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Overholonomicity of overconvergent F-isocrystals over smooth varieties

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Abstract

We prove the overholonomicity of overconvergent F-isocrystals over smooth varieties. This implies that the notions of overholonomicity and devissability in overconvergent F-isocrystals are equivalent. Then the overholonomicity is stable under tensor products. So, the overholonomicity gives a p-adic cohomology stable under Grothendieck's cohomological operations.

Contents

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Introduction

Let $\mathcal{V}$ be a complete discrete valuation ring of characteristic 0, with perfect residue field k of characteristic $p > 0$ and field of fractions K. In order to define a good category of p-adic coefficients over k-varieties (i.e., separated schemes of finite type over $\operatorname{Spec} k$) stable under cohomological operations, Berthelot introduced the notion of arithmetic $\mathcal{D}$-modules and their cohomological operations (see [Ber90], [Ber02], [Ber96b], [Ber00]). These arithmetic $\mathcal{D}$-modules over k-varieties correspond to an arithmetic analogue of the classical theory of $\mathcal{D}$-modules over complex varieties. Also, he defined holonomic F-complexes of arithmetic $\mathcal{D}$-modules. Virrion checked the stability of holonomicity under the dual functor (see [Vir00]). Berthelot conjectured its stability under the other Grothendieck's operations: direct images (to be precise, morphisms should be proper at the level of formal $\mathcal{V}$-schemes), extraordinary direct images, inverse images, extraordinary inverse images, tensor products (see [Ber02, 5.3.6]). We checked that the conjecture on the stability of holonomicity under inverse images implies the others ones (see [Car09c]).

In order to avoid these conjectures and to get a category of F-complexes of arithmetic $\mathcal{D}$-modules that satisfies these stability conditions, the first step was to introduce the notion of overcoherence as follows. A coherent F-complex of arithmetic $\mathcal{D}$-modules is overcoherent (in fact, the 'F,' i.e., the Frobenius structure, is not necessary) if its coherence is stable under extraordinary inverse image. (See [Car04] for the definition and [Car09d] for this characterization.) We checked that this notion of overcoherence is stable under extraordinary inverse image, direct image (by a proper morphism at the level of formal $\mathcal{V}$-schemes) and local cohomological functors. This stability allows us, for instance, to define canonically overcoherent arithmetic $\mathcal{D}$-modules over k-varieties. (Otherwise, we work on formal $\mathcal{V}$-schemes.) To improve the stability properties, we defined the category of overholonomic F-complexes over k-varieties that is, roughly speaking, the smallest subcategory of overcoherent F-complexes such that it is moreover stable by dual functors. (More precisely, see the definition [Car09c, 3.1].) We got the stability of overholonomicity by direct images, extraordinary direct images, extraordinary inverse images, and inverse images. Moreover, it is already known that this category of p-adic coefficients is not zero since it contains unit-root overconvergent F-isocrystals (see [Car09c]) and, in particular, the constant coefficient associated to a k-variety (i.e., that gives, for example, the corresponding Weil's zeta functions). Because an overholonomic arithmetic F-$\mathcal{D}$-module is holonomic (which is not obvious), these gave new examples of holonomicity. This was checked by descent of the overholonomicity property (this descent is technically possible thanks to its stability) using de Jong's desingularization theorem. Now, it remains to check the stability of overholonomicity by (internal or external) tensor products.

The second step was to construct an equivalence between the category of overconvergent F-isocrystals over a smooth k-variety Y (which is the category of p-adic coefficients associated to Berthelot's rigid cohomology; see [LS07]) and the category of overcoherent F-isocrystals on Y, where this last one is a subcategory of arithmetic F-$\mathcal{D}$-modules over Y. (See [Car06a] and [Car07a] for the general case.) Next, from this equivalence, we got the notion of F-complexes of arithmetic $\mathcal{D}$-modules dévissable in overconvergent F-isocrystals. We proved first that overholonomic (see [Car06a]) and next overcoherent (see [Car07a]) F-complexes of arithmetic $\mathcal{D}$-modules are dévissable in overconvergent F-isocrystals. Since overconvergent F-isocrystals are stable under tensor products, we established that F-complexes dévissable in overconvergent F-isocrystals are also stable under tensor products (see [Car07b]).

The third step is to prove that the notions (still with Frobenius structures) of overcoherence, overholonomicity and devissability in overconvergent F-isocrystals are identical. With what we have proved in the first and second steps, the equality between the overholonomicity and the devissability in overconvergent F-isocrystals implies that the overholonomicity is stable under Grothendieck's aforesaid six cohomological operations and is wide enough since it contains overconvergent F-isocrystals on smooth k-varieties. Also, for this purpose, it is enough to prove the overholonomicity of overconvergent F-isocrystals on smooth k-varieties. Fortunately, Kedlaya has just checked that Shiho's semistable reduction conjecture is exact, i.e., that given an overconvergent F-isocrystal on a smooth k-variety, one can pull back along a suitable generically finite cover to obtain an isocrystal that extends, with logarithmic singularities and nilpotent residues, to some complete variety (see [Ked07], [Ked08], [Ked09], and at last [Ked11]). Kedlaya's semistable reduction theorem gives us a very important tool since we come down by descent (indeed, overholonomicity behaves well by proper generically étale descent thanks to its stability by extraordinary inverse images and direct images) to study the case of the overconvergent F-isocrystals that extend with logarithmic singularities and nilpotent residues to some complete variety. We began this study in [Car09a]. We proceed in this article and check the overholonomicity of these log-extendable overconvergent F-isocrystals, which finish the check of our third step. The technical key point of this overholonomicity is a comparison theorem between relative logarithmic rigid cohomology and rigid cohomology and above all, in a more general essential context, the fact that both cohomologies are not so different. This fundamental key point was checked by the second author, and the fact that this implies the overholonomicity of log-extendable overconvergent F-isocrystals was checked by the first one.

Now, let us describe the contents. Let $g : \mathfrak{X} \to \mathfrak{T}$ be a smooth morphism of smooth formal $\mathcal{V}$-schemes of pure relative dimension d, let $\mathcal{Z}$ be a relative strict normal crossings divisor of $\mathfrak{X}$ over $\mathcal{T}$, let $\mathcal{Y}$ be the complement of $\mathcal{Z}$ in $\mathfrak{X}$, let D be a closed subscheme of X and U the complement of D in X. Let $\mathfrak{X}^{\#} = (\mathfrak{X}, \mathcal{Z})$ be the logarithmic formal $\mathcal{V}$-scheme with the logarithmic structure associated to $\mathcal{Z}$ and $u : \mathfrak{X}^{\#} \to \mathfrak{X}$ be the canonical morphism.

In the first chapter, we compare logarithmic rigid cohomology and rigid cohomology with overconvergent coefficients in the relative situations. Let E be a log-isocrystal on $U^{\#}/\mathcal{T}_K$ overconvergent along D (see the definition in (1.1.0.2)). Suppose that, along each irreducible component of Z that is not included in D,

(a) none of differences of exponents is a p-adic Liouville number and
(b′) any exponent is neither a p-adic Liouville number nor a positive integer.

Then the natural comparison map

$$\mathbb{R}g_{K*}(j_U^{\dagger}\Omega^{\bullet}_{\mathfrak{X}^{\#}_K/\mathcal{T}_K} \otimes_{j_U^{\dagger}\mathcal{O}_{]X[_{\mathfrak{X}}}} E) \to \mathbb{R}g_{K*}(j_{Y\cap U}^{\dagger}\Omega^{\bullet}_{\mathfrak{X}_K/\mathcal{T}_K} \otimes_{j^{\dagger}_{Y\cap U}\mathcal{O}_{]X[_{\mathfrak{X}}}} j^{\dagger}_{Y\cap U}E)$$

is an isomorphism (see 1.1.1). Let us consider the case where g has a section which is identified with $\mathcal{Z}$ such that $Z \not\subset D$. If one assumes (a) above and

(b) none of exponents is a p-adic Liouville number,

then the cone of the above comparison map is given by a complex that consists of overconvergent log-isocrystals on the divisor (see 1.1.4). In the second section we develop a notion of quasi-coherence on formal log-schemes, which was studied by Berthelot in the case of formal schemes (see [Ber02]), and cohomological operators such as direct images and extraordinary inverse images by morphisms of smooth formal $\mathcal{V}$-log-schemes. Furthermore, in the third section, we translate this comparison in the language of arithmetic $\mathcal{D}$-modules.

In the first section of the second chapter, we recall Kedlaya's semistable reduction theorem. Let $\mathcal{E}$ be a coherent $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\#},\mathbb{Q}}$-module that is a locally projective $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}$-module of finite type that satisfies conditions (a) and (b′) above. Then, using the comparison theorem of the first section, we check that the canonical morphism $u_+(\mathcal{E}) \to \mathcal{E}(^{\dagger}Z)$ is an isomorphism (see 2.2.9). This implies that the canonical morphism $\Omega^{\bullet}_{\mathfrak{X}^{\#}/\mathcal{T},\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X},\mathbb{Q}}} \mathcal{E} \to \Omega^{\bullet}_{\mathfrak{X}/\mathcal{T},\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X},\mathbb{Q}}} \mathcal{E}(^{\dagger}Z)$ is a quasi-isomorphism (see (2.2.12)). In the third section, we prove that if

(c) none of elements of $\mathrm{Exp}(\mathcal{E})^{\mathrm{gr}}$ (the group generated by all exponents of $\mathcal{E}$) is a p-adic Liouville number,

then $u_+(\mathcal{E})$ is overholonomic, which implies that $\mathcal{E}(^{\dagger}Z)$ (the isocrystal on Y overconvergent along Z associated to $\mathcal{E}$) is overholonomic. The principal reason why we need to replace conditions (a) and (b′) by condition (c) is because we need here something stable under duality and because the log-relative duality isomorphism is of the form (see [Car09a, 5.25.2] and [Car09a, 5.22]) $\mathbb{D}_{\mathfrak{X}} \circ u_+(\mathcal{E}) \xrightarrow{\sim} u_+(\mathcal{E}^{\vee}(-\mathcal{Z}))$, where "$\mathbb{D}_{\mathfrak{X}}$" means the dual as $\mathcal{D}^{\dagger}_{\mathfrak{X},\mathbb{Q}}$-module and "$\vee$" is the dual as a convergent log-isocrystal; e.g., even if $\mathcal{E}$ is a convergent log-F-isocrystal, then unfortunately $\mathcal{E}^{\vee}(-\mathcal{Z})$ have positive exponents. Hence, using Kedlaya's semistable reduction theorem, we obtain by descent the overholonomicity of overconvergent F-isocrystals on smooth k-varieties. Thus, the notion of overholonomicity, overcoherence and devissability in overconvergent F-isocrystals are the same. Also, the overholonomicity behaves as well as the holonomicity in the classical theory. Finally, in the case of curves, we extend some results of [Car06b]. (We can mention here that we have similar holonomicity results of Crew in [Cre06] and of Noot-Huyghe and Trihan in [NHT07].) More precisely, let $\mathfrak{X}$ be a smooth separated formal $\mathcal{V}$-scheme of dimension 1, Z a divisor of X, $\mathfrak{Y} := \mathfrak{X} \setminus Z$ and $\mathcal{E}$ a complex of F-$D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{X}}({}^{\dagger}Z)_{\mathbb{Q}})$. Then, firstly we prove that $\mathcal{E}$ is holonomic if and only if $\mathcal{E}$ is overholonomic. Secondly, if the restriction of $\mathcal{E}$ on $\mathfrak{Y}$ is a holonomic F-$\mathcal{D}^{\dagger}_{\mathfrak{Y},\mathbb{Q}}$-module, then we check that $\mathcal{E}$ is a holonomic F-$\mathcal{D}^{\dagger}_{\mathfrak{X},\mathbb{Q}}$-module. Both results should be true in higher dimensions but are still conjectures. Besides, this second conjecture implies the first one and is the strongest Berthelot's conjecture on the stability of holonomicity (see [Ber02, 5.3.6.D]).

Notation. Let $\mathcal{V}$ be a complete valuation ring of characteristic 0, k its residue field of characteristic $p > 0$, K its fractions field with a multiplicative valuation $|\text{-}|$, $\mathcal{S} := \operatorname{Spf} \mathcal{V}$. From Section 1.2 on, we assume furthermore that K is discrete, π is a uniformizer and the residue field k is perfect. We also fix σ: $\mathcal{V} \to \mathcal{V}$ a lifting of the a-th power Frobenius.

If $\mathfrak{X} \to \mathcal{T}$ is a morphism of smooth formal schemes over $\mathcal{S}$ and if $\mathcal{Z}$ is a relative strict normal crossings divisor of $\mathfrak{X}$ over $\mathcal{T}$, we denote by $\mathfrak{X}^{\#} = (\mathfrak{X}, \mathcal{Z})$ the smooth log-formal $\mathcal{V}$-scheme whose underlying smooth formal $\mathcal{V}$-scheme is $\mathfrak{X}$ and whose logarithmic structure is the canonical one induced by $\mathcal{Z}$. To indicate the corresponding special fibers, we use roman letters, e.g., X, Z and T are the special fibers of $\mathfrak{X}$, $\mathcal{Z}$ and $\mathcal{T}$. Similarly, $X^{\#} = (X, Z)$ means the canonical log-scheme induced by any smooth scheme X and any strict normal crossing divisor Z of X. We denote by d_X or simply d the dimension of X. The subscript $\mathbb{Q}$ means that we have applied the functor $- \otimes_{\mathbb{Z}} \mathbb{Q}$. Modules over a noncommutative ring are left modules, unless otherwise indicated.

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1. A comparison theorem between relative log-rigid cohomology and relative rigid cohomology

1.1. *Proof of the comparison theorem.* In this section we only suppose that K is a complete field of characteristic 0 under the valuation |-| and the residue field k of the integer ring $\mathcal{V}$ is of characteristic $p > 0$. Let us fix several notations in rigid cohomology. For a formal $\mathcal{V}$-scheme $\mathcal{P}$ of finite type, let $\mathcal{P}_K$ be the Raynaud generic fiber of $\mathcal{P}$ that is a quasi-compact and quasi-separated rigid analytic K-space, $\mathrm{sp} : \mathcal{P}_K \to \mathcal{P}$ the specialization map, and $]T[_{\mathcal{P}}= \mathrm{sp}^{-1}(T)$ the tube of a locally closed subscheme T in $P = \mathcal{P} \times_{\mathrm{Spf}\,\mathcal{V}} \mathrm{Spec}\, k$. For a morphism $u : \mathcal{P} \to \mathcal{Q}$, we denote by $u_K : \mathcal{P}_K \to \mathcal{Q}_K$ the morphism of rigid analytic spaces associated to u. Let X be a closed subscheme of P, Z a closed subscheme of X, and Y the complement of Z in X. For any admissible open subset $V \subset]X[_{\mathcal{P}}$, we denote by $\alpha_V : V \to]X[_{\mathcal{P}}$ the canonical inclusion. Let $\mathcal{A}$ be a sheaf of rings on $]X[_{\mathcal{P}}$. For an $\mathcal{A}$-module $\mathcal{H}$, let $j^\dagger_Y\mathcal{H} = \varinjlim_V \alpha_{V*}(\mathcal{H}|_V)$ denote the sheaf of sections of $\mathcal{H}$ overconvergent along Z, where V runs over all strict neighborhoods of $]Y[_{\mathcal{P}}$ in $]X[_{\mathcal{P}}$. The functor $j^\dagger_Y$ is exact, and the natural morphism $\mathcal{H} \to j^\dagger_Y\mathcal{H}$ is an epimorphism [Ber96a, 2.1.3]. The sheaf $\underline{\Gamma}^\dagger_{]Z[_{\mathcal{P}}}(\mathcal{H})$ of sections of $\mathcal{H}$ whose supports are included in $]Z[_{\mathcal{P}}$ is defined by the exact sequence

$$0 \longrightarrow \underline{\Gamma}^\dagger_{]Z[_{\mathcal{P}}}(\mathcal{H}) \longrightarrow \mathcal{H} \longrightarrow j^\dagger_Y\mathcal{H} \longrightarrow 0. \tag{1.1.0.1}$$

Then $\underline{\Gamma}^\dagger_{]Z[_{\mathcal{P}}}$ is an exact functor by the snake lemma [Ber96a, 2.1.6].

We will fix some notation. Let $g : \mathfrak{X} \to \mathcal{T}$ be a smooth morphism of smooth formal schemes over $\mathcal{S}$, of pure relative dimension d, let $\mathcal{Z}$ be a relative strict normal crossings divisor of $\mathfrak{X}$ over $\mathcal{T}$, let $\mathcal{Y}$ be the complement of $\mathcal{Z}$ in $\mathfrak{X}$, let D be a closed subscheme of X and $\mathfrak{U}$ the complement of D in $\mathfrak{X}$. Let $\mathfrak{X}^\# = (\mathfrak{X}, \mathcal{Z})$ be the logarithmic formal $\mathcal{V}$-scheme with the logarithmic structure associated to $\mathcal{Z}$, and $\mathfrak{U}^\#$ the restriction of $\mathfrak{X}^\#$ on $\mathfrak{U}$. Let $\mathfrak{X}^\#_K = (\mathfrak{X}_K, \mathcal{Z}_K)$ be the rigid analytic space endowed with the logarithmic structure associated to $\mathcal{Z}_K$ and $\Omega^\bullet_{\mathfrak{X}^\#_K/\mathcal{T}_K}$ the de Rham complex of logarithmic Kähler differential forms on $\mathfrak{X}^\#_K$. Then the underlying analytic space of $\mathfrak{X}^\#_K$ is $]X[_{\mathfrak{X}}= \mathfrak{X}_K$ and $\Omega^\bullet_{\mathfrak{X}^\#_K/\mathcal{T}_K} \cong \mathrm{sp}^*\Omega^\bullet_{\mathfrak{X}^\#/\mathcal{T},\mathbb{Q}}$.

We recall the definition of logarithmic connection with the overconvergence condition ([Car09a, 4.2] and [Ked07, 6.5.4]). By the gluing lemma [Ber96a, 2.1.12], it is sufficient to give a definition locally on $\mathfrak{X}$ and U. So we may suppose that $\mathcal{T}$ is affine, $\mathfrak{X}$ is sufficiently small affine, and D is a divisor that is defined by $f = 0$ in X for $f \in \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$. Let $z_1, z_2, \ldots, z_d$ be relative local coordinates of $\mathfrak{X}$ over $\mathcal{T}$ such that the irreducible component $\mathcal{Z}_i$ of the relative strict normal crossings divisor $\mathcal{Z} = \cup_{i=1}^s \mathcal{Z}_i$ is defined by $z_i = 0$. An integrable logarithmic connection $\nabla : E \to j^\dagger_U\Omega^1_{\mathfrak{X}^\#_K/\mathcal{T}_K} \otimes_{j^\dagger_U\mathcal{O}_{]X[_{\mathfrak{X}}}} E$ is overconvergent if there exist a strict neighborhood V of $]U[_{\mathfrak{X}}$ in $]X[_{\mathfrak{X}}$ and a locally free

$\mathcal{O}_V$-module $\mathcal{E}$ of finite type furnished with an integrable logarithmic connection $\nabla : \mathcal{E} \to (\Omega^1_{\mathfrak{X}^\#_K/\mathcal{T}_K}|_V) \otimes_{\mathcal{O}_V} \mathcal{E}$ such that $j^\dagger_U(\mathcal{E}, \nabla) = (E, \nabla)$, which satisfies the following overconvergence condition. For any $\xi \in |K^\times|_{\mathbb{Q}} \cap]0,1[$, there exists an affinoid strict neighborhood $W \subset V$ of $]U[_{\mathfrak{X}}$ in $]X[_{\mathfrak{X}}$ such that

$$||\underline{\partial}^{[\underline{n}]}_\#(e)||\xi^{|\underline{n}|} \to 0 \ \text{(as } |\underline{n}| \to \infty) \tag{1.1.0.2}$$

for any section $e \in \Gamma(W, \mathcal{E})$. Here $||\text{-}||$ is a Banach $\Gamma(W, \mathcal{O}_{]X[_{\mathfrak{X}}})$-norm on $\Gamma(W, \mathcal{E})$, $\partial_{\# i} = \nabla(z_i \frac{\partial}{\partial z_i})$ for $1 \le i \le s$, $\partial_i = \nabla(\frac{\partial}{\partial z_i})$ for $s+1 \le i \le d$, and $|\underline{n}| = n_1 + \cdots + n_d$, $\underline{n}! = n_1! \cdots n_d!$ and $\underline{\partial}^{[\underline{n}]}_\# = \frac{1}{\underline{n}!}\left(\prod_{i=1}^s \prod_{j=0}^{n_i-1}(\partial_{\# i} - j)\right)\partial_{s+1}^{n_{s+1}} \cdots \partial_d^{n_d}$ for a multi-index $\underline{n} = (n_1, \ldots, n_d)$. (E, ∇) is called a log-isocrystal on $U^\#/\mathcal{T}_K$ overconvergent along D (simply denoted by E and called an overconvergent log-isocrystal).

Let (E, ∇) be a log-isocrystal on $U^\#/\mathcal{T}_K$ overconvergent along D, and let Z_i be an irreducible component of Z that is not included in D. The eigenvalues of the *residue* of ∇ along $\mathcal{Z}_{iK}$, i.e., the eigenvalues of the matrix $\nabla(\partial_{\# i}) \pmod{z_i}$ contained in an algebraic closure of the field of fractions of $\Gamma(\mathcal{Z}_{iK}, j^\dagger_{Z_i \cap U}\mathcal{O}_{\mathcal{Z}_{iK}})$, are called "exponent" of E along Z_i (For a definition of the residue, see, for example, [Ked07, 2.3.9].) This is related to the definition in [AB01, 1, §6]. Any exponent is contained in $\mathbb{Z}_p$ by (1.1.0.2).

Let $\mathcal{I}_{\mathcal{Z}}$ be the sheaf of ideals of $\mathcal{Z}$ in $\mathfrak{X}$. Since $\mathcal{I}_{\mathcal{Z}}$ is invertible, $\mathcal{I}_{\mathcal{Z},\mathbb{Q}}$ is a coherent $\mathcal{D}^\dagger_{\mathfrak{X}^\#,\mathbb{Q}}$-module that is an invertible $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}$-module. Hence, $I_{\mathcal{Z},\mathbb{Q}} = \mathrm{sp}^*\mathcal{I}_{\mathcal{Z},\mathbb{Q}}$ is a convergent isocrystal on X/K with logarithmic poles along Z. Let E be a log-isocrystal on $U^\#/\mathcal{T}_K$ overconvergent along D. For an integer m, we put

$$E(m\mathcal{Z}) = E \otimes_{j^\dagger_U \mathcal{O}_{]X[_{\mathfrak{X}}}} j^\dagger_U I^{\otimes -m}_{\mathcal{Z},\mathbb{Q}}.$$

$E(m\mathcal{Z})$ is an overconvergent log-isocrystal, and the exponents of $E(m\mathcal{Z})$ are the exponents of E minus m. Then there is a natural commutative diagram

$$\begin{array}{ccc} E & \stackrel{\subset}{\longrightarrow} & E(m\mathcal{Z}) \\ =\downarrow & & \downarrow \\ E & \longrightarrow & j^\dagger_{Y \cap U} E \end{array} \tag{1.1.0.3}$$

for any nonnegative integer m.

We recall that a p-adic integer α is a "p-adic Liouville number" if the radius of convergence of formal power series, either $\sum_{n \in \mathbb{Z}_{\ge 0}, n \ne \alpha} x^n/(n-\alpha)$ or $\sum_{n \in \mathbb{Z}_{\ge 0}, n \ne -\alpha} x^n/(n+\alpha)$, is less than 1. Note that (1) a p-adic integer that is an algebraic number is not a p-adic Liouville number and (2) a p-adic integer α is a p-adic Liouville number if and only if so is $-\alpha$ (resp. $\alpha + m$ for any integer m). For p-adic Liouville numbers, we refer to [DGS94, VI, 1] and [BC92, 1.2].

Theorem 1.1.1. *With the above notation, let E be a log-isocrystal on $U^{\#}/\mathcal{T}_K$ overconvergent along D. Suppose that*

(a) *none of differences of exponents of E is a p-adic Liouville number, and*
(b) *none of exponents of E is a p-adic Liouville number*

along each irreducible component Z_i of Z such that $Z_i \not\subset D$. Let c be the nonnegative integer defined by

$$c = \max\{e \mid e \textit{ is a positive integral exponent of } E \textit{ along some irreducible component } Z_i \textit{ of } Z \textit{ such that } Z_i \not\subset D\} \cup \{0\}.$$

Then the diagram (1.1.0.3) *induces an isomorphism*

(1.1.1.1)

$$\mathbb{R}g_{K*}\underline{\Gamma}^{\dagger}_{]Z[_{\mathfrak{X}}}(j^{\dagger}_U\Omega^{\bullet}_{\mathfrak{X}^{\#}_K/\mathcal{T}_K} \otimes_{j^{\dagger}_U\mathcal{O}_{]X[_{\mathfrak{X}}}} E) \cong \mathbb{R}g_{K*}\mathrm{Cone}\left(j^{\dagger}_U\Omega^{\bullet}_{\mathfrak{X}^{\#}_K/\mathcal{T}_K} \otimes_{j^{\dagger}_U\mathcal{O}_{]X[_{\mathfrak{X}}}} E \to j^{\dagger}_U\Omega^{\bullet}_{\mathfrak{X}^{\#}_K/\mathcal{T}_K} \otimes_{j^{\dagger}_U\mathcal{O}_{]X[_{\mathfrak{X}}}} E(m\mathcal{Z})\right)[-1]$$

for any $m \geq c$. In particular, if none of exponents along each irreducible component Z_i of Z such that $Z_i \not\subset D$ is a positive integer, then the restriction induces an isomorphism

(1.1.1.2)

$$\mathbb{R}g_{K*}(j^{\dagger}_U\Omega^{\bullet}_{\mathfrak{X}^{\#}_K/\mathcal{T}_K} \otimes_{j^{\dagger}_U\mathcal{O}_{]X[_{\mathfrak{X}}}} E) \xrightarrow{\sim} \mathbb{R}g_{K*}(j^{\dagger}_{Y\cap U}\Omega^{\bullet}_{\mathfrak{X}_K/\mathcal{T}_K} \otimes_{j^{\dagger}_{Y\cap U}\mathcal{O}_{]X[_{\mathfrak{X}}}} j^{\dagger}_{Y\cap U}E).$$

Remarks 1.1.2. (1) In fact, we will see in 2.2.12 that the comparison homomorphism corresponding to (1.1.1.2) is an isomorphism on the formal scheme side without the functor $g^{\#}_{+}$. But the first step towards this result is to establish 1.1.1.

(2) Note that $j^{\dagger}_{Y\cap U}E$ is an isocrystal on $Y \cap U/\mathcal{T}_K$ overconvergent along $Z \cup D$ and the right-hand side of the isomorphism in the theorem above is a relative rigid cohomology with respect to the closed immersion $T \to \mathcal{T}$. It is independent of the choice of $\mathfrak{X}$ that is smooth over $\mathcal{T}$ around U [CT03, §10]. The left-hand side of (1.1.1.2) is regarded as a relative logarithmic rigid cohomology.

(3) This type of comparison theorem between p-adic cohomology with logarithmic poles and rigid cohomology was studied in [BC94, 3.1], [Tsu99, 3.5.1], [Shi02, 2.2.4 and 2.2.13] (see also the definition [Shi02, 2.1.5]), and [BB04, A.1]. They suppose that an overconvergent isocrystal is locally free on the formal side or for [Shi02, 2.2.4 and 2.2.13] it concerns the absolute case. In the theorem above we relax this assumption and suppose that an overconvergent isocrystal is locally free only on the analytic side.

(4) One can also prove the comparison theorem in the case g is smooth around U replacing 1.1.8 and 1.1.18 (the weak fibration theorem) by the strong forms (the strong fibration theorem) with modifications.

Remarks 1.1.3. For a log-isocrystal E on $U^{\#}/\mathcal{T}_K$ overconvergent along D, we denote by $\mathrm{Exp}(E) \subset \mathbb{Z}_p$ (resp. $\mathrm{Exp}(E)^{\mathrm{gr}} \subset \mathbb{Z}_p$) the monoid (resp. abelian group) generated by all exponents along irreducible components Z_i of Z such that $Z_i \not\subset D$. $\mathrm{Exp}(E)$ and $\mathrm{Exp}(E)^{\mathrm{gr}}$ do not depend on the choice of local coordinates.

(1) Let $\mathfrak{X}^{\#} = (\mathfrak{X}, \mathcal{Z})$ and $\mathfrak{X}'^{\#} = (\mathfrak{X}', \mathcal{Z}')$ be smooth formal $\mathcal{V}$-schemes with relative strict normal crossings divisors over $\mathcal{T}$, let $\mathfrak{U}, D, \mathfrak{U}^{\#}, \mathfrak{U}', D', \mathfrak{U}'^{\#}$ as above, and let $h : \mathfrak{X}' \to \mathfrak{X}$ be a morphism over $\mathcal{T}$ such that $h^{-1}(D \cup Z) \subset D' \cup Z'$. Suppose that h induces a log-morphism $(h|_{\mathfrak{U}'})^{\#} : \mathfrak{U}'^{\#} \to \mathfrak{U}^{\#}$. Then the inverse image $h_K^{\#*}E$ is a log-isocrystal on $U'^{\#}/\mathcal{T}_K$ overconvergent along D' because h_K induces a log-morphism of rigid analytic spaces between suitable strict neighborhoods by our assumption. Suppose, furthermore, that none of elements in $\mathrm{Exp}(E)$ (resp. $\mathrm{Exp}(E)^{\mathrm{gr}}$) is a p-adic Liouville number. Then the same holds for the inverse image $h_K^{\#*}E$. Indeed, for a suitable choice of local coordinates $z_i\,(1 \leq i \leq s)$ and $z'_j\,(1 \leq j \leq s')$ along the normal crossings divisors $\mathcal{Z}$ and $\mathcal{Z}'$ of $\mathfrak{X}$ and $\mathfrak{X}'$ respectively, we have $z_i = u_i {z'_1}^{m_{i1}} \cdots {z'_{s'}}^{m_{is'}}$ locally at a generic point of $\mathcal{Z}'$. Here u_i is a unit of $\mathcal{O}_{\mathfrak{U}'}$ and m_{ij} is a nonnegative integer. Since the residues of E with respect to Z_{i_1} and Z_{i_2} commute with each other by the integrability of the log-connection and $dz_i/z_i \equiv \sum_j m_{ij} dz'_j/z'_j \,(\mathrm{mod}\, \Omega^1_{\mathfrak{U}'/\mathcal{T}})$, $\mathrm{Exp}(h_K^{\#*}E)$ is a submonoid of $\mathrm{Exp}(E)$ (see [AB01, 6.2.5]).

Even if any exponent of E is not a positive integer, it might happen that some exponent of the inverse image $h_K^{\#*}E$ is a positive integer. Since $\mathrm{Exp}(E) \cap \mathbb{Q}_{\geq 0}$ is finitely generated as a monoid where $\mathbb{Q}_{\geq 0}$ is the monoid consisting of nonnegative rational numbers, $\mathrm{Exp}(E(m\mathcal{Z}))$ does not contain any positive rational numbers for a sufficiently large integer m. Therefore, none of exponents of an arbitrary inverse image $h_K^{\#*}E(m\mathcal{Z})$ is a positive integer.

(2) Let $h^{\#} : \mathfrak{X}'^{\#} \to \mathfrak{X}^{\#}$ be a log-morphism such that $h^{-1}(D) = D'$ and $h^{-1}(\mathcal{Z}) = \mathcal{Z}'$. Suppose that the underlying morphism h is finite étale. Note that local parameters of $\mathfrak{X}^{\#}$ becomes local parameters of $\mathfrak{X}'^{\#}$. Then, for a log-isocrystal E' on $U'^{\#}/\mathcal{T}_K$ overconvergent along D', $h^{\#}_{K*}E'$ is a log-isocrystal on $U^{\#}/\mathcal{T}_K$ overconvergent along D. Moreover, for an irreducible component Z_i of Z such that $Z_i \not\subset D$, the exponents of $h^{\#}_{K*}E'$ along Z_i coincide with the exponents of E' along $h^{-1}(Z)$ (including multiplicities). In particular, $\mathrm{Exp}(h^{\#}_{K*}E') = \mathrm{Exp}(E')$ (see [AB01, 6.5.4]). The first part easily follows from our geometric situation, and we have $\mathrm{rank}_{j^\dagger \mathcal{O}_{]\overline{X}[_{\mathfrak{X}}}} h^{\#}_{K*}E' = \deg(h)\mathrm{rank}_{j^\dagger \mathcal{O}_{]\overline{X}'[_{\mathfrak{X}'}}} E'$, where $\deg(h)$ is the degree of the underlying morphism of h. The second part is a problem only along the generic point of $\mathcal{Z}_i$. We may assume that $\mathfrak{X}$ and $\mathfrak{X}'$ are affine, Z is

irreducible and is not included in D. Let $(j^\dagger\mathcal{O}_{]\overline{X}[_\mathfrak{X}})^{\widehat{}}_{\mathcal{Z}}$ be the completion along $\mathcal{Z}_K$. Then there is a natural K-algebra homomorphism from the ring of global sections of $(j^\dagger\mathcal{O}_{]\overline{X}[_\mathfrak{X}})^{\widehat{}}_{\mathcal{Z}}$ into $K(\mathcal{Z})[[z]]$, where $K(\mathcal{Z})$ is the field of fractions of $\Gamma(\mathcal{Z}, j^\dagger\mathcal{O}_\mathcal{Z})$ and z is a local coordinate of $\mathcal{Z}$. This K-algebra homomorphism naturally extends to a K-algebra homomorphism from the ring of global sections of $(j^\dagger\mathcal{O}_{]\overline{X}'[_{\mathfrak{X}'}})^{\widehat{}}_{\mathcal{Z}'}$ into a direct sum of finite unramified extensions of $K(\mathcal{Z})[[z]]$. We may replace the residue field $K(\mathcal{Z})$ of $K(\mathcal{Z})[[z]]$ by its algebraic closure $\overline{K(\mathcal{Z})}$ since all exponents are contained in $\mathbb{Z}_p$ and invariant under any automorphism of $\overline{K(\mathcal{Z})}$. Hence, $(j^\dagger\mathcal{O}_{]\overline{X}'[_{\mathfrak{X}'}})^{\widehat{}}_{\mathcal{Z}'}$ goes to a direct sum of $\deg(h)$ copies of $\overline{K(\mathcal{Z})}[[z]]$. Now our second assertion is clear.

First we prove a special case.

Proposition 1.1.4. *Under the hypothesis in* 1.1.1, *suppose that* $\mathcal{Z}$ *is irreducible such that* $Z \not\subset D$ *and that the composition* $g \circ i : \mathcal{Z} \to \mathcal{T}$ *of the closed immersion* $i : \mathcal{Z} \to \mathfrak{X}$ *and* $g : \mathfrak{X} \to \mathcal{T}$ *is an isomorphism. If we define* $T \cap U = Z \cap U$ *through the isomorphism* $g \circ i : Z \to T$, *then* $g_{K*}\nabla : g_{K*}(E(m\mathcal{Z})/E) \to g_{K*}(j_U^\dagger\Omega^1_{\mathfrak{X}_K^\#/\mathcal{T}_K} \otimes_{j_U^\dagger\mathcal{O}_{]X[_\mathfrak{X}}} E(m\mathcal{Z})/E)$ *is a* $j^\dagger_{T\cap U}\mathcal{O}_{]T[_\mathcal{T}}$-*homomorphism of locally free* $j^\dagger_{T\cap U}\mathcal{O}_{]T[_\mathcal{T}}$-*modules of finite type and the natural morphism* (1.1.1.1) *induces an isomorphism*

(1.1.4.1)
$$\begin{aligned}&\mathbb{R}g_{K*}\underline{\Gamma}^\dagger_{]Z[_\mathfrak{X}}(j_U^\dagger\Omega^\bullet_{\mathfrak{X}_K^\#/\mathcal{T}_K} \otimes_{j_U^\dagger\mathcal{O}_{]X[_\mathfrak{X}}} E)\\ &\quad\cong \left[g_{K*}(E(m\mathcal{Z})/E) \xrightarrow{g_{K*}\nabla} g_{K*}(j_U^\dagger\Omega^1_{\mathfrak{X}_K^\#/\mathcal{T}_K} \otimes_{j_U^\dagger\mathcal{O}_{]X[_\mathfrak{X}}} E(m\mathcal{Z})/E)\right][-1]\end{aligned}$$

for any $m \geq c$ *in the derived category of complexes of* $j^\dagger_{T\cap U}\mathcal{O}_{]T[_\mathcal{T}}$-*modules. Here* $[A \to B]$ *means a complex consisting of the terms of degree* 0 *and degree* 1.

We will see that, in 1.1.22, the overconvergence of the induced Gauss-Manin connection on $g_{K*}(E(m\mathcal{Z})/E)$ holds in the relative case. An example such that the cokernel of

$$g_{K*}\nabla : g_{K*}(E(m\mathcal{Z})/E) \to g_{K*}(j_U^\dagger\Omega^1_{\mathfrak{X}_K^\#/\mathcal{T}_K} \otimes_{j_U^\dagger\mathcal{O}_{]X[_\mathfrak{X}}} E(m\mathcal{Z})/E)$$

is not locally free is also given in 1.1.23.

Proof. At first we shall define $j^\dagger_{T\cap U}\mathcal{O}_{]T[_\mathcal{T}}$-module structures on both sides of (1.1.4.1).

We shall prove that $\mathbb{R}^q g_{K*}(E(\mathcal{Z})/E) = 0$ for $q \neq 0$ and the locally freeness of $g_{K*}(E(\mathcal{Z})/E)$. Since $i^{-1}(X \setminus U) = Z \setminus U$ as underlying topological spaces, $i_K^*E(\mathcal{Z}) = j^\dagger_{Z\cap U}\mathcal{O}_{]Z[_\mathcal{Z}} \otimes_{i_K^{-1}j_U^\dagger\mathcal{O}_{]X[_\mathfrak{X}}} i_K^{-1}E(\mathcal{Z})$ is a locally free $j^\dagger_{Z\cap U}\mathcal{O}_{]Z[_\mathcal{Z}}$-module of finite type and the adjoint gives an isomorphism $i_{K*}i_K^*E(\mathcal{Z}) \cong E(\mathcal{Z})/E$. Because i is a closed immersion, $i_K :]Z[_{\mathcal{Z}} \to]X[_{\mathfrak{X}}$ is an affinoid morphism. Hence, $\mathbb{R}i_{K*}\mathcal{M} = i_{K*}\mathcal{M}$ for any coherent $j^\dagger_{Z\cap U}\mathcal{O}_{]Z[_{\mathcal{Z}}}$-module $\mathcal{M}$ by $i^{-1}(X\setminus U) = Z\setminus U$ [CT03, 5.2.2]. Since $g\circ i$ is an isomorphism, we have

$$\begin{aligned}\mathbb{R}g_{K*}(E(\mathcal{Z})/E) &= \mathbb{R}g_{K*}(i_{K*}i_K^*E(\mathcal{Z}))\\ &= \mathbb{R}g_{K*}\mathbb{R}i_{K*}i_K^*E(\mathcal{Z}) = \mathbb{R}(g\circ i)_{K*}i_K^*E(\mathcal{Z}) = (g\circ i)_{K*}i_K^*E(\mathcal{Z})\end{aligned}$$

and the two assertions above. Therefore, we show, for $m\geq 0$, $\mathbb{R}^q g_{K*}(E(m\mathcal{Z})/E) = 0$ for $q\neq 0$ and $g_{K*}(E(m\mathcal{Z})/E)$ is a locally free $j^\dagger_{T\cap U}\mathcal{O}_{]T[_{\mathcal{T}}}$-module of finite type by induction on m.

For a $j^\dagger_U\mathcal{O}_{]X[_{\mathfrak{X}}}$-module $\mathcal{H}$, the $j^\dagger_U\mathcal{O}_{]X[_{\mathfrak{X}}}$-module $\underline{\Gamma}^\dagger_{]Z[_{\mathfrak{X}}}(j^\dagger_U\mathcal{H})$ is not *a priori* a $g_K^{-1}(j^\dagger_{T\cap U}\mathcal{O}_{]T[_{\mathcal{T}}})$-module because $U\subset g^{-1}(T\cap U)$ might not hold. The following lemma says that $\underline{\Gamma}^\dagger_{]Z[_{\mathfrak{X}}}(j^\dagger_U\mathcal{H})$ has a $g_K^{-1}(j^\dagger_{T\cap U}\mathcal{O}_{]T[_{\mathcal{T}}})$-module structure. Hence, the left-hand side of (1.1.4.1) belongs to the derived category of complexes of $j^\dagger_{T\cap U}\mathcal{O}_{]T[_{\mathcal{T}}}$-modules.

LEMMA 1.1.5. *Under the hypothesis in* 1.1.4, *let us put* $U' = g^{-1}(T\cap U)\cap U$. *If* $\mathcal{A}$ *is a sheaf of rings on* $]X[_{\mathfrak{X}}$, *then the restriction morphism*

$$\underline{\Gamma}^\dagger_{]Z[_{\mathfrak{X}}}(j^\dagger_U\mathcal{H}) \to \underline{\Gamma}^\dagger_{]Z[_{\mathfrak{X}}}(j^\dagger_{U'}\mathcal{H})$$

is an isomorphism for any $\mathcal{A}$*-module* $\mathcal{H}$.

Proof. Since $T\cap U = Z\cap U$ via $g\circ i$ and $Y\cap U = U\setminus Z$, we have $(Z\cap U)\subset U'$ and $U = U'\cup(Y\cap U)$. Hence, the natural morphism $[j^\dagger_U\mathcal{H}\to j^\dagger_{Y\cap U}\mathcal{H}] \to [j^\dagger_{U'}\mathcal{H}\to j^\dagger_{Y\cap U'}\mathcal{H}]$ of complexes is an isomorphism by [Ber96a, 2.1.8]. □

We divide the proof of 1.1.4 into seven parts.

0° *Reduce to the case where none of the exponents of* E *along* $\mathcal{Z}$ *is a positive integer*; *that is*, $c = 0$. Since the natural morphism $j^\dagger_{Y\cap U}E\to j^\dagger_{Y\cap U}E(m\mathcal{Z})$ is an isomorphism, the natural morphism of complexes induces a triangle

$$\begin{array}{c}\mathbb{R}g_{K*}\mathrm{Cone}\left(j^\dagger_U\Omega^\bullet_{\mathfrak{X}^\#_K/\mathcal{T}_K}\otimes_{j^\dagger_U\mathcal{O}_{]X[_{\mathfrak{X}}}} E\to j^\dagger_U\Omega^\bullet_{\mathfrak{X}^\#_K/\mathcal{T}_K}\otimes_{j^\dagger_U\mathcal{O}_{]X[_{\mathfrak{X}}}} E(m\mathcal{Z})\right)[-1]\\ \swarrow \qquad\qquad\qquad \nwarrow +1\\ \mathbb{R}g_{K*}\underline{\Gamma}^\dagger_{]Z[_{\mathfrak{X}}}(j^\dagger_U\Omega^\bullet_{\mathfrak{X}^\#_K/\mathcal{T}_K}\otimes_{j^\dagger_U\mathcal{O}_{]X[_{\mathfrak{X}}}} E)\to\mathbb{R}g_{K*}\underline{\Gamma}^\dagger_{]Z[_{\mathfrak{X}}}(j^\dagger_U\Omega^\bullet_{\mathfrak{X}^\#_K/\mathcal{T}_K}\otimes_{j^\dagger_U\mathcal{O}_{]X[_{\mathfrak{X}}}} E(m\mathcal{Z}))\end{array}$$

for any $m\geq 0$. If we prove the vanishing $\mathbb{R}g_{K*}\underline{\Gamma}^\dagger_{]Z[_{\mathfrak{X}}}(j^\dagger_U\Omega^\bullet_{\mathfrak{X}^\#_K/\mathcal{T}_K}\otimes_{j^\dagger_U\mathcal{O}_{]X[_{\mathfrak{X}}}} E) = 0$ for $c = 0$, then, for any c, the triangle above induces the desired isomorphism when $m\geq c$. Hence, we may assume $m = c = 0$ and we shall prove the vanishing.

1° *Local problem on X and U.* By the Čech spectral sequences associated to a finite open covering $\{\mathfrak{X}_i\}$ of $\mathfrak{X}$ (resp. a finite open covering $\{\mathfrak{U}_{ij}\}$ of each $\mathfrak{X}_i \cap \mathfrak{U}$) [Ber90, 4.1.3], [CT03, 8.3.3], the vanishing is local on X and U. Since the vanishing of $\mathbb{R}g_{K*}\underline{\Gamma}^{\dagger}_{]Z[_{\mathfrak{X}}}(j^{\dagger}_U\Omega^{\bullet}_{\mathfrak{X}^{\#}_K/\mathcal{T}_K} \otimes_{j^{\dagger}_U\mathcal{O}_{]X[_{\mathfrak{X}}}} E)$ is trivial in the case where $Z = \emptyset$, we may assume that $\mathfrak{X}$ is affine, D is defined by a single equation $f = 0$ in X for some $f \in \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$, and there is a coordinate z of $\mathfrak{X}$ over $\mathcal{T}$ such that $\mathcal{Z}$ is defined by $z = 0$ in $\mathfrak{X}$. Indeed, it is enough to take a certain covering consisting of $\mathfrak{X} \setminus \mathcal{Z}$ and a covering of $\mathcal{Z}$.

2° *Reduction to the local case by rigid analytic geometry.* Let us add some notation. Let us put $]U[_{\mathfrak{X},\lambda} = \{x \in]X[_{\mathfrak{X}} \mid |f(x)| \geq \lambda\}$ (resp. $]Y[_{\mathfrak{X},\lambda} = \{x \in]X[_{\mathfrak{X}} \mid |z(x)| \geq \lambda\}$, resp. $]Z \cap U[_{\mathcal{Z},\lambda} = \{x \in]Z[_{\mathcal{Z}} \mid |\overline{f}(x)| \geq \lambda\}$, resp. $[Z]_{\mathfrak{X},\lambda} = \{x \in]Z[_{\mathfrak{X}} \mid |z(x)| \leq \lambda\}$) for $\lambda \in |K^{\times}|_{\mathbb{Q}} \cap]0,1[$, where $\overline{f}$ is the reduction of f in $\Gamma(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}})$. We define $]T \cap U[_{\mathcal{T},\lambda} =]Z \cap U[_{\mathcal{Z},\lambda}$ by the identification through $g \circ i$. Note that the set $\{]U[_{\mathfrak{X},\lambda}\}_{\lambda \in |K^{\times}|_{\mathbb{Q}} \cap]0,1[}$ forms a fundamental system of strict neighborhoods of $]U[_{\mathfrak{X}}$ in $]X[_{\mathfrak{X}}$. Let $\alpha_V : V \to]X[_{\mathfrak{X}}$ denote the canonical morphism for admissible open sets V in $]X[_{\mathfrak{X}}$.

Take $\nu \in |K^{\times}|_{\mathbb{Q}} \cap]0,1[$ such that there is a locally free $\mathcal{O}_{]U[_{\mathfrak{X},\nu}}$-module $\mathcal{E}$ of finite type endowed with a logarithmic connection

$$\nabla : \mathcal{E} \to (\Omega^1_{\mathfrak{X}^{\#}_K/\mathcal{T}_K}|_{]U[_{\mathfrak{X},\nu}}) \otimes_{\mathcal{O}_{]U[_{\mathfrak{X},\nu}}} \mathcal{E}$$

that satisfies the overconvergence condition (1.1.0.2). Hence, there exist a strictly increasing sequence $\underline{\xi} = (\xi_l)$ in $|K^{\times}|_{\mathbb{Q}} \cap]0,1[$ with $\xi_l \to 1^-$ as $l \to \infty$ and an increasing sequence $\underline{\lambda} = (\lambda_l)$ in $|K^{\times}|_{\mathbb{Q}} \cap [\nu, 1[$ such that, for any l,

$$(1.1.5.1) \qquad ||\partial^{[n]}_{\#}(e)||\xi^n_l \to 0 \ \text{(as } n \to \infty)$$

for any section $e \in \Gamma(]U[_{\mathfrak{X},\lambda_l}, \mathcal{E})$. Here $\partial_{\#} = \nabla(z\frac{d}{dz})$ and $\partial^{[l]}_{\#} = \frac{1}{l!}\prod_{j=0}^{l-1}(\partial_{\#} - j)$.

Let $\mathcal{A}$ be a sheaf of rings on $]X[_{\mathfrak{X}}$. Let $\eta \in |K^{\times}|_{\mathbb{Q}} \cap]0,1[$. We define a functor $\underline{\Gamma}^{\dagger}_{]Z[_{\mathfrak{X},\eta}}$ from the category of $\mathcal{A}$-modules to itself by the exact sequence

$$(1.1.5.2) \qquad 0 \longrightarrow \underline{\Gamma}^{\dagger}_{]Z[_{\mathfrak{X},\eta}}(\mathcal{H}) \longrightarrow \mathcal{H} \longrightarrow \lim_{\mu \to \eta^-} \alpha_{]Y[_{\mathfrak{X},\mu}*}(\mathcal{H}|_{]Y[_{\mathfrak{X},\mu}}) \longrightarrow 0$$

for any $\mathcal{A}$-module $\mathcal{H}$. Here the morphism $\mathcal{H} \to \lim_{\mu \to \eta^-} \alpha_{]Y[_{\mathfrak{X},\mu}*}(\mathcal{H}|_{]Y[_{\mathfrak{X},\mu}})$ is an epimorphism for the same reason as for the epimorphism $\mathcal{H} \to j^{\dagger}_Y\mathcal{H}$. One can easily see that $\underline{\Gamma}^{\dagger}_{]Z[_{\mathfrak{X},\eta}}(\mathcal{H})|_{]Y[_{\mathfrak{X},\eta}} = 0$ and $\underline{\Gamma}^{\dagger}_{]Z[_{\mathfrak{X},\eta}}$ is an exact functor by the snake lemma. For $\xi \in |K^{\times}|_{\mathbb{Q}} \cap [\eta, 1[$, the restriction induces a morphism

$$\underline{\Gamma}^{\dagger}_{]Z[_{\mathfrak{X},\eta}}(\mathcal{H}) \to \underline{\Gamma}^{\dagger}_{]Z[_{\mathfrak{X},\xi}}(\mathcal{H})$$

of $\mathcal{A}$-modules. By definition, we have

Proposition 1.1.6. *With the same notation as above, the inductive system induces an isomorphism*

$$\varinjlim_{\eta\to 1^-} \underline{\Gamma}^\dagger_{]Z[_{\mathfrak{X},\eta}}(\mathcal{H}) \cong \underline{\Gamma}^\dagger_{]Z[_{\mathfrak{X}}}(\mathcal{H}).$$

Proposition 1.1.7. *Let $\lambda \in |K^\times|_{\mathbb{Q}} \cap]0,1[$.*

(1) *The functor $\underline{\Gamma}^\dagger_{]Z[_{\mathfrak{X},\eta}}$ commutes with direct limits. Also, for any $\mathcal{A}$-module $\mathcal{H}$, the natural morphism*

$$\alpha_{]U[_{\mathfrak{X},\lambda}*}(\underline{\Gamma}^\dagger_{]Z[_{\mathfrak{X},\eta}}(\mathcal{H})|_{]U[_{\mathfrak{X},\lambda}}) \to \underline{\Gamma}^\dagger_{]Z[_{\mathfrak{X},\eta}}(\alpha_{]U[_{\mathfrak{X},\lambda}*}(\mathcal{H}|_{]U[_{\mathfrak{X},\lambda}}))$$

is an isomorphism. Moreover, $j^\dagger_U \underline{\Gamma}^\dagger_{]Z[_{\mathfrak{X},\eta}} = \underline{\Gamma}^\dagger_{]Z[_{\mathfrak{X},\eta}} j^\dagger_U$.

(2) *For any coherent $\mathcal{O}_{]U[_{\mathfrak{X},\lambda}}$-module $\mathcal{H}_\lambda$ and any $q \geq 1$, we have*

$$\mathbb{R}^q\alpha_{]U[_{\mathfrak{X},\lambda}*}(\underline{\Gamma}^\dagger_{]Z[_{\mathfrak{X},\eta}}(\alpha_{]U[_{\mathfrak{X},\lambda}*}\mathcal{H}_\lambda)|_{]U[_{\mathfrak{X},\lambda}}) = 0.$$

Proof. (1) Since the morphism $\alpha_{]Y[_{\mathfrak{X},\mu}}$ is quasi-compact and quasi-separated, we obtain from (1.1.5.2) the first assertion. By applying the functor $\alpha_{]U[_{\mathfrak{X},\lambda}*}\alpha^{-1}_{]U[_{\mathfrak{X},\lambda}}$ to the exact sequence (1.1.5.2), we get the sequence

$$\begin{aligned} 0 &\longrightarrow \alpha_{]U[_{\mathfrak{X},\lambda}*}(\underline{\Gamma}^\dagger_{]Z[_{\mathfrak{X},\eta}}(\mathcal{H})|_{]U[_{\mathfrak{X},\lambda}}) \longrightarrow \alpha_{]U[_{\mathfrak{X},\lambda}*}(\mathcal{H}|_{]U[_{\mathfrak{X},\lambda}}) \\ &\longrightarrow \alpha_{]U[_{\mathfrak{X},\lambda}*}\left(\left(\varinjlim_{\mu\to\eta^-} \alpha_{]Y[_{\mathfrak{X},\mu}*}(\mathcal{H}|_{]Y[_{\mathfrak{X},\mu}})\right)|_{]U[_{\mathfrak{X},\lambda}}\right) \longrightarrow 0, \end{aligned}$$

which is exact by a similar proof to that of [Ber96a, 2.1.3(i)]. The quasi-compactness and quasi-separateness of $\alpha_{]U[_{\mathfrak{X},\lambda}}$ implies the assertions.

(2) Because $\mathcal{H}_\lambda$ is a coherent $\mathcal{O}_{]U[_{\mathfrak{X},\lambda}}$-module and both $]U[_{\mathfrak{X},\lambda}$ and $]Y[_{\mathfrak{X},\mu}$ are affinoid subdomains of the affinoid $]X[_{\mathfrak{X}}$, then $\mathbb{R}^q\alpha_{]U[_{\mathfrak{X},\lambda}*}(\mathcal{H}_\lambda) = 0$ and

$$\mathbb{R}^q\alpha_{]U[_{\mathfrak{X},\lambda}*}\left(\left(\varinjlim_{\mu\to\eta^-} \alpha_{]Y[_{\mathfrak{X},\mu}*}(\mathcal{H}_\lambda|_{]Y[_{\mathfrak{X},\mu}})\right)|_{]U[_{\mathfrak{X},\lambda}}\right) = 0$$

for $q \geq 1$ by Kiehl's Theorem B [Kie67, 2.4]. These facts and the exactness of the sequence in the proof of (1) imply the vanishing of higher direct images. □

Since g_K is an affinoid morphism, it is quasi-compact and $\mathbb{R}g_{K*}$ commutes with direct limits [Ber96a, 0.1.8]. Hence, we have

$$\begin{aligned} &\mathbb{R}^q g_{K*}\underline{\Gamma}^\dagger_{]Z[_{\mathfrak{X}}}(j^\dagger_U\Omega^\bullet_{\mathfrak{X}^\#_K/\mathcal{T}_K} \otimes_{j^\dagger_U\mathcal{O}_{]X[_{\mathfrak{X}}}} E) \\ &\cong \mathbb{R}^q g_{K*}\left(\varinjlim_{\eta\to 1^-} \underline{\Gamma}^\dagger_{]Z[_{\mathfrak{X},\eta}}(j^\dagger_U(\Omega^\bullet_{\mathfrak{X}^\#_K/\mathcal{T}_K} \otimes_{\mathcal{O}_{]X[_{\mathfrak{X}}}} \alpha_{]U[_{\mathfrak{X},\nu}*}\mathcal{E}))\right) \\ &\cong \varinjlim_{\eta\to 1^-} \mathbb{R}^q g_{K*}\underline{\Gamma}^\dagger_{]Z[_{\mathfrak{X},\eta}}\left(\varinjlim_{\lambda\to 1^-} \alpha_{]U[_{\mathfrak{X},\lambda}*}((\Omega^\bullet_{\mathfrak{X}^\#_K/\mathcal{T}_K}|_{]U[_{\mathfrak{X},\lambda}}) \otimes_{\mathcal{O}_{]U[_{\mathfrak{X},\lambda}}} \mathcal{E}|_{]U[_{\mathfrak{X},\lambda}})\right) \\ &\cong \varinjlim_{\eta\to 1^-} \varinjlim_{\lambda\to 1^-} \mathbb{R}^q g_{K*}\underline{\Gamma}^\dagger_{]Z[_{\mathfrak{X},\eta}}(\alpha_{]U[_{\mathfrak{X},\lambda}*}((\Omega^\bullet_{\mathfrak{X}^\#_K/\mathcal{T}_K}|_{]U[_{\mathfrak{X},\lambda}}) \otimes_{\mathcal{O}_{]U[_{\mathfrak{X},\lambda}}} \mathcal{E}|_{]U[_{\mathfrak{X},\lambda}})) \\ &\cong \varinjlim_{\eta,\lambda\to 1^-} \mathbb{R}^q g_{K*}\underline{\Gamma}^\dagger_{]Z[_{\mathfrak{X},\eta}}(\alpha_{]U[_{\mathfrak{X},\lambda}*}((\Omega^\bullet_{\mathfrak{X}^\#_K/\mathcal{T}_K}|_{]U[_{\mathfrak{X},\lambda}}) \otimes_{\mathcal{O}_{]U[_{\mathfrak{X},\lambda}}} \mathcal{E}|_{]U[_{\mathfrak{X},\lambda}})) \end{aligned}$$

for any q. Indeed, the first isomorphism follows from 1.1.6 and the other ones from the commutation of the functors $\mathbb{R}g_{K*}$ and $\underline{\Gamma}^{\dagger}_{]Z[_{\mathfrak{X},\eta}}$ (by 1.1.7) with direct limits. We will consider the family of open subsets indexed by the directed set

$$(1.1.7.1)\qquad \Lambda_{\underline{\xi},\underline{\lambda}} = \left\{(\lambda,\eta)\in\left(|K^{\times}|_{\mathbb{Q}}\cap]0,1[\right)^2 \;\middle|\; \begin{array}{l}\lambda>\eta, \lambda\geq\max\{\lambda_l,\nu\},\\ \eta<\xi_l \text{ for some } l\end{array}\right\}.$$

Here the condition $\lambda > \eta$ comes from 1.1.8(2). This family is cofinal for $\eta,\lambda\to 1^-$, so that the limit with respect to $\Lambda_{\underline{\xi},\underline{\lambda}}$ is the same as the original one.

Let $g_\lambda :]U[_{\mathfrak{X},\lambda}\to]T[_{\mathcal{T}}$ and $g_{\lambda,\eta} :]U[_{\mathfrak{X},\lambda}\cap[Z]_{\mathfrak{X},\eta}\to]T[_{\mathcal{T}}$ denote the restrictions of g for $(\lambda,\eta)\in\Lambda_{\underline{\xi},\underline{\lambda}}$. Then

$$\mathbb{R}g_{K*}\underline{\Gamma}^{\dagger}_{]Z[_{\mathfrak{X},\eta}}(\alpha_{]U[_{\mathfrak{X},\lambda}*}((\Omega^{\bullet}_{\mathfrak{X}^{\#}_K/\mathcal{T}_K}|_{]U[_{\mathfrak{X},\lambda}})\otimes_{\mathcal{O}_{]U[_{\mathfrak{X},\lambda}}}\mathcal{E}|_{]U[_{\mathfrak{X},\lambda}}))$$
$$\cong \mathbb{R}g_{\lambda*}(\underline{\Gamma}^{\dagger}_{]Z[_{\mathfrak{X},\eta}}(\alpha_{]U[_{\mathfrak{X},\lambda}*}((\Omega^{\bullet}_{\mathfrak{X}^{\#}_K/\mathcal{T}_K}|_{]U[_{\mathfrak{X},\lambda}})\otimes_{\mathcal{O}_{]U[_{\mathfrak{X},\lambda}}}\mathcal{E}|_{]U[_{\mathfrak{X},\lambda}}))|_{]U[_{\mathfrak{X},\lambda}})$$

by 1.1.7. Since $\underline{\Gamma}^{\dagger}_{]Z[_{\mathfrak{X},\eta}}((\Omega^{\bullet}_{\mathfrak{X}^{\#}_K/\mathcal{T}_K}|_{]U[_{\mathfrak{X},\lambda}})\otimes_{\mathcal{O}_{]U[_{\mathfrak{X},\lambda}}}\mathcal{E})|_{]Y[_{\mathfrak{X},\eta}}=0$, and since

$$\{]U[_{\mathfrak{X},\lambda}\cap]Y[_{\mathfrak{X},\eta},]U[_{\mathfrak{X},\lambda}\cap[Z]_{\mathfrak{X},\eta}\}$$

is an admissible covering of $]U[_{\mathfrak{X},\lambda}$, we have

$$\mathbb{R}g_{\lambda*}(\underline{\Gamma}^{\dagger}_{]Z[_{\mathfrak{X},\eta}}(\alpha_{]U[_{\mathfrak{X},\lambda}*}((\Omega^{\bullet}_{\mathfrak{X}^{\#}_K/\mathcal{T}_K}|_{]U[_{\mathfrak{X},\lambda}})\otimes_{\mathcal{O}_{]U[_{\mathfrak{X},\lambda}}}\mathcal{E}|_{]U[_{\mathfrak{X},\lambda}}))|_{]U[_{\mathfrak{X},\lambda}})$$
$$\cong \mathbb{R}g_{\lambda,\eta*}(\underline{\Gamma}^{\dagger}_{]Z[_{\mathfrak{X},\eta}}(\alpha_{]U[_{\mathfrak{X},\lambda}*}((\Omega^{\bullet}_{\mathfrak{X}^{\#}_K/\mathcal{T}_K}|_{]U[_{\mathfrak{X},\lambda}})\otimes_{\mathcal{O}_{]U[_{\mathfrak{X},\lambda}}}\mathcal{E}|_{]U[_{\mathfrak{X},\lambda}}))|_{]U[_{\mathfrak{X},\lambda}\cap[Z]_{\mathfrak{X},\eta}}).$$

Hence, in order to prove the vanishing $\mathbb{R}g_{K*}\underline{\Gamma}^{\dagger}_{]Z[_{\mathfrak{X}}}(j^{\dagger}_U\Omega^{\bullet}_{\mathfrak{X}^{\#}_K/\mathcal{T}_K}\otimes_{j^{\dagger}_U\mathcal{O}_{]X[_{\mathfrak{X}}}}E)=0$, we have only to prove the vanishing

(1.1.7.2)
$$\mathbb{R}g_{\lambda,\eta*}(\underline{\Gamma}^{\dagger}_{]Z[_{\mathfrak{X},\eta}}(\alpha_{]U[_{\mathfrak{X},\lambda}*}((\Omega^{\bullet}_{\mathfrak{X}^{\#}_K/\mathcal{T}_K}|_{]U[_{\mathfrak{X},\lambda}})\otimes_{\mathcal{O}_{]U[_{\mathfrak{X},\lambda}}}\mathcal{E}|_{]U[_{\mathfrak{X},\lambda}}))|_{]U[_{\mathfrak{X},\lambda}\cap[Z]_{\mathfrak{X},\eta}})=0$$

for any $(\lambda,\eta)\in\Lambda_{\underline{\xi},\underline{\lambda}}$.

3° *Reduce to the local computations.* Let us denote the 1-dimensional open (resp. closed) unit disk over $\mathrm{Spm}\,K$ of radius $\eta\in|K^{\times}|_{\mathbb{Q}}$ by $D(0,\eta^-)$ (resp. $D(0,\eta^+)$). Since $Z\not\subset D$, we have the lemma below by the weak fibration theorem [Ber96a, 1.3.1, 1.3.2]; see also [BC94, 4.3].

LEMMA 1.1.8. *With the notation as above, we have*

(1) *There is an admissible covering* $\{V_\beta\}_\beta$ *of* $]T[_{\mathcal{T}}$ *such that there exists an isomorphism*

$$g_K^{-1}(V_\beta)\cap]Z[_{\mathfrak{X}}\cong V_\beta\times_{\mathrm{Spm}\,K}D(0,1^-)$$

of rigid analytic K*-spaces, under which the coordinate of* $D(0,1^-)$ *is* z *as above.*

(2) *Under the isomorphism in* (1),

$$g_{\lambda,\eta}^{-1}(V_\beta) \cong (V_\beta \cap]T \cap U[_{\mathcal{T},\lambda}) \times_{\mathrm{Spm}\, K} D(0, \eta^+)$$

for any $\lambda, \eta \in |K^\times|_{\mathbb{Q}} \cap]0,1[$ *with* $\lambda > \eta$.

In order to prove 1.1.8(2), the condition $\lambda > \eta$ is needed because of using $\overline{f}$ for the definition of $]T \cap U[_{\mathcal{T},\lambda}$.

Let $S = \mathrm{Spm}\, R$ be an integral smooth K-affinoid subdomain of $V_\beta \cap]T \cap U[_{\mathcal{T},\lambda}$ with a complete K-algebra norm $|\text{-}|_R$ on R. Since R is an integral K-Banach algebra, all complete K-algebra norms are equivalent [BGR84, 3.8.2, Cor. 4]. In order to prove the vanishing (1.1.7.2), it is sufficient to prove the vanishing

$$\mathbb{R}\Gamma\left(g_{\lambda,\eta}^{-1}(S),\, \underline{\Gamma}^\dagger_{]Z[_{\mathfrak{X},\eta}}\left(\left[\mathcal{E} \xrightarrow{\nabla} (\Omega^1_{\mathfrak{X}_K^\#/\mathcal{T}_K}|_{]U[_{\mathfrak{X},\nu}}) \otimes_{\mathcal{O}_{]U[_{\mathfrak{X},\nu}}} \mathcal{E}\right]\right)\right) = \mathbb{R}\Gamma\left(g_{\lambda,\eta}^{-1}(S),\, \underline{\Gamma}^\dagger_{]Z[_{\mathfrak{X},\eta}}\left(\left[\mathcal{E} \xrightarrow{\partial_\#} \mathcal{E}\right]\right)\right) = 0$$

of hypercohomology for any such S by 1.1.8(2) since $]T[_{\mathcal{T}} =]Z[_{\mathcal{Z}}$ is integral and smooth and $\Omega^1_{\mathfrak{X}_K^\#/\mathcal{T}_K}$ is a free $\mathcal{O}_{]X[_{\mathfrak{X}}}$-module of rank 1 generated by $\frac{dz}{z}$. The hypercohomology above can be calculated by

$$\mathbb{R}^q\Gamma\left(g_{\lambda,\eta}^{-1}(S), \underline{\Gamma}^\dagger_{]Z[_{\mathfrak{X},\eta}}\left(\left[\mathcal{E} \xrightarrow{\partial_\#} \mathcal{E}\right]\right)\right) \cong H^q\left(\mathrm{Tot}\begin{bmatrix} \Gamma(g_{\lambda,\eta}^{-1}(S), \mathcal{E}) & \to & \lim\limits_{\mu \to \eta^-} \Gamma(g_{\lambda,\eta}^{-1}(S) \cap]Y[_{\mathfrak{X},\mu}, \mathcal{E}) \\ \partial_\# \downarrow & & \downarrow \partial_\# \\ \Gamma(g_{\lambda,\eta}^{-1}(S), \mathcal{E}) & \to & \lim\limits_{\mu \to \eta^-} \Gamma(g_{\lambda,\eta}^{-1}(S) \cap]Y[_{\mathfrak{X},\mu}, {}^*\mathcal{E}) \end{bmatrix}\right).$$

Here Tot means the total complex induced by the commutative bicomplex, the left top item in the bicomplex is located at degree $(0,0)$, and the horizontal arrows in the bicomplex are the natural injections. Indeed, the cohomological functor commutes with filtered direct limits since $g_{\lambda,\eta}$ is an affinoid morphism, and the vanishings $H^q(g_{\lambda,\eta}^{-1}(S), \mathcal{E}) = 0$ and $H^q(g_{\lambda,\eta}^{-1}(S) \cap]Y[_{\mathfrak{X},\mu}, \mathcal{E}) = 0$ for $q \geq 1$ hold by Kiehl's Theorem B [Kie67, 2.4] since $g_{\lambda,\eta}^{-1}(S)$ and $g_{\lambda,\eta}^{-1}(S) \cap]Y[_{\mathfrak{X},\mu}$ are affinoid.

More explicitly, the following formula (1.1.8.1) holds when $\mathcal{E}|_{g_{\lambda,\eta}^{-1}(S)}$ is a free $\mathcal{O}_{g_{\lambda,\eta}^{-1}(S)}$-module of rank r. We will prove the freeness in the next step 4°. Put R-algebras

$$\begin{aligned} \mathcal{A}_R(\eta) &= \Gamma(g_{\lambda,\eta}^{-1}(S), \mathcal{O}_{]X[_{\mathfrak{X}}}) \\ &= \left\{ \sum_{n=0}^{\infty} a_n z^n \,\middle|\, a_n \in R,\ |a_n|_R \eta^n \to 0 \text{ as } n \to \infty \right\}, \end{aligned}$$

$$\begin{aligned}\mathcal{A}_R(\eta^-) &= \Gamma\left(\underset{\mu<\eta}{\cup} g_{\lambda,\mu}^{-1}(S), \mathcal{O}_{]X[_{\mathfrak{X}}}\right)\\ &= \left\{\sum_{n=0}^{\infty} a_n z^n \,\middle|\, a_n \in R,\ |a_n|_R \mu^n \to 0 \text{ as } n \to \infty \text{ for any } \mu < \eta\right\},\\ \mathcal{R}_R(\eta) &= \lim_{\mu\to\eta^-} \Gamma(g_{\lambda,\eta}^{-1}(S), \alpha_{]Y[_{\mathfrak{X},\mu}*}\mathcal{O}_{]Y[_{\mathfrak{X}.\mu}})\\ &= \left\{\sum_{n=-\infty}^{\infty} a_n z^n \,\middle|\, a_n \in R,\ \begin{array}{l}|a_n|_R \eta^n \to 0 \text{ as } n \to \infty\\ |a_n|_R \mu^n \to 0 \text{ as } n \to -\infty \text{ for some } \mu < \eta\end{array}\right\},\end{aligned}$$

and define a norm on $\mathcal{A}_R(\eta)$ by $|\sum_n a_n z^n|_{\mathcal{A}_R(\eta)} = \sup_n |a_n|\eta^n$. It follows that $\mathcal{A}_R(\eta), \mathcal{A}_R(\eta^-)$ and $\mathcal{R}_R(\eta)$ are independent of the choice of complete K-algebra norms on R since there exist positive real numbers ρ_1 and ρ_2 such that $\rho_1|\text{-}| \leq |\text{-}|' \leq \rho_2|\text{-}|$ for equivalent norms $|\text{-}|$ and $|\text{-}|'$ by [BGR84, 2.1.8, Cor. 4]. Let $\underline{v}$ be a basis of vectors of $\Gamma(g_{\lambda,\eta}^{-1}(S), \mathcal{E})$ over $\mathcal{A}_R(\eta)$ such that the derivation along z is given by $\partial_\#(\underline{v}) = \underline{v}G$ for a matrix G with entries in $\mathcal{A}_R(\eta)$. Then we have

(1.1.8.1)
$$\begin{aligned}&\mathbb{R}^q\Gamma\left(g_{\lambda,\eta}^{-1}(S), \underline{\Gamma}^\dagger_{]Z[_{\mathfrak{X},\eta}}\left(\left[\mathcal{E} \xrightarrow{\partial_\#} \mathcal{E}\right]\right)\right)\\ &\quad\cong H^q\left(\operatorname{Tot}\begin{bmatrix} \mathcal{A}_R(\eta)^r & \to & \mathcal{R}_R(\eta)^r\\ \partial_\# + G \downarrow & & \downarrow \partial_\# + G\\ \mathcal{A}_R(\eta)^r & \to & \mathcal{R}_R(\eta)^r\end{bmatrix}\right)\\ &\quad\cong H^q\left(\left[(\mathcal{R}_R(\eta)/\mathcal{A}_R(\eta))^r \xrightarrow{\partial_\#+G} (\mathcal{R}_R(\eta)/\mathcal{A}_R(\eta))^r\right][-1]\right).\end{aligned}$$

4° *Local classification of logarithmic connections along a smooth divisor.*

PROPOSITION 1.1.9. *Let $S = \operatorname{Spm} R$ be a smooth integral K-affinoid variety, and let $W = S \times_{\operatorname{Spm} K} D(0, \xi^-)$ be a quasi-Stein space over S for some $\xi \in |K^\times|_{\mathbb{Q}} \cap]0, 1]$. Let $\mathcal{M}$ be a locally free $\mathcal{O}_W$-module of finite type furnished with an R-derivation $\partial_\# = z\frac{d}{dz} : M \to M$, where $M = \Gamma(W, \mathcal{M})$, such that*

- (i) *for any $\eta \in |K^\times|_{\mathbb{Q}} \cap]0, \xi[$, if $W_\eta = S \times_{\operatorname{Spm} K} D(0, \eta^+)$ is an affinoid subdomain of W and if $||\text{-}||$ is a Banach $\mathcal{A}_R(\eta)$-norm on $M_\eta = \Gamma(W_\eta, \mathcal{M})$, then $||\frac{1}{n!}\prod_{j=0}^{n-1}(\partial_\# - j)(e)||\mu^n \to 0\,(n \to \infty)$ for any $e \in M_\eta$ and $0 < \mu < 1$; and*
- (ii) *any difference of exponents of $(\mathcal{M}, \partial_\#)$ along $z = 0$ is neither a p-adic Liouville number nor a nonzero integer.*

Then there are a projective R-module L of finite type furnished with a linear R-operator $N : L \to L$ such that $||\frac{1}{n!}\prod_{j=0}^{n-1}(N - j)(e)||\mu^n \to 0\,(n \to \infty)$ for any $e \in L$ and $0 < \mu < 1$, where $||\text{-}||$ is a Banach R-norm on L, and an isomorphism $(\mathcal{M}, \partial_\#) \cong (\mathcal{O}_W \otimes_R L, \partial_{\#N})$ in which the R-derivation $\partial_{\#N}$ on $\mathcal{O}_W \otimes_R L$ is defined by $\partial_{\#N}(a \otimes e) = \partial_\#(a) \otimes e + a \otimes N(e)$.

If $\mathcal{M}$ is a free $\mathcal{O}_W$-module in the proposition above, then the assertion is a part of Christol's transfer theorem [Chr84, Th. 2] and its generalization in [BC92]. Christol's transfer theorem is in the case where R is a field K. By the argument in [BC92, 4.1], the transfer theorem also works on an integral K-affinoid algebra R. 'A part' means that we consider solutions not in meromorphic functions but only in holomorphic functions. When M is free, one has a formal matrix solution by the hypothesis that any difference of exponents is not an integer except 0, and then all entries are contained in $\mathcal{A}_R(\xi^-)$ because of conditions (i) and (ii).

LEMMA 1.1.10. *Let R be an integral K-affinoid algebra.*

(1) *There exists a finite injective morphism $T_l \to R$ of K-affinoid algebras from a free Tate K-algebra T_l of some dimension l.*
(2) *Suppose, furthermore, that R is Cohen-Macaulay. Then, for any finite injective morphism $T_l \to R$ of K-affinoid algebras, R is projective of finite type over T_l. Moreover, if M is a projective R-module of finite type, then M is free over T_l.*

Proof. (1) The assertion is the Noether normalization theorem [BGR84, 6.1.2 Cor. 2].

(2) Since T_l is regular and R is Cohen-Macaulay, R is projective over T_l by [Nag62, 25.16]. If M is a projective R-module of finite type, then M is also projective of finite type over T_l; hence, M is free over T_l by [Ked04, 6.5]. □

With the notation as in 1.1.9, let us fix a finite injective morphism $T_l \to R$ of K-affinoid algebras 1.1.10(1). Considering the norm on R that is defined by the maximum of norms of tuples under an identification $R \cong T_l^m$ by 1.1.10(2), we regard M_η as an $\mathcal{A}_{T_l}(\eta)[\partial_\#]$-module by the natural finite injective morphism $\mathcal{A}_{T_l}(\eta) \to \mathcal{A}_R(\eta)$ of K-affinoid algebras for $\eta \in |K^\times|_{\mathbb{Q}} \cap]0, \xi[$. Moreover, $\mathcal{A}_{T_l}(\eta)[\partial_\#]$-module M_η satisfies the hypothesis in 1.1.9 (see 2) and M_η is a free $\mathcal{A}_{T_l}(\eta)$-module 1.1.10(2). Fix a basis $\underline{v}$ of M_η over $\mathcal{A}_{T_l}(\eta)$ and let G_η be a matrix with entries in $\mathcal{A}_{T_l}(\eta)$ such that $\partial_\#(\underline{v}) = \underline{v}G_\eta$. By applying a generalization of Christol's transfer theorem (as we explain after 1.1.9), there is an invertible matrix Y with entries in $\mathcal{A}_{T_l}(\eta^-)$ such that

$$\partial_\# Y + G_\eta Y = Y G_\eta(0), \tag{1.1.10.1}$$

where $G_\eta(0) = G_\eta \,(\mathrm{mod}\, z\mathcal{A}_{T_l}(\eta))$ is a matrix with entries in T_l. Then there is a free T_l-module L_η with a T_l-linear homomorphism N_η defined by the matrix $G_\eta(0)$ such that $(\mathcal{A}_{T_l}(\eta^-) \otimes_{\mathcal{A}_{T_l}(\eta)} M_\eta, \partial_\#) \cong (\mathcal{A}_{T_l}(\eta^-) \otimes_{T_l} L_\eta, \partial_{\# N_\eta})$. If we put $H^0(M_\eta) = \ker(\partial_\# : M_\eta \to M_\eta)$, then $H^0(M_\eta) \cong \ker(N_\eta : L_\eta \to L_\eta)$.

LEMMA 1.1.11. *With the notation as above, the following hold.*

(1) *The pair* (L_η, N_η) *is independent of the choices of* $\eta \in |K^\times|_{\mathbb{Q}} \cap]0, \xi[$ *up to canonical isomorphisms. Moreover,* $(M, \partial_\#) \cong (\mathcal{A}_{T_l}(\xi^-) \otimes_{T_l} L_\eta, \partial_{\# N_\eta})$ *for any* η.

(2) *If we put* $H^0(M) = \ker(\partial_\# : M \to M)$, *then the natural* R*-homomorphism* $H^0(M) \to H^0(M_\eta)$ (*not only the* T_l *structure*) *induced by the restriction is an isomorphism.*

Proof. (1) For $\eta' \leq \eta$, there is an invertible matrix Q with entries in $\mathcal{A}_{T_l}(\eta')$ such that $\partial_\# Q + G_{\eta'}(0)Q = QG_\eta(0)$ by the restriction. Since none of the differences of exponents is an integer except 0, Q is an invertible matrix with entries in T_l. Hence, the pair is independent of the choices of η. Note that $\{W_\eta\}_{\eta \in |K^\times|_{\mathbb{Q}} \cap]0,\xi[}$ is an affinoid covering of the quasi-Stein space W and M is the projective limit of M_η ($\eta \in |K^\times|_{\mathbb{Q}} \cap]0, \xi[$). Therefore, the assertion holds.

(2) follows from (1). □

LEMMA 1.1.12. *Let* R *be an integral domain over* $\mathbb{Q}_p$ *with field of fractions* F, *and let* (L, N) *be a pair such that* L *is a free* R*-module of finite rank and* $N : L \to L$ *is an* R*-linear endomorphism. Suppose that* $e_1, \ldots, e_s$ *are distinct eigenvalues of* $N \otimes F$ *with multiplicities* $m_1, \ldots, m_s$, *respectively, such that* $e_1, \ldots, e_s$ *are contained in* $\mathbb{Z}_p$, *and let* $\varphi_N(x) = (x-e_1)^{m_1} \cdots (x-e_s)^{m_s} \in \mathbb{Z}_p[x]$ *be the characteristic polynomial of* N. *If we put* $L(e_i) = \varphi_i(N)L$ *where* $\varphi_i(x) = \varphi_N(x)/(x-e_i)^{m_i}$, *then* L *is a direct sum of* R*-submodules* $L(e_1), \ldots, L(e_s)$ *of* L *such that all eigenvalues of* $N|_{L(e_i)} \otimes F$ *are* e_i *for any* i. *Such a decomposition is unique.*

LEMMA 1.1.13. *With the notation in* 1.1.9, *let* $e_1, \ldots, e_s$ *be distinct exponents of* $(M, \partial_\#)$ *along* $z = 0$. *Then* M *is a direct sum of* $\mathcal{A}_R(\xi^-)[\partial_\#]$*-submodules* $M(e_1), \ldots, M(e_s)$ *of* M *such that all exponents of* $(M(e_i), \partial_\#)$ *are* e_i *for any* i.

Proof. With the notation in 1.1.11 and 1.1.12, take a free T_l-module L of finite type furnished with a T_l-linear homomorphism N such that $(M, \partial_\#) \cong (\mathcal{A}_{T_l}(\xi^-) \otimes_{T_l} L, \partial_{\# N})$. Since $L(e_i)$ is a direct summand of the free T_l-module L, $L(e_i)$ is free. Put $M(e_i) = (\mathcal{A}_{T_l}(\xi^-) \otimes_{T_l} L(e_i), \partial_{\# N|_{L(e_i)}})$. Then M is a direct sum of $M(e_1), \ldots, M(e_s)$ as $\mathcal{A}_{T_l}(\xi^-)[\partial_\#]$-modules. Since any $\mathcal{A}_{T_l}(\xi^-)[\partial_\#]$-homomorphism between $M(e_i)$ and $M(e_j)$ for $i \neq j$ is a zero map, $M(e_i)$ is an $\mathcal{A}_R(\xi^-)[\partial_\#]$-module for all i. Hence, the decomposition is the desired one. □

LEMMA 1.1.14. *Let* $S = \operatorname{Spm} R$ *be a* K*-affinoid variety,* $W = S \times_{\operatorname{Spm} K} D(0, \xi^+)$ *for some* $\xi \in |K^\times|_{\mathbb{Q}}$, *and let* $\mathcal{M}$ *be a locally free* $\mathcal{O}_W$*-module of finite type. Then there exist a finite affinoid covering* $\{S_i\}$ *of* S *and a real number* $\xi' \in$

$|K^\times|_{\mathbb{Q}} \cap]0,\xi]$ *such that if* $W_{S_i,\xi'}$ *denotes the affinoid subdomain* $S_i \times D(0,\xi'^+)$ *of* W, *then* $\mathcal{M}|_{W_{S_i,\xi'}}$ *is a free* $\mathcal{O}_{W_{S_i,\xi'}}$*-module for all* i.

Proof. Since $\mathcal{M}/z\mathcal{M}$ is regarded as a locally free $\mathcal{O}_S$-module, there is a finite affinoid covering $\{S_i\}$ of S such that $(\mathcal{M}/z\mathcal{M})|_{S_i}$ is a free $\mathcal{O}_{S_i}$-module for all i. Since $W_i = S_i \times_{\mathrm{Spm}\,K} D(0,\xi^+)$ is an affinoid, $\mathcal{M}/z\mathcal{M}$ is generated by $\Gamma(W_i,\mathcal{M})$ by Kiehl's Theorems A and B [Kie67, 2.4]. Let $v_1,\ldots,v_r \in \Gamma(W_i,\mathcal{M})$ be elements whose reductions form a basis of $(\mathcal{M}/z\mathcal{M})|_{S_i}$ over $\mathcal{O}_{S_i}$. The support of $\mathcal{M}|_{W_i}/(v_1,\ldots,v_r)$ is an analytic closed subset of W_i that does not intersect with the closed subspace defined by $z=0$. Since $\mathcal{M}$ is locally free, there is a real number $\xi'_i \in |K^\times|_{\mathbb{Q}} \cap]0,\xi[$ such that $\mathcal{M}|_{S_i\times_{\mathrm{Spm}\,K} D(0,\xi_i'^+)}$ is free and is generated by $v_1,\ldots,v_r$ because of the maximum modulus principle [BGR84, 6.2.1, Prop.4]. Then it is enough to take $\xi' = \min_i \xi'_i$. □

Proof of 1.1.9. We may assume that any exponent of $\mathcal{M}$ along $z=0$ is 0 by 1.1.13 and by twisting by an object of rank 1 with a suitable exponent. We may also assume that $\mathcal{M}|_{W_{\xi'}}$ is a free $\mathcal{O}_{W_{\xi'}}$-module for some $\xi' \in |K^\times|_{\mathbb{Q}} \cap]0,\xi[$ by 1.1.14. By applying the transfer Theorem 1.1.9 for the free cases with conditions (i) and (ii), if one takes an $\eta \in |K^\times|_{\mathbb{Q}} \cap]0,\xi'[$, then there is a free R-module L furnished with an R-linear operator $N : L \to L$ such that $\beta_\eta : (\mathcal{M},\partial_\#)|_{W_\eta} \xrightarrow{\sim} (\mathcal{O}_{W_\eta} \otimes_R L, \partial_{\# N})$. Denote the dual of $\mathcal{M}$ by $(\mathcal{M}^\vee, -\partial_\#)$. Then we have a natural commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{O}_W[\partial_\#]}(\mathcal{M}, \mathcal{O}_W \otimes_R L) & \longrightarrow & \mathrm{Hom}_{\mathcal{O}_{W_\eta}[\partial_\#]}(\mathcal{M}|_{W_\eta}, \mathcal{O}_{W_\eta} \otimes_R L) \\
\cong \big\downarrow & & \big\downarrow \cong \\
H^0(M^\vee \otimes_R L) & \xrightarrow{\sim} & H^0(M^\vee_\eta \otimes_R L),
\end{array}$$

where the vertical arrows are isomorphisms since $\mathcal{M}$ is locally free and the bottom horizontal arrow is an isomorphism by 1.1.11(2) since all differences of exponents of $(\mathcal{M}^\vee \otimes_R L, -\partial_\# \otimes 1 + 1 \otimes \partial_N)$ along $z=0$ are 0.

Let $\beta : (\mathcal{M},\partial_\#) \to (\mathcal{O}_W \otimes_R L, \partial_{\# N})$ be the $\mathcal{O}_W[\partial_\#]$-homomorphism corresponding to β_η via the isomorphisms above. We will prove that β is an isomorphism. In the case where R is a field, β is an isomorphism since the support of an $\mathcal{A}_R(\xi^-)[\partial_\#]$-module, which is finitely generated over $\mathcal{A}_R(\xi^-)$, is either W or one point $z=0$ by Bézout property of $\mathcal{A}_R(\xi^-)$ [Cre98, 4.6]. Let us return to the case of general R. For a maximal ideal x of R, the induced homomorphism $\beta\,(\mathrm{mod}\,x)$ is an isomorphism by the case where R is a field. Hence, β is an isomorphism around $x \times_{\mathrm{Spm}\,K} D(0,\xi^-)$ by Nakayama's lemma. Since both sides of β are coherent, β is an isomorphism [BGR84, 9.4.2, Cor. 7]. □

5° *The vanishing* (1.1.7.2) *in special cases*: *any difference of exponents is neither a p-adic Liouville number nor an integer except* 0. Let us first suppose that (ii) in 1.1.9 and $c=0$ for the exponents along $z=0$ by 0°.

LEMMA 1.1.15. *With the notation in* 1.1.12, *the following hold.*

(1) *Let j be an integer. Then there is a monic polynomial $g_j(x) \in \mathbb{Z}_p[x]$ of degree $r-1$ such that $(N-j)g_j(N)+\varphi_N(j)I_L=0$. Here I_L is the identity of L.*
(2) *If all of $e_1, \ldots, e_s$ are neither p-adic Liouville numbers nor positive integers, then $(N-j)$ is invertible and, for any $0 < \eta < 1$, $|\varphi_N(j)^{-1}|\eta^j \to 0$ as $j \to \infty$*

Take $(\lambda, \eta) \in \Lambda_{\underline{\xi},\underline{\lambda}}$ such that $\lambda \geq \lambda_m$ and $\eta < \xi_m$ for some m. Then the restriction $(\mathcal{E}, \partial_\#)$ on $S \times_{\mathrm{Spm}K} D(0, \xi_m^-)$ for an integral smooth K-affinoid $S = \mathrm{Spm}\, R$ in $V_\beta \cap]Z \cap U[_{\mathcal{Z},\lambda_m}$ satisfies the assumption of 1.1.9 by the overconvergence condition in 2°. Considering an admissible affinoid covering of S, we may assume that there is a basis of $\Gamma(g_{\lambda,\eta}^{-1}(S), \mathcal{E})$ over $\mathcal{A}_R(\eta)$ such that G is a matrix with entries in R.

Since any eigenvalue of G is not a positive integer, $\partial_\# + G$ is injective on $(\mathcal{R}_R(\eta)/\mathcal{A}_R(\eta))^r$. Since any eigenvalue of G is neither a p-adic Liouville nor a positive integer, $\partial_\# + G$ is surjective on $(\mathcal{R}_R(\eta)/\mathcal{A}_R(\eta))^r$. Indeed, with the notation in 1.1.15(1), $\partial_\# + G$ maps $-\sum_{j=1}^{\infty} \varphi_G(j)^{-1} g_j(G)\underline{a}_j z^{-j}$ to $\sum_{j=1}^{\infty} \underline{a}_j z^{-j}$ and $\sum_{j=1}^{\infty} \varphi_G(j)^{-1} g_j(G)\underline{a}_j z^{-j}$ is contained in $(\mathcal{R}_R(\eta)/\mathcal{A}_R(\eta))^r$ by (1.1.15)(2). Hence, the cohomology groups in (1.1.8.1) vanish for any q and it implies the vanishing (1.1.7.2).

6° *The vanishing* (1.1.7.2) *in general cases*: *any difference of exponents is not a p-adic Liouville number.* Let us suppose conditions (a) 1.1.1 and $c = 0$ for the exponents along $z = 0$ by 0°.

PROPOSITION 1.1.16. *With the notation as in* 1.1.9, *we assume conditions* (i) *in* 1.1.9, (a) *in* 1.1.1, *and $c = 0$ for exponents of $(\mathcal{M}, \partial_\#)$ along $z = 0$. Then there is a locally free $\mathcal{O}_W$-submodule $\mathcal{M}'$ of $\mathcal{M}$ that is stable under $\partial_\#$ such that* (1) *$(\mathcal{M}', \partial_\#)$ satisfies conditions* (i) *and* (ii) *in* 1.1.9, (2) *none of exponents of $(\mathcal{M}', \partial_\#)$ along $z = 0$ is a positive integer*, (3) *the support of $\mathcal{M}/\mathcal{M}'$ is included in the closed subset defined by $z = 0$ and it is a locally free O_S-module of finite type, and* (4) *the induced homomorphism $\overline{\partial}_\# : \mathcal{M}/\mathcal{M}' \to \mathcal{M}/\mathcal{M}'$ is an isomorphism.*

LEMMA 1.1.17. *Let R be an integral K-affinoid algebra, and let $\eta \in |K^\times|_{\mathbb{Q}}$. Suppose that M is a free $\mathcal{A}_R(\eta)$-module of finite rank furnished with an R-derivation $\partial_\# = z\frac{d}{dz} : M \to M$ such that $e_1, \ldots, e_s$ are distinct exponents of $(M, \partial_\#)$ along $z = 0$ with multiplicities $m_1, \ldots, m_s$, respectively.*

(1) *There exists a basis $\underline{v}$ of M such that if G is the matrix with entries in $\mathcal{A}_R(\xi)$ defined by $\partial_\#(\underline{v}) = \underline{v}G$, then $G(0) = \begin{pmatrix} G_1(0) & & 0 \\ & \ddots & \\ 0 & & G_s(0) \end{pmatrix}$ and all eigenvalues of the R-matrix $G_i(0)$ of degree m_i are e_i for any i.*

(2) *Let* $\underline{v}_i$ *be the part of the basis as in* (1) *corresponding to the* i*-th direct summand modulo* z*; that is,* $\partial_\#(\underline{v}_i) \equiv \underline{v}_i G_i(0) \,(\operatorname{mod} z\mathcal{A}_R(\eta))$*. Let* M' *be the* $\mathcal{A}_R(\eta)$*-submodule of* M *generated by* $z\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_s$*. Then* M' *is stable under* $\partial_\#$ *with exponents* $e_1+1, e_2, \ldots, e_s$ *and multiplicities* $m_1, m_2, \ldots, m_s$*, respectively. Moreover,* M/M' *is a free* R*-module of rank* m_1*, and, if* $e_1 \neq 0$*, then the induced* R*-homomorphism* $\overline{\partial}_\# : M/M' \to M/M'$ *is an isomorphism.*

Proof. (1) follows from 1.1.12.

(2) The stability follows from (1). If we denote the matrix that represents the derivation of M' by G', then

$$\begin{aligned} G' &= P^{-1} z \frac{d}{dz} P + P^{-1} G P \\ &\equiv \begin{pmatrix} G_1(0) + I_{m_1} & & & * \\ & G_2(0) & & \\ & & \ddots & \\ 0 & & & G_s(0) \end{pmatrix} (\operatorname{mod} z\mathcal{A}_R(\eta)) \end{aligned}$$

for $P = \begin{pmatrix} zI_{m_1} & 0 \\ 0 & I_{r-m_1} \end{pmatrix}$. Here $r = m_1 + \cdots + m_s$ and I_t is the identity matrix of degree t. The induced R-homomorphism $\overline{\partial}_\# : M/M' \to M/M'$ is given by the matrix $G_1(0)$. □

Proof of 1.1.16. We use the induction on the largest integral difference of exponents and its multiplicity. By 1.1.14 we may assume that $\mathcal{M}|_{W_\eta}$ is free for some $\eta \in |K^\times|_{\mathbb{Q}} \cap]0, \xi[$. We have an $\mathcal{O}_{W_\eta}$-submodule $\mathcal{M}'_\eta$ of $\mathcal{M}|_{W_\eta}$ such that exponents are improved by 1.1.17. Indeed, we apply 1.1.17 to an exponent that is neither a positive integer nor 0 because of the condition $c = 0$. Since the support of $\mathcal{M}|_{W_\eta}/\mathcal{M}'_\eta$ is included in $z = 0$, one can glue $\mathcal{M}'_\eta$ and $\mathcal{M}|_{W\setminus\{z=0\}}$. Hence, the induction works. □

We use the same notation as in 5°. Considering an admissible affinoid covering of S, we may assume that $\mathcal{E}|_{g^{-1}_{\lambda,\mu}(S)}$ is free for some $\mu \in |K^\times|_{\mathbb{Q}} \cap]0, \xi_m]$ by 1.1.14, and then we can apply 1.1.16. Let $\mathcal{E}'$ be a locally free $\mathcal{O}_{g^{-1}_{\lambda,\xi_m}(S)}$-submodule of $\mathcal{E}|_{g^{-1}_{\lambda,\xi_m}(S)}$ that is stable under $\partial_\#$ such that it satisfies conditions (1), (2), and (3) in 1.1.16. Now we calculate the difference of the local computation of cohomology between $\mathcal{E}$ and $\mathcal{E}'$ by the module version of the second form of (1.1.7.2). If $E_\eta = \Gamma(g^{-1}_{\lambda,\eta}(S), \mathcal{E})$ and $E'_\eta = \Gamma(g^{-1}_{\lambda,\eta}(S), \mathcal{E})$, then $E' \otimes \mathcal{R}_R(\eta) = E \otimes \mathcal{R}_R(\eta)$ by condition (2) on the support of $\mathcal{E}/\mathcal{E}'$. The difference is calculated by the complex

$$\operatorname{Tot} \begin{bmatrix} E'_\eta & \to & E_\eta \\ \partial_\# \downarrow & & \downarrow \partial_\# \\ E'_\eta & \to & E_\eta \end{bmatrix} \cong \left[E_\eta / E'_\eta \xrightarrow{\partial_\#} E_\eta / E'_\eta \right],$$

and it is 0 by (3). Hence, the vanishing (1.1.7.2) for $\mathcal{E}$ follows from the vanishing for $\mathcal{E}'$ by 5°.

This completes the proof of Proposition 1.1.4. □

Proof of Theorem 1.1.1. By the same reason as 0° in the proof of 1.1.4, we may assume $c = 0$ and have only to prove the vanishing

$$\mathbb{R}g_{K*}\underline{\Gamma}^{\dagger}_{]Z[_{\mathfrak{X}}}(j^{\dagger}_U\Omega^{\bullet}_{\mathfrak{X}^{\#}_K/\mathcal{T}_K} \otimes_{j^{\dagger}_U\mathcal{O}_{]X[_{\mathfrak{X}}}} E) = 0.$$

By the Čech spectral sequence the problem of the vanishing is local on X and U as in 1° in the proof of 1.1.4. We may assume that $\mathfrak{X}$ is affine, D is defined by a single equation $f = 0$ in X for some $f \in \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$, and there is a system of relative local coordinates $z_1, z_2, \ldots, z_d \in \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ of $\mathfrak{X}$ over $\mathcal{T}$ such that each irreducible component $\mathcal{Z}_i$ of the relative strict normal crossings divisor $\mathcal{Z} = \cup_{i=1}^{s}\mathcal{Z}_i$ is defined by $z_i = 0$. Let us denote by Z_i (resp. Y_i) the closed subscheme of X defined by $z_i = 0$ (resp. the complement of Z_i in X).

Let us define $]U[_{\mathfrak{X},\lambda}$ (resp. $]Y_i[_{\mathfrak{X},\lambda}$, resp. $[Z_i]_{\mathfrak{X},\lambda})$ as in 2° of the proof of 1.1.4 (resp. replacing $\mathcal{Z}$, Z by $\mathcal{Z}_i$, Z_i).

By the hypothesis on (E, ∇) there exist a strict neighborhood $]U[_{\mathfrak{X},\nu}$ of $]U[_{\mathfrak{X}}$ in $]X[_{\mathfrak{X}}$ for some $\nu \in |K^{\times}|_{\mathbb{Q}} \cap]0, 1[$ and a locally free $\mathcal{O}_{]U[_{\mathfrak{X},\nu}}$-module $\mathcal{E}$ of finite type furnished with a logarithmic connection

$$\nabla : \mathcal{E} \to (\Omega^1_{\mathfrak{X}^{\#}_K/\mathcal{T}_K}|_{]U[_{\mathfrak{X},\nu}}) \otimes_{\mathcal{O}_{]U[_{\mathfrak{X},\nu}}} \mathcal{E}$$

such that $j^{\dagger}_U(\mathcal{E}, \nabla) = (E, \nabla)$, which satisfies the overconvergence condition (1.1.0.2).

7° *Induction on the number s of irreducible components of the strict normal crossings divisor Z.* If $s = 0$, then the assertion is trivial. Put $Z' = \cup_{i=2}^{s} Z_i$. Applying the natural exact sequence

$$0 \longrightarrow \underline{\Gamma}^{\dagger}_{]Z_1[_{\mathfrak{X}}}(\mathcal{H}) \longrightarrow \underline{\Gamma}^{\dagger}_{]Z[_{\mathfrak{X}}}(\mathcal{H}) \longrightarrow \underline{\Gamma}^{\dagger}_{]Z'[_{\mathfrak{X}}}(j^{\dagger}_{Y_1}\mathcal{H}) \longrightarrow 0$$

for a sheaf $\mathcal{H}$ of abelian groups on $]X[_{\mathfrak{X}}$ (see the proof of [Ber96a, 2.1.7]), we have a triangle

$$\begin{array}{ccc} & \mathbb{R}g_{K*}\underline{\Gamma}^{\dagger}_{]Z'[_{\mathfrak{X}}}(j^{\dagger}_{Y_1\cap U}\Omega^{\bullet}_{\mathfrak{X}^{\#}_K/\mathcal{T}_K} \otimes_{j^{\dagger}_{Y_1\cap U}\mathcal{O}_{]X[_{\mathfrak{X}}}} j^{\dagger}_{Y_1\cap U}E) & \\ {}^{+1}\swarrow & & \nwarrow \\ \mathbb{R}g_{K*}\underline{\Gamma}^{\dagger}_{]Z_1[_{\mathfrak{X}}}(j^{\dagger}_U\Omega^{\bullet}_{\mathfrak{X}^{\#}_K/\mathcal{T}_K} \otimes_{j^{\dagger}_U\mathcal{O}_{]X[_{\mathfrak{X}}}} E) & \longrightarrow & \mathbb{R}g_{K*}\underline{\Gamma}^{\dagger}_{]Z[_{\mathfrak{X}}}(j^{\dagger}_U\Omega^{\bullet}_{\mathfrak{X}^{\#}_K/\mathcal{T}_K} \otimes_{j^{\dagger}_U\mathcal{O}_{]X[_{\mathfrak{X}}}} E). \end{array}$$

Hence, we have only to prove the vanishing

$$\mathbb{R}g_{K*}\underline{\Gamma}^{\dagger}_{]Z_1[_{\mathfrak{X}}}(j^{\dagger}_U\Omega^{\bullet}_{\mathfrak{X}^{\#}_K/\mathcal{T}_K} \otimes_{j^{\dagger}_U\mathcal{O}_{]X[_{\mathfrak{X}}}} E) = 0$$

by the induction on s. If $Z_1 \subset D$, the vanishing is trivial. Hence, we may assume that Z_1 is not included in D.

8° *Reduction to the case of sections.* Let us denote the formal affine space of relative dimension r over $\mathcal{T}$ by $\widehat{\mathbb{A}}^r_{\mathcal{T}}$. By our hypothesis there is a commutative diagram

$$(1.1.17.1)\qquad \begin{array}{ccccc} \mathcal{Z}_1 & \longrightarrow & \mathfrak{X} & & \\ \downarrow & & \downarrow & & \\ \widehat{\mathbb{A}}^{d-1}_{\mathcal{T}} & \longrightarrow & \widehat{\mathbb{A}}^{d}_{\mathcal{T}} & \longrightarrow & \widehat{\mathbb{A}}^{d-1}_{\mathcal{T}} \end{array}$$

of formal $\mathcal{V}$-schemes such that the vertical arrow $\mathfrak{X} \to \widehat{\mathbb{A}}^d_{\mathcal{T}}$, which is étale, (resp. $\mathcal{Z}_1 \to \widehat{\mathbb{A}}^{d-1}_{\mathcal{T}}$) is induced by $z_1, \ldots, z_d$ (resp. $z_2, \ldots, z_d$) and the composite of bottom arrows is the identity. Since the diagonal morphism $\Delta : \mathcal{Z}_1 \to \mathcal{Z}_1 \times_{\widehat{\mathbb{A}}^{d-1}_{\mathcal{T}}} \mathcal{Z}_1$ is étale and a closed immersion, $\widetilde{\mathfrak{X}} = \mathcal{Z}_1 \times_{\widehat{\mathbb{A}}^{d-1}_{\mathcal{T}}} \mathfrak{X} \backslash (\mathcal{Z}_1 \times_{\widehat{\mathbb{A}}^{d-1}_{\mathcal{T}}} \mathcal{Z}_1 \backslash \Delta(\mathcal{Z}_1))$ is an open formal subscheme of $\mathcal{Z}_1 \times_{\widehat{\mathbb{A}}^{d-1}_{\mathcal{T}}} \mathfrak{X}$. Let us now consider the commutative diagram

$$(1.1.17.2)\qquad \begin{array}{ccccc} & & \widetilde{\mathfrak{X}} & & \\ & \overset{\Delta}{\nearrow} & \downarrow & \overset{h}{\searrow} & \\ \mathcal{Z}_1 & \overset{\Delta}{\longrightarrow} & \mathcal{Z}_1 \times_{\widehat{\mathbb{A}}^{d-1}_{\mathcal{V}}} \mathfrak{X} & \underset{\mathrm{pr}_2}{\longrightarrow} & \mathfrak{X} \\ =\downarrow & & \downarrow & & \\ \mathcal{Z}_1 & \underset{\mathrm{pr}_1}{\longleftarrow} & \mathcal{Z}_1 \times_{\widehat{\mathbb{A}}^{d-1}_{\mathcal{V}}} \widehat{\mathbb{A}}^d_{\mathcal{V}} & & \end{array}$$

of formal $\mathcal{T}$-schemes and define $h : \widetilde{\mathfrak{X}} \to \mathfrak{X}$ (resp. $\widetilde{g}_1 : \widetilde{\mathfrak{X}} \to \mathcal{Z}_1$, $\widetilde{g}' : \mathcal{Z}_1 \to \mathcal{T}$, resp. $\widetilde{g} = \widetilde{g}' \circ \widetilde{g}_1$) as in the diagram (resp. by the composition $\widetilde{\mathfrak{X}} \to \mathcal{Z}_1 \times_{\widehat{\mathbb{A}}^{d-1}_{\mathcal{V}}} \mathfrak{X} \to \mathcal{Z}_1 \times_{\widehat{\mathbb{A}}^{d-1}_{\mathcal{V}}} \widehat{\mathbb{A}}^d_{\mathcal{V}} \to \mathcal{Z}_1$, resp. by the canonical morphism, resp. by the composition). We identify $\Delta(\mathcal{Z}_1)$ (resp. $\Delta(Z_1)$) with $\mathcal{Z}_1$ (resp. Z_1) and denote the special fiber of $\widetilde{\mathfrak{X}}$ (resp. the complement of Z_1, resp. the inverse image of U by h) by $\widetilde{X}$ (resp. $\widetilde{Y}_1$, resp. $\widetilde{U}$). $\mathcal{Z}_1$ is a smooth divisor over $\mathcal{T}$. Note that, étale locally, $h^{-1}(\mathcal{Z})$ is a relative normal crossings divisor. $\widetilde{\mathfrak{X}}^{\#}_K$ denotes the formal $\mathcal{V}$-scheme with the logarithmic structure over $\mathcal{T}_K$ that is induced by the logarithmic structure of $\mathfrak{X}^{\#}_K$, and $\Omega^1_{\widetilde{\mathfrak{X}}^{\#}_K/\mathcal{T}_K}$ denotes the sheaf of logarithmic Kähler differentials on $\widetilde{\mathfrak{X}}^{\#}_K$ over $\mathcal{T}_K$. Then $h^*_K \Omega^\bullet_{\mathfrak{X}^{\#}_K/\mathcal{T}_K} \cong \Omega^\bullet_{\widetilde{\mathfrak{X}}^{\#}_K/\mathcal{T}_K}$.

Let us define $]\widetilde{U}[_{\widetilde{\mathfrak{X}},\lambda}$ (resp. $]\widetilde{Y}_1[_{\widetilde{\mathfrak{X}},\lambda}$, resp. $[Z_1]_{\widetilde{\mathfrak{X}},\lambda}$) as in 2° of the proof of 1.1.4.

Lemma 1.1.18. *With the notation as above, we have*

(1) $h_K^{-1}(]Z_1[_{\mathfrak{X}}) =]Z_1[_{\widetilde{\mathfrak{X}}}$.

(2) *The restriction of* h_K *gives an isomorphism* $]Z_1[_{\widetilde{\mathfrak{X}}} \xrightarrow{\sim}]Z_1[_{\mathfrak{X}}$.

(3) *Under the isomorphism in* (2),

$$]\widetilde{U}[_{\widetilde{\mathfrak{X}},\lambda} \cap [Z_1]_{\widetilde{\mathfrak{X}},\eta} \xrightarrow{\sim}]U[_{\mathfrak{X},\lambda} \cap [Z_1]_{\mathfrak{X},\eta}$$

for any $\lambda, \eta \in |K^\times|_{\mathbb{Q}} \cap]0,1[$.

Proof. Since $(\mathcal{Z}_1 \times_{\widehat{\mathbb{A}}^{d-1}_{\mathcal{T}}} \mathcal{Z}_1 \setminus \Delta(\mathcal{Z}_1))$ is removed, we get (1). The other assertion (2) (resp. (3)) follows from [Ber96a, 1.3.1] and the fact that h is étale (resp. and $Z_1 \not\subset D$). □

Proposition 1.1.19. *With the notation as above, we have the following.*

(1) *If* $\mathcal{H}$ *is a sheaf of Abelian groups on* $]\widetilde{X}[_{\widetilde{\mathfrak{X}}}$, *then*

$$\mathbb{R}h_{K*}\underline{\Gamma}^{\dagger}_{]Z_1[_{\widetilde{\mathfrak{X}}}}(\mathcal{H}) \cong h_{K*}\underline{\Gamma}^{\dagger}_{]Z_1[_{\widetilde{\mathfrak{X}}}}(\mathcal{H}).$$

(2) *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be a sheaf of rings on* $]X[_{\mathfrak{X}}$ *and* $]\widetilde{X}[_{\widetilde{\mathfrak{X}}}$, *respectively, with a morphism* $h_K^{-1}\mathcal{A} \to \mathcal{B}$ *of rings such that* $\mathcal{A}|_{]Z_1[_{\mathfrak{X}}} \xrightarrow{\sim} \mathcal{B}|_{]Z_1[_{\widetilde{\mathfrak{X}}}}$ *under the isomorphism in* 1.1.18(2). *If* $\mathcal{H}$ *is an* $\mathcal{A}$*-module, then the adjoint map*

$$\underline{\Gamma}^{\dagger}_{]Z_1[_{\widetilde{\mathfrak{X}}}}(\mathcal{H}) \to h_{K*}\underline{\Gamma}^{\dagger}_{]Z_1[_{\widetilde{\mathfrak{X}}}}(\mathcal{B} \otimes_{h_K^{-1}\mathcal{A}} h_K^{-1}\mathcal{H})$$

is an isomorphism of $\mathcal{A}$*-modules.*

Proof. Let us define a functor

$$\underline{\Gamma}^{\dagger}_{]Z_1[_{\widetilde{X}},\eta}(\mathcal{H}) = \ker\left(\mathcal{H} \to \lim_{\mu\to\eta^-} \alpha_{]\widetilde{Y}_1[_{\widetilde{\mathfrak{X}},\mu}*}(\mathcal{H}|_{]\widetilde{Y}_1[_{\widetilde{\mathfrak{X}},\mu}})\right)$$

as in 2° of the proof of 1.1.4, where $\alpha_{]\widetilde{Y}_1[_{\widetilde{\mathfrak{X}},\mu}} :]\widetilde{Y}_1[_{\widetilde{\mathfrak{X}},\mu} \to]\widetilde{X}[_{\widetilde{\mathfrak{X}}}$ is the canonical open immersion. Then the analogues of 1.1.6 and 1.1.7 hold.

(1) Since $\underline{\Gamma}^{\dagger}_{]Z_1[_{\widetilde{X}},\eta}(\mathcal{H})|_{]Y_1[_{\widetilde{X},\eta}} = 0$, we have $\mathbb{R}^q h_{K*}\underline{\Gamma}^{\dagger}_{]Z_1[_{\widetilde{X}},\eta}(\mathcal{H}) = 0$ for any $q \geq 1$ by 1.1.18(2). Because the cohomological functor $\mathbb{R}^q h_{K*}$ commutes with filtered inductive limits by the quasi-compactness and quasi-separateness of h_K, we have

$$\mathbb{R}^q h_{K*}\underline{\Gamma}^{\dagger}_{]Z_1[_{\widetilde{\mathfrak{X}}}}(\mathcal{H}) \cong \mathbb{R}^q h_{K*}\left(\lim_{\eta\to 1^-}\underline{\Gamma}^{\dagger}_{]Z_1[_{\widetilde{\mathfrak{X}}},\eta}(\mathcal{H})\right) \cong \lim_{\eta\to 1^-}\mathbb{R}^q h_{K*}\underline{\Gamma}^{\dagger}_{]Z_1[_{\widetilde{\mathfrak{X}}},\eta}(\mathcal{H}) = 0$$

for any $q \geq 1$ by 1.1.6.

(2) Since $\mathcal{H}|_{[Z_1]_{\mathfrak{X},\eta}} \xrightarrow{\sim} (\mathcal{B} \otimes_{h_K^{-1}\mathcal{A}} h_K^{-1}\mathcal{H})|_{[Z_1]_{\widetilde{\mathfrak{X}},\eta}}$, the assertion follows from 1.1.6 and 1.1.18. □

Let $(\widetilde{E}, \widetilde{\nabla})$ be the inverse image of (E, ∇) by h_k; i.e.,

$$\widetilde{E} = h_K^* E = j^{\dagger}_{\widetilde{U}}\mathcal{O}_{]\widetilde{X}[_{\widetilde{\mathfrak{X}}}} \otimes_{h_K^{-1}(j^{\dagger}_U\mathcal{O}_{]X[_{\mathfrak{X}}})} h_K^{-1}E$$
$$\widetilde{\nabla} : \widetilde{E} \to j^{\dagger}_{\widetilde{U}}\Omega^1_{\widetilde{\mathfrak{X}}^{\#}_K/\mathcal{T}_K} \otimes_{j^{\dagger}_{\widetilde{U}}\mathcal{O}_{]\widetilde{X}[_{\widetilde{\mathfrak{X}}}}} \widetilde{E},$$

where $\widetilde{\nabla}$ is the induced $\mathcal{O}_{]T[_{\mathcal{T}}}$-linear connection by ∇ because of the étaleness of h. We also denote the induced basis of $\Omega^1_{\widetilde{\mathfrak{X}}^{\#}_K/\mathcal{T}_K}$ by $\frac{dz_1}{z_1}, \ldots, \frac{dz_s}{z_s}, dz_{s+1}, \ldots, dz_d$ and the dual basis of derivations by $z_1\frac{\partial}{\partial z_1}, \ldots, z_s\frac{\partial}{\partial z_s}, \frac{\partial}{\partial z_{s+1}}, \ldots, \frac{\partial}{\partial z_d}$.

PROPOSITION 1.1.20. (1) *If we put* $(\widetilde{\mathcal{E}}, \widetilde{\nabla}) = h_K^*(\mathcal{E}, \nabla)$, *then the natural morphism* $j_{\widetilde{U}}^\dagger(\widetilde{\mathcal{E}}, \widetilde{\nabla}) \to (\widetilde{E}, \widetilde{\nabla})$ *is an isomorphism.*

(2) *The derivation* $\widetilde{\partial}_{\#1} = \nabla(z_1 \frac{\partial}{\partial z_1})$ *on* $\widetilde{\mathcal{E}}$ *satisfies the overconvergence condition* (1.1.5.1).

Proof. (1) easily follows from the fact $\mathcal{E}$ is locally free.

(2) It is enough to check the overconvergence condition for $\mathrm{pr}_{2K}^*(\mathcal{E}, \nabla)$ along $z_1 = 0$. Fix a complete K-algebra norm on the affinoid algebra associated to $]X[_{\mathfrak{X}}$. Then one can take a contractive complete K-algebra norm on the affinoid algebra associated to $]Z_1 \times_{\mathbb{A}_k^{d-1}} X[_{\mathcal{Z}_1 \times_{\widehat{\mathbb{A}}_{\mathcal{T}}^{d-1}} \mathfrak{X}}$ [BGR84, 6.1.3, Prop. 3]. The induced norms $||\text{-}||_{\mathfrak{X}}$ on $\Gamma(]U[_{\mathfrak{X},\lambda}, \mathcal{E})$ and $||\text{-}||_{\mathcal{Z}_1 \times \mathfrak{X}}$ on $\Gamma\left(\mathrm{pr}_{2K}^{-1}(]U[_{\mathfrak{X},\lambda}), \mathrm{pr}_{2K}^*\mathcal{E}\right)$ satisfy the inequality $||e||_{\mathcal{Z}_1 \times \mathfrak{X}} \leq ||e||_{\mathfrak{X}}$ for any $e \in \Gamma(]U[_{\mathfrak{X},\lambda}, \mathcal{E})$. The overconvergence condition for $\mathrm{pr}_{2K}^*(\mathcal{E}, \nabla)$ along $z_1 = 0$ follows from the inequality. $\square$

Remarks 1.1.21. The connection $(\widetilde{\mathcal{E}}, \widetilde{\nabla})$ satisfies the overconvergence condition (1.1.0.2). It should be called a log-isocrystal on $\widetilde{U}^\#/\mathcal{T}_K$ overconvergent along $\widetilde{D}$.

Since $(j_U^\dagger \mathcal{O}_{]X[_{\mathfrak{X}}})|_{]Z_1[_{\mathfrak{X}}} \xrightarrow{\sim} (j_{\widetilde{U}}^\dagger \mathcal{O}_{]\widetilde{X}[_{\widetilde{\mathfrak{X}}}})|_{]Z_1[_{\widetilde{\mathfrak{X}}}}$, we have

$$
\begin{aligned}
&\mathbb{R}g_{K*}\underline{\Gamma}^\dagger_{]Z_1[_{\mathfrak{X}}}(j_U^\dagger \Omega^\bullet_{\mathfrak{X}_K^\#/\mathcal{T}_K} \otimes_{j_U^\dagger \mathcal{O}_{]X[_{\mathfrak{X}}}} E) \\
&\quad \cong \mathbb{R}g_{K*}(h_{K*}\underline{\Gamma}^\dagger_{]Z_1[_{\widetilde{\mathfrak{X}}}}(j_{\widetilde{U}}^\dagger \Omega^\bullet_{\widetilde{\mathfrak{X}}_K^\#/\mathcal{T}_K} \otimes_{j_{\widetilde{U}}^\dagger \mathcal{O}_{]\widetilde{X}[_{\widetilde{\mathfrak{X}}}}} \widetilde{E})) \\
&\quad \cong \mathbb{R}g_{K*}\mathbb{R}h_{K*}\underline{\Gamma}^\dagger_{]Z_1[_{\widetilde{\mathfrak{X}}}}(j_{\widetilde{U}}^\dagger \Omega^\bullet_{\widetilde{\mathfrak{X}}_K^\#/\mathcal{T}_K} \otimes_{j_{\widetilde{U}}^\dagger \mathcal{O}_{]\widetilde{X}[_{\widetilde{\mathfrak{X}}}}} \widetilde{E}) \\
&\quad \cong \mathbb{R}\widetilde{g}_{K*}\underline{\Gamma}^\dagger_{]Z_1[_{\widetilde{\mathfrak{X}}}}(j_{\widetilde{U}}^\dagger \Omega^\bullet_{\widetilde{\mathfrak{X}}_K^\#/\mathcal{T}_K} \otimes_{j_{\widetilde{U}}^\dagger \mathcal{O}_{]\widetilde{X}[_{\widetilde{\mathfrak{X}}}}} \widetilde{E})
\end{aligned}
$$

by 1.1.19. Hence, we have only to prove the vanishing

$$\mathbb{R}\widetilde{g}_{K*}\underline{\Gamma}^\dagger_{]Z_1[_{\widetilde{\mathfrak{X}}}}(j_{\widetilde{U}}^\dagger \Omega^\bullet_{\widetilde{\mathfrak{X}}_K^\#/\mathcal{T}_K} \otimes_{j_{\widetilde{U}}^\dagger \mathcal{O}_{]\widetilde{X}[_{\widetilde{\mathfrak{X}}}}} \widetilde{E}) = 0.$$

9° *An argument of Gauss-Manin type.* Let Ω_0^q (resp. Ω_1^q) be the free $\mathcal{O}_{]\widetilde{X}[_{\widetilde{\mathfrak{X}}}}$-submodule of $\Omega^q_{\widetilde{\mathfrak{X}}_K^\#/\mathcal{T}_K}$ generated by wedge products of the terms of the form $\frac{dz_2}{z_2}, \ldots, \frac{dz_s}{z_s}, dz_{s+1}, \ldots, dz_d$ (resp. $\frac{dz_1}{z_1} \wedge \omega$ for $\omega \in \Omega^{q-1}_{\widetilde{\mathfrak{X}}_K^\#/\mathcal{T}_K}$). Then $\Omega_0^q \xrightarrow{\sim} \Omega_1^{q+1}$ by $\omega \mapsto \frac{dz_1}{z_1} \wedge \omega$. Define

$$
\begin{aligned}
(1.1.21.1) \qquad &\widetilde{\nabla}_0 = \sum_{i=2}^{s} \frac{dz_i}{z_i} \otimes \partial_{\#i} + \sum_{i=s+1}^{d} dz_i \otimes \partial_i : \widetilde{E} \to j_{\widetilde{U}}^\dagger \Omega_0^1 \otimes_{j_{\widetilde{U}}^\dagger \mathcal{O}_{]\widetilde{X}[_{\widetilde{\mathfrak{X}}}}} \widetilde{E}, \\
&\widetilde{\nabla}_1 = \mathrm{id} \otimes \partial_{\#1} : j_{\widetilde{U}}^\dagger \Omega_0^q \otimes_{j_{\widetilde{U}}^\dagger \mathcal{O}_{]\widetilde{X}[_{\widetilde{\mathfrak{X}}}}} \widetilde{E} \to j_{\widetilde{U}}^\dagger \Omega_1^q \otimes_{j_{\widetilde{U}}^\dagger \mathcal{O}_{]\widetilde{X}[_{\widetilde{\mathfrak{X}}}}} \widetilde{E},
\end{aligned}
$$

where id is the identity of $j^\dagger_{\widetilde{U}}\Omega^q_0$. The definition of $\widetilde{\nabla}_0$ and $\widetilde{\nabla}_1$ is independent of the choices of local parameters $z_1, z_2, \dots, z_d$ of $\mathfrak{X}$ over $\mathfrak{T}$ as above. Then the exterior power of $j^\dagger_{\widetilde{U}}\Omega^1_0$ induces a complex $(j^\dagger_{\widetilde{U}}\Omega^\bullet_0 \otimes_{j^\dagger_{\widetilde{U}}\mathcal{O}_{]\widetilde{X}[_{\widetilde{\mathfrak{X}}}}} \widetilde{E}, \widetilde{\nabla}_0)$ and there is an isomorphism

(1.1.21.2)

$$j^\dagger_{\widetilde{U}}\Omega^\bullet_{\widetilde{\mathfrak{X}}^\#_K/\mathfrak{T}_K} \otimes_{j^\dagger_{\widetilde{U}}\mathcal{O}_{]\widetilde{X}[_{\widetilde{\mathfrak{X}}}}} \widetilde{E} \xrightarrow{\sim} \left[(j^\dagger_{\widetilde{U}}\Omega^\bullet_0 \otimes_{j^\dagger_{\widetilde{U}}\mathcal{O}_{]\widetilde{X}[_{\widetilde{\mathfrak{X}}}}} \widetilde{E}, \widetilde{\nabla}_0) \xrightarrow{\widetilde{\nabla}_1} (j^\dagger_{\widetilde{U}}\Omega^\bullet_1 \otimes_{j^\dagger_{\widetilde{U}}\mathcal{O}_{]\widetilde{X}[_{\widetilde{\mathfrak{X}}}}} \widetilde{E}, \frac{dz_1}{z_1} \wedge \widetilde{\nabla}_0) \right]$$

of complexes of $\mathcal{O}_{]T[_{\mathfrak{T}}}$-modules. Note that $\widetilde{\nabla}_1$ is the relative connection $\widetilde{E} \to j^\dagger_{\widetilde{U}}\Omega^1_{\widetilde{\mathfrak{X}}^\#_K/\mathcal{Z}^\#_{1K}} \otimes_{j^\dagger_{\widetilde{U}}\mathcal{O}_{]\widetilde{X}[_{\widetilde{\mathfrak{X}}}}} \widetilde{E}$ induced by $\widetilde{\nabla}$.

One can easily see that $(\widetilde{E}, \widetilde{\nabla}_1)$ satisfies the hypotheses (a) and (b) along $z_1 = 0$ in 1.1.1 by 1.1.18 and the overconvergence condition in 1.1.4, so that

$$\mathbb{R}\widetilde{g}_{1K*}\underline{\Gamma}^\dagger_{]Z_1[_{\widetilde{\mathfrak{X}}}} \left(\left[j^\dagger_{\widetilde{U}}\Omega^q_0 \otimes_{j^\dagger_{\widetilde{U}}\mathcal{O}_{]\widetilde{X}[_{\widetilde{\mathfrak{X}}}}} \widetilde{E} \xrightarrow{\widetilde{\nabla}_1} j^\dagger_{\widetilde{U}}\Omega^{q+1}_1 \otimes_{j^\dagger_{\widetilde{U}}\mathcal{O}_{]\widetilde{X}[_{\widetilde{\mathfrak{X}}}}} \widetilde{E} \right] \right) = 0$$

for any q by 1.1.4. Hence,

$$\mathbb{R}\widetilde{g}_{K*}\underline{\Gamma}^\dagger_{]Z_1[_{\widetilde{\mathfrak{X}}}}(j^\dagger_{\widetilde{U}}\Omega^\bullet_{\widetilde{\mathfrak{X}}^\#_K/\mathfrak{T}_K} \otimes_{j^\dagger_{\widetilde{U}}\mathcal{O}_{]\widetilde{X}[_{\widetilde{\mathfrak{X}}}}} \widetilde{E}) = \mathbb{R}\widetilde{g}'_{K*}\mathbb{R}\widetilde{g}_{1K*}\underline{\Gamma}^\dagger_{]Z_1[_{\widetilde{\mathfrak{X}}}}(j^\dagger_{\widetilde{U}}\Omega^\bullet_{\widetilde{\mathfrak{X}}^\#_K/\mathfrak{T}_K} \otimes_{j^\dagger_{\widetilde{U}}\mathcal{O}_{]\widetilde{X}[_{\widetilde{\mathfrak{X}}}}} \widetilde{E}) = 0.$$

This completes the proof of 1.1.1. □

PROPOSITION 1.1.22. *With the notation in as* 1.1.1, *we assume furthermore that* $g : \mathfrak{X} \to \mathfrak{T}$ *factors through an irreducible component* $\mathcal{Z}_1$ *of* $\mathcal{Z}$ *by a smooth morphism* $g_1 : \mathfrak{X} \to \mathcal{Z}_1$ *over* $\mathfrak{T}$ *such that the composite* $g_1 \circ i_1 : \mathcal{Z}_1 \to \mathcal{Z}_1$ *of the closed immersion* $i_1 : \mathcal{Z}_1 \to \mathfrak{X}$ *and* g_1 *is the identity of* $\mathcal{Z}_1$ *and that the inverse image of the relative strict normal crossings divisor* $\mathcal{Z}'_1 = \cup_{i=2}^s \mathcal{Z}_1 \cap \mathcal{Z}_i$ *of* $\mathcal{Z}_1$ *by* g_1 *is* $\cup_{i=2}^s \mathcal{Z}_i$. *Let* E *be a log-isocrystal on* $U^\#/\mathfrak{T}_K$ *overconvergent along* D. *Then, for any nonnegative integer* m, $g_{1K*}\widetilde{\nabla}_0$ (*resp.* $g_{1K*}(\frac{dz_1}{z_1} \wedge \widetilde{\nabla}_0)$) *in* (1.1.21.1) *induces an integrable logarithmic* $\mathcal{O}_{]T[_{\mathfrak{T}}}$*-connection of the locally free* $j^\dagger_{Z_1\cap U}\mathcal{O}_{]Z_1[_{\mathcal{Z}_1}}$*-module* $g_{1K*}(E(m\mathcal{Z}_1)/E)$ (*resp.* $g_{1K*}(j^\dagger_U\Omega^1_{\mathfrak{X}^\#_K/\mathcal{Z}^\#_{1K}} \otimes_{j^\dagger_U\mathcal{O}_{]X[_{\mathfrak{X}}}} E(m\mathcal{Z}_1)/E)$) *of finite type on* $(\mathcal{Z}_{1K}, \mathcal{Z}'_{1K})/\mathfrak{T}_K$ *that satisfies the overconvergence condition as a log-isocrystal on* $(Z_1\cap U)^\#/\mathfrak{T}_K$ *overconvergent along* $Z_1 \cap D$.

Suppose, furthermore, that $Z_1 \not\subset D$ *and that* E *satisfies conditions* (a) *and* (b) *in* 1.1.1. *Then*

(1.1.22.1)

$$\begin{aligned}&\mathbb{R}g_{1K*}\underline{\Gamma}^{\dagger}_{]Z_1[_{\mathfrak{X}}}(j_U^{\dagger}\Omega^{\bullet}_{\mathfrak{X}^{\#}_K/\mathcal{Z}^{\#}_{1K}} \otimes_{j_U^{\dagger}\mathcal{O}_{]X[_{\mathfrak{X}}}} E)\\ &\quad\cong \left[g_{1K*}(E(m\mathcal{Z}_1)/E) \stackrel{g_{1K*}\nabla}{\longrightarrow} g_{1K*}(j_U^{\dagger}\Omega^1_{\mathfrak{X}^{\#}_K/\mathcal{Z}^{\#}_{1K}} \otimes_{j_U^{\dagger}\mathcal{O}_{]X[_{\mathfrak{X}}}} E(m\mathcal{Z}_1)/E)\right][-1]\end{aligned}$$

and $g_{1K*}(E(m\mathcal{Z}_1)/E)$ (*resp.* $g_{1K*}(j_U^{\dagger}\Omega^1_{\mathfrak{X}^{\#}_K/\mathcal{Z}^{\#}_{1K}} \otimes_{j_U^{\dagger}\mathcal{O}_{]X[_{\mathfrak{X}}}} E(m\mathcal{Z}_1)/E)$) *also satisfies the same conditions* (a) *and* (b) *for any* $m \geq \max\{e \,|\, e$ *is a positive integral exponent of* ∇ *along* $Z_1\} \cup \{0\}$.

Proof. The locally freeness has been already proved at the beginning of the proof of 1.1.4. From the definition of $\widetilde{\nabla}_0$ in (1.1.21.1), it induces an integrable connection. Since $\mathcal{Z}_1$ is a section of $\mathfrak{X}$ over $\mathcal{T}$, one can use on an affinoid open subset of $]Z_1[_{\mathcal{Z}_1}$ a Banach norm induced by a Banach norm on some affinoid open subset of $]X[_{\mathfrak{X}}$. Hence, the logarithmic connections on $g_{1K*}(E(m\mathcal{Z}_1)/E)$ and $g_{1K*}(j_U^{\dagger}\Omega^1_{\mathfrak{X}^{\#}_K/\mathcal{Z}^{\#}_{1K}} \otimes_{j_U^{\dagger}\mathcal{O}_{]X[_{\mathfrak{X}}}} E(m\mathcal{Z}_1)/E)$ satisfy the overconvergence condition. Their exponents along Z_i are m copies of those of E by the definition of $\widetilde{\nabla}_0$ for $i \neq 1$. Therefore, conditions (a) and (b) also hold. □

Examples 1.1.23. Let $\mathfrak{X}$ be the formal projective scheme $\widehat{\mathbb{P}}^1_{\mathcal{V}} \times_{\mathrm{Spf}\,\mathcal{V}} \widehat{\mathbb{P}}^1_{\mathcal{V}}$ over $\mathcal{S} = \mathrm{Spf}\,\mathcal{V}$ with homogeneous coordinates $(x_0, x_1), (y_0, y_1)$, let $\mathcal{Z}_1$ (resp. $\mathcal{Z}_2$) be the divisor defined by $x_1 = 0$ (resp. $y_1 = 0$) in $\mathfrak{X}$, and put $\mathcal{Z} = \mathcal{Z}_1 \cup \mathcal{Z}_2$ and $\mathfrak{X}^{\#} = (\mathfrak{X}, \mathcal{Z})$. Let X (resp. Z, resp. Z_1, resp. Z_2) be the special fiber of $\mathfrak{X}$ (resp. $\mathcal{Z}$, resp. $\mathcal{Z}_1$, resp. $\mathcal{Z}_2$), let D be a closed subscheme of X defined by $x_0 = 0$ or $y_0 = 0$, put $U = X \setminus D$, and let $z_1 = x_1/x_0, z_2 = y_1/y_0$ be the affine coordinates. For integers $e > 0$ and $h \geq 0$, we define a log-isocrystal E on $U^{\#}/\mathcal{S}_K$ of rank 2 overconvergent along D ($E = j_U^{\dagger}\mathcal{O}_{]X[_{\mathfrak{X}}}v_1 \oplus j_U^{\dagger}\mathcal{O}_{]X[_{\mathfrak{X}}}v_2$) by

$$\nabla(v_1, v_2) = (v_1, v_2)\begin{pmatrix} e & z_2^h \\ 0 & e\end{pmatrix}\frac{dz_1}{z_1} + (v_1, v_2)\begin{pmatrix} 0 & 0 \\ 0 & h\end{pmatrix}\frac{dz_2}{z_2}$$

for some strict neighborhood of $]U[_{\mathfrak{X}}$ in $]X[_{\mathfrak{X}}$. Indeed, since the exponents along Z_1 (resp. Z_2) are e and e (resp. 0 and h), the logarithmic connection satisfies the overconvergence condition and is overconvergent along D. Moreover, it satisfies conditions (a) and (b) in 1.1.1. If $g_1 : \mathfrak{X} \to \mathcal{Z}_1$ is the second projection (note that the coordinate of $\mathcal{Z}_1 \cap \mathfrak{U}$ is z_2), then

$$\begin{aligned}&\mathbb{R}g_{1K*}\underline{\Gamma}^{\dagger}_{]Z_1[_{\mathfrak{X}}}(j_U^{\dagger}\Omega^{\bullet}_{\mathfrak{X}^{\#}_K/\mathcal{Z}^{\#}_{1K}} \otimes_{j_U^{\dagger}\mathcal{O}_{]X[_{\mathfrak{X}}}} E)\\ &\quad\cong \left[g_{1K*}(E(m\mathcal{Z}_1)/E) \stackrel{g_{K*}(\frac{dz_1}{z_1}\otimes\partial_{\#1})}{\longrightarrow} g_{1K*}(j_U^{\dagger}\Omega^1_{\mathfrak{X}^{\#}_K/\mathcal{Z}^{\#}_{1K}} \otimes_{j_U^{\dagger}\mathcal{O}_{]X[_{\mathfrak{X}}}} E(m\mathcal{Z}_1)/E)\right][-1]\end{aligned}$$

for $m \geq e$ by 1.1.4. Hence, $\mathbb{R}^q g_{1K*}\underline{\Gamma}^\dagger_{]Z_1[_{\mathfrak{X}}}(j^\dagger_U \Omega^\bullet_{\mathfrak{X}^\#_K/\mathcal{Z}^\#_{1K}} \otimes_{j^\dagger_U \mathcal{O}_{]X[_{\mathfrak{X}}}} E) = 0$ for $q \neq 1, 2$ and

$$\mathbb{R}^q g_{1K*}\underline{\Gamma}^\dagger_{]Z_1[_{\mathfrak{X}}}(j^\dagger_U \Omega^\bullet_{\mathfrak{X}^\#_K/\mathcal{Z}^\#_{1K}} \otimes_{j^\dagger_U \mathcal{O}_{]X[_{\mathfrak{X}}}} E) \cong \begin{cases} j^\dagger_{Z_1\cap U}\mathcal{O}_{]Z_1[_{\mathcal{Z}_1}} z_1^{-e} v_1 & \text{if } q = 1, \\ (j^\dagger_{Z_1\cap U}\mathcal{O}_{]Z_1[_{\mathcal{Z}_1}}/z_2^h j^\dagger_{Z_1\cap U}\mathcal{O}_{]Z_1[_{\mathcal{Z}_1}}) z_1^{-e} v_1 \oplus j^\dagger_{Z_1\cap U}\mathcal{O}_{]Z_1[_{\mathcal{Z}_1}} z_1^{-e} v_2 & \text{if } q = 2. \end{cases}$$

Therefore, $\mathbb{R}^2 g_{1K*}\underline{\Gamma}^\dagger_{]Z_1[_{\mathfrak{X}}}(j^\dagger_U \Omega^\bullet_{\mathfrak{X}^\#_K/\mathcal{Z}^\#_{1K}} \otimes_{j^\dagger_U \mathcal{O}_{]X[_{\mathfrak{X}}}} E)$ is not always locally free. By (1.1.21.2) and using a spectral sequence, the dimensions of total cohomology groups are as follows:

$$\dim_K \mathbb{H}^q(]X[_{\mathfrak{X}}, \underline{\Gamma}^\dagger_{]Z_1[_{\mathfrak{X}}}(j^\dagger_U \Omega^\bullet_{\mathfrak{X}^\#_K/\mathcal{S}_K} \otimes_{j^\dagger_U \mathcal{O}_{]X[_{\mathfrak{X}}}} E)) = \begin{cases} 1 & \text{if } q = 1, \\ 2 & \text{if } q = 2, \\ 1 & \text{if } q = 3, \\ 0 & \text{if } q \neq 1, 2, 3. \end{cases}$$

1.2. *Cohomological operations on arithmetic log-$\mathcal{D}$-modules.* Later, we will need some basic properties of cohomological operations such as direct images and extraordinary inverses images by morphisms of smooth log-formal $\mathcal{V}$-schemes. Here, we follow Berthelot's procedure for the study of arithmetic $\mathcal{D}$-modules. We recall that in order to come down from the case of formal schemes to the case of schemes (the latter case is technically much better), the strategy of Berthelot was to develop a notion of *quasi-coherence* for complexes on formal schemes (see [Ber02]). Below we naturally extend (see 1.2.2 and 1.2.3) Berthelot's notion of quasi-coherence in the case of formal log-schemes. This will allow us, for instance, to check the transitivity of direct images and extraordinary inverse images (see 1.2.6), which is essential for our work.

First, let us fix some notation that we will keep in this section. Let $\mathcal{T}$ be a smooth formal scheme over $\mathcal{V}$, h: $\mathfrak{X}' \to \mathfrak{X}$ be a morphism of smooth formal schemes over $\mathcal{T}$, let $\mathcal{Z}$ (resp. $\mathcal{Z}'$) be a relative strict normal crossings divisor of $\mathfrak{X}$ (resp. $\mathfrak{X}'$) over $\mathcal{T}$ such that $h^{-1}(\mathcal{Z}) \subset \mathcal{Z}'$, let D (resp. D') be a divisor of X (resp. X') such that $h^{-1}(D) \subset D'$. We denote by $U := X \setminus D$, $\mathfrak{X}^\# := (\mathfrak{X}, \mathcal{Z})$, $\mathfrak{X}'^\# := (\mathfrak{X}', \mathcal{Z}')$, u: $\mathfrak{X}^\# \to \mathfrak{X}$, $g^\#$: $\mathfrak{X}^\# \to \mathcal{T}$ the canonical morphisms, and $h^\#$: $\mathfrak{X}'^\# \to \mathfrak{X}^\#$ the induced morphism of smooth formal log-schemes over $\mathcal{T}$. We denote by $h_i^\#$: $X_i'^\# \to X_i^\#$ the reduction of $h^\#$ modulo π^{i+1}. Berthelot has constructed in [Ber96b, 4.2.3] the $\mathcal{O}_{X_i}$-algebra $\mathcal{B}^{(m)}_{X_i}(D)$ that is endowed with a compatible structure of left $\mathcal{D}^{(m)}_{X_i^\#}$-module. We recall that when $f \in \mathcal{O}_{X_i}$ is a lifting of an equation of D in X, then $\mathcal{B}^{(m)}_{X_i}(D) = \mathcal{O}_{X_i}[T]/(f^{p^{m+1}}T - p)$. By abuse of notation, we pose $\mathcal{D}^{(m)}_{X_i^\#}(D) := \mathcal{B}^{(m)}_{X_i}(D) \otimes_{\mathcal{O}_{X_i}} \mathcal{D}^{(m)}_{X_i^\#}$, $\mathcal{D}^{(m)}_{X_i'^\#}(D') := \mathcal{B}^{(m)}_{X_i'}(D') \otimes_{\mathcal{O}_{X_i'}} \mathcal{D}^{(m)}_{X_i'^\#}$. For any $\mathcal{O}_{X_i}$-module $\mathcal{M}_i$, we pose

$\mathcal{M}_i(Z_i) := \mathcal{O}_{X_i}(Z_i) \otimes_{\mathcal{O}_{X_i}} \mathcal{M}_i$, where $\mathcal{O}_{X_i}(Z_i) := \mathcal{H}om_{\mathcal{O}_{X_i}}(\omega_{X_i}, \omega_{X_i^\#})$. When $\mathcal{M}_i$ is even a $\mathcal{D}^{(m)}_{X_i^\#}(D)$-module, then $\mathcal{M}_i(Z_i)$ has a canonical structure of $\mathcal{D}^{(m)}_{X_i^\#}(D)$-module (see [Car09a, 5.1]).

We check by functoriality that the sheaf $\mathcal{B}^{(m)}_{X'_i}(D') \otimes_{\mathcal{O}_{X'_i}} h_i^*(\mathcal{D}^{(m)}_{X_i^\#})$ is a $(\mathcal{D}^{(m)}_{X_i'^\#}(D'), h_i^{-1}\mathcal{D}^{(m)}_{X_i^\#}(D))$-bimodule, which will be denoted by $\mathcal{D}^{(m)}_{X_i'^\# \to X_i^\#}(D', D)$. Also, we get a $(h_i^{-1}\mathcal{D}^{(m)}_{X_i^\#}(D), \mathcal{D}^{(m)}_{X_i'^\#}(D'))$-bimodule with $\mathcal{D}^{(m)}_{X_i^\# \leftarrow X_i'^\#}(D, D') := \mathcal{B}^{(m)}_{X'_i}(D') \otimes_{\mathcal{O}_{X_i}} (\omega_{X_i'^\#} \otimes_{\mathcal{O}_{X'_i}} h_i^{*l}(\mathcal{D}^{(m)}_{X_i^\#} \otimes_{\mathcal{O}_{X_i}} \omega^{-1}_{X_i^\#})$, where the symbol '$l$' means that to compute the inverse image by h_i we choose the left structure of left $\mathcal{D}^{(m)}_{X_i^\#}$-module of $\mathcal{D}^{(m)}_{X_i^\#} \otimes_{\mathcal{O}_{X_i}} \omega^{-1}_{X_i^\#}$.

Before proceeding, let us state the following lemma needed to define the local cohomological functor with support in a closed subscheme (see 1.2.5).

LEMMA 1.2.1. *Let $\mathcal{E}$ be a $\mathcal{D}^{(m)}_{X_i}$-module and $\mathcal{F}$ be a $\mathcal{D}^{(m)}_{X_i^\#}$-module. Then $\mathcal{H}om_{\mathcal{O}_{X_i}}(\mathcal{E}, \mathcal{F})$ is endowed with a unique structure of $\mathcal{D}^{(m)}_{X_i^\#}$-module such that, for any morphism ϕ of $\mathcal{H}om_{\mathcal{O}_{X_i}}(\mathcal{E}, \mathcal{F})$, for any section x on $\mathcal{E}$, we have*

$$(\underline{\partial}^{\langle \underline{k} \rangle}_\# \cdot \phi)(x) = \sum_{\underline{h} \leq \underline{k}} (-1)^{|\underline{h}|} \left\{ \begin{smallmatrix} \underline{k} \\ \underline{h} \end{smallmatrix} \right\} \underline{t}^{\underline{h}} \underline{\partial}^{\langle \underline{k} - \underline{h} \rangle}_\# \cdot (\phi(\underline{\partial}^{<\underline{h}>} \cdot x)). \tag{1.2.1.1}$$

Proof. We denote by $\mathcal{P}^n_{X_i^\#, (m)}$ the m-PD-envelop of order n of the diagonal immersion of $X_i^\#$ and denote by $d^n_{1*}\mathcal{P}^n_{X_i^\#, (m)}$ (resp. $d^n_{2*}\mathcal{P}^n_{X_i^\#, (m)}$) the induced $\mathcal{O}_{X_i}$-algebra for the left (resp. right) structure. Using the isomorphisms

$$\mathcal{H}om_{\mathcal{O}_{X_i}}(\mathcal{E}, \mathcal{F}) \otimes_{\mathcal{O}_{X_i}} d^n_{i*}\mathcal{P}^n_{X_i^\#, (m)} \xrightarrow{\sim} \mathcal{H}om_{\mathcal{P}^n_{X_i, (m)}}(\mathcal{E} \otimes_{\mathcal{O}_{X_i}} d^n_{i*}\mathcal{P}^n_{X_i, (m)}, \mathcal{F} \otimes_{\mathcal{O}_{X_i}} d^n_{i*}\mathcal{P}^n_{X_i^\#, (m)}),$$

we pose $\varepsilon_n^{\mathcal{H}om_{\mathcal{O}_{X_i}}(\mathcal{E}, \mathcal{F})} := \mathcal{H}om_{\mathcal{P}^n_{X_i, (m)}}((\varepsilon_n^{\mathcal{E}})^{-1}, \varepsilon_n^{\mathcal{F}})$, where $\varepsilon_n^{\mathcal{E}}$ is the m-PD-stratification of $\mathcal{E}$ with respect to $X_i^\#/S_i$ corresponding to its structure of $\mathcal{D}^{(m)}_{X_i}$-module and $\varepsilon_n^{\mathcal{F}}$ is the m-PD-stratification of $\mathcal{F}$ with respect to $X_i^\#/S_i$ corresponding to its structure of $\mathcal{D}^{(m)}_{X_i^\#}$-module (see [Car09a, 1.8])

To compute $(\varepsilon_n^{\mathcal{E}})^{-1}$ and $\varepsilon_n^{\mathcal{F}}$, we use respectively [Ber96b, 2.3.2.3] (note that this formula is not any more true with logarithmic structure) and [Car09a, 1.8.1]. □

1.2.2 (Quasi-coherence, step I). Let $\mathcal{B}$ be a sheaf of $\mathcal{O}_{\mathfrak{X}}$-algebras, $\mathcal{E} \in D^-(\mathcal{B}^{\mathrm{r}})$, $\mathcal{F} \in D^-({}^{\mathrm{l}}\mathcal{B})$; i.e., $\mathcal{E}$ (resp. $\mathcal{F}$) is a bounded above complex of right (resp. left) $\mathcal{D}$-modules. We pose $\mathcal{B}_i := \mathcal{B}/\pi^{i+1}\mathcal{B}$, $\mathcal{E}_i := \mathcal{E} \otimes^{\mathbb{L}}_{\mathcal{B}} \mathcal{B}_i$, $\mathcal{F}_i := \mathcal{B}_i \otimes^{\mathbb{L}}_{\mathcal{B}} \mathcal{F}$, $\mathcal{E} \widehat{\otimes}^{\mathbb{L}}_{\mathcal{B}} \mathcal{F} := \mathbb{R}\varprojlim_i \mathcal{E}_i \otimes^{\mathbb{L}}_{\mathcal{B}_i} \mathcal{F}_i$.

• We say that $\mathcal{E}$ (resp. $\mathcal{F}$) is $\mathcal{B}$-*quasi-coherent* if $\mathcal{E}_0 \in D^-_{\mathrm{qc}}(\mathcal{B}_0{}^{\mathrm{r}})$ (resp. $\mathcal{F}_0 \in D^-_{\mathrm{qc}}({}^{\mathrm{l}}\mathcal{B}_0)$) and if the canonical morphism $\mathcal{E} \to \mathcal{E}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{B}}\mathcal{B}$ (resp. $\mathcal{F} \to \mathcal{B}\widehat{\otimes}^{\mathbb{L}}_{\mathcal{B}}\mathcal{F}$) is an isomorphism. We denote by $D^-_{\mathrm{qc}}({}^*\mathcal{B})$ (resp. $D^{\mathrm{b}}_{\mathrm{qc}}({}^*\mathcal{B})$) the full subcategory of quasi-coherent complexes of $D^-({}^*\mathcal{B})$ (resp. $D^{\mathrm{b}}({}^*\mathcal{B})$), where '$*$' is either 'r' or 'l'.

• We pose $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\#}(D) := \varprojlim_i \mathcal{D}^{(m)}_{X_i^\#}(D)$. Since $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\#}(D)$ is a flat $\widehat{\mathcal{B}}^{(m)}_{\mathfrak{X}}(D)$-module (for the right or the left structures), a complex of $D^*({}^*\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\#}(D))$ is $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\#}(D)$-quasi-coherent (and in particular when $\mathfrak{X}^\#$ is replaced by $\mathfrak{X}$) if and only if it is $\widehat{\mathcal{B}}^{(m)}_{\mathfrak{X}}(D)$-quasi-coherent. Then, the forgetful functor $D^*({}^*\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}}(D)) \to D^*({}^*\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\#}(D))$ induces $D^*_{\mathrm{qc}}({}^*\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}}(D)) \to D^*_{\mathrm{qc}}({}^*\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\#}(D))$. Also, it follows from [Ber96b, 4.3.3(i)] that $\widehat{\mathcal{B}}^{(m)}_{\mathfrak{X}}(D) \otimes^{\mathbb{L}}_{\mathcal{V}} \mathcal{V}/\pi^{i+1} \xrightarrow{\sim} \widehat{\mathcal{B}}^{(m)}_{\mathfrak{X}}(D) \otimes_{\mathcal{V}} \mathcal{V}/\pi^{i+1} \xrightarrow{\sim} \mathcal{B}^{(m)}_{X_i}(D)$. Hence, a complex of $D^*({}^*\widehat{\mathcal{B}}^{(m)}_{\mathfrak{X}}(D))$ is $\widehat{\mathcal{B}}^{(m)}_{\mathfrak{X}}(D)$-quasi-coherent if and only if it is $\mathcal{O}_{\mathfrak{X}}$-quasi-coherent, if and only if it is $\mathcal{V}$-quasi-coherent.

• We get a $(\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'^\#}(D'), h^{-1}\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\#}(D))$-bimodule by posing $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'^\# \to \mathfrak{X}^\#}(D', D) := \varprojlim_i \mathcal{D}^{(m)}_{X_i'^\# \to X_i^\#}(D', D)$. Also, we have the $(h^{-1}\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\#}(D), \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'^\#}(D'))$-bimodule $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\# \leftarrow \mathfrak{X}'^\#}(D, D') := \varprojlim_i \mathcal{D}^{(m)}_{X_i^\# \leftarrow X_i'^\#}(D, D')$.

1.2.3 (Quasi-coherence, step II). Let $\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\#}(D) := (\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\#}(D))_{m\in\mathbb{N}}$ be the canonical inductive system. Localizing twice $D^{\mathrm{b}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\#}(D))$ (these localizations replace respectively the functor $-\otimes_{\mathbb{Z}}\mathbb{Q}$ and the inductive limit on the level m), we construct similarly to [Ber02, 4.2.1, 4.2.2] and [Car06b, 1.1.3] a category denoted by $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\#}(D))$. Let $\mathcal{E}^{(\bullet)} = (\mathcal{E}^{(m)})_{m\in\mathbb{N}} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\#}(D))$. As for [Ber02, 4.2.3] and [Car06b, 1.1.3], we say that $\mathcal{E}^{(\bullet)}$ is quasi-coherent if for any m, $\mathcal{E}^{(m)}$ is $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\#}(D)$-quasi-coherent. We denote the subcategory of quasi-coherent sheaves by $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\#}(D))$. With the second point of 1.2.2, we check that the canonical functor $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}}(D)) \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\#}(D))$ induces the following one: $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}}(D)) \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\#}(D))$.

1.2.4 (Extraordinary inverse image, direct image, tensor product). Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\#}(D))$, $\mathcal{E}'^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\#}(D'))$. The following functors extend those which were already defined without log-structure.

• The extraordinary inverse image of $\mathcal{E}^{(\bullet)}$ by $h^\#$ is defined as follows:

$$h^{\#!}_{D',D}(\mathcal{E}^{(\bullet)}) := (\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'^\# \to \mathfrak{X}^\#}(D', D)\widehat{\otimes}^{\mathbb{L}}_{h^{-1}\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\#}(D)} h^{-1}\mathcal{E}^{(m)}[d_{X'/X}])_{m\in\mathbb{N}} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\#}(D')). \tag{1.2.4.1}$$

- The direct image by $h^{\#}$ of $\mathcal{E}'^{(\bullet)}$ is defined as follows:

(1.2.4.2)
$$h^{\#}_{D,D'+}(\mathcal{E}'^{(\bullet)}) := (\mathbb{R}h_*(\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^{\#}\leftarrow\mathfrak{X}'^{\#}}(D,D')\widehat{\otimes}^{\mathbb{L}}_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'^{\#}}(D')}\mathcal{E}'^{(m)}))_{m\in\mathbb{N}} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\#}}(D)).$$

- Let $\widetilde{D}$ be a divisor of X containing D. We pose

(1.2.4.3)
$$({}^{\dagger}\widetilde{D},D)(\mathcal{E}^{(\bullet)}) := (\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^{\#}}(\widetilde{D})\widehat{\otimes}^{\mathbb{L}}_{\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^{\#}}(D)}\mathcal{E}^{(m)})_{m\in\mathbb{N}} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\#}}(\widetilde{D})).$$

  We denote by $\mathrm{Forg}_{D,\widetilde{D}}$: $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\#}}(\widetilde{D})) \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\#}}(D))$ the forgetful functor.
- When D or D' are empty, we remove them in the notation. Also, when $D' = h^{-1}(D)$, we remove D' in the notation.

Using the remark [Ber96b, 2.3.5(iii)], we get the isomorphism in the category $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\#}}(\widetilde{D}))$:

(1.2.4.4)
$$\mathcal{O}_{\mathfrak{X}}({}^{\dagger}\widetilde{D})_{\mathbb{Q}}\overset{\mathbb{L}}{\otimes}{}^{\dagger}_{\mathcal{O}_{\mathfrak{X}}({}^{\dagger}D)_{\mathbb{Q}}}\mathcal{E}^{(\bullet)} := (\widehat{\mathcal{B}}^{(m)}_{\mathfrak{X}}(\widetilde{D})\widehat{\otimes}^{\mathbb{L}}_{\widehat{\mathcal{B}}^{(m)}_{\mathfrak{X}}(D)}\mathcal{E}^{(m)})_{m\in\mathbb{N}} \xrightarrow{\sim} ({}^{\dagger}\widetilde{D},D)(\mathcal{E}^{(\bullet)}).$$

Since a flat $\mathcal{D}^{(m)}_{X_i^{\#}}$-module (resp. a flat $\mathcal{D}^{(m)}_{X_i}$-module) is also a flat $\mathcal{O}^{(m)}_{X_i}$-module, we check that the functor $({}^{\dagger}\widetilde{D},D)$ commutes with the forgetful functor

$$\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}}(D)) \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\#}}(D)).$$

Hence, by [Car06b, 1.1.8] and the associativity of tensor products, we deduce from (1.2.4.4) that we have a canonical isomorphism $({}^{\dagger}\widetilde{D},D) \xrightarrow{\sim} ({}^{\dagger}\widetilde{D})\circ\mathrm{Forg}_D$. Similarly, if D_1 and D_2 are two divisors of X, then $({}^{\dagger}D_1)\circ({}^{\dagger}D_2) \xrightarrow{\sim} ({}^{\dagger}D_1\cup D_2)$ (we have omitted the forgetful functor). Then we notice that $({}^{\dagger}D_1)$ and $({}^{\dagger}D_1\cup D_2)$ are canonically isomorphic on $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\#}}(D_2))$.

1.2.5 (Local cohomological functor with support in a closed subscheme). Let $\widetilde{X}$ be a closed subscheme of X, $\mathcal{E}^{(\bullet)},\mathcal{F}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\#}}(D))$. Let $\mathfrak{I}_i$ be the ideal of $\mathcal{O}_{X_i}$ defined by $\widetilde{X}\subset X_i$, $\mathcal{P}_{(m)}(\mathfrak{I}_i)$ the m-PD-envelop of $\mathfrak{I}_i$ (resp. $\mathcal{P}^n_{(m)}(\mathfrak{I}_i)$ the m-PD-envelop of order n of $\mathfrak{I}_i$), and $\overline{\mathfrak{I}}_i^{\{n\}_{(m)}}$ its m-PD filtration (see [Ber96b, 1.3–4]). From [Ber02, 4.4.4], $\mathcal{P}_{(m)}(\mathfrak{I}_i)$ is a $\mathcal{D}^{(m)}_{X_i}$-module such that, for any integers n and n', for any $P\in\mathcal{D}^{(m)}_{X_i,n}$, $x\in\overline{\mathfrak{I}}_i^{\{n'\}_{(m)}}$, we have $P\cdot x\in\overline{\mathfrak{I}}_i^{\{n'-n\}_{(m)}}$. With formula (1.2.1.1), this implies that the sub-sheaf

$$\underline{\Gamma}^{(m)}_{\widetilde{X}}(\mathcal{E}_i) := \varinjlim_n \mathcal{H}om_{\mathcal{O}_{X_i}}(\mathcal{P}^n_{(m)}(\mathfrak{I}_i),\mathcal{E}_i)$$

of $\mathcal{H}om_{\mathcal{O}_{X_i}}(\mathcal{P}_{(m)}(\mathcal{I}_i), \mathcal{E}_i)$ has an induced structure of $\mathcal{D}^{(m)}_{X_i^\#}$-module. We get a functor $\mathbb{R}\underline{\Gamma}^{(m)}_{\widetilde{X}} : D^+(\mathcal{D}^{(m)}_{X_i^\#}) \to D^+(\mathcal{D}^{(m)}_{X_i^\#})$, which is computed using a resolution by injective $\mathcal{D}^{(m)}_{X_i^\#}$-modules. When $\mathcal{Z}$ is empty (i.e., without log-poles), we retrieve the usual local cohomological functor (e.g., see [Ber02, 4.4.4] or [Car04, 1.1.3]). Since $\mathcal{D}^{(m)}_{X_i^\#}$ is flat as an $\mathcal{O}_{X_i}$-module, we notice that an injective $\mathcal{D}^{(m)}_{X_i^\#}$-module (resp. an injective $\mathcal{D}^{(m)}_{X_i}$-module) is also an injective $\mathcal{O}^{(m)}_{X_i}$-module. Then, this functor $\mathbb{R}\underline{\Gamma}^{(m)}_{\widetilde{X}}$ commutes with the forgetful functor $D^+(\mathcal{D}^{(m)}_{X_i}) \to D^+(\mathcal{D}^{(m)}_{X_i^\#})$.

Then we construct $\mathbb{R}\underline{\Gamma}^{\dagger}_{\widetilde{X}}\colon \underset{\longrightarrow}{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\#}) \to \underset{\longrightarrow}{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\#})$, the local cohomology with strict compact support in $\widetilde{X}$, similarly to [Car04, 2.1–2]. Also, as for [Car04, 2.2.6.1], we have the canonical isomorphism

$$\mathbb{R}\underline{\Gamma}^{\dagger}_{\widetilde{X}}(\mathcal{E}^{(\bullet)}) \overset{\mathbb{L}}{\otimes}{}^{\dagger}_{\mathcal{O}_{\mathfrak{X},\mathbb{Q}}} \mathcal{F}^{(\bullet)} \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^{\dagger}_{\widetilde{X}}(\mathcal{E}^{(\bullet)} \overset{\mathbb{L}}{\otimes}{}^{\dagger}_{\mathcal{O}_{\mathfrak{X},\mathbb{Q}}} \mathcal{F}^{(\bullet)}). \tag{1.2.5.1}$$

Finally, since it is known (e.g., see [Car04, 2.2.1]) when $\mathcal{E}^{(\bullet)} = \mathcal{O}^{(\bullet)}_{\mathfrak{X}}$ (in the category $\underset{\longrightarrow}{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}})$ and then in $\underset{\longrightarrow}{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\#})$ via the forgetful functor) for any divisor $\widetilde{X}$ of X, we get from (1.2.5.1) and (1.2.4.4) the exact triangle of localization of $\mathcal{E}^{(\bullet)}$ with respect to $\widetilde{X}$ as follows:

$$\mathbb{R}\underline{\Gamma}^{\dagger}_{\widetilde{X}}(\mathcal{E}^{(\bullet)}) \to \mathcal{E}^{(\bullet)} \to ({}^\dagger \widetilde{X})(\mathcal{E}^{(\bullet)}) \to \mathbb{R}\underline{\Gamma}^{\dagger}_{\widetilde{X}}(\mathcal{E}^{(\bullet)})[1]. \tag{1.2.5.2}$$

Similarly, we deduce from (1.2.5.1) that the usual rules of composition of local cohomological functors and Mayer-Vietoris exact triangles holds (more precisely, see [Car04, 2.2.8, 2.2.16]).

1.2.6 (Transitivity). Let $h'\colon \mathfrak{X}'' \to \mathfrak{X}'$ be a second morphism of smooth formal schemes over $\mathcal{T}$, let $\mathcal{Z}''$ be a relative strict normal crossings divisor of $\mathfrak{X}''$ over $\mathcal{T}$ such that $h'^{-1}(\mathcal{Z}') \subset \mathcal{Z}''$, and let D'' be a divisor of X'' such that $h'^{-1}(D') \subset D''$. We denote by $\mathfrak{X}''^\# := (\mathfrak{X}'', \mathcal{Z}'')$ and $h'^\#\colon \mathfrak{X}''^\# \to \mathfrak{X}'^\#$ the induced morphism of smooth formal log-schemes over $\mathcal{T}$.

Then, we have the isomorphisms of functors

$$h^{\#}_{D,D'+} \circ h'^{\#}_{D',D''+} \xrightarrow{\sim} (h^{\#} \circ h'^{\#})_{D,D''+}, \tag{1.2.6.1}$$

$$h'^{\#!}_{D'',D'} \circ h^{\#!}_{D',D} \xrightarrow{\sim} (h^{\#} \circ h'^{\#})^{!}_{D'',D}. \tag{1.2.6.2}$$

Indeed, thanks to Berthelot's notion of quasi-coherence, we come down to the case of log-schemes, which is classical.

1.2.7. Similarly to [Car06b, 1.1.9], we check the canonical isomorphisms of functors

$$\text{(1.2.7.1)}\quad \mathrm{Forg}_D \circ h^{\#}_{D,D'+} \xrightarrow{\sim} h^{\#}_{+} \circ \mathrm{Forg}_{D'}, \quad ({}^\dagger D') \circ h^{\#!} \xrightarrow{\sim} h^{\#!}_{D',D} \circ ({}^\dagger D).$$

1.2.8 (Coherence and quasi-coherence). Let $\mathcal{D}^\dagger_{\mathfrak{X}^\#}({}^\dagger D)_{\mathbb{Q}} := \varinjlim_m \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\#}(D)_{\mathbb{Q}}$.

We get a $(\mathcal{D}^\dagger_{\mathfrak{X}'^\#}({}^\dagger D')_{\mathbb{Q}},\, h^{-1}\mathcal{D}^\dagger_{\mathfrak{X}^\#}({}^\dagger D)_{\mathbb{Q}})$-bimodule with

$$\mathcal{D}^\dagger_{\mathfrak{X}'^\# \to \mathfrak{X}^\#}({}^\dagger D', D)_{\mathbb{Q}} := \varinjlim_m \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}'^\# \to \mathfrak{X}^\#}(D', D)_{\mathbb{Q}}.$$

We get a $(h^{-1}\mathcal{D}^\dagger_{\mathfrak{X}^\#}({}^\dagger D)_{\mathbb{Q}},\, \mathcal{D}^\dagger_{\mathfrak{X}'^\#}({}^\dagger D')_{\mathbb{Q}})$-bimodule with

$$\mathcal{D}^\dagger_{\mathfrak{X}^\# \leftarrow \mathfrak{X}'^\#}({}^\dagger D, D')_{\mathbb{Q}} := \varinjlim_m \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^\# \leftarrow \mathfrak{X}'^\#}(D, D')_{\mathbb{Q}}.$$

We have also the canonical functor $\varinjlim\colon \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\#}(D)) \to D(\mathcal{D}^\dagger_{\mathfrak{X}^\#}({}^\dagger D)_{\mathbb{Q}})$ (see [Ber02, 4.2.2]). Remark that by abuse of notation this functor is in fact the composition of the inductive limit on the level with the functor $-\otimes_{\mathbb{Z}} \mathbb{Q}$. This functor $\varinjlim$ induces an equivalence of categories between a subcategory of $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\#}(D))$, denoted by $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\#}(D))$, and $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}^\#}({}^\dagger D)_{\mathbb{Q}})$ (similarly to [Ber02, 4.2.4]). Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\#}(D)), \mathcal{E}'^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'^\#}(D'))$. We denote by $\mathcal{E} := \varinjlim \mathcal{E}^{(\bullet)}$, $\mathcal{E}' := \varinjlim \mathcal{E}'^{(\bullet)}$. Then we get

$$\text{(1.2.8.1)}\quad \varinjlim \circ h^{\#!}_{D',D}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \mathcal{D}^\dagger_{\mathfrak{X}'^\# \to \mathfrak{X}^\#}({}^\dagger D', D)_{\mathbb{Q}} \otimes^{\mathbb{L}}_{h^{-1}\mathcal{D}^\dagger_{\mathfrak{X}^\#}({}^\dagger D)_{\mathbb{Q}}} h^{-1}\mathcal{E}[d_{X'/X}] =: h^{\#!}_{D',D}(\mathcal{E}),$$

$$\text{(1.2.8.2)}\quad \varinjlim \circ h^{\#}_{D,D'+}(\mathcal{E}'^{(\bullet)}) \xrightarrow{\sim} \mathbb{R}h_*(\mathcal{D}^\dagger_{\mathfrak{X}^\# \leftarrow \mathfrak{X}'^\#}({}^\dagger D, D')_{\mathbb{Q}} \otimes^{\mathbb{L}}_{\mathcal{D}^\dagger_{\mathfrak{X}'^\#}({}^\dagger D')_{\mathbb{Q}}} \mathcal{E}') =: h^{\#}_{D,D'+}(\mathcal{E}'),$$

$$\text{(1.2.8.3)}\quad \varinjlim \circ ({}^\dagger \widetilde{D}, D)(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \mathcal{D}^\dagger_{\mathfrak{X}^\#}({}^\dagger \widetilde{D})_{\mathbb{Q}} \otimes_{\mathcal{D}^\dagger_{\mathfrak{X}^\#}({}^\dagger D)_{\mathbb{Q}}} \mathcal{E} =: ({}^\dagger \widetilde{D}, D)(\mathcal{E}).$$

In the last isomorphism, we have removed the symbol "$\mathbb{L}$" since the extension $\mathcal{D}^\dagger_{\mathfrak{X}^\#}({}^\dagger D)_{\mathbb{Q}} \to \mathcal{D}^\dagger_{\mathfrak{X}^\#}({}^\dagger \widetilde{D})_{\mathbb{Q}}$ is flat. (This a consequence of [Car09a, 4.7].) Also, we can write $\mathcal{E}({}^\dagger \widetilde{D}, D) := ({}^\dagger \widetilde{D}, D)(\mathcal{E})$.

We pose $\mathcal{O}_{\mathfrak{X}}(\mathcal{Z}) := \mathcal{H}om_{\mathcal{O}_{\mathfrak{X}}}(\omega_{\mathfrak{X}}, \omega_{\mathfrak{X}^\#})$ and $\mathcal{E}(\mathcal{Z}) = \mathcal{O}_{\mathfrak{X}}(\mathcal{Z}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E}$. This functor $(-)(\mathcal{Z})$ preserves $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}^\#}({}^\dagger D)_{\mathbb{Q}})$ (see [Car09a, 5.1]). Moreover, because this is true when $\mathcal{E} = \mathcal{D}^\dagger_{\mathfrak{X}^\#}({}^\dagger D)_{\mathbb{Q}}$, we check by functoriality the isomorphism in $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}^\#}({}^\dagger D)_{\mathbb{Q}})$:

$$\text{(1.2.8.4)}\qquad \mathcal{E}(\mathcal{Z})({}^\dagger D) \xrightarrow{\sim} \mathcal{E}({}^\dagger D)(\mathcal{Z}).$$

Also, when $Z \subset D$, we compute $\mathcal{E}({}^\dagger D) \xrightarrow{\sim} \mathcal{E}(\mathcal{Z})({}^\dagger D)$.

1.2.9. Let $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{X}^{\#},\mathbb{Q}})$. The $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\#},\mathbb{Q}}$-linear dual of $\mathcal{E}$ is well defined as follows (see [Car09a, 5.6]):

$$\mathbb{D}_{\mathfrak{X}^{\#}}(\mathcal{E}) = \mathbb{R}\mathcal{H}om_{\mathcal{D}^{\dagger}_{\mathfrak{X}^{\#},\mathbb{Q}}}(\mathcal{E}, \mathcal{D}^{\dagger}_{\mathfrak{X}^{\#},\mathbb{Q}}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega^{-1}_{\mathfrak{X}^{\#}}[d_X]. \tag{1.2.9.1}$$

1.2.10 (Direct image by a log-smooth morphism). We suppose here that $h^{\#}$ is log-smooth. Then, as for [Ber02, 4.2.1.1], we have the canonical quasi-isomorphism $\Omega^{\bullet}_{\mathfrak{X}'^{\#}/\mathfrak{X}^{\#},\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X}',\mathbb{Q}}} \mathcal{D}^{\dagger}_{\mathfrak{X}'^{\#},\mathbb{Q}}[d_{\mathfrak{X}'^{\#}/\mathfrak{X}^{\#}}] \xrightarrow{\sim} \mathcal{D}^{\dagger}_{\mathfrak{X}^{\#}\leftarrow\mathfrak{X}'^{\#},\mathbb{Q}}$. This implies $\Omega^{\bullet}_{\mathfrak{X}'^{\#}/\mathfrak{X}^{\#},\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X}',\mathbb{Q}}} \mathcal{D}^{\dagger}_{\mathfrak{X}'^{\#}}({}^{\dagger}D')_{\mathbb{Q}}[d_{\mathfrak{X}'^{\#}/\mathfrak{X}^{\#}}] \xrightarrow{\sim} \mathcal{D}^{\dagger}_{\mathfrak{X}^{\#}\leftarrow\mathfrak{X}'^{\#}}({}^{\dagger}D, D')_{\mathbb{Q}}$. Then, for any $\mathcal{E}' \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{X}'^{\#}}({}^{\dagger}D')_{\mathbb{Q}})$,

$$\begin{aligned} h^{\#}_{D,D'+}(\mathcal{E}') &:= \mathbb{R}h_*(\mathcal{D}^{\dagger}_{\mathfrak{X}^{\#}\leftarrow\mathfrak{X}'^{\#}}({}^{\dagger}D, D')_{\mathbb{Q}} \otimes^{\mathbb{L}}_{\mathcal{D}^{\dagger}_{\mathfrak{X}'^{\#}}({}^{\dagger}D')_{\mathbb{Q}}} \mathcal{E}') \\ &\xrightarrow{\sim} \mathbb{R}h_*(\Omega^{\bullet}_{\mathfrak{X}'^{\#}/\mathfrak{X}^{\#},\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X}',\mathbb{Q}}} \mathcal{E}')[d_{\mathfrak{X}'^{\#}/\mathfrak{X}^{\#}}]. \end{aligned} \tag{1.2.10.1}$$

1.3. *Interpretation of the comparison theorem with arithmetic log-$\mathcal{D}$-modules.* We keep the notation of 1.2. First, in this section we give the following interpretation of convergent (F-)log-isocrystals on (X, Z) over $\mathcal{S}$. Moreover, we translate Theorem 1.1.1 and finally Proposition 1.1.22, which will be respectively fundamental for Sections 2.2 and 2.3.

PROPOSITION 1.3.1. (1) *The functors* sp^* *and* sp_* *induce quasi-inverse equivalences between the category of coherent* $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\#}}({}^{\dagger}D)_{\mathbb{Q}}$*-modules, locally projective of finite type over* $\mathcal{O}_{\mathfrak{X}}({}^{\dagger}D)_{\mathbb{Q}}$ *and the category of locally free* $j^{\dagger}_U\mathcal{O}_{]X[_{\mathfrak{X}}}$*-modules of finite type with an integrable logarithmic connection* $\nabla : E \to j^{\dagger}_U\Omega^1_{\mathfrak{X}^{\#}_K/\mathcal{S}_K} \otimes_{j^{\dagger}_U\mathcal{O}_{]X[_{\mathfrak{X}}}} E$ *satisfying the overconvergence condition of* (1.1.0.2).

(2) *Denote by* $I_{\mathrm{conv,et}}((X,Z)/\mathrm{Spf}\,\mathcal{V})$ *the category of convergent log-isocrystals on* (X,Z) *over* $\mathcal{S}$ *in the sense of Shiho* (*see* [Shi02, 2.1.5, 2.1.6] *and* [Shi00]). *There exists an equivalence between* $I_{\mathrm{conv,et}}((X,Z)/\mathrm{Spf}\,\mathcal{V})$ *and the category of coherent* $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\#},\mathbb{Q}}$*-modules, locally projective of finite type over* $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}$.

Proof. We check the first equivalence of categories similarly to [Ber96b, 4.4.12] (see also [Car09a, 4.19]). We deduce the next one by Kedlaya's theorem [Ked07, 6.4.1] (see also his definition [Ked07, 2.3.7]). □

Remarks 1.3.2. • With the notation 1.3.1, since D is a divisor, for any locally free $j^{\dagger}_U\mathcal{O}_{]X[_{\mathfrak{X}}}$-module E of finite type, for any integer $j \neq 0$, $\mathcal{H}^j\mathbb{R}\mathrm{sp}_*(E) = 0$.

• Moreover, it follows from 1.3.1(1) that for any coherent $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\#}}({}^{\dagger}D)_{\mathbb{Q}}$-module, locally projective of finite type over $\mathcal{O}_{\mathfrak{X}}({}^{\dagger}D)_{\mathbb{Q}}$, $E := \mathrm{sp}^*(\mathcal{E})$ is a locally free $j^{\dagger}_U\mathcal{O}_{]X[_{\mathfrak{X}}}$-module of finite type with a logarithmic connection $\nabla : E \to$

$j_U^\dagger \Omega^1_{\mathfrak{X}^\#_K/\mathcal{T}_K} \otimes_{j_U^\dagger \mathcal{O}_{]X[_\mathfrak{X}}} E$ satisfying the overconvergence condition of (1.1.0.2). Of course, the converse is not true unless $\mathcal{T}=\mathcal{S}$.

1.3.3 (Inverse image). Let $\mathcal{V} \to \mathcal{V}'$ be a morphism of mixed characteristic complete discrete valuation rings, $k \to k'$ be the induced morphism of perfect residue fields, $\mathfrak{X}$ be a smooth formal $\mathcal{V}$-scheme, $\mathfrak{X}'$ be a smooth formal $\mathcal{V}'$-scheme, and $\mathcal{Z}$ (resp. $\mathcal{Z}'$) be a relative strict normal crossings divisor of $\mathfrak{X}$ over $\operatorname{Spf}\mathcal{V}$ (resp. $\mathfrak{X}'$ over $\operatorname{Spf}\mathcal{V}'$). Let f_0: $(X', Z') \to (X, Z)$ be a morphism of log-schemes over $\operatorname{Spec} k$. We have a canonical inverse image functor under f_0 denoted by f_0^*: $I_{\mathrm{conv,et}}((X,Z)/\operatorname{Spf}\mathcal{V}) \to I_{\mathrm{conv,et}}((X',Z')/\operatorname{Spf}\mathcal{V}')$. (This is obvious from the definition [Shi02, 2.1.5, 2.1.6].) We get from 1.3.1(2) an inverse image functor under f_0, also denoted by f_0^*, from the category of coherent $\mathcal{D}^\dagger_{(\mathfrak{X},\mathcal{Z}),\mathbb{Q}}$-modules, locally projective of finite type over $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}$ to the category of coherent $\mathcal{D}^\dagger_{(\mathfrak{X}',\mathcal{Z}'),\mathbb{Q}}$-modules, locally projective of finite type over $\mathcal{O}_{\mathfrak{X}',\mathbb{Q}}$. When there exists a lifting f: $(\mathfrak{X}', \mathcal{Z}') \to (\mathfrak{X}, \mathcal{Z})$ of $(X', Z') \to (X, Z)$, then f_0^* is canonically isomorphic to the usual functor f^*.

1.3.4 (Frobenius structure). Suppose now that $\mathcal{V} \to \mathcal{V}'$ is σ (which is a fixed lifting of the a-th Frobenius power of k) and f_0 is $F_{(X,Z)}$ (or simply F) the a-th power of the absolute Frobenius of (X, Z). A "coherent F-$\mathcal{D}^\dagger_{(\mathfrak{X},\mathcal{Z}),\mathbb{Q}}$-module, locally projective of finite type over $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}$" or "coherent $\mathcal{D}^\dagger_{(\mathfrak{X},\mathcal{Z}),\mathbb{Q}}$-module, locally projective of finite type over $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}$ and endowed with a Frobenius structure" is a coherent $\mathcal{D}^\dagger_{(\mathfrak{X},\mathcal{Z}),\mathbb{Q}}$-module $\mathcal{E}$, locally projective of finite type over $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}$ and endowed with a $\mathcal{D}^\dagger_{(\mathfrak{X},\mathcal{Z}),\mathbb{Q}}$-linear isomorphism $\mathcal{E} \xrightarrow{\sim} F^*(\mathcal{E})$. This notion is compatible (via the equivalence of categories 1.3.1(2)) with Shiho's notion of convergent F-log-isocrystal on (X, Z) (see [Shi02, 2.4.2]). By [Shi02, 2.4.3], an F-log-isocrystal on (X, Z) is strikingly locally free.

The following lemma indicates that the equivalence of categories of 1.3.1(1) is compatible with the most useful functors (see also 2.3.10 for inverse images).

LEMMA 1.3.5. *Let $D \subset D'$ be a second divisor of X and $U' := X \setminus D'$. Let $\mathcal{E}$ be a coherent $\mathcal{D}^\dagger_{\mathfrak{X}^\#}({}^\dagger D)_\mathbb{Q}$-module that is a locally projective $\mathcal{O}_\mathfrak{X}({}^\dagger D)_\mathbb{Q}$-module of finite type and $E := \mathrm{sp}^*(\mathcal{E})$. Then*

$$\mathcal{E}({}^\dagger D') = \mathcal{D}^\dagger_{\mathfrak{X}^\#}({}^\dagger D')_\mathbb{Q} \otimes_{\mathcal{D}^\dagger_{\mathfrak{X}^\#}({}^\dagger D)_\mathbb{Q}} \mathcal{E} \xrightarrow{\sim} \mathrm{sp}_*(j^\dagger_{U'} E), \tag{1.3.5.1}$$

$$\mathbb{R}\underline{\Gamma}^\dagger_{D'}(\mathcal{E}) \xrightarrow{\sim} \mathbb{R}\mathrm{sp}_* \circ \underline{\Gamma}^\dagger_{]D'[_\mathfrak{X}}(E). \tag{1.3.5.2}$$

Proof. We have the canonical isomorphism

$$\mathrm{sp}_*(j^\dagger_{U'} E) \xrightarrow{\sim} \mathcal{O}_\mathfrak{X}({}^\dagger D')_\mathbb{Q} \otimes_{\mathcal{O}_\mathfrak{X}({}^\dagger D)_\mathbb{Q}} \mathcal{E}.$$

Since $j^{\dagger}_{U'}E$ satisfies the overconvergence condition, $\mathcal{O}_{\mathfrak{X}}({}^{\dagger}D')_{\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X}}({}^{\dagger}D)_{\mathbb{Q}}} \mathcal{E}$ is then a coherent $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\#}}({}^{\dagger}D')_{\mathbb{Q}}$-module that is also a locally projective $\mathcal{O}_{\mathfrak{X}}({}^{\dagger}D')_{\mathbb{Q}}$-module of finite type. Then, we get a morphism of coherent $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\#}}({}^{\dagger}D')_{\mathbb{Q}}$-modules: $\mathcal{O}_{\mathfrak{X}}({}^{\dagger}D')_{\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X}}({}^{\dagger}D)_{\mathbb{Q}}} \mathcal{E} \to \mathcal{D}^{\dagger}_{\mathfrak{X}^{\#}}({}^{\dagger}D')_{\mathbb{Q}} \otimes_{\mathcal{D}^{\dagger}_{\mathfrak{X}^{\#}}({}^{\dagger}D)_{\mathbb{Q}}} \mathcal{E}$. Since this morphism is an isomorphism outside D', this is an isomorphism (see [Car09a, 4.8]). Thus, we have proved (1.3.5.1).

By applying the functor $\mathbb{R}\mathrm{sp}_*$ to an exact sequence of the form (1.1.0.1), we get the exact triangle (and with the first remark of 1.3.2)

$$\mathbb{R}\mathrm{sp}_* \circ \underline{\Gamma}^{\dagger}_{]D'[_{\mathfrak{X}}}(E) \longrightarrow \mathrm{sp}_*(E) \longrightarrow \mathrm{sp}_*(j^{\dagger}_{U'}(E)) \longrightarrow \mathbb{R}\mathrm{sp}_* \circ \underline{\Gamma}^{\dagger}_{]D'[_{\mathfrak{X}}}(E)[1].$$

Since $\mathrm{sp}_*(E) \longrightarrow \mathrm{sp}_*(j^{\dagger}_{U'}(E))$ is canonically isomorphic to $\mathcal{E} \to \mathcal{E}({}^{\dagger}D')$, it follows from the exact triangle of localization of $\mathcal{E}$ with respect to D' (see (1.2.5.2)) that $\mathbb{R}\underline{\Gamma}^{\dagger}_{D'}(\mathcal{E}) \xrightarrow{\sim} \mathbb{R}\mathrm{sp}_* \circ \underline{\Gamma}^{\dagger}_{]D'[_{\mathfrak{X}}}(E)$. □

An exponent of a coherent $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\#}}({}^{\dagger}D)_{\mathbb{Q}}$-module, locally projective of finite type over $\mathcal{O}_{\mathfrak{X}}({}^{\dagger}D)_{\mathbb{Q}}$-module means an exponent of the associated overconvergent log-isocrystal by 1.3.1(1). The comparison Theorem 1.1.1 can be reformulated as follows.

THEOREM 1.3.6. *Let $\mathcal{E}$ be a coherent $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\#}}({}^{\dagger}D)_{\mathbb{Q}}$-module that is a locally projective $\mathcal{O}_{\mathfrak{X}}({}^{\dagger}D)_{\mathbb{Q}}$-module of finite type. Suppose that*

(a) *none of differences of exponents is a p-adic Liouville number*, *and*
(b′) *any exponent is neither a p-adic Liouville number nor a positive integer*

along each irreducible component Z_i of Z such that $Z_i \not\subset D$. Then the natural morphism

$$\mathbb{R}g_*\left(\Omega^{\bullet}_{\mathfrak{X}^{\#}/\mathcal{T},\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X},\mathbb{Q}}} \mathcal{E}\right) \to \mathbb{R}g_*\left(\Omega^{\bullet}_{\mathfrak{X}/\mathcal{T},\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X},\mathbb{Q}}} \mathcal{E}({}^{\dagger}Z)\right) \tag{1.3.6.1}$$

is an isomorphism.

Proof. Using 1.3.1 (and the first remark 1.3.2), we have only to apply the functor sp_* in 1.1.1 (with $E := \mathrm{sp}^*(\mathcal{E})$). □

Remarks 1.3.7. With the notation of 1.3.6, since

$$\mathbb{R}g_*\left(\Omega^{\bullet}_{\mathfrak{X}^{\#}/\mathcal{T},\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X},\mathbb{Q}}} \mathcal{E}({}^{\dagger}Z)\right) = \mathbb{R}g_*\left(\Omega^{\bullet}_{\mathfrak{X}/\mathcal{T},\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X},\mathbb{Q}}} \mathcal{E}({}^{\dagger}Z)\right),$$

it follows from (1.2.10.1) and (1.2.5.2) that the fact that the morphism (1.3.6.1) is an isomorphism is equivalent to the fact that $g^{\#}_{D,+} \circ \mathbb{R}\underline{\Gamma}^{\dagger}_{Z}(\mathcal{E}) = 0$. We will see also that this is equivalent to the fact that $g_+(\rho)$ is an isomorphism. But first, we need to recall the construction of ρ.

1.3.8 (The morphism ρ). Let $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{X}^{\#}}({}^{\dagger}D)_{\mathbb{Q}})$.

• From [Car09a, 5.2.4], we get the isomorphism of $(\mathcal{D}^{\dagger}_{\mathfrak{X}}({}^{\dagger}D)_{\mathbb{Q}}, \mathcal{D}^{\dagger}_{\mathfrak{X}^{\#}}({}^{\dagger}D)_{\mathbb{Q}})$-bimodules: $\mathcal{D}^{\dagger}_{\mathfrak{X}\leftarrow\mathfrak{X}^{\#}}({}^{\dagger}D)_{\mathbb{Q}} \xrightarrow{\sim} \mathcal{D}^{\dagger}_{\mathfrak{X}}({}^{\dagger}D)_{\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\mathfrak{X}}(\mathcal{Z})$, where to compute the tensor product we take the right structure of $\mathcal{D}^{\dagger}_{\mathfrak{X}}({}^{\dagger}D)_{\mathbb{Q}}$-module (and then the right structure of $\mathcal{O}_{\mathfrak{X}}$-module) of $\mathcal{D}^{\dagger}_{\mathfrak{X}}({}^{\dagger}D)_{\mathbb{Q}}$. Hence, the canonical inclusion $\mathcal{D}^{\dagger}_{\mathfrak{X}}({}^{\dagger}D)_{\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\mathfrak{X}}(\mathcal{Z}) \subset \mathcal{D}^{\dagger}_{\mathfrak{X}}({}^{\dagger}D \cup Z)_{\mathbb{Q}}$ induces the morphism

$$u_{D+}(\mathcal{E}) = \mathcal{D}^{\dagger}_{\mathfrak{X}\leftarrow\mathfrak{X}^{\#}}({}^{\dagger}D)_{\mathbb{Q}} \otimes^{\mathbb{L}}_{\mathcal{D}^{\dagger}_{\mathfrak{X}^{\#}}({}^{\dagger}D)_{\mathbb{Q}}} \mathcal{E} \to \mathcal{D}^{\dagger}_{\mathfrak{X}}({}^{\dagger}D \cup Z)_{\mathbb{Q}} \otimes_{\mathcal{D}^{\dagger}_{\mathfrak{X}^{\#}}({}^{\dagger}D)_{\mathbb{Q}}} \mathcal{E} = \mathcal{E}({}^{\dagger}Z).$$

This canonical morphism is denoted by ρ: $u_{D+}(\mathcal{E}) \to \mathcal{E}({}^{\dagger}Z)$.

• From $\mathcal{D}^{\dagger}_{\mathfrak{X}\leftarrow\mathfrak{X}^{\#}}({}^{\dagger}D)_{\mathbb{Q}} \xrightarrow{\sim} \mathcal{D}^{\dagger}_{\mathfrak{X}}({}^{\dagger}D)_{\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\mathfrak{X}}(\mathcal{Z})$ (and also [Car09a, 6.2.1]), we get

$$u_{D+}(\mathcal{E}) \xrightarrow{\sim} \mathcal{D}^{\dagger}_{\mathfrak{X}}({}^{\dagger}D)_{\mathbb{Q}} \otimes^{\mathbb{L}}_{\mathcal{D}^{\dagger}_{\mathfrak{X}^{\#}}({}^{\dagger}D)_{\mathbb{Q}}} \mathcal{E}(\mathcal{Z}). \tag{1.3.8.1}$$

• Finally, by [Car09a, 5.25], when $\mathcal{E}$ is furthermore a log-isocrystal on $\mathfrak{X}^{\#}$ overconvergent along D, for any $j \neq 0$, $\mathcal{H}^j(u_{D+}(\mathcal{E})) = 0$; i.e., $u_{D+}(\mathcal{E}) \xrightarrow{\sim} \mathcal{D}^{\dagger}_{\mathfrak{X}}({}^{\dagger}D)_{\mathbb{Q}} \otimes_{\mathcal{D}^{\dagger}_{\mathfrak{X}^{\#}}({}^{\dagger}D)_{\mathbb{Q}}} \mathcal{E}(\mathcal{Z})$. This will be essential in the proof of 2.3.4.

Remarks 1.3.9. With the notation 1.3.8, since the canonical morphism $({}^{\dagger}Z) \circ u_{+}(\mathcal{E}) \to \mathcal{E}({}^{\dagger}Z)$ of coherent $\mathcal{D}^{\dagger}_{\mathfrak{X}}({}^{\dagger}D \cup Z)_{\mathbb{Q}}$-modules is an isomorphism (this is obvious outside $D \cup Z$ and so we can apply [Ber96b, 4.3.12]), the localization triangle of $u_{D+}(\mathcal{E})$ with respect to Z is canonically isomorphic to

$$\mathbb{R}\underline{\Gamma}^{\dagger}_{Z} \circ u_{D+}(\mathcal{E}) \to u_{D+}(\mathcal{E}) \xrightarrow{\rho} \mathcal{E}({}^{\dagger}Z) \to \mathbb{R}\underline{\Gamma}^{\dagger}_{Z} \circ u_{D+}(\mathcal{E})[1]. \tag{1.3.9.1}$$

Hence, $\mathbb{R}\underline{\Gamma}^{\dagger}_{Z} \circ u_{+}(\mathcal{E}) = 0$ if and only if ρ is an isomorphism.

We will need the following two commutativity lemmas.

LEMMA 1.3.10. *Let $\widetilde{D}$ be a second divisor of X, $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\#}}(D))$. We have the following isomorphisms in $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}}(D))$ (and then in the category $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}}(\widetilde{D}))$):*

$$(u_{D+}(\mathcal{E}^{(\bullet)}))({}^{\dagger}\widetilde{D}) \xrightarrow{\sim} u_{D+}(\mathcal{E}^{(\bullet)}({}^{\dagger}\widetilde{D})) \xrightarrow{\sim} u_{\widetilde{D}+}(\mathcal{E}^{(\bullet)}({}^{\dagger}\widetilde{D})). \tag{1.3.10.1}$$

Proof. Since over $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\#}}(D))$, $({}^{\dagger}\widetilde{D}) \xrightarrow{\sim} ({}^{\dagger}D \cup \widetilde{D})$, we can suppose that $D \subset \widetilde{D}$. According to our notation (see the beginning of 1.2), u_i: $X^{\#}_i \to X_i$ denotes the reduction modulo π^{i+1} of u and $\mathcal{E}^{(m)}_i := \mathcal{O}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}^{\#}}} \mathcal{E}^{(m)}$. By posing $\mathcal{F}^{(\bullet)} := \mathcal{E}^{(\bullet)}({}^{\dagger}\widetilde{D})$, we get $\mathcal{F}^{(m)}_i \xrightarrow{\sim} \mathcal{D}^{(m)}_{X^{\#}_i}(\widetilde{D}) \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{X^{\#}_i}(D)} \mathcal{E}^{(m)}_i$. By [Car09a, 5.2.4], $\mathcal{D}^{(m)}_{X_i\leftarrow X^{\#}_i}(D) \xrightarrow{\sim} \mathcal{D}^{(m)}_{X_i}(D) \otimes_{\mathcal{O}_{X_i}} \mathcal{O}_{X_i}(Z_i)$. Hence, using [Car09a, 5.1.2], we obtain $\mathcal{D}^{(m)}_{X_i \leftarrow X_i^\#}(D) \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{X_i^\#}(D)} \mathcal{F}_i \xrightarrow{\sim} \mathcal{D}^{(m)}_{X_i}(D) \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{X_i^\#}(D)} (\mathcal{F}_i^{(m)}(Z_i))$. Via the canonical transposition isomorphism γ: $\mathcal{D}^{(m)}_{X_i^\#}(\widetilde{D}) \otimes_{\mathcal{O}_{X_i}} \mathcal{O}_{X_i}(Z_i) \xrightarrow{\sim} \mathcal{O}_{X_i}(Z_i) \otimes_{\mathcal{O}_{X_i}} \mathcal{D}^{(m)}_{X_i^\#}(\widetilde{D})$ (see [Car09a, 1.24]) and via [Car09a, 5.1.2], we get

$$\mathcal{F}_i^{(m)}(Z_i) \xrightarrow{\sim} \mathcal{D}^{(m)}_{X_i^\#}(\widetilde{D}) \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{X_i^\#}(D)} (\mathcal{E}_i^{(m)}(Z_i)).$$

Thus,

$$\mathcal{D}^{(m)}_{X_i \leftarrow X_i^\#}(D) \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{X_i^\#}(D)} \mathcal{F}_i \xrightarrow{\sim} \mathcal{D}^{(m)}_{X_i}(D) \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{X_i^\#}(D)} \mathcal{D}^{(m)}_{X_i^\#}(\widetilde{D}) \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{X_i^\#}(D)} (\mathcal{E}_i^{(m)}(Z_i)).$$

Since $\mathcal{D}^{(m)}_{X_i}(D)$ and $\mathcal{D}^{(m)}_{X_i^\#}(D)$ are $\mathcal{B}^{(m)}_{X_i}(D)$-flat, we check

$$\mathcal{D}^{(m)}_{X_i^\#}(\widetilde{D}) \xrightarrow{\sim} \mathcal{D}^{(m)}_{X_i^\#}(D) \otimes^{\mathbb{L}}_{\mathcal{B}^{(m)}_{X_i}(D)} \mathcal{B}^{(m)}_{X_i}(\widetilde{D})$$

(and also without #). This gives the following $(\mathcal{D}^{(m)}_{X_i}(D), \mathcal{D}^{(m)}_{X_i^\#}(\widetilde{D}))$-linear isomorphism: $\mathcal{D}^{(m)}_{X_i}(D) \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{X_i^\#}(D)} \mathcal{D}^{(m)}_{X_i^\#}(\widetilde{D}) \xrightarrow{\sim} \mathcal{D}^{(m)}_{X_i}(\widetilde{D})$, which furnishes the second isomorphism

(1.3.10.2)

$$\begin{aligned}\mathcal{D}^{(m)}_{X_i \leftarrow X_i^\#}(D) \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{X_i^\#}(D)} \mathcal{F}_i &\xrightarrow{\sim} \mathcal{D}^{(m)}_{X_i}(D) \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{X_i^\#}(D)} \mathcal{D}^{(m)}_{X_i^\#}(\widetilde{D}) \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{X_i^\#}(D)} (\mathcal{E}_i^{(m)}(Z_i)) \\ &\xrightarrow{\sim} \mathcal{D}^{(m)}_{X_i}(\widetilde{D}) \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{X_i^\#}(D)} \mathcal{E}_i^{(m)}(Z_i) \\ &\xrightarrow{\sim} \mathcal{D}^{(m)}_{X_i}(\widetilde{D}) \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{X_i}(D)} (\mathcal{D}^{(m)}_{X_i}(D) \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{X_i^\#}(D)} \mathcal{E}_i^{(m)}(Z_i)).\end{aligned}$$

So we have checked $u_{D+}(\mathcal{E}^{(\bullet)}({}^\dagger \widetilde{D})) \xrightarrow{\sim} (u_{D+}(\mathcal{E}^{(\bullet)}))({}^\dagger \widetilde{D})$. By (1.2.7.1), the second isomorphism was known (we can also use the second isomorphism of (1.3.10.2)). □

LEMMA 1.3.11. *Let $\widetilde{D}$ be a second divisor of X, $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^\#}(D))$. We have*

$$u_{D+} \circ \mathbb{R}\underline{\Gamma}^\dagger_{\widetilde{D}}(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^\dagger_{\widetilde{D}} \circ u_{D+}(\mathcal{E}^{(\bullet)}). \quad (1.3.11.1)$$

Proof. This is a consequence of 1.3.10. Indeed, following (1.2.5.2), the mapping cone of $\mathbb{R}\underline{\Gamma}^\dagger_{\widetilde{D}} \circ u_{D+} \circ \mathbb{R}\underline{\Gamma}^\dagger_{\widetilde{D}}(\mathcal{E}^{(\bullet)}) \to u_{D+} \circ \mathbb{R}\underline{\Gamma}^\dagger_{\widetilde{D}}(\mathcal{E}^{(\bullet)})$ is isomorphic to $({}^\dagger \widetilde{D}) \circ u_{D+} \circ \mathbb{R}\underline{\Gamma}^\dagger_{\widetilde{D}}(\mathcal{E}^{(\bullet)}) = 0$ by 1.3.10. Also, the mapping cone of $\mathbb{R}\underline{\Gamma}^\dagger_{\widetilde{D}} \circ u_{D+} \circ \mathbb{R}\underline{\Gamma}^\dagger_{\widetilde{D}}(\mathcal{E}^{(\bullet)}) \to \mathbb{R}\underline{\Gamma}^\dagger_{\widetilde{D}} \circ u_{D+}(\mathcal{E}^{(\bullet)})$ is isomorphic to $\mathbb{R}\underline{\Gamma}^\dagger_{\widetilde{D}} \circ u_{D+} \circ ({}^\dagger \widetilde{D})(\mathcal{E}^{(\bullet)}) = 0$ by 1.3.10. □

COROLLARY 1.3.12. *Let $\mathcal{E}$ be a coherent $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\#}}({}^{\dagger}D)_{\mathbb{Q}}$-module that is a locally projective $\mathcal{O}_{\mathfrak{X}}({}^{\dagger}D)_{\mathbb{Q}}$-module of finite type and that satisfies conditions* (a) *and* (b′) *of* 1.3.6. *Then, the morphism $g_{D,+}(u_{D+}(\mathcal{E})) \underset{g_+(\rho)}{\longrightarrow} g_{D\cup Z,+}(\mathcal{E}({}^{\dagger}Z))$ is an isomorphism and $g_+\mathbb{R}\underline{\Gamma}^{\dagger}_Z \circ u_{D,+}(\mathcal{E}) = 0$.*

Proof. By the exact triangle (1.3.9.1), it is sufficient to check that $g_{D,+} \circ \mathbb{R}\underline{\Gamma}^{\dagger}_Z \circ u_{D+}(\mathcal{E}) = 0$. But $g^{\#}_{D,+} \xrightarrow{\sim} g_{D,+} \circ u_{D,+}$ (see (1.2.6.1)). Hence, by 1.3.7, we get $g_{D,+} \circ u_{D,+} \circ \mathbb{R}\underline{\Gamma}^{\dagger}_Z(\mathcal{E}) = 0$. We finish the proof by using (1.3.11.1). □

Finally, we finish with the following version of 1.1.22.

THEOREM 1.3.13. *We assume that $g : \mathfrak{X} \to \mathcal{T}$ factors through an irreducible component $\mathcal{Z}_1$ of $\mathcal{Z}$ by a smooth morphism $g_1 : \mathfrak{X} \to \mathcal{Z}_1$ over $\mathcal{T}$ such that the composite $g_1 \circ i_1 : \mathcal{Z}_1 \to \mathcal{Z}_1$ of the closed immersion $i_1 : \mathcal{Z}_1 \to \mathfrak{X}$ and g_1 is the identity. Moreover, we suppose that $D \cap Z_1$ is a divisor of Z_1. Let $\mathcal{Z}'_1 = \cup_{i=2}^{s} \mathcal{Z}_1 \cap \mathcal{Z}_i$ be a strict normal crossings divisor of $\mathcal{Z}_1$, $\mathcal{Z}_1^{\#} := (\mathcal{Z}_1, \mathcal{Z}'_1)$. We suppose that $g_1^{-1}(\mathcal{Z}'_1) = \cup_{i=2}^{s} \mathcal{Z}_i$ and let $g_1^{\#} : \mathfrak{X}^{\#} \to \mathcal{Z}_1^{\#}$ be the canonical induced morphism.*

Let $\mathcal{E}$ be a coherent $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\#}}({}^{\dagger}D)_{\mathbb{Q}}$-module that is a locally projective $\mathcal{O}_{\mathfrak{X}}({}^{\dagger}D)_{\mathbb{Q}}$-module of finite type and that satisfies conditions (a) *and* (b) *in* 1.1.1. *Then the complex*

(1.3.13.1)
$$\mathrm{Cone}\left(g^{\#}_{1+}(\mathcal{E}) \to g^{\#}_{1+}(\mathcal{E}({}^{\dagger}Z_1))\right)$$

is isomorphic to a complex of coherent $\mathcal{D}^{\dagger}_{\mathcal{Z}_1^{\#}}({}^{\dagger}D\cap Z_1)_{\mathbb{Q}}$-modules, locally projective of finite type as $\mathcal{O}_{\mathcal{Z}_1}({}^{\dagger}D \cap Z_1)_{\mathbb{Q}}$-modules and satisfying conditions (a) *and* (b) *of* 1.1.1.

Proof. We pose $E := \mathrm{sp}^*(\mathcal{E})$ and $Y_1 := X \setminus Z_1$. Then, since the functor $\underline{\Gamma}^{\dagger}_{]Z_1[_{\mathfrak{X}}}$ is exact, since the functor $\mathbb{R}g_{1K*}(\Omega^{\bullet}_{\mathfrak{X}^{\#}_K/\mathcal{Z}^{\#}_{1K}} \otimes_{\mathcal{O}_{]X[_{\mathfrak{X}}}} -)$ commutes with the mapping cones and $j^{\dagger}_U\Omega^{\bullet}_{\mathfrak{X}^{\#}_K/\mathcal{Z}^{\#}_{1K}} \otimes_{j^{\dagger}_U\mathcal{O}_{]X[_{\mathfrak{X}}}} E \cong \Omega^{\bullet}_{\mathfrak{X}^{\#}_K/\mathcal{Z}^{\#}_{1K}} \otimes_{\mathcal{O}_{]X[_{\mathfrak{X}}}} E$, we obtain

(1.3.13.2)
$$\mathbb{R}g_{1K*}\underline{\Gamma}^{\dagger}_{]Z_1[_{\mathfrak{X}}}\left(j^{\dagger}_U\Omega^{\bullet}_{\mathfrak{X}^{\#}_K/\mathcal{Z}^{\#}_{1K}} \otimes_{j^{\dagger}_U\mathcal{O}_{]X[_{\mathfrak{X}}}} E\right)$$
$$\cong \mathrm{Cone}\left(\mathbb{R}g_{1K*}(\Omega^{\bullet}_{\mathfrak{X}^{\#}_K/\mathcal{Z}^{\#}_{1K}} \otimes_{\mathcal{O}_{]X[_{\mathfrak{X}}}} E) \to \mathbb{R}g_{1K*}(\Omega^{\bullet}_{\mathfrak{X}^{\#}_K/\mathcal{Z}^{\#}_{1K}} \otimes_{\mathcal{O}_{]X[_{\mathfrak{X}}}} j^{\dagger}_{Y_1}E)\right)[-1].$$

By applying the functor $\mathbb{R}\mathrm{sp}_*$ in the right term of (1.3.13.2), since $\mathbb{R}\mathrm{sp}_* \circ \mathbb{R}g_{1K*} \xrightarrow{\sim} \mathbb{R}g_{1*} \circ \mathbb{R}\mathrm{sp}_*$ and using the first remark of 1.3.2, we get the complex

(1.3.13.3)
$$\mathrm{Cone}\Big(\mathbb{R}g_{1*}(\Omega^{\bullet}_{\mathfrak{X}^{\#}/\mathcal{Z}_1^{\#},\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X},\mathbb{Q}}} \mathrm{sp}_*(E)) \to \mathbb{R}g_{1*}(\Omega^{\bullet}_{\mathfrak{X}^{\#}/\mathcal{Z}_1^{\#},\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X},\mathbb{Q}}} \mathrm{sp}_*(j^{\dagger}_{Y_1}E))\Big)[-1].$$

Following (1.2.10.1), 1.3.1(1) and (1.3.5.1), the complex (1.3.13.3) is isomorphic (up to a shift) to (1.3.13.1).

On the other hand, by applying the functor $\mathbb{R}\mathrm{sp}_*$ in the left term of (1.3.13.2), using the isomorphism (1.1.22.1) and the first remark of 1.3.2 (and of course 1.3.1(1)), we get a complex isomorphic to a complex of coherent $\mathcal{D}^\dagger_{\mathcal{Z}_1^\#}({}^\dagger D \cap Z_1)_{\mathbb{Q}}$-modules, locally projective of finite type as $\mathcal{O}_{\mathcal{Z}_1}({}^\dagger D \cap Z_1)_{\mathbb{Q}}$-modules and satisfying conditions (a) and (b) in 1.1.1 $\square$

Remarks 1.3.14. With the notation 1.3.13, we have the isomorphism (see (1.2.5.2))

(1.3.14.1) $$g^{\#}_{1+} \circ \mathbb{R}\underline{\Gamma}^\dagger_{Z_1}(\mathcal{E}) \xrightarrow{\sim} \mathrm{Cone}\left(g^{\#}_{1+}(\mathcal{E}) \to g^{\#}_{1+}(\mathcal{E}({}^\dagger Z_1))\right)[-1].$$

2. Application to the study of overconvergent F-isocrystals and arithmetic $\mathcal{D}$-modules

2.1. *Kedlaya's semi-stable reduction theorem.* We recall the following definitions of Kedlya (see [Ked08, 3.2.1, 3.2.4]).

Definition 2.1.1. Let X be a smooth irreducible variety over $\mathrm{Spec}\, k$, Z be a strict normal crossings divisor of X, and let E be a convergent isocrystal on $X \setminus Z$. We say that E is *log-extendable* on X if there exists a log-isocrystal with nilpotent residues convergent on the log-scheme (X, Z) (see [Shi02, 2.1.5, 2.1.6]) whose induced convergent isocrystal on $X \setminus Z$ is E. When E is even an isocrystal on $X \setminus Z$ overconvergent along Z, then E is log-extendable if and only if E has unipotent monodromy along Z (see definition [Ked07, 4.4.2] and theorem [Ked07, 6.4.5]).

Definition 2.1.2. Let Y be a smooth irreducible variety over $\mathrm{Spec}\, k$, let X be a partial compactification of Y, and let E be an F-isocrystal on Y overconvergent along $X \setminus Y$. We say that E *admits semistable reduction* if there exists

(1) a proper, surjective, generically étale morphism $f\colon X_1 \to X$;
(2) an open immersion $X_1 \hookrightarrow \overline{X}_1$ into a smooth projective variety over k such that $D_1 := f^{-1}(X \setminus Y) \cup (\overline{X}_1 \setminus X_1)$ is a strict normal crossings divisor of $\overline{X}_1$

such that the isocrystal $f^*(E)$ on $Y_1 := f^{-1}(Y)$ overconvergent along $D_1 \cap X_1$ is log-extendable on X_1 (see 2.1.1).

With the previous definitions, Kedlaya has proved in [Ked11, 2.4.4] (see also [Ked07], [Ked08], [Ked09]) the following theorem, which answers positively to Shiho's conjecture in [Shi02, 3.1.8].

THEOREM 2.1.3 (Kedlaya). *Let Y be a smooth irreducible k-variety, X be a partial compactification of Y, $Z := X \setminus Y$, and let E be an F-isocrystal on Y overconvergent along Z. Then E admits semistable reduction.*

Remarks 2.1.4. This conjecture was previously checked by Tsuzuki when E is unit-root in [Tsu02] and by Kedlaya in the case of curves (see [Ked03]).

2.2. *A comparison theorem between log-de Rham complexes and de Rham complexes.* Let $\mathfrak{X}$ be a smooth formal $\mathcal{V}$-scheme, D be a divisor of X, $Y := X \setminus D$, $\mathcal{Z}$ be a strict normal crossings divisor of $\mathfrak{X}$, $\mathfrak{X}^{\#} := (\mathfrak{X}, \mathcal{Z})$ be the induced smooth logarithmic formal $\mathcal{V}$-scheme, and u: $\mathfrak{X}^{\#} \to \mathfrak{X}$ be the canonical morphism.

LEMMA 2.2.1. *Let $\mathcal{Z}'$ be a strict normal crossings divisor of $\mathfrak{X}$ such that $\mathcal{Z} \cup \mathcal{Z}'$ is a strict normal crossings divisor of $\mathfrak{X}$ and such that $Z \cap Z'$ is of codimension 2 in X (i.e., the irreducible components of Z and Z' are different). We pose $\mathfrak{X}^{\#\prime} = (\mathfrak{X}, \mathcal{Z} \cup \mathcal{Z}')$. Then the canonical morphism $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\#\prime}}({}^{\dagger}D \cup Z')_{\mathbb{Q}} \to \mathcal{D}^{\dagger}_{\mathfrak{X}^{\#}}({}^{\dagger}D \cup Z')_{\mathbb{Q}}$ is an isomorphism.*

Proof. The assertion is local in $\mathfrak{X}$. We can suppose that there exists local coordinates $t_1, \ldots, t_d$ of $\mathfrak{X}$ such that $\mathcal{Z} \cup \mathcal{Z}' = V(t_1 \ldots t_r)$ and $\mathcal{Z} = V(t_{s+1} \ldots t_r)$ for some $0 \leq s \leq r$. For any integer m, we have the canonical inclusion $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^{\#\prime}}(D \cup Z')_{\mathbb{Q}} \subset \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}^{\#}}(D \cup Z')_{\mathbb{Q}}$ (see the notation of 1.2.2). *A fortiori*, by direct limit on the level, we obtain $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\#\prime}}({}^{\dagger}D \cup Z')_{\mathbb{Q}} \subset \mathcal{D}^{\dagger}_{\mathfrak{X}^{\#}}({}^{\dagger}D \cup Z')_{\mathbb{Q}}$.

Less obviously, let us check the converse. For any integer k, we denote by $q_k^{(m)}$, $q_k^{(m+1)}$, $r_k^{(m)}$, $r_k^{(m+1)}$, $\tilde{r}_k^{(m)}$ the integers satisfying the following conditions: $k = p^m q_k^{(m)} + r_k^{(m)}$, $0 \leq r_k^{(m)} < p^m$, $k = p^{m+1} q_k^{(m+1)} + r_k^{(m+1)}$, $0 \leq r_k^{(m+1)} < p^{m+1}$, $q_k^{(m)} = p q_k^{(m+1)} + \tilde{r}_k^{(m)}$, $0 \leq \tilde{r}_k^{(m)} < p$. We recall that the p-adic valuation of $k!$ is $v_p(k!) = (k - \sigma(k))/(p-1)$, where $\sigma(k) = \sum_i a_i$ if $k = \sum_i a_i p^i$ with $0 \leq a_i < p$. We compute

$$v_p(q_k^{(m)}!) - v_p(q_k^{(m+1)}!) = (q_k^{(m)} - q_k^{(m+1)} - \tilde{r}_k^{(m)})/(p-1) = q_k^{(m+1)}.$$

By [Ber96b, 2.2.3.1] (and $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X},\mathbb{Q}} \subset \widehat{\mathcal{D}}^{(m+1)}_{\mathfrak{X},\mathbb{Q}}$), we have

$$\partial_i^{\langle k \rangle_{(m)}} = q_k^{(m)}!/q_k^{(m+1)}! \partial_i^{\langle k \rangle_{(m+1)}}.$$

Then, there exists a unit u of $\mathbb{Z}_p$ such that for every $0 \leq i \leq s$, we get

$$\partial_i^{\langle k \rangle_{(m)}} = u p^{q_k^{(m+1)}} \partial_i^{\langle k \rangle_{(m+1)}} = \frac{u}{t_i^{r_k^{(m+1)}}} \left(\frac{p}{t_i^{p^{m+1}}} \right)^{q_k^{(m+1)}} t_i^k \partial_i^{\langle k \rangle_{(m+1)}}.$$

Since for any k we have $\frac{u}{t_i^{r_k^{(m+1)}}}\left(\frac{p}{t_i^{p^{m+1}}}\right)^{q_k^{(m+1)}} \in \frac{1}{t_i^{(p^{m+1}-1)}}\widehat{\mathcal{B}}_{\mathfrak{X}}^{(m)}(D\cup Z')$, we obtain the inclusion $\widehat{\mathcal{D}}_{\mathfrak{X}^\#}^{(m)}(D\cup Z')_{\mathbb{Q}} \subset \frac{1}{t_i^{(p^{m+1}-1)}}\widehat{\mathcal{D}}_{\mathfrak{X}^{\#\prime}}^{(m+1)}(D\cup Z')_{\mathbb{Q}}$. Since $\frac{1}{t_i^{(p^{m+1}-1)}}$ is invertible in $\widehat{\mathcal{D}}_{\mathfrak{X}^{\#\prime}}^{(m+1)}(D\cup Z')_{\mathbb{Q}}$, this implies $\widehat{\mathcal{D}}_{\mathfrak{X}^\#}^{(m)}(D\cup Z')_{\mathbb{Q}} \subset \widehat{\mathcal{D}}_{\mathfrak{X}^{\#\prime}}^{(m+1)}(D\cup Z')_{\mathbb{Q}}$. Then, by taking the direct limit on the level,

$$\mathcal{D}^\dagger_{\mathfrak{X}^\#}({}^\dagger D\cup Z')_{\mathbb{Q}} \subset \mathcal{D}^\dagger_{\mathfrak{X}^{\#\prime}}({}^\dagger D\cup Z')_{\mathbb{Q}}. \qquad \square$$

LEMMA 2.2.2. *With the same notation as in* 2.2.1, *let* $v\colon \mathfrak{X}^{\#\prime}\to\mathfrak{X}$ *be the canonical morphism. For any* $\mathcal{E}\in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}^\#}({}^\dagger D)_{\mathbb{Q}})$ *and* $\mathcal{E}'\in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}^{\#\prime}}({}^\dagger D)_{\mathbb{Q}})$, *we have the following isomorphisms in* $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}}({}^\dagger D\cup Z')_{\mathbb{Q}})$:

$$v_{D\cup Z'+}(\mathcal{E}({}^\dagger Z')) \xrightarrow{\sim} u_{D\cup Z'+}(\mathcal{E}({}^\dagger Z')) \xrightarrow{\sim} (u_{D+}(\mathcal{E}))({}^\dagger Z'), \tag{2.2.2.1}$$

$$u_{D\cup Z'+}(\mathcal{E}'({}^\dagger Z')) \xrightarrow{\sim} v_{D\cup Z'+}(\mathcal{E}'({}^\dagger Z')) \xrightarrow{\sim} (v_{D+}(\mathcal{E}'))({}^\dagger Z'). \tag{2.2.2.2}$$

Proof. First, since $\mathcal{D}^\dagger_{\mathfrak{X}^{\#\prime}}({}^\dagger D\cup Z')_{\mathbb{Q}} = \mathcal{D}^\dagger_{\mathfrak{X}^\#}({}^\dagger D\cup Z')_{\mathbb{Q}}$ (see 2.2.1), the left terms of (2.2.2.1) and (2.2.2.2) are well defined. Also, as the proof of (2.2.2.2) is similar, we will only check (2.2.2.1).

By (1.3.8.1), $u_{D+}(\mathcal{E}) \xrightarrow{\sim} \mathcal{D}^\dagger_{\mathfrak{X}}({}^\dagger D)_{\mathbb{Q}} \otimes^{\mathbb{L}}_{\mathcal{D}^\dagger_{\mathfrak{X}^\#}({}^\dagger D)_{\mathbb{Q}}} \mathcal{E}(\mathcal{Z})$. Then, by associativity of the tensor product, we get

$$\begin{aligned}(u_{D+}(\mathcal{E}))({}^\dagger Z') &\xrightarrow{\sim} \mathcal{D}^\dagger_{\mathfrak{X}}({}^\dagger D\cup Z')_{\mathbb{Q}} \otimes^{\mathbb{L}}_{\mathcal{D}^\dagger_{\mathfrak{X}^\#}({}^\dagger D)_{\mathbb{Q}}} \mathcal{E}(\mathcal{Z})\\ &\xrightarrow{\sim} \mathcal{D}^\dagger_{\mathfrak{X}}({}^\dagger D\cup Z')_{\mathbb{Q}} \otimes^{\mathbb{L}}_{\mathcal{D}^\dagger_{\mathfrak{X}^\#}({}^\dagger D\cup Z')_{\mathbb{Q}}} \mathcal{E}(\mathcal{Z})({}^\dagger Z').\end{aligned}$$

On the other hand, by (1.3.8.1) (and, for the second isomorphism, since $\mathcal{D}^\dagger_{\mathfrak{X}^{\#\prime}}\cdot({}^\dagger D\cup Z')_{\mathbb{Q}} = \mathcal{D}^\dagger_{\mathfrak{X}^\#}({}^\dagger D\cup Z')_{\mathbb{Q}}$), we get

$$u_{D\cup Z'+}(\mathcal{E}({}^\dagger Z')) \xrightarrow{\sim} \mathcal{D}^\dagger_{\mathfrak{X}}({}^\dagger D\cup Z')_{\mathbb{Q}} \otimes^{\mathbb{L}}_{\mathcal{D}^\dagger_{\mathfrak{X}^\#}({}^\dagger D\cup Z')_{\mathbb{Q}}} \mathcal{E}({}^\dagger Z')(\mathcal{Z}),$$

$$v_{D\cup Z'+}(\mathcal{E}({}^\dagger Z')) \xrightarrow{\sim} \mathcal{D}^\dagger_{\mathfrak{X}}({}^\dagger D\cup Z')_{\mathbb{Q}} \otimes^{\mathbb{L}}_{\mathcal{D}^\dagger_{\mathfrak{X}^\#}({}^\dagger D\cup Z')_{\mathbb{Q}}} \mathcal{E}({}^\dagger Z')(\mathcal{Z}\cup\mathcal{Z}').$$

Since $\mathcal{E}({}^\dagger Z')(\mathcal{Z}\cup\mathcal{Z}') \xrightarrow{\sim} \mathcal{E}({}^\dagger Z')(\mathcal{Z}')(\mathcal{Z}) \xrightarrow{\sim} \mathcal{E}({}^\dagger Z')(\mathcal{Z}) \xrightarrow{\sim} \mathcal{E}(\mathcal{Z})({}^\dagger Z')$ (see (1.2.8.4)), we conclude the proof of (2.2.2.1). $\square$

PROPOSITION 2.2.3. *Let* $\mathfrak{A} = \operatorname{Spf}\mathcal{V}\{t_1,\dots,t_n\}$, D *be a divisor of the affine space* $\operatorname{Spec} k[t_1,\dots,t_n]$ *and for* $i=1,\dots,n$, *let* $\mathfrak{H}_i$ *be the formal closed subscheme of* $\mathfrak{A}$ *defined by* $t_i=0$, *i.e.*, $\mathfrak{H}_i = \operatorname{Spf}\mathcal{V}\{t_1,\dots,\widehat{t_i},\dots,t_n\}$. *Fix an integer* $r\in\{0,\dots,n\}$, *and pose* $\mathfrak{H} := \cup_{1\le i\le r}\mathfrak{H}_i$. *Let* $\mathfrak{A}^\# := (\mathfrak{A},\mathfrak{H})$ *and* $w:\mathfrak{A}^\#\to\mathfrak{A}$ *be the canonical morphism. Let* $\mathcal{E}$ *be a coherent* $\mathcal{D}^\dagger_{\mathfrak{A}^\#}({}^\dagger D)_{\mathbb{Q}}$-*module that is a locally projective* $\mathcal{O}_{\mathfrak{A}}({}^\dagger D)_{\mathbb{Q}}$-*module of finite type such that conditions*

(a) *and* (b′) *in* 1.3.6 *hold. Then the canonical morphism* ρ: $w_{D+}(\mathcal{E}) \to \mathcal{E}(^\dagger H)$ (*see* 1.3.8) *is an isomorphism.*

Proof. We have to check $\mathbb{R}\underline{\Gamma}^\dagger_H w_{D+}(\mathcal{E}) = 0$ (thanks to the exact triangle (1.3.9.1)). To prove it, we will proceed by induction on r. When $r = 0$, this is obvious. Suppose $r \geq 1$, and pose $\mathfrak{H}' = \cup_{r\geq i\geq 2}\mathfrak{H}_i$ (when $r = 1$, $\mathfrak{H}'$ is empty) and $\mathcal{G} := w_{D+}(\mathcal{E})$. We get the Mayer-Vietoris exact triangle (see [Car04, 2.2.16])

$$\text{(2.2.3.1)} \qquad \mathbb{R}\underline{\Gamma}^\dagger_{H_1\cap H'}\mathcal{G}(^\dagger H_1) \to \mathbb{R}\underline{\Gamma}^\dagger_{H_1}\mathcal{G}(^\dagger H_1) \oplus \mathbb{R}\underline{\Gamma}^\dagger_{H'}\mathcal{G}(^\dagger H_1) \to \mathbb{R}\underline{\Gamma}^\dagger_{H_1\cup H'}\mathcal{G}(^\dagger H_1) \to \mathbb{R}\underline{\Gamma}^\dagger_{H_1\cap H'}\mathcal{G}(^\dagger H_1)[1].$$

Since $\mathbb{R}\underline{\Gamma}^\dagger_{H_1}\mathcal{G}(^\dagger H_1) = 0$ and $\mathbb{R}\underline{\Gamma}^\dagger_{H_1\cap H'}\mathcal{G}(^\dagger H_1) = 0$, we obtain $\mathbb{R}\underline{\Gamma}^\dagger_{H'}\mathcal{G}(^\dagger H_1) \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^\dagger_{H}\mathcal{G}(^\dagger H_1)$.

Let $\mathfrak{A}^{\#\prime} := (\mathfrak{A}, \mathfrak{H}')$, $w' : \mathfrak{A}^{\#\prime} \to \mathfrak{A}$ be the canonical map, and let $E := \mathrm{sp}^*(\mathcal{E})$. By 1.3.5, $\mathcal{E}(^\dagger H_1) \xrightarrow{\sim} \mathrm{sp}_*(j^\dagger_{Y_1\cap U}E)$, where $U = \mathbb{A}^n_k \setminus D$ and $Y_1 = \mathbb{A}^n_k \setminus H_1$. Moreover, from 2.2.1, $\mathcal{D}^\dagger_{\mathfrak{A}^{\#\prime}}(^\dagger D \cup H_1)_{\mathbb{Q}} = \mathcal{D}^\dagger_{\mathfrak{A}^{\#}}(^\dagger D \cup H_1)_{\mathbb{Q}}$. Then $\mathcal{E}(^\dagger H_1)$ is a coherent $\mathcal{D}^\dagger_{\mathfrak{A}^{\#\prime}}(^\dagger D \cup H_1)_{\mathbb{Q}}$-module that is a locally projective of finite type $\mathcal{O}_{\mathfrak{A}}(^\dagger D \cup H_1)_{\mathbb{Q}}$-module satisfying both conditions (a) and (b′). Using the induction hypothesis, this implies $\mathbb{R}\underline{\Gamma}^\dagger_{H'} w'_{D\cup H_1,+}(\mathcal{E}(^\dagger H_1)) = 0$. We get from (2.2.2.2) the isomorphism $w'_{D\cup H_1,+}(\mathcal{E}(^\dagger H_1)) \xrightarrow{\sim} (w_{D+}(\mathcal{E}))(^\dagger H_1)$. Since $\mathbb{R}\underline{\Gamma}^\dagger_{H'}\mathcal{G}(^\dagger H_1) \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^\dagger_{H}\mathcal{G}(^\dagger H_1)$, we obtain $\mathbb{R}\underline{\Gamma}^\dagger_{H}\mathcal{G}(^\dagger H_1) = 0$. Symmetrically, for any $i = 1, \ldots, r$, we check that $\mathbb{R}\underline{\Gamma}^\dagger_{H}\mathcal{G}(^\dagger H_i) = 0$. With the exact triangle of localization of $\mathbb{R}\underline{\Gamma}^\dagger_{H}\mathcal{G}$ with respect to H_i, this means that the canonical morphism $\mathbb{R}\underline{\Gamma}^\dagger_{H_i}\mathbb{R}\underline{\Gamma}^\dagger_{H}\mathcal{G} \to \mathbb{R}\underline{\Gamma}^\dagger_{H}\mathcal{G}$ is an isomorphism. By [Car04, 2.2.8], this implies $\mathbb{R}\underline{\Gamma}^\dagger_{H_1\cap\cdots\cap H_r}\mathcal{G} \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^\dagger_{H}\mathcal{G}$.

It remains to prove that $\mathbb{R}\underline{\Gamma}^\dagger_{H_1\cap\cdots\cap H_r}\mathcal{G} = 0$. When D contains $H_1 \cap \cdots \cap H_r$, this is obvious. This reduces us to the case where $D \cap (H_1 \cap \cdots \cap H_r)$ is a divisor of $H_1 \cap \cdots \cap H_r$.

Let ι be the canonical closed immersion $\mathfrak{H}_1 \cap \cdots \cap \mathfrak{H}_r = \mathrm{Spf}\,\mathcal{V}\{t_{r+1}, \ldots, t_n\} \hookrightarrow \mathrm{Spf}\,\mathcal{V}\{t_1, \ldots, t_n\} = \mathfrak{A}$ and g: $\mathfrak{A} \to \mathrm{Spf}\,\mathcal{V}\{t_{r+1}, \ldots, t_n\}$ be the canonical projection. We notice that $g \circ \iota$ is the identity. Since $\mathcal{E}$ satisfies conditions (a) and (b′) and $\mathcal{G} = w_{D+}(\mathcal{E})$, from 1.3.12 it follows that $g_{D+}\mathbb{R}\underline{\Gamma}^\dagger_{H}(\mathcal{G}) = 0$. (Notice that we do need here the relative case of 1.3.12, i.e., $\mathcal{T}$ is not necessary equal to $\mathcal{S}$.) Hence, $g_{D+}\mathbb{R}\underline{\Gamma}^\dagger_{H_1\cap\cdots\cap H_r}(\mathcal{G}) = 0$. By [Ber02, 4.4.5], $\mathbb{R}\underline{\Gamma}^\dagger_{H_1\cap\cdots\cap H_r}(\mathcal{G}) \xrightarrow{\sim} \iota_+\iota^!(\mathcal{G})$. Then $g_{D+}\mathbb{R}\underline{\Gamma}^\dagger_{H_1\cap\cdots\cap H_r}(\mathcal{G}) \xrightarrow{\sim} g_+\iota_+\iota^!(\mathcal{G}) \xrightarrow{\sim} \iota^!(\mathcal{G})$. Hence, $\iota^!(\mathcal{G}) = 0$ and then $\mathbb{R}\underline{\Gamma}^\dagger_{H_1\cap\cdots\cap H_r}(\mathcal{G}) = 0$, which finishes the proof. □

We will need to extend [Car09a, 6.11], which will be essential (in the proof of 2.2.9 or 2.3.13). As for [Car09a, 6.11], we need a preliminary result.

Lemma 2.2.4. *With the same notation as in* 2.2.1, *let* $X_i^{\#}$ *and* $X_i^{\#\prime}$ *be respectively the reductions of* $\mathfrak{X}^{\#}$ *and* $\mathfrak{X}^{\#\prime}$ *modulo* π^{i+1}. *Let* $\mathcal{B}_{X_i}$ *be a* $\mathcal{D}^{(m)}_{X_i^{\#}}$-*module endowed with a compatible structure of* $\mathcal{O}_{X_i}$-*algebra. We pose* $\widetilde{\mathcal{D}}^{(m)}_{X_i^{\#}} := \mathcal{B}_{X_i} \otimes_{\mathcal{O}_{X_i}} \mathcal{D}^{(m)}_{X_i^{\#}}$, $\widetilde{\mathcal{D}}^{(m)}_{X_i^{\#\prime}} := \mathcal{B}_{X_i} \otimes_{\mathcal{O}_{X_i}} \mathcal{D}^{(m)}_{X_i^{\#\prime}}$. *Let* $\mathcal{E}'$ *be a left* $\widetilde{\mathcal{D}}^{(m)}_{X_i^{\#\prime}}$-*module and* $\mathcal{E}$ *be a left* $\widetilde{\mathcal{D}}^{(m)}_{X_i^{\#}}$-*module. Then the canonical morphism of* $\widetilde{\mathcal{D}}^{(m)}_{X_i^{\#}}$-*modules*

$$\widetilde{\mathcal{D}}^{(m)}_{X_i^{\#}} \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X_i^{\#\prime}}} (\mathcal{E}' \otimes_{\mathcal{B}_{X_i}} \mathcal{E}) \to (\widetilde{\mathcal{D}}^{(m)}_{X_i^{\#}} \otimes_{\widetilde{\mathcal{D}}^{(m)}_{X_i^{\#\prime}}} \mathcal{E}') \otimes_{\mathcal{B}_{X_i}} \mathcal{E} \tag{2.2.4.1}$$

is an isomorphism.

Proof. Similar to [Car09a, 3.6]. □

Proposition 2.2.5. *With the same notation as in* 2.2.1, *let* $\tilde{u}$: $\mathfrak{X}^{\#\prime} \to \mathfrak{X}^{\#}$ *be the canonical morphism. Let* $\mathcal{E}$ *be a coherent* $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\#}}({}^{\dagger}D)_{\mathbb{Q}}$-*module that is a locally projective* $\mathcal{O}_{\mathfrak{X}}({}^{\dagger}D)_{\mathbb{Q}}$-*module of finite type. Then* $\mathcal{E}$ *is also a coherent* $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\#\prime}}({}^{\dagger}D)_{\mathbb{Q}}$-*module that is a locally projective* $\mathcal{O}_{\mathfrak{X}}({}^{\dagger}D)_{\mathbb{Q}}$-*module of finite type. Furthermore, we have the isomorphism of* $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\#}}({}^{\dagger}D)_{\mathbb{Q}}$-*modules*

$$\tilde{u}_{D+}(\mathcal{E}) \xrightarrow{\sim} \mathcal{D}^{\dagger}_{\mathfrak{X}^{\#}}({}^{\dagger}D \cup Z')_{\mathbb{Q}} \otimes_{\mathcal{D}^{\dagger}_{\mathfrak{X}^{\#}}({}^{\dagger}D)_{\mathbb{Q}}} \mathcal{E} = \mathcal{E}({}^{\dagger}Z'). \tag{2.2.5.1}$$

In particular, $\tilde{u}_{D+}(\mathcal{E})$ (*resp.* $\mathcal{E}({}^{\dagger}Z')$) *can be endowed with a canonical structure of coherent* $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\#}}({}^{\dagger}D \cup Z')_{\mathbb{Q}}$-*module* (*resp. coherent* $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\#}}({}^{\dagger}D)_{\mathbb{Q}}$-*module*).

Proof. By 1.3.1, $\mathrm{sp}^*(\mathcal{E})$ is a locally free $j^{\dagger}_U\mathcal{O}_{]X[_{\mathfrak{X}}}$-module of finite type with a logarithmic connection $\nabla : E \to j^{\dagger}_U \Omega^1_{\mathfrak{X}^{\#}_K/\mathcal{S}_K} \otimes_{j^{\dagger}_U\mathcal{O}_{]X[_{\mathfrak{X}}}} E$ satisfying the overconvergence condition (see 1.3.1). Then, we check that the induced logarithmic connection $\nabla' : E \to j^{\dagger}_U \Omega^1_{\mathfrak{X}^{\#\prime}_K/\mathcal{S}_K} \otimes_{j^{\dagger}_U\mathcal{O}_{]X[_{\mathfrak{X}}}} E$ satisfies the overconvergence condition. So, $\mathcal{E}$ is a coherent $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\#\prime}}({}^{\dagger}D)_{\mathbb{Q}}$-module that is a locally projective $\mathcal{O}_{\mathfrak{X}}({}^{\dagger}D)_{\mathbb{Q}}$-module of finite type.

As for [Car09a, 6.8], we compute $\tilde{u}_{D+}(\mathcal{O}_{\mathfrak{X}}({}^{\dagger}D)_{\mathbb{Q}}) \xrightarrow{\sim} \mathcal{O}_{\mathfrak{X}}({}^{\dagger}D \cup Z')_{\mathbb{Q}}$. Then, in the same way as for the proof of [Car09a, 6.11], we deduce from 2.2.4 that the isomorphism (2.2.5.1) holds. □

Remarks 2.2.6. With the notation of 2.2.5, it comes from (1.2.4.4) and (1.2.8.3) that there is no ambiguity in writing $\mathcal{E}({}^{\dagger}Z')$. More precisely,

$$\mathcal{D}^{\dagger}_{\mathfrak{X}^{\#\prime}}({}^{\dagger}D \cup Z')_{\mathbb{Q}} \otimes_{\mathcal{D}^{\dagger}_{\mathfrak{X}^{\#\prime}}({}^{\dagger}D)_{\mathbb{Q}}} \mathcal{E} \xrightarrow{\sim} \mathcal{D}^{\dagger}_{\mathfrak{X}^{\#}}({}^{\dagger}D \cup Z')_{\mathbb{Q}} \otimes_{\mathcal{D}^{\dagger}_{\mathfrak{X}^{\#}}({}^{\dagger}D)_{\mathbb{Q}}} \mathcal{E} \xrightarrow{\sim} \mathcal{E}({}^{\dagger}Z').$$

Lemma 2.2.7. *Let* h: $\mathfrak{X}' \to \mathfrak{X}$ *be a finite étale morphism of smooth formal* $\mathcal{V}$-*schemes,* $D' = h^{-1}(D)$, $\mathfrak{X}'^{\#} := (\mathfrak{X}', h^{-1}(\mathcal{Z}))$, *and let* $h^{\#}$: $\mathfrak{X}'^{\#} \to \mathfrak{X}^{\#}$ *be the induced morphism by* h. *Let* $\mathcal{E}'$ *be a coherent* $\mathcal{D}^{\dagger}_{\mathfrak{X}'^{\#}}({}^{\dagger}D')_{\mathbb{Q}}$-*module that is a* *locally projective* $\mathcal{O}_{\mathfrak{X}'}({}^\dagger D')_{\mathbb{Q}}$*-module of finite type. Then* $h^{\#}_{D+}(\mathcal{E}')$ *is a coherent* $\mathcal{D}^\dagger_{\mathfrak{X}^\#}({}^\dagger D)_{\mathbb{Q}}$*-module that is a locally projective* $\mathcal{O}_{\mathfrak{X}}({}^\dagger D)_{\mathbb{Q}}$*-module of finite type. Furthermore, if* $\mathcal{E}'$ *satisfies conditions* (a) *and* (b′) *of* 1.3.6, *so is* $h^{\#}_{D+}(\mathcal{E}')$.

Proof. Since $h^{\#}$ is smooth, we have the canonical isomorphism

$$\Omega^\bullet_{\mathfrak{X}'^\#/\mathfrak{X}^\#,\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X},\mathbb{Q}}} \mathcal{D}^\dagger_{\mathfrak{X}'^\#,\mathbb{Q}}[d_{\mathfrak{X}'^\#/\mathfrak{X}^\#}] \xrightarrow{\sim} \mathcal{D}^\dagger_{\mathfrak{X}^\# \leftarrow \mathfrak{X}'^\#,\mathbb{Q}}$$

(see 1.2.10). Since h is even étale, we get $\Omega^1_{\mathfrak{X}'^\#/\mathfrak{X}^\#} = 0$ and then $\mathcal{D}^\dagger_{\mathfrak{X}'^\#,\mathbb{Q}} \xrightarrow{\sim} \mathcal{D}^\dagger_{\mathfrak{X}^\# \leftarrow \mathfrak{X}'^\#,\mathbb{Q}}$. But $\mathbb{R}h_* = h_*$ because h is finite. This implies that $h^{\#}_{D+}(\mathcal{E}')$ is canonically isomorphic to $h_*(\mathcal{E}')$. Pose $U' := \mathfrak{X}' \setminus D'$. Recall that by 1.3.1, $E' := \mathrm{sp}^*(\mathcal{E}')$ is a locally free $j^\dagger_{U'}\mathcal{O}_{]X'[_{\mathfrak{X}'}}$-module of finite type endowed with a logarithmic connection $\nabla : E' \to j^\dagger_{U'}\Omega^1_{\mathfrak{X}'^\#_K/\mathcal{S}_K} \otimes_{j^\dagger_{U'}\mathcal{O}_{]X'[_{\mathfrak{X}'}}} E'$ satisfying the overconvergence condition of (1.1.0.2). By hypothesis, E' satisfies conditions (a) and (b′) of 1.3.6. By 1.1.3(2), then so is $h_*(E')$. We conclude with the isomorphism $\mathrm{sp}_*h_*(E') \xrightarrow{\sim} h_*\mathrm{sp}_*(E') \xrightarrow{\sim} h_*(\mathcal{E}')$. □

LEMMA 2.2.8. *Let* $h\colon \mathcal{P} \to \mathcal{P}'$ *be a finite and étale morphism of smooth formal* $\mathcal{V}$*-schemes,* D' *be a divisor of* X', $D := h^{-1}(D')$, $\mathcal{E} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathcal{P}}(D))$. *Then* $h_+(\mathcal{E}) = 0$ *if and only if* $\mathcal{E} = 0$.

THEOREM 2.2.9. *Let* $\mathcal{E}$ *be a coherent* $\mathcal{D}^\dagger_{\mathfrak{X}^\#,\mathbb{Q}}$*-module that is a locally projective* $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}$*-module of finite type such that conditions* (a) *and* (b′) *in* 1.3.6 *hold. Then the canonical morphism* $\rho\colon u_+(\mathcal{E}) \to \mathcal{E}({}^\dagger Z)$ (*see* 1.3.8) *is an isomorphism.*

Proof. This is equivalent to proving that $\mathbb{R}\underline{\Gamma}^\dagger_Z u_+(\mathcal{E}) = 0$ (see (1.3.9.1)). We proceed by induction on the dimension of X.

1° *How to use the case* 2.2.3 *of affine spaces.* Let x be a point of $\mathfrak{X}$, and let $\mathcal{Z}_1, \ldots, \mathcal{Z}_r$ be the irreducible components of $\mathcal{Z}$ that contain x. By [Ked05, Th. 2], there exist an open dense subset $\mathfrak{U}$ of $\mathfrak{X}$ containing x and a finite étale morphism $h_0\colon U \to \mathbb{A}^n_k$ such that $\mathcal{Z} \cap \mathfrak{U} = (\mathcal{Z}_1 \cup \cdots \cup \mathcal{Z}_r) \cap \mathfrak{U}$ and $Z_1 \cap U, \ldots, Z_r \cap U$ map by h_0 to coordinate hyperplanes $H_1, \ldots, H_r$. Since the theorem is local in $\mathfrak{X}$, we can suppose that $\mathfrak{U} = \mathfrak{X}$.

Let $h\colon \mathfrak{X} \to \mathrm{Spf}\,\mathcal{V}\{t_1, \ldots, t_n\}$ be a lifting of h_0. Denote by $\mathfrak{H}_1, \ldots, \mathfrak{H}_n$ the coordinate hyperplanes of $\mathrm{Spf}\,\mathcal{V}\{t_1, \ldots, t_n\}$, $\mathfrak{H} := \mathfrak{H}_1 \cup \cdots \cup \mathfrak{H}_r$, $\mathcal{Z}'' := h^{-1}(\mathfrak{H})$. Let $\mathcal{Z}'$ be the union of the irreducible components of $\mathcal{Z}''$ that is not an irreducible component of $\mathcal{Z}$. Denote by $\mathfrak{X}^{\#\prime} = (\mathfrak{X}, \mathcal{Z}'')$, $\widehat{\mathbb{A}}^n_{\mathcal{V}} = \mathrm{Spf}\,\mathcal{V}\{t_1, \ldots, t_n\}$, $\widehat{\mathbb{A}}^{n\#}_{\mathcal{V}} = (\mathrm{Spf}\,\mathcal{V}\{t_1, \ldots, t_n\}, \mathfrak{H})$, $h^{\#}\colon \mathfrak{X}^{\#\prime} \to \widehat{\mathbb{A}}^{n\#}_{\mathcal{V}}$, $w\colon \widehat{\mathbb{A}}^{n\#}_{\mathcal{V}} \to \widehat{\mathbb{A}}^n_{\mathcal{V}}$, $v\colon \mathfrak{X}^{\#\prime} \to \mathfrak{X}$.

We get the following commutative diagram:

$$\begin{array}{ccccc} \mathfrak{X} & = & \mathfrak{X} & \xrightarrow{h} & \widehat{\mathbb{A}}^n_{\mathcal{V}} \\ {\scriptstyle u}\uparrow & & {\scriptstyle v}\uparrow & & {\scriptstyle w}\uparrow \\ (\mathfrak{X}, \mathcal{Z}) & \xleftarrow{\tilde{u}} & (\mathfrak{X}, \mathcal{Z}'') & \xrightarrow{h^{\#}} & \widehat{\mathbb{A}}^{n\#}_{\mathcal{V}}. \end{array}$$

2° *The canonical morphism* $\mathbb{R}\underline{\Gamma}^{\dagger}_{Z\cap Z'} u_+(\mathcal{E}) \to \mathbb{R}\underline{\Gamma}^{\dagger}_{Z} u_+(\mathcal{E})$ *is an isomorphism.* We notice (for example, see 2.2.5) that $\mathcal{E}$ is also a coherent $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\#'},\mathbb{Q}}$-module that is a locally projective $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}$-module of finite type. By 2.2.7, since h is finite and étale, $h^{\#}_{+}(\mathcal{E})$ is a coherent $\mathcal{D}^{\dagger}_{\widehat{\mathbb{A}}^{n\#}_{\mathcal{V}},\mathbb{Q}}$-module that is a locally projective $\mathcal{O}_{\widehat{\mathbb{A}}^{n}_{\mathcal{V}},\mathbb{Q}}$-module of finite type and that satisfies both conditions (a) and (b$'$) of 1.3.6. Hence, by 2.2.3, $\mathbb{R}\underline{\Gamma}^{\dagger}_{H} w_+ h^{\#}_{+}(\mathcal{E}) = 0$. We have $h_+(\mathbb{R}\underline{\Gamma}^{\dagger}_{Z''} v_+(\mathcal{E})) \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^{\dagger}_{H} h_+ v_+(\mathcal{E}) \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^{\dagger}_{H} w_+ h^{\#}_{+}(\mathcal{E})$. (See [Car04, 2.2.18.2] for the first isomorphism and (1.2.6.1) for the second one.) Then, by 2.2.8, $\mathbb{R}\underline{\Gamma}^{\dagger}_{Z''} v_+(\mathcal{E}) = 0$. Since $Z \subset Z''$, we get $\mathbb{R}\underline{\Gamma}^{\dagger}_{Z} v_+(\mathcal{E}) = 0$.

It follows from (2.2.5.1) that $\mathcal{E}({}^{\dagger}Z') \xrightarrow{\sim} \tilde{u}_+(\mathcal{E})$. Then, by (1.2.6.1), $u_+(\mathcal{E}({}^{\dagger}Z')) \xrightarrow{\sim} u_+\tilde{u}_+(\mathcal{E}) \xrightarrow{\sim} v_+(\mathcal{E})$. This implies $\mathbb{R}\underline{\Gamma}^{\dagger}_{Z} u_+(\mathcal{E}({}^{\dagger}Z')) = 0$. By (1.3.10.1), $u_+(\mathcal{E}({}^{\dagger}Z')) \xrightarrow{\sim} (u_+(\mathcal{E}))({}^{\dagger}Z')$. Hence, $\mathbb{R}\underline{\Gamma}^{\dagger}_{Z}({}^{\dagger}Z') u_+(\mathcal{E}) = 0$. Using the exact triangle of localization of $\mathbb{R}\underline{\Gamma}^{\dagger}_{Z} u_+(\mathcal{E})$ with respect to Z', this means that the canonical morphism $\mathbb{R}\underline{\Gamma}^{\dagger}_{Z}\mathbb{R}\underline{\Gamma}^{\dagger}_{Z'} u_+(\mathcal{E}) \to \mathbb{R}\underline{\Gamma}^{\dagger}_{Z} u_+(\mathcal{E})$ is an isomorphism. Since $\mathbb{R}\underline{\Gamma}^{\dagger}_{Z\cap Z'} u_+(\mathcal{E}) \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^{\dagger}_{Z}\mathbb{R}\underline{\Gamma}^{\dagger}_{Z'} u_+(\mathcal{E})$ (see [Car04, 2.2.8]), we come down to prove $\mathbb{R}\underline{\Gamma}^{\dagger}_{Z'\cap Z} u_+(\mathcal{E}) = 0$.

3° *We check that* $\mathbb{R}\underline{\Gamma}^{\dagger}_{Z'\cap Z} u_+(\mathcal{E}) = 0$. When $Z \cap Z'$ is empty, this is obvious. It remains to deal with the case where $Z \cap Z'$ is not empty. Let x be a closed point of $Z \cap Z'$, let $\mathcal{Z}_1, \dots, \mathcal{Z}_r$ be the irreducible components of $\mathcal{Z}$ containing x, and let $\mathcal{Z}_{r+1}, \dots, \mathcal{Z}_s$ be the irreducible components of $\mathcal{Z}'$ containing x. Since $\mathbb{R}\underline{\Gamma}^{\dagger}_{Z'\cap Z} u_+(\mathcal{E})$ is zero outside $Z \cap Z'$, it is sufficient to prove its nullity around x. Then, we can suppose that $\mathcal{Z} = \mathcal{Z}_1 \cup \cdots \cup \mathcal{Z}_r$ and $\mathcal{Z}' = \mathcal{Z}_{r+1} \cup \cdots \cup \mathcal{Z}_s$.

To end the proof, we need the following lemma.

Lemma 2.2.9.1. *With the above notation, let* $\mathfrak{X}'$ *be an intersection of some irreducible components of* $\mathcal{Z}'$. *Let* $\mathfrak{X}'^{\#} := (\mathfrak{X}', \mathfrak{X}' \cap \mathcal{Z})$, $\iota\colon \mathfrak{X}' \hookrightarrow \mathfrak{X}$, $\iota^{\#}\colon \mathfrak{X}'^{\#} \hookrightarrow \mathfrak{X}^{\#}$, $u'\colon \mathfrak{X}'^{\#} \to \mathfrak{X}'$ *be the canonical morphisms. For any* $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}^{\#}})$, *we have the canonical isomorphism* $\iota^{!} u_+(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} u'_{+}\iota^{\#!}(\mathcal{E}^{(\bullet)})$.

Proof. We keep the notation of Section 1.2; e.g., X'_i means the reduction modulo π^{i+1} of $\mathfrak{X}'$, etc. From $\mathcal{D}^{(m)}_{X_i \leftarrow X_i^{\#}} \xrightarrow{\sim} \mathcal{D}^{(m)}_{X_i}(Z_i)$ (see [Car09a, 5.2.4]) and by [Car09a, 5.1.2], we get $\mathcal{D}^{(m)}_{X_i \leftarrow X_i^{\#}} \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{X_i^{\#}}} \mathcal{E}^{(m)}_i \xrightarrow{\sim} \mathcal{D}^{(m)}_{X_i} \otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{X_i^{\#}}} \mathcal{E}^{(m)}_i(Z_i)$.

Thus,

$$\mathcal{D}^{(m)}_{X'_i\to X_i}\otimes^{\mathbb{L}}_{\iota^{-1}\mathcal{D}^{(m)}_{X_i}}\iota^{-1}(\mathcal{D}^{(m)}_{X_i\leftarrow X_i^\#}\otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{X_i^\#}}\mathcal{E}^{(m)}_i)\xrightarrow{\sim}\mathcal{D}^{(m)}_{X'_i\to X_i}\otimes^{\mathbb{L}}_{\iota^{-1}\mathcal{D}^{(m)}_{X_i^\#}}\iota^{-1}\mathcal{E}^{(m)}_i(Z_i).$$

The canonical morphism $\mathcal{D}^{(m)}_{X_i^\#}\to\mathcal{D}^{(m)}_{X_i}$ induces canonically the morphism of $(\mathcal{D}^{(m)}_{X_i'^\#},\iota^{-1}\mathcal{D}^{(m)}_{X_i^\#})$-bimodules $\mathcal{D}^{(m)}_{X_i'^\#\to X_i^\#}\to\mathcal{D}^{(m)}_{X'_i\to X_i}$. We get $\mathcal{D}^{(m)}_{X'_i}\otimes_{\mathcal{D}^{(m)}_{X_i'^\#}}\mathcal{D}^{(m)}_{X_i'^\#\to X_i^\#}\to\mathcal{D}^{(m)}_{X'_i\to X_i}$. By a computation in local coordinates, we check that this morphism is an isomorphism. Since $\mathcal{D}^{(m)}_{X_i'^\#\to X_i^\#}$ is locally free over $\mathcal{D}^{(m)}_{X_i'^\#}$, we obtain $\mathcal{D}^{(m)}_{X'_i}\otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{X_i'^\#}}\mathcal{D}^{(m)}_{X_i'^\#\to X_i^\#}\xrightarrow{\sim}\mathcal{D}^{(m)}_{X'_i\to X_i}$. This implies

$$\mathcal{D}^{(m)}_{X'_i\to X_i}\otimes^{\mathbb{L}}_{\iota^{-1}\mathcal{D}^{(m)}_{X_i^\#}}\iota^{-1}\mathcal{E}^{(m)}_i(Z_i)\xrightarrow{\sim}(\mathcal{D}^{(m)}_{X'_i}\otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{X_i'^\#}}\mathcal{D}^{(m)}_{X_i'^\#\to X_i^\#})\otimes^{\mathbb{L}}_{\iota^{-1}\mathcal{D}^{(m)}_{X_i^\#}}\iota^{-1}(\mathcal{E}^{(m)}_i(Z_i)).$$

Moreover,

$$\mathcal{D}^{(m)}_{X_i'^\#\to X_i^\#}\otimes^{\mathbb{L}}_{\iota^{-1}\mathcal{D}^{(m)}_{X_i^\#}}\iota^{-1}(\mathcal{E}^{(m)}_i(Z_i))\xrightarrow{\sim}(\mathcal{D}^{(m)}_{X_i'^\#\to X_i^\#}\otimes^{\mathbb{L}}_{\iota^{-1}\mathcal{D}^{(m)}_{X_i^\#}}\iota^{-1}\mathcal{E}^{(m)}_i)(Z_i\cap X'_i).$$

From $\mathcal{D}^{(m)}_{X'_i\leftarrow X_i'^\#}\xrightarrow{\sim}\mathcal{D}^{(m)}_{X'_i}(Z_i\cap X'_i)$ (see [Car09a, 5.2.4]) and using the commutation of the functor '$-(Z_i\cap X'_i)$' with '$-\otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{X_i'^\#}}-$' (see [Car09a, 5.1.2]), we obtain

$$\mathcal{D}^{(m)}_{X'_i}\otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{X_i'^\#}}\left((\mathcal{D}^{(m)}_{X_i'^\#\to X_i^\#}\otimes^{\mathbb{L}}_{\iota^{-1}\mathcal{D}^{(m)}_{X_i^\#}}\iota^{-1}\mathcal{E}^{(m)}_i)(Z_i\cap X'_i)\right)$$
$$\xrightarrow{\sim}\mathcal{D}^{(m)}_{X'_i\leftarrow X_i'^\#}\otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{X_i'^\#}}(\mathcal{D}^{(m)}_{X_i'^\#\to X_i^\#}\otimes^{\mathbb{L}}_{\iota^{-1}\mathcal{D}^{(m)}_{X_i^\#}}\iota^{-1}\mathcal{E}^{(m)}_i).$$

Then, by composition we get $\mathcal{D}^{(m)}_{X'_i\to X_i}\otimes^{\mathbb{L}}_{\iota^{-1}\mathcal{D}^{(m)}_{X_i}}\iota^{-1}(\mathcal{D}^{(m)}_{X_i\leftarrow X_i^\#}\otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{X_i^\#}}\mathcal{E}^{(m)}_i)\xrightarrow{\sim}\mathcal{D}^{(m)}_{X'_i\leftarrow X_i'^\#}\otimes^{\mathbb{L}}_{\mathcal{D}^{(m)}_{X_i'^\#}}(\mathcal{D}^{(m)}_{X_i'^\#\to X_i^\#}\otimes^{\mathbb{L}}_{\iota^{-1}\mathcal{D}^{(m)}_{X_i^\#}}\iota^{-1}\mathcal{E}^{(m)}_i)$, which is up to a shift the required isomorphism at the level m. □

In particular, let $\mathcal{Z}_s^\#:=(\mathcal{Z}_s,\mathcal{Z}_s\cap\mathcal{Z})$, ι: $\mathcal{Z}_s\hookrightarrow\mathfrak{X}$, $\iota^\#$: $\mathcal{Z}_s^\#\hookrightarrow\mathfrak{X}^\#$, u': $\mathcal{Z}_s^\#\to\mathcal{Z}_s$ be the canonical morphisms. We obtain

$$\mathbb{R}\underline{\Gamma}^\dagger_{Z_s\cap Z}u_+(\mathcal{E})\xrightarrow{\sim}\mathbb{R}\underline{\Gamma}^\dagger_{Z}\iota_+\iota^!u_+(\mathcal{E})$$
$$\underset{(2.2.9.1)}{\xrightarrow{\sim}}\mathbb{R}\underline{\Gamma}^\dagger_{Z}\iota_+u'_+\iota^{\#!}(\mathcal{E})\xrightarrow{\sim}\iota_+\mathbb{R}\underline{\Gamma}^\dagger_{Z\cap Z_s}u'_+\iota^{\#!}(\mathcal{E}).$$

(See [Ber02, 4.4.5] for the first isomorphism.) Since $\mathcal{E}$ is flat over $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}$, then $\iota^{\#!}(\mathcal{E})[1] \xrightarrow{\sim} \iota^{\#*}(\mathcal{E})$. Since $\iota^{\#*}(\mathcal{E})$ is a coherent $\mathcal{D}^\dagger_{\mathcal{Z}_s^{\#},\mathbb{Q}}$-module which is a locally projective $\mathcal{O}_{\mathcal{Z}_s,\mathbb{Q}}$-module of finite type and which satisfies conditions (a) and (b′) of 1.3.6 (see the proof of 1.1.22), since $\dim Z_s < \dim X$, the induction hypothesis implies that $\mathbb{R}\underline{\Gamma}^\dagger_{Z\cap Z_s} u'_+ \iota^{\#!}(\mathcal{E}) = 0$. Then $\mathbb{R}\underline{\Gamma}^\dagger_{Z_s\cap Z} u_+(\mathcal{E}) = 0$. Similarly, we check that, for any j between $r+1$ and s, $\mathbb{R}\underline{\Gamma}^\dagger_{Z_j\cap Z} u_+(\mathcal{E}) = 0$. Hence, using Mayer-Vietoris exact triangles (see [Car04, 2.2.16]), $\mathbb{R}\underline{\Gamma}^\dagger_{Z'\cap Z} u_+(\mathcal{E}) = 0$. □

Examples 2.2.10. The exponents of an overconvergent isocrystal with nilpotent residues (see 2.1.1) are zero. Then the holonomicity of overconvergent isocrystals with unipotent monodromy along Z follows from 2.2.9.

PROPOSITION 2.2.11. *Let $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}^\#}({}^\dagger D)_{\mathbb{Q}})$. Suppose that there exists a smooth morphism $\mathfrak{X} \to \mathfrak{T}$ of smooth formal schemes over $\mathcal{S}$ such that $\mathcal{Z}$ is a relative strict normal crossings divisor of $\mathfrak{X}$ over $\mathfrak{T}$. Then, we have the canonical quasi-isomorphism*

$$\Omega^\bullet_{\mathfrak{X}^\#/\mathfrak{T},\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X},\mathbb{Q}}} \mathcal{E} \xrightarrow{\sim} \Omega^\bullet_{\mathfrak{X}/\mathfrak{T},\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X},\mathbb{Q}}} u_{D+}(\mathcal{E}). \tag{2.2.11.1}$$

Proof. The proof is similar to that of [Car09a, 6.3]. □

The second part of the next corollary improves the statements of 1.1.1 (or 1.3.6).

THEOREM 2.2.12. *Let $\mathcal{E}$ be a coherent $\mathcal{D}^\dagger_{\mathfrak{X}^\#,\mathbb{Q}}$-module that is a locally projective $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}$-module of finite type and that satisfies conditions* (a) *and* (b′) *of* 1.3.6. *Then $\mathcal{E}({}^\dagger Z)$ is a holonomic $\mathcal{D}^\dagger_{\mathfrak{X},\mathbb{Q}}$-module.*

Moreover, suppose that there exists a smooth morphism $\mathfrak{X} \to \mathfrak{T}$ of smooth formal schemes over $\mathcal{S}$ such that $\mathcal{Z}$ is a relative strict normal crossings divisor of $\mathfrak{X}$ over $\mathfrak{T}$. Then the canonical morphism $\Omega^\bullet_{\mathfrak{X}^\#/\mathfrak{T},\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X},\mathbb{Q}}} \mathcal{E} \to \Omega^\bullet_{\mathfrak{X}/\mathfrak{T},\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X},\mathbb{Q}}} \mathcal{E}({}^\dagger Z)$ is a quasi-isomorphism.

Proof. The first assertion is a consequence of [Car09a, 5.25] and the second one follows from 2.2.9 and 2.2.11. □

We finish this section by checking that the conclusions of Theorem 2.2.9 (and then Theorem 2.2.12) are stable under inverse image by smooth morphisms.

PROPOSITION 2.2.13. *Let $f\colon \mathfrak{X}' \to \mathfrak{X}$ be a smooth morphism of smooth formal $\mathcal{V}$-schemes, $\mathcal{Z}' := f^{-1}(\mathcal{Z})$, $\mathfrak{X}'^\# = (\mathfrak{X}', \mathcal{Z}')$, $u'\colon \mathfrak{X}'^\# \to \mathfrak{X}'$ be the canonical morphisms, and let $f^\#\colon \mathfrak{X}'^\# \to \mathfrak{X}^\#$ be the morphism induced by f. Let $\mathcal{E}$ be a coherent $\mathcal{D}^\dagger_{\mathfrak{X}^\#,\mathbb{Q}}$-module that is a locally projective $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}$-module of finite type.*

Then we have the canonical isomorphism

$$f^* u_+(\mathcal{E}) \xrightarrow{\sim} u'_+ f^{\#*}(\mathcal{E}). \tag{2.2.13.1}$$

Proof. We have

$$u'_+ f^{\#*}(\mathcal{E}) \xrightarrow{\sim} \mathcal{D}^\dagger_{\mathfrak{X}',\mathbb{Q}} \otimes_{\mathcal{D}^\dagger_{\mathfrak{X}'^\#,\mathbb{Q}}} (\mathcal{D}^\dagger_{\mathfrak{X}'^\# \to \mathfrak{X}^\#,\mathbb{Q}} \otimes_{f^{-1}\mathcal{D}^\dagger_{\mathfrak{X}^\#,\mathbb{Q}}} f^{-1}\mathcal{E})(\mathcal{Z}')$$

(see 1.3.8 for the direct image). The canonical morphism $\mathcal{D}^\dagger_{\mathfrak{X}'^\# \to \mathfrak{X}^\#,\mathbb{Q}} \to \mathcal{D}^\dagger_{\mathfrak{X}' \to \mathfrak{X},\mathbb{Q}}$ induces the morphism of coherent $\mathcal{D}^\dagger_{\mathfrak{X}',\mathbb{Q}}$-modules (which are also $(\mathcal{D}^\dagger_{\mathfrak{X}',\mathbb{Q}}, f^{-1}\mathcal{D}^\dagger_{\mathfrak{X}^\#,\mathbb{Q}})$-bimodules) $\mathcal{D}^\dagger_{\mathfrak{X}',\mathbb{Q}} \otimes_{\mathcal{D}^\dagger_{\mathfrak{X}'^\#,\mathbb{Q}}} \mathcal{D}^\dagger_{\mathfrak{X}'^\# \to \mathfrak{X}^\#,\mathbb{Q}} \to \mathcal{D}^\dagger_{\mathfrak{X}' \to \mathfrak{X},\mathbb{Q}}$. We compute that this morphism is an isomorphism. (We come down to the case of log-schemes which corresponds to a computation in local coordinates.) Then

$$u'_+ f^{\#*}(\mathcal{E}) \xrightarrow{\sim} \mathcal{D}^\dagger_{\mathfrak{X}' \to \mathfrak{X},\mathbb{Q}} \otimes_{f^{-1}\mathcal{D}^\dagger_{\mathfrak{X}^\#,\mathbb{Q}}} f^{-1}\mathcal{E}(\mathcal{Z}')$$
$$\xrightarrow{\sim} \mathcal{D}^\dagger_{\mathfrak{X}' \to \mathfrak{X},\mathbb{Q}} \otimes_{f^{-1}\mathcal{D}^\dagger_{\mathfrak{X},\mathbb{Q}}} f^{-1}(\mathcal{D}^\dagger_{\mathfrak{X},\mathbb{Q}} \otimes_{\mathcal{D}^\dagger_{\mathfrak{X}^\#,\mathbb{Q}}} \mathcal{E}(\mathcal{Z})) \xrightarrow{\sim} f^* u_+(\mathcal{E}). \quad \square$$

COROLLARY 2.2.14. *With the notation of* 2.2.13, *if the morphism* $u_+(\mathcal{E}) \to \mathcal{E}({}^\dagger Z)$ *is an isomorphism, then so is* $u'_+(f^{\#*}(\mathcal{E})) \to f^{\#*}(\mathcal{E})({}^\dagger Z')$.

2.3. *Overholonomicity of overconvergent F-isocrystals.*

Definition 2.3.1. Let $\mathfrak{X}$ be a smooth formal $\mathcal{V}$-scheme.

(1) Let $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}})$. Let Y be a subscheme of X such that there exists a divisor T of X satisfying $Y = \overline{Y} \setminus T$, where $\overline{Y}$ is the closure of Y in X. The complex $\mathcal{E}^{(\bullet)}$ *is smoothly dévissable over* Y *in partially overconvergent isocrystals* if there exist some divisors $T_1, \ldots, T_r$ containing T with $T_r = T$ such that, for any $i := 0, \ldots, r-1$ and posing $T_0 := \overline{Y}$, $Y_i := T_0 \cap T_1 \cap \cdots \cap T_i \setminus T_{i+1}$, we have $\overline{Y}_i$ smooth and the cohomological spaces of $\varinjlim \mathbb{R}\underline{\Gamma}^\dagger_{Y_i}(\mathcal{E}^{(\bullet)})$ (see [Car07a, 3.2.1]) are in the essential image of the functor $\mathrm{sp}_{\overline{Y}_i \hookrightarrow \mathfrak{X}, T_{i+1},+}$, where $\mathrm{sp}_{\overline{Y}_i \hookrightarrow \mathfrak{X}, T_{i+1},+}$ is the canonical fully faithful functor from the category of isocrystals on Y_i overconvergent along $\overline{Y}_i \setminus Y_i$ to the category of coherent $\mathcal{D}^\dagger_{\mathfrak{X}}({}^\dagger T_{i+1})_{\mathbb{Q}}$-modules (see [Car09b]). To simplify the notation, we shall sometimes suppress $\varinjlim$.

More precisely, we can say that the complex $\mathcal{E}^{(\bullet)}$ *is smoothly dévissable over the stratification* $Y = \sqcup_{i=0,\ldots,r-1} Y_i$ *in partially overconvergent isocrystals* or $(T_1, \ldots, T_r)$ *gives a smooth dévissage over* Y *of* $\mathcal{E}^{(\bullet)}$ *in partially overconvergent isocrystals*.

We point out that this notion of smooth devissability of $\mathcal{E}^{(\bullet)}$ over the stratification $Y = \sqcup_{i=0,\ldots,r-1} Y_i$ in partially overconvergent isocrystals is well defined since this does not depend on the choice of the divisor T of X such that $Y = \overline{Y} \setminus T$. Indeed, let T' be another divisor of X such that $Y = \overline{Y} \setminus T'$. For $i := 1, \ldots, r-1$, we pose $T'_i = T_i \cup T'$. We pose $T'_0 = T_0$ and $T'_r := T'$. For $i := 0, \ldots, r-1$, we check that $Y_i := T_0 \cap T_1 \cap \cdots \cap T_i \setminus T_{i+1} = T'_0 \cap T'_1 \cap \cdots \cap T'_i \setminus T'_{i+1}$. Then $(T'_1, \ldots, T'_r)$ gives a smooth dévissage over the stratification $Y = \sqcup_{i=0,\ldots,r-1} Y_i$ of $\mathcal{E}^{(\bullet)}$ in partially overconvergent isocrystals

(2) Let D be a divisor of X, $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{X}}({}^{\dagger}D)_{\mathbb{Q}})$ and $\mathcal{E}^{(\bullet)} \in \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}}(D))$ such that $\varinjlim(\mathcal{E}^{(\bullet)}) \xrightarrow{\sim} \mathcal{E}$. (This has a meaning since $\varinjlim$ induces the equivalence of categories $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{coh}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}}(D)) \cong D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{X}}({}^{\dagger}D)_{\mathbb{Q}})$.)

We say that $\mathcal{E}$ is smoothly dévissable in partially overconvergent isocrystals if $\mathcal{E}^{(\bullet)}$ is smoothly dévissable over $X \setminus D$ in partially overconvergent isocrystals.

Let $T_1, \ldots, T_r$ be some divisors of X such that T_r is empty. We pose, for $i = 0, \ldots, r$, $T'_i := T_i \cup D$. We say that $(T_1, \ldots, T_r)$ (*resp.* $(T'_1, \ldots, T'_r)$) *gives a smooth dévissage of* $\mathcal{E}$ *over* X (*resp.* $X \setminus D$) *in partially overconvergent isocrystals* if $(T_1, \ldots, T_r)$ (resp. $(T'_1, \ldots, T'_r)$) gives a smooth dévissage over X (resp. $X \setminus D$) of $\mathcal{E}^{(\bullet)}$ in partially overconvergent isocrystals.

Remarks 2.3.2. (1) With the notation 2.3.1(1), for any $i = 0, \ldots, r$, let $X_i := T_0 \cap T_1 \cap \cdots \cap T_i$. Then, for any $i = 0, \ldots, r-1$, the exact triangle of localization of $\mathbb{R}\underline{\Gamma}^{\dagger}_{X_i}(\mathcal{E}^{(\bullet)})$ with respect to T_{i+1} is

$$\mathbb{R}\underline{\Gamma}^{\dagger}_{X_{i+1}}(\mathcal{E}^{(\bullet)}) \to \mathbb{R}\underline{\Gamma}^{\dagger}_{X_i}(\mathcal{E}^{(\bullet)}) \to \mathbb{R}\underline{\Gamma}^{\dagger}_{Y_i}(\mathcal{E}^{(\bullet)}) \to \mathbb{R}\underline{\Gamma}^{\dagger}_{X_{i+1}}(\mathcal{E}^{(\bullet)})[1],$$

which explains the word "dévissage."

(2) With the notation 2.3.1(2), $\mathcal{E}$ is smoothly dévissable over X in partially overconvergent isocrystals if and only if it is so over $X \setminus D$. Indeed, if $(T_1, \ldots, T_r = \emptyset)$ gives a smooth dévissage of $\mathcal{E}$ over X in partially overconvergent isocrystals, then $(T'_1, \ldots, T'_r = D)$ gives a smooth dévissage of $\mathcal{E}$ over $X \setminus D$. Conversely, if $(T'_1, \ldots, T'_r = D)$ gives a smooth dévissage of $\mathcal{E}$ over $X \setminus D$, then $(T'_1, \ldots, T'_r, T'_{r+1} := \emptyset)$ gives a smooth dévissage of $\mathcal{E}$ over X in partially overconvergent isocrystals.

2.3.3. Similar to [Car07a, 3.2.7–8], we have the following result. Let $\mathfrak{X}$ be a smooth formal $\mathcal{V}$-scheme and Y a subscheme of X. We suppose that there exists a divisor T of X such that $Y = \overline{Y} \setminus T$. Let $\mathcal{E} \in F\text{-}\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}({}^{\mathrm{g}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathcal{P}})$. Let $T_1, \ldots, T_r$ be some divisors of P containing T with $T_r = T$ and, for any $i := 0, \ldots, r-1$, $Y_i := T_0 \cap T_1 \cap \cdots \cap T_i \setminus T_{i+1}$ where $T_0 := \overline{Y}$.

If, for any $i := 0, \ldots, r-1$, $\mathcal{E}$ is smoothly dévissable over Y_i in partially overconvergent isocrystals, then so is $\mathcal{E}$ over Y.

More precisely, for any $i = 0, \ldots, r-1$, let $T_{(i,1)}, \ldots, T_{(i,r_i)}$ be some divisors containing T_{i+1} with $T_{(i,r_i)} = T_{i+1}$ such that if $T_{(i,0)} := \overline{Y}_i$ and, for any $h =$

$0, \ldots, r_i - 1$, $Y_{(i,h)} := T_{(i,0)} \cap \cdots \cap T_{(i,h)} \setminus T_{(i,h+1)}$, then $\overline{Y_{(i,h)}}$ is smooth and, for any integer j, $\mathcal{H}^j(\underrightarrow{\lim}\mathbb{R}\underline{\Gamma}^{\dagger}_{Y_{(i,h)}}\mathcal{E})$ is in the essential image of $\mathrm{sp}_{\overline{Y_{(i,h)}} \hookrightarrow X, T_{(i,h+1)},+}$.

Then $(T_{(0,1)}, \ldots, T_{(0,r_0)}, T_{(1,1)}, \ldots, T_{(1,r_1)}, \ldots, T_{(r-1,1)}, \ldots, T_{(r-1,r_{r-1})})$ gives a smooth dévissage of $\mathcal{E}$ in partially overconvergent isocrystals over the stratification

(2.3.3.1)

$$Y = Y_{(0,0)} \sqcup \cdots \sqcup Y_{(0,r_0-1)} \bigsqcup Y_{(1,0)} \sqcup \cdots \sqcup Y_{(1,r_1-1)} \bigsqcup \cdots \bigsqcup Y_{(r-1,0)} \sqcup \cdots \sqcup Y_{(r-1,r_{r-1}-1)}.$$

PROPOSITION 2.3.4. *Let* $\mathfrak{A} = \mathrm{Spf}\,\mathcal{V}\{t_1, \ldots, t_n\}$ *and, for* $i = 1, \ldots, n$, *let* $\mathfrak{H}_i$ *be the formal closed subscheme of* $\mathfrak{A}$ *defined by* $t_i = 0$; *i.e.,* $\mathfrak{H}_i = \mathrm{Spf}\,\mathcal{V}\{t_1, \ldots, \widehat{t_i}, \ldots, t_n\}$. *We fix* I *and* I' *two subsets of* $\{1, \ldots, n\}$ *such that* $I \cap I'$ *is empty. We pose* $\mathfrak{H} := \cup_{i \in I}\mathfrak{H}_i$ *and* $\mathfrak{H}' := \cup_{i' \in I'}\mathfrak{H}_{i'}$. *Let* $\mathfrak{A}^{\#} := (\mathfrak{A}, \mathfrak{H})$ *and* $w\colon \mathfrak{A}^{\#} \to \mathfrak{A}$ *be the canonical morphism.*

Then there exist some divisors $T_1, \ldots, T_N$ *that satisfy the following property. If* $\mathcal{E}^{\bullet}$ *is any bounded complex of coherent* $\mathcal{D}^{\dagger}_{\mathfrak{A}^{\#}}({}^{\dagger}H')_{\mathbb{Q}}$*-modules, locally projective of finite type as* $\mathcal{O}_{\mathfrak{A}}({}^{\dagger}H')_{\mathbb{Q}}$*-module, and such that conditions* (a) *and* (b) *of* 1.1.1 *hold, then* $T_1, \ldots, T_N$ *gives a smooth dévissage of* $w_{H'+}(\mathcal{E}^{\bullet})$ *in partially overconvergent isocrystals over* $\mathbb{A}^n_k$.

Moreover, one may assume that, for $1 \le i \le N-1$, *the divisor* T_i *is such that* $H' \subset T_i \subset H \cup H'$, *and that* $T_1 = H \cup H'$, $T_{N-1} = H'$ *and* $T_N = \emptyset$.

Proof.

0° *Induction.* For the sake of convenience, we add the case $n = 0$ where $\mathfrak{A} = \mathrm{Spf}\,\mathcal{V}$ (and then I and I' are empty). We proceed by induction on the lexicographic order $(n, |I|)$, with $n \ge 0$. The case $n = 0$ is obvious. So we can suppose that $n \ge 1$ and the proposition is checked for $n - 1$. Moreover, the case where $|I| = 0$ means that H is empty. This case is thus straightforward. So, we come down to treat the case $|I| \ge 1$. Up to a re-indexation, we can suppose $1 \in I$.

1° *We come down to the case where* $\mathcal{E}^{\bullet}$ *is a module.* So, suppose here that there exist some divisors $T_1, \ldots, T_N$ such that, for any coherent $\mathcal{D}^{\dagger}_{\mathfrak{A}^{\#}}({}^{\dagger}H')_{\mathbb{Q}}$-module $\mathcal{E}$, locally projective of finite type as $\mathcal{O}_{\mathfrak{A}}({}^{\dagger}H')_{\mathbb{Q}}$-module and satisfying conditions (a) and (b) above, $T_1, \ldots, T_N$ give a smooth dévissage of $w_{H'+}(\mathcal{E})$ in partially overconvergent isocrystals over $\mathbb{A}^n_k$.

Following [Car09a, 5.25.1], for any coherent $\mathcal{D}^{\dagger}_{\mathfrak{A}^{\#}}({}^{\dagger}H')_{\mathbb{Q}}$-module $\mathcal{E}$, locally projective of finite type as $\mathcal{O}_{\mathfrak{A}}({}^{\dagger}H')_{\mathbb{Q}}$-module, for any $j \ne 0$, $\mathcal{H}^j(w_{H'+}(\mathcal{E})) = 0$. We pose $\mathcal{F}^{\bullet} := w_{H'+}(\mathcal{E}^{\bullet})$. Then, for any integer r, $\mathcal{F}^r = w_{H'+}(\mathcal{E}^r)$.

For any $i := 0, \ldots, r-1$, let $Y_i := T_0 \cap T_1 \cap \cdots \cap T_i \setminus T_{i+1}$ (with $T_0 := \overline{Y}$) and pose $\Phi := \underline{\Gamma}^{\dagger}_{Y_i} = \underline{\Gamma}^{\dagger}_{T_0 \cap T_1 \cap \cdots \cap T_i} \circ ({}^{\dagger}T_{i+1})$. Then, the first spectral sequence of hypercohomology of Φ gives $E_1^{r,s} = \mathcal{H}^s(\mathbb{R}\phi(\mathcal{F}^r)) \Rightarrow \mathcal{H}^n(\mathbb{R}\phi(\mathcal{F}^{\bullet}))$. If for any r, s, $\mathcal{H}^s(\mathbb{R}\phi(\mathcal{F}^r))$ is an isocrystal on Y_i overconvergent along $\overline{Y_i} \setminus Y_i$, then so is

$\mathcal{H}^n(\mathbb{R}\phi(\mathcal{F}^\bullet))$. Then we can suppose that $\mathcal{F}^\bullet$ has only one term. Thus, $\mathcal{E}^\bullet$ has only one term. From now, we will write $\mathcal{E}$ instead of $\mathcal{E}^\bullet$.

2° *Dévissage.* Via the exact triangle of localization of $w_{H'+}(\mathcal{E})$ with respect to H, it is sufficient to check that $\mathbb{R}\underline{\Gamma}^\dagger_H w_{H'+}(\mathcal{E})$ is smoothly dévissable in partially overconvergent isocrystals.

The exact triangle of localization of $\mathbb{R}\underline{\Gamma}^\dagger_H w_{H'+}(\mathcal{E})$ with respect to H_1 is of the form

(2.3.4.1)
$$\mathbb{R}\underline{\Gamma}^\dagger_{H_1} w_{H'+}(\mathcal{E}) \to \mathbb{R}\underline{\Gamma}^\dagger_H w_{H'+}(\mathcal{E}) \to ({}^\dagger H_1)\mathbb{R}\underline{\Gamma}^\dagger_H w_{H'+}(\mathcal{E}) \to \mathbb{R}\underline{\Gamma}^\dagger_{H_1} w_{H'+}(\mathcal{E})[1].$$

From the exact triangle (2.3.4.1) and using 2.3.3, it is sufficient to check the following two last steps.

3° $({}^\dagger H_1)\mathbb{R}\underline{\Gamma}^\dagger_H w_{H'+}(\mathcal{E})$ *is smoothly dévissable in partially overconvergent isocrystals.* Let $\widetilde{\mathfrak{H}} := \cup \mathfrak{H}_{i\in I\setminus\{1\}}$, $\widetilde{w} : (\mathfrak{A}, \widetilde{\mathfrak{H}}) \to \mathfrak{A}$ be the canonical map. Similarly to the beginning of the proof of 2.2.3 (i.e., using a Mayer-Vietoris exact triangle), we get the second isomorphism $({}^\dagger H_1)\mathbb{R}\underline{\Gamma}^\dagger_H w_{H'+}(\mathcal{E}) \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^\dagger_H \circ ({}^\dagger H_1)\circ w_{H'+}(\mathcal{E}) \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^\dagger_{\widetilde{H}} \circ ({}^\dagger H_1) \circ w_{H'+}(\mathcal{E})$. We get from (2.2.2.2) the isomorphism $({}^\dagger H_1)(w_{H'+}(\mathcal{E})) \xleftarrow{\sim} \widetilde{w}_{H'\cup H_1,+}(\mathcal{E}({}^\dagger H_1))$. Thus, $({}^\dagger H_1)\mathbb{R}\underline{\Gamma}^\dagger_H w_{H'+}(\mathcal{E}) \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^\dagger_{\widetilde{H}} \widetilde{w}_{H'\cup H_1,+}(\mathcal{E}({}^\dagger H_1))$. By the induction hypothesis, $\mathbb{R}\underline{\Gamma}^\dagger_{\widetilde{H}} \widetilde{w}_{H'\cup H_1,+}(\mathcal{E}({}^\dagger H_1))$ is smoothly dévissable in partially overconvergent isocrystals.

4° $\mathbb{R}\underline{\Gamma}^\dagger_{H_1} w_{H'+}(\mathcal{E})$ *is smoothly dévissable in partially overconvergent isocrystals.* Let $\mathfrak{H}_1^\# = (\mathfrak{H}_1, \mathfrak{H}_1 \cap \widetilde{\mathfrak{H}})$, $i_1 : \mathfrak{H}_1 \hookrightarrow \mathfrak{A}$, $g_1 : \mathfrak{A} \to \mathfrak{H}_1$, $g_1^\# : \mathfrak{A}^\# \to \mathfrak{H}_1^\#$, $w_1 : \mathfrak{H}_1^\# \to \mathfrak{H}_1$ be the canonical morphisms.

By 1.3.13 (and with the remark 1.3.14), $g^\#_{1+} \circ \mathbb{R}\underline{\Gamma}^\dagger_{H_1}(\mathcal{E})$ is a complex of coherent $\mathcal{D}^\dagger_{\mathfrak{H}_1^\#}({}^\dagger H_1 \cap H')_{\mathbb{Q}}$-modules, locally projective of finite type as $\mathcal{O}_{\mathfrak{H}_1}({}^\dagger H_1 \cap H')_{\mathbb{Q}}$-modules and satisfying conditions (a) and (b). Then, by induction hypothesis, $w_{1+} \circ g^\#_{1+} \circ \mathbb{R}\underline{\Gamma}^\dagger_{H_1}(\mathcal{E})$ is smoothly dévissable in partially overconvergent isocrystals. Moreover,

$$\begin{aligned}(2.3.4.2)\quad w_{1,+} \circ g^\#_{1+} \circ \mathbb{R}\underline{\Gamma}^\dagger_{H_1}(\mathcal{E}) &\xrightarrow{\sim} g_{1+} \circ w_{H'+} \circ \mathbb{R}\underline{\Gamma}^\dagger_{H_1}(\mathcal{E}) \\ &\underset{1.3.11}{\xrightarrow{\sim}} g_{1+} \circ \mathbb{R}\underline{\Gamma}^\dagger_{H_1} \circ w_{H'+}(\mathcal{E}) \xrightarrow{\sim} i_1^! w_{H'+}(\mathcal{E}).\end{aligned}$$

Thus, $i_1^! w_{H'+}(\mathcal{E})$ is smoothly dévissable in partially overconvergent isocrystals and so is $\mathbb{R}\underline{\Gamma}^\dagger_{H_1} w_{H'+}(\mathcal{E})$. □

In order to prove Theorem 2.3.13, we need the following definition.

Definition 2.3.5. Let $\mathfrak{X}$ be a smooth formal $\mathcal{V}$-scheme, D a divisor of X and $\mathcal{E} \in D(\mathcal{D}^\dagger_{\mathfrak{X}}({}^\dagger D)_{\mathbb{Q}})$. Let n, r be integers such that $n \geq 0$ and $r \geq -1$. We define by induction on r the notion of (n, r)-overholonomicity as follows.

- To avoid confusion with the coherence over $\mathcal{D}^{\dagger}_{\mathfrak{X}}({}^{\dagger}D)_{\mathbb{Q}}$, we will say that $\mathcal{E}$ is $(n,-1)$-overholonomic if $\mathcal{E}\in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{X},\mathbb{Q}})$ and $\dim X\leq n$.
- We say that $\mathcal{E}$ is $(n,0)$-overholonomic if $\mathcal{E}$ is $(n,-1)$-overholonomic and for any smooth morphism of formal $\mathcal{V}$-schemes of the form $f:\mathfrak{X}'\to\mathfrak{X}$ with $\dim X'\leq n$, for any divisor T' of X', we have $({}^{\dagger}T')f^{!}(\mathcal{E})\in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{X}',\mathbb{Q}})$.
- Suppose $r\geq 1$. We say that $\mathcal{E}$ is (n,r)-overholonomic if $\mathcal{E}$ is $(n,r-1)$-overholonomic and for any smooth morphism of formal $\mathcal{V}$-schemes of the form $f:\mathfrak{X}'\to\mathfrak{X}$ with $\dim X'\leq n$, for any divisor T' of X', the complex $\mathbb{D}({}^{\dagger}T')f^{!}(\mathcal{E})$ is $(n,r-1)$-overholonomic.

We say that $\mathcal{E}$ is r-overholonomic if $\mathcal{E}$ is (n,r)-overholonomic for any $n\in\mathbb{N}$, which is exactly (when $r\geq 0$) the previous definition of r-overholonomic that appears in [Car09c, 3.1].

2.3.6 (Stability of the (n,r)-overholonomicity). We keep the above notation 2.3.5.

- We will use freely the easy properties of the stability of the (n,r)-overholonomicity similar to [Car09c, 3.3] (i.e., stability under extension, overconvergent cohomological local functors, duality. . .).
- Let $f:\mathfrak{X}'\to\mathfrak{X}$ be a proper morphism of smooth formal $\mathcal{V}$-schemes of relative dimension d_f and such that $\dim X\leq n$. For any $\mathcal{E}'\in D(\mathcal{D}^{\dagger}_{\mathfrak{X}',\mathbb{Q}})$, if $\mathcal{E}'$ is $(n+d_f,r)$-overholonomic, then $f_{+}(\mathcal{E}')$ is (n,r)-overholonomic. (The proof is the same than [Car09c, 3.9].)
- Let $f:\mathfrak{X}'\to\mathfrak{X}$ be a morphism of smooth formal $\mathcal{V}$-schemes of relative dimension d_f and $\mathcal{E}\in D(\mathcal{D}^{\dagger}_{\mathfrak{X},\mathbb{Q}})$ be a (n,r)-overholonomic complex. When f is smooth and $\dim X'\leq n$, it is clear that $f^{!}(\mathcal{E})$ is (n,r)-overholonomic. But, we will be careful with the fact that when f is a closed immersion, it is not obvious that $f^{!}(\mathcal{E})$ is $(n+d_f,r)$-overholonomic. (In the proof of [Car09c, 3.8], we do not control the dimension of the smooth formal $\mathcal{V}$-schemes.) So, we will avoid using this latter fact.

LEMMA 2.3.7. *Let $\mathfrak{A}=\operatorname{Spf}\mathcal{V}\{t_1,\ldots,t_n\}$ and, for $i=1,\ldots,n$, let $\mathfrak{H}_i$ be the formal closed subscheme of $\mathfrak{A}$ defined by $t_i=0$. Let I be a subset of $\{1,\ldots,n\}$. We pose $\mathfrak{H}:=\cup_{i\in I}\mathfrak{H}_i$. Let $\mathfrak{A}^{\#}:=(\mathfrak{A},\mathfrak{H})$, w: $\mathfrak{A}^{\#}\to\mathfrak{A}$ be the canonical morphism. Let $\mathcal{E}$ be coherent $\mathcal{D}^{\dagger}_{\mathfrak{A}^{\#},\mathbb{Q}}$-module, locally projective of finite type as $\mathcal{O}_{\mathfrak{A},\mathbb{Q}}$-module and satisfying conditions* (a) *and* (b) *of* 1.1.1.

Let $T_1,\ldots,T_N$ be some sub-divisors of H such that $T_1=H$, $T_N=\emptyset$, and $(T_1,\ldots,T_N)$ give a smooth dévissage of $w_{+}(\mathcal{E})$ in partially overconvergent isocrystals over $\mathbb{A}^n_k$ (by 2.3.4, such divisors exist). Then the partially overconvergent isocrystals that appear in the smooth dévissage of $w_{+}(\mathcal{E})$ given by the divisors $(T_1,\ldots,T_N)$ are -1-overholonomic.

Proof. First, we prove by induction on n that, for any subset $J \subset I$, $\mathbb{R}\underline{\Gamma}^{\dagger}_{H_J} w_+(\mathcal{E}) \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{X},\mathbb{Q}})$, where $\mathfrak{H}_J = \cap_{j \in J} \mathfrak{H}_j$.

Let J a subset of I. The case where J is empty is obvious. So, we come down to treat the case $|J| \geq 1$. Up to a re-indexation, we can suppose $1 \in J$. From (2.3.4.2) and with its notation, we get $w_{1,+} \circ g^{\#}_{1+} \circ \mathbb{R}\underline{\Gamma}^{\dagger}_{H_1}(\mathcal{E}) \xrightarrow{\sim} i^!_1 w_+(\mathcal{E})$, where $g^{\#}_{1+} \circ \mathbb{R}\underline{\Gamma}^{\dagger}_{H_1}(\mathcal{E})$ is a complex of coherent $\mathcal{D}^{\dagger}_{\mathfrak{H}^{\#}_1,\mathbb{Q}}$-modules, locally projective of finite type as $\mathcal{O}_{\mathfrak{H}_1,\mathbb{Q}}$-modules and satisfying conditions (a) and (b). Then, by the induction hypothesis, $\mathbb{R}\underline{\Gamma}^{\dagger}_{H_J} i^!_1 w_+(\mathcal{E}) \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{H}_1,\mathbb{Q}})$. Since $\mathbb{R}\underline{\Gamma}^{\dagger}_{H_J} w_+(\mathcal{E}) \xrightarrow{\sim} i_{1+} i^!_1 \mathbb{R}\underline{\Gamma}^{\dagger}_{H_J} w_+(\mathcal{E}) \xrightarrow{\sim} i_{1+} \mathbb{R}\underline{\Gamma}^{\dagger}_{H_J} i^!_1 w_+(\mathcal{E})$, it follows that $\mathbb{R}\underline{\Gamma}^{\dagger}_{H_J} w_+(\mathcal{E}) \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{X},\mathbb{Q}})$.

Secondly, let J and J' be two subsets of I. Then, using a Mayer-Vietoris exact sequence, since $H_J \cap H_{J'} = H_{J \cup J'}$, we check that $\mathbb{R}\underline{\Gamma}^{\dagger}_{H_J \cup H_{J'}} w_+(\mathcal{E}) \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{X},\mathbb{Q}})$. Similarly, we obtain by induction on $r \geq 1$ that, for any subsets $J_1, \ldots, J_r$ of I, the complex $\mathbb{R}\underline{\Gamma}^{\dagger}_{\cup_{s=1,\ldots,r} H_{J_s}} w_+(\mathcal{E})$ belongs to $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{X},\mathbb{Q}})$. If D_1 and D_2 are some closed subschemes that are a finite union of some closed subschemes of the form H_J with J as subset of I, by the exact triangle of localization of $\mathbb{R}\underline{\Gamma}^{\dagger}_{D_1} w_+(\mathcal{E})$ with respect to D_2, we get $({}^{\dagger}D_2) \circ \mathbb{R}\underline{\Gamma}^{\dagger}_{D_1} w_+(\mathcal{E}) \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{X},\mathbb{Q}})$. Moreover, since any divisor $T_1, \ldots, T_N$ of 2.3.4 is a sub-divisor of H, then the partially overconvergent isocrystals that appear in the smooth dévissage of $w_+(\mathcal{E})$ given by the divisors $T_1, \ldots, T_N$ are of the form of $({}^{\dagger}D_2) \circ \mathbb{R}\underline{\Gamma}^{\dagger}_{D_1} w_+(\mathcal{E}) \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{X},\mathbb{Q}})$ such as above, which finishes the proof. □

LEMMA 2.3.8. *Let $r \geq -1$ and $n \geq 0$ be two integers, h: $\mathfrak{X} \to \mathfrak{X}'$ be a finite and étale morphism of smooth formal $\mathcal{V}$-schemes, and D' be a divisor of X', $D := h^{-1}(D')$, $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{X}}({}^{\dagger}D)_{\mathbb{Q}})$. If $h_+(\mathcal{E})$ is (n, r)-overholonomic* (*see the definition of* 2.3.5) *and smoothly dévissable in partially overconvergent isocrystals, then $\mathcal{E}$ is (n, r)-overholonomic and smoothly dévissable in partially overconvergent isocrystals.*

Proof. Let Z' be a smooth closed subscheme of X', T' a divisor which contains D' such that $T' \cap X'$ is a divisor of Z' and the cohomological spaces of $\mathbb{R}\underline{\Gamma}^{\dagger}_{Z'}({}^{\dagger}T')(h_+(\mathcal{E}))$ are in the essential image of the functor $\mathrm{sp}_{Z' \hookrightarrow \mathfrak{X}', T', +}$. Pose $T := h^{-1}(T')$ and $Z := h^{-1}(Z')$. Then, $h_+(\mathbb{R}\underline{\Gamma}^{\dagger}_{Z}({}^{\dagger}T)(\mathcal{E})) \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^{\dagger}_{Z'}({}^{\dagger}T')(h_+(\mathcal{E}))$. By smooth dévissage, we come down to the case where $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{X}}({}^{\dagger}T)_{\mathbb{Q}})$, $\mathcal{E} \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^{\dagger}_{Z}({}^{\dagger}T)(\mathcal{E})$ and the cohomological spaces of $h_+(\mathcal{E})$ are in the essential image of the functor $\mathrm{sp}_{Z' \hookrightarrow \mathfrak{X}', T', +}$.

(1) First, we prove that in that case the cohomological spaces of $\mathcal{E}$ are in the essential image of the functor $\mathrm{sp}_{Z \hookrightarrow \mathfrak{X}, T, +}$. Since h_+ is exact, we can suppose that $\mathcal{E}$ is a coherent $\mathcal{D}^{\dagger}_{\mathfrak{X}}({}^{\dagger}T)_{\mathbb{Q}}$-module. Since this is local in $\mathfrak{X}$ and h is affine, we can suppose $\mathfrak{X}$ and $\mathfrak{X}'$ affine. Then, there exists respectively some liftings a :

$\mathcal{Z} \to \mathcal{Z}'$, $\iota : \mathcal{Z} \hookrightarrow \mathfrak{X}$, $\iota' : \mathcal{Z}' \hookrightarrow \mathfrak{X}'$ of $Z \to Z'$, $Z \hookrightarrow X$, $Z' \hookrightarrow X'$. Since h_+ commutes with the overconvergent local cohomology, $\iota'_+ \iota'^!(h_+(\mathcal{E})) \xrightarrow{\sim} h_+ \iota_+ \iota^!(\mathcal{E})$. Because the direct images of arithmetic $\mathcal{D}$-modules do not depend (up to a canonical isomorphism) on the choice of the lifting, $h_+ \iota_+ \iota^!(\mathcal{E}) \xrightarrow{\sim} \iota'_+ a_+ \iota^!(\mathcal{E})$. Hence, $\iota'_+ \iota'^!(h_+(\mathcal{E})) \xrightarrow{\sim} \iota'_+ a_+ \iota^!(\mathcal{E})$. Since $\iota'^! \iota'_+ \xrightarrow{\sim} \mathrm{Id}$, $\iota'^!(h_+(\mathcal{E})) \xrightarrow{\sim} a_+ \iota^!(\mathcal{E})$. This means that $\iota^!(\mathcal{E})$ is a coherent $\mathcal{D}^\dagger_{\mathcal{Z}}({}^\dagger T \cap Z)_{\mathbb{Q}}$-module (because $\mathcal{E}$ has its support in Z) such that $a_+ \iota^!(\mathcal{E})$ is $\mathcal{O}_{\mathcal{Z}'}({}^\dagger T' \cap Z')_{\mathbb{Q}}$-coherent. Let $\mathcal{Y} := \mathcal{Z} \setminus T$, $\mathcal{Y}' := \mathcal{Z}' \setminus T'$. Since the morphism $\mathcal{Y} \to \mathcal{Y}'$ induced by a is finite (and étale), the fact that $a_+ \iota^!(\mathcal{E})$ is $\mathcal{O}_{\mathcal{Z}'}({}^\dagger T' \cap Z')_{\mathbb{Q}}$-coherent implies that $\Gamma(\mathcal{Y}, \iota^!(\mathcal{E}))$ is of finite type over $\Gamma(\mathcal{Y}, \mathcal{O}_{\mathcal{Y},\mathbb{Q}})$. Then, by [Car06b, 2.2.12–13], $\iota^!(\mathcal{E})$ is associated to an isocrystal on Y overconvergent along $T \cap Z$. Since $\mathcal{E} \xrightarrow{\sim} \iota_+ \iota^!(\mathcal{E})$, then we have checked that $\mathcal{E}$ is in the essential image of $\mathrm{sp}_{Z \hookrightarrow \mathfrak{X}, T, +}$.

(2) It remains to check the (n, r)-overholonomicity of $\mathcal{E}$. Since a is finite and étale, $a_+ = a_*$, $a^! = a^*$, and thus $\iota^!(\mathcal{E})$ is a direct factor of $a^! a_+ \iota^!(\mathcal{E})$. Then, $\mathcal{E}$ is a direct factor of $\iota_+ a^! a_+ \iota^!(\mathcal{E})$. By [Car04, 3.1.8], $\iota_+ a^! \xrightarrow{\sim} h^! \iota'_+$. Hence, $\iota_+ a^! a_+ \iota^!(\mathcal{E}) \xrightarrow{\sim} h^! \iota'_+ a_+ \iota^!(\mathcal{E}) \xrightarrow{\sim} h^! h_+ \iota_+ \iota^!(\mathcal{E}) \xrightarrow{\sim} h^! h_+(\mathcal{E})$. Since h is in particular smooth and $h_+(\mathcal{E})$ is (n, r)-overholonomic, then $h^! h_+(\mathcal{E})$ is (n, r)-overholonomic. Because $\mathcal{E}$ is a direct factor of $h^! h_+(\mathcal{E})$, this implies that $\mathcal{E}$ is (n, r)-overholonomic. □

Notation 2.3.9. Let $\mathfrak{X}$, $\mathfrak{X}'$ be two smooth formal $\mathcal{V}$-schemes, f_0: $X' \to X$ a morphism of k-schemes, Z (resp. Z') a divisor of X (resp. X') such that $f_0^{-1}(Z) \subset Z'$.

From [Ber00, 2.1.6], we have a functor $f_0^!$: $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}}) \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'})$. We obtain $f^!_{0,Z',Z} := ({}^\dagger Z') \circ f_0^! \circ \mathrm{Forg}_Z$: $\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}}(Z)) \to \underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathcal{D}}^{(\bullet)}_{\mathfrak{X}'}(Z'))$. When there exists a lifting f: $\mathfrak{X}' \to \mathfrak{X}$ of f_0, we retrieve $f^!_{Z',Z}$. We pose $f^*_{0,Z',Z} = \mathcal{H}^0 \circ f^!_{0,Z',Z}[-d_{X'/X}]$ and $f^*_{Z',Z} = \mathcal{H}^0 \circ f^!_{Z',Z}[-d_{X'/X}]$, where $d_{X'/X}$ is the relative dimension of X' over X. We keep the previous notation when we work with coherent complexes. Remark that if $f_0^{-1}(Z) = Z'$, then $f^*_{Z',Z} = f^*$, where f^* is the usual inverse image functor (as $\mathcal{O}_{\mathfrak{X}}$-modules).

LEMMA 2.3.10. *Let $\mathfrak{X}$, $\mathfrak{X}'$ be two smooth formal $\mathcal{V}$-schemes, $\mathcal{Z}$ (resp. $\mathcal{Z}'$) be a strict normal crossings divisor of $\mathfrak{X}$ (resp. $\mathfrak{X}'$). Let f_0: $X' \to X$ be a morphism of k-schemes such that $f_0^{-1}(Z) \subset Z'$. We note that $f_0^{\#}$: $(X', Z') \to (X, Z)$ the induced morphism. Let $\mathcal{E}$ (resp. $\mathcal{F}$) be a coherent F-$\mathcal{D}^\dagger_{(\mathfrak{X},\mathcal{Z}),\mathbb{Q}}$-module (resp. $\mathcal{D}^\dagger_{(\mathfrak{X},\mathcal{Z}),\mathbb{Q}}$-module), locally projective of finite type over $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}$ (see* 1.3.4).

(1) *We have the isomorphism of coherent F-$\mathcal{D}^\dagger_{\mathfrak{X}'}({}^\dagger Z')_{\mathbb{Q}}$-modules, $\mathcal{O}_{\mathfrak{X}'}({}^\dagger Z')_{\mathbb{Q}}$-coherent*:

$$({}^\dagger Z')(f_0^{\#*}(\mathcal{E})) \xrightarrow{\sim} f^*_{0,Z',Z}(\mathcal{E}({}^\dagger Z)), \tag{2.3.10.1}$$

where the first (*resp. second*) *inverse image is defined in* 1.3.3 (*resp.* 2.3.9).

(2) *Suppose that there exists a lifting* $f\colon \mathfrak{X}' \to \mathfrak{X}$ *of* f_0 *that induces a lifting* $f^{\#}\colon (\mathfrak{X}', \mathcal{Z}') \to (\mathfrak{X}, \mathcal{Z})$ *of* $f_0^{\#}$. *Then, we have the isomorphism of coherent* $\mathcal{D}^{\dagger}_{\mathfrak{X}'}({}^{\dagger}Z')_{\mathbb{Q}}$*-modules,* $\mathcal{O}_{\mathfrak{X}'}({}^{\dagger}Z')_{\mathbb{Q}}$*-coherent*:

$$({}^{\dagger}Z')(f^{\#*}(\mathcal{F})) \xrightarrow{\sim} f^{*}_{Z',Z}(\mathcal{F}({}^{\dagger}Z)). \tag{2.3.10.2}$$

Proof. The sheaf $f_0^{\#*}(\mathcal{E})$ is a coherent F-$\mathcal{D}^{\dagger}_{(\mathfrak{X}',\mathcal{Z}'),\mathbb{Q}}$-module, locally projective of finite type over $\mathcal{O}_{\mathfrak{X}',\mathbb{Q}}$. By both Kedlaya's full faithfulness theorems [Ked07, 6.4.5] and [Ked08, 4.2.1], it is sufficient to check the isomorphism (2.3.10.1) outside Z', which is obvious. Using (1.3.5.1), isomorphism (2.3.10.2) becomes straightforward. □

Remarks 2.3.11. In the proof of (2.3.10.1) we use the Frobenius structure. (More precisely, the second Kedlaya's full faithfulness theorem, i.e., [Ked08, 4.2.1], needs a Frobenius structure.) But, the isomorphism (2.3.10.1) should be true without a Frobenius structure on $\mathcal{E}$. This check is technical (we have to paste local isomorphisms), and we avoid it because this is not really useful in this paper.

2.3.12 (log-relative duality isomorphism). We recall in this paragraph the isomorphism [Car09a, 5.25.2] and give a version of this. This isomorphism will be essential in the next theorem. Let $\mathfrak{X}$ be a smooth formal $\mathcal{V}$-scheme, $\mathcal{Z}$ a strict normal crossings divisor of $\mathfrak{X}$, $\mathfrak{X}^{\#} := (\mathfrak{X}, \mathcal{Z})$ the induced smooth logarithmic formal $\mathcal{V}$-scheme, $u\colon \mathfrak{X}^{\#} \to \mathfrak{X}$ the canonical morphism. Let $\mathcal{E}$ be a coherent $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\#},\mathbb{Q}}$-module that is a locally projective $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}$-module of finite type. It follows from [Car09a, 5.25.2] that $\mathbb{D}_{\mathfrak{X}} \circ u_{+}(\mathcal{E}) \xrightarrow{\sim} u_{+} \circ \mathbb{D}_{\mathfrak{X}^{\#}}(\mathcal{E}(\mathcal{Z}))$ (see the notation 1.2.9). By [Car09a, 5.22], $\mathbb{D}_{\mathfrak{X}^{\#}}(\mathcal{E}(\mathcal{Z})) \xrightarrow{\sim} (\mathcal{E}(\mathcal{Z}))^{\vee} \xrightarrow{\sim} \mathcal{E}^{\vee}(-\mathcal{Z})$. Then

$$\mathbb{D}_{\mathfrak{X}} \circ u_{+}(\mathcal{E}) \xrightarrow{\sim} u_{+}(\mathcal{E}^{\vee}(-\mathcal{Z})). \tag{2.3.12.1}$$

THEOREM 2.3.13. *Let* $\mathfrak{X}$ *be a smooth formal* $\mathcal{V}$*-scheme,* $\mathcal{Z}$ *a strict normal crossings divisor of* $\mathfrak{X}$, $\mathfrak{X}^{\#} := (\mathfrak{X}, \mathcal{Z})$ *the induced smooth formal* $\mathcal{V}$*-scheme,* u: $\mathfrak{X}^{\#} \to \mathfrak{X}$ *the canonical morphism. Let* $\mathcal{E}$ *be a coherent* $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\#},\mathbb{Q}}$*-module that is a locally projective* $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}$*-module of finite type satisfying the following condition*:

(c) *none of elements of* $\mathrm{Exp}(\mathcal{E})^{\mathrm{gr}}$ (*see the definition in* 1.1.3) *is a p-adic Liouville number.*

Then $u_{+}(\mathcal{E})$ *is overholonomic.*

Proof. Let $r \geq -1$, $n \geq 0$ be two integers, and let us consider the next properties:

($P_{n,r}$) For any $\mathfrak{X}$, $\mathcal{Z}$, $\mathcal{E}$ that satisfy the assumptions of the theorem, the module $u_{+}(\mathcal{E})$ is (n, r)-overholonomic (see 2.3.5);

$(Q_{n,r})$ For any $\mathfrak{X}$, $\mathcal{Z}$, $\mathcal{E}$ that satisfy the assumptions of the theorem, the complex $\mathbb{R}\underline{\Gamma}^{\dagger}_{Z} u_{+}(\mathcal{E})$ is (n,r)-overholonomic;

$(R_{n,r})$ For any $\mathfrak{X}$, $\mathcal{Z}$, $\mathcal{E}$ that satisfy the assumptions of the theorem, the module $\mathcal{E}({}^{\dagger}Z)$ is (n,r)-overholonomic.

(I) *First, for any $n \geq 1$, $r \geq -1$, we check that $(P_{n-1,r}) \Rightarrow (Q_{n,r})$.*

1° *How to use the case* 2.3.7 *of affine spaces.* Let $\mathfrak{H}_1, \ldots, \mathfrak{H}_n$ be the coordinate hyperplanes of $\operatorname{Spf} \mathcal{V}\{t_1, \ldots, t_n\}$, $\mathfrak{H} := \mathfrak{H}_1 \cup \cdots \cup \mathfrak{H}_r$ for some $r \leq n$, $\widehat{\mathbb{A}}^n_{\mathcal{V}} := \operatorname{Spf} \mathcal{V}\{t_1, \ldots, t_n\}$ and $\widehat{\mathbb{A}}^{n\#}_{\mathcal{V}} := (\operatorname{Spf} \mathcal{V}\{t_1, \ldots, t_n\}, \mathfrak{H})$. Since (n,r)-overholonomicity is local in $\mathfrak{X}$, similarly to the first step of the proof of Theorem 2.2.9, we come down to the case where there exists a commutative diagram of the form

$$\begin{array}{ccccc} \mathfrak{X} & = & \mathfrak{X} & \xrightarrow{h} & \widehat{\mathbb{A}}^n_{\mathcal{V}} \\ \uparrow{\scriptstyle u} & & \uparrow{\scriptstyle v} & & \uparrow{\scriptstyle w} \\ (\mathfrak{X}, \mathcal{Z}) & \xleftarrow{\tilde{u}} & (\mathfrak{X}, \mathcal{Z}'') & \xrightarrow{h^{\#}} & \widehat{\mathbb{A}}^{n\#}_{\mathcal{V}}, \end{array}$$

where h is a finite étale morphism, $\mathcal{Z}'' := h^{-1}(\mathfrak{H})$, and where $h^{\#}$, w, v, $\tilde{u}$ are the canonical induced morphisms. Moreover, denote by $\mathfrak{X}^{\#\prime} := (\mathfrak{X}, \mathcal{Z}'')$ and $\mathcal{Z}'$ the union of the irreducible components of $\mathcal{Z}''$ that are not an irreducible component of $\mathcal{Z}$.

2° $\mathbb{R}\underline{\Gamma}^{\dagger}_{H} w_{+} h^{\#}_{+}(\mathcal{E})$ *is (n,r)-overholonomic and smoothly dévissable in partially overconvergent isocrystals.* The case where $r = -1$ is already known from 2.3.7. Suppose now $r \geq 0$. We notice (for example, see 2.2.5) that $\mathcal{E}$ is also a coherent $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\#\prime},\mathbb{Q}}$-module that is a locally projective $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}$-module of finite type. Since h is finite and étale, $h^{\#}_{+}(\mathcal{E})$ is a coherent $\mathcal{D}^{\dagger}_{\widehat{\mathbb{A}}^{n\#}_{\mathcal{V}},\mathbb{Q}}$-module that is a locally projective $\mathcal{O}_{\widehat{\mathbb{A}}^{n}_{\mathcal{V}},\mathbb{Q}}$-module of finite type and such that condition (c) holds (see 1.1.3(2)). Hence, by 2.3.4, $\mathbb{R}\underline{\Gamma}^{\dagger}_{H} w_{+} h^{\#}_{+}(\mathcal{E})$ is smoothly dévissable in partially overconvergent isocrystals. Also, in the proof of 2.3.4 (see (2.3.4.2)) and with its notation, we have checked that $i^{!}_{1} w_{+} h^{\#}_{+}(\mathcal{E})$ is isomorphic to the image by w_{1+} of a complex of coherent $\mathcal{D}^{\dagger}_{\mathfrak{H}^{\#}_{1},\mathbb{Q}}$-module that are locally projective $\mathcal{O}_{\mathfrak{H}_1,\mathbb{Q}}$-modules of finite type satisfying condition (c) by 1.1.22. The hypothesis $(P_{n-1,r})$ implies that $i^{!}_{1} w_{+} h^{\#}_{+}(\mathcal{E})$ is $(n-1,r)$-overholonomic. Then by using 2.3.6, the complex $i_{1+} i^{!}_{1} w_{+} h^{\#}_{+}(\mathcal{E}) \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^{\dagger}_{H_1} w_{+} h^{\#}_{+}(\mathcal{E})$ is (n,r)-overholonomic. Symmetrically, we obtain for any $i = 1, \ldots, r$ that $\mathbb{R}\underline{\Gamma}^{\dagger}_{H_i} w_{+} h^{\#}_{+}(\mathcal{E})$ is (n,r)-overholonomic. Using Mayer-Vietoris exact triangles and the stability of the (n,r)-overholonomicity under local cohomological functors, this implies that $\mathbb{R}\underline{\Gamma}^{\dagger}_{H} w_{+} h^{\#}_{+}(\mathcal{E})$ is (n,r)-overholonomic.

3° $({}^\dagger Z')\mathbb{R}\underline{\Gamma}^\dagger_Z u_+(\mathcal{E})$ *is* (n,r)-*overholonomic.* We have $h_+(\mathbb{R}\underline{\Gamma}^\dagger_{Z''} v_+(\mathcal{E})) \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^\dagger_H h_+ v_+(\mathcal{E}) \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^\dagger_H w_+ h^\#_+(\mathcal{E})$. (See [Car04, 2.2.18.2] for the first isomorphism and (1.2.6.1) for the second one.) Then, by 2.3.8 and the second step, $\mathbb{R}\underline{\Gamma}^\dagger_{Z''} v_+(\mathcal{E})$ is (n,r)-overholonomic. We have checked in the proof of 2.2.9 that $u_+(\mathcal{E}({}^\dagger Z')) \xrightarrow{\sim} v_+(\mathcal{E})$. This implies $\mathbb{R}\underline{\Gamma}^\dagger_{Z''}({}^\dagger Z') u_+(\mathcal{E})$ is (n,r)-overholonomic. Using a Mayer-Vietoris exact triangle (similarly to (2.2.3.1)), we obtain

$$\mathbb{R}\underline{\Gamma}^\dagger_{Z''}({}^\dagger Z') u_+(\mathcal{E}) \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^\dagger_{Z}({}^\dagger Z') u_+(\mathcal{E}).$$

Using the exact triangle of localization of $\mathbb{R}\underline{\Gamma}^\dagger_Z u_+(\mathcal{E})$ with respect to Z', we come down to prove $\mathbb{R}\underline{\Gamma}^\dagger_{Z'\cap Z} u_+(\mathcal{E})$ is (n,r)-overholonomic, which is the last step of the proof of (I).

4° $\mathbb{R}\underline{\Gamma}^\dagger_{Z'\cap Z} u_+(\mathcal{E})$ *is* (n,r)-*overholonomic.* When $Z\cap Z'$ is empty, this is obvious. It remains to deal with the case where $Z\cap Z'$ is not empty. Let x be a closed point of $Z\cap Z'$, $\mathcal{Z}_1,\dots,\mathcal{Z}_r$ be the irreducible components of $\mathcal{Z}$ containing x, $\mathcal{Z}_{r+1},\dots,\mathcal{Z}_s$ be the irreducible components of $\mathcal{Z}'$ containing x. Since $\mathbb{R}\underline{\Gamma}^\dagger_{Z'\cap Z} u_+(\mathcal{E})$ is zero outside $Z\cap Z'$, it is sufficient to prove its (n,r)-overholonomicity around x. Then, we can suppose that $\mathcal{Z} = \mathcal{Z}_1\cup\dots\cup\mathcal{Z}_r$ and $\mathcal{Z}' = \mathcal{Z}_{r+1}\cup\dots\cup\mathcal{Z}_s$.

Let I be a nonempty subset of $\{r+1,\dots,s\}$, $\mathfrak{X}' := \cap_{i\in I}\mathcal{Z}_i$, $\mathfrak{X}'^\# := (\mathfrak{X}',\mathfrak{X}'\cap\mathcal{Z})$, $\iota\colon \mathfrak{X}'\hookrightarrow\mathfrak{X}$. Let $\iota^\#\colon \mathfrak{X}'^\#\hookrightarrow\mathfrak{X}^\#$, $u'\colon \mathfrak{X}'^\#\to\mathfrak{X}'$ be the canonical morphisms. Then, $\mathbb{R}\underline{\Gamma}^\dagger_{X'\cap Z} u_+(\mathcal{E}) \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^\dagger_Z \iota_+\iota^! u_+(\mathcal{E}) \underset{(2.2.9.1)}{\xrightarrow{\sim}} \mathbb{R}\underline{\Gamma}^\dagger_Z \iota_+ u'_+\iota^{\#!}(\mathcal{E}) \xrightarrow{\sim} \iota_+\mathbb{R}\underline{\Gamma}^\dagger_{Z\cap X'} u'_+\iota^{\#!}(\mathcal{E})$. From $(P_{n-1,r})$, we get that $\mathbb{R}\underline{\Gamma}^\dagger_{Z\cap X'} u'_+\iota^{\#!}(\mathcal{E})$ is $(n-1,r)$-overholonomic. Hence, using the second point of 2.3.6 and the above isomorphisms, $\mathbb{R}\underline{\Gamma}^\dagger_{X'\cap Z} u_+(\mathcal{E})$ is (n,r)-overholonomic. Using Mayer-Vietoris exact triangles, we get that if $\mathfrak{X}''$ is the union of some intersections of some irreducible components of $\mathcal{Z}'$, then $\mathbb{R}\underline{\Gamma}^\dagger_{X''\cap Z} u_+(\mathcal{E})$ is (n,r)-overholonomic. In particular, $\mathbb{R}\underline{\Gamma}^\dagger_{Z'\cap Z} u_+(\mathcal{E})$ is (n,r)-overholonomic.

(II) *We prove* $(P_{n,r-1}) + (Q_{n,r}) \Rightarrow (R_{n,r})$ *for any* $n\geq 0$, $r\geq 0$.

We suppose $r=0$ (resp. $r\geq 1$). By (2.3.10.2), it is sufficient to prove that for any divisor D of X, $\mathcal{E}({}^\dagger Z\cup D)$ is $\mathcal{D}^\dagger_{\mathfrak{X},\mathbb{Q}}$-coherent (resp. $\mathbb{D}_{\mathfrak{X}}(\mathcal{E}({}^\dagger Z\cup D))$ is $(n,r-1)$-overholonomic). Using de Jong's desingularization theorem ([dJ96]), there exist a proper smooth morphism $f\colon \mathcal{P}'\to\mathfrak{X}$ of smooth formal $\mathcal{V}$-schemes, a smooth scheme X' over k, a closed immersion $\iota'_0\colon X'\hookrightarrow\mathcal{P}'$, a projective, surjective, generically finite and étale morphism $a_0\colon X'\to X$ such that $a_0 = f_0\circ\iota'_0$ and $Z'' := a_0^{-1}(Z\cup D)$ is a strict normal crossings divisor of X'. Since $\mathcal{E}({}^\dagger Z\cup D)$ is associated to an isocrystal on $X\setminus(Z\cup D)$ overconvergent along $Z\cup D$ (i.e., is a coherent $\mathcal{D}^\dagger_{\mathfrak{X}}({}^\dagger Z\cup D)_{\mathbb{Q}}$-module, $\mathcal{O}_{\mathfrak{X}}({}^\dagger Z\cup D)_{\mathbb{Q}}$-coherent), by [Car06a, 6.1.4 and 6.3.1], this implies that $\mathcal{E}({}^\dagger Z\cup D)$ is a direct factor of $f_+\mathbb{R}\underline{\Gamma}^\dagger_{X'} f^!(\mathcal{E}({}^\dagger Z\cup D))$. Using the second point of 2.3.6 (resp. and the fact that

f_+ commutes with $\mathbb{D}_{\mathfrak{X}}$), it remains to prove that $\mathbb{R}\underline{\Gamma}^\dagger_{X'} f^!(\mathcal{E}(^\dagger Z \cup D))$ is $\mathcal{D}^\dagger_{\mathcal{P}',\mathbb{Q}}$-coherent (resp. $\mathbb{D}_{\mathcal{P}'} \circ \mathbb{R}\underline{\Gamma}^\dagger_{X'} \circ f^!(\mathcal{E}(^\dagger Z \cup D))$ is $(n+d_f, r-1)$-overholonomic, where d_f is the relative dimension of f). This is local in $\mathcal{P}'$. Then, we can suppose that there exists a lifting ι': $\mathfrak{X}' \hookrightarrow \mathcal{P}'$ of ι'_0 and that Z'' lifts to a relative strict normal crossings divisor $\mathcal{Z}''$ of $\mathfrak{X}'$ over $\mathcal{V}$. We pose $a = f \circ \iota'$ and denote by $u' : (\mathfrak{X}', \mathcal{Z}'') \to \mathfrak{X}'$ and $a^\#$: $(\mathfrak{X}', \mathcal{Z}'') \to (\mathfrak{X}, \mathcal{Z})$ the canonical morphisms.

By [Ber02, 4.4.5],

$$\mathbb{R}\underline{\Gamma}^\dagger_{X'} f^!(\mathcal{E}(^\dagger Z \cup D)) \xrightarrow{\sim} \iota'_+ \iota'^! f^!(\mathcal{E}(^\dagger Z \cup D)) \xrightarrow{\sim} \iota'_+ a^!(\mathcal{E}(^\dagger Z \cup D)).$$

Then, using the second point of 2.3.6, we come down to prove that the module $a^!(\mathcal{E}(^\dagger Z \cup D)) = a^*(\mathcal{E}(^\dagger Z \cup D))$ (by flatness) is $\mathcal{D}^\dagger_{\mathfrak{X}',\mathbb{Q}}$-coherent (resp. $\mathbb{D}_{\mathfrak{X}'} a^*(\mathcal{E}(^\dagger Z \cup D))$ is $(n, r-1)$-overholonomic). We have $a^*(\mathcal{E}(^\dagger Z \cup D)) \xrightarrow{\sim} (^\dagger Z'') \circ a^*(\mathcal{E}(^\dagger Z)) \xrightarrow{\sim} a^*_{Z'',Z}(\mathcal{E}(^\dagger Z))$. We get from (2.3.10.2) the following isomorphism:

$$a^*_{Z'',Z}(\mathcal{E}(^\dagger Z)) \xrightarrow{\sim} (^\dagger Z'')(a^{\#*}(\mathcal{E})).$$

Thus, it remains to prove that $(^\dagger Z'')(a^{\#*}(\mathcal{E}))$ is $\mathcal{D}^\dagger_{\mathfrak{X}',\mathbb{Q}}$-coherent (resp. $\mathbb{D}_{\mathfrak{X}'} \circ (^\dagger Z'')(a^{\#*}(\mathcal{E}))$ is $(n, r-1)$-overholonomic). We check this separately.

Nonrespective case. By $(Q_{n,0})$, since $a^{\#*}(\mathcal{E})$ satisfies condition (c) (see 1.1.3(1)), the complex $\mathbb{R}\underline{\Gamma}^\dagger_{Z''} u'_+(a^{\#*}(\mathcal{E}))$ is overcoherent. By (1.3.9.1), using the exact triangle of localization of $u'_+(a^{\#*}(\mathcal{E}))$ with respect to Z'', this implies that $(^\dagger Z'')(a^{\#*}(\mathcal{E}))$ is $\mathcal{D}^\dagger_{\mathfrak{X}',\mathbb{Q}}$-coherent.

Respective case. By applying the functor $\mathbb{D}_{\mathfrak{X}'}$ to the exact triangle of localization of $u'_+(a^{\#*}(\mathcal{E}))$ with respect to Z'' (see (1.3.9.1)), we get

$$\mathbb{D}_{\mathfrak{X}'} \circ (^\dagger Z'')(a^{\#*}(\mathcal{E})) = \mathrm{Cone}\Big(\mathbb{D}_{\mathfrak{X}'} \circ u'_+(a^{\#*}(\mathcal{E})) \to \mathbb{D}_{\mathfrak{X}'} \circ \mathbb{R}\underline{\Gamma}^\dagger_{Z''} \circ u'_+(a^{\#*}(\mathcal{E}))\Big)[-1].$$

Since $a^{\#*}(\mathcal{E})$ satisfies condition (c) (see 1.1.3(1)), using $(Q_{n,r})$ hypothesis, we get that $\mathbb{D}_{\mathfrak{X}'} \circ \mathbb{R}\underline{\Gamma}^\dagger_{Z''} \circ u'_+(a^{\#*}(\mathcal{E}))$ is $(n, r-1)$-overholonomic. Also, the log-relative duality isomorphism of (2.3.12.1) gives

$$\mathbb{D}_{\mathfrak{X}'} \circ u'_+(a^{\#*}(\mathcal{E})) \xrightarrow{\sim} u'_+((a^{\#*}(\mathcal{E}))^\vee(-\mathcal{Z}'')).$$

Since $(a^{\#*}(\mathcal{E}))^\vee(-\mathcal{Z}'')$ satisfies also condition (c) (see 1.1.3(1)) of our theorem, using $(P_{n,r-1})$ we obtain that $u'_+((a^{\#*}(\mathcal{E}))^\vee(-\mathcal{Z}''))$ is $(n, r-1)$-overholonomic. Hence, $\mathbb{D}_{\mathfrak{X}'} \circ (^\dagger Z'')(a^{\#*}(\mathcal{E}))$ is $(n, r-1)$-overholonomic.

(III) *Conclusion.*

For any $n \geq 0$, we know that $(P_{n,-1})$ is true. Also, for any $r \geq -1$, $(P_{0,r})$ is already known (see [Car09c, 7.3]).

We get from the two previous steps that, for any $r \geq 0$ and $n \geq 1$, $(P_{n,r-1}) + (P_{n-1,r}) \Rightarrow (Q_{n,r}) + (R_{n,r})$. Using the exact triangle of localization of $u_+(\mathcal{E})$ with respect to Z, we get $(Q_{n,r})+(R_{n,r}) \Rightarrow (P_{n,r})$. Thus, $(P_{n,r-1})+(P_{n-1,r}) \Rightarrow (P_{n,r})$. This implies that $(P_{n,r})$ is true for any $r \geq -1$ and $n \geq 0$. □

Remarks 2.3.14. We have used in (step (II) of) the proof of 2.3.13, the stability of condition (c) by inverse image and above all by the functor $\mathcal{E} \mapsto \mathcal{E}^\vee(-\mathcal{Z})$. Since condition (b′) of 1.3.6 is not stable by the functor $\mathcal{E} \mapsto \mathcal{E}^\vee(-\mathcal{Z})$, we do need the strong version of Theorem 1.1.1 and Proposition 1.1.22.

THEOREM 2.3.15. *Let $\mathcal{P}$ be a separated smooth formal scheme over $\mathcal{V}$, T a divisor of P, X a closed smooth subscheme such that $Z := T \cap X$ is a divisor of X, $Y := X \setminus Z$. Let E be an F-isocrystal on Y overconvergent along Z. Then $\mathrm{sp}_{X\hookrightarrow\mathcal{P},T,+}(E)$ is overholonomic.*

Proof. Since E admits a semi-stable reduction (see 2.1.3), there exists a commutative diagram of the form

$$\begin{array}{ccccc} Y' & \longrightarrow & X' & \xrightarrow{\iota'_0} & \mathcal{P}' \\ \downarrow b_0 & & \downarrow a_0 & & \downarrow f \\ Y & \longrightarrow & X & \xrightarrow{\iota_0} & \mathcal{P}, \end{array} \tag{2.3.15.1}$$

such that f is a proper smooth morphism of smooth formal $\mathcal{V}$-schemes, the left square is cartesian, X' is a smooth scheme over k, ι'_0 is a closed immersion, a_0 is a projective, surjective, generically finite and étale morphism, $a_0^{-1}(Z)$ is a strict normal crossings divisor of X', and the F-isocrystal $a_0^*(E)$ on Y' overconvergent along $a_0^{-1}(Z)$ is log-extendable on X'. We pose $\mathcal{E} := \mathrm{sp}_{X\hookrightarrow\mathcal{P},T,+}(E)$. We have $\mathbb{R}\underline{\Gamma}^\dagger_{X'} f^!_T(\mathcal{E}) \xrightarrow{\sim} \mathrm{sp}_{X'\hookrightarrow\mathcal{P}',f^{-1}(T),+}(a_0^*(E))$. By [Car06a, 6.1.4]), $\mathcal{E} \in F\text{-Isoc}^{\dagger\dagger}(\mathcal{P},T,X/K)$. Then by [Car06a, 6.3.1], we check that $\mathcal{E}$ is a direct factor of $f_{T,+}\mathrm{sp}_{X'\hookrightarrow\mathcal{P}',f^{-1}(T),+}(a_0^*(E))$. Since the overholonomicity is stable under direct image by a proper morphism, then it is sufficient to prove that the isocrystal $\mathrm{sp}_{X'\hookrightarrow\mathcal{P}',f^{-1}(T),+}(a_0^*(E))$ is overholonomic. This last statement is local in $\mathcal{P}'$. Then, we can suppose that there exists a lifting $\iota'\colon \mathfrak{X}' \hookrightarrow \mathcal{P}'$ of ι'_0 and that $a_0^{-1}(Z)$ lifts to a strict normal crossings divisor $\mathcal{Z}'$ of $\mathfrak{X}'$ over $\mathcal{S}$. Then, $\mathrm{sp}_{X'\hookrightarrow\mathcal{P}',f^{-1}(T),+}(a_0^*(E)) \xrightarrow{\sim} \iota'_+\mathrm{sp}_*(a_0^*(E))$, where sp: $\mathfrak{X}'_K \to \mathfrak{X}'$ is the specialization morphism of $\mathfrak{X}'$. It remains to check that $\mathrm{sp}_*(a_0^*(E))$ is overholonomic. But since $a_0^*(E)$ is an F-isocrystal on Y' overconvergent along $a_0^{-1}(Z)$ that is log-extendable on X', it follows from 2.3.13 that $\mathrm{sp}_*(a_0^*(E))$ is overholonomic. □

The following theorem was the conjecture [Car07a, 3.2.25.1].

THEOREM 2.3.16. *Let Y be a smooth separated scheme of finite type over k. Let E be an overconvergent F-isocrystal on Y. Then $\mathrm{sp}_{Y+}(E)$ is an overholonomic arithmetic $\mathcal{D}_Y$-module* (*see* [Car04, 3.2.10]), *where* $\mathrm{sp}_{Y,+}$:

$F\text{-Isoc}^{\dagger}(Y/K) \cong F\text{-Isoc}^{\dagger\dagger}(Y/K)$ *is the canonical equivalence from the category of overconvergent F-isocrystals on Y into the category of overcoherent F-isocrystals on Y (see* [Car07a, 2.3.1]).

Proof. The theorem is local in Y. We can suppose Y affine and then that there exists an immersion of Y into in proper smooth formal $\mathcal{V}$-scheme $\mathcal{P}$, a divisor T of P such that $Y = X \setminus T$, where X is the closure of Y in P. Let $Z := X \cap T$ and $\mathcal{E} := \mathrm{sp}_{Y+}(E) \in F\text{-Isoc}^{\dagger\dagger}(Y/K) = F\text{-Isoc}^{\dagger\dagger}(\mathcal{P}, T, X/K)$ (notation of [Car06a, 6.2.1] and [Car07a, 2.2.4]).

Using de Jong's desingularization, we come down to the case where X is smooth (similarly to the proof of 2.3.15), which was already checked in 2.3.15. □

THEOREM 2.3.17. *Let $\mathcal{P}$ be a proper smooth formal scheme over $\mathcal{V}$, T a divisor of P, $\mathcal{E} \in F\text{-}D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathcal{P}}({}^{\dagger}T)_{\mathbb{Q}})$. Then the following assertion are equivalent:*

(1) *The F-complex $\mathcal{E}$ is $\mathcal{D}^{\dagger}_{\mathcal{P}}({}^{\dagger}T)_{\mathbb{Q}}$-overcoherent.*
(2) *The F-complex $\mathcal{E}$ is $\mathcal{D}^{\dagger}_{\mathcal{P},\mathbb{Q}}$-overcoherent.*
(3) *The F-complex $\mathcal{E}$ is overholonomic.*
(4) *The F-complex $\mathcal{E}$ is dévissable in overconvergent F-isocrystals.*

Proof. By [Car07a, 3.1.2], if $\mathcal{E}$ is F-$\mathcal{D}^{\dagger}_{\mathcal{P}}({}^{\dagger}T)_{\mathbb{Q}}$-overcoherent, then there exists a dévissage of $\mathcal{E}$ in overconvergent F-isocrystals. By 2.3.16, if there exists a dévissage of $\mathcal{E}$ in overconvergent F-isocrystals, then $\mathcal{E}$ is overholonomic. Finally, it is obvious that if $\mathcal{E}$ is overholonomic, then $\mathcal{E}$ is $\mathcal{D}^{\dagger}_{\mathcal{P},\mathbb{Q}}$-overcoherent and that if $\mathcal{E}$ is $\mathcal{D}^{\dagger}_{\mathcal{P},\mathbb{Q}}$-overcoherent then $\mathcal{E}$ is $\mathcal{D}^{\dagger}_{\mathcal{P}}({}^{\dagger}T)_{\mathbb{Q}}$-overcoherent. □

We end this section with the following consequences of 2.3.17, explained respectively in [Car07a, 3.2.26.1] and [Car07b, 5.8].

COROLLARY 2.3.18. *Let $\mathcal{P}$ be a proper smooth formal scheme over $\mathcal{V}$, T a divisor of P, Y a subscheme of P.*

(1) *We have an equivalence between the category of quasi-coherent F-complexes dévissable in overconvergent F-isocrystals and the category of coherent F-complexes dévissable in overconvergent F-isocrystals; i.e.,*

$$F\text{-}\underrightarrow{LD}^{\mathrm{b}}_{\mathbb{Q},\mathrm{dev}}({}^{\mathrm{g}}\widehat{\mathcal{D}}^{(\bullet)}_{\mathcal{P}}(T)) \cong F\text{-}D^{\mathrm{b}}_{\mathrm{dev}}(\mathcal{D}^{\dagger}_{\mathcal{P}}({}^{\dagger}T)_{\mathbb{Q}}).$$

(2) *Denoting by $F\text{-}D^{\mathrm{b}}_{\mathrm{ovhol}}(\mathcal{D}_Y)$, the category of overholonomic F-complexes of arithmetic $\mathcal{D}_Y$-modules, we get a canonical tensor product:*

$$-\overset{\mathbb{L}}{\otimes}{}^{\dagger}_{\mathcal{O}_Y}- \ : \ F\text{-}D^{\mathrm{b}}_{\mathrm{ovhol}}(\mathcal{D}_Y) \times F\text{-}D^{\mathrm{b}}_{\mathrm{ovhol}}(\mathcal{D}_Y) \to F\text{-}D^{\mathrm{b}}_{\mathrm{ovhol}}(\mathcal{D}_Y). \tag{2.3.18.1}$$

2.4. *Some precisions for the case of curves.* In this section, $i\colon \mathcal{Z} \hookrightarrow \mathfrak{X}$ is a closed immersion of separated smooth formal $\mathcal{V}$-schemes such that $\dim X = 1$

and Z is a divisor of X. Let $\mathcal{Y} := \mathfrak{X} \setminus \mathcal{Z}$, $\mathfrak{X}^{\#} := (\mathfrak{X}, \mathcal{Z})$, $u\colon \mathfrak{X}^{\#} \to \mathfrak{X}$, $f\colon \mathfrak{X} \to \mathcal{S}$ be the canonical morphisms and $f^{\#} := f \circ u \;:\; \mathfrak{X}^{\#} \to \mathcal{S}$.

The next theorem is slightly better for curves than 2.2.9 because we have another divisor D.

PROPOSITION 2.4.1. *Let D be a divisor of X, $\mathcal{E}$ be a coherent $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\#}}({}^{\dagger}D)_{\mathbb{Q}}$-module that is a locally projective $\mathcal{O}_{\mathfrak{X}}({}^{\dagger}D)_{\mathbb{Q}}$-module of finite type. Suppose that $\mathcal{E}$ satisfies conditions* (a) *and* (b′) (*see* 1.3.6). *Then the canonical morphism $\rho\colon u_{D+}(\mathcal{E}) \to \mathcal{E}({}^{\dagger}Z)$* (*see* 1.3.8) *is an isomorphism.*

Proof. By (1.3.9.1), this is equivalent to checking that $\mathbb{R}\underline{\Gamma}^{\dagger}_{Z} \circ u_{+}(\mathcal{E}) = 0$. By applying the functor f_{+} to the localization triangle of $u_{D+}(\mathcal{E})$ with respect to Z, we get

(2.4.1.1)
$$f_{+} \circ \mathbb{R}\underline{\Gamma}^{\dagger}_{Z} \circ u_{+}(\mathcal{E}) \longrightarrow f_{+} \circ u_{+}(\mathcal{E}) \overset{f_{+}(\rho)}{\longrightarrow} f_{+}(\mathcal{E}({}^{\dagger}Z)) \longrightarrow f_{+} \circ \mathbb{R}\underline{\Gamma}^{\dagger}_{Z} \circ u_{+}(\mathcal{E})[1].$$

Following 1.3.12, the morphism $f_{+} \circ u_{+}(\mathcal{E}) \to f_{+}(\mathcal{E}({}^{\dagger}Z))$ is an isomorphism. Then, by 2.4.1.1, $f_{+} \circ \mathbb{R}\underline{\Gamma}^{\dagger}_{Z} \circ u_{+}(\mathcal{E}) = 0$. Furthermore, since $\mathbb{R}\underline{\Gamma}^{\dagger}_{Z} \xrightarrow{\sim} i_{+} \circ i^{!}$ (by [Ber02, 4.4.5]), we get $(f \circ i)_{+} \circ i^{!} \circ u_{+}(\mathcal{E}) \xrightarrow{\sim} f_{+} \circ \mathbb{R}\underline{\Gamma}^{\dagger}_{Z} \circ u_{+}(\mathcal{E}) = 0$. Because $f \circ i$ is finite and étale, by 2.2.8 this implies $i^{!} \circ u_{+}(\mathcal{E}) = 0$ and then $\mathbb{R}\underline{\Gamma}^{\dagger}_{Z} \circ u_{+}(\mathcal{E}) = 0$. □

Remarks 2.4.2. Even if the assertions look different, the proof of 2.4.1 is the same as that of [Car06b, 2.3.2]. Here the coherent $\mathcal{D}^{\dagger}_{\mathfrak{X},\mathbb{Q}}$-module is $u_{+}(\mathcal{E})$ and we have replaced the finiteness theorem of rigid cohomology (this requires the properness of $\mathfrak{X}$ and a Frobenius structure) by 1.3.12.

LEMMA 2.4.3. *Let $\mathcal{P}$ be a smooth formal $\mathcal{V}$-scheme, let $\mathcal{E} \in F\text{-}D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathcal{P},\mathbb{Q}})$ with finite extraordinary fibers* (*see the definition* [Car09c, 2.1]); *i.e., for any closed point x of P, for any lifting i_x of the canonical closed immersion induced by x, the cohomology spaces of $\mathbb{L}i^{*}_{x}(\mathcal{E})$ have finite dimension as K-vector spaces. Then there exists a divisor T of P such that the complex $\mathcal{E}({}^{\dagger}T)$ is $\mathcal{O}_{\mathcal{P}}({}^{\dagger}T)_{\mathbb{Q}}$-coherent.*

Proof. From [Car06b, 2.2.12], it is enough to check that there exists a dense open set $\mathfrak{U}$ of $\mathcal{P}$ such that $\mathcal{E}|\mathfrak{U} \in F\text{-}D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_{\mathfrak{U},\mathbb{Q}})$. We proceed by induction on the cardinal number of the set $\{n \in \mathbb{N} \,|\, \mathcal{H}^{n}(\mathcal{E}) \neq 0\}$. Let N be the greater number of this set. Then, for any closed point x of P, for any lifting i_x of the canonical closed immersion induced by x, $i^{*}_{x}(\mathcal{H}^{N}(\mathcal{E}))$ has finite dimension as K-vector space (because i^{*}_{x} is right exact). We notice that the theorem [Car06b, 2.2.17] is still true by replacing $i^{!}_{x}$ by i^{*}_{x}. (In the proof, we only use the fact that $i^{*}_{x}(\mathcal{E})$ has finite dimension as K-vector space.) Then there exists a dense open $\mathfrak{U}$ such that $\mathcal{H}^{N}(\mathcal{E})|\mathfrak{U}$ is $\mathcal{O}_{\mathfrak{U},\mathbb{Q}}$-coherent. We can suppose

$\mathfrak{X} = \mathfrak{U}$; i.e., $\mathcal{H}^N(\mathcal{E})$ is $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}$-coherent. Then, for any closed point x of P, for any integer $j \neq 0$, $\mathcal{H}^j \mathbb{L} i_x^*(\mathcal{H}^N(\mathcal{E})) = 0$. This implies that the truncated complex $\tau_{\leq N-1}(\mathcal{E})$ has finite extraordinary fibers. We conclude using the induction hypothesis. □

The following theorem extends [Car06b, 2.3] (e.g., notice that here $\mathfrak{X}$ does not need to be proper).

THEOREM 2.4.4. *Let* $\mathcal{E} \in F\text{-}D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X},\mathbb{Q}})$. *The following assertions are equivalent*:

(1) *The F-complex $\mathcal{E}$ has finite extraordinary fibers.*
(2) *For any divisor T of X, the F-complex $\mathcal{E}(^\dagger T)$ belongs to $F\text{-}D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X},\mathbb{Q}})$.*
(3) *The complex $\mathcal{E}$ is holonomic.*
(4) *The F-complex $\mathcal{E}$ is smoothly dévissable in partially overconvergent F-isocrystals.*
(5) *The F-complex $\mathcal{E}$ is $\mathcal{D}^\dagger_{\mathfrak{X},\mathbb{Q}}$-overcoherent.*
(6) *The F-complex $\mathcal{E}$ is overholonomic.*

Proof. To check the equivalence between the three first assertions, we have only to rewrite the proof of [Car06b, 2.3.3] where we replace [Car06b, 2.3.2] by 2.3.15.

Proof of 1⇒4: suppose $\mathcal{E}$ satisfies 1. By 2.4.3, there exists a divisor T of X such that the cohomology spaces of $\mathcal{E}(^\dagger T)$ are isocrystals on $X \setminus T$ overconvergent along T. Let $i : \mathfrak{T} \hookrightarrow \mathfrak{X}$ be a lifting of the $T \subset X$. Then, by hypothesis, $i^!(\mathcal{E})$ is $\mathcal{O}_{\mathfrak{T},\mathbb{Q}}$-coherent. Hence, $\mathcal{E}$ is smoothly dévissable in partially overconvergent F-isocrystals. The implication 4⇒6 is a consequence of 2.3.15. Finally, 6⇒5⇒1 are obvious. □

For curves, the following statement answers positively to Berthelot's conjecture of [Ber02, 5.3.6.D] in the case of curves.

THEOREM 2.4.5. *Let* $\mathcal{E} \in F\text{-}D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^\dagger_{\mathfrak{X}}(^\dagger Z)_{\mathbb{Q}})$ *whose restriction on* $\mathcal{Y}$ *is a holonomic* $F\text{-}\mathcal{D}^\dagger_{\mathcal{Y},\mathbb{Q}}$*-module. Then* $\mathcal{E}$ *is a holonomic* $F\text{-}\mathcal{D}^\dagger_{\mathfrak{X},\mathbb{Q}}$*-module.*

Proof. Replacing [Car06b, 4.3.4] by 2.3.15 and [Car06b, 2.3.3] by 2.4.4, it is sufficient to rewrite the proof of [Car06b, 4.3.5]. □

Remarks 2.4.6. This Berthelot's conjecture above (of [Ber02, 5.3.6.D]) leads to Berthelot's conjecture on the stability of the holonomicity under inverse image. This latter conjecture, following [Car09c], implies that holonomicity equals overholonomicity.

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