Annals of Mathematics **176** (2012), 1373–1380
http://dx.doi.org/10.4007/annals.2012.176.2.12

Addendum to: Subelliptic $\mathrm{Spin}_{\mathbb{C}}$ Dirac operators, III

By Charles L. Epstein

Abstract

We prove the relative index conjecture, which in turn implies that the set of embeddable deformations of a strictly pseudoconvex CR-structure on a compact 3-manifold is closed in the $\mathcal{C}^\infty$-topology.

1. Proof of the Relative Index Conjecture

Let Y denote an oriented, compact, 3-dimensional manifold, with $H \subset TY$ a plane field, defining a contact structure. A strictly pseudoconvex CR-structure on Y is defined by a complex structure on the fibers of H, which we can represent as the bundle of $-i$-eigenspaces, denoted $T_b^{0,1}Y$. The CR-structure, in turn, defines a differential operator,

$$\bar{\partial}_b f = df \restriction_{T_b^{0,1}Y} . \tag{1}$$

The space of CR-functions on Y is the null-space of $\bar{\partial}_b$. A Szegő projector is an L^2-orthogonal projection onto the L^2-closure of the $\ker \bar{\partial}_b$, defined by the choice of a smooth, nondegenerate density on Y. None of our results depend upon the choice of this density.

A CR-structure is embeddable, or fillable if the $\ker \bar{\partial}_b$ contains sufficiently many functions to embed Y into $\mathbb{C}^N$ for some N. This is equivalent to the requirement that the CR-manifold $(Y, T_b^{0,1}Y)$ arises as the boundary of a compact normal Stein space; see pp. 4 and 5 of [2].

Recall that the deformations of a reference CR-structure, ${}^0T_b^{0,1}Y$, on (Y, H) are parametrized by

$$\mathrm{Def}(Y, H, \mathcal{S}_0) = \{\Phi \in \mathcal{C}^\infty(Y; \mathrm{Hom}({}^0T_b^{0,1}Y, {}^0T_b^{1,0}Y)) : \|\Phi\|_{L^\infty} < 1\} \tag{2}$$

via the prescription

$${}^\Phi T_{b,y}^{0,1}Y = \{\overline{Z}_y + \Phi_y(\overline{Z}_y) : \overline{Z}_y \in {}^0T_{b,y}^{0,1}Y\}. \tag{3}$$

Research partially supported by NSF grants DMS02-03795, DMS06-03973, and the Thomas A. Scott chair.

Here and in the sequel we often use the Szegő projector (instead of Φ) to label a CR-structure. From now on we assume that the reference CR-structure, with Szegő projector $\mathcal{S}_0$, is fillable.

Let $\mathcal{E} \subset \mathrm{Def}(Y, H, \mathcal{S}_0)$ be the subset consisting of the fillable deformations. In Theorem A of [2], [3] we showed that if $\mathcal{S}_0$ is the Szegő projector defined by the (fillable) reference CR-structure and $\mathcal{S}_1$ that defined by a deformation, then the deformed structure is fillable if and only if the restriction

$$\mathcal{S}_1 : \operatorname{Im} \mathcal{S}_0 \longrightarrow \operatorname{Im} \mathcal{S}_1 \tag{4}$$

is a Fredholm operator. Let $\text{R-Ind}(\mathcal{S}_0, \mathcal{S}_1)$ denote its Fredholm index, which we call the *relative index*. For each $m \in \mathbb{N} \cup \{0\}$ and any $\delta > 0$, let

$$\mathfrak{S}_m^\delta = \Big\{ \mathcal{S}_1 \in \mathrm{Def}(Y, H, \mathcal{S}_0) : \ -\infty < \text{R-Ind}(\mathcal{S}_0, \mathcal{S}_1) \le m \quad \text{and } \|\Phi\|^2_{L^\infty} \le \frac{1}{2} - \delta \Big\}. \tag{5}$$

Proposition 10.1 in [2] shows that there is an integer k_0, so that if a sequence $\langle \Phi_n \rangle \subset \mathfrak{S}_m^\delta$ converges to Φ in the $\mathcal{C}^{k_0}$-norm, then the structure defined by Φ is fillable.

In this addendum to [5], we show how the formula for the relative index between the Szegő projectors $\mathcal{S}_0, \mathcal{S}_1$, defined by two fillable CR-structures on a contact 3-manifold (Y, H), gives a proof of the Relative Index Conjecture.

Theorem 1. *Let (Y, H) be a compact 3-dimensional co-oriented, contact manifold, and let $\mathcal{S}_0$ be the Szegő projector defined by an fillable* CR*-structure on Y, with underlying plane field H. There is a nonnegative integer M such that for the Szegő projector $\mathcal{S}_1$ defined by any fillable deformation of the reference structure, with underlying plane field, H, we have the upper bound*

$$\text{R-Ind}(\mathcal{S}_0, \mathcal{S}_1) \le M. \tag{6}$$

Combining (6) with Proposition 10.1 of [2] we prove

Corollary 1. *Under the hypotheses of Theorem* 1, *the set of fillable deformations of the* CR*-structure on Y is closed in the $\mathcal{C}^\infty$-topology.*

Proof of the corollary. Suppose that $\langle \Phi_n \rangle$ is a sequence of fillable deformations in $\mathcal{E} \subset \mathrm{Def}(Y, H, \mathcal{S}_0)$ converging to $\Phi \in \mathrm{Def}(Y, H, \mathcal{S}_0)$, in the $\mathcal{C}^\infty$-topology. Recall that, by definition, $\|\Phi\|_{L^\infty} < 1$.

Let Ψ_1 and Ψ_2 be deformations of the reference structure, with local representations

$$\Psi_j = \psi_j Z \otimes \bar{\omega}, \tag{7}$$

where Z locally spans ${}^0T_b^{1,0}Y$ and $\bar{\omega}$ is the $(0,1)$-form dual to $\overline{Z}$; see page 12 in [2]. The analogous local coordinate representation of Ψ_2 as a deformation of Ψ_1 is given by

$$\psi_{21} = \frac{\psi_2 - \psi_1}{1 - \overline{\psi_1}\psi_2}; \tag{8}$$

see equation (5.5) in [2]. We can represent Φ as a deformation of any of the structures in the sequence. From equation (8) it is clear that there is an integer N so that, as deformations of Φ_N, a tail of the sequence and its limit lie in the L^∞-ball in $\mathrm{Def}(Y, H, \mathcal{S}_N)$, centered at 0, of radius $\frac{1}{4}$. Theorem 1 shows that there is an M so that

$$\text{R-Ind}(\mathcal{S}_N, \mathcal{S}_n) \leq M \text{ for all } n \in \mathbb{N}. \tag{9}$$

Proposition 10.1 from [2] then implies that the limiting structure Φ is also fillable, completing the proof of the corollary. □

Before proving Theorem 1 we recall the formula for the relative index, which is Theorem 13 in [5]. This formula involves topological and analytic invariants, which we now define, of the complex manifolds that fill the pair of CR-structures. Let X be a 4-dimensional manifold with boundary, and let $\widehat{H}^2(X)$ denote the image of $H^2(X, bX)$ in $H^2(X)$ under the natural map. The signature of the nondegenerate quadratic form on $\widehat{H}^2(X)$, defined by

$$([\alpha], [\beta]) \mapsto \int_X \alpha \wedge \beta, \tag{10}$$

is denoted $\mathrm{sig}[X]$, and $\chi[X]$ is the topological Euler characteristic

$$\chi[X] = \sum_{j=0}^{4} b_j(X)(-1)^j, \text{ where } b_j(X) = \dim H_j(X; \mathbb{Q}). \tag{11}$$

The final element needed for the proof of Theorem 1 is the relative index formula itself.

Theorem 2. *Let (Y, H) be a compact 3-dimensional co-oriented, contact manifold, and let $\mathcal{S}_0, \mathcal{S}_1$ be Szegő projectors for fillable CR-structures with underlying plane field H. Suppose that (X_0, J_0), (X_1, J_1) are strictly pseudoconvex complex manifolds with boundary $(Y, H, \mathcal{S}_0)$, $(Y, H, \mathcal{S}_1)$, respectively. Then*

$$\begin{aligned}\text{R-Ind}(\mathcal{S}_0, \mathcal{S}_1) &= \dim H^{0,1}(X_0, J_0) - \dim H^{0,1}(X_1, J_1) \\ &\quad + \frac{\mathrm{sig}[X_0] - \mathrm{sig}[X_1] + \chi[X_0] - \chi[X_1]}{4}.\end{aligned} \tag{12}$$

If $(Y, T_b^{0,1}Y)$ is fillable, then the normal Stein space, X, that it bounds is unique. By the definition of a normal singularity, the algebra of CR-functions on $(Y, T_b^{0,1}Y)$ is isomorphic to the algebra of holomorphic functions on X. If $\widehat{X}$ is obtained from X by resolving the singularities, then the algebras of holomorphic functions on X and $\widehat{X}$ are isomorphic, and therefore the Szegő projector defined by this CR-structure is the projection onto the boundary values of holomorphic functions on X, or any resolution of X.

Proof of Theorem 1. Recall that $\mathcal{S}_0, \mathcal{S}_1$ are Szegő projectors defined by fillable CR-structures on (Y, H). We let X_0 and X_1 denote complex manifolds with strictly pseudoconvex boundaries, obtained as the minimal resolutions of the normal Stein spaces bounded by $(Y, \mathcal{S}_0)$ and $(Y, \mathcal{S}_1)$ respectively. In Theorem $2'$ of [1], Bogomolov and De Oliveira prove that there are small perturbations of the complex structures on X_0 and X_1 making them into Stein manifolds. Hence it follows that X_0 and X_1, with the deformed complex structures, have strictly plurisubharmonic exhaustion functions. Therefore both X_0 and X_1 have the homotopy type of 2-dimensional CW-complexes. This implies that the Betti numbers $b_3(X_i)$ and $b_4(X_i)$ are zero.

The long exact sequence of the pair (X_i, bX_i) in homology, reads, in part

$$\cdots \longrightarrow H_1(bX_i) \longrightarrow H_1(X_i) \longrightarrow H_1(X_i, bX_i) \longrightarrow \cdots . \tag{13}$$

Poincaré-Lefschetz duality states that $H_1(X_i, bX_i) \simeq H^3(X_i)$, for $i = 0, 1$. As X_0 and X_1 have the homotopy type of 2-complexes, and the singular cohomology groups are homotopy invariant, it follows that $H^3(X_i) = 0$, and therefore, as $bX_i = Y$,

$$\dim H_1(X_i) \leq \dim H_1(Y), \text{ for } i = 0, 1; \tag{14}$$

see also page 328 in [9]. Poincaré-Lefschetz duality implies the isomorphism $H^2(X_i, bX_i) \simeq H_2(X_i)$. If $b_2^+(X_i)$ ($b_2^-(X_i)$) is the dimension of the maximal subspace on which the pairing in (10) is positive definite (negative definite), and $b_2^0(X_i)$ is the dimension of the null-space of the map $H^2(X_i, bX_i) \to H^2(X_i)$, then we see that

$$\begin{gathered} \dim H_2(X_i) = b_2(X_i) = b_2^+(X_i) + b_2^-(X_i) + b_2^0(X_i) = \dim H^2(X_i, bX_i) \\ \text{and } \operatorname{sig}[X_i] = b_2^+(X_i) - b_2^-(X_i). \end{gathered} \tag{15}$$

Taking advantage of these facts we can rewrite the formula in (12) as

$$\text{R-Ind}(\mathcal{S}_0, \mathcal{S}_1) = C_0 - C_1, \tag{16}$$

where C_i denotes the contribution of the terms from X_i:

$$C_i = \dim H^{0,1}(X_i, J_i) + \frac{2b_2^+(X_i) + b_2^0(X_i) - b_1(X_i)}{4}. \tag{17}$$

From equations (16) and (17), and the fact that $b_1(X_1) \leq b_1(Y)$, we conclude that

$$\text{R-Ind}(\mathcal{S}_0, \mathcal{S}_1) \leq C_0 + \frac{b_1(Y)}{4}. \tag{18}$$

This completes the proof of the theorem. □

1.1. *A new proof of Lempert's stability.* It is a consequence of Theorem D in [2] that $\text{R-Ind}(\mathcal{S}_0, \mathcal{S}_1) \geq 0$ for sufficiently small deformations. If $Y = S^3$ and $X_0 \subset \mathbb{C}^2$ is diffeomorphic to the 4-ball, then (14) shows that $b_1(X_0) = b_1(X_1) = 0$ and $C_0 = 0$ in (16). The relative index formula takes the very simple form:

$$\text{R-Ind}(\mathcal{S}_0, \mathcal{S}_1) = -\left[\dim H^{0,1}(X_1, J_1) + \frac{2b_2^+(X_1) + b_2^0(X_1)}{4}\right]. \tag{19}$$

The nonnegativity of $\text{R-Ind}(\mathcal{S}_0, \mathcal{S}_1)$ for small deformations and (19) show, in the present circumstance that for small deformations, the relative index $\text{R-Ind}(\mathcal{S}_0, \mathcal{S}_1)$ must vanish. When this is so, then a small extension of the results in Section 5 of [2] shows that for any nonnegative integer k, there is an integer l_k and a constant M_k so that the $\mathcal{C}^k$-operator norm of the difference, $\|\mathcal{S}_0 - \mathcal{S}_1\|_{\mathcal{C}^k}$, is bounded by $M_k\|\Phi\|_{\mathcal{C}^{l_k}}$. Here Φ is the deformation tensor for the CR-structure defining $\mathcal{S}_1$ as a deformation of that defining $\mathcal{S}_0$.

The coordinate functions $z_1 \restriction_{bX_0}, z_2 \restriction_{bX_0}$ define a CR-embedding of $(Y, \mathcal{S}_0)$ into $\mathbb{C}^2$. By definition of the Szegő projector, the functions

$$\varphi_i = \mathcal{S}_1[z_i \restriction_{bX_0}], \text{ for } i = 1, 2, \tag{20}$$

are CR-functions relative to the deformed structure. If $\|\mathcal{S}_0 - \mathcal{S}_1\|_{\mathcal{C}^1}$ is sufficiently small, then $y \mapsto (\varphi_1(y), \varphi_2(y))$ defines a CR-embedding of $(Y, {}^{\Phi}T^{0,1}Y)$ into $\mathbb{C}^2$, which is a $\mathcal{C}^1$-small deformation of bX_0. This completes the proof of the following proposition.

PROPOSITION 1. *Suppose that X_0 is an embedding of the standard 4-ball into $\mathbb{C}^2$ with a smooth strictly pseudoconvex boundary diffeomorphic to S^3. There is an $\varepsilon > 0$ and an l so that any embeddable deformation of the induced CR-structure on bX_0 with deformation tensor Φ, satisfying $\|\Phi\|_{\mathcal{C}^l} < \varepsilon$, arises as a small deformation of bX_0 in $\mathbb{C}^2$.*

This gives a new proof of a generalization of Lempert's first stability theorem, Theorem 4.5 in [6]. Lempert's original result assumes that X_0 is a strictly linearly convex domain. He uses the existence of "inner and outer S^1-actions" to verify that the deformed structure can be embedded as a small perturbation of the reference structure. In particular, Lempert's argument makes extensive usage of a "pseudoconcave cap" to compactify the deformed Stein space. This type of compactification is not needed for our analysis, but our results also say nothing about the existence of inner S^1-actions.

Suppose that X_0 is strictly linearly convex. As noted above, if the deformation tensor is sufficiently small in the $\mathcal{C}^{l_2}$-norm, then the $\mathcal{C}^2$-operator norm of the difference $\|\mathcal{S}_0 - \mathcal{S}_1\|_{\mathcal{C}^2}$ will also be small. From this it follows, as in [6], that the deformed structure has an embedding that is also strictly linearly convex. In a subsequent paper, [7], Lempert removed the hypothesis of strict linear convexity and extended his stability result to the boundaries of smoothly bounded, strictly pseudoconvex domains in $\mathbb{C}^2$.

1.2. *Remarks on the Ozbagci-Stipsicz Conjecture.* As noted above, $\operatorname{sig}[X_1] + b_2(X_1) = 2b_2^+(X_1) + b_2^0(X_1)$. A global bound on $|\operatorname{R-Ind}(\mathcal{S}_0, \mathcal{S}_1)|$, among all Szegő projectors $\mathcal{S}_1$ defined by elements of $\mathcal{E}$, is therefore equivalent to an upper bound for the quantity

$$b_2^+(X_1) + b_2^0(X_1) + \dim H^{0,1}(X_1),$$

among all Stein spaces, X_1 filling (Y, H). The existence of an upper bound on $b_2^+(X_1) + b_2^0(X_1)$ was conjectured by Ozbagci and Stipsicz, and proved in some special cases; see [9].

The fact, noted above, that $\operatorname{R-Ind}(\mathcal{S}_0, \mathcal{S}_1) \geq 0$, for sufficiently small deformations and (16) show that for such deformations,

$$\begin{aligned} \dim H^{0,1}(X_1) &+ \frac{2b_2^+(X_1) + b_2^0(X_1)}{4} \\ &\leq \dim H^{0,1}(X_0) + \frac{2b_2^+(X_0) + b_2^0(X_0) + b_1(Y) - b_1(X_0)}{4}. \end{aligned} \tag{21}$$

On page 328 of [9], Stipsicz proves the existence of a constant $K_{(Y,H)}$ (which may be positive or negative) so that for any Stein filling of (Y, H), we have the estimate

$$b_2^-(X_1) \leq 5b_2^+(X_1) + 2 - K_{(Y,H)} + 2b_1(Y). \tag{22}$$

These estimates, along with (15) and (21), prove a "germ" form of the Ozbagci–Stipsicz conjecture.

Proposition 2. *With (Y, H) as above, let $\mathcal{S}_0$ be a fillable reference* CR-*structure. Among sufficiently small, fillable deformations of this* CR-*structure the set of numbers*

$$\{b_1(X_1), \operatorname{sig}(X_1), \chi(X_1)\}$$

is finite. Here X_1 ranges over the minimal resolutions of the normal Stein spaces bounded by the deformed structures $(Y, H, \mathcal{S}_1)$.

The notion of smallness here depends on the size of the gap at 0 in the spectrum of the $\Box_b$-operator of the reference CR-structure. This can vary quite dramatically from fillable structure to fillable structure, which is why we call this a germ form of the Ozbagci–Stipsicz conjecture.

1.3. *Open problems and a possible strategy*: Our results suggest a strategy for proving a lower bound on $\operatorname{R-Ind}(\mathcal{S}_0, \mathcal{S}_1)$, among deformations Φ with $\|\Phi\|_{L^\infty} < 1 - \varepsilon$, for an $\varepsilon > 0$. Suppose that no such bound exists, one could then choose a sequence $\langle \Phi_n \rangle \subset \mathcal{E}$ for which $\operatorname{R-Ind}(\mathcal{S}_0, \mathcal{S}_n)$ tends to $-\infty$. A contradiction would follow immediately if we could show that $\langle \Phi_n \rangle$ is bounded in the $\mathcal{C}^{k_0+1}$-norm.

While such an *a priori* bound seems unlikely for the original sequence, it would suffice to replace the sequence $\langle \Phi_n \rangle$ with a "wiggle-equivalent" sequence. Let M_n denote a projective surface containing $(Y, {}^{\Phi_n}T_b^{0,1}Y)$ as a separating hypersurface; see Theorem 8.1 in [8]. An equivalent sequence with better regularity might be obtained by wiggling the hypersurfaces defined by $(Y, {}^{\Phi_n}T_b^{0,1}Y)$ within M_n, perhaps using some sort of heat-flow. After composing the resultant deformations with suitable contact transformations, we might be able to obtain a sequence $\langle \Phi_n' \rangle$ with $\text{R-Ind}(\mathcal{S}_0, \mathcal{S}_n') = \text{R-Ind}(\mathcal{S}_0, \mathcal{S}_n)$ that does satisfy an *a priori* $\mathcal{C}^{k_0+1}$-bound. Such an argument would seem to require an improved understanding of the metric geometry of $\mathrm{Def}(Y, H, \mathcal{S}_0)$, as well as the relationship of an abstract deformation to the local extrinsic geometry of Y as a hypersurface in M_n.

Acknowledgment. I would like to thank Fedya Bogomolov for asking me to speak about the relative index, Sylvain Cappell for helping me with some topological calculations, and the Courant Institute for their hospitality during the completion of this work. I would also like to thank the referee for many, many suggestions, which markedly improved the exposition in this note.

References

[1] F. A. Bogomolov and B. de Oliveira, Stein small deformations of strictly pseudoconvex surfaces, in *Birational Algebraic Geometry* (Baltimore, MD, 1996), *Contemp. Math.* **207**, Amer. Math. Soc., Providence, RI, 1997, pp. 25–41. MR 1462922. Zbl 0889.32021. http://dx.doi.org/10.1090/conm/207/02717.

[2] C. L. Epstein, A relative index on the space of embeddable CR-structures. I, *Ann. of Math.* **147** (1998), 1–59. MR 1609455. Zbl 0942.32025. http://dx.doi.org/10.2307/120982.

[3] ______, A relative index on the space of embeddable CR-structures. II, *Ann. of Math.* (2) **147** (1998), 61–91. MR 1609451. Zbl 0942.32026. http://dx.doi.org/10.2307/120983. Available at http://dx.doi.org/10.2307/120983.

[4] ______, Erratum: A relative index on the space of embeddable CR-structures. I, *Ann. of Math.* **154** (2001), 223–226. MR 1847595. Zbl 0983.32036. http://dx.doi.org/10.2307/3062117.

[5] ______, Subelliptic $\mathrm{Spin}_{\mathbb{C}}$ Dirac operators. III. The Atiyah-Weinstein conjecture, *Ann. of Math.* **168** (2008), 299–365. MR 2415404. Zbl 1169.32008. http://dx.doi.org/10.4007/annals.2008.168.299.

[6] L. Lempert, On three-dimensional Cauchy-Riemann manifolds, *J. Amer. Math. Soc.* **5** (1992), 923–969. MR 1157290. Zbl 0781.32014. http://dx.doi.org/10.2307/2152715.

[7] L. Lempert, Embeddings of three-dimensional Cauchy-Riemann manifolds, *Math. Ann.* **300** (1994), 1–15. MR 1289827. Zbl 0817.32009. http://dx.doi.org/10.1007/BF01450472.

[8] ______, Algebraic approximations in analytic geometry, *Invent. Math.* **121** (1995), 335–353. MR 1346210. Zbl 0837.32008. http://dx.doi.org/10.1007/BF01884302.

[9] A. I. Stipsicz, On the geography of Stein fillings of certain 3-manifolds, *Michigan Math. J.* **51** (2003), 327–337. MR 1992949. Zbl 1043.53066. http://dx.doi.org/10.1307/mmj/1060013199.

(Received: March 2, 2012)
(Revised: April 30, 2012)

University of Pennsylvania, Philadelphia, PA
E-mail: cle@math.upenn.edu