\text{opp}}$. This defines an isomorphism $\text{Hk}_{\gamma}: \text{Bun}_{\mathcal{G}(m,n)} \rightarrow \text{Bun}_{\mathcal{G}(m,n)}$. Moreover, [\[23, Lemma 14\]](#) implies that the restriction of this map to $\text{Bun}_{\mathcal{G}(m,n)}^0$ induces an isomorphism $\text{Bun}_{\mathcal{G}(m,n)}^0 \rightarrow \text{Bun}_{\mathcal{G}(m,n)}^{\gamma}$. \square

COROLLARY 1.3. (1) *Denote by $\mathcal{G}(\mathbb{P}^1)$ the automorphism group of the trivial \mathcal{G} -bundle. Then the inclusion $B\mathcal{G}(\mathbb{P}^1) \rightarrow \text{Bun}_{\mathcal{G}}$ is an affine, open embedding.*

(2) *Denote by $\mathcal{T}(\mathbb{P}^1) = H^0(\mathbb{P}^1, \mathcal{T}) = H^0(\mathbb{P}^1, \mathcal{G}(0, 0))$ the automorphism group of the trivial $\mathcal{G}(0, 0)$ -bundle. The inclusion of the trivial bundle defines an affine open embedding $B(\mathcal{T}(\mathbb{P}^1)) \hookrightarrow \text{Bun}_{\mathcal{G}(0,0)}$.*

(3) *The trivial $\mathcal{G}(0, 1)$ -bundle defines an affine, open embedding $\text{Spec}(k) \hookrightarrow \text{Bun}_{\mathcal{G}(0,1)}^0$.*

(4) *Applying the action of $I(1)/I(2)$ on $\text{Bun}_{\mathcal{G}(0,2)}$ to the trivial $\mathcal{G}(0, 2)$ -bundle, we obtain a canonical map $j_0: I(1)/I(2) \hookrightarrow \text{Bun}_{\mathcal{G}(0,2)}$. This is an affine open embedding.*

(5) *For any $\gamma \in \Omega$, the map*

$$j_{\gamma} := \text{Hk}_{\gamma} \circ j_0: I(1)/I(2) \hookrightarrow \text{Bun}_{\mathcal{G}(0,2)}$$

is an affine open embedding, called the big cell.

(6) *Applying the action of $\mathcal{T}_0^{\text{red}} \times I(1)/I(2)$ on $\text{Bun}_{\mathcal{G}(1,2)}$ to the trivial $\mathcal{G}(1, 2)$ -bundle, we obtain canonical affine embeddings*

$$\tilde{j}_0: \mathcal{T}_0^{\text{red}} \times I(1)/I(2) \hookrightarrow \text{Bun}_{\mathcal{G}(1,2)} \text{ and}$$

$$\tilde{j}_{\gamma} := \text{Hk}_{\gamma} \circ \tilde{j}_0: \mathcal{T}_0^{\text{red}} \times I(1)/I(2) \hookrightarrow \text{Bun}_{\mathcal{G}(1,2)}.$$

Proof. Let us first recall why this corollary holds for GL_n . For (1), note that the only vector bundle on \mathbb{P}^1 of rank n with trivial cohomology is the bundle $\mathcal{O}(-1)^n$. Therefore, the inverse of the determinant of cohomology line bundle on Bun_{GL_n} has a section vanishing precisely on the trivial bundle. This proves (1) in this case. Next, recall that a $\text{GL}_n(0, 0)$ bundle is a vector bundle together with full flags at 0 and ∞ . Let us denote by $\text{Mod}_{i,0}: \text{Bun}_{\text{GL}_n(0,0)}^0 \rightarrow \text{Bun}_{\text{GL}_n(0,0)}^i$ the i -th upper modification along the flag at 0. The inverse of this map is given by the i -th lower modification, which we will denote by $\text{Mod}_{-i,0}$.

To prove (2), denote by $\text{Mod}_{i,\infty}$ the i -th modification at ∞ . The trivial $\text{GL}_n(0, 0)$ -bundle is the bundle given by \mathcal{O}^n and the canonical opposite flags $(V_{i,0})_{i=0,\dots,n}$ of the fiber $(\mathcal{O}^n)_0$ at 0 and $(V_{i,\infty})_{i=0,\dots,n}$ at ∞ . This is the only $\text{GL}_n(0, 0)$ -bundle $(\mathcal{E}, V_{i,0}, V_{i,\infty})$ of degree 0 on \mathbb{P}^1 such that the complex $\mathcal{E} \rightarrow \mathcal{E}_0/V_{i,0} \oplus \mathcal{E}_{\infty}/V_{n-i,\infty}$ has trivial cohomology for all i . Thus again, the inclusion

of the trivial $\mathrm{GL}_n(0, 0)$ bundle in $\mathrm{Bun}_{\mathrm{GL}_n(0,0)}$ is defined by the nonvanishing of sections of line bundles.

For general \mathcal{G} , we pick a faithful representation $\rho: \mathcal{G} \rightarrow \mathrm{GL}_n \times \mathbb{P}^1$. This defines a map $\mathrm{Bun}_{\mathcal{G}} \rightarrow \mathrm{Bun}_{\mathrm{GL}_n}$. The Birkhoff-Grothendieck decomposition implies that a \mathcal{G} -bundle is trivial if and only if the associated GL_n -bundle is trivial. Moreover, in order to check that the reductions to \mathcal{B} at $0, \infty$ are opposite, it is also sufficient to check this on the induced GL_n -bundle. This proves (1) and (2).

One can show that $\mathcal{T}(\mathbb{P}^1) = H^0(\mathbb{P}^1, \mathcal{T}) \cong \mathcal{T}_0^{\mathrm{red}}$; see the proof of the claim in [Appendix A](#). Now $\mathrm{Bun}_{\mathcal{G}(0,1)}$ is a $\mathcal{T}_0^{\mathrm{red}}$ -torsor over $\mathrm{Bun}_{\mathcal{G}(0,0)}$. Thus the preimage of the affine open embedding $B(\mathcal{T}(\mathbb{P}^1)) \hookrightarrow \mathrm{Bun}_{\mathcal{G}(0,0)}$ from (2) is the point $\mathrm{Spec}(k)$ defined by the trivial $\mathcal{G}(0, 1)$ -bundle. This proves (3) .

(4) follows from (3) because the map $\mathrm{Bun}_{\mathcal{G}(0,2)} \rightarrow \mathrm{Bun}_{\mathcal{G}(0,1)}$ is an $I(1)/I(2)$ -torsor. By [Corollary 1.2](#), (5) follows from (4). Finally (6) follows from (5) since $\mathrm{Bun}_{\mathcal{G}(1,2)} \rightarrow \mathrm{Bun}_{\mathcal{G}(0,2)}$ is a $\mathcal{T}_0^{\mathrm{red}}$ -torsor. \square

2. Eigensheaf and eigenvalues: Statement of main results

In this section we will construct the automorphic sheaf A_ϕ . We will also state our main results about the local and global monodromy of the Kloosterman sheaf $\mathrm{Kl}_{L\mathcal{G}}(\phi, \chi)$, which will be defined to be the eigenvalue of A_ϕ .

2.1. *The eigenfunction.* In this subsection we give a simple formula for an eigenfunction in Gross’s automorphic representation π mentioned in the introduction. Surprisingly, this calculation turns out to be independent of Gross’s result.

We set k to be a finite field and $K = k(t)$. The completions of K at $t = 0$ and $t = \infty$ are denoted by $k((t))$ and $k((s))$. We will use the generic affine character $\psi \circ \phi: I(1)/I(2) \rightarrow \overline{\mathbb{Q}}_\ell$ chosen in [Section 1.3](#).

By [Proposition 1.1](#), we know

$$\begin{aligned} \mathrm{Bun}_{\mathcal{G}(0,2)}(k) &= \mathcal{G}(K) \backslash \mathcal{G}(0, 2)(\mathbb{A}_K) / \prod_x \mathcal{G}(0, 2)(\mathcal{O}_x) \\ &= I^-(0) \backslash \mathcal{G}(k((s))) / I(2) \end{aligned}$$

and

$$\mathcal{G}(k((s))) = \prod_{w \in \widetilde{W}} I^-(0) w I(1).$$

Suppose we were given, as in the introduction, an automorphic representation $\pi = \otimes' \pi_\nu$ of $\mathcal{G}(\mathbb{A}_K)$ such that for $\nu \notin \{0, \infty\}$, the local representation π_ν is unramified, π_0 is the Steinberg representation and such that π_∞ occurs in $\mathrm{c}\text{-Ind}_{Z(\mathcal{G}(0,1)) \times I(1)}^{\mathcal{G}(k((s)))}(\psi \circ \phi)$. Then there exists a function f on $\mathrm{Bun}_{\mathcal{G}(0,2)}(k)$ such that $f(gh) = \psi(\phi(h))f(g)$ for all $h \in I(1)$.

The following lemma characterizes such functions in an elementary way.

LEMMA 2.1. *Let $f: \text{Bun}_{\mathcal{G}(0,2)}(k) \rightarrow \overline{\mathbb{Q}}_\ell$ be a function such that for all $h \in I(1), g \in \mathcal{G}(0,2)(k((s)))$, we have $f(gh) = \psi(\phi(h))f(g)$. Then f is uniquely determined by the values $f(\gamma)$ for $\gamma \in \Omega$. Moreover, $f(w) = 0$ for all $w \in \widetilde{W} - \Omega$.*

Proof. For any $w \in \widetilde{W}$ with $l(w) > 0$, there exists a simple affine root α_i such that $w(\alpha_i)$ is negative. This implies that $wU_{\alpha_i}w^{-1} \subset I^-(0)$. This implies that for all $u \in U_{\alpha_i}$, we have $\psi(\phi(u))f(w) = f(wu) = f(w)$ so $f(w) = 0$. For w with $l(w) = 0$, i.e., $w \in \Omega$, the value of $f(\gamma)$ can be arbitrary by Corollary 1.3. □

Remark 2.2. Any function f in the above lemma is automatically cuspidal. In fact, for any parabolic $Q \subset G$ with unipotent radical N_Q , the constant term function

$$f_{N_Q}(g) = \int_{N_Q(K) \backslash N_Q(\mathbb{A}_K)} f(ng)dn$$

is left invariant under $N_Q(k((s)))$ and right equivariant under $I_\infty(1)$ against the character $\psi \circ \phi$. A similar argument as in Lemma 2.1 shows that such a function must be zero.

2.2. *The automorphic sheaf $A_{\phi,\chi}$.* Let us reformulate the preceding observation geometrically. Denote by AS_ψ the Artin-Schreier sheaf on \mathbb{A}^1 defined by the character ψ . We set $\text{AS}_\phi := \phi^*(\text{AS}_\psi)$, the pull-back of the Artin-Schreier sheaf to $I(1)/I(2) \cong \mathbb{A}^d$.

Let us denote by $\text{Perv}(\text{Bun}_{\mathcal{G}(0,2)})^{I(1), \text{AS}_\phi}$ the category of perverse sheaves on $\text{Bun}_{\mathcal{G}(0,2)}$ that are $(I(1), \text{AS}_\phi)$ -equivariant. We will use this notation more generally for any stack with an action of $I(1)/I(2)$.

For any $\gamma \in \Omega$, denote by $j_\gamma: I(1)/I(2) \hookrightarrow \text{Bun}_{\mathcal{G}(0,2)}^\gamma$ the embedding of the big cell and by $i_\gamma: \text{Spec}(k) \rightarrow \text{Bun}_{\mathcal{G}(0,2)}^\gamma$, the map given by the $\mathcal{G}(0,2)$ bundle defined by γ .

The following is the geometric analog of Lemma 2.1.

LEMMA 2.3. *The sheaf AS_ϕ satisfies $j_{\gamma,!} \text{AS}_\phi = j_{\gamma,*} \text{AS}_\phi$. Moreover, for any $\gamma \in \Omega$, the functor*

$$\begin{aligned} \text{Perv}(\mathbb{G}_m) &\rightarrow \text{Perv}(\text{Bun}_{\mathcal{G}(0,2)}^\gamma \times \mathbb{G}_m), \\ F &\mapsto j_{\gamma,!} \text{AS}_\phi \boxtimes F[\dim(\text{Bun}_{\mathcal{G}(0,2)})] \end{aligned}$$

is an equivalence of categories. An inverse is given by

$$K \mapsto (i_\gamma \times \text{id}_{\mathbb{P}^1_{\setminus \{0,\infty\}}})^* K[-\dim(\text{Bun}_{\mathcal{G}(0,2)})].$$

Proof. For any $w \in \widetilde{W} - \Omega$, we pick a representative in $\mathcal{N}(\mathcal{T}((t)))$, again denoted by w . Consider the $\mathcal{G}(0, 2)$ -bundle P_w defined by w . Let $U_\alpha \subset I(1)$ be a root subgroup corresponding to a simple affine root α such that $w \cdot \alpha$ is negative, i.e., such that $U_{w \cdot \alpha} \subset I^-(0)$. This defines an inclusion $U_\alpha \hookrightarrow \text{Aut}(P_w)$. Thus we get a commutative square

$$\begin{array}{ccc} U_\alpha \times \text{Spec}(k) & \longrightarrow & \text{Spec}(k) \\ \downarrow (\text{id}, P_w) & & \downarrow P_w \\ U_\alpha \times \text{Bun}_{\mathcal{G}(0,2)} & \xrightarrow{\text{act}} & \text{Bun}_{\mathcal{G}(0,2)}. \end{array}$$

This implies that $(\text{AS}_\phi|_{U_\alpha}) \boxtimes K|_{P_w} \cong \overline{\mathbb{Q}}_\ell|_{U_\alpha} \boxtimes K|_{P_w}$. Since we assumed that $\text{AS}_\phi|_{U_\alpha}$ is defined by a nontrivial character of U_α , it follows that the stalk of K at P_w vanishes. Dually, the same result holds for the costalk of K at P_w .

This proves our first claim. Also for our second claim, this implies that any $(I(1), \text{AS}_\phi)$ -equivariant perverse sheaf on $\text{Bun}_{\mathcal{G}(0,2)}^\gamma$ is its $!$ -extension from the substack $j_\gamma(I(1)/I(2))$. On this substack, tensoring with the local system AS_ϕ gives an equivalence between $(I(1)/I(2))$ -equivariant sheaves and $(I(1)/I(2), \text{AS}_\phi)$ -equivariant sheaves. This proves our claim. \square

Using this lemma we can now define our automorphic sheaf.

Definition 2.4. We define $A_\phi \in \text{Perv}(\text{Bun}_{\mathcal{G}(0,2)})^{I(1), \text{AS}_\phi}$ to be the perverse sheaf given on the component $\text{Bun}_{\mathcal{G}(0,2)}^\gamma$ by $j_{\gamma,!} \text{AS}_\phi[\dim(\text{Bun}_{\mathcal{G}(0,2)})]$. We will denote by A_ϕ^γ the restriction of A_ϕ to the component $\text{Bun}_{\mathcal{G}(0,2)}^\gamma$.

Remark 2.5 (A variant with multiplicative characters). We can generalize the above construction of A_ϕ slightly. Recall that the open cell in $\text{Bun}_{\mathcal{G}(1,2)}^\gamma$ is canonically isomorphic to $\mathcal{T}_0^{\text{red}} \times I(1)/I(0)$. Any character $\chi: \mathcal{T}_0^{\text{red}}(k) \rightarrow \overline{\mathbb{Q}}_\ell$ defines a rank-one local system Kum_χ on the torus $\mathcal{T}_0^{\text{red}}$. Then [Lemma 2.3](#) also holds for $(\mathcal{T}_0^{\text{red}} \times I(1)/I(0), \text{Kum}_\chi \boxtimes \text{AS}_\phi)$ -equivariant sheaves on $\text{Bun}_{\mathcal{G}(1,2)}$.

Definition 2.6. We define $A_{\phi, \chi} \in \text{Perv}(\text{Bun}_{\mathcal{G}(0,2)})$ to be the perverse sheaf given on the component $\text{Bun}_{\mathcal{G}(0,2)}^\gamma$ by $j_{\gamma,!}(\text{Kum}_\chi \boxtimes \text{AS}_\phi)[\dim(\text{Bun}_{\mathcal{G}(0,2)})]$.

2.3. The geometric Hecke operators. In order to state our main result, we need to recall the definition of the geometric Hecke operators. The stack of Hecke modifications is the stack

$$\text{Hecke}_{\mathcal{G}(m,n)}^{\mathbb{A}^1}(S) := \left\langle (\mathcal{E}_1, \mathcal{E}_2, x, \varphi) \mid \begin{array}{l} \mathcal{E}_i \in \text{Bun}_{\mathcal{G}(m,n)}(S), x: S \rightarrow \mathbb{A}^1, \\ \varphi: \mathcal{E}_1|_{\mathbb{P}^1-x \times S} \xrightarrow{\cong} \mathcal{E}_2|_{\mathbb{P}^1-x} \end{array} \right\rangle.$$

This stack has natural forgetful maps:

$$(2.7) \quad \begin{array}{ccc} & \text{Hecke}_{\mathcal{G}(m,n)}^{\mathbb{A}^1} & \xrightarrow{\text{pr}_{\mathbb{A}^1}} \mathbb{A}^1 \\ \text{pr}_1 \swarrow & & \searrow \text{pr}_2 \\ \text{Bun}_{\mathcal{G}(m,n)} & & \text{Bun}_{\mathcal{G}(m,n)} \times \mathbb{A}^1. \end{array}$$

Set $\text{Hecke}_{\mathcal{G}(m,n)}^{\mathbb{P}^1 \setminus \{0,\infty\}} := \text{pr}_{\mathbb{A}^1}^{-1}(\mathbb{P}^1_{\setminus \{0,\infty\}})$.

- Remark 2.8.* (1) The fiber of pr_1 over the trivial bundle $\mathcal{G}(m,n) \in \text{Bun}_{\mathcal{G}}$ is called the Beilinson-Drinfeld Grassmannian. It will be denoted by $\text{GR}_{\mathcal{G}(m,n)}$. The fibers of $\text{GR}_{\mathcal{G}} \rightarrow \mathbb{A}^1$ over a point $x \in \mathbb{P}^1_{\setminus \{0,\infty\}}$ are isomorphic to the affine Grassmannian $\text{Gr}_{\mathcal{G},x}$, the quotient $\mathcal{G}(K_x)/\mathcal{G}(\mathcal{O}_x)$.
- (2) The geometric fibers of pr_2 over $\text{Bun}_{\mathcal{G}(m,n)} \times \mathbb{P}^1_{\setminus \{0,\infty\}}$ are (noncanonically) isomorphic to the affine Grassmannian $\text{Gr}_{\mathcal{G}}$. Locally in the smooth topology on $\text{Bun}_{\mathcal{G}} \times \mathbb{P}^1_{\setminus \{0,\infty\}}$, this fibration is trivial (e.g., [Remark 4.1](#)).
- (3) The diagram (2.7) has a large group of symmetries. The group $I(0)/I(2) = (I(1)/I(2)) \rtimes \mathcal{T}_{\infty}^{\text{red}}$ acts on $\text{Bun}_{\mathcal{G}(m,2)}$ by changing the $I(2)$ -level structures at ∞ , and this action extends to the diagram (2.7). (That is, it also acts on $\text{Hecke}_{\mathcal{G}(m,2)}^{\mathbb{A}^1}$ and the maps pr_i are equivariant under these actions.) The 1-dimensional torus $\mathbb{G}_m^{\text{rot}}$ acts on the curve \mathbb{P}^1 fixing the points $\{0\}$ and $\{\infty\}$; hence, it also acts on (2.7). Finally, the pinned automorphisms $\text{Aut}^{\dagger}(G)$ act on (2.7). So we see that the group $I(0)/I(2) \rtimes (\mathbb{G}_m^{\text{rot}} \times \text{Aut}^{\dagger}(G))$ acts on the diagram (2.7).

Let us first recall the Hecke-operators for constant group schemes. For this, we collect some facts about the geometric Satake equivalence (see [\[30\]](#), [\[18\]](#) and [\[31\]](#)).

Let $\text{Gr}_{\mathcal{G}} = G((\tau))/G[[\tau]]$ be the abstract affine Grassmannian, without reference to any point on $\mathbb{P}^1_{\setminus \{0,\infty\}}$. Let $\mathcal{O} = k[[\tau]]$, and let $\text{Aut}_{\mathcal{O}}$ be the pro-algebraic k -group of continuous (under the τ -adic topology) automorphisms of \mathcal{O} . Then $G[[\tau]] \rtimes \text{Aut}_{\mathcal{O}}$ acts on $\text{Gr}_{\mathcal{G}}$ from the left. The $G[[\tau]]$ orbits on $\text{Gr}_{\mathcal{G}}$ are indexed by dominant cocharacter $\mu \in \mathbb{X}_*(T)^+$. The orbits are denoted by $\text{Gr}_{G,\mu}$, and their closures (the Schubert varieties) are denoted $\text{Gr}_{G,\leq\mu}$. We denote the intersection cohomology sheaf of on $\text{Gr}_{G,\leq\mu}$ by IC_{μ} . We will normalize IC_{μ} to be of weight 0 (see [Remark 2.10](#)).

The *Satake category* $\text{Sat} = \text{Perv}_{\text{Aut}_{\mathcal{O}}}(G[[\tau]] \backslash G((\tau))/G[[\tau]])$ is the category of $G[[\tau]] \rtimes \text{Aut}_{\mathcal{O}}$ -equivariant perverse sheaves (with finite-type support) on $\text{Gr}_{\mathcal{G}}$. Similarly, we define Sat^{geo} by considering the base change of the situation to \bar{k} . Finally we define the *normalized semisimple Satake category* \mathcal{S} to be the full subcategory of Sat consisting of direct sums of IC_{μ} 's.

In [\[31\]](#) and [\[18\]](#), it was shown that

- Sat^{geo} carries a natural tensor structure (which is also defined for Sat) such that the global cohomology functor $h = H^*(\text{Gr}_G, -) : \text{Sat}^{\text{geo}} \rightarrow \text{Vec}$ is a fiber functor.
- $\underline{\text{Aut}}^{\otimes}(h)$ is a connected reductive group over $\overline{\mathbb{Q}}_{\ell}$, which is Langlands dual to G . Let $\check{G} := \underline{\text{Aut}}^{\otimes}(h)$. Then the Tannakian formalism gives the *geometric Satake equivalence* of tensor categories

$$\text{Sat}^{\text{geo}} \cong \text{Rep}(\check{G}).$$

- By construction, \check{G} is equipped with a maximal torus \check{T} , and a natural isomorphism $\mathbb{X}^*(\check{T}) \cong \mathbb{X}_*(T)$. (In fact, \check{G} is equipped with a canonical pinning; see [Lemma B.3](#).) The geometric Satake equivalence sends IC_{μ} to the irreducible representation V_{μ} of extremal weight μ . We denote the inverse of this equivalence by $V \mapsto \text{IC}_V$.

Remark 2.9. In [[1](#), §3.5], it was argued that \mathcal{S} is closed under the tensor structure on Sat . Therefore \mathcal{S} is naturally a tensor category.

The pull-back along $\text{Gr}_G \otimes_k \bar{k} \rightarrow \text{Gr}_G$ gives a tensor functor $\mathcal{S} \rightarrow \text{Sat}^{\text{geo}}$, which is easily seen to be an equivalence because both categories are semisimple with explicit simple objects. Therefore, the above results in [[31](#)] and [[18](#)] all apply to \mathcal{S} . In particular, we have the (semisimplified k -version of) the geometric Satake equivalence

$$\mathcal{S} \cong \text{Sat}^{\text{geo}} \cong \text{Rep}(\check{G}).$$

Remark 2.10 (Normalization of weights). We use the normalization making the complex IC_{μ} pure of weight 0; i.e., we choose a square root of q in $\overline{\mathbb{Q}}_{\ell}$ and denote by IC_{μ} the intersection complex, tensored by $\overline{\mathbb{Q}}_{\ell}(\frac{1}{2} \dim(\text{Gr}_{\mu}))$.

As was pointed out in [[15](#)], it is not necessary to make this rather unnatural choice, which is made to obtain the group \check{G} from the category \mathcal{S} . Alternatively, we can enlarge the category \mathcal{S} by including all Tate-twists of the intersection cohomology sheaves $\text{IC}_{\mu}(n)$. By the previous remark, this is still a neutral tensor category, defining group \check{G}_1 , which is an extension of \check{G} by a central, 1-dimensional torus.

The stack $\text{Hecke}_{G(m,n)}^{\mathbb{P}^1 \setminus \{0,\infty\}}$ is a locally trivial fibration over $\text{Bun}_{G(m,n)} \times \mathbb{P}^1 \setminus \{0,\infty\}$ with fiber Gr_G , and the $G[[\tau]]$ -orbits Gr_{μ} on Gr_G define substacks $\text{Hecke}_{\mu}^{\mathbb{P}^1 \setminus \{0,\infty\}} \subset \text{Hecke}_{G(m,n)}^{\mathbb{P}^1 \setminus \{0,\infty\}}$. By abuse of notation we will also denote by IC_{μ} the intersection cohomology complex of $\text{Hecke}_{\mu}^{\mathbb{P}^1 \setminus \{0,\infty\}}$, shifted in degree such that IC_{μ} restricts to the intersection complex on every fiber.

One defines the geometric Hecke operators as a functor (see [[17](#)]):

$$\begin{aligned} \text{Hk} : \text{Rep}(\check{G}) \times D^b(\text{Bun}_{G(m,n)}) &\rightarrow D^b(\text{Bun}_{G(m,n)} \times \mathbb{P}^1 \setminus \{0,\infty\}), \\ (V, K) &\mapsto \text{Hk}_V(K) := \text{pr}_{2,1} \text{pr}_1^*(K \otimes \text{IC}_V). \end{aligned}$$

In order to compose these operators, one extends Hk_V to an operator

$$D^b(\mathrm{Bun}_{G(m,n)} \times \mathbb{P}_{\setminus\{0,\infty\}}^1) \rightarrow D^b(\mathrm{Bun}_{G(m,n)} \times \mathbb{P}_{\setminus\{0,\infty\}}^1),$$

defined as $K \mapsto \mathrm{pr}_2!((\mathrm{pr}_1 \times \mathrm{pr}_{\mathbb{P}_{\setminus\{0,\infty\}}^1})^* K \otimes \mathrm{IC}_V)$.

2.4. *Local systems as eigenvalues.* By a \check{G} -local system E on $\mathbb{P}_{\setminus\{0,\infty\}}^1$, we mean a tensor functor

$$E : \mathrm{Rep}(\check{G}) \rightarrow \mathrm{Loc}(\mathbb{P}_{\setminus\{0,\infty\}}^1).$$

For such a \check{G} -local system E , we denote its value on $V \in \mathrm{Rep}(\check{G})$ by E_V , which is a $\overline{\mathbb{Q}}_\ell$ -local system on $\mathbb{P}_{\setminus\{0,\infty\}}^1$ in the usual sense.

Definition 2.11 (See [17] for details). Let E be a \check{G} -local system on $\mathbb{P}_{\setminus\{0,\infty\}}^1$. A *Hecke eigensheaf with eigenvalue E* is a perverse sheaf $K \in \mathrm{Perv}(\mathrm{Bun}_{\mathcal{G}(m,n)})$ together with isomorphisms $\mathrm{Hk}_V(K) \xrightarrow{\sim} K \boxtimes E^V$, which are compatible with the symmetric tensor structure on $\mathrm{Rep}(\check{G})$ and composition of Hecke correspondences.

Remark 2.12. Since local systems are usually introduced differently, let us briefly recall how the tensor functor E allows us to reconstruct the monodromy representation of the local system. This will be useful to fix notations for the monodromy representation. Choose a geometric point $\bar{\eta}$ over $\mathrm{Spec} K_0 \in \mathbb{P}_{\setminus\{0,\infty\}}^1$. The restriction to $\bar{\eta}$ defines a tensor functor

$$(2.13) \quad \omega_E : \mathcal{S} \xrightarrow{E} \mathrm{Loc}(\mathbb{P}_{\setminus\{0,\infty\}}^1) \xrightarrow{j_{\bar{\eta}}^*} \mathrm{Vec},$$

i.e., (see [12, Th. 3.2]) a \check{G} torsor with an action of $\pi_1(\mathbb{P}_{\setminus\{0,\infty\}}^1, \bar{\eta})$. Choosing a point of the torsor, this defines the monodromy representation

$$(2.14) \quad \varphi : \pi_1(\mathbb{P}_{\setminus\{0,\infty\}}^1, \bar{\eta}) \rightarrow \check{G}(\overline{\mathbb{Q}}_\ell).$$

We denote by φ^{geo} the restriction of φ to $\pi_1^{\mathrm{geo}}(\mathbb{P}_{\setminus\{0,\infty\}}^1, \bar{\eta}) = \pi_1(\mathbb{P}_{\setminus\{0,\infty\}}^1 \otimes_k \bar{k}, \bar{\eta})$ and call it the *global geometric monodromy representation* of E .

The analog of this construction for twisted groups \mathcal{G} will produce ${}^L\mathcal{G}$ -local systems. Here we define the L -group of \mathcal{G} to be ${}^L\mathcal{G} = \check{G} \rtimes \langle \sigma \rangle$, where σ is the automorphism of order N of G used in the definition of \mathcal{G} . This automorphism induces an automorphism of \check{G} via the geometric Satake isomorphism. In [Lemma B.3](#) we will check that this automorphism indeed preserves the pinning of \check{G} .

Let us give a definition of ${}^L\mathcal{G}$ -local systems that is sufficient for our purposes. The reason why we cannot immediately apply the geometric Satake isomorphism is that the fibers of pr_2 are not constant along $\mathbb{P}_{\setminus\{0,\infty\}}^1$, so only some of the Hecke operators Hk_V will define global Hecke operators over

$\mathbb{P}^1_{\setminus\{0,\infty\}}$. However, we can pull back the convolution diagram by the map $[N] : \mathbb{P}^1_{\setminus\{0,\infty\}} \rightarrow \mathbb{P}^1_{\setminus\{0,\infty\}}$. We will denote this covering by $\widetilde{\mathbb{P}}^1_{\setminus\{0,\infty\}} \rightarrow \mathbb{P}^1_{\setminus\{0,\infty\}}$. After pull-back we can, as before, define Hecke operators on $\text{Hecke}_{G(m,n)}^{\widetilde{\mathbb{P}}^1_{\setminus\{0,\infty\}}}$:

$$\text{Hk} : \text{Rep}(\check{G}) \times D^b(\text{Bun}_{G(m,n)}) \cong \mathcal{S} \times D^b(\text{Bun}_{G(m,n)}) \rightarrow D^b(\text{Bun}_{G(m,n)} \times \widetilde{\mathbb{P}}^1_{\setminus\{0,\infty\}}).$$

Moreover, the covering group μ_N acts on the convolution diagram over $\widetilde{\mathbb{P}}^1_{\setminus\{0,\infty\}}$. This defines a μ_N -equivariant structure on the functor Hk . On the source, μ_N acts on \mathcal{S} via $\sigma : \mu_N \rightarrow \text{Aut}^\dagger(G)$, which can be identified with the action of $\text{Aut}^\dagger(\check{G})$ on $\text{Rep}(\check{G})$ (Lemma B.3); on the target μ_N acts on $\widetilde{\mathbb{P}}^1_{\setminus\{0,\infty\}}$.

Let E be a \check{G} -local system on $\widetilde{\mathbb{P}}^1_{\setminus\{0,\infty\}}$ together with compatible isomorphisms $\zeta^* E \cong E \times^{\check{G},\sigma(\zeta)} \check{G}$ for $\zeta \in \mu_N$. We view E as a tensor functor

$$E : \text{Rep}(\check{G}) \rightarrow \text{Loc}(\widetilde{\mathbb{P}}^1_{\setminus\{0,\infty\}})$$

together with a μ_N -equivariant structure. We can define a Hecke eigensheaf $K \in \text{Perv}(\text{Bun}_{G(m,n)})$ with eigenvalue E as before, but now we have to specify an isomorphism of functors $\epsilon(V) : \text{Hk}_V(K) \xrightarrow{\sim} K \boxtimes E^V$ compatible with the tensor structure on $\text{Rep}(\check{G})$ that commutes with the μ_N -equivariant structures of the functor $V \mapsto \text{Hk}_V(K)$ and the functor E .

Note that if $\sigma : \mu_N \rightarrow \text{Aut}(G)$ is trivial, the isomorphisms $\zeta^* E \cong E \times^{\check{G},\sigma(\zeta)} \check{G}$ define a descent datum for E . So in this case the definition coincides with the definition for constant groups.

With these definitions we can state our first main result.

- THEOREM 1.** (1) *The sheaves $A = A_\phi$ and $A_{\phi,\chi}$ (see Definitions 2.4 and 2.6) are Hecke eigensheaves. We will denote the eigenvalue of A_ϕ (resp. $A_{\phi,\chi}$) by $\text{Kl}_{L_G}(\phi)$ (resp. $\text{Kl}_{L_G}(\phi,\chi)$).*
- (2) *If $\mathcal{G} = \mathbb{P}^1 \times G$ is a constant group scheme, the local system $\text{Kl}_{\check{G}}(\phi)$ is tamely ramified at 0. The monodromy action at 0 on $\text{Kl}_{\check{G}}(\phi)$ is given by a principal unipotent element in \check{G} .*
- (3) *For any irreducible representation $V \in \text{Rep}(\check{G})$, the sheaf $\text{Kl}_{L_G}(\phi,\chi)^V$ is pure.*

Since we defined $\text{Kl}_{\check{G}}(\phi)$ using geometric Hecke operators, for any point $x \in \mathbb{P}^1_{\setminus\{0,\infty\}}$ the \check{G} -conjugacy class of Frob_x defined by the local system is given by the Satake parameter of Gross's automorphic form $\pi(\phi)$ at x .

2.5. The monodromy representation. In this section, we assume $\mathcal{G} = G \times \mathbb{P}^1$, where G is an almost simple split group over k . As in Remark 2.12, we will denote by φ the monodromy representation for $\text{Kl}_{\check{G}}(\phi)$ and by φ^{geo} its geometric monodromy representation.

Our next result is about the geometric monodromy of $\text{Kl}_{\check{G}}(\phi, \chi)$ at ∞ . Let $\text{Kl}_{\check{G}}^{\text{Ad}}(\phi, \chi)$ be the local system on $\mathbb{P}^1_{\setminus\{0, \infty\}}$ induced from the adjoint representation $V = \check{\mathfrak{g}}$. Fixing an embedding of the local Galois group $\text{Gal}(K_{\infty}^{\text{sep}}/K_{\infty})$ into $\pi_1(\mathbb{P}^1_{\setminus\{0, \infty\}}, \bar{\eta})$, we get an action of $\text{Gal}(K_{\infty}^{\text{sep}}/K_{\infty})$ on the geometric stalk of $\text{Kl}_{\check{G}}^{\text{Ad}}(\phi, \chi)$ at the formal punctured discs $\text{Spec} K_{\infty}^{\text{sep}}$. Choosing an identification of this stalk with $\check{\mathfrak{g}}$, we get an action of $\text{Gal}(K_{\infty}^{\text{sep}}/K_{\infty})$ on $\check{\mathfrak{g}}$ (well defined up to \check{G} -conjugacy).

Let $\mathcal{J}_{\infty} \subset \text{Gal}(K_{\infty}^{\text{sep}}/K_{\infty})$ be the inertia group. Let $\mathcal{J}_{\infty}^+ \subset \mathcal{J}_{\infty}$ be the wild inertia group and $\mathcal{J}_{\infty}^t = \mathcal{J}_{\infty}/\mathcal{J}_{\infty}^+$ be the tame inertia group.

THEOREM 2. *Suppose $p = \text{char}(k)$ is good for G if G is not simply-laced (i.e., $p > 2$ when G is of type B_n, C_n and $p > 3$ when G is of type F_4, G_2). Then*

- $\text{Swan}_{\infty}(\text{Kl}_{\check{G}}^{\text{Ad}}(\phi, \chi)) = r(\check{G})$, the rank of \check{G} ;
- $\check{\mathfrak{g}}^{\mathcal{J}_{\infty}^t} = 0$.

The Swan equality will be proved in [Corollary 5.1](#); the vanishing of \mathcal{J}_{∞} -invariants will be proved in [Proposition 5.3\(2\)](#). Combining this theorem with a result of Gross and Reeder [[22](#), Prop. 5.6], we get an explicit description of the geometric monodromy of $\text{Kl}_{\check{G}}(\phi, \chi)$ at ∞ .

COROLLARY 2.15 (Gross-Reeder). *Suppose $p = \text{char}(k)$ does not divide $\#W$. Then the local geometric Galois representation $\varphi_{\infty}^{\text{geo}} : \mathcal{J}_{\infty} \rightarrow \check{G}$ is a simple wild parameter defined in [[22](#), §6]. More precisely, up to \check{G} -conjugation, we have a commutative diagram of exact sequences*

$$\begin{array}{ccccc}
 \mathcal{J}_{\infty}^+ & \longrightarrow & \mathcal{J}_{\infty} & \longrightarrow & \mathcal{J}_{\infty}^t \\
 \downarrow \varphi_{\infty}^{\text{geo}, +} & & \downarrow \varphi_{\infty}^{\text{geo}} & & \downarrow \\
 \check{T} & \longrightarrow & N(\check{T}) & \longrightarrow & W,
 \end{array}$$

where

- A topological generator of the tame inertia \mathcal{J}_{∞}^t maps to a Coxeter element $\text{Cox} \in W$ (well defined up to W -conjugacy; see [[6](#), Ch.V, §6]).
- The wild inertia \mathcal{J}_{∞}^+ maps onto a subgroup $\check{T}(\zeta) \subset \check{T}[p]$. Here $\zeta \in \overline{\mathbb{F}}_p^{\times}$ is a primitive h -th root of unity (h is the Coxeter number) and $\check{T}(\zeta) \subset \check{T}[p]$ is the unique $\mathbb{F}_p[\text{Cox}]$ -submodule of $\check{T}[p]$ isomorphic to $\mathbb{F}_p[\zeta]$ (on which Cox acts by multiplication by ζ).
- The nonzero breaks of the \mathcal{J}_{∞} -representation $\text{Kl}_{\check{G}}^{\text{Ad}}(\phi, \chi)$ are equal to $1/h$.

Finally, we can state our result on the global geometric monodromy of $\text{Kl}_{\check{G}}(\phi)$. Let $\check{G}_{\text{geo}} \subset \check{G}$ be the Zariski closure of the image of the global geometric monodromy representation φ^{geo} .

THEOREM 3. *Suppose $\text{char}(k) > 2$. Then the geometric monodromy group \check{G}_{geo} for $\text{Kl}_{\check{G}}(\phi)$ is connected and*

- $\check{G}_{\text{geo}} = \check{G}^{\text{Aut}^\dagger(\check{G}), \circ}$ if \check{G} is not of type A_{2n} ($n \geq 2$) or B_3 ,
- $\check{G}_{\text{geo}} = \check{G}$ if \check{G} is of type A_{2n} ,
- $\check{G}_{\text{geo}} = G_2$ if \check{G} is of type B_3 and $\text{char}(k) > 3$.

The proof will be given in [Section 6.2](#).

2.6. Variant. There is a variant of our construction using D -modules instead of ℓ -adic sheaves. The base field is then taken to be $k = \mathbb{C}$. The Artin-Schreier local system AS_ψ is replaced by the exponential D -module $\mathbb{C}\langle x, \partial_x \rangle / (\partial_x - 1)$ on $\mathbb{A}_{\mathbb{C}}^1 = \text{Spec} \mathbb{C}[x]$. All the rest of the construction carries through, and we get a tensor functor

$$\text{Kl}_{L\mathcal{G}}(\phi)_{\text{dR}} : \text{Rep}({}^L\mathcal{G}_{\mathbb{C}}) \rightarrow \text{Conn}(\mathbb{P}_{\setminus\{0, \infty\}, \mathbb{C}}^1),$$

where $\text{Conn}(\mathbb{P}_{\setminus\{0, \infty\}, \mathbb{C}}^1)$ is the tensor category of vector bundles with connections on $\mathbb{P}_{\setminus\{0, \infty\}, \mathbb{C}}^1$.

We conjecture that our construction should give the same connection as the Frenkel-Gross construction. To state this precisely, let G be almost simple and \mathcal{G} be a quasi-split form of G over $\mathbb{P}_{\setminus\{0, \infty\}}^1$ given by $\sigma : \mu_N \rightarrow \text{Aut}^\dagger(G)$. Recall from [[15](#), §5] the Frenkel-Gross connection on the trivial ${}^L\mathcal{G}$ -bundle on $\mathbb{P}_{\setminus\{0, \infty\}, \mathbb{C}}^1$:

$$\nabla_{L\mathcal{G}}(\check{X}_0, \dots, \check{X}_{r_\sigma}) = d + \sum_{i=0}^{r_\sigma} \check{X}_i \frac{d}{dz},$$

where r_σ is the rank of G^σ and \check{X}_i is a basis of the $(-\alpha_i)$ -root space of the (twisted) affine Kac-Moody Lie algebra associate to $\check{\mathfrak{g}}$ and σ .

CONJECTURE 2.16. *There is a bijection between the set of generic linear functions $\phi : I(1)/I(2) \rightarrow \mathbb{G}_{a, \mathbb{C}}$ and the set of bases $(\check{X}_0, \dots, \check{X}_{r_\sigma})$ such that whenever ϕ corresponds to $(\check{X}_0, \dots, \check{X}_{r_\sigma})$ under this bijection, there is a natural isomorphism between ${}^L\mathcal{G}$ -connections on $\mathbb{P}_{\setminus\{0, \infty\}, \mathbb{C}}^1$:*

$$\text{Kl}_{L\mathcal{G}}(\phi)_{\text{dR}} \cong ({}^L\mathcal{G}, \nabla_{L\mathcal{G}}(\check{X}_0, \dots, \check{X}_{r_\sigma})).$$

After the paper was written, we learned that Xinwen Zhu [[38](#)] has obtained a proof of this conjecture.

3. Example: Kloosterman sheaf for GL_n

In this section, we calculate the Kloosterman sheaf $\text{Kl}_{\text{GL}_n}(\phi, \chi)$ for the constant group $G = \text{GL}_n$ over \mathbb{P}^1 . Its Langlands dual is $\text{GL}_{n, \overline{\mathbb{Q}}_\ell}$ and we will denote the standard representation by Std . Describing this GL_n -local system over $\mathbb{P}_{\setminus\{0, \infty\}}^1$ is then the same as describing the rank- n local system

$\mathrm{Kl}_{\mathrm{GL}_n}^{\mathrm{Std}}(\phi, \chi)$. We will see that this rank- n local system coincides with the classical Kloosterman sheaf defined by Deligne in [10].

3.1. *Another modular interpretation.* We want to interpret $\mathrm{GL}_n(1, 2)$ -bundles in terms of vector bundles. We first define a variant of $\mathrm{Bun}_{\mathrm{GL}_n(1,2)}$. Let $\mathrm{Bun}_{n,1,2}$ be the stack classifying the data $(\mathcal{E}, F^*\mathcal{E}, \{v^i\}, F_*\mathcal{E}, \{v_i\})$ where

- (1) \mathcal{E} is a vector bundle of rank n on \mathbb{P}^1 ;
- (2) a decreasing filtration $F^*\mathcal{E}$ giving a complete flag of the fiber of \mathcal{E} at 0:

$$\mathcal{E} = F^0\mathcal{E} \supset F^1\mathcal{E} \supset \dots \supset F^n\mathcal{E} = \mathcal{E}(-\{0\});$$

- (3) a nonzero vector $v^i \in F^{i-1}\mathcal{E}/F^i\mathcal{E}$ for each $i = 1, \dots, n$;
- (4) an increasing filtration $F_*\mathcal{E}$ giving a complete flag of the fiber of \mathcal{E} at ∞ :

$$\mathcal{E}(-\{\infty\}) = F_0\mathcal{E} \subset F_1\mathcal{E} \subset \dots \subset F_n\mathcal{E} = \mathcal{E};$$

- (5) a vector $v_i \in F_i\mathcal{E}/F_{i-2}\mathcal{E}$ that does not lie in $F_{i-1}\mathcal{E}/F_{i-2}\mathcal{E}$ for $i = 1, \dots, n$. (We understand $F_{-1}\mathcal{E}$ as $(F_{n-1}\mathcal{E})(-\{\infty\})$.)

Note that $\mathrm{Bun}_{n,1,2}$ is the moduli stack of \mathcal{G} -torsors over \mathbb{P}^1 , where \mathcal{G} is the Bruhat-Tits group scheme over \mathbb{P}^1 such that

- $\mathcal{G}|_{\mathbb{P}^1 \setminus \{0, \infty\}} = \mathrm{GL}_n \times (\mathbb{P}^1 \setminus \{0, \infty\})$,
- $\mathcal{G}(\mathcal{O}_0) = I_{\mathrm{GL}_n}(1)^{\mathrm{opp}}$,
- $\mathcal{G}(\mathcal{O}_\infty) = Z_{\mathrm{GL}_n}(1)(k[[s]]) \cdot I_{\mathrm{SL}_n}(2) \supset I_{\mathrm{GL}_n}(2)$.

The only difference between \mathcal{G} and $\mathrm{GL}_n(1, 2)$ is that they take different level structures for the center $\mathbb{G}_m = Z_{\mathrm{GL}_n}$ at ∞ . Therefore we have a natural morphism $\mathrm{GL}_n(1, 2) \rightarrow \mathcal{G}$, hence a natural morphism $\mathrm{Bun}_{\mathrm{GL}_n(1,2)} \rightarrow \mathrm{Bun}_{n,1,2}$, which is a \mathbb{G}_a -torsor.

Choosing a trivialization of the bundle \mathcal{E} over $\mathbb{P}^1 \setminus \{0, \infty\}$, we can rewrite the moduli problem for $\mathrm{Bun}_{n,1,2}$ in the following way. Let Λ be the free $k[t, t^{-1}]$ -module with basis $\{e_1, \dots, e_n\}$. Let $e_{i+jn} = t^j e_i$ for $1 \leq i \leq n$ and $j \in \mathbb{Z}$. Then Λ is a k -vector space with basis $\{e_i\}_{i \in \mathbb{Z}}$. For any k -algebra R , an $R[t]$ -lattice in $R \otimes \Lambda$ is an $R[t]$ -submodule $\Lambda' \subset R \otimes \Lambda$ such that there exists $M \in \mathbb{Z}_{>0}$ such that

$$\mathrm{Span}_R\{e_i | i > M\} \subset \Lambda' \subset \mathrm{Span}_R\{e_i | i \geq -M\}$$

and that both $\Lambda'/\mathrm{Span}_R\{e_i | i > M\}$ and $\mathrm{Span}_R\{e_i | i \geq -M\}/\Lambda'$ are projective R -modules. Similarly, we can define the notion of $R[t^{-1}]$ -lattices in $R \otimes \Lambda$.

Let $\widetilde{\mathrm{Bun}}_{n,1,2}(R)$ classify the data $(\Lambda^*, \{v^i\}_{0 \leq i \leq n-1}, \Lambda_*, \{v_i\}_{1 \leq i \leq n})$.

- (1) $R \otimes \Lambda \supset \Lambda^0 \supset \Lambda^1 \supset \dots \supset \Lambda^n = t\Lambda^0$ is a chain of $R[t]$ -lattices such that Λ^i/Λ^{i+1} is a projective R -module of rank 1. We let $\Lambda^{i+jn} = t^j \Lambda^i$ for any $1 \leq i \leq n$ and $j \in \mathbb{Z}$.
- (2) $v^i \in \Lambda^i/\Lambda^{i+1}$ is an R -basis for $i = 0, \dots, n - 1$.

- (3) $R \otimes \Lambda \supset \Lambda_n \supset \Lambda_{n-1} \supset \dots \supset \Lambda_0 = t^{-1}\Lambda_n$ is a chain of $R[t^{-1}]$ -lattices such that Λ_i/Λ_{i+1} is a projective R -module of rank 1. We let $\Lambda_{i+jn} = t^j\Lambda_i$ for any $1 \leq i \leq n$ and $j \in \mathbb{Z}$.
- (4) $v_i \in \Lambda_i/\Lambda_{i-2}$ whose image in Λ_i/Λ_{i-1} is an R -basis, for $i = 1, \dots, n$.

The group $\mathrm{GL}_n(k[t, t^{-1}])$ (viewed as an ind-group scheme over k) acts on Λ , and hence it acts on the stack $\widetilde{\mathrm{Bun}}_{n,1,2}$. Moreover, $\mathrm{Bun}_{n,1,2}$ is naturally isomorphic to the quotient stack $\widetilde{\mathrm{Bun}}_{n,1,2}/\mathrm{GL}_n(k[t, t^{-1}])$.

Associated with the chains of lattices (Λ^*, Λ_*) is the locally constant integer-valued function on $\mathrm{Spec}R$:

$$\mathrm{deg}(\Lambda^*, \Lambda_*) := \chi_R(\Lambda^0 \oplus \Lambda_0 \xrightarrow{(\iota^0, \iota_0)} R \otimes \Lambda).$$

Here ι^0, ι_0 are inclusion maps and the Euler characteristic on the right-hand side is defined as $\mathrm{rk}_R \ker(\iota^0, \iota_0) - \mathrm{rk}_R \mathrm{coker}_R(\iota^0, \iota_0)$.

For $d \in \mathbb{Z}$, let $\mathrm{Bun}_{n,1,2}^d \subset \mathrm{Bun}_{n,1,2}$ be the substack classifying $\{\Lambda^*, v^i, \Lambda_*, v_i\}$ with $\mathrm{deg}(\Lambda^*, \Lambda_*) = d$.

The embedding of the big cell $j^d : T \times \mathbb{G}_a^n \hookrightarrow \mathrm{Bun}_{n,1,2}^d$ can be fixed as follows. For $\underline{a} = (a_1, \dots, a_n) \in T(R) = (R^\times)^n$ and $\underline{b} = (b_1, \dots, b_n) \in R^n$, the point $j^d(\underline{a}, \underline{b}) \in \mathrm{Bun}_{n,1,2}^d(R)$ is given by the $\mathrm{GL}_n(R[t, t^{-1}])$ -orbit of the following data:

- (1) $\Lambda^i = \mathrm{Span}_R\{e_j | j > i\} \subset R \otimes \Lambda$,
- (2) $v^i = a_{i+1}e_{i+1} \in \Lambda^i/\Lambda^{i+1}$,
- (3) $\Lambda_i = \mathrm{Span}_R\{e_j | j \leq i + d\} \subset R \otimes \Lambda$,
- (4) $v_i = e_{i+d} + b_i e_{i+d-1} \in \Lambda_i/\Lambda_{i-2}$.

In particular, we have a base point $\star_d = j^d(\underline{1}, \underline{0}) \in \mathrm{Bun}_{n,1,2}^d$. We denote the $k[t]$ -chain (resp. $k[t^{-1}]$ -chain) of \star_d by $\Lambda^*(\star_d)$ (resp. $\Lambda_*(\star_d)$). Notice that the underlying vector bundle for any point $j^d(\underline{a}, \underline{b})$ is the bundle

$$\mathcal{E}_d = \mathcal{O}(m+1)^{\oplus r} \oplus \mathcal{O}(m)^{\oplus n-r},$$

where $m \in \mathbb{Z}$ and $0 \leq r \leq n-1$ is uniquely determined by $d = mn + r$.

3.2. *The Kloosterman sheaf associated with the standard representation of GL_n .* Let $\chi : T(k) \rightarrow \overline{\mathbb{Q}}_\ell^\times$ be a character, which defines a Kummer local system Kum_χ on T . The perverse sheaves

$$A_{\phi, \chi}^d = j_!^d(\mathrm{Kum}_\chi[n] \boxtimes \phi^* \mathrm{AS}_\psi[n])$$

on $\mathrm{Bun}_{n,1,2}^d$ form a Hecke eigensheaf on $\mathrm{Bun}_{n,1,2}$ by [Theorem 1](#).

Consider the Hecke correspondence Hecke_{ω_1} given by the coweight $\omega_1 = (1, 0, \dots, 0)$. (ω_1 defines the representation Std .) We restrict it to the components of Bun^0 and Bun^1 and the curve $\mathbb{P}^1_{\setminus\{0,\infty\}}$:

$$(3.1) \quad \begin{array}{ccc} & \text{Hecke}_{\omega_1} & \\ \swarrow pr_1 & & \searrow pr_2 \\ \text{Bun}^0_{n,1,2} & & \text{Bun}^1_{n,1,2} \times (\mathbb{P}^1_{\setminus\{0,\infty\}}). \end{array}$$

We would like to evaluate the eigenvalue local system $\text{Kl}_{\text{GL}_n}^{\text{Std}}(\phi, \chi)$ on $\mathbb{P}^1_{\setminus\{0,\infty\}}$ characterized by

$$pr_{2,!}pr_1^*A_{\phi,\chi}^0[n-1] \cong A_{\phi,\chi}^1 \boxtimes \text{Kl}_{\text{GL}_n}^{\text{Std}}(\phi, \chi).$$

The intersection cohomology sheaf IC_{ω_1} on Hk_{ω_1} is simply $\overline{\mathbb{Q}}_\ell[n-1]$ because $pr_1: \text{Hk}_{\omega_1} \rightarrow \text{Bun}_{n,1,2}$ is a \mathbb{P}^{n-1} -bundle.¹ We restrict the diagram (3.1) to the fiber of $\star_1 \times (\mathbb{P}^1_{\setminus\{0,\infty\}}) \subset \text{Bun}^1_{n,1,2} \times (\mathbb{P}^1_{\setminus\{0,\infty\}})$ under pr_2 and the fiber $\text{GR}_{\omega_1}^\circ$ over the big cell in $\text{Bun}^0_{n,1,2}$ under pr_1 . We get

$$(3.2) \quad \begin{array}{ccccc} & & \text{GR}_{\omega_1}^\circ & \longrightarrow & \text{GR}_{\omega_1} \\ & \swarrow pr_1 & & \swarrow pr_1 & \searrow \pi \\ & T \times \mathbb{G}_a^n & \xrightarrow{j^0} & \text{Bun}^0_{n,1,2} & & \mathbb{P}^1_{\setminus\{0,\infty\}} \end{array}$$

The R -points of the open subscheme $\text{GR}_{\omega_1}^\circ \subset \text{GR}_{\omega_1}$ classifies $R[t, t^{-1}]$ -homomorphisms $M: R \otimes \Lambda \rightarrow R \otimes \Lambda$ sending the chains $(\Lambda^*(\star_0), \Lambda_*(\star_0))$ to $(\Lambda^*(\star_1), \Lambda_*(\star_1))$ up to pre-composing with automorphisms of $(R \otimes \Lambda, \Lambda^*(\star_0), \Lambda_*(\star_0))$. With respect to the $R[t, t^{-1}]$ -basis $\{e_1, \dots, e_n\}$ of $R \otimes \Lambda$, any such M takes the form

$$\begin{pmatrix} a_1 & & & & & & & & tb_n \\ b_1 & a_2 & & & & & & & \\ & b_2 & a_3 & & & & & & \\ & & & \ddots & \ddots & & & & \\ & & & & & b_{n-1} & a_n & & \end{pmatrix}$$

¹In this calculation, we will *not* normalized IC_{ω_1} to be of weight 0, as we did in [Remark 2.10](#). So strictly speaking, the sheaf $\text{Kl}_{\text{GL}_n}^{\text{Std}}(\phi, \chi)$ differs from the one defined in [Section 2.3](#) by a Tate twist $(\frac{n-1}{2})$.

up to right multiplication by diagonal matrices. (Here $a_i, b_i \in R^\times$.) Therefore, we can normalize the matrix of M to be

$$(3.3) \quad M = \begin{pmatrix} a_1 & & & & t \\ & 1 & a_2 & & \\ & & \ddots & \ddots & \\ & & & & 1 & a_n \end{pmatrix}.$$

That is, we get an identification $U \cong \mathbb{G}_m^n$ by sending M to $(-a_1, \dots, -a_n)$.

We now describe the projections pr_1 and π using this identification. Using the isomorphism M we can pull back the vectors $\{v^i = e_{i+1}\}, \{v_i = e_{i+1}\}$ in the data of \star_1 to get the corresponding vectors for the point $\text{pr}_1(M) = (\Lambda, \Lambda^*(\star_0), \{v^i\}, \Lambda_*(\star_0), \{v_i\}) \in \text{Bun}_{n,1,2}^0$:

$$\begin{aligned} v^i &= a_{i+1}^{-1} e_{i+1}, & i &= 0, \dots, n-1, \\ v_i &= e_i - a_i e_{i-1}, & i &= 1, \dots, n. \end{aligned}$$

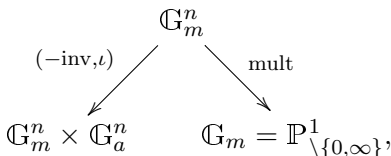
This means that $\text{pr}_1(M)$ for M as in (3.3) has coordinates

$$\text{pr}_1(M) = j^0(a_1^{-1}, \dots, a_n^{-1}, -a_1, \dots, -a_n).$$

On the other hand, the point $\pi(M) \in \mathbb{P}^1_{\setminus\{0,\infty\}}$ is the value of t such that $\det(M) = 0$; i.e.,

$$\pi(M) = (-1)^n a_1 \cdots a_n.$$

In summary, the maps pr_1 and π from $\text{GR}_{\omega_1}^\circ$ can be identified with the following maps:



where $\text{inv} : \mathbb{G}_m^n \rightarrow \mathbb{G}_m^n$ is the coordinate-wise inverse and $\iota : \mathbb{G}_m^n \hookrightarrow \mathbb{G}_a^n$ is the natural inclusion. We thus get

PROPOSITION 3.4. *The Kloosterman sheaf $\text{Kl}_{\text{GL}_n}^{\text{Std}}(\phi, \chi)$ associated with the Hecke eigensheaf $A_{\phi, \chi}$ and the standard representation of the dual group $\text{GL}_{n, \overline{\mathbb{Q}}_\ell} = \widehat{\text{GL}}_n$ takes the form*

$$\text{Kl}_{\text{GL}_n}^{\text{Std}}(\phi, \chi) \cong \text{mult}_!((- \text{inv})^* \text{Kum}_\chi \otimes \phi^* \text{AS}_\psi)[n-1].$$

Here we denote the restriction of $\phi : \mathbb{G}_a^n \rightarrow \mathbb{G}_a$ to \mathbb{G}_m^n still by ϕ .

Remark 3.5. When $\phi = \text{add} : \mathbb{G}_a^n \rightarrow \mathbb{G}_a$ is the addition of the coordinates and χ is written as n -multiplicative characters (χ_1, \dots, χ_n) , the Kloosterman sheaf $\text{Kl}_{\text{GL}_n}^{\text{Std}}(\phi, \chi)$ is the same as the Kloosterman sheaf

$$\text{Kl}(\psi; \chi_1, \dots, \chi_n; 1, \dots, 1)$$

defined by Katz in [26, §4.1], which is a generalization of Deligne’s Kloosterman sheaves [10, §7].

4. Proof of Theorem 1

In this section we prove [Theorem 1](#).

4.1. *First Step: $\text{Hk}_V(A)$ is perverse.* We want to show that for every $V \in \text{Rep}(\tilde{G})$, the complex $\text{Hk}_V(A_{\phi,\chi})[-1]$ is a perverse sheaf. In order to simplify notation, we will only consider the sheaf $A = A_\phi$. The proof for $A_{\phi,\chi}$ is identical; one only needs to replace $\mathcal{G}(0, 2)$ by $\mathcal{G}(1, 2)$ everywhere in the argument.

Let us recall the convolution diagram from [Section 2.2](#):

$$\begin{array}{ccc}
 & \text{Hecke}_{\mathcal{G}(0,2)}^{\mathbb{A}^1} & \xrightarrow{\text{pr}_{\mathbb{A}^1}} \mathbb{A}^1 \\
 \text{pr}_1 \swarrow & & \searrow \text{pr}_2 \\
 \text{Bun}_{\mathcal{G}(0,2)} & & \text{Bun}_{\mathcal{G}(0,2)} \times \mathbb{A}^1.
 \end{array}$$

Our proof is based on a few simple geometric observations. First we need to recall that the maps pr_i in the above diagram are locally trivial fibrations.

Remark 4.1. The map pr_1 is a locally trivial fibration; i.e., there exists a smooth atlas $U \rightarrow \text{Bun}_{\mathcal{G}(0,2)}$ such that

$$U \times_{\text{Bun}_{\mathcal{G}(0,2)}} \text{Hecke}_{\mathcal{G}(0,2)}^{\mathbb{A}^1} \cong U \times \text{GR}_{\mathcal{G}(0,2)}.$$

Furthermore, the map pr_2 is also locally trivial on this atlas; i.e.,

$$(U \times \mathbb{A}^1) \times_{\text{Bun}_{\mathcal{G}(0,2)} \times \mathbb{A}^1} \text{Hecke}_{\mathcal{G}(0,2)}^{\mathbb{A}^1} \cong U \times \text{GR}_{\mathcal{G}(0,2)}.$$

Proof. In order to find an atlas $p: U \rightarrow \text{Bun}_{\mathcal{G}(0,2)}$ satisfying these conditions, we only need that the family of $\mathcal{G}(0, 2)$ -bundles corresponding to p on $U \times \mathbb{P}^1$ is trivial over $U \times \mathbb{A}^1$. By [Proposition 1.1](#), for any U this condition is satisfied locally in the étale topology on U . □

For $\gamma \in \Omega$, we denoted by $j_\gamma: I(1)/I(2) \hookrightarrow \text{Bun}_{\mathcal{G}(0,2)}$ the canonical embedding. Recall that $A^\gamma = j_{\gamma,!} \text{AS}_\phi[\dim(\text{Bun}_{\mathcal{G}(0,2)})]$. Denote by $j'_\gamma: \text{pr}_1^{-1}(I(1)/I(2)) \hookrightarrow \text{Hecke}_{\mathbb{A}^1}^{\mathcal{G}(0,2)}$ the inverse image of this embedding into the Hecke stack.

Remark 4.2. The restriction of pr_2 to $\text{pr}_1^{-1}(I(1)/I(2))$,

$$\text{pr}_2: \text{pr}_1^{-1}(I(1)/I(2)) \rightarrow \text{Bun}_{\mathcal{G}(0,2)} \times \mathbb{A}^1,$$

is affine.

Proof. By [Corollary 1.3](#) the open subset $I(1)/I(2) \hookrightarrow \text{Bun}_{\mathcal{G}(0,2)}$ is defined by the nonvanishing of sections of line bundles \mathcal{L}_i , which can be defined as the pull-back of the corresponding sections for GL_n under a faithful representation $\mathcal{G} \rightarrow \text{GL}_n$. If $\mathcal{G} = \text{GL}_n$, then for any $x \in \mathbb{A}^1$, the pull-backs of these bundles generate of the Picard-group $\text{Gr}_{\text{GL}_n(0,2),x}$, which is an ind-projective scheme (e.g., [\[14, Th. 7 and 8\]](#)). Thus the preimage of $I(1)/I(2)$ in $\text{GR}_{\text{GL}_n(0,2)}$ is affine over \mathbb{A}^1 .

For general \mathcal{G} , the ind-scheme $\text{Gr}_{\mathcal{G}(0,2),x}$ is usually constructed as a closed sub-scheme of Gr_{GL_n} for a suitable faithful representation ([33]). So again, the claim follows from the case $\mathcal{G} = \text{GL}_n$. \square

Now we can prove the first step. Fix any $V \in \text{Rep}(\check{G})$ and the corresponding perverse sheaf IC_V on $\text{Hecke}_{\mathcal{G}(0,2)}^{\mathbb{A}^1}$. We claim that Remark 4.1 implies

$$j'_!(\text{pr}_1^* \text{AS}_\phi \otimes \text{IC}_V) \cong j'_*(\text{pr}_1^* \text{AS}_\phi \otimes \text{IC}_V).$$

To see this, recall that the formation of $j_!$ and j_* commutes with smooth base-change. Thus, to prove the claimed isomorphism, we may choose a smooth atlas $p: U \rightarrow \text{Bun}_{\mathcal{G}(0,2)}$ such that the pull-back of pr_1 is isomorphic to the projection $U \times \text{GR}_{\mathcal{G}} \rightarrow U$. But in this case the claim follows from the isomorphism $j_! \text{AS}_\phi = j_* \text{AS}_\phi$ (Lemma 2.3).

In particular, we find that $j'_!(\text{pr}_1^* A \otimes \text{IC}_V)[1]$ is a perverse sheaf and

$$\begin{aligned} \text{Hk}_V(A) &= (\text{pr}_2 \circ j')_!(\text{pr}_1^*(A) \otimes \text{IC}_V) \\ &= (\text{pr}_2 \circ j')_*(\text{pr}_1^*(A) \otimes \text{IC}_V). \end{aligned}$$

By Remark 4.2, the map $(\text{pr}_2 \circ j')$ is affine. Therefore $(\text{pr}_2 \circ j')_*$ is right-exact for the perverse t-structure ([3, §4.1.1]). Thus $\text{Hk}_V(A)[1]$ must be perverse.

4.2. *Second step: A is an eigensheaf.* We already noted (Remark 2.8) that the action of $I(1)/I(2)$ on $\text{Bun}_{\mathcal{G}(0,2)}$ extends to an action on the convolution diagram. Thus $\text{Hk}_V(A)$ is $(I(1)/I(2), \phi)$ -equivariant. By Lemma 2.3 this implies that for any $\gamma \in \pi_0(\text{Bun}_{\mathcal{G}(0,2)})$, we have

$$\text{Hk}_V(A)|_{\text{Bun}_{\mathcal{G}(0,2)}^\gamma \times X} = A^\gamma \boxtimes E_V^\gamma,$$

where $E_V^\gamma[1]$ is a perverse sheaf on X .

We claim that the sheaf E_V^γ does not depend on γ . To show this, we may assume that IC_V is supported only on one connected component $\text{GR}_{\mathcal{G}}^\gamma \subset \text{GR}_{\mathcal{G}}$, because the functor Hk_V is isomorphic to the direct sum of the functors defined by the restriction of IC_V to the connected components.

Note that for any $\gamma' \in \Omega$, the Hecke operator $\text{Hk}_{\gamma'}$ commutes with Hk_V , because Hecke operators supported at different points of \mathbb{P}^1 commute. By definition A is an eigensheaf for the operators $\text{Hk}_{\gamma'}$. Thus,

$$\begin{aligned} A^0 \boxtimes E_V^0 &= \text{Hk}_V(A^{-\gamma}) = \text{Hk}_V(\text{Hk}_{-\gamma}(A^0)) \\ &= \text{Hk}_{-\gamma}(\text{Hk}_V(A^0)) = \text{Hk}_{-\gamma}(A^\gamma \boxtimes E_V^\gamma) \\ &= A^0 \boxtimes E_V^\gamma. \end{aligned}$$

This implies that the sheaves E_V^γ are canonically isomorphic to E_V^0 , and so we may drop the index γ .

The Hk_V are compatible with the tensor product of representations so, in particular, we have

$$\text{Hk}_{V \otimes V}(A) = \text{Hk}_V \circ \text{Hk}_V(A) = A \boxtimes (E_V \otimes E_V).$$

This implies that $E_V \otimes E_V[-1]$ is again a perverse sheaf. Therefore, E_V must be a perverse sheaf concentrated in cohomological degree 0. So for any V , the complex E_V is a sheaf such that $E_V = j_{\eta,*}E_{V,\eta}$, where we denoted by $j_\eta: \eta \hookrightarrow \mathbb{P}^1$ the inclusion of the generic point.

Also, we obtain a tensor functor $V \mapsto (E_V)_\eta$ with values in the category of local systems on η . This is a rigid tensor category, so by [12, Th. 3.2] this defines a \check{G} -local system over η . We need to show that $E_{V,\eta}$ extends to a local system on $\mathbb{P}^1_{\setminus\{0,\infty\}}$.

For the trivial Hecke operator Hk_1 , we have canonical isomorphisms $E_1 = \overline{\mathbb{Q}}_\ell$ together with maps $E_1 \rightarrow E_V \otimes E_{V^*} \rightarrow E_1$ such that the composition is equal to multiplication by $\dim(V)$. We already know that $E_V = j_{\eta,*}E_{V,\eta}$. Thus for any geometric point \bar{x} of \mathbb{G}_m , the fiber $E_{V,\bar{x}}$ is a subspace of the geometric generic fiber $E_{V,\bar{\eta}}$ and the canonical map $\text{id}_{\bar{\eta}}: \overline{\mathbb{Q}}_{\ell,\bar{\eta}} \rightarrow (E_V \otimes E_{V^*})_{\bar{\eta}}$ factors through $E_{V,\bar{x}} \otimes E_{V^*,\bar{x}}$. This implies that the sheaves E_V are locally constant because $\text{id}_{\bar{\eta}}$ corresponds to the identity of $E_{V,\bar{\eta}}$.

Therefore, the E_V define a tensor functor from $\text{Rep}(\check{G})$ to the category $\text{Loc}(\mathbb{P}^1_{\setminus\{0,\infty\}})$ of local systems on $\mathbb{P}^1_{\setminus\{0,\infty\}}$. Since again this is a rigid tensor category, this defines a \check{G} -local system on $\mathbb{P}^1_{\setminus\{0,\infty\}}$.

4.3. *Third step: The monodromy at the tame point.* As in the statement of Theorem 1 (2) we now assume $\mathcal{G} = G \times \mathbb{P}^1$ is a constant split group. To compute the monodromy of $\text{Kl}_{\check{G}}(\phi)$ at $\{0\}$ we rephrase an argument of Bezrukavnikov [4]. His argument relies on results on central sheaves of Gaitsgory [16] and on Gabber’s result that the monodromy filtration on nearby cycles coincides with the weight filtration [2].

In order to explain the argument we need to recall Gaitsgory’s construction ([16]). He considered the diagram

$$\begin{array}{ccccc} \text{GR}_{\mathcal{G}}^{\mathbb{P}^1 \setminus \{0,\infty\}} & \xrightarrow{j_{GR}} & \text{GR}_{\mathcal{G}} & \longleftarrow & \text{GR}_{\mathcal{G},0} = \text{Fl}_{\mathcal{G}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{P}^1_{\setminus\{0,\infty\}} & \xrightarrow{j} & \mathbb{A}^1 & \longleftarrow & \{0\} \end{array}$$

and the induced nearby cycles functor $\Psi: \text{Perv}(\text{GR}_{\mathcal{G}}^{\mathbb{P}^1 \setminus \{0,\infty\}}) \rightarrow \text{Perv}_{I_0}(\text{Fl}_{\mathcal{G}})$. He showed that the monodromy action on the sheaves $\Psi(\text{IC}_V)$ is unipotent.

Since the map $\text{pr}_1: \text{Hecke}_{\check{G}(0,2)}^{\mathbb{A}^1} \rightarrow \text{Bun}_{\mathcal{G}(0,2)}$ is locally isomorphic to the product with fibers isomorphic to $\text{GR}_{\mathcal{G}(0,2)}$, we know that $\Psi(\text{pr}_1^*(A) \otimes \text{IC}_V) =$

$\mathrm{pr}_1^*(A) \otimes \Psi(\mathrm{IC}_V)$. In particular, by Gaitsgory’s result [16, Th. 2] we find that the monodromy action on this sheaf is unipotent. Therefore ([19, Lemma 5.6]) the monodromy action on $\mathrm{pr}_{2,!}(\Psi(\mathrm{pr}_1^*(A) \otimes \mathrm{IC}_V))$ is again unipotent. Since taking nearby-cycles commutes with proper push-forward, the monodromy action on

$$(4.3) \quad \mathrm{pr}_{2,!}(\mathrm{pr}_1^*(A) \otimes \Psi(\mathrm{IC}_V)) = \mathrm{pr}_{2,!}(\Psi(\mathrm{pr}_1^*(A) \otimes \mathrm{IC}_V)) = \Psi(A \boxtimes E_V) = A \boxtimes \Psi(E_V)$$

is also unipotent. By definition, $\Psi(E_V)$ is the stalk of E_V at the geometric point $\mathrm{Spec}K_0^{\mathrm{sep}}$ over the punctured formal neighborhood of 0, carrying the $\mathrm{Gal}(K_0^{\mathrm{sep}}/K_0)$ -action as monodromy. Restricting (4.3) to the trivial $\mathcal{G}(0, 2)$ -bundle, we therefore get

$$(4.4) \quad \mathrm{Kl}_G^V|_{\mathrm{Spec}K_0^{\mathrm{sep}}} = R\Gamma_c(\mathrm{pr}_1^*A \otimes \Psi(\mathrm{IC}_V))$$

with the inertia group $\mathcal{J}_0 \subset \mathrm{Gal}(K_0^{\mathrm{sep}}/K_0)$ acting tamely and unipotently.

We have to show that the monodromy action is given by a principal unipotent element. Recall that the $\mathcal{G}(0, 2)(k[[t]]) = I(0)^{\mathrm{opp}}$ -orbits on the affine flag manifold $\mathrm{Fl}_G = \mathrm{GR}_{\mathcal{G},0}$ are parametrized by the Iwahori-Weyl group \widetilde{W} ([33, Prop.8.1]). The intersection cohomology sheaves of the closures of these orbits will be denoted by $\mathrm{IC}_{\widetilde{w}}$. The convolution with these sheaves defines Hecke operators $\mathrm{Hk}_{\widetilde{w}}(A) := \mathrm{pr}_{2,!}(\mathrm{pr}_1^*(A) \otimes \mathrm{IC}_{\widetilde{w}})$.

By a result of Görtz and Haines [19, Corollary 1.2], the sheaf $\Psi(\mathrm{IC}_V)$ has a filtration such that the associated graded sheaves are isomorphic to $\mathrm{IC}_w(i)$ for some $w \in \widetilde{W}, i \in \mathbb{Z}$ and the multiplicity of the $\mathrm{IC}_1(i)$ is equal to $\dim H^{2i}(\mathrm{Gr}_G, \mathrm{IC}_V)$.

Now, for any $\widetilde{w} \in \widetilde{W}$ of length $l(\widetilde{w}) > 0$, there exists a simple reflection s such that $\widetilde{w} = \widetilde{w}'s$ and $l(\widetilde{w}') < l(\widetilde{w})$. Write $P_s := I(0)^{\mathrm{opp}} \cup I(0)^{\mathrm{opp}}sI(0)^{\mathrm{opp}}$ for the parahoric subgroup generated by $I(0)^{\mathrm{opp}}$ and s , so that the projection $\mathrm{pr}_s : \mathrm{Fl} \rightarrow G((t))/P_s$ is a \mathbb{P}^1 -bundle. From this we see that the sheaf $\mathrm{IC}_{\widetilde{w}}$ is of the form $\mathrm{pr}_s^*(\mathrm{IC}_{\widetilde{w}'})$. Therefore the complex $\mathrm{Hk}_{\widetilde{w}}(A)$ is invariant under the larger parahoric subgroup $P_s \supset I(0)^{\mathrm{opp}}$. Since $\mathrm{Hk}_{\widetilde{w}}(A)$ is also $(I(1)/I(2), \mathrm{AS}_\phi)$ -equivariant, this implies that $\mathrm{Hk}_{\widetilde{w}}(A) = 0$.

Thus only the Hecke operators of length 0 act nontrivially on A . Therefore, by the result of Görtz and Haines, the dimensions of the weight filtration of $\Psi(\mathrm{Kl}_G^V) = \mathrm{Kl}_{G,\overline{\eta}}^V$ are given by the dimensions of $H^{2i}(\mathrm{Gr}_G, \mathrm{IC}_V)$. Since the monodromy filtration agrees with the weight filtration [2], the monodromy must act as a principal nilpotent element. This proves our claim.

Remark 4.5. A key ingredient needed in the article of Görtz and Haines is the formula for the trace function of the sheaves $\Psi(\mathrm{IC}_\mu)$. This trace is given by Bernstein’s formula for the central elements of the Iwahori-Hecke algebra (see [19, §2.7]). One can also deduce our proposition directly from this formula.

5. Cohomological properties of Kloosterman sheaves

In this section, let G be a split, almost simple group over k , viewed as a constant group scheme over \mathbb{P}^1 . Recall that for each root $\alpha \in \Phi$, U_α is the root group in G . We fix an isomorphism $u_\alpha : \mathbb{G}_a \xrightarrow{\sim} U_\alpha$ for each root α .

Notation. For a scheme X defined over k and a $\overline{\mathbb{Q}}_\ell$ -complex F of sheaves on X , $\chi_c(X, F)$ and $H^*(X, F)$ mean the Euler characteristic and the cohomology of the pull-back of F to $X \otimes_k \bar{k}$.

5.1. *The Euler characteristics, Swan conductors and cohomological rigidity.* There are two uniform choices of $V \in \text{Rep}(\check{G})$ that have small dimensions. One is $V = \text{Ad} = \check{\mathfrak{g}}$, the adjoint representation; the other is $V = V_{\theta^\vee}$, the representation whose nonzero weights consist of short roots of $\check{\mathfrak{g}}$. We call V_{θ^\vee} the *quasi-minuscule* representation of \check{G} (which coincides with the adjoint representation when \check{G} is simply-laced). Basic facts about V_{θ^\vee} are summarized in [Appendix C](#). We will denote the number of long (resp. short) simple roots of G by $r_\ell(G)$ (resp. $r_s(G)$) and call it the *long rank* (resp. the *short rank* of G). See [Lemma C.1](#) for equivalent descriptions of these numbers.

Let $\text{Kl}_G^{\theta^\vee}(\phi, \chi)$ (resp. $\text{Kl}_G^{\text{Ad}}(\phi, \chi)$) be the local system associated to $\text{Kl}_{\check{G}}(\phi, \chi)$ and the quasi-minuscule representation (resp. the adjoint representation) of \check{G} .

THEOREM 4.

- (1) $-\chi_c(\mathbb{P}^1_{\setminus\{0, \infty\}}, \text{Kl}_G^{\theta^\vee}(\phi, \chi))$ equals the number of long simple roots of G .
- (2) Suppose $\text{char}(k)$ is good for G when G is not simply-laced. Then

$$-\chi_c(\mathbb{P}^1_{\setminus\{0, \infty\}}, \text{Kl}_G^{\text{Ad}}(\phi, \chi))$$

equals the rank of G .

By the Grothendieck-Ogg-Shafarevich formula, we get

COROLLARY 5.1.

- (1) $\text{Swan}_\infty(\text{Kl}_G^{\theta^\vee}(\phi, \chi)) = r_s(\check{G})$.
- (2) $\text{Swan}_\infty(\text{Kl}_G^{\text{Ad}}(\phi, \chi)) = r(\check{G})$, under the same assumption on $\text{char}(k)$ as in [Theorem 4](#)(2).

The following subsections will be devoted to the proof of [Theorem 4](#). We first draw some consequences.

LEMMA 5.2. *Suppose the principal nilpotent element acts on V without trivial Jordan block, then $H^0(\mathbb{P}^1_{\setminus\{0, \infty\}}, \text{Kl}_G^V(\phi)) = 0$.*

Proof. Suppose the contrary, then $\text{Kl}_G^V(\phi)$ contains the constant sheaf as a sub-local-system. By construction, $\text{Kl}_G^V(\phi)$ is pure. By [\[3, Th.5.3.8\]](#), over $\mathbb{P}^1_{\setminus\{0, \infty\}} \otimes_k \bar{k}$, $\text{Kl}_G^V(\phi)$ is a direct sum of simple perverse sheaves. Hence the constant sub-sheaf must be a direct summand.

On the other hand, by assumption, V does not contain any direct summand under which the tame monodromy at $\{0\} \in \mathbb{P}^1$ (i.e., a principal unipotent element, by [Theorem 1](#)) acts trivially. This yields a contradiction. \square

Let $i_0: \{0\} \hookrightarrow \mathbb{P}^1; i_\infty: \{\infty\} \hookrightarrow \mathbb{P}^1$ and $j: \mathbb{P}^1_{\setminus\{0,\infty\}} \hookrightarrow \mathbb{P}^1$ be the inclusions. For any local system L on $\mathbb{P}^1_{\setminus\{0,\infty\}}$, we abuse the notation $j_{!*}L$ to mean $j_{!*}(L[1])[-1]$.

As in the discussion before [Theorem 2](#), the Galois groups $\text{Gal}(K_\infty^{\text{sep}}/K_\infty)$ and $\text{Gal}(K_0^{\text{sep}}/K_0)$, hence the inertia groups J_0 and J_∞ , act on the corresponding geometric stalks of $\text{Kl}_G^V(\phi)$, defining representations on V up to conjugacy.

PROPOSITION 5.3. *Under the same assumption on $\text{char}(k)$ as in [Theorem 4\(2\)](#), we have*

(1) (Cohomological rigidity) $H^*(\mathbb{P}_k^1, j_{!*}\text{Kl}_G^{\text{Ad}}(\phi)) = 0$.

(2) $\check{g}^{J_\infty} = 0$. (Note that \check{g} is the space of $V = \text{Ad}$.)

The same statements hold for $\text{Kl}_G^{\theta^\vee}(\phi)$ in place of $\text{Kl}_G^{\text{Ad}}(\phi)$, with no restriction on $\text{char}(k)$.

Proof. The statements being geometric, we will ignore Tate twists in this proof. Since the adjoint representation \check{g} is self-dual, the local system $\text{Kl}_G^{\text{Ad}}(\phi)$ is also self-dual. Fixing such an isomorphism $\text{Kl}_G^{\text{Ad}}(\phi) \xrightarrow{\sim} (\text{Kl}_G^{\text{Ad}}(\phi))^\vee$, we get isomorphisms

$$H^0(\mathbb{P}^1, j_{!*}\text{Kl}_G^{\text{Ad}}(\phi)) \cong H^2(\mathbb{P}^1, j_{!*}\text{Kl}_G^{\text{Ad}}(\phi))^\vee$$

and

$$H^0(\mathbb{P}^1_{\setminus\{0,\infty\}}, \text{Kl}_G^{\text{Ad}}(\phi)) \cong H_c^2(\mathbb{P}^1_{\setminus\{0,\infty\}}, \text{Kl}_G^{\text{Ad}}(\phi))^\vee.$$

We have a distinguished triangle in $D_c^b(\mathbb{P}^1, \overline{\mathbb{Q}}_\ell)$,

$$j_!\text{Kl}_G^{\text{Ad}}(\phi) \rightarrow j_{!*}\text{Kl}_G^{\text{Ad}}(\phi) \rightarrow H^0 i_0^* j_* \text{Kl}_G^{\text{Ad}}(\phi) \oplus H^0 i_\infty^* j_* \text{Kl}_G^{\text{Ad}}(\phi) \rightarrow,$$

which induces a long exact sequence

$$(5.4) \quad 0 = H_c^0(\mathbb{P}^1_{\setminus\{0,\infty\}}, \text{Kl}_G^{\text{Ad}}(\phi)) \rightarrow H^0(\mathbb{P}^1, j_{!*}\text{Kl}_G^{\text{Ad}}(\phi)) \rightarrow \check{g}^{J_0} \oplus \check{g}^{J_\infty} \xrightarrow{d},$$

$$(5.5) \quad \xrightarrow{d} H_c^1(\mathbb{P}^1_{\setminus\{0,\infty\}}, \text{Kl}_G^{\text{Ad}}(\phi)) \rightarrow H_c^1(\mathbb{P}^1, j_{!*}\text{Kl}_G^{\text{Ad}}(\phi)) \rightarrow 0 \rightarrow,$$

$$(5.6) \quad \rightarrow H_c^2(\mathbb{P}^1_{\setminus\{0,\infty\}}, \text{Kl}_G^{\text{Ad}}(\phi)) \rightarrow H_c^2(\mathbb{P}^1, j_{!*}\text{Kl}_G^{\text{Ad}}(\phi)) \rightarrow 0$$

By (5.6) we first conclude

$$H^2(\mathbb{P}^1, j_{!*}\text{Kl}_G^{\text{Ad}}(\phi)) \cong H_c^2(\mathbb{P}^1_{\setminus\{0,\infty\}}, \text{Kl}_G^{\text{Ad}}(\phi)) \cong H^0(\mathbb{P}^1_{\setminus\{0,\infty\}}, \text{Kl}_G^{\text{Ad}}(\phi))^\vee = 0,$$

where the vanishing of the last term follows by applying [Lemma 5.2](#) to $V = \check{g}$. By duality, $H^0(\mathbb{P}^1, j_{!*}\text{Kl}_G^{\text{Ad}}(\phi)) = 0$. By (5.4) and the vanishing of H^0 's, the connecting homomorphism d is injective.

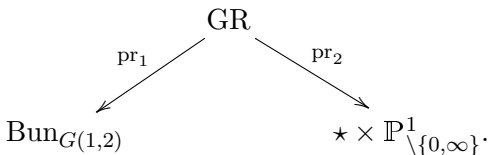
On the one hand, by [Theorem 1\(2\)](#), $\dim \check{\mathfrak{g}}^{J_0} = r(\check{G})$ because J_0 acts on $\check{\mathfrak{g}}$ through a principal unipotent element. On the other hand, by [Theorem 4](#), $\dim H_c^1(\mathbb{P}^1_{\setminus\{0,\infty\}}, \mathrm{Kl}_G^{\mathrm{Ad}}(\phi)) = -\chi_c(\mathbb{P}^1_{\setminus\{0,\infty\}}, \mathrm{Kl}_G^{\mathrm{Ad}}(\phi)) = r(\check{G})$. (Here we again used the vanishing of $H_c^i(\mathbb{P}^1_{\setminus\{0,\infty\}}, \mathrm{Kl}_G^{\mathrm{Ad}}(\phi))$ for $i = 0, 2$.) Therefore d must be an isomorphism. This implies $\check{\mathfrak{g}}^{J_\infty} = 0$. By [\(5.5\)](#), we conclude that $H^1(\mathbb{P}^1, j_* \mathrm{Kl}_G^{\mathrm{Ad}}(\phi)) = 0$.

The statement for $\mathrm{Kl}_G^{\theta_V}(\phi)$ is proved in the same way. We may apply [Lemma 5.2](#) to V_{θ_V} because of [equation \(C.2\)](#) from the proof of [Lemma C.1](#). \square

Remark 5.7. Following Katz [[28](#), §5.0], we call a \check{G} -local system L on $\mathbb{P}^1_{\setminus\{0,\infty\}} \otimes_k \bar{k}$ *cohomologically rigid* if it satisfies $H^1(\mathbb{P}^1, j_* L^{\mathrm{Ad}}) = 0$. We think of $H^1(\mathbb{P}^1, j_* L^{\mathrm{Ad}})$ as the space of infinitesimal deformations of the \check{G} -local system L with fixed isomorphism type on the formal punctured discs around 0 and ∞ , although the notion of such deformations has not been defined. [Proposition 5.3](#) implies that $\mathrm{Kl}_G(\phi)$ is cohomologically rigid, which provides evidence for its *physical rigidity*. For more precise conjectures, see [Section 7](#).

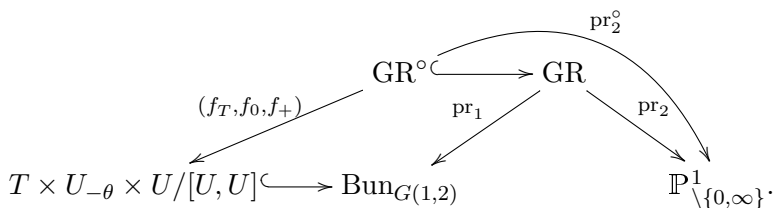
5.2. General method of calculation. In the following calculation, we will only consider the neutral component of $\mathrm{Bun}_{G(1,2)}$. Hence we may assume that G is simply-connected.

We denote by $\star \in \mathrm{Bun}_{G(1,2)}$ the point corresponding to the trivial bundle. In order to compute the sheaf $\mathrm{Kl}_G^V(\phi, \chi)$, we restrict the convolution diagram [\(2.7\)](#) to $\star \times \mathbb{P}^1_{\setminus\{0,\infty\}} \subset \mathrm{Bun}_{G(1,2)} \times \mathbb{P}^1_{\setminus\{0,\infty\}}$:



By proper base change, we have $\mathrm{Kl}_G^V(\phi, \chi) = \mathrm{pr}_{2,!}(\mathrm{pr}_1^* A \otimes \mathrm{IC}_V)$. Furthermore, A is supported only on the big cell $j: T \times I(1)/I(2) \subset \mathrm{Bun}_{G(1,2)}$. Let us denote by $\mathrm{Gr}^\circ \subset \mathrm{Gr}$ the inverse image of the big cell and write $I(1)/I(2) \cong U/[U, U] \times U_{-\theta}$. We write $\phi = (\phi_+, \phi_0)$ according to this decomposition.

We obtain a diagram



By proper base change,

$$(5.8) \quad \text{Kl}_G^V(\phi, \chi) = \text{pr}_{2,!}^\circ(f_T^* \text{Kum}_\chi f_0^* \text{AS}_{\phi_0} \otimes f_+^* \text{AS}_{\phi_+} \otimes \text{IC}_V).$$

So we need to describe the subspace GR° and compute the maps f_T, f_0, f_+ .

First, let us denote by $\text{GR}^{\text{triv}} \subset \text{GR}$ the preimage of the trivial G -bundle under the forgetful map $\text{GR} \rightarrow \text{Bun}_G$. Let us fix a point $x \in \mathbb{P}^1_{\setminus\{0,\infty\}}$ and chose $t_x := 1 - \frac{t}{x}$ as a local parameter at x . So $t_x = 1$ corresponds to $t = 0$ and $t_x = \infty$ corresponds to $t = \infty$.

The ind-scheme $\text{Gr}_x^{\text{triv}}$ classifies isomorphism classes of pairs (\mathcal{E}, φ) such that \mathcal{E} is a trivial G -bundle on \mathbb{P}^1 and $\varphi: \mathcal{E}|_{\mathbb{P}^1 \setminus \{x\}} \xrightarrow{\sim} G \times (\mathbb{P}^1 \setminus \{x\})$ is an isomorphism. We can rigidify this moduli problem so that $\text{Gr}_x^{\text{triv}}$ classifies an automorphism of the trivial G -bundle on $\mathbb{P}^1 \setminus \{x\}$ that is identity at ∞ . Hence we can identify

$$(5.9) \quad \text{Gr}_x^{\text{triv}} \xrightarrow{\sim} G[t_x^{-1}]_1 := \ker(G[t_x^{-1}] \xrightarrow{\text{ev}(t_x=\infty)} G),$$

where $G[t_x^{-1}]$ is the ind-scheme whose R -points are $G(R[t_x^{-1}])$.

Remark 5.10. There is a dilation action of $\mathbb{G}_m^{\text{rot}}$ on $\mathbb{P}^1_{\setminus\{0,\infty\}}$, which extends to GR : $\lambda \in \mathbb{G}_m^{\text{rot}}$ sends $(x, \mathcal{E}, \varphi) \mapsto (\lambda x, \lambda^{-1,*} \mathcal{E}, \lambda^{-1,*} \varphi)$ and stabilizes GR^{triv} . Note that the local coordinate t_x is invariant under the simultaneous dilation on t and x .

Let $\text{Gr}^{\text{triv}} \subset \text{Gr} = G((\tau))/G[[\tau]]$ be an abstract copy of the affine Grassmannian, not referring to any point on $\mathbb{P}^1_{\setminus\{0,\infty\}}$. The dilation action, together with (5.9), gives a trivialization of the family GR^{triv} over $\mathbb{P}^1_{\setminus\{0,\infty\}}$:

$$(5.11) \quad \begin{aligned} \text{GR}^{\text{triv}} &\cong (\mathbb{P}^1_{\setminus\{0,\infty\}}) \times G[\tau^{-1}]_1 \cong (\mathbb{P}^1_{\setminus\{0,\infty\}}) \times \text{Gr}^{\text{triv}}, \\ (x, g(t_x^{-1})) &\mapsto (x, g(\tau^{-1})). \end{aligned}$$

By definition, the big cell in $\text{Bun}_{G(1,2)}$ is obtained from the action of $T \times I(1)/I(0)$ on the trivial bundle. Let $\widehat{\text{ev}}_0: G[t_x^{-1}] \rightarrow G[[t]]$ and $\widehat{\text{ev}}_\infty: G[t_x^{-1}] \rightarrow G[[t^{-1}]]$ denote the expansions around 0 and ∞ . Let $\text{ev}_0, \text{ev}_\infty: G[t_x^{-1}] \rightarrow G$ denote the evaluation at $t = 0$ and $t = \infty$.

The $G(1, 2)$ -level structure on the G -bundle $(\mathcal{E}, \varphi) \in \text{Gr}_x$ is obtained from the trivialization φ . For $g \in G[t_x^{-1}]$, the composition $\mathcal{E}|_{\mathbb{P}^1 \setminus \{x\}} \xrightarrow{\varphi} G \times \mathbb{P}^1 \setminus \{x\} \xrightarrow{g} G \times \mathbb{P}^1 \setminus \{x\}$ extends to an isomorphism $\mathcal{E} \rightarrow G \times \mathbb{P}^1$. Thus g defines the level structure on the trivial bundle defined by $\text{ev}_0(g), \text{ev}_\infty(g)$. We have a commutative diagram

$$\begin{array}{ccc} G[t_x^{-1}] & \xrightarrow{\widehat{\text{ev}}_0, \widehat{\text{ev}}_\infty} & G \backslash (G[[t]] \times G[[t^{-1}]]) \\ & \searrow & \swarrow \\ & & \text{Bun}_{G(1,2)}. \end{array}$$

Here $G = \text{Aut}(G \times \mathbb{P}^1)$ acts diagonally on $G[[t]] \times G[[t^{-1}]]$. Thus we find that $g \in G[t_x^{-1}]$ lies in Gr_x° if and only if $(\text{ev}_0(g), \text{ev}_\infty(g)) \in G \backslash G(B^{\text{opp}} \times U)$. Using the identifications (5.9) and (5.11), we get

$$(5.12) \quad \begin{aligned} \text{Gr}_x^\circ &\xrightarrow{\sim} \{g(t_x^{-1}) \in G[t_x^{-1}]_1 | \text{ev}_0(g) \in UB^{\text{opp}}\}, \\ \text{GR}^\circ &\xrightarrow{\sim} (\mathbb{P}_{\setminus\{0, \infty\}}^1) \times \{g(\tau^{-1}) \in G[\tau^{-1}]_1 | g(1) \in UB^{\text{opp}}\} \\ &\subset \text{GR}^{\text{triv}} = (\mathbb{P}_{\setminus\{0, \infty\}}^1) \times G[\tau^{-1}]_1. \end{aligned}$$

LEMMA 5.13. *For $(x, g(\tau^{-1})) \in \text{GR}^\circ$ under the parametrization in (5.12), write $g(1) = ub^{\text{opp}}$ with $u \in U$ and $b^{\text{opp}} \in B^{\text{opp}}$. Then we have*

$$\begin{aligned} f_T(x, g) &= b^{\text{opp}} \pmod{U^{\text{opp}}} \in T, \\ f_+(x, g) &= u^{-1} \pmod{[U, U]} \in U/[U, U], \\ f_0(x, g) &= xa_{-\theta}(g) \in U_{-\theta} \cong \mathfrak{g}_{-\theta}, \end{aligned}$$

where $a_{-\theta} : G[\tau^{-1}] \rightarrow \mathfrak{g}_{-\theta}$ sends g to the $\mathfrak{g}_{-\theta}$ -part of the tangent vector $\left. \frac{dg(\tau^{-1})}{d(\tau^{-1})} \right|_{\tau^{-1}=0} \in \mathfrak{g}$.

Proof. The formulas for f_T, f_+ follow from our description in (5.12). By definition $f_0(x, g)$ is obtained by expanding $g(t_x^{-1})$ at $t = \infty$ using the local parameter t^{-1} and taking the $\mathfrak{g}_{-\theta}$ -part of the coefficient of t^{-1} . Note that

$$\left. \frac{dg(t_x^{-1})}{d(t^{-1})} \right|_{t^{-1}=0} = x \left. \frac{dg(t_x^{-1})}{d(t_x^{-1})} \right|_{t_x^{-1}=0} \in \mathfrak{g}.$$

Moreover, under the identification (5.11), the parameter t_x corresponds to τ , therefore $f_0(x, g) = xa_{-\theta}(g)$. This proves the lemma. \square

Let $\text{AS}_{U, -\phi_+}$ be the pull-back of AS_ψ via $U \rightarrow U/[U, U] \xrightarrow{-\phi_+} \mathbb{G}_a$. Let $\text{Kum}_{B^{\text{opp}}, \chi}$ be the pull-back of the Kummer local system Kum_χ via $B^{\text{opp}} \rightarrow T$. Let $j_{UB^{\text{opp}}} : UB^{\text{opp}} \hookrightarrow G$ be the inclusion, and denote

$$J = J_{-\phi_+, \chi} := j_{UB^{\text{opp}}, !}(\text{AS}_{U, -\phi_+} \boxtimes \text{Kum}_{B^{\text{opp}}, \chi}) \in D_c^b(G, \overline{\mathbb{Q}}_\ell).$$

Remark 5.14. According to [5], or the argument of Lemma 2.3 (in this paper), we have the cleanness property of J :

$$j_{UB^{\text{opp}}, !}(\text{AS}_{U, -\phi_+} \boxtimes \text{Kum}_{B^{\text{opp}}, \chi}) \xrightarrow{\sim} j_{UB^{\text{opp}}, *}(\text{AS}_{U, -\phi_+} \boxtimes \text{Kum}_{B^{\text{opp}}, \chi}).$$

We can view $J = J_{-\phi_+, \chi} \in D_c^b(G, \overline{\mathbb{Q}}_\ell)$ as a finite-field analog of the automorphic sheaf $A_{\phi, \chi}$ in Definition 2.6 and Remark 2.5.

With this notation, and using the identification (5.11), formula (5.8) becomes

$$(5.15) \quad \text{Kl}_G^V(\phi, \chi) \xrightarrow{\sim} \pi_1^{\text{triv}}(\text{IC}_V^{\text{triv}} \otimes f_0^* \text{AS}_{\phi_0} \otimes \text{ev}_{\tau=1}^* J),$$

where $\pi^{\text{triv}} : \text{Gr}^{\text{triv}} = \mathbb{P}^1_{\setminus\{0,\infty\}} \times G[\tau^{-1}]_1 \rightarrow \mathbb{P}^1_{\setminus\{0,\infty\}}$ is the projection. In the sequel, we often write $\text{ev}_{\tau=1}$ simply as ev .

The ultimate goal of this section is to calculate the Euler characteristics of $\text{Kl}_G^V(\phi, \chi)$ for $V = V_{\theta^\vee}$ and \mathfrak{g} . Here we make a few reduction steps for general V . By (5.15), we need to calculate

$$(5.16) \quad \chi_c(\mathbb{P}^1_{\setminus\{0,\infty\}}, \text{Kl}_G^V(\phi, \chi)) = \chi_c(\mathbb{P}^1_{\setminus\{0,\infty\}} \times \text{Gr}^{\text{triv}}, \text{IC}_V^{\text{triv}} \otimes f_0^* \text{AS}_{\phi_0} \otimes \text{ev}^* J).$$

According to whether $a_{-\theta}$ vanishes or not, we decompose $\text{Gr}^{\text{triv}} \cong G[\tau^{-1}]_1$ into $\text{Gr}^{\text{triv}, a \neq 0}$ and $\text{Gr}^{\text{triv}, a=0}$. Over $\mathbb{P}^1_{\setminus\{0,\infty\}} \times \text{Gr}^{\text{triv}, a=0}$, the complex in (5.16) is constant along $\mathbb{P}^1_{\setminus\{0,\infty\}}$; hence, the Euler characteristic is 0. On the other hand, we have a change of variable isomorphism

$$\mathbb{P}^1_{\setminus\{0,\infty\}} \times \text{Gr}^{\text{triv}, a \neq 0} \ni (x, g) \mapsto (xa_{-\theta}(g), g) \in \mathbb{G}_m \times \text{Gr}^{\text{triv}, a \neq 0}.$$

Under this isomorphism, $f_0 = xa_{-\theta}$ becomes the projection to the \mathbb{G}_m -factor, and we can apply the Künneth formula. Summarizing these steps, we get

$$(5.17) \quad \begin{aligned} \chi_c(\mathbb{P}^1_{\setminus\{0,\infty\}}, \text{Kl}_G^V(\phi, \chi)) &= \chi_c(\mathbb{P}^1_{\setminus\{0,\infty\}} \times \text{Gr}^{\text{triv}, a \neq 0}, \text{IC}_V^{\text{triv}} \otimes f_0^* \text{AS}_{\phi_0} \otimes \text{ev}^* J) \\ &= \chi_c(\mathbb{G}_m, \text{AS}_{\phi_0}) \chi_c(\text{Gr}^{\text{triv}, a \neq 0}, \text{IC}_V^{\text{triv}} \otimes \text{ev}^* J) \\ &= -\chi_c(\text{Gr}^{\text{triv}, a \neq 0}, \text{IC}_V^{\text{triv}} \otimes \text{ev}^* J) \\ &= -\chi_c(\text{Gr}^{\text{triv}} \setminus \star, \text{IC}_V^{\text{triv}} \otimes \text{ev}^* J) + \chi_c(\text{Gr}^{\text{triv}, a=0} \setminus \star, \text{IC}_V^{\text{triv}} \otimes \text{ev}^* J). \end{aligned}$$

The last equality is because the base point $\star \in \text{Gr}$ belongs to $\text{Gr}^{\text{triv}, a=0}$.

LEMMA 5.18. $\chi_c(\text{Gr}^{\text{triv}} \setminus \star, \text{IC}_V^{\text{triv}} \otimes \text{ev}^* J) = 0$.

Proof. Since IC_V is constant along the strata Gr_λ , it suffices to show that $\chi_c(\text{Gr}_\lambda^{\text{triv}}, \text{ev}^* J) = 0$ for dominant coweights $\lambda \neq 0$. Denote by $\overset{G}{*}$ the convolution product on $D_c^b(G, \overline{\mathbb{Q}}_\ell)$. For $K_1, K_2 \in D_c^b(G, \overline{\mathbb{Q}}_\ell)$,

$$K_1 \overset{G}{*} K_2 := m_!(K_1 \boxtimes K_2),$$

where $m : G \times G \rightarrow G$ is the multiplication map. Let $K_\lambda := \text{ev}_! \overline{\mathbb{Q}}_{\ell, \text{Gr}_\lambda^{\text{triv}}} \in D_c^b(G)$. Then $R\Gamma_c(\text{Gr}_\lambda^{\text{triv}}, \text{ev}^* J)$ is the stalk at $e \in G$ of the convolution $\text{AS}_{U, \phi_+} \overset{G}{*} K_\lambda \overset{G}{*} \text{Kum}_{B^{\text{opp}}, \chi^{-1}}$. Since $\text{ev} = \text{ev}_{\tau=1}$ is G -equivariant (under conjugation), K_λ carries a natural G -equivariant structure. Hence,

$$K_\lambda \overset{G}{*} \text{Kum}_{B^{\text{opp}}, \chi^{-1}} \cong \text{Kum}_{B^{\text{opp}}, \chi^{-1}} \overset{G}{*} K_\lambda.$$

In particular, over each Bruhat stratum $B^{\text{opp}} w B^{\text{opp}}$, $K_\lambda \overset{G}{*} \text{Kum}_{B^{\text{opp}}, \chi^{-1}}$ has the form $S_w \otimes L_w$, where L_w is a local system on $U^{\text{opp}} w B^{\text{opp}}$ and S_w (a complex of $\overline{\mathbb{Q}}_\ell$ -vector spaces) is the stalk of $K_\lambda \overset{G}{*} \text{Kum}_{B^{\text{opp}}, \chi^{-1}}$ at $\dot{w} \in N_G(T)$ (a representative of $w \in W$).

Therefore, $\text{AS}_{U, \phi_+} \overset{G}{*} K_\lambda \overset{G}{*} \text{Kum}_{B^{\text{opp}}, \chi^{-1}}$ is a successive extension of $S_w \otimes (\text{AS}_{U, \phi_+} \overset{G}{*} L_w)$ for various $w \in W$. To prove the vanishing of the Euler characteristic of the stalks of $\text{AS}_{U, \phi_+} \overset{G}{*} K_\lambda \overset{G}{*} \text{Kum}_{B^{\text{opp}}, \chi^{-1}}$, it suffices to show that $\chi(S_w) = 0$ for all w .

By definition,

$$\begin{aligned}
 (5.19) \quad S_w &= (K_\lambda \overset{G}{*} \text{Kum}_{B^{\text{opp}}, \chi^{-1}})_w \\
 &= R\Gamma_c(wB^{\text{opp}}, K_\lambda \otimes \text{Kum}_{B^{\text{opp}}, \chi}) \\
 &= R\Gamma_c(\text{Gr}_\lambda^{\text{triv}} \cap \text{ev}^{-1}(wB^{\text{opp}}), \text{ev}^* \text{Kum}_{B^{\text{opp}}, \chi}).
 \end{aligned}$$

The T -action on $\text{Gr}^{\text{triv}} = G[\tau^{-1}]_1$ by conjugation preserves $\text{Gr}_\lambda^{\text{triv}} \cap \text{ev}^{-1}(wB^{\text{opp}})$. The only T -fixed points on Gr are τ^μ for $\mu \in \mathbb{X}_*(T)$, which do not belong to any $\text{Gr}_\lambda^{\text{triv}}$ ($\lambda \neq 0$). Moreover, the local system $\text{ev}^* \text{Kum}_{B^{\text{opp}}, \chi}$ is monodromic under T -conjugation: there exists $m \geq 1$ (prime to p) such that $\text{ev}^* \text{Kum}_{B^{\text{opp}}, \chi}$ is equivariant under the m -th power of the T -conjugation. Since the m -th power of T -conjugation still has no fixed point on $\text{Gr}_\lambda^{\text{triv}} \cap \text{ev}^{-1}(wB^{\text{opp}})$, the Euler characteristic in (5.19) is zero. This proves the lemma. \square

COROLLARY 5.20. *Let $\mathbb{X}_*(T)^+ \subset \mathbb{X}_*(T)$ be the dominant coweights and $V(\lambda) \subset V$ be the weight space for $\lambda \in \mathbb{X}_*(T)$. Then*

$$\chi_c(\mathbb{P}^1_{\setminus \{0, \infty\}}, \text{Kl}_G^V(\phi, \chi)) = \sum_{\lambda \in \mathbb{X}_*(T)^+, \lambda \neq 0} \dim V(\lambda) \chi_c(\text{Gr}_\lambda^{\text{triv}, a=0}, \text{ev}^* J).$$

Proof. By [30, Th. 6.1], the Euler characteristic of the stalks of IC_V along Gr_λ is $\dim V(\lambda)$. Therefore,

$$\chi_c(\text{Gr}^{\text{triv}, a=0} \setminus \star, \text{IC}_V^{\text{triv}} \otimes \text{ev}^* J) = \sum_{\lambda \in \mathbb{X}_*(T)^+, \lambda \neq 0} \dim V(\lambda) \chi_c(\text{Gr}_\lambda^{\text{triv}, a=0}, \text{ev}^* J).$$

Combining this with (5.17) and Lemma 5.18 yields the desired identity. \square

5.3. Quasi-minuscule Schubert variety. For a coweight $\lambda \in \mathbb{X}_*(T)$, let $P_\lambda \subset G$ be the parabolic generated by T and the root spaces U_α for $\langle \alpha, \lambda \rangle \geq 0$. For each root α and $i \in \mathbb{Z}$, let $U_{\alpha, \geq i} \subset U_\alpha((\tau))$ be the subgroup whose R -points are $U_\alpha(\tau^i R[[\tau]])$. Let $U_{\alpha, i} = U_{\alpha, \geq i} / U_{\alpha, \geq i+1}$.

Now let $\lambda \in \mathbb{X}_*(T)^+$, and consider the (open) Schubert variety $\text{Gr}_\lambda \subset \text{Gr}$. By [32, Lemme 2.3 and discussions preceding it], we have (fixing a total ordering of Φ^+)

$$\begin{aligned}
 (5.21) \quad G \overset{P_{-\lambda}}{\times} \left(\prod_{\langle \alpha, \lambda \rangle \geq 2} U_{\alpha, \geq 1} / U_{\alpha, \geq \langle \alpha, \lambda \rangle} \right) &\xrightarrow{\sim} \text{Gr}_\lambda \\
 (g, u) &\mapsto gu\tau^\lambda.
 \end{aligned}$$

The action of $P_{-\lambda}$ on the product in (5.21) is given by the adjoint action of $P_{-\lambda}$ on $\prod_{\alpha} U_{\alpha, \geq 1}$ followed by the projection onto factors $U_{\alpha, \geq 1}$ for $\langle \alpha, \lambda \rangle \geq 2$.

Now consider the special case $\lambda = \theta^\vee$, the dominant short coroot. We need to recall the description of the quasi-minuscule Schubert variety Gr_{θ^\vee} from [32]. We write $P_{-\theta}$ for $P_{-\theta^\vee}$.

LEMMA 5.22. *The open locus $\text{Gr}_{\theta^\vee}^{\text{triv}} \subset \text{Gr}_{\theta^\vee}$ consists of a single G -orbit. More precisely, the morphism*

$$G \times^{P_{-\theta}} U_{-\theta, -1}^\times \rightarrow G[\tau^{-1}]_1 = \text{Gr}^{\text{triv}},$$

$$(g, u_{-\theta}(c\tau^{-1})) \mapsto \text{Ad}(g)u_{-\theta}(c\tau^{-1})$$

gives an isomorphism onto $\text{Gr}_{\theta^\vee}^{\text{triv}}$. (Here $P_{-\theta}$ acts on $U_{-\theta, -1}$ through adjoint action and $U_{\alpha, i}^\times = U_{\alpha, i} \setminus \{1\}$.)

Proof. This is essentially [32, Lemme 7.2]. Applying (5.21) to $\lambda = \theta^\vee$, we find that the product in (5.21) consists of only one term $U_{\theta, 1}$ (since $\langle \alpha, \theta^\vee \rangle \leq 1$ for any root $\alpha \neq \theta$). Therefore, $\text{Gr}_{\theta^\vee}^{\text{triv}} \cong G \times^{P_{-\theta}} U_{\theta, 1}^{\text{triv}} \cdot \tau^{\theta^\vee}$ for some proper open subset $U_{\theta, 1}^{\text{triv}} \subset U_{\theta, 1}$ stable under $P_{-\theta}$. ($\text{Gr}_{\theta^\vee}^{\text{triv}}$ cannot be equal to Gr_{θ^\vee} because $\tau^{\theta^\vee} \notin \text{Gr}^{\text{triv}}$.) Since $P_{-\theta}$ acts on $U_{\theta, 1}$ via dilation, $U_{\theta, 1}^{\text{triv}}$ must be $U_{\theta, 1}^\times$. Hence,

$$\text{Gr}_{\theta^\vee}^{\text{triv}} \cong G \times^{P_{-\theta}} U_{\theta, 1}^\times \cdot \tau^{\theta^\vee}.$$

On the other hand, the following calculation in the SL_2 -subgroup defined by θ ,

$$(5.23) \quad \begin{pmatrix} 1 & c\tau \\ & 1 \end{pmatrix} \begin{pmatrix} \tau & \\ & \tau^{-1} \end{pmatrix} = \begin{pmatrix} \tau & c \\ & \tau^{-1} \end{pmatrix} = \begin{pmatrix} 1 & \\ c^{-1}\tau^{-1} & 1 \end{pmatrix} \begin{pmatrix} \tau & c \\ -c^{-1} & \end{pmatrix},$$

shows that $u_\theta(c\tau)\tau^{\theta^\vee} \in \text{Gr}$ equals $u_{-\theta}(c^{-1}\tau^{-1}) \in U_{-\theta, -1}^\times \subset G[\tau^{-1}]_1$ for c invertible. This proves the lemma. \square

The Bruhat decomposition for $G = \sqcup_{w \in W/W_\theta} UwP_{-\theta}$ gives a decomposition

$$G \times^{P_{-\theta}} U_{-\theta, -1}^\times = \bigsqcup_{w \in W/W_\theta} UwP_{-\theta} \times^{P_{-\theta}} U_{-\theta, -1}^\times.$$

The stratum $UwP_{-\theta} \times^{P_{-\theta}} U_{-\theta, -1}^\times$ has image $\text{Ad}(U)U_{-w\theta, -1}^\times$ in $G[\tau^{-1}]_1$. Since $w \mapsto -w\theta$ sets up a bijection between W/W_θ and the set of long roots of G , we can rewrite the above decomposition as

$$\text{Gr}_{\theta^\vee}^{\text{triv}} = \bigsqcup_{\beta \text{ long root}} \text{Ad}(U)U_{\beta, -1}^\times.$$

For each $w \in W/W_\theta$, we have an isomorphism (fixing total ordering on Φ^+)

$$\prod_{\alpha \in \Phi^+, \langle w^{-1}\alpha, \theta^\vee \rangle > 0} U_\alpha \cong U w P_{-\theta} / P_{-\theta}.$$

Therefore, for $\beta = -w\theta$, the stratum $\text{Ad}(U)U_{\beta,-1}^\times$ can be written as

$$(5.24) \quad \text{Ad}(U)U_{\beta,-1}^\times \cong \prod_{\alpha \in \Phi^+, \langle \alpha, \beta^\vee \rangle < 0} U_\alpha \times U_{\beta,-1}^\times,$$

$$\text{Ad}\left(\prod u_\alpha\right)u_\beta(c\tau^{-1}) \leftrightarrow \left(\prod u_\alpha, u_\beta(c\tau^{-1})\right).$$

LEMMA 5.25. *The function $a_{-\theta} : \text{Gr}_{\theta^\vee}^{\text{triv}} \rightarrow \mathfrak{g}_{-\theta}$ in Lemma 5.13 restricted on each stratum $\text{Ad}(U)U_{\beta,-1}^\times \subset \text{Gr}_{\theta^\vee}^{\text{triv}}$ is given by*

$$a_{-\theta}(\text{Ad}(u)u_\beta(c\tau^{-1})) = \begin{cases} c\mathfrak{x}_{-\theta} & \text{if } \beta = -\theta, \\ 0 & \text{otherwise.} \end{cases}$$

Here $\mathfrak{x}_{-\theta} \in \mathfrak{g}_{-\theta}$ corresponds to $u_{-\theta}(1) \in U_{-\theta}$. In particular,

$$(5.26) \quad \text{Gr}_{\theta^\vee}^{\text{triv}, a=0} = \bigsqcup_{\beta \text{ long}, \beta \neq -\theta} \text{Ad}(U)U_{\beta,-1}^\times.$$

Proof. By definition, the value of $a_{-\theta}$ on $\text{Ad}(u)u_\beta(c\tau^{-1})$ is

$$\frac{d}{d(\tau^{-1})} \text{Ad}(u)u_\beta(c\tau^{-1})|_{\tau^{-1}=0} \in \text{Ad}(U)\mathfrak{g}_\beta.$$

If $\beta \neq -\theta$, $\text{Ad}(u)\mathfrak{g}_\beta$ only involves roots $\geq \beta$. If $\beta = -\theta$, then the above derivative equals $\text{Ad}(u)c\mathfrak{x}_{-\theta} \in \mathfrak{g}$, whose $\mathfrak{g}_{-\theta}$ -part is $c\mathfrak{x}_{-\theta} \in \mathfrak{g}_{-\theta}$. \square

5.4. *Proof of Theorem 4(1).* Applying Corollary 5.20 to $V = V_{\theta^\vee}$, which only has one nonzero dominant weight θ^\vee , we get

$$\chi_c(\mathbb{P}_{\{0, \infty\}}^1, \text{Kl}_G^{\theta^\vee}(\phi, \chi)) = \chi_c(\text{Gr}_{\theta^\vee}^{\text{triv}, a=0}, \text{ev}^* J).$$

By the decomposition (5.26) in Lemma 5.25, we only need to calculate

$$\chi_c(\text{Ad}(U)U_{\beta,-1}^\times, \text{ev}^* J)$$

for long roots $\beta \neq -\theta$. Theorem 4(1) thus follows from

CLAIM. *Suppose $\beta \neq -\theta$ is a long root. Then*

$$\chi_c(\text{Ad}(U)U_{\beta,-1}^\times, \text{ev}^* J) = \begin{cases} -1 & \beta \text{ is a simple long root,} \\ 0 & \text{otherwise.} \end{cases}$$

To prove the claim, we distinguish three cases.

Case I: β is positive but not simple. Since $\beta > 0$, we have $\text{ev}(\text{Ad}(U)U_{\beta,-1}^\times) \subset U$. Since β is not simple, the image of $\text{ev}(\text{Ad}(U)U_{\beta,-1}^\times)$ in $U/[U, U]$ is trivial; hence, ev^*J is the constant sheaf on $\text{Ad}(U)U_{\beta,-1}^\times$. By (5.24), $\text{Ad}(U)U_{\beta,-1}^\times$ has a factor $U_{\beta,-1}^\times \cong \mathbb{G}_m$; hence,

$$\chi_c(\text{Ad}(U)U_{\beta,-1}^\times, \text{ev}^*J) = \chi_c(\text{Ad}(U)U_{\beta,-1}^\times) = 0.$$

Case II: $\beta = \alpha_i$ is a simple long root. For $u \in U$, $\text{ev}(\text{Ad}(u)u_\beta(c\tau^{-1}))$ has image $u_{\alpha_i}(c) \in U/[U, U]$. In terms of the coordinates in (5.24), ev^*J is the pull-back of AS_{ϕ_i} from the $U_{\beta,-1}^\times \cong U_{\alpha_i}^\times$ -factor. Hence,

$$\chi_c(\text{Ad}(U)U_{\beta,-1}^\times, \text{ev}^*J) = \chi_c(U_{\alpha_i}^\times, \text{AS}_{\phi_i}) = -1.$$

Case III: β is negative and $\beta \neq -\theta$. Since $\beta \neq -\theta$, there exists a simple root α_i such that $\beta - \alpha_i$ is still a root. Then $\alpha = -\beta + \alpha_i$ is a positive root. Moreover, since $\langle \alpha_i, \beta^\vee \rangle \leq 1$ (because $\alpha_i \neq \beta$), we have $\langle \alpha, \beta^\vee \rangle = \langle -\beta + \alpha_i, \beta^\vee \rangle = -2 + \langle \alpha_i, \beta^\vee \rangle < 0$. Hence U_α appears in the decomposition (5.24).

Using (5.24), we write an element in $\text{Ad}(U)U_{\beta,-1}^\times$ as $\text{Ad}(u)u_\beta(c_\beta\tau^{-1})$, where $u = u^\alpha u_\alpha(c_\alpha)$ and $u^\alpha \in \mathbb{A} := \prod_{\alpha' > 0, \langle \alpha', \beta^\vee \rangle < 0, \alpha' \neq \alpha} U_{\alpha'}$. Note that

$$(5.27) \quad \text{ev}(\text{Ad}(u)u_\beta(c_\beta\tau^{-1})) = uu_\beta(c_\beta)u^{-1} = u^\alpha[u_\alpha(c_\beta), u_\beta(c_\beta)]u_\beta(c_\beta)u^{\alpha,-1}.$$

Since $\alpha \neq \pm\beta$, we can apply Chevalley's commutator relation [9, p. 36, (4)] to conclude that $[u_\alpha(c_\beta), u_\beta(c_\beta)]$ is a product of elements in the root groups $U_{i\alpha+j\beta}$ for $i, j \in \mathbb{Z}_{>0}$. Our assumptions that

- (i) $\alpha + \beta$ is simple and
- (ii) β is a long root

imply that any such $i\alpha + j\beta$ is positive (if it is a root), and the only simple root of this form is $\alpha + \beta$. Therefore, $[u_\alpha(c_\beta), u_\beta(c_\beta)] \in U$, and its image in $U/[U, U]$ is $u_{\alpha_i}(\epsilon c_\alpha c_\beta)$, where $\epsilon = \pm 1$.

By (5.27), $\text{Ad}(u)u_\beta(c_\beta) \in UB^{\text{opp}}$ if and only if $u_\beta(c_\beta)u^{\alpha,-1} \in UB^{\text{opp}}$. Moreover, when $\text{Ad}(u)u_\beta(c_\beta) \in UB^{\text{opp}}$, its image in $U/[U, U]$ is $u_{\alpha_i}(\epsilon c_\alpha c_\beta)$ times another element that only depends on c_β and u^α .

Under the decomposition (5.24), $\text{Ad}(U)U_{\beta,-1}^\times \cong U_\alpha \times \mathbb{A}$. Let $\text{pr}_\mathbb{A} : \text{Ad}(U)U_{\beta,-1}^\times \rightarrow \mathbb{A}$ be the projection. By the above discussion, ev^*J restricted to the fibers of $\text{pr}_\mathbb{A}$ are isomorphic to Artin-Schreier sheaves on U_α . Therefore $\text{pr}_{\mathbb{A},!}\text{ev}^*J = 0$; hence,

$$H_c^*(\text{Ad}(U)U_{\beta,-1}^\times, \text{ev}^*J) = H_c^*(\mathbb{A}, \text{pr}_{\mathbb{A},!}\text{ev}^*J) = 0.$$

This proves the claim and completes the proof of Theorem 4 (1).

5.5. *The adjoint Schubert variety.* In this subsection, let G be nonsimply-laced. We always assume that $\text{char}(k)$ is a good prime for G . So $\text{char}(k) > 2$ when G is of type B_n, C_n and $\text{char}(k) > 3$ when G is of type F_4, G_2 .

Let γ be the short dominant root of G and γ^\vee be the corresponding long coroot. We define $P_{-\gamma}, W_\gamma$, etc. in the same way as $P_{-\theta}, W_\theta$, etc. According to the possible values of $\langle \alpha, \gamma^\vee \rangle$ for roots $\alpha \in \Phi$, we have two cases:

- Type (B_n, C_n, F_4) : $|\langle \alpha, \gamma^\vee \rangle| = 0, 1, 2$;
- Type (G_2) : $|\langle \alpha, \gamma^\vee \rangle| = 0, 1, 2, 3$.

Let $\Phi_n^\gamma := \{\alpha \in \Phi \mid \langle \alpha, \gamma^\vee \rangle = n\}$.

LEMMA 5.28. *Suppose G is of type B_n, C_n or F_4 . Then $\text{Gr}_{\gamma^\vee}^{\text{triv}}$ consists of a single G -orbit. More precisely, let*

$$V_{-\gamma} = \prod_{\langle \alpha, \gamma^\vee \rangle = 2} U_{-\alpha, -1}.$$

We identify $V_{-\gamma}$ with its Lie algebra. The adjoint action of $P_{-\gamma}$ on $V_{-\gamma}$ (which factors through its Levi factor L_γ) stabilizes a unique quadric $Q_{-\gamma}$ defined by a nondegenerate quadratic form $q_{-\gamma}$ on $V_{-\gamma}$, so that the action of L_γ factors through $\text{GO}(V_{-\gamma}, q_{-\gamma})$,² with a dense orbits $V_{-\gamma} - Q_{-\gamma}$.

The natural morphism

$$(5.29) \quad G \times^{P_{-\gamma}} (V_{-\gamma} - Q_{-\gamma}) \ni (g, v) \mapsto \text{Ad}(g)v \in G[\tau^{-1}]_1$$

gives an isomorphism $G \times^{P_{-\gamma}} (V_{-\gamma} - Q_{-\gamma}) \cong \text{Gr}_{\gamma^\vee}^{\text{triv}}$.

Proof. In the case G is of type B, C or F_4 , the product in (5.21) becomes $\Lambda = \prod_{\langle \alpha, \gamma^\vee \rangle = 2} U_{\alpha, 1}$ (a commutative unipotent group), which can be identified with its Lie algebra. Since $\text{Gr}_{\gamma^\vee}^{\text{triv}}$ is stable under G -conjugation, it takes the form $G \times^{P_{-\gamma}} \Lambda^{\text{triv}}$ for some $P_{-\gamma}$ -stable open subset $\Lambda^{\text{triv}} \subset \Lambda$. The action of $P_{-\gamma}$ on Λ factors through the Levi quotient L_γ .

The vector space Λ carries a bilinear form

$$(5.30) \quad (x, y)_\Lambda := (x, \text{Ad}(w_\gamma)y)_\mathfrak{g},$$

where w_γ is the image of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ under the homomorphism $\text{SL}_2 \rightarrow G$ corresponding to the root γ , and $(\cdot, \cdot)_\mathfrak{g}$ is an $\text{Ad}(G)$ -invariant nondegenerate symmetric bilinear pairing on \mathfrak{g} (which exists when $\text{char}(k)$ is good; see [8, §1.16]). It is easy to check that $(\cdot, \cdot)_\Lambda$ is a nondegenerate symmetric bilinear pairing, and the L_γ -action on Λ preserves this pairing up to scalar. Let q_Λ be the quadratic form associated to $(\cdot, \cdot)_\Lambda$.

² $\text{GO}(V_{-\gamma}, q_{-\gamma})$ consists of invertible linear automorphisms of $V_{-\gamma}$ preserving $q_{-\gamma}$ up to a scalar.

By a quick case-by-case analysis, one checks that the action map $L_\gamma \rightarrow \text{GO}(\Lambda, q_\Lambda)$ is surjective. Therefore, Λ contains a unique L_γ -stable irreducible divisor $Q_\Lambda = \{q_\Lambda = 0\}$ whose complement is a single L_γ -orbit.

On the other hand, the complement $\text{Gr}_{\leq \gamma^\vee} - \text{Gr}_{\leq \gamma^\vee}^{\text{triv}}$ is an ample divisor representing the class of the determinant line bundle and hence has codimension 1. Since $\text{Gr}_{\leq \gamma^\vee} - \text{Gr}_{\gamma^\vee}$ has codimension at least 2, $\text{Gr}_{\gamma^\vee} - \text{Gr}_{\gamma^\vee}^{\text{triv}}$ also has codimension 1 in Gr_{γ^\vee} . Hence $\Lambda - \Lambda^{\text{triv}}$ also has codimension 1 in Λ ; therefore, it must be the irreducible divisor Q_Λ . This implies $\Lambda^{\text{triv}} = \Lambda - Q_\Lambda$, which is a single $P_{-\gamma}$ -orbit, therefore $\text{Gr}_{\gamma^\vee}^{\text{triv}}$ is a single G -orbit.

By the same SL_2 -calculation as in (5.23), we have

$$u_\gamma(\tau)\tau^{\gamma^\vee} = u_{-\gamma}(\tau^{-1}) \in \text{Gr}^{\text{triv}}.$$

Therefore, $\Lambda^{\text{triv}}\tau^{\gamma^\vee} = \text{Ad}(P_{-\gamma})u_\gamma(\tau)\tau^{\gamma^\vee} = \text{Ad}(P_{-\gamma})u_{-\gamma}(\tau^{-1}) \subset \text{Gr}^{\text{triv}}$. By a similar argument as above, $\text{Ad}(P_{-\gamma})u_{-\gamma}(\tau^{-1})$ is the open subset of $V_{-\gamma} = \prod_{\langle \alpha, \gamma^\vee \rangle = 2} U_{-\alpha, -1}$ defined by the complement of a quadric $Q_{-\gamma}$. Therefore,

$$\text{Gr}_{\gamma^\vee}^{\text{triv}} = G \times^{P_{-\gamma}} (\Lambda - Q_\Lambda)\tau^{\gamma^\vee} = G \times^{P_{-\gamma}} (V_{-\gamma} - Q_{-\gamma}) \subset G[\tau^{-1}]_1. \quad \square$$

LEMMA 5.31. *The morphism (5.29) extends to a resolution*

$$\nu : G \times^{P_{-\gamma}} V_{-\gamma} \rightarrow \text{Gr}_{\leq \gamma^\vee}^{\text{triv}},$$

which is an isomorphism over $\text{Gr}_{\gamma^\vee}^{\text{triv}}$ by Lemma 5.28.

- (1) *The fiber $\nu^{-1}(\star) \cong G/P_{-\gamma}$.*
- (2) *The fibers over $\text{Gr}_{\theta^\vee}^{\text{triv}}$ are isomorphic to $L_\theta/L_\theta \cap P_{-\gamma}$.*

Proof. Since $G \times^{P_{-\gamma}} (V_{-\gamma} - Q_{-\gamma})$ is dense in $G \times^{P_{-\gamma}} V_{-\gamma}$ and $\text{Gr}_{\leq \gamma^\vee}^{\text{triv}}$ is the closure of $\text{Gr}_{\gamma^\vee}^{\text{triv}}$ in Gr^{triv} , the morphism (5.29) extends to ν . By Lemma 5.28, there are three $\text{Ad}(P_{-\gamma})$ (or $\text{Ad}(L_\gamma)$)-orbits on $V_{-\gamma}$: $V_{-\gamma} - Q_{-\gamma}$, $Q_{-\gamma}^\times := Q_{-\gamma} - \{0\}$ and $\{0\}$, which give three G -orbits of $G \times^{P_{-\gamma}} V_{-\gamma}$.

The orbit $G \times^{P_{-\gamma}} \{0\}$ maps to $\text{Gr}_0^{\text{triv}} = \star$, which proves (1).

The orbit $G \times^{P_{-\gamma}} Q_{-\gamma}^\times$ must then map to $\text{Gr}_{\theta^\vee}^{\text{triv}}$. The G -stabilizer of $u_{-\theta}(\tau^{-1})$ in $G \times^{P_{-\gamma}} Q_{-\gamma}^\times$ and in $\text{Gr}_{\theta^\vee}^{\text{triv}} = G \times^{P_{-\theta}} U_{-\theta, -1}^\times$ are $P_{-\gamma} \cap P_{-\theta}^1$ and $P_{-\theta}^1$ respectively, where $P_{-\gamma}^1 = \ker(\theta : P_{-\theta} \rightarrow \mathbb{G}_m)$. The fiber $\nu^{-1}(u_{-\theta}(\tau^{-1}))$ thus equals $P_{-\theta}^1/P_{-\theta}^1 \cap P_{-\gamma}$. It is easy to check that the inclusions $L_\theta \hookrightarrow P_{-\theta}$ and $P_{-\theta}^1 \hookrightarrow P_{-\theta}$ induce isomorphisms

$$L_\theta/L_\theta \cap P_{-\gamma} \xrightarrow{\sim} P_{-\theta}/P_{-\theta} \cap P_{-\gamma} \xleftarrow{\sim} P_{-\theta}^1/P_{-\theta}^1 \cap P_{-\gamma}.$$

Since $\text{Gr}_{\theta^\vee}^{\text{triv}}$ is a single G -orbit by Lemma 5.22, all the fibers of ν over $\text{Gr}_{\theta^\vee}^{\text{triv}}$ are isomorphic to $\nu^{-1}(u_{-\theta}(\tau^{-1})) \cong L_\theta/L_\theta \cap P_{-\gamma}$. □

The Bruhat decomposition $G = \sqcup_{W/W_\gamma} UwP_{-\gamma}$ gives a decomposition

$$G \times^{P_{-\gamma}} V_{-\gamma} = \bigsqcup_{W/W_\gamma} UwP_{-\gamma} \times^{P_{-\gamma}} V_{-\gamma}.$$

The map $w \mapsto -w\gamma$ sets up a bijection between W/W_γ and the set of short roots of G . For $\beta = -w\gamma$, let $V_\beta = \text{Ad}(w)V_{-\gamma}$. Then the above decomposition can be rewritten as

$$(5.32) \quad G \times^{P_{-\gamma}} V_{-\gamma} = \bigsqcup_{\beta \text{ short root}} \text{Ad}(U)V_\beta.$$

As in the quasi-minuscule case, we can further write each stratum as

$$(5.33) \quad \begin{aligned} \text{Ad}(U)V_\beta &\cong \prod_{\alpha \in \Phi^+, \langle \alpha, \beta^\vee \rangle < 0} U_\alpha \times \prod_{\langle \beta', \beta^\vee \rangle = 2} U_{\beta', -1}, \\ \text{Ad}\left(\prod u_\alpha\right) \prod u_{\beta'}(c_{\beta'}\tau^{-1}) &\leftrightarrow \left(\prod u_\alpha, u_{\beta'}(c_{\beta'}\tau^{-1})\right). \end{aligned}$$

Remark 5.34. (1) For each short root β , the set Φ_2^β (those roots that appear in the factors of V_β) is *totally ordered* according to their heights. To see this, we only need to show that for different $\beta', \beta'' \in \Phi_2^\beta$, $\langle \rho^\vee, \beta' - \beta'' \rangle \neq 0$. In the proof of [Lemma 5.28](#), we remarked that $L_\gamma \rightarrow \text{GO}(\Lambda, q_\Lambda)$ is surjective, which means that for any two different roots $\alpha', \alpha'' \in \Phi_2^\gamma$ (i.e., they appear in the factors of Λ), the difference $\alpha' - \alpha''$ is a nonzero multiple of a root of L_γ . Since $\Phi_2^\beta = w\Phi_2^\gamma$ if $\beta = w\gamma$, the difference $\beta' - \beta''$ is also a nonzero multiple of a root of L_β ; therefore, $\langle \rho^\vee, \beta' - \beta'' \rangle \neq 0$.

The set Φ_2^β carries an involution $\beta' \mapsto 2\beta - \beta'$, with β the only fixed point. This involution is order-reversing.

(2) Again since $L_\gamma \rightarrow \text{GO}(\Lambda, q_\Lambda)$ in the proof of [Lemma 5.28](#), all roots $\gamma' \in \Phi_2^\gamma \setminus \{\gamma\}$ are permuted by W_γ . A similar statement holds if γ is replaced by any short root β . In particular, all roots in Φ_2^β are long roots except β itself.

LEMMA 5.35. *For a short root β , $\prod u_{\beta'}(c_{\beta'}\tau^{-1}) \in V_\beta$ (the product over Φ_2^β), and $u \in U$, we have*

$$a_{-\theta}(\text{Ad}(u)(\prod u_{\beta'}(c_{\beta'}\tau^{-1}))) = \begin{cases} c_{-\theta}x_{-\theta} & \text{if } -\theta \in \Phi_2^\beta, \\ 0 & \text{otherwise.} \end{cases}$$

The proof is similar to that of [Lemma 5.25](#).

5.6. Proof of Theorem 4(2). Here we assume G is of type B, C or F_4 . For G of type A, D, E , the statement (2) is identical to (1) in [Theorem 4](#); the proof for $G = G_2$ will be given in [Section D](#).

Applying [Corollary 5.20](#) to $V = \mathfrak{g}$, which has two nonzero dominant weights θ^\vee and γ^\vee , each with multiplicity one, we conclude that

$$\chi_c(\mathbb{P}^1_{\setminus\{0,\infty\}}, \mathrm{Kl}_G^{\mathrm{Ad}}(\phi, \chi)) = \chi_c(\mathrm{Gr}_{\theta^\vee}^{\mathrm{triv}, a=0}, \mathrm{ev}^* J) + \chi_c(\mathrm{Gr}_{\gamma^\vee}^{\mathrm{triv}, a=0}, \mathrm{ev}^* J).$$

We already know from (1) that $\chi_c(\mathrm{Gr}_{\theta^\vee}^{\mathrm{triv}, a=0}, \mathrm{ev}^* J) = r_\ell(G)$; we only need to show that $\chi_c(\mathrm{Gr}_{\gamma^\vee}^{\mathrm{triv}, a=0}, \mathrm{ev}^* J) = r_s(G)$.

For any subset $S \subset \Phi$, let

$$V_\beta^S = \left\{ \prod u_{\beta''} (c_{\beta''} \tau^{-1}) \in V_\beta \mid c_{\beta''} = 0 \text{ for } \beta'' \notin S \right\}.$$

Using this definition and the standard partial ordering on Φ (which restricts to the total ordering on Φ_2^β , by [Remark 5.34\(1\)](#)), the meanings of $V_\beta^{>-\theta}$, $V_\beta^{>0}$, etc. are obvious.

Using [Lemma 5.31](#), [Lemma 5.35](#) and the decomposition (5.32), the resolution ν restricted to $\mathrm{Gr}_{\leq \gamma^\vee}^{\mathrm{triv}, a=0}$ reads

$$\nu^{a=0} : \bigsqcup_{\beta \text{ short root}} \mathrm{Ad}(U)(V_\beta^{>-\theta}) \rightarrow \mathrm{Gr}_{\leq \gamma^\vee}^{\mathrm{triv}, a=0}.$$

By [Lemma 5.31](#), the Euler characteristic of $\mathrm{ev}^* J$ on the target of $\nu^{a=0}$ has contributions from \star , $\mathrm{Gr}_{\theta^\vee}^{\mathrm{triv}, a=0}$ and $\mathrm{Gr}_{\gamma^\vee}^{\mathrm{triv}, a=0}$:

$$\begin{aligned} (5.36) \quad & \sum_{\beta \text{ short root}} \chi_c(\mathrm{Ad}(U)V_\beta^{>-\theta}, \mathrm{ev}^* J) \\ &= \chi_c(G/P_{-\gamma}) + \chi_c(\mathrm{Gr}_{\theta^\vee}^{\mathrm{triv}, a=0}, \mathrm{ev}^* J) \chi_c(L_\theta/L_\theta \cap P_{-\gamma}) \\ & \quad + \chi_c(\mathrm{Gr}_{\gamma^\vee}^{\mathrm{triv}, a=0}, \mathrm{ev}^* J). \end{aligned}$$

The Bruhat-decomposition implies

$$(5.37) \quad \chi_c(G/P_{-\gamma}) = \#(W/W_\gamma) = \#\{\text{short roots in } \Phi\},$$

$$(5.38) \quad \chi_c(L_\theta/L_\theta \cap P_{-\gamma}) = \#(W_\theta/W_\theta \cap W_\gamma).$$

CLAIM. For a short root β ,

$$\chi_c(\mathrm{Ad}(U)(V_\beta^{>-\theta}), \mathrm{ev}^* J) = \begin{cases} 0 & \Phi_2^\beta \text{ contains a simple root,} \\ 1 & \text{otherwise.} \end{cases}$$

Admitting the claim first, we finish the proof. Combining (5.36), (5.37), (5.38) and the claim, we get

$$\begin{aligned}
 (5.39) \quad -\chi_c(\mathrm{Gr}_{\gamma^v}^{\mathrm{triv}, a=0}, \mathrm{ev}^* J) &= \#\{\text{short roots}\} - r_\ell(G) \#(W_\theta/W_\theta \cap W_\gamma) \\
 &\quad - \#\{\text{short roots } \beta \mid \Phi_2^\beta \text{ does not contain simple root}\} \\
 &= \#\{\text{short roots } \beta \mid \Phi_2^\beta \text{ contains a simple root}\} \\
 &\quad - r_\ell(G) \#(W_\theta/W_\theta \cap W_\gamma).
 \end{aligned}$$

By Remark 5.34(1), Φ_2^β is totally ordered by heights and hence contains at most one simple root. Therefore

$$\#\{\text{short roots } \beta \mid \Phi_2^\beta \text{ contains a simple root}\} = N_s + N_\ell,$$

where N_s (resp. N_ℓ) is the number of short roots β such that Φ_2^β contains a short (resp. long) simple root.

- If Φ_2^β contains a short simple root, this simple root must be β since β is the only short root in Φ_2^β by Remark 5.34(2). Therefore $N_s = r_s(G)$.
- Any simple long root is in the W -orbit of θ ; therefore $N_\ell = r_\ell \cdot N_\theta$, where $N_\theta = \#\{\text{short roots } \beta \mid \theta \in \Phi_2^\beta\}$. Such short roots β are in the W_θ -orbit (this follows by applying Remark 5.34(2) to the dual root system), and γ is one of them; hence, $N_\theta = \#(W_\theta/W_\theta \cap W_\gamma)$. Hence, $N_\ell = r_\ell(G) \#(W_\theta/W_\theta \cap W_\gamma)$.

Combining these calculations and (5.39), we conclude that $-\chi_c(\mathrm{Gr}_{\gamma^v}^{\mathrm{triv}, a=0}, \mathrm{ev}^* J) = r_s(G)$, hence proving Theorem 4(2).

It remains to prove the claim.

Case I: Φ_2^β contains a simple root α_i . For $u \in U$ and $v = u_{\alpha_i}(c_{\alpha_i} \tau^{-1})v^{\alpha_i} \in V_\beta$, where $v^{\alpha_i} \in \prod_{\beta' \neq \alpha_i} U_{\beta', 1}$, we have

$$\mathrm{ev}(\mathrm{Ad}(u)v) = \mathrm{Ad}(u)u_{\alpha_i}(c_{\alpha_i}) \cdot \mathrm{Ad}(u)\mathrm{ev}(v^{\alpha_i}).$$

This means that $\mathrm{Ad}(u)v \in UB^{\mathrm{opp}}$ if and only if $\mathrm{Ad}(u)\mathrm{ev}(v^{\alpha_i}) \in UB^{\mathrm{opp}}$, and in case this happens, its image in $U/[U, U]$ is $u_{\alpha_i}(c_{\alpha_i})$ times the image of $\mathrm{Ad}(u)\mathrm{ev}(v^{\alpha_i})$ in $U/[U, U]$. Using the decomposition (5.33), we can write $\mathrm{Ad}(U)V_\beta^{>-\theta} = U_{\alpha_i, -1} \times \mathbb{A}$ for some affine space \mathbb{A} (the product of other factors), and the sheaf $\mathrm{ev}^* J$ is an exterior product $\mathrm{ev}^* \mathrm{AS}_{\phi_i} \boxtimes L$ for some local system L on \mathbb{A} . By Künneth formula,

$$H_c^*(\mathrm{Ad}(U)V_\beta^{>-\theta}, \mathrm{ev}^* J) = H_c^*(U_{-\alpha_i, -1}, \mathrm{ev}^* \mathrm{AS}_{\phi_i}) \otimes H_c^*(\mathbb{A}, L) = 0.$$

Case II. Φ_2^β does not contain any simple root. We stratify the vector space $V_\beta^{>-\theta}$ into

$$V_\beta^{>-\theta} = V_\beta^{>0} \sqcup \left(\bigsqcup_{\beta' \in \Phi_2^\beta, -\theta < \beta' < 0} (V_\beta^{\geq \beta'} - V_\beta^{> \beta'}) \right).$$

First, we show that $\chi_c(\text{Ad}(U)V_\beta^{>0}, \text{ev}^*J) = 1$. In fact, since $\Phi_2^\beta \cap \Phi^+$ contains no simple root, $\text{ev}(\text{Ad}(U)V_\beta^{>0}) \subset [U, U]$; hence, ev^*J is the constant sheaf on $\text{Ad}(U)V_\beta^{>0}$. Since $\text{Ad}(U)V_\beta^{>0}$ is an affine space by (5.33), we get the conclusion.

Second, we prove that $\chi_c(\text{Ad}(U)(V_\beta^{\geq \beta'} - V_\beta^{> \beta'}), \text{ev}^*J) = 0$ for each $\beta' \in \Phi_2^\beta, -\theta < \beta' < 0$. Since $\beta' \neq -\theta, \beta' - \alpha_i$ is still a root for some simple α_i . Then $\alpha = \alpha_i - \beta'$ is a positive root. By assumption, $\alpha_i \notin \Phi_2^\beta$, therefore $\langle \alpha, \beta^\vee \rangle = \langle \alpha_i, \beta^\vee \rangle - 2 < 0$, i.e., α appears in the first product in (5.33).

LEMMA 5.40. Let $a, b \in \mathbb{Z}_{>0}$ and $\beta' \leq \beta_1, \dots, \beta_b$ be roots in Φ_2^β (not necessarily distinct). Then $a\alpha + \sum_{i=1}^b \beta_i \neq 0$. If $a\alpha + \sum_{i=1}^b \beta_i$ is a root, then one of the following situations happens:

- (1) $a\alpha + \sum_{i=1}^b \beta_i$ is a negative root. Then $a = b - 1, a\alpha + \sum_{i=1}^b \beta_i \in \Phi_2^\beta$ and is larger than any of the β_i 's (in the total order of Φ_2^β);
- (2) $a\alpha + \sum_{i=1}^b \beta_i$ is positive but not simple;
- (3) $a = b = 1, \beta_1 = \beta'$ and $a\alpha + \sum_{i=1}^b \beta_i = \alpha_i$.

Proof. If $a \geq b$, since $\alpha + \beta' > 0$ and $\beta_i \geq \beta', \alpha + \beta_i$ must have positive height. Therefore,

$$a\alpha + \sum_{i=1}^b \beta_i = (a - b)\alpha + \sum_{i=1}^b (\alpha + \beta_i) > 0.$$

Since each term on right-hand side of the above sum has positive height, it is a simple root if there is only one summand, which must be the case (3).

If $a < b$, then $\langle a\alpha + \sum_{i=1}^b \beta_i, \beta^\vee \rangle \geq 2b - 2a \geq 2$. Therefore, if $a\alpha + \sum_{i=1}^b \beta_i$ is a root, we must have $a = b - 1$ and $a\alpha + \sum_{i=1}^b \beta_i \in \Phi_2^\beta$. Since Φ_2^β does not contain simple roots, we get either case (1) or case (2).

In any case, we have $a\alpha + \sum_{i=1}^b \beta_i \neq 0$. □

For $u = \prod u_{\alpha'}(c_{\alpha'})$ (the product over $\alpha' \in \Phi^+, \langle \alpha, \beta^\vee \rangle < 0$), let $u^\alpha = \prod_{\alpha' \neq \alpha} u_{\alpha'}(c_{\alpha'})$. Similarly, for $v \in V_\beta^{\geq \beta'} - V_\beta^{> \beta'}$, write

$$v = u_{\beta'}(c_{\beta'}\tau^{-1})v^{\beta'} \prod u_{\beta''}(c_{\beta''}\tau^{-1})$$

(product over $\beta' < \beta'' \in \Phi_2^\beta$, and $c_{\beta'}$ invertible). Then

$$(5.41) \quad \begin{aligned} & \text{ev}(\text{Ad}(u)v) \\ &= u^\alpha [u_\alpha(c_\alpha), u_{\beta'}(c_{\beta'})] u_{\beta'}(c_{\beta'}) \left(\prod_{\beta'' > \beta'} [u_\alpha(c_\alpha), u_{\beta''}(c_{\beta''})] u_{\beta''}(c_{\beta''}) \right) u^{\alpha, -1}. \end{aligned}$$

To calculate (5.41), we use Chevalley’s commutator relation to write each $[u_\alpha(c_\alpha), u_{\beta''}(c_{\beta''})]$ into products of positive and negative root factors. We try to pass the negative root factors (which necessarily appear in Φ_2^β by Lemma 5.40(1)) to the right. By Chevalley’s commutator relation, each time we will produce new factors of the form $a\alpha + \beta_1 + \dots + \beta_b$ for $\beta_i \in \Phi_2^\beta$, $a, b > 0$. We keep the positive factors and pass the negative factors further to the right. The only thing we need to make sure of in this process is that when we do commutators $[U_{\alpha'}, U_{\gamma'}]$, we always have that α' and γ' are linearly independent so that Chevalley’s commutator relation is applicable. In fact, in the process, we only encounter the case where $\alpha' \in \Phi^+$, $\gamma' \in \Phi^-$ and $\alpha' + \gamma'$ has the form $a\alpha + \beta_1 + \dots + \beta_b$ for $\beta' \leq \beta_i \in \Phi_2^\beta$ ($a, b \in \mathbb{Z}_{>0}$). The only possibility for α' and γ' to be linearly dependent is $\alpha' + \gamma' = 0$, which was eliminated by Lemma 5.40.

In the end of the process, we get

$$(5.42) \quad \text{ev}(\text{Ad}(u)v) = u^\alpha u_{\alpha_i}(\epsilon c_\alpha c_{\beta'}) u^+ \left(\prod_{\beta'' \geq \beta', \beta'' \in \Phi_2^\beta} u_{\beta''}(c_{\beta''} + \tilde{c}_{\beta''}) \right) u^{\alpha, -1}.$$

The term $u_{\alpha_i}(\epsilon c_\alpha c_{\beta'})$ (where $\epsilon = \pm 1$) comes from $[u_\alpha(c_\alpha), u_{\beta'}(c_{\beta'})]$. The term u^+ is the product all the other *positive* factors in $U_{a\alpha + \beta_1 + \dots + \beta_b}$ (i.e., $a\alpha + \beta_1 + \dots + \beta_b \in \Phi^+$). By Lemma 5.40(2)(3), these $a\alpha + \beta_1 + \dots + \beta_b$ are never simple; therefore $u^+ \in [U, U]$. Finally, the extra coefficient $\tilde{c}_{\beta''}$ comes from the negative factors in $U_{a\alpha + \beta_1 + \dots + \beta_b}$, which is a polynomial functions in c_α and $c_{\beta''}$ for $\beta''' \in \Phi_2^\beta$. By Lemma 5.40(1), $c_{\beta''}$ only involves those $c_{\beta''}$ such that $\beta''' < \beta''$.

Therefore, we can make a change of variables

$$(5.43) \quad \begin{aligned} \text{Ad}(U)(V_\beta^{\geq \beta'} - V_\beta^{> \beta'}) &\cong U_\alpha \times U_{\beta', -1}^\times \times \prod_{\langle \alpha', \beta^\vee \rangle < 0, \alpha' \neq \alpha} U_{\alpha'} \times \prod_{\beta'' \in \Phi_2^\beta, \beta'' > \beta'} U_{\beta'', -1}, \\ \text{Ad}(u)(v) &\leftrightarrow \left(u_\alpha(c_\alpha), u_{\beta'}(c_{\beta'} \tau^{-1}), u^\alpha, \prod u_{\beta''}(c_{\beta''} + \tilde{c}_{\beta''}) \right). \end{aligned}$$

Let \mathbb{A} be the product of the last three terms in (5.43). Let $\text{pr}_\mathbb{A} : \text{Ad}(U)(V_\beta^{\geq \beta'} - V_\beta^{> \beta'}) \rightarrow \mathbb{A}$ be the projection. In view of (5.42), the restriction of $\text{ev}^* J$ on

the fibers of $\text{pr}_{\mathbb{A}}$ are isomorphic to Artin-Schreier sheaves on U_α . Therefore $\text{pr}_{\mathbb{A},!}\text{ev}^*J = 0$; hence,

$$H_c^*(\text{Ad}(U)U_{\beta,-1}^\times, \text{ev}^*J) = H_c^*(\mathbb{A}, \text{pr}_{\mathbb{A},!}\text{ev}^*J) = 0.$$

This completes the proof of Case II of the claim.

6. Global monodromy

This section is devoted to the proof of [Theorem 3](#). Let G be split, almost simple over k .

6.1. *Dependence on the additive character.* Recall from [Remark 2.8\(3\)](#) that $T \rtimes \text{Aut}^\dagger(G) \times \mathbb{G}_m^{\text{rot}}$ acts on $\text{Bun}_G(0,2)$ and the Hecke correspondence (2.7). The group $T(k) \rtimes \text{Aut}^\dagger(G) \times \mathbb{G}_m^{\text{rot}}(k)$ also acts on $I_\infty(1)/I_\infty(2)$, hence on the space of generic additive characters. Let S_ϕ be the stabilizer of ϕ under the action of $T \rtimes \text{Aut}^\dagger(G) \times \mathbb{G}_m^{\text{rot}}$. This is a finite group scheme over k .

When G is of adjoint type, for each $\sigma \in \text{Aut}^\dagger(G)$, there is a unique $(t, s) \in (T \times \mathbb{G}_m^{\text{rot}})(k)$ such that (t, σ, s) fixes ϕ . Therefore, in this case, the projection $S_\phi \rightarrow \text{Aut}^\dagger(G) \xrightarrow{\sim} \text{Out}(G)$ is an isomorphism (as discrete groups over k). In general, $S_\phi \rightarrow \text{Aut}^\dagger(G)$ is a ZG -torsor but may not be surjective on k -points.

The following lemma follows immediately from the definition of the geometric Hecke operators.

LEMMA 6.1. *The tensor functor defining $\text{Kl}_{\check{G}}(\phi)$,*

$$\text{Hk}_\phi : \mathcal{S} \ni \text{IC}_V \mapsto \text{Hk}_V(A_\phi)|_{\star \times \mathbb{P}_{\setminus\{0,\infty\}}^1} \in \text{Loc}(\mathbb{P}_{\setminus\{0,\infty\}}^1),$$

carries a natural S_ϕ -equivariant structure. Here $S_\phi(k)$ acts on \mathcal{S} via its image in $\text{Aut}^\dagger(G)$ and $S_\phi(k)$ acts on $\text{Loc}(\mathbb{P}_{\setminus\{0,\infty\}}^1)$ via its action on $\mathbb{P}_{\setminus\{0,\infty\}}^1$ through $S_\phi \rightarrow \mathbb{G}_m^{\text{rot}}$.

In [Lemma B.3\(1\)](#) we will check that under the equivalence $\text{Rep}(\check{G}) \cong \mathcal{S}$, the $\text{Aut}^\dagger(G)$ -action on \mathcal{S} coincides with the action of $\text{Aut}^\dagger(G) \cong \text{Aut}^\dagger(\check{G})$ with respect to the chosen pinning of the dual group.

Let $S_\phi^1(k) = \ker(S_\phi(k) \rightarrow \mathbb{G}_m^{\text{rot}}(k))$, and let $S_\phi^{\text{rot}}(k)$ be the image of $S_\phi(k)$ in $\mathbb{G}_m^{\text{rot}}(k)$, which is a finite cyclic group. We would like to use the above equivariance to conclude that the monodromy representation of $\text{Kl}_{\check{G}}(\phi)$ can be chosen to take values in $\check{G}^{S_\phi^1(k)}$. However, since this representation is only defined up to inner automorphisms, this requires an extra argument.

Recall that we have chosen a geometric generic point $\bar{\eta}$ over $\text{Spec}K_0$, the formal punctured disc at 0. We defined $\text{Kl}_{\check{G}}(\phi, 1)$ as a functor from the Satake category $\mathcal{S} \cong \text{Rep}(\check{G})$ to $\text{Loc}(\mathbb{P}_{\setminus\{0,\infty\}}^1)$ so that restriction to $\bar{\eta}$ is a tensor

functor

$$(6.2) \quad \omega_\phi : \mathcal{S} \xrightarrow{\text{Kl}_{\check{G}(\phi)}} \text{Loc}(\mathbb{P}^1_{\setminus\{0,\infty\}}) \xrightarrow{j_{\bar{\eta}}^*} \text{Vec}.$$

As this is a fiber functor of \mathcal{S} , we can define ${}^\phi\check{G} = \underline{\text{Aut}}^\otimes(\omega_\phi)$. Of course, ${}^\phi\check{G}$ is isomorphic to \check{G} but not canonically so. The group ${}^\phi\check{G}$ has the advantage that by Tannakian formalism, we get a canonical homomorphism $\underline{\text{Aut}}^\otimes(j_{\bar{\eta}}^*) \rightarrow \underline{\text{Aut}}^\otimes(\omega_\phi)$, where $\underline{\text{Aut}}^\otimes(j_{\bar{\eta}}^*)$ is the pro-algebraic envelope of the fundamental group $\pi_1(\mathbb{P}^1_{\setminus\{0,\infty\}}, \bar{\eta})$. Therefore, we get a homomorphism

$$(6.3) \quad \varphi : \pi_1(\mathbb{P}^1_{\setminus\{0,\infty\}}, \bar{\eta}) \rightarrow {}^\phi\check{G}(\overline{\mathbb{Q}}_\ell).$$

By Lemma 6.1 we get a homomorphism

$$(6.4) \quad S_\phi^1(k) \rightarrow \text{Aut}^\otimes(\mathcal{S}, \text{Hk}_\phi) \xrightarrow{(j_{\bar{\eta}}^*)^*} \text{Aut}^\otimes(\mathcal{S}, \omega_\phi) = \text{Aut}({}^\phi\check{G}).$$

In other words, we have an action of S_ϕ^1 on ${}^\phi\check{G}$. We will prove in Lemma B.4 that ${}^\phi\check{G}$ also carries a natural pinning \ddagger and that the above action preserves this pinning. The pinnings \ddagger and \dagger define a canonical isomorphism $\text{can} : {}^\phi\check{G} \cong \check{G}$. Using this identification, we obtain

COROLLARY 6.5. *The monodromy representation φ extends to a homomorphism between exact sequences:*

$$(6.6) \quad \begin{array}{ccccc} \pi_1(\mathbb{P}^1_{\setminus\{0,\infty\}}, \bar{\eta}) & \xrightarrow{[\#S_\phi^{\text{rot}}(k)]} & \pi_1(\mathbb{P}^1_{\setminus\{0,\infty\}}, \bar{\eta}) & \longrightarrow & S_\phi^{\text{rot}}(k) \\ \downarrow \varphi & & \downarrow \varphi \sim & & \parallel \\ \check{G}^{S_\phi^1(k)} & \longrightarrow & \check{G}^{S_\phi^1(k)} \times S_\phi^{\text{rot}}(k) & \longrightarrow & S_\phi^{\text{rot}}(k). \end{array}$$

Proof. By equation (6.4), the $S_\phi^1(k)$ -action on ${}^\phi\check{G}$ factors through $S_\phi^1(k) \rightarrow \text{Aut}^\otimes(\mathcal{S}, \text{Hk}_\phi)$. By Lemma B.2, the monodromy representation φ thus factors through $\pi_1(\mathbb{P}^1_{\setminus\{0,\infty\}}, \bar{\eta}) \rightarrow {}^\phi\check{G}^{S_\phi^1(k)} \subset {}^\phi\check{G}$; i.e.,

$$\text{Hk}_\phi : \mathcal{S} \cong \text{Rep}({}^\phi\check{G}) \xrightarrow{\text{Res}} \text{Rep}({}^\phi\check{G}^{S_\phi^1(k)}) \xrightarrow{\kappa} \text{Loc}(\mathbb{P}^1_{\setminus\{0,\infty\}})$$

for some tensor functor κ .

We identified $\check{G} \cong {}^\phi\check{G}$ using the pinned isomorphism “can,” so we can view the functor κ as $\text{Rep}(\check{G}^{S_\phi^1(k)}) \rightarrow \text{Loc}(\mathbb{P}^1_{\setminus\{0,\infty\}})$. Then $S_\phi^{\text{rot}}(k) = S_\phi(k)/S_\phi^1(k)$ acts on $\check{G}^{S_\phi^1(k)}$ via $S_\phi(k) \rightarrow \text{Out}(G) \xrightarrow{\iota} \text{Aut}^\dagger(\check{G})$, hence acting on the source and target of κ . Lemma 6.1 implies that κ carries a natural $S_\phi^{\text{rot}}(k)$ -equivariant

structure. Taking $S_\phi^{\text{rot}}(k)$ -invariants of both tensor categories, we get a functor $\tilde{\kappa}$:

$$\begin{CD} \text{Rep}(\check{G}^{S_\phi^1(k)} \rtimes S_\phi^{\text{rot}}(k)) @>\tilde{\kappa}>> \text{Loc}(\mathbb{P}^1_{\setminus\{0,\infty\}}/S_\phi^{\text{rot}}(k)) \\ @VV\wr V @VV\wr V \\ \text{Rep}(\check{G}^{S_\phi^1(k)})^{S_\phi^{\text{rot}}(k)} @>(\kappa)^{S_\phi^{\text{rot}}(k)}>> \text{Loc}(\mathbb{P}^1_{\setminus\{0,\infty\}})^{S_\phi^{\text{rot}}(k)}. \end{CD}$$

Since the quotient map $\mathbb{P}^1_{\setminus\{0,\infty\}} \rightarrow \mathbb{P}^1_{\setminus\{0,\infty\}}/S_\phi^{\text{rot}}(k)$ can be identified with the $\#S_\phi^{\text{rot}}(k)$ -th power map of $\mathbb{P}^1_{\setminus\{0,\infty\}} = \mathbb{G}_m$, we arrive at the diagram (6.6). \square

Remark 6.7. Corollary 6.5 remains true if k is replaced by an extension k' and $\mathbb{P}^1_{\setminus\{0,\infty\}}$ is replaced by $\mathbb{P}^1_{\setminus\{0,\infty\}} \otimes_k k'$. In particular, to get information about the geometric monodromy, we take for $k' = \bar{k}$.

In the case $\text{Out}(\check{G})$ is nontrivial, we have

\check{G}	$\text{Aut}^\dagger(\check{G}) \xrightarrow{\sim} \text{Out}(\check{G})$	$\check{G}^{\text{Aut}^\dagger(\check{G})}$	$S_\phi^1(\bar{k})$	$S_\phi^{\text{rot}}(\bar{k})$
$A_{2n-1} \ (n \geq 2)$	$\mathbb{Z}/2$	C_n	$\text{Out}(\check{G})$	1
A_{2n}	$\mathbb{Z}/2$	B_n	1	$\mathbb{Z}/2$
D_4	S_3	G_2	$\text{Out}(\check{G})$	1
$D_n \ (n \geq 5)$	$\mathbb{Z}/2$	B_{n-1}	$\text{Out}(\check{G})$	1
E_6	$\mathbb{Z}/2$	F_4	$\text{Out}(\check{G})$	1

Table 2. Outer automorphisms and stabilizers of ϕ .

6.2. *Zariski closure of global monodromy.* Let $\check{G}_{\text{geo}} \subset \check{G}$ be the Zariski closure of the image of the geometric monodromy representation

$$\varphi^{\text{geo}} : \pi_1(\mathbb{P}^1_{\setminus\{0,\infty\}} \otimes_k \bar{k}, \bar{\eta}) \rightarrow \phi \check{G} \cong \check{G}.$$

We first show that \check{G}_{geo} is not too small.

PROPOSITION 6.8 (B. Gross). *If $\text{char}(k) > 2$, and the rank of G is at least 2, the \check{G}_{geo} is not contained in any principal $\text{PGL}_2 \subset \check{G}$.*

Proof. Suppose instead $\check{G}_{\text{geo}} \subset \text{PGL}_2 \subset \check{G}$, where PGL_2 contains a principal unipotent element (image of \mathcal{J}_0^t) of \check{G} . The image of the wild inertia \mathcal{J}_∞^+ must be nontrivial because $\text{Kl}_{\check{G}}^{\theta_V}(\phi)$ has nonzero Swan conductor at ∞ .

Since $p = \text{char}(k) > 2$, $\varphi(\mathcal{J}_\infty^+)$ lies in a maximal torus $\mathbb{G}_m \subset \text{PGL}_2$ and contains $\mu_p \subset \mathbb{G}_m$. Since $\varphi(\mathcal{J}_\infty)$ normalizes $\varphi(\mathcal{J}_\infty^+)$, it must be contained in the normalizer $N(\mathbb{G}_m) \subset \text{PGL}_2$ of the torus \mathbb{G}_m .

For any irreducible representation $S^{2\ell} = \text{Sym}^{2\ell}(\text{Std})$ of PGL_2 (where Std is the 2-dimensional representation of SL_2), every pair of weight spaces $S^{2\ell}(n) \oplus S^{2\ell}(-n)$ ($0 \leq n \leq \ell$) is stable under $N(\mathbb{G}_m)$, hence under \mathcal{J}_∞ . If the weight

n does not divide p , the Swan conductor of $S^{2\ell}(n) \oplus S^{2\ell}(-n)$ is at least 1. Therefore,

$$\text{Swan}_\infty(S^{2\ell}) \geq \ell - [\ell/p].$$

Consider the action of J_∞ on the quasi-minuscule representation V_{θ^\vee} of \check{G} . By Lemma C.1, V_{θ^\vee} decomposes into $r_s(\check{G})$ irreducible representations of the principal PGL_2 : $V_{\theta^\vee} = S^{2\ell_1} \oplus \dots \oplus S^{2\ell_{r_s(\check{G})}}$. Therefore,

$$\text{Swan}_\infty(V_{\theta^\vee}) \geq \sum_{i=1}^{r_s(\check{G})} (\ell_i - [\ell_i/p]) \geq (1 - 1/p) \frac{hr_s(\check{G})}{2}.$$

Here we used $\sum_i \ell_i = hr_s(\check{G})/2$. (See the proof of Lemma C.1.) On the other hand, we have $\text{Swan}_\infty(V_{\theta^\vee}) = r_s(\check{G})$ by Corollary 5.1(1). So as long as $h > 3$ or $p > 3$, we get contradiction with . Even when $h = 3$ and $p = 3$, then $\check{G} = \text{PGL}_3$ with $V_{\theta^\vee} = S^4 \oplus S^2$, we still have $\text{Swan}_\infty(V_{\theta^\vee}) \geq 2 + 1 = 3 > r_s(\check{G})$, a contradiction! \square

Proof of Theorem 3. Since $\text{Kl}_G^V(\phi)$ is a pure of weight 0 for every $V \in \text{Rep}(\check{G})$, it is geometrically semisimple. By Deligne [11, Corollaire 1.3.9], the neutral component $\check{G}_{\text{geo}}^\circ$ of \check{G}_{geo} is a semisimple group.

Step I. Assume G is not of type A_1, A_{2n} or B_3 . We first determine $\check{G}_{\text{geo}}^\circ$, or equivalently, its Lie algebra $\check{\mathfrak{g}}_{\text{geo}}$. On the one hand, by Theorem 1(2), $\check{G}_{\text{geo}}^\circ$ contains a principal unipotent element; hence it contains a principal PGL_2 . Proposition 6.8 says that \check{G}_{geo} cannot be equal to PGL_2 . Since a principal PGL_2 is its own normalizer in \check{G} , we conclude that $\check{G}_{\text{geo}}^\circ$ cannot be equal to PGL_2 , i.e., $\mathfrak{sl}_2 \subsetneq \check{\mathfrak{g}}_{\text{geo}}$. On the other hand, since G is not of type A_{2n} , $S_\phi^1(\bar{k}) = \text{Aut}^\dagger(\check{G}) \xrightarrow{\sim} \text{Out}(\check{G})$. In this case, Corollary 6.5 implies $\check{G}_{\text{geo}} \subset \check{G}^{\text{Aut}^\dagger(\check{G})}$.

Dynkin classified all Lie subalgebras $\check{\mathfrak{g}}_{\text{geo}} \subset \check{\mathfrak{g}}$ that contain a principal \mathfrak{sl}_2 . If $\check{\mathfrak{g}} \neq \mathfrak{so}_7$, then either $\check{\mathfrak{g}}_{\text{geo}} = \mathfrak{sl}_2$ or $\check{\mathfrak{g}}_{\text{geo}}$ is the fixed point algebra of some pinned automorphism of \check{G} . In our case, if G is not of type A_{2n} , we can already conclude that $\check{\mathfrak{g}}_{\text{geo}} = \check{\mathfrak{g}}^{\text{Aut}^\dagger(\check{G})}$.

Step II. Suppose \check{G} is of type B_3 and $\text{char}(k) > 3$. We claim that $\check{G}_{\text{geo}}^\circ = G_2$. For this it suffices to show that $\text{Kl}_{\text{SO}_7}^{\wedge^3 V_7}(\phi)$ contains a global section over $\mathbb{P}_{\setminus\{0,\infty\}}^1 \otimes_k \bar{k}$, where V_7 is the 7-dimensional standard representation of SO_7 . Suppose the contrary. Then the long exact sequence (5.4)–(5.6) with $\check{\mathfrak{g}}$ replaced by $\wedge^3 V_7$ would imply

$$(6.9) \quad \dim(\wedge^3 V_7)^{J_0} + \dim(\wedge^3 V_7)^{J_\infty} \leq \dim H_c^1(\mathbb{P}_{\setminus\{0,\infty\}}^1, \text{Kl}_{\text{SO}_7}^{\wedge^3 V_7}(\phi)) = \text{Swan}_{J_\infty}(\wedge^3 V_7).$$

We use the standard basis $\{e_{-3}, \dots, e_0, \dots, e_3\}$ for V_7 . (The quadratic form is $e_0^2 + e_1 e_{-1} + e_2 e_{-2} + e_3 e_{-3}$.) The 0-weight space of $\wedge^3 V_7$ under the principal

\mathfrak{sl}_2 is spanned by $\{e_1 \wedge e_{-2} \wedge e_{-3}, e_{-1} \wedge e_2 \wedge e_3, e_0 \wedge e_i \wedge e_{-i}, i = 1, 2, 3\}$ and hence has dimension 5. Since J_0 acts on V_7 as a principal unipotent element by [Theorem 1\(3\)](#), one concludes that $\dim(\wedge^3 V_7)^{J_0} = 5$.

Since $\text{char}(k) > 3$, it does not divide the Coxeter number of \check{G} . Therefore [Corollary 2.15](#) is applicable. Since the breaks of the J_∞^+ -action on V_7 are $1/h = 1/6$, the breaks of J_∞^+ on the nonzero weight spaces of $\wedge^3 V_7$ are $\leq 1/6$. Therefore $\text{Swan}_{J_\infty}(\wedge^3 V_7) \leq [\frac{1}{6} \dim \wedge^3 V_7] = [35/6] = 5$. Moreover, by the description of $\varphi(J_\infty^+)$ in [Corollary 2.15](#), J_∞^+ acts trivially on $\text{Span}\{e_1 \wedge e_2 \wedge e_3, e_{-1} \wedge e_{-2} \wedge e_{-3}\}$, and the Coxeter permutation $e_1 \rightarrow e_2 \rightarrow e_3 \rightarrow e_{-1} \rightarrow e_{-2} \rightarrow e_{-3} \rightarrow e_1$ permutes $e_1 \wedge e_2 \wedge e_3$ and $e_{-1} \wedge e_{-2} \wedge e_{-3}$. We get $\dim(\wedge^3 V_7)^{J_\infty} \geq 1$. Thus,

$$\dim(\wedge^3 V_7)^{J_0} + \dim(\wedge^3 V_7)^{J_\infty} \geq 5 + 1 > 5 \geq \text{Swan}_{J_\infty}(\wedge^3 V_7),$$

which contradicts [\(6.9\)](#). This proves that $\check{G}_{\text{geo}} = G_2$ in the case $\check{G} = \text{SO}_7$.

Step III: When \check{G} is of type A_{2n} . In this case, $\text{Kl}_{\check{G}}(\phi)$ comes from the classical Kloosterman sheaf $\text{Kl}_n(\phi)$, whose global monodromy is treated by Katz in [\[26\]](#); see [\(0.2\)](#).

Step IV. It remains to prove that \check{G}_{geo} is connected. The case of A_n is treated by Katz in loc. cit. In the case \check{G} is of type E_6 , one checks that $\check{G}^{\text{Aut}^\dagger(\check{G})}$ is already connected.

In general, we have $\check{G}_{\text{geo}} \subset N_{\check{G}}(\check{G}_{\text{geo}}^\circ) = \check{G}_{\text{geo}}^\circ Z\check{G}$. Consider the surjective homomorphism

$$\overline{\varphi}^{\text{geo}} : \pi_1^{\text{geo}}(\mathbb{P}^1_{\setminus\{0,\infty\}}, \bar{\eta}) \xrightarrow{\varphi^{\text{geo}}} \check{G}_{\text{geo}} \rightarrow \pi_0(\check{G}_{\text{geo}}) = Z\check{G}/Z\check{G} \cap \check{G}_{\text{geo}}^\circ.$$

Assuming \check{G} is not of type A_n or E_6 , then $\pi_0(\check{G})$ is a 2-group. Since $\text{char}(k) > 2$, $\overline{\varphi}^{\text{geo}}$ factors through the tame quotient. On the other hand, the tame generator in $J_0 \cong \pi^{\text{geo}}(\mathbb{P}^1_{\setminus\{0,\infty\}}, \bar{\eta})$ must map to a unipotent element in \check{G}_{geo} , hence inside $\check{G}_{\text{geo}}^\circ$. Therefore the map $\overline{\varphi}^{\text{geo}}$ is trivial; i.e., \check{G}_{geo} is connected. □

7. Functoriality of Kloosterman sheaves: Conjectures

In this section, we offer some conjectures for a further study on Kloosterman sheaves. In particular, according to a rigidity property that is known in the case of GL_n according to Katz and Gabber, the Kloosterman sheaves with the same geometric monodromy tabulated in [Table 1](#) should be isomorphic after matching the additive characters ϕ . We will also give a conjectural description of the local and global monodromy of $\text{Kl}_{L\mathcal{G}}(\phi)$ for certain quasi-split groups \mathcal{G} .

7.1. *Rigidity.*

CONJECTURE 7.1 (Physical rigidity of Kloosterman sheaves). *Suppose L is a \check{G} -local system on $\mathbb{P}^1_{\setminus\{0,\infty\}} \otimes_k \bar{k}$ that, as \mathcal{J}_0 and \mathcal{J}_∞ -representations, has the same isomorphism types as $\text{Kl}_{\check{G}}(\phi, \chi)$. Then $L \cong \text{Kl}_{\check{G}}(\phi, \chi)$ over $\mathbb{P}^1_{\setminus\{0,\infty\}} \otimes_k \bar{k}$.*

Even stronger, we expect

CONJECTURE 7.2. *Suppose L is a \check{G} -local system on $\mathbb{P}^1_{\setminus\{0,\infty\}}$ satisfying*

- *L is tame at $\{0\}$; the semisimple part of the image of a topological generator of \mathcal{J}_0^t is conjugate to an element in $\check{T}[q-1]$ that corresponds to a multiplicative character $\chi : T(k) \rightarrow \overline{\mathbb{Q}}_\ell^\times$.*
- *$\text{Swan}_\infty(L^{\text{Ad}}) = r$, and $(L^{\text{Ad}})^{\mathcal{J}_\infty} = 0$.*

Then there exists a generic linear function $\phi : I(1)/I(2) \rightarrow \mathbb{G}_a$ such that $L \cong \text{Kl}_{\check{G}}(\phi, \chi)$ up to an unramified twist (given by $\text{Gal}(\bar{k}/k) \rightarrow Z\check{G}$).

Inspired by the above conjectures, and the calculation of the local and global monodromy of Kloosterman sheaves given in [Theorem 1\(2\)](#), [Corollary 2.15](#) and [Theorem 3](#), we conjecture that there should be functorial relationship between the Kloosterman sheaves.

CONJECTURE 7.3. *Let G, G' be split, almost simple groups over k whose dual groups $\check{G} \supset \check{G}'$ appear in the same line of [Table 1](#). Then for every generic linear function ϕ of G , there exists a generic linear function ϕ' of G' such that over $\mathbb{P}^1_{\setminus\{0,\infty\}} \otimes_k \bar{k}$, the Kloosterman sheaf $\text{Kl}_{\check{G}}(\phi)$ is the pushout of the Kloosterman sheaf $\text{Kl}_{\check{G}'}(\phi')$.*

7.2. *Quasi-split groups.* Let G be a split, almost simple and simply-connected group over k . Let \mathcal{G} be the quasi-split group scheme on \mathbb{P}^1 whose restriction to $\mathbb{P}^1_{\setminus\{0,\infty\}}$ is given by the twisting $\sigma : \mu_N \hookrightarrow \text{Aut}^\dagger(G)$ as in [Section 1.1](#). Recall that N is assumed to be prime to $\text{char}(k)$, and $N = 2$ unless G is of type D_4 , in which case $N = 3$. We abuse the notation to denote still by σ the image of a generator of μ_N in $\text{Aut}^\dagger(G)$. We identify $\text{Aut}^\dagger(G)$ with $\text{Aut}^\dagger(\check{G})$ using the isomorphism in [Lemma B.3\(2\)](#). We write $\langle \sigma \rangle \subset \text{Aut}^\dagger(G) \cong \text{Aut}^\dagger(\check{G})$ for the subgroup generated by σ . The associated L -group can be taken as ${}^L\mathcal{G} = \check{G} \rtimes \langle \sigma \rangle$. The Kloosterman sheaf $\text{Kl}_{L\mathcal{G}}(\phi)$ constructed in [Theorem 1](#) gives a monodromy representation

$$\varphi : \pi_1(\mathbb{P}^1_{\setminus\{0,\infty\}}, \bar{\eta}) \rightarrow {}^L\mathcal{G} = \check{G} \rtimes \langle \sigma \rangle.$$

The adjoint representation $\text{Ad} : \check{G} \rightarrow \text{GL}(\check{\mathfrak{g}})$ can be extended to a homomorphism

$$\text{Ad}_\sigma : {}^L\mathcal{G} \rightarrow \text{GL}(\check{\mathfrak{g}})$$

so that the Kloosterman sheaf $\text{Kl}_{L\mathcal{G}}(\phi)$ induces a local system $\text{Kl}_{L\mathcal{G}}^{\text{Ad}}(\phi)$.

We have the following predictions for the local monodromy of $\mathrm{Kl}_{L\mathcal{G}}(\phi)$. The tame monodromy at $\{0\}$ should be generated by the element $(u, \sigma) \in {}^L\mathcal{G} = \check{G} \rtimes \langle \sigma \rangle$, where $u \in \check{G}^\sigma$ is a principal unipotent element. The Swan conductor $\mathrm{Swan}_\infty(\mathrm{Kl}_{L\mathcal{G}}(\phi)) = -\chi_c(\mathbb{P}^1_{\setminus\{0, \infty\}}, \mathrm{Kl}_{L\mathcal{G}}^{\mathrm{Ad}}(\phi))$ should equal the rank of the neutral component of \check{G}^σ . Moreover, we also expect that the analog of [Corollary 2.15](#) holds for $\mathrm{Kl}_{L\mathcal{G}}(\phi)$, with the Coxeter element replaced by the σ -twisted Coxeter element $\mathrm{Cox}_\sigma \in W \times \{\sigma\} \subset W \rtimes \langle \sigma \rangle$ (see [\[36\]](#) and [\[35, §5\]](#)) and the Coxeter number replaced by the σ -twisted Coxeter number h_σ (the order of the Cox_σ).

As for the global monodromy, we expect that for $\mathrm{char}(k)$ not too small, the global geometric monodromy representation φ^{geo} for $\mathrm{Kl}_{L\mathcal{G}}(\phi)$ has Zariski dense image.

We observe from [Table 1](#) that for simply-laced split groups G not of type A_{2n} , the Zariski closure \check{G}_{geo} of the geometric monodromy of $\mathrm{Kl}_{\check{G}}(\phi)$ is smaller than \check{G} . Now pick a quasi-split form \mathcal{G} of G built out of a nontrivial $\sigma \in \mathrm{Aut}^\dagger(G)$ of order N . Then according to our expectation, the \check{G} -local system $[N]^*\mathrm{Kl}_{L\mathcal{G}}(\phi)$ on $\check{\mathbb{P}}^1_{\setminus\{0, \infty\}}$ (the N -th Kummer cover of $\mathbb{P}^1_{\setminus\{0, \infty\}}$, which is still isomorphic to $\mathbb{P}^1_{\setminus\{0, \infty\}}$) should have Zariski dense geometric monodromy in \check{G} . This compensates the smallness of \check{G}_{geo} for simply-laced G .

When G is of type A_{2n} , by [Corollary 6.5](#), φ extends to

$$\pi_1(\mathbb{P}^1_{\setminus\{0, \infty\}}, \bar{\eta}) \rightarrow \check{G} \rtimes \mu_2,$$

hence giving a $\check{G} \rtimes \mu_2$ -local system $\overline{\mathrm{Kl}}_{\check{G}}(\phi)$ on $\mathbb{P}^1_{\setminus\{0, \infty\}}/\mu_2$. On the other hand, let \mathcal{G} be the quasi-split unitary group given by the nontrivial $\sigma \in \mathrm{Aut}^\dagger(G)$. We have a Kloosterman sheaf $\mathrm{Kl}_{L\mathcal{G}}(\phi')$ on $\mathbb{P}^1_{\setminus\{0, \infty\}}$. After identifying ${}^L\mathcal{G}$ with $\check{G} \rtimes \mu_2$ and matching ϕ with ϕ' , we expect that $\mathrm{Kl}_{L\mathcal{G}}(\phi) \cong \overline{\mathrm{Kl}}_{\check{G}}(\phi')$ over $\mathbb{P}^1_{\setminus\{0, \infty\}} \otimes_k \bar{k} \cong (\mathbb{P}^1_{\setminus\{0, \infty\}}/\mu_2) \otimes_k \bar{k}$.

Appendix A. Proof of Proposition 1.1

In this appendix, we give a proof of [Proposition 1.1](#) on the geometry of moduli spaces of \mathcal{G} -bundles on \mathbb{P}^1 .

Proof. The proof of the first two claims uses the same argument as in [\[23, Prop. 3\]](#). We will first assume that the base field k is algebraically closed.

(A) Assume that $\mathcal{G} = \mathcal{T}$; i.e., $\mathcal{G}|\mathbb{P}^1_{\setminus\{0, \infty\}}$ is a torus. In this case, (2) has been proved in [\[25, Lemma 16\]](#). Let us prove (1). Since \mathcal{T} is abelian, for any \mathcal{T} -torsor \mathcal{P} , we have $H^1(\mathbb{P}^1, \mathcal{P} \times^{\mathcal{T}} \mathrm{Lie}(\mathcal{T})) = H^1(\mathbb{P}^1, \mathrm{Lie}(\mathcal{T}))$. We claim that this group vanishes. By construction we know that $\mathrm{Lie}(\mathcal{T}) \subset \mathrm{Lie}([N_*](\mathbb{G}_m^r \times \mathbb{P}^1))$ is a direct summand. Also $\mathrm{Lie}([N_*](\mathbb{G}_m^r \times \mathbb{P}^1)) = [N]_*\mathcal{O}_{\mathbb{P}^1}^r$ and $H^1(\mathbb{P}^1, [N]_*\mathcal{O}_{\mathbb{P}^1}) = H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = 0$. Thus, $H^1(\mathbb{P}^1, \mathcal{P} \times^{\mathcal{T}} \mathrm{Lie}(\mathcal{T})) = 0$

for any \mathcal{P} so that there are no nontrivial deformations of \mathcal{T} -bundles. Since k is algebraically closed, $H^1(k(\mathbb{P}^1), \mathcal{T}) = 0$; i.e., all \mathcal{T} -bundles are generically trivial. Thus the map $\bigoplus_{x \in \mathbb{P}^1} \mathcal{T}(K_x) / \mathcal{T}(\mathcal{O}_x) \rightarrow \text{Bun}_{\mathcal{T}}(k)$ is surjective. By construction, the isomorphism $\pi_0(\text{Bun}_{\mathcal{T}}) \cong \pi_1(\mathcal{T})_{\pi_1(\mathbb{G}_m)}$ is induced from the Kottwitz homomorphism $\pi_0(\text{Gr}_{\mathcal{T}, x}) \rightarrow \mathbb{X}_*(\mathcal{T})_{\text{Gal}(K_x^{\text{sep}}/K_x)}$. Thus for any x , the map $\mathcal{T}(K_x) \rightarrow \text{Bun}_{\mathcal{T}}(k)$ is surjective. This proves (1) in the case of tori. Finally (1) and (2) imply (4) for tori.

(B) In case $\mathcal{G}|_{\mathbb{P}^1 \setminus \{0, \infty\}}$ is a semisimple, simply connected group, statements (1) and (2) have been proved in [25, Th. 4], again assuming k to be algebraically closed.

(C) Suppose that the derived group $\mathcal{G}_{\text{der}}|_{\mathbb{P}^1 \setminus \{0, \infty\}}$ of $\mathcal{G}|_{\mathbb{P}^1 \setminus \{0, \infty\}}$ is simply connected, and denote $\mathcal{D}|_{\mathbb{P}^1 \setminus \{0, \infty\}} := \mathcal{G} / \mathcal{G}_{\text{der}}|_{\mathbb{P}^1 \setminus \{0, \infty\}}$. Denote by \mathcal{G}_{der} and \mathcal{D} the corresponding group schemes over \mathbb{P}^1 constructed as in Section 1.2. Then Haines and Rapoport show ([23, proof of Prop. 3]) that there is an induced exact sequence of group schemes over \mathbb{P}^1 :

$$1 \rightarrow \mathcal{G}_{\text{der}} \rightarrow \mathcal{G} \rightarrow \mathcal{D} \rightarrow 1.$$

By (A) any \mathcal{D} -bundle is trivial over $\mathbb{P}^1 \setminus \{x\}$, and by (B) we know the same for all forms of \mathcal{G}_{der} as well. This implies (1) for \mathcal{G} . Moreover, we know from [23] that $\mathcal{G}(k((s))) \rightarrow \mathcal{D}(k((s)))$ is surjective. Thus $\pi_0(\text{Bun}_{\mathcal{G}}) \cong \pi_0(\text{Bun}_{\mathcal{D}})$, which proves (2) for \mathcal{G} .

(D) For general groups \mathcal{G} we can choose a z -extension

$$1 \rightarrow \mathcal{Z}|_{\mathbb{P}^1 \setminus \{0, \infty\}} \rightarrow \mathcal{G}'|_{\mathbb{P}^1 \setminus \{0, \infty\}} \rightarrow \mathcal{G}|_{\mathbb{P}^1 \setminus \{0, \infty\}} \rightarrow 1$$

such that $\mathcal{Z}|_{\mathbb{P}^1 \setminus \{0, \infty\}}$ is an induced torus, which is induced from a covering $[N]: \mathbb{G}_m \rightarrow \mathbb{G}_m$ as in Section 1, and $\mathcal{G}'_{\text{der}}|_{\mathbb{P}^1 \setminus \{0, \infty\}}$ is simply connected. As in Section 1.2, the induced torus $\mathcal{Z}|_{\mathbb{P}^1 \setminus \{0, \infty\}}$ extends to \mathbb{P}^1 as the Weil restriction of a split torus via the map $[N]: \mathbb{P}^1 \rightarrow \mathbb{P}^1$ and $\mathcal{G}'|_{\mathbb{P}^1 \setminus \{0, \infty\}}$ extends naturally as to a group scheme \mathcal{G} over \mathbb{P}^1 as in Section 1.2. By the local computation in [23, d) in proof of Prop. 3], this defines an exact sequence

$$1 \rightarrow \mathcal{Z} \rightarrow \mathcal{G}' \rightarrow \mathcal{G} \rightarrow 1.$$

Since \mathcal{Z} is the Weil restriction of a split torus, via the tamely ramified, finite map $[N]: \mathbb{P}^1 \rightarrow \mathbb{P}^1$, Tsen's theorem implies that $H^2(\mathbb{P}^1, \mathcal{Z}) = 0$. Therefore, any \mathcal{G} bundle can be lifted to a \mathcal{G}' bundle. Since we know (1) for \mathcal{G}' , this proves (1) for \mathcal{G} . To prove (2) denote by $\mathcal{Z}_{\bar{\eta}}$ a geometric generic fiber of \mathcal{Z} . Under our assumption, $(\mathbb{X}_*(\mathcal{Z}_{\bar{\eta}}))_{\pi_1(\mathbb{P}^1 \setminus \{0, \infty\})}$ is torsion free, and therefore as in [23], we have an exact sequence

$$0 \rightarrow \mathbb{X}^*(\check{\mathcal{Z}}_{\bar{\eta}}^{\pi_1(\mathbb{P}^1 \setminus \{0, \infty\})}) \rightarrow \mathbb{X}^*(Z(\check{\mathcal{G}}'_{\bar{\eta}})^{\pi_1(\mathbb{P}^1 \setminus \{0, \infty\})}) \rightarrow \mathbb{X}^*(Z(\check{\mathcal{G}}_{\bar{\eta}})^{\pi_1(\mathbb{P}^1 \setminus \{0, \infty\})}) \rightarrow 0.$$

This implies (2).

To prove the third claim we follow the arguments of Harder [24], Ramanathan [34] or Faltings [14, Lemma 4]. Recall from Section 1.2 that we fixed $S \subset \mathcal{T} \subset \mathcal{B} \subset \mathcal{G}$, which are extensions of a maximal split torus, a maximal torus and a Borel subgroup over $\mathbb{P}^1_{\setminus\{0,\infty\}}$ to \mathbb{P}^1 .

We need to fix notation for dominant weights. First $\mathbb{X}^*(\mathcal{T}_{\bar{\eta}})^{\pi_1(\mathbb{G}_m)}_{\mathbb{Q}} = \mathbb{X}^*(S)_{\mathbb{Q}}$. Moreover, the relative roots $\Phi(\mathcal{G}, S)$ span the subspace of characters of S that are trivial on the center of \mathcal{G} . We denote by $\mathbb{X}_*(\mathcal{T}_{\bar{\eta}})_{\pi_1(\mathbb{G}_m),+} \subset \pi_0(\text{Bun}_{\mathcal{T}})$ the subset of $\gamma \in \mathbb{X}_*(\mathcal{T})_{\pi_1(\mathbb{G}_m)}$ such that for any positive, relative root $a \in \Phi(\mathcal{G}, S)^+$, we have $a(\gamma) \geq 0$.

First, we want to prove that

$$\mathcal{G}(k((s))) = \prod_{\lambda \in \mathbb{X}_*(\mathcal{T})_{\pi_1(\mathbb{G}_m),+}} \mathcal{G}(k[t]^{\lambda(s)} \mathcal{G}(k[[s]]).$$

From (1) we conclude that every \mathcal{G} -bundle is trivial outside any point. In particular, any \mathcal{G} bundle admits a reduction to \mathcal{B} . Let \mathcal{E} be a \mathcal{G} -bundle and choose a reduction $\mathcal{E}_{\mathcal{B}}$ of \mathcal{E} to \mathcal{B} .

For any character $\alpha: \mathcal{T} \rightarrow \mathbb{G}_m$, we denote by $\mathcal{E}_{\mathcal{B}}(\alpha)$ the associated line bundle on \mathbb{P}^1 . Since $\mathbb{X}^*(\mathcal{T})_{\mathbb{Q}}^{\pi_1(\mathbb{G}_m)} \cong \mathbb{X}^*(S)_{\mathbb{Q}}$, the degree $\text{deg}(\mathcal{E}_{\mathcal{B}}(\alpha)) \in \mathbb{Q}$ is also defined for $\alpha \in \mathbb{X}^*(S)$.

We claim that if for all positive, simple roots $a_i \in \Phi(\mathcal{G}, S)^+$ we have $\text{deg}(\mathcal{E}_{\mathcal{B}}(a_i)) \geq 0$, then the bundle $\mathcal{E}_{\mathcal{B}}$ admits a reduction to \mathcal{T} . To show this, denote by $\mathcal{U} \subset \mathcal{B}$ the closure of the unipotent radical of $\mathcal{B}|_{\mathbb{P}^1_{\setminus\{0,\infty\}}}$ and $\mathcal{E}_{\mathcal{T}} := \mathcal{E}_{\mathcal{B}}/\mathcal{U}$ the induced \mathcal{T} -bundle. In order to show that $\mathcal{E}_{\mathcal{B}}$ is induced from $\mathcal{E}_{\mathcal{T}}$, we only need to show that $H^1(\mathbb{P}^1, \mathcal{E}_{\mathcal{T}} \times^{\mathcal{T}} \mathcal{U}) = 0$. The group \mathcal{U} has a filtration such that the subquotients are given by root subgroups.

Consider a positive, relative root $a \in \Phi(\mathcal{G}, S)$. The root subgroup \mathcal{U}_a is a direct summand of $[N]_*(\oplus \mathcal{U}_{a'})$ where the sum is over those roots $a' \in \Phi(\mathcal{G}, S)$ that restrict to a on S . Thus \mathcal{U}_a is a direct summand of a vector bundle \mathcal{V} satisfying $H^1(\mathbb{P}^1, \mathcal{V}) = 0$.

Similarly, $\mathcal{E}_{\mathcal{T}} \times^{\mathcal{T}} \mathcal{U}_a$ is a direct summand of $\mathcal{E}_{\mathcal{T}} \times^{\mathcal{T}} [N]_*(\oplus \mathcal{U}_{a'})$. Since $[N]^* \mathcal{E}_{\mathcal{T}}$ is $\pi_1(\mathbb{G}_m)$ -invariant, this implies that H^1 of this bundle is 0 if $\text{deg}(\mathcal{E}_{\mathcal{B}}(a)) \geq 0$. Thus we also find $H^1(\mathbb{P}^1, \mathcal{E}_{\mathcal{T}} \times^{\mathcal{T}} \mathcal{U}) = 0$.

If the reduction $\mathcal{E}_{\mathcal{B}}$ does not satisfy the condition $\text{deg}(\mathcal{E}_{\mathcal{B}}(a_i)) \geq 0$ for some simple root a_i , we want to modify the reduction $\mathcal{E}_{\mathcal{B}}$. Consider the parabolic subgroup $\mathcal{P}_{a_i} \subset \mathcal{G}$ generated by \mathcal{B} and \mathcal{U}_{-a_i} . The root subgroups $\mathcal{U}_{\pm a_i}$ define a subgroup \mathcal{L} of \mathcal{P} such that the simply connected cover of \mathcal{L} (extended from $\mathbb{P}^1_{\setminus\{0,\infty\}}$ to \mathbb{P}^1 as in §1.2) is either isomorphic to $[n]_* \text{SL}_2$ or isomorphic to $[n]_* \text{SU}_3$ for some n dividing N [7, §4.1.4]. The semisimple quotient $\mathcal{P}^{ss}/Z(\mathcal{P}^{ss})$ is isomorphic to \mathcal{L}^{ad} and furthermore $\mathcal{P}/\mathcal{B} \cong \mathcal{L}/(\mathcal{L} \cap \mathcal{B})$. We claim that the result holds for \mathcal{L}^{ad} bundles because these bundles can be described in terms of vector bundles. If $\mathcal{L}^{\text{ad}} = [n]_* \text{PGL}_2$, then the result follows from the result

for vector bundles of rank 2. The case of unitary groups is similar; we will explain it below in [Lemma A.1](#).

Thus we can find a new reduction $\mathcal{E}'_{\mathcal{B}}$ to \mathcal{B} such that $\mathcal{E}_{\mathcal{P}_{a_i}}$ is unchanged, but $\mathcal{E}'_{\mathcal{B}}(a_i) \geq 0$. This implies that for the fundamental weights ϵ_k (multiples of the determinant of the adjoint representation of the maximal parabolic subgroups \mathcal{P}^k , generated by \mathcal{B} and all \mathcal{U}_{a_j} with $j \neq k$), the degree of $\mathcal{E}_{\mathcal{B}}(\epsilon_k)$ for $k \neq i$ is unchanged but the degree $\mathcal{E}'_{\mathcal{B}}(\epsilon_i)$ is larger than the degree of $\mathcal{E}_{\mathcal{B}}(\epsilon_i)$.

However, $\mathcal{E}'_{\mathcal{B}}(\epsilon_i)$ is a subbundle of $\wedge^{\dim(\text{Lie}(\mathcal{P}^i))} \mathcal{E}(\text{Lie}(\mathcal{G}))$, so the degree of all of these line bundles is bounded. Thus the procedure must eventually produce a \mathcal{B} -reduction satisfying $\deg(\mathcal{E}_{\mathcal{B}}(a_i)) \geq 0$ for all simple roots. This proves (3).

Let us deduce the Birkhoff decomposition (4). In the case of constant groups, this is usually deduced from the decomposition $G = B^-W_0B$ and (3). For general \mathcal{G} , the analog of this is provided by [7, Th. 4.6.33]. Denote by $G_0 := \mathcal{G}_0^{\text{red}}$ the reductive quotient of the fiber of \mathcal{G} over 0 and $B_0 \subset G_0$ the image of \mathcal{B} . Denote by U_0 the unipotent radical of B_0 so that $G_0 = U_0W_0B_0$. The quoted result says that the inverse image of B_0 in \mathcal{G} is $I(0)$. By construction of \mathcal{G} , elements of $U_0(k)$ can be lifted to $\mathcal{U}(\mathbb{P}^1)$. Thus we have $\mathcal{G}(k[[s]]) = U_0W_0I(0)$.

Similarly, by our construction of \mathcal{G} , the evaluation $\mathcal{G}(\mathbb{A}^1) \rightarrow G_0(k)$ is surjective so that $\mathcal{G}(\mathbb{A}^1) = I^-(0)W_0U_0$.

For dominant $t \in \mathcal{T}(k((s)))$ and $b \in U_0$, we have $t^{-1}bt \in I(0)$. Thus using (3), we find $\mathcal{G}(k((s))) = I^-(0)\mathcal{T}(k((s)))\mathcal{G}(k[[s]])$. Now we want to argue in the same way, decomposing $\mathcal{G}(k[[s]])$. For any t , we can choose $w \in W_0$ such that wtw^{-1} is dominant. Then choose $B'_0 \subset G_0$ as wB_0w^{-1} and write $G_0 = B'_0W_0B_0$. Then we can use the same argument as before to deduce (4) (still assuming k to be algebraically closed).

Let us deduce the case that k is a finite field. First assume that \mathcal{G} splits over the totally ramified covering $[N]: \mathbb{P}^1 \rightarrow \mathbb{P}^1$. The embedding $\widetilde{W} \rightarrow \text{Gr}_{\mathcal{G},x}$ is then defined over k so that all geometric points of $\text{Bun}_{\mathcal{G}}$ are defined over k . Once we show the following claim, (1), (3) and (4) follow over k by Lang's theorem.

CLAIM. The automorphism group of any $\mathcal{G}(0,0)$ -bundle is connected.

Proof of the claim. We first treat the case $\mathcal{G} = \mathcal{T}$. Since \mathcal{T} is constructed as a subgroup of a Weil restriction $[N]_*(\mathbb{G}_m^r)$, we have

$$H^0(\mathbb{P}^1, \mathcal{T}) \hookrightarrow H^0(\mathbb{P}^1, [N]_*(\mathbb{G}_m^r)),$$

and the same will hold after any base change $S \rightarrow \text{Spec}(k)$. Thus, if we let $p: \mathbb{P}^1 \rightarrow \text{Spec}(k)$ denote the projection, we find that $p_*\mathcal{T} \hookrightarrow p_*([N]_*(\mathbb{G}_m^r))$ as sheaves in the fppf-topology, both of which are represented by affine group schemes over k . Since the evaluation at $0 \in \mathbb{P}^1$ defines an isomorphism

$p_*([N]_*(\mathbb{G}_m^r)) \xrightarrow{\sim} ([N]_*\mathbb{G}_m^r)_0^{\text{red}} = \mathbb{G}_m^r$, this implies that also for \mathcal{T} the evaluation $p_*\mathcal{T} \rightarrow \mathcal{T}_0^{\text{red}}$ is injective. Since \mathcal{T} was constructed as $(([N]_*(\mathbb{G}_m^r))^\sigma)^\circ$ we also see that $\mathcal{T}_0^{\text{red}} \hookrightarrow ([N]_*\mathbb{G}_m^r)_0^{\text{red}} = \mathbb{G}_m^r \cong p_*([N]_*(\mathbb{G}_m^r))$ defines a section $\mathcal{T}_0^{\text{red}} \rightarrow p_*\mathcal{T}$ so that $p_*(\mathcal{T}) \cong \widetilde{\mathcal{T}_0^{\text{red}}}$, which is a connected group.

In general, for any $w \in \widetilde{W}$, the automorphism group of the corresponding $\mathcal{G}(0,0)$ -bundle over \bar{k} is $I^-(0) \cap wI(0)w^{-1}$. This group admits $p_*(\mathcal{T})$ as a quotient, and the kernel is a product of root subgroups for affine roots, which are connected as well. □

The general case follows from the previous case (i.e., \mathcal{G} splits by pulling back via $[N]$) by Galois descent, using [23, Remark 9] that the Iwahori-Weyl group can be computed as the Galois invariants in the Iwahori-Weyl group over the separable closure of k . Part (5) follows from the definition of $\mathcal{G}(m,n)$. □

In the above proof we used the following special case. Denote by SU_3 the quasi-split unitary group for the covering [2]: $\mathbb{P}^1 \rightarrow \mathbb{P}^1$. This can be described as the special unitary group for the hermitian form $h(x_1, x_2, x_3) = x_1x_3^\sigma + x_2x_2^\sigma + x_3x_1^\sigma$. Denote by PSU_3 the corresponding adjoint group.

LEMMA A.1. *Any PSU_3 bundle \mathcal{P} has a reduction $\mathcal{P}_{\mathcal{B}}$ to \mathcal{B} such that for the positive root α we have $\text{deg}(\mathcal{P}_{\mathcal{B}}(\alpha)) \geq 0$.*

Proof. Define GU_3 to be the group obtained from SU_3 by extending the center of SU_3 to $[2]_*\mathbb{G}_m$ so that there is an exact sequence $1 \rightarrow [2]_*\mathbb{G}_m \rightarrow \text{GU}_3 \rightarrow \text{PSU}_3 \rightarrow 1$. Again, every PSU_3 bundle is induced from a GU_3 -bundle. Such a bundle can be viewed as a rank-3 vector bundle \mathcal{E} on the covering $\mathbb{P}^1 \xrightarrow{[2]} \mathbb{P}^1$ with a hermitian form with values in a line bundle of the form $[2]^*\mathcal{L}$. In this case, to give a reduction to \mathcal{B} it is sufficient to give an isotropic line subbundle $\mathcal{E}_1 \rightarrow \mathcal{E}$, as this defines a flag $\mathcal{E}_1 \subset \mathcal{E}_1^\perp \subset \mathcal{E}$. If \mathcal{E} is not semistable, then the canonical subbundle of \mathcal{E} defines an isotropic subbundle of positive degree, so in this case a reduction exists. If \mathcal{E} is semistable, then the hermitian form defines a global isomorphism $\mathcal{E} \xrightarrow{\cong} \sigma^*\mathcal{E}^\vee \otimes [2]^*\mathcal{L}$. But such an isomorphism must be constant so that we can find an isotropic subbundle of degree $\frac{\text{deg}(\mathcal{E})}{3}$. □

Appendix B. Geometric Satake and pinnings of the dual group

We first need some general properties of tensor functors. Let \mathcal{C} be a rigid tensor category with a fiber functor $\omega : \mathcal{C} \rightarrow \text{Vec}_F$, where F is a field. Let $H = \underline{\text{Aut}}^\otimes(\omega)$ be the algebraic group over F determined by (\mathcal{C}, ω) .

For any F -algebra R , let $\omega_R : \mathcal{C} \xrightarrow{\omega} \text{Vec}_F \xrightarrow{\otimes_F R} \text{Mod}_R$. Let $\text{Aut}^\otimes(\mathcal{C}, \omega_R)$ be isomorphism classes of pairs (σ, α) , where $\sigma : \mathcal{C} \xrightarrow{\sim} \mathcal{C}$ is a tensor auto-equivalence, and $\alpha : \omega_R \circ \sigma \Rightarrow \omega_R$ is a natural isomorphism of functors. Then

$\text{Aut}^\otimes(\mathcal{C}, \omega_R)$ has a natural group structure. Denote by $\underline{\text{Aut}}^\otimes(\mathcal{C}, \omega)$ the functor $R \mapsto \text{Aut}^\otimes(\mathcal{C}, \omega_R)$, which defines a fppf sheaf of groups. On the other hand, let $\underline{\text{Aut}}(H)$ be the fppf sheaf of automorphisms of the pro-algebraic group H over F .

LEMMA B.1. *There is a natural isomorphism of fppf sheaves of groups $\underline{\text{Aut}}^\otimes(\mathcal{C}, \omega) \xrightarrow{\sim} \underline{\text{Aut}}(H)$. In particular, we have a natural isomorphism of groups $\text{Aut}^\otimes(\mathcal{C}, \omega) \xrightarrow{\sim} \text{Aut}(H)$, which induces an isomorphism of groups $[\text{Aut}^\otimes(\mathcal{C})] \xrightarrow{\sim} \text{Out}(H)$, where $[\text{Aut}^\otimes(\mathcal{C})]$ is the set of isomorphism classes of tensor auto-equivalences of \mathcal{C} .*

On the level of F -points, a pair $(\sigma, \alpha) \in \text{Aut}^\otimes(\mathcal{C}, \omega)$ gives the following automorphism of $H = \underline{\text{Aut}}^\otimes(\omega)$: it sends $h : \omega \Rightarrow \omega$ to the natural transformation

$$\omega \xrightarrow{\alpha} \omega \circ \sigma \xrightarrow{\text{hoid}_\sigma} \omega \circ \sigma \xrightarrow{\alpha^{-1}} \omega.$$

More generally, suppose we are given a tensor functor $\Phi : \mathcal{C} \rightarrow \mathcal{C}'$ into another rigid tensor category \mathcal{C}' ; we can similarly define a sheaf of groups $\underline{\text{Aut}}^\otimes(\mathcal{C}, \Phi)$.

Let $\omega' : \mathcal{C}' \rightarrow \text{Vec}$ be a fiber functor and $\omega = \omega' \circ \Phi$. Then there is a natural homomorphism $\omega'_* : \underline{\text{Aut}}^\otimes(\mathcal{C}, \Phi) \rightarrow \underline{\text{Aut}}^\otimes(\mathcal{C}, \omega) = \underline{\text{Aut}}(H)$ by sending $(\sigma, \alpha) \in \underline{\text{Aut}}^\otimes(\mathcal{C}, \Phi)$ to $(\sigma, \text{id}_\omega \circ \alpha) \in \underline{\text{Aut}}^\otimes(\mathcal{C}, \omega)$. In other words, $\underline{\text{Aut}}^\otimes(\mathcal{C}, \Phi)$ acts on the pro-algebraic group H . On the other hand, we have natural homomorphism of pro-algebraic groups $\Phi^* : H' = \underline{\text{Aut}}^\otimes(\omega') \rightarrow H = \underline{\text{Aut}}^\otimes(\omega)$.

LEMMA B.2. *The homomorphism $\Phi^* : H' \rightarrow H$ factors through $H' \rightarrow H \underline{\text{Aut}}^\otimes(\mathcal{C}, \Phi) \subset H$.*

Now we consider the normalized semisimple Satake category \mathcal{S} in Section 2.3. Following [31] and [18], we use the global section functor h to define $\check{G} = \underline{\text{Aut}}^\otimes(h)$ and get the geometric Satake equivalence $\mathcal{S} \cong \text{Rep}(\check{G})$.

LEMMA B.3.

- (1) *There is a natural homomorphism $\text{Aut}(G) \rightarrow \text{Aut}^\otimes(\mathcal{S}, h) \cong \text{Aut}(\check{G})$ that factors through $\tilde{\iota} : \text{Out}(G) \rightarrow \text{Aut}(\check{G})$.*
- (2) *There is a natural pinning $\dagger = (\check{B}, \check{T}, \{\mathfrak{x}_{\alpha_i^\vee}\})$ of \check{G} preserved by the $\text{Out}(G)$ -action via $\tilde{\iota}$. Let $\text{Aut}^\dagger(\check{G})$ be the automorphism group of \check{G} fixing this pinning. Then $\tilde{\iota}$ induces an isomorphism $\iota : \text{Out}(G) \xrightarrow{\sim} \text{Aut}^\dagger(\check{G})$.*

Proof. (1) The $\text{Aut}(G)$ -action on (\mathcal{S}, h) is induced from its action on Gr_G . Since objects in \mathcal{S} carry G -equivariant structures under the conjugation action of G on Gr_G , inner automorphisms of G acts trivially on (\mathcal{S}, h) ; i.e., the $\text{Aut}(G)$ -action on (\mathcal{S}, h) factors through $\text{Out}(G)$.

(2) We first need to exhibit a pinning of \check{G} that is preserved by the $\text{Aut}(G)$ -action. For this, we need to give a maximal torus $\check{T} \subset \check{G}$, a cocharacter $2\rho \in \mathbb{X}^*(T) = \mathbb{X}_*(\check{T})$ (half the sum of positive coroots in the pinning) and a principal nilpotent element $e \in \check{\mathfrak{g}}$ (the sum of simple root vectors). Equivalently, we need to

(P1) Factor the fiber functor h into a tensor functor

$$h = \bigoplus_{\mu \in \mathbb{X}_*(T)} h^\mu : \mathcal{S} \rightarrow \text{Vec}^{\mathbb{X}_*(T)} \xrightarrow{\text{forget}} \text{Vec}.$$

(P2) Find a tensor derivation $e : h \rightarrow h$ (i.e., $e(K_1 * K_2) = e(K_1) \otimes \text{id}_{K_2} + \text{id}_{K_1} \otimes e(K_2)$) that sends h^μ to $\bigoplus_i h^{\mu+\alpha_i}$ such that for each simple root α_i , the component $h^\mu \rightarrow h^{\mu+\alpha_i}$ is nonzero as a functor.

The factorization h^μ is given by Mirković-Vilonen’s “weight functors” [31, Ths. 3.5, 3.6]. They also proved that $h = \bigoplus_i h^i$ (where h^i is the sum of h^μ with $\langle 2\rho, \mu \rangle = i$) coincides with the cohomological grading of $h = H^*(\text{Gr}_G, -)$. The tensor derivation e is given by the cup product with $c_1(\mathcal{L}_{\det}) \in H^2(\text{Gr}_G, \overline{\mathbb{Q}}_\ell)$, where \mathcal{L}_{\det} is the determinant line bundle on Gr_G .

The action of $\text{Aut}^\dagger(G) \subset \text{Aut}(G)$ on \mathcal{S} permutes the weight functors h^μ in the same way as it permutes $\mu \in \mathbb{X}_*(T)$, preserves the $\langle 2\rho, \mu \rangle$ (hence preserves $2\rho \in \mathbb{X}^*(T) = \mathbb{X}_*(\check{T})$) and commutes with $e = c_1(\mathcal{L}_{\det})$. Therefore, the $\text{Aut}^\dagger(G) \xrightarrow{\sim} \text{Out}(G)$ -action on \mathcal{S} preserves the above pinning.

An element $\sigma \in \text{Aut}^\dagger(G)$ induces a dual automorphism $\check{\sigma}$ of the Dynkin diagram of \check{G} . Since $\sigma^* \text{IC}_\mu \cong \text{IC}_{\sigma^{-1}(\mu)}$, the self-equivalence σ^* of $\mathcal{S} \cong \text{Rep}(\check{G})$ is isomorphic to the self-equivalence of $\text{Rep}(\check{G})$ induced by the pinned automorphisms of \check{G} given by the dual automorphism $\check{\sigma}^{-1}$ on the Dynkin diagram of \check{G} . This proves

$$\text{Out}(G) \rightarrow [\text{Aut}^\otimes(\mathcal{S})] \cong [\text{Aut}^\otimes(\text{Rep}(\check{G}))] \cong \text{Out}(\check{G})$$

is an isomorphism. (The last isomorphism follows from Lemma B.1.) Hence $\iota : \text{Out}(G) \rightarrow \text{Aut}^\dagger(\check{G})$ is also an isomorphism. \square

For our purpose in Section 6, we shall also need a different fiber functor ω_ϕ defined in (2.13). Recall that S_ϕ is the stabilizer of ϕ under the action of $T \rtimes \text{Aut}^\dagger(G) \times \mathbb{G}_m^{\text{rot}}$, and $S_\phi^1(k) = \ker(S_\phi(k) \rightarrow \mathbb{G}_m^{\text{rot}}(k))$. Recall from (6.4) that we have an action of $S_\phi^1(k)$ on ${}^\phi\check{G}$.

LEMMA B.4. *There is a natural pinning $\ddagger = (\check{B}, \check{T}, \{\phi_{\mathbb{X}\alpha_i}\})$ of ${}^\phi\check{G}$, which is preserved by the $S_\phi^1(k)$ -action.*

Proof. Using (4.4), we can rewrite the fiber functor ω_ϕ as

$$\mathcal{S} \xrightarrow{\Psi} \text{Perv}_{I_0}(\text{Fl}_G) \xrightarrow{V_\phi} \text{Vec},$$

where $\text{Fl}_G = G((t))/I_0$ is the affine flag variety at $\{0\}$,

$$\Psi : \mathcal{S} \rightarrow \text{Perv}(I_0 \backslash G((t))/I_0) = \text{Perv}_{I_0}(\text{Fl})$$

is the nearby cycles functor of Gaitsgory [16] and $V_\phi(K) := R\Gamma_c(\text{Fl}_G, K \otimes \text{pr}_1^* A_\phi)$ as in (4.4). To exhibit a pinning of ${}^\phi \check{G}$, we need to find analogs of (P1) and (P2) as in the proof of Lemma B.3(2).

According to Arkhipov-Bezrukavnikov [1, Th. 4], each object $\Psi(K)$ admits a $\mathbb{X}_*(T)$ -filtration with Wakimoto sheaves as associated graded pieces. More precisely, they constructed a functor

$$(B.5) \quad \bigoplus_{\mu \in \mathbb{X}_*(T)} W^\mu \circ \Psi(-) : \mathcal{S} \rightarrow \text{Vec}^{\mathbb{X}_*(T)}.$$

Here for $\Psi(K) \in \text{Perv}_{I_0}(\text{Fl}_G)$ with Wakimoto filtration $\Psi(K)_{\leq \mu}$, we write $\text{gr}_\mu \Psi(K) = \Psi(K)_{\leq \mu} / \Psi(K)_{< \mu} = J_\mu \otimes W^\mu(K)$, where J_μ is the Wakimoto sheaf [1, §3.2] and $W^\mu(K)$ is a vector space with Frobenius action. In [1, Th. 6], it is proved that (B.5) is tensor.

For each μ , let $j_\mu : \text{Fl}_\mu = I_0 t^\mu I_0 / I_0 \hookrightarrow \text{Fl}_G$ be the inclusion. We fix the Frobenius structure of J_μ in the following way: for μ regular dominant, let $J_\mu = j_{\mu,*} \overline{\mathbb{Q}}_\ell\langle 2\rho, \mu \rangle$; for μ regular anti-dominant, let $J_\mu = j_{\mu,!} \overline{\mathbb{Q}}_\ell[\langle 2\rho, \mu \rangle]$; for general $\mu = \mu_1 + \mu_2$ where μ_1 is regular dominant and μ_2 is regular anti-dominant, let $J_\mu = J_{\mu_1} \overset{I_0}{*} J_{\mu_2}$ (where $\overset{I_0}{*}$ is the convolution on Fl_G). It follows from [1, Lemma 8, Cor. 1] that J_μ is well defined. This normalization makes sure that in the composition series of J_μ , $\delta = \text{IC}_1$ appears exactly once, with multiplicity space $\overline{\mathbb{Q}}_\ell$ as a trivial Frobenius module. (See [1, Lemma 3(a)], with obvious adjustment to the mixed setting.) Since all $\text{IC}_{\tilde{w}}$ are killed by V_ϕ except $\tilde{w} = 1$, we conclude that

$$(B.6) \quad V_\phi(J_\mu) = \overline{\mathbb{Q}}_\ell \text{ as a trivial Frobenius module for all } \mu \in \mathbb{X}_*(T).$$

CLAIM. For $K \in \mathcal{S}$, $W^\mu(K)$ is pure of weight $\langle 2\rho, \mu \rangle$. In fact, we have a natural isomorphism of functors $h^\mu \cong W^\mu$. (h^μ is the weight functor in [31, Ths. 3.5, 3.6], which was used in the proof of Lemma B.3.)

Proof. We first recall the definition of the weight functors in [31]. For every $\mu \in \mathbb{X}_*(T)$, let $\mathfrak{S}_\mu \subset \text{Gr}_G$ be the $U((t))$ -orbit containing t^μ . The weight function defined in loc. cit. is $h^\mu(K) = H_c^*(\mathfrak{S}_\mu, K)$, which is concentrated in degree $\langle 2\rho, \mu \rangle$.

Let $\pi : \text{Fl}_G \rightarrow \text{Gr}_G$ be the projection. Then $\pi^{-1}(\mathfrak{S}_\mu) = \sqcup_{w \in W} \tilde{\mathfrak{S}}_{\mu w}$, where $\tilde{\mathfrak{S}}_{\mu w} \subset \text{Fl}_G$ is the $U((t))$ -orbit containing $t^\mu w$. We have natural isomorphisms (for $K \in \mathcal{S}$)

$$h^\mu(K) = H_c^*(\mathfrak{S}_\mu, \pi_! \Psi(K)) \cong H_c^*(\pi^{-1}(\mathfrak{S}_\mu), \Psi(K)) \xrightarrow{\sim} H_c^*(\tilde{\mathfrak{S}}_\mu, \Psi(K)) \xrightarrow{\sim} W^\mu(K).$$

Here, the first equality follows from $\pi_! \Psi(K) = K$ and the last two isomorphisms follow from [1, Th. 4(2)] (with extra care about the Frobenius structure).

Since $h^i(K) = H^i(\mathrm{Gr}_G, K)$ is pure of weight i and $h^\mu(K)$ is a direct summand of $h^{\langle 2\rho, \mu \rangle}$, $h^\mu(K)$ is pure of weight $\langle 2\rho, \mu \rangle$. Hence, $W^\mu(K)$ is also pure of weight $\langle 2\rho, \mu \rangle$. \square

Now we construct a natural isomorphism $\bigoplus_\mu W^\mu \cong \omega_\phi$. For each $K \in \mathcal{S}$, the Wakimoto filtration on $\Psi(K)$ gives a spectral sequence calculating $V_\phi(\Psi(K))$ with E_1 -page $V_\phi(\mathrm{gr}_\mu \Psi(K))$. By (B.6),

$$(B.7) \quad V_\phi(\mathrm{gr}_\mu \Psi(K)) = V_\phi(J_\mu) \otimes W^\mu(K) = W^\mu(K)$$

is concentrated in degree 0; the spectral sequence degenerates at E_1 . The limit of the spectral sequence gives a Wakimoto filtration on the Frobenius module $\omega_\phi(K) = V_\phi(\Psi(K))$, which we denote by $w_{\leq \mu}$. By the claim and (B.7), this filtration refines the weight filtration $w_{\leq i}$ on $\omega_\phi(K)$:

$$\mathrm{gr}_i^w \omega_\phi(K) = \bigoplus_{\langle 2\rho, \mu \rangle = i} \mathrm{gr}_\mu^w \omega_\phi(K).$$

Since $\omega_\phi(K)$ is a Frobenius module in $\overline{\mathbb{Q}}_\ell$ -vector spaces, the weight filtration splits canonically. Therefore, the Wakimoto filtration $w_{\leq \mu} \omega_\phi(K)$ also splits canonically. This gives a canonical isomorphism $\bigoplus_\mu W^\mu \cong \omega_\phi$.

Finally, the principal nilpotent element is given by the logarithm of the monodromy action (of a topological generator of the tame inertia group at 0) on the nearby cycles ([16, Th. 2]) $M_K : \Psi(K) \rightarrow \Psi(K)(-1)$.

The action of $S_\phi^1(k)$ commutes with the nearby cycle functor and therefore commutes with the monodromy M_K . It permutes the Wakimoto sheaves, hence permutes the functors W^μ through the action of $S_\phi^1(k) \rightarrow \mathrm{Aut}^\dagger(G)$ on $\mathbb{X}_*(T)$, and it preserves $\langle 2\rho, \mu \rangle$. Therefore the $S_\phi^1(k)$ -action preserves the pinning \ddagger . \square

Appendix C. Quasi-minuscule combinatorics

We assume G is almost simple of rank at least 2. Let θ be the highest root (which is a long root) and θ^\vee the corresponding coroot (which is a short coroot). The set of roots Φ is partitioned into $\Phi_n^\theta = \{\alpha \in \Phi \mid \langle \alpha, \theta^\vee \rangle = n\}$, $n = 0, \pm 1, \pm 2$, with $\Phi_{\pm 2}^\theta = \{\pm \theta\}$. The roots in Φ_1^θ are coupled into pairs (α, β) with $\alpha + \beta = \theta$.

Let V_{θ^\vee} be the irreducible representation of \check{G} with highest weight θ^\vee . The nonzero weights of V_{θ^\vee} are the short roots of \check{G} , each with multiplicity 1. Let $\{e = \sum_i x_i, 2\rho, f = \sum_i y_i\} \in \check{\mathfrak{g}}$ be the principal \mathfrak{sl}_2 -triple, where $x_i \in \check{\mathfrak{g}}_{\alpha_i^\vee}, y_i \in \check{\mathfrak{g}}_{-\alpha_i^\vee}$ are nonzero and $2\rho \in \check{\mathfrak{t}}$ is the sum of positive coroots of \check{G} .

LEMMA C.1. *The following numbers are the same:*

- (1) $\dim V_{\theta^\vee}(0)$, where $V_{\theta^\vee}(0)$ is the zero weight space under the 2ρ -action, and is also the zero weight space under the \check{T} -action;
- (2) $\dim V_{\theta^\vee}^e$, where $V_{\theta^\vee}^e = \ker(\text{ad}(e)|V_{\theta^\vee})$;
- (3) the number of short simple roots of \check{G} (the short rank $r_s(\check{G})$);
- (4) $\#(W\theta^\vee)/h = \#\{\text{short roots of } \check{G}\}/h$. (Here h is the Coxeter number of W .)

Proof. Under the principal \mathfrak{sl}_2 action, V_{θ^\vee} can be decomposed as a sum of irreducible representations of \mathfrak{sl}_2 :

$$V_{\theta^\vee} = \sum_{i=1}^{r_s(\check{G})} \text{Sym}^{2\ell_i}(\text{Std}),$$

where Std is the 2-dimensional representation of \mathfrak{sl}_2 . Since the weights of the 2ρ -action on V_{θ^\vee} are even, only even symmetric powers of Std appear in V_{θ^\vee} . Since each $\text{Sym}^{2\ell_i}(\text{Std})$ contributes 1-dimension to both $V_{\theta^\vee}(0)$ and $V_{\theta^\vee}^e$, we have $r_s(\check{G}) = \dim V_{\theta^\vee}(0) = \dim V_{\theta^\vee}^e$. This proves the equality of the numbers in (1) and (2).

Let $V_{\theta^\vee}(n)$ be the weight n -eigenspace of the ρ -action. Then

$$V_{\theta^\vee}(n) = \sum_{\alpha^\vee \text{ short}, \langle \rho, \alpha^\vee \rangle = n} V_{\theta^\vee}(\alpha^\vee).$$

In particular, $\dim V_{\theta^\vee}(1)$ is the number of short simple roots of \check{G} . The map $e : V_{\theta^\vee}(0) \rightarrow V_{\theta^\vee}(1)$ is clearly surjective. It is also injective, because if $v \in V_{\theta^\vee}(0)$, $ev = 0$ means $\check{\mathfrak{g}}_{\alpha_i^\vee} v = 0$ for all simple α_i^\vee ; i.e., v is a highest weight, a contradiction. This proves

$$(C.2) \quad \dim V_{\theta^\vee}(0) = \dim V_{\theta^\vee}(1) = r_s(\check{G}).$$

It remains prove that (4) is the same as the rest. The argument is similar to that of [29, §6.7], where Kostant considered the adjoint representation instead of V_{θ^\vee} . We only give a sketch. Let ζ be a primitive h -th root of unity, and let $P = \rho(\zeta) \in \check{G}$. Let γ^\vee be the highest root of \check{G} and $z = e + \mathfrak{x}_{-\gamma^\vee}$. Then z is a regular semisimple element in $\check{\mathfrak{g}}$ and $\text{Ad}(P)z = \zeta z$. Let \check{G}_z be the centralizer of z (which is a maximal torus). Then $P \in N_{\check{G}}(\check{G}_z)$, and its image in the Weyl group is a Coxeter element. Choosing a basis u_i for the highest weight line in $\text{Sym}^{2\ell_i}(\text{Std}) \subset V_{\theta^\vee}$, there exists a unique $v_i \in V_{\theta^\vee}(\ell_i - h)$ such that $u_i + v_i \in V_{\theta^\vee}^z$ (kernel of the z -action on V_{θ^\vee}). Then $\{u_i + v_i\}_{1 \leq i \leq r_s(\check{G})}$ form a basis of $V_{\theta^\vee}^z$, with eigenvalues ζ^{ℓ_i} under the action of $\text{Ad}(P)$.

The representation V_{θ^\vee} of \check{G} is clearly self-dual. According to zero and nonzero weights under $\check{\mathfrak{g}}_z$, we can write $V_{\theta^\vee} = V_{\theta^\vee}^z \oplus V'$ as $N_{\check{G}}(\check{G}_z)$ -modules. Any self-duality $V_{\theta^\vee} \cong V_{\theta^\vee}^*$ of \check{G} -modules necessarily restricts to a self-duality $V_{\theta^\vee}^z \cong (V_{\theta^\vee}^z)^*$ as $N_{\check{G}}(\check{G}_z)$ -modules. Hence the eigenvalues of $\text{Ad}(P)$ on $V_{\theta^\vee}^z$ are

invariant under inversion; i.e., the multi-set $\{\zeta^{\ell_i}\}_{1 \leq i \leq r_s(\check{G})}$ is invariant under inversion. This implies $\sum_i \ell_i = r_s(\check{G})h/2$. Hence $\dim V_{\theta^\vee} = \sum_i (2\ell_i + 1) = (h + 1)r_s(\check{G})$. Since the nonzero weight spaces of V_{θ^\vee} are indexed by short roots of \check{G} , hence by the orbit $W\theta^\vee$, we get $\#(W\theta^\vee) = \dim V_{\theta^\vee} - \dim V_{\theta^\vee}(0) = hr_s(\check{G})$. This proves that (4) coincides with the rest of the numbers. \square

Appendix D. The adjoint Schubert variety for G_2

In this section we assume $\text{char}(k) > 3$. Recall (see tables in [8]) that G_2 has four nonregular unipotent orbits:

- the *subregular* orbit containing a generic element in the unipotent radical of $P_{-\theta}$, which has dimension 10;
- the orbit containing $U_{-\gamma}^\times$, which has dimension 8;
- the orbit containing $U_{-\theta}^\times$, which has dimension 6;
- the identity orbit, which has dimension 0.

The G -orbits of $\text{Gr}_{\leq \gamma^\vee}^{\text{triv}}$ for G_2 turns out to be closely related to these unipotent orbits. More precisely,

LEMMA D.1. (1) *There are four $\text{Ad}(G)$ -orbits on $\text{Gr}_{\leq \gamma^\vee}^{\text{triv}}$,*

$$\text{Gr}_{\leq \gamma^\vee}^{\text{triv}} = \text{Gr}_{\text{subr}} \sqcup \text{Ad}(G)U_{-\gamma, -1}^\times \sqcup \text{Gr}_{\theta^\vee}^{\text{triv}} \sqcup \{\star\},$$

*of dimensions 10, 8, 6 and 0 respectively, which, under the evaluation map $\text{ev}_{\tau=1}$, map onto the four nonregular unipotent orbits.*³

(2) *The morphism*

$$\begin{aligned} \nu : G \times^{P_{-\theta}} \left(\prod_{\langle \beta, -\theta^\vee \rangle \geq 1} U_{\beta, -1} \right) &\rightarrow \text{Gr}_{\leq \gamma^\vee}^{\text{triv}}, \\ \left(g, \prod u_\beta (c_\beta \tau^{-1}) \right) &\mapsto \text{Ad}(g) \left(\prod u_\beta (c_\beta \tau^{-1}) \right) \end{aligned}$$

is a resolution. Its fibers over the G -orbits are

- ν is an isomorphism over Gr_{subr} ,
- $\nu^{-1}(u_{-\gamma}(\tau^{-1})) \cong \mathbb{P}^1$,
- $\nu^{-1}(u_{-\theta}(\tau^{-1}))$ is a projective cone over \mathbb{P}^1 (it contains a point, whose complement is a line bundle over \mathbb{P}^1),
- $\nu^{-1}(\star) \cong G/P_{-\theta}$.

(3) *The morphism*

$$\nu' : G \times^{P_{-\gamma}} \left(\prod_{\langle \beta, -\gamma^\vee \rangle \geq 2} U_{\beta, -1} \right) \rightarrow \text{Gr}_{\leq \gamma^\vee}^{\text{triv}},$$

³In fact, Gr_{subr} is an étale double cover of the subregular orbit; the other G -orbits map isomorphically to the corresponding unipotent orbits. We do not need this more precise statement in this paper.

defined similarly as ν , is a resolution of the closure of $\text{Ad}(G)U_{-\gamma,-1}^\times$. Its fibers over the G -orbits are

- ν is an isomorphism over $\text{Ad}(G)U_{-\gamma,-1}^\times$,
- $\nu^{-1}(u_{-\theta}(\tau^{-1})) \cong \mathbb{P}^1$,
- $\nu^{-1}(\star) \cong G/P_{-\gamma}$.

Proof of Theorem 4(2) for $G = G_2$. By the same reduction steps as in the case of other types, we reduce to showing

$$(D.2) \quad \chi_c(\text{Gr}_{\gamma^\vee}^{\text{triv}, a=0}, \text{ev}^* J) = -r_s(G) = -1.$$

Step I. $\chi_c((\text{Ad}(G)U_{-\gamma,-1}^\times)^{a=0}, \text{ev}^* J) = -1$. For any short root β , let $V_\beta = \prod_{\langle \beta', \beta^\vee \rangle \geq 2} U_{\beta', -1}$. Then the source of ν' has a Bruhat decomposition

$$G \times^{P_{-\gamma}} V_{-\gamma} = \bigsqcup_{\beta \text{ short root}} \text{Ad}(U)V_\beta.$$

The following Claim can be proved similarly as the claim in Section 5.6.

CLAIM. For a short root β ,

$$\chi_c(\text{Ad}(U)V_\beta^{>-\theta}, \text{ev}^* J) = \begin{cases} 0 & \Phi_{\geq 2}^\beta \text{ contains a simple root,} \\ 1 & \text{otherwise.} \end{cases}$$

Looking at the root system G_2 , there are 3 short roots β such that $\Phi_{\geq 2}^\beta$ does not contain a simple root. Therefore $\chi_c((G \times^{P_{-\gamma}} V_{-\gamma})^{a=0}, \nu'^* \text{ev}^* J) = 3$. On the other hand, by Lemma D.1(3), we have

$$\begin{aligned} 3 &= \chi_c((G \times^{P_{-\gamma}} V_{-\gamma})^{a=0}, \nu'^* \text{ev}^* J) \\ &= \chi_c(G/P_{-\gamma}) + \chi_c(\mathbb{P}^1)\chi_c(\text{Gr}_{\theta^\vee}^{\text{triv}, a=0}, \text{ev}^* J) + \chi_c((\text{Ad}(G)U_{-\gamma,-1}^\times)^{a=0}, \text{ev}^* J). \end{aligned}$$

Plugging in $\chi_c(G/P_{-\gamma}) = \#W/W_{\gamma^\vee} = 6$, $\chi_c(\mathbb{P}^1) = 2$ and $\chi_c(\text{Gr}_{\theta^\vee}^{\text{triv}, a=0}, \text{ev}^* J) = -1$ from the proof of Theorem 4(1), we conclude that

$$\chi_c((\text{Ad}(G)U_{-\gamma,-1}^\times)^{a=0}, \text{ev}^* J) = -1.$$

Step II. $\chi_c(\text{Gr}_{\text{subr}}^{a=0}, \text{ev}^* J) = 0$. For any long root α , let

$$V_\alpha = \prod_{\langle \alpha', \alpha^\vee \rangle \geq 1} U_{\alpha', -1}.$$

Then the source of ν has a Bruhat decomposition

$$G \times^{P_{-\theta}} V_{-\theta} = \bigsqcup_{\alpha \text{ long root}} \text{Ad}(U)V_\alpha.$$

The following claim can be proved similarly as the claim in Section 5.6.

CLAIM. For a long root α ,

$$\chi_c(\mathrm{Ad}(U)V_\alpha^{>-\theta}, \mathrm{ev}^*J) = \begin{cases} 0 & \Phi_{\geq 1}^\alpha \text{ contains a simple root,} \\ 1 & \text{otherwise.} \end{cases}$$

Looking at the root system G_2 , $\alpha = -\theta$ is the only long root for which $\Phi_{\geq 1}^\alpha$ does not contain a simple root. Therefore $\chi_c((G^{P_{-\theta}} \times V_{-\theta})^{a=0}, \nu^* \mathrm{ev}^*J) = 1$. On the other hand, by Lemma D.1(2), we have

$$\begin{aligned} 1 &= \chi_c((G^{P_{-\theta}} \times V_{-\theta})^{a=0}, \nu^* \mathrm{ev}^*J) \\ &= \chi_c(G/P_{-\theta}) + \chi_c(\mathbb{P}^1)\chi_c((\mathrm{Ad}(G)U_{-\gamma,-1}^\times)^{a=0}, \mathrm{ev}^*J) \\ &\quad + \chi_c(\nu^{-1}(u_{-\theta}(\tau^{-1})))\chi_c(\mathrm{Gr}_{\theta\nu}^{\mathrm{triv},a=0}, \mathrm{ev}^*J) + \chi_c(\mathrm{Gr}_{\mathrm{subr}}^{a=0}, \mathrm{ev}^*J). \end{aligned}$$

Plugging in $\chi_c(G/P_{-\theta}) = \#W/W_\theta = 6$, $\chi_c(\mathbb{P}^1) = 2$, $\chi_c(\nu^{-1}(u_{-\theta}(\tau^{-1}))) = 3$ from Lemma D.1(2), $\chi_c((\mathrm{Ad}(G)U_{-\gamma,-1}^\times)^{a=0}, \mathrm{ev}^*J) = -1$ from Step I and $\chi_c(\mathrm{Gr}_{\theta\nu}^{\mathrm{triv},a=0}, \mathrm{ev}^*J) = -1$ from the proof of Theorem 4(1), we conclude that $\chi_c(\mathrm{Gr}_{\mathrm{subr}}^{a=0}, \mathrm{ev}^*J) = -1$.

Combining Step I and II, since $\mathrm{Gr}_{\gamma\nu}^{\mathrm{triv},a=0} = \mathrm{Gr}_{\mathrm{subr}}^{a=0} \sqcup (\mathrm{Ad}(G)U_{-\gamma,-1}^\times)^{a=0}$, we get (D.2). This proves Theorem 4(2) in the case of G_2 . \square

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