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# Isoparametric hypersurfaces with $(g, m) = (6, 2)$

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**Abstract**

We prove that isoparametric hypersurfaces with $(g, m) = (6, 2)$ are homogeneous, which answers Dorfmeister-Neher's conjecture affirmatively and solves Yau's problem in the case $g = 6$.

## 1. Introduction

A one-parameter family of isoparametric hypersurfaces is a particularly beautiful object that fills space by means of the evolution of wave fronts for a certain kind of wave equation, the solutions of which are called isoparametric functions.

These hypersurfaces were studied systematically by E. Cartan [Car38], [Car39a], [Car39b], [Car40] and classified completely in the euclidean and the hyperbolic spaces as homogeneous hypersurfaces with one or two principal curvatures. On the other hand, in the sphere, Cartan showed the existence of more examples. Münzner [Mün80], [Mün81] proved then, by a topological argument, that the number of principal curvatures $g$ is limited to $g = 1, 2, 3, 4$ and 6. While Cartan had already shown that they are all homogeneous if $g \leq 3$, a surprising discovery was made by Ozeki-Takeuchi [OT76], in which they found infinitely many nonhomogeneous isoparametric hypersurfaces with $g = 4$, by using the Clifford algebra. Since many more examples were constructed by Ferus-Karcher-Münzner [FKM81], the case $g = 4$ seems to be very special. Nevertheless, Cecil-Chi-Jensen [CCJ07] obtained a remarkable result to the effect that isoparametric hypersurfaces with $g = 4$ are exhausted by these examples and homogeneous ones, except for four cases with lower multiplicities. Later on, Immervoll [Imm08] gave a new proof of the result in [CCJ07], based on Dorfmeister-Neher's work [DN83]. Recently, Q. S. Chi made further progress for $g = 4$, and at this stage, only the case $(m_1, m_2) = (7, 8)$ is remaining [Chi09], [Chi11b], [Chi11a], [Chi12], [Chi11c].

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We should mention that the isoparametric submanifolds in $\mathbb{R}^{n+1}$ ([Ter85]) with codimension greater than 2 are homogeneous [Tho91], [Olm93]. Isoparametric hypersurfaces in $S^n$ are the case of codimension 2 in $\mathbb{R}^{n+1}$, and the classification problem turns out to be most difficult.

As for the case $g = 6$, Abresch [Abr83] shows that the multiplicity of each principal curvature is the same number $m$, which takes only the values 1 or 2. In the former case, Dorfmeister-Neher [DN85] proved the homogeneity of such hypersurfaces and conjectured that it is true for the case $m = 2$. Because their proof depends on a very intricate algebraic calculation, it seems hard to extend it to the case $(g, m) = (6, 2)$. This was the motivation when the author studied the case $(g, m) = (6, 1)$ in [Miy93] and characterized the homogeneity by the invariant kernel of the shape operators of its focal submanifolds (which is called “Condition A” in the case $g = 4$ by Ozeki-Takeuchi and Chi). In this context, we give a new proof for Dorfmeister-Neher’s theorem in [Miy09]. In this paper, in the same principle, we solve the conjecture affirmatively, which settles Yau’s 34-th problem [Yau92] for $g = 6$.

THEOREM 1.1. *Isoparametric hypersurfaces in the sphere with* $(g, m) = (6, 2)$ *are homogeneous.*

The homogeneous hypersurfaces with $(g, m) = (6, 2)$ are given by the adjoint orbits of $G_2$ in its Lie algebra $\mathfrak{g}$ [Miy11]. These orbits sweep out the unit sphere $S^{13} \subset \mathfrak{g} \cong \mathbb{R}^{14}$ as a family of isoparametric hypersurfaces $M_t$, $-1 < t < 1$, and two focal submanifolds $M_\pm = M_{\pm 1}$. The former are principal, and the latter are singular orbits, respectively. In [Miy11], we describe the structure of the $G_2$ orbits in detail, which turns out to be closely related to Bryant’s twistor fibration of symmetric spaces $S^6$ and $G_2/\mathrm{SO}(4)$.

The strategy of the proof and the organization of this papar are as follows. For the principal curvatures $\lambda_1 > \cdots > \lambda_6$, we denote the curvature distributions of $M$ by $D_1(p), \ldots, D_6(p)$. Let $M_-$ and $M_+$ be the focal submanifolds obtained by making $D_1(p)$ and $D_6(p)$ collapse, respectively. The shape operators of $M_\pm$ are known to be isospectral. In our case, the eigenvalues are given by $\pm\sqrt{3}, \pm 1/\sqrt{3}$ and 0, each of which has multiplicity two (Section 2). As in the case $(g, m) = (6, 1)$ [Miy93], the homogeneity follows if we show that the kernel of these operators is independent of normal directions (Section 15).

When $(g, m) = (6, 2)$, the normal space $T^\perp M_+$ is of dimension three, and unit normal vectors are parametrized by 2-sphere $S^2$ in $T^\perp M_+$. In order to carry out the calculation, we take a geodesic $c = N(t)$ of $S^2$ and consider the one-parameter family of shape operators $L(t) = B_{N(t)}$. Then $L(t)$ is expressed as $L(t) = \cos t B_\eta + \sin t B_\zeta$, where $\eta$ and $\zeta$ are mutually orthogonal unit normals. On the other hand, $M$ is an $S^2$ bundle over $M_+$ with fiber consisting of the leaf $L_6$ corresponding to the curvature distribution $D_6$. Naturally, $L_6$

is identified with the space of the unit normals to $M_+$ at a point. Under this identification, the kernel of $L(t)$ turns out to coincide with $D_3(t) = D_3(p(t))$, where $p(t) \in c$ is the point corresponding to $\cos t\eta + \sin t\zeta$. Now, consider the space $E(c)$ spanned by the kernel of $L(t)$ for all $t$. If we suppose that the kernel changes with $t$, then $d = \dim E(c) \geq 3$. On the other hand, we can show that each $L(t)$ maps $E(c)$ into its orthogonal complement $E(c)^\perp$ in $TM_+$ (Section 5), and this implies $d \leq 6$. Then, with respect to the decomposition $TM_+ = E(c) \oplus E(c)^\perp$, we can express (Section 6)

$$L(t) = \begin{pmatrix} 0 & R \\ {}^tR & S \end{pmatrix}, \tag{1}$$

which plays an important role in the whole argument. Namely, if we express an eigenvector of $L(t)$ with eigenvalue $\mu$ by $\boldsymbol{e} = \left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right)$, we obtain

$$\begin{cases} R(t)Y = \mu X, \\ {}^tR(t)X + S(t)Y = \mu Y. \end{cases}$$

Thus for $\mu \neq 0$, a solution $Y$ to

$$({}^tR(t)R(t) + \mu S(t) - \mu^2)Y = 0 \tag{2}$$

gives an eigenvector $\boldsymbol{e} = \left(\begin{smallmatrix} (1/\mu)R(t)Y \\ Y \end{smallmatrix}\right)$ for $\mu$. In this way, the equation $L(t)\boldsymbol{e} = \mu\boldsymbol{e}$ reduces to equation (2), and it makes it possible to carry out the calculation. Actually, in our calculation in Section 13, 10-by-10 matrices are reduced to 4-by-4 matrices.

Taking a suitable moving frame of $\ker L(t)$ along $c$, we can show $d = 6$ if $d > 2$ (Section 8). The description of $E(c)$ in terms of principal vectors is given in Sections 9–12, and we find many possibilities of $E(c)$ with continuous parameters. However, using that $E(c)$ is parallel along $c$, we can show that the eigenvalues of $T(t) = {}^tR(t)R(t)$ and $S(t)$ are constant, so that these operators become again isospectral (Section 12). Then calculating the characteristic polynomials of $T(t)$ and $S(t)$, which are 4-by-4 matrices, we show that some eigenvalues of $S(t)$ should vanish (Section 13). This makes it possible to restrict $E(c)$ to only two types (Theorem 13.11).

In these arguments, there are two main difficulties. One is caused by the nonlinear motion of a kernel vector $e_3(t)$, and another by $m = 2$. In fact, on the supposition that $\ker L(t)$ depends on $t$, it turns out that we must investigate the derivatives of $e_3(t)$ up to at least second order, and $e_3(t)$ behaves nonlinearly. Moreover, when $m = 2$, if, for instance, $M$ is Kähler (as in the homogeneous case), the principal vectors $e_i(t), e_{\bar{i}}(t) \in D_i(t)$ move in a unified way. However, in our case, we have no way to choose a frame of curvature spaces in a canonical way. These considerations make it much more difficult to determine the space $E(c)$ than in the case $m = 1$ [Miy09].

Using these difficulties as an advantage, we find a natural choice of a basis $e_3(t), e_{\bar{3}}(t)$ of $D_3(t)$ by "rotating" them in $D_3(t)$, so that they become an "even" or "odd" vector, by which we mean $e_3(t+\pi) = e_3(t)$, or $e_3(t+\pi) = -e_3(t)$, respectively. We use an argument such that odd-dimensional parallel space cannot have a continuous frame consisting of odd vectors for the reason of orientation (Section 8). Such investigation is essential because the "spin action" of the orthogonal group is always a concern. In fact, since the shape operators are isospectral, $L(t)$ is expressed as $L(t) = U(t)L(0)^tU(t)$ for some $U(t) \in O(10)$, and it causes the signature ambiguity. Moreover, the isospectrality is, in some sense, a weak condition for dimension as high as $\dim O(10) = 45$. Much stronger is the condition that $L(t)$ is expressed in a linear combination $\cos tB_\eta + \sin tB_\zeta$. Using this combination in the computation of the characteristic polynomials (in the reduced size), we can restrict $E(c)$ to two types at last. Then by using mainly the Gauss equation and taking both the focal submanifolds into account, we show that these cases are impossible (Section 14). Thus we know that the kernel of the shape operators of the focal submanifolds does not depend on the choice of normal directions.

Once we show that the shape operators have an invariant kernel, many components of the matrix expression of the shape operators vanish at the same time, and we can express them explicitly, which turn out to coincide with those of the homogeneous case given in [Miy11].

Even if we do not know the homogeneous data, we can show the homogeneity by using Singer's strongly curvature-homogeneous theorem. By definition ([KN69, p. 357]), a Riemannian manifold $X$ is strongly curvature-homogeneous if, for any two points $x, y \in X$, there is a linear isomorphism of $T_xX$ onto $T_yX$ that maps $g_x$ (the metric at $x$) and $(\nabla^k R)_x$ (higher covariant derivatives of the curvature tensor $R$), $k = 0, 1, 2, \ldots$ upon $g_y$ and $(\nabla^k R)_y$, $k = 0, 1, 2, \ldots$ .

THEOREM 1.2 ([Sin60], [Nom62], [KN69, Th. 2, p. 357]). *If a connected Riemannian manifold $X$ is strongly curvature-homogeneous, then it is locally homogeneous. Moreover, if $M$ is complete and simply connected, it is homogeneous.*

In our case, the shape operators are expressed in terms of the structure coefficients $\Lambda_{\alpha\beta}^\gamma$ of $M$ with respect to a frame $e_{\underline{i}}$ consisting of principal vectors. This frame defines an isometry between $T_pM$ and $T_qM$. The explicit expression of the shape operators implies that the structure coefficients $\Lambda_{\alpha\beta}^\gamma$ are locally constant. Then the components of $(\nabla^k R)_x$ are given by polynomials in $\Lambda_{\alpha\beta}^\gamma$ and again are all locally constant. Moreover, since $M$ is complete and simply connected, applying Theorem 1.2, we know $M$ is intrinsically homogeneous. Finally by using the rigidity theorem of hypersurfaces with type number larger than two [KN69, p. 45], we conclude that $M$ is extrinsically homogeneous.

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## 2. Preliminaries

2.1. *Isoparametric hypersurfaces.* We refer the readers to [Tho00] for a nice survey of isoparametric hypersurfaces. In this subsection as well as in the next subsection, we review fundamental facts and the notation given in [Miy93].

A hypersurface $M$ in the unit sphere $S^{n+1}$ is called isoparametric when all the principal curvatures are constant. Obviously, homogeneous hypersurfaces [HL71] are isoparametric hypersurfaces. Throughout the paper, we assume $M$ to be isoparametric. Let $\xi$ be a unit normal vector field of $M$. We denote the Riemannian connection on $S^{n+1}$ by $\tilde{\nabla}$ and the induced connection on $M$ by $\nabla$. Let $\lambda_1 \geq \cdots \geq \lambda_n$ be the constant principal curvatures of $M$, and let $D_\lambda(p)$ be the curvature distribution of $\lambda \in \{\lambda_i, i = 1, 2 \ldots, n\}$. We denote the multiplicity of $\lambda$ by $m_\lambda$. Then $D_\lambda$ is completely integrable and the leaf $L_\lambda$ of $D_\lambda$ is an $m_\lambda$-dimensional sphere [Rec76]. Choose a local orthonormal frame $e_1, \ldots, e_n$ consisting of unit principal vectors corresponding to $\lambda_1, \ldots, \lambda_n$. We express

$$\tilde{\nabla}_{e_\alpha} e_\beta = \Lambda^\sigma_{\alpha\beta} e_\sigma + \lambda_\alpha \delta_{\alpha\beta} \xi, \tag{3}$$

where $1 \leq \alpha, \beta, \sigma \leq n$, using the Einstein convention. We have

$$\Lambda^\gamma_{\alpha\beta} = -\Lambda^\beta_{\alpha\gamma}, \tag{4}$$

and the curvature tensor $R_{\alpha\beta\gamma\delta}$ of $M$ is given by

$$\begin{aligned} R_{\alpha\beta\gamma\delta} &= (1 + \lambda_\alpha\lambda_\beta)(\delta_{\beta\gamma}\delta_{\alpha\delta} - \delta_{\alpha\gamma}\delta_{\beta\delta}) \\ &= e_\alpha(\Lambda^\delta_{\beta\gamma}) - e_\beta(\Lambda^\delta_{\alpha\gamma}) + \Lambda^\sigma_{\beta\gamma}\Lambda^\delta_{\alpha\sigma} - \Lambda^\sigma_{\alpha\gamma}\Lambda^\delta_{\beta\sigma} - \Lambda^\sigma_{\alpha\beta}\Lambda^\delta_{\sigma\gamma} + \Lambda^\sigma_{\beta\alpha}\Lambda^\delta_{\sigma\gamma}. \end{aligned} \tag{5}$$

The covariant derivative of the coefficients of the second fundamental tensor $h_{\alpha\beta} = \lambda_\alpha\delta_{\alpha\beta}$ is given by

$$\begin{aligned} h_{\alpha\beta,\gamma} &= e_\gamma(h_{\alpha\beta}) - \Lambda^\sigma_{\gamma\alpha} h_{\sigma\beta} - \Lambda^\sigma_{\gamma\beta} h_{\alpha\sigma} \\ &= e_\gamma(\lambda_\alpha)\delta_{\alpha\beta} + \Lambda^\beta_{\gamma\alpha}(\lambda_\alpha - \lambda_\beta). \end{aligned} \tag{6}$$

From the equation of Codazzi,

$$h_{\alpha\beta,\gamma} = h_{\beta\gamma,\alpha} = h_{\gamma\alpha,\beta}, \tag{7}$$

we obtain

$$e_\beta(\lambda_\alpha) = \Lambda^\beta_{\alpha\alpha}(\lambda_\alpha - \lambda_\beta) \qquad \text{for} \quad \alpha \neq \beta. \tag{8}$$

If $\lambda_\alpha, \lambda_\beta, \lambda_\gamma$ are distinct, we have

$$\Lambda^\gamma_{\alpha\beta}(\lambda_\beta - \lambda_\gamma) = \Lambda^\beta_{\gamma\alpha}(\lambda_\alpha - \lambda_\beta) = \Lambda^\alpha_{\beta\gamma}(\lambda_\gamma - \lambda_\alpha). \tag{9}$$

Moreover,

$$\Lambda^\gamma_{ab} = 0, \quad \Lambda^\gamma_{aa} = \Lambda^\gamma_{bb}, \qquad \text{if} \quad \lambda_a = \lambda_b \neq \lambda_\gamma \quad \text{and} \quad a \neq b, \tag{10}$$

hold, and since $\lambda_\alpha$ is constant on $M$, it follows from (8)

$$\Lambda^\gamma_{\alpha\alpha} = 0 \qquad \text{if} \quad \lambda_\gamma \neq \lambda_\alpha. \tag{11}$$

*Remark* 2.1. Formula (9) shows that if $\lambda_\alpha, \lambda_\beta, \lambda_\gamma$ are distinct, $\Lambda^\gamma_{\alpha\beta}$ is determined by $e_\alpha, e_\beta, e_\gamma$ at a point and is independent of the extension of these vectors.

*Remark* 2.2. In (10), note that $\Lambda^b_{\gamma a} \neq 0$ in general, where $\lambda_a = \lambda_b \neq \lambda_\gamma$. In fact, we can "rotate" $e_a$ arbitrarily in $D_a$, which makes $\Lambda^b_{\gamma a} \neq 0$. We call a frame such that $\Lambda^b_{\gamma a} = 0$ for $\lambda_a = \lambda_b \neq \lambda_\gamma$ *admissible* (see (32)).

2.2. *The focal submanifolds.* Let $M$ be an isoparametric hypersurface with $(g, m) = (6, 2)$, i.e., a hypersurface with six constant principal curvatures, each of which has multiplicity two. As is well known [Mün80], $\lambda_i = \cot(\theta_1 + \frac{(i-1)\pi}{6})$, $1 \leq i \leq 6$, $0 < \theta_1 < \frac{\pi}{6}$. Since the homogeneity is independent of the choice of $\theta_1$, and cotangent has the period $\pi$, we take

$$\theta_1 = \frac{\pi}{12} = -\theta_6, \quad \theta_2 = \frac{\pi}{4} = -\theta_5, \quad \theta_3 = \frac{5\pi}{12} = -\theta_4$$

so that

$$\lambda_1 = -\lambda_6 = 2 + \sqrt{3}, \quad \lambda_2 = -\lambda_5 = 1, \quad \lambda_3 = -\lambda_4 = 2 - \sqrt{3}. \tag{12}$$

In particular, we have chosen $\theta_i \in (-\frac{\pi}{2}, \frac{\pi}{2})$, so that the first focal point in the direction $\pm\xi$ is nearest to $p$; see (13). Denote $D_i = D_{\lambda_i}$. We choose a local orthonormal frame field $e_1, e_{\bar{1}}, \ldots, e_6, e_{\bar{6}}$, where $\{e_i, e_{\bar{i}}\}$ is an orthonormal frame of $D_i$. For convenience, we put $\lambda_{\bar{i}} = \lambda_i$, and $\underline{i}$ always stands for $i$ or $\bar{i}$. By (10) and (11), a leaf $L_i = L_i(p)$ of $D_i$ is a totally geodesic 2-sphere in the corresponding curvature sphere $S_i$ since $T^\perp_p L_i \cap T_p S_i = \oplus_{j \neq i} D_j(p)$. For $a = 6$ or 1, define the focal map $f_a \colon M \to S^{13}$ by

$$f_a(p) = \cos\theta_a p + \sin\theta_a \xi_p, \tag{13}$$

where $L_a(p)$ shrinks into a point $\bar{p} = f_a(p)$. Then we have

$$df_a(e_j) = \sin\theta_a(\lambda_a - \lambda_j)e_j \quad \text{and} \quad df_a(e_{\bar{j}}) = \sin\theta_a(\lambda_a - \lambda_j)e_{\bar{j}}, \tag{14}$$

where the right-hand side is considered as a vector in $T_{\bar{p}}S^{13}$ by a parallel translation in $S^{13}$. In the following, we always use such identification. The rank of $f_a$ is constant and we obtain the focal submanifold $M_a$ of $M$:

$$M_a = \{\cos\theta_a p + \sin\theta_a \xi_p \mid p \in M\}.$$

We denote $M_+ = M_6$ and $M_- = M_1$. It follows $T_{\bar{p}} M_a = \oplus_{j \neq a} D_j(q)$ from (14) for any $q \in {f_a}^{-1}(\bar{p})$. An orthonormal basis of the normal space of $M_a$ at $\bar{p}$ is given by

$$\eta_q = -\sin\theta_a q + \cos\theta_a \xi_q, \quad \zeta_q = e_a(q), \quad \bar{\zeta}_q = e_{\bar{a}}(q)$$

for any $q \in L_a(p) = f_a^{-1}(\bar{p})$.

We consider the connection $\bar{\nabla}$ on $M_a$ induced from the connection $\tilde{\nabla}$ of $S^{13}$; that is,

$$\tilde{\nabla}_{e_{\underline{j}}} X = \bar{\nabla}_{e_{\underline{j}}} \tilde{X} + \bar{\nabla}^{\perp}_{e_j} \tilde{X}, \quad \lambda_j \neq \lambda_a, \tag{15}$$

where $X$ is a tangent field on $S^{13}$ in a neighborhood of $p$ and $\tilde{X}$ is the one near $\bar{p}$ obtained by the parallel transport from $X$. We denote by $\bar{\nabla}^{\perp}_{e_j} \tilde{X}$ the normal component in $S^{13}$ at $\bar{p}$. In particular, we have for $j \neq a$,

$$\tilde{\nabla}_{e_{\underline{j}}} e_{\underline{k}} = \frac{1}{\sin\theta_a(\lambda_a - \lambda_j)} \left\{ \sum \Lambda^{l}_{\underline{jk}} e_{\underline{l}} + \delta_{\underline{jk}} (\lambda_j \xi_p - p) \right\}$$

and hence

$$\bar{\nabla}_{e_{\underline{j}}} \tilde{e}_{\underline{k}} = \frac{1}{\sin\theta_a(\lambda_a - \lambda_j)} \sum_{l \neq a} \Lambda^{l}_{\underline{jk}} e_{\underline{l}}, \tag{16}$$

$$\bar{\nabla}^{\perp}_{e_{\underline{j}}} \tilde{e}_{\underline{k}} = \frac{1}{\sin\theta_a(\lambda_a - \lambda_j)} \left( \Lambda^{a}_{\underline{jk}} e_a + \Lambda^{\bar{a}}_{\underline{jk}} e_{\bar{a}} \right) + \frac{1 + \lambda_j \lambda_a}{\lambda_a - \lambda_j} \delta_{\underline{jk}} \eta_p, \tag{17}$$

where we use $\langle \lambda_j \xi_p - p, \eta_p \rangle = \sin\theta_a (1 + \lambda_a \lambda_j)$. In the following, we identify $\tilde{e}_i$ with $e_i$. Denote by $B_N$ the shape operator of $M_a$ with respect to the normal vector $N$. Then from (16) and (17), we obtain

Lemma 2.3. *When we identify $T_{\bar{p}} M_a$ with $\oplus_{j=1}^{5} D_{a+j}(p)$ where the indices are modulo 6, the shape operators $B_{\eta_p}$, $B_{\zeta_p}$ and $B_{\bar{\zeta}_p}$ at $\bar{p}$ with respect to the basis of $T_{\bar{p}} M_a$ given by $e_{a+1}, e_{\overline{a+1}}, \ldots, e_{a+5}, e_{\overline{a+5}}$ at $p$ are expressed respectively by symmetric matrices:*

$$B_{\eta_p} = \begin{pmatrix} \sqrt{3} I & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{\sqrt{3}} I & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{3} I \end{pmatrix},$$

$$B_{\zeta_p} = \begin{pmatrix} 0 & B_{a+1\,a+2} & B_{a+1\,a+3} & B_{a+1\,a+4} & B_{a+1\,a+5} \\ B_{a+2\,a+1} & 0 & B_{a+2\,a+3} & B_{a+2\,a+4} & B_{a+2\,a+5} \\ B_{a+3\,a+1} & B_{a+3\,a+2} & 0 & B_{a+3\,a+4} & B_{a+3\,a+5} \\ B_{a+4\,a+1} & B_{a+4\,a+2} & B_{a+4\,a+3} & 0 & B_{a+4\,a+5} \\ B_{a+5\,a+1} & B_{a+5\,a+2} & B_{a+5\,a+3} & B_{a+5\,a+4} & 0 \end{pmatrix},$$

$$B_{\bar\zeta_p} = \begin{pmatrix} 0 & \bar B_{a+1\,a+2} & \bar B_{a+1\,a+3} & \bar B_{a+1\,a+4} & \bar B_{a+1\,a+5} \\ \bar B_{a+2\,a+1} & 0 & \bar B_{a+2\,a+3} & \bar B_{a+2\,a+4} & \bar B_{a+2\,a+5} \\ \bar B_{a+3\,a+1} & \bar B_{a+3\,a+2} & 0 & \bar B_{a+3\,a+4} & \bar B_{a+3\,a+5} \\ \bar B_{a+4\,a+1} & \bar B_{a+4\,a+2} & \bar B_{a+4\,a+3} & 0 & \bar B_{a+4\,a+5} \\ \bar B_{a+5\,a+1} & \bar B_{a+5\,a+2} & \bar B_{a+5\,a+3} & \bar B_{a+5\,a+4} & 0 \end{pmatrix},$$

*where $I$ (0, resp.) is the $2\times 2$ unit (zero, resp.) matrix, and*

$$B_{ij} = \frac{1}{\sin\theta_a(\lambda_i - \lambda_a)} \begin{pmatrix} \Lambda^j_{ia} & \Lambda^{\bar j}_{ia} \\ \Lambda^j_{\bar i a} & \Lambda^{\bar j}_{\bar i a} \end{pmatrix} = {}^t B_{ji}, \tag{18}$$

$$\bar B_{ij} = \frac{1}{\sin\theta_a(\lambda_i - \lambda_a)} \begin{pmatrix} \Lambda^j_{i\bar a} & \Lambda^{\bar j}_{i\bar a} \\ \Lambda^j_{\bar i\bar a} & \Lambda^{\bar j}_{\bar i\bar a} \end{pmatrix} = {}^t \bar B_{ji}.$$

*Proof.* First, consider the case $a = 6$. From (17), it follows $B_{\eta_p}(e_j) = \mu_j e_j$, where

$$\mu_j = \frac{1+\lambda_j\lambda_6}{\lambda_6 - \lambda_j}, \quad \mu_1 = \sqrt3 = -\mu_5, \quad \mu_2 = 1/\sqrt3 = -\mu_4, \quad \mu_3 = 0. \tag{19}$$

When $a = 1$, $T_{\bar p}M_1 = \oplus_{j=1}^5 D_{1+j}(p)$ holds, and if we denote $B_{\eta_p}(e_{1+j}) = \nu_j e_{1+j}$, we have

$$\nu_j = \frac{1+\lambda_{1+j}\lambda_1}{\lambda_1 - \lambda_{1+j}}, \quad \nu_1 = \sqrt3 = -\nu_5, \quad \nu_2 = 1/\sqrt3 = -\nu_4,\ \nu_3 = 0 \tag{20}$$

and obtain the matrix $B_{\eta_p}$. Next from

$$B_{ij} = \begin{pmatrix} \langle \bar\nabla^\perp_{e_j} e_i, e_a\rangle & \langle \bar\nabla^\perp_{e_{\bar j}} e_i, e_a\rangle \\ \langle \bar\nabla^\perp_{e_j} e_{\bar i}, e_a\rangle & \langle \bar\nabla^\perp_{e_{\bar j}} e_{\bar i}, e_a\rangle \end{pmatrix}, \ \bar B_{ij} = \begin{pmatrix} \langle \bar\nabla^\perp_{e_j} e_i, e_{\bar a}\rangle & \langle \bar\nabla^\perp_{e_{\bar j}} e_i, e_{\bar a}\rangle \\ \langle \bar\nabla^\perp_{e_j} e_{\bar i}, e_{\bar a}\rangle & \langle \bar\nabla^\perp_{e_{\bar j}} e_{\bar i}, e_{\bar a}\rangle \end{pmatrix},$$

we obtain (18) by using (17) because

$$\frac{1}{\sin\theta_a(\lambda_a - \lambda_j)}\Lambda^a_{\underline{j}\underline{i}} = \frac{1}{\sin\theta_a(\lambda_j - \lambda_a)}\Lambda^i_{\underline{j}a} = \frac{1}{\sin\theta_a(\lambda_i - \lambda_a)}\Lambda^j_{\underline{i}a},$$

where we use (9). Moreover, $B_{ii} = 0$ follows from (10). □

By this lemma, $M_a$ is minimal [Nom75]. In fact, $\mathrm{tr} B_\eta = 0$ in the expression (19) is nothing but the Cartan formula [Car38, eq. (21)]. The following is important.

Lemma 2.4 ([Mün81], [Miy93]). *For any unit normal vector $N$ of $M_a$ at $\bar p$, $B_N$ is isospectral, i.e., the eigenvalues of $B_N$ are $\pm\sqrt3$, $\pm\frac{1}{\sqrt3}$, 0, and each eigenspace is of dimension 2.*

*Proof.* For any $q \in L_a(p)$, Lemma 2.3 implies that $B_{\eta_q}$ has eigenvalues $\pm\sqrt{3}$, $\pm 1/\sqrt{3}$, 0 with 2-dimensional eigenspaces. It is easy to see that the map given by $S^2 \cong L_a(p) \ni q \mapsto -\sin\theta_a q + \cos\theta_a \xi_q \in S^2(1) \subset T^\perp_{\bar{q}} M_a = T^\perp_{\bar{p}} M_a$ is of full rank and one-to-one, and hence bijective, so any unit normal vector $N \in T^\perp_{\bar{p}} M$ is expressed as $N = \eta_q = -\sin\theta_a q + \cos\theta_a \xi_q$ for some $q \in L_a(p)$. □

*Remark* 2.5. Since $\Lambda^{\bar{a}}_{\underline{j}a}$ does not vanish in general by Remark 2.3, we should express the covariant derivative of a normal vector $e_a$ of $M_a$ as

$$\tilde{\nabla}_{e_{\underline{j}}} e_a = \bar{\nabla}_{e_{\underline{j}}} e_a + \frac{1}{\sin\theta_a(\lambda_a - \lambda_j)} \Lambda^{\bar{a}}_{\underline{j}a} e_{\bar{a}}, \tag{21}$$

see (15), and $\langle \tilde{\nabla}_{e_{\underline{j}}} e_a, \eta\rangle = -\langle e_a, \tilde{\nabla}_{e_{\underline{j}}} \eta\rangle = 0$.

## 3. Isospectral operators and Gauss equation

From now on, we take $a = 6$ and consider the focal submanifold $M_+$. A similar argument holds for $M_-$ with a suitable change of indices.

By Lemma 2.4, $L(t) = \cos tB_\eta + \sin tB_\zeta$ is isospectral and so can be written as

$$L(t) = U(t)L(0)U^{-1}(t) \tag{22}$$

for some $U(t) \in O(10)$. Moreover, this implies the Lax equation

$$L_t(t) = \frac{d}{dt}L(t) = [H(t), L(t)], \tag{23}$$

where

$$H(t) = U_t(t)U(t)^{-1} \in \mathfrak{o}(10).$$

In particular, we have $L(0) = B_\eta$, and

$$L_t(t) = -\sin tB_\eta + \cos tB_\zeta = L(t + \pi/2). \tag{24}$$

Hence for $L_t(0) = B_\zeta = (B_{ij})$, where $B_{ij} = {}^tB_{ji}$, putting $H(0) = (H_{ij})$, $H_{ji} = -{}^tH_{ij}$, we can express

$$\begin{aligned} B_\zeta &= L(\pi/2) = [H(0), B_\eta] \\ &= \begin{pmatrix} 0 & -\frac{2}{\sqrt{3}}H_{12} & -\sqrt{3}H_{13} & -\frac{4}{\sqrt{3}}H_{14} & -2\sqrt{3}H_{15} \\ \frac{2}{\sqrt{3}}H_{21} & 0 & -\frac{1}{\sqrt{3}}H_{23} & -\frac{2}{\sqrt{3}}H_{24} & -\frac{4}{\sqrt{3}}H_{25} \\ \sqrt{3}H_{31} & \frac{1}{\sqrt{3}}H_{32} & 0 & -\frac{1}{\sqrt{3}}H_{34} & -\sqrt{3}H_{35} \\ \frac{4}{\sqrt{3}}H_{41} & \frac{2}{\sqrt{3}}H_{42} & \frac{1}{\sqrt{3}}H_{43} & 0 & -\frac{2}{\sqrt{3}}H_{45} \\ 2\sqrt{3}H_{51} & \frac{4}{\sqrt{3}}H_{52} & \sqrt{3}H_{53} & \frac{2}{\sqrt{3}}H_{54} & 0 \end{pmatrix}. \end{aligned} \tag{25}$$

Note that the eigenvectors of $L(t)$ are given by

$$e_{\underline{i}}(t) = U(t)e_{\underline{i}}(0), \tag{26}$$

which implies

$$\nabla_{\frac{d}{dt}} e_{\underline{j}}(t) = H(t) e_{\underline{j}}(t). \tag{27}$$

In the proof of Lemma 2.4, we identify $L_6(p)$ with the unit sphere of the normal space of $M_+$ at $\bar{p}$. In particular, we identify the one-parameter family of $L(t)$, or more precisely, of the normal directions $\cos t\eta_p + \sin t\zeta_p$, with the geodesic of $L_6(p)$ through $p$ in the direction $\zeta_p = e_6(p)$. Then we have

$$\nabla_{\frac{d}{dt}} = c_0 \nabla_{e_6}, \quad c_0 = |\sin\theta_6| = \sqrt{2}(\sqrt{3}-1)/4. \tag{28}$$

*Remark* 3.1. Because of $\sin\theta_6 < 0$ by our definition, $\cos t\eta_p + \sin t\zeta_p$ corresponds to the geodesic $p(t)$ of $L_6$ parametrized by

$$p(t) - \cos\theta_6\bar{p} = \cos t(p - \cos\theta_6\bar{p}) - \sin t \sin\theta_6 \zeta_p. \tag{29}$$

In fact, from $\bar{p} = \cos\theta_6 p + \sin\theta_6 \xi_p$, we obtain

$$p - \cos\theta_6\bar{p} = -\sin\theta_6\eta_p, \quad \dot{p}(0) = -\sin\theta_6\zeta_p = -\sin\theta_6 e_6(p), \tag{30}$$

which is in the positive direction of $\eta_p$ and $\zeta_p = e_6(p)$, and $L(t)$ is compatible with $p(t) \in L_6$ parametrized in this way. Thus $\nabla_{\frac{d}{dt}}$ is the derivation in the positive direction of $e_6(p)$, and (28) follows. The signature of $c_0$ is important in the proof of Lemma 5.1.

Now we obtain $H(0) = \big(H_{ij}(0)\big)$, where

$$H_{ij}(0) = c_0 \begin{pmatrix} \Lambda^i_{6j}(0) & \Lambda^i_{6\bar{j}}(0) \\ \\ \Lambda^{\bar{i}}_{6j}(0) & \Lambda^{\bar{i}}_{6\bar{j}}(0) \end{pmatrix} = -c_0 \begin{pmatrix} \Lambda^j_{6i}(0) & \Lambda^{\bar{j}}_{6i}(0) \\ \\ \Lambda^j_{6\bar{i}}(0) & \Lambda^{\bar{j}}_{6\bar{i}}(0) \end{pmatrix}. \tag{31}$$

For a suitable frame, we may consider $H_{ii}(0) = 0$. In fact, if we "rotate" a moving frame $e_i(t), e_{\bar{i}}(t)$ in $D_i(t)$, so that

$$\begin{aligned} v_i(t) &= (\cos\varphi(t)) e_i(t) + (\sin\varphi(t)) e_{\bar{i}}(t), \\ v_{\bar{i}}(t) &= -(\sin\varphi(t)) e_i(t) + (\cos\varphi(t)) e_{\bar{i}}(t) \end{aligned} \tag{32}$$

along $c$, we have

$$\langle \nabla_{e_6} v_i(t), v_{\bar{i}}(t)\rangle = \Lambda^{\bar{i}}_{6i}(t) + \dot{\varphi}(t).$$

Thus if we choose $\varphi(t)$ (locally) so that $\dot{\varphi}(t) = -\Lambda^{\bar{i}}_{6i}(t)$, we obtain $\Lambda^{\bar{i}}_{6i} = 0$ with respect to $v_i(t), v_{\bar{i}}(t)$. We call such a frame *admissible*.

*Remark* 3.2. Note that $B_{ii} = 0$ holds for any frame, but $H_{ii} = 0$ holds only for an admissible frame.

Now, denoting the $(i, j)$ block of $L(t + \pi/2)$ by $B_{ij} = \begin{pmatrix} b_{ij} & b_{i\bar{j}} \\ b_{\bar{i}j} & b_{\bar{i}\bar{j}} \end{pmatrix}$, where $b_{\underline{i}\underline{j}} = b_{\underline{j}\underline{i}}$ (note that this is *not* the component of $L(t)$ but of $L(t + \pi/2)$), we have at $p(t)$,

$$L_t(t+\pi/2)_{\underline{ij}} = c_0\nabla_{e_6}(b_{\underline{ij}}) = c_0\{e_6(b_{\underline{ij}}) - b_{\underline{kj}}\Lambda^k_{6\underline{i}}(t) - b_{\underline{ik}}\Lambda^k_{6\underline{j}}(t)\},$$

and hence putting $t=0$ and noting that $L_t(\pi/2) = -B_\eta$, $L(\pi/2) = B_\zeta$, we obtain

$$B_\eta = -c_0 e_6(B_\zeta) - [H(0), B_\zeta]. \tag{33}$$

With respect to an *admissible* frame, we can rewrite (25) as

$$H(0) = \begin{pmatrix} 0 & -\frac{\sqrt{3}}{2}B_{12} & -\frac{1}{\sqrt{3}}B_{13} & -\frac{\sqrt{3}}{4}B_{14} & -\frac{1}{2\sqrt{3}}B_{15} \\ \frac{\sqrt{3}}{2}B_{21} & 0 & -\sqrt{3}B_{23} & -\frac{\sqrt{3}}{2}B_{24} & -\frac{\sqrt{3}}{4}B_{25} \\ \frac{1}{\sqrt{3}}B_{31} & \sqrt{3}B_{32} & 0 & -\sqrt{3}B_{34} & -\frac{1}{\sqrt{3}}B_{35} \\ \frac{\sqrt{3}}{4}B_{41} & \frac{\sqrt{3}}{2}B_{42} & \sqrt{3}B_{43} & 0 & -\frac{\sqrt{3}}{2}B_{45} \\ \frac{1}{2\sqrt{3}}B_{51} & \frac{\sqrt{3}}{4}B_{52} & \frac{1}{\sqrt{3}}B_{53} & \frac{\sqrt{3}}{2}B_{54} & 0 \end{pmatrix}.$$

Substituting this into (33), for each $[i.j]$, we have

[1.1] $\sqrt{3}I = 2(\frac{\sqrt{3}}{2}B_{12}B_{21} + \frac{1}{\sqrt{3}}B_{13}B_{31} + \frac{\sqrt{3}}{4}B_{14}B_{41} + \frac{1}{2\sqrt{3}}B_{15}B_{51})$

[2.2] $\frac{1}{\sqrt{3}}I = 2(-\frac{\sqrt{3}}{2}B_{21}B_{12} + \sqrt{3}B_{23}B_{32} + \frac{\sqrt{3}}{2}B_{24}B_{42} + \frac{\sqrt{3}}{4}B_{25}B_{52})$

[3.3] $0 = 2(-\frac{1}{\sqrt{3}}B_{31}B_{13} - \sqrt{3}B_{32}B_{23} + \sqrt{3}B_{34}B_{43} + \frac{1}{\sqrt{3}}B_{35}B_{53})$

[4.4] $-\frac{1}{\sqrt{3}}I = 2(-\frac{\sqrt{3}}{4}B_{41}B_{14} - \frac{\sqrt{3}}{2}B_{42}B_{24} - \sqrt{3}B_{43}B_{34} + \frac{\sqrt{3}}{2}B_{45}B_{54})$

[5.5] $-\sqrt{3}I = -2(\frac{1}{2\sqrt{3}}B_{51}B_{15} + \frac{\sqrt{3}}{4}B_{52}B_{25} + \frac{1}{\sqrt{3}}B_{53}B_{35} + \frac{\sqrt{3}}{2}B_{54}B_{45})$

[1.2] $0 = -c_0e_6(B_{12}) + \frac{4}{\sqrt{3}}B_{13}B_{32} + \frac{3\sqrt{3}}{4}B_{14}B_{42} + \frac{5}{4\sqrt{3}}B_{15}B_{52}$

[1.3] $0 = -c_0e_6(B_{13}) - \frac{\sqrt{3}}{2}B_{12}B_{23} + \frac{5\sqrt{3}}{4}B_{14}B_{43} + \frac{\sqrt{3}}{2}B_{15}B_{53}$

[1.4] $0 = -c_0e_6(B_{14}) - \frac{2}{\sqrt{3}}B_{13}B_{34} + \frac{2}{\sqrt{3}}B_{15}B_{54}$

[1.5] $0 = -c_0e_6(B_{15}) + \frac{\sqrt{3}}{4}B_{12}B_{25} - \frac{\sqrt{3}}{4}B_{14}B_{45}$

[2.3] $0 = -c_0e_6(B_{23}) - \frac{5}{2\sqrt{3}}B_{21}B_{13} + \frac{3\sqrt{3}}{2}B_{24}B_{43} + \frac{7}{4\sqrt{3}}B_{25}B_{53}$

[2.4] $0 = -c_0e_6(B_{24}) - \frac{3\sqrt{3}}{4}B_{21}B_{14} + \frac{3\sqrt{3}}{4}B_{25}B_{54}$

[2.5] $0 = -c_0e_6(B_{25}) - \frac{2}{\sqrt{3}}B_{21}B_{15} + \frac{2}{\sqrt{3}}B_{23}B_{35}$

[3.4] $0 = -c_0e_6(B_{34}) - \frac{7}{4\sqrt{3}}B_{31}B_{14} - \frac{3\sqrt{3}}{2}B_{32}B_{24} + \frac{5}{2\sqrt{3}}B_{35}B_{54}$

[3.5] $0 = -c_0e_6(B_{35}) - \frac{\sqrt{3}}{2}B_{31}B_{15} - \frac{5\sqrt{3}}{4}B_{32}B_{25} + \frac{\sqrt{3}}{2}B_{34}B_{45}$

[4.5] $0 = -c_0e_6(B_{45}) - \frac{5}{4\sqrt{3}}B_{41}B_{15} - \frac{3\sqrt{3}}{4}B_{42}B_{25} - \frac{4}{\sqrt{3}}B_{43}B_{35}$

*Remark* 3.3. These are nothing but another description of the Gauss equations (5) where $b_{ij}$, say, corresponds to $R_{6ij6}/\sin\theta_6(\lambda_6-\lambda_i)$ [Miy08]. Since (5) is a bit messy, we can use the above formula, taking an admissible frame. Although $[i.j]$, $i\neq j$ holds only for an admissible frame, $[i.i]$ holds for any orthonormal frame of $D_j$'s. In fact, if we "rotate" an orthonormal frame of $D_j$ by $U_j(t) \in O(2)$, $B_{ij}$ changes into $U_i(t)B_{ij}{}^tU_j(t)$, and hence $B_{ij}B_{ji}$ changes into $U_i(t)(B_{ij}B_{ji})^tU_i(t)$. Thus the relation $[i.i]$ is preserved.

## 4. **Global symmetry**

Any isoparametric hypersurface $M$ can be uniquely extended to a closed one [Car38]. We now treat global properties of $M$.

Let $p \in M$, and let $\gamma$ be the normal geodesic at $p$. We know that $\gamma \cap M$ consists of twelve points $p_1, \ldots, p_{12}$ that are vertices of certain dodecagon; see Figure 1, where indices are changed from [M1, Lemma 6] and [M2, p. 197]. At $p_1$, the segment joining $p_1$ with $p_2, p_4, p_6, p_8, p_{10}, p_{12}$ corresponds, respectively, to the leaf $L_1, L_2, L_3, L_4, L_5, L_6$. Leaves are expressed in a similar way at each point. A remarkable fact is that the leaves expressed by parallel segments in Figure 1 are really parallel with respect to the connection of $S^{13}$.

Lemma 4.1 ([Miy89, Lemma 6]). *We have the relations*

$$D_i(p_1) = D_{2-i}(p_2) = D_{i+4}(p_3) = D_{4-i}(p_4) = D_{i+2}(p_5) = D_{6-i}(p_6),$$
$$D_i(p_j) = D_i(p_{j+6}), \quad j = 1, \ldots, 6,$$

*where the equality means "be parallel to with respect to the connection of $S^{13}$," and the indices are modulo* 6.

The author uses tautness to prove this in [Miy89]. Since $D_6(p_1) = D_2(p_2)$ holds by Lemma 4.1, choosing $e_6(p_1)$ parallel with $e_2(p_2)$, let $p(t)$ be the geodesic of $L_6(p_1)$ in the direction $e_6(p_1)$ such that $p_1 = p(0)$, prametrized by the center angle, where the center means that of a circle on a plane. Similarly, let $q(t)$ be the geodesic of $L_2(p_2)$ in the direction $e_2(p_2)$ parametrized from $p_2 = q(0)$. Extend $e_6$ and $e_2$ as the unit tangent vectors of $p(t)$ and $q(t)$, respectively. Consider the normal geodesic $\gamma_t$ at $p_1' = p(t)$, and let $p_2' = q(t) \cap \gamma_t$. By Lemma 4.1, we can take $e_{\underline{3}}(p_1')$ parallel with $e_{\underline{5}}(p_2')$. Then we obtain

$$\frac{1}{\sin\theta_6}\nabla_{\frac{d}{dt}} e_{\underline{3}}(p_1') = \frac{\sin\theta_2}{\sin\theta_6}\frac{1}{\sin\theta_2}\nabla_{\frac{d}{dt}} e_{\underline{5}}(p_2').$$

Thus the $D_j$ component of $(\tilde{\nabla}_{e_{\underline{6}}} e_{\underline{3}})(p_{\underline{1}})$ is the $D_{2-j}$ component of $(\tilde{\nabla}_{e_{\underline{2}}} e_{\underline{5}})(p_{\underline{2}})$ multiplied by $\sin\theta_2/\sin\theta_6$. We denote such a relation by

$$\Lambda^{j}_{\underline{63}}(p_1) \sim \Lambda^{2-j}_{\underline{25}}(p_2), \quad \Lambda^{\bar{j}}_{\underline{63}}(p_1) \sim \Lambda^{\overline{2-j}}_{\underline{25}}(p_2).$$

A similar argument at every $p_m$ implies the global correspondence among $\Lambda^{\gamma}_{\alpha\beta}$'s. Here, the vanishing of $\Lambda^{\gamma}_{\alpha\beta}$ concerns us later, and we do not care about coefficients.

LEMMA 4.2. *For suitable frames around $p_m$, we have the correspondences $\Lambda^{i}_{\underline{j}\underline{k}}(p_m) \sim \Lambda^{i'}_{\underline{j}'\underline{k}'}(p_n)$, where $i, j, k$ at $p_m$ correspond to $i', j', k'$ at $p_n$ in Table 1.*

| $p_1$ | $p_2$ | $p_3$ | $p_4$ | $p_5$ | $p_6$ |
|---|---|---|---|---|---|
| 1 | 1 | 5 | 3 | 3 | 5 |
| 2 | 6 | 6 | 2 | 4 | 4 |
| 3 | 5 | 1 | 1 | 5 | 3 |
| 4 | 4 | 2 | 6 | 6 | 2 |
| 5 | 3 | 3 | 5 | 1 | 1 |
| 6 | 2 | 4 | 4 | 2 | 6 |

*Table* 1

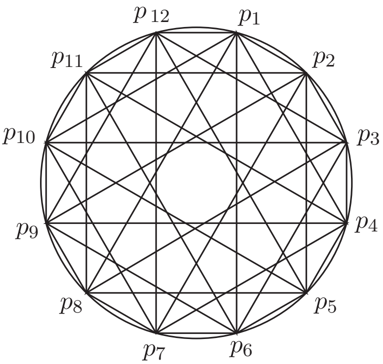


*Figure* 1

*Remark* 4.3. A local frame $e_i, e_{\bar{i}}$ $i = 1, \ldots, 6$ determines $\Lambda^{\gamma}_{\alpha\beta}$ locally, and at the same time, it determines $\Lambda^{\gamma}_{\alpha\beta}$ globally in some sense, by the correspondence along normal geodesics through each point in a neighborhood of $M$.

## 5. **The kernel of the shape operators**

Fix $\bar{p} \in M_+$ and let $L_6 = f^{-1}(\bar{p})$, denoting $f = f_6$. At $p \in L_6$, we consider $B_{\eta_p}$ and $B_{\zeta_p}$, where $\zeta_p = e_6(p) \in D_6(p)$ is arbitrarily chosen. Define the subspace $E(p, \zeta_p)$ of $T_{\bar{p}}M_+$ to be the space spanned by the kernels of all the shape operators of the form $L(t) = \cos tB_{\eta_p} + \sin tB_{\zeta_p}$; i.e.,

$$E(p, \zeta_p) = \mathrm{span}\{\mathrm{Ker}L(t) \mid t \in [0, 2\pi)\}.$$

By definition, $E(p, \zeta_p)$ is determined by the geodesic $c$ of $L_6$ through $p$ in the direction $e_6(p)$, and hence we can express

$$E(c) = E(p, \zeta_p) = \mathrm{span}\ \{D_3(q) \mid q \in c\}.$$

In Section 15, we show that $M$ is homogeneous if and only if $\dim E(c) = 2$ holds for all $c$.

Recall (19): $B_\eta(e_{\underline{i}}) = \mu_i e_{\underline{i}}$, $\mu_1 = -\mu_5 = \sqrt{3}$, $\mu_2 = -\mu_4 = \dfrac{1}{\sqrt{3}}$, where

$$\mu_i = \frac{1 + \lambda_i\lambda_6}{\lambda_6 - \lambda_i} = \frac{1 - \lambda_i\lambda_1}{\lambda_6 - \lambda_i} = \lambda_1\frac{\lambda_3 - \lambda_i}{\lambda_6 - \lambda_i} \tag{34}$$

because of $1/\lambda_1 = \lambda_3$. Recall (16) and (28), and we obtain

$$\sin\theta_6(\lambda_6 - \lambda_3) = 4c_0 = \sqrt{2}(\sqrt{3} - 1). \tag{35}$$

Put

$$c_1 = 4c_0\lambda_1 = \sqrt{2}(\sqrt{3}+1) = 1/c_0.$$

The following lemma is important.

LEMMA 5.1. *Take $p \in f^{-1}(\bar{p})$, and identify $T_{\bar{p}}M_+$ with $\oplus_{j=1}^5 D_j(p)$. Then for fixed $e_{\underline{3}} \in D_3(p)$ and $e_{\underline{6}} \in D_6(p)$, we have*

$$B_\eta(\nabla_{e_{\underline{6}}} e_{\underline{3}}) = c_1 \bar{\nabla}_{e_{\underline{3}}} e_{\underline{6}}, \tag{36}$$

$$B_\zeta(e_{\underline{3}}) = -\bar{\nabla}_{e_{\underline{3}}} e_6, \quad B_{\bar{\zeta}}(e_{\underline{3}}) = -\bar{\nabla}_{e_{\underline{3}}} e_{\bar{6}}, \tag{37}$$

$$B_\eta(\nabla^2_{e_{\underline{6}}} e_{\underline{3}}) = 2c_1 \nabla_{e_{\underline{6}}} \bar{\nabla}_{e_{\underline{3}}} e_{\underline{6}}, \tag{38}$$

$$B_\zeta(\nabla_{e_6} e_{\underline{3}}) = -\nabla_{e_6} \bar{\nabla}_{e_{\underline{3}}} e_6, \quad B_{\bar{\zeta}}(\nabla_{e_{\bar{6}}} e_{\underline{3}}) = -\nabla_{e_{\bar{6}}} \bar{\nabla}_{e_{\underline{3}}} e_{\bar{6}}. \tag{39}$$

*Remark* 5.2. Note that (36) implies $\nabla_{e_{\underline{6}}} e_{\underline{3}} \equiv 0$ modulo $D_3(p)$ if and only if $\bar{\nabla}_{e_{\underline{3}}} e_{\underline{6}} = 0$, and (37) implies that the kernel is independent of the normal direction only when $\bar{\nabla}_{e_{\underline{3}}} e_{\underline{6}} = 0$, thus, when $\nabla_{e_{\underline{6}}} e_{\underline{3}} \equiv 0$ modulo $D_3(p)$. See Remark 2.5.

*Proof.* Using (9), (17) and noting (35), we have

$$\begin{aligned} B_\eta(\nabla_{e_{\underline{6}}} e_{\underline{3}}) &= \Lambda^{i}_{\underline{63}} \mu_i e_{\underline{i}} = \Lambda^{i}_{\underline{63}} \lambda_1 \frac{\lambda_3 - \lambda_i}{\lambda_6 - \lambda_i} e_{\underline{i}} \\ &= \lambda_1 \Lambda^{i}_{\underline{36}} e_{\underline{i}} = \lambda_1 (4c_0) \bar{\nabla}_{e_{\underline{3}}} e_{\underline{6}} = c_1 \bar{\nabla}_{e_{\underline{3}}} e_{\underline{6}}, \end{aligned} \tag{40}$$

where $i$ is summed over $i \neq 6$. On the other hand, by the definition of the shape operators, we have

$$B_\zeta(e_{\underline{3}}) = -\bar{\nabla}_{e_{\underline{3}}} e_6, \quad B_{\bar{\zeta}}(e_{\underline{3}}) = -\bar{\nabla}_{e_{\underline{3}}} e_{\bar{6}}.$$

Recall (24) and (27), namely, $L(t+\pi/2) = L_t(t) = c_0 \nabla_{e_6} L(t)$, where $\nabla_{\frac{d}{dt}} = c_0 \nabla_{e_6}$. Taking the covariant derivative of (37), we have

$$-\nabla_{e_6} \bar{\nabla}_{e_3} e_6 = \nabla_{e_6} \left( B_\zeta(e_3) \right) = -1/c_0 B_\eta(e_3) + B_\zeta(\nabla_{e_6} e_3) = B_\zeta(\nabla_{e_6} e_3).$$

Finally taking the covariant derivative of (36), and using (39), we have

$$\begin{aligned} c_1 \nabla_{e_6} \bar{\nabla}_{e_3} e_6 &= \nabla_{e_6} \left( B_\eta(\nabla_{e_6} e_3) \right) \\ &= 1/c_0 B_\zeta(\nabla_{e_6} e_3) + B_\eta(\nabla^2_{e_6} e_3) \\ &= -1/c_0 \nabla_{e_6} \bar{\nabla}_{e_3} e_6 + B_\eta(\nabla^2_{e_6} e_3). \end{aligned}$$

Then from $c_1 + 1/c_0 = 2c_1$, (38) follows. Similar formulas hold for indices with a bar. □

Let $E^\perp(c)$ be the orthogonal complement of $E(c)$ in $T_{\bar{p}}M_+$, and let

$$W(c) = \operatorname{span}\{\bar{\nabla}_{e_3} e_6(q), \bar{\nabla}_{e_{\bar{3}}} e_6(q), \mid q \in c\} \subset T_{\bar{p}}M_+,$$

where $e_6(q)$ is the unit tangent vector of $c$ at $q$. Note that it does not depend on the choice of the frame $e_3, e_{\bar{3}}$ of $D_3(q)$.

LEMMA 5.3. $W(c) \subset E^{\perp}(c)$.

*Proof.* Take any $q \in c$, and express $L(t) = \cos t B_\eta + \sin t B_\zeta$ with respect to $e_i(q), e_{\bar{i}}(q)$, $i = 1, \ldots 5$, as in Lemma 2.3:

$$L(t) = \begin{pmatrix} \sqrt{3}c & sB_{12} & sB_{13} & sB_{14} & sB_{15} \\ sB_{21} & \frac{1}{\sqrt{3}}c & sB_{23} & sB_{24} & sB_{25} \\ sB_{31} & sB_{32} & 0 & sB_{34} & sB_{35} \\ sB_{41} & sB_{42} & sB_{43} & -\frac{1}{\sqrt{3}}c & sB_{45} \\ sB_{51} & sB_{52} & sB_{53} & sB_{54} & -\sqrt{3}c \end{pmatrix}, \quad \begin{cases} c = \cos t, \\ s = \sin t. \end{cases}$$

Let $e_{\underline{3}}(t) = {}^t(u_1(t), u_{\bar{1}}(t), \ldots, u_5(t), u_{\bar{5}}(t))$ belong to the kernel of $L(t)$. Then the third block of $L(t)(e_{\underline{3}}(t))$ must satisfy

$$\frac{\sin t}{\sin\theta_6}\frac{1}{\lambda_3 - \lambda_6}\sum_{j=1}^{5}\begin{pmatrix} \Lambda_{36}^{j}(q) & \Lambda_{36}^{\bar{j}}(q) \\ \Lambda_{\bar{3}6}^{j}(q) & \Lambda_{\bar{3}6}^{\bar{j}}(q) \end{pmatrix}\begin{pmatrix} u_j(t) \\ u_{\bar{j}}(t) \end{pmatrix} = 0.$$

Thus we obtain

$$\langle \bar{\nabla}_{e_3} e_6(q), e_{\underline{3}}(t)\rangle = 0, \quad \langle \bar{\nabla}_{e_{\bar{3}}} e_6(q), e_{\underline{3}}(t)\rangle = 0 \tag{41}$$

for all $t$ and any $q \in c$. This means $\bar{\nabla}_{e_3} e_6(q), \bar{\nabla}_{e_{\bar{3}}} e_6(q) \in E^{\perp}(c)$. □

By the analyticity and the definition of $E(c)$ and $W(c)$, we can express for a fixed $q \in c$,

$$\begin{aligned} E(c) &= \mathrm{span}\{e_{\underline{3}}(q), \nabla_{e_6}^{k} e_{\underline{3}}(q),\ k = 1, 2, \ldots\}, \\ W(c) &= \mathrm{span}\{\bar{\nabla}_{e_{\underline{3}}} e_6(q), \nabla_{e_6}^{k}\bar{\nabla}_{e_{\underline{3}}} e_6(q),\ k = 1, 2, \ldots\}, \end{aligned} \tag{42}$$

which do not depend on the choice of $q$. Thus for any frame of $D_3(q)$, we have

$$\langle \nabla_{e_6}^{k} e_{\underline{3}}, \nabla_{e_6}^{l}\bar{\nabla}_{e_{\underline{3}}} e_6\rangle = 0, \tag{43}$$

where $k, l = 0, 1, 2, \ldots$ and $e_{\underline{3}} \in D_3$.

LEMMA 5.4. *For any $t$, $L(t)$ maps $E(c)$ onto $W(c) \subset E^{\perp}(c)$.*

*Proof.* We can express $L(t) = \cos t L(\tau) + \sin t L_t(\tau)$ for any $\tau$. Then $L(\tau)(e_3(\tau)) = 0$ and $L_t(\tau)(e_3(\tau)) = -\bar{\nabla}_{e_3} e_6(\tau)$ (see (37)) imply

$$\begin{aligned} L(t)(e_3(\tau)) &= (\cos t L(\tau) + \sin t L_t(\tau))(e_3(\tau)) \\ &= -\sin t\bar{\nabla}_{e_3} e_6(\tau) \in W(c). \end{aligned} \tag{44}$$

Since $e_3(\tau)$ for $\tau \in [0, 2\pi)$ spans $E(c)$, $L(t)(E(c))$ is a subset of $W(c)$. Surjectivity follows from (37). □

LEMMA 5.5. $\dim E(c) \leq 6$ *holds for any geodesic $c$ of $L_6$.*

*Proof.* Take any $p\in c$. Since $\operatorname{Ker} B_{\eta_p} = D_3(p) \subset E(c)$, we have $\dim B_\eta(E(c)) = \dim E(c) - 2$. Because $B_{\eta_p}(E(c))$ is a subspace of $E^\perp(c)$, the lemma follows from $\mathbb{R}^{10} \cong T_{\bar{p}}M_+ = E(c) \oplus E^\perp(c)$. □

## 6. **Reduction of the matrix size**

Fix a geodesic $c$ of $L_6(p)$, and let $\zeta = e_6(p)$ be its unit tangent vector at $p$. Consider $L(t) = \cos t B_\eta + \sin t B_\zeta$. The following lemma is fundamental.

Lemma 6.1. *When $\dim E(c) = d$ where $2 \le d \le 6$, we can express $L = L(t)$ as*

$$L = \begin{pmatrix} 0 & R \\ {}^tR & S \end{pmatrix},$$

*with respect to the decomposition $T_{\bar{p}}M_+ = E(c) \oplus E^\perp(c)$, where $0$ is $d$ by $d$, $R$ is $d$ by $10-d$ and $S$ is $10-d$ by $10-d$ matrices. The kernel of $L$ is given by*

$$\begin{pmatrix} X \\ 0 \end{pmatrix} \in E(c), \quad {}^tRX = 0, \tag{45}$$

*and*

$$\operatorname{rank} {}^tR = \operatorname{rank} R = d-2. \tag{46}$$

*The eigenvectors with respect to $\mu_i$, $i = 1, 2, 4, 5$ are given by*

$$\begin{pmatrix} \frac{1}{\mu_i} RY \\ Y \end{pmatrix}, \tag{47}$$

*where $Y \in E(c)^\perp$ is a solution of*

$$({}^tRR + \mu_i S - \mu_i^2 I)Y = 0. \tag{48}$$

*Proof.* The first part follows from Lemma 5.4. Let $\left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right)$ be an eigenvector of $L$ with respect to $\mu_i$, where $X \in E(c)$ and $Y \in E(c)^\perp$, abusing the notation $X = \left(\begin{smallmatrix} X \\ 0 \end{smallmatrix}\right)$ and $Y = \left(\begin{smallmatrix} 0 \\ Y \end{smallmatrix}\right)$. Then we have

$$\begin{pmatrix} 0 & R \\ {}^tR & S \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} RY \\ {}^tRX + SY \end{pmatrix} = \mu_i \begin{pmatrix} X \\ Y \end{pmatrix},$$

and hence

$$\begin{cases} RY = \mu_i X, \\ {}^tRX + SY = \mu_i Y. \end{cases}$$

For $\mu_3 = 0$, $Y = 0$ and ${}^tRX = 0$ hold since the kernel belongs to $E(c)$. Thus the image of $E(c)$ under ${}^tR$ is of dimension $d-2$, which implies (46). When $\mu_i \neq 0$, multiplying $\mu_i$ to the second equation, and substituting the first one into it, we obtain

$${}^tRRY + \mu_i SY = \mu_i^2 Y.$$

Then the eigenvector of $L$ for an eigenvalue $\mu_i$ is given by (47). □

Proposition 6.2. *When $d = 6$, we have for any $t$,*

$$\det({}^tR(t)R(t)) = 1. \tag{49}$$

*In particular, ${}^tR(t)R(t)$ is positive definite.*

*Proof.* When $d=6$, ${}^tRR$ is a 4-by-4 matrix. Equation (48) has 2-dimensional solutions for $\mu \in \{\pm\sqrt{3}, \pm 1/\sqrt{3}\}$ (see (19)), and hence we obtain

$$\det({}^tR(t)R(t) + \mu S(t) - \mu^2 I) = (\mu^2 - 3)^2 \left(\mu^2 - \frac{1}{3}\right)^2. \tag{50}$$

If we put $\mu = 0$, then (49) follows. □

## 7. Basic investigation

### 7.1. *Behavior of $D_3(t)$.*

Lemma 7.1. *Let $c$ be a geodesic of $L_6(p)$, and let $p, q \in c$, which are not antipodal. If $e_3(p) = \pm e_3(q)$ holds, then $e_3(p) \in D_3(t)$ holds for all $t$, and $e_3(t) = e_3(p)$ is parallel along $c$. In particular, if $\nabla_{e_6} e_3(p) \equiv 0$ modulo $D_3(p)$, then $e_3(t) = e_3(p)$ is parallel along $c$.*

*Proof.* If two linear operators have a common kernel vector $v$, all the linear combinations of these operators have $v$ as a kernel vector. Thus when $e_3(p) = \pm e_3(q)$ holds, $L(t) = \cos tB_\eta + \sin tB_\zeta$ has a kernel $e_3(t) = e_3(p)$, independent of $t$. If $\nabla_{e_6} e_3(p) \equiv 0$ modulo $D_3(p)$, then $B_\zeta(e_3) = -\bar{\nabla}_{e_3} e_6(p) = -1/c_1 B_\eta(\nabla_{e_6} e_3(p)) = 0$ follows from (36) and (37), which means $e_3(\pi/2) = e_3(p)$. Thus $e_3(t) = e_3(p)$ is parallel along $c$. □

Corollary 7.2. *Let $c$ be a geodesic of $L_6(p)$, and let $p, q \in c$, which are not antipodal. If $D_3(p) = D_3(q)$ holds, then* $\dim E(c) = 2$.

Lemma 7.3. *Let* $\dim E(c) = 2$ *hold for at least two distinct geodesics of $L_6(p)$. Then* $\dim E(\gamma) = 2$ *holds for any geodesic $\gamma$, and $E(\gamma) = D_3(p)$ is parallel along $L_6(p)$.*

*Proof.* When $\dim E(c_1) = 2 = \dim E(c_2)$, let $p \in c_1 \cap c_2$. Because $D_3(p) \subset E(c_i)$, $i = 1, 2$, $\dim E(c_i) = 2$ implies $E(c_1) = D_3(p) = E(c_2)$. Then for any geodesic $\gamma$, taking $q \in \gamma \cap c_1$ and $r \in \gamma \cap c_2$, we obtain $D_3(q) = D_3(r) = D_3(p)$. Thus the lemma follows from Corollary 7.2. □

*Remark* 7.4. Therefore, $\dim E(c) = 2$ holds either for at most one geodesic of $L_6$ or for all the geodesics of $L_6$.

In the following, we identify $L_6(p)$ with the unit sphere $S^2 \subset T^\perp_{\bar{p}} M_+$ by the correspondence $L_6(p) \ni q \mapsto \eta_q \in S^2$ given in the proof of Lemma 2.4. Through this identification, a geodesic $c$ of $L_6(p)$ corresponds to the one-parameter family of the shape operators $L(t)$. Note that the space of oriented geodesics of $S^2$ is identified with $S^2$ itself, by assigning $c$ to the point $p_c \in S^2$ normal to the plane on which $c$ lies, where we distinguish the orientation of $c$. Let $T_1^\perp M_+$ be the unit normal bundle of $M_+$. When we regard $T_1^\perp M_+$ as the sphere bundle with fibers consisting of oriented geodesics of $S^2$, we denote it by $G_+ \to M_+$. It is easy to see that the total space of $G_+$ is diffeomorphic to $M$. When the complement of a subset $\mathcal{U}$ of $G_+$ is of measure zero, we call elements of $\mathcal{U}$ generic, where $G_+$ is equipped with the natural metric. Since $\dim E(c)$ is a lower semi-continuous function on $G_+$, $\dim E(c) > 2$ is an open condition. More precisely, using the analyticity, we have (see also the remark above)

Lemma 7.5. *When* $\dim E(c) > 2$ *holds for some* $c \in G_+$, $\dim E(c') > 2$ *holds for generic* $c' \in G_+$.

Now consider the other focal submanifold $M_-$. We denote by $G_- \to M_-$ the $S^2$ bundle of which the fiber is the space of oriented geodesics of $S^2 \subset T_{\bar{q}}^\perp M_-$. Let $\gamma \in G_-$, and define

$$F(\gamma) = \operatorname{span}\{D_4(q) \mid q \in \gamma\},$$

where $D_4(q)$ is the kernel of the shape operator of $M_-$ in the normal direction $\eta_q$. The argument on $M_+$ can be applied to $M_-$ if we replace $E(c)$ by $F(\gamma)$ and change indices suitably. Moreover, if $\dim E(c) = 2$ holds on an open subset of $G_+$, then $\Lambda^\alpha_{\underline{36}} = 0$, $\alpha \neq \underline{6}$, holds identically on $M$ by the analyticity. Thus $\Lambda^\alpha_{\underline{14}} = 0$, $\alpha \neq \underline{4}$, follows by the global correspondence in Section 4, and $\dim F(\gamma) = 2$ holds for any $\gamma$. As a conclusion, we have

Lemma 7.6. *If* $\dim E(c) = 2$ *holds on an open subset of* $G_+$, *then this holds over all* $G_+$, *and moreover,* $\dim F(\gamma) = 2$ *holds over all* $G_-$. *The same is true if we replace* $E(c)$ *by* $F(\gamma)$ *and* $G_+$ *by* $G_-$.

7.2. *Behavior at* $p(t+\pi)$. Take a point $p \in M$, and let $e_1(p), e_{\bar{1}}(p), \ldots, e_5(p)$, $e_{\bar{5}}(p)$ be an orthonormal basis of $T_{\bar{p}}M_+$. Let $c$ be a geodesic of $L_6(p)$ through $p$, and let $q \in L_6(p)$ be not on $c$. Since $D_i \to L_6(p)$ is a vector bundle over $L_6(p) \cong S^2$, it is trivial on $L_6(p) \setminus \{q\}$, and we can extend the frame $e_i(p), e_{\bar{i}}(p) \in D_i(p)$ over $L_6(p) \setminus \{q\}$ continuously, or more strongly, analytically since $D_i$ is analytic. In particular, along $c = c(t)$, we obtain an analytic frame $e_i(t), e_{\bar{i}}(t) \in D_i(c(t))$ such that

$$e_i(2\pi) = e_i(0), \quad e_{\bar{i}}(2\pi) = e_{\bar{i}}(0). \tag{51}$$

This is an advantage of $m = 2$ since when $m = 1$, $e_i(2\pi)$ equals to $e_i(0)$ only up to sign. As for $D_3(t)$, we have more.

LEMMA 7.7. *Along a geodesic $c$ of $L_6(p)$, we have an analytic frame $e_3(t)$, $e_{\bar{3}}(t)$ of $D_3(t)$ such that*

$$e_3(\pi) = \varepsilon e_3(0), \quad e_{\bar{3}}(\pi) = \varepsilon e_{\bar{3}}(0), \quad \varepsilon = \pm 1.$$

*Proof.* By the above argument, we may choose a frame of $D_1(t), D_2(t)$ so that

$$e_{\underline{1}}(t + 2\pi) = e_{\underline{1}}(t), \quad e_{\underline{2}}(t + 2\pi) = e_{\underline{2}}(t).$$

Since $D_{6-i}(t + \pi) = D_i(t)$ holds by the global symmetry (see Section 4), we may define

$$e_{\underline{5}}(t) = e_{\underline{1}}(t + \pi), \quad e_{\underline{4}}(t) = e_{\underline{2}}(t + \pi),$$

and we may choose a frame of $D_3(t)$ so that

$$e_3(t + \pi) = \varepsilon e_3(t), \quad e_{\bar{3}}(t + \pi) = \varepsilon' e_{\bar{3}}(t), \quad \varepsilon, \varepsilon' = \pm 1.$$

Let $U(t)$ be such that $e_{\underline{i}}(t) = U(t)e_{\underline{i}}(0)$. Since $U(0) = I \in \mathrm{SO}(10)$, $U(t) \in \mathrm{SO}(10)$ follows by the continuity. Then from

$$\begin{aligned} &e_{\underline{1}}(\pi) = e_{\underline{5}}(0), && e_{\underline{2}}(\pi) = e_{\underline{4}}(0), \\ &e_{\underline{5}}(\pi) = e_{\underline{1}}(2\pi) = e_1(0), && e_{\underline{4}}(\pi) = e_{\underline{2}}(2\pi) = e_2(0), \end{aligned}$$

and because $U(\pi) \in \mathrm{SO}(10)$, we obtain

$$\varepsilon = \varepsilon'. \tag{52}$$

□

*Remark* 7.8. In Section 11, we construct such a frame explicitly.

## 8. **Dimension of $E(c)$**

The purpose of this section is to prove the following crucial proposition.

PROPOSITION 8.1. $\dim E(c) = 6$ *holds if* $\dim E(c) > 2$.

To show this, we need a special frame of $D_3(t)$ along $c$. For a vector field $v(t)$ on $c$, we call $v(t)$ *even* when $v(t+\pi) = v(t)$ and *odd* when $v(t+\pi) = -v(t)$. We sometimes denote $v(0) = v(p)$.

Put $d = \dim E(c)$, and let $E'$ be the orthogonal complement of $e_{\bar{3}}(0)$ in $E(c)$. Note that $D_3(t)$ depends on $t$ analytically, and $\dim D_3(t) \cap E' \geq 2 + (d-1) - d = 1$ holds for each $t$. Here the equality holds for small $t$ as $e_{\bar{3}}(p)$ is orthogonal to $E'$. Thus we have an analytic field $e_3(t) \in D_3(t) \cap E'$ for $t$ in some interval $I$ containing 0. At this moment, we are not sure if $I$ covers $c$ or not.

LEMMA 8.2. $\dim D_3(t) \cap E' = 1$ *holds for all $t$, and we have an analytic field $e_3(t) \in D_3(t)$ on $c$, which is always orthogonal to $e_{\bar{3}}(0)$. If we put $S = \mathrm{span}_t\{e_3(t)\}$, then the space $L(t)(S)$ does not depend on $t$, which we denote by $V$. In particular,* $\dim V = \dim S - 1$ *holds.*

*Proof.* Put $\tilde{S} = \mathrm{span}_t\,(D_3(t) \cap E') \subset E'$. For any $e_3(\tau) \in \tilde{S}$, we can express $L(t) = \cos t L(\tau) + \sin t L_t(\tau)$, and so $L(\tau)(e_3(\tau)) = 0$ and $L_t(\tau)(e_3(\tau)) = -\bar{\nabla}_{e_3} e_6(\tau)$ (see (37)) imply

$$L(t)(e_3(\tau)) = (\cos t L(\tau) + \sin t L_t(\tau))(e_3(\tau)) = -\sin t \bar{\nabla}_{e_3} e_6(\tau), \tag{53}$$

of which direction is independent of $t$. Therefore,

$$\tilde{V} = L(t)(\tilde{S}) = \mathrm{span}\{\bar{\nabla}_{e_3} e_6(\tau) \mid e_3(\tau) \in \tilde{S}\}$$

is independent of $t$. Suppose $\dim \tilde{V} = \dim \tilde{S} - 2$. Then $\tilde{S}$ contains $\ker L(t)$; namely, $D_3(t) \subset \tilde{S} \subset E'$ holds for all $t \in I$, which contradicts that $e_{\bar{3}}(0)$ is orthogonal to $E'$. Thus $\dim \tilde{V} = \dim \tilde{S} - 1$. This means $D_3(t) \cap E'$ is of dimension one for all $t$, and we obtain $I = [0, 2\pi)$. Moreover, $\tilde{S} = S$ and $\tilde{V} = V$ follow. ☐

Next, take $\hat{e}_3(t) \in D_3(t)$ orthogonal to $e_3(0)$.

Claim. *For each $t$, $e_3(t)$ and $\hat{e}_3(t)$ are independent.*

In fact, suppose these are dependent for some $t_0$. Let $E'_0$ be the orthogonal complement of $e_3(t_0) = \pm\hat{e}_3(t_0)$ in $E(c)$. Then $D_3(0) \subset E'_0$ follows. However, applying the above argument to $E'_0$, we have a contradition.

Because $\dim D_3(t) \cap E' = 1$ holds for all $t \in [0, 2\pi)$, any $\tilde{e}_3(t) \in D_3(t)$ that is independent of $e_3(t)$ does not belong to $E'$, namely, is not orthogonal to $e_{\bar{3}}(0) = \hat{e}_3(0)$ for each $t$. Thus $\hat{e}_3(t) \in D_3(t)$ satisfies

$$\langle \hat{e}_3(0), \hat{e}_3(t) \rangle \neq 0, \tag{54}$$

and hence $\hat{e}_3(t)$ is an even vector since we have $\hat{e}_3(t+\pi) = \pm\hat{e}_3(t)$. This is also true for $e_3(t)$.

Lemma 8.3. *If we choose $e_3(t)$ orthogonal to $e_{\bar{3}}(0)$, then $e_3(t)$, $\nabla_{e_6} e_3(t)$, $\nabla^2_{e_6} e_3(t), \ldots$ are even vectors in $S$. On the other hand, $\bar{\nabla}_{e_3} e_6(t)$, $\nabla_{e_6} \bar{\nabla}_{e_3} e_6(t)$, $\nabla^2_{e_6} \bar{\nabla}_{e_3} e_6(t), \ldots$ are odd vectors in $V$. These are true if we replace $e_3(t)$ by $\hat{e}_3(t)$.*

*Proof.* The former is clear from $\nabla^k_{e_6} e_3(t+\pi) = \nabla^k_{e_6} e_3(t)$. The latter follows from $L(t+\pi) = -L(t)$ and $L(t)(\nabla_{e_6} e_3(t)) = c_1 \bar{\nabla}_{e_3} e_6(t)$. Then its derivatives in the direction $e_6(t)$ are all odd. ☐

Since $D_3(t) = \mathrm{span}\{e_3(t), \hat{e}_3(t)\}$ at each $t$, putting $\hat{S} = \mathrm{span}_t\{\hat{e}_3(t)\}$ and $\hat{V} = L(t)(\hat{S})$, we have

$$E(c) = S + \hat{S}, \quad W(c) = V + \hat{V}.$$

As $S$ ($\hat{S}$, resp.) is orthogonal to $e_{\bar{3}}(0)$ ($e_3(0)$, resp.), we have

$$\dim S, \dim \hat{S} \leq 5, \quad \dim V, \dim \hat{V} \leq 4. \tag{55}$$

For the same reason, $\dim E(c) = 6$ follows if $\dim S = 5$ or $\dim \hat{S} = 5$ holds. Thus to prove Proposition 8.1, we may consider the cases $\dim S, \dim \hat{S} \in \{1, 2, 3, 4\}$. First, we prove

LEMMA 8.4. *For any $c$, and for* any *continuous vector field $e_3(t) \in D_3(t)$ along $c$,* $\dim \operatorname{span}_t\{e_3(t)\} = 2$ *implies* $\dim E(c) = 2$. *Thus* $\dim E(c) > 2$ *implies* $\dim \operatorname{span}_t\{e_3(t)\} > 2$, *unless* $\dim \operatorname{span}_t\{e_3(t)\} = 1$.

*Proof.* This lemma holds for *any* continous $e_3(t)$, and so we put $K = \operatorname{span}_t\{e_3(t)\}$ instead of $S$. Assume $\dim K = 2$. Then it follows $\nabla_{e_6} e_3(p) \not\equiv 0$ modulo $D_3(p)$. For $q = p(\pi/2)$, we have $K = \operatorname{span}\{e_3(p), e_3(q)\}$ by Lemma 7.1. Thus we may express

$$e_3(t) = a(t)e_3(p) + b(t)e_3(q) \in K.$$

Recall (37)

$$B_\zeta(e_3(p)) = -\bar{\nabla}_{e_3} e_6(p)$$

and, because $e_3(q) \in \operatorname{Ker} L(\pi/2) = \ker B_\zeta$, exchanging $p$ and $q$, we have

$$B_\eta(e_3(q)) = \varepsilon \bar{\nabla}_{e_3} e_6(q), \quad \varepsilon = \pm 1.$$

Therefore, denoting $c = \cos t$, $s = \sin t$, $a = a(t)$ and $b = b(t)$, we have

$$\begin{aligned} 0 = L(t)e_3(t) &= (cB_\eta + sB_\zeta)(ae_3(p) + be_3(q)) \\ &= bcB_\eta(e_3(q)) + asB_\zeta(e_3(p)) = -bc\varepsilon\bar{\nabla}_{e_3} e_6(q) - as\bar{\nabla}_{e_3} e_6(p), \end{aligned}$$

from which it follows

$$\bar{\nabla}_{e_3} e_6(q) = u\bar{\nabla}_{e_3} e_6(p) \tag{56}$$

for some nonzero $u$. Thus multiplying by $1/\mu_i$ on both sides of

$$\Lambda^{\underline{i}}_{36}(q)e_{\underline{i}}(q) = u\Lambda^{\underline{i}}_{36}(p)e_{\underline{i}}(p),$$

and summing up in $i \neq 6$, via (36) and

$$\langle \nabla_{e_6} e_3, e_3 \rangle = 0, \tag{57}$$

we obtain

$$\nabla^\perp_{e_6} e_3(q) = u\nabla^\perp_{e_6} e_3(p), \tag{58}$$

where $\nabla^\perp_{e_6} e_3$ is the component of $\nabla_{e_6} e_3$ orthogonal to $D_3$. Note that (58) implies $\nabla^\perp_{e_6} e_3(p)$ is orthogonal to $D_3(q)$, too. Thus we can express

$$\begin{aligned} &0 \neq \nabla_{e_6} e_3(p) = \nabla^\perp_{e_6} e_3(p) + ke_{\bar{3}}(p) \in K, \\ &0 \neq \nabla_{e_6} e_3(q) = \nabla^\perp_{e_6} e_3(q) + le_{\bar{3}}(q) = u\nabla^\perp_{e_6} e_3(p) + le_{\bar{3}}(q) \in K; \end{aligned} \tag{59}$$

see Remark 5.2. On the other hand, by (57), we can express

$$K = \mathbb{R}e_3(p) \oplus \mathbb{R}\nabla_{e_6} e_3(p) = \mathbb{R}e_3(q) \oplus \mathbb{R}\nabla_{e_6} e_3(q). \tag{60}$$

Thus if $K$ is orthogonal to $\nabla^{\perp}_{e_6} e_3(p)$, then $\nabla^{\perp}_{e_6} e_3(p) = 0$ follows, and from (59) and (60), we obtain $K = D_3(p) = D_3(q)$, which implies $\dim E(c) = 2$ by Corollary 7.2. When $\nabla^{\perp}_{e_6} e_3(p) \neq 0$, $K$ is not orthogonal to $\nabla^{\perp}_{e_6} e_3(p)$. Thus an element of $K$ orthogonal to $\nabla^{\perp}_{e_6} e_3(p)$ lies in the 1-dimensional space, which we may express as

$$e'_3(p) = ue_3(p) + ve_{\bar{3}}(p) = we_3(q) + ze_{\bar{3}}(q) = e''_3(q) \in K \tag{61}$$

for some $u, v, w, z$, $v^2 + z^2 \neq 0$. Therefore, $e'_3(p)$ turns out to be parallel along $c$ by Lemma 7.1. Since $e_3(t) \in K$ is independent of $e'_3(p)$ for generic $t$, $e_3(t)$ and $e'_3(p)$ span $D_3(t)$, and we conclude that

$$E(c) = \operatorname{span}\{e_3(t), e'_3(p)\} = K$$

and $\dim E(c) = 2$. □

Even if we assume $\dim E(c) > 2$, there might exist $e_3$ parallel along $c$. The following proposition, based on the previous lemma, implies this is not the case.

Proposition 8.5. *When* $\dim E(c) > 2$, *for a generic geodesic* $c$ *through* $p$, *there does* not *exist* $e_3$ *parallel along* $c$.

*Proof.* Let $c_s$ be a geodesic through $p$ in the direction $e^s_6(p) = \cos s e_6(p) + \sin s e_{\bar{6}}(p)$. Suppose there exists an interval $J$ containing $s = 0$ such that for each $s \in J$, there exists $e^s_3(p)$ parallel along $c_s$. For $0 < s < \pi$, $e^0_3(p)$ and $e^s_3(p)$ are independent. In fact, if $e^0_3(p) = e^s_3(p)$ holds for some $s$, then $\nabla_{e^0_6} e^0_3(p) \equiv 0 \equiv \nabla_{e^s_6} e^0_3(p)$ modulo $D_3(p)$ holds for this $s$, which implies $\nabla_{e_{\bar{6}}} e^0_3(p) \equiv 0$. Hence $\bar{\nabla}_{e^0_3} e_6(p) = 0 = \bar{\nabla}_{e^0_3} e_{\bar{6}}(p)$ holds by (37), and by the global correspondence (see Figure 1 in Section 4), we have $\bar{\nabla}^-_{e_1} e_4(p_3) = \bar{\nabla}^-_{e_1} e_{\bar{4}}(p_3) = 0$, where $\bar{\nabla}^-$ is the connection of $M_-$, which implies $\nabla_{e_1} e_4(p_3) \equiv \nabla_{e_1} e_{\bar{4}}(p_3) \equiv 0$ modulo $D_4(p_3)$. Thus the kernel vector of the shape operator of $M_-$ at $f_1(p_3)$ is parallel, which does not occur generically under our assumption (Lemma 7.6).

Thus $e^0_3(p)$ and $e^s_3(p)$ are independent in $D_3(p)$ for $s \neq 0$ modulo $\pi$. Let $c'$ be a geodesic of $L_6$ intersecting both $c$ and $\bar{c} = c_{\pi/2}$. Since $e^s_3(p)$ is parallel along $c_s$, $e^s_3(p)$ lies in $D_3(p_s)$ where $p_s \in c_s \cap c'$. Hence $e_3(s) = e^s_3(p) \in E(c')$ spans a 2-dimensional space $D_3(p)$ along $c'$. This implies $\dim E(c') = 2$ by Lemma 8.4, and since $c'$ is arbitrarily chosen, Lemma 7.3 implies $\dim E(c) = 2$, a contradiction. (It is sufficient to consider a family of geodesics through a point, since an open set of $G_+$ always contains such a family.) □

Corollary 8.6. *When* $\dim E(c) > 2$, $\dim S, \dim S' \geq 3$, *and hence* $e_3(t)$ *and* $\hat{e}_3(t)$ *move nonlinearly.*

Lemma 8.7. $\dim S = 4$ *or* $\dim \hat{S} = 4$ *never occurs*; *i.e.*, $\dim S, \dim \hat{S} \in \{3, 5\}$.

*Proof.* Suppose $\dim S = 4$, and let $S' \subset S$ be the 3-dimensional subspace orthogonal to $e_3(0)$. Then $L(t)$ is of rank 3 on $S'$ for all $t$ because its kernel $e_3(t)$ has a nontrivial $e_3(0)$ component (see (54)). Thus for a fixed frame $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$ of $S'$, we obtain a continuous frame of $V$ by

$$\boldsymbol{v}_1(t) = L(t)(\boldsymbol{u}_1), \quad \boldsymbol{v}_2(t) = L(t)(\boldsymbol{u}_2), \quad \boldsymbol{v}_3(t) = L(t)(\boldsymbol{u}_3).$$

However, these are odd vectors as before, and they reverse the orientation of $V$, contradicting that $\dim V = 3$ and $V$ is parallel. □

Thus by the statement before Lemma 8.4, it is sufficient for the proof of Proposition 8.1 to consider the case $\dim S = 3 = \dim \hat{S}$. In this case, it is obvious that $\dim E(c) = \dim(S + \hat{S}) \geq 4$ since $S$ is orthogonal to $e_{\bar{3}}(p) \in \hat{S} \subset E(c)$. Now we show

LEMMA 8.8. $\dim E(c) = 5$ *does not occur.*

*Proof.* If it occurs, $\dim W(c) = 3$ follows, where $W(c) = L(t)(E(c))$. Let $E' \subset E(c)$ be the 3-dimensional subspace orthogonal to $D_3(0)$. Then $L(t)$ is of rank 3 on $E'$ for all $t$ since $e_3(t)$ has a nontrivial $e_3(0)$ component, and $\hat{e}_3(t)$ has a nontrivial $\hat{e}_3(0)$ component by (54). Thus for a fixed frame $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$ of $E'$, we obtain a continuous frame of $W(c)$:

$$\boldsymbol{v}_1(t) = L(t)(\boldsymbol{u}_1), \quad \boldsymbol{v}_2(t) = L(t)(\boldsymbol{u}_2), \quad \boldsymbol{v}_3(t) = L(t)(\boldsymbol{u}_3).$$

However, these are odd vectors as before, and they reverse the orientation of $W(c)$, a contradiction. Thus $\dim E(c) \neq 5$. □

The following depends on Proposition 8.5, and the proof is similar to that of Lemma 8.4.

LEMMA 8.9. $\dim E(c) = 4$ *does not occur.*

*Proof.* Suppose $\dim E(c) = 4$ occurs on a nonempty open set of $G_+$. Then, denoting by $\nabla^{\perp}_{e_6} e_3$ the component orthogonal to $D_3$, for any independent $e_3(0), e'_3(0)$, we can express

$$E(c) = D_3(0) \oplus \operatorname{span}\{\nabla^{\perp}_{e_6} e_3(0), \nabla^{\perp}_{e_6} e'_3(0)\}. \tag{62}$$

In fact, if $\nabla^{\perp}_{e_6} e_3(0)$ and $\nabla^{\perp}_{e_6} e'_3(0)$ are dependent, we have $\tilde{e}_3(0)$ such that $\nabla^{\perp}_{e_6} \tilde{e}_3(0) = 0$; namely, $\nabla_{e_6} \tilde{e}_3(0) \equiv 0$ modulo $D_3(0)$. However, then by Remark 5.2, $\tilde{e}_3(0)$ is parallel along $c$, contradicting Proposition 8.5. Thus (62) holds, and we have orthogonal decompositions at $p = p(0)$ and $q = p(\pi/2)$:

$$\begin{aligned} E(c) &= D_3(p) \oplus \operatorname{span}\{\nabla^{\perp}_{e_6} e_3(p), \nabla^{\perp}_{e_6} e'_3(p)\} \\ &= D_3(q) \oplus \operatorname{span}\{\nabla^{\perp}_{e_6} e_3(q), \nabla^{\perp}_{e_6} e'_3(q)\}. \end{aligned} \tag{63}$$

Also $E(c) = D_3(p) + D_3(q)$ holds since $D_3(p) \cap D_3(q) = \{0\}$ (Lemma 7.1 and Proposition 8.5). Thus we may express

$$e_3(t) = ae_3(p) + be_3(q), \quad e_{\bar{3}}(t) = \bar{a}(t)e_3'(p) + \bar{b}(t)e_3'(q)$$

for some $e_3(p), e_3'(p) \in D_3(p)$, $e_3(q), e_3'(q) \in D_3(q)$, which are independent for generic $t$. Just as we obtain (56) in the proof of Lemma 8.4, we have

$$\nabla^{\perp}_{e_6} e_3(q) = u\nabla^{\perp}_{e_6} e_3(p), \quad \nabla^{\perp}_{e_6} e_3'(q) = v\nabla^{\perp}_{e_6} e_3'(p)$$

for some nonzero $u, v$. Thus it follows

$$\mathrm{span}\{\nabla^{\perp}_{e_6} e_3(p), \nabla^{\perp}_{e_6} e_3'(p)\} = \mathrm{span}\{\nabla^{\perp}_{e_6} e_3(q), \nabla^{\perp}_{e_6} e_3'(q)\}.$$

However, because of (63), this implies $D_3(p) = D_3(q)$, a contradiction. □

Finally, Proposition 8.1 is proved.

## 9. Investigation of $E(c)$ when $\dim E(c) = 6$

9.1. *Description of $T$ and $S$.* When $\dim E(c) = 6$, $E(c)^{\perp} = L(t)(E(c))$ holds for all $t$ by Lemma 5.4. Using the notation in Section 6, we can express each eigenvector $e_{\underline{i}}$ of $L$ as

$$e_{\underline{i}} = \begin{pmatrix} \frac{1}{\mu_i} RY_{\underline{i}} \\ Y_{\underline{i}} \end{pmatrix}, \quad i = 1, 2, 4, 5,$$

where $Y_{\underline{i}} \in E(c)^{\perp}$ is a solution of (48). Obviously, $Y_i$ and $Y_{\bar{i}}$ are independent. Let $\Pi_i$ be the 2-dimensional subspace in $E(c)^{\perp}$ spanned by $Y_i$ and $Y_{\bar{i}}$. Since $T = {}^tRR$ is positive definite (see Proposition 6.2), we have

$$TY_1 + Y_1 \neq 0, \quad TY_{\bar{1}} + Y_{\bar{1}} \neq 0.$$

Moreover, these two vectors in $E(c)^{\perp}$ are independent since, otherwise,

$$TY_1 + Y_1 = a(TY_{\bar{1}} + Y_{\bar{1}}), \quad a \neq 0$$

implies

$$T(Y_1 - aY_{\bar{1}}) + Y_1 - aY_{\bar{1}} = 0,$$

and hence $Y_1 = aY_{\bar{1}}$, a contradiction. From $\langle e_{\underline{i}}, e_{\underline{j}} \rangle = 0$ for $i \neq j$, we have

$$0 = \left\langle \frac{1}{\mu_i} RY_i, \frac{1}{\mu_j} RY_j \right\rangle + \langle Y_i, Y_j \rangle = \left\langle \frac{1}{\mu_i \mu_j} {}^tRRY_i + Y_i, Y_j \right\rangle, \tag{64}$$

namely, for $\underline{i} \in \{i, \bar{i}\}$,

$$\langle Y_{\underline{1}}, TY_{\underline{2}} + Y_{\underline{2}} \rangle = \langle Y_{\underline{1}}, -TY_{\underline{4}} + Y_{\underline{4}} \rangle = \langle Y_{\underline{1}}, -\frac{1}{3} TY_{\underline{5}} + Y_{\underline{5}} \rangle = 0, \tag{65}$$

$$\langle Y_{\underline{2}}, TY_{\underline{1}} + Y_{\underline{1}} \rangle = \langle Y_{\underline{2}}, -3TY_{\underline{4}} + Y_{\underline{4}} \rangle = \langle Y_{\underline{2}}, -TY_{\underline{5}} + Y_{\underline{5}} \rangle = 0, \tag{66}$$

$$\langle Y_{\underline{4}}, TY_{\underline{5}} + Y_{\underline{5}} \rangle = \langle Y_{\underline{4}}, -TY_{\underline{1}} + Y_{\underline{1}} \rangle = \langle Y_{\underline{4}}, -3TY_{\underline{2}} + Y_{\underline{2}} \rangle = 0, \tag{67}$$

$$\langle Y_{\underline{5}}, TY_{\underline{4}} + Y_{\underline{4}} \rangle = \left\langle Y_{\underline{5}}, -\frac{1}{3} TY_{\underline{1}} + Y_{\underline{1}} \right\rangle = \langle Y_{\underline{5}}, -TY_{\underline{2}} + Y_{\underline{2}} \rangle = 0. \tag{68}$$

Lemma 9.1. *When* $\dim E(c) = 6$, *the four vectors* $Y_1, Y_{\bar{1}}, Y_2, Y_{\bar{2}} \in E(c)^\perp$ *give a basis of* $E(c)^\perp$. *Similarly,* $Y_4, Y_{\bar{4}}, Y_5, Y_{\bar{5}} \in E(c)^\perp$ *give a basis of* $E(c)^\perp$; *i.e.,* $\Pi_1 + \Pi_2 = E(c)^\perp = \Pi_4 + \Pi_5$ *holds.*

*Proof.* Since $Y_i$ and $Y_{\bar{i}}$ are independent, we may show that any vector $Y_1$ in $\Pi_1$ is independent of any vector $Y_2$ in $\Pi_2$. This follows from

$$0 = \langle e_1, e_2 \rangle = \langle RY_1, RY_2 \rangle + \langle Y_1, Y_2 \rangle$$

because $Y_2 = kY_1$ implies $k = 0$. Similarly, we have $E(c)^\perp = \Pi_4 + \Pi_5$. □

Now we investigate how $\Pi_1, \Pi_2$ are related to $\Pi_4, \Pi_5$. Diagonalize $T$ as $T = \mathrm{diag}(\nu_1, \nu_2, \nu_3, \nu_4)$, and let $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4$ be the corresponding unit eigenvectors.

9.2. *Easy case.* When $Y_i$ is an eigenvector of $T$, the argument is simple and basic.

Lemma 9.2. *When* $\dim E(c) = 6$ *and if* $Y_1$ *is an eigenvector of* $T$, *say* $Y_1 = \boldsymbol{v}_1$, *then one of the following occurs*:

(i) $TY_1 = Y_1$, *i.e.,* $\nu_1 = 1$, *and there exists* $Y_4 \in \Pi_4$ *such that* $Y_1 = Y_4$.
(ii) $TY_1 = 3Y_1$, *i.e.,* $\nu_1 = 3$, *and there exists* $Y_5 \in \Pi_5$ *such that* $Y_1 = Y_5$.

*When* $Y_2$ *is an eigenvector of* $T$, *say* $Y_2 = \boldsymbol{v}_2$, *one of the following occurs*:

(iii) $TY_2 = Y_2$, *i.e.,* $\nu_2 = 1$, *and there exists* $Y_5 \in \Pi_5^\perp$ *such that* $Y_2 = Y_5$.
(iv) $TY_2 = \frac{1}{3}Y_2$, *i.e.,* $\nu_2 = \frac{1}{3}$, *and there exists* $Y_4 \in \Pi_4$ *such that* $Y_2 = Y_4$.

*When* $Y_4$ *is an eigenvector of* $T$, *the conclusion of* (i) *or* (iv) *occurs. When* $Y_5$ *is an eigenvector of* $T$, *the conclusion of* (ii) *or* (iii) *occurs.*

*Proof.* When $Y_1 = \boldsymbol{v}_1$, from

$$\langle Y_{\underline{4}}, TY_1 - Y_1 \rangle = 0, \quad \langle Y_{\underline{5}}, TY_1 - 3Y_1 \rangle = 0,$$

we have either $TY_1 - Y_1 = 0$ or $TY_1 - 3Y_1 = 0$ because $Y_4, Y_{\bar{4}}, Y_5, Y_{\bar{5}}$ span $E(c)^\perp$. In the former case, $Y_1$ satisfies

$$TY_1 + \sqrt{3}SY_1 - 3Y_1 = 0 \tag{69}$$

by (48), and we have

$$SY_1 = \frac{2}{\sqrt{3}}Y_1.$$

Thus $\boldsymbol{v}_1 = Y_1$ satisfies

$$T\boldsymbol{v}_1 - \frac{1}{\sqrt{3}}S\boldsymbol{v}_1 - \frac{1}{3}\boldsymbol{v}_1 = 0$$

and hence belongs to $\Pi_4$. In the case $TY_1 = 3Y_1$, (69) implies $SY_1 = 0$, and hence $\boldsymbol{v}_1 = Y_1$ belongs also to $\Pi_5$. When $Y_2 = \boldsymbol{v}_2$, from

$$\langle Y_{\underline{5}}, TY_2 - Y_2 \rangle = 0, \quad \left\langle Y_{\underline{4}}, TY_2 - \frac{1}{3}Y_2 \right\rangle = 0,$$

we have either $TY_2 - Y_2 = 0$ or $TY_2 - \frac{1}{3}Y_2 = 0$ because $Y_4, Y_{\bar{4}}, Y_5, Y_{\bar{5}}$ span $E(c)^{\perp}$. In the former case, $Y_2$ satisfies

$$TY_2 + \frac{1}{\sqrt{3}}SY_2 - \frac{1}{3}Y_2 = 0 \tag{70}$$

and we have

$$SY_2 = -\frac{2}{\sqrt{3}}Y_2.$$

Thus $\boldsymbol{v}_2 = Y_2$ satisfies

$$T\boldsymbol{v}_2 - \sqrt{3}S\boldsymbol{v}_2 - 3\boldsymbol{v}_2 = 0$$

and belongs to $\Pi_5$. In the latter case, from (70) we obtain $SY_2 = 0$. Then $\boldsymbol{v}_2 = Y_2$ belongs to $\Pi_4$. The proof of the remaining part is obtained similarly. □

We put $W_1 = \mathrm{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ and $W_2 = \mathrm{span}\{\boldsymbol{v}_3, \boldsymbol{v}_4\}$, i.e., $E^{\perp}(c) = W_1 \oplus W_2$, where $\boldsymbol{v}_1, \boldsymbol{v}_2$ are some two fixed eigenvectors of $T$. Certainly, $W_1 = \Pi_i$, $i \in \{1, 2, 4, 5\}$ means that we can take $Y_i = \boldsymbol{v}_1$ and $Y_{\bar{i}} = \boldsymbol{v}_2$.

Lemma 9.3. *If $W_1 = \Pi_i$ holds for some $i \in \{1, 2, 4, 5\}$, then $\nu_1, \nu_2 \in \{1/3, 1, 3\}$, and one of the following occurs. In particular, all $Y_{\underline{i}}$ are eigenvectors of $T$.*

(0) $T = I_4$ *and* $S = \begin{pmatrix} \frac{2}{\sqrt{3}}I_2 & 0 \\ 0 & -\frac{2}{\sqrt{3}}I_2 \end{pmatrix}$ *where $E(c)^{\perp} = \Pi_1 \oplus \Pi_2$, the orthogonal direct sum, and $\Pi_1 = \Pi_4$, $\Pi_2 = \Pi_5$.*

(I) $T = \begin{pmatrix} 3I_2 & 0 \\ 0 & \frac{1}{3}I_2 \end{pmatrix}$ *and $S = 0_4$ where $E(c)^{\perp} = \Pi_1 \oplus \Pi_2$, the orthogonal direct sum, and $\Pi_1 = \Pi_5$, $\Pi_2 = \Pi_4$.*

(II) $T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$, *where* $T_1 = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$, $T_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1/3 \end{pmatrix}$, $S = \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}$, *where* $S_1 = \begin{pmatrix} 2/\sqrt{3} & 0 \\ 0 & 0 \end{pmatrix}$ *and* $S_2 = \begin{pmatrix} -2/\sqrt{3} & 0 \\ 0 & 0 \end{pmatrix}$.

*Proof.* We treat the case $W_1 = \Pi_1$. Other cases follow similarly. In this case, we may assume $Y_1 = \boldsymbol{v}_1, Y_{\bar{1}} = \boldsymbol{v}_2$. Since $\{Y_2, Y_{\bar{2}}\}$ is orthogonal to $W_1$ by (66), $W_2 = \Pi_2$ follows, where we may consider $Y_2 = \boldsymbol{v}_3, Y_{\bar{2}} = \boldsymbol{v}_4$. Thus using Lemma 9.2, we have either of the following:

(0) $\Pi_1 = W_1 = \Pi_4$ and $\nu_1 = \nu_2 = 1$. In this case, $\Pi_2 = W_2 = \Pi_5$ follows and $\nu_3 = \nu_4 = 1$.

(I) $\Pi_1 = W_1 = \Pi_5$ and $\nu_1 = \nu_2 = 3$. In this case, $\Pi_2 = W_2 = \Pi_4$ follows and $\nu_3 = \nu_4 = 1/3$.

(II) $\Pi_1 = W_1 = \{Y_4, Y_{\bar{5}}\}$ and $\nu_1 = 1$, $\nu_2 = 3$. In this case, $\Pi_2 = W_2 = \{Y_5, Y_{\bar{4}}\}$ follows and $\nu_3 = 1$ and $\nu_4 = 1/3$.

Thus $T$ is given by (0), (I), or the mixture of these, (II). □

Corollary 9.4. *When $W_1 = \Pi_i$ holds for some $i \in \{1, 2, 4, 5\}$, we can rechoose $W_1$, $W_2$ so that $Y_1, Y_2, Y_4, Y_5 \in W_1$ and $Y_{\bar{1}}, Y_{\bar{2}}, Y_{\bar{4}}, Y_{\bar{5}} \in W_2$.*

*Proof.* Take $W_1$ spanned by (0) $Y_1 = Y_4$ and $Y_2 = Y_5$, (I) $Y_1 = Y_5$ and $Y_2 = Y_4$, and (II) $Y_1 = Y_4$ and $Y_2 = Y_5$. □

In the next section, we show that we can choose $W_1, W_2$ as in the corollary, even when $Y_{\underline{i}}$'s are not eigenvectors of $T$ (Proposition 10.3).

## 10. **General case**

Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4$ be an orthonormal frame of $E(c)^\perp$ consisting of eigenvectors of $T$. In general, $Y_{\underline{i}}$ is not an eigenvector of $T$, and $\nu_i \notin \{1/3, 1, 3\}$. For $W_1 = \mathrm{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ and $W_2 = \mathrm{span}\{\boldsymbol{v}_3, \boldsymbol{v}_4\}$, put $E_1^R = \mathrm{span}\{R\boldsymbol{v}_1, R\boldsymbol{v}_2\}$ and $E_2^R = \mathrm{span}\{R\boldsymbol{v}_3, R\boldsymbol{v}_4\}$, where we consider $R : E(c)^\perp \to E(c)$. Then $E_1^R$ and $E_2^R$ are orthogonal to each other because $\langle R\boldsymbol{v}_1, R\boldsymbol{v}_3\rangle = \langle T\boldsymbol{v}_1, \boldsymbol{v}_3\rangle = 0$, etc. The situation of the following proposition will be shown to hold in Proposition 10.3.

PROPOSITION 10.1. *When $W_1$ contains $Y_1, Y_2, Y_4, Y_5$, we can take $Y_{\bar{1}}, Y_{\bar{2}}, Y_{\bar{4}}, Y_{\bar{5}}$ in $W_2$, and $T$ has eigenvalues in pairs $\sigma, 1/\sigma$ and $\tau, 1/\tau$, which belong to the interval $[1/3, 3]$. Moreover, with respect to the decomposition $E(c)^\perp = W_1 \oplus W_2$, we can express*

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}, \quad S = \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}, \tag{71}$$

*where $T = \mathrm{diag}(\nu_1, \nu_2, \nu_3, \nu_4)$, and*

$$S_1 = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}, \quad \sigma + \frac{1}{\sigma} + a^2 = \frac{10}{3}, \tag{72}$$

$$S_2 = \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}, \quad \tau + \frac{1}{\tau} + b^2 = \frac{10}{3}.$$

*Remark* 10.2. The decomposition $W_1 \oplus W_2$ depends on $p(t) \in c$.

*Proof.* Since $T$ maps $W_1$ onto itself, from

$$TY_i + \mu_i SY_i - \mu_i^2 Y_i = 0, \quad Y_i \in W_1, \tag{73}$$

we know that $S$ maps $W_1$ into itself. Here, $S$ is *symmetric*, and so we have (71). This implies the splitting of (48) into

$$\begin{cases} T_1 Y + \mu_i S_1 Y - \mu_i^2 Y = 0, & \mu_i \neq 0, \\ T_2 Y + \mu_i S_2 Y - \mu_i^2 Y = 0, & i = 1, 2, 4, 5. \end{cases} \tag{74}$$

Since the former has solutions $Y_1, Y_2, Y_4, Y_5$ for each $\mu_i$ by our assumption, and since the solution space of (48) for each $\mu_i$ is of dimension two, the second equation must have solutions $Y_{\bar{1}}, Y_{\bar{2}}, Y_{\bar{4}}, Y_{\bar{5}}$ for each $\mu_i$, which span $W_2$. Then the following argument can be applied to both $W_1$ and $W_2$. Put $S_1 = \left(\begin{smallmatrix} s_1 & s_2 \\ s_2 & s_3 \end{smallmatrix}\right)$ and $S_2 = \left(\begin{smallmatrix} t_1 & t_2 \\ t_2 & t_3 \end{smallmatrix}\right)$. Let

$$Y = x\boldsymbol{v}_1 + y\boldsymbol{v}_2 \quad \text{or} \quad \begin{pmatrix} x \\ y \end{pmatrix} \in W_1 \subset E^{\perp}(c) \tag{75}$$

be a nontrivial solution of

$$T_1 Y + \mu_i S_1 Y - \mu_i^2 Y = 0, \quad \mu_i \neq 0. \tag{76}$$

Then this is rewritten as

$$(x\nu_1\boldsymbol{v}_1 + y\nu_2\boldsymbol{v}_2) + \mu_i\{(xs_1 + ys_2)\boldsymbol{v}_1 + (xs_2 + ys_3)\boldsymbol{v}_2\} - \mu_i^2(x\boldsymbol{v}_1 + y\boldsymbol{v}_2) = 0.$$

Taking the coefficients of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, we have

$$\begin{aligned} (\nu_1 - \mu_i^2 + \mu_i s_1)x + \mu_i s_2 y &= 0, \\ \mu_i s_2 x + (\nu_2 - \mu_i^2 + \mu_i s_3)y &= 0. \end{aligned} \tag{77}$$

Thus $(x, y) \neq (0, 0)$ implies

$$(\nu_1 - \mu_i^2 + \mu_i s_1)(\nu_2 - \mu_i^2 + \mu_i s_3) - \mu_i^2 s_2^2 = 0; \tag{78}$$

i.e.,

$$(\nu_1 - \mu_i^2)(\nu_2 - \mu_i^2) + \mu_i^2 \det S_1 - \mu_i(s_3\nu_1 + s_1\nu_2) = 0.$$

As this holds for $\mu_i = \pm\sqrt{3}, \pm\frac{1}{\sqrt{3}}$, we have

$$(\nu_1 - 3)(\nu_2 - 3) + 3\det S_1 = 0, \tag{79}$$

$$s_3\nu_1 + s_1\nu_2 = 0, \tag{80}$$

$$\nu_1\nu_2 = 1. \tag{81}$$

By the last formula, we may put $\nu_1 = \sigma$ and $\nu_2 = 1/\sigma$. Applying a similar argument to $W_2$, we obtain $\nu_3 = \tau$ and $\nu_4 = 1/\tau$.

Next, when $S_1 = 0_2$ ($S_2 = 0_2$, respectively), (79) implies $\sigma = 3$ ($\tau = 3$, respectively) and (72) holds. In general, from (79), we have

$$\begin{aligned} \sigma + \frac{1}{\sigma} - s_1 s_3 + s_2^2 &= \frac{10}{3}, \\ \tau + \frac{1}{\tau} - t_1 t_3 + t_2^2 &= \frac{10}{3}. \end{aligned} \tag{82}$$

On the other hand, from

$$\|L\|^2 = 2\mathrm{Tr}\, T + \|S\|^2 = \frac{40}{3}, \tag{83}$$

it follows

$$2\left(\sigma + \frac{1}{\sigma} + \tau + \frac{1}{\tau}\right) + s_1^2 + 2s_2^2 + s_3^2 + t_1^2 + 2t_2^2 + t_3^2 = \frac{40}{3}.$$

Thus using (82), we obtain

$$(s_1 + s_3)^2 + (t_1 + t_3)^2 = 0.$$

When $s_1 = -s_3 = 0$, putting $a = s_2$ ($t_1 = -t_3 = 0$ putting $b = t_2$, respectively) in (77), we have (72). When $s_1 = -s_3 \neq 0$ ($t_1 = -t_3 \neq 0$, respectively), (80) and (81) imply $T_1 = I_2$ ($T_2 = I_2$, respectively), and by (79), the eigenvalue of $S_1$ is $\pm a$ where $a^2 = 4/3$ (of $S_2$ is $\pm b$ where $b^2 = 4/3$, respectively). Since all the vectors in $W_1$ ($W_2$, respectively) are eigenvectors of $T_1 = I_2$ ($T_2 = I_2$ respectively), we can choose a basis of $W_i$ so that $S_i$ is expressed as in (72). □

In fact, the situation of Proposition 10.1 is always satisfied.

Proposition 10.3. *For a suitable choice of $W_1$ and $W_2$, $W_1$ contains $Y_1, Y_2, Y_4, Y_5$ and $W_2$ contains $Y_{\bar{1}}, Y_{\bar{2}}, Y_{\bar{4}}, Y_{\bar{5}}$. Thus the eigenvalues of $T$ are given by $\sigma, 1/\sigma$, $\tau, 1/\tau$, where $1/3 \leq \sigma, \tau \leq 3$, and with respect to a suitable choice of $W_1$ and $W_2$, $T$ and $S$ are given as in Proposition 10.1.*

*Proof.* By Corollary 9.4 and Proposition 10.1, it is sufficient to show that either $W_1 = \Pi_i$ for some $i$ or $Y_1, Y_2, Y_4, Y_5 \in W_1$ occurs. Take an eigenvector $\boldsymbol{v}_4$ of $T$, and put $V = \mathrm{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$, the orthogonal complement of $\boldsymbol{v}_4$ in $W(c)$. Since $\dim \Pi_i = 2$, $\dim \Pi_i \cap V \geq 2 + 3 - 4 = 1$, and we can choose $Y_i$, $i = 1, 2, 4, 5$ orthogonal to $\boldsymbol{v}_4$. This implies

$$\langle TY_i, \boldsymbol{v}_4\rangle = \langle Y_i, T\boldsymbol{v}_4\rangle = 0,$$

and hence

$$TY_i + xY_i \in V, \quad x \in \mathbb{R} \tag{84}$$

holds. Denote the $V$ component of $Y_{\bar{i}}$ by $Y_{\bar{i}}^V$. If $Y_1$ and $Y_{\bar{1}}^V$ are dependent in $V$, i.e., if $Y_{\bar{1}}^V = kY_1$ holds for some $k$, then $\tilde{Y}_1 = Y_{\bar{1}} - kY_1$ should be a nonzero multiple of $\boldsymbol{v}_4$, and $\boldsymbol{v}_4 \in \Pi_1$. Similarly if $Y_2$ and $Y_{\bar{2}}^V$ are dependent in $V$, we have $\boldsymbol{v}_4 \in \Pi_2$. Note that $\Pi_1 \cap \Pi_2 = \{0\}$ since $Y_{\underline{1}}$ and $Y_{\underline{2}}$ are independent. Thus, we have three cases:

(i) $\boldsymbol{v}_4 \notin \Pi_1, \Pi_2$, and $Y_1$ and $Y_1^V$, $Y_2$ and $Y_2^V$ are independent, respectively.
(ii) $\boldsymbol{v}_4 = Y_{\bar{1}} \in \Pi_1$, and $Y_2$ and $Y_{\bar{2}}^V$ are independent.
(iii) $\boldsymbol{v}_4 = Y_{\bar{2}} \in \Pi_2$, and $Y_1$ and $Y_{\bar{1}}^V$ are independent.

(i) In this case, the orthogonal complement of $\mathrm{span}\{Y_1, Y_{\bar{1}}^V\}$ in $V$ is of dimension one. Thus, from

$$\begin{aligned} \langle Y_1, TY_2 + Y_2\rangle = 0 = \langle Y_{\bar{1}}, TY_2 + Y_2\rangle = \langle Y_{\bar{1}}^V, TY_2 + Y_2\rangle, \\ \langle Y_1, TY_4 - Y_4\rangle = 0 = \langle Y_{\bar{1}}, TY_4 - Y_4\rangle = \langle Y_{\bar{1}}^V, TY_4 - Y_4\rangle, \\ \langle Y_1, TY_5 - 3Y_5\rangle = 0 = \langle Y_{\bar{1}}, TY_5 - 3Y_5\rangle = \langle Y_{\bar{1}}^V, TY_5 - 3Y_5\rangle, \end{aligned} \tag{85}$$

where we use (84), we obtain

$$\begin{aligned} TY_4 - Y_4 = k(TY_2 + Y_2), \\ TY_5 - 3Y_5 = l(TY_2 + Y_2) \end{aligned} \tag{86}$$

for some $k$ and $l$. Similarly, the orthogonal complement of $\operatorname{span}\{Y_2, Y_{\bar{2}}^V\}$ in $V$ is of one dimension, and from

$$\begin{aligned}\langle Y_2, TY_1+Y_1\rangle = 0 = \langle Y_{\bar{2}}, TY_1+Y_1\rangle = \langle Y_{\bar{2}}^V, TY_1+Y_1\rangle,\\ \langle Y_2, 3TY_4-Y_4\rangle = 0 = \langle Y_{\bar{2}}, 3TY_4-Y_4\rangle = \langle Y_{\bar{2}}^V, 3TY_4-Y_4\rangle,\\ \langle Y_2, TY_5-Y_5\rangle = 0 = \langle Y_{\bar{2}}, TY_5-Y_5\rangle = \langle Y_{\bar{2}}^V, TY_5-Y_5\rangle,\end{aligned} \tag{87}$$

we obtain

$$\begin{aligned}3TY_4 - Y_4 = m(TY_1+Y_1),\\ TY_5 - Y_5 = n(TY_1+Y_1)\end{aligned} \tag{88}$$

for some $m$ and $n$. Now from (86) and (88), it follows

$$\begin{aligned}T(lY_4 - kY_5) - lY_4 + 3kY_5 = 0,\\ T(3nY_4 - mY_5) - nY_4 + mY_5 = 0.\end{aligned} \tag{89}$$

Thus we obtain

$$\begin{aligned}T((lm-3kn)Y_4) = (lm-kn)Y_4 - 2kmY_5,\\ T(-3kn+lm)Y_5) = 2lnY_4 + (lm-9kn)Y_5.\end{aligned} \tag{90}$$

When $lm = 3kn$, i.e., the left-hand sides vanish, it is easy to see that $l = k = 0$ or $m = n = 0$ holds since $Y_4$ is independent of $Y_5$. Thus $Y_4$, $Y_5$ are eigenvectors of $T$, and we may put $W_1 = \operatorname{span}\{Y_4, Y_5\} = \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$. Then we have either one of the following:

(a) $Y_1 = Y_4, Y_2 = Y_5$, i.e., $Y_1, Y_2, Y_4, Y_5 \in W_1$;

(b) $Y_1 = Y_4, Y_{\bar{1}} = Y_5$, i.e., $W_1 = \Pi_1$;

(c) $Y_2 = Y_4, Y_{\bar{2}} = Y_5$, i.e., $W_1 = \Pi_2$.

Thus we have shown the first sentence of this proof.

When $lm \neq 3kn$ in (90), $T$ maps $\operatorname{span}\{Y_4, Y_5\}$ onto itself, where onto follows because rank $T = 4$. As $Y_4$ and $Y_5$ are independent, the orthogonal complement of $\operatorname{span}\{Y_4, Y_5\}$ in $V$ is of one dimension, which is preserved by $T$. Thus this is an eigenspace, of which vector we denote by $\boldsymbol{v}_3$. Then $\operatorname{span}\{Y_4, Y_5\} = \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ follows, which we denote by $W_1$. When $km = 0$, $Y_4$ is an eigenvector of $T$. Then $Y_5$ is orthogonal to $Y_4$ by (68), and $Y_5 = \boldsymbol{v}_2$ follows. As before, we are done. When $k \neq 0$ and $m \neq 0$ hold in (86) and (88), $TY_1 + Y_1, TY_2 + Y_2 \in W_1$ holds, and this implies $Y_1, Y_2$ have no $\boldsymbol{v}_3$ component since $\nu_i > 0$. Thus $Y_1, Y_2, Y_4, Y_5$ belong to $W_1$, and we are done.

(ii) In this case, $Y_2$ and $Y_{\bar{2}}$ are independent, and (88) holds. If $m = 0$, we may put $Y_4 = \boldsymbol{v}_1$ since $Y_4$ is orthogonal to $\boldsymbol{v}_4$. Therefore, applying Lemma 9.2 to $W_1 = \operatorname{span}\{Y_{\bar{1}}, Y_4\} = \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_4\}$, we have either one of the following:

(d) $Y_{\bar{1}} = Y_{\bar{5}}, Y_2 = Y_4$, i.e, $Y_{\bar{1}}, Y_2, Y_4, Y_{\bar{5}} \in W_1$;

(e) $Y_{\bar{1}}=Y_{\bar{5}}, Y_1=Y_4$, i.e., $W_1=\Pi_1$;

(f) $Y_2=Y_4, Y_{\bar{1}}=Y_{\bar{4}}$, i.e., $W_1=\Pi_4$, $Y_{\bar{1}}, Y_2, Y_4, Y_{\bar{4}} \in \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_4\}$,

and we are done.

When $n=0$, a similar argument can be applied, which we omit. When $mn \neq 0$, we consider as follows. By Lemma 9.2, either $Y_{\bar{1}}=Y_{\bar{4}}$ or $Y_{\bar{1}}=Y_{\bar{5}}$ occurs. In the former case, i.e., when $\nu_4=1$, from $\langle Y_{\underline{5}}, -3TY_{\bar{1}}+Y_{\bar{1}}\rangle=0$, $Y_5$ and $Y_{\bar{5}}$ are contained in $V$, and we may assume $Y_5$ has no $\boldsymbol{v}_3$ component. Thus $Y_5 \in \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ follows, which we put $W_1$. Then $TY_5$ has no $\boldsymbol{v}_3$ component, and by (88), $Y_1$, and hence $Y_4$ cannot have a $\boldsymbol{v}_3$ component. Moreover, $\langle Y_{\underline{2}}, TY_{\bar{1}}+Y_{\bar{1}}\rangle = 0$ implies $Y_2, Y_{\bar{2}} \in V$, and so we may assume $Y_2 \in W_1$. Therefore we obtain $Y_1, Y_2, Y_4, Y_5 \in W_1$, and we are done. The latter case when $Y_{\bar{1}}=Y_{\bar{5}}$ can be treated similarly.

(iii) This case is similar to Case (ii), and we omit it. □

## 11. Frames of $E(c)$ and $E(c)^\perp$

Proposition 11.1. *An orthonormal basis of $E(c)$, and $E(c)^\perp$, respectively, is given by*

(91)
$$\begin{aligned}
&e_3, e_{\bar{3}},\\
&X_1=\alpha(e_1+e_5)+\beta(e_2+e_4),\\
&X_2=\frac{1}{\sqrt{\sigma}}\left(\frac{\beta}{\sqrt{3}}(e_1-e_5)-\sqrt{3}\alpha(e_2-e_4)\right),\\
&X_{\bar{1}}=\gamma(e_{\bar{1}}+e_{\bar{5}})+\delta(e_{\bar{2}}+e_{\bar{4}}),\\
&X_{\bar{2}}=\frac{1}{\sqrt{\tau}}\left(\frac{\delta}{\sqrt{3}}(e_{\bar{1}}-e_{\bar{5}})-\sqrt{3}\gamma(e_{\bar{2}}-e_{\bar{4}})\right)
\end{aligned}$$

*and*

(92)
$$\begin{aligned}
&Z_1=\frac{1}{\sqrt{\sigma}}\left(\sqrt{3}\alpha(e_1-e_5)+\frac{\beta}{\sqrt{3}}(e_2-e_4)\right)\\
&Z_2=\beta(e_1+e_5)-\alpha(e_2+e_4),\\
&Z_{\bar{1}}=\frac{1}{\sqrt{\tau}}\left(\sqrt{3}\gamma(e_{\bar{1}}-e_{\bar{5}})+\frac{\delta}{\sqrt{3}}(e_{\bar{2}}-e_{\bar{4}})\right),\\
&Z_{\bar{2}}=\delta(e_{\bar{1}}+e_{\bar{5}})-\gamma(e_{\bar{2}}+e_{\bar{4}}),
\end{aligned}$$

*where* $(3-\sigma)\alpha^2=(\sigma-1/3)\beta^2$, $(3-\tau)\gamma^2=(\tau-1/3)\delta^2$, *and* $\alpha^2+\beta^2=1/2=\gamma^2+\delta^2$.

*Proof.* By Proposition 10.3, we may consider $Y_1, Y_2, Y_4, Y_5 \in W_1$, and by Proposition 10.1, we may put $s_1=s_3=0=t_1=t_3$ and $s_2=a$ and $t_2=b$. First, consider the case $a \neq 0$; then $1/3<\sigma<3$ follows from (72). Thus by

(77), we can express

$$Y_1 = \begin{pmatrix} -\sqrt{3}a \\ \sigma - 3 \end{pmatrix}, \quad Y_5 = \begin{pmatrix} \sqrt{3}a \\ \sigma - 3 \end{pmatrix}, \tag{93}$$
$$Y_2 = \begin{pmatrix} -\frac{a}{\sqrt{3}} \\ \sigma - \frac{1}{3} \end{pmatrix}, \quad Y_4 = \begin{pmatrix} \frac{a}{\sqrt{3}} \\ \sigma - \frac{1}{3} \end{pmatrix}$$

in $W_1$, and in $E_1^R \oplus W_1 \subset T_{\bar{p}}M_+$,

$$\hat{e}_1 = \begin{pmatrix} \frac{1}{\sqrt{3}}RY_1 \\ -\sqrt{3}a \\ \sigma - 3 \end{pmatrix}, \quad \hat{e}_5 = \begin{pmatrix} -\frac{1}{\sqrt{3}}RY_5 \\ \sqrt{3}a \\ \sigma - 3 \end{pmatrix}, \tag{94}$$
$$\hat{e}_2 = \begin{pmatrix} \frac{1}{\sqrt{3}}RY_2 \\ -\frac{a}{\sqrt{3}} \\ \sigma - \frac{1}{3} \end{pmatrix}, \quad \hat{e}_4 = \begin{pmatrix} -\frac{1}{\sqrt{3}}RY_4 \\ \frac{a}{\sqrt{3}} \\ \sigma - \frac{1}{3} \end{pmatrix}.$$

Claim. $|\hat{e}_1| = |\hat{e}_5|, \quad |\hat{e}_2| = |\hat{e}_4|$.

In fact, we have $|Y_1| = |Y_5|$ and $|Y_2| = |Y_4|$. On the other hand, using $|RY_i|^2 = \langle RY_i, RY_i\rangle = \langle TY_i, Y_i\rangle$, we obtain

$$|RY_1|^2 = \langle TY_1, Y_1\rangle = \left\langle \begin{pmatrix} -\sqrt{3}a\sigma \\ \frac{\sigma-3}{\sigma} \end{pmatrix}, \begin{pmatrix} -\sqrt{3}a \\ \sigma - 3 \end{pmatrix} \right\rangle$$
$$= \left\langle \begin{pmatrix} \sqrt{3}a\sigma \\ \frac{\sigma-3}{\sigma} \end{pmatrix}, \begin{pmatrix} \sqrt{3}a \\ \sigma - 3 \end{pmatrix} \right\rangle = \langle TY_5, Y_5\rangle = |RY_5|^2,$$

and hence $|\hat{e}_1| = |\hat{e}_5|$ follows. Similarly, we have $|\hat{e}_2| = |\hat{e}_4|$.

In order that $X = x\hat{e}_1 + y\hat{e}_2 + z\hat{e}_4 + w\hat{e}_5$ belongs to $E$, we have

$$\sqrt{3}a(-x + w) - \frac{a}{\sqrt{3}}(y - z) = 0, \tag{95}$$

$$(\sigma - 3)(x + w) + \left(\sigma - \frac{1}{3}\right)(y + z) = 0. \tag{96}$$

Since we can describe

$$X = \frac{x + w}{2}(\hat{e}_1 + \hat{e}_5) + \frac{x - w}{2}(\hat{e}_1 - \hat{e}_5) + \frac{y + z}{2}(\hat{e}_2 + \hat{e}_4) + \frac{y - z}{2}(\hat{e}_2 - \hat{e}_4)$$

in the case $a \neq 0$, i.e., $\sigma \neq 3, 1/3$, (95) and (96) imply

$$X = k\left\{\left(\sigma - \frac{1}{3}\right)(\hat{e}_1 + \hat{e}_5) - (\sigma - 3)(\hat{e}_2 + \hat{e}_4)\right\} + l\left\{\frac{a}{\sqrt{3}}(\hat{e}_1 - \hat{e}_5) - \sqrt{3}a(\hat{e}_2 - \hat{e}_4)\right\}$$

for any $k, l$. Thus putting

$$\alpha = \left(\sigma - \frac{1}{3}\right)|\hat{e}_1|, \quad \beta = (3 - \sigma)|\hat{e}_2|, \tag{97}$$

we can express $X$ as a combination of

$$\hat{X}_1 = \alpha(e_1 + e_5) + \beta(e_2 + e_4)$$

and

$$\begin{aligned}\hat{X}_2 &= \frac{|\hat{e}_1|}{\sqrt{3}}(e_1 - e_5) - \sqrt{3}|\hat{e}_2|(e_2 - e_4)\\ &= \frac{\alpha}{\sqrt{3}(\sigma - 1/3)}(e_1 - e_5) - \frac{\sqrt{3}\beta}{3-\sigma}(e_2 - e_4),\end{aligned}$$

where $e_{\underline{i}}$ is normalized from $\hat{e}_{\underline{i}}$. On the other hand, $\langle B_\eta \hat{X}_1, \hat{X}_2\rangle = 0$ implies

$$\frac{\alpha^2}{\sigma - 1/3} = \frac{\beta^2}{3-\sigma}. \tag{98}$$

Thus we may express

$$\hat{X}_2 = \frac{\beta}{\sqrt{3}}(e_1 - e_5) - \sqrt{3}\alpha(e_2 - e_4).$$

If we normalize $\hat{X}_1$, (98) implies

$$\sigma = \frac{3\alpha^2 + \beta^2/3}{\alpha^2 + \beta^2} = 2(3\alpha^2 + \beta^2/3) = \|\hat{X}_2\|^2. \tag{99}$$

A similar argument holds for $W_2$ when $t_1 = 0$ and $t_2 = b \neq 0$.

When $a = 0$, $\sigma = 3$ or $1/3$ follows, i.e., $T_1 = \begin{pmatrix} 3 & \\ & 1/3 \end{pmatrix}$ and $S_1 = 0$ follow. Then (76) becomes $TY_i = \mu_i^2 Y_i$, and we may consider $Y_1 = \binom{1}{0}$, $Y_2 = \binom{0}{1} \in W_1$, where $Y_1$ and $Y_5$ ($Y_2$ and $Y_4$, respectively) coincide up to sign. If we put $Y_1 = -Y_5$ and $Y_2 = -Y_4$, then it follows

$$\hat{e}_1 = \begin{pmatrix} \frac{1}{\sqrt{3}}RY_1 \\ Y_1 \end{pmatrix}, \hat{e}_5 = \begin{pmatrix} \frac{1}{\sqrt{3}}RY_1 \\ -Y_1 \end{pmatrix}, \hat{e}_2 = \begin{pmatrix} \sqrt{3}RY_2 \\ Y_2 \end{pmatrix}, \hat{e}_4 = \begin{pmatrix} \sqrt{3}RY_2 \\ -Y_2 \end{pmatrix}, \tag{100}$$

and after normalization, we obtain

$$E_1^R = \text{span}\{e_1 + e_5, e_2 + e_4\}, \quad W_1 = \text{span}\{e_1 - e_5, e_2 - e_4\}. \tag{101}$$

A similar argument holds for $E_2, W_2$ when $b = 0$. □

In the following, *we restrict our argument to the case when* $ab \neq 0$, *i.e., when* $1/3 < \sigma, \tau < 3$. This is also the case when $\alpha\beta\gamma\delta \neq 0$.

*Remark* 11.2. When $a(t)b(t) \not\equiv 0$, applying above argument to each $L(t) = \cos tB_\eta + \sin tB_\zeta$, and noting that

$$R(t+\pi) = -R(t),\ a(t+\pi) = -a(t),\ \sigma(t+\pi) = \sigma(t),\ \tau(t+\pi) = \tau(t)$$

hold in (94), we have

$$e_{\underline{1}}(t+\pi) = e_{\underline{5}}(t), \quad e_{\underline{2}}(t+\pi) = e_{\underline{4}}(t), \tag{102}$$

and it follows

$$e_{\underline{3}}(t+\pi) = \epsilon e_{\underline{3}}(t), \quad \epsilon = \pm 1. \tag{103}$$

The normalization of $\hat{e}_{\underline{i}}(t)$ does not effect their directions. In particular,

$$e_{\underline{i}}(t+2\pi) = e_{\underline{i}}(t) \tag{104}$$

holds for any $1 \le i \le 5$, and we have an analytic frame along $c$.

Using the frame along $c$ mentioned above, we obtain

LEMMA 11.3. *When* $1/3 < \sigma, \tau < 3$, *choose* $e_{\underline{i}}(t)$ *as in* (94). *Then* $e_{\underline{1}}(t) + e_{\underline{5}}(t), e_{\underline{2}}(t) + e_{\underline{4}}(t)$ *are even vector fileds and* $e_{\underline{1}}(t) - e_{\underline{5}}(t), e_{\underline{2}}(t) - e_{\underline{4}}(t)$ *are odd vector fields along* $c$.

## 12. Invariance of $\sigma, \tau$ when $ab \neq 0$

When we apply the previous argument to various points $p(t) \in c$, we use the moving frame $e_{\underline{3}}(t), X_{\underline{i}}(t), Z_{\underline{i}}(t)$, with respect to which, the relations satisfied by $B_\eta$ and $B_\zeta$ hold for $L(t)$ and $L_t(t)$.

In the next section we will prove $a(t)b(t) \equiv 0$. By (72), $a = 0$ is equivalent to $\sigma = 1/3$ or 3, which is also equivalent to $\alpha\beta = 0$ by (97). When we argue at various points $p(t)$ of $c$, a choice of $\alpha(t), \beta(t)$ in (97) seems unnatural since they are always nonnegative. The purpose of this section is to show, in fact, $\alpha(t), \beta(t), \gamma(t), \delta(t)$, and hence $\sigma(t), \tau(t)$ are constant along $c$. When $a(t)b(t) \equiv 0$ holds on an open interval, $\sigma, \tau = 1/3$ or 3 holds over all $c$. Therefore, we consider *what happens when* $a(t)b(t) \not\equiv 0$.

12.1. *Description of* $H(0) = U_t(0)$. With respect to the frame in (91) and (92), we can express

$$L(0) = B_\eta = \begin{pmatrix} 0 & 0 & 0 & \vdots & 0 & 0 \\ 0 & 0 & 0 & \vdots & A_1 & 0 \\ 0 & 0 & 0 & \vdots & 0 & A_2 \\ \cdots & \cdots & \cdots & & \cdots & \cdots \\ 0 & {}^tA_1 & 0 & \vdots & D_1 & 0 \\ 0 & 0 & {}^tA_2 & \vdots & 0 & D_2 \end{pmatrix}, \tag{105}$$

$$A_1 = \begin{pmatrix} \sqrt{\sigma} & 0 \\ 0 & \dfrac{1}{\sqrt{\sigma}} \end{pmatrix}, \quad A_2 = \begin{pmatrix} \sqrt{\tau} & 0 \\ 0 & \dfrac{1}{\sqrt{\tau}} \end{pmatrix},$$

$$D_1 = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix},$$

where by (82), we have

$$\sigma + \frac{1}{\sigma} + a^2 = \frac{10}{3} = \tau + \frac{1}{\tau} + b^2.$$

Now recall the argument in Section 3, where we put $e_{\underline{i}}(t) = U(t)e_{\underline{i}}(p)$, $U(t) \in O(10)$.

LEMMA 12.1. *With respect to* (91) *and* (92) *at* $p$, *we can express*

$$H(0) = U_t(0) = \begin{pmatrix} H_0 & X & Y & \vdots & 0 & 0 \\ -{}^tX & H_1 & Z & \vdots & K_1 & 0 \\ -{}^tY & -{}^tZ & H_2 & \vdots & 0 & K_2 \\ \cdots & \cdots & \cdots & & \cdots & \cdots \\ 0 & -{}^tK_1 & 0 & \vdots & H_3 & V \\ 0 & 0 & -{}^tK_2 & \vdots & -{}^tV & H_4 \end{pmatrix}, \tag{106}$$

*where* $H_i$, $i = 0, 1, 2, 3, 4$ *are skew, and*

$$K_1 = \begin{pmatrix} 0 & k_1 \\ -k_1/\sigma & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & k_2 \\ -k_2/\tau & 0 \end{pmatrix}. \tag{107}$$

*Proof.* First, we can put

$$H(0) = U_t(0)^tU(0) = \begin{pmatrix} H_0 & X & Y & \vdots & 0 & 0 \\ -{}^tX & H_1 & Z & \vdots & K_1 & G_1 \\ -{}^tY & -{}^tZ & H_2 & \vdots & G_2 & K_2 \\ \cdots & \cdots & \cdots & & \cdots & \cdots \\ 0 & -{}^tK_1 & -{}^tG_2 & \vdots & H_3 & V \\ 0 & -{}^tG_1 & -{}^tK_2 & \vdots & -{}^tV & H_4 \end{pmatrix} \tag{108}$$

because $H(0)$ maps $D_3(p)$ to $\{\nabla_{e_6} e_3(p), \nabla_{e_6} e_{\bar{3}}(p)\} \subset E(c)$. In general, $H(0)X_i \neq c_0 \nabla_{e_6} X_i$ because $\alpha(t), \beta(t), \gamma(t), \delta(t)$ as well as $\sigma(t), \tau(t)$ are not neccesarily constant. In fact, from (27), it follows

$$\nabla_{\frac{d}{dt}} X_1 = \dot{\alpha}(e_1 + e_5) + \dot{\beta}(e_2 + e_4) + H(0)X_1, \tag{109}$$

$$\nabla_{\frac{d}{dt}} X_2 = \frac{d}{dt}\left(\frac{\beta}{\sqrt{3}\sigma}\right)(e_1 - e_5) + \frac{d}{dt}\left(\frac{\sqrt{3}\alpha}{\sqrt{\sigma}}\right)(e_2 - e_4) + H(0)X_2. \tag{110}$$

However, we know $\nabla_{\frac{d}{dt}} X_1 \in D_3 \oplus \mathrm{span}\{X_2, X_{\bar{1}}, X_{\bar{2}}\}$ and $\nabla_{\frac{d}{dt}} X_2 \in D_3 \oplus \mathrm{span}\{X_1, X_{\bar{1}}, X_{\bar{2}}\}$ because $X_i$ is a unit vector. Thus in view of (92), $H(0)X_1$ cannot have components in $E(c)^\perp$ except for $Z_2$. Similarly, $H(0)X_2$ has no components in $E(c)^\perp$ except for $Z_1$. This implies $G_1 = 0$, $K_1 = \left(\begin{smallmatrix} 0 & k_1 \\ l_1 & 0 \end{smallmatrix}\right)$, and similarly, $G_2 = 0$, $K_2 = \left(\begin{smallmatrix} 0 & k_2 \\ l_2 & 0 \end{smallmatrix}\right)$. Now, if we denote

$$H(0) = \begin{pmatrix} J_1 & J_3 \\ -{}^tJ_3 & J_2 \end{pmatrix}$$

with respect to the decomposition $E(c) \oplus E(c)^{\perp}$, we have

$$\begin{aligned} B_\zeta &= [H(0), B_\eta] \\ &= \begin{pmatrix} J_1 & J_3 \\ -{}^tJ_3 & J_2 \end{pmatrix} \begin{pmatrix} 0 & A \\ {}^tA & D \end{pmatrix} - \begin{pmatrix} 0 & A \\ {}^tA & D \end{pmatrix} \begin{pmatrix} J_1 & J_3 \\ -{}^tJ_3 & J_2 \end{pmatrix} \\ &= \begin{pmatrix} J_3{}^tA + A^tJ_3 & * \\ * & * \end{pmatrix}, \end{aligned}$$

where $J_3 = \left(\begin{smallmatrix} 0 & 0 \\ K_1 & 0 \\ 0 & K_2 \end{smallmatrix}\right)$. Then from

$$J_3{}^tA = \begin{pmatrix} 0 & 0 & 0 \\ 0 & K_1{}^tA_1 & 0 \\ 0 & 0 & K_2{}^tA_2 \end{pmatrix},$$

we obtain

$$K_1{}^tA_1 = \begin{pmatrix} 0 & k_1 \\ l_1 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{\sigma} & 0 \\ 0 & 1/\sqrt{\sigma} \end{pmatrix} = \begin{pmatrix} 0 & k_1/\sqrt{\sigma} \\ l_1\sqrt{\sigma} & 0 \end{pmatrix}.$$

Since $J_3{}^tA + A^tJ_3 = 0$, i.e., $J_3{}^tA$ is skew, $l_1 = -k_1/\sigma$ follows. A similar argument holds for $K_2$. □

12.2. *Splitting of $U(t)$*. In the following discussion, it is again important that a vector field $v(t)$ along $c$ is even or odd.

PROPOSITION 12.2. *When $a(t)b(t) \not\equiv 0$, in the expression (108) of $H$ at any fixed point of $c$, $K_1 = K_2 = 0$ holds, and the orthogonal group $U(t)$ such that $e_{\underline{i}}(t) = U(t)e_{\underline{i}}$ splits into*

$$U(t) = \begin{pmatrix} U_1(t) & 0 \\ 0 & U_2(t) \end{pmatrix}, \quad U_1(t) \in O(6), U_2(t) \in O(4), \tag{111}$$

*with respect to the decomposition $E(c) \oplus E(c)^{\perp}$.*

*Proof.* Recall

$$L(t) = U(t)B_\eta{}^tU(t) = \begin{pmatrix} 0 & R(t) \\ {}^tR(t) & S(t) \end{pmatrix}. \tag{112}$$

However, the splitting of $U(t)$ never follows from this. Now, since $D_3(t) = U(t)D_3(p)$ belongs to $E(c)$ where

$$D_3(p) = \begin{pmatrix} e_3(p) & e_{\bar{3}}(p) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix},$$

we can express $U(t)$ with respect to the decomposition $(D_3(p) \oplus E_1) \oplus E(c)^\perp$, where $E_1 = \mathrm{span}\{X_1, X_2, X_{\bar{1}}, X_{\bar{2}}\}$, as

$$U(t) = \begin{pmatrix} V_1(t) & V_2(t) & \vdots & 0 \\ V_3(t) & V_4(t) & \vdots & V_5(t) \\ \cdots & \cdots & \cdot & \cdots \\ 0 & V_6(t) & \vdots & V_7(t) \end{pmatrix}, \quad \begin{array}{l} V_1(t) \in M_2(\mathbb{R}), \\ V_2(t), {}^tV_3(t) \in M_{2,4}(\mathbb{R}), \\ V_4(t), V_5(t), V_6(t), V_7(t) \in M_4(\mathbb{R}), \end{array} \tag{113}$$

where $M_{i,j}(\mathbb{R})$ denotes the space of $i \times j$ matrices and $M_i(\mathbb{R}) = M_{i,i}(\mathbb{R})$. Here, we have an expansion of the analytic $U(t)$:

$$U(t) = I + U_t(0)t + [t^2] = I + H(0)t + [t^2],$$

denoting $[t^j]$ the term of order not less than $j$. In particular, it follows

$$V_3(t) = t\begin{pmatrix} -{}^tX \\ -{}^tY \end{pmatrix} + [t^2], \quad V_5(t) = t\begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix} + [t^2].$$

On the other hand, taking the $(1,3)$ block of ${}^tU(t)U(t) = I_{10}$, we have

$${}^tV_3(t)V_5(t) = 0. \tag{114}$$

Then it follows

$$0 = \left(t\begin{pmatrix} -X & -Y \end{pmatrix} + [t^2]\right)\left(t\begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix} + [t^2]\right),$$

and from the coefficient of $t^2$, we obtain

$$\begin{pmatrix} XK_1 & YK_2 \end{pmatrix} = 0.$$

By (107), $K_1$ and $K_2$ are of rank either 0 or 2. Moreover, rank $\begin{pmatrix} X & Y \end{pmatrix} = 2$ because

$$\mathrm{span}\{\nabla_{e_6} e_{\underline{3}}(p)\} = HD_3(p) \subset \mathrm{span}\left\{\begin{pmatrix} H_0 \\ -{}^tX \\ -{}^tY \\ 0 \\ 0 \end{pmatrix}\right\} \subset E(c),$$

and $\dim \mathrm{span}\{\nabla^\perp_{e_6} e_3(p), \nabla^\perp_{e_6} e_{\bar{3}}(p)\} = 2$, where $\nabla^\perp$ denotes the component orthogonal to $D_3$; see Lemma 7.1 and Proposition 8.5. Thus we have either

(i) $K_1 = K_2 = 0$;
(ii) $K_2 \neq 0$, $K_1 = Y = 0$; or
(iii) $K_1 \neq 0$, $X = K_2 = 0$.

The above argument can be applied at any point $p(t)$ on $c$ with respect to the moving frame $e_{\underline{3}}(t), X_{\underline{i}}(t), Z_{\underline{i}}(t)$ at $p(t)$. In this case, although $K_i(t), X(t), Y(t)$ depends on $t$, the decomposition $E(c) \oplus E(c)^\perp$ is independent of $t$. Thus if we show (i) occurs over all $c$ with respect to the moving frame, the right upper $6 \times 4$ part of $H(t)$ always vanishes, and this proves the proposition.

If (ii) occurs at a point, it occurs on an open interval because $K_2(t) \neq 0$ is an open condition. Moreover by the analyticity, $K_1(t) = Y(t) = 0$ holds over all $c$. Thus we have

$$\text{span}\{\nabla^{\perp}_{e_6} e_3(t), \nabla^{\perp}_{e_6} e_{\bar{3}}(t)\} = \text{span}\{X_1(t), X_2(t)\} \tag{115}$$

for each $t$. By (103), the orientation of the left-hand side is preserved at $t = \pi$. When $a(t)b(t) \not\equiv 0$, namely, when $\alpha\beta\gamma\delta \not\equiv 0$, we can see $X_1(t)$ is even (odd, resp.) if and only if $X_2(t)$ is odd (even, resp.), which depends on evenness and oddness of $(\alpha(t), \beta(t))$ and $(\gamma(t), \delta(t))$, by Lemma 11.3 and by (91). Thus the orientation of the right-hand side of (115) is reversed at $t = \pi$, a contradiction. In the same way, we can show that (iii) does not occur. □

Therefore, we have the following fundamental result.

Corollary 12.3. *When $a(t)b(t) \not\equiv 0$, with respect to the frame (91) and (92) of $E(c) \oplus E(c)^{\perp}$ at any point, we have*

$$H(0) = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix}, \ J_1 = \begin{pmatrix} H_0 & X & Y \\ -{}^tX & H_1 & Z \\ -{}^tY & -{}^tZ & H_2 \end{pmatrix}, \ J_2 = \begin{pmatrix} H_3 & V \\ -{}^tV & H_4 \end{pmatrix}. \tag{116}$$

*Moreover, $\sigma, \tau$, as well as $\alpha, \beta, \gamma, \delta$ are constant along $c$. Therefore, $X_{\underline{1}}(t), Z_{\underline{2}}(t)$ are even, and $X_{\underline{2}}(t), Z_{\underline{1}}(t)$ are odd.*

*Proof.* Since $K_1 = K_2 = 0$, we know from (109) and similar formulas for $X_{\underline{i}}$ that $\nabla_{e_6} X_{\underline{i}}$ belongs to $E(c)$ if and only if $\dot{\alpha} = \dot{\beta} = \dot{\gamma} = \dot{\delta} = 0$. Then $\dot{\sigma} = \dot{\tau} = 0$ follows from (99) and a similar formula for $\tau$. This holds at any point of $c$, and the conclusion follows. The last assertion follows from Lemma 11.3. □

*Remark* 12.4. If we know $\nabla_{e_6} e_3(t)$ is even and $\bar{\nabla}_{e_3} e_6(t)$ is odd as in Lemma 8.3, these *never mean* that $\nabla_{e_6} e_3(t)$ is a combination of $X_1(t)$ and $X_{\bar{1}}(t)$ nor that $\bar{\nabla}_{e_3} e_6(t)$ is a combination of $Z_1(t)$ and $Z_{\bar{1}}(t)$. This is because even vectors multiplied by odd functions are odd, and odd vectors multiplied by odd functions are even.

A final consequence obtained from the constantness of $\alpha, \beta$ is

Corollary 12.5. *When $a(t)b(t) \not\equiv 0$, let $U(t) = \begin{pmatrix} U_1(t) & 0 \\ 0 & U_2(t) \end{pmatrix}$ be such that $e_{\underline{i}}(t) = U(t)e_{\underline{i}}(0)$. Then*

$$X_{\underline{i}}(t) = U_1(t)X_{\underline{i}}(0), \quad Z_{\underline{i}}(t) = U_2(t)Z_{\underline{i}}(0) \tag{117}$$

*holds for $i = 1, 2$.*

By (111), $L(t) = \left(\begin{smallmatrix} 0_6 & R(t) \\ {}^tR(t) & S(t) \end{smallmatrix}\right) = U(t)B_\eta{}^tU(t)$ is given by

$$R(t) = U_1(t)A\,{}^tU_2(t), \quad S(t) = U_2(t)D\,{}^tU_2(t), \tag{118}$$

where $A = R(0)$ and $D = S(0)$, i.e., $B_\eta = \left(\begin{smallmatrix} 0_6 & A \\ {}^tA & D \end{smallmatrix}\right)$. In particular,

$${}^tR(t)R(t) = U_2(t)({}^tAA){}^tU_2(t) \tag{119}$$

holds. Thus we obtain the following proposition.

Proposition 12.6. *When $a(t)b(t) \not\equiv 0$, in the expressions*

$$B_\eta = \begin{pmatrix} 0 & A \\ {}^tA & D \end{pmatrix}, \quad L(t) = \begin{pmatrix} 0_6 & R(t) \\ {}^tR(t) & S(t) \end{pmatrix},$$

*$T(t) = {}^tR(t)R(t)$ is isospectral with ${}^tAA$ and $S(t)$ is isospectral with $D$.*

## 13. Properties of $T(t)$ and $S(t)$

13.1. *The case $a^2 \neq b^2$ and $ab \neq 0$.* Now, we consider what occurs when $ab \neq 0$, equivalently, when $1/3 < \sigma, \tau < 3$. First, assume $a^2 \neq b^2$. With respect to the decomposition $T_{\bar{p}}M_+ = E \oplus E^\perp$, we express

$$B_\eta = \begin{pmatrix} 0 & A \\ {}^tA & D \end{pmatrix}, \quad B_\zeta = \begin{pmatrix} 0 & M \\ {}^tM & N \end{pmatrix}.$$

In particular, by Propositions 10.1 and 10.3, we have

$$T = {}^tAA = \operatorname{diag}\begin{pmatrix} \sigma & \frac{1}{\sigma} & \tau & \frac{1}{\tau} \end{pmatrix}, \tag{120}$$

$$D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}, D_2 = \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}$$

with respect to the orthonormal basis $Z_1, Z_2, Z_{\bar{1}}, Z_{\bar{2}}$ of $E(c)^\perp$ at the point. From (116) and from $B_\zeta = [H(0), B_\eta]$, we have

$$M = J_1A - AJ_2, \tag{121}$$

and from (108), we have

$$N = [J_2, D] = \begin{pmatrix} H_3 & V \\ -{}^tV & H_4 \end{pmatrix}\begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} - \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}\begin{pmatrix} H_3 & V \\ -{}^tV & H_4 \end{pmatrix} \tag{122}$$

$$= \begin{pmatrix} H_3D_1 - D_1H_3 & VD_2 - D_1V \\ -{}^tVD_1 + D_2{}^tV & H_4D_2 - D_2H_4 \end{pmatrix}.$$

Moreover, if we put

$$H_3 = \begin{pmatrix} 0 & h_3 \\ -h_3 & 0 \end{pmatrix}, \quad H_4 = \begin{pmatrix} 0 & h_4 \\ -h_4 & 0 \end{pmatrix},$$

we obtain

$$N=\begin{pmatrix} d & 0 & f & g \\ 0 & -d & k & l \\ f & k & m & 0 \\ g & l & 0 & -m \end{pmatrix}, \quad d=2ah_3, m=2bh_4. \tag{123}$$

LEMMA 13.1. *When* $a^2 \neq b^2$ *and* $ab \neq 0$, $d=m=0$ *holds.*

*Proof.* Since $\cos tD + \sin tN$ has eigenvalues $\pm a, \pm b$, we have

$$\det(\cos tD + \sin tN - xI) = (x^2-a^2)(x^2-b^2).$$

Then, putting $c=\cos t$, $s=\sin t$, we calculate the the left-hand side (by Mathematica):

$$\begin{aligned}
&\det(\cos tD + \sin tN - xI) \\
&\quad = \det\begin{pmatrix} sd-x & ca & sf & sg \\ ca & -sd-x & sk & sl \\ sf & sk & sm-x & cb \\ sg & sl & cb & -sm-x \end{pmatrix} \\
&\quad = x^4 - x^2\{c^2(a^2+b^2)+s^2(d^2+f^2+g^2+k^2+l^2+m^2)\} \\
&\qquad - 2xs^2\{c(a(fk+gl)+b(fg+kl)) \\
&\qquad\qquad + s(d(f^2+g^2-k^2-l^2)+m(f^2-g^2+k^2-l^2))\} \\
&\qquad + s^2\{c^2(b^2d^2-2ab(fl+gk)+a^2m^2) \\
&\qquad\qquad + 2cs(am(fk-gl)+bd(fg-kl)) \\
&\qquad\qquad + s^2((fl-gk)^2+d^2m^2+dm(-f^2+g^2+k^2-l^2))\}.
\end{aligned}$$

We obtain

$$d^2+f^2+g^2+k^2+l^2+m^2 = a^2+b^2, \tag{124}$$

$$a(fk+gl)+b(fg+kl)=0, \tag{125}$$

$$d(f^2+g^2-k^2-l^2)+m(f^2-g^2+k^2-l^2)=0, \tag{126}$$

$$b^2d^2-2ab(fl+gk)+a^2m^2 = 2a^2b^2, \tag{127}$$

$$am(fk-gl)+bd(fg-kl)=0, \tag{128}$$

$$(fl-gk)^2+d^2m^2+dm(-f^2+g^2+k^2-l^2)=a^2b^2, \tag{129}$$

which are, respectively, the coeffcients of $s^2x^2$, $cs^2x$, $s^3x$, $c^2s^2$, $cs^3$ and $s^4$. Note that there *exist* many matrices which satisfy these equations.

Consider the *moving frame* $Z_1(t), Z_2(t), Z_{\bar{1}}(t), Z_{\bar{2}}(t)$ along $c$ consisting of eigenvectors of the isospectral operator $T(t)$ for eigenvalues $\sigma, 1/\sigma, \tau, 1/\tau$, respectively. Then the argument before the lemma holds for each $L(t) =$

$\begin{pmatrix} 0 & A(t) \\ {}^tA(t) & D(t) \end{pmatrix}$ with respect to this moving frame. On the other hand, $h_3(t) = \langle H(t)Z_1(t), Z_2(t)\rangle = \langle \nabla_{e_6} Z_1(t), Z_2(t)\rangle$ is odd since $Z_1(t)$ is odd and $Z_2(t)$ is even. Thus there exists $t_0$ at which $h_3(t_0) = 0$. If we take $p = p(t_0)$, $d = 0$ follows from (123). Now putting $d = 0$ in (124) and (126) $\sim$ (129), we obtain

$$f^2 + g^2 + k^2 + l^2 + m^2 = a^2 + b^2, \tag{130}$$

$$a(fk + gl) + b(fg + kl) = 0, \tag{131}$$

$$m(f^2 - g^2 + k^2 - l^2) = 0, \tag{132}$$

$$-2ab(fl + gk) + a^2m^2 = 2a^2b^2, \tag{133}$$

$$am(fk - gl) = 0, \tag{134}$$

$$(fl - gk)^2 = a^2b^2. \tag{135}$$

Claim. *Let $a^2 \neq b^2$, $ab \neq 0$. If $d = 0$ and $m \neq 0$ hold, then we have $a = 3\varepsilon b$ for $\varepsilon = \pm 1$.*

In fact, if $m \neq 0$, from (132) and (134) follows $(f \pm k)^2 = (g \pm l)^2$, and hence we may put

$$f + k = \varepsilon(g + l), \quad f - k = \varepsilon'(g - l), \quad \varepsilon, \varepsilon' = \pm 1,$$

which imply

$$f = \frac{\varepsilon + \varepsilon'}{2}g + \frac{\varepsilon - \varepsilon'}{2}l, \quad k = \frac{\varepsilon - \varepsilon'}{2}g + \frac{\varepsilon + \varepsilon'}{2}l. \tag{136}$$

Then

$$\begin{aligned} fl - gk &= \left(\frac{\varepsilon + \varepsilon'}{2}g + \frac{\varepsilon - \varepsilon'}{2}l\right)l - g\left(\frac{\varepsilon - \varepsilon'}{2}g + \frac{\varepsilon + \varepsilon'}{2}l\right) \\ &= \frac{\varepsilon - \varepsilon'}{2}(l^2 - g^2) \end{aligned}$$

follows. Since the right-hand side of (135) does not vanish, $\varepsilon \neq \varepsilon'$ and $g^2 \neq l^2$ follow. Thus we obtain $f = \varepsilon l$, $k = \varepsilon g$ from (136). Substituting these into (131), we have

$$(a + b\varepsilon)gl = 0,$$

and from $a^2 \neq b^2$, $gl = 0$ follows. When $l = 0$, we have $f = 0$, and (130), (133) and (135) imply

$$\begin{aligned} 2k^2 + m^2 &= a^2 + b^2, \\ -2\varepsilon abk^2 + a^2m^2 &= 2a^2b^2, \\ k^4 &= a^2b^2. \end{aligned}$$

If we put $k^2 = \epsilon ab$, $\varepsilon = \epsilon$ follows from the second one since $am \neq 0$, and so

$$m^2 = 4b^2.$$

On the other hand, from the first one follows

$$m^2 = (a - \varepsilon b)^2,$$

and we have

$$a - \varepsilon b = \pm 2b.$$

Now from $a^2 \neq b^2$, we obtain $a = 3\varepsilon b$. When $g = 0$, a parallel argument holds, and we also obtain $a = 3\varepsilon b$.

In the above argument, we choose a point $p(t_0)$ at which $h_3(t_0) = 0$, and we obtain $a = 3\varepsilon b$ when $m \neq 0$. Similarly, if we use the oddness of

$$h_4(t) = \langle H(t) Z_{\bar{1}}(t), Z_{\bar{2}}(t) \rangle = \langle \nabla_{e_6} Z_{\bar{1}}(t), Z_{\bar{2}}(t) \rangle,$$

there exists $t_1$ such that $h_4(t_1) = 0$. Although the frame at $p(t_1)$ differs from the one at $p(t_0)$, we can apply a similar argument at $p(t_1)$ with respect to the frame at $p(t_1)$. Note that (124) $\sim$ (129) are preserved if we exchange the triple $(a, d, g)$ with $(b, m, k)$. Thus putting $m = 0$ in (123) at $p = p(t_1)$, we obtain $b = 3\varepsilon' a$ under the assumption $d \neq 0$. However, since $a$ and $b$ are constant, i.e., independent of a choice of the frame, $a = 3\varepsilon b$ and $b = 3\varepsilon' a$ imply $a = b = 0$, a contradiction. Therefore, at $p(t_0)$ and $p(t_1)$, $d = m = 0$ holds. □

Thus taking $p = p(t_0)$, we may put $N = \begin{pmatrix} 0 & N_1 \\ {}^t N_1 & 0 \end{pmatrix}$.

Lemma 13.2. *When $a^2 \neq b^2$ and $ab \neq 0$, we have either one of the following*:

(i) $N_1 = \epsilon \begin{pmatrix} a & 0 \\ 0 & -b \end{pmatrix}$;

(ii) $N_1 = \epsilon \begin{pmatrix} b & 0 \\ 0 & -a \end{pmatrix}$;

(iii) $N_1 = \epsilon \begin{pmatrix} 0 & a \\ -b & 0 \end{pmatrix}$;

(iv) $N_1 = \epsilon \begin{pmatrix} 0 & b \\ -a & 0 \end{pmatrix}$, $\epsilon = \pm 1$.

*Proof.* Since $d = m = 0$ and $ab \neq 0$, dividing (133) by $2ab$ and deleting its square from (135), we obtain $fgkl = 0$. When $g = 0$, $fl = -ab$ holds by (133), and $f = \varepsilon b, l = -\varepsilon a$ follows from (131) unless $k = 0$. However then (130) implies $k = 0$, a contradiction. Thus we have $g = k = 0$. Similarly $k = 0$ implies $g = 0$, and (i) or (ii) follows from (130) and (133). When $gk \neq 0$, $f = l = 0$ follows by a similar argument, and we obtain (iii) or (iv). □

Proposition 13.3. *When $a^2 \neq b^2$ and $ab \neq 0$, only Case* (iv) $N_1 = \begin{pmatrix} 0 & b \\ -a & 0 \end{pmatrix}$ *is possible, and $U_2(t)$ is given by*

$$U_2(t) = \begin{pmatrix} c & 0 & s & 0 \\ 0 & 1 & 0 & 0 \\ -s & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad c = \cos t, s = \sin t.$$

*Proof.* Consider the case $\epsilon = 1$. The case $\epsilon = -1$ is similarly treated. Recall Corollary 12.5, where $U(t) = \begin{pmatrix} U_1(t) & 0 \\ 0 & U_2(t) \end{pmatrix}$, and $Z_{\underline{i}}(t) = U_2(t) Z_{\underline{i}}(0)$ holds for $i = 1, 2$. The eigenvectors of $S(t) = \cos tD + \sin tN = U_2(t) D^t U_2(t)$ for eigenvalues $a, -a, b, -b$ are given by $\boldsymbol{v}_i(t) = U_2(t)\boldsymbol{v}_i$, where

$$\boldsymbol{v}_1 = Z_1 + Z_2, \quad \boldsymbol{v}_2 = Z_1 - Z_2, \quad \boldsymbol{v}_3 = Z_{\bar{1}} + Z_{\bar{2}}, \quad \boldsymbol{v}_4 = Z_{\bar{1}} - Z_{\bar{2}} \tag{137}$$

are eigenvectors of $D$, $Z_{\underline{i}} = Z_{\underline{i}}(0)$. Conversely, we know $U_2(t)$ from $\boldsymbol{v}_1(t)$, $\boldsymbol{v}_2(t)$, $\boldsymbol{v}_3(t)$, $\boldsymbol{v}_4(t)$. For instance, in Case (i), from $S(t) = \begin{pmatrix} 0 & ca & sa & 0 \\ ca & 0 & 0 & -sb \\ sa & 0 & 0 & cb \\ 0 & -sb & cb & 0 \end{pmatrix}$, it is easy to see

$$\boldsymbol{v}_1(t) = \begin{pmatrix} 1 \\ c \\ s \\ 0 \end{pmatrix}, \quad \boldsymbol{v}_2(t) = \begin{pmatrix} 1 \\ -c \\ -s \\ 0 \end{pmatrix}, \quad \boldsymbol{v}_3(t) = \begin{pmatrix} 0 \\ -s \\ c \\ 1 \end{pmatrix}, \quad \boldsymbol{v}_4(t) = \begin{pmatrix} 0 \\ -s \\ c \\ -1 \end{pmatrix},$$

and $U_2(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c & -s & 0 \\ 0 & s & c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ follows. In this way, we conclude that

(i) When $N_1 = \begin{pmatrix} a & 0 \\ 0 & -b \end{pmatrix}$, $U_2(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c & -s & 0 \\ 0 & s & c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Thus the odd vector $Z_1$ is parallel along $c$, a contradiction.

(ii) When $N_1 = \begin{pmatrix} b & 0 \\ 0 & -a \end{pmatrix}$, $U_2(t) = \begin{pmatrix} c & 0 & 0 & s \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s & 0 & 0 & c \end{pmatrix}$. Thus the odd vector $Z_{\bar{1}}$ is parallel along $c$, a contradiction.

(iii) When $N_1 = \begin{pmatrix} 0 & a \\ -b & 0 \end{pmatrix}$, $U_2(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c & 0 & -s \\ 0 & 0 & 1 & 0 \\ 0 & s & 0 & c \end{pmatrix}$. Thus the odd vector $Z_1$ is parallel along $c$, a contradiction.

(iv) When $N_1 = \begin{pmatrix} 0 & b \\ -a & 0 \end{pmatrix}$, we have

$$U_2(t) = \begin{pmatrix} c & 0 & s & 0 \\ 0 & 1 & 0 & 0 \\ -s & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{138}$$

In this case, we have no contradiction up to here. □

13.2. *The case $a^2 = b^2 \neq 0$.* Now, we consider the case $a^2 = b^2$.

PROPOSITION 13.4. *$a^2 = b^2 \neq 0$ implies case* (iv) *with $a = \varepsilon b$, $\varepsilon = \pm 1$.*

*Proof.* The argument in the proof of Lemma 13.1 implies that we can choose a suitable $t_0$ so that $d = 0$ holds in (122). Taking $p = p(t_0)$ and putting $a = \varepsilon b$, $\varepsilon = \pm 1$, in (124) $\sim$ (129), we have

$$f^2 + g^2 + k^2 + l^2 + m^2 = 2a^2, \tag{139}$$

$$a(f + \varepsilon l)(k + \varepsilon g) = 0, \tag{140}$$

$$m(f^2 - g^2 + k^2 - l^2) = 0, \tag{141}$$

$$-2\varepsilon a^2(fl + gk) + a^2 m^2 = 2a^4, \tag{142}$$

$$am(fk - gl) = 0, \tag{143}$$

$$(fl - gk)^2 = a^4. \tag{144}$$

When $a \neq 0$, from (139), and (142) divided by $a^2$, it follows $(f+\varepsilon l)^2+(g+\varepsilon k)^2 = 0$, and we obtain $f = -\varepsilon l$ and $g = -\varepsilon k$. Then from (139) and (144), it follows

$$f^2 + g^2 + m^2/2 = a^2 = \pm(f^2 - g^2). \tag{145}$$

Thus we obtain $g = m = 0$ (and $f^2 = a^2$) or $f = m = 0$ (and $g^2 = a^2$). When $\varepsilon = 1$, (i) $\sim$ (iv) of Lemma 13.2 with $a = b$ follow just as before, and $S(t) = \cos tD + \sin tN = U_2(t)D^tU_2(t)$ holds for each $U_2(t)$ given there. Then we can apply the argument on evenness and oddness of the eigenvectors of $S(t)$ as in the proof of Proposition 13.3 to conclude that only Case (iv) is possible. When $a = -b$, with respect to $Z_1(t), Z_2(t), Z_{\bar{1}}(t)$ and $-Z_{\bar{2}}(t)$, we may consider $a = b$, and applying the same argument, we can show that only Case (iv) is possible since the evenness and oddness of $Z_{\underline{i}}(t)$ are not changed. □

*Remark* 13.5. The assumption $ab \neq 0$ is essential in the above argument. In fact, when $ab = 0$, in particular, when $a = b = 0$, we have no information on $U_2$ since $D = 0$. Thus we need another argument (see Section 14.2).

13.3. *Case* (iv). We need a more detailed argument to eliminate Case (iv). The following argument is independent of the choice of the signature of $Z_{\bar{2}}$, and so we may consider $a = b$ when $a^2 = b^2$. Recall (iv) occurs under the assumption $a(t)b(t) \not\equiv 0$.

PROPOSITION 13.6. *Let $N$ be as in* (iv) *where we allow $a = b$. Then $a(t)b(t) \equiv 0$ follows, and hence* (iv) *cannot occur.*

*Proof.* When $a(t)b(t) \not\equiv 0$, $Z_1(t)$ and $Z_{\bar{1}}(t)$ are odd and $Z_2(t)$ and $Z_{\bar{2}}(t)$ are even vectors (Corollary 12.3). It is easy to see that $S(t) = \cos tD + \sin tN = U_2(t)D^tU_2(t)$ holds for $U_2(t)$ in (138), and hence $Z_{\underline{2}}(t) = U_2(t)Z_{\underline{2}} = Z_{\underline{2}}$ is parallel along $c$. Let $W'$ be the orthogonal complement of $Z_{\bar{2}}$ in $E(c)^{\perp}$, and put $W(t) = \operatorname{span}\{\bar{\nabla}_{e_3}e_6(t), \bar{\nabla}_{e_{\bar{3}}}e_6(t)\}$ for fixed $t$. Then we have $\dim W' \cap W(t) \geq 3 + 2 - 4 = 1$ for each $t$. Since $W(t)$ spans $E(c)^{\perp}$, not all of $W(t)$ is contained in $W'$, namely, there exists an interval $I$ on which $\dim W' \cap W(t) = 1$. On this interval, $e_3(t)$ so that $\nabla^{\perp}_{e_3}e_6(t) \in W'$ can be continuously defined.

LEMMA 13.7. $\dim W' \cap W(t) = 1$ *holds for all $t$, and we have an analytic field $e_3(t) \in D_3(t)$ on $c$, satisfying $\nabla^{\perp}_{e_3}e_6(t) \in W'$. If we put $K = \operatorname{span}_t\{e_3(t)\}$, then all $L(t)$ map $K$ into $W'$, and $W = L(t)(K)$ is independent of $t$. In particular*, $\dim W = \dim K - 1$.

*Proof.* Put $\tilde{K} = \mathrm{span}_t\{e_3(t) \mid \nabla^{\perp}_{e_3} e_6(t) \in W'\}$. For any $\tau$, we can express $L(t) = \cos t L(\tau) + \sin t L_t(\tau)$, and so $L(\tau)(e_3(\tau)) = 0$ and $L_t(\tau)(e_3(\tau)) = \bar{\nabla}_{e_3} e_6(\tau)$ (see (37)) imply

$$L(t)(e_3(\tau)) = (\cos t L(\tau) + \sin t L_t(\tau))(e_3(\tau)) = \sin t \bar{\nabla}_{e_3} e_6(\tau),$$

of which direction is independent of $t$. Therefore,

$$\tilde{W} = L(t)(\tilde{K}) = \mathrm{span}\{\bar{\nabla}_{e_3} e_6(\tau) \mid e_3(\tau) \in \tilde{K}\} \subset W'$$

is independent of $t$. Suppose $\dim \tilde{W} = \dim \tilde{K} - 2$. Then $\tilde{K}$ contains all $\ker L(t)$, namely, $\tilde{W} = E(c)^{\perp}$, contradicting $\tilde{W} \subset W'$. Thus $\dim \tilde{W} = \dim \tilde{K} - 1$. This means $\dim W' \cap W(t) = 1$ for all $t$, and we have $I = [0, 2\pi)$. Thus $\tilde{K} = K$ and $\tilde{W} = W$ hold, and the lemma is proved. □

Corollary 13.8. *$K$ is orthogonal to $X_{\bar{2}}(t)$ for each $t$. In particular,* $\dim K \leq 5$.

*Proof.* If a vector $v$ in $K$ has nonzero $X_{\bar{2}}(t_1)$ component (and thus not a kernel vector of $L(t_1)$) for some $t_1$, then $L(t_1)(v)$ has nonzero $Z_{\bar{2}}$ component, a contradiction. □

Lemma 13.9. $\dim K \neq 4, 5$.

*Proof.* Since $K$ is orthogonal to $X_{\bar{2}}(\tau)$ for any fixed $\tau$, $\dim K = 5$ implies both $e_3(\tau), e_{\bar{3}}(\tau)$ belong to $K$, which contradicts Lemma 13.7. If $\dim K = 4$, then $\dim W = 3$ follows, and we can express $W = \mathrm{span}\{Z_1(t), Z_{\bar{1}}(t), Z_2\}$ for each $t$. Thus $K$ contains $e_3(t), X_1(t), X_{\bar{1}}(t), X_2(t)$, and hence $K = \mathrm{span}\{e_3(t), X_1(t), X_{\bar{1}}(t), X_2(t)\}$ holds for each $t$. Then the orthogonal complement of $K$ in $E(c)$ is given by $K^{\perp} = \mathrm{span}_t\{e_{\bar{3}}(t), X_{\bar{2}}(t)\}$ for each $t$, which is parallel along $c$. However, then $\mathrm{span}_t\{e_{\bar{3}}(t)\} \subset K^{\perp}$ is of dimension at most 2, contradicting Lemma 8.4 and Proposition 8.5. □

Lemma 13.10. $\dim K \neq 3$.

*Proof.* When $\dim K = 3$, $W (\subset W')$ is of dimension 2, and it contains a vector in $\mathrm{span}\{Z_1(t), Z_{\bar{1}}(t)\}$ by the dimension count. Since (138) implies

$$Z_1(t) = \cos t Z_1(0) - \sin t Z_{\bar{1}}(0),\ Z_{\bar{1}}(t) = \sin t Z_1(0) + \cos t Z_{\bar{1}}(0),$$

$W = \mathrm{span}\{Z_1(t), Z_{\bar{1}}(t)\}$ follows. Then $K = \mathrm{span}\{e_3(t), X_1(t), X_{\bar{1}}(t)\}$ holds for each $t$. Therefore, $e_{\bar{3}}(t)$ is orthogonal to $K$, in particular, orthogonal to $e_3(p)$, and hence we have

$$\mathrm{span}\{e_{\bar{3}}(t)\} = K^{\perp} = \mathrm{span}\{e_{\bar{3}}(t_1), X_2(t_1), X_{\bar{2}}(t_1)\}$$

for any $t_1$ (see Section 8). Thus Lemma 8.2 implies

$$\dim L(t)(K^{\perp}) = 2. \tag{146}$$

On the other hand, $c_0\nabla_{e_6}X_{\underline{i}} = H(0)X_{\underline{i}}$ holds for $i = 1, 2$, and we have

$$\langle H(0)X_{\underline{1}}, X_{\underline{2}}\rangle = 0$$

since $K$ and $K^\perp$ are parallel. Then we can express

$$J_1 = \begin{pmatrix} H_0 & X & Y \\ -{}^tX & 0 & Z \\ -{}^tY & -{}^tZ & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix},$$

and because $He_3 \in K$ and $He_{\bar{3}} \in K^\perp$ hold, we can put

$$X = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}, \quad Y = \begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix}, \tag{147}$$

where $(x_1, y_1), (x_2, y_2) \neq (0, 0)$. Recall from (138) and $H(0) = U_t(0)$ that

$$J_2 = \begin{pmatrix} 0 & V \\ -{}^tV & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore, from (121), we have

$$\begin{aligned} M &= J_1A - AJ_2 \qquad (148) \\ &= \begin{pmatrix} H_0 & X & Y \\ -{}^tX & 0 & Z \\ -{}^tY & -{}^tZ & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ A_1 & 0 \\ 0 & A_2 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ A_1 & 0 \\ 0 & A_2 \end{pmatrix} \begin{pmatrix} 0 & V \\ -{}^tV & 0 \end{pmatrix} \\ &= \begin{pmatrix} XA_1 & YA_2 \\ 0 & ZA_2 - A_1V \\ -{}^tZA_1 + A_2{}^tV & 0 \end{pmatrix} \\ &= \begin{pmatrix} x_1\sqrt{\sigma} & 0 & y_1\sqrt{\tau} & 0 \\ 0 & x_2/\sqrt{\sigma} & 0 & y_2/\sqrt{\tau} \\ 0 & 0 & z_1\sqrt{\tau} - \sqrt{\sigma} & 0 \\ 0 & 0 & 0 & z_2/\sqrt{\tau} \\ -\sqrt{\sigma}z_1 + \sqrt{\tau} & 0 & 0 & 0 \\ 0 & -z_2/\sqrt{\sigma} & 0 & 0 \end{pmatrix}. \end{aligned}$$

On the other hand, from $B_\zeta = UB_\eta{}^tU$ where $U = U(\pi/2) = \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix}$, and $M = U_1A^tU_2$ where $U_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, we have, putting $A = \begin{pmatrix} a_1 & a_2 & a_4 & a_4 \end{pmatrix}$ and $M = \begin{pmatrix} m_1 & m_2 & m_3 & m_4 \end{pmatrix}$,

$$m_1 = -U_1a_3,\ m_2 = U_1a_2,\ m_3 = U_1a_1,\ m_4 = U_1a_4.$$

In particular, $\langle m_i, m_j\rangle = 0$ holds for $i \neq j$. Then from (148), we obtain

$$x_1y_1 = 0, \quad x_2y_2 = 0,$$

and either

(1) $X=0$,
(2) $Y=0$,
(3) $\begin{pmatrix} X & Y \end{pmatrix} = \begin{pmatrix} x_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & y_2 \end{pmatrix}$, or
(4) $\begin{pmatrix} 0 & 0 & y_1 & 0 \\ 0 & x_2 & 0 & 0 \end{pmatrix}$

occurs. Since these are mutually exclusive cases, only one of the cases occurs on $c$ where we may apply the argument at any $p(t)$. In Cases (1) and (3), $H(t)e_{\bar{3}}(t) = c_0\nabla_{e_6}e_{\bar{3}}(t)$ is in the direction of $X_{\bar{2}}(t)$, and hence $\bar{\nabla}_{e_{\bar{3}}}e_6(t)$ is in the direction of $Z_{\bar{2}}$ that is parallel, contradicting (146). In Cases (2) and (4), $H(t)e_{\bar{3}}(t)$ is in the direction of $X_2(t)$, and hence $\bar{\nabla}_{e_{\bar{3}}}e_6(t)$ is in the direction of $Z_2$ that is parallel, a contradiction. Thus $\dim K \neq 3$ follows. Since $\dim K \neq 2$ by Lemma 8.4, we have a contradiction caused by $a(t)b(t) \not\equiv 0$. □

As a summary, we conclude

THEOREM 13.11. *When* $\dim E(c) = 6$, *any shape operator* $\begin{pmatrix} 0 & R \\ {}^tR & S \end{pmatrix}$ *satisfies either one of the following, where* $T = {}^tRR$:

(I) $T = \begin{pmatrix} 3I_2 & 0 \\ 0 & \frac{1}{3}I_2 \end{pmatrix}$ *and* $S = 0_4$.

(II) $T = \begin{pmatrix} T_1 & \\ & T_2 \end{pmatrix}$, $T_2 = \begin{pmatrix} 3 & \\ & \frac{1}{3} \end{pmatrix}$ *and* $S = \begin{pmatrix} S_1 & \\ & 0 \end{pmatrix}$.

*Proof.* By Proposition 13.6, we obtain $ab \equiv 0$. When $a = b = 0$, $T = \begin{pmatrix} 3I_2 & \\ & 1/3I_2 \end{pmatrix}$ follows. When $a \neq 0$ and $b = 0$, for instance, (II) occurs. (Thus Case (0) in Lemma 9.3 cannot occur). □

## 14. Investigation of the remaining cases

14.1. *Case* (II). We investigate Case (II) first.

PROPOSITION 14.1. *Case* (II) *does not occur.*

*Proof.* Note that we cannot apply the argument in Section 12 as we are dealing with the case $ab = 0$. However, we can use the argument in Section 11. First, we have

$$T(0) = {}^tAA = \begin{pmatrix} T_1 & \\ & T_2 \end{pmatrix}, \; T_2 = \begin{pmatrix} 3 & \\ & 1/3 \end{pmatrix}, \; D = \begin{pmatrix} D_1 & 0 \\ 0 & 0 \end{pmatrix},$$

where $\operatorname{rank} D = 2$. Similarly, we have $\operatorname{rank} S(t) = 2$, where $S(t) = \cos tD + \sin tN$. Therefore, $N$ should be of the form $N = \begin{pmatrix} N_1 & 0 \\ 0 & 0 \end{pmatrix}$, and we have a parallel decomposition $E(c)^\perp = W_1 \oplus W_2$, where $W_2$ is spanned by eigenvectors $Z_{\bar{1}}(t), Z_{\bar{2}}(t)$ of $T(t)$ for eigenvalues 3 and 1/3.

Next, we can take eigenvectors $Z_1(t), Z_2(t)$ for $\sigma(t)$ and $1/\sigma(t)$ continuously along $c$, even where $\sigma(t_0) = 1$, so that $S_1(t) = \begin{pmatrix} 0 & a(t) \\ a(t) & 0 \end{pmatrix}$ holds with respect to this moving frame. We have $L(\pi) = -L(0)$ from $L(t) = \cos tB_\eta +$

$\sin tB_\zeta$ and $T(\pi) = T(0)$ from $T(t) = {}^tR(t)R(t)$. The latter implies $\sigma = \sigma(\pi) = \sigma(0)$. As an eigenvector of $T_1(0)$ for $\sigma$, $Z_1(\pi)$ is parallel to $Z_1(0)$. Then from

$$\begin{cases} L(\pi)(X_1(\pi)) = \sqrt{\sigma} Z_1(\pi), \\ L(0)(X_1(0)) = \sqrt{\sigma} Z_1(0), \end{cases}$$

we have

$$X_1(\pi) = \varepsilon X_1(0), \quad Z_1(\pi) = -\varepsilon Z_1(0), \quad \varepsilon = \pm 1.$$

Similarly from

$$\begin{cases} L(\pi)(X_2(\pi)) = 1/\sqrt{\sigma} Z_2(\pi), \\ L(0)(X_2(0)) = 1/\sqrt{\sigma} Z_2(0), \end{cases}$$

when $\alpha\beta \not\equiv 0$, equivalently, $a(t) \not\equiv 0$, we have

$$X_2(\pi) = -\varepsilon X_2(0), \quad Z_2(\pi) = \varepsilon Z_2(0),$$

where we use $e_i(\pi) \in D_{6-i}(0)$ by the global correspondence in (91) and (92). However, since $W_1$ is parallel along $c$ and the pair $Z_1(t), Z_2(t)$ is a continuous orthonormal frame of $W_1$, this contradicts the fact that a continuous frame preserves the orientation. Therefore, $\alpha\beta \equiv 0$, namely $a(t) \equiv 0$, follows, a contradiction. □

14.2. *Case* (I): *Autoparallel distribution.* To eliminate Case (I), we need an argument using both $M_+$ and $M_-$. In this case, using a frame at a point $p \in c$, we can express (see (101))

$$E(c) = D_3 \oplus \mathrm{span}\{e_1 + e_5, e_{\bar 1} + e_{\bar 5}, e_2 + e_4, e_{\bar 2} + e_{\bar 4}\}, \tag{149}$$
$$E(c)^\perp = \mathrm{span}\{e_1 - e_5, e_{\bar 1} - e_{\bar 5}, e_2 - e_4, e_{\bar 2} - e_{\bar 4}\}.$$

From these, we easily see $B_{31} = -B_{35}$ and $B_{32} = -B_{34}$. Moreover, $B_{15}$ and $B_{24}$ are skew because $\langle B_\zeta(e_1 + e_5), e_{\bar 1} + e_{\bar 5}\rangle = 0$, etc. Recall $B_\zeta = (B_{ij})$ depends on $\zeta = e_6 \in T^\perp M_+ \cong G_+$ (see Section 7).

Lemma 14.2. *In Case* (I), $B_{31} = 0$ *or* $B_{32} = 0$ *does not occur for generic* $e_6 \in G_+$.

*Proof.* First, suppose $B_{23} = B_{34} = 0$ occurs on an open subset of $G_+$; namely,

$$\Lambda^2_{\underline{36}} = 0 = \Lambda^4_{\underline{36}}.$$

Then this holds over all $G_+$ by the analyticity. Thus by the global symmetry in Section 4, we have

$$B_{14} = B_{25} = 0. \tag{150}$$

In the following, we use the Gauss equation $[i.j]$, and so we need an admissible frame. From $\langle\nabla_{e_6}(e_1+e_5), e_{\bar{1}}-e_{\bar{5}}\rangle=0$, we obtain

$$0=\Lambda_{61}^{\bar{1}}-\Lambda_{65}^{\bar{5}}-\Lambda_{61}^{\bar{5}}+\Lambda_{65}^{\bar{1}}=\Lambda_{61}^{\bar{1}}-\Lambda_{65}^{\bar{5}}-\Lambda_{61}^{\bar{5}}-\Lambda_{6\bar{1}}^{5}=\Lambda_{61}^{\bar{1}}-\Lambda_{65}^{\bar{5}},$$

where the last equality follows since $B_{15}$ is skew, where $b_{15}$, etc., is related to $\Lambda_{61}^{5}$ by (9). Thus $e_1(t), e_{\bar{1}}(t)\in D_1(t)$ is admissible if and only if $e_5(t), e_{\bar{5}}(t)\in D_5(t)$ is admissible in our pair $e_1+e_5, e_{\bar{1}}+e_{\bar{5}}$. Similarly, $e_2(t), e_{\bar{2}}(t)\in D_2(t)$ is admissible if and only if $e_4(t), e_{\bar{4}}(t)\in D_4(t)$ is admissible. Thus taking an admissible $e_i(t), e_{\bar{i}}(t)\in D_i(t)$, we obtain an admissible frame compatible with the expression of (149).

Now from [1.4] and [2.5], we obtain $B_{15}B_{54}=B_{15}B_{12}=0$. Since $B_{15}$ is skew, rank $B_{15}=0$ or 2. In the latter case, we have $B_{12}=B_{54}=0$. However, this means

$$\langle\nabla_{e_6}(e_{\underline{1}}+e_{\underline{5}}), e_{\underline{2}}\rangle=0=\langle\nabla_{e_6}(e_{\underline{1}}+e_{\underline{5}}), e_{\underline{4}}\rangle,$$

which holds everywhere. Then $D_3\oplus\operatorname{span}\{e_1+e_5, e_{\bar{1}}+e_{\bar{5}}\}$ is parallel, which implies $\dim E(c)=4$, a contradiction. Thus $B_{15}=0$ follows. In this case, from [2.3] it follows

$$B_{21}B_{13}=0.$$

If rank $B_{13}<2$, we may choose $e_1, e_{\bar{1}}$ and $e_3, e_{\bar{3}}$ so that $B_{31}=\left(\begin{smallmatrix} * & 0\\ 0 & 0\end{smallmatrix}\right)=-B_{35}$, namely, $e_{\bar{3}}$ is parallel along $c$, contradicting Proposition 8.5. Thus we obtain $B_{12}=0$, which implies $B_{54}=0$ by the global symmetry, but this cannot occur as before.

Next, suppose $B_{31}=B_{35}=0$ occurs in a neighborhood of $G_+$, which implies

(151) $$\Lambda_{\underline{63}}^{\underline{1}}=0=\Lambda_{\underline{63}}^{\underline{5}}.$$

Now consider $M_-$, of which shape operators we now denote by

$$C_\zeta=(C_{ij})_{2\le i,j\le 6},\quad \zeta=e_1(p)$$

with respect to $D_2(p)\oplus D_3(p)\oplus D_4(p)\oplus D_5(p)\oplus D_6(p)$. From (151), it follows by the global symmetry that

$$0=\Lambda_{\underline{25}}^{\underline{1}}=\Lambda_{\underline{41}}^{\underline{5}}=\Lambda_{\underline{41}}^{\underline{3}}.$$

Hence, we have

$$C_\zeta=\begin{pmatrix} 0 & C_{23} & C_{24} & 0 & C_{26}\\ C_{32} & 0 & 0 & C_{35} & 0\\ C_{42} & 0 & 0 & 0 & C_{46}\\ 0 & C_{53} & 0 & 0 & C_{56}\\ C_{62} & 0 & C_{42} & C_{65} & 0\end{pmatrix}.$$

This corresponds to the case when $B_{32}=B_{34}=0$ on $M_+$, where the Gauss equations $[i.j]$ holds if we replace $B_{ij}$ by $C_{i+1\,j+1}$, because the eigenspaces of $C_\eta$ ($\eta = -\sin\theta_1 p + \cos\theta_1 \xi_p$) for eigenvectors $\sqrt{3}, 1/\sqrt{3}, 0, -1/\sqrt{3}, -\sqrt{3}$ are shifted to $D_2(p), D_3(p), D_4(p), D_5(p), D_6(p)$, respectively; see (20). Therefore, a similar argument as before implies a contradiction. □

Proposition 14.3. *When Case* (I) *occurs, $E(c)$ is independent of $c$, and so is $F(\gamma)$ of $\gamma$. Let $p = p_1$ and $q = p_3$ in Figure* 1. *Then with respect to the basis at $p = p_1$, we have*

$$F(\gamma) = D_6(p) \oplus E(c)^\perp, \quad E(c) = F(\gamma)^\perp \oplus D_3(p). \tag{152}$$

*Proof.* It is sufficient to show $E(c) = E(c_s)$ for any geodesic $c_s$ through $p$ in the direction $e_6^s = \cos s e_6 + \sin s e_{\bar{6}}$. In fact, then for any geodesic $c'$ not through $p$, a point $p' \in c'$ lies on some $c_s$, and so $D_3(p') \subset E(c_s) = E(c)$, and $\dim E(c') = 6$ implies $E(c') = E(c)$.

For generic $e_3 \in D_3(p)$, by Lemma 14.2 we may express

$$\nabla_{e_3} e_6(p) = u(e_1 - e_5) + v(e_2 - e_4), \quad uv \neq 0, \tag{153}$$

where we use $e_{\underline{i}} = e_{\underline{i}}(p)$. Because $\nabla_{e_3} e_6(p) = \nabla_{e_1} e_4(q)$ holds up to a scalar multiple, denoting by $\gamma$ the geodesic of $L_1(q)$ through $q$ in the direction $e_1(q) = e_3(p)$, we obtain $\nabla_{e_3} e_6(p) \in F(\gamma)$. Since only Case (I) is possible for $M_-$ too, using the frame $e_{\underline{i}}$ at $p$ (not $q$), we can express

$$F(\gamma) = D_6(p) \oplus \operatorname{span}\{e_1 - e_5, e_2 - e_4, e_{\bar{1}} - \epsilon_1 e_{\bar{5}}, e_{\bar{2}} - \epsilon_2 e_{\bar{4}}\},$$

where $\epsilon_i = \pm 1$. Next, for any $e_6^s$, $s \neq 0$ modulo $\pi$, $\nabla_{e_3} e_6^s(p)$, identified with $\nabla_{e_1} e_4^s(q)$, belongs to $F(\gamma)$. If this has $e_{\bar{1}} - \epsilon_1 e_{\bar{5}}, e_{\bar{2}} - \epsilon_2 e_{\bar{4}}$ components, $\nabla_{e_6^s} e_3$ has $e_{\bar{1}} + \epsilon_1 e_{\bar{5}}, e_{\bar{2}} + \epsilon_2 e_{\bar{4}}$ components, which belong to $E(c_s)$. As $s$ tends to 0, $E(c_s)$ tends to $E(c)$, and by continuity, we have $\epsilon_i = 1$. Thus, when $\nabla_{e_3} e_6^s$ has $e_{\bar{i}}$ components, $e_{\bar{1}} + e_{\bar{5}}, e_{\bar{2}} + e_{\bar{4}}$ belong to $E(c_s)$, and

$$\begin{aligned} F(\gamma) &= D_6(p) \oplus \operatorname{span}\{e_1 - e_5, e_2 - e_4, e_{\bar{1}} - e_{\bar{5}}, e_{\bar{2}} - e_{\bar{4}}\} \\ &= D_6(p) \oplus E(c)^\perp \end{aligned}$$

follows. Then two elements of $E(c_s)$ orthogonal to $D_3(p)$ and $e_{\bar{1}} + e_{\bar{5}}, e_{\bar{2}} + e_{\bar{4}}$ are given by $e_1 + \varepsilon_1 e_5, e_2 + \varepsilon_2 e_4$, and $\varepsilon_i = 1$ follows by continuity as before, and $E(c_s)$ does not depend on $s$.

On the other hand, when $\nabla_{e_3} e_6^s$ has no $e_{\bar{i}}(p)$ components, namely, belong to $\operatorname{span}\{e_1 - e_5, e_2 - e_4\}$, $\nabla_{e_6^s} e_3(p) \in \operatorname{span}\{e_1 + e_5, e_2 + e_4\}$ follows, and

$$E(c_s) = D_3(p) \oplus \operatorname{span}\{e_1 + e_5, e_2 + e_4, e_{\bar{1}} + \varepsilon_1 e_{\bar{5}}, e_{\bar{2}} + \varepsilon_2 e_{\bar{4}}\},$$

where $\varepsilon_i = \pm 1$. Again, as $E(c_s)$ tends to $E(c)$, we have $\varepsilon_i = 1$ by continuity. Thus we conclude that $E(c)$ is independent of $c$. Then $\nabla_{e_3} e_6^s \in F(\gamma)$ implies $F(\gamma) = D_6(p) \oplus E(c)^\perp$. □

By Proposition 14.3, $E(c)$ depends only on $\bar{p} \in M_+$, and we express it as $E(\bar{p})$. Now we prove

PROPOSITION 14.4. *Case* (I) *does not occur.*

For the proof, define a distribution $\tilde{E}$ on $M$ by

$$\tilde{E}(p) = E(\bar{p}), \quad p \in M;$$

namely, for $p \in f_6^{-1}(\bar{p})$, $\tilde{E}(p)$ is the parallel transport of $E(\bar{p})$ along the normal geodesic at $p$ of $M$ with respect to the connection of $S^{13}$. Similarly, we define a distribution $\tilde{F}$ on $M$ by $\tilde{F}(q) = F(\bar{q})$, $q \in M$.

LEMMA 14.5. $E(\bar{p}) = \tilde{F}(\bar{q})^{\perp}$ *is parallel in the direction* $D_6(p)$ *and* $D_3(p)$.

*Proof.* $E(\bar{p}) = E(c)$ is parallel along $c$, i.e., in the direction of $D_6(p)$. Moreover, $E(\bar{p})^{\perp} = F(\gamma) = F(\bar{q})$ is parallel along $D_1(q) = D_3(p)$, and the lemma follows. □

*Proof of Proposition 14.4.* Now, we may express $E(\bar{p}) = \text{span}\{D_3(x_j) \mid x_1, \ldots, x_k \in L_6(p)\}$, where $k \geq 3$. In fact, $E(\bar{p}) = D_3(x_1) + D_3(x_2) + D_3(x_3)$ holds if $(D(x_1) + D_3(x_2)) \cap D_3(x_3) = \{0\}$. At worst, we can find $k$ finite. Then, a vector $X \in \tilde{E}(p)$ is expressed as

$$X = \sum_{j=1}^{k} (u_j e_3(x_j) + v_j e_{\bar{3}}(x_j)).$$

Since $\bar{x}_j = f(x_j) = \bar{p}$, $E(\bar{x}_j)$ is identified with $\tilde{E}(p)$. Moreover, since $E(\bar{x}_j)$ is parallel in the direction $D_3(x_j)$ by Lemma 14.5, for any $Y \in \tilde{E}(p)$,

$$\nabla_X Y = \frac{1}{c_2} \sum (u_j \bar{\nabla}_{e_3(x_j)} Y + v_j \bar{\nabla}_{e_{\bar{3}}(x_j)} Y)$$

belongs to $\text{span}_j\{E(\bar{x}_j)\} = \tilde{E}(p)$. Thus $\tilde{E}$ is autoparallel, by which we mean $\nabla_X Y \in \tilde{E}$ for any $X, Y \in \tilde{E}$ with respect to the connection of $M$. In other words, $\tilde{E}$ is a totally geodesic distribution on $M$. On the other hand, with respect to the connection $\tilde{\nabla}$ of $S^{13}$, we have

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y)\xi_p,$$

where $\xi_p$ is the unit normal of $M$ at $p$, and $h(\,,\,)\xi_p$ is the second fundamental tensor of $M$ in $S^{13}$. In particular, for $e_3 \in D_3(x)$, $x \in L_6(p)$, we have

$$\tilde{\nabla}_{e_3} e_3 = \lambda_3 \xi_x,$$

where we use (3) and (11). Here, $\tilde{E}(p)$ contains six independent $e_{\underline{3}}(x_j)$, $x_j \in L_6(p)$, and so all the eigenvalues of the shape operators $h(\,,\,)$ of a leaf $\mathcal{L}$ of $\tilde{E}$ are $\lambda_3$, and $\tilde{E}$ is totally umbilic in $S^{13}$. Hence, $\mathcal{L}$ is a 6-sphere $S^6$, which is totally geodesic in $M$. Now the same is true for $\tilde{F}$, and we obtain $M = S^6 \times S^6$, which is an isoparametric hypersurface in $S^{13}$ with two principal curvatures, contradicting our assumption. □

Finally, we obtain

THEOREM 14.6. *The focal submanifolds of an isoparametric hypersurface with* $(g, m) = (6, 2)$ *have the shape operators* $B_{\boldsymbol{n}}$ *whose kernel does not depend on* $\boldsymbol{n}$.

## 15. Homogeneity

In this section, we prove Theorem 1.1. The shape operators of $M_+$ have the invariant kernel, and so

$$\Lambda^j_{\underline{63}} = 0, \quad j = 1, 2, 3, 4, 5 \tag{154}$$

holds over all $M$. Then by the global correspondence, we have

$$\Lambda^j_{\underline{14}} = 0, \quad \Lambda^j_{\underline{25}} = 0. \tag{155}$$

Note that the former implies that the kernel of the shape operators $C_N$ of $M_-$ is also independent of $N$. By (155), for the shape operator $B_N$ of $M_+$, we have

$$B_N = \begin{pmatrix} 0 & B_{12} & 0 & 0 & B_{15} \\ B_{21} & 0 & 0 & B_{24} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & B_{42} & 0 & 0 & B_{45} \\ B_{51} & 0 & 0 & B_{54} & 0 \end{pmatrix} \tag{156}$$

for any $N = \cos t\zeta + \sin s\bar{\zeta}$, where we use the expression with respect to the frame $e_1, e_{\bar{1}}, \dots, e_5, e_{\bar{5}}$ at $p$ as in Lemma 2.3. Here, we may choose an *admissible* frame, with respect to which the Gauss equation $[i.j]$ holds in each direction $e_6 \in D_6(p)$.

PROPOSITION 15.1. *Either* $B_{12} = B_{45} = 0$, *or* $B_{15} = 0$ *occurs.*

For the proof, note that in (33), $e_6(B_{ij})$ vanishes for $i = j$, $i = 3$, $j = 3$, $(i, j) = (1, 4)$ and $(i, j) = (2, 5)$. Using these, we rewrite some of the Gauss equation $[i.j]$:

[1.1] $\sqrt{3}I = 2(\frac{\sqrt{3}}{2}B_{12}B_{21} + \frac{1}{2\sqrt{3}}B_{15}B_{51})$,
[2.2] $\frac{1}{\sqrt{3}}I = 2(-\frac{\sqrt{3}}{2}B_{21}B_{12} + \frac{\sqrt{3}}{2}B_{24}B_{42})$,
[4.4] $-\frac{1}{\sqrt{3}}I = 2(-\frac{\sqrt{3}}{2}B_{42}B_{24} + \frac{\sqrt{3}}{2}B_{45}B_{54})$,
[5.5] $-\sqrt{3}I = -2(\frac{1}{2\sqrt{3}}B_{51}B_{15} + \frac{\sqrt{3}}{2}B_{54}B_{45})$,
[1.4] $0 = \frac{2}{\sqrt{3}}B_{15}B_{54}$,
[2.5] $0 = -\frac{2}{\sqrt{3}}B_{21}B_{15}$.

Obviously, $\operatorname{rank} B_{ij}$ is independent of the choice of the frame of $D_i(p)$ and $D_j(p)$. Here, $B_{ij}$ depends on $e_6 \in D_6$, and we denote it by $B_{ij}(e_6)$. By [1.4], [2.5], and because $\operatorname{rank} B_\zeta = 8$, $\operatorname{rank} B_{15}(e_6) = 2$ holds if and only if $B_{12}(e_6) = B_{45}(e_6) = 0$. Since the former is an open, and the latter is a closed condition, rank $B_{15}(e_6) = 2$ holds for all $e_6 \in D_6$, or never holds on $D_6$. Similarly, $B_{15} = 0$ holds or never holds on $D_6$, and rank$B_{15} = 1$ holds or never holds on $D_6$. Therefore, rank $B_{15}$ is either 0, 1 or 2 over all $D_6$ at all $p \in M$. In more detail, we have the following.

LEMMA 15.2. *For any* $N = \cos s\zeta + \sin s\bar{\zeta}$ *and* $B_N = \left(B_{ij}\right)$, rank$B_{ij}$ *is independent of* $s$. *Moreover, choosing a suitable basis of* $D_i$ *for each* $s$, *we have one of the following*:

(i) $B_{15} = \sqrt{3}J$, $B_{12} = B_{45} = 0$ *and* $B_{24} = 1/\sqrt{3}J$, *where* $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$;
(ii) $B_{15} = 0$ *and* $B_{12} = J$, $B_{24} = -(2/\sqrt{3})J$, *and* $B_{45} = J$;
(iii) $B_{15} = \left(\begin{smallmatrix} \sqrt{3} & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $\quad B_{45} = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $\quad B_{12} = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$,

$$B_{24} = \begin{pmatrix} \varepsilon/\sqrt{3} & 0 \\ 0 & 2\varepsilon'/\sqrt{3} \end{pmatrix}, \qquad \varepsilon, \varepsilon' = \pm 1.$$

*Proof.* (i) When $B_{15}$ is of rank 2, choose $e_5$ parallel with $\nabla_{e_6} e_{\bar{1}}$ so that $B_{15} = \left(\begin{smallmatrix} u & v \\ w & 0 \end{smallmatrix}\right)$ holds. Then from [1.1], we have

$$3I = B_{15}B_{51} = \begin{pmatrix} u^2 + v^2 & uw \\ uw & w^2 \end{pmatrix},$$

and hence $u = 0$ follows. Therefore, we can express $B_{15} = \sqrt{3}J$. Similarly, choosing $e_4$ parallel with $\nabla_{e_6} e_{\bar{2}}$, we obtain (i) by [2.2].

(ii) When $B_{15} = 0$, we may put $B_{12} = J$ by [1.1], choosing $e_2$ parallel with $\nabla_{e_6} e_{\bar{1}}$. In view of [2.2], this implies $B_{24} = -\frac{2}{\sqrt{3}}J$, with respect to a suitable basis of $D_4$. Then from [4.4] and [5.5], we may consider $B_{45} = J$ by a suitable choice of a basis of $D_5$.

(iii) When rank $B_{15} = 1$, taking a suitable basis of $D_1(p)$ and $D_5(p)$, we may assume $B_{15} = \left(\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Then choosing $e_{\bar{4}}$ parallel with the $D_4$ component of $\nabla_{e_6} e_{\bar{5}}$, we have $B_{54} = \left(\begin{smallmatrix} b_1 & b_2 \\ 0 & b \end{smallmatrix}\right)$. Substituting this into [1.4], we have $b_1 = b_2 = 0$. Moreover, choosing $e_{\bar{2}}$ parallel with the $D_2$ component of $\nabla_{e_6} e_{\bar{1}}$, we have $B_{12} = \left(\begin{smallmatrix} c_1 & c_2 \\ 0 & c \end{smallmatrix}\right)$. Then [2.5] implies $c_1 = c_2 = 0$. From [1.1] and [5.5], we obtain $a^2 = 3, b^2 = c^2 = 1$. Now put $B_{24} = \left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right)$. Since it follows from [2.2] and [4.4] that

$$B_{24}B_{42} = B_{42}B_{24} = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{4}{3} \end{pmatrix}.$$

we obtain

$$\begin{cases} x^2 + y^2 = x^2 + z^2 = \frac{1}{3}, \\ y^2 + w^2 = z^2 + w^2 = \frac{4}{3}, \\ xz + yw = 0, \\ xy + zw = 0, \end{cases}$$

and solving these, we have

$$B_{24} = \begin{pmatrix} \frac{\varepsilon}{\sqrt{3}} & 0 \\ 0 & \frac{2\varepsilon'}{\sqrt{3}} \end{pmatrix}, \quad \varepsilon, \varepsilon' = \pm 1. \qquad \square$$

*Proof of Proposition 15.1.* We show the mixed case (iii) in the above lemma does not occur. To investigate $B_{\bar\zeta}$, using (9), we calculate

$$0 = R_{\bar{1}6\bar{6}\bar{1}} = \Lambda^{\underline{i}}_{\bar{1}6}\Lambda^{\bar{1}}_{\bar{6}\underline{i}} - \Lambda^{\underline{i}}_{\bar{1}6}\Lambda^{\bar{1}}_{\bar{6}\underline{i}} - \Lambda^{\underline{i}}_{\bar{1}6}\Lambda^{\bar{1}}_{\underline{i}\bar{6}} + \Lambda^{\underline{i}}_{\bar{6}\bar{1}}\Lambda^{\bar{1}}_{\underline{i}\bar{6}} = \Lambda^{\underline{i}}_{\bar{1}6}\Lambda^{\underline{i}}_{\bar{6}\bar{1}},$$

where the repetition of $\underline{i}$ means taking sum over $i$ and $\bar{i}$. Thus, we have

$$\Lambda^{2}_{\bar{1}6}\Lambda^{2}_{\bar{6}\bar{1}} + \Lambda^{5}_{\bar{1}6}\Lambda^{5}_{\bar{6}\bar{1}} = 0.$$

Then from $\Lambda^{\bar{2}}_{\bar{1}6} \neq 0$ and $\Lambda^{5}_{\bar{1}6} = 0$, we have

$$\Lambda^{\bar{2}}_{\bar{1}\bar{6}} = 0. \tag{157}$$

Next, from $0 = R_{\bar{2}6\bar{6}\bar{2}} = c'\Lambda^{\bar{4}}_{\bar{2}6}\Lambda^{\bar{4}}_{\bar{6}\bar{2}}$, it follows $\Lambda^{\bar{4}}_{\bar{2}\bar{6}} = 0$, and from $0 = R_{2\bar{6}6\bar{2}} = c\Lambda^{4}_{\bar{2}6}\Lambda^{4}_{\bar{6}2}$, it follows $\Lambda^{4}_{2\bar{6}} = 0$. Thus we may put $\bar{B}_{24} = \left(\begin{smallmatrix} 0 & k \\ l & 0 \end{smallmatrix}\right)$, where $kl \neq 0$ since rank $\bar{B}_{24} = 2$ follows from Lemma 15.2. On the other hand, since $\operatorname{rank}(\cos sB_{12} + \sin s\bar{B}_{12}) = 1$ holds for any $s$, $\Lambda^{2}_{1\bar{6}}$ must vanish, and using (157), we may put $\bar{B}_{12} = \left(\begin{smallmatrix} 0 & m \\ n & 0 \end{smallmatrix}\right)$, where $mn = 0$. On the other hand, from

$$\begin{aligned} 0 &= R_{\bar{1}6\bar{6}4} = -\Lambda^{\underline{k}}_{\bar{1}\bar{6}}\Lambda^{4}_{6\underline{k}} - \Lambda^{\underline{k}}_{\bar{1}6}\Lambda^{4}_{\underline{k}\bar{6}}\frac{\lambda_1 - \lambda_6}{\lambda_1 - \lambda_k} \\ &= -\Lambda^{2}_{\bar{1}\bar{6}}\Lambda^{4}_{62} - \Lambda^{\bar{2}}_{\bar{1}6}\Lambda^{4}_{\bar{2}\bar{6}}\frac{\lambda_1 - \lambda_6}{\lambda_1 - \lambda_2}, \end{aligned}$$

$n \neq 0$ follows from $l \neq 0$, and we obtain $m = 0$. Therefore, we have

$$B_{12} = c\begin{pmatrix} \Lambda^{2}_{16} & \Lambda^{\bar{2}}_{16} \\ \Lambda^{2}_{\bar{1}6} & \Lambda^{\bar{2}}_{\bar{1}6} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \bar{B}_{12} = c\begin{pmatrix} \Lambda^{2}_{1\bar{6}} & \Lambda^{\bar{2}}_{1\bar{6}} \\ \Lambda^{2}_{\bar{1}\bar{6}} & \Lambda^{\bar{2}}_{\bar{1}\bar{6}} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix}. \tag{158}$$

Next, consider the shape operators $C_N$ of $M_-$. We denote $C = C_{e_1} = \left(C_{ij}\right)$ and $\bar{C} = C_{e_{\bar{1}}} = \left(\bar{C}_{ij}\right)$, with respect to the decomposition $TM_- = D_2 \oplus \cdots \oplus D_6$. Then by (158), $C_{26}$ and $\bar{C}_{26}$ are given by

$$C_{26} = c'\begin{pmatrix} \Lambda^{6}_{21} & \Lambda^{\bar{6}}_{21} \\ \Lambda^{6}_{\bar{2}1} & \Lambda^{\bar{6}}_{\bar{2}1} \end{pmatrix} = 0, \quad \bar{C}_{26} = c'\begin{pmatrix} \Lambda^{6}_{2\bar{1}} & \Lambda^{\bar{6}}_{2\bar{1}} \\ \Lambda^{6}_{\bar{2}\bar{1}} & \Lambda^{\bar{6}}_{\bar{2}\bar{1}} \end{pmatrix} \neq 0.$$

However, this contradicts that $\operatorname{rank}(\cos sC_{26} + \sin s\bar{C}_{26})$ is independent of $s$, which follows from Lemma 15.2 applied to $M_-$. Thus we obtain Proposition 15.1. $\square$

*Proof of Theorem 1.1*. In Case (i), with respect to a suitable basis, we have

$$B_\zeta = \begin{pmatrix} 0 & 0 & 0 & 0 & \sqrt{3}J \\ 0 & 0 & 0 & \frac{1}{\sqrt{3}}J & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{3}}J & 0 & 0 & 0 \\ -\sqrt{3}J & 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_{\bar\zeta} = \begin{pmatrix} 0 & 0 & 0 & 0 & \sqrt{3}I \\ 0 & 0 & 0 & \frac{1}{\sqrt{3}}I & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}}I & 0 & 0 & 0 \\ \sqrt{3}I & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{159}$$

using rank $B_{ij}$ =rank $\bar{B}_{ij}$, applying the Gauss equation, and using that $\cos sB_\zeta + \sin sB_{\bar\zeta}$ is isospectral. Next we show

$$C = \begin{pmatrix} 0 & J & 0 & 0 & 0 \\ -J & 0 & 0 & -\frac{2}{\sqrt{3}}J & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{\sqrt{3}}J & 0 & 0 & J \\ 0 & 0 & 0 & -J & 0 \end{pmatrix}, \bar{C} = \begin{pmatrix} 0 & -I & 0 & 0 & 0 \\ -I & 0 & 0 & \frac{2}{\sqrt{3}}I & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{\sqrt{3}}I & 0 & 0 & -I \\ 0 & 0 & 0 & -I & 0 \end{pmatrix}. \tag{160}$$

Because of $C_{26} = \bar{C}_{26} = 0$, $C$ or $\bar{C}$ is of type (ii) in Lemma 15.2, where $B_{ij}$ corresponds to $C_{i+1\,j+1}$. Moreover, since

$$C_{56} = \frac{1}{\sin\theta_1(\lambda_5 - \lambda_1)} \begin{pmatrix} 0 & \Lambda_{51}^{\bar 6} \\ \Lambda_{51}^{6} & 0 \end{pmatrix}, \bar{C}_{56} = \frac{1}{\sin\theta_1(\lambda_5 - \lambda_1)} \begin{pmatrix} \Lambda_{5\bar 1}^{6} & 0 \\ 0 & \Lambda_{5\bar 1}^{\bar 6} \end{pmatrix},$$

$C_{56} = J$ follows. Then, it is not difficult to show (160), by using $R_{\underline{i}6\bar{6}\underline{j}}$ as well as the global correspondence, with respect to our frame $e_{\underline{i}}(p)$.

Next, to show that $M$ is homogeneous, consider those $\Lambda_{\underline{ij}}^{\underline{k}}$ that do not appear above. Though they are those without indices $\underline{1}$ and $\underline{6}$, we can determine these by the global correspondence in Section 4. Namely, $\Lambda_{\underline{23}}^{\underline{5}} = 0$ and $\Lambda_{\underline{24}}^{\underline{5}} = 0$ follow from (155), $\Lambda_{\underline{23}}^{\underline{4}} = 0$ follows from $\Lambda_{\underline{65}}^{\underline{4}} = 0$, and $\Lambda_{\underline{34}}^{\underline{5}}$ is determined by $\Lambda_{\underline{16}}^{\underline{5}}$. In this way, all the structure coefficients are determined from the coefficients of the shape operators of the focal submanifolds $M_\pm$ and turn out to be locally constant.

In Case (ii), we can exchange $M_+$ with $M_-$ and apply the same argument to determine all $\Lambda_{\underline{ij}}^{\underline{k}}$. Thus in both cases, we have a local frame with respect to which all the structure coefficients are constant.

Now recall Singer's strongly curvature-homogeneous theorem. By definition ([KN69, p. 357]), a Riemannian manifold $X$ is strongly curvature-homogeneous if, for any two points $x, y \in X$, there is a linear isomorphism of $T_xX$ onto $T_yX$ that maps $g_x$ (the metric at $x$) and $(\nabla^k R)_x$ (higher covariant derivatives of the curvature tensor $R$), $k = 0, 1, 2, \ldots$ upon $g_y$ and $(\nabla^k R)_y$, $k = 0, 1, 2, \ldots$ .

Theorem 15.3 ([Sin60], [Nom62], [KN69, Th. 2, p. 357]). *If a connected Riemannian manifold $X$ is strongly curvature-homogeneous, then it is locally homogeneous. Moreover, if $M$ is complete and simply connected, it is homogeneous.*

In our case, the local frame $e_{\underline{i}}$ defines an isometry between $T_pM$ and $T_qM$, and since $\Lambda_{\alpha\beta}^{\gamma}$ are locally constant, components of $(\nabla^k R)_x$ are given by polynomials in $\Lambda_{\alpha\beta}^{\gamma}$ (see (5)), and so are all locally constant. Moreover, since $M$ is complete and simply connected, where the latter holds since $M$ is an iterated $S^2$ bundle over $S^2$, applying Theorem 15.3, we know that $M$ is intrinsically homogeneous. Finally by using the rigidity theorem of hypersurfaces with type number larger than two [KN69, p. 45], we conclude that $M$ is extrinsically homogeneous. □

In [Miy11], we calculate all the structure constants of the $G_2$ orbits, which coincide with those calculated above, and corroborate the proof.

## References


[Abr83] U. Abresch, Isoparametric hypersurfaces with four or six distinct principal curvatures. Necessary conditions on the multiplicities, *Math. Ann.* **264** (1983), 283–302. MR 0714104. Zbl 0505.53027. http://dx.doi.org/10.1007/BF01459125.

[Car38] É. Cartan, Familles de surfaces isoparamétriques dans les espaces à courbure constante, *Ann. Mat. Pura Appl.* **17** (1938), 177–191. MR 1553310. Zbl 0020.06505. http://dx.doi.org/10.1007/BF02410700.

[Car39a] É. Cartan, Sur des familles remarquables d'hypersurfaces isoparamétriques dans les espaces sphériques, *Math. Z.* **45** (1939), 335–367. MR 0000169. Zbl 0021.15603. http://dx.doi.org/10.1007/BF01580289.

[Car39b] É. Cartan, Sur quelques familles remarquables d'hypersurfaces, *C. R. Congrès Math. Liège* (1939), 30–41. Zbl 0025.36601.

[Car40] É. Cartan, Sur des familles d'hypersurfaces isoparamétriques des espaces sphériques à 5 et à 9 dimensions, *Univ. Nac. Tucumán. Revista A.* **1** (1940), 5–22. MR 0004519. Zbl 0025.22603.

[CCJ07] T. E. Cecil, Q.-S. Chi, and G. R. Jensen, Isoparametric hypersurfaces with four principal curvatures, *Ann. of Math.* **166** (2007), 1–76. MR 2342690. Zbl 1143.53058. http://dx.doi.org/10.4007/annals.2007.166.1.

[Chi09] Q.-S. Chi, Isoparametric hypersurfaces with four principal curvatures revisited, *Nagoya Math. J.* **193** (2009), 129–154. MR 2502911. Zbl 1165.53032. Available at http://projecteuclid.org/euclid.nmj/1236089984.

[Chi11a] Q.-S. Chi, Isoparametric hypersurfaces with four principal curvatures, II, *Nagoya Math. J.* **204** (2011), 1–18. MR 2863363. Zbl 1243.53094. Available at http://projecteuclid.org/euclid.nmj/1323107835.

[Chi11b] Q.-S. Chi, A note on the paper "Isoparametric hypersurfaces with four principal curvatures" [MR2342690], *Pac. J. Appl. Math.* **3** (2011), 127–134. MR 2918561.

[Chi11c] Q.-S. Chi, Isoparametric hypersurfaces with four principal curvatures, III, 2011, preprint. arXiv 1104.3249.

[Chi12] Q.-S. Chi, A new look at condition A, *Osaka J. of Math.* (2012), 133–166. MR 2903258. Zbl 06029031. Available at http://projecteuclid.org/euclid.ojm/1332337242.

[DN83] J. Dorfmeister and E. Neher, An algebraic approach to isoparametric hypersurfaces in spheres. I, *Tôhoku Math. J.* **35** (1983), 187–224. MR 0699927. Zbl 0507.53038. http://dx.doi.org/10.2748/tmj/1178229051.

[DN85] J. Dorfmeister and E. Neher, Isoparametric hypersurfaces, case $g = 6$, $m = 1$, *Comm. Algebra* **13** (1985), 2299–2368. MR 0807479. Zbl 0578.53041. http://dx.doi.org/10.1080/00927878508823278.

[FKM81] D. Ferus, H. Karcher, and H. F. Münzner, Cliffordalgebren und neue isoparametrische Hyperflächen, *Math. Z.* **177** (1981), 479–502. MR 0624227. Zbl 0443.53037. http://dx.doi.org/10.1007/BF01219082.

[HL71] W.-y. Hsiang and H. B. Lawson, Jr., Minimal submanifolds of low cohomogeneity, *J. Differential Geometry* **5** (1971), 1–38. MR 0298593. Zbl 0219.53045. Available at http://projecteuclid.org/euclid.jdg/1214429775.

[Imm08] S. Immervoll, On the classification of isoparametric hypersurfaces with four distinct principal curvatures in spheres, *Ann. of Math.* **168** (2008), 1011–1024. MR 2456889. Zbl 1176.53057. http://dx.doi.org/10.4007/annals.2008.168.1011.

[KN69] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry. Vol. II*, *Interscience Tracts in Pure and Applied Math.* **15**, Interscience Publishers John Wiley & Sons, Inc., New York, 1969. MR 0238225. Zbl 0175.48504.

[Miy89] R. Miyaoka, Dupin hypersurfaces with six principal curvatures, *Kodai Math. J.* **12** (1989), 308–315. MR 1023536. Zbl 0711.53049. http://dx.doi.org/10.2996/kmj/1138039098.

[Miy93] R. Miyaoka, The linear isotropy group of $G_2/\mathrm{SO}(4)$, the Hopf fibering and isoparametric hypersurfaces, *Osaka J. Math.* **30** (1993), 179–202. MR 1233508. Zbl 0815.53070. Available at http://projecteuclid.org/euclid.ojm/1200784357.

[Miy08] R. Miyaoka, Isoparametric geometry and related fields, in *Surveys on Geometry and Integrable Systems*, *Adv. Stud. Pure Math.* **51**, Math. Soc. Japan, Tokyo, 2008, pp. 315–337. MR 2509798. Zbl 1161.53047.

[Miy09] R. Miyaoka, The Dorfmeister-Neher theorem on isoparametric hypersurfaces, *Osaka J. Math.* **46** (2009), 695–715. MR 2583325. Zbl 1185.53059. Available at http://projecteuclid.org/euclid.ojm/1256564202.

[Miy11] R. Miyaoka, Geometry of $G_2$ orbits and isoparametric hypersurfaces, *Nagoya Math. J.* **203** (2011), 175–189. MR 2834253. Zbl 1231.53051. Available at http://projecteuclid.org/euclid.nmj/1313682316.

[Mün80] H. F. Münzner, Isoparametrische Hyperflächen in Sphären. I, *Math. Ann.* **251** (1980), 57–71. MR 0583825. Zbl 0417.53030. http://dx.doi.org/10.1007/BF01420281.

[Mün81] H. F. Münzner, Isoparametrische Hyperflächen in Sphären. II. Über die Zerlegung der Sphäre in Ballbündel, *Math. Ann.* **256** (1981), 215–232. MR 0620709. Zbl 0438.53050. http://dx.doi.org/10.1007/BF01450799.

[Nom62] K. Nomizu, Sur les algèbres de Lie de générateurs de Killing et l'homogénéité d'une variété riemannienne, *Osaka Math. J.* **14** (1962), 45–51. MR 0141054. Zbl 0103.38603. Available at http://projecteuclid.org/euclid.ojm/1200690496.

[Nom75] K. Nomizu, Élie Cartan's work on isoparametric families of hypersurfaces, in *Differential Geometry* (Stanford Univ., Stanford, CA 1973), *Proc. Symp. Pure Math.* **27**, Amer. Math. Soc., Providence, R.I., 1975, pp. 191–200. MR 0423260. Zbl 0318.53052.

[Olm93] C. Olmos, Isoparametric submanifolds and their homogeneous structures, *J. Differential Geom.* **38** (1993), 225–234. MR 1237484. Zbl 0791.53051. Available at http://projecteuclid.org/euclid.jdg/1214454294.

[OT76] H. Ozeki and M. Takeuchi, On some types of isoparametric hypersurfaces in spheres. II, *Tôhoku Math. J.* **28** (1976), 7–55. MR 0454889. Zbl 0359.53012. http://dx.doi.org/10.2748/tmj/1178240877.

[Rec76] H. Reckziegel, Krümmungsflächen von isometrischen Immersionen in Räume konstanter Krümmung, *Math. Ann.* **223** (1976), 169–181. MR 0425846. Zbl 0319.53042.

[Sin60] I. M. Singer, Infinitesimally homogeneous spaces, *Comm. Pure Appl. Math.* **13** (1960), 685–697. MR 0131248. Zbl 0171.42503. http://dx.doi.org/10.1002/cpa.3160130408.

[Ter85] C.-L. Terng, Isoparametric submanifolds and their Coxeter groups, *J. Differential Geom.* **21** (1985), 79–107. MR 0806704. Zbl 0615.53047. Available at http://projecteuclid.org/euclid.jdg/1214439466.

[Tho91] G. Thorbergsson, Isoparametric foliations and their buildings, *Ann. of Math.* **133** (1991), 429–446. MR 1097244. Zbl 0727.57028. http://dx.doi.org/10.2307/2944343.

[Tho00] G. Thorbergsson, A survey on isoparametric hypersurfaces and their generalizations, in *Handbook of Differential Geometry, Vol. I*, North-Holland, Amsterdam, 2000, pp. 963–995. MR 1736861. Zbl 0979.53002. http://dx.doi.org/10.1016/S1874-5741(00)80013-8.

[Yau92] S.-T. Yau, Open problems in geometry, in *Chern—A Great Geometer of the Twentieth Century*, Int. Press, Hong Kong, 1992, pp. 275–319. MR 1201369. Zbl 0826.01010. http://dx.doi.org/10.1007/BF01360880.

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