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Complex multiplication cycles and Kudla-Rapoport divisors

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Abstract

We study the intersections of special cycles on a unitary Shimura variety of signature $(n-1,1)$ and show that the intersection multiplicities of these cycles agree with Fourier coefficients of Eisenstein series. The results are new cases of conjectures of Kudla and suggest a Gross-Zagier theorem for unitary Shimura varieties.

1. Introduction

1.1. *Overview.* In [26], Kudla and Rapoport define a family of divisors $Z(m)$ on a unitary Shimura variety M of dimension $n-1$, all defined over a quadratic imaginary field K_0. The variety and the divisors have integral models $\mathcal{M}$ and $\mathcal{Z}(m)$ over $\mathcal{O}_{K_0}$. The program begun in [25], [26], [42] seeks to compute the n-fold intersection multiplicity of a tuple $\mathcal{Z}(m_1),\ldots,\mathcal{Z}(m_n)$ and to relate the intersection multiplicity to Fourier coefficients of Eisenstein series. In this article we intersect the Kudla-Rapoport divisors with a different cycle on $\mathcal{M}$, formed by points with complex multiplication. By fixing a CM field K of degree n over K_0 and a CM type Φ satisfying a suitable signature condition, we obtain a 0-cycle X_Φ on M defined over the reflex field of Φ, representing points with complex multiplication by $\mathcal{O}_K$ and CM type Φ. Passing to integral models yields a cycle $\mathcal{X}_\Phi$ on $\mathcal{M}$ of absolute dimension 1, and our main results relate the intersection multiplicity of $\mathcal{Z}(m)$ and $\mathcal{X}_\Phi$ with Fourier coefficients of an Eisenstein series.

The intersection $\mathcal{Z}(m)\cap\mathcal{X}_\Phi$ naturally decomposes as a disjoint union of 0-dimensional stacks $\mathcal{Z}_\Phi(\alpha)$, where the index α ranges over those totally positive elements of the maximal totally real subfield $F\subset K$ that satisfy $\mathrm{Tr}_{F/\mathbb{Q}}(\alpha)=m$. In the body of the paper we allow K to be a product of CM fields, in which case some $\mathcal{Z}_\Phi(\alpha)$ may have dimension one; i.e., the cycles $\mathcal{X}_\Phi$ and $\mathcal{Z}(m)$ may intersect improperly. This does not happen when K is a field. The *Arakelov degree* of $\mathcal{Z}_\Phi(\alpha)$ is (essentially) defined to be the sum of the

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lengths of the local rings of all geometric points, and our main result shows that, as α varies, these degrees are the Fourier coefficients of the derivative of a weight one Hilbert modular Eisenstein series $\mathcal{E}_\Phi(\tau, s)$ at the center $s = 0$ of its functional equation.

Returning to the original problem, the intersection multiplicity of $\mathcal{X}_\Phi$ with $\mathcal{Z}(m)$ is obtained by adding together the degrees of those $\mathcal{Z}_\Phi(\alpha)$ with $\mathrm{Tr}_{F/\mathbb{Q}}(\alpha) = m$. This intersection multiplicity is equal to the m^{th} Fourier coefficient of the central derivative of the pullback of $\mathcal{E}_\Phi(\tau, s)$ via the diagonal embedding of the upper half plane into a product of upper half planes.

1.2. *Statement of the results.* Fix a quadratic imaginary field $K_0 \subset \mathbb{C}$, denote by ι the inclusion $K_0 \to \mathbb{C}$, and let $\bar{\iota}$ be the conjugate embedding. For nonnegative integers r, s, let $\mathcal{M}_{(r,s)}$ be the algebraic stack over $\mathrm{Spec}(\mathcal{O}_{K_0})$ whose functor of points assigns to every $\mathcal{O}_{K_0}$-scheme S the groupoid of triples (A, κ, λ) in which

- $A \to S$ is an abelian scheme of relative dimension $r + s$,
- $\kappa : \mathcal{O}_{K_0} \to \mathrm{End}(A)$ is an action of $\mathcal{O}_{K_0}$ on A,
- $\lambda : A \to A^\vee$ is a principal polarization.

(Throughout this paper *scheme* means *locally Noetherian scheme* and *algebraic stack* means *Deligne-Mumford stack*.) We require that the polarization λ be $\mathcal{O}_{K_0}$-linear, in the sense that

$$\lambda \circ \kappa(\overline{x}) = \kappa(x)^\vee \circ \lambda$$

for all $x \in \mathcal{O}_{K_0}$. We further require that the action of $\mathcal{O}_{K_0}$ satisfy the (r, s)-*signature condition*: for any $x \in \mathcal{O}_{K_0}$, locally on S the determinant of $T - x$ acting on $\mathrm{Lie}(A)$ is equal to the image of

$$(T - \iota(x))^r (T - \bar{\iota}(x))^s \in \mathcal{O}_{K_0}[T]$$

in $\mathcal{O}_S[T]$. Our stack $\mathcal{M}_{(r,s)}$ is the stack denoted $\mathcal{M}(r, s)^{\mathrm{naive}}$ in [26]; it is smooth of relative dimension rs over $\mathcal{O}_{K_0}[\mathrm{disc}(K_0)^{-1}]$. The generic fiber of $\mathcal{M}_{(r,s)}$ is a union of Shimura varieties associated to the unitary similitude groups of finitely many Hermitian spaces over K_0, but for us the interpretation as a moduli space is paramount.

Note that $\mathcal{M}_{(1,0)}$ is simply the moduli stack of elliptic curves $A_0 \to S$ over $\mathcal{O}_{K_0}$-schemes with complex multiplication by $\mathcal{O}_{K_0}$, normalized so that the action of $\mathcal{O}_{K_0}$ on $\mathrm{Lie}(A_0)$ is through the structure morphism $\mathcal{O}_{K_0} \to \mathcal{O}_S$.

For the remainder of the introduction, we fix a positive integer n and focus on the case of signature $(n - 1, 1)$. We will construct two types of cycles on the stack

$$\mathcal{M} = \mathcal{M}_{(1,0)} \times_{\mathcal{O}_{K_0}} \mathcal{M}_{(n-1,1)}.$$

For an $\mathcal{O}_{K_0}$-scheme S, an S-valued point of $\mathcal{M}$ is a sextuple $(A_0, \kappa_0, \lambda_0, A, \kappa, \lambda)$ with

$$(A_0, \kappa_0, \lambda_0) \in \mathcal{M}_{(1,0)}(S), \qquad (A, \kappa, \lambda) \in \mathcal{M}_{(n-1,1)}(S),$$

but we will usually abbreviate this sextuple to (A_0, A).

The first family of cycles on $\mathcal{M}$ are the *Kudla-Rapoport divisors* of [26]. If S is connected and $(A_0, A) \in \mathcal{M}(S)$, the projective $\mathcal{O}_{K_0}$-module of finite rank

$$L(A_0, A) = \mathrm{Hom}_{\mathcal{O}_{K_0}}(A_0, A)$$

comes equipped with a positive definite $\mathcal{O}_{K_0}$-valued Hermitian form

$$\langle f_1, f_2 \rangle = \lambda_0^{-1} \circ f_2^\vee \circ \lambda \circ f_1. \tag{1.2.1}$$

(The right-hand side lies in $\mathcal{O}_{K_0} = \mathrm{End}_{\mathcal{O}_{K_0}}(A_0)$.) For an integer $m \neq 0$, let $\mathcal{Z}(m)$ be the moduli stack over $\mathcal{O}_{K_0}$ whose S-valued points are triples (A_0, A, f) with $(A_0, A) \in \mathcal{M}(S)$ and $f \in L(A_0, A)$ satisfying $\langle f, f \rangle = m$. There is an obvious forgetful morphism

$$\mathcal{Z}(m) \to \mathcal{M}.$$

In terms of Shimura varieties, these divisors correspond roughly to inclusions of algebraic groups of the form $\mathrm{H} \to \mathrm{GU}(V)$, where V is a Hermitian space over K_0 of signature $(n-1, 1)$, and H is the stabilizer of a vector of positive length. But, once again, to us it is the moduli interpretation that matters most.

The second type of cycle is constructed from abelian varieties with complex multiplication. Let F be a totally real étale $\mathbb{Q}$-algebra (in other words, a product of totally real number fields) with $[F : \mathbb{Q}] = n$, and fix a CM type Φ of

$$K = F \otimes_{\mathbb{Q}} K_0$$

of signature $(n-1, 1)$. This means that there are $n-1$ elements of Φ whose restriction to K_0 is ι, and a unique element whose restriction to K_0 is $\bar{\iota}$. Let

$$K_\Phi \subset \mathbb{C}$$

be a number field containing both K_0 and the reflex field of Φ, and set $\mathcal{O}_\Phi = \mathcal{O}_{K_\Phi}$. Let $\mathcal{CM}_\Phi$ be the algebraic stack over $\mathcal{O}_\Phi$ classifying principally polarized abelian schemes with complex multiplication by $\mathcal{O}_K$ and CM type Φ. See Section 3.1 for the precise definition. For an $\mathcal{O}_\Phi$-scheme S, an S-valued point of

$$\mathcal{X}_\Phi = \mathcal{M}_{(1,0)/\mathcal{O}_\Phi} \times_{\mathcal{O}_\Phi} \mathcal{CM}_\Phi$$

is a pair $(A_0, A) \in \mathcal{M}(S)$ together with an extension of the $\mathcal{O}_{K_0}$-action on A to complex multiplication by $\mathcal{O}_K$, and as such there is an evident forgetful morphism

$$\mathcal{X}_\Phi \to \mathcal{M}_{/\mathcal{O}_\Phi}.$$

The stack $\mathcal{X}_\Phi$ is étale and proper over $\mathcal{O}_\Phi$ and, in particular, is regular of dimension 1. In terms of Shimura varieties, the map $\mathcal{X}_\Phi \to \mathcal{M}_{/\mathcal{O}_\Phi}$ corresponds roughly to $T \to \mathrm{GU}(V)$, where V is a Hermitian space over K_0 of signature $(n-1,1)$, and T is the torus with $\mathbb{Q}$-points

$$T(\mathbb{Q}) = \{x \in K^\times : \mathrm{Nm}_{K/F}(x) \in \mathbb{Q}^\times\}.$$

Now we come to the central problem of this paper: to compute the intersection multiplicity on $\mathcal{M}_{/\mathcal{O}_\Phi}$ of the Kudla-Rapoport divisor $\mathcal{Z}(m)_{/\mathcal{O}_\Phi}$ with the complex multiplication cycle $\mathcal{X}_\Phi$. Consider the cartesian diagram (this is the definition of the upper left corner)

$$\begin{tikzcd} \mathcal{X}_\Phi \cap \mathcal{Z}(m) \arrow[r] \arrow[d] & \mathcal{Z}(m)_{/\mathcal{O}_\Phi} \arrow[d] \\ \mathcal{X}_\Phi \arrow[r] & \mathcal{M}_{/\mathcal{O}_\Phi}. \end{tikzcd}$$

Let S be a connected $\mathcal{O}_\Phi$-scheme. Given a point

$$(A_0, A) \in \mathcal{X}_\Phi(S),$$

we may consider the $\mathcal{O}_{K_0}$-module $L(A_0,A)$ attached to the image $(A_0,A) \in \mathcal{M}(S)$. The fact that the pair (A_0,A) comes from $\mathcal{X}_\Phi(S)$ endows $L(A_0,A)$ with obvious extra structure: the action of $\mathcal{O}_K$ on A makes $L(A_0,A)$ into an $\mathcal{O}_K$-module. Slightly less obviously, there is a unique K-valued totally positive definite $\mathcal{O}_K$-Hermitian form $\langle f_1, f_2\rangle_{\mathrm{CM}}$ on $L(A_0,A)$, which refines $\langle f_1, f_2\rangle$, in the sense that

$$\langle f_1, f_2\rangle = \mathrm{Tr}_{K/K_0}\langle f_1, f_2\rangle_{\mathrm{CM}}.$$

By contemplation of the the moduli problems, there is a decomposition

$$\mathcal{X}_\Phi \cap \mathcal{Z}(m) = \bigsqcup_{\substack{\alpha \in F \\ \mathrm{Tr}_{F/\mathbb{Q}}(\alpha)=m}} \mathcal{Z}_\Phi(\alpha), \tag{1.2.2}$$

where $\mathcal{Z}_\Phi(\alpha)$ is the moduli space of triples (A_0, A, f) over $\mathcal{O}_\Phi$-schemes S, in which

$$(A_0, A) \in \mathcal{X}_\Phi(S)$$

and $f \in L(A_0,A)$ satisfies $\langle f, f\rangle_{\mathrm{CM}} = \alpha$. If $\alpha \in F^\times$, the stack $\mathcal{Z}_\Phi(\alpha)$ has dimension 0, and is nonempty only if α is totally positive ($\alpha \gg 0$). If $\alpha \notin F^\times$, then $\mathcal{Z}_\Phi(\alpha)$ may have irreducible components of dimension 1, in which case the intersection (1.2.2) is improper.

For a prime $\mathfrak{p}$ of K_Φ, let $k_{\Phi,\mathfrak{p}}$ denote the residue field of $\mathfrak{p}$. When $\mathcal{Z}_\Phi(\alpha)$ has dimension 0, its *Arakelov degree*

$$\widehat{\deg}\, \mathcal{Z}_\Phi(\alpha) = \sum_{\mathfrak{p} \subset \mathcal{O}_\Phi} \frac{\log(\mathrm{N}(\mathfrak{p}))}{[K_\Phi : \mathbb{Q}]} \sum_{z \in \mathcal{Z}_\Phi(\alpha)(k^{\mathrm{alg}}_{\Phi,\mathfrak{p}})} \frac{\mathrm{length}(\mathcal{O}^{\mathrm{sh}}_{\mathcal{Z}_\Phi(\alpha),z})}{\#\mathrm{Aut}(z)}$$

is finite and is independent of the choice of K_Φ. (Here $\mathcal{O}^{\mathrm{sh}}_{\mathcal{Z}_\Phi(\alpha),z}$ is the strictly Henselian local ring of $\mathcal{Z}_\Phi(\alpha)$ at z, i.e., the local ring for the étale topology.) Our first main result is a formula for the Arakelov degree. To state it we need some notation. Let $\varphi^{\mathrm{sp}} \in \Phi$ be the *special element*, determined by $\varphi^{\mathrm{sp}}|_{K_0} = \bar{\iota}$. Recalling that K is a product of CM fields, there is a unique factor $K^{\mathrm{sp}} \subset K$ such that $\varphi^{\mathrm{sp}} : K \to \mathbb{C}$ factors through the projection $K \to K^{\mathrm{sp}}$. Denote by F^{sp} the maximal totally real subfield of K^{sp}, so that F^{sp} is a direct summand of F. If $\mathfrak{p}$ is a prime of F^{sp}, then we denote again by $\mathfrak{p}$ the prime of F determined by pullback through the projection $F \to F^{\mathrm{sp}}$. If $\mathfrak{b}$ is a fractional $\mathcal{O}_F$-ideal, define

$$\rho(\mathfrak{b}) = \#\{\mathfrak{B} \subset \mathcal{O}_K : \mathfrak{B}\overline{\mathfrak{B}} = \mathfrak{b}\mathcal{O}_K\}. \tag{1.2.3}$$

In particular, $\rho(\mathfrak{b}) = 0$ if $\mathfrak{b} \not\subset \mathcal{O}_F$. For any prime p, set

$$\varepsilon_p = \begin{cases} 1 & \text{if } K_0/\mathbb{Q} \text{ is unramified at } p \\ 0 & \text{if } K_0/\mathbb{Q} \text{ is ramified at } p. \end{cases} \tag{1.2.4}$$

The following theorem appears in the text as Theorem 3.6.3.

Theorem A. *Assume the discriminants of $K_0/\mathbb{Q}$ and $F/\mathbb{Q}$ are odd and relatively prime. If $\alpha \in F^{\gg 0}$, then $\mathcal{Z}_\Phi(\alpha)$ has dimension zero, and*

$$\widehat{\deg}\, \mathcal{Z}_\Phi(\alpha) = \frac{h(K_0)}{w(K_0)} \sum_{\mathfrak{p}} \frac{\log(\mathrm{N}(\mathfrak{p}))}{[K^{\mathrm{sp}} : \mathbb{Q}]} \cdot \mathrm{ord}_{\mathfrak{p}}(\alpha\mathfrak{p}\mathfrak{d}_F) \cdot \rho(\alpha\mathfrak{p}^{-\varepsilon_p}\mathfrak{d}_F),$$

where the sum is over all primes $\mathfrak{p}$ of F^{sp} nonsplit in K^{sp}, p is the rational prime below $\mathfrak{p}$, $\mathfrak{d}_F$ is the different of $F/\mathbb{Q}$, $h(K_0)$ is the class number of K_0, $w(K_0)$ is number of roots of unity in K_0, and $\mathrm{N}(\mathfrak{p})$ is the cardinality of the residue field of $\mathfrak{p}$.

Fix a prime $\mathfrak{p} \subset \mathcal{O}_\Phi$, and let $W_{\Phi,\mathfrak{p}}$ be the completion of the ring of integers in the maximal unramified extension of $\mathcal{O}_{\Phi,\mathfrak{p}}$. The most difficult part of the proof of Theorem A is the calculation of the length of the local ring at a geometric point $z \in \mathcal{Z}_\Phi(\alpha)(k^{\mathrm{alg}}_{\Phi,\mathfrak{p}})$, corresponding to a triple (A_0, A, f). This calculation proceeds in two steps. First we show that the formal deformation space of the pair (A_0, A) is isomorphic to the formal spectrum of $W_{\Phi,\mathfrak{p}}$. In more concrete terms, this means that (A_0, A) lifts uniquely to any complete local Noetherian $W_{\Phi,\mathfrak{p}}$-algebra with residue field $k^{\mathrm{alg}}_{\Phi,\mathfrak{p}}$ and, in particular, has a unique lift to $W_{\Phi,\mathfrak{p}}$ called the *canonical lift.* Let $(A_0^{(k)}, A^{(k)})$ be the reduction of the canonical lift to the quotient $W_{\Phi,\mathfrak{p}}/\mathfrak{m}^k$, where $\mathfrak{m}$ is the maximal ideal of $W_{\Phi,\mathfrak{p}}$. The length of the local ring of $\mathcal{Z}_\Phi(\alpha)$ at z is then equal to the largest k such that f lifts to a map $A_0^{(k)} \to A^{(k)}$. In other words, the $\mathcal{O}_K$-module $L(A_0, A)$ has a filtration

$$\cdots \subset L^{(3)} \subset L^{(2)} \subset L^{(1)} = L(A_0, A)$$

in which

$$L^{(k)} = \mathrm{Hom}_{\mathcal{O}_{K_0}}(A_0^{(k)}, A^{(k)}),$$

and the problem is to compute the largest k such that $f \in L^{(k)}$. We show that this k is

$$k = \frac{1}{2} \cdot e_{\mathfrak{p}} \cdot \mathrm{ord}_{\mathfrak{p}_F}(\alpha \mathfrak{p}_F \mathfrak{d}_F),$$

where $\mathfrak{p}_F$ is the pullback of $\mathfrak{p}$ under the map $\varphi^{\mathrm{sp}} : F \to K_\Phi$, and $e_{\mathfrak{p}}$ is the ramification degree of $\mathfrak{p}_F$ in K. All of this is done in Section 2.3, using the Grothendieck-Messing deformation theory of p-divisible groups, with the final application to lengths of local rings appearing as Theorem 3.6.2. In the case $n = 1$, so that A_0 and A are supersingular elliptic curves, all of these calculations reduce to calculations of Gross, as explained at the end of Section 2.3.

In Section 4 we construct a Hilbert modular Eisenstein series $\mathcal{E}_\Phi(\tau, s)$ of parallel weight one. The Eisenstein series $\mathcal{E}_\Phi(\tau, s)$ satisfies a function equation in $s \mapsto -s$ that forces $\mathcal{E}_\Phi(\tau, 0) = 0$, and the central derivative has a Fourier expansion

$$\mathcal{E}'_\Phi(\tau, 0) = \sum_{\alpha \in F} b_\Phi(\alpha, y) \cdot q^\alpha,$$

where $\tau = x + iy \in \mathcal{H}^n$ lies in the product of n upper half planes. The Fourier coefficients $b_\Phi(\alpha, y)$ can easily be computed using explicit formulas of Yang [45], and the result is stated as Corollary 4.2.2. Comparison with Theorem A shows that, for $\alpha \in F^{\gg 0}$,

$$\widehat{\deg}\, \mathcal{Z}_\Phi(\alpha) = -\frac{h(K_0)}{w(K_0)} \cdot \frac{\sqrt{\mathrm{N}(d_{K/F})}}{2^{r-1}[K^{\mathrm{sp}} : \mathbb{Q}]} \cdot b_\Phi(\alpha, y), \tag{1.2.5}$$

where $d_{K/F}$ is the relative discriminant of K/F, and r is the number of places of F ramified in K, including the archimedean places. In particular, the right-hand side is independent of y.

Of course the right-hand side of (1.2.5) makes sense for all $\alpha \in F$, while at the moment the left-hand side is only defined for $\alpha \gg 0$. To remedy this asymmetry we introduce in Section 3.7 the Gillet-Soulé *arithmetic Chow group* $\widehat{\mathrm{CH}}^1(\mathcal{X}_\Phi)$ of the 1-dimensional stack $\mathcal{X}_\Phi$. Elements of the arithmetic Chow group are rational equivalence classes of pairs $(\mathtt{Z}, \mathtt{Gr})$, where $\mathtt{Z}$ is a 0-cycle on $\mathcal{X}_\Phi$ with rational coefficients, and $\mathtt{Gr}$ is a Green function for $\mathtt{Z}$. As $\mathtt{Z}$ has no points in characteristic 0, $\mathtt{Gr}$ is just a function on the finite set of complex points of $\mathcal{X}_\Phi$. In Section 3.7 we construct a divisor class

$$\widehat{\mathtt{Z}}_\Phi(\alpha, y) \in \widehat{\mathrm{CH}}^1(\mathcal{X}_\Phi)$$

for every $\alpha \in F^\times$ and every $y \in F_{\mathbb{R}}^{\gg 0}$. If $\alpha \gg 0$, then this class is $(\mathtt{Z}_\Phi(\alpha), 0)$, where the 0-cycle $\mathtt{Z}_\Phi(\alpha)$ is the image of the map $\mathcal{Z}_\Phi(\alpha) \to \mathcal{X}_\Phi$, with points counted with appropriate multiplicities. If $\alpha \not\gg 0$, then our divisor class has the form $(0, \mathtt{Gr}_\Phi(\alpha, y, \cdot))$ for a particular function $\mathtt{Gr}_\Phi(\alpha, y, \cdot)$ on the complex points of $\mathcal{X}_\Phi$. There is a canonical *arithmetic degree*

$$\widehat{\deg} : \widehat{\mathrm{CH}}^1(\mathcal{X}_\Phi) \to \mathbb{R}, \tag{1.2.6}$$

and (1.2.5) has the following generalization, which appears in the text as Theorem 4.2.3.

THEOREM B. *Assume the discriminants of $K_0/\mathbb{Q}$ and $F/\mathbb{Q}$ are odd and relatively prime. If $\alpha \in F^\times$ and $y \in F_{\mathbb{R}}^{\gg 0}$, then*

$$\widehat{\deg}\, \widehat{\mathtt{Z}}_\Phi(\alpha, y) = -\frac{h(K_0)}{w(K_0)} \cdot \frac{\sqrt{\mathrm{N}(d_{K/F})}}{2^{r-1}[K^{\mathrm{sp}} : \mathbb{Q}]} \cdot b_\Phi(\alpha, y).$$

We now return to our original motivating problem: the calculation of the intersection multiplicity of $\mathcal{X}_\Phi$ and $\mathcal{Z}(m)$ on $\mathcal{M}$. Assume that $m \neq 0$ and that F is a field. This guarantees that $\mathcal{X}_\Phi \cap \mathcal{Z}(m)$ is 0 dimensional. The intersection multiplicity $I(\mathcal{X}_\Phi : \mathcal{Z}(m))$ is defined as the Arakelov degree of the 0-dimensional stack $\mathcal{X}_\Phi \cap \mathcal{Z}(m)$. It is a more or less formal consequence of (1.2.2), see Theorem 3.8.4, that

$$I(\mathcal{X}_\Phi : \mathcal{Z}(m)) = \sum_{\substack{\alpha \in F^\times, \alpha \gg 0 \\ \mathrm{Tr}_{F/\mathbb{Q}}(\alpha) = m}} \widehat{\deg}\, \widehat{\mathtt{Z}}_\Phi(\alpha, y) \tag{1.2.7}$$

for all $y \in \mathbb{R}^{>0}$. In Section 3.8 we define a Green function $\mathtt{Gr}(m, y, \cdot)$ for the Kudla-Rapoport divisor $\mathcal{Z}(m)$. It is a smooth function on $\mathcal{M}(\mathbb{C})$, except for a logarithmic singularity along the divisor $\mathcal{Z}(m)(\mathbb{C})$, and depends on a parameter $y \in \mathbb{R}^{>0}$. If $m < 0$, then $\mathcal{Z}(m) = \emptyset$, and $\mathtt{Gr}(m, y, \cdot)$ is simply a smooth function on $\mathcal{M}(\mathbb{C})$. This function may be evaluated at the finite set of complex points of $\mathcal{X}_\Phi$, and the result, Theorem 3.8.6, is

$$\mathtt{Gr}(m, y, \mathcal{X}_\Phi) = \sum_{\substack{\alpha \in F^\times, \alpha \not\gg 0 \\ \mathrm{Tr}_{F/\mathbb{Q}}(\alpha) = m}} \widehat{\deg}\, \widehat{\mathtt{Z}}_\Phi(\alpha, y). \tag{1.2.8}$$

Let $i_F : \mathcal{H} \to \mathcal{H}^n$ be the diagonal embedding of the upper half plane. The restriction $\mathcal{E}_\Phi(i_F(\tau), s)$ of $\mathcal{E}_\Phi(\tau, s)$ to $\mathcal{H}$ vanishes at $s = 0$, and the derivative has a Fourier expansion

$$\mathcal{E}'_\Phi(i_F(\tau), 0) = \sum_{m \in \mathbb{Z}} c_\Phi(m, y) \cdot q^m,$$

where now $\tau = x + iy \in \mathcal{H}$, and

$$c_\Phi(m, y) = \sum_{\substack{\alpha \in F^\times \\ \mathrm{Tr}_{F/\mathbb{Q}}(\alpha) = m}} b_\Phi(m, y).$$

Combining this with Theorem B and the decompositions (1.2.7) and (1.2.8) gives an arithmetic interpretation of the Fourier coefficients $c_\Phi(m, y)$.

Theorem C. *Assume the discriminants of $K_0/\mathbb{Q}$ and $F/\mathbb{Q}$ are odd and relatively prime. If F is a field and m is nonzero, then*

$$I(\mathcal{X}_\Phi : \mathcal{Z}(m)) + \mathtt{Gr}(m, y, \mathcal{X}_\Phi) = -\frac{h(K_0)}{w(K_0)} \cdot \frac{\sqrt{\mathrm{N}(d_{K/F})}}{2^{r-1}[K:\mathbb{Q}]} \cdot c_\Phi(m, y)$$

for all $y \in \mathbb{R}^{>0}$.

When $n = 1$ or 2, our results have precedents in the literature, albeit in very different language. The case $n = 1$ is essentially treated by Kudla-Rapoport-Yang in [27]. In this case $F = \mathbb{Q}$, $\mathcal{Z}(m)$ is a divisor on the 1-dimensional stack

$$\mathcal{M} = \mathcal{M}_{(1,0)} \times_{\mathcal{O}_{K_0}} \mathcal{M}_{(0,1)},$$

and $\mathcal{X}_\Phi = \mathcal{M}$. Is this degenerate case the intersection $I(\mathcal{X}_\Phi : \mathcal{Z}(m))$ is simply the Arakelov degree of the 0-cycle $\mathcal{Z}(m)$, which is not quite what is computed in [27]. For every triple $(A_0, \kappa_0, \lambda_0)$ in $\mathcal{M}_{(1,0)}$, there is a conjugate triple $(A_0, \overline{\kappa}_0, \lambda_0)$, where $\overline{\kappa}_0(x) = \kappa_0(\overline{x})$. The functor taking a triple to its conjugate defines an isomorphism $\mathcal{M}_{(1,0)} \to \mathcal{M}_{(0,1)}$, which allows us to define the substack $\mathcal{M}^\Delta \to \mathcal{M}$ as the image of $\mathcal{M}_{(1,0)}$ under the diagonal embedding. The intersection $\mathcal{Z}^\Delta(m) = \mathcal{Z}(m) \cap \mathcal{M}^\Delta$ is then the moduli space of triples (E, κ, f) where E is an elliptic curve, $\kappa : \mathcal{O}_{K_0} \to \mathrm{End}(E)$ is an action of $\mathcal{O}_{K_0}$ (suitable normalized), and $f \in \mathrm{End}(E)$ is a degree m isogeny satisfying $\kappa(x) \circ f = f \circ \kappa(\overline{x})$ for all $x \in \mathcal{O}_{K_0}$. It is the Arakelov degree of $\mathcal{Z}^\Delta(m)$ that is computed in [27] and is shown to agree with the Fourier coefficients of the central derivative of a weight one Eisenstein series.

When $n = 2$, so that F is either $\mathbb{Q} \times \mathbb{Q}$ or a real quadratic field, our results are closely related to the work of Gross-Zagier on prime factorizations of singular moduli [14], and heights of Heegner points [15]. In this case the moduli space $\mathcal{M}$ is a union of Shimura varieties attached to groups of type $\mathrm{GU}(1,1)$. Such Shimura varieties are, roughly, unions of Shimura curves parametrizing abelian surfaces with quaternionic multiplication, including the classical modular curves. This is worked out in detail in [26, §14]. The author has not worked out carefully the translation of the results of this paper into the language of moduli of elliptic curves, but the picture should look roughly like this. Both cycles $\mathcal{X}_\Phi$ and $\mathcal{Z}(m)$ are divisors on $\mathcal{M}$ representing points with complex multiplication, i.e., Heegner points. In the case where F is a real quadratic field, the compositum $K = K_0 \cdot F$ contains another quadratic imaginary field K_1, and $\mathcal{X}_\Phi$ is the divisor formed by elliptic curves with complex multiplication by $\mathcal{O}_{K_1}$. The divisor $\mathcal{Z}(1)$ is formed by elliptic curves with complex multiplication by $\mathcal{O}_{K_0}$, and $\mathcal{Z}(m)$ is the translate of $\mathcal{Z}(1)$ by the m^{th} Hecke correspondence. The calculation of the intersection multiplicity $I(\mathcal{X}_\Phi, \mathcal{Z}(m))$, which amounts to computing congruences between values of the j-function at CM points, and the observation that these intersection multiplicities appear as the Fourier coefficients of the diagonal restriction of a Hilbert modular Eisenstein series, is the content of [14]. See also [17].

If $n = 2$ and $F = \mathbb{Q} \times \mathbb{Q}$, then our calculations should be closely related to the more famous result of Gross-Zagier [15]. In this case the calculation of $I(\mathcal{X}_\Phi, \mathcal{Z}(m))$ amounts to the calculation of the intersection multiplicity, on the modular curve, of the divisor of elliptic curves with complex multiplication by $\mathcal{O}_{K_0}$ with the same divisor translated by the m^{th} Hecke correspondence. This is the key calculation performed in [15], although those authors deal with instances of improper intersection (and other serious complications), which we have avoided. Our results assert that these intersections agree with the Fourier coefficients of the central derivative of the diagonal pullback of an Eisenstein series on $\mathrm{GL}_2 \times \mathrm{GL}_2$; that is to say, the derivative of a *product* of two weight one Eisenstein series on $\mathcal{H}$, say $E_1(\tau, s)E_2(\tau, s)$. On the other hand, the results of Gross-Zagier assert that these same intersection multiplicities are the Fourier coefficients of the product of the central derivative of an Eisenstein series with a weight one theta series. One of our Eisenstein series, say $E_1(\tau, s)$, vanishes at $s = 0$, while the other does not, and so the central derivative of the product is $E_1'(\tau, 0) \cdot E_2(\tau, 0)$. But the Siegel-Weil formula then asserts that the central value $E_2(\tau, 0)$ is actually a weight one theta series, so our results are compatible with those of [15].

1.3. *Speculation.* We would like to interpret Theorem C in terms of the arithmetic intersection theory of Gillet-Soulé [10], [11], [40]. Let $\widehat{\mathrm{CH}}^1(\mathcal{M})$ be the codimension one arithmetic Chow group, so that $\mathcal{Z}(m)$ (now viewed as a divisor on $\mathcal{M}$), with its Green function $\mathtt{Gr}(m, y, \cdot)$, defines a class

$$\widehat{\mathtt{Z}}(m, y) \in \widehat{\mathrm{CH}}^1(\mathcal{M}) \tag{1.3.1}$$

for every $m \neq 0$ and $y \in \mathbb{R}^+$. The composition of pullback by $\mathcal{X}_\Phi \to \mathcal{M}_{/\mathcal{O}_\Phi}$ with (1.2.6) defines a linear functional

$$\widehat{\mathrm{CH}}^1(\mathcal{M}_{/\mathcal{O}_\Phi}) \to \widehat{\mathrm{CH}}^1(\mathcal{X}_\Phi) \to \mathbb{R}.$$

Composing with base change from $\mathcal{O}_{K_0}$ to $\mathcal{O}_\Phi$, we obtain a linear functional

$$\widehat{\deg}_{\mathcal{X}_\Phi} : \widehat{\mathrm{CH}}^1(\mathcal{M}) \to \mathbb{R},$$

called the *arithmetic degree along* $\mathcal{X}_\Phi$. What our Theorem C essentially shows is that (ignoring the uninteresting constants appearing in the theorem)

$$\widehat{\deg}_{\mathcal{X}_\Phi} \widehat{\mathtt{Z}}(m, y) = c_\Phi(m, y). \tag{1.3.2}$$

There are several gaps in the above interpretation of Theorem C: to have a good theory of arithmetic Chow groups one needs to work on a stack that is flat, regular, and proper. The stack $\mathcal{M}$ has none of these properties. The stack $\mathcal{M}$ is only flat and regular after inverting $\mathrm{disc}(K_0)$, but Pappas [37] and Krämer [19] have modified the moduli problem defining $\mathcal{M}$ in order to obtain a flat and regular moduli stack that agrees with $\mathcal{M}$ over $\mathcal{O}_{K_0}[\mathrm{disc}(K_0)^{-1}]$. As for properness, the theory of toroidal compactifications of the complex fiber of $\mathcal{M}$ is well understood [1], [6], [35], and Lan's thesis [30] gives a complete theory of the arithmetic toroidal compactifications of $\mathcal{M}$ over $\mathcal{O}_{K_0}[\mathrm{disc}(K_0)^{-1}]$. See also [31] for the signature $(2,1)$ case. It seems likely that the results of Lan's thesis can be extended to give a compactification of the integral model of Pappas and Krämer, but the details have not been written down. In any case, let us suppose that we have replaced $\mathcal{M}$ by a stack that is flat, regular, and proper.

In order to define the class $\widehat{\mathtt{Z}}(m,y)$ for $m \neq 0$, one needs to understand the behavior of the Green function $\mathtt{Gr}(m,y,\cdot)$ near the boundary components of the newly compactified $\mathcal{M}$. Some preliminary calculations suggest that if one adds a particular linear combination of boundary components to the divisor $\mathcal{Z}(m)$, the function $\mathtt{Gr}(m,y,\cdot)$ becomes a Green function for the modified divisor, but with log-log singularities along the boundary. Thus one expects to obtain a class in the generalized arithmetic Chow group of Burgos-Gil–Kramer–Kühn [4], [5]. Assume this is the case, so that (1.3.1) is defined for all $m \neq 0$.

The next step is to define the class $\widehat{\mathtt{Z}}(0,y)$. The definition of the stack $\mathcal{Z}(m)$ makes sense when $m = 0$, but the map $\mathcal{Z}(0) \to \mathcal{M}$ is surjective, and so this is clearly not the way to proceed. To find the correct definition of $\widehat{\mathtt{Z}}(0,y)$, one should interpret $\widehat{\mathrm{CH}}^1(\mathcal{M})$ as the group of isomorphism classes of metrized line bundles on $\mathcal{M}$. Based on work of Kudla-Rapoport-Yang [28], [29] and conjectures of Kudla [23], the class $\widehat{\mathtt{Z}}(0,y)$ should be defined as the Hodge bundle on $\mathcal{M}$, endowed with a particular choice of metric (which will depend on the parameter $y \in \mathbb{R}^{>0}$).

One should also seek a natural definition of

$$\widehat{\mathtt{Z}}_\Phi(\alpha,y) \in \widehat{\mathrm{CH}}^1(\mathcal{X}_\Phi)$$

for all $\alpha \in F$, not just for $\alpha \notin F^\times$, for which Theorem B continues to hold. There are two cases, depending on whether or not $\varphi^{\mathrm{sp}}(\alpha) = 0$. If $\varphi^{\mathrm{sp}}(\alpha) \neq 0$, then Theorem 3.6.2 shows that the stack $\mathcal{Z}_\Phi(\alpha)$ has dimension zero; if $\varphi^{\mathrm{sp}}(\alpha) = 0$, then the proof of that same theorem shows that every irreducible component of $\mathcal{Z}_\Phi(\alpha)$ has dimension 1. The upshot is that if $\varphi^{\mathrm{sp}}(\alpha) \neq 0$, the definition of $\widehat{\mathtt{Z}}_\Phi(\alpha,y)$ should be close to the definition we have given for $\alpha \in F^\times$. If $\varphi^{\mathrm{sp}}(\alpha) = 0$, then the correct definition should be in terms of the metrized Hodge bundle in $\widehat{\mathrm{CH}}^1(\mathcal{X}_\Phi)$. These classes should satisfy two properties. First, the pullback map

$$\widehat{\mathrm{CH}}^1(\mathcal{M}_{/\mathcal{O}_\Phi}) \to \widehat{\mathrm{CH}}^1(\mathcal{X}_\Phi)$$

should satisfy

$$\widehat{\mathsf{Z}}(m,y) \mapsto \sum_{\substack{\alpha\in F \\ \mathrm{Tr}_{F/\mathbb{Q}}(\alpha)=m}} \widehat{\mathsf{Z}}_\Phi(\alpha,y)$$

for all m and all $y \in \mathbb{R}^{>0}$. Second, the relation

$$\widehat{\deg}\, \widehat{\mathsf{Z}}_\Phi(\alpha,y) = b_\Phi(\alpha,y)$$

should hold for all $\alpha \in F$ and $y \in F_{\mathbb{R}}^{\gg 0}$. Given these two properties, one can deduce (1.3.2) for all $m \in \mathbb{Z}$ and all $y \in \mathbb{R}^{>0}$.

The final step in the program laid out by Kudla [23] is to form the vector-valued generating series

$$\widehat{\theta}(\tau) = \sum_{m\in\mathbb{Z}} \widehat{\mathsf{Z}}(m,y)\cdot q^m \in \widehat{\mathrm{CH}}^1(\mathcal{M})[[q]]$$

for $\tau = x+iy \in \mathcal{H}$. The equality (1.3.2) amounts to

$$\widehat{\deg}_{\mathcal{X}_\Phi}\widehat{\theta}(\tau) = \mathcal{E}'_\Phi(i_F(\tau),0).$$

Given the results of Kudla-Rapoport-Yang [29] on CM cycles on Shimura curves, and the results of Bruinier–Burgos-Gil–Kühn [4] on Hirzebruch-Zagier divisors on Hilbert modular surfaces, it is reasonable to expect that the above generating series is a vector-valued nonholomorphic modular form of weight n. If f is a weight n cuspform on $\mathcal{H}$, we may therefore form the Petersson inner product of $f(\tau)$ with $\widehat{\theta}(\tau)$ and so define the *arithmetic theta lift*

$$\widehat{\theta}_f = \langle f, \widehat{\theta}\rangle^{\mathrm{Pet}} \in \widehat{\mathrm{CH}}^1(\mathcal{M}).$$

Moving the linear functional $\widehat{\deg}_{\mathcal{X}_\Phi}$ inside the integral defining the Petersson inner product, one finds the Gross-Zagier style formula

$$\begin{aligned} \widehat{\deg}_{\mathcal{X}_\Phi}\widehat{\theta}_f &= \langle f, \widehat{\deg}_{\mathcal{X}_\Phi}\widehat{\theta}\rangle^{\mathrm{Pet}} \\ &= \langle f(\tau), \mathcal{E}'_\Phi(i_F(\tau),0)\rangle^{\mathrm{Pet}} \\ &= \mathcal{L}'_\Phi(f,0), \end{aligned} \tag{1.3.3}$$

where

$$\mathcal{L}_\Phi(f,s) = \langle f(\tau), \mathcal{E}_\Phi(i_F(\tau),s)\rangle^{\mathrm{Pet}}.$$

In the case where F is a real quadratic field (so $n=2$), a function very much like $\mathcal{L}_\Phi(f,s)$ appears in the work of Gross-Kohnen-Zagier [13], and is shown to be closely related to the usual L-function of f. When F is a field of degree >2, there seems to be no literature at all on the function $\mathcal{L}_\Phi(f,s)$, and the author is at a loss as to its properties and significance.

However, there are interesting cases where $n>2$, and one has some hope of better understanding $\mathcal{L}_\Phi(f,s)$. For example, consider the totally degenerate case of $F = \mathbb{Q}\times\cdots\times\mathbb{Q}$ and $K = K_0\times\cdots\times K_0$. Modulo details, one should expect the following. Our Hilbert modular Eisenstein series on $\mathcal{H}^n$ is just a product of classical weight one Eisenstein series

$$\mathcal{E}_\Phi(\tau, s) = \mathcal{F}_1(\tau_1, s) \cdots \mathcal{F}_n(\tau_n, s).$$

Each factor will satisfy a functional equation in $s \to -s$, and the sign of the functional equation will be 1 for all factors but one. Say the last factor has sign -1. The first $n-1$ factors are *coherent*, the last one is the *incoherent* factor. The Siegel-Weil formula implies that the value at $s = 0$ of each coherent factor is a theta function Θ attached to the extension $K_0/\mathbb{Q}$, and so

$$\mathcal{E}'_\Phi(i_F(\tau), 0) = \Theta^{n-1}(\tau)\mathcal{F}'_n(\tau, 0).$$

But the Petersson inner product

$$L(f \times \Theta^{n-1}, s) = \langle f(\tau), \Theta^{n-1}(\tau)\mathcal{F}_n(\tau, s)\rangle^{\mathrm{Pet}}$$

is, up to rescaling and shifting in the variable s, just the Rankin-Selberg convolution L-function of f with Θ^{n-1}, and hence the mysterious function $\mathcal{L}_\Phi(f, s)$ has the less mysterious central derivative

$$\mathcal{L}'_\Phi(f, 0) = L'(f \times \Theta^{n-1}, 0).$$

When $n = 2$, so that $F = \mathbb{Q} \times \mathbb{Q}$, the Rankin-Selberg L-function on the right is the one appearing in the work of Gross-Zagier [15], as we have noted earlier.

Finally, it may be helpful to put the above results and conjectures into the context of seesaw dual pairs and the Siegel-Weil formula, which are among the guiding principles of Kudla's conjectures [23]. Suppose we start with free K-module W of rank 1, equipped with a totally positive definite Hermitian form $\langle\cdot,\cdot\rangle_{\mathrm{CM}}$. Let V denote the underlying K_0-vector space with the K_0-Hermitian form $\langle v_1, v_2\rangle = \mathrm{Tr}_{K/K_0}\langle w_1, w_2\rangle_{\mathrm{CM}}$. Define a torus $T = \mathrm{Res}_{F/\mathbb{Q}}U(W)$ so that $T \subset U(V)$. The dual reductive pairs $(\mathrm{SL}_2, U(V))$ and $(\mathrm{Res}_{F/\mathbb{Q}}\mathrm{SL}_2, T)$ can be arranged into the *seesaw diagram*

$$\begin{tikzcd} \mathrm{Res}_{F/\mathbb{Q}}\mathrm{SL}_2 \arrow[d, dash] \arrow[dr, dash] & U(V) \arrow[d, dash] \arrow[dl, dash] \\ \mathrm{SL}_2 & T. \end{tikzcd}$$

Starting with a cusp form f on SL_2, one can theta lift to an automorphic form θ_f on $U(V)$, then restrict to T and integrate against the constant function 1. By tipping the seesaw, this is the same as theta lifting the constant function 1 on T to a Hilbert modular theta series on $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{SL}_2$, restricting that theta series to the diagonally embedded SL_2, and integrating against f. The Siegel-Weil formula implies that the Hilbert modular theta series appearing in this process is in fact the central value of a Hilbert modular Eisenstein series, say $E(g,s)$, at $s=0$. Thus

$$\int_{T(\mathbb{A})} \theta_f(t)\,dt = \int_{\mathrm{SL}_2(\mathbb{A})} f(g)E(g,0)\,dg. \tag{1.3.4}$$

The conjectural picture described (and largely proved) above, is formally similar. Automorphic forms on $U(V)$ are replaced by elements of $\widehat{\mathrm{CH}}^1(\mathcal{M})$, the theta lift $f \mapsto \theta_f$ is replaced by the arithmetic theta lift $f \mapsto \widehat{\theta}_f$, and the linear functional "integrate over the torus T" is replaced by the linear functional "arithmetic degree along $\mathcal{X}_\Phi$." On the other side of the seesaw, the Hilbert modular theta series (which is the central *value* of an Eisenstein series) is replaced by the central *derivative* of an Eisenstein series, and the integral over SL_2 is replaced by the Petersson inner product. In this way, (1.3.3) can be seen as an arithmetic version of (1.3.4).

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2. **Barsotti-Tate groups with complex multiplication**

This section contains the technical deformation theory calculations that will eventually be used in the proof of Theorem 3.6.2 to compute the lengths of the local rings of the 0-dimensional stack $\mathcal{Z}_\Phi(\alpha)$. The reader might prefer to begin with the global theory of Section 3, and refer back to this section as needed.

Fix a prime p, and let $\mathbb{F}$ be an algebraic closure of the field of p elements. Let W be the ring of Witt vectors of $\mathbb{F}$, let $\mathrm{Frac}(W)$ be the fraction field of W, and let $\mathbb{C}_p$ be any algebraically closed field containing $\mathrm{Frac}(W)$. If L is a product of finite extensions of $\mathbb{Q}_p$, denote by L^u the maximal unramified extension of $\mathbb{Q}_p$ in L and by $\mathcal{O}_L^u$ the ring of integers of L^u. A p-divisible group over $\mathbb{F}$ is *supersingular* if all slopes of its Dieudonné module are equal to $1/2$. Here and throughout, *Dieudonné module* means *covariant Dieudonné module.*

2.1. *Deformations of Barsotti-Tate groups with complex multiplication.* Let F be a field extension of $\mathbb{Q}_p$ of degree n, and let K be a quadratic étale F-algebra (so K is a either a quadratic field extension of F, or $K \cong F \times F$). Denote by $x \mapsto \overline{x}$ the nontrivial automorphism of K/F, and for any $\mathbb{Q}_p$-algebra map $\varphi : K \to \mathbb{C}_p$, define the conjugate map $\overline{\varphi}(x) = \varphi(\overline{x})$. A *p-adic CM type* of K is a set Φ of $\mathbb{Q}_p$-algebra maps $K \to \mathbb{C}_p$ such that $\mathrm{Hom}(K, \mathbb{C}_p)$ is the disjoint union of Φ and $\overline{\Phi}$. Fix such a Φ, and let $K_\Phi \subset \mathbb{C}_p$ be any finite extension of $\mathbb{Q}_p$ large enough that

$$\sigma \in \mathrm{Aut}(\mathbb{C}_p/K_\Phi) \implies \Phi^\sigma = \Phi. \tag{2.1.1}$$

Denote by

$$W_\Phi \subset \mathbb{C}_p$$

the ring of integers in the completion of the maximal unramified extension of K_Φ, so that $\mathrm{Frac}(W_\Phi)$ is the compositum of K_Φ and $\mathrm{Frac}(W)$.

Let **ART** be the category of local Artinian W_Φ-algebras with residue field $\mathbb{F}$. If R is an object of **ART** and A is a p-divisible group over R, an action $\kappa : \mathcal{O}_K \to \mathrm{End}(A)$ satisfies the Φ-*determinant condition* if for every $x_1, \ldots, x_r \in \mathcal{O}_K$, the determinant of $T_1x_1 + \cdots + T_rx_r$ acting on $\mathrm{Lie}(A)$ is equal to the image of

$$\prod_{\varphi \in \Phi} (T_1\varphi(x_1) + \cdots + T_r\varphi(x_r)) \in W_\Phi[T_1, \ldots, T_r] \tag{2.1.2}$$

in $R[T_1, \ldots, T_r]$. In particular, this implies

$$\dim(A) = [F : \mathbb{Q}_p].$$

For the remainder of this subsection, fix a triple (A, κ, λ) in which

- A is a p-divisible group over $\mathbb{F}$.
- $\kappa : \mathcal{O}_K \to \mathrm{End}(A)$ satisfies the Φ-determinant condition.
- $\lambda : A \to A^\vee$ is an $\mathcal{O}_K$-linear polarization of A (which is not assumed to be principal). The condition of $\mathcal{O}_K$-linearity means that

$$\lambda \circ \kappa(\overline{x}) = \kappa(x)^\vee \circ \lambda$$

  for every $x \in \mathcal{O}_K$.

Let $\mathrm{Def}_\Phi(A, \kappa, \lambda)$ be the functor that assigns to every object R of **ART** the set of isomorphism classes of deformations of (A, κ, λ) to R, where the deformations are again required to satisfy the Φ-determinant condition. The goal of this subsection is to prove that $\mathrm{Def}_\Phi(A, \kappa, \lambda)$ is pro-represented by W_Φ.

PROPOSITION 2.1.1. *Let (A, κ, λ) be the triple fixed above.*

(1) *The Dieudonné module $D(A)$ is free of rank one over $\mathcal{O}_K \otimes_{\mathbb{Z}_p} W$.*
(2) *The image of $\mathcal{O}_K$ in $\mathrm{End}(A)$ is equal to its own centralizer.*
(3) *If K is a field, then A is supersingular.*

Proof. The first claim follows from the argument of [38, Lemma 1.3], and the second claim follows easily from the first. The category of Dieudonné modules over $\mathbb{F}$ up to isogeny is semisimple. If

$$D(A) \sim D_1^{m_1} \times \cdots \times D_r^{m_r}$$

with $D_1, \ldots, D_r$ simple and pairwise nonisogenous, then

$$\mathrm{End}(D(A)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong M_{m_1}(H_1) \times \cdots \times M_{m_r}(H_r)$$

with each H_i a division algebra over $\mathbb{Q}_p$. The only way this product can contain a field equal to its own centralizer is if $r = 1$. Therefore, $D(A)$ is isoclinic:

it is isogenous to a power of a simple Dieudonné module, and hence its slope sequence is constant. By hypothesis, $D(A)$ admits a polarization, so its slope sequence is symmetric in $s \mapsto 1 - s$. Therefore, $1/2$ is the unique slope of $D(A)$. $\square$

Given a $\mathbb{Q}_p$-algebra map $\varphi : K \to \mathbb{C}_p$, let $\mathbb{C}_p(\varphi)$ denote $\mathbb{C}_p$, with K acting through φ. There is a unique W_Φ-algebra map

$$\eta_\Phi : \mathcal{O}_K \otimes_{\mathbb{Z}_p} W_\Phi \to \prod_{\varphi \in \Phi} \mathbb{C}_p(\varphi)$$

sending $x \otimes 1$ to the Φ-tuple $(\varphi(x))_{\varphi \in \Phi}$. The kernel and image of η_Φ are denoted J_Φ and Lie_Φ, respectively, so that there is an exact sequence of $\mathcal{O}_K \otimes_{\mathbb{Z}_p} W_\Phi$-modules

$$0 \to J_\Phi \to \mathcal{O}_K \otimes_{\mathbb{Z}_p} W_\Phi \to \mathrm{Lie}_\Phi \to 0. \tag{2.1.3}$$

We will make repeated use of the isomorphism

$$\mathcal{O}_K^u \otimes_{\mathbb{Z}_p} W \cong \prod_{\psi : \mathcal{O}_K^u \to W} W$$

sending $x \otimes 1 \mapsto (\psi(x))_\psi$. For each factor on the right-hand side there is a corresponding idempotent $e_\psi \in \mathcal{O}_K^u \otimes_{\mathbb{Z}_p} W$ characterized by

$$(x \otimes 1) e_\psi = (1 \otimes \psi(x)) e_\psi$$

for all $x \in \mathcal{O}_K^u$.

LEMMA 2.1.2. *The ideal $J_\Phi \subset \mathcal{O}_K \otimes_{\mathbb{Z}_p} W_\Phi$ enjoys the following properties.*

(a) *As W_Φ-modules, J_Φ and Lie_Φ are each free of rank n. Furthermore, for any tuple $x_1, \ldots, x_r \in \mathcal{O}_K$, the determinant of $T_1 x_1 + \cdots + T_r x_r$ acting on Lie_Φ is equal to* (2.1.2).

(b) *The ideal J_Φ is generated by the set of all elements of the form*

$$j_\Phi(x, \psi) = e_\psi \prod_{\substack{\varphi \in \Phi \\ \varphi|_{\mathcal{O}_K^u} = \psi}} (x \otimes 1 - 1 \otimes \varphi(x)) \in \mathcal{O}_K \otimes_{\mathbb{Z}_p} W_\Phi$$

with $x \in \mathcal{O}_K$ and $\psi : \mathcal{O}_K^u \to W$.

(c) *Suppose R is an object of* **ART**, *M is a free $\mathcal{O}_K \otimes_{\mathbb{Z}_p} R$-module of rank one, and $M_1 \subset M$ is an $\mathcal{O}_K$-stable R-direct summand such that for any $x_1, \ldots, x_r \in \mathcal{O}_K$, the determinant of $T_1 x_1 + \cdots + T_r x_r$ acting on M/M_1 is* (2.1.2). *Then $M_1 = J_\Phi M$.*

Proof. The first claim is elementary linear algebra, and the proof is left to the reader. For the second claim, $j_\Phi(x, \psi) \in J_\Phi$ is obvious from the definitions. To prove the other inclusion, fix a $\varpi \in \mathcal{O}_K$ such that $\mathcal{O}_K = \mathcal{O}_K^u[\varpi]$, and let

$\mu(z) \in \mathcal{O}_K^u[z]$ be the minimal polynomial of ϖ. Using $\varpi \mapsto z$ to identify $\mathcal{O}_K \cong \mathcal{O}_K^u[z]/(\mu)$, we obtain an isomorphism

$$\mathcal{O}_K \otimes_{\mathbb{Z}_p} W_\Phi \cong (\mathcal{O}_K^u \otimes_{\mathbb{Z}_p} W_\Phi)[z]/(\mu) \cong \prod_{\psi:\mathcal{O}_K^u \to W} W_\Phi[z]/(\mu_\psi),$$

where μ_ψ is the image of μ under $\psi : \mathcal{O}_K^u[z] \to W[z]$. Under these isomorphisms, the element $j_\Phi(\varpi, \psi)$ on the left is identified with the tuple on the right whose ψ-coordinate is the polynomial

$$\prod_{\substack{\varphi \in \Phi \\ \varphi|_{\mathcal{O}_K^u} = \psi}} (z - \varphi(\varpi)) \tag{2.1.4}$$

and all other coordinates are 0. Now suppose $j \in J_\Phi$. Under the above isomorphism, j corresponds to a tuple of polynomials $j_\psi(z) \in W_\Phi[z]/(\mu_\psi)$, and the assumption that $j \in J_\Phi$ means precisely that the polynomial $j_\psi(z)$ vanishes at $z = \varphi(\varpi)$ for each $\varphi \in \Phi$ whose restriction to $\mathcal{O}_K^u$ is ψ. Such a $j_\psi(z)$ is obviously divisible, in $W_\Phi[z]$, by (2.1.4). It follows that $e_\psi j$ is a multiple of $j_\Phi(\varpi, \psi)$ and hence that J_Φ is contained in the ideal generated by the elements $j_\Phi(\varpi, \psi)$ as ψ varies.

For the final claim, extend each $\varphi \in \Phi$ to a W-linear map $\varphi : \mathcal{O}_K \otimes_{\mathbb{Z}_p} W \to \mathbb{C}_p$. The determinant condition imposed on M/M_1 implies that for every $x \in \mathcal{O}_K$,

$$e_\psi(x \otimes 1) \in \mathcal{O}_K \otimes_{\mathbb{Z}_p} W$$

acts on M/M_1 with characteristic polynomial

$$\prod_{\varphi \in \Phi} (T - \varphi(e_\psi)\varphi(x)) = T^r \prod_{\substack{\varphi \in \Phi \\ \varphi|_{\mathcal{O}_K^u} = \psi}} (T - \varphi(x)) \in W_\Phi[T],$$

where $r = \#\{\varphi \in \Phi : \varphi|_{\mathcal{O}_K^u} \neq \psi\}$, and hence acts on $e_\psi(M/M_1)$ with characteristic polynomial

$$\prod_{\substack{\varphi \in \Phi \\ \varphi|_{\mathcal{O}_K^u} = \psi}} (T - \varphi(x)) \in W_\Phi[T].$$

Therefore, $x \otimes 1$ acts on $e_\psi(M/M_1)$ with this same characteristic polynomial, and the Cayley-Hamilton theorem implies that $e_\psi(M/M_1)$ is annihilated by

$$\prod_{\substack{\varphi \in \Phi \\ \varphi|_{\mathcal{O}_K^u} = \psi}} (x \otimes 1 - 1 \otimes \varphi(x)) \in \mathcal{O}_K \otimes_{\mathbb{Z}_p} W_\Phi.$$

Hence M/M_1 is annihilated by $j_\Phi(x, \psi)$. By the second claim of the lemma, $J_\Phi M \subset M_1$, and as $J_\Phi M$ and M_1 are R-module direct summands of M of the same rank, they must be equal. $\square$

We now make use of the theory of Grothendieck-Messing crystals. Standard references include [2], [16], [32]; for Zink's reconstruction of the theory by different means, see [33], [46]. Associated to A is a short exact sequence

$$0 \to \mathrm{Fil}^1 \mathcal{D}_A(\mathbb{F}) \to \mathcal{D}_A(\mathbb{F}) \to \mathrm{Lie}(A) \to 0 \tag{2.1.5}$$

of $\mathbb{F}$-modules, in which $\mathcal{D}_A(\mathbb{F})$ is the covariant Grothendieck-Messing crystal of A evaluated at $\mathbb{F}$, and the submodule $\mathrm{Fil}^1 \mathcal{D}_A(\mathbb{F})$ is its Hodge filtration. Proposition 2.1.1 and the isomorphisms

$$\mathcal{D}_A(\mathbb{F}) \cong \mathcal{D}_A(W) \otimes_W \mathbb{F} \cong D(A) \otimes_W \mathbb{F},$$

the second by [2, Th. 4.2.14], imply that

$$\mathcal{D}_A(\mathbb{F}) \cong \mathcal{O}_K \otimes_{\mathbb{Z}_p} \mathbb{F},$$

and the final claim of Lemma 2.1.2 implies that

$$\mathrm{Fil}^1 \mathcal{D}_A(\mathbb{F}) = J_\Phi \mathcal{D}_A(\mathbb{F}).$$

In particular, (2.1.5) is obtained from (2.1.3) by applying $\otimes_{W_\Phi} \mathbb{F}$, which explains our choice of notation Lie_Φ. Similarly, if R is an object of **ART** and

$$(A', \kappa', \lambda') \in \mathrm{Def}_\Phi(A, \kappa, \lambda)(R),$$

there is an associated short exact sequence of free R-modules

$$0 \to \mathrm{Fil}^1 \mathcal{D}_{A'}(R) \to \mathcal{D}_{A'}(R) \to \mathrm{Lie}(A') \to 0. \tag{2.1.6}$$

Applying $\otimes_R \mathbb{F}$ to (2.1.6) recovers (2.1.5), and an easy Nakayama's lemma argument then shows that

$$\mathcal{D}_{A'}(R) \cong \mathcal{O}_K \otimes_{\mathbb{Z}_p} R.$$

Another application of Lemma 2.1.2 shows that

$$\mathrm{Fil}^1 \mathcal{D}_{A'}(R) = J_\Phi \mathcal{D}_{A'}(R),$$

and so (2.1.6) is obtained from (2.1.3) by applying $\otimes_{W_\Phi} R$. In this sense, (2.1.3) is the universal Hodge short exact sequence of deformations of (A, κ, λ). As the following theorem demonstrates, this information is enough to deduce the existence and uniqueness of deformations of (A, κ, λ).

THEOREM 2.1.3. *The functor* $\mathrm{Def}_\Phi(A, \kappa, \lambda)$ *is pro-represented by* W_Φ. *Equivalently, the triple* (A, κ, λ) *admits a unique deformation to every object of* **ART**.

Proof. Let $S \to R$ be a surjective morphism in **ART** whose kernel $\mathcal{I}$ satisfies $\mathcal{I}^2 = 0$. In particular, $\mathcal{I}$ comes equipped with its trivial divided power structure. Suppose we have already lifted the triple (A, κ, λ) over $\mathbb{F}$ to a triple

(A', κ', λ') over R. Let $\mathcal{D}_{A'}$ be the Grothendieck-Messing crystal of A', so that $\mathcal{D}_{A'}(R) \cong \mathcal{O}_K \otimes_{\mathbb{Z}_p} R$, and

$$\mathrm{Fil}^1 \mathcal{D}_{A'}(R) = J_\Phi \mathcal{D}_{A'}(R).$$

Now evaluate the crystal $\mathcal{D}_{A'}$ on S. As $\mathcal{D}_{A'}(S) \otimes_S \mathbb{F} \cong \mathcal{D}_A(\mathbb{F})$, a Nakayama's lemma argument shows that $\mathcal{D}_{A'}(S)$ is free of rank one over $\mathcal{O}_K \otimes_{\mathbb{Z}_p} S$. By the main result of Grothendieck-Messing theory, deformations of the pair (A', κ') to S (satisfying the Φ-determinant condition, as always) are in bijection with $\mathcal{O}_K$-stable S-direct summands $M_1 \subset \mathcal{D}_{A'}(S)$ for which the action of $\mathcal{O}_K$ on $\mathcal{D}_{A'}(S)/M_1$ satisfies the Φ-determinant condition. The final claim of Lemma 2.1.2 shows that $M_1 = J_\Phi \mathcal{D}_{A'}(S)$ is the unique such summand, and so (A', κ') admits a unique deformation (A'', κ'') to S.

By the results of [2, Ch. 5.3], the polarization λ' of A' induces an alternating S-bilinear form

$$\lambda' : \mathcal{D}_{A'}(S) \times \mathcal{D}_{A'}(S) \to S$$

satisfying $\lambda(xv, w) = \lambda(v, \overline{x}w)$ for every $x \in \mathcal{O}_K \otimes_{\mathbb{Z}_p} W_\Phi$. But every $x \in J_\Phi$ satisfies

$$x\overline{x} \in \ker(\eta_\Phi) \cap \ker(\eta_{\overline{\Phi}}) = 0,$$

and hence $J_\Phi \mathcal{D}_{A'}(S)$ is isotropic for the pairing λ'. This implies that the polarization λ' lifts (uniquely) to a polarization λ'' of (A'', κ''), and so (A', κ', λ') admits a unique deformation to S. Induction on the length now shows that (A, κ, λ) lifts uniquely to every object of **ART**. □

Remark 2.1.4. The proof of Theorem 2.1.3 actually shows that something slightly stronger is true: the pair (A, κ) deforms uniquely to every object of **ART**, and λ automatically lifts to that deformation.

Remark 2.1.5. Instead of the Φ-determinant condition imposed on the action $\mathcal{O}_K \to \mathrm{End}(A)$ at the beginning of this subsection, we might have imposed the (seemingly) weaker condition that every $x \in \mathcal{O}_K$ acts on $\mathrm{Lie}(A)$ with characteristic polynomial equal to the image of

$$\prod_{\varphi \in \Phi} (T - \varphi(x)) \in W_\Phi[T]$$

in $R[T]$. The advantage of the stronger Φ-determinant condition is that it determines not only the characteristic polynomial of every element of $\mathcal{O}_K$, but of every element of $\mathcal{O}_K \otimes_{\mathbb{Z}_p} R$. This was needed in the proof of part (c) of Lemma 2.1.2. It would be interesting to know whether the two conditions are equivalent.

2.2. *Deformations of* CM *abelian varieties.* Theorem 2.1.3, when combined with the Serre-Tate theorem, gives information about the formal deformation spaces of CM abelian varieties over $\mathbb{F}$, and in much greater generality than is needed in this paper. Because the result, Theorem 2.2.1, is of independent interest, we state it in full generality.

Let $K = \prod K_i$ be any product of CM fields, and let $F = \prod F_i$ be its maximal totally real subalgebra. Let Φ be any CM type of K, and let $K_\Phi \subset \mathbb{C}$ be a number field containing the reflex field of Φ. Set $\mathcal{O}_\Phi = \mathcal{O}_{K_\Phi}$. Suppose $\mathfrak{o} \subset K$ is an order and S is a locally Noetherian $\mathcal{O}_\Phi$-scheme. A *polarized* $(\mathfrak{o}, \Phi)$-*CM abelian scheme over* S is a triple (A, κ, λ) in which

- $A \to S$ is an abelian scheme over S of relative dimension n;
- $\kappa : \mathfrak{o} \to \mathrm{End}(A)$ is an action of $\mathfrak{o}$ on A such that, locally on S, for any tuple $x_1, \ldots, x_r \in \mathfrak{o}$, the determinant of $T_1 x_1 + \cdots + T_r x_r$ on $\mathrm{Lie}(A)$ is equal to the image of
$$\prod_{\varphi \in \Phi} (T_1 \varphi(x_1) + \cdots + T_r \varphi(x_r)) \in \mathcal{O}_\Phi[T_1, \ldots, T_r]$$
in $\mathcal{O}_S[T_1, \ldots, T_r]$;
- $\lambda : A \to A^\vee$ is a polarization of A satisfying $\lambda \circ \kappa(\overline{x}) = \kappa(x)^\vee \circ \lambda$ for every $x \in \mathfrak{o}$.

Given a prime $\mathfrak{p}$ of K_Φ, let $W_{\Phi,\mathfrak{p}}$ be the completion of the ring of integers in the maximal unramified extension of $K_{\Phi,\mathfrak{p}}$, and let $k^{\mathrm{alg}}_{\Phi,\mathfrak{p}}$ be its residue field. Let p be the rational prime below $\mathfrak{p}$.

THEOREM 2.2.1. *Let R be a complete local Noetherian $W_{\Phi,\mathfrak{p}}$-algebra with residue field $k^{\mathrm{alg}}_{\Phi,\mathfrak{p}}$. If $\mathfrak{o}$ is maximal at p, then every polarized $(\mathfrak{o}, \Phi)$-*CM *abelian scheme (A, κ, λ) over $k^{\mathrm{alg}}_{\Phi,\mathfrak{p}}$ admits a unique deformation to a polarized $(\mathfrak{o}, \Phi)$-*CM *abelian scheme over R.*

Proof. Let $\mathbb{C}_\mathfrak{p}$ be an algebraically closed field containing $W_{\Phi,\mathfrak{p}}$, and fix an isomorphism between the algebraic closures of K_Φ in $\mathbb{C}$ and $\mathbb{C}_\mathfrak{p}$. This allows us to view elements of Φ as maps $\varphi : K \to \mathbb{C}_\mathfrak{p}$. For any prime $\mathfrak{P}$ of F above p let $\Phi(\mathfrak{P}) \subset \Phi$ be the subset consisting of those φ whose restriction to F induces the prime $\mathfrak{P}$. There is a decomposition of p-divisible groups
$$A[p^\infty] \cong \prod_{\mathfrak{P}} A[\mathfrak{P}^\infty],$$
where the product is over the primes of F lying above p. A similar decomposition holds for any deformation of the triple (A, κ, λ). Each factor $A[\mathfrak{P}^\infty]$ has an action
$$\kappa[\mathfrak{P}^\infty] : \mathcal{O}_{K,\mathfrak{P}} \to \mathrm{End}(A[\mathfrak{P}^\infty])$$

satisfying the $\Phi(\mathfrak{P})$-determinant condition, and an $\mathcal{O}_{K,\mathfrak{P}}$-linear polarization $\lambda[\mathfrak{P}^\infty]$. If R is Artinian, we may apply Theorem 2.1.3 to see that each triple $(A[\mathfrak{P}^\infty], \kappa[\mathfrak{P}^\infty], \lambda[\mathfrak{P}^\infty])$ admits a unique deformation to R. By the Serre-Tate theorem [32], the same is true of the triple (A, κ, λ). This proves the claim if R is Artinian, and the general case follows from Grothendieck's formal existence theorem as in [7, §3]. ☐

Theorem 2.2.1 is false if one omits the hypothesis that $\mathfrak{o}$ is maximal at p, even for elliptic curves. This is clear from the theory of *quasi-canonical lifts* of elliptic curves, due to Serre-Tate in the ordinary case, and Gross in the supersingular case [12], [34], [44].

2.3. *Lifting homomorphisms*: *the signature* $(n-1,1)$ *case*. In this subsection K_0 is a quadratic field extension of $\mathbb{Q}_p$, $F/\mathbb{Q}_p$ is a field extension of degree n, and

$$K = K_0 \otimes_{\mathbb{Q}_p} F.$$

We assume that K_0 does not embed into F, so that K is a field. Let $\mathfrak{D}_0$ and $\mathfrak{D}$ be the differents of $K_0/\mathbb{Q}_p$ and $K/\mathbb{Q}_p$, respectively, and let $\mathfrak{p}_F$ be the maximal ideal of $\mathcal{O}_F$.

Fix an embedding $\iota : K_0 \to \mathbb{C}_p$, so that $\Phi_0 = \{\iota\}$ is a p-adic CM type of K_0. A p-adic CM type Φ of K is said to be of *signature* $(n-1,1)$ if there is a unique $\varphi^{\mathrm{sp}} \in \Phi$ satisfying $\varphi^{\mathrm{sp}}|_{K_0} = \bar{\iota}$. The distinguished element φ^{sp} is the *special element* of Φ, and this element determines Φ uniquely. Fix a Φ of signature $(n-1,1)$, and define

$$K_\Phi = \varphi^{\mathrm{sp}}(K).$$

For any $\sigma \in \mathrm{Aut}(\mathbb{C}_p/K_\Phi)$, the p-adic CM type Φ^σ is again of signature $(n-1,1)$, and still contains φ^{sp}. Thus $\Phi = \Phi^\sigma$, and condition (2.1.1) is satisfied. Let W_Φ be as in Section 2.1.

Fix a triple (A, κ, λ) in which

- A is a p-divisible group over $\mathbb{F}$ of dimension n,
- $\kappa : \mathcal{O}_K \to \mathrm{End}(A)$ satisfies the Φ-determinant condition,
- $\lambda : A \to A^\vee$ is an $\mathcal{O}_K$-linear polarization with kernel $A[\mathfrak{a}]$ for some ideal $\mathfrak{a} \subset \mathcal{O}_F$.

Fix a second triple $(A_0, \kappa_0, \lambda_0)$ in which

- A_0 is a p-divisible group over $\mathbb{F}$ of dimension 1,
- $\kappa_0 : \mathcal{O}_{K_0} \to \mathrm{End}(A_0)$ satisfies the Φ_0-determinant condition,
- $\lambda_0 : A_0 \to A_0^\vee$ is an $\mathcal{O}_{K_0}$-linear principal polarization.

By Proposition 2.1.1, both A_0 and A are supersingular.

The $\mathcal{O}_K$-module

$$L(A_0, A) = \mathrm{Hom}_{\mathcal{O}_{K_0}}(A_0, A)$$

has a natural $\mathcal{O}_{K_0}$-Hermitian form $\langle f_1, f_2\rangle$ defined by

$$\langle f_1, f_2\rangle = \lambda_0^{-1} \circ f_2^\vee \circ \lambda \circ f_1.$$

This Hermitian form is compatible with the action of $\mathcal{O}_K$ on $L(A_0, A)$, in the sense that

$$\langle x \cdot f_1, f_2\rangle = \langle f_1, \overline{x} \cdot f_2\rangle$$

for every $x \in \mathcal{O}_K$, and it follows that there is a unique K-valued $\mathcal{O}_K$-Hermitian form $\langle f_1, f_2\rangle_{\mathrm{CM}}$ on $L(A_0, A)$ satisfying

$$\langle f_1, f_2\rangle = \mathrm{Tr}_{K/K_0}\langle f_1, f_2\rangle_{\mathrm{CM}}.$$

It is not easy to give a description of $\langle \cdot, \cdot\rangle_{\mathrm{CM}}$, other than "the Hermitian form whose trace is $\langle \cdot, \cdot\rangle$." Nevertheless, the structure of the Hermitian space $\big(L(A_0, A), \langle \cdot, \cdot\rangle_{\mathrm{CM}}\big)$ will be described quite explicitly in Proposition 2.3.3.

Set

$$\mathcal{S} = \mathcal{O}_K \otimes_{\mathbb{Z}_p} W.$$

If $\mathrm{Fr} \in \mathrm{Aut}(W)$ is the Frobenius automorphism, there is an induced automorphism of $\mathcal{S}$ defined by

$$(x \otimes w)^{\mathrm{Fr}} = x \otimes w^{\mathrm{Fr}}.$$

As in Section 2.1, for each $\psi : \mathcal{O}_K^u \to W$, there is an idempotent $e_\psi \in \mathcal{S}$ satisfying

$$(x \otimes 1)e_\psi = (1 \otimes \psi(x))e_\psi$$

for every $x \in \mathcal{O}_K^u$. These idempotents satisfy $(e_\psi)^{\mathrm{Fr}} = e_{\mathrm{Fr}\circ\psi}$, and

$$\mathcal{S} = \prod_{\psi:\mathcal{O}_K^u \to W} e_\psi \mathcal{S},$$

where each factor on the right is isomorphic to the ring of integers in the completion of the maximal unramified extension of K. In particular, each factor is a discrete valuation ring, whose valuation determines a surjection

$$\mathrm{ord}_\psi : \mathcal{S} \to \mathbb{Z}^{\geq 0} \cup \{\infty\}.$$

Denote by

$$m(\psi, \Phi) = \#\{\varphi \in \Phi : \varphi|_{\mathcal{O}_K^u} = \psi\}$$

the *multiplicity of* ψ in Φ. Similarly, if we set

$$\mathcal{S}_0 = \mathcal{O}_{K_0} \otimes_{\mathbb{Z}_p} W,$$

there is a decomposition of W-algebras

$$\mathcal{S}_0 = \prod_{\psi_0:\mathcal{O}_{K_0}^u \to W} e_{\psi_0} \mathcal{S}_0$$

in which each factor is isomorphic to the integers in the completion of the maximal unramified extension of K_0. For each ψ_0, there is an associated valuation

$$\mathrm{ord}_{\psi_0} : \mathcal{S}_0 \to \mathbb{Z}^{\geq 0} \cup \{\infty\},$$

and the *multiplicity of* ψ_0 *in* Φ_0 is

$$m(\psi_0, \Phi_0) = \#\{\varphi \in \Phi_0 : \varphi|_{\mathcal{O}^u_{K_0}} = \psi_0\}.$$

Let $D(A_0)$ and $D(A)$ be the Dieudonné modules of A_0 and A, respectively. The following lemma makes the structure of these Dieudonné modules more explicit.

LEMMA 2.3.1. *There is an isomorphism of* $\mathcal{S}$*-modules* $D(A) \cong \mathcal{S}$. *Under any such isomorphism, the operators* F *and* V *on* $D(A)$ *take the form* $F = a \circ \mathrm{Fr}$ *and* $V = b \circ \mathrm{Fr}^{-1}$ *for some* $a, b \in \mathcal{S}$ *satisfying* $ab^{\mathrm{Fr}} = p$, *and satisfying*

$$\mathrm{ord}_\psi(b) = m(\psi, \Phi)$$

for every $\psi : \mathcal{O}^u_K \to W$.

Similarly, there is an isomorphism of $\mathcal{S}_0$*-modules* $D(A_0) \cong \mathcal{S}_0$. *Under any such isomorphism, the operators* F *and* V *on* $D(A_0)$ *take the form* $F = a_0 \circ \mathrm{Fr}$ *and* $V = b_0 \circ \mathrm{Fr}^{-1}$ *for some* $a_0, b_0 \in \mathcal{S}_0$ *satisfying* $a_0 b_0^{\mathrm{Fr}} = p$, *and satisfying*

$$\mathrm{ord}_{\psi_0}(b_0) = m(\psi_0, \Phi_0)$$

for every $\psi_0 : \mathcal{O}^u_{K_0} \to W$.

Proof. We give the proof only for $D(A)$, as the proof for $D(A_0)$ is identical. The only assertion that is not obvious from Proposition 2.1.1 is the formula for $\mathrm{ord}_\psi(b)$. The Lie algebra of A is canonically identified with

$$D(A)/VD(A) \cong \mathcal{S}/b\mathcal{S},$$

and the e_ψ component of $\mathcal{S}/b\mathcal{S}$ is an $\mathbb{F}$-vector space of dimension $\mathrm{ord}_\psi(b)$. It follows that the characteristic polynomial of any $x \in \mathcal{O}^u_K$ acting on $\mathrm{Lie}(A)$ is equal to

$$\prod_{\psi:\mathcal{O}^u_K \to W} (T - \psi(x))^{\mathrm{ord}_\psi(b)} \in \mathbb{F}[T].$$

On the other hand, the Φ-determinant condition imposed on (A, κ, λ) implies that this characteristic polynomial is equal to

$$\prod_{\varphi \in \Phi} (T - \varphi(x)) = \prod_{\psi:\mathcal{O}^u_K \to W} (T - \psi(x))^{m(\psi,\Phi)}.$$

It follows that $\mathrm{ord}_\psi(b) = m(\psi, \Phi)$ for every ψ. □

Fix isomorphisms $D(A_0) \cong \mathcal{S}_0$ and $D(A) \cong \mathcal{S}$ as in the lemma, so that $L(A_0, A)$ is identified with an $\mathcal{O}_K$-submodule of $\mathrm{Hom}_{\mathcal{S}_0}(\mathcal{S}_0, \mathcal{S}) \cong \mathcal{S}$. Of course an element of $\mathrm{Hom}_{\mathcal{S}_0}(\mathcal{S}_0, \mathcal{S})$ lies in the submodule $L(A_0, A)$ if and only if it respects the V (equivalently, F) operators on $D(A_0)$ and $D(A)$. In the notation of Lemma 2.3.1 this amounts to

$$L(A_0, A) \cong \{s \in \mathcal{S} : (b_0 s)^{\mathrm{Fr}} = b^{\mathrm{Fr}} s\}. \tag{2.3.1}$$

Let $s \mapsto \overline{s}$ be the automorphism of $\mathcal{S}$ induced by the nontrivial automorphism of K/F

LEMMA 2.3.2. *Under the isomorphism* (2.3.1), *the Hermitian form* $\langle\cdot,\cdot\rangle_{\mathrm{CM}}$ *on* $L(A_0, A)$ *is identified with the Hermitian form*

$$\langle s_1, s_2\rangle_{\mathrm{CM}} = \xi s_1 \overline{s}_2$$

on the right-hand side of (2.3.1), *for some* $\xi \in \mathcal{S} \otimes_{\mathbb{Z}} \mathbb{Q}$ *satisfying*

(1) $\overline{\xi} = \xi$,
(2) $\xi\mathcal{S} = \mathfrak{a}\mathfrak{D}_0\mathfrak{D}^{-1}\mathcal{S}$, *and*
(3) $(b_0\overline{b}_0)^{\mathrm{Fr}}\xi = \xi^{\mathrm{Fr}}(b\overline{b})^{\mathrm{Fr}}$.

(*The last condition guarantees that* $\xi s_1 \overline{s}_2$ *lies in* $K = (\mathcal{S}\otimes_{\mathbb{Z}}\mathbb{Q})^{\mathrm{Fr}=1}$, *as it must.*)

Proof. This is an exercise in linear algebra. The polarization λ induces a W-symplectic form

$$\lambda : D(A) \times D(A) \to W$$

satisfying $\lambda(sx, y) = \lambda(x, \overline{s}y)$ for all $s \in \mathcal{S}$, and $\lambda(Fx, y) = \lambda(x, Vy)^{\mathrm{Fr}}$. The first property implies that the induced pairing

$$\lambda : \mathcal{S} \times \mathcal{S} \to W$$

has the form

$$\lambda(s_1, s_2) = \mathrm{Tr}_{K/\mathbb{Q}_p}(\zeta s_1 \overline{s}_2)$$

for some $\zeta \in \mathcal{S} \otimes_{\mathbb{Z}} \mathbb{Q}$ satisfying $\overline{\zeta} = -\zeta$. The second property implies that $p\zeta = (\zeta b\overline{b})^{\mathrm{Fr}}$. The assumption that $\lambda : A \to A^\vee$ has kernel $A[\mathfrak{a}]$ implies that $\zeta\mathcal{S} = \mathfrak{a}\mathfrak{D}^{-1}\mathcal{S}$. Similarly, the principal polarization λ_0 induces a perfect pairing

$$\lambda_0 : \mathcal{S}_0 \times \mathcal{S}_0 \to W$$

of the form

$$\lambda_0(s_1, s_2) = \mathrm{Tr}_{K_0/\mathbb{Q}_p}(\zeta_0 s_1 \overline{s}_2)$$

for some $\zeta_0 \in \mathcal{S}_0 \otimes_{\mathbb{Z}} \mathbb{Q}$ satisfying $\overline{\zeta}_0 = -\zeta_0$, $p\zeta_0 = (\zeta_0 b_0 \overline{b}_0)^{\mathrm{Fr}}$, and $\zeta_0\mathcal{S}_0 = \mathcal{S}_0$.

The $\mathcal{O}_{K_0}$-Hermitian form $\langle f_1, f_2\rangle$ on (2.3.1) is then given by the explicit formula

$$\langle s_1, s_2\rangle = \mathrm{Tr}_{K/K_0}(\zeta_0^{-1}\zeta s_1 \overline{s}_2).$$

It follows that $\langle s_1, s_2\rangle_{\mathrm{CM}} = \xi s_1 \overline{s}_2$, where $\xi = \zeta_0^{-1}\zeta$. □

Armed with the above explicit coordinates, we may describe the Hermitian space $L(A_0, A)$ attached to our fixed triples $(A_0, \kappa_0, \lambda_0)$ and (A, κ, λ).

PROPOSITION 2.3.3. *For some* $\beta \in F^\times$ *satisfying*

$$\beta\mathcal{O}_K = \begin{cases} \mathfrak{a}\mathfrak{p}_F\mathfrak{D}_0\mathfrak{D}^{-1}\mathcal{O}_K & \text{if } K_0/\mathbb{Q} \text{ is unramified} \\ \mathfrak{a}\mathfrak{D}_0\mathfrak{D}^{-1}\mathcal{O}_K & \text{if } K_0/\mathbb{Q} \text{ is ramified,} \end{cases}$$

there is an isomorphism of $\mathcal{O}_K$-modules $L(A_0, A) \cong \mathcal{O}_K$ identifying $\langle\cdot,\cdot\rangle_{\mathrm{CM}}$ with the Hermitian form $\langle x, y\rangle_{\mathrm{CM}} = \beta x\overline{y}$ on $\mathcal{O}_K$.

Proof. First we show that $L(A_0, A)$ is free of rank one over $\mathcal{O}_K$. Let

$$H = \mathrm{End}(A_0) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p,$$

so that H is a quaternion division algebra over $\mathbb{Q}_p$. As A_0 and A are supersingular there is an isogeny $A \to A_0 \times \cdots \times A_0$ (n factors). The Noether-Skolem theorem implies that any two maps $K_0 \to M_n(H)$ are conjugate, and it follows that the above isogeny may be chosen to be $\mathcal{O}_{K_0}$-linear. A choice of such isogeny allows us to identify

$$L(A_0, A) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong \mathrm{Hom}_{\mathcal{O}_{K_0}}(A_0, A_0 \times \cdots \times A_0) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong K_0 \times \cdots K_0$$

as K_0-vector spaces. Thus $L(A_0, A)$ is free of rank n over $\mathcal{O}_{K_0}$, and hence $L(A_0, A)$ is free of rank one over $\mathcal{O}_K$.

Fix an $\mathcal{O}_K$-module generator s of (2.3.1), so that $x \cdot s \mapsto x$ defines an isomorphism $L(A_0, A) \to \mathcal{O}_K$ identifying $\langle\cdot,\cdot\rangle_{\mathrm{CM}}$ with $\beta x\overline{y}$ where, in the notation of Lemma 2.3.2,

$$\beta = \xi s\overline{s}.$$

We know that $\xi\mathcal{S} = \mathfrak{a}\mathfrak{D}_0\mathfrak{D}^{-1}\mathcal{S}$, and so it only remains to determine the ideal $s\overline{s}\mathcal{S}$.

Let $e(K/K_0)$ be the ramification degree of K/K_0. Set $d = [K^u : \mathbb{Q}_p]$, and enumerate the maps $\mathcal{O}_K^u \to W$ as $\{\psi^i : i \in \mathbb{Z}/d\mathbb{Z}\}$ in such a way that $\psi^{i+1} = \mathrm{Fr} \circ \psi^i$. Let ψ_0^i be the restriction of ψ^i to $\mathcal{O}_{K_0}^u$. The relation $(b_0 s)^{\mathrm{Fr}} = b^{\mathrm{Fr}} s$ implies

$$\begin{aligned}
\mathrm{ord}_{\psi^{i+1}}(s) &= \mathrm{ord}_{\psi^i}(s) - \mathrm{ord}_{\psi^i}(b) + \mathrm{ord}_{\psi^i}(b_0) \\
&= \mathrm{ord}_{\psi^i}(s) - \mathrm{ord}_{\psi^i}(b) + e(K/K_0) \cdot \mathrm{ord}_{\psi_0^i}(b_0) \\
&= \mathrm{ord}_{\psi^i}(s) - m(\psi^i, \Phi) + e(K/K_0) \cdot m(\psi_0^i, \Phi_0),
\end{aligned}$$

where the final equality is by Lemma 2.3.1. Note that there is at least one $\psi : \mathcal{O}_K^u \to W$ for which $\mathrm{ord}_\psi(s) = 0$; otherwise s would be divisible in $L(A_0, A)$ by a uniformizing parameter of $\mathcal{O}_K$. This observation and the above relation between $\mathrm{ord}_{\psi^{i+1}}(s)$ and $\mathrm{ord}_{\psi^i}(s)$ will allow us to compute $\mathrm{ord}_\psi(s)$ for all $\psi : \mathcal{O}_K^u \to W$.

If $K_0/\mathbb{Q}_p$ is ramified, then $m(\psi_0^i, \Phi_0) = 1$. Each $\psi^i : \mathcal{O}_K^u \to W$ admits

$$[K : K^u] = 2 \cdot e(K/K_0)$$

extensions to a map $K \to \mathbb{C}_p$. Exactly half of these extensions lie in Φ, and so $m(\psi^i, \Phi) = e(K/K_0)$. It follows that $\mathrm{ord}_{\psi^{i+1}}(s) = \mathrm{ord}_{\psi^i}(s)$ for every $i \in \mathbb{Z}/d\mathbb{Z}$, hence $s \in \mathcal{S}^\times$ and

$$\beta\mathcal{S} = \xi s\overline{s}\mathcal{S} = \mathfrak{a}\mathfrak{D}_0\mathfrak{D}^{-1}\mathcal{S}$$

as desired.

Now suppose that $K_0/\mathbb{Q}_p$ is unramified. As K_0 does not embed into F, this implies $[F^u : \mathbb{Q}_p] = 2f+1$ for some $f \in \mathbb{Z}^{\geq 0}$, and so $d = 4f+2$. Assume the ψ^i have been enumerated in such a way that ψ^0 is the restriction of φ^{sp} to $\mathcal{O}_K^u$. This implies $\psi_0^0 = \bar{\iota}$, and so

$$m(\psi_0^i, \Phi_0) = \begin{cases} 1 & \text{if } i \text{ is odd} \\ 0 & \text{if } i \text{ is even.} \end{cases}$$

If φ is any extension of ψ^0 to a map $K \to \mathbb{C}_p$, then the restriction of φ to $\mathcal{O}_{K_0}$ is $\bar{\iota}$. Therefore, $\varphi \in \Phi$ if and only if $\varphi = \varphi^{\mathrm{sp}}$, and so $m(\psi^0, \Phi) = 1$. This shows

$$\mathrm{ord}_{\psi^1}(s) = \mathrm{ord}_{\psi^0}(s) - 1.$$

The automorphism $x \mapsto \overline{x}$ of K restricts to Fr^{2f+1} on K^u, and so the restriction of $\overline{\varphi^{\mathrm{sp}}}$ to $\mathcal{O}_K^u$ is ψ^{2f+1}. The map $\psi^{2f+1} : \mathcal{O}_K^u \to W$ then admits $[K : K^u] = e(K/K_0)$ distinct extensions to a map $K \to \mathbb{C}_p$, every one of which except $\overline{\varphi^{\mathrm{sp}}}$ is contained in Φ. Therefore, $m(\psi^{2f+1}, \Phi) = e(K/K_0) - 1$, from which we deduce

$$\mathrm{ord}_{\psi^{2f+2}}(s) = \mathrm{ord}_{\psi^{2f+1}}(s) + 1.$$

Now suppose $i \in \mathbb{Z}/d\mathbb{Z}$ is not equal to 0 or $2f+1$, so that ψ^i is not the restriction to $\mathcal{O}_K^u$ of either φ^{sp} or $\overline{\varphi^{\mathrm{sp}}}$. Similar reasoning to the above shows that if i is even, then $m(\psi^i, \Phi)$ and $m(\psi_0^i, \Phi_0)$ are both 0, while if i is odd, then $m(\psi^i, \Phi) = e(K/K_0)$ and $m(\psi_0^i, \Phi_0) = 1$. In either case, $\mathrm{ord}_{\psi^{i+1}}(s) = \mathrm{ord}_{\psi^i}(s)$. Recalling that $\mathrm{ord}_{\psi}(s) = 0$ for at least one $\psi : \mathcal{O}_K^u \to W$, we deduce first

$$\mathrm{ord}_{\psi^i}(s) = \begin{cases} 0 & \text{if } 1 \leq i \leq 2f+1 \\ 1 & \text{if } 2f+2 \leq i \leq d \end{cases}$$

and then $\mathrm{ord}_{\psi^i}(s\overline{s}) = \mathrm{ord}_{\psi^i}(s) + \mathrm{ord}_{\psi^{i+2f+1}}(s) = 1$. Thus $s\overline{s}\mathcal{S} = \mathfrak{p}_F\mathcal{S}$ and

$$\beta\mathcal{S} = \mathfrak{a}\mathfrak{p}_F\mathfrak{D}_0\mathfrak{D}^{-1}\mathcal{S}. \qquad \square$$

Remark 2.3.4. Proposition 2.3.3 specifies β up to multiplication by $\mathcal{O}_F^\times$, but to determine the isomorphism class of $(\mathcal{O}_K, \beta x\overline{y})$ one needs to know β up to multiplication by $\mathrm{Nm}_{K/F}(\mathcal{O}_K^\times)$. If K/F is unramified, there is no difference, and so Proposition 2.3.3 completely determines the isomorphism class of the pair $\big(L(A_0, A), \langle\cdot,\cdot\rangle_{\mathrm{CM}}\big)$. If K/F is ramified, there is some remaining ambiguity, as Proposition 2.3.3 only narrows down the isomorphism class of the pair $\big(L(A_0, A), \langle\cdot,\cdot\rangle_{\mathrm{CM}}\big)$ to two possibilities.

Let $\mathfrak{m}$ be the maximal ideal of W_Φ, and for every $k \in \mathbb{Z}^{>0}$, set

$$R^{(k)} = W_\Phi/\mathfrak{m}^k.$$

By Theorem 2.1.3 there is a unique deformation $(A^{(k)}, \kappa^{(k)}, \lambda^{(k)})$ of (A, κ, λ) to $R^{(k)}$ and a unique deformation $(A_0^{(k)}, \kappa_0^{(k)}, \lambda_0^{(k)})$ of $(A_0, \kappa_0, \lambda_0)$ to $R^{(k)}$. The image of the reduction map

$$\mathrm{Hom}_{\mathcal{O}_{K_0}}(A_0^{(k)}, A^{(k)}) \to L(A_0, A)$$

is an $\mathcal{O}_K$-submodule $L^{(k)}$, and

$$\cdots \subset L^{(3)} \subset L^{(2)} \subset L^{(1)} = L(A_0, A)$$

is a decreasing filtration of $L(A_0, A)$.

The following theorem, which shows that the filtration on $L(A_0, A)$ is completely determined by the Hermitian form $\langle\cdot,\cdot\rangle_{\mathrm{CM}}$, generalizes a result of Gross [12], as explained in the remarks at the end of this subsection. Gross's original proof, which can be found in an expanded form in the ARGOS volume [44], is based on Lubin-Tate groups and the theory of formal group laws. Our proof will be based on crystalline deformation theory and is closer in spirit to Zink's proof of Gross's result, found in [46, Prop. 77].

Theorem 2.3.5. *Assume that at least one of the following hypotheses is satisfied*:

(1) $K/\mathbb{Q}_p$ *is unramified*,

(2) $p \neq 2$ *and one of* $K_0/\mathbb{Q}_p$ *or* $F/\mathbb{Q}_p$ *is unramified.*

For any nonzero $f \in L(A_0, A)$, f *is in* $L^{(k)}$ *but not* $L^{(k+1)}$ *where* $\alpha = \langle f, f\rangle_{\mathrm{CM}}$ *and*

$$k = \frac{1}{2} \cdot \mathrm{ord}_{\mathcal{O}_K}(\alpha \mathfrak{p}_F \mathfrak{D}\mathfrak{D}_0^{-1}\mathfrak{a}^{-1}).$$

The proof, which occupies the remainder of this subsection, is by induction on the divisibility of f by a uniformizing parameter of $\mathcal{O}_K$. Proposition 2.3.6 serves as the base case, and Proposition 2.3.7 forms the inductive step.

Fix an injective ring homomorphism $\mathcal{O}_F \to M_n(\mathbb{Z}_p)$. If B_0 is a p-divisible group defined over some base scheme S, denote by $B_0 \otimes \mathcal{O}_F$ the p-divisible group B_0^n, and let $\mathcal{O}_F$ act through the embedding $\mathcal{O}_F \to M_n(\mathbb{Z}_p)$ just chosen. This construction has a more intrinsic characterization: the functor of points of $B_0 \otimes \mathcal{O}_F$ is

$$(B_0 \otimes \mathcal{O}_F)(T) = B_0(T) \otimes_{\mathbb{Z}_p} \mathcal{O}_F$$

for any S-scheme T. If B_0 has an action of $\mathcal{O}_{K_0}$, then $B_0 \otimes \mathcal{O}_F$ inherits an action of the subring $\mathcal{O}_{K_0} \otimes_{\mathbb{Z}_p} \mathcal{O}_F \subset \mathcal{O}_K$. If B is a p-divisible group over S with an action of $\mathcal{O}_K$, then every $\mathcal{O}_{K_0}$-linear homomorphism $f : B_0 \to B$ induces an $\mathcal{O}_{K_0} \otimes_{\mathbb{Z}_p} \mathcal{O}_F$-linear homomorphism $f : B_0 \otimes \mathcal{O}_F \to B$.

Proposition 2.3.6. *Suppose* f *is an* $\mathcal{O}_K$*-module generator of* $L(A_0, A)$.

(1) *If* $K_0/\mathbb{Q}_p$ *is unramified, then* f *is in* $L^{(1)}$ *but not* $L^{(2)}$.

(2) *If* $K_0/\mathbb{Q}_p$ *is ramified and* $F/\mathbb{Q}_p$ *is unramified, then* f *is in* $L^{(d)}$ *but not* $L^{(d+1)}$, *where* $d = \mathrm{ord}_{\mathcal{O}_{K_0}}(\mathfrak{D}_0)$.

Proof. Let $\mathcal{D}_0$ and $\mathcal{D}$ be the Grothendieck-Messing crystals of $A_0 = A_0^{(1)}$ and $A = A^{(1)}$, respectively. First assume $K_0/\mathbb{Q}_p$ is unramified. The kernel $\mathcal{I}$ of $R^{(2)} \to R^{(1)}$ can be equipped with a divided power structure compatible with the canonical divided powers on $pR^{(2)}$ (take the trivial divided powers on $\mathcal{I}$ if W_Φ/W is ramified, and the canonical divided powers on $\mathcal{I} = pR^{(2)}$ otherwise), and once such divided powers are chosen we may identify, using [46, Cor. 56],

$$\mathcal{D}_0(R^{(2)}) \cong D(A_0) \otimes_W R^{(2)} \cong \mathcal{S}_0 \otimes_W R^{(2)}$$

and

$$\mathcal{D}(R^{(2)}) \cong D(A) \otimes_W R^{(2)} \cong \mathcal{S} \otimes_W R^{(2)}.$$

As in the proof of Theorem 2.1.3 the lifts of the Hodge filtrations of $\mathcal{D}_0(R^{(1)})$ and $\mathcal{D}(R^{(1)})$ corresponding to the deformations $A_0^{(2)}$ and $A^{(2)}$ are $J_{\Phi_0}\mathcal{D}_0(R^{(2)})$ and $J_\Phi\mathcal{D}(R^{(2)})$, and f lifts to a map $A_0^{(2)} \to A^{(2)}$ if and only if

$$J_{\Phi_0}\mathcal{D}_0(R^{(2)}) \xrightarrow{f} \mathcal{D}(R^{(2)})/J_\Phi\mathcal{D}(R^{(2)})$$

is trivial. If f corresponds to $s \in \mathcal{S}$ under the isomorphism (2.3.1), we must therefore prove that the map

$$J_{\Phi_0}(\mathcal{S}_0 \otimes_W W_\Phi) \xrightarrow{s\cdot} (\mathcal{S} \otimes_W W_\Phi)/J_\Phi(\mathcal{S} \otimes_W W_\Phi)$$

is nonzero modulo $\mathfrak{m}^2$. But this is clear from the proof of Proposition 2.3.3: if ψ denotes the restriction of φ^{sp} to $\mathcal{O}_K^u \to W$, then we have already seen that $\mathrm{ord}_\psi(s) = 1$, and so the image of s under the surjection $\varphi^{\mathrm{sp}} : \mathcal{S} \to W_\Phi$ is a uniformizing parameter. The assumption that $K_0/\mathbb{Q}_p$ is unramified implies that $\mathcal{S}_0 \otimes_W W_\Phi \cong W_\Phi \times W_\Phi$ and that the composition

$$W_\Phi \cong J_{\Phi_0}(\mathcal{S}_0 \otimes_W W_\Phi) \xrightarrow{s\cdot} (\mathcal{S} \otimes_W W_\Phi)/J_\Phi(\mathcal{S} \otimes_W W_\Phi) \xrightarrow{\varphi^{\mathrm{sp}}} W_\Phi$$

is multiplication by $\varphi^{\mathrm{sp}}(s)$.

Now assume $K_0/\mathbb{Q}_p$ is ramified and $F/\mathbb{Q}_p$ is unramified, so that

$$\mathcal{O}_K = \mathcal{O}_{K_0} \otimes_{\mathbb{Z}_p} \mathcal{O}_F.$$

Let $s \in \mathcal{S}$ correspond to f under the isomorphism (2.3.1), and recall from the proof of Proposition 2.3.3 that $s \in \mathcal{S}^\times$. This implies that the induced map

$$f : D(A_0) \otimes_{\mathbb{Z}_p} \mathcal{O}_F \to D(A)$$

is an isomorphism of Dieudonné modules and, in particular, f induces an isomorphism of Lie algebras

$$\mathrm{Lie}(A_0) \otimes_{\mathbb{Z}_p} \mathcal{O}_F \cong \mathrm{Lie}(A).$$

If f lifts to a map $f^{(k)} : A_0^{(k)} \to A^{(k)}$, then Nakayama's lemma implies that the induced map

$$\mathrm{Lie}(A_0^{(k)}) \otimes_{\mathbb{Z}_p} \mathcal{O}_F \cong \mathrm{Lie}(A^{(k)})$$

is again an isomorphism, and comparing the $\mathcal{O}_K$-action on each side we find the equality in $R^{(k)}[T]$

$$\prod_{\varphi\in\Phi^*}(T-\varphi(x)) = \prod_{\varphi\in\Phi}(T-\varphi(x)) \tag{2.3.2}$$

for every $x \in \mathcal{O}_K$, where

$$\Phi^* = \{\varphi \in \mathrm{Hom}(K, \mathbb{C}_p) : \varphi|_{K_0} = \iota\}$$

is the p-adic CM type of K obtained by replacing φ^{sp} by $\overline{\varphi}^{\mathrm{sp}}$. Comparing the coefficients of T^{n-1} shows that $\varphi^{\mathrm{sp}}, \overline{\varphi}^{\mathrm{sp}} : \mathcal{O}_K \to W_\Phi$ are congruent modulo $\mathfrak{m}^k$, which implies $k \le \mathrm{ord}_{\mathcal{O}_{K_0}}(\mathfrak{D}_0)$.

Conversely, if $k \le \mathrm{ord}_{\mathcal{O}_{K_0}}(\mathfrak{D}_0)$ then the polynomials (2.3.2) in $R^{(k)}[T]$ are equal, and so the natural $\mathcal{O}_K$-action on the p-divisible group $A_0^{(k)} \otimes_{\mathbb{Z}_p} \mathcal{O}_F$ over $R^{(k)}$ satisfies the Φ-determinant condition. The map $f : A_0 \otimes_{\mathbb{Z}_p} \mathcal{O}_F \to A$ is an isomorphism of p-divisible groups (because it induces an isomorphism of Dieudonné modules), and this allows us to view $A_0^{(k)} \otimes_{\mathbb{Z}_p} \mathcal{O}_F$ as a deformation of A with its $\mathcal{O}_K$-action. By the uniqueness of such deformations (see Remark 2.1.4) there is an $\mathcal{O}_K$-linear isomorphism

$$A_0^{(k)} \otimes_{\mathbb{Z}_p} \mathcal{O}_F \to A^{(k)}$$

lifting $f : A_0 \otimes_{\mathbb{Z}_p} \mathcal{O}_F \to A$, and precomposing with the inclusion

$$A_0^{(k)} \to A_0^{(k)} \otimes_{\mathbb{Z}_p} \mathcal{O}_F$$

gives the desired lift of $f : A_0 \to A$. This shows that f lifts to $A_0^{(k)} \to A^{(k)}$ if and only if $k \le \mathrm{ord}_{\mathcal{O}_{K_0}}(\mathfrak{D}_0)$. □

PROPOSITION 2.3.7. *Let π_K be a uniformizer of $\mathcal{O}_K$. If $f \in L^{(k)}$, then $\pi_K f \in L^{(k+1)}$. Furthermore, if any one of the conditions*

(1) $k > 1$,
(2) $p \ne 2$,
(3) $K/\mathbb{Q}_p$ *is unramified*

is satisfied, then the map $\pi_K : L^{(k)}/L^{(k+1)} \to L^{(k+1)}/L^{(k+2)}$ is injective.

Proof. The essential observation is that if

$$j_0 \in J_{\Phi_0} = \ker\big(\mathcal{O}_{K_0} \otimes_{\mathbb{Z}_p} W_\Phi \to \mathbb{C}_p(\iota)\big),$$

then

$$(x \otimes 1 - 1 \otimes \varphi^{\mathrm{sp}}(x)) \cdot j_0 \in J_\Phi = \ker\Big(\mathcal{O}_K \otimes_{\mathbb{Z}_p} W_\Phi \to \prod_{\varphi\in\Phi} \mathbb{C}_p(\varphi)\Big)$$

for every $x \in \mathcal{O}_K$. So, given an $\mathcal{O}_{K_0} \otimes_{\mathbb{Z}_p} W_\Phi$-module M_0, an $\mathcal{O}_K \otimes_{\mathbb{Z}_p} W_\Phi$-module M, and an $\mathcal{O}_{K_0}$-linear map $f : M_0 \to M$, the induced map

$$f : J_{\Phi_0} M_0 \to M/J_\Phi M$$

satisfies

$$(x \otimes 1 - 1 \otimes \varphi^{\mathrm{sp}}(x)) \circ f = 0$$

for all $x \in \mathcal{O}_K$.

Now suppose $f : A_0 \to A$ lifts to a map

$$f^{(k)} : A_0^{(k)} \to A^{(k)}.$$

Let $\mathcal{D}_0$ and $\mathcal{D}$ be the Grothendieck-Messing crystals of $A_0^{(k)}$ and $A^{(k)}$, respectively. By equipping the kernel of $R^{(k+1)} \to R^{(k)}$ with its trivial divided power structure, $f^{(k)}$ induces a commutative diagram

$$\begin{tikzcd}
J_{\Phi_0}\mathcal{D}_0(R^{(k+1)}) \arrow[r] \arrow[d] & \mathcal{D}_0(R^{(k+1)}) \arrow[r, "f^{(k)}"] \arrow[d] & \mathcal{D}(R^{(k+1)}) \arrow[r] \arrow[d] & \mathcal{D}(R^{(k+1)})/J_\Phi\mathcal{D}(R^{(k+1)}) \arrow[d] \\
J_{\Phi_0}\mathcal{D}_0(R^{(k)}) \arrow[r] & \mathcal{D}_0(R^{(k)}) \arrow[r, "f^{(k)}"] & \mathcal{D}(R^{(k)}) \arrow[r] & \mathcal{D}(R^{(k)})/J_\Phi\mathcal{D}(R^{(k)}).
\end{tikzcd}$$

As the middle arrow of the bottom row must preserve the Hodge filtrations of the crystals, the composition along the bottom row is trivial (see the proof of Theorem 2.1.3). Therefore, the composition along the top row

$$J_{\Phi_0}\mathcal{D}_0(R^{(k+1)}) \to \mathcal{D}(R^{(k+1)})/J_\Phi\mathcal{D}(R^{(k+1)})$$

becomes trivial after applying $\otimes_{R^{(k+1)}} R^{(k)}$, and so it has image annihilated by $\mathfrak{m}$. By the comments of the previous paragraph,

$$\pi_K f^{(k)} = \varphi^{\mathrm{sp}}(\pi_K) f^{(k)}$$

when viewed as maps

$$J_{\Phi_0}\mathcal{D}_0(R^{(k+1)}) \to \mathcal{D}(R^{(k+1)})/J_\Phi\mathcal{D}(R^{(k+1)}),$$

and as $\varphi^{\mathrm{sp}}(\pi_K) \in \mathfrak{m}$ we deduce that these maps are trivial. Therefore,

$$\pi_K f^{(k)} : \mathcal{D}_0(R^{(k+1)}) \to \mathcal{D}(R^{(k+1)})$$

takes the submodule $J_{\Phi_0}\mathcal{D}_0(R^{(k+1)})$ into $J_\Phi\mathcal{D}(R^{(k+1)})$. By the proof of Theorem 2.1.3, these submodules are the lifts of the Hodge filtrations defining $A_0^{(k+1)}$ and $A^{(k+1)}$, and so $\pi_K f^{(k)}$ lifts to a map $A_0^{(k+1)} \to A^{(k+1)}$.

Now suppose f is in $L^{(k)}$ but not $L^{(k+1)}$, and let $\mathcal{I}$ be the kernel of $R^{(k+2)} \to R^{(k)}$. If $k > 1$, then $\mathcal{I}^2 = 0$, and we may equip $\mathcal{I}$ with its trivial divided powers. If $p \neq 2$, then $\mathcal{I}^3 = 0$ allows us to equip $\mathcal{I}$ with its trivial divided powers. If $K/\mathbb{Q}_p$ is unramified, then $W_\Phi = W$, and we may equip

$\mathcal{I} = p^k W_\Phi$ with its canonical divided powers. In any case there is some divided power structure on $\mathcal{I}$, and so we may add a third row

$$\begin{array}{ccccccc}
J_{\Phi_0}\mathcal{D}_0(R^{(k+2)}) & \longrightarrow & \mathcal{D}_0(R^{(k+2)}) & \xrightarrow{f^{(k)}} & \mathcal{D}(R^{(k+2)}) & \longrightarrow & \mathcal{D}(R^{(k+2)})/J_\Phi\mathcal{D}(R^{(k+2)}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
J_{\Phi_0}\mathcal{D}_0(R^{(k+1)}) & \longrightarrow & \mathcal{D}_0(R^{(k+1)}) & \xrightarrow{f^{(k)}} & \mathcal{D}(R^{(k+1)}) & \longrightarrow & \mathcal{D}(R^{(k+1)})/J_\Phi\mathcal{D}(R^{(k+1)}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
J_{\Phi_0}\mathcal{D}_0(R^{(k)}) & \longrightarrow & \mathcal{D}_0(R^{(k)}) & \xrightarrow{f^{(k)}} & \mathcal{D}(R^{(k)}) & \longrightarrow & \mathcal{D}(R^{(k)})/J_\Phi\mathcal{D}(R^{(k)})
\end{array}$$

to the diagram above. If $\pi_K f^{(k)}$ lifts to a map $A_0^{(k+2)} \to A^{(k+2)}$, then

$$\pi_K f^{(k)} : J_{\Phi_0}\mathcal{D}_0(R^{(k+2)}) \to \mathcal{D}(R^{(k+2)})/J_\Phi\mathcal{D}(R^{(k+2)})$$

is trivial. By the comments of the first paragraph this implies that

$$\varphi^{\mathrm{sp}}(\pi_K) f^{(k)} : J_{\Phi_0}\mathcal{D}_0(R^{(k+2)}) \to \mathcal{D}(R^{(k+2)})/J_\Phi\mathcal{D}(R^{(k+2)})$$

is also trivial, and so

$$f^{(k)} : J_{\Phi_0}\mathcal{D}_0(R^{(k+2)}) \to \mathcal{D}(R^{(k+2)})/J_\Phi\mathcal{D}(R^{(k+2)})$$

takes values in $\mathfrak{m}^{k+1} \cdot \mathcal{D}(R^{(k+2)})/J_\Phi\mathcal{D}(R^{(k+2)})$. But this implies that

$$f^{(k)} : J_{\Phi_0}\mathcal{D}_0(R^{(k+1)}) \to \mathcal{D}(R^{(k+1)})/J_\Phi\mathcal{D}(R^{(k+1)})$$

is trivial, contradicting our hypothesis that $f^{(k)}$ does not lift to a map $A_0^{(k+1)} \to A^{(k+1)}$. Therefore, $\pi_K f^{(k)}$ lifts to $A_0^{(k+1)} \to A^{(k+1)}$ but not to $A_0^{(k+2)} \to A^{(k+2)}$. $\square$

Proof of Theorem 2.3.5. Let $\beta \in F^\times$ be as in Proposition 2.3.3. Fix a uniformizer $\pi_K \in \mathcal{O}_K$ and write $f = \pi_K^m f_0$ with f_0 an $\mathcal{O}_K$-module generator of $L(A_0, A)$, so that

$$\langle f_0, f_0 \rangle_{\mathrm{CM}} \mathcal{O}_F = \beta \mathcal{O}_F.$$

If $K_0/\mathbb{Q}_p$ is unramified, then

$$\alpha \mathcal{O}_K = \mathfrak{p}_F^{2m} \beta \mathcal{O}_K = \mathfrak{D}_0 \mathfrak{D}^{-1} \mathfrak{a} \mathfrak{p}_F^{2m+1}.$$

Using induction on m, Propositions 2.3.6 and 2.3.7 imply that f is in $L^{(m+1)}$ but not $L^{(m+2)}$, and the claim follows. If $K_0/\mathbb{Q}_p$ is ramified, then

$$\alpha \mathcal{O}_K = \mathfrak{p}_F^{m} \beta \mathcal{O}_K = \mathfrak{D}_0 \mathfrak{D}^{-1} \mathfrak{a} \mathfrak{p}_F^{m}.$$

Using induction on m, Proposition 2.3.6 (with $d = 1$, as p is odd) and Proposition 2.3.7 imply that f is in $L^{(m+1)}$ but not $L^{(m+2)}$, and again the claim follows. $\square$

Consider the special case of $F = \mathbb{Q}_p$, so that $K = K_0$ and W_Φ is the completion of the ring of integers in the maximal unramified extension of K. We end this subsection by explaining how, in this special case, Theorem 2.3.5 reduces to a well-known formula of Gross [12], which plays a crucial role in the proof of the famous Gross-Zagier formula [15]. Assume for simplicity that $p \neq 2$.

Keep $(A_0, \kappa_0, \lambda_0)$ as above, but now take $(A, \kappa, \lambda) = (A_0, \overline{\kappa}_0, \lambda_0)$, where $\overline{\kappa}_0(x) = \kappa_0(\overline{x})$. Suppressing κ_0 from the notation, the $\mathcal{O}_K$-module $L(A_0, A)$ now sits inside of $\mathrm{End}(A_0)$ as the set of $j \in \mathrm{End}(A_0)$ satisfying $j \circ x = \overline{x} \circ j$ for all $x \in \mathcal{O}_K$, and

$$\mathrm{End}(A_0) = \mathcal{O}_K \oplus L(A_0, A).$$

Furthermore,

$$\mathrm{End}(A_0^{(k)}) = \mathcal{O}_K \oplus L^{(k)},$$

and so in this special case Theorem 2.3.5 amounts to an explicit description of how the ring $\mathrm{End}(A_0^{(k)})$ shrinks as k grows. Fix an $\mathcal{O}_K$-module generator $f \in L(A_0, A)$. If π_K is a uniformizing parameter of $\mathcal{O}_K$, then our results prove

$$\mathrm{End}(A_0^{(k)}) = \mathcal{O}_K \oplus \mathcal{O}_K \pi_K^{k-1} f,$$

which is exactly Gross's formula.

2.4. *Lifting homomorphisms*: *the signature* $(n, 0)$ *case.* As in the previous subsection, K_0 is a quadratic field extension of $\mathbb{Q}_p$, $F/\mathbb{Q}_p$ is a field extension of degree n, and

$$K = K_0 \otimes_{\mathbb{Q}_p} F.$$

We now allow the possibility $K \cong F \times F$. Fix an embedding $\iota : K_0 \to \mathbb{C}_p$, so that $\Phi_0 = \{\iota\}$ is a p-adic CM type of K_0. A p-adic CM type Φ of K is of *signature* $(n, 0)$ if $\varphi|_{K_0} = \iota$ for every $\varphi \in \Phi$. Fix such a Φ (in fact, it's unique). If we let $K_\Phi \subset \mathbb{C}_p$ be any subfield containing $\iota(K_0)$, then condition (2.1.1) is satisfied. Let W_Φ and **ART** be as in Section 2.1.

Fix a triple (A, κ, λ) in which

- A is a p-divisible group over $\mathbb{F}$ of dimension n,
- $\kappa : \mathcal{O}_K \to \mathrm{End}(A)$ satisfies the Φ-determinant condition,
- $\lambda : A \to A^\vee$ is an $\mathcal{O}_K$-linear polarization.

Fix a second triple $(A_0, \kappa_0, \lambda_0)$ in which

- A_0 is a p-divisible group over $\mathbb{F}$ of dimension 1,
- $\kappa_0 : \mathcal{O}_{K_0} \to \mathrm{End}(A_0)$ satisfies the Φ_0-determinant condition,
- $\lambda_0 : A_0 \to A_0^\vee$ is an $\mathcal{O}_{K_0}$-linear polarization.

By Theorem 2.1.3 each of $(A_0, \kappa_0, \lambda_0)$ and (A, κ, λ) admits a unique deformation to any object of **ART**. The following is the signature $(n, 0)$ version of Theorem 2.3.5. Now the situation is drastically simplified.

Proposition 2.4.1. *Let R be an object of* **ART**, *and let $(A_0', \kappa_0', \lambda_0')$ and (A', κ', λ') be the unique deformations of $(A_0, \kappa_0, \lambda_0)$ and (A, κ, λ) to R. The reduction map*

$$\mathrm{Hom}_{\mathcal{O}_{K_0}}(A_0', A') \to \mathrm{Hom}_{\mathcal{O}_{K_0}}(A_0, A)$$

is a bijection.

Proof. Let $R^{(2)} \to R^{(1)}$ be any surjection in **ART** whose kernel $\mathcal{I}$ satisfies $\mathcal{I}^2 = 0$, and equip $\mathcal{I}$ with its trivial divided power structure. Let $(A_0^{(i)}, \kappa_0^{(i)}, \lambda_0^{(i)})$ and $(A^{(i)}, \kappa^{(i)}, \lambda^{(i)})$ be the unique deformations of $(A_0, \kappa_0, \lambda_0)$ and (A, κ, λ) to $R^{(i)}$, and suppose we are given an $\mathcal{O}_{K_0}$-linear map $f : A_0^{(1)} \to A^{(1)}$. Let $\mathcal{D}_0$ and $\mathcal{D}$ be the Grothendieck-Messing crystals of $A_0^{(1)}$ and $A^{(1)}$. The map f induces an $\mathcal{O}_{K_0} \otimes_{\mathbb{Z}_p} W_\Phi$-linear map on crystals $f : \mathcal{D}_0(R^{(2)}) \to \mathcal{D}(R^{(2)})$. The hypothesis that Φ has signature $(n, 0)$ implies that

$$J_{\Phi_0}(\mathcal{O}_K \otimes_{\mathbb{Z}_p} W_\Phi) \subset J_\Phi,$$

and therefore f satisfies

$$f(J_{\Phi_0}\mathcal{D}_0(R^{(2)})) = J_{\Phi_0} f(\mathcal{D}_0(R^{(2)})) \subset J_\Phi \mathcal{D}(R^{(2)}).$$

By the proof of Theorem 2.1.3, the deformations $A_0^{(2)}$ and $A^{(2)}$ of $A_0^{(1)}$ and $A^{(1)}$ correspond to the lifts

$$J_{\Phi_0}\mathcal{D}_0(R^{(2)}) \subset \mathcal{D}_0(R^{(2)}) \qquad J_\Phi \mathcal{D}(R^{(2)}) \subset \mathcal{D}(R^{(2)})$$

of the Hodge filtrations of $\mathcal{D}_0(R^{(1)})$ and $\mathcal{D}(R^{(1)})$. As f preserves these filtrations, it follows that f lifts (uniquely) to a map $A_0^{(2)} \to A^{(2)}$. The claim follows by induction on the length of R. □

3. Arithmetic intersection theory

Throughout Section 3 we fix the following data, as in the introduction:

- $K_0 \subset \mathbb{C}$ is a quadratic imaginary field, and $\iota : K_0 \to \mathbb{C}$ is the inclusion;
- F is a totally real étale $\mathbb{Q}$-algebra of degree n;
- $K = K_0 \otimes_{\mathbb{Q}} F$;
- Φ is a CM type of K of signature $(n-1, 1)$ (this means there is a unique $\varphi^{\mathrm{sp}} \in \Phi$ whose restriction to K_0 is $\bar{\iota} : K_0 \to \mathbb{C}$);
- $K_\Phi \subset \mathbb{C}$ is a finite extension of K_0 containing the reflex field of Φ;
- $\mathcal{O}_\Phi$ is the ring of integers of K_Φ;
- $\mathfrak{a} \subset \mathcal{O}_F$ is an ideal (eventually we will take $\mathfrak{a} = \mathcal{O}_F$).

The CM type Φ is uniquely determined by its *special element* φ^{sp}, and thus $\sigma \in \mathrm{Aut}(\mathbb{C}/K_0)$ fixes Φ if and only if it fixes φ^{sp}. It follows that $\varphi^{\mathrm{sp}}(K) \subset K_\Phi$ and that we may take $K_\Phi = \varphi^{\mathrm{sp}}(K)$ if we choose. In any case, every prime $\mathfrak{p}$ of K_Φ restricts, via the map

$$\varphi^{\mathrm{sp}} : K \to K_\Phi, \tag{3.0.1}$$

to a prime $\mathfrak{p}_K$ of K. The prime of F below $\mathfrak{p}_K$ is denoted $\mathfrak{p}_F$. The special element $\varphi^{\mathrm{sp}} \in \Phi$ determines an archimedean place of K, whose restriction to F is denoted ∞^{sp}.

Let $\pi_0(F)$ denote the set of connected components of $\mathrm{Spec}(F)$. The algebra F is a product of totally real number fields indexed by $\pi_0(F)$, and each connected component has the form $\mathrm{Spec}(F')$ for a subfield $F' \subset F$. There is a quadratic character of $(F'_{\mathbb{A}})^\times$ associated to the CM field $K' = K_0 \otimes_{\mathbb{Q}} F'$, and by collecting together the quadratic characters of the different components of $\mathrm{Spec}(F)$ we obtain a generalized character

$$\chi_{K/F} : F_{\mathbb{A}}^\times \to \{\pm 1\}^{\pi_0(F)} \tag{3.0.2}$$

associated to the extension K/F.

For each $\mathfrak{p} \subset \mathcal{O}_\Phi$, fix an algebraic closure $K^{\mathrm{alg}}_{\Phi,\mathfrak{p}}$ of $K_{\Phi,\mathfrak{p}}$, let $\mathbb{C}_\mathfrak{p}$ be its completion, and let

$$W_{\Phi,\mathfrak{p}} \subset \mathbb{C}_\mathfrak{p}$$

be the ring of integers of the completion of the maximal unramified extension of $K_{\Phi,\mathfrak{p}}$. Denote by $k^{\mathrm{alg}}_{\Phi,\mathfrak{p}}$ the common residue field of $K^{\mathrm{alg}}_{\Phi,\mathfrak{p}}$, $\mathbb{C}_\mathfrak{p}$, and $W_{\Phi,\mathfrak{p}}$ Let $\mathfrak{D}_0$ and $\mathfrak{D}$ be the differents of $K_0/\mathbb{Q}$ and $K/\mathbb{Q}$, respectively, and let $\mathfrak{d}_F$ be the different of $F/\mathbb{Q}$.

3.1. *The stack* $\mathcal{CM}^{\mathfrak{a}}_\Phi$. Recall the moduli stack $\mathcal{M}_{(r,s)}$ of the introduction. We now define the cycle of points of $\mathcal{M}_{(n-1,1)}$ with complex multiplication by $\mathcal{O}_K$ and CM type Φ. Taking $\mathfrak{a} = \mathcal{O}_F$ in the following definition gives the stack $\mathcal{CM}_\Phi$ of the introduction.

Definition 3.1.1. Let $\mathcal{CM}^{\mathfrak{a}}_\Phi$ be the algebraic stack over $\mathcal{O}_\Phi$ whose functor of points assigns to a connected $\mathcal{O}_\Phi$-scheme S the groupoid of triples (A, κ, λ) in which

- $A \to S$ is an abelian scheme of relative dimension n,
- $\kappa : \mathcal{O}_K \to \mathrm{End}(A)$ satisfies the Φ-determinant condition,
- $\lambda : A \to A^\vee$ is an $\mathcal{O}_K$-linear polarization with kernel $A[\mathfrak{a}]$.

The condition of $\mathcal{O}_K$*-linearity* means that

$$\lambda \circ \kappa(\overline{x}) = \kappa(x)^\vee \circ \lambda$$

for every $x \in \mathcal{O}_K$. The Φ*-determinant condition*, introduced by Kottwitz [18], is the following: locally on S, for any $x_1, \ldots, x_r \in \mathcal{O}_K$, the determinant of $T_1 x_1 + \cdots + T_r x_r$ acting on $\mathrm{Lie}(A)$ is equal to the image of

$$\prod_{\varphi \in \Phi} (T_1 \varphi(x_1) + \cdots + T_r \varphi(x_r)) \in \mathcal{O}_\Phi[T_1, \ldots, T_r]$$

in $\mathcal{O}_S[T_1,\ldots,T_r]$. Note that this condition implies that every $x\in\mathcal{O}_K$ acts on $\mathrm{Lie}(A)$ with characteristic polynomial

$$\prod_{\varphi\in\Phi}(T-\varphi(x))\in\mathcal{O}_\Phi[T]$$

and, in particular, the action of $\mathcal{O}_{K_0}$ on A satisfies the signature $(n-1,1)$-condition of the introduction. If we take $\mathfrak{a}=\mathcal{O}_F$, then restricting the action from $\mathcal{O}_K$ to $\mathcal{O}_{K_0}$ defines a morphism

$$\mathcal{CM}_\Phi^{\mathcal{O}_F}\to\mathcal{M}_{(n-1,1)/\mathcal{O}_\Phi}.$$

For an $\mathcal{O}_\Phi$-scheme S, an S-valued point of

$$\mathcal{M}_{(1,0)}\times\mathcal{CM}_\Phi^{\mathfrak{a}}=\mathcal{M}_{(1,0)/\mathcal{O}_\Phi}\times_{\mathcal{O}_\Phi}\mathcal{CM}_\Phi^{\mathfrak{a}}$$

is a sextuple $(A_0,\kappa_0,\lambda_0,A,\kappa,\lambda)$ with $(A_0,\kappa_0,\lambda_0)\in\mathcal{M}_{(1,0)}(S)$ and $(A,\kappa,\lambda)\in\mathcal{CM}_\Phi^{\mathfrak{a}}(S)$. We usually abbreviate this sextuple to (A_0,A).

Proposition 3.1.2. *Let $\mathfrak{p}$ be a prime of K_Φ, and let $\mathcal{Y}_\Phi$ denote one of $\mathcal{M}_{(1,0)/\mathcal{O}_\Phi}$, $\mathcal{CM}_\Phi^{\mathfrak{a}}$, or $\mathcal{M}_{(1,0)}\times\mathcal{CM}_\Phi^{\mathfrak{a}}$.*

(1) *If R is any complete local Noetherian $W_{\Phi,\mathfrak{p}}$-algebra with residue field $k_{\Phi,\mathfrak{p}}^{\mathrm{alg}}$, the reduction map*

$$\mathcal{Y}_\Phi(R)\to\mathcal{Y}_\Phi(k_{\Phi,\mathfrak{p}}^{\mathrm{alg}})$$

(on isomorphism classes) is a bijection.

(2) *The completed strictly Henselian local ring of $\mathcal{Y}_\Phi$ at any geometric point $z\in\mathcal{Y}_\Phi(k_{\Phi,\mathfrak{p}}^{\mathrm{alg}})$ is isomorphic to $W_{\Phi,\mathfrak{p}}$.*

(3) *The structure morphism $\mathcal{Y}_\Phi\to\mathrm{Spec}(\mathcal{O}_\Phi)$ is étale and proper. In particular, $\mathcal{Y}_\Phi$ is a regular stack of dimension one.*

Proof. The first claim follows easily from Theorem 2.2.1. The second follows from the first, as the completed strictly Henselian local ring of a geometric point z represents the functor of deformations of z to complete local Noetherian $W_{\Phi,\mathfrak{p}}$-algebras. The étaleness part of the third claim can be checked on the level of completed strictly Henselian local rings, and so follows from the second claim. Properness follows from the valuative criterion of properness for stacks, together with the fact that CM abelian varieties over discrete valuation rings have potentially good reduction. □

Remark 3.1.3. If $\mathcal{Y}_\Phi$ is as above, it follows from Proposition 3.1.2 that there is a canonical bijection

$$\mathcal{Y}_\Phi(\mathbb{C}_\mathfrak{p})\to\mathcal{Y}_\Phi(k_{\Phi,\mathfrak{p}}^{\mathrm{alg}})$$

on isomorphism classes. Indeed, each object of $\mathcal{Y}_\Phi(\mathbb{C}_\mathfrak{p})$ is a polarized abelian variety with complex multiplication (or a pair of such things). By the theory of complex multiplication such an abelian variety admits a model with good reduction defined over some finite extension of $K_{\Phi,\mathfrak{p}}$. Reducing this model modulo $\mathfrak{p}$ and then base changing to $k^{\mathrm{alg}}_{\Phi,\mathfrak{p}}$ defines the desired reduction map. For the inverse: each object of $\mathcal{Y}_\Phi(k^{\mathrm{alg}}_{\Phi,\mathfrak{p}})$ lifts uniquely to $\mathcal{Y}_\Phi(W_{\Phi,\mathfrak{p}})$, and we base change this lift to $\mathbb{C}_\mathfrak{p}$.

Definition 3.1.4. Let $\mathfrak{p}$ be a prime of $\mathcal{O}_\Phi$. The unique lift of a triple

$$(A,\kappa,\lambda) \in \mathcal{CM}^{\mathfrak{a}}_\Phi(k^{\mathrm{alg}}_{\Phi,\mathfrak{p}})$$

to $W_{\Phi,\mathfrak{p}}$ is its *canonical lift*

$$(A^{\mathrm{can}},\kappa^{\mathrm{can}},\lambda^{\mathrm{can}}) \in \mathcal{CM}^{\mathfrak{a}}_\Phi(W_{\Phi,\mathfrak{p}}).$$

Similarly, the unique lift of a triple

$$(A_0,\kappa_0,\lambda_0) \in \mathcal{M}_{(1,0)}(k^{\mathrm{alg}}_{\Phi,\mathfrak{p}})$$

to $W_{\Phi,\mathfrak{p}}$ is its *canonical lift*

$$(A_0^{\mathrm{can}},\kappa_0^{\mathrm{can}},\lambda_0^{\mathrm{can}}) \in \mathcal{M}_{(1,0)}(W_{\Phi,\mathfrak{p}}).$$

An abelian variety A over an algebraically closed field of nonzero characteristic is *supersingular* if A is isogenous to a product of supersingular elliptic curves. Equivalently, by [36, Th. 4.2], A is supersingular if its p-divisible group is supersingular, in the sense of Section 2.

PROPOSITION 3.1.5. *Fix a prime* $\mathfrak{p}$ *of* K_Φ, *let* $\mathfrak{p}_F$ *be the prime of* F *defined after* (3.0.1), *let* p *be the rational prime below* $\mathfrak{p}$, *and assume that* p *is nonsplit in* K_0. *For any* $(A,\kappa,\lambda) \in \mathcal{CM}^{\mathfrak{a}}_\Phi(k^{\mathrm{alg}}_{\Phi,\mathfrak{p}})$, *the following hold*:

(1) *If* $\mathfrak{q} \subset \mathcal{O}_F$ *is a prime above* p *different from* $\mathfrak{p}_F$, *then* $A[\mathfrak{q}^\infty]$ *is supersingular*;
(2) $A[\mathfrak{p}_F^\infty]$ *is supersingular if and only if* $\mathfrak{p}_F$ *is nonsplit in* K;
(3) A *is supersingular if and only if* $\mathfrak{p}_F$ *is nonsplit in* K.

Proof. It suffices to prove the first two claims, as the third is a trivial consequence. Following Remark 3.1.3, let $(A^*,\kappa^*,\lambda^*) \in \mathcal{CM}^{\mathfrak{a}}_\Phi(\mathbb{C}_\mathfrak{p})$ be the unique lift of (A,κ,λ). Fix an isomorphism of K_Φ-algebras $\mathbb{C}_\mathfrak{p} \cong \mathbb{C}$, and view each $\varphi : K \to \mathbb{C}$ as taking values in $\mathbb{C}_\mathfrak{p}$. Fix a prime $\mathfrak{q} \subset \mathcal{O}_F$ above p.

For a prime $\mathfrak{Q}$ of K above $\mathfrak{q}$, let $H_\mathfrak{Q}$ be the set of all $\mathbb{Q}$-algebra maps $K \to \mathbb{C}_\mathfrak{p}$ inducing $\mathfrak{Q}$. For a map $\varphi : K \to \mathbb{C}_\mathfrak{p}$, we define the conjugate by $\overline{\varphi}(x) = \varphi(\overline{x})$, so that $H_{\overline{\mathfrak{Q}}} = \overline{H_\mathfrak{Q}}$. The proof of the Shimura-Taniyama formula, for example [8, Cor. 4.3], shows that

$$\dim A[\mathfrak{Q}^\infty] = \dim A^*[\mathfrak{Q}^\infty] = \#(\Phi \cap H_\mathfrak{Q})$$

and

$$\text{height } A[\mathfrak{Q}^\infty] = \text{height } A^*[\mathfrak{Q}^\infty] = \#H_\mathfrak{Q}.$$

The argument used in the proof of Proposition 2.1.1 shows that the Dieudonné module of $A[\mathfrak{Q}^\infty]$ is isoclinic. If the slope sequence consists of s/t repeated k times, then dim $A[\mathfrak{Q}^\infty] = sk$ and height $A[\mathfrak{Q}^\infty] = tk$. It follows that

$$A[\mathfrak{Q}^\infty] \text{ is supersingular} \iff \frac{1}{2} = \frac{\#(\Phi \cap H_\mathfrak{Q})}{\#H_\mathfrak{Q}}.$$

First consider the easy case in which $\mathfrak{q}$ is nonsplit in K. Then $\mathfrak{Q} = \overline{\mathfrak{Q}}$, and so $H_\mathfrak{Q}$ is the disjoint union of $\Phi \cap H_\mathfrak{Q}$ with

$$\overline{\Phi} \cap H_\mathfrak{Q} = \overline{\Phi} \cap H_{\overline{\mathfrak{Q}}} = \overline{\Phi \cap H_\mathfrak{Q}},$$

and it follows from the preceding paragraph that $A[\mathfrak{q}^\infty]$ is supersingular.

Now assume $\mathfrak{q}$ is split in K. This implies that $K_{0,p}$ embeds into $F_\mathfrak{q}$, and so $[F_\mathfrak{q} : \mathbb{Q}_p] = [K_\mathfrak{Q} : \mathbb{Q}_p] = \#H_\mathfrak{Q}$ is even, say $\#H_\mathfrak{Q} = 2d$. Each of the sets

$$\begin{aligned}
H_\mathfrak{Q}(\iota) &= \{\varphi \in H_\mathfrak{Q} : \varphi|_{K_0} = \iota\},\\
H_\mathfrak{Q}(\bar\iota) &= \{\varphi \in H_\mathfrak{Q} : \varphi|_{K_0} = \bar\iota\},\\
H_{\overline{\mathfrak{Q}}}(\iota) &= \{\varphi \in H_{\overline{\mathfrak{Q}}} : \varphi|_{K_0} = \iota\},\\
H_{\overline{\mathfrak{Q}}}(\bar\iota) &= \{\varphi \in H_{\overline{\mathfrak{Q}}} : \varphi|_{K_0} = \bar\iota\}
\end{aligned}$$

has d elements. If $\mathfrak{q} \neq \mathfrak{p}_F$, then $\Phi \cap H_\mathfrak{Q}(\bar\iota)$ and $\Phi \cap H_{\overline{\mathfrak{Q}}}(\bar\iota)$ are empty, and so

$$H_\mathfrak{Q}(\iota) = [\Phi \cap H_\mathfrak{Q}(\iota)] \cup [\overline{\Phi} \cap H_\mathfrak{Q}(\iota)] = [\Phi \cap H_\mathfrak{Q}(\iota)] \cup [\overline{\Phi \cap H_{\overline{\mathfrak{Q}}}(\bar\iota)}] = \Phi \cap H_\mathfrak{Q}(\iota).$$

This implies

$$\Phi \cap H_\mathfrak{Q} = [\Phi \cap H_\mathfrak{Q}(\iota)] \cup [\Phi \cap H_\mathfrak{Q}(\bar\iota)] = H_\mathfrak{Q}(\iota),$$

and so

$$\frac{\#(\Phi \cap H_\mathfrak{Q})}{\#H_\mathfrak{Q}} = \frac{d}{2d}.$$

This proves that $A[\mathfrak{Q}^\infty]$ is supersingular. The same argument shows that $A[\overline{\mathfrak{Q}}^\infty]$ is supersingular, and hence so is $A[\mathfrak{q}^\infty] = A[\mathfrak{Q}^\infty] \times A[\overline{\mathfrak{Q}}^\infty]$. If $\mathfrak{q} = \mathfrak{p}_F$, then one of $\Phi \cap H_\mathfrak{Q}(\bar\iota)$ and $\Phi \cap H_{\overline{\mathfrak{Q}}}(\bar\iota)$ is empty, and the other is $\{\varphi^{\mathrm{sp}}\}$. After possibly interchanging $\mathfrak{Q}$ and $\overline{\mathfrak{Q}}$ we may assume that $\Phi \cap H_\mathfrak{Q}(\bar\iota) = \{\varphi^{\mathrm{sp}}\}$ and $\Phi \cap H_{\overline{\mathfrak{Q}}}(\bar\iota) = \emptyset$. The argument above shows first that $H_\mathfrak{Q}(\iota) = \Phi \cap H_\mathfrak{Q}(\iota)$ and then that

$$\Phi \cap H_\mathfrak{Q} = [\Phi \cap H_\mathfrak{Q}(\iota)] \cup [\Phi \cap H_\mathfrak{Q}(\bar\iota)] = H_\mathfrak{Q}(\iota) \cup \{\varphi^{\mathrm{sp}}\}.$$

Therefore

$$\frac{\#(\Phi \cap H_\mathfrak{Q})}{\#H_\mathfrak{Q}} = \frac{d+1}{2d} \neq \frac{1}{2}.$$

Therefore, $A[\mathfrak{Q}^\infty]$ is not supersingular, and so neither is $A[\mathfrak{q}^\infty]$. $\square$

The following proposition tells us that $\mathcal{CM}_\Phi^\mathfrak{a}$ is typically nonempty.

PROPOSITION 3.1.6. *There is a unique fractional $\mathcal{O}_F$-ideal $\mathfrak{s}$ for which*

$$\mathfrak{s}\mathcal{O}_K = \mathfrak{D}_0\mathfrak{D}^{-1}.$$

If the discriminants of $K_0/\mathbb{Q}$ and $F/\mathbb{Q}$ are relatively prime, then the category $\mathcal{CM}_\Phi^{\mathfrak{a}}(\mathbb{C})$ is nonempty, and furthermore $\mathfrak{s}^{-1} = \mathfrak{d}_F$.

Proof. Let $\delta_0 \in \widehat{K}_0^\times$ satisfy $\overline{\delta}_0 = -\delta_0$ and $\delta_0\mathcal{O}_{K_0} = \mathfrak{D}_0$, and let $\delta \in \widehat{K}^\times$ satisfy $\overline{\delta} = -\delta$ and $\delta\mathcal{O}_K = \mathrm{Diff}(K/F)$. There is $c \in \widehat{F}^\times$ such that $\delta_0 = c\delta$, and setting $\mathfrak{c} = c\mathcal{O}_F$ the existence of the ideal $\mathfrak{s}$ follows from $\mathfrak{D}_0\mathfrak{D}^{-1} = \mathfrak{c} \cdot \mathrm{Diff}(F/\mathbb{Q})^{-1}$, and the uniqueness is clear. Now assume that $K_0/\mathbb{Q}$ and $F/\mathbb{Q}$ have relatively prime discriminants. This implies that $\mathfrak{D}_0\mathcal{O}_K = \mathrm{Diff}(K/F)$, and the equality $\mathfrak{s}^{-1} = \mathfrak{d}_F$ follows easily.

Taking the product over all $\varphi \in \Phi$ yields an isomorphism of $\mathbb{R}$-vector spaces $K_\mathbb{R} \cong \mathbb{C}^n$ and allows us to view $K_\mathbb{R}$ as a complex vector space. Let $\zeta \in K^\times$ be any element satisfying $\overline{\zeta} = -\zeta$. Using weak approximation we may multiply ζ by an element of $F^\times$ in order to assume that $\varphi(\zeta) \cdot i > 0$ for every $\varphi \in \Phi$. Then $\lambda(x, y) = \mathrm{Tr}_{K/\mathbb{Q}}(\zeta x\overline{y})$ defines an $\mathbb{R}$-symplectic form on $K_\mathbb{R}$, and $\lambda(i \cdot x, x)$ is positive definite. Class field theory implies that the norm map from the ideal class group of K to the narrow ideal class group of F is surjective. ($K_0/\mathbb{Q}$ and $F/\mathbb{Q}$ have relatively prime discriminants, and so K/F is ramified at some finite prime; therefore the Hilbert class field of K and the narrow Hilbert class field of F are linearly disjoint over F.) It follows that there is a fractional $\mathcal{O}_K$-ideal $\mathfrak{A}$ and a $u \in F^{\gg 0}$ satisfying $u\mathfrak{A}\overline{\mathfrak{A}} = \zeta^{-1}\mathfrak{D}^{-1}\mathfrak{a}$. Replacing ζ by ζu^{-1} we may therefore assume $\zeta\mathfrak{A}\overline{\mathfrak{A}} = \mathfrak{a}\mathfrak{D}^{-1}$, and so

$$\mathfrak{a}^{-1}\mathfrak{A} = \{x \in K_\mathbb{R} : \lambda(x, \mathfrak{A}) \subset \mathbb{Z}\}.$$

The Riemann form λ defines a polarization of the complex torus $K_\mathbb{R}/\mathfrak{A}$, and the kernel of this polarization is the subgroup $\mathfrak{a}^{-1}\mathfrak{A}/\mathfrak{A}$ of $\mathfrak{a}$-torsion points. This proves that $\mathcal{CM}_\Phi^{\mathfrak{a}}(\mathbb{C}) \neq \emptyset$. □

3.2. *The space $L(A_0, A)$: first results.* Suppose we are given a connected $\mathcal{O}_\Phi$-scheme S and a pair

$$(A_0, A) \in (\mathcal{M}_{(1,0)} \times \mathcal{CM}_\Phi^{\mathfrak{a}})(S).$$

The $\mathcal{O}_{K_0}$-module

$$L(A_0, A) = \mathrm{Hom}_{\mathcal{O}_{K_0}}(A_0, A)$$

carries a natural positive definite $\mathcal{O}_{K_0}$-Hermitian form [26, Lemma 2.8] defined by

$$\langle f_1, f_2 \rangle = \lambda_0^{-1} \circ f_2^\vee \circ \lambda \circ f_1,$$

and the action of $\mathcal{O}_K$ on A determines an action of $\mathcal{O}_K$ on $L(A_0, A)$ satisfying

$$\langle x \cdot f_1, f_2 \rangle = \langle f_1, \overline{x} \cdot f_2 \rangle$$

for every $x \in \mathcal{O}_K$. It follows that there is a unique K-valued totally positive definite $\mathcal{O}_K$-Hermitian form $\langle f_1, f_2\rangle_{\mathrm{CM}}$ on $L(A_0, A)$ for which

$$\langle f_1, f_2\rangle = \mathrm{Tr}_{K/K_0}\langle f_1, f_2\rangle_{\mathrm{CM}}.$$

Set

$$V(A_0, A) = L(A_0, A) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Recall Serre's twisting construction, as in [7, §7]. Suppose we are given a scheme S, an abelian scheme $B \to S$, an action $\mathcal{O} \to \mathrm{End}(B)$ of an order in a number field, and a projective $\mathcal{O}$-module $\mathfrak{Z}$. To this data we may attach a new abelian scheme $\mathfrak{Z} \otimes_{\mathcal{O}} B$ over S. This abelian scheme is determined by its functor of points

$$(\mathfrak{Z} \otimes_{\mathcal{O}} B)(T) = \mathfrak{Z} \otimes_{\mathcal{O}} B(T)$$

for any S-scheme T.

The following proposition shows that $V(A_0, A)$ is rather small, unless A_0 and A are supersingular.

Proposition 3.2.1. *Suppose k is an algebraically closed field, and*

$$(A_0, A) \in (\mathcal{M}_{(1,0)} \times \mathcal{CM}_{\Phi}^{\mathfrak{a}})(k).$$

If there is an $f \in V(A_0, A)$ such that $\langle f, f\rangle_{\mathrm{CM}} \in F^{\times}$, then k has nonzero characteristic, and A_0 and A are supersingular.

Proof. The map f induces an $\mathcal{O}_{K_0}$-linear map $f_F : \mathcal{O}_F \otimes_{\mathbb{Z}} A_0 \to A$. Fix a prime $\ell \nmid \mathrm{char}(k)$, and for any abelian variety B over k, let

$$\mathrm{Ta}_\ell^0(B) = \mathrm{Ta}_\ell(B) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

be the rational ℓ-adic Tate module. The polarization λ_0 induces a perfect $\mathbb{Q}_\ell$-linear pairing

$$\lambda_0 : \mathrm{Ta}_\ell^0(A_0) \times \mathrm{Ta}_\ell^0(A_0) \to \mathbb{Q}_\ell(1),$$

and tensoring with F_ℓ results in a perfect F_ℓ-linear pairing

$$\Lambda_0 : \mathrm{Ta}_\ell^0(\mathcal{O}_F \otimes_{\mathbb{Z}} A_0) \times \mathrm{Ta}_\ell^0(\mathcal{O}_F \otimes_{\mathbb{Z}} A_0) \to F_\ell(1).$$

The polarization λ induces a perfect pairing

$$\lambda : \mathrm{Ta}_\ell^0(A) \times \mathrm{Ta}_\ell^0(A) \to \mathbb{Q}_\ell(1),$$

which has the form $\lambda = \mathrm{Tr}_{F/\mathbb{Q}}\Lambda$ for a unique F_ℓ-linear

$$\Lambda : \mathrm{Ta}_\ell^0(A) \times \mathrm{Ta}_\ell^0(A) \to F_\ell(1).$$

The *adjoint* of

$$f_F : \mathrm{Ta}_\ell^0(\mathcal{O}_F \otimes_{\mathbb{Z}} A_0) \to \mathrm{Ta}_\ell^0(A) \tag{3.2.1}$$

is the unique $f_F^\dagger : \mathrm{Ta}_\ell^0(A) \to \mathrm{Ta}_\ell^0(\mathcal{O}_F \otimes_{\mathbb{Z}} A_0)$ for which $\Lambda_0(x, f_F^\dagger y) = \Lambda(f_F x, y)$, and some linear algebra shows that $\langle f, f\rangle_{\mathrm{CM}} = f_F^\dagger \circ f_F$ as elements of

$$F_\ell \subset \mathrm{End}_{\mathbb{Q}_\ell}(\mathrm{Ta}_\ell^0(\mathcal{O}_F \otimes_{\mathbb{Z}} A_0)).$$

The hypothesis $\langle f, f\rangle_{\mathrm{CM}} \in F^\times$ now implies that (3.2.1) is injective, and it follows that $f_F : \mathcal{O}_F \otimes_{\mathbb{Z}} A_0 \to A$ is an isogeny. Thus we have $\mathcal{O}_{K_0}$-linear isogenies

$$A \sim \mathcal{O}_F \otimes_{\mathbb{Z}} A_0 \sim \underbrace{A_0 \times \cdots \times A_0}_{n \text{ times}}.$$

As in the proof of [26, Lemma 2.22], the signature conditions imposed on A_0 and A now imply that $\mathrm{char}(k) > 0$ and that A_0 and A are supersingular. $\square$

PROPOSITION 3.2.2. *Suppose k is an algebraically closed field of nonzero characteristic, and suppose*

$$(A_0, A) \in (\mathcal{M}_{(1,0)} \times \mathcal{CM}_\Phi^{\mathfrak{a}})(k),$$

with A_0 and A supersingular. Then $L(A_0, A)$ is a projective $\mathcal{O}_K$-module of rank one. Furthermore, if q is a rational prime (which may or may not equal the characteristic of k), and $\mathfrak{q}$ is a prime of F above q, then the natural map

$$L(A_0, A) \otimes_{\mathcal{O}_F} \mathcal{O}_{F,\mathfrak{q}} \to \mathrm{Hom}_{\mathcal{O}_{K_0}}(A_0[q^\infty], A[\mathfrak{q}^\infty])$$

is an isomorphism. Here $A_0[q^\infty]$ and $A[\mathfrak{q}^\infty]$ are the q-divisible groups of q-power and $\mathfrak{q}$-power torsion in A_0 and A.

Proof. An argument using the Noether-Skolem theorem (as in the beginning of the proof of Proposition 2.3.3) shows there is an $\mathcal{O}_{K_0}$-linear isogeny

$$A \to \underbrace{A_0 \times \cdots \times A_0}_{n \text{ times}}.$$

Fixing such an isogeny determines an injection with finite cokernel

$$\mathrm{Hom}_{\mathcal{O}_{K_0}}(A_0, A) \to \mathrm{Hom}_{\mathcal{O}_{K_0}}(A_0, A_0 \times \cdots \times A_0) \cong \mathcal{O}_{K_0} \times \cdots \times \mathcal{O}_{K_0},$$

and we deduce that $\mathrm{Hom}_{\mathcal{O}_{K_0}}(A_0, A)$ is a projective $\mathcal{O}_{K_0}$-module of rank n. For the same reason $\mathrm{Hom}_{\mathcal{O}_{K_0}}(A_0[q^\infty], A[q^\infty])$ is a projective $\mathcal{O}_{K_0,q}$-module of rank n. The natural map

$$\mathrm{Hom}_{\mathcal{O}_{K_0}}(A_0, A) \otimes_{\mathbb{Z}} \mathbb{Z}_q \to \mathrm{Hom}_{\mathcal{O}_{K_0}}(A_0[q^\infty], A[q^\infty])$$

is injective with $\mathbb{Z}_q$-torsion free cokernel, and hence is an isomorphism, as both sides have the same $\mathbb{Z}_q$-rank. It follows easily that

$$L(A_0, A) \otimes_{\mathcal{O}_F} \mathcal{O}_{F,\mathfrak{q}} \to \mathrm{Hom}_{\mathcal{O}_{K_0}}(A_0[q^\infty], A[\mathfrak{q}^\infty])$$

is an isomorphism.

We now prove that $L(A_0, A)$ is a projective $\mathcal{O}_K$-module of rank one. Of course if K is a field, this is obvious, as we know from the previous paragraph that $L(A_0, A)$ is a torsion-free $\mathbb{Z}$-module of the same rank as $\mathcal{O}_K$. The point is to rule out the possibility that the action of $\mathcal{O}_K$ on $L(A_0, A)$ factors through projection to a proper direct summand of $\mathcal{O}_K$. Fix a prime $q \neq \mathrm{char}(k)$. The argument used in the proof of [38, Lemma 1.3] shows that the q-adic Tate modules $\mathrm{Ta}_q(A_0)$ and $\mathrm{Ta}_q(A)$ are free of rank one over $\mathcal{O}_{K_0,q}$ and $\mathcal{O}_{K,q}$, respectively, and combining this with the paragraph above shows that

$$L(A_0, A) \otimes_{\mathbb{Z}} \mathbb{Z}_q \cong \mathrm{Hom}_{\mathcal{O}_{K_0}}(\mathrm{Ta}_q(A_0), \mathrm{Ta}_q(A)) \cong \mathcal{O}_{K,q}.$$

As $L(A_0, A)$ is $\mathbb{Z}$-torsion free, this is enough to show that $L(A_0, A)$ is projective of rank one. $\square$

3.3. *Twisting Hermitian spaces.* In the next subsection we will determine the structure of the Hermitian space $L(A_0, A)$ of Proposition 3.2.2 more explicitly. In this subsection we first recall some elementary properties of Hermitian spaces. Suppose L is a projective $\mathcal{O}_K$-module of rank one, $V = L \otimes_{\mathcal{O}_K} K$, and H is a nondegenerate K-Hermitian form on V. By fixing a K-linear isomorphism $V \cong K$ we see that

$$(L, H) \cong (\mathfrak{A}, \alpha x\overline{y})$$

for some $\alpha \in F^\times$ and some fractional $\mathcal{O}_K$-ideal $\mathfrak{A}$. Of course $\alpha x\overline{y}$ is shorthand for the Hermitian form $(x, y) \mapsto \alpha x\overline{y}$. For any place v of F, let

$$\chi_v : F_v^\times \to \{\pm 1\}$$

be the quadratic character associated to the extension K_v/F_v. The *local invariant* of (L, H) at v is $\chi_v(\alpha)$. If v is archimedean, then knowing the local invariant at v is equivalent to knowing the signature of (V, H) at v. The collection of local invariants determines the space (V, H) up to isomorphism. If we choose an $\widehat{\mathcal{O}}_K$-linear isomorphism $\widehat{L} \cong \widehat{\mathcal{O}}_K$, then

$$(\widehat{L}, H) \cong (\widehat{\mathcal{O}}_K, \beta x\overline{y})$$

for some $\beta \in \widehat{F}^\times$ satisfying $\chi_v(\alpha) = \chi_v(\beta)$ for all finite v, and satisfying

$$\beta\mathcal{O}_F = \alpha\mathfrak{A}\overline{\mathfrak{A}}.$$

Define the *ideal* of (L, H) to be the fractional $\mathcal{O}_F$-ideal $\beta\mathcal{O}_F$. Given another projective $\mathcal{O}_K$-module of rank one L' and a Hermitian form H', we say that the pairs (L, H) and (L', H') *belong to the same genus* if they have the same signature at every archimedean place and if $(\widehat{L}, H) \cong (\widehat{L}', H')$. It is not hard to see that the genus of (L, H) is completely determined by

- the ideal $\beta\mathcal{O}_F$,
- the local invariant at every finite prime of F ramified in K,
- the signature at every archimedean place.

There is a natural group action on the set of all isomorphism classes of pairs (L, H). Let I_K be the set of all pairs $\mathfrak{z} = (\mathfrak{Z}, \zeta)$ where $\mathfrak{Z}$ is a fractional $\mathcal{O}_K$-ideal and $\zeta \in F^{\gg 0}$ satisfies $\zeta \mathfrak{Z}\overline{\mathfrak{Z}} = \mathcal{O}_K$. The set I_K is a group under componentwise multiplication, and has a natural subgroup

$$P_K = \{(z^{-1}\mathcal{O}_K, z\overline{z}) : z \in K^\times\}.$$

Denote by $C_K = I_K/P_K$ the quotient group. Given a $\mathfrak{z} \in C_K$ and a pair (L, H) as above, define a new pair

$$\mathfrak{z} \bullet (L, H) = (\mathfrak{Z}L, \zeta H).$$

The ideal and signature of (L, H) are obviously unchanged by this action, and the finite group C_K acts simply transitively on the set of isomorphism classes of pairs with the same ideal and signature as (L, H).

The action of C_K does not preserve the genus of (L, H), but it has a natural subgroup that does. Define an algebraic group over F

$$H = \ker(\mathrm{Nm} : K^\times \to F^\times),$$

a compact open subgroup

$$U = \ker(\mathrm{Nm} : \widehat{\mathcal{O}}_K^\times \to \widehat{\mathcal{O}}_F^\times) \subset H(\widehat{F}),$$

and a finite group

$$C_K^0 = H(F)\backslash H(\widehat{F})/U.$$

The rule $h \mapsto (h\mathcal{O}_K, 1)$ defines an injection $C_K^0 \to C_K$ whose image is the *genus subgroup* of C_K. Let

$$\eta : \widehat{\mathcal{O}}_F^\times/\mathrm{Nm}_{K/F}(\widehat{\mathcal{O}}_K^\times) \to \{\pm 1\}^{\pi_0(F)} \tag{3.3.1}$$

be the restriction to $\widehat{\mathcal{O}}_F^\times$ of the character (3.0.2). There is an exact sequence

$$1 \to C_K^0 \to C_K \xrightarrow{\mathrm{gen}} \widehat{\mathcal{O}}_F^\times/\mathrm{Nm}_{K/F}(\widehat{\mathcal{O}}_K^\times) \xrightarrow{\eta} \{\pm 1\}^{\pi_0(F)}, \tag{3.3.2}$$

where the middle arrow (the *genus invariant*) is defined as follows: given $\mathfrak{z} \in I_K$, choose a finite idele $z \in \widehat{K}^\times$ such that $z\mathcal{O}_K = \mathfrak{Z}$ and set

$$\mathrm{gen}(\mathfrak{z}) = \zeta z\overline{z}.$$

A simple calculation shows that

$$\mathfrak{z} \bullet (\widehat{L}, H) \cong (\widehat{L}, \mathrm{gen}(\mathfrak{z}) \cdot H),$$

and it follows easily that C_K^0 acts simply transitively on the genus of (L, H).

For us, the usefulness of the action of C_K on Hermitian spaces is that it is compatible with the twisting construction of Serre. Suppose S is a connected $\mathcal{O}_\Phi$-scheme and

$$(A, \kappa, \lambda) \in \mathcal{CM}_\Phi^{\mathfrak{a}}(S).$$

Given $\mathfrak{z} = (\mathfrak{Z}, \zeta) \in I_K$, the abelian scheme

$$A^{\mathfrak{z}} = \mathfrak{Z} \otimes_{\mathcal{O}_K} A$$

carries a natural $\mathcal{O}_K$-action $\kappa^{\mathfrak{z}} : \mathcal{O}_K \to \mathrm{End}(A^{\mathfrak{z}})$ defined by $\kappa^{\mathfrak{z}}(x) = \mathrm{id} \otimes \kappa(x)$, which again satisfies the Φ-determinant condition. There is a quasi-isogeny

$$s \in \mathrm{Hom}_{\mathcal{O}_K}(A^{\mathfrak{z}}, A) \otimes_{\mathbb{Z}} \mathbb{Q}$$

defined by $s(z \otimes a) = \kappa(z) \cdot a$, and the composition

$$\lambda^{\mathfrak{z}} = s^{\vee} \circ \lambda \circ \kappa(\zeta) \circ s$$

is an $\mathcal{O}_K$-linear polarization of $A^{\mathfrak{z}}$ with kernel $A^{\mathfrak{z}}[\mathfrak{a}]$. For a proof that $\lambda \circ \kappa(\zeta)$, and hence $\lambda^{\mathfrak{z}}$, is a polarization, see [38, Prop. 1.17]; it is here where we must assume $\zeta \gg 0$. We obtain a new object

$$(A^{\mathfrak{z}}, \kappa^{\mathfrak{z}}, \lambda^{\mathfrak{z}}) \in \mathcal{CM}^{\mathfrak{a}}_{\Phi}(S),$$

and in this way the group C_K acts on the set of isomorphism classes of objects in $\mathcal{CM}^{\mathfrak{a}}_{\Phi}(S)$.

Now fix a pair

$$(A_0, A) \in (\mathcal{M}_{(1,0)} \times \mathcal{CM}^{\mathfrak{a}}_{\Phi})(S).$$

Let $\langle f_1, f_2 \rangle^{\mathfrak{z}}_{\mathrm{CM}}$ denote the $\mathcal{O}_K$-Hermitian form on $L(A_0, A^{\mathfrak{z}})$. By [7, Lemma 7.14], the function $f \mapsto s \circ f$ defines an isomorphism of $\mathcal{O}_K$-modules

$$L(A_0, A^{\mathfrak{z}}) \cong \mathfrak{Z} \cdot L(A_0, A) \tag{3.3.3}$$

identifying $\langle \cdot, \cdot \rangle^{\mathfrak{z}}_{\mathrm{CM}} = \zeta \cdot \langle \cdot, \cdot \rangle_{\mathrm{CM}}$. In other words,

$$\mathfrak{z} \bullet (L(A_0, A), \langle \cdot, \cdot \rangle_{\mathrm{CM}}) \cong (L(A_0, A^{\mathfrak{z}}), \langle \cdot, \cdot \rangle^{\mathfrak{z}}_{\mathrm{CM}})$$

(at least assuming that $L(A_0, A)$ is projective of rank one, the only case in which we have defined the action $\mathfrak{z}\bullet$).

Here is the form in which these results will be used.

PROPOSITION 3.3.1. *Suppose S is a connected $\mathcal{O}_{\Phi}$-scheme and*

$$(A_0, A) \in (\mathcal{M}_{(1,0)} \times \mathcal{CM}^{\mathfrak{a}}_{\Phi})(S).$$

For any $\mathfrak{z} \in C_K$, there is an isomorphism of $\widehat{\mathcal{O}}_K$-modules

$$\widehat{L}(A_0, A^{\mathfrak{z}}) \cong \widehat{L}(A_0, A)$$

identifying the Hermitian form $\langle \cdot, \cdot \rangle^{\mathfrak{z}}_{\mathrm{CM}}$ on the left with the form $\mathrm{gen}(\mathfrak{z})\langle \cdot, \cdot \rangle_{\mathrm{CM}}$ on the right.

Proof. Fix a representative $(\mathfrak{Z}, \zeta) \in I_K$ of $\mathfrak{z}$, and a $z \in \widehat{K}^{\times}$ satisfying $z\mathcal{O}_K = \mathfrak{Z}$. Using multiplication by z to identify

$$\widehat{L}(A_0, A) \cong \mathfrak{Z} \cdot \widehat{L}(A_0, A),$$

and using (3.3.3), we obtain an isomorphism

$$\widehat{L}(A_0, A) \cong \widehat{L}(A_0, A^{\mathfrak{z}})$$

denoted $f \mapsto f^{\mathfrak{z}}$, where $f^{\mathfrak{z}} = s^{-1} \circ \kappa(z^{-1}) \circ f$. This isomorphism satisfies

$$\langle f_1^{\mathfrak{z}}, f_2^{\mathfrak{z}} \rangle_{\mathrm{CM}}^{\mathfrak{z}} = \zeta z\overline{z} \cdot \langle f_1, f_2 \rangle_{\mathrm{CM}},$$

as desired. □

3.4. *Calculation of* $L(A_0, A)$. We now proceed to compute $L(A_0, A)$ in particular cases, the most important being the case where A_0 and A are supersingular.

First consider the situation in characteristic 0. For a pair

$$(A_0, A) \in (\mathcal{M}_{(1,0)} \times \mathcal{CM}_{\Phi}^{\mathfrak{a}})(\mathbb{C}),$$

the space $L(A_0, A)$ is rather small. For example if F is a field, it follows from Proposition 3.2.1 that $L(A_0, A) = 0$. As a substitute for this space, we replace A_0 and A by their first homology groups

$$H_1(A_0) = H_1(A_0(\mathbb{C}), \mathbb{Z}), \qquad H_1(A) = H_1(A(\mathbb{C}), \mathbb{Z}),$$

and define

$$L_{\mathrm{B}}(A_0, A) = \mathrm{Hom}_{\mathcal{O}_{K_0}}(H_1(A_0), H_1(A)). \tag{3.4.1}$$

The polarizations of A_0 and A induce symplectic forms on $H_1(A_0)$ and $H_1(A)$, which we view as $\mathbb{Z}$-module maps from $H_1(A_0)$ and $H_1(A)$ to their $\mathbb{Z}$-duals. The $\mathcal{O}_K$-module $L_B(A_0, A)$ is then endowed with an $\mathcal{O}_{K_0}$-Hermitian form $\langle \cdot, \cdot \rangle$, and an $\mathcal{O}_K$-Hermitian form $\langle \cdot, \cdot \rangle_{\mathrm{CM}}$ defined exactly as for $L(A_0, A)$. One may think of $L_B(A_0, A)$ as the space of $\mathcal{O}_{K_0}$-linear maps of *real* Lie groups $A_0(\mathbb{C}) \to A(\mathbb{C})$, and so there is an obvious injection of Hermitian $\mathcal{O}_K$-modules

$$L(A_0, A) \to L_{\mathrm{B}}(A_0, A).$$

Abbreviate

$$V_B(A_0, A) = L_B(A_0, A) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

The structure of $L_B(A_0, A)$ is quite easy to describe. Recall that the fractional $\mathcal{O}_F$-ideal $\mathfrak{s}$ was defined in Proposition 3.1.6.

PROPOSITION 3.4.1. *Suppose*

$$(A_0, A) \in (\mathcal{M}_{(1,0)} \times \mathcal{CM}_{\Phi}^{\mathfrak{a}})(\mathbb{C}).$$

There is a $\beta \in \widehat{F}^{\times}$ *satisfying* $\beta \mathcal{O}_F = \mathfrak{a}\mathfrak{s}$, *and an isomorphism*

$$\left(\widehat{L}_B(A_0, A), \langle \cdot, \cdot \rangle_{\mathrm{CM}}\right) \cong \left(\widehat{\mathcal{O}}_K, \beta x\overline{y}\right).$$

Furthermore, the $\mathcal{O}_K$*-Hermitian form* $\langle \cdot, \cdot \rangle_{\mathrm{CM}}$ *is negative definite at the place* ∞^{sp} *defined after* (3.0.1), *and positive definite at all other archimedean places of* F.

Proof. This follows from the classical theory of CM abelian varieties over $\mathbb{C}$. For some fractional $\mathcal{O}_K$-ideal $\mathfrak{A}$, there is an isomorphism of $\mathcal{O}_K$-modules $\mathfrak{A} \cong H_1(A)$, and the polarization λ determines a symplectic pairing on $\mathfrak{A}$ of the form

$$\lambda(x, y) = \mathrm{Tr}_{K/\mathbb{Q}}(\zeta x \overline{y}),$$

where $\zeta \in K^\times$ satisfies $\overline{\zeta} = -\zeta$ and $\zeta \mathfrak{A}\overline{\mathfrak{A}} = \mathfrak{a}\mathfrak{D}^{-1}$. The real vector space $\mathfrak{A}_\mathbb{R}$ is canonically identified with $\mathrm{Lie}(A)$, and hence comes with a complex structure for which the quadratic form $\lambda(ix, x)$ is positive definite. This last condition is equivalent to $\varphi(\zeta) \cdot i > 0$ for every $\varphi \in \Phi$.

Similarly, for some fractional $\mathcal{O}_{K_0}$-ideal $\mathfrak{A}_0$, there is an isomorphism of $\mathcal{O}_{K_0}$-modules $\mathfrak{A}_0 \cong H_1(A_0)$, and the symplectic form on $\mathfrak{A}_0$ induced by the polarization λ_0 has the form

$$\lambda_0(x, y) = \mathrm{Tr}_{K_0/\mathbb{Q}}(\zeta_0 x \overline{y})$$

for some $\zeta_0 \in K_0^\times$ satisfying $\overline{\zeta}_0 = -\zeta_0$, $\zeta_0 \mathfrak{A}_0 \overline{\mathfrak{A}}_0 = \mathfrak{D}_0^{-1}$, and $\iota(\zeta_0) \cdot i > 0$.

There are now isomorphisms of $\mathcal{O}_K$-modules

$$\mathfrak{A}_0^{-1}\mathfrak{A} \cong \mathrm{Hom}_{\mathcal{O}_{K_0}}(\mathfrak{A}_0, \mathfrak{A}) \cong L_B(A_0, A),$$

and under these identifications the $\mathcal{O}_K$-Hermitian form on $L_B(A_0, A)$ is identified with the $\mathcal{O}_K$-Hermitian form $\zeta_0^{-1}\zeta x \overline{y}$ on $\mathfrak{A}_0^{-1}\mathfrak{A}$. If $\varphi \in \Phi$ with $\varphi \neq \varphi^{\mathrm{sp}}$, then $\varphi(\zeta_0^{-1}\zeta) > 0$, while $\varphi^{\mathrm{sp}}(\zeta_0^{-1}\zeta) < 0$. This shows that $\langle f, f \rangle_{\mathrm{CM}}$ is negative definite at ∞^{sp} and positive definite at all other archimedean places of F. The rest follows by fixing an $\widehat{\mathcal{O}}_K$-linear isomorphism $\mathfrak{A}_0^{-1}\mathfrak{A}\widehat{\mathcal{O}}_K \cong \widehat{\mathcal{O}}_K$. □

Remark 3.4.2. Of course Proposition 3.4.1 does not determine the Hermitian space $L_B(A_0, A)$ up to isomorphism, nor does it even determine the genus of $L_B(A_0, A)$. In the terminology of Section 3.3, Proposition 3.4.1 tells us the ideal of $L_B(A_0, A)$, and the signature at every archimedean place, and so only determines the C_K-orbit of $L_B(A_0, A)$. Let $\mathcal{L}_B$ denote the set of isomorphism classes of pairs (L, H) where

- L is a projective $\mathcal{O}_K$-module of rank one,
- H is a K-valued $\mathcal{O}_K$-Hermitian form on L,
- (L, H) has ideal $\mathfrak{a}\mathfrak{s}$, in the terminology of Section 3.3,
- (L, H) is negative definite at ∞^{sp} and positive definite at all other archimedean places of F.

This is a transitive C_K-set, and Proposition 3.4.1 tells us that every $L_B(A_0, A)$ lies in $\mathcal{L}_B$. The discussion preceding Proposition 3.3.1 applies equally well to $L_B(A_0, A)$ and shows that

$$\mathfrak{z} \bullet (L_B(A_0, A), \langle \cdot, \cdot \rangle_{\mathrm{CM}}) \cong (L_B(A_0, A^{\mathfrak{z}}), \langle \cdot, \cdot \rangle_{\mathrm{CM}}^{\mathfrak{z}})$$

for any $\mathfrak{z} \in C_K$. Thus as the pair (A_0, A) varies, we obtain every element of $\mathcal{L}_B$. In this sense, Proposition 3.4.1 is as sharp as possible.

The remainder of this subsection is devoted to the proof of the following theorem, which similarly determines the C_K-orbit of the Hermitian space $(L(A_0, A), \langle\cdot,\cdot\rangle_{\mathrm{CM}})$ at a supersingular point.

THEOREM 3.4.3. *Suppose $\mathfrak{p}$ is a prime of K_Φ for which $\mathfrak{p}_F$ is nonsplit in K, and suppose*

$$(A_0, A) \in (\mathcal{M}_{(1,0)} \times \mathcal{CM}_\Phi^{\mathfrak{a}})(k_{\Phi,\mathfrak{p}}^{\mathrm{alg}}).$$

There is an isomorphism

$$\big(\widehat{L}(A_0, A), \langle\cdot,\cdot\rangle_{\mathrm{CM}}\big) \cong \big(\widehat{\mathcal{O}}_K, \beta x\overline{y}\big)$$

for some $\beta \in \widehat{F}^\times$ satisfying

$$\beta\mathcal{O}_F = \mathfrak{a}\mathfrak{s}\mathfrak{p}_F^{\varepsilon_p}.$$

Here p is the rational prime below $\mathfrak{p}$ and ε_p is defined by (1.2.4). *Furthermore, if we view $\beta \in F_{\mathbb{A}}^\times$ with trivial archimedean components, then $\chi_{K/F}(\beta) = 1$.*

Proof. The pair (A_0, A) is necessarily supersingular: A is supersingular by Proposition 3.1.5 and A_0 is supersingular as p is nonsplit in K_0. We will determine the structure of $\big(L(A_0, A), \langle\cdot,\cdot\rangle_{\mathrm{CM}}\big)$ by exploiting the fact that the pair (A_0, A) has a canonical lift, in the sense of Definition 3.1.4. This will allow us to reduce most of the calculation of $L(A_0, A)$ to a calculation in characteristic 0, where Proposition 3.4.1 applies.

By Remark 3.1.3 there is a unique lift of (A_0, A) to a pair

$$(A_0', A') \in (\mathcal{M}_{(1,0)} \times \mathcal{CM}_\Phi^{\mathfrak{a}})(\mathbb{C}_{\mathfrak{p}}).$$

After fixing an isomorphism of K_Φ-algebras $\mathbb{C}_{\mathfrak{p}} \cong \mathbb{C}$, we may view (A_0', A') also as a pair

$$(A_0', A') \in (\mathcal{M}_{(1,0)} \times \mathcal{CM}_\Phi^{\mathfrak{a}})(\mathbb{C}). \tag{3.4.2}$$

The comparison between $L(A_0, A)$ and $L_B(A_0', A')$ now proceeds by replacing A_0, A, A_0', and A' by their Barsotti-Tate groups. Suppose $\mathfrak{q} \subset \mathcal{O}_F$ is a prime lying above a rational prime q (which may or may not equal p). The $\mathcal{O}_{K,\mathfrak{q}}$-module

$$L_{\mathfrak{q}}(A_0, A) = \mathrm{Hom}_{\mathcal{O}_{K_0}}(A_0[q^\infty], A[\mathfrak{q}^\infty])$$

comes equipped with a $K_{\mathfrak{q}}$-valued $\mathcal{O}_{K,\mathfrak{q}}$-Hermitian form $\langle f_1, f_2\rangle_{\mathrm{CM}}$ defined exactly as above. Similarly, define an $\mathcal{O}_{K,\mathfrak{q}}$-Hermitian space

$$L_{\mathfrak{q}}(A_0', A') = \mathrm{Hom}_{\mathcal{O}_{K_0}}(A_0'[q^\infty], A'[\mathfrak{q}^\infty]).$$

There are isomorphism of Hermitian $\mathcal{O}_{K,\mathfrak{q}}$-modules

$$\begin{aligned} L_{\mathrm{B}}(A_0', A') \otimes_{\mathcal{O}_K} \mathcal{O}_{K,\mathfrak{q}} &\cong L_{\mathfrak{q}}(A_0', A'), \\ L(A_0, A) \otimes_{\mathcal{O}_K} \mathcal{O}_{K,\mathfrak{q}} &\cong L_{\mathfrak{q}}(A_0, A). \end{aligned} \tag{3.4.3}$$

The first is obvious, as the q-divisible groups of A'_0 and A' are constant, and isomorphic to $H_1(A'_0) \otimes_{\mathbb{Z}} \mathbb{Q}_q/\mathbb{Z}_q$ and $H_1(A') \otimes_{\mathbb{Z}} \mathbb{Q}_q/\mathbb{Z}_q$, respectively. The second isomorphism is part of the statement of Proposition 3.2.2. These isomorphisms, together with the following lemma, allow us to convert information about $L_B(A'_0, A')$ to information about $L(A_0, A)$.

LEMMA 3.4.4. *Suppose $\mathfrak{q} \subset \mathcal{O}_F$ is a prime with $\mathfrak{q} \neq \mathfrak{p}_F$. There is an $\mathcal{O}_K$-linear isomorphism*

$$L_{\mathfrak{q}}(A'_0, A') \cong L_{\mathfrak{q}}(A_0, A) \tag{3.4.4}$$

respecting the Hermitian forms.

Proof. Let q be the rational prime below $\mathfrak{q}$. If $q \neq p$, then the q-adic Tate modules of A' and A are canonically isomorphic, and similarly for the q-adic Tate modules of A'_0 and A_0. Therefore

$$\mathrm{Hom}_{\mathcal{O}_{K_0}}(A'_0[q^\infty], A'[q^\infty]) \cong \mathrm{Hom}_{\mathcal{O}_{K_0}}(A_0[q^\infty], A[q^\infty])$$

and (3.4.4) follows by taking $\mathfrak{q}$-parts.

Now suppose $q = p$, so $\mathfrak{q}$ lies above p. Let $\Phi(\mathfrak{q})$ be the set of all $\varphi \in \Phi$ that, when viewed as a map $K \to \mathbb{C}_{\mathfrak{p}}$, induce the prime $\mathfrak{q}$. The hypothesis that $\mathfrak{q} \neq \mathfrak{p}_F$ implies that $\varphi^{\mathrm{sp}} \notin \Phi(\mathfrak{q})$, and so every $\varphi \in \Phi(\mathfrak{q})$ satisfies $\varphi|_{K_0} = \iota$. In the terminology of Section 2.4, $\Phi(\mathfrak{q})$ is a p-adic CM type of $K_{\mathfrak{q}}$ of signature $(m, 0)$, where $m = [F_{\mathfrak{q}} : \mathbb{Q}_p]$. Furthermore, the p-divisible group $A[\mathfrak{q}^\infty]$, with its action of $\mathcal{O}_{K,\mathfrak{q}}$, satisfies the $\Phi(\mathfrak{q})$-determinant condition of Section 2.1. By Proposition 2.4.1, the reduction map

$$\mathrm{Hom}_{\mathcal{O}_{K_0}}(A_0^{\mathrm{can}}[p^\infty], A^{\mathrm{can}}[\mathfrak{q}^\infty]) \to \mathrm{Hom}_{\mathcal{O}_{K_0}}(A_0[p^\infty], A[\mathfrak{q}^\infty])$$

is an isomorphism. Strictly speaking, Proposition 2.4.1 deals with deformations to Artinian quotients of $W_{\Phi,\mathfrak{p}}$, but one may pass to the limit by applying [7, Th. 3.4] to truncated p-divisible groups.

The pair (A'_0, A') is the image of $(A_0^{\mathrm{can}}, A^{\mathrm{can}})$ under base change through $W_{\Phi,\mathfrak{p}} \to \mathbb{C}_{\mathfrak{p}}$, and base change defines an injection

$$\mathrm{Hom}_{\mathcal{O}_{K_0}}(A_0^{\mathrm{can}}[p^\infty], A^{\mathrm{can}}[\mathfrak{q}^\infty]) \to \mathrm{Hom}_{\mathcal{O}_{K_0}}(A'_0[p^\infty], A'[\mathfrak{q}^\infty])$$

whose image is, by Tate's theorem [41, p. 181], the submodule of invariants for the action of $\mathrm{Aut}(\mathbb{C}_{\mathfrak{p}}/W_{\Phi,\mathfrak{p}})$. In particular, the cokernel is $\mathbb{Z}_p$-torsion free. We have now constructed an injection

$$L_{\mathfrak{q}}(A_0, A) \to L_{\mathfrak{q}}(A'_0, A')$$

with $\mathbb{Z}_p$-torsion free cokernel. But Propositions 3.4.1 and 3.2.2, together with the isomorphisms (3.4.3), imply that the domain and codomain are free of rank one over $\mathcal{O}_{K,\mathfrak{q}}$, and so this map is an isomorphism. It is clear from the construction that it respects the Hermitian forms. □

It only remains to collect the pieces together. Let $\mathfrak{q}$ be a prime of F. If $\mathfrak{q} \neq \mathfrak{p}_F$, then (3.4.3), and Lemma 3.4.4 tell us that

$$L_B(A_0', A') \otimes_{\mathcal{O}_F} \mathcal{O}_{F,\mathfrak{q}} \cong L(A_0, A) \otimes_{\mathcal{O}_F} \mathcal{O}_{F,\mathfrak{q}},$$

and so by Proposition 3.4.1 there is an isomorphism

$$L(A_0, A) \otimes_{\mathcal{O}_F} \mathcal{O}_{F,\mathfrak{q}} \cong \mathcal{O}_{K,\mathfrak{q}}$$

identifying $\langle \cdot, \cdot \rangle_{\mathrm{CM}}$ with $\beta_\mathfrak{q} x\overline{y}$ for some $\beta_\mathfrak{q} \in F_\mathfrak{q}^\times$ satisfying $\beta_\mathfrak{q}\mathcal{O}_{F,\mathfrak{q}} = \mathfrak{as}\mathcal{O}_{F,\mathfrak{q}}$.

If $\mathfrak{q} = \mathfrak{p}_F$ then, as in the proof of Lemma 3.4.4, let $\Phi(\mathfrak{q})$ be the set of all $\varphi \in \Phi$ which, when viewed as a map $K \to \mathbb{C}_\mathfrak{p}$, induce the prime $\mathfrak{q}$. The assumption that $\mathfrak{q} = \mathfrak{p}_F$ implies that $\varphi^{\mathrm{sp}} \in \Phi(\mathfrak{q})$, and the p-adic CM type $\Phi(\mathfrak{q})$ of $K_\mathfrak{q}$ has signature, in the terminology of Section 2.3, $(m-1, 1)$ where $m = [F_\mathfrak{q} : \mathbb{Q}_p]$. The p-divisible group $A[\mathfrak{q}^\infty]$, with its action of $\mathcal{O}_{K,\mathfrak{q}}$, satisfies the $\Phi(\mathfrak{q})$-determinant condition, and so the results of Section 2.3 apply. In particular, Proposition 2.3.3 and (3.4.3) give isomorphisms

$$L(A_0, A) \otimes_{\mathcal{O}_F} \mathcal{O}_{F,\mathfrak{q}} \cong L_\mathfrak{q}(A_0, A) \cong \mathcal{O}_{K,\mathfrak{q}},$$

which identify $\langle f_1, f_2 \rangle_{\mathrm{CM}}$ with $\beta_\mathfrak{q} x\overline{y}$ for some $\beta_\mathfrak{q} \in F_\mathfrak{q}^\times$ satisfying $\beta_\mathfrak{q}\mathcal{O}_{F,\mathfrak{q}} = \mathfrak{as}\mathfrak{p}_F^{\varepsilon_p}\mathcal{O}_{F,\mathfrak{q}}$.

Setting $\beta = \prod_\mathfrak{q} \beta_\mathfrak{q}$, we have now shown that there is an isomorphism

$$\widehat{L}(A_0, A) \cong \widehat{\mathcal{O}}_K$$

identifying $\langle \cdot, \cdot \rangle_{\mathrm{CM}}$ with $\beta x\overline{y}$. It only remains to show that $\chi_{K/F}(\beta) = 1$. We know that $V(A_0, A)$ is a free K-module of rank one, equipped with a positive definite Hermitian form. It follows that for some $\beta^* \in F^{\gg 0}$, there is an isomorphism $V(A_0, A) \cong K$ identifying $\langle \cdot, \cdot \rangle_{\mathrm{CM}}$ with $\beta^* x\overline{y}$. Certainly $\chi_{K/F}(\beta^*) = 1$, and β and β^* differ everywhere locally by a norm from $K_\mathbb{A}^\times$. Therefore, also $\chi_{K/F}(\beta) = 1$, completing the proof of Theorem 3.4.3. □

The following proposition is not needed in the proofs of our main results, but it is illuminating, and follows easily from what has been said.

Proposition 3.4.5. *Let (A_0, A) be as in Theorem* 3.4.3, *and let (A_0', A') be as in* (3.4.2). *The K-Hermitian spaces $V_B(A_0', A')$ and $V(A_0, A)$ are isomorphic locally at a place v of F if and only if $v \notin \{\infty^{\mathrm{sp}}, \mathfrak{p}_F\}$.*

Proof. The set of places of F at which the Hermitian spaces in question are not isomorphic is finite of even cardinality. As the second is totally positive definite, Proposition 3.4.1 implies that they are isomorphic at all archimedean places except ∞^{sp}. Therefore, the set of finite places of F at which they are not isomorphic has odd cardinality. By Lemma 3.4.4 they are isomorphic at all finite places $\mathfrak{q} \neq \mathfrak{p}_F$, and it follows that $\mathfrak{p}_F$ is the unique finite place at which they are not isomorphic. □

One may interpret Proposition 3.4.5 as follows. Recall the collection of $\mathcal{O}_K$-Hermitian spaces $\mathcal{L}_B$ of Remark 3.4.2, and define a collection of rank one K-Hermitian spaces

$$\mathcal{V}_B = \{(L \otimes_{\mathcal{O}_K} K, H) : (L, H) \in \mathcal{L}_B\}.$$

This is precisely the collection of Hermitian spaces $V_B(A_0', A')$ that appear as the pair (A_0, A) varies in Theorem 3.4.3. A rank one Hermitian space is determined by the collection of local invariants at all places of F, and for each space in $\mathcal{V}_B$ one can construct a new Hermitian space by changing the invariant both at ∞^{sp} and at $\mathfrak{p}_F$. If we denote by $\mathcal{V}_B(\mathfrak{p})$ the set of Hermitian spaces obtained from $\mathcal{V}_B$ in this way, then as the pair (A_0, A) varies in Theorem 3.4.3, the Hermitian spaces $V(A_0, A)$ vary over $\mathcal{V}_B(\mathfrak{p})$.

3.5. *The stack $\mathcal{Z}^{\mathfrak{a}}_{\Phi}(\alpha)$.* If S is an $\mathcal{O}_\Phi$-scheme, then to each S-valued point

$$(A_0, A) \in (\mathcal{M}_{(1,0)} \times \mathcal{CM}^{\mathfrak{a}}_{\Phi})(S)$$

we have associated an $\mathcal{O}_K$-module $L(A_0, A)$ equipped with an $\mathcal{O}_K$-Hermitian form $\langle \cdot, \cdot \rangle_{\mathrm{CM}}$.

Definition 3.5.1. For any $\alpha \in F$, let $\mathcal{Z}^{\mathfrak{a}}_{\Phi}(\alpha)$ be the algebraic stack over $\mathcal{O}_\Phi$ classifying triples (A_0, A, f) over $\mathcal{O}_\Phi$-schemes S in which

- $(A_0, A) \in (\mathcal{M}_{(1,0)} \times \mathcal{CM}^{\mathfrak{a}}_{\Phi})(S)$,
- $f \in L(A_0, A)$ satisfies $\langle f, f \rangle_{\mathrm{CM}} = \alpha$.

If $\alpha = \mathcal{O}_F$, we omit it from the notation.

The evident forgetful morphism

$$\mathcal{Z}^{\mathfrak{a}}_{\Phi}(\alpha) \to \mathcal{M}_{(1,0)} \times \mathcal{CM}^{\mathfrak{a}}_{\Phi}$$

is finite and unramified, by the proof of [26, Prop. 2.10].

PROPOSITION 3.5.2. *Suppose $\alpha \in F^\times$.*

(1) *The stack $\mathcal{Z}^{\mathfrak{a}}_{\Phi}(\alpha)$ has dimension zero, is supported in nonzero characteristic, and every geometric point is supersingular. Furthermore, $\mathcal{Z}^{\mathfrak{a}}_{\Phi}(\alpha)$ is empty unless α is totally positive.*
(2) *If $\mathfrak{p}$ is a prime of K_Φ for which $\mathcal{Z}^{\mathfrak{a}}_{\Phi}(\alpha)(k^{\mathrm{alg}}_{\Phi,\mathfrak{p}}) \neq \emptyset$, then $\mathfrak{p}_F$ is nonsplit in K.*

Proof. Suppose $(A_0, A, f) \in \mathcal{Z}^{\mathfrak{a}}_{\Phi}(\alpha)(k)$ with $\alpha \in F^\times$ and k an algebraically closed field. As $\langle f, f \rangle_{\mathrm{CM}} = \alpha$, Proposition 3.2.1 shows that k has nonzero characteristic and that A_0 and A are supersingular. The supersingularity of A_0 implies that p is nonsplit in K_0, and Proposition 3.1.5 then implies $\mathfrak{p}_F$ is nonsplit in K. Next we show that $\mathcal{Z}^{\mathfrak{a}}_{\Phi}(\alpha)$ has dimension 0. Suppose $\mathfrak{p}$ is a prime of $\mathcal{O}_\Phi$ and $z \in \mathcal{Z}^{\mathfrak{a}}_\Phi(\alpha)(k^{\mathrm{alg}}_{\Phi,\mathfrak{p}})$ is a geometric point. The forgetful morphism

$$\mathcal{Z}^{\mathfrak{a}}_\Phi(\alpha) \to \mathcal{M}_{(1,0)} \times \mathcal{CM}^{\mathfrak{a}}_\Phi$$

is unramified, and so induces a surjection on completed strictly Henselian local rings. Proposition 3.1.2 now implies that $\widehat{\mathcal{O}}_{\mathcal{Z}^{\mathfrak{a}}_\Phi(\alpha),z}$ is a quotient of $W_{\Phi,\mathfrak{p}}$. As $\mathcal{Z}^{\mathfrak{a}}_\Phi(\alpha)$ has no geometric points in characteristic 0, this quotient has dimension 0.

The only thing left to prove is that $\mathcal{Z}^{\mathfrak{a}}_\Phi(\alpha) = \emptyset$ unless $\alpha \gg 0$. This is clear from the fact that $\langle \cdot, \cdot \rangle_{\mathrm{CM}}$ is totally positive definite. $\square$

The following theorem essentially counts the number of geometric points of $\mathcal{Z}^{\mathfrak{a}}_\Phi(\alpha)$.

THEOREM 3.5.3. *Suppose $\alpha \in F^{\gg 0}$, and assume $\mathcal{CM}^{\mathfrak{a}}_\Phi(\mathbb{C}) \neq \emptyset$. If $\mathfrak{p}$ is a prime of K_Φ for which $\mathfrak{p}_F$ is nonsplit in K, then*

$$\sum_{(A_0,A,f) \in \mathcal{Z}^{\mathfrak{a}}_\Phi(\alpha)(k^{\mathrm{alg}}_{\Phi,\mathfrak{p}})} \frac{1}{\#\mathrm{Aut}(A_0,A,f)} = \frac{h(K_0)}{w(K_0)} \cdot \rho\left(\frac{\alpha \mathcal{O}_F}{\mathfrak{a}\mathfrak{s}\mathfrak{p}_F^{\varepsilon_p}} \right),$$

where p is the rational prime below $\mathfrak{p}$. Recall that $\mathfrak{s}$ was defined in Proposition 3.1.6, ε_p was defined by (1.2.4), ρ was defined by (1.2.3), $h(K_0)$ is the class number of K_0, and $w(K_0)$ is the number of roots of unity in K_0.

Proof. As an abelian variety over $\mathbb{C}$ with complex multiplication admits a model over a number field having everywhere good reduction, the hypothesis $\mathcal{CM}^{\mathfrak{a}}_\Phi(\mathbb{C}) \neq \emptyset$ implies that $\mathcal{CM}^{\mathfrak{a}}_\Phi(k^{\mathrm{alg}}_{\Phi,\mathfrak{p}}) \neq \emptyset$. As $\mathcal{M}_{(1,0)}(\mathbb{C})$ has $h(K_0)$ elements, we similarly have $\mathcal{M}_{(1,0)}(k^{\mathrm{alg}}_{\Phi,\mathfrak{p}}) \neq \emptyset$. Fix a pair

$$(A_0, A) \in (\mathcal{M}_{(1,0)} \times \mathcal{CM}^{\mathfrak{a}}_\Phi)(k^{\mathrm{alg}}_{\Phi,\mathfrak{p}}).$$

Using (3.3.3), we compute

$$\begin{aligned}
\sum_{\mathfrak{z} \in C_K^0} \#\{f \in L(A_0, A^{\mathfrak{z}}) : \langle f, f \rangle^{\mathfrak{z}}_{\mathrm{CM}} = \alpha\} &= \sum_{\mathfrak{z} \in C_K^0} \sum_{\substack{x \in V(A_0,A) \\ \langle x,x \rangle_{\mathrm{CM}} = \alpha}} \mathbf{1}_{\mathfrak{z} L(A_0,A)}(x) \\
&= \sum_{h \in H(F) \backslash H(\widehat{F})/U} \sum_{\substack{x \in V(A_0,A) \\ \langle x,x \rangle_{\mathrm{CM}} = \alpha}} \mathbf{1}_{\widehat{L}(A_0,A)}(h^{-1}x) \\
&= \#(H(F) \cap U) \sum_{h \in H(\widehat{F})/U} \sum_{\substack{x \in H(F) \backslash V(A_0,A) \\ \langle x,x \rangle_{\mathrm{CM}} = \alpha}} \mathbf{1}_{\widehat{L}(A_0,A)}(h^{-1}x).
\end{aligned}$$

Here and elsewhere, $\mathbf{1}$ means characteristic function. If $\mu(K)$ denotes the group of roots of unity in $\mathcal{O}_K$, then $\mathrm{Aut}(z) \cong \mu(K)$ for any $z \in \mathcal{CM}^{\mathfrak{a}}_\Phi(k^{\mathrm{alg}}_{\Phi,\mathfrak{p}})$, and so

$$\mathrm{Aut}(A_0, A^{\mathfrak{z}}) \cong \mu(K_0) \times \mu(K).$$

Furthermore, $\mu(K) \cong H(F) \cap U$, and we have now proved

$$\sum_{\mathfrak{z} \in C_K^0} \sum_{\substack{f \in L(A_0, A^{\mathfrak{z}}) \\ \langle f, f \rangle_{\mathrm{CM}}^{\mathfrak{z}} = \alpha}} \frac{w(K_0)}{\#\mathrm{Aut}(A_0, A^{\mathfrak{z}})} = \sum_{h \in H(\widehat{F})/U} \sum_{\substack{x \in H(F) \backslash V(A_0, A) \\ \langle x, x \rangle_{\mathrm{CM}} = \alpha}} \mathbf{1}_{\widehat{L}(A_0, A)}(h^{-1}x).$$

If there are no $x \in V(A_0, A)$ satisfying $\langle x, x \rangle_{\mathrm{CM}} = \alpha$, then of course the right-hand side is 0. If there are such x, then they are permuted simply transitively by $H(F)$, and so

$$\sum_{\mathfrak{z} \in C_K^0} \sum_{\substack{f \in L(A_0, A^{\mathfrak{z}}) \\ \langle f, f \rangle_{\mathrm{CM}}^{\mathfrak{z}} = \alpha}} \frac{1}{\#\mathrm{Aut}(A_0, A^{\mathfrak{z}})} = \frac{1}{w(K_0)} \sum_{h \in H(\widehat{F})/U} \mathbf{1}_{\widehat{L}(A_0, A)}(h^{-1}x), \tag{3.5.1}$$

where on the right we have fixed one $x \in V(A_0, A)$ satisfying $\langle x, x \rangle_{\mathrm{CM}} = \alpha$.

We interrupt the proof for a definition.

Definition 3.5.4. For any $\alpha \in \widehat{F}^{\times}$, define the *orbital integral*

$$O_\alpha(A_0, A) = \sum_{h \in H(\widehat{F})/U} \mathbf{1}_{\widehat{L}(A_0, A)}(h^{-1} \cdot x),$$

where $x \in \widehat{V}(A_0, A)$ satisfies $\langle x, x \rangle_{\mathrm{CM}} = \alpha$. If such x exist, then $H(\widehat{F})$ permutes them simply transitively, so the orbital integral is independent of the choice. If no such x exists, then set $O_\alpha(A_0, A) = 0$.

Using this new notation, (3.5.1) may be rewritten as

$$\sum_{\mathfrak{z} \in C_K^0} \sum_{\substack{f \in L(A_0, A^{\mathfrak{z}}) \\ \langle f, f \rangle_{\mathrm{CM}}^{\mathfrak{z}} = \alpha}} \frac{1}{\#\mathrm{Aut}(A_0, A^{\mathfrak{z}})} = \frac{1}{w(K_0)} \cdot O_\alpha(A_0, A).$$

It follows from Proposition 3.3.1 that

$$O_\alpha(A_0, A^{\mathfrak{z}}) = O_{\mathrm{gen}(\mathfrak{z})^{-1}\alpha}(A_0, A)$$

for any $\mathfrak{z} \in C_K$, and so summing over $\mathfrak{z} \in C_K/C_K^0$ and using the exactness of (3.3.2) shows that

$$\sum_{\mathfrak{z} \in C_K} \sum_{\substack{f \in L(A_0, A^{\mathfrak{z}}) \\ \langle f, f \rangle_{\mathrm{CM}}^{\mathfrak{z}} = \alpha}} \frac{1}{\#\mathrm{Aut}(A_0, A^{\mathfrak{z}})} = \frac{1}{w(K_0)} \sum_{\xi \in \ker(\eta)} O_{\xi\alpha}(A_0, A), \tag{3.5.2}$$

where the sum is over ξ in the kernel of (3.3.1).

Assuming that $\widehat{V}(A_0, A)$ represents α, Theorem 3.4.3 reduces the calculation of $O_\alpha(A_0, A)$ to a pleasant exercise, as in [17, §2.5]. We interrupt our proof yet again to state the result as a lemma.

LEMMA 3.5.5. *Let β be as in the statement of Theorem* 3.4.3. *For any* $\alpha \in \widehat{F}^\times$,

$$O_\alpha(A_0, A) = \begin{cases} \rho(\alpha\beta^{-1}\mathcal{O}_F) & \text{if } \widehat{V}(A_0, A) \text{ represents } \alpha \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Assume that $\widehat{V}(A_0, A)$ represents α, and fix an $x \in \widehat{K}$ such that $\alpha = \beta x\overline{x}$. The orbital integral factors as product of local integrals $O_{\alpha,v}(A_0, A)$, one for each finite place v of F, defined by

$$O_{\alpha,v}(A_0, A) = \sum_{h \in H(F_v)/U_v} \mathbf{1}_{\mathcal{O}_{K,v}}(h^{-1}x_v).$$

If v is nonsplit in K, then $H(F_v)/U_v = 1$ and

$$O_{\alpha,v}(A_0, A) = \begin{cases} 1 & \text{if } \alpha_v\beta_v^{-1} \in \mathcal{O}_{F,v} \\ 0 & \text{otherwise.} \end{cases}$$

If v is split in K, then $K_v \cong F_v \times F_v$. After fixing a uniformizer $\varpi \in F_v$ we find that $H(F_v)/U_v$ is the cyclic group generated by $(\varpi, \varpi^{-1}) \in F_v^\times \times F_v^\times$, and

$$O_{\alpha,v}(A_0, A) = \begin{cases} 1 + \mathrm{ord}_v(\alpha_v\beta_v^{-1}) & \text{if } \alpha_v\beta_v^{-1} \in \mathcal{O}_{F,v} \\ 0 & \text{otherwise.} \end{cases}$$

In either case, $O_{\alpha,v}(A_0, A)$ is the number of ideals $\mathfrak{C}_v \subset \mathcal{O}_{K,v}$ satisfying

$$\beta_v\mathfrak{C}_v\overline{\mathfrak{C}}_v = \alpha\mathcal{O}_{F,v},$$

and therefore, recalling the definition (1.2.3) of $\rho(\mathfrak{b})$, we have proved

$$O_\alpha(A_0, A) = \rho(\alpha\beta^{-1}\mathcal{O}_F)$$

completing the proof of the lemma. □

Now go back to our fixed $\alpha \in F^{\gg 0}$, and assume that $\rho(\alpha\beta^{-1}\mathcal{O}_F) \neq 0$. This implies that $\alpha\mathcal{O}_F = \beta\mathfrak{C}\overline{\mathfrak{C}}$ for some $\mathcal{O}_K$-ideal $\mathfrak{C}$, and it follows that there is a unique

$$\xi \in \widehat{\mathcal{O}}_F^\times/\mathrm{Nm}_{K/F}\widehat{\mathcal{O}}_K^\times$$

such that $\xi\alpha$ is represented by the quadratic form $\beta x\overline{x}$ on $\widehat{K}$. Recalling that $\chi_{K/F}(\beta) = 1$ and that $\alpha \gg 0$, this ξ lies in the kernel of (3.3.1). In other words, there is a unique $\xi \in \ker(\eta)$ such that $\widehat{V}(A_0, A)$ represents $\xi\alpha$. Using $\beta\mathcal{O}_F = \mathfrak{asp}_F^{\varepsilon_p}$, we now deduce

$$\sum_{\xi \in \ker(\eta)} O_{\xi\alpha}(A_0, A) = \rho\left(\frac{\alpha\mathcal{O}_F}{\mathfrak{asp}_F^{\varepsilon_p}}\right). \tag{3.5.3}$$

If, on the other hand, $\rho(\alpha\beta^{-1}\mathcal{O}_F) = 0$, then $\widehat{V}(A_0, A)$ does not represent $\xi\alpha$ for any $\xi \in \widehat{\mathcal{O}}_F^\times$, and both sides of (3.5.3) are zero. Comparing with (3.5.2) shows that

$$\sum_{\mathfrak{z}\in C_K} \sum_{\substack{f\in L(A_0,A^{\mathfrak{z}}) \\ \langle f,f\rangle^{\mathfrak{z}}_{\mathrm{CM}}=\alpha}} \frac{1}{\#\mathrm{Aut}(A_0, A^{\mathfrak{z}})} = \frac{1}{w(K_0)} \cdot \rho\left(\frac{\alpha\mathcal{O}_F}{\mathfrak{a}\mathfrak{s}\mathfrak{p}_F^{\varepsilon_p}}\right). \tag{3.5.4}$$

The action of C_K on the set of isomorphism classes of $\mathcal{CM}^{\mathfrak{a}}_\Phi(k^{\mathrm{alg}}_{\Phi,\mathfrak{p}})$ is simply transitive. For example, one can first prove this in characteristic 0 using the complex uniformization of CM abelian varieties, and then use Remark 3.1.3 to deduce the result over $k^{\mathrm{alg}}_{\Phi,\mathfrak{p}}$. The same argument shows that there are $h(K_0)$ isomorphism classes of objects in $\mathcal{M}_{(1,0)}(k^{\mathrm{alg}}_{\Phi,\mathfrak{p}})$. Therefore, (3.5.4) implies

$$\sum_{\substack{A_0\in\mathcal{M}_{(1,0)}(k^{\mathrm{alg}}_{\Phi,\mathfrak{p}}) \\ A\in\mathcal{CM}^{\mathfrak{a}}_\Phi(k^{\mathrm{alg}}_{\Phi,\mathfrak{p}})}} \sum_{\substack{f\in L(A_0,A) \\ \langle f,f\rangle_{\mathrm{CM}}=\alpha}} \frac{1}{\#\mathrm{Aut}(A_0, A)} = \frac{h(K_0)}{w(K_0)} \cdot \rho\left(\frac{\alpha\mathcal{O}_F}{\mathfrak{a}\mathfrak{s}\mathfrak{p}_F^{\varepsilon_p}}\right),$$

and Theorem 3.5.3 follows. □

3.6. *The degree of* $\mathcal{Z}^{\mathfrak{a}}_\Phi(\alpha)$. Throughout this subsection we assume that *the discriminants of* $K_0/\mathbb{Q}$ *and* $F/\mathbb{Q}$ *are odd and relatively prime.* The primary reason for this assumption is so that we may apply Theorem 2.3.5, the secondary reason is so that Proposition 3.1.6 applies.

Definition 3.6.1. For any $\alpha \in F$ for which $\mathcal{Z}^{\mathfrak{a}}_\Phi(\alpha)$ has dimension 0, define the *Arakelov degree*

$$\widehat{\deg}\,\mathcal{Z}^{\mathfrak{a}}_\Phi(\alpha) = \sum_{\mathfrak{p}\subset\mathcal{O}_\Phi} \frac{\log(\mathrm{N}(\mathfrak{p}))}{[K_\Phi : \mathbb{Q}]} \sum_{z\in\mathcal{Z}_\Phi(\alpha)(k^{\mathrm{alg}}_{\Phi,\mathfrak{p}})} \frac{\mathrm{length}(\mathcal{O}^{\mathrm{sh}}_{\mathcal{Z}_\Phi(\alpha),z})}{\#\mathrm{Aut}(z)}.$$

Our goal is to compute the Arakelov degree of $\mathcal{Z}^{\mathfrak{a}}_\Phi(\alpha)$ for $\alpha \gg 0$. The degree has been normalized in such a way that it is unchanged if the field K_Φ is enlarged. By the comments at the beginning of Section 3, we may therefore make the minimal choice $K_\Phi = \varphi^{\mathrm{sp}}(K)$. This will ease comparison with the notation of Section 2.3. As in the introduction, let K^{sp} be the factor of K on which $\varphi^{\mathrm{sp}} : K \to \mathbb{C}$ is nonzero. Let F^{sp} be the maximal totally real subfield of K^{sp}. We henceforth use φ^{sp} to identify

$$K^{\mathrm{sp}} = K_\Phi.$$

For any prime $\mathfrak{p}$ of K_Φ, we have $\mathfrak{p}_K = \mathfrak{p}$ under this identification, and $\mathfrak{p}_F$ is the prime of F^{sp} below $\mathfrak{p}$. Let $e_\mathfrak{p}$ be the ramification degree of $K^{\mathrm{sp}}_{\mathfrak{p}_K}/F^{\mathrm{sp}}_{\mathfrak{p}_F}$.

Theorem 3.6.2. *Fix* $\alpha \in F$ *with* $\varphi^{\mathrm{sp}}(\alpha) \neq 0$. *Let* $\mathfrak{p}$ *be a prime of* K_Φ *such that* $\mathfrak{p}_F$ *is nonsplit in* K. *The strictly Henselian local ring of* $\mathcal{Z}^{\mathfrak{a}}_\Phi(\alpha)$ *at* *any geometric point $z \in \mathcal{Z}^{\mathfrak{a}}_{\Phi}(\alpha)(k^{\mathrm{alg}}_{\Phi,\mathfrak{p}})$ is Artinian of length*

$$\mathrm{length}(\mathcal{O}^{\mathrm{sh}}_{\mathcal{Z}^{\mathfrak{a}}_{\Phi}(\alpha),z}) = \frac{1}{2} \cdot e_{\mathfrak{p}} \cdot \mathrm{ord}_{\mathfrak{p}_F}(\alpha \mathfrak{p}_F \mathfrak{a}^{-1} \mathfrak{d}_F).$$

In particular, the length does not depend on z. Note that $\varphi^{\mathrm{sp}}(\alpha) \neq 0$ guarantees that α has nonzero projection to the factor $F^{\mathrm{sp}} \subset F$, and so $\mathrm{ord}_{\mathfrak{p}_F}(\alpha) < \infty$. Thus the right-hand side is finite.

Proof. Let p be the rational prime below $\mathfrak{p}$, and recall that $W_{\Phi,\mathfrak{p}}$ is the completed integer ring of the maximal unramified extension of $K_{\Phi,\mathfrak{p}}$. Let **ART** be the category of Artinian local $W_{\Phi,\mathfrak{p}}$-algebras with residue field $k^{\mathrm{alg}}_{\Phi,\mathfrak{p}}$. If

$$(A_0, A, f) \in \mathcal{Z}^{\mathfrak{a}}_{\Phi}(\alpha)(k^{\mathrm{alg}}_{\Phi,\mathfrak{p}})$$

is the triple corresponding to z, then the completed strictly Henselian local ring $\widehat{\mathcal{O}}^{\mathrm{sh}}_{\mathcal{Z}^{\mathfrak{a}}_{\Phi}(\alpha),z}$ pro-represents the functor of deformations of (A_0, A, f) to objects of **ART**. By the Serre-Tate theorem this is the same as the corresponding deformation functor of p-divisible groups $(A_0[p^\infty], A[p^\infty], f[p^\infty])$.

We argue as in the proofs of Lemma 3.4.4 and Theorem 3.4.3. There is a decomposition $A[p^\infty] \cong \prod_{\mathfrak{q}} A[\mathfrak{q}^\infty]$ over the primes $\mathfrak{q} \subset \mathcal{O}_F$ above p, and similarly for any deformation of (A, κ, λ). Fix an isomorphism of K_Φ-algebras $\mathbb{C}_{\mathfrak{p}} \cong \mathbb{C}$, and let $\Phi(\mathfrak{q})$ be the set of all $\varphi \in \Phi$ whose restriction to $F \to \mathbb{C}_{\mathfrak{p}}$ induces the prime $\mathfrak{q}$. The triple $(A[\mathfrak{q}^\infty], \kappa[\mathfrak{q}^\infty], \lambda[\mathfrak{q}^\infty])$ satisfies the $\Phi(\mathfrak{q})$-determinant condition of Section 2.1. Set $m = [F_{\mathfrak{q}} : \mathbb{Q}_p]$.

If $\mathfrak{q} \neq \mathfrak{p}_F$, then $\varphi^{\mathrm{sp}} \notin \Phi(\mathfrak{q})$, and $\Phi(\mathfrak{q})$ has signature $(m, 0)$ in the sense of Section 2.4. Theorem 2.1.3 implies that $(A_0[p^\infty], A[\mathfrak{q}^\infty])$ lifts uniquely to every object of **ART**, and Proposition 2.4.1 implies that the homomorphism

$$f[\mathfrak{q}^\infty] : A_0[p^\infty] \to A[\mathfrak{q}^\infty]$$

lifts uniquely as well. It follows that the deformation functors of the triples (A_0, A, f) and $(A_0[p^\infty], A[\mathfrak{p}_F^\infty], f[\mathfrak{p}_F^\infty])$ are canonically isomorphic. As $\varphi^{\mathrm{sp}} \in \Phi(\mathfrak{p}_F)$, the p-adic CM type $\Phi(\mathfrak{p}_F)$ has signature $(m-1, 1)$ in the sense of Section 2.3. By Theorem 2.1.3, the deformation functor of the pair $(A_0[p^\infty], A[\mathfrak{p}_F^\infty])$ is pro-represented by $W_{\Phi,\mathfrak{p}}$, and Theorem 2.3.5 implies that the deformation functor of $(A_0[p^\infty], A[\mathfrak{p}_F^\infty], f[\mathfrak{p}_F^\infty])$ is pro-represented by $W_{\Phi,\mathfrak{p}}/\mathfrak{m}^k$, where $\mathfrak{m}$ is the maximal ideal of $W_{\Phi,\mathfrak{p}}$ and

$$k = \frac{1}{2} \cdot \mathrm{ord}_{\mathfrak{p}_K}(\alpha \mathfrak{p}_F \mathfrak{D} \mathfrak{D}_0^{-1} \mathfrak{a}^{-1}).$$

Therefore, the length of $\widehat{\mathcal{O}}^{\mathrm{sh}}_{\mathcal{Z}^{\mathfrak{a}}_{\Phi}(\alpha),z}$ is

$$k = \frac{1}{2} \cdot e_{\mathfrak{p}} \cdot \mathrm{ord}_{\mathfrak{p}_F}(\alpha \mathfrak{p}_F \mathfrak{s}^{-1} \mathfrak{a}^{-1}).$$

□

Theorem 3.6.2 computes the lengths of the local rings of $\mathcal{Z}^{\mathfrak{a}}_{\Phi}(\alpha)$, while Theorem 3.5.3 counts the number geometric points. The calculation of the Arakelov degree is now an easy corollary of these results. Theorem A of the introduction is the case $\mathfrak{a} = \mathcal{O}_F$ of the following theorem.

THEOREM 3.6.3. *If $\alpha \in F^{\gg 0}$, then $\mathcal{Z}^{\mathfrak{a}}_{\Phi}(\alpha)$ has dimension 0, and*

$$\widehat{\deg}\, \mathcal{Z}^{\mathfrak{a}}_{\Phi}(\alpha) = \frac{h(K_0)}{w(K_0)} \sum_{\mathfrak{p}} \frac{\log(\mathrm{N}(\mathfrak{p}))}{[K^{\mathrm{sp}} : \mathbb{Q}]} \cdot \mathrm{ord}_{\mathfrak{p}}(\alpha\mathfrak{p}\mathfrak{d}_F\mathfrak{a}^{-1}) \cdot \rho(\alpha\mathfrak{p}^{-\varepsilon_p}\mathfrak{d}_F\mathfrak{a}^{-1}),$$

where the sum is over all primes $\mathfrak{p}$ of F^{sp} nonsplit in K^{sp}, and p is the prime of $\mathbb{Q}$ below $\mathfrak{p}$.

Proof. The first claim is Proposition 3.5.2. If $\mathfrak{p}$ is a prime of K_Φ for which $\mathfrak{p}_F$ is nonsplit in K, then combining Theorem 3.5.3 and Theorem 3.6.2 shows that

$$\sum_{z \in \mathcal{Z}^{\mathfrak{a}}_{\Phi}(\alpha)(k^{\mathrm{alg}}_{\Phi,\mathfrak{p}})} \frac{\mathrm{length}(\mathcal{O}^{\mathrm{sh}}_{\mathcal{Z}_\Phi(\alpha),z})}{\#\mathrm{Aut}(z)} = \frac{e_{\mathfrak{p}} h(K_0)}{2w(K_0)} \cdot \mathrm{ord}_{\mathfrak{p}_F}(\alpha\mathfrak{p}_F\mathfrak{d}_F\mathfrak{a}^{-1}) \cdot \rho(\alpha\mathfrak{p}_F^{-\varepsilon_p}\mathfrak{d}_F\mathfrak{a}^{-1}).$$

If $\mathfrak{p}$ is a prime of K_Φ for which $\mathfrak{p}_F$ is split in K, then the left-hand side is zero by Proposition 3.5.2. Summing over all primes $\mathfrak{p}$ of $K_\Phi \cong K^{\mathrm{sp}}$ yields the result. □

3.7. *Arithmetic divisors on $\mathcal{X}_\Phi$.* In this subsection we fix an $\alpha \in F^\times$ and restrict to the case $\mathfrak{a} = \mathcal{O}_F$. Abbreviate

$$\mathcal{Z}_\Phi(\alpha) = \mathcal{Z}^{\mathcal{O}_F}_\Phi(\alpha) \qquad \mathcal{CM}_\Phi = \mathcal{CM}^{\mathcal{O}_F}_\Phi,$$

and define a regular 1-dimensional stack

$$\mathcal{X}_\Phi = \mathcal{M}_{(1,0)/\mathcal{O}_\Phi} \times_{\mathcal{O}_\Phi} \mathcal{CM}_\Phi$$

over $\mathcal{O}_\Phi$. We know from Section 3.5 that $\mathcal{Z}_\Phi(\alpha)$ is 0-dimensional and that the natural map

$$\mathcal{Z}_\Phi(\alpha) \to \mathcal{X}_\Phi$$

is finite and unramified. This allows us to view $\mathcal{Z}_\Phi(\alpha)$ as a divisor on $\mathcal{X}_\Phi$, which we denote by $\mathtt{Z}_\Phi(\alpha)$. To give the precise definition, it suffices to describe the pullback of $\mathtt{Z}_\Phi(\alpha)$ to an atlas $\gamma : X \to \mathcal{X}_\Phi$. Let Z be the cartesian product

$$\begin{tikzcd} Z \arrow[r, "\phi"] \arrow[d] & X \arrow[d, "\gamma"] \\ \mathcal{Z}_\Phi(\alpha) \arrow[r] & \mathcal{X}_\Phi, \end{tikzcd}$$

so that $\phi : Z \to X$ is a finite unramified morphism of *schemes*. The divisor $\mathtt{Z}_\Phi(\alpha)$ on $\mathcal{X}_\Phi$ is defined as the unique divisor whose pullback to X is

$$\gamma^* \mathtt{Z}_\Phi(\alpha) = \sum_{z \in Z} [k(z) : k(\phi(z))] \cdot \mathrm{length}(\mathcal{O}_{Z,z}) \cdot \phi(z).$$

The reader may consult [9], [43] for the general theory of divisors and cycles on stacks. Of course $\mathtt{Z}_\Phi(\alpha) = 0$ unless $\alpha \gg 0$.

An *arithmetic divisor* on $\mathcal{X}_\Phi$ is a pair $(\mathtt{Z}, \mathtt{Gr})$, where $\mathtt{Z}$ is a Weil divisor on $\mathcal{X}_\Phi$ with rational coefficients, and $\mathtt{Gr}$ is a Green function for $\mathtt{Z}$. As $\mathtt{Z}$ has no points in characteristic 0, this simply means that $\mathtt{Gr}$ is *any* function on the finite set of points

$$\bigsqcup_{\substack{\sigma : K_\Phi \to \mathbb{C} \\ \sigma|_{K_0} = \iota}} \mathcal{X}_\Phi^\sigma(\mathbb{C}),$$

where $\mathcal{X}_\Phi^\sigma$ is the stack over $\mathbb{C}$ obtained from $\mathcal{X}_\Phi$ by base change. To each rational function f on $\mathcal{X}_\Phi$ there is an associated *principal arithmetic divisor* $(\mathrm{div}(f), -\log|f|^2)$. The quotient group of arithmetic divisors modulo principal arithmetic divisors is the *codimension* 1 *arithmetic Chow group* $\widehat{\mathrm{CH}}^1(\mathcal{X}_\Phi)$ of Gillet-Soulé [5], [10], [11], [29].

We will construct a Green function $\mathtt{Gr}_\Phi(\alpha, y, \cdot)$ for the divisor $\mathtt{Z}_\Phi(\alpha)$, depending on an auxiliary parameter $y \in F_\mathbb{R}^{\gg 0}$. For $t \in \mathbb{R}^{>0}$, define

$$\beta_1(t) = \int_1^\infty e^{-tu} u^{-1}\, du. \tag{3.7.1}$$

First suppose that $\sigma : K_\Phi \to \mathbb{C}$ is the inclusion, so that a point $z \in \mathcal{X}_\Phi^\sigma(\mathbb{C})$ corresponds to a pair

$$(A_0, A) \in \mathcal{M}_{(1,0)}(\mathbb{C}) \times \mathcal{CM}_\Phi(\mathbb{C}).$$

To each such pair we attach, exactly as in (3.4.1), an $\mathcal{O}_K$-module

$$L_B(A_0, A) = \mathrm{Hom}_{\mathcal{O}_{K_0}}(H_1(A_0), H_1(A))$$

equipped with a Hermitian form $\langle \cdot, \cdot \rangle_{\mathrm{CM}}$. By Proposition 3.4.1, $L_B(A_0, A)$ is a projective $\mathcal{O}_K$-module of rank one, is negative definite at the archimedean place ∞^{sp} of F determined by $\varphi^{\mathrm{sp}} : K \to \mathbb{C}$, and is positive definite at the other archimedean places. We define

$$\mathtt{Gr}_\Phi(\alpha, y, z) = \sum_{\substack{f \in L_B(A_0, A) \\ \langle f, f \rangle_{\mathrm{CM}} = \alpha}} \beta_1(4\pi |y\alpha|_{\infty^{\mathrm{sp}}}). \tag{3.7.2}$$

To complete the definition of $\mathtt{Gr}_\Phi(\alpha, y, \cdot)$ we must generalize this construction to an arbitrary $\mathbb{Q}$-algebra map $\sigma : K_\Phi \to \mathbb{C}$ whose restriction to K_0 is ι. If we extend σ in some way to an automorphism of $\mathbb{C}$, we obtain a new CM type Φ^σ, which does not depend on how σ was extended. It is not hard to see that $\mathcal{X}_\Phi^\sigma(\mathbb{C}) \cong \mathcal{X}_{\Phi^\sigma}(\mathbb{C})$, and so points $z \in \mathcal{X}_\Phi^\sigma(\mathbb{C})$ correspond to a pairs

$$(A_0, A) \in \mathcal{M}_{(1,0)}(\mathbb{C}) \times \mathcal{CM}_{K,\Phi^\sigma}(\mathbb{C}).$$

Define

$$\mathtt{Gr}_\Phi(\alpha, y, z) = \sum_{\substack{f \in L_B(A_0, A) \\ \langle f, f\rangle_{\mathrm{CM}} = \alpha}} \beta_1(4\pi |y\alpha|_{\infty^{\mathrm{sp},\sigma}}) \tag{3.7.3}$$

as above, where now $\infty^{\mathrm{sp},\sigma}$ is the archimedean place of F induced by the special element $\sigma \circ \varphi^{\mathrm{sp}} : K \to \mathbb{C}$ of Φ^σ. As $\langle \cdot, \cdot \rangle_{\mathrm{CM}}$ is negative definite at $\infty^{\mathrm{sp},\sigma}$, and positive definite at the remaining archimedean places, the function $\mathtt{Gr}(\alpha, y, \cdot)$ is identically 0 if $\alpha \gg 0$.

Definition 3.7.1. For every $\alpha \in F^\times$ and $y \in F_{\mathbb{R}}^{\gg 0}$, define an arithmetic divisor

$$\widehat{\mathtt{Z}}_\Phi(\alpha, y) = \big(\mathtt{Z}_\Phi(\alpha), \mathtt{Gr}_\Phi(\alpha, y, \cdot)\big) \in \widehat{\mathrm{CH}}^1(\mathcal{X}_\Phi).$$

Note that if $\alpha \gg 0$, then

$$\widehat{\mathtt{Z}}_\Phi(\alpha, y) = \big(\mathtt{Z}_\Phi(\alpha), 0\big),$$

while if $\alpha \not\gg 0$, then

$$\widehat{\mathtt{Z}}_\Phi(\alpha, y) = \big(0, \mathtt{Gr}_\Phi(\alpha, y, \cdot)\big).$$

If $\alpha \not\gg 0$, then our definition of $\widehat{\mathtt{Z}}_\Phi(\alpha, y)$ is, at the moment, rather unmotivated, although the use of the function $\beta_1(t)$ in the definition follows Kudla [21], [29]. The particular choice of Green function will be justified in Section 4, when we show that, for all $\alpha \in F^\times$, the arithmetic divisor $\widehat{\mathtt{Z}}_\Phi(\alpha, y)$ is closely related to the Fourier coefficient of a Hilbert modular Eisenstein series.

There is a canonical linear functional

$$\widehat{\deg} : \widehat{\mathrm{CH}}^1(\mathcal{X}_\Phi) \to \mathbb{R}$$

defined as the composition

$$\widehat{\mathrm{CH}}^1(\mathcal{X}_\Phi) \to \widehat{\mathrm{CH}}^1(\mathrm{Spec}(\mathcal{O}_\Phi)) \to \mathbb{R},$$

where the first arrow is push-forward by the structure map $\mathcal{X}_\Phi \to \mathrm{Spec}(\mathcal{O}_\Phi)$ and the second is $[K_\Phi : \mathbb{Q}]^{-1}$ times the degree of [11, §3.4.3]. If $\mathtt{Z}$ is a prime Weil divisor on $\mathcal{X}_\Phi$, then

$$\widehat{\deg}\,(\mathtt{Z}, 0) = \frac{1}{[K_\Phi : \mathbb{Q}]} \sum_{\mathfrak{p} \subset \mathcal{O}_\Phi} \log(\mathrm{N}(\mathfrak{p})) \sum_{z \in \mathtt{Z}(k_{\Phi,\mathfrak{p}}^{\mathrm{alg}})} \frac{1}{\#\mathrm{Aut}(z)}.$$

If $\mathtt{Gr}$ is a Green function on $\mathcal{X}_\Phi$, then

$$\widehat{\deg}\,(0,\mathtt{Gr}) = \frac{1}{[K_\Phi:\mathbb{Q}]} \sum_{\substack{\sigma:K_\Phi\to\mathbb{C}\\ \sigma|_{K_0}=\iota}} \sum_{z\in\mathcal{X}_\Phi^\sigma(\mathbb{C})} \frac{\mathtt{Gr}(z)}{\#\mathrm{Aut}(z)}. \tag{3.7.4}$$

THEOREM 3.7.2. *Suppose the discriminants of K_0 and K are odd and relative prime. Fix $\alpha\in F^\times$ and $y\in F_{\mathbb{R}}^{\gg 0}$.*

(1) *If $\alpha\gg 0$, then*

$$\widehat{\deg}\,\widehat{\mathtt{Z}}_\Phi(\alpha,y) = \frac{h(K_0)}{w(K_0)} \sum_{\mathfrak{p}} \frac{\log(\mathrm{N}(\mathfrak{p}))}{[K^{\mathrm{sp}}:\mathbb{Q}]} \cdot \mathrm{ord}_{\mathfrak{p}}(\alpha\mathfrak{p}\mathfrak{d}_F)\cdot\rho(\alpha\mathfrak{p}^{-\varepsilon_p}\mathfrak{d}_F),$$

where the sum is over all primes $\mathfrak{p}$ of F^{sp} nonsplit in K^{sp}, and p is the prime of $\mathbb{Q}$ below $\mathfrak{p}$.

(2) *Suppose $\alpha\not\gg 0$. If α is negative at exactly one archimedean place v of F, and if the corresponding map $F\to\mathbb{R}$ factors through the summand F^{sp} of F, then*

$$\widehat{\deg}\,\widehat{\mathtt{Z}}_\Phi(\alpha,y) = \frac{h(K_0)}{w(K_0)} \frac{1}{[K^{\mathrm{sp}}:\mathbb{Q}]} \cdot \beta_1(4\pi|y\alpha|_v)\cdot\rho(\alpha\mathfrak{d}_F).$$

If no such v exists, then the left-hand side is 0.

Proof. If $\alpha\gg 0$, then

$$\widehat{\deg}\,\widehat{\mathtt{Z}}_\Phi(\alpha,y) = \widehat{\deg}\,\mathcal{Z}_\Phi(\alpha), \tag{3.7.5}$$

where the right-hand side is the Arakelov degree of Definition 3.6.1. Hence the first claim is just a restatement of Theorem 3.6.3, in the special case $\mathfrak{a}=\mathcal{O}_F$.

Now suppose $\alpha\not\gg 0$, and fix a $\sigma:K_\Phi\to\mathbb{C}$. If α is negative at $\infty^{\mathrm{sp},\sigma}$ and positive at all other archimedean places of F, then repeating the proof of Theorem 3.5.3 shows that

$$\sum_{\substack{A_0\in\mathcal{M}_{(1,0)}(\mathbb{C})\\ A\in\mathcal{CM}_{K,\Phi^\sigma}(\mathbb{C})}} \sum_{\substack{f\in L_B(A_0,A)\\ \langle f,f\rangle_{\mathrm{CM}}=\alpha}} \frac{1}{\#\mathrm{Aut}(A_0,A)} = \frac{h(K_0)}{w(K_0)}\cdot\rho(\alpha\mathfrak{s}^{-1}).$$

The only difference is in the calculation of the orbital integral (Lemma 3.5.5), where one replaces the β of Theorem 3.4.3 with the β of Proposition 3.4.1. The inner sum on the left is empty if α is positive at $\infty^{\mathrm{sp},\sigma}$ or negative at some other archimedean place.

As $\sigma:K_\Phi\to\mathbb{C}$ varies over all embeddings whose restriction to K_0 is ι, $\infty^{\mathrm{sp},\sigma}$ varies over all archimedean places of F^{sp}, each counted with multiplicity $[K_\Phi:\mathbb{Q}]/[K^{\mathrm{sp}}:\mathbb{Q}]$. If α is negative at exactly one archimedean place v of F, and if this place v lies on F^{sp}, then we compute

$$\begin{aligned}\widehat{\deg}\,\widehat{\mathtt{Z}}_\Phi(\alpha,y) &= \frac{1}{[K_\Phi:\mathbb{Q}]}\sum_{\substack{\sigma:K_\Phi\to\mathbb{C}\\ \sigma|_{K_0}=\iota}}\sum_{z\in\mathcal{X}^\sigma_\Phi(\mathbb{C})}\frac{\mathtt{Gr}_\Phi(\alpha,y,z)}{\#\mathrm{Aut}(z)}\\ &= \frac{1}{[K_\Phi:\mathbb{Q}]}\sum_{\substack{\sigma:K_\Phi\to\mathbb{C}\\ \sigma|_{K_0}=\iota}}\sum_{\substack{A_0\in\mathcal{M}_{(1,0)}(\mathbb{C})\\ A\in\mathcal{CM}_{K,\Phi^\sigma}(\mathbb{C})}}\sum_{\substack{f\in L_B(A_0,A)\\ \langle f,f\rangle_{\mathrm{CM}}=\alpha}}\frac{\beta_1(4\pi|y\alpha|_{\infty^{\mathrm{sp},\sigma}})}{\#\mathrm{Aut}(A_0,A)}\\ &= \frac{1}{[K^{\mathrm{sp}}:\mathbb{Q}]}\frac{h(K_0)}{w(K_0)}\cdot\beta_1(4\pi|y\alpha|_v)\cdot\rho(\alpha\mathfrak{s}^{-1}).\end{aligned}$$

If no such v exists, then the inner sum on the third line is empty. To complete the proof, recall from Proposition 3.1.6 that, under our hypotheses on the discriminants of K_0 and K, $\mathfrak{s}=\mathfrak{d}_F^{-1}$. □

3.8. *Arithmetic divisors on* $\mathcal{M}$. In this subsection we study arithmetic intersection theory on the $\mathcal{O}_{K_0}$-stack

$$\mathcal{M}=\mathcal{M}_{(1,0)}\times_{\mathcal{O}_{K_0}}\mathcal{M}_{(n-1,1)}$$

of the introduction. Recall that $\mathcal{M}$ is smooth of relative dimension $n-1$ over $\mathcal{O}_{K_0}[\mathrm{disc}(K_0)^{-1}]$. If S is a connected $\mathcal{O}_{K_0}$-scheme, then to every point $(A_0,A)\in\mathcal{M}(S)$ we have attached an $\mathcal{O}_{K_0}$-module

$$L(A_0,A)=\mathrm{Hom}_{\mathcal{O}_{K_0}}(A_0,A)$$

and an $\mathcal{O}_{K_0}$-Hermitian form $\langle\cdot,\cdot\rangle$ defined by (1.2.1).

Definition 3.8.1. For any nonzero $m\in\mathbb{Z}$, let $\mathcal{Z}(m)$ be algebraic stack over $\mathcal{O}_{K_0}$ whose functor of points assigns to any connected $\mathcal{O}_{K_0}$-scheme S the groupoid of triples (A_0,A,f), where $(A_0,A)\in\mathcal{M}(S)$, and $f\in L(A_0,A)$ satisfies $\langle f,f\rangle=m$.

We call the stacks $\mathcal{Z}(m)$ the *Kudla-Rapoport divisors*. By [26, Prop. 2.10], the natural map $\mathcal{Z}(m)\to\mathcal{M}$ is finite and unramified. As in Section 3.7, we abbreviate $\mathcal{X}_\Phi$ for the 1-dimensional stack

$$\mathcal{X}_\Phi=\mathcal{M}_{(1,0)/\mathcal{O}_\Phi}\times_{\mathcal{O}_\Phi}\mathcal{CM}_\Phi.$$

The map $\mathcal{CM}_\Phi\to\mathcal{M}_{(n-1,1)/\mathcal{O}_\Phi}$ defined by restricting the action of $\mathcal{O}_K$ to $\mathcal{O}_{K_0}$ induces a map

$$\mathcal{X}_\Phi\to\mathcal{M}_{/\mathcal{O}_\Phi}.$$

A point in the intersection of $\mathcal{X}_\Phi$ and $\mathcal{Z}(m)$, defined over some $\mathcal{O}_\Phi$-scheme S, is a triple (A_0,A,f) in which $(A_0,A,f)\in\mathcal{Z}(m)$, and A is endowed with complex multiplication by $\mathcal{O}_K$. We know from Section 3.2 that the induced $\mathcal{O}_K$-action on $L(A_0,A)$ then endows $L(A_0,A)$ with additional structure: a totally positive definite $\mathcal{O}_K$-Hermitian form $\langle\cdot,\cdot\rangle_{\mathrm{CM}}$ whose trace is the original Hermitian form $\langle\cdot,\cdot\rangle$. Thus $\langle f,f\rangle_{\mathrm{CM}}$ must satisfy

$$m = \mathrm{Tr}_{F/\mathbb{Q}}\langle f,f\rangle_{\mathrm{CM}}.$$

In this way we see that the stack theoretic intersection

$$\mathcal{X}_\Phi \cap \mathcal{Z}(m) = \mathcal{X}_\Phi \times_{\mathcal{M}_{/\mathcal{O}_\Phi}} \mathcal{Z}(m)_{/\mathcal{O}_\Phi}$$

admits a decomposition

$$\mathcal{X}_\Phi \cap \mathcal{Z}(m) = \bigsqcup_{\substack{\alpha\in F\\ \mathrm{Tr}_{F/\mathbb{Q}}(\alpha)=m}} \mathcal{Z}_\Phi(\alpha), \tag{3.8.1}$$

where $\mathcal{Z}_\Phi(\alpha) = \mathcal{Z}_\Phi^{\mathcal{O}_F}(\alpha)$ is the stack of Section 3.5.

Definition 3.8.2. Define the *intersection multiplicity*

$$I(\mathcal{X}_\Phi : \mathcal{Z}(m)) = \sum_{\mathfrak{p}\subset\mathcal{O}_\Phi} \frac{\log(\mathrm{N}(\mathfrak{p}))}{[K_\Phi:\mathbb{Q}]} \sum_{z\in(\mathcal{X}_\Phi\cap\mathcal{Z}(m))(k^{\mathrm{alg}}_{\Phi,\mathfrak{p}})} \frac{\mathrm{length}\big(\mathcal{O}^{\mathrm{sh}}_{\mathcal{X}_\Phi\cap\mathcal{Z}(m),z}\big)}{\#\mathrm{Aut}(z)}.$$

This is finite if $\mathcal{X}_\Phi \cap \mathcal{Z}(m)$ has dimension 0.

Remark 3.8.3. From the point of view of arithmetic intersection theory, Definition 3.8.2 is a bit naive. The more natural definition is the *Serre intersection multiplicity* of [40, Ch. I.2] or [39, Ch. V.3], which takes into account higher Tor terms of the structure sheaves $\mathcal{O}_{\mathcal{X}_\Phi}$ and $\mathcal{O}_{\mathcal{Z}(m)}$. We have not done this, as the stack $\mathcal{M}$ in which the intersection is taking place is neither flat nor regular, and so is itself a rather naive place to be doing arithmetic intersection theory. See the comments of Section 1.3. For the reader's benefit, we only point out that [39, p. 111] shows that under modest hypotheses these higher Tor terms vanish, and Serre's intersection multiplicity agrees with the naive intersection multiplicity.

Theorem 3.8.4. *Let m be any nonzero integer. If F is a field, then* (3.8.1) *has dimension* 0, *and*

$$I(\mathcal{X}_\Phi : \mathcal{Z}(m)) = \sum_{\substack{\alpha\in F^\times,\alpha\gg0\\ \mathrm{Tr}_{F/\mathbb{Q}}(\alpha)=m}} \widehat{\deg}\,\widehat{\mathsf{Z}}_\Phi(\alpha,y)$$

for any $y\in\mathbb{R}^{>0}$.

Proof. The assumption that F is a field implies that every $\alpha\in F$ with $\mathrm{Tr}_{F/\mathbb{Q}}(\alpha)=m$ must satisfy $\alpha\in F^\times$. By Proposition 3.5.2 the right-hand side of (3.8.1) has dimension zero, and the only nonempty contribution comes from totally positive α. Therefore, (3.8.1) implies

$$I(\mathcal{X}_\Phi : \mathcal{Z}(m)) = \sum_{\substack{\alpha \in F^\times \\ \mathrm{Tr}_{F/\mathbb{Q}}(\alpha) = m}} \widehat{\deg}\, \mathcal{Z}_\Phi(\alpha),$$

where $\widehat{\deg}$ is the Arakelov degree of Definition 3.6.1, and the claim follows from (3.7.5). ☐

In order to construct a Green function for the divisors $\mathcal{Z}(m)$, we first describe the complex uniformizations of the algebraic stacks $\mathcal{M}$ and $\mathcal{Z}(m)$, following [26]. Recall that K_0 comes with a fixed embedding $\iota : K_0 \to \mathbb{C}$. Let $\delta \in K_0$ be the unique square root of $\mathrm{disc}(K_0)$ for which $\delta = i \cdot |\delta|$. If W is any K_0-vector space, then $W_\mathbb{R} = W \otimes_\mathbb{Q} \mathbb{R}$ is a $\mathbb{C}$-vector space.

Definition 3.8.5. A *principal Hermitian lattice* of signature (r, s) is a projective $\mathcal{O}_{K_0}$-module $\mathfrak{A}$ of rank $r + s$ together with a Hermitian form H of signature (r, s) under which $\mathfrak{A}$ is self-dual.

Let $\mathfrak{A}$ and $\mathfrak{A}_0$ be a principal Hermitian lattices of signature $(n-1, 1)$ and $(1, 0)$, respectively, with Hermitian forms H and H_0. Define a $\mathbb{Q}$-symplectic form λ on $\mathfrak{A}_\mathbb{Q}$ by

$$\delta \cdot \lambda(v, w) = H(v, w) - H(w, v), \tag{3.8.2}$$

and a $\mathbb{Q}$-symplectic form λ_0 on $\mathfrak{A}_{0\mathbb{Q}}$ by the same formula, with H replaced by H_0. The $\mathcal{O}_{K_0}$-module

$$L_B(\mathfrak{A}_0, \mathfrak{A}) = \mathrm{Hom}_{\mathcal{O}_{K_0}}(\mathfrak{A}_0, \mathfrak{A})$$

carries a natural Hermitian form $\langle f_1, f_2 \rangle = f_2^* \circ f_1$, where for any $f \in L_B(\mathfrak{A}_0, \mathfrak{A})$ we define $f^* : \mathfrak{A} \to \mathfrak{A}_0$ by the relation $H(fv, w) = H_0(v, f^* w)$. The Hermitian forms H_0, H, and $\langle \cdot, \cdot \rangle$ are related by

$$H(f_1 v_1, f_2 v_2) = H_0(v_1, v_2) \cdot \langle f_1, f_2 \rangle.$$

Abbreviate

$$V = L_B(\mathfrak{A}_0, \mathfrak{A}) \otimes_\mathbb{Z} \mathbb{Q},$$

and let $\mathcal{D}$ be the set of negative $\mathbb{C}$-lines in the Hermitian space $V_\mathbb{R}$. Given a nonzero isotropic vector $e \in V_\mathbb{R}$, there is an isotropic $e' \in V_\mathbb{R}$ such that $\langle e, e' \rangle = \delta$. The restriction of $\langle \cdot, \cdot \rangle$ to the orthogonal complement of the $\mathbb{C}$-span of $\{e, e'\}$ is positive definite, and so we may extend $\{e, e'\}$ to a $\mathbb{C}$-basis $e, e_1 \ldots, e_{n-2}, e' \in V_\mathbb{R}$ in such a way that the Hermitian form is given by

$$\langle x, y \rangle = {}^t x \cdot \begin{pmatrix} & & \delta \\ & A & \\ -\delta & & \end{pmatrix} \cdot \overline{y}$$

for a diagonal matrix $A \in M_{n-2}(\mathbb{R})$ with positive diagonal entries. There is a bijection

$$\mathcal{D} \cong \{(w,u) \in \mathbb{C} \times \mathbb{C}^{n-2} : \mathrm{Tr}(\delta w) + {}^t u A \overline{u} < 0\} \tag{3.8.3}$$

defined by associating (w,u) to the negative $\mathbb{C}$-line spanned by

$$\begin{bmatrix} w \\ u \\ 1 \end{bmatrix} \in \mathbb{C}^n \cong V_{\mathbb{R}}.$$

We say that the basis $e, e_1, \ldots, e_{n-2}, e'$ and the coordinates (w,u) are *adapted* to the isotropic vector e, which should be thought of as the limit as $w \to i \cdot \infty$. The coordinates (w,u) make $\mathcal{D}$ into a complex manifold. If $n = 1$, then $V_{\mathbb{R}}$ has signature $(0,1)$ and so has no nonzero isotropic vector. In this degenerate case, $\mathcal{D}$ consists of a single point.

Any choice of nonzero vector in $\mathfrak{A}_{0\mathbb{Q}}$ determines an isomorphism (evaluation at the chosen vector) of K-vector spaces $V \to \mathfrak{A}_{\mathbb{Q}}$, which identifies H with a positive rational multiple of $\langle \cdot, \cdot \rangle$ and identifies $\mathcal{D}$ with the space of negative lines in $\mathfrak{A}_{\mathbb{R}}$. This identification does not depend on the choice of vector used in its definition. Any $\mathtt{h} \in \mathcal{D}$, viewed as a negative line in the complex vector space $\mathfrak{A}_{\mathbb{R}}$, determines an endomorphism $J_{\mathtt{h}}$ of $\mathfrak{A}_{\mathbb{R}}$ by

$$J_{\mathtt{h}} v = \begin{cases} -iv & \text{if } v \in \mathtt{h} \\ iv & \text{if } v \in \mathtt{h}^{\perp}, \end{cases}$$

where $\mathtt{h}^{\perp}$ is the orthogonal complement of $\mathtt{h}$ with respect to H. Of course

$$J_{\mathtt{h}} \circ J_{\mathtt{h}} = -1,$$

and it is easy to see that the quadratic form $\lambda(J_{\mathtt{h}} v, v)$ on $\mathfrak{A}_{\mathbb{R}}$ is positive definite. A little linear algebra shows that every $\mathbb{R}$-linear endomorphism of $\mathfrak{A}_{\mathbb{R}}$ satisfying these two properties is of the form $J_{\mathtt{h}}$ for a unique $\mathtt{h} \in \mathcal{D}$.

We now describe the complex uniformization of $\mathcal{M}(\mathbb{C})$, following [26]. The complex elliptic curve $A_0(\mathbb{C}) = \mathfrak{A}_{0\mathbb{R}}/\mathfrak{A}_0$, with its principal polarization determined by λ_0, and its natural $\mathcal{O}_{K_0}$-action, determines a point of $\mathcal{M}_{(1,0)}(\mathbb{C})$. To each $\mathtt{h} \in \mathcal{D}$ there is an associated $(A_{\mathtt{h}}, \kappa_{\mathtt{h}}, \lambda_{\mathtt{h}}) \in \mathcal{M}_{(n-1,1)}(\mathbb{C})$ in which

- $A_{\mathtt{h}}(\mathbb{C}) = \mathfrak{A}_{\mathbb{R}}/\mathfrak{A}$ with the complex structure determined by $J_{\mathtt{h}}$,
- $\kappa_{\mathtt{h}} : \mathcal{O}_{K_0} \to \mathrm{End}(A_{\mathtt{h}})$ is induced by the $\mathcal{O}_{K_0}$-module structure on $\mathfrak{A}$,
- $\lambda_{\mathtt{h}} : A_{\mathtt{h}} \to A_{\mathtt{h}}^{\vee}$ is the polarization induced by the symplectic form λ.

The rule $\mathtt{h} \mapsto (A_0, A_{\mathtt{h}})$ defines a morphism of complex orbifolds

$$\mathcal{D} \to \mathcal{M}(\mathbb{C}).$$

Let $\Gamma_{\mathfrak{A}}$ be the automorphism group of $(\mathfrak{A}, H)$, and let $\Gamma_{\mathfrak{A}_0}$ be the automorphism group of $(\mathfrak{A}_0, H_0)$ (so that $\Gamma_{\mathfrak{A}_0}$ is just the group of roots of unity in K_0). The group $\Gamma = \Gamma_{\mathfrak{A}_0} \times \Gamma_{\mathfrak{A}}$ acts on $L_B(\mathfrak{A}_0, \mathfrak{A})$ through automorphisms preserving the

Hermitian form $\langle\cdot,\cdot\rangle$, and so acts on the space $\mathcal{D}$. The pair $(A_0, A_{\mathtt{h}})$ depends only on the Γ-orbit of $\mathtt{h}$, and we obtain a morphism of complex orbifolds

$$[\Gamma\backslash\mathcal{D}] \to \mathcal{M}(\mathbb{C})$$

identifying $[\Gamma\backslash\mathcal{D}]$ with a connected component of $\mathcal{M}(\mathbb{C})$. The other connected components are obtained by repeating this construction for each of the finitely many isomorphism classes of pairs $(\mathfrak{A}_0, \mathfrak{A})$.

Given a nonisotropic $f \in L_B(\mathfrak{A}_0, \mathfrak{A})$, define

$$\mathcal{D}(f) = \{\mathtt{h} \in \mathcal{D} : f \perp \mathtt{h}\}.$$

Following [21] or [3] there is a standard way to construct a smooth function on $\mathcal{D} \smallsetminus \mathcal{D}(f)$ with a logarithmic singularity along $\mathcal{D}(f)$. Let $f_{\mathtt{h}}$ be the orthogonal projection of f to $\mathtt{h}$, and set

$$R(f, \mathtt{h}) = -\langle f_{\mathtt{h}}, f_{\mathtt{h}}\rangle,$$

a nonnegative real analytic function on $\mathcal{D}$ whose zero set is $\mathcal{D}(f)$. If we write

$$f = ae + b_1e_1 + \cdots + b_{n-2}e_{n-2} + ce'$$

in terms of a basis adapted to an isotropic $e \in \mathfrak{A}_{\mathbb{R}}$, then this function is given by the explicit formula

$$R(f, \mathtt{h}) = \frac{|\delta(\overline{c}w - \overline{a}) + {}^t\overline{b}Au|^2}{|\delta(w - \overline{w}) + {}^tuA\overline{u}|} \tag{3.8.4}$$

in the coordinates (3.8.3), where ${}^tb = [b_1 \cdots b_{n-2}]$. This calculation shows that $\mathcal{D}(f)$ is a complex analytic divisor on $\mathcal{D}$, defined by the equation

$$\delta(\overline{c}w - \overline{a}) + {}^t\overline{b}Au = 0.$$

If $\langle f, f\rangle < 0$, then $\mathcal{D}(f) = \emptyset$, and $R(f, \mathtt{h})$ is a positive function on $\mathcal{D}$. In the degenerate case $n = 1$, the set $\mathcal{D}(f)$ is empty, and $R(f, \mathtt{h}) = -\langle f, f\rangle$. For any $\mathtt{h} \in \mathcal{D}$, each $f \in L_B(\mathfrak{A}_0, \mathfrak{A})$ induces a homomorphism of *real* Lie groups

$$f : A_0(\mathbb{C}) \to A_{\mathtt{h}}(\mathbb{C}),$$

and linear algebra shows that this map is complex analytic if and only if $\mathtt{h} \in \mathcal{D}(f)$. In this way we obtain a morphism of orbifolds

$$\Big[\Gamma\backslash \bigsqcup_{\substack{f \in L_B(\mathfrak{A}_0, \mathfrak{A}) \\ \langle f, f\rangle = m}} \mathcal{D}(f)\Big] \to \mathcal{Z}(m)(\mathbb{C})$$

defined by sending $\mathtt{h} \in \mathcal{D}(f)$ to the triple $(A_0, A_{\mathtt{h}}, f)$. The image is an open and closed suborbifold of $\mathcal{Z}(m)(\mathbb{C})$, and taking the disjoint union over all isomorphism classes of pairs $(\mathfrak{A}_0, \mathfrak{A})$ gives a complex uniformization of $\mathcal{Z}(m)(\mathbb{C})$.

The function $\beta_1(x)$ of (3.7.1) has a logarithmic singularity at $x = 0$, in the sense that $\beta_1(x) + \log(x)$ can be extended smoothly to $\mathbb{R}$. Furthermore, $\beta_1(x)$

decays exponentially as $x \to \infty$. Given a positive parameter $y \in \mathbb{R}$, define a smooth function

$$\mathtt{Gr}(f, y, \mathtt{h}) = \beta_1\Big(4\pi y R(f, \mathtt{h})\Big)$$

on $\mathcal{D} \smallsetminus \mathcal{D}(f)$. If $g(\mathtt{h}) = 0$ is any equation for the divisor $\mathcal{D}(f)$ on some open subset U of $\mathcal{D}$, then (3.8.4) shows that $\mathtt{Gr}(f, y, \mathtt{h}) + \log|g(\mathtt{h})|^2$ extends smoothly to all of U. For nonzero $m \in \mathbb{Z}$, the sum

$$\mathtt{Gr}(m, y, \mathtt{h}) = \sum_{\substack{f \in L_B(\mathfrak{A}_0, \mathfrak{A}) \\ \langle f, f\rangle = m}} \mathtt{Gr}(f, y, \mathtt{h})$$

defines a Green function, in the sense of [11], [40], for the orbifold divisor

$$\Big[\Gamma \backslash \bigsqcup_{\substack{f \in L_B(\mathfrak{A}_0, \mathfrak{A}) \\ \langle f, f\rangle = m}} \mathcal{D}(f)\Big] \to [\Gamma \backslash \mathcal{D}].$$

Using the complex uniformizations of $\mathcal{Z}(m)(\mathbb{C})$ and $\mathcal{M}(\mathbb{C})$ described above, the function $\mathtt{Gr}(m, y, \cdot)$, constructed now for every isomorphism class of pairs $(\mathfrak{A}_0, \mathfrak{A})$, defines a Green function for the divisor $\mathcal{Z}(m)$ on $\mathcal{M}$. If $m < 0$, then $\mathtt{Gr}(m, y, \cdot)$ is a smooth function on $\mathcal{M}(\mathbb{C})$.

Using the forgetful map $\mathcal{X}_\Phi \to \mathcal{M}_{/\mathcal{O}_\Phi}$, it makes sense to evaluate $\mathtt{Gr}(m, y, \cdot)$ on the finite set of points of the complex fiber of $\mathcal{X}_\Phi$. More precisely, we define

$$\mathtt{Gr}(m, y, \mathcal{X}_\Phi) = \frac{1}{[K_\Phi : \mathbb{Q}]} \sum_{\substack{\sigma : K_\Phi \to \mathbb{C} \\ \sigma|_{K_0} = \iota}} \sum_{z \in \mathcal{X}_\Phi^\sigma(\mathbb{C})} \frac{\mathtt{Gr}(m, y, z)}{\#\mathrm{Aut}(z)}.$$

Here $\mathcal{X}_\Phi^\sigma$ is the $\mathbb{C}$-scheme obtained from $\mathcal{X}_\Phi$ by base change through σ. There is a slight abuse of notation on the right-hand side, as we are confusing $z \in \mathcal{X}_\Phi^\sigma(\mathbb{C})$ with its image in $(\mathcal{M}_{/\mathcal{O}_\Phi})^\sigma(\mathbb{C}) \cong \mathcal{M}(\mathbb{C})$. The right-hand side is only defined if the images of $\mathcal{X}_\Phi^\sigma$ and $\mathcal{Z}(m)$ have no common points in the complex fiber of $\mathcal{M}$. This is equivalent to (3.8.1) being 0-dimensional.

THEOREM 3.8.6. *Let m be any nonzero integer. If F is a field, then* (3.8.1) *has dimension* 0, *and*

$$\mathtt{Gr}(m, y, \mathcal{X}_\Phi) = \sum_{\substack{\alpha \in F^\times, \alpha \gg 0 \\ \mathrm{Tr}_{F/\mathbb{Q}}(\alpha) = m}} \widehat{\deg}\, \widehat{\mathtt{Z}}_\Phi(\alpha, y)$$

for any $y \in \mathbb{R}^{>0}$.

Proof. We already saw in Theorem 3.8.4 that (3.8.1) has dimension 0. Suppose we have a point $z \in \mathcal{X}_\Phi(\mathbb{C})$ representing a pair (A_0, A). Set $\mathfrak{A}_0 = H_1(A_0(\mathbb{C}), \mathbb{Z})$ and $\mathfrak{A} = H_1(A(\mathbb{C}), \mathbb{Z})$, viewed as principal Hermitian lattices using the Hermitian forms H_0 and H determined, using (3.8.2), by the polarizations λ_0 and λ. The canonical isomorphism $\mathfrak{A}_\mathbb{R} \cong \mathrm{Lie}(A)$ determines a complex structure on $\mathfrak{A}_{\mathbb{R}}$, and under this complex structure multiplication by i has the form $J_{\mathtt{h}}$ for a unique $\mathtt{h} \in \mathcal{D}$. If $\varepsilon^{\mathrm{sp}} \in F_{\mathbb{R}}$ is the idempotent corresponding to the place ∞^{sp} determined by the restriction of $\varphi^{\mathrm{sp}} : K \to \mathbb{C}$ to F, then the negative line $\mathtt{h}$ is none other than $\mathtt{h} = \varepsilon^{\mathrm{sp}} \cdot \mathfrak{A}_{\mathbb{R}}$.

It follows easily that for any $f \in L_B(A_0, A)$, we have

$$R(f, \mathtt{h}) = -\langle f_{\mathtt{h}}, f_{\mathtt{h}} \rangle = -\langle \varepsilon^{\mathrm{sp}} f, \varepsilon^{\mathrm{sp}} f \rangle = |\langle f, f \rangle_{\mathrm{CM}}|_{\infty^{\mathrm{sp}}}.$$

Therefore, $\mathtt{Gr}(f, y, \mathtt{h}) = \beta_1\big(4\pi y |\langle f, f\rangle_{\mathrm{CM}}|_{\infty^{\mathrm{sp}}}\big)$ and

$$\begin{aligned}
\sum_{z \in \mathcal{X}_\Phi(\mathbb{C})} \frac{\mathtt{Gr}(m, y, z)}{\#\mathrm{Aut}(z)} &= \sum_{(A_0, A) \in \mathcal{X}_\Phi(\mathbb{C})} \sum_{\substack{f \in L_B(A_0, A) \\ \langle f, f \rangle = m}} \frac{\beta_1\big(4\pi y |\langle f, f\rangle_{\mathrm{CM}}|_{\infty^{\mathrm{sp}}}\big)}{\#\mathrm{Aut}(A_0, A)} \\
&= \sum_{\substack{\alpha \in F \\ \mathrm{Tr}_{F/\mathbb{Q}}(\alpha) = m}} \sum_{(A_0, A) \in \mathcal{X}_\Phi(\mathbb{C})} \sum_{\substack{f \in L_B(A_0, A) \\ \langle f, f \rangle_{\mathrm{CM}} = \alpha}} \frac{\beta_1\big(4\pi y |\alpha|_{\infty^{\mathrm{sp}}}\big)}{\#\mathrm{Aut}(A_0, A)} \\
&= \sum_{\substack{\alpha \in F \\ \mathrm{Tr}_{F/\mathbb{Q}}(\alpha) = m}} \sum_{z \in \mathcal{X}_\Phi(\mathbb{C})} \frac{\mathtt{Gr}_\Phi(\alpha, y, z)}{\#\mathrm{Aut}(z)},
\end{aligned}$$

where the final equality is by the definition (3.7.2) of $\mathtt{Gr}_\Phi(\alpha, y, z)$ at a point $z \in \mathcal{X}_\Phi(\mathbb{C})$. As F is a field and $m \neq 0$, we may restrict to $\alpha \in F^\times$ in the final sum. As the Hermitian form $\langle \cdot, \cdot \rangle_{\mathrm{CM}}$ is negative definite at ∞^{sp} by Proposition 3.4.1, we may further restrict to $\alpha \not\gg 0$.

Now suppose $z \in \mathcal{X}_\Phi^\sigma(\mathbb{C})$. As in the discussion preceding (3.7.3), we may identify $\mathcal{X}_\Phi^\sigma(\mathbb{C}) \cong \mathcal{X}_{\Phi^\sigma}(\mathbb{C})$. Repeating the argument above with Φ replaced by Φ^σ and ∞^{sp} replaced by $\infty^{\mathrm{sp},\sigma}$, and using (3.7.3) instead of (3.7.2), shows that

$$\mathtt{Gr}(m, y, \mathcal{X}_\Phi) = \frac{1}{[K_\Phi : \mathbb{Q}]} \sum_{\substack{\alpha \in F^\times, \alpha \not\gg 0 \\ \mathrm{Tr}_{F/\mathbb{Q}}(\alpha) = m}} \sum_{\substack{\sigma : K_\Phi \to \mathbb{C} \\ \sigma|_{K_0} = \iota}} \sum_{z \in \mathcal{X}_\Phi^\sigma(\mathbb{C})} \frac{\mathtt{Gr}_\Phi(\alpha, y, z)}{\#\mathrm{Aut}(z)}.$$

Comparing with (3.7.4) completes the proof. □

4. **Eisenstein series**

Keep K_0, F, K, Φ, and K_Φ as in Section 3. In this section we construct a Hilbert modular Eisenstein series $\mathcal{E}_\Phi(\tau, s)$ on $\mathcal{H}_F$. This Eisenstein series is *incoherent* in the sense of Kudla [21], and so vanishes at $s = 0$. We use formulas of Yang [45] to compute the Fourier coefficients of the derivative at $s = 0$, and show that these coefficients agree with the arithmetic degrees appearing in Theorem 3.7.2.

4.1. *A Hilbert modular Eisenstein series.* In this subsection we attach to every $\mathbf{c} \in F_{\mathbb{A}}^{\times}$ a Hilbert modular Eisenstein series $\mathcal{E}(\tau, s; \mathbf{c}, \psi_F)$ of the type considered in [45].

First we quickly recall some of the local theory of [21], [24], [27], [45]. Fix a place v of F and a $\mathbf{c} \in F_v^{\times}$, and let χ_v be the character of $F_v^{\times}$ associated to the quadratic extension K_v/F_v. Let ψ be an additive character $F_v \to \mathbb{C}^{\times}$. Associated to the F_v-quadratic space $(K_v, \mathbf{c}x\overline{x})$ and the character ψ is a Weil representation $\omega_{\mathbf{c},\psi}$ of $\mathrm{SL}_2(F_v)$ on the space of Schwartz functions $\mathfrak{S}(K_v)$ on K_v; see [20, Ch. II.4]. For $s \in \mathbb{C}$, let $I(\chi_v, s)$ be the space of the induced representation of the character $\chi_v(x) \cdot |x|_v^s$. There is an $\mathrm{SL}_2(F_v)$-intertwining operator

$$\lambda_{\mathbf{c},\psi} : \mathfrak{S}(K_v) \to I(\chi_v, 0)$$

defined by

$$\lambda_{\mathbf{c},\psi}(\varphi)(g) = (\omega_{\mathbf{c},\psi}(g)\varphi)(0).$$

For any $\varphi \in \mathfrak{S}(K_v)$, there is an associated section $\Phi(g, s) \in I(\chi_v, s)$ characterized by the properties

- $\Phi(\cdot, 0) = \lambda_{\mathbf{c},\psi}(\varphi)$,
- $\Phi(g, s)$ is *standard* in the sense that $\Phi(k, s)$ is independent of s for all k in the usual maximal compact subgroup of $\mathrm{SL}_2(F_v)$.

It will always be clear from context whether Φ refers to a section of $I(\chi_v, s)$, or to the fixed CM type of K.

If v is a finite place of F, let $\mathbf{1}_{\mathcal{O}_{K,v}} \in \mathfrak{S}(K_v)$ be the characteristic function of $\mathcal{O}_{K,v}$, and let

$$\Phi_{\mathbf{c},\psi}(g, s) \in I(\chi_v, s)$$

be the standard section satisfying $\Phi_{\mathbf{c},\psi}(\cdot, 0) = \lambda_{\mathbf{c},\psi}(\mathbf{1}_{\mathcal{O}_{K,v}})$. If v is archimedean, let $\varphi(x) = \exp(-2\pi|\mathbf{c}x\overline{x}|_v)$ be the Gaussian, and let $\Phi_{\mathbf{c},\psi}$ be the corresponding standard section satisfying $\Phi_{\mathbf{c},\psi}(\cdot, 0) = \lambda_{\mathbf{c},\psi}(\varphi)$. If

$$\mathrm{sign}(\mathbf{c}) = \mathbf{c}/|\mathbf{c}|_v$$

denotes the sign of $\mathbf{c}$, then $\Phi_{\mathbf{c},\psi}$ is the *normalized standard section of weight* $\mathrm{sign}(\mathbf{c})$, characterized by the property

$$\Phi_{\mathbf{c},\psi}\left(\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, s\right) = e^{\mathrm{sign}(\mathbf{c})\cdot i\theta}$$

for every $\theta \in \mathbb{R}$; see for example [24, (4.29)].

For each $\alpha \in F_v^{\times}$ and $\Phi \in I(\chi_v, s)$, define the local Whittaker function

$$W_\alpha(g, s; \Phi, \psi) = \int_{F_v} \Phi\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) g, s\right) \psi_v(-\alpha x)\, dx,$$

where $g \in \mathrm{SL}_2(F_v)$, and the Haar measure on F_v is self-dual with respect to ψ. When $\Phi = \Phi_{\mathbf{c},\psi}$ as above, we abbreviate

$$W_\alpha(g, s; \mathbf{c}, \psi) = W_\alpha(g, s; \Phi_{\mathbf{c},\psi}, \psi). \tag{4.1.1}$$

If we fix a $\delta \in F_v^\times$ and set $(\delta\psi)(x) = \psi(\delta x)$, then $\Phi_{\mathbf{c},\delta\psi} = \Phi_{\delta\mathbf{c},\psi}$ and

$$W_\alpha(g, s; \mathbf{c}, \delta\psi) = |\delta|_v^{1/2} \cdot W_{\delta\alpha}(g, s; \delta\mathbf{c}, \psi) \tag{4.1.2}$$

for all $\alpha \in F_v$. Indeed, the first equality is clear from explicit formulas for the Weil representation, as in [20, Ch. II.4], and the second is clear from the first.

Now we switch to the global setting. Let $\psi_{\mathbb{Q}} : \mathbb{Q}\backslash\mathbb{Q}_{\mathbb{A}} \to \mathbb{C}^\times$ be the usual additive character, whose archimedean component satisfies $\psi_{\mathbb{Q}}(x) = e^{2\pi i x}$ for all $x \in \mathbb{R}$, and whose nonarchimedean components are unramified. Set

$$\psi_F(x) = \psi_{\mathbb{Q}}(\mathrm{Tr}_{F/\mathbb{Q}}(x)).$$

Let

$$\chi : F_{\mathbb{A}}^\times \to \{\pm 1\}$$

be the composition of (3.0.2) with the product map $\{\pm 1\}^{\pi_0(F)} \to \{\pm 1\}$, so that $\chi = \prod_v \chi_v$, and let $I(\chi, s) = \otimes_v I(\chi_v, s)$ be the representation of $\mathrm{SL}_2(F_{\mathbb{A}})$ induced by the character χ. Given any $\mathbf{c} \in F_{\mathbb{A}}^\times$, we define a section of $I(\chi, s)$ by $\Phi_{\mathbf{c},\psi_F} = \otimes_v \Phi_{\mathbf{c}_v,\psi_{F,v}}$ and an Eisenstein series

$$E(g, s; \mathbf{c}, \psi_F) = \sum_{\gamma \in B(F)\backslash \mathrm{SL}_2(F)} \Phi_{\mathbf{c},\psi_F}(\gamma g, s)$$

on $\mathrm{SL}_2(F_{\mathbb{A}})$, where $B \subset \mathrm{SL}_2$ is the subgroup of upper triangular matrices.

Let

$$\mathcal{H}_F = \{x + iy \in F_{\mathbb{C}} : x, y \in F_{\mathbb{R}}, y \gg 0\}$$

be the F-upper half plane. A choice of isomorphism $F_{\mathbb{R}} \cong \mathbb{R}^n$, which we do not make, identifies $\mathcal{H}_F$ with a product of n complex upper half planes. For $\tau = x + iy \in \mathcal{H}_F$, set

$$g_\tau = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix} \in \mathrm{SL}_2(F_{\mathbb{R}}),$$

viewed as an element of $\mathrm{SL}_2(F_{\mathbb{A}})$ with trivial nonarchimedean components, and set (recall that $\mathfrak{d}_F$ is the different of $F/\mathbb{Q}$)

$$\mathcal{E}(\tau, s; \mathbf{c}, \psi_F) = \mathrm{N}(\mathfrak{d}_F)^{\frac{s+1}{2}} \cdot \frac{L(s+1, \chi)}{\mathrm{Norm}_{F/\mathbb{Q}}(y)^{1/2}} \cdot E(g_\tau, s; \mathbf{c}, \psi_F),$$

where $L(s, \chi) = \prod_v L(s, \chi_v)$ is the Dirichlet L-function of χ, including the Γ-factor

$$L(s, \chi_v) = \pi^{-(s+1)/2} \cdot \Gamma\left(\frac{s+1}{2}\right)$$

for archimedean v. Thus $\mathcal{E}(\tau, s; \mathbf{c}, \psi_F)$ is a Hilbert modular form, and it admits a Fourier expansion

$$\mathcal{E}(\tau, s; \mathbf{c}, \psi_F) = \sum_{\alpha \in F} \mathcal{E}_\alpha(\tau, s; \mathbf{c}, \psi_F)$$

in which

$$\mathcal{E}_\alpha(\tau, s; \mathbf{c}, \psi_F) = \mathrm{Norm}_{F/\mathbb{Q}}(y)^{-1/2} \int_{F \backslash F_\mathbb{A}} E\left(\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} g_\tau, s; \mathbf{c}, \psi_F(-b\alpha) \right) db.$$

Assume that $\mathbf{c} \in F_\mathbb{A}^\times$ satisfies $\mathbf{c}\mathcal{O}_F = \mathfrak{d}_F^{-1}$, $\mathbf{c}_v = 1$ for every archimedean v, and $\chi(\mathbf{c}) = -1$. The second condition implies that $\mathcal{E}(\tau, s; \mathbf{c}, \psi_F)$ has parallel weight one. The third condition implies that the $F_\mathbb{A}$-quadratic space $(K_\mathbb{A}, \mathbf{c}x\overline{x})$ is not the adelization of any F-quadratic space, and so the Eisenstein series $\mathcal{E}(\tau, s; \mathbf{c}, \psi_F)$ is *incoherent* in the sense of [21]. In particular,

$$\mathcal{E}(\tau, 0; \mathbf{c}, \psi_F) = 0$$

by [21, Th. 2.2]. Strictly speaking, the notion of an incoherent Eisenstein series only makes sense if F is a field. In general, we write

$$F = \prod_j F_j$$

as a product of totally real fields. There are corresponding factorizations $K = \prod_j K_j$, where each K_j is a quadratic totally imaginary extension of F_j, and

$$\mathcal{H}_F = \prod_j \mathcal{H}_{F_j},$$

where $\mathcal{H}_{F_j}$ is the F_j-upper half plane. The element $\mathbf{c}$ factors as $\mathbf{c} = \prod_j \mathbf{c}_j$, where each $\mathbf{c}_j \in F_{j\mathbb{A}}^\times \subset F_\mathbb{A}^\times$ has trivial components away from the factor $F_{j\mathbb{A}}^\times$. Similarly $\psi_F = \prod_j \psi_{F_j}$, and there is a factorization of Eisenstein series

$$\mathcal{E}(\tau, s; \mathbf{c}, \psi_F) = \prod_j \mathcal{E}_j(\tau_j, s; \mathbf{c}_j, \psi_{F_j}). \tag{4.1.3}$$

Each Eisenstein series in the factorization is then either *coherent* or *incoherent*, depending on whether $\chi(\mathbf{c}_j) = 1$ or -1. All incoherent factors vanish at $s = 0$, and as $\prod_j \chi(\mathbf{c}_j) = \chi(\mathbf{c}) = -1$, there is at least one incoherent factor.

Recall that the fixed CM type Φ has a distinguished element $\varphi^{\mathrm{sp}} : K \to \mathbb{C}$, which determines a direct factor K^{sp} of K with maximal totally real subfield F^{sp}. Recall also that the restriction of φ^{sp} to F determines an archimedean place denoted ∞^{sp}. We will be restricting our attention to Eisenstein series $\mathcal{E}(\tau, s; \mathbf{c}, \psi_F)$ with $\mathbf{c}$ chosen so that all factors on the right-hand side of (4.1.3) are coherent, except for the incoherent factor corresponding to F^{sp}.

Definition 4.1.1. Define a Hilbert modular Eisenstein series of weight one

$$\mathcal{E}_\Phi(\tau,s)=\sum_{\mathbf{c}\in\Xi}\mathcal{E}(\tau,s;\mathbf{c},\psi_F),$$

where the sum is over the finite set Ξ of $\mathrm{Nm}_{K/F}(\widehat{\mathcal{O}}_K^\times)$-orbits of $\mathbf{c}\in\mathbb{A}_F^\times$ satisfying

- $\mathbf{c}\mathcal{O}_F=\mathfrak{d}_F^{-1}$,
- $\mathbf{c}_v=1$ for every archimedean v,
- for every factor F_j of F

$$\chi(\mathbf{c}_j)=\begin{cases}1 & \text{if } F_j\neq F^{\mathrm{sp}}\\ -1 & \text{if } F_j=F^{\mathrm{sp}}.\end{cases}$$

To understand the motivation behind the particular set Ξ, reconsider the collection of Hermitian spaces $\mathcal{L}_B$ of Remark 3.4.2. Thus $\mathcal{L}_B$ consists of all isomorphism classes of Hermitian spaces $(L_B(A_0,A),\langle\cdot,\cdot\rangle_{\mathrm{CM}})$ as

$$(A_0,A)\in(\mathcal{M}_{(1,0)}\times\mathcal{CM}_\Phi^{\mathfrak{a}})(\mathbb{C})$$

varies. Take $\mathfrak{a}=\mathcal{O}_F$, and assume that $\mathfrak{s}=\mathfrak{d}_F^{-1}$ (which is the case if K_0 and K have relatively prime discriminants, by Proposition 3.1.6). The elements of $\mathcal{L}_B$ are alternately characterized as the isomorphism classes of pairs (L,H) in which L is a projective $\mathcal{O}_K$-module of rank 1, H is a K-valued $\mathcal{O}_K$-Hermitian form on L, the ideal of (L,H) is $\mathfrak{d}_F^{-1}$, and (L,H) is negative definite at ∞^{sp} and positive definite at all other archimedean places. For any such (L,H), there is a $\beta\in F_\mathbb{A}^\times$ and an isomorphism of $K_\mathbb{A}$-Hermitian spaces

$$(L\otimes_{\mathcal{O}_K}K_\mathbb{A},H)\cong(K_\mathbb{A},\beta x\overline{y})$$

identifying $L\otimes_{\mathcal{O}_K}\widehat{\mathcal{O}}_K\cong\widehat{\mathcal{O}}_K$. This β must satisfy $\beta\mathcal{O}_F=\mathfrak{d}_F^{-1}$, be negative at ∞^{sp} and positive at all other archimedean places, and satisfy $\chi(\beta_j)=1$ for each factor F_j of F. The finite part of β is well defined up to multiplication by a norm from $\widehat{\mathcal{O}}_K^\times$, and each archimedean component is well defined up to sign. This makes clear the connection between Ξ and $\mathcal{L}_B$: the elements of Ξ arise by taking the β's corresponding to elements of $\mathcal{L}_B$ and replacing the negative component at ∞^{sp} by a positive component. The corresponding $K_\mathbb{A}$-Hermitian spaces $(K_\mathbb{A},\mathbf{c}x\overline{y})$ from which the Eisenstein series $\mathcal{E}(\tau,s;\mathbf{c},\psi_F)$ are constructed are therefore incoherent at the factor F^{sp} and coherent at all other factors. That is to say, the $K_{j\mathbb{A}}$-Hermitian space $(K_{j\mathbb{A}},\mathbf{c}_jx\overline{y})$ arises as the adelization of a K_j-Hermitian space if any only if $F_j\neq F^{\mathrm{sp}}$.

4.2. *Fourier coefficients.* Of course $\mathcal{E}_\Phi(\tau,0)=0$, and so we study the derivative at $s=0$, which has a Fourier expansion

$$\frac{d}{ds}\mathcal{E}_\Phi(\tau,s)\Big|_{s=0}=\sum_{\alpha\in F}b_\Phi(\alpha,y)\cdot q^\alpha$$

in which

$$q^\alpha = \exp(2\pi i \mathrm{Tr}_{F/\mathbb{Q}}(\alpha\tau)).$$

We will give an explicit formula for the coefficients, at least when $\alpha \in F^\times$, and compare them with the formulas of Theorem 3.7.2.

For $\alpha \in F^\times$ and $\mathbf{c} \in \Xi$, define a finite set of places of F

$$\mathrm{Diff}(\alpha, \mathbf{c}) = \{v : \chi_v(\alpha\mathbf{c}) = -1\}.$$

Note that every $v \in \mathrm{Diff}(\alpha, \mathbf{c})$ is nonsplit in K and that there is a disjoint union

$$\mathrm{Diff}(\alpha, \mathbf{c}) = \bigsqcup_j \{\text{places } v \text{ of } F_j : \chi_v(\alpha\mathbf{c}) = -1\}.$$

Our hypotheses on $\mathbf{c}$ imply that every set in the disjoint union has even cardinality, except for

$$\mathrm{Diff}^{\mathrm{sp}}(\alpha, \mathbf{c}) = \{\text{places } v \text{ of } F^{\mathrm{sp}} : \chi_v(\alpha\mathbf{c}) = -1\},$$

which has odd cardinality. In particular, $\mathrm{Diff}(\alpha, \mathbf{c})$ has odd cardinality, and if it contains a unique place of F, that place must lie on the factor F^{sp}.

If v is a finite place of F and $\mathfrak{b}$ is a fractional $\mathcal{O}_{F,v}$-ideal, let

$$\rho_v(\mathfrak{b}) = \#\{\mathfrak{B} \subset \mathcal{O}_{K,v} : \mathfrak{B}\overline{\mathfrak{B}} = \mathfrak{b}\mathcal{O}_{K,v}\}.$$

If $\mathfrak{b}$ is a fractional $\mathcal{O}_F$-ideal, set $\rho(\mathfrak{b}) = \prod_v \rho_v(\mathfrak{b}_v)$, as in the introduction. The following proposition follows from calculations of Yang [45].

Proposition 4.2.1. *Suppose $\alpha \in F^\times$, let $d_{K/F}$ be the relative discriminant of K/F, and let r denote the number of places of F ramified in K (including the archimedean places). Suppose $\mathbf{c} \in \Xi$.*

(1) *If $\#\mathrm{Diff}(\alpha, \mathbf{c}) > 1$, then $\mathrm{ord}_{s=0}\, \mathcal{E}_\alpha(\tau, s; \mathbf{c}, \psi_F) > 1$.*

(2) *If $\mathrm{Diff}(\alpha, \mathbf{c}) = \{\mathfrak{p}\}$ with $\mathfrak{p}$ finite prime of F, then*

$$\frac{d}{ds}\mathcal{E}_\alpha(\tau, s; \mathbf{c}, \psi_F)\Big|_{s=0} = \frac{-2^{r-1}}{\mathrm{N}(d_{K/F})^{1/2}} \cdot \rho(\alpha\mathfrak{d}_F\mathfrak{p}^{-\varepsilon_\mathfrak{p}}) \cdot \mathrm{ord}_\mathfrak{p}(\alpha\mathfrak{d}_F\mathfrak{p}) \cdot \log(\mathrm{N}(\mathfrak{p})) \cdot q^\alpha,$$

where $\varepsilon_\mathfrak{p} = 0$ if $\mathfrak{p}$ ramifies in K, and $\varepsilon_\mathfrak{p} = 1$ if $\mathfrak{p}$ is unramified in K.

(3) *If $\mathrm{Diff}(\alpha, \mathbf{c}) = \{v\}$ with v an archimedean place of F, then*

$$\frac{d}{ds}\mathcal{E}_\alpha(\tau, s; \mathbf{c}, \psi_F)\Big|_{s=0} = \frac{-2^{r-1}}{\mathrm{N}(d_{K/F})^{1/2}} \cdot \rho(\alpha\mathfrak{d}_F) \cdot \beta_1(4\pi|y\alpha|_v) \cdot q^\alpha.$$

Recall that $\beta_1(t)$ was defined by (3.7.1).

Proof. Returning briefly to the local setting of (4.1.1), define the normalized local Whittaker function

$$W^*_{\alpha_v}(g_v, s; \mathbf{c}_v, \psi_v) = L(s+1, \chi_v) \cdot W_{\alpha_v}(g_v, s; \mathbf{c}_v, \psi_v).$$

Here ψ_v is any local additive character. The Fourier coefficient factors as a product

$$\mathcal{E}_\alpha(\tau, s; \mathbf{c}, \psi_F) = \mathrm{N}(\mathfrak{d}_F)^{(s+1)/2}\mathrm{Norm}_{F/\mathbb{Q}}(y)^{-1/2}\prod_v W^*_{\alpha_v}(g_{\tau,v}, s; \mathbf{c}_v, \psi_{F,v}).$$

The character $\psi^{\mathrm{unr}}(x) = \psi_F(\mathbf{c}x)$ is an unramified character of $F_{\mathbb{A}}^\times$, and (4.1.2) shows that

$$\mathcal{E}_\alpha(\tau, s; \mathbf{c}, \psi_F) = \mathrm{N}(\mathfrak{d}_F)^{s/2}\mathrm{Norm}_{F/\mathbb{Q}}(y)^{-1/2}\prod_v W^*_{\delta_v\alpha_v}(g_{\tau,v}, s; 1, \psi_v^{\mathrm{unr}}). \tag{4.2.1}$$

Let v be a nonarchimedean place of F, fix a uniformizing parameter $\varpi \in F_v$, let $f_v = \mathrm{ord}_v(d_{K/F})$, and let $q_v = \#\mathcal{O}_{F,v}/(\varpi)$. We now invoke [45, Prop. 2.1] and [45, Prop. 2.3]. If $\chi_v(\alpha\mathbf{c}) = 1$, then

$$\begin{aligned}&W^*_{\delta_v\alpha_v}(g_{\tau,v}, 0; 1, \psi_v^{\mathrm{unr}})\\ &= \chi_v(-1)\varepsilon(1/2, \chi_v, \psi_v^{\mathrm{unr}})\rho_v(\alpha\mathfrak{d}_F)\cdot\begin{cases}2q_v^{-f_v/2} & \text{if } v \text{ is ramified in } K\\ 1 & \text{if } v \text{ is unramified in } K.\end{cases}\end{aligned}$$

If instead $\chi_v(\alpha\mathbf{c}) = -1$, then $W^*_{\delta_v\alpha_v}(g_{\tau,v}, s; 1, \psi_v^{\mathrm{unr}})$ vanishes at $s = 0$, and

$$\begin{aligned}\frac{d}{ds}W^*_{\delta_v\alpha_v}(g_{\tau,v}, s; 1, \psi_v^{\mathrm{unr}})\Big|_{s=0} &= \chi_v(-1)\varepsilon(1/2, \chi_v, \psi_v^{\mathrm{unr}})\log(q_v)\cdot\frac{\mathrm{ord}_v(\alpha\mathfrak{d}_F)+1}{2}\\ &\times\begin{cases}2q_v^{-f_v/2}\cdot\rho_v(\alpha\mathfrak{d}_F) & \text{if } v \text{ is ramified in } K\\ \rho_v(\alpha\mathfrak{d}_F\mathfrak{p}_v^{-1}) & \text{if } v \text{ is unramified in } K,\end{cases}\end{aligned}$$

where $\mathfrak{p}_v$ is the prime ideal associated to v.

Now suppose v is an archimedean place of F. In this case we cite [45, Prop. 2.4]. If $\chi_v(\alpha\mathbf{c}) = 1$, then

$$W^*_{\delta_v\alpha_v}(g_{\tau,v}, 0; 1, \psi_v^{\mathrm{unr}}) = 2\chi_v(-1)\varepsilon(1/2, \chi_v, \psi_v^{\mathrm{unr}})\cdot y_v^{1/2}e^{2\pi i\alpha_v\tau_v}.$$

If $\chi_v(\alpha\mathbf{c}) = -1$, then $W^*_{\delta_v\alpha_v}(g_{\tau,v}, 0; 1, \psi_v^{\mathrm{unr}}) = 0$ and

$$\frac{d}{ds}W^*_{\delta_v\alpha_v}(g_{\tau,v}, s; 1, \psi_v^{\mathrm{unr}})\Big|_{s=0} = \chi_v(-1)\varepsilon(1/2, \chi_v, \psi_v^{\mathrm{unr}})\cdot y_v^{1/2}e^{2\pi i\alpha_v\tau_v}\beta_1(4\pi|y\alpha|_v).$$

Everything now follows easily. The above formulas show that when $v \in \mathrm{Diff}(\alpha, \mathbf{c})$, the v factor on the right-hand side of (4.2.1) vanishes at $s = 0$, and so the order of vanishing of $\mathcal{E}_\alpha(\tau, s; \mathbf{c}, \psi_F)$ is at least $\#\mathrm{Diff}(\alpha, \mathbf{c})$. If $\mathrm{Diff}(\alpha, \mathbf{c}) = \{w\}$, then differentiating (4.2.1) at $s = 0$ shows that

$$\begin{aligned}\frac{d}{ds}\mathcal{E}_\alpha(\tau, s; \mathbf{c}, \psi_F)\Big|_{s=0} &= \mathrm{Norm}_{F/\mathbb{Q}}(y)^{-1/2}\cdot\frac{d}{ds}W^*_{\delta_w\alpha_w}(g_{\tau,w}, s; 1, \psi_w^{\mathrm{unr}})\Big|_{s=0}\\ &\times\prod_{v\neq w}W^*_{\delta_v\alpha_v}(g_{\tau,v}, 0; 1, \psi_v^{\mathrm{unr}}),\end{aligned}$$

and the claim follows from the formulas above and the root number calculation

$$\prod_v \varepsilon(1/2, \chi_v, \psi_v^{\mathrm{unr}}) = \chi(\mathbf{c}) \cdot \prod_v \varepsilon(1/2, \chi_v, \psi_{F,v}) = -1.$$

(The first equality follows from [22, (3.29)], the second follows from the functional equation of $L(s, \chi)$, which shows that $\varepsilon(1/2, \chi) = 1$.) □

By the first claim of the proposition, for any $\alpha \in F^\times$, we have

$$\begin{aligned} b_\Phi(\alpha, y) \cdot q^\alpha &= \sum_{\mathbf{c} \in \Xi} \mathcal{E}'_\alpha(\tau, 0; \mathbf{c}, \psi_F) \\ &= \sum_v \sum_{\substack{\mathbf{c} \in \Xi \\ \mathrm{Diff}(\alpha, \mathbf{c}) = \{v\}}} \mathcal{E}'_\alpha(\tau, 0; \mathbf{c}, \psi_F), \end{aligned}$$

where the outer sum is over all places v of F. This sum is unchanged if we restrict further to places v of F^{sp} that are nonsplit in K^{sp}, as these are the only places for which the relation $\mathrm{Diff}(\alpha, \mathbf{c}) = \{v\}$ can ever hold.

Corollary 4.2.2. *Suppose $\alpha \in F^\times$ and $y \in F_{\mathbb{R}}^{\gg 0}$.*

(1) *If α is totally positive, then*

$$b_\Phi(\alpha, y) = \frac{-2^{r-1}}{\mathrm{N}(d_{K/F})^{1/2}} \cdot \sum_{\mathfrak{p}} \mathrm{ord}_{\mathfrak{p}}(\alpha \mathfrak{d}_F \mathfrak{p}) \cdot \rho(\alpha \mathfrak{d}_F \mathfrak{p}^{-\varepsilon_{\mathfrak{p}}}) \cdot \log(\mathrm{N}(\mathfrak{p})),$$

where the sum is over all primes $\mathfrak{p}$ of F^{sp} nonsplit in K^{sp}. In particular, $b_\Phi(\alpha, y)$ is independent of y.

(2) *If α is negative at exactly one archimedean place, v, of F, and if this v lies on the factor F^{sp}, then*

$$b_\Phi(\alpha, y) = \frac{-2^{r-1}}{\mathrm{N}(d_{K/F})^{1/2}} \cdot \rho(\alpha \mathfrak{d}_F) \cdot \beta_1(4\pi |y\alpha|_v).$$

(3) *In all other cases, $b_\Phi(\alpha, y) = 0$.*

Proof. Suppose first that α is totally positive, so that $\mathrm{Diff}(\alpha, \mathbf{c})$ contains only finite places of F. Proposition 4.2.1 implies

$$b_\Phi(\alpha, y) = \frac{-2^{r-1}}{\mathrm{N}(d_{K/F})^{1/2}} \sum_{\mathfrak{p}} \sum_{\substack{\mathbf{c} \in \Xi \\ \mathrm{Diff}(\alpha, \mathbf{c}) = \{\mathfrak{p}\}}} \rho(\alpha \mathfrak{d}_F \mathfrak{p}^{-\varepsilon_{\mathfrak{p}}}) \cdot \mathrm{ord}_{\mathfrak{p}}(\alpha \mathfrak{d}_F \mathfrak{p}) \cdot \log(\mathrm{N}(\mathfrak{p})),$$

where the first sum is over all primes of F^{sp} that are nonsplit in K^{sp}. Obviously, we may further restrict to those $\mathfrak{p}$ for which $\rho(\alpha \mathfrak{d}_F \mathfrak{p}^{-\varepsilon_{\mathfrak{p}}}) \neq 0$, and for each $\mathfrak{p}$, there is a unique choice of $\mathbf{c} \in \Xi$ for which $\mathrm{Diff}(\alpha, \mathbf{c}) = \{\mathfrak{p}\}$. This proves the first claim, and the proofs of the remaining claims are similar. □

Comparing Theorem 3.7.2 and Corollary 4.2.2 proves the following result.

THEOREM 4.2.3. *Assume the discriminants of $K_0/\mathbb{Q}$ and $F/\mathbb{Q}$ are odd and relatively prime. If $\alpha \in F^\times$ and $y \in F_{\mathbb{R}}^{\gg 0}$, then*

$$\widehat{\deg}\,\widehat{\mathtt{Z}}_\Phi(\alpha, y) = -\frac{h(K_0)}{w(K_0)} \cdot \frac{\sqrt{\mathrm{N}(d_{K/F})}}{2^{r-1}[K^{\mathrm{sp}} : \mathbb{Q}]} \cdot b_\Phi(\alpha, y).$$

Let $i_F : \mathcal{H} \to \mathcal{H}_F$ be the diagonal embedding of the usual complex upper half plane into $\mathcal{H}_F$. The restriction $\mathcal{E}_\Phi(i_F(\tau), s)$ of $\mathcal{E}_\Phi(\tau, s)$ to $\mathcal{H}$ vanishes at $s = 0$, and the derivative has a Fourier expansion

$$\frac{d}{ds}\mathcal{E}_\Phi(i_F(\tau), s)\Big|_{s=0} = \sum_{m \in \mathbb{Z}} c_\Phi(m, y) \cdot q^m$$

in which

$$c_\Phi(m, y) = \sum_{\substack{\alpha \in F \\ \mathrm{Tr}_{F/\mathbb{Q}}(\alpha) = m}} b_\Phi(\alpha, y).$$

Here $\tau = x + iy \in \mathcal{H}$ and $q = \exp(2\pi i \tau)$, as usual.

COROLLARY 4.2.4. *Assume the discriminants of $K_0/\mathbb{Q}$ and $F/\mathbb{Q}$ are odd and relatively prime. If F is a field and m is nonzero, then*

$$I(\mathcal{X}_\Phi : \mathcal{Z}(m)) + \mathtt{Gr}(m, y, \mathcal{X}_\Phi) = -\frac{h(K_0)}{w(K_0)} \cdot \frac{\sqrt{\mathrm{N}(d_{K/F})}}{2^{r-1}[K : \mathbb{Q}]} \cdot c_\Phi(m, y)$$

for all $y \in \mathbb{R}^{>0}$.

Proof. Theorems 3.8.4 and 3.8.6 imply

$$I(\mathcal{X}_\Phi : \mathcal{Z}(m)) + \mathtt{Gr}(m, y, \mathcal{X}_\Phi) = \sum_{\substack{\alpha \in F \\ \mathrm{Tr}_{F/\mathbb{Q}}(\alpha) = m}} \widehat{\deg}\,\widehat{\mathtt{Z}}_\Phi(\alpha, y),$$

and so the claim is clear from Theorem 4.2.3. □

References

[1] A. ASH, D. MUMFORD, M. RAPOPORT, and Y.-S. TAI, *Smooth Compactifications of Locally Symmetric Varieties*, second ed., *Cambridge Math. Library*, Cambridge Univ. Press, Cambridge, 2010, with the collaboration of Peter Scholze. MR 2590897. Zbl 1209.14001. http://dx.doi.org/10.1017/CBO9780511674693.

[2] P. BERTHELOT, L. BREEN, and W. MESSING, *Théorie de Dieudonné Cristalline. II*, *Lecture Notes in Math.* **930**, Springer-Verlag, New York, 1982. MR 0667344. Zbl 0516.14015. http://dx.doi.org/10.1007/BFb0093025.

[3] J. H. BRUINIER, Borcherds products and Chern classes of Hirzebruch-Zagier divisors, *Invent. Math.* **138** (1999), 51–83. MR 1714336. Zbl 1011.11027. http://dx.doi.org/10.1007/s002220050341.

[4] J. H. Bruinier, J. I. Burgos Gil, and U. Kühn, Borcherds products and arithmetic intersection theory on Hilbert modular surfaces, *Duke Math. J.* **139** (2007), 1–88. MR 2322676. Zbl 1208.11077. http://dx.doi.org/10.1215/S0012-7094-07-13911-5.

[5] J. I. Burgos Gil, J. Kramer, and U. Kühn, Cohomological arithmetic Chow rings, *J. Inst. Math. Jussieu* **6** (2007), 1–172. MR 2285241. Zbl 1115.14013. http://dx.doi.org/10.1017/S1474748007000011.

[6] J. W. Cogdell, Arithmetic cycles on Picard modular surfaces and modular forms of Nebentypus, *J. Reine Angew. Math.* **357** (1985), 115–137. MR 0783537. Zbl 0547.10028. http://dx.doi.org/10.1515/crll.1985.357.115.

[7] B. Conrad, Gross-Zagier revisited, in *Heegner Points and Rankin L-Series*, *Math. Sci. Res. Inst. Publ.* **49**, Cambridge Univ. Press, Cambridge, 2004, with an appendix by W. R. Mann, pp. 67–163. MR 2083211. Zbl 1072.11040. http://dx.doi.org/10.1017/CBO9780511756375.006.

[8] ———, Shimura-Taniyama formula, 2005, in *Notes on complex multiplication*: 2004-05 *VIGRE number theory working group*. Available at http://math.stanford.edu/~conrad.

[9] H. Gillet, Intersection theory on algebraic stacks and Q-varieties, in *Proceedings of the Luminy Conference on Algebraic K-Theory* (Luminy, 1983), **34** J. Pure Appl. Algebra no. 2-3, 1984, pp. 193–240. MR 0772058. Zbl 0607.14004. http://dx.doi.org/10.1016/0022-4049(84)90036-7.

[10] ———, Arithmetic intersection theory on Deligne-Mumford stacks, in *Motives and Algebraic Cycles*, *Fields Inst. Commun.* **56**, Amer. Math. Soc., Providence, RI, 2009, pp. 93–109. MR 2562454. Zbl 1196.14022.

[11] H. Gillet and C. Soulé, Arithmetic intersection theory, *Inst. Hautes Études Sci. Publ. Math.* (1990), 93–174 (1991). MR 1087394. Zbl 0741.14012. Available at http://www.numdam.org/item?id=PMIHES_1990__72__93_0.

[12] B. H. Gross, On canonical and quasi-canonical liftings, *Invent. Math.* **84** (1986), 321–326. MR 0833193. Zbl 0597.14044. http://dx.doi.org/10.1007/BF01388810.

[13] B. H. Gross, W. Kohnen, and D. Zagier, Heegner points and derivatives of L-series. II, *Math. Ann.* **278** (1987), 497–562. MR 0909238. Zbl 0641.14013. http://dx.doi.org/10.1007/BF01458081.

[14] B. H. Gross and D. B. Zagier, On singular moduli, *J. Reine Angew. Math.* **355** (1985), 191–220. MR 0772491. Zbl 0545.10015. http://dx.doi.org/10.1515/crll.1985.355.191.

[15] ———, Heegner points and derivatives of L-series, *Invent. Math.* **84** (1986), 225–320. MR 0833192. Zbl 0608.14019. http://dx.doi.org/10.1007/BF01388809.

[16] A. Grothendieck, *Groupes de Barsotti-Tate et cristaux de Dieudonné*, Les Presses de l'Université de Montréal, Montreal, Que., 1974, Séminaire de Mathématiques Supérieures, No. 45 (Été, 1970). MR 0417192. Zbl 0331.14021.

[17] B. Howard and T. Yang, Singular moduli refined, in *Arithmetic Geometry and Automorphic Forms*, *Adv. Lectures in Math.* **19**, Higher Education Press, Beijing and International Press, Boston, 2011, pp. 367–406.

[18] R. E. Kottwitz, Points on some Shimura varieties over finite fields, *J. Amer. Math. Soc.* **5** (1992), 373–444. MR 1124982. Zbl 0796.14014. http://dx.doi.org/10.2307/2152772.

[19] N. Krämer, Local models for ramified unitary groups, *Abh. Math. Sem. Univ. Hamburg* **73** (2003), 67–80. MR 2028507. Zbl 1053.14024. http://dx.doi.org/10.1007/BF02941269.

[20] S. S. Kudla, Notes on the local theta correspondence, 1996. Available at http://www.math.toronto.edu/~skudla/castle.pdf.

[21] ———, Central derivatives of Eisenstein series and height pairings, *Ann. of Math.* **146** (1997), 545–646. MR 1491448. Zbl 0990.11032. http://dx.doi.org/10.2307/2952456.

[22] ———, Tate's thesis, in *An Introduction to the Langlands Program* (Jerusalem, 2001), Birkhäuser, Boston, MA, 2003, pp. 109–131. MR 1990377. Zbl 1111.11315.

[23] ———, Special cycles and derivatives of Eisenstein series, in *Heegner Points and Rankin L-series*, *Math. Sci. Res. Inst. Publ.* **49**, Cambridge Univ. Press, Cambridge, 2004, pp. 243–270. MR 2083214. Zbl 1073.11042. http://dx.doi.org/10.1017/CBO9780511756375.009.

[24] S. S. Kudla and S. Rallis, On the Weil-Siegel formula, *J. Reine Angew. Math.* **387** (1988), 1–68. MR 0946349. Zbl 0644.10021. http://dx.doi.org/10.1515/crll.1988.391.65.

[25] S. S. Kudla and M. Rapoport, Special cycles on unitary Shimura varieties I. Unramified local theory, *Invent. Math.* **184** (2011), 629–682. MR 2800697. Zbl 1229.14020. http://dx.doi.org/10.1007/s00222-010-0298-z.

[26] ———, Special cycles on unitary Shimura varieties II. Global theory, 2011. arXiv 0912.3758.

[27] S. S. Kudla, M. Rapoport, and T. Yang, On the derivative of an Eisenstein series of weight one, *Internat. Math. Res. Notices* (1999), 347–385. MR 1683308. Zbl 1009.11032. http://dx.doi.org/10.1155/S1073792899000185.

[28] ———, Derivatives of Eisenstein series and Faltings heights, *Compos. Math.* **140** (2004), 887–951. MR 2059224. Zbl 1088.11050. http://dx.doi.org/10.1112/S0010437X03000459.

[29] ———, *Modular Forms and Special Cycles on Shimura Curves*, *Ann. of Math. Studies* **161**, Princeton Univ. Press, Princeton, NJ, 2006. MR 2220359. Zbl 1157.11027.

[30] K.-W. Lan, Arithmetic compactifications of PEL-type Shimura varieties, ProQuest LLC, Ann Arbor, MI, 2008, Ph.D. thesis, Harvard University. MR 2711676. Available at http://gateway.proquest.com/openurl?url_ver=Z39.88-2004&rft_val_fmt=info:ofi/fmt:kev:mtx:dissertation&res_dat=xri:pqdiss&rft_dat=xri:pqdiss:3312427.

[31] M. J. Larsen, Arithmetic compactification of some Shimura surfaces, in *The Zeta Functions of Picard Modular Surfaces*, Univ. Montréal, Montreal, QC, 1992, pp. 31–45. MR 1155225. Zbl 0760.14009.

[32] W. Messing, *The Crystals Associated to Barsotti-Tate Groups: With Applications to Abelian Schemes*, *Lecture Notes in Math.* **264**, Springer-Verlag, New York, 1972. MR 0347836. Zbl 0243.14013. http://dx.doi.org/10.1007/BFb0058301.

[33] ———, Travaux de Zink, in *Séminaire Bourbaki. Vol.* 2005/2006, *Astérisque* **311**, 2007, pp. Exp. No. 964, ix, 341–364. MR 2359049. Zbl 1197.14050.

[34] V. Meusers, Canonical and quasi-canonical liftings in the split case, *Astérisque* (2007), 87–98. MR 2340373. Zbl 1223.14051.

[35] A. Miller, Compactifications of universal abelian threefolds with CM, *Geom. Dedicata* **143** (2009), 155–179. MR 2576300. Zbl 1198.14032. http://dx.doi.org/10.1007/s10711-009-9380-3.

[36] F. Oort, Subvarieties of moduli spaces, *Invent. Math.* **24** (1974), 95–119. MR 0424813. Zbl 0259.14011. http://dx.doi.org/10.1007/BF01404301.

[37] G. Pappas, On the arithmetic moduli schemes of PEL Shimura varieties, *J. Algebraic Geom.* **9** (2000), 577–605. MR 1752014. Zbl 0978.14023.

[38] M. Rapoport, Compactifications de l'espace de modules de Hilbert-Blumenthal, *Compositio Math.* **36** (1978), 255–335. MR 0515050. Zbl 0386.14006. Available at http://www.numdam.org/item?id=CM_1978__36_3_255_0.

[39] J.-P. Serre, *Local Algebra*, *Springer Monogr. Math.*, Springer-Verlag, New York, 2000, translated from the French by CheeWhye Chin and revised by the author. MR 1771925. Zbl 0959.13010.

[40] C. Soulé, *Lectures on Arakelov Geometry*, *Cambridge Studies in Adv. Math.* **33**, Cambridge Univ. Press, Cambridge, 1992, with the collaboration of D. Abramovich, J.-F. Burnol, and J. Kramer. MR 1208731. Zbl 0812.14015. http://dx.doi.org/10.1017/CBO9780511623950.

[41] J. T. Tate, p-divisible groups, in *Proc. Conf. Local Fields* (Driebergen, 1966), Springer-Verlag, New York, 1967, pp. 158–183. MR 0231827. Zbl 0157.27601.

[42] U. Terstiege, Intersections of special cycles on the Shimura variety for $\mathrm{GU}(1,2)$, 2010, preprint. arXiv 1006.2106.

[43] A. Vistoli, Intersection theory on algebraic stacks and on their moduli spaces, *Invent. Math.* **97** (1989), 613–670. MR 1005008. Zbl 0694.14001. http://dx.doi.org/10.1007/BF01388892.

[44] S. Wewers, *Canonical and Quasi-Canonical Liftings*, *Astérisque* **312**, 2007. MR 2340372. Zbl 1156.14323.

[45] T. Yang, CM number fields and modular forms, *Pure Appl. Math. Q.* **1** (2005), 305–340. MR 2194727. Zbl 1146.11028.

[46] T. Zink, The display of a formal p-divisible group, in *Cohomologies p-Adiques et Applications Arithmétiques, I*, *Astérisque* **278**, 2002, pp. 127–248. MR 1922825. Zbl 1008.14008.

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