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The existence of an abelian variety over $\overline{\mathbb{Q}}$ isogenous to no Jacobian

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Abstract

We prove the existence of an abelian variety A of dimension g over $\overline{\mathbb{Q}}$ that is not isogenous to any Jacobian, subject to the necessary condition $g > 3$. Recently, C. Chai and F. Oort gave such a proof assuming the André-Oort conjecture. We modify their proof by constructing a special sequence of CM points for which we can avoid any unproven hypotheses. We make use of various techniques from the recent work of Klingler-Yafaev et al.

1. Introduction

This article is motivated by the following question of Nicholas Katz and Frans Oort: *Does there exist an abelian variety of genus g over $\overline{\mathbb{Q}}$ that is not isogenous to a Jacobian of a stable curve*?

For $g \leq 3$, the answer is no because every principally polarized abelian variety is a Jacobian, while for $g \geq 4$, the answer is expected to be yes. In [1], C. Chai and F. Oort establish this under the André-Oort conjecture, which we recall in Section 2. In fact, they prove the following stronger statement.

Theorem 1.1 ([1]). *Denote by $A_{g,1}/\overline{\mathbb{Q}}$ the coarse moduli space of principally polarized abelian varieties of dimension g defined over $\overline{\mathbb{Q}}$, and let $X \subsetneq A_{g,1}$ be a proper closed subvariety. Then assuming the André-Oort conjecture, there exists a closed point $y = [A, \lambda]$ in $A_{g,1}(\overline{\mathbb{Q}})$ such that for all closed points $x = [B, \lambda']$ in X, the abelian varieties A and B are not isogenous.*

The question about Jacobians follows by taking for X the closed Torelli locus.

The way Chai and Oort proved Theorem 1.2 is roughly by looking at the sequence of all CM points y and using the fact that CM type is preserved under isogeny. Hence, if Theorem 1.2 is false, X must contain points with every possible CM type. One then applies the André-Oort conjecture to conclude that X contains a finite set of Shimura subvarieties containing CM points of each possible CM type. In [1], this is ruled out using algebraic methods, finishing the proof.

In [11], the André-Oort conjecture is proven assuming the Generalized Riemann Hypothesis for Dedekind zeta functions of CM fields, henceforth referred to as GRH. The reason GRH is used is that they need to produce, for the CM fields K that occur, many small[1] split primes. Our idea is to construct an infinite sequence of CM fields that we can prove have many small split primes. (Of course, assuming GRH, they *all do*.)

We do this in Section 3 by using a powerful equidistribution theorem from Chavdarov[2], which is due to Nick Katz. We then go into the proof of André-Oort in [11] and carry it through for our sequence of CM points without assuming GRH. Finally, in Section 4 we apply the arguments in [1] to our sequence. Thus, our main result is

THEOREM 1.2. *Denote by $A_{g,1}/\overline{\mathbb{Q}}$ the coarse moduli space of principally polarized abelian varieties of dimension g over $\overline{\mathbb{Q}}$, and let $X \subsetneq A_{g,1}$ be a proper closed subvariety. Then there exists a closed point $y = [A, \lambda]$ in $A_{g,1}$ such that for all closed points $x = [B, \lambda']$ in X, the abelian varieties A and B are not isogenous.*

We point out that we make no progress on the André-Oort conjecture itself, as the conjecture is about the 'worst' possible sequence of CM points, whereas we only show that it holds for certain carefully constructed sequences.

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2. Notation and background

2.1. *Weyl* CM *fields.* Following [1], we say that a field L over $\mathbb{Q}$ of degree $2g$ is CM if it is a totally complex quadratic extension of a totally real field F. Given a CM field L, a CM type for L is a set $S_L = \{\phi_1, \phi_2, \dots \phi_g\}$ of g distinct embeddings of L into $\mathbb{C}$ such that no two of them are complex conjugates. We let W_g denote the Weyl group

$$W_g := (\mathbb{Z}/2\mathbb{Z})^g \ltimes S_g.$$

Suppose we are given a CM field L with a CM type S_L, and let M denote the normal closure of L over $\mathbb{Q}$. Then there is a natural embedding of $\mathrm{Gal}(M/\mathbb{Q})$

[1] Here "small" is with respect to the discriminant D_K.

into W_g given as follows. An element $h \in \mathrm{Gal}(M/\mathbb{Q})$ permutes the pairs of embeddings $P_i = (\phi_i, \overline{\phi_i})$, and for each i, it either takes ϕ_i to some ϕ_j *or* to $\overline{\phi_j}$. We thus get a signed permutation for each element, and hence an element of W_g.

Definition. We say that L is a Weyl CM field if the Galois group $\mathrm{Gal}(M/\mathbb{Q})$ is isomorphic to W_g.

2.2. *Shimura varieties and the André-Oort conjecture.* Here we recall some of the basic theory of Shimura varieties. For more details, we refer to [4] and [5]. A Shimura variety is a pair (G, X), where G is a reductive algebraic group defined over $\mathbb{Q}$ and X is a conjugacy class of maps from $\mathbb{S}^1 = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$ to G satisfying the axioms of a Shimura datum, together with an open compact subgroup K of $G(\mathbb{A}_f)$, where $\mathbb{A}_f$ are the finite adeles. The X then acquires the structure of a hermitian symmetric space, and we define the space $\mathrm{Sh}(G, X)_K := G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K$, which is then naturally endowed with the structure of a quasi-projective algebraic variety over $\overline{\mathbb{Q}}$. Given another Shimura variety $\mathrm{Sh}(G_1, X_1)_{K_1}$ and homomorphism $\phi : G_1 \to G$ that send X_1 to X and send K_1 to K, we get a map $\tilde{\phi} : \mathrm{Sh}(G_1, X_1)_{K_1} \to \mathrm{Sh}(G, X)_K$. A *special subvariety* of $\mathrm{Sh}(G, X)_K$ is defined to be an irreducible component of a "Hecke translate" by an element of $G(\mathbb{A}_f)$ of such an image. A Shimura subvariety of dimension 0 is called a *CM point*.

An important special case of a Shimura variety is the moduli space of principally polarized abelian varieties $A_{g,1}$. It corresponds to the pair $(\mathrm{GSp}_{2g}, \mathbb{H}_g)$ together with the standard maximal compact subgroup of $\mathrm{GSp}_{2g}(\mathbb{A}_f)$. In this case special points correspond exactly to abelian varieties with complex multiplication.

Conjecture 2.1 (André-Oort). *Let S be a Shimura variety and $\Gamma \subset S$ be a set of special points in S. Then the Zariski closure of Γ is a finite union of Shimura subvarieties.*

We call a point $x \in A_{g,1}$ a *Weyl CM point* if the associated abelian variety has complex multiplication by a Weyl CM field of degree $2g$.

2.3. *Siegel zeroes and totally split primes.* Later on we shall need to produce totally split primes in algebraic number fields, so we collect the relevant analytic results here for convenience. Fix $d > 0$ throughout this section. Take K to be a Galois extension of $\mathbb{Q}$ of degree d and discriminant D_K. For a positive real number, define $N_K(X)$ to be the number of primes $p < X$ such that p is a totally split prime in K. By Chebotarev's density theorem, we know that

$$N_K(X) \sim \frac{X}{d \cdot \log X}$$

as $X \to \infty$. However, we shall need a quantified version of this result. For this, we introduce the concept of an exceptional (Siegel) zero.

Theorem 2.1. *For $d > 1$, there exists a $C_d > 0$ depending only on d such that the Dedekind zeta function $\zeta_K(s)$ has at most one real zero in the range*

$$1 - \frac{C_d}{\log(D_K)} \leq \sigma < 1.$$

Such a zero, if it exists, is called an exceptional zero, or Siegel zero.

Note that the definition of an exceptional zero is with reference to a constant C_d (more precisely, to a family of constants, parametrized by d). We will make use of this later on by picking a small enough family of constants C_d.

Exceptional zeroes, though conjectured to not exist, must be entertained all over analytic number theory, and the reason they are important for us is the following result, due to Lagarias and Odlyzko [12].

Theorem 2.2. *For K a Galois number field of degree d, we have*

$$N_K(X) = \frac{X}{d\log X} + O\left(\frac{X^\beta}{\log X}\right) + O\left(\frac{\sqrt{|D_K|}Xe^{-C_d\sqrt{\log X}}}{\log X}\right),$$

where β is the possible exceptional zero of $\zeta_K(s)$. The $O\left(\frac{X^\beta}{\log X}\right)$ term can be removed if there is no exceptional zero.

It is a well-established principle that exceptional zeroes, if they exist at all, are very rare. We recall this below, and later we shall construct our CM fields so as to avoid exceptional zeroes. By the following result of Heilbronn [9], exceptional zeroes can genuinely show up only in degree-2 extensions.

Theorem 2.3. *If K is a Galois number field with β as an exceptional zero of $\zeta_K(s)$, then there is a quadratic field $F \subset K$ with $\zeta_F(\beta) = 0$, so that β is an exceptional zero of $\zeta_F(s)$.*

For quadratic fields, we have the following repulsion result.

Theorem 2.4. *Let F_1, F_2 be two distinct quadratic number fields of discriminants D_1, D_2 respectively, and let β_1, β_2 be real zeroes of $\zeta_{F_1}(s)$, $\zeta_{F_2}(s)$ respectively. There exists an absolute constant $c > 0$ such that*

$$\min(\beta_1, \beta_2) < 1 - \frac{c}{\log(D_1 D_2)}.$$

From now on, when speaking of an exceptional zero, we will be speaking with reference to a family of constants C_d satisfying

$$C_{2^d \cdot d!} \leq \frac{c}{70 \cdot 2^{d+1} \cdot d! \cdot d^5}, \tag{1}$$

where c is the constant in Theorem 2.4.

The proof of the Theorem 2.4 can be found in Theorem 5.27 of [10]. Chapter 5 of [10] is also a great introduction to Siegel zeroes and the analytic theory of L-functions in general.

3. Producing Weyl CM fields

In [11], the André-Oort conjecture (2.1) was proven under the assumption of GRH. The reason for their assuming of GRH was to guarantee that certain CM fields have many small split primes. As such, our first task is to produce a sequence of Weyl CM fields of fixed degree g containing many small split primes. This is a problem in algebraic number theory. We construct our CM fields by using zeta functions of families of curves over finite fields. One advantage of our approach is that we immediately produce CM fields, without having to filter out the CM condition. In the next section, we follow the methods of [11] and prove conjecture (2.1) about Zariski closures for our sequence of CM points unconditionally.

We fix an integer $g > 1$ and pick a prime number $q > 2g$, which shall remain fixed for the rest of the section.

In [2], N. Chavdarov studies the following situation. Consider a family of proper, smooth curves of genus g, $\psi : C \to U$, where U is a smooth affine curve over $\mathbb{F}_q$. Assume that for $l \neq 2, q$, the mod-l monodromy of $R^1\psi_!\mathbb{Z}_l$ is the full symplectic group $\mathrm{Sp}_{2g}(\mathbb{F}_l)$. Such a family can be constructed by taking the family of curves,

$$\left\{y^2 = (x-t)\prod_{i=1}^{2g}(x-i)\right\},$$

parametrized by $t \in \mathbb{A}^1_{\mathbb{F}_q}$, as was proven by J. K. Yu (unpublished). The result was also reproven and generalized by Hall in [8]. Fix a symplectic pairing $\langle\,,\rangle$ and define

$$\mathrm{GSp}_{2g}(\mathbb{F}_l) = \{A \in M_{2g}(\mathbb{F}_l) \mid \langle Av, Aw\rangle = \gamma\langle v, w\rangle \text{ for some } \gamma \in \mathbb{F}_l^\times\}.$$

For each prime l, fix a symplectic isomorphism $H^1_{\mathrm{et}}(\overline{C}_0, \mathbb{F}_l) \cong \mathbb{F}_l^{2g}$. We shall use heavily the following theorem from [2], where it is attributed to N. Katz.

THEOREM 3.1 ([2, Thm. 4.1]). *With notation as above, let $l_1, l_2, \ldots, l_r$ be a distinct set of primes not equal to* 2 *or* q. *Set*

$$G_0 = \prod_{i=1}^r \mathrm{Sp}_{2g}(\mathbb{F}_{l_i}), G = \prod_{i=1}^r \mathrm{GSp}_{2g}(\mathbb{F}_{l_i}).$$

Then we have the following commutative diagram, where the rows are exact:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1^{\mathrm{geom}}(U,0) & \longrightarrow & \pi_1(U,0) & \xrightarrow{\deg} & \hat{\mathbb{Z}} & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \lambda_0} & & \downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle 1\to\gamma^{-1}} & & \\
1 & \longrightarrow & G_0 & \longrightarrow & G & \xrightarrow{\text{mult}} & \Gamma & \longrightarrow & 1,
\end{array}$$

and λ, λ_0 denote monodromy actions on $H^1_{\mathrm{et}}(\overline{C}_0, \prod_i \mathbb{F}_{l_i})$.

For each conjugacy class C of G, we have

$$\left|\mathrm{Prob}\{u \in U(\mathbb{F}_{q^n}) \mid \mathrm{Frob}_u \in C\} - \frac{|C \cap \mathrm{mult}^{-1}(\gamma^n)|}{|G_0|}\right| \ll_\psi |G| q^{-n/2}.$$

In the above theorem the notation $\ll_\psi |G|q^{-n/2}$ means there exists some constant $c(\psi) > 0$ depending only on the family ψ such that the left-hand side is at most $c(\psi)|G|q^{-n/2}$. It is critical for us to have the uniform dependence on G as the group itself varies.

For each $u \in U(\mathbb{F}_{q^n})$, we consider the numerator $P_u(T)$ of the zeta function of C_u. Theorem 2.3 of [2] says that $P_u(T)$ is irreducible for a density 1 subset of $U(\overline{\mathbb{F}_q})$, where the density of a set S is defined by

$$\lim_{n\to\infty} \frac{|S \cap U(\mathbb{F}_{q^n})|}{|U(\mathbb{F}_{q^n})|}.$$

Moreover, the field $\mathbb{K}_u = \mathbb{Q}(\pi_u)$ is a Weyl CM field for a subset of density 1, where π_u is a root of $P_u(T)$. We remind the reader that by the Weil conjectures for curves, all conjugates of π_u have absolute value $q^{n/2}$. We shall use the fact that how a prime $l \neq q$ factors in $\mathbb{K}_u$ can be read off from the image in $\mathrm{GSp}_{2g}(\mathbb{F}_l)$ of Frob_u.

The idea of the proof is that a conjugacy class mod l tells us how $P_u(T)$ reduces mod l. It is proven in [2] that by fixing a finite set of primes $m_1, m_2, \ldots, m_h$ and conjugacy classes C_i in $\mathrm{GSp}_{2g}(\mathbb{F}_{m_i})$, one can force $P_u(T)$ to be irreducible and for the associated field to be a Weyl CM field. (See [2, Lemmas 5.5, 5.6].)

We will now use Theorem 3.1 to construct Weyl CM fields $\mathbb{K}_u$ with many small split primes. Throughout the rest of this section, n will be an integer parameter that will be tending to infinity, and we shall be picking primes l_i to depend on n. First, note that since the ring of integers $O_{\mathbb{K}_u}$ contains $\mathbb{Z}[\pi_u]$ as a subring of finite index, we have $\mathrm{Disc}(\mathbb{K}_u) \leq \mathrm{Disc}(\mathbb{Z}[\pi_u]) \ll q^{ng^2}$, where the last inequality follows from the fact that all conjugates of π_u have absolute value $q^{n/2}$. Fix a prime l such that $n^5 < l < 2n^5$. Applying Theorem 3.1 to this prime, we see that l splits completely in $|U(\mathbb{F}_{q^n})|(\frac{1}{2^g g!} + o_n(1))$ fields $\mathbb{K}_u$. Since this is true for each prime l, we see that on average, each field $\mathbb{K}_u$ has

$$\frac{n^5}{2^g g! \log(n^5)} \cdot (1 + o_n(1))$$

primes between n^5 and $2n^5$ split completely. (Note that since most fields are Weyl CM fields, this is what is expected from Chebatorev's density theorem.) In particular, there exists at least one CM field $\mathbb{K}_u$ with at least $\frac{n^5}{2^{g+1} g! \log(n^5)}$ primes between n^5 and $2n^5$ that split completely in $\mathbb{K}_u$. By varying over n, we can thus create an infinite such sequence.

We are almost done, but there is still an issue to deal with: We have produced a sequence of Weyl CM fields with lots of split primes, but for these primes to be 'small' compared to the discriminant, we need to ensure that the discriminant of $\mathbb{K}_u$ is large. For that, we have the following bound by W. Schmidt [13].

THEOREM 3.2. *Let $N_g(X)$ be the number of fields K of degree g over $\mathbb{Q}$ with $D_K \leq X$. Then*

$$N_g(X) \ll_g X^{\frac{g+2}{4}}.$$

We mention that it is conjectured that $N_g(X) \sim c_g X$, and better bounds towards this conjecture have been obtained by Ellenberg and Venkatesh [7]. As a corollary of Theorem 3.2, we have the following useful lemma, which we shall use to rule out $\mathbb{K}_u$ having small discriminant.

LEMMA 3.3. *There exists a prime $l' \ll_g q^{\frac{n}{32g^2}}$ and a conjugacy class $C \subset GSp_g(\mathbb{F}_l')$ with $\operatorname{mult}(C) = \gamma^n$ such that if $u \in U(\mathbb{F}_{q^n})$ and $\operatorname{Frob}_u \in C$, then $\operatorname{Disc} \mathbb{K}_u \gg q^{\frac{n}{64g^3}}$.*

Note that we have made no attempt to optimize exponents.

Proof. Suppose not. Corresponding to u there is an algebraic integer π_u with all its conjugates of absolute value $q^{n/2}$ such that $\pi_u \in \mathbb{K}_u$. Note that $N_{K/\mathbb{Q}}(u) = q^{ng}$. Now, each $\mathbb{K}_u$ contains at most $(ng)^{2g}$ different ideals I with norm q^{ng}, and each such ideal is generated by at most $O_g(1)$ different algebraic integers π such that all of the conjugates of π have the same norm. Thus, by Theorem 3.2 the number of algebraic integers π having all their conjugates of absolute value $q^{n/2}$ that generate a field of degree $2g$ over K with $\operatorname{Disc} K \leq q^{\frac{n}{64g^3}}$ is at most $O_g(1) \cdot n^{2g} \cdot q^{\frac{(2g+2)n}{256g^3}} \leq O_g(1) \cdot q^{\frac{n}{64g^2}}$. Thus, there are at most $O_g(1) \cdot q^{\frac{n}{64g^2}}$ characteristic polynomials of those algebraic integers modulo l'. Now simply pick C with a distinct characteristic polynomial. $\square$

We can now prove the main result of this section.

LEMMA 3.4. *For each g, there exists a sequence of distinct Weyl* CM *fields K_i with discriminant D_i satisfying the following properties*:

(1) *There exists a constant c_g such that at least $c_g \frac{\log^5 D_i}{\log\log D_i}$ primes $p \leq 2\log(D_i)^5$ split completely in K_i.*
(2) *For each number field L, the Galois closure of K_i does not contain L for $i \gg_L 0$.*
(3) *There exist c_1, c_2 such that $c_1 q^{\frac{n}{64g^3}} \leq D_n \leq c_2 q^{ng^2}$.*

Proof. We build the K_n in a few steps. First, we pick a finite set of primes $m_1, m_2, \ldots, m_h$ and conjugacy classes C_i in the corresponding groups

$\mathrm{GSp}_{2g}(\mathbb{F}_{m_i})$ such that $\mathrm{mult}(C_i) = \gamma^n$ and that for any u with $\lambda(\mathrm{Frob}_u) \in C_i$, the polynomial $P_u(T)$ is irreducible and $\mathbb{K}_u$ is a Weyl CM field. Next, pick for each n a prime l' distinct from the m_i such that $l' \ll_g q^{\frac{n}{32g^2}}$, and a conjugacy class $C \subset GSp_g(\mathbb{F}'_l)$ as in Lemma 3.3. Finally, we pick an auxiliary prime l such that $n^5 < l < 2n^5$ and let E_l denote the union of all conjugacy classes in $\mathrm{GSp}_{2g}(\mathbb{F}_l)$ such that $\mathrm{mult}(E_l) = \gamma^n$ and also the characteristic polynomials of all elements E_l split completely over $\mathbb{F}_l$. We now apply Theorem 3.1 to the primes m_i, l' with the union of conjugacy classes $C = \prod_{i=1}^h C_i \times C$. In the notation of Theorem 3.1, we have $G = \prod_{i=1}^h \mathrm{GSp}_{2g}(\mathbb{F}_{m_i}) \times \mathrm{GSp}_{2g}(\mathbb{F}_{l'})$, $G_0 = \prod_{i=1}^h \mathrm{Sp}_{2g}(\mathbb{F}_{m_i}) \times \mathrm{Sp}_{2g}(\mathbb{F}_{l'})$, and

$$\begin{aligned}\mathrm{Prob}\{u \in U(\mathbb{F}_{q^n}) \mid \mathrm{Frob}_u \in C\} &= \frac{|C \cap G^{\mathrm{mult}\gamma^n}|}{|G_0|} + O(|G|q^{-n/2}) \\ &= \frac{|C \cap G^{\mathrm{mult}\gamma^n}|}{|G_0|} + O(q^{-3n/8}).\end{aligned}$$

As $|G_0| \ll q^{n/4}$ and $U(\mathbb{F}_{q^n}) \asymp q^n$, we see that we have a set $S \subset U(\mathbb{F}_{q^n})$ of points u with $\mathrm{Frob}_u \in C$ of size

$$|S| = \frac{U(\mathbb{F}_{q^n}) \times |C \cap G^{\mathrm{mult}\gamma^n}|}{|G_0|} + O(q^{5n/8})$$

such that $\mathbb{Q}(\pi_u)$ is a Weyl CM field $\mathbb{K}_u$ with discriminant

$$q^{\frac{n}{64g^3}} \le \mathrm{Disc}(\mathbb{K}_u) \ll q^{ng^2}$$

so that (3) holds.

Now, we apply a similar calculation to the primes m_i, l', l, where now we take the union of conjugacy classes

$$C^0 = \prod_{i=1}^h C_i \times C \times E_l.$$

This shows that the number of points $u \in S$ such that the prime l splits completely in $\mathbb{K}_u$ is

$$\frac{|E_l|}{|\mathrm{Sp}_{2g}(\mathbb{F}_l)|} \times \frac{U(\mathbb{F}_{q^n}) \times |C^0 \cap G^{\mathrm{mult}\gamma^n}|}{|G_0|} + O(|\mathrm{GSp}_{2g}(\mathbb{F}_l)|q^{5n/8}).$$

By [2, Thm. 3.5], it follows that $\frac{|E_l|}{|\mathrm{Sp}_{2g}(\mathbb{F}_l)|} \longrightarrow \frac{1}{2^g \times g!}$. Averaging over l between n^5 and $2n^5$, we see that at least one of the $\mathbb{K}_u$ satisfies conditions (1) and (3).

Finally, for condition (2), enumerate all number fields $L_1, L_2, \ldots, L_n, \ldots$ and pick a totally inert prime p_i in each. We can then repeat the above construction of the K_i, insisting that K_n is eventually totally split at each of $p_1, p_2, \ldots p_m, \ldots$ by picking appropriate conjugacy classes. This will ensure that (2) holds. $\square$

In order to produce primes later on, we shall need a subsequence of the K_i that has no exceptional zeroes.

LEMMA 3.5. *There is an infinite subsequence W_j of the K_i such that for V_j the Galois closure of W_j, $\zeta_{V_j}(s)$ has no exceptional zero.*

Proof. Assume not. Since there are only finitely many number fields with a fixed degree and bounded discriminant, there is some real number r such that for $D_i \geq r$, the Dedekind zeta function $\zeta_{L_i}(s)$ has an exceptional zero, where L_i is the Galois closure of K_i. By Theorem 2.3, this implies that there is a quadratic subfield $F_i \subset L_i$ such that $\zeta_{F_i}(s)$ has a zero β_i such that

$$1 - \frac{C_{2^g \cdot g!}}{\log \operatorname{Disc}(F_i)} < 1 - \frac{C_{2^g \cdot g!}}{\log D_i} < \beta_i < 1$$

for any sufficiently large i. By (3) of Lemma 3.4, there is some K_j with $D_j > D_i > r$ such that

$$1 - \frac{70g^5 \cdot C_{2^g \cdot g!}}{\log D_j} < \beta_i < 1 - \frac{C_{2^g \cdot g!}}{\log D_j}.$$

Hence there is some quadratic field $F_j \subset L_j$ such that $\zeta_{F_j}(s)$ has a zero β_j with

$$1 - \frac{C_{2^g \cdot g!}}{\log D_j} < \beta_j < 1.$$

However, note that

$$\log\left(|\operatorname{Disc}(F_i)| \cdot |\operatorname{Disc}(F_j)|\right) \leq 2\log(|\operatorname{Disc}(L_j)|) \leq 2^{g+1} \cdot g! \log(D_j).$$

Applying Theorem 2.4, we arrive at

$$\begin{aligned} 1 - \frac{70g^5 C_{2^g \cdot g!}}{\log D_j} < \min(\beta_1, \beta_2) &< 1 - \frac{c}{\log\left(|\operatorname{Disc}(F_i)| \cdot |\operatorname{Disc}(F_j)|\right)} \\ &< 1 - \frac{c}{2^{g+1} \cdot g! \log(D_j)}. \end{aligned}$$

This last statement contradicts equation (1), completing the proof. □

4. **Proof of Theorem 1.2**

In this section we combine the arguments of [1] with our Lemma 3.4 to prove Theorem 1.2. First, we recall the following bound of Yafaev.

LEMMA 4.1 (Yafaev). *Fix a Shimura variety* $\operatorname{Sh}(G, X)_K$ *defined over a number field F. For any $\varepsilon > 0$ and $N > 0$, there exist $c_1, c_2 > 0$ such that the following holds.*

Let s be a special point, with CM *by a field K, in* $\operatorname{Sh}(G, X)$. *Let K have discriminant D_K, and suppose there are at least $\varepsilon \frac{\log(D_K)}{\log(\log(D_K))}$ primes $p <$*

$\frac{1}{\varepsilon}(\log D_K)^5$ *that split completely in* K. *Then*

$$|\mathrm{Gal}(\overline{\mathbb{Q}}/F)\cdot s| \geq c_1 \cdot (\log D_K)^N \cdot \prod_{\substack{p \text{ prime} \\ \mathrm{MT(s)}_{/\mathbb{F}_p} \text{ is not a torus}}} c_2 p,$$

where $\mathrm{MT}(s)$ *denotes the Mumford-Tate group associated to* s.

Proof. The above is Theorem 2.1 in [15]. The theorem is stated with the assumptions of GRH, but this assumption is only used in Theorem 2.15 to produce small split primes, whose existence we are assuming in the statement of the lemma. In [15], the theorem is stated in terms of D_L, the discriminant of the normal closure of K. Since $\frac{\log(D_L)}{\log(D_K)} = O_{[K:\mathbb{Q}]}(1)$, the version above follows at a cost of enlarging the constant c_1. □

Before proceeding with the proof of Theorem 1.2, we make a definition: Following [1], we define a Hilbert modular variety attached to a totally real field F of degree g over $\mathbb{Q}$ to be any irreducible component of a closed subvariety $A_{g,1}^{\mathcal{O}} \subset A_{g,1}$ over $\overline{\mathbb{Q}}$. Here $\mathcal{O}$ is an order in F and $A_{g,1}^{\mathcal{O}}$ is the locus of all points $[A,\lambda]$, where the endomorphisms ring of A contains $\mathcal{O}$ as a subring. Note that each Hilbert modular variety is a Shimura subvariety of $A_{g,1}$ corresponding to the pair $(\mathcal{G}_F, \mathbb{H}^g)$, where $\mathcal{G}_F$ is the subgroup of $\mathrm{Res}_{F/\mathbb{Q}}\,\mathrm{GL}_2$ with determinant in $\mathbb{Q}^\times$.

LEMMA 4.2. *If* $S \subsetneq A_{g,1}$ *is a positive dimensional Shimura subvariety that contains a Weyl* CM *point*, *then* S *is a Hilbert modular variety.*

Proof. This is Lemma 3.5 in [1]. □

Proof of Theorem 1.2. Pick a sequence of points $y_i \in \mathbb{A}_{g,1}(\overline{\mathbb{Q}})$ such that y_i corresponds to a principally polarized abelian variety with complex multiplication by a subring of the field W_i, where W_i are the Weyl CM fields constructed in Lemma 3.5. That one can do this is a standard fact in the theory of abelian varieties; see [14] for details. Assume the statement of the theorem is false. Then X contains x_i such that x_i is isogenous to y_i and therefore has complex multiplication by W_i. If Theorem 8.3.1 in [11] holds for $Z = X$ and $V = x_i$, then for $i \gg 0$ we can conclude that X contains a Shimura subvariety S_i containing x_i. By Lemma 4.2, S_i must be a Hilbert modular variety. Moreover, the S_i form an infinite set since the W_i eventually have distinct totally real subfields by (2) of Lemma 3.4. However, by Theorem 1.2 of [3], some subsequence S_{n_i} becomes equidistributed for the unique homogeneous measure corresponding to a Shimura subvariety $S \subset A_{g,1}$ that must contain S_{n_i} for large enough i. We can thus conclude that S is not a finite union of Hilbert modular varieties, and so by Lemma 4.2, this means that S must be all of $A_{g,1}$. The S_i thus become equidistributed for the natural measure in $A_{g,1}$, which is a contradiction to $S_i \subset X$. Hence, its enough to verify Theorem 8.3.1 of [11] in our case.

Now, the assumption of GRH in Theorem 8.3.1 is used only in Proposition 9.1 of [11] to produce a small prime l as in the following Proposition 4.3. By proving the following proposition unconditionally, we complete the proof of Theorem 1.2.

Definition. Fix a positive constant $B > 0$. Define β_i to be $\beta_i = \prod_p (Bp)$ where the product goes over all primes p such that $\mathrm{MT}(x_i)_{/\mathbb{F}_p}$ is not a torus, where $\mathrm{MT}(x_i)$ denotes the Mumford-Tate group associated to x_i.

From now on, D_i will denote the discriminant of W_i.

PROPOSITION 4.3. *Fix $\varepsilon > 0, c > 0$. Then for each $i \gg 0$, there exists a prime l such that*

(1) *l is totally split in W_i,*
(2) *$\mathrm{MT}(x_i)_{/\mathbb{F}_l}$ is a torus,*
(3) *$l < c\log(D_i)^6 \beta_i^\varepsilon$.*

Proof. By construction, there is a constant c_g such that there are at least

$$c_g \frac{(\log D_i)^5}{\log(\log(D_i))}$$

primes $p \leq 2\log(D_i)^5$ split completely in W_i. Since β_i is bounded from below (there are only finitely many primes less than $\frac{1}{B}$), we see that for $i \gg 0$, all these primes satisfy conditions (1) and (3). We are thus done unless $\mathrm{MT}(x_i)_{/\mathbb{F}_p}$ is not a torus for all these primes p. Assume this is the case from now on. We thus have

$$\beta_i \gg e^{(\log D_i)^4}. \tag{2}$$

By Theorem 2.2, for $X \gg e^{(\log D_i)^3}$, the number of totally split primes in W_i less than X is

$$\pi_{W_i}(X) = \frac{1}{2^g \cdot g!} \cdot \frac{X}{\log X} + o\left(\frac{X}{\log X}\right)$$

since by construction the Dedekind zeta function $\zeta_{V_i}(s)$ has no exceptional zero, where we define V_i to be the Galois closure of W_i. Thus, for $i \gg 0$, we have

$$\pi_{W_i}(X) \gg \frac{X}{\log X}.$$

Since for large enough i we have $e^{(\log D_i)^3} < c\log(D_i)^6 \beta_i^\varepsilon$, there are at least $\frac{\beta_i^\varepsilon}{\varepsilon \log(\beta_i)}$ totally split primes l in W_i such that $l < c\log(D_i)^6 \beta_i^\varepsilon$ for large enough i.

Now, one of these primes l must be such that $\mathrm{MT}(x_i)_{/\mathbb{F}_l}$ is a torus, since otherwise we would have

$$\beta_i \gg 2^{\frac{\beta_i^{\varepsilon}}{\varepsilon \log(\beta_i)}},$$

which is false for large enough i since $\beta_i \to \infty$. This completes the proof. □

5. Final remarks: Effectivity

We end by discussing the effectivity of Theorem 1.2. Specifically, we show how given enough computing power, one could actually provably produce an abelian variety with no isogeny to a point in $X \subset A_{g,1}$. The proof of Theorem 1.2 consists of two parts:

- getting a Weyl CM point x such that if $x \in X$, then x is contained in a Shimura subvariety of X, which must be a Hilbert modular variety;
- getting a bound for the number of Hilbert modular varieties in X.

To get a sequence of points x_i satisfying (i) effectively is easy: We simply find a curve $C \to U/F_q$ and look at the sequence of all CM fields we get in this way: $K_1, K_2, \ldots$. As we proved, there is an effective constant $\delta(n)$ such that there are at least n Weyl CM fields W_j among the $K_i, i \leq \delta(n)$ such that if $x \in X$ corresponds to an abelian variety with endomorphism algebra W_j, then x lies on a Hilbert modular variety inside X. But given a field K_i, we can check if it works manually in the following way. All we need to check is that sufficiently many primes l split in K_i up to a given number Y for Proposition 4.3 to go through. Since by Brauer's work there exist effective (though very poor) bounds on the smallest nontrivial zeroes of L-functions, the explicit Chebotarev Density in [12] shows that given a field, we only need to check up to an effectively finite Y before Chebotarev guarantees sufficiently many small split primes.

For (ii), we appeal to a theorem of Einsiedler, Margulis, and Venkatesh [6] that gives an effective rate on how quickly Hilbert modular varieties equidistribute in terms of their covolume. For Hilbert modular varieties corresponding to $\mathrm{SL}_2(F)$, the volume is, up to some simple constants, equal to $\zeta_F(2)D_F^{\frac{3}{2}}$, and so it grows, effectively, like a power of the discriminant D_F. Thus, for high enough D_F, we can guarantee that X does not contain a Hilbert modular variety corresponding to F. This finishes the proof.

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