

The second fundamental theorem of invariant theory for the orthogonal group

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Abstract

Let $V = \mathbb{C}^n$ be endowed with an orthogonal form and $G = O(V)$ be the corresponding orthogonal group. Brauer showed in 1937 that there is a surjective homomorphism $\nu : B_r(n) \rightarrow \text{End}_G(V^{\otimes r})$, where $B_r(n)$ is the r -string Brauer algebra with parameter n . However the kernel of ν has remained elusive. In this paper we show that, in analogy with the case of $\text{GL}(V)$, for $r \geq n + 1$, ν has a kernel which is generated by a single idempotent element E , and we give a simple explicit formula for E . Using the theory of cellular algebras, we show how E may be used to determine the multiplicities of the irreducible representations of $O(V)$ in $V^{\otimes r}$. We also show how our results extend to the case where \mathbb{C} is replaced by an appropriate field of positive characteristic, and we comment on quantum analogues of our results.

1. Introduction

Let K be a field of characteristic zero, let $V = K^n$ be an n -dimensional vector space with a nondegenerate symmetric bilinear form $(-, -)$, and assume that with respect to some basis of V , the form has matrix equal to the identity matrix. Equivalently, there is a basis $\{b_1, \dots, b_n\}$ such that $(b_i, b_j) = \delta_{ij}$; such a basis is called orthonormal. The *orthogonal group* $O(V)$ is the isometry group of this form, defined as $O(V) = \{g \in \text{GL}(V) \mid (gv, gw) = (v, w) \forall v, w \in V\}$. In [3], Brauer showed that the first fundamental theorem of invariant theory for $O(V)$ implies that there is a surjective map ν from the Brauer algebra $B_r(n)$ over K to $\text{End}_{O(V)}(V^{\otimes r})$, but the fact that $B_r(n)$ is semisimple if and only if $r \leq n + 1$ [5], [20] has complicated the determination of the kernel of ν and, therefore, limited the use of this fact.

In this work we determine $\ker(\nu)$. More specifically, we show that $\ker(\nu)$ is generated as an ideal of $B_r(n)$ by a single idempotent E , which we describe explicitly. Using the fact that $B_r(n)$ has a cellular structure [6], we show how this fact may be used to illuminate the Schur-Weyl duality between the actions of $O(V)$ and $B_r(n)$ on $V^{\otimes r}$, by using E to describe the radicals of the canonical forms on the relevant cell modules of $B_r(n)$.

The special case $n = 3$ has been treated in [12], [14], as has its quantum analogue for the Birman-Murukami-Wenzl (BMW) algebra [2]. This latter work was done in the context of the 3-dimensional irreducible representation of \mathfrak{sl}_2 . The case of the symplectic group $\mathrm{Sp}_{2n}(K)$ and its quantum analogue, which seems rather different from the present case, has been treated by Hu and Xiang in [9].

The present work could be considered analogous to that of Jones [10] concerning the Temperley-Lieb algebra, in a nonsemisimple context.

2. The Brauer algebra

2.1. *Generalities.* Let K be a field of characteristic zero and let $\delta \in K$. For any positive integer r , the Brauer algebra $B_r(\delta)$ [3] is the K -algebra with basis the set of diagrams with $2r$ nodes, or vertices, labelled as in Figure 1, in which each node is joined to just one other one.

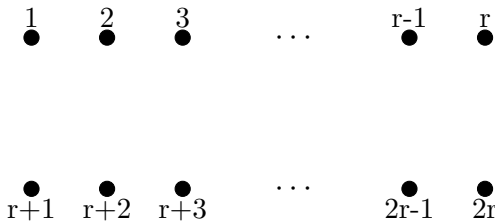


Figure 1.

Note that each diagram in $B_r(\delta)$ may be thought of as a graph with vertices $\{1, \dots, 2r\}$ in which each vertex is joined to just one other one. We will often refer to the “edges” of such a diagram, and speak of “horizontal edges” and “vertical edges” (the latter also known as “through strings”) as respectively those joining vertices in the same row, or in different rows.

The composite $D_1 \circ D_2$ of two diagrams D_1 and D_2 is obtained by concatenation of diagrams, placing D_1 above D_2 , with the intermediate nodes and any free loops being erased. The product $D_1 D_2$ in $B_r(\delta)$ is $\delta^{l(D_1, D_2)} D_1 \circ D_2$, where $l(D_1, D_2)$ is the number of deleted free loops.

We shall need to consider certain special elements of $B_r(\delta)$, which we now describe.

For $i = 1, \dots, r - 1$, s_i is the diagram shown in Figure 2. For each pair i, j with $1 \leq i < j \leq r$, define the diagram $e_{i,j}$ as depicted in Figure 3.

The following facts are all well known.

LEMMA 2.1. (i) *The elements s_1, \dots, s_{r-1} generate a subalgebra of $B_r(\delta)$, isomorphic to the group algebra $K\mathrm{Sym}_r$ of the symmetric group Sym_r .*

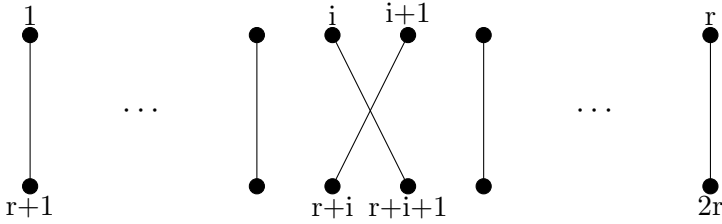


Figure 2.

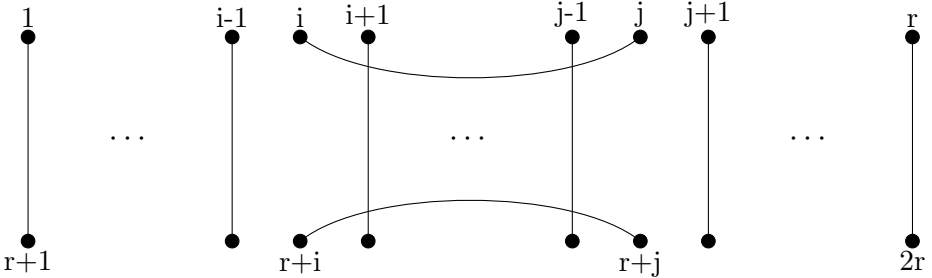


Figure 3.

- (ii) The elements $e_{i,j}$ satisfy $e_{i,j}^2 = \delta e_{i,j}$, and if i, j, k and ℓ are distinct, $e_{i,j}$ commutes with $e_{k,\ell}$.
- (iii) If we write $e_i = e_{i,i+1}$ for $i = 1, \dots, r-1$, then $B_r(\delta)$ has a presentation as K -algebra with generators $\{s_1, \dots, s_{r-1}; e_1, \dots, e_{r-1}\}$ and relations $s_i^2 = 1$, $e_i^2 = \delta e_i$, $s_i e_i = e_i s_i = e_i$ for all i , $s_i s_j = s_j s_i$, $s_i e_j = e_j s_i$, $e_i e_j = e_j e_i$ if $|i - j| \geq 2$, and $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$, $e_i e_{i\pm 1} e_i = e_i$ and $s_i e_{i+1} e_i = s_{i+1} e_i$, $e_{i+1} e_i s_{i+1} = e_{i+1} s_i$ and $e_i s_{i\pm 1} e_i = e_i$ for all applicable i .

2.2. *Some special notation.* For positive integers k, l with $k \leq l$, define $[k, l] := \{k, k + 1, k + 2, \dots, l\}$. For any subset $S \subseteq [1, r]$, $\text{Sym}(S)$ is the subgroup of Sym_r that fixes each element of $[1, r] \setminus S$. For any subgroup $H \leq \text{Sym}_r$, define $a(H) = \sum_{h \in H} \varepsilon(h) h \in B_r(\delta)$, where ε is the alternating character of Sym_r . This is referred to as the “alternating element” of KH . The following elementary observation is well known but very useful.

LEMMA 2.2. *Suppose the subgroup $H \leq \text{Sym}_r$ contains the simple transposition s_{ij} that interchanges i and j . If $e_{i,j}$ is the element defined above, then $e_{i,j} a(H) = a(H) e_{i,j} = 0$.*

Proof. Since $s_{ij} a(H) = -a(H) = a(H) s_{ij}$, we have $a(H) = \frac{1}{2}(1 - s_{ij})a(H) = \frac{1}{2}a(H)(1 - s_{ij})$. But $e_{i,j} s_{ij} = s_{ij} e_{i,j} = e_{i,j}$, and the result is clear. \square

3. The fundamental theorems of invariant theory for $O(n)$

3.1. *First (linear) formulation.* For any positive integer t , the space $V^{\otimes t}$ (also denoted $T^t(V)$) is an $O(V)$ -module in the usual way: $g(v_1 \otimes \cdots \otimes v_t) = gv_1 \otimes gv_2 \otimes \cdots \otimes gv_t$. Moreover the given form on V provides a nondegenerate symmetric bilinear form $[-, -]$ on $V^{\otimes t}$, given by $[v_1 \otimes \cdots \otimes v_t, w_1 \otimes \cdots \otimes w_t] := \prod_{i=1}^t (v_i, w_i)$, which permits the identification of $V^{\otimes t}$ with its dual space $V^{\otimes t*}$.

The space of invariants $(V^{\otimes t*})^{O(V)}$ is defined as the space of linear functions that are constant on $O(V)$ orbits. One formulation of the first fundamental theorem of invariant theory for $O(V)$ [21], [1] is as follows (see [15], [13], [17], [18], [19, Prop. 21]).

THEOREM 3.1. *The space $(V^{\otimes t*})^{O(V)}$ is zero if t is odd. If $t = 2r$ is even, then any element of $(V^{\otimes t*})^{O(V)}$ is a linear combination of maps of the form*

$$\gamma_D : v_1 \otimes \cdots \otimes v_{2r} \mapsto \prod_{(i,j) \text{ is an edge of } D} (v_i, v_j),$$

where D is a diagram in $B_r(n)$.

The second fundamental theorem provides a description of all linear relations among these functions γ_D . Let us begin by describing some obvious linear relations. Suppose $r \geq n + 1$. (Recall that $\dim V = n$.)

Let S and S' be disjoint subsets of $[1, 2r]$ such that $|S| = |S'| = n + 1$ and $S \cap S' = \emptyset$, and let β be any pairing of the vertices $\{1, \dots, 2r\} \setminus (S \amalg S')$.

Definition 3.2. For $\pi \in \text{Sym}_{n+1}$, $S = \{i_1, \dots, i_{n+1}\}$, $S' = \{j_1, \dots, j_{n+1}\}$, let $D_\pi(S, S', \beta)$ be the Brauer diagram with edges $\{(i_k, j_{\pi(k)}) \mid k = 1, 2, \dots, n + 1\} \amalg \beta$, and denote by $\gamma_{D_\pi(S, S', \beta)}$ the corresponding linear functional on $V^{\otimes 2r}$ as above.

Define $\gamma(S, S', \beta) := \sum_{\pi \in \text{Sym}_{n+1}} \varepsilon(\pi) \gamma_{D_\pi(S, S', \beta)}$.

The next statement describes some obvious linear relations among the γ_D .

LEMMA 3.3. *We have, for each S, S', β as above, $\gamma(S, S', \beta) = 0 (\in V^{\otimes 2r*})$.*

Proof. If $S = i_1 < i_2 < \cdots < i_{n+1}$ and $S' = j_1 < j_2 < \cdots < j_{n+1}$, then for any $v_1 \otimes \cdots \otimes v_{2r} \in V^{\otimes 2r}$, clearly the $(n + 1) \times (n + 1)$ matrix with k, l entry (v_{i_k}, v_{j_l}) is singular, since the rows are linearly dependent, as by dimension, there is a linear relation among the v_{i_k} . The lemma follows by observing that $\gamma(S, S', \beta)$ is a multiple of the function $v_1 \otimes \cdots \otimes v_{2r} \mapsto \det(v_{i_k}, v_{j_l})$, which is zero. □

The second fundamental theorem for $O(V)$ may be stated as follows [19, Prop. 21].

THEOREM 3.4. *If $r \leq n$, the γ_D form a basis of the space of $O(V)$ -invariants on $(V^{\otimes 2r})^*$. If $r \geq n + 1$, then any linear relation among the functionals γ_D is a linear consequence of the relations in Lemma 3.3.*

3.2. Second formulation. Our objective is to reinterpret the first and second fundamental theorems in terms of the Brauer algebra $B_r(n)$, which is the algebra described above, with δ replaced by $n = \dim V$. For this purpose we consider some maps, which we now define. There is a canonical map $\xi : V \otimes V \rightarrow \text{End}(V)$ given by $\xi(v \otimes w) : x \mapsto (w, x)v$ (where $v, w, x \in V$). Define $A : V^{\otimes 2r} \rightarrow \text{End}(V^{\otimes r})(\simeq (\text{End}(V))^{\otimes r})$ by

$$A(v_1 \otimes \cdots \otimes v_{2r}) = \xi(v_1 \otimes v_{r+1}) \otimes \xi(v_2 \otimes v_{r+2}) \otimes \cdots \otimes \xi(v_r \otimes v_{2r}).$$

This map respects the action of $O(V)$ on its domain and range, where $O(V)$ acts on $\text{End}(V)^{\otimes r}$ by conjugation: for $g \in O(V)$ and $\alpha_1 \otimes \cdots \otimes \alpha_r \in \text{End}(V)^{\otimes r}$, $g \cdot (\alpha_1 \otimes \cdots \otimes \alpha_r) := g\alpha_1g^{-1} \otimes \cdots \otimes g\alpha_rg^{-1}$.

Next observe that if b_1, \dots, b_n is an orthonormal basis of V , the element $\gamma_0 := \sum_{i=1}^n b_i \otimes b_i$ is $O(V)$ -invariant and independent of the basis. Hence the linear map $\phi : V \otimes V \rightarrow V \otimes V$ defined by $\phi(v \otimes w) = (v, w)\gamma_0$ commutes with $O(V)$, i.e., $\phi \in \text{End}_{O(V)}(V^{\otimes 2})$. This map ϕ is called the contraction map. For $i = 1, 2, \dots, r - 1$, define $\phi_i \in \text{End}(V^{\otimes r})$ as the endomorphism that is the contraction ϕ on the tensor product of the i^{th} and $(i + 1)^{\text{st}}$ factors and is the identity on all other factors. It was proved by Brauer [3] that if $\text{Sym}_r \subset B_r(n)$ acts via place permutations on $V^{\otimes r}$, i.e., if we define $\nu(\sigma) \cdot (v_1 \otimes \cdots \otimes v_r) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(r)}$ for $\sigma \in \text{Sym}_r$, then ν extends to a homomorphism of associative algebras

$$\nu : B_r(n) \rightarrow \text{End}_{O(V)}(V^{\otimes r}) \subseteq \text{End}(V^{\otimes r})$$

by defining $\nu(e_i) = \phi_i$. Note that if $\sigma \in \text{Sym}_r$, then the diagram representing σ in $B_r(n)$ has edges $(\sigma(i), r + i)$ for $i = 1, 2, \dots, r$.

Finally, we define a linear map $\tau : B_r(n) \rightarrow V^{\otimes 2r}$ as follows. For any diagram $D \in B_r(n)$, define $\tau(D) = t_D$, where

$$t_D = \sum_{\substack{i_1, \dots, i_{2r}=1 \\ i_j=i_k \text{ if } (j, k) \text{ is an edge of } D}}^n b_{i_1} \otimes \cdots \otimes b_{i_{2r}},$$

where b_1, \dots, b_n is an orthonormal basis of V . Notice that t_D is the unique element of $V^{\otimes 2r}$ such that in the notation above,

$$[-, t_D] = \gamma_D.$$

Now consider the following diagram of linear maps:

$$(3.5) \quad \begin{array}{ccc} & B_r(n) & \\ \swarrow \tau & & \searrow \nu \\ V^{\otimes 2r} & \xrightarrow{A} & \text{End}(V^{\otimes r}). \end{array}$$

The next statement is crucial for understanding the second fundamental theorem in the context of the Brauer algebra.

PROPOSITION 3.6. *The diagram (3.5) commutes.*

Proof. We start by observing that the group $\text{Sym}_r \times \text{Sym}_r$ acts on each of the three spaces in the diagram as follows. Let $(\sigma_1, \sigma_2) \in \text{Sym}_r \times \text{Sym}_r$. Then for a diagram $D \in B_r(n)$, we have $(\sigma_1, \sigma_2) \cdot D := \sigma_1 D \sigma_2^{-1}$.

For $v_1 \otimes \cdots \otimes v_r \otimes w_1 \otimes \cdots \otimes w_r \in V^{\otimes 2r}$, we have

$$(\sigma_1, \sigma_2) \cdot (v_1 \otimes \cdots \otimes v_r \otimes w_1 \otimes \cdots \otimes w_r) := \nu(\sigma_1)(v_1 \otimes \cdots \otimes v_r) \otimes \nu(\sigma_2)(w_1 \otimes \cdots \otimes w_r)$$

and for $T \in \text{End}(V^{\otimes r})$, we have $(\sigma_1, \sigma_2) \cdot T := \nu(\sigma_1) T \nu(\sigma_2^{-1})$.

Next, a straightforward computation shows that each of the maps ν, τ and A respects the action of $\text{Sym}_r \times \text{Sym}_r$.

Now each diagram $D \in B_r(n)$ may be written in the form $D = \sigma_1 l(s) \sigma_2^{-1}$ for some $(\sigma_1, \sigma_2) \in \text{Sym}_r \times \text{Sym}_r$ and some diagram $l(s)$, where $s \in \{1, 2, \dots, \lfloor \frac{r+1}{2} \rfloor\}$, and the diagram $l(s)$ is as shown in Figure 4.

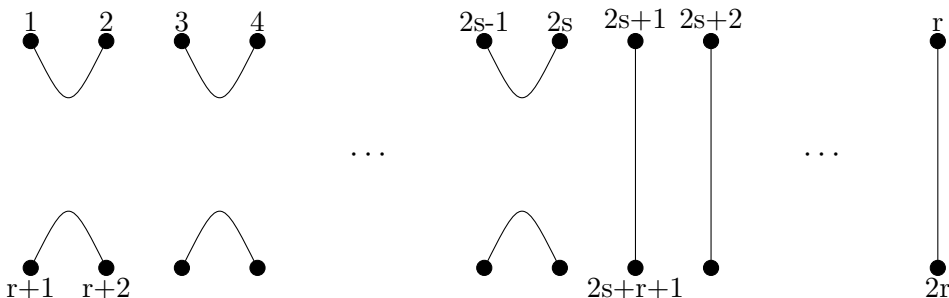


Figure 4.

Hence if we were able to prove that for each s we have

$$(3.7) \quad A\tau(l(s)) = \nu(l(s)),$$

we would have

$$A\tau(\sigma_1 l(s) \sigma_2^{-1}) = (\sigma_1, \sigma_2) \cdot A\tau(l(s)) = (\sigma_1, \sigma_2) \cdot \nu(l(s)) = \nu(\sigma_1 l(s) \sigma_2^{-1})$$

for $(\sigma_1, \sigma_2) \in \text{Sym}_r \times \text{Sym}_r$ and the Proposition would follow. Hence it remains to prove (3.7), and this may be checked directly, given the identities

$$\sum_{i=1}^n \xi(b_i \otimes b_i) = \text{Id}_V$$

and

$$\sum_{i,j=1}^n \xi(b_i \otimes b_j) \otimes \xi(b_i \otimes b_j) = \phi : V \otimes V \rightarrow V \otimes V,$$

where ϕ is the contraction introduced above. □

COROLLARY 3.8. *The map ν maps $B_r(n)$ surjectively to $\text{End}_{O(V)}(V^{\otimes r})$, and $\ker(\nu) = \ker(\tau)$.*

Proof. Since A is an $O(V)$ -equivariant isomorphism, it restricts to an isomorphism from the $O(V)$ -invariants of $V^{\otimes 2r}$ to those of $\text{End}(V^{\otimes r})$. It follows from [Proposition 3.6](#) that since τ has image $(V^{\otimes 2r})^{O(V)}$, $\text{im}(\nu) = \text{End}(V^{\otimes r})^{O(V)} = \text{End}_{O(V)}(V^{\otimes r})$. The fact that $\ker(\nu) = \ker(\tau)$ is clear from the commutativity of the diagram. \square

3.3. Third formulation: determinantal ideals. We regard V as canonically identified with its dual V^* through the nondegenerate form $(-, -)$. Then the coordinate ring $\mathbb{C}[V]$, defined as the symmetric algebra $S(V^*)$, is canonically identified with $S(V)$ and has the usual grading $S(V) = \bigoplus_{i=0}^{\infty} S_i(V)$. Hence the coordinate ring $\mathbb{C}[V^{\oplus 2r}] \xrightarrow{\cong} S(V)^{\otimes 2r}$ is \mathbb{Z}^{2r} -graded, with graded component $S_{m_1}(V) \otimes \cdots \otimes S_{m_{2r}}(V)$ of degree (m_1, \dots, m_{2r}) .

Now for any positive integer r , $O(V)$ acts on $\mathbb{C}[V^{\oplus 2r}]$ in the usual way, and the third formulation of the fundamental theorems, which is in terms of the commutative algebra $\mathbb{C}[V^{\oplus 2r}]$, is as follows. A general reference for this is [\[18\]](#).

THEOREM 3.9. (i) *(First fundamental theorem) The subalgebra of $O(V)$ -invariants in $\mathbb{C}[V^{\oplus 2r}] \simeq S(V)^{\otimes 2r}$ is generated by the elements $\{t_{i,j} \mid 1 \leq i \leq j \leq 2r\}$, where, if b_1, \dots, b_n is an orthonormal basis of V , then $t_{i,j}$ is defined by*

$$t_{i,j} = \begin{cases} \sum_{\alpha=1}^n 1 \otimes \cdots \otimes b_{\alpha} \otimes \cdots \otimes b_{\alpha} \otimes 1 \otimes \cdots \otimes 1 & \text{if } i \neq j, \\ \sum_{\alpha=1}^n 1 \otimes \cdots \otimes 1 \otimes b_{\alpha}^2 \otimes \cdots \otimes 1 & \text{if } i = j. \end{cases}$$

Here the nonunit factors in the tensors are in positions i and j , and we define $t_{i,j} = t_{j,i}$.

(ii) *(Second fundamental theorem) The kernel of the surjection*

$$\mathbb{C}[x_{1,1}, x_{1,2}, \dots, x_{2r,2r}] \xrightarrow{\eta} (S(V)^{\otimes 2r})^{O(V)},$$

where the $x_{i,j}$ are indeterminates satisfying $x_{i,j} = x_{j,i}$ and $\eta(x_{i,j}) = t_{i,j}$, is the determinantal ideal generated by the $(n + 1) \times (n + 1)$ minors of the “generic” matrix $(x_{i,j})_{1 \leq i,j \leq 2r}$.

This may be summarised by the following diagram. Denote by \mathcal{D} the determinantal ideal referred to above; then we have a short exact sequence of commutative algebras

$$0 \longrightarrow \mathcal{D} \longrightarrow \mathbb{C}[x_{i,j}] \xrightarrow{\eta} (S(V)^{\otimes 2r})^{O(V)} \longrightarrow 0.$$

The connection between this classical view and our investigation may best be described as follows. Define a *multiplier monomial* as a monomial

$X = \prod_{k=1}^r x_{i_k, j_k}$, where $\{i_1, i_2, \dots, i_r, j_1, \dots, j_r\} = \{1, 2, \dots, 2r\}$. Clearly the multiplier monomials are precisely those monomials whose image lies in the graded component of $S(V)^{\otimes 2r}$ of degree $(1, 1, \dots, 1)$, which is the subspace $V^{\otimes 2r}$ of $S(V)^{\otimes 2r}$. Let M_r be the subspace of $\mathbb{C}[x_{i,j}]$ spanned by the multiplier monomials. Then we have the following commutative diagram of linear maps, in which the rows are exact and the vertical arrows are (linear) isomorphisms:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathcal{D} \cap M_r & \xrightarrow{\iota} & M_r & \xrightarrow{\eta} & (V^{\otimes 2r})^{O(V)} \longrightarrow 0 \\
 (3.10) & & \downarrow \simeq & & \mu \downarrow \simeq & & A \downarrow \simeq \\
 0 & \longrightarrow & \ker(\nu) & \xrightarrow{\text{incl}} & B_r(n) & \xrightarrow{\nu} & \text{End}_{O(V)}(V^{\otimes r}) \longrightarrow 0.
 \end{array}$$

In the diagram (3.10), μ takes the monomial $\prod_{k=1}^r x_{i_k, j_k}$ to the diagram in which the vertices i_k, j_k are joined. The maps A and ν are precisely those in the diagram (3.5), and just as in the latter case, the last line of (3.10) is an exact sequence of noncommutative algebras; we shall show that this extra structure permits a simple description of $\ker(\nu)$ (and therefore $\mathcal{D} \cap M_r$) as the ideal generated by a single idempotent of the Brauer algebra $B_r(n)$.

In Section 4.3 we give a sketch of our proof in terms of the classical theory. We note also that the top row of the diagram (3.10) is a sequence of CSym_{2r} -modules, and so there is a description of it in terms of partitions, tableaux, and when the characteristic is varied, filtrations. In particular, the space $M_r \simeq \mathbb{C}[\text{Sym}_{2r}/(\text{Sym}_r \times (\mathbb{Z}/2\mathbb{Z})^r)]$ is multiplicity free, so that $\ker(\nu)$ may be completely described in terms of partitions. We do not go into these matters in this work, but see [4].

4. Statement of the main result

4.1. *Overview.* We shall translate Theorem 3.4, which uses only the linear structure, into an explicit description of the ideal $\ker(\nu)$ above. We start with the following easy observation.

LEMMA 4.1. *For each triple (S, S', β) as in Definition 3.2, define the element*

$$b(S, S', \beta) = \sum_{\pi \in \text{Sym}_{n+1}} \varepsilon(\pi) D_\pi(S, S', \beta) \in B_r(n).$$

Then the elements $b(S, S', \beta)$ span $\ker(\nu)$.

Proof. Writing $t(S, S', \beta) = \tau(b(S, S', \beta))$, it is clear that the functional

$$x \mapsto [x, t(S, S', \beta)] \quad (x \in V^{\otimes r})$$

on $V^{\otimes 2r}$ is equal to $\gamma(S, S', \beta)$, which is zero by Lemma 3.3. Hence $b(S, S', \beta) \in \ker(\tau) = \ker(\nu)$. It follows from Theorem 3.4 that these elements span the kernel. □

Next, we identify a small subset of the elements $b(S, S', \beta)$ of $B_r(n)$, which are such that the ideal of $B_r(n)$ which they generate contains, for each triple S, S', β as in [Definition 3.2](#), the element $b(S, S', \beta)$. By [Lemma 4.1](#) the ideal they generate is the whole kernel. We then show that this ideal is in fact generated by one of those elements.

4.2. Formulation. We begin by defining certain elements of $B_r(n)$. For this purpose, the following notation will be convenient. If k, l are integers such that $1 \leq k < l$, write $a(k, l) := a(\text{Sym}\{k, k + 1, \dots, l\})$ (see [Section 2.2](#)). By convention, if $k \geq l$, then $a(k, l) = 1$.

Definition 4.2. For $i = 0, 1, \dots, [\frac{n+1}{2}]$, define the following elements of $B_r(n)$:

- (i) $F_i := a(1, i)a(i + 1, n + 1)$, where F_0 is interpreted as $a(1, n + 1)$.
- (ii) For $j = 0, 1, 2, \dots, i$, define $e_i(j) = e_{i, i+1}e_{i-1, i+2} \dots e_{i-j+1, i+j}$. This is a diagram with j top (resp. bottom) horizontal arcs. Note that $e_i(0) = 1$ by convention.
- (iii) Using the above notation, define elements E_i ($i = 0, 1, \dots, [\frac{n+1}{2}]$) as follows:

$$E_i = \sum_{j=0}^i (-1)^j c_i(j) F_i e_i(j) F_i,$$

where $c_i(j) = ((i - j)!(n + 1 - i - j)!(j!)^2)^{-1}$.

Note that the leading term ($j = 0$) of E_i is $(i!(n + 1 - i)!)^{-1} F_i^2 = F_i$.

Our main result is

THEOREM 4.3. *In the notation of [Definition 4.2](#), write $E = E_{[\frac{n+1}{2}]}$. Then $E^2 = ([\frac{n+1}{2}]!(n + 1 - [\frac{n+1}{2}]!)E$. If $r \leq n$, the map $\nu : B_r(n) \rightarrow \text{End}_{O(V)}(V^{\otimes r})$ is an isomorphism. If $r \geq n + 1$, $\ker(\nu)$ is generated as an ideal of $B_r(n)$ by E .*

We begin with the following result, whose proof will require arguments involving the geometry of Brauer diagrams.

PROPOSITION 4.4. *Assume $r \geq n + 1$. For each $i = 0, 1, \dots, [\frac{n+1}{2}]$, $E_i \in \ker(\nu)$.*

Proof. We shall show that each element E_i is of the form $b(S, S', \beta)$ for some triple S, S', β . For this, let $S_i = \{1, 2, \dots, i, i + 1 + r, i + 2 + r, \dots, n + 1 + r\}$ and $S'_i = \{i + 1, i + 2, \dots, n + 1, r + 1, r + 2, \dots, r + i\}$. Then $|S_i| = |S'_i| = n + 1$ and $S_i \cap S'_i = \emptyset$. In [Figure 5](#), the points of S_i are denoted by \circ , those of S'_i by $*$ and the others by \bullet .

With S, S' as above, $\{1, \dots, 2r\} \setminus (S_i \amalg S'_i) = \{n + 2, n + 3, \dots, r, n + 2 + r, n + 3 + r, \dots, 2r\}$, and we take β_i to be the pairing $(n + 2, n + 2 + r), \dots, (r, 2r)$. We shall show that

$$(4.5) \quad E_i = b(S_i, S'_i, \beta_i).$$

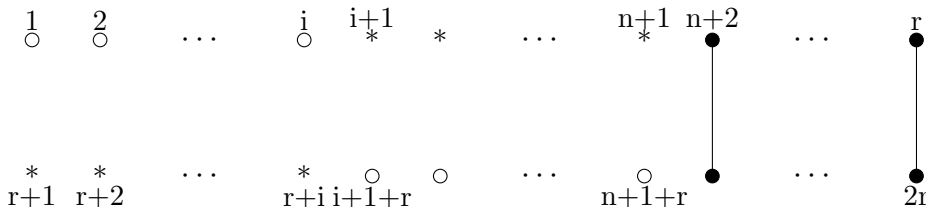


Figure 5.

Now $b(S_i, S'_i, \beta_i)$ is the alternating sum of the set \mathcal{S}_i of $(n + 1)!$ diagrams of the form depicted in Figure 5, in which each point of S_i is joined to a point of S'_i . Let $H = \text{Sym}\{1, \dots, i\} \times \text{Sym}\{i + 1, \dots, n + 1\}$, regarded as a subgroup of the algebra $B_r(n)$. Clearly $H \times H$ acts on this set \mathcal{S}_i of diagrams, via $(h, h').D = hDh'^{-1}$, preserving the number of horizontal arcs. Moreover each diagram in \mathcal{S}_i may be transformed by $H \times H$ into a unique diagram $e_i(j)$ for some $j = 0, 1, \dots, i$. That is, $H \times H$ has $i + 1$ orbits on \mathcal{S}_i , and the $e_i(j)$ form a set of orbit representatives. It is therefore clear that the alternating sum of the diagrams in the orbit of $e_i(j)$ is a scalar times $F_i e_i(j) F_i$. If the trivial diagram (in the orbit of $e_i(0) = 1$) has sign $+1$, then observing that $e_i(j + 1)$ is obtained from $e_i(j)$ by a simple interchange in S'_i (of $r + i - j$ with $i + j + 1$), we see that $b(S_i, S'_i, \beta_i) = \sum_j (-1)^j c_i(j) F_i e_i(j) F_i$, where $c_i(j)$ is the inverse of $|\{(h, h') \in H \times H \mid h e_i(j) h' = e_i(j)\}|$. This proves (4.5), and we are done. \square

4.3. *Summary of the argument in classical terms.* We outline here how the proof of Theorem 4.3 goes in terms of the classical theory. Define the degree of a monomial in $\mathbb{C}[x_{i,j}]$ as $(m_1, m_2, \dots, m_{2r})$, where m_i is the number of occurrences of i as a subscript in the relevant monomial. This makes $\mathbb{C}[x_{i,j}]$ into a \mathbb{Z}^{2r} -graded commutative algebra, and the subspace M_r spanned by the multiplier monomials (cf. Section 3.3) is then the graded component of degree $(1, 1, \dots, 1)$ of $\mathbb{C}[x_{i,j}]$. Now observe that any element X of \mathcal{D} is of the form $X = \sum_i f_i D_i$, where D_i is an $(n + 1) \times (n + 1)$ minor of $(x_{i,j})$ and $f_i \in \mathbb{C}[x_{i,j}]$. Hence if $X \in M_r \cap \mathcal{D}$ (see Section 3.3), then by degree, the determinants D_i such that $f_i \neq 0$ are homogeneous of degree $(m_1, m_2, \dots, m_{2r})$, where $m_i = 0$ or 1 for each i , and $\sum_i m_i = 2(n + 1)$. Each such determinant may be multiplied by an appropriate monomial f_i to obtain an element of $\mathcal{D} \cap M_r$.

The elements $b(S, S', \beta) \in B_r(n)$ defined in Lemma 4.1 correspond under the isomorphism μ to precisely these elements, and we have shown that they span $\ker(\nu)$ as linear space.

Our proof of Theorem 4.3 runs as follows. First, in Section 6 (Proposition 6.4) we show that the elements E_{ij} and E_{ij}^* are precisely certain of the $b(S, S', \beta)$, and hence are in $\ker(\nu)$. Moreover we show also in Section 6 that

those elements generate $\ker(\nu)$ as an ideal of $B_r(n)$. This is a straightforward consequence of the linear description of the kernel. Next, we show that each of the generators E_{ij}, E_{ij}^* belongs to the ideal generated by E_k for one of the elements E_k defined in Definition 4.2, and it follows (Corollary 6.10) that $\ker(\nu)$ is generated by the elements E_k , which may also be described as $E_k = E_{k,n+1-k}$, which are the E_{ij} of deficiency zero.

In Section 5 we develop a calculus of the elements E_k . In particular, we prove (Corollary 5.14) that they are quasi-idempotent. This calculus, together with an observation (Lemma 5.9) concerning the representations of Sym_r , enables us to complete the proof in Section 7 by showing that for all k , E_k lies in the ideal generated by $E_{\lfloor \frac{n+1}{2} \rfloor}$.

5. Some computations in the Brauer algebra

In this section we carry out some necessary computations in the Brauer algebra $B_r(\delta)$, where δ is arbitrary, and apply them to prove a key annihilation result (Lemma 5.10).

5.1. Arcs in the Brauer algebra.

LEMMA 5.1. (i) *In the group algebra $K\text{Sym}_r$, we have, in the notation of Section 2.2,*

$$a(\text{Sym}_n) = a(\text{Sym}_{n-1}) - |\text{Sym}_{n-2}|^{-1} a(\text{Sym}_{n-1}) s_{n-1} a(\text{Sym}_{n-1}).$$

(ii) *In the K -algebra $B_r(\delta)$, let*

$$F = a(1, i) a(i + 1, s) \quad \text{and} \quad F' = a(1, i - 1) a(i + 2, s).$$

Then for $2 \leq i \leq s - 2$,

$$e_{i,i+1} F e_{i,i+1} = (\delta - s + 2) F' e_{i,i+1} + [(i - 2)! (s - i - 2)!]^{-1} e_{i,i+1} F' e_{i-1,i+2} F'.$$

(iii) *Statement (ii) above remains true if $i = 1$, provided that $e_{0,3}$ is interpreted as 0, and $a(k, l) = 1$ whenever $k \geq l$.*

Proof. The first statement is a simple consequence of the double coset decomposition $\text{Sym}_n = \text{Sym}_{n-1} \amalg \text{Sym}_{n-1} s_{n-1} \text{Sym}_{n-1}$. For the second, observe that from (i), we have

$$a(1, i) = a(1, i - 1) - (i - 2)!^{-1} a(1, i - 1) s_{i-1} a(1, i - 1)$$

and

$$a(i + 1, s) = a(i + 2, s) - (s - i - 2)!^{-1} a(i + 2, s) s_{i+1} a(i + 2, s).$$

One now computes directly, using the relations in $B_r(\delta)$, the key relation here being $e_i s_{i\pm 1} e_i = e_i$. The third statement, concerning the case $i = 1$, follows from the above argument, but it may also be computed directly. □

The computation above may be usefully iterated as follows.

COROLLARY 5.2. *Assume that $0 \leq i \leq s - i$ and that $0 \leq j \leq i - 1$. For $k = 0, 1, \dots, i$, write $J_k = a(1, i - k)a(i + k + 1, s)$ so that $J_0 = F$ in [Lemma 5.1](#), $J_{i-1} = a(2i, s)$ and $J_i = a(2i + 1, s)$, interpreted as 1 if $i = 2s$. Write $e(j) = e_{i-j+1, i+j}$ for $j = 0, 1, \dots, i$; by convention, $e(j) = 0$ for $j > i$, and we note that $e(j)J_k = J_k e(j)$ for $j \leq k$. Then*

(i) *We have, for all i, s as above and for j such that $0 \leq j \leq i - 1$,*

$$(5.3) \quad e(j + 1)J_j e(j + 1) = (\delta - s + 2j + 2)J_{j+1}e(j + 1) + ((i - j - 2)!(s - i - j - 2)!)^{-1}e(j + 1)J_{j+1}e(j + 2)J_{j+1}.$$

(ii) *The case $j = i - 1$ of (i) is given by*

$$(5.4) \quad e(i)J_{i-1}e(i) = (\delta - s + 2i)J_i e(i).$$

This is consistent with (5.3) if $e_{k,l}$ is interpreted as 0 for $k < 0$.

(iii) *With the above notation, we have, for $k = 0, 1, \dots, i - 1$,*

$$(5.5) \quad e(1)J_0 e(1)e(2) \dots e(k) = A_k J_1 e(1)e(2) \dots e(k) + B_k J_1 e(1)e(2) \dots e(k + 1)J_k,$$

where

$$(5.6) \quad A_k = k(\delta - s + 2) + k(k - 1) \text{ and } B_k = \frac{1}{(i - k - 1)!(s - i - k - 1)!}.$$

(iv) *Statement (iii) remains true for $k = i$, given the conventions for interpreting $a(p, q)(= 1)$ when $p \geq q$ and $e_{p,q}(= 0)$ when $p < 0$. That is,*

$$(5.7) \quad e(1)J_0 e(1)e(2) \dots e(i) = A_i J_1 e(1)e(2) \dots e(i),$$

where $A_i = i(\delta - s + 2) + i(i - 1)$ is as given by the formula (5.6).

Proof. For $j < i - 1$, statement (i) is simply a translation of [Lemma 5.1](#) into the present context. When $j = i - 1$ or i , straightforward calculation shows that the formula (5.3) remains true, given the specified conventions. This proves (i) and (ii).

Assertion (iii) is proved by induction on k . First observe that the case $k = 0$ asserts that $e(1)J_0 = B_0 J_1 e(1)J_0$, where $B_0 = \frac{1}{(i-1)!(s-i-1)!}$, which is easily checked. Now suppose that for some fixed k with $0 \leq k \leq i - 1$, we have

$$e(1)J_0 e(1) \dots e(k) = A_k J_1 e(1) \dots e(k) + B_k J_1 e(1) \dots e(k + 1)J_k.$$

Multiplying on the right by $e(k + 1)$ and applying (5.3) to the second term, which terminates with $e(k + 1)J_k e(k + 1)$, a short calculation shows that we obtain

$$e(1)J_0 e(1) \dots e(k + 1) = A_{k+1} J_1 e(1) \dots e(k + 1) + B_{k+1} J_1 e(1) \dots e(k + 2)J_{k+1},$$

where

$$(5.8) \quad \begin{aligned} A_{k+1} &= A_k + B_k(\delta - s + 2k + 2)(i - k - 1)(s - i - k - 1)! \quad \text{and} \\ B_{k+1} &= B_k(i - k - 1)(s - i - k - 1). \end{aligned}$$

The recursion (5.8) has the unique solution given in (5.6).

Finally, assertion (iv) follows from the case $k = i - 1$ of (iii), noting that $J_i = a(2i + 1, s)$ and that the case $j = i - 1$ of (i) yields that $e(i)J_{i-1}e(i) = (\delta - s + 2i)e(i)J_i$. The proof of this last formula involves separate consideration of the cases $s = 2i$ and $s > 2i$. □

5.2. *A computation in the symmetric group algebra.* Fix an integer s and for i such that $0 \leq i \leq s - i$, and define $F_i(s) := a(1, i)a(i + 1, s)$, with the usual conventions.

LEMMA 5.9. *Write $K\text{Sym}_r = \bigoplus_{\lambda} I_{\lambda}$ for the usual canonical decomposition of the group algebra of Sym_r into simple two-sided ideals. Then*

- (i) *The ideal $\langle F_i(s) \rangle$ of $K\text{Sym}_r$ generated by $F_i(s)$ is the sum of those I_{λ} such that λ has at least $s - i$ boxes in its first column and s boxes in its first and second column.*
- (ii) *We have, for $1 \leq i \leq \frac{s}{2}$, $\langle F_{i-1}(s) \rangle \subseteq \langle F_i(s) \rangle$.*
- (iii) *There exist elements $\alpha_{i\ell}, \beta_{i\ell} \in B_r(\delta)$ such that for i as in (ii), $F_{i-1} = \sum_{\ell} \alpha_{i\ell} F_i \beta_{i\ell}$*

Proof. The first statement follows easily from the Littlewood-Richardson rule. In fact, one only requires the (dual of) the Pieri rule. The second and third statements follow easily from (i). □

5.3. *The annihilation lemma.* We shall prove the following result.

LEMMA 5.10. *Let $E_i \in B_r(n)$, $i = 0, 1, \dots, \lfloor \frac{n+1}{2} \rfloor$ be the elements defined in Definition 4.2(iii), and assume that $r \geq n + 1$. Then for $j = 1, 2, \dots, n$, we have $e_j E_i = E_i e_j = 0$.*

Proof. It is clear from Lemma 2.2 that $e_j F_i = F_i e_j = 0$ for $j \in \{1, 2, \dots, n + 1\}$, $j \neq i$. Hence to prove the lemma it suffices to prove that

$$(5.11) \quad e_i E_i = E_i e_i = 0.$$

Moreover, since we have $E_i^* = E_i$ and $e_i^* = e_i$ where $*$ is the cellular involution of $B_r(n)$ (reflection in a horizontal), to prove (5.11), it suffices to prove that $e_i E_i = 0$, since this implies that $(e_i E_i)^* = E_i^* e_i^* = E_i e_i = 0$. Thus we are reduced to proving that $e_i E_i = 0$.

Maintaining the notation of Definition 4.2, define elements

$$F_i(k) = a(1, i - k)a(i + k + 1, n + 1)$$

for $k = 0, 1, \dots, i$, with the usual conventions applying. Thus $F_i(0) = F_i$. The $F_i(k)$ are analogues of the elements J_k of Corollary 5.2, and translating (iii) of that corollary into the notation of the elements in Definition 4.2, bearing in mind that here $\delta = n$ and $s = n + 1$, we obtain

$$e_i F_i e_i(k) = k^2 F_i(1) e_i(k) + \frac{1}{(i - k - 1)!(n - i - k)!} F_i(1) e_i(k + 1) F_i(k)$$

for $k = 0, 1, 2, \dots, i$. Note that when $k = 0$, the first term vanishes and when $k = i$, the second term vanishes, given our conventions.

It follows that for $k = 0, 1, 2, \dots, i$, with the usual notational conventions, (5.12) $e_i F_i e_i(k) F_i = k^2 F_i(1) e_i(k) F_i + (i - k)(n + 1 - i - k) F_i(1) e_i(k + 1) F_i$.

It follows from (5.12) that $e_i E_i$ is a linear combination of $F_i(1) e_i(j) F_i$ for $j = 1, 2, \dots, i$. Moreover, also by (5.12), the coefficient of $F_i(1) e_i(k) F_i$ in $e_i E_i$ is, in the notation of Definition 4.2(iii),

$$(-1)^k (k^2 c_i(k) - (i - k + 1)(n + 2 - i - k) c_i(k - 1)),$$

and using the explicit values of the $c_i(j)$, this is equal to

$$(-1)^k \left\{ \frac{1}{(i - k)!(n + 1 - i - k)!(k - 1)!^2} - (i - k + 1)(n + 2 - i - k) \frac{1}{(i - k + 1)!(n + 2 - i - k)!(k - 1)!^2} \right\},$$

which is equal to zero.

This shows that $e_i E_i = 0$ and hence completes the proof of the lemma. □

COROLLARY 5.13. *Let D be any diagram in $B_{n+1}(n) \subseteq B_r(n)$ that has fewer than $n + 1$ through strings (i.e., that has a horizontal arc). Then for $i = 0, 1, 2, \dots, \lfloor \frac{n+1}{2} \rfloor$, $DE_i = E_i D = 0$.*

Proof. Fix i as above. If $\sigma \in \text{Sym}_i \times \text{Sym}_{n+1-i} \subset B_{n+1}(n)$, then $\sigma E_i = \pm E_i$. Now it is clear that for any diagram D as above, there is an element $\sigma \in \text{Sym}_i \times \text{Sym}_{n+1-i}$ such that for some $j \in \{1, 2, \dots, n\}$, $D\sigma = D'e_j$ for some diagram $D' \in B_{n+1}(n)$. Hence $DE_i = \pm D\sigma E_i = \pm D'e_j E_i$, which is zero by Lemma 5.10. The proof that $E_i D = 0$ is similar. □

COROLLARY 5.14. *The elements E_i are quasi-idempotent. Specifically, for $i = 0, 1, \dots, \lfloor \frac{n+1}{2} \rfloor$, we have*

$$E_i^2 = i!(n + 1 - i)! E_i.$$

Proof. Recall that $E_i = \sum_{j=0}^i (-1)^j c_i(j) F_i e_i(j) F_i$. Now it follows from Corollary 5.13 that for $j > 0$, $E_i F_i e_i(j) F_i = 0$, since the second factor is a sum of diagrams with at least one horizontal arc. Hence

$$E_i^2 = E_i \sum_{j=0}^i (-1)^j c_i(j) F_i e_i(j) F_i = E_i F_i = i!(n + 1 - i)! E_i. \quad \square$$

6. Generators of the kernel

In this section we shall prove

PROPOSITION 6.1. *The ideal $\ker(\nu)$ is generated by $E_0, E_1, \dots, E_{\lfloor \frac{n+1}{2} \rfloor}$.*

6.1. *The cellular anti-involution on $B_r(\delta)$.* We shall make use of the cellular anti-involution [6] on $B_r(\delta)$. This is the unique algebra anti-involution $*$: $B_r(\delta) \rightarrow B_r(\delta)$ satisfying $s_i^* = s_i$ and $e_i^* = e_i$ for each i . Then for $\sigma \in \text{Sym}_r$, we have $\sigma^* = \sigma^{-1}$. Geometrically, $*$ may be thought of as reflecting diagrams in a horizontal line.

The proof of Proposition 6.1 will proceed by showing that each of the elements $b(S, S', \beta)$ lies in the ideal of $B_r(n)$ generated by a certain explicit element E_{ij} or E_{ij}^* , and that each of the elements E_{ij} and E_{ij}^* lies in the ideal generated by E_k for some k .

6.2. *Generators and deficiency.* Let S, S' be disjoint subsets of $\{1, \dots, 2r\}$ such that $|S| = |S'| = n + 1$. If $|S \cap \{1, \dots, r\}| = i$ and $|S' \cap \{1, \dots, r\}| = j$, then after pre and post multiplying $b(S, S', \beta)$ by elements of $\text{Sym}_r \subset B_r(n)$ and possibly interchanging S with S' and $\{1, \dots, r\}$ with $\{r + 1, \dots, 2r\}$, we may assume that $i \leq j$ and $i + j \geq n + 1$; the case $i + j = n + 1$ leads to the elements E_i . We write $d_{ij} = i + j - (n + 1)$ and refer to this as the *deficiency* of the pair S, S' . Fix i, j as above.

For $k = 0, 1, \dots, n + 1 - j$, let $D_{ij}(k)$ be the diagram depicted in Figure 6, in which the points of S are denoted by \circ , those of S' by $*$ and those of $\{1, \dots, 2r\} \setminus (S \amalg S')$ by \bullet .

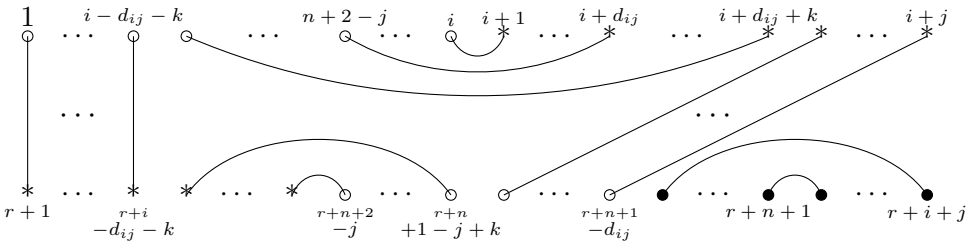


Figure 6.

The diagram $D_{ij}(k)$ is regarded as an element of $B_r(n)$ through the natural inclusion $B_l(n) \hookrightarrow B_r(n)$ for any $l \leq r$. Note that in the deficiency zero case, where $i + j = n + 1$, the diagram $D_{ij}(k)$ coincides with the diagram $e_i(k)$ of Definition 4.2(ii).

Definition 6.2. For i, j such that $0 \leq i \leq j \leq n + 1$ and $i + j \geq n + 1$, define $E_{ij} \in B_r(n)$ by

$$(6.3) \quad E_{ij} = \sum_{k=0}^{n+1-j} (-1)^j c_{ij}(k) a(1, i) a(i+1, i+j) D_{ij}(k) a(1, n+1-j) a(n+2-j, n+1-d_{ij}),$$

where $c_{ij}(k) = ((n + 1 - j - k)!(n + 1 - i - k)!k!(d_{ij} + k)!)^{-1}$.

- PROPOSITION 6.4. (i) *The elements E_{ij} and E_{ij}^* are in $\ker(\nu)$.*
 (ii) *The kernel $\ker(\nu)$ is generated as an ideal of $B_r(n)$ by the elements E_{ij} and E_{ij}^* , where $0 \leq i \leq j \leq n + 1$, $i + j \geq n + 1$. Here $*$ denotes the cellular involution of $B_r(n)$, discussed above.*

Proof. The kernel $\ker(\nu)$ is spanned by the elements $b(S, S', \beta)$, each of which is an alternating sum over Sym_{n+1} . To see that $E_{ij} \in \ker(\nu)$, we shall show that E_{ij} is precisely one of the elements $b(S, S', \beta)$, where $S = S_{ij} := \{1, \dots, i, r + i + 1 - d_{ij}, \dots, r + n + 1 - d_{ij}\}$, $S' = S'_{ij} := \{i + 1, \dots, i + j, r + 1, \dots, r + i - d_{ij}\}$, and β is the pairing of $\{1, \dots, 2r\} \setminus (S \amalg S')$ depicted in the diagram $D_{ij}(k)$ for any k .

Observe that from the formula (6.3), E_{ij} is alternating with respect to both $\text{Sym}(S)$ and $\text{Sym}(S')$, for if t is any transposition in $\text{Sym}(S)$, then $t \cdot E_{ij} = -E_{ij}$, and similarly for S' . In fact, the constants $c_{ij}(k)$ are chosen so that E_{ij} is precisely the alternating sum of $(n + 1)!$ diagrams, obtained from $D_{ij}(0)$ by permuting the elements of S . This shows that $E_{ij} \in \ker(\nu)$. Since $\ker(\nu)$ is evidently invariant under $*$, this proves (i).

It is straightforward to see that using the action of Sym_r on the right and left, any pair S, S' of subsets as above may be transformed into a pair S_{ij}, S'_{ij} or S'_{ij}, S_{ij} , where these sets are as above, with $i \leq j$ and $i + j \geq n + 1$.

It follows, since any summand of an element $b(S, S', \beta)$, where the pair S, S' has deficiency d , has at least d horizontal edges, that any element $b(S, S', \beta)$ may be transformed by $\text{Sym}_r \times \text{Sym}_r$ into an element of $B_r(n)$, each of whose diagram summands satisfies the condition that its leftmost $i + j$ part coincides with that of E_{ij} or E_{ij}^* for some i, j and whose rightmost $r - (i + j)$ part is constant for each such summand. Hence $\pi b(S, S', \beta) \pi' = E_{ij} D_\alpha$ or $E_{ij}^* D_\alpha$ for some $\pi, \pi' \in \text{Sym}_r$ and $D_\alpha \in B_r(n)$. This proves (ii). \square

LEMMA 6.5. *We have, for each i, j and k as above,*

$$D_{ij}(k) = e_i(k + d_{ij}) \pi_{ij} = e_{i,i+1} e_{i-1,i+2} \cdots e_{i-k-d_{ij}+1,i+k+d_{ij}} \pi_{ij}$$

and

$$E_{ij} = \sum_{k=0}^{n+1-j} (-1)^j c_{ij}(k) a(1, i) a(i + 1, i + j) e_i(k + d_{ij}) \cdot a(1, n + 1 - j) a(2i + j - n, i + j) \pi_{ij},$$

where $\pi_{ij} \in \text{Sym}_r \subset B_r(n)$ is the permutation defined by

$$\pi_{ij}(l) = \begin{cases} l & \text{if } 1 \leq l \leq n + 1 - j \text{ or } l > i + j \\ l + n + 1 - i & \text{if } i - d_{ij} + 1 \leq l \leq i + d_{ij} \\ l - 2d_{ij} & \text{if } i + d_{ij} + 1 \leq l \leq i + j. \end{cases}$$

The permutation π_{ij} is independent of k .

Proof. The first statement may be directly verified, and the second follows easily, by computing $\pi_{ij}a(1, n + 1 - j)a(2i + j - n, i + j)\pi_{ij}^{-1}$. □

The next result is required for the proof of [Proposition 6.1](#).

PROPOSITION 6.6. *We have, in the above notation, $e_\ell E_{ij} = 0$ for all ℓ such that $1 \leq \ell \leq i + j - 1$.*

Proof. It is clear by [Lemma 2.2](#) that the proposition holds for $\ell \neq i$. It therefore suffices to prove that

$$(6.7) \quad e_i E_{ij} = 0.$$

To apply the computations of [Section 5.1](#), it is convenient to rewrite the E_{ij} as follows. For d in the range $0 \leq d \leq i$, write $F_{ij}(d) = a(1, i - d)a(i + 1 + d, i + j)$. Noting that $n + 1 - j = i + d_{ij}$, etc., we may rewrite the expression for E_{ij} in [Lemma 6.5](#) as

$$(6.8) \quad E_{ij} = \sum_{k=0}^{n+1-j} (-1)^j c_{ij}(k) F_{ij}(0) e_i(k + d_{ij}) F_{ij}(d_{ij}) \pi_{ij}.$$

Note that the elements $F_{ij}(d)$ are special cases of the elements J_d of [Corollary 5.2](#), which may now be applied directly, replacing δ, s and k respectively by $n, i + j$ and $k + d_{ij} = i + j + k - (n + 1)$.

We obtain

$$\begin{aligned} e_i F_{ij}(0) e_i(k + d_{ij}) F_{ij}(d_{ij}) &= A_{k+d_{ij}} F_{ij}(1) e_i(k + d_{ij}) F_{ij}(d_{ij}) \\ &\quad + B_{k+d_{ij}} F_{ij}(1) e_i(k + 1 + d_{ij}) F_{ij}(k + d_{ij}) F_{ij}(d_{ij}) \\ &= A_{k+d_{ij}} F_{ij}(1) e_i(k + d_{ij}) F_{ij}(d_{ij}) \\ &\quad + (i - k - d_{ij})!(j - k - d_{ij})! B_{k+d_{ij}} F_{ij}(1) e_i(k + 1 + d_{ij}) F_{ij}(d_{ij}). \end{aligned}$$

It follows that in the expression for $e_i E_{ij}$ as a sum of the elements $G_k := F_{ij}(1) e_i(k) F_{ij}(d_{ij}) \pi_{ij}$, the coefficient of G_{k+1} is

$$(-1)^{k+1} \left(A_{k+1+d_{ij}} c_{ij}(k + 1) - (i - k - d_{ij})!(j - k - d_{ij})! B_{k+d_{ij}} c_{ij}(k) \right).$$

To evaluate this we substitute the actual values of A_ℓ and B_ℓ . We have

$$\begin{aligned} A_{k+d_{ij}} &= (k + d_{ij})(n - (i + j) + 2) + (k + d_{ij})(k + d_{ij} - 1) \\ &= k(k + d_{ij}), \end{aligned}$$

while

$$B_{k+d_{ij}} = [(i - 1 - (k + d_{ij}))!(j - 1 - (k + d_{ij}))]^{-1}.$$

Moreover,

$$c_{ij}(k) = [(d_{ij} + k)!k!(n + 1 - j - k)!n + 1 - i - k!]^{-1}.$$

Substituting these values into the expression above, we obtain

$$\begin{aligned} A_{k+1+d_{ij}}c_{ij}(k + 1) &= \frac{1}{(d_{ij} + k)!k!(n - j - k)!(n - i - k)!} \\ &= (i - k - d_{ij})!(j - k - d_{ij})!B_{k+d_{ij}}c_{ij}(k). \end{aligned}$$

It follows that the coefficient of G_k in e_iE_{ij} is zero for $k = 0, 1, \dots, i - d_{ij}$. Hence $e_iE_{ij} = 0$, and the proposition is proved. \square

COROLLARY 6.9. *If D is any diagram in $B_{i+j}(n)$ with at least one horizontal edge, then $DE_{ij} = 0$.*

Proof. For any such diagram D , there is a permutation $\sigma \in \text{Sym}_i \times \text{Sym}_j$ such that $D\sigma = D'e_\ell$ for some diagram $D' \in B_{i+j}(n)$ and ℓ satisfying $1 \leq \ell \leq i + j - 1$. The result now follows from [Proposition 6.6](#). \square

COROLLARY 6.10. *For each pair i, j with $i \leq j$ and $i + j \geq n + 1$, we have $E_{ij} \in \langle E_k \rangle$ for some k with $0 \leq k \leq \lfloor \frac{n+1}{2} \rfloor$. We also have $E_{ij}^* \in \langle E_k \rangle$ for the same k .*

Proof. It follows from [Corollary 6.9](#) that for any elements $x, y \in B_{i+j}(n)$ and any k such that $0 \leq k \leq \lfloor \frac{n+1}{2} \rfloor$, we have $x E_k y E_{ij} = x F_k y E_{ij}$, since the other summands of the product vanish. Write $F_{ij} = F_{ij}(0)$ in the notation of the proof of [Proposition 6.6](#). By the above observation, since $F_{ij} E_{ij}$ is a nonzero multiple of E_{ij} , in order to show that $E_{ij} \in \langle E_k \rangle$, it will suffice to show that there are elements $x, y \in B_{i+j}(n)$ such that $x F_k y = F_{ij}$ for some k . We shall in fact show that there are elements x, y of $K\text{Sym}_{i+j} \subset B_{i+j}(n) \subseteq B_r(n)$ which have the desired property.

Now we have seen that in $K\text{Sym}_{i+j}$, the ideal $\langle F_{ij} \rangle = \oplus_\lambda I_\lambda$, where λ runs over partitions of $i + j$ whose first column has at least j elements and whose first two columns have at least $i + j$ elements. But for $k = 0, 1, \dots, \lfloor \frac{n+1}{2} \rfloor$, $\langle F_k \rangle = \oplus_\mu I_\mu$, where μ runs over partitions whose first column contains at least $n + 1 - k$ elements and whose first two columns contain at least $n + 1$ elements. Since $i + j \geq n + 1$, it follows that for $k \geq n + 1 - j$ (note that $i + j \geq n + 1, i \leq j \implies n + 1 - j \leq \lfloor \frac{n+1}{2} \rfloor$), $E_{ij} \in \langle F_k \rangle$, whence there are elements $x, y \in K\text{Sym}_{i+j}$ such that $E_{ij} = x F_k y$, whence $E_{ij} \in \langle E_k \rangle$.

To show that $E_{ij}^* \in \langle E_k \rangle$, observe that by taking the $*$ of [Corollary 6.9](#), we have $E_{ij}^* D = 0$ of any diagram in $B_{i+j}(n)$ with at least one horizontal edge. Hence, as above, we see that for any elements $x, y \in B_{i+j}(n)$, $E_{ij}^* x E_k y = E_{ij}^* x F_k y$, and the argument proceeds as above. This completes the proof of the corollary. \square

We may now complete the

Proof of Proposition 6.1. It follows from Proposition 6.4 that $\ker(\nu)$ is generated by the E_{ij} and E_{ij}^* . But by Corollary 6.10, each of the elements E_{ij} and E_{ij}^* is in the ideal generated by $E_0, E_1, \dots, E_{\lfloor \frac{n+1}{2} \rfloor}$. Proposition 6.1 follows. \square

7. Proof of the main theorem

In this section we complete the proof of Theorem 4.3. The arguments are similar to the ones employed in the last section.

Proof of Theorem 4.3. The first assertion of the theorem is a special case of Corollary 5.14. To prove that $E = E_{\lfloor \frac{n+1}{2} \rfloor}$ generates $\ker(\nu)$, we proceed as follows. By Proposition 4.4, the E_i are in $\ker(\nu)$, and by Proposition 6.1 $\ker(\nu)$ is generated by the E_i . The result will therefore follow if we show that

$$(7.1) \quad \text{For } i = 1, \dots, \lfloor \frac{n+1}{2} \rfloor, E_{i-1} \text{ is in the ideal generated by } E_i.$$

For if (7.1) holds, then writing $\langle y \rangle$ for the ideal of $B_r(n)$ generated by any element $y \in B_r(n)$, we would have $\langle E \rangle \supseteq \langle E_{\lfloor \frac{n+1}{2} \rfloor - 1} \rangle \supseteq \dots \supseteq \langle E_1 \rangle \supseteq \langle E_0 \rangle$.

To prove (7.1), let $\alpha_{i\ell}, \beta_{i\ell}$ be elements of $B_r(n)$ as in Lemma 5.9. Then for any i such that $i \leq \frac{n+1}{2}$, $F_{i-1} = \sum_{\ell} \alpha_{i\ell} F_i \beta_{i\ell}$. Consider the element $x := \sum_{\ell} E_{i-1} \alpha_{i\ell} E_i \beta_{i\ell} \in \langle E_i \rangle$. Now for each ℓ , $\alpha_{i\ell} E_i \beta_{i\ell} = \sum_{j=0}^i \alpha_{i\ell} E_i(j) \beta_{i\ell}$, where $E_i(j) = (-1)^j c_i(j) F_i e_i(j) F_i$ is a sum of diagrams in $B_{n+1}(n)$ with at least j horizontal arcs. Hence by Corollary 5.13, $E_{i-1} \alpha_{i\ell} E_i(j) \beta_{i\ell} = 0$ if $j > 0$.

It follows that $x = \sum_{\ell} E_{i-1} \alpha_{i\ell} F_i \beta_{i\ell}$, since $E_i(0) = F_i$. Hence $x = E_{i-1} F_{i-1} = (i-1)!(n-i)! E_{i-1} \in \langle E_i \rangle$. This proves (7.1) and completes the proof of Theorem 4.3. \square

8. Cellular structure

It is well known that $B_r(n)$ has a cellular structure [6, §4] in which the cells are indexed by the set Λ of partitions $\lambda = (\lambda_1 \geq \dots \geq \lambda_p)$, with $|\lambda| = \sum_{i=1}^p \lambda_i \in \mathcal{T}$, where $\mathcal{T} = \{t \in \mathbb{Z} \mid 0 \leq t \leq r; t \equiv r \pmod{2}\}$. The partial order on Λ is given by the rule that $\lambda < \mu$ if $|\lambda| < |\mu|$, or $|\lambda| = |\mu|$ and $\lambda < \mu$ in the dominance order.

We therefore have cell modules $W(\lambda)$ ($\lambda \in \Lambda$) for $B_r(n)$. These are endowed with a canonical invariant form, whose radical $\text{Rad}(\lambda)$ has irreducible quotient, which we write here as I_λ . It is part of the general theory of cellular algebras that the nonzero I_λ form a complete set of representatives of the isomorphism classes of $B_r(n)$. Now $B_r(\delta)$ is quasi-hereditary [22] whenever $\delta \neq 0$, from which it follows that the I_λ are each nonzero and, therefore, that the irreducible $B_r(n)$ -modules are indexed by Λ .

Now we have assumed that the characteristic of K is zero. In consequence, $V^{\otimes r}$ is semisimple, both as $O(V)$ -module and as $B_r(n)$ -module.

LEMMA 8.1. *As $O(V) \times B_r(n)$ -module, we have*

$$V^{\otimes r} \simeq \bigoplus_{\lambda \in \Lambda^0} L_\lambda \otimes I_\lambda,$$

where Λ^0 is the subset of Λ consisting of partitions whose first and second columns have fewer than $n + 1$ boxes in total. Here L_λ and I_λ are respectively the simple $O(V)$ -module and the simple $B_r(V)$ -module corresponding to λ .

Proof. The statement that there is a decomposition of the type shown, for some subset of Λ , follows from generalities about double centraliser theory. The identification of Λ^0 in our case follows easily from our [Theorem 4.3](#) but is well known in any case. □

One consequence of this lemma is that the multiplicity of L_λ in $V^{\otimes r}$ is the dimension of I_λ . However, in consequence of the nonsemisimple nature of $B_r(n)$ for $r \geq n + 2$, these dimensions are not given by purely combinatorial (cellular) data. Nonetheless, our main theorem is relevant to this decomposition through the following result.

PROPOSITION 8.2. *Let $E = E_{[\frac{n+1}{2}]}$ be the element of $B_r(n)$ defined in [Definition 4.2\(iii\)](#). Then for $\lambda \in \Lambda^0$, the submodule $\text{Rad}(\lambda)$ of $W(\lambda)$ is given by $\text{Rad}(\lambda) = B_r(n)EW(\lambda)$. That is, $\text{Rad}(\lambda)$ is generated by $EW(\lambda)$.*

Proof. First, note that by [Lemma 8.1](#), I_λ is a summand of $V^{\otimes r}$ if and only if $\lambda \in \Lambda^0$. Hence Λ^0 consists of those λ such that I_λ is annihilated by $\ker(\nu)$ and hence by E since by [Theorem 4.3](#) E generates $\ker(\nu)$.

It follows that $EW(\lambda) \subseteq \text{Rad}(\lambda)$. But the ideal $\langle E \rangle$ contains the radical of the algebra $B_r(n)$. Hence by the local criterion proved in [[12](#), Th. 5.4(3)] for a self-dual ideal to contain the radical, it follows that for all $\lambda \in \Lambda$, $\langle E \rangle W(\lambda) \supseteq \text{Rad}(\lambda)$. It follows that for $\lambda \in \Lambda^0$, $\langle E \rangle W(\lambda) = \text{Rad}(\lambda)$, and the Proposition follows. □

We conclude this section with the remark that by the above proposition, we have, for $\lambda \in \Lambda^0$,

$$I_\lambda \simeq \frac{W(\lambda)}{\text{Rad}(\lambda)} \simeq \frac{W(\lambda)}{\langle E \rangle W(\lambda)},$$

and this makes it possible in principle to compute the dimension of I_λ by identifying the subspace $EW(\lambda)$ of $W(\lambda)$.

9. Change of base field, and remarks about the quantum case

In this section we discuss the situation when the field K has positive characteristic, as well as the quantum analogue of our result, which applies to the Birman-Wenzl-Murakami (BMW) algebra. We assume throughout that K is infinite.

9.1. *The case of positive characteristic.* When K is an infinite field of characteristic not equal to 2, our basic setup remains the same. We still have the map $\nu : B_r(n) \rightarrow \text{End}_{\mathcal{O}(V)}(V^{\otimes r})$, and it is still surjective. [Proposition 3.6](#) also remains true.

It is important to note that although the elements E_i and E_{ij} have denominators in their definitions, they are actually linear combinations of diagrams with coefficients ± 1 . Therefore they are elements of the Brauer algebra over \mathbb{Z} and may be thought of independently of the ground field. Many of the results above remain true for arbitrary K . For any commutative ring R , with $\delta \in R$, write $B_r^R(\delta)$ for the Brauer algebra over R , which may be defined by its presentation as given in [Lemma 2.1\(iii\)](#); $B_r^R(\delta)$ is free over R , with basis the set of Brauer diagrams. As usual, $B_r(n) = B_r^K(n)$.

LEMMA 9.1. *Let K be a field of characteristic other than two. The elements $b(S, S', \beta) \in B_r^K(n)$ of [Lemma 4.1](#) span $\ker(\nu)$.*

This is because the version of the second fundamental theorem in [[19](#), Prop. 21] is valid in this generality, and our statement follows from the commutativity of the diagram ([3.5](#)).

Next we have

PROPOSITION 9.2. *Let K be a field of characteristic other than two. Let $E_{ij} \in B_r(n)$ be the elements defined in [Definition 6.2](#). Then $\ker(\nu)$ is generated as ideal of $B_r(n)$ by the E_{ij}, E_{ij}^* .*

Proof. Note that although the definition of E_{ij} as given involves denominators, since E_{ij} is actually one of the elements $b(S, S', \beta)$, it is a \mathbb{Z} -linear combination of diagrams. Hence it may be interpreted as an element of $B_r^{\mathbb{Z}}(n)$ and hence of $B_r^R(n)$ for any ring R .

The proof of the current proposition involves merely the observation that the proof of [Proposition 6.4\(ii\)](#) remains valid in this more general setting. \square

PROPOSITION 9.3. *Let K be any ring, and let $E_{ij} \in B_r^K(n)$ be as in the previous proposition. Then for ℓ with $1 \leq \ell \leq i + j - 1$, we have $e_\ell E_{ij} = 0$.*

Proof. Although the proof of [Proposition 6.6](#) involves denominators, it is clear that it may be restated (and proved in the same way) as a result in $B_r^{\mathbb{Z}}(n)$. Applying the specialisation functor $K \otimes_{\mathbb{Z}}$, we obtain the present statement. \square

THEOREM 9.4. *Let K be a field of characteristic $p > n + 1$. Then $\ker(\nu) = \langle E \rangle$, where E is the element $E_{\lfloor \frac{n+1}{2} \rfloor}$ of [Theorem 4.3](#).*

Proof. The proofs of [Corollary 6.10](#) and of ([7.1](#)) involve computations in the group algebra $K\text{Sym}_{i+j}$. But if the characteristic of K is greater than

$i + j$, this algebra is semisimple, and hence the arguments in those proofs apply without change. The result follows. \square

Remark 9.5. It is likely that the conclusion of the above theorem is valid for any characteristic other than two.

9.2. *The quantum case.* Let $U_q := U_q(\mathfrak{o}_n)$ be the smash product of the quantised enveloping algebra corresponding to the complex Lie algebra $\mathfrak{so}_n(\mathbb{C})$ with the group algebra of \mathbb{Z}_2 (see [11, §8]). Let \mathcal{C}_q be the category of finite dimensional type $(1, 1, \dots, 1)$ representations of U_q . Using Lusztig’s integral form [16] of U_q and lattices in the simple U_q -modules, we have a specialisation functor $S : M_q \mapsto M$ taking modules M_q in \mathcal{C}_q to their “classical limit.”

Let V_q be the “natural” representation of the quantum group $U_q(\mathfrak{o}_n)$, that is, the representation that corresponds to the natural representation V of O_n under the specialisation above. It is well known (cf., e.g., [11]) that there is a surjective homomorphism $\psi : \mathbb{C}(q)\mathcal{B}_r \rightarrow \text{End}_{U_q(\mathfrak{o}_n)}(V_q^{\otimes r})$, where \mathcal{B}_r is the r -string braid group, acting through the generalised R -matrices.

Moreover this action factors through $\text{BMW}_r(q) = \text{BMW}_r(q^{2(1-n)}, q^2 - q^{-2})$, the Birman-Murukami-Wenzl algebra over $\mathbb{C}(q)$ with the indicated parameters. The specialisation at $q = 1$ (see [12, Lemma 4.2]) of $\text{BMW}_r(q)$ is $B_r(n)$. It follows from the results of [11], [12], [14] that we have a commutative diagram of specialisations as depicted below, which compare the classical case, treated above, with the quantum case:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \ker(\psi) & \longrightarrow & \text{BMW}_r(q) & \xrightarrow{\psi} & \text{End}_{U_q}(V_q^{\otimes r}) & \longrightarrow & 0 \\
 (9.6) & & \downarrow S & & \downarrow S & & \downarrow S & & \\
 0 & \longrightarrow & \ker(\nu) & \longrightarrow & B_r(n) & \xrightarrow{\nu} & \text{End}_U(V^{\otimes r}) & \longrightarrow & 0.
 \end{array}$$

This diagram naturally leads to the

CONJECTURE. *In the above notation, there is an element $\Phi_q \in \text{BMW}_r(q)$ such that $\ker(\psi) = \langle \Phi_q \rangle$. The specialisation at $q = 1$ of Φ is E .*

This was proved for the case $n = 3$ in [14, Th. 2.6], where an explicit formula was given for Φ_q .

There is another way to generalise the result [14, Th. 2.7], which is the case $n = 3$ of our current work. It was shown in [11] that the map $\mathbb{C}(q)\mathcal{B}_r \rightarrow \text{End}_{U_q(\mathfrak{sl}_2)}(V_{d,q}^{\otimes r})$ is surjective, where $V_{d,q}$ is the q -analogue of the d -dimensional representation of $U_q(\mathfrak{sl}_2)$. It is natural to ask for presentations of finite dimensional algebras through which this map factors (cf. [16]), and whether these algebras have a cellular structure (cf. [7], [8]). This is not likely to be straightforward, because in this case, the generators satisfy a polynomial equation of degree d .

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