

# Rational points over finite fields for regular models of algebraic varieties of Hodge type $\geq 1$

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## Abstract

Let  $R$  be a discrete valuation ring of mixed characteristics  $(0, p)$ , with finite residue field  $k$  and fraction field  $K$ , let  $k'$  be a finite extension of  $k$ , and let  $X$  be a regular, proper and flat  $R$ -scheme, with generic fibre  $X_K$  and special fibre  $X_k$ . Assume that  $X_K$  is geometrically connected and of Hodge type  $\geq 1$  in positive degrees. Then we show that the number of  $k'$ -rational points of  $X$  satisfies the congruence  $|X(k')| \equiv 1 \pmod{|k'|}$ . We deduce such congruences from a vanishing theorem for the Witt cohomology groups  $H^q(X_k, W\mathcal{O}_{X_k, \mathbb{Q}})$  for  $q > 0$ . In our proof of this last result, a key step is the construction of a trace morphism between the Witt cohomologies of the special fibres of two flat regular  $R$ -schemes  $X$  and  $Y$  of the same dimension, defined by a surjective projective morphism  $f : Y \rightarrow X$ .

## Contents

1. Introduction and first reductions	414
2. Application of $p$ -adic Hodge theory	420
3. An injectivity theorem for coherent cohomology	435
4. Koszul resolutions and local description of the trace morphism $\tau_f$	437
5. Preliminaries on the relative de Rham-Witt complex	441
6. The Hodge-Witt trace morphism for projective spaces	454
7. A Hodge-Witt local class for regularly embedded subschemes	459
8. Proof of the injectivity theorem for Witt vector cohomology	468
9. An example	477
Appendix: Complete intersection morphisms of virtual relative dimension 0	484
A. The canonical section of the relative dualizing sheaf	484
B. The trace morphism $\tau_f$ on $Rf_*(\mathcal{O}_Y)$	494
References	505

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## 1. Introduction and first reductions

Let  $R$  be a discrete valuation ring of mixed characteristics  $(0, p)$ , with perfect residue field  $k$  and fraction field  $K$ . The main goal of this article is to prove the following theorem.

**THEOREM 1.1.** *Let  $X$  be a proper and flat  $R$ -scheme, with generic fibre  $X_K$ , such that the following conditions hold:*

- (a)  $X$  is a regular scheme.
- (b)  $X_K$  is geometrically connected.
- (c)  $H^q(X_K, \mathcal{O}_{X_K}) = 0$  for all  $q \geq 1$ .

*If  $k$  is finite, then, for any finite extension  $k'$  of  $k$ , the number of  $k'$ -rational points of  $X$  satisfies the congruence*

$$(1.1.1) \quad |X(k')| \equiv 1 \pmod{|k'|}.$$

Condition (c) should be viewed as a Hodge theoretic property of  $X_K$ , which can be stated by saying that  $X_K$  has Hodge type  $\geq 1$  in positive degrees. From this point of view, this theorem fits in the general analogy between the vanishing of Hodge numbers for varieties over a field of characteristic 0 and congruences on the number of rational points with values in finite extensions for varieties over a finite field. This analogy came to light with the coincidence between the numerical values in Deligne's theorem on smooth complete intersections in a projective space [SGA7II, Exposé XI, Th. 2.5] and in the Ax-Katz theorem on congruences on the number of solutions of systems of algebraic equations [Ktz71, Th. 1.0]. It has been made effective by Katz's conjecture [Ktz71, Conj. 2.9] relating the Newton and Hodge polygons associated to the cohomology of a proper and smooth variety (and generalizing earlier results of Dwork for hypersurfaces [Dwo64]). For varieties in characteristic  $p$ , this conjecture was proved by Mazur ([Maz72], [Maz73]) and Ogus [BO78, Th. 8.39]. In the mixed characteristic case, where a stronger form can be given using the Hodge polygon of the generic fibre, it is a consequence of the fundamental results in  $p$ -adic Hodge theory. Our proof of Theorem 1.1 makes essential use of the inequality between these two polygons, but the setup of the theorem is actually more general, since the scheme  $X$  is not supposed to be semi-stable over  $R$ .

Let us also recall that a result similar to Theorem 1.1 has been proved by the second author [Esn06, Th. 1.1] by  $\ell$ -adic methods, with condition (c) replaced by a coniveau condition: for any  $q \geq 1$ , any cohomology class in  $H_{\text{ét}}^q(X_{\bar{K}}, \mathbb{Q}_\ell)$  vanishes in  $H_{\text{ét}}^q(U_{\bar{K}}, \mathbb{Q}_\ell)$  for some nonempty open subset  $U \subset X_K$ . It is easy to see, using [Del71], that this coniveau condition implies that the Hodge level of  $X_K$  is  $\geq 1$  in degree  $q \geq 1$  (see [Ill06, 4.4 (d)] for a more general discussion). It would actually follow from Grothendieck's generalized Hodge

conjecture [Grt69] that the two conditions are equivalent. In this article, the use of  $p$ -adic methods, and in particular of  $p$ -adic Hodge theory, allows us to derive congruence (1.1.1) directly from Hodge theoretic hypotheses.

1.2. As explained by Ax [Ax64], congruences such as (1.1.1) can be expressed in terms of the zeta function of the special fibre  $X_k$  of  $X$ . We recall that the rationality of the zeta function  $Z(X_k, t)$  allows us to define the slope  $< 1$  part  $Z^{<1}(X_k, t)$  of  $Z(X_k, t)$  as follows [BBE07, 6.1]. Let  $|k| = p^a$ , and write

$$Z(X_k, t) = \prod_i (1 - \alpha_i t) / \prod_j (1 - \beta_j t),$$

with  $\alpha_i, \beta_j \in \overline{\mathbb{Q}}_p$  and  $\alpha_i \neq \beta_j$  for all  $i, j$ . Normalizing the  $p$ -adic valuation  $v$  of  $\overline{\mathbb{Q}}_p$  by  $v(p^a) = 1$ , one sets

$$Z^{<1}(X_k, t) = \prod_{v(\alpha_i) < 1} (1 - \alpha_i t) / \prod_{v(\beta_j) < 1} (1 - \beta_j t).$$

Then the congruences (1.1.1) are equivalent to

$$(1.2.1) \quad Z^{<1}(X_k, t) = \frac{1}{1 - t}$$

[BBE07, Prop. 6.3].

On the other hand, let  $W(\mathcal{O}_{X_k})$  be the sheaf of Witt vectors with coefficients in  $\mathcal{O}_{X_k}$ , and  $W\mathcal{O}_{X_k, \mathbb{Q}} = W(\mathcal{O}_{X_k}) \otimes \mathbb{Q}$ . Then the identification of the slope  $< 1$  part of rigid cohomology with Witt vector cohomology provides the cohomological interpretation

$$(1.2.2) \quad Z^{<1}(X_k, t) = \prod_i \det(1 - tF^a | H^i(X_k, W\mathcal{O}_{X_k, \mathbb{Q}}))^{(-1)^{i+1}},$$

where  $F$  is induced by the Frobenius endomorphism of  $W(\mathcal{O}_{X_k})$  [BBE07, Cor. 1.3]. Therefore, Theorem 1.1 is a consequence of the following theorem, where  $k$  is only assumed to be perfect.

**THEOREM 1.3.** *Let  $X$  be a regular, proper and flat  $R$ -scheme. Assume that  $H^q(X_K, \mathcal{O}_{X_K}) = 0$  for some  $q \geq 1$ . Then*

$$(1.3.1) \quad H^q(X_k, W\mathcal{O}_{X_k, \mathbb{Q}}) = 0.$$

*Proof of Theorem 1.1, assuming Theorem 1.3.* Let us prove here this implication, which is easy and does not use the regularity assumption on  $X$ . Let  $W = W(k)$ , and  $K_0 = \text{Frac}(W)$ . Thanks to (1.2.1) and (1.2.2), Theorem 1.3 implies that it suffices to prove that the homomorphism  $K_0 \rightarrow H^0(X_k, W\mathcal{O}_{X_k, \mathbb{Q}})$  is an isomorphism.

As  $X$  is proper and flat over  $R$ ,  $H^0(X, \mathcal{O}_X)$  is a free finitely generated  $R$ -module. Since the generic fibre  $X_K$  is geometrically connected and geometrically reduced, the rank of  $H^0(X, \mathcal{O}_X)$  is 1. The homomorphism  $R \rightarrow H^0(X, \mathcal{O}_X)$  maps 1 to 1, hence Nakayama's lemma implies that it is

an isomorphism. Applying Zariski's connectedness theorem, it follows that  $X_k$  is connected, and even geometrically connected, since the same argument can be applied after any base change from  $R$  to  $R'$ , where  $R'$  is the ring of integers of a finite extension of  $K$ .

On the other hand, let  $\bar{k}$  be an algebraic closure of  $k$ , and let  $k'$  be a finite extension of  $k$  such that  $X_{\bar{k}\text{red}}$  is defined over  $k'$ . As  $k'$  is separable over  $k$ , the homomorphisms  $W_n(k) \rightarrow W_n(k')$  are finite étale liftings of  $k \rightarrow k'$  and the homomorphisms  $W_n(k') \otimes_{W_n(k)} W_n(\mathcal{O}_{X_k}) \rightarrow W_n(\mathcal{O}_{X_{k'}})$  are isomorphisms [Ill79, I, Prop. 1.5.8]. It follows that the homomorphism  $W(k') \otimes_{W(k)} H^0(X_k, W(\mathcal{O}_{X_k})) \rightarrow H^0(X_{k'}, W(\mathcal{O}_{X_{k'}}))$  is an isomorphism and that it suffices to prove the claim for  $X_{k'}$ . Using the fact that

$$H^0(X_{k'}, W\mathcal{O}_{X_{k'}, \mathbb{Q}}) \xrightarrow{\sim} H^0(X_{k'\text{red}}, W\mathcal{O}_{X_{k'\text{red}}, \mathbb{Q}})$$

by [BBE07, Prop. 2.1 (i)], it suffices to check that, if  $Z$  is a proper, geometrically connected and geometrically reduced  $k$ -scheme, the homomorphism  $W(k) \rightarrow H^0(Z, W(\mathcal{O}_Z))$  is an isomorphism.

Under these assumptions, the homomorphism  $k \rightarrow H^0(Z, \mathcal{O}_Z)$  is an isomorphism. As the homomorphism  $R : W_n(\mathcal{O}_Z) \rightarrow W_{n-1}(\mathcal{O}_Z)$  is the projection of a product onto one of its factors, the homomorphisms  $H^0(Z, W_n(\mathcal{O}_Z)) \rightarrow H^0(Z, W_{n-1}(\mathcal{O}_Z))$  are surjective, and one gets by induction that the homomorphism  $W_n(k) \rightarrow H^0(Z, W_n(\mathcal{O}_Z))$  is an isomorphism for all  $n$ . Taking inverse limits, the claim follows.  $\square$

1.4. Theorem 1.3 is deeper, and most of our paper is devoted to developing the techniques used in its proof. We may observe though that, in the context of Theorem 1.1, there is a case where (1.3.1) is trivial: namely, if we replace the condition on the Hodge numbers of  $X_K$ , which is equivalent to requiring that the modules  $H^q(X, \mathcal{O}_X)$  be  $p$ -torsion modules, by the stronger condition that  $H^q(X, \mathcal{O}_X)$  vanishes for all  $q \geq 1$ . Indeed, the flatness of  $X$  over  $R$  allows us to apply the derived base change formula for coherent cohomology and to conclude that  $H^q(X_k, \mathcal{O}_{X_k}) = 0$  for all  $q \geq 1$ . By induction on  $n$ , one gets that  $H^q(X_k, W_n(\mathcal{O}_{X_k})) = 0$  for all  $n, q \geq 1$ , and (1.3.1) follows for all  $q \geq 1$  (even before tensoring with  $\mathbb{Q}$ ).

In the general case, where the  $H^q(X, \mathcal{O}_X)$  are  $p$ -torsion modules, we do not know any direct argument to derive the vanishing property stated in (1.3.1). Our strategy is then to use the results of  $p$ -adic Hodge theory relating the Hodge and Newton polygons of certain filtered  $F$ -isocrystals on  $k$ , which allow us to study separately the cohomology groups for a given  $q$  as in Theorem 1.3. In particular, when  $X$  is semi-stable on  $R$ , a straightforward argument using the fundamental comparison theorems of  $p$ -adic Hodge theory allows us to deduce (1.3.1) from the inequality between the two polygons defined by the log crystalline cohomology of  $X_k$ . We explain this argument in Theorem 2.1.

In the rest of Section 2, we show that this argument can be modified to prove the vanishing of  $H^q(X_k, W\mathcal{O}_{X_k, \mathbb{Q}})$  in the general case. For any finite extension  $K'$  of  $K$ , with ring of integers  $R'$ , let  $X_{R'}$  be deduced from  $X$  by base change from  $R$  to  $R'$ . After reducing to the case where  $R$  is complete, the first step is to apply de Jong's alteration theorem to construct for any  $m$  an  $m$ -truncated simplicial scheme  $Y_\bullet$  over the ring of integers  $R'$  of a suitable extension  $K'$  of  $K$ , endowed with an augmentation morphism  $Y_0 \rightarrow X_{R'}$ , such that the  $Y_i$ 's are pullbacks of proper semi-stable schemes, and  $Y_\bullet \rightarrow X_{R'}$  induces an  $m$ -truncated proper hypercovering of  $X_{K'}$  (see Lemma 2.2 for a precise statement). Then, using Tsuji's extension of the comparison theorems to truncated simplicial schemes [Tsu98], we show that, in this situation, the cohomology group  $H^q(Y_{\bullet, k}, W\mathcal{O}_{Y_{\bullet, k}, \mathbb{Q}})$  vanishes. However, due to the possible presence of vertical components in the coskeletons, the special fibre  $Y_{\bullet, k}$  of the  $m$ -truncated simplicial scheme  $Y_\bullet$  may not be a proper hypercovering of  $X_k$ , and it is unclear how the groups  $H^q(Y_{\bullet, k}, W\mathcal{O}_{Y_{\bullet, k}, \mathbb{Q}})$  are related to the groups  $H^q(X_k, W\mathcal{O}_{X_k, \mathbb{Q}})$ . Therefore another ingredient will be necessary to complete the proof. It will be provided by the following injectivity theorem, the proof of which will be given in Section 8.

**THEOREM 1.5.** *Let  $X, Y$  be two flat, regular  $R$ -schemes of finite type, of the same dimension, and let  $f : Y \rightarrow X$  be a projective and surjective  $R$ -morphism, with reduction  $f_k$  over  $\text{Spec } k$ . Then, for all  $q \geq 0$ , the functoriality homomorphism*

$$(1.5.1) \quad f_k^* : H^q(X_k, W\mathcal{O}_{X_k, \mathbb{Q}}) \longrightarrow H^q(Y_k, W\mathcal{O}_{Y_k, \mathbb{Q}})$$

*is injective.*

1.6. We will deduce Theorem 1.5 from the existence of a trace morphism

$$(1.6.1) \quad \tau_{i, \pi} : Rf_*(W\mathcal{O}_{Y_k, \mathbb{Q}}) \longrightarrow W\mathcal{O}_{X_k, \mathbb{Q}},$$

defined by means of a factorization  $f = \pi \circ i$ , where  $\pi$  is the projection of a projective space  $\mathbb{P}_X^d$  on  $X$  and  $i$  is a closed immersion. The key fact used in the construction of this trace morphism is that, under the assumptions of Theorem 1.5,  $i$  is a regular immersion of codimension  $d$ , or, said otherwise, that  $f$  is a complete intersection morphism of virtual relative dimension 0, in the sense of [SGA6, Exposé VIII].

Sections 3 to 7 are devoted to the construction of  $\tau_{i, \pi}$ . In Section 3, we state a similar result for  $\mathcal{O}_X$ , providing a canonical trace morphism

$$\tau_f : Rf_*(\mathcal{O}_Y) \rightarrow \mathcal{O}_X,$$

whenever  $X$  is a noetherian scheme with a relative dualizing complex, and  $f : Y \rightarrow X$  is a proper complete intersection morphism of virtual relative dimension 0 (see Theorem 3.1). The existence of  $\tau_f$  has been observed by

El Zein as a particular case of his construction of the relative fundamental class [EZ78, IV, Prop. 6]. However, in the literature there does not seem to be a complete proof of the properties listed in Theorem 3.1. Due to the many corrections and complements to [Har66] made by Conrad in [Con00], we have included in an appendix the details of a proof of Theorem 3.1 based on [Con00]. We refer to B.7 for the definition of  $\tau_f$  and to B.9 for the proof of Theorem 3.1. When  $Y$  is finite locally free of rank  $r$  over  $X$ , the composition of the functoriality morphism  $\mathcal{O}_X \rightarrow Rf_*(\mathcal{O}_Y)$  with  $\tau_f$  is multiplication by  $r$  on  $\mathcal{O}_X$ . This has striking consequences for the functoriality maps induced by  $f$  on coherent cohomology (see Theorem 3.2). For example, if  $r$  is invertible on  $X$ , one obtains an injectivity theorem that may be of independent interest. An outline of the construction of  $\tau_f$  is given in the introduction to the appendix.

To construct the trace morphism  $\tau_{i,\pi}$ , we consider more generally a projective complete intersection morphism  $f : Y \rightarrow X$  of virtual relative dimension 0 between two noetherian  $\mathbb{F}_p$ -schemes with dualizing complexes. Under these assumptions, we construct a compatible family of morphisms

$$\tau_{i,\pi,n} : Rf_*(W_n(\mathcal{O}_Y)) \rightarrow W_n(\mathcal{O}_X)$$

for  $n \geq 1$ , with  $\tau_{i,\pi,1} = \tau_f$ . Our main tool here is the theory of the relative de Rham-Witt complex developed by Langer and Zink [LZ04]. In Section 5, we recall some basic facts about their construction, and we extend to the relative case some structure theorems proved by Illusie [Ill79] when the base scheme is perfect (see, in particular, Proposition 5.7 and Theorem 5.13). Then we define  $\tau_{i,\pi,n}$  by combining two morphisms. On the one hand, we consider a projective space  $P := \mathbb{P}_X^d$  with projection  $\pi$  on  $X$ , and in Section 6 we define a trace morphism

$$\mathrm{Trp}_{\pi,n} : R\pi_*(W_n\Omega_{P/X}^d[d]) \rightarrow W_n(\mathcal{O}_X),$$

using the  $d$ -th power of the Chern class of the canonical bundle  $\mathcal{O}_P(1)$ . On the other hand, we consider a regularly embedded closed subscheme  $Y$  of a smooth  $X$ -scheme  $P$ , and in Section 7 we define a relative Hodge-Witt local class for  $Y$  in  $P$ , which is a section of  $\mathcal{H}_Y^d(W_n\Omega_{P/X}^d)$  and defines a morphism

$$\gamma_{i,\pi,n} : i_*W_n(\mathcal{O}_Y) \rightarrow W_n\Omega_{P/X}^d[d],$$

with  $i : Y \hookrightarrow P$  and  $d = \mathrm{codim}_P(Y)$ . This allows us to define the morphism  $\tau_{i,\pi,n}$  as being the composition  $\mathrm{Trp}_{\pi,n} \circ R\pi_*(\gamma_{i,\pi,n})$ . The proof of Theorem 1.5 is then completed in Section 8 thanks to a theorem relating the morphisms  $\tau_{i,\pi,n}$  defined by the reduction mod  $p$  of a factorization of the given morphism  $f : Y \rightarrow X$  over  $R$  and the morphism  $\tau_f$  defined by  $f$ .

It may be worth pointing out here that these results seem to indicate that Grothendieck's relative duality theory for coherent  $\mathcal{O}$ -modules can be generalized to some extent to the Hodge-Witt sheaves, as was already apparent

from [Eke84] when the base scheme is a perfect field. We do not try to develop such a generalization in this article, and we limit ourselves to the properties needed for the proof of Theorem 1.1. For example, it is very likely that the morphisms  $\tau_{i,\pi,n}$  only depend on  $f$ , and not on the chosen factorization  $f = \pi \circ i$ , but this is not needed here, and we do not prove it in this article. A natural context one might think of for developing our results is the theory of the trace map for projectively embeddable morphisms outlined in [Har66, III, 10.5 and §11]. However, as discussed by Conrad in [Con00, pp. 103–104], the foundational work needed for the definition of such a theory has not really been done even for coherent  $\mathcal{O}$ -modules. We hope to return to these questions in another article.

Finally, in Section 9 we conclude by giving a family of examples to which Theorem 1.1 can be applied but that are not covered by earlier results, nor by cases where Theorem 1.3 can be proved directly, such as the trivial case where  $H^i(X, \mathcal{O}_X) = 0$  for all  $i \geq 1$ , or the semi-stable case. These examples are obtained for  $p \geq 7$  and are quotients of an hypersurface of degree  $p$  in a projective space  $\mathbb{P}_R^{p-2}$  by a free  $(\mathbb{Z}/p\mathbb{Z})$ -action. Their generic fibre is a smooth variety of general type, and their special fibre has isolated singularities, at least when  $p$  is not a Fermat number.

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*General conventions.* 1) All schemes under consideration are supposed to be separated. By a projective morphism  $f : Y \rightarrow X$ , we always mean a morphism that can be factorized as  $f = \pi \circ i$ , where  $i$  is a closed immersion in some projective space  $\mathbb{P}_X^n$  and  $\pi$  is the natural projection  $\mathbb{P}_X^n \rightarrow X$ .

2) In this paper, we use the terminology of [SGA6] for complete intersection morphisms: a morphism of schemes  $f : Y \rightarrow X$  is said to be a *complete intersection morphism* if, for any  $y \in Y$ , there exists an open neighbourhood  $U$  of  $y$  in  $Y$  such that the restriction of  $f$  to  $U$  can be factorized as  $f|_U = \pi \circ i$ , where  $\pi$  is a smooth morphism and  $i$  a regular immersion [SGA6, VIII, 1.1]. Note that this notion of complete intersection morphism is more general than the notion of “local complete intersection map” used in [Har66] and [Con00], where “lci map” is only used for regular immersions.

If  $d$  is the codimension of  $i$  at  $y$  and  $n$  the relative dimension of  $\pi$  at  $i(y)$ , the integer  $m = n - d$  does not depend upon the local factorization  $f|_U = \pi \circ i$  and is called the *virtual relative dimension* of  $f$  at  $y$  [SGA6, VIII, 1.9]. One says that  $f$  has *constant virtual relative dimension*  $m$  if the integer  $m$  does not depend upon  $y$ . In this paper, we will always assume that the virtual relative

dimension of the morphisms under consideration is constant. (However, the dimension of the fibres of such morphisms can vary.)

3) Apart from the previous remark, we will use the definitions and sign conventions from Conrad's book [Con00]. In particular, when  $i : Y \hookrightarrow P$  is a regular immersion of codimension  $d$  defined by an ideal  $\mathcal{I} \subset \mathcal{O}_P$ , we define  $\omega_{Y/P}$  by

$$\omega_{Y/P} = \wedge^d((\mathcal{I}/\mathcal{I}^2)^\vee)$$

rather than  $(\wedge^d(\mathcal{I}/\mathcal{I}^2))^\vee$  as in [Har66, III, p. 141] (see [Con00, p. 7]). The canonical identification between both definitions is given by [Bou70, III, §11, Prop. 7].

4) If  $R, S$  are commutative rings,  $R \rightarrow S$  a ring homomorphism, and  $X$  an  $R$ -scheme, then we denote by  $X_S$  the  $S$ -scheme  $\mathrm{Spec} S \times_{\mathrm{Spec} R} X$ .

5) If  $\mathcal{E}^\bullet$  is a complex, we denote by  $(\sigma_{\geq i} \mathcal{E}^\bullet)_{i \in \mathbb{Z}}$  the naive filtration on  $\mathcal{E}^\bullet$ , i.e., the filtration defined by  $\sigma_{\geq i} \mathcal{E}^n = 0$  if  $n < i$ ,  $\sigma_{\geq i} \mathcal{E}^n = \mathcal{E}^n$  if  $n \geq i$ .

6) If  $X$  is a scheme (resp. locally noetherian scheme), we denote by  $D_{\mathrm{qc}}^{\mathrm{b}}(\mathcal{O}_X)$  (resp.  $D_{\mathrm{coh}}^{\mathrm{b}}(\mathcal{O}_X)$ ) the full subcategory of the derived category  $D(\mathcal{O}_X)$  that has as objects the bounded complexes with  $\mathcal{O}_X$ -quasi-coherent (resp.  $\mathcal{O}_X$ -coherent) cohomology sheaves. We denote by  $D_{\mathrm{fTd}}^{\mathrm{b}}(\mathcal{O}_X) \subset D(\mathcal{O}_X)$  the full subcategory of complexes that are isomorphic to a bounded complex of flat  $\mathcal{O}_X$ -modules. Adding several of the indices to  $D^{\mathrm{b}}(\mathcal{O}_X)$ , as in  $D_{\mathrm{qc}, \mathrm{fTd}}^{\mathrm{b}}(\mathcal{O}_X)$ , means taking the intersection of the corresponding subcategories.

When relevant, we will use similar notation for the analogous subcategories of  $D(\mathcal{O}_X)$  and  $D^+(\mathcal{O}_X)$ .

## 2. Application of $p$ -adic Hodge theory

In this section, we explain how the fundamental results of  $p$ -adic Hodge theory can be used to prove Theorem 1.3. We begin with the semi-stable case, where  $p$ -adic Hodge theory suffices to conclude and which will serve as a model for the general case. We use the notation  $R, K, k$  as in the introduction.

**THEOREM 2.1.** *Let  $X$  be a proper and semi-stable  $R$ -scheme, with generic fibre  $X_K$  and special fibre  $X_k$ , and let  $q \geq 0$  be an integer. If  $H^q(X_K, \mathcal{O}_{X_K}) = 0$ , then  $H^q(X_k, W\mathcal{O}_{X_k, \mathbb{Q}}) = 0$ .*

*Proof.* We may assume that  $R$  is a complete discrete valuation ring. Indeed, if  $\widehat{R}$  is the completion of  $R$ ,  $\widehat{K} = \mathrm{Frac}(\widehat{R})$  and  $\widetilde{X} = X_{\widehat{R}}$ , then  $\widetilde{X}$  is proper and semi-stable over  $\widehat{R}$ ,  $H^q(\widetilde{X}_{\widehat{R}}, \mathcal{O}_{\widetilde{X}_{\widehat{R}}}) = \widehat{K} \otimes_K H^q(X_K, \mathcal{O}_{X_K}) = 0$ , and  $X$  and  $\widetilde{X}$  have isomorphic special fibres. So the theorem for  $\widetilde{X}$  implies the theorem for  $X$ .

We endow  $S = \mathrm{Spec} R$  with the log structure defined by the divisor  $\mathrm{Spec} k \subset S$ ,  $S_0 = \mathrm{Spec} k$  with the induced log structure, and we denote by



$\underline{S}$ ,  $\underline{S}_0$  the corresponding log schemes. Similarly, we endow  $X$  with the log structure defined by the special fibre  $X_k$ ,  $X_k$  with the induced log structure, and we denote by  $\underline{X}$ ,  $\underline{X}_k$  the corresponding log schemes. Then  $\underline{X}$  is smooth over  $\underline{S}$  and  $\underline{X}_k$  is smooth of Cartier type [Kat89, (4.8)] over  $\underline{S}_0$ .

Let  $W_n = W_n(k)$  (resp.  $W = W(k)$ ), and let  $\underline{\Sigma}_n$  (resp.  $\underline{\Sigma}$ ) be the log scheme obtained by endowing  $\Sigma_n = \text{Spec } W_n$  (resp.  $\Sigma = \text{Spec } W$ ) with the log structure associated to the pre-log structure defined by the morphism  $M_{S_0} \rightarrow \mathcal{O}_{S_0} = \mathcal{O}_{\Sigma_1} \rightarrow \mathcal{O}_{\Sigma_n}$  (resp.  $\mathcal{O}_{\Sigma}$ ) provided by composition with the Teichmüller representative map. We can then consider the log crystalline cohomology groups  $H_{\text{crys}}^q(\underline{X}/\underline{\Sigma}_n)$ , which are finitely generated  $W_n$ -modules endowed with a Frobenius action  $\varphi$  and a monodromy operator  $N$ . The log scheme  $\underline{X}_k$  also carries a logarithmic de Rham-Witt complex  $W\Omega_{\underline{X}_k}^\bullet = \varprojlim_n W_n\Omega_{\underline{X}_k}^\bullet$ , constructed by Hyodo [Hyo91] in the semi-stable case, and generalized by Hyodo and Kato [HK94, (4.1)] to the case of smooth  $\underline{S}_0$ -log schemes of Cartier type. In degree 0, we have

$$(2.1.1) \quad W_n\Omega_{\underline{X}_k}^0 = W_n(\mathcal{O}_{X_k}),$$

by [HK94, Prop. (4.6)].

It follows from [HK94, Th. (4.19)] that, for all  $q$ , there are canonical isomorphisms

$$(2.1.2) \quad H_{\text{crys}}^q(\underline{X}_k/\underline{\Sigma}_n) \xrightarrow{\sim} H^q(\underline{X}_k, W_n\Omega_{\underline{X}_k}^\bullet),$$

which are compatible when  $n$  varies and commute with the Frobenius actions. As  $X_k$  is proper over  $S_0$ , these cohomology groups are artinian  $W$ -modules. Therefore, one can apply the Mittag-Leffler criterium to get canonical isomorphisms

$$(2.1.3) \quad \begin{aligned} H_{\text{crys}}^q(\underline{X}_k/\underline{\Sigma}) &\xrightarrow{\sim} \varprojlim_n H_{\text{crys}}^q(\underline{X}_k/\underline{\Sigma}_n) \\ &\xrightarrow{\sim} \varprojlim_n H^q(\underline{X}_k, W_n\Omega_{\underline{X}_k}^\bullet) \xleftarrow{\sim} H^q(\underline{X}_k, W\Omega_{\underline{X}_k}^\bullet) \end{aligned}$$

compatible with the Frobenius actions. Using the naive filtration of  $W\Omega_{\underline{X}_k}^\bullet$  and tensoring by  $K_0$ , one obtains a spectral sequence

$$(2.1.4) \quad E_1^{i,j} = H^j(\underline{X}_k, W\Omega_{\underline{X}_k}^i) \otimes K_0 \implies H_{\text{crys}}^{i+j}(\underline{X}_k/\underline{\Sigma}) \otimes K_0$$

endowed by functoriality with a Frobenius action  $F^*$ . The operators  $d$ ,  $F$ , and  $V$  on the logarithmic de Rham-Witt complex satisfy the same relations as on the usual de Rham-Witt complex [HK94, (4.1)], and the structure theorems of [Ill79] remain valid in the logarithmic case [HK94, Th. (4.4) and Cor. (4.5)]. It follows that, for all  $i, j$ , the  $K_0$ -vector space  $H^j(\underline{X}_k, W\Omega_{\underline{X}_k}^i) \otimes K_0$  is finite dimensional, and the action of  $F^*$  on this space has slopes in  $[i, i + 1[$  [Lor02, 3.1]. Therefore, the spectral sequence (2.1.4) degenerates at  $E_1$  and yields, in particular, an isomorphism  $(H_{\text{crys}}^q(\underline{X}_k/\underline{\Sigma}) \otimes K_0)^{<1} \xrightarrow{\sim} H^q(\underline{X}_k, W\Omega_{\underline{X}_k}^0) \otimes K_0$ ,

the source being the part of  $H_{\text{crys}}^q(\underline{X}_k/\underline{\Sigma}) \otimes K_0$  where Frobenius acts with slope  $< 1$ . Thanks to (2.1.1), we finally get a canonical isomorphism

$$(2.1.5) \quad (H_{\text{crys}}^q(\underline{X}_k/\underline{\Sigma}) \otimes K_0)^{<1} \xrightarrow{\sim} H^q(X_k, W\mathcal{O}_{X_k, \mathbb{Q}}).$$

On the other hand, the choice of a uniformizer of  $R$  determines a Hyodo-Kato isomorphism [HK94, Th. (5.1)]

$$(2.1.6) \quad \rho : H_{\text{crys}}^q(\underline{X}_k/\underline{\Sigma}) \otimes_W K \xrightarrow{\sim} H^q(X_K, \Omega_{X_K/K}^\bullet).$$

This allows us to endow  $H_{\text{crys}}^q(\underline{X}_k/\underline{\Sigma}) \otimes_W K$  with the filtration deduced via  $\rho$  from the Hodge filtration of  $H^q(X_K, \Omega_{X_K/K}^\bullet)$ . Together with its Frobenius action and monodromy operator,  $H_{\text{crys}}^q(\underline{X}_k/\underline{\Sigma}) \otimes_W K$  is then a filtered  $(\varphi, N)$ -module as defined by Fontaine [Fon94, 4.3.2 and 4.4.8]. As such, it has both a Newton polygon, built as usual from the slopes of the Frobenius action, and a Hodge polygon, built as usual from the Hodge numbers of  $H^q(X_K, \Omega_{X_K/K}^\bullet)$ . Now, let  $\bar{K}$  be an algebraic closure of  $K$ , and let  $B_{\text{st}}, B_{\text{dR}}$  be the Fontaine  $p$ -adic period rings. Then Tsuji’s comparison theorem [Tsu99, Th. 0.2] provides a  $B_{\text{st}}$ -linear isomorphism

$$(2.1.7) \quad B_{\text{st}} \otimes_{K_0} H_{\text{crys}}^q(\underline{X}_k/\underline{\Sigma}) \xrightarrow{\sim} B_{\text{st}} \otimes_K H_{\text{ét}}^q(X_{\bar{K}}, \mathbb{Q}_p),$$

compatible with the natural Galois, Frobenius, and monodromy actions on both sides and with the natural Hodge filtrations defined on both sides after scalar extension from  $B_{\text{st}}$  to  $B_{\text{dR}}$ . Thus,  $H_{\text{crys}}^q(\underline{X}_k/\underline{\Sigma}) \otimes K_0$  is an admissible filtered  $(\varphi, N)$ -module [Fon94, 5.3.3], and therefore it is weakly admissible [Fon94, 5.4.2]. This implies that its Newton polygon lies above its Hodge polygon [Fon94, 4.4.6]. In particular, either  $H_{\text{crys}}^q(\underline{X}_k/\underline{\Sigma}) \otimes K_0 = 0$ , or the smallest slope of its Newton polygon is bigger than the smallest slope of its Hodge polygon. By assumption, the latter is at least 1, which forces the part of slope  $< 1$  of  $H_{\text{crys}}^q(\underline{X}_k/\underline{\Sigma}) \otimes K_0$  to vanish. Thanks to (2.1.5), this implies the theorem.  $\square$

In the general case, we will use truncated simplicial log schemes satisfying the conditions of the next lemma. We will assume that all the log schemes under consideration are fine log schemes [Kat89, (2.3)], and all constructions involving log schemes will be done in the category of fine log schemes. For any finite extension  $K'$  of  $K$ , with ring of integers  $R'$ , we will endow  $\text{Spec } R'$  with the log structure defined by its closed point, and pullbacks of log schemes to  $\text{Spec } R'$  will mean pullbacks in the category of log schemes. Note that, because of [Kat89, (4.4)(ii) and (4.3.1)], the underlying scheme of such a pullback is the usual pullback in the category of schemes. We will denote log schemes by underlined letters and drop the underlining to denote the underlying schemes.

LEMMA 2.2. *Assume that  $R$  is complete and that  $X$  is an integral, flat  $R$ -scheme of finite type. Let  $m \geq 0$  be an integer. Then there exists a finite*

extension  $K'$  of  $K$ , with ring of integers  $R'$ , a split  $m$ -truncated simplicial  $R'$ -log scheme  $\underline{Y}_\bullet = (Y_\bullet, M_{Y_\bullet})$  [SGA4II, Vbis, 5.1.1], and an augmentation morphism  $u : Y_0 \rightarrow X_{R'}$  over  $R'$  such that the following conditions hold:

(a) For each  $r$ ,  $Y_r$  is projective over  $X_{R'}$  and  $\underline{Y}_r$  is a disjoint union of pullbacks to  $R'$  of semi-stable schemes over the integers of sub- $K$ -extensions of  $K'$  endowed with the log structure defined by their special fibre.

(b) Via the augmentation morphism induced by  $u$ ,  $Y_{\bullet, K'}$  is an  $m$ -truncated proper hypercovering of  $X_{K'}$ .

(c) There exists a projective  $R$ -alteration  $f : Y \rightarrow X$ , where  $Y$  is semi-stable over the ring of integers  $R_1$  of a sub- $K$ -extension  $K_1$  of  $K'$ , and there exists finitely many  $R$ -embeddings  $\sigma_i : R_1 \hookrightarrow R'$  such that if  $u_1 : Y \rightarrow X_{R_1}$  denotes the  $R_1$ -morphism defined by  $f$  and if  $Y_{\sigma_i}$  (resp.  $u_{\sigma_i} : Y_{\sigma_i} \rightarrow X_{R'}$ ) denotes the  $R'$ -scheme (resp.  $R'$ -morphism) deduced by base change via  $\sigma_i$  from  $Y$  (resp.  $u_1$ ), then  $Y_0 = \coprod_i Y_{\sigma_i}$  and  $u|_{Y_{\sigma_i}} = u_{\sigma_i}$ .

Therefore, we obtain the following commutative diagram:

$$(2.2.1) \quad \begin{array}{ccccc} Y_0 = \coprod_i Y_{\sigma_i} & \longleftarrow & Y_{\sigma_i} & \longrightarrow & Y \\ & \searrow u & \downarrow u_{\sigma_i} & & \downarrow u_1 \\ & & X_{R'} & \longrightarrow & X_{R_1} \longrightarrow X \end{array}$$

*Proof.* This is a well-known consequence of de Jong’s alteration theorem [DJ96, Th. 6.5]. For the sake of completeness, we briefly recall how to construct such a simplicial log scheme. For  $r \geq 0$ , we denote by  $[r]$  the ordered set  $\{0, \dots, r\}$ , and by  $\Delta$  (resp.  $\Delta[m]$ ) the category that has the sets  $[r]$  (resp. with  $r \leq m$ ) as objects, the set of morphisms from  $[r]$  to  $[s]$  being the set of nondecreasing maps  $[r] \rightarrow [s]$ .

One proceeds by induction on  $m$ . Assume first that  $m = 0$ . de Jong’s theorem provides a finite extension  $K_1$  of  $K$ , an integral semi-stable scheme  $Y$  over the ring of integers  $R_1$  of  $K_1$ , and an  $R$ -morphism  $f : Y \rightarrow X$  that is a projective alteration. Let  $u_1 : Y \rightarrow X_{R_1}$  be the morphism defined by  $f$ . Let  $K'$  be a finite extension of  $K_1$  such that  $K'/K$  is Galois, and let  $R'$  be its ring of integers. For any  $g \in \text{Gal}(K'/K)$ , let  $\sigma_g$  be the composition  $K_1 \rightarrow K' \xrightarrow{g} K'$ , and let  $\underline{Y}_g$  (resp.  $u_g : Y_g \rightarrow X_{R'}$ ) be the  $R'$ -log scheme (resp.  $R'$ -morphism) deduced from  $\underline{Y}$  (resp.  $u_1$ ) by base change via  $\sigma_g : R_1 \rightarrow R'$ . Then one defines  $\underline{Y}_0$  and  $u$  by setting

$$\underline{Y}_0 = \coprod_{g \in \text{Gal}(K'/K)} \underline{Y}_g, \quad u|_{Y_g} = u_g.$$

One easily checks by Galois descent that  $Y_{0, K'} \rightarrow X_{K'}$  is surjective; conditions (a)–(c) are then satisfied.

Assume now that the lemma has been proved for  $m - 1$ . Over the ring of integers  $R''$  of some finite extension  $K''$  of  $K$ , this provides a split  $(m - 1)$ -truncated simplicial log scheme  $\underline{Y}''$ , together with an augmentation morphism  $u'' : Y''_0 \rightarrow X_{R''}$ , so as to satisfy conditions (a)–(c). Note that these conditions remain satisfied after a base change to the ring of integers of any finite extension of  $K''$ . Let  $\text{cosk}_{m-1}(\underline{Y}''_m)$  be the coskeleton of  $\underline{Y}''_m$  in the category of simplicial fine  $R''$ -log schemes and  $\underline{Z} = \text{cosk}_{m-1}(\underline{Y}''_m)$  its component of index  $m$ . Denote by  $Z_1, \dots, Z_c$  those irreducible components of  $\underline{Z}$  that are flat over  $R''$ , and endow each  $Z_j$  with the log structure induced by the log structure of  $\underline{Z}$ . As a consequence of condition (a), this log structure induces the trivial log structure on the generic fibre  $\underline{Z}_{j,K''}$ . Applying de Jong’s theorem to  $Z_j$ , one can find a finite extension  $K'_j$  of  $K''$ , with ring of integers  $R'_j$ , an integral semi-stable scheme  $T_j$  over  $R'_j$ , and a projective alteration  $f_j : T_j \rightarrow Z_j$ . One endows  $T_j$  with the log structure defined by its special fibre. Because the log structure of the generic fibre  $\underline{Z}_{j,K''}$  is trivial, the morphism  $f_j$  extends uniquely to a log morphism  $f_j : \underline{T}_j \rightarrow \underline{Z}_j$ . Let  $K'$  be a Galois extension of  $K''$  containing  $K'_j$  for all  $j$ ,  $1 \leq j \leq c$ , and let  $R'$  be its ring of integers. Arguing as in the case  $m = 0$  above, one can deduce from the alterations  $f_j$  an  $R'$ -morphism

$$(2.2.2) \quad \underline{T} \longrightarrow \coprod_{j=1}^c \underline{Z}_{j,R'} \longrightarrow \underline{Z}_{R'} \xrightarrow{\sim} \text{cosk}_{m-1}(\underline{Y}''_{\bullet,R'})_m,$$

where  $\underline{T}$  satisfies condition (a) and  $T_{K'} \rightarrow \text{cosk}_{m-1}(Y''_{\bullet,K'})_m$  is projective and surjective. (Note that since all log structures are trivial on the generic fibres, the generic fibre of the coskeleton computed in the category of fine log schemes is the coskeleton of the generic fibres computed in the category of schemes.) One can then follow the method of Saint-Donat [SGA4II, Vbis, 5.1.3] and Deligne [Del74, (6.2.5)] to extend  $\underline{Y}''_{\bullet,R'}$  as a split  $m$ -truncated simplicial log scheme  $\underline{Y}_\bullet$  over  $R'$ . The  $R'$ -log scheme  $\underline{Y}_m$  is defined by

$$\underline{Y}_m = \underline{T} \amalg \coprod_{[m] \rightarrow [l], l < m} N(\underline{Y}_l),$$

where  $N(\underline{Y}_l)$  is the complement of the union of the images of the degeneracy morphisms with target  $\underline{Y}_l$ . It satisfies condition (a) because  $\underline{T}$  does and  $\underline{Y}_\bullet$  is split. Similarly, the morphism  $Y_{m,K'} \rightarrow \text{cosk}_{m-1}(Y_{\bullet,K'})_m$  is proper and surjective because the morphism  $T_{K'} \rightarrow \text{cosk}_{m-1}(Y''_{\bullet,K'})_m$  is proper and surjective. Thus the  $m$ -truncated simplicial scheme  $Y_{\bullet,K'}$  is an  $m$ -truncated proper hypercovering of  $X_{K'}$ . Finally, condition (c) is satisfied thanks to the induction hypothesis.  $\square$

2.3. We recall how to associate cohomological invariants to simplicial schemes and truncated simplicial schemes (see [SGA4II, Vbis, 2.3], [Del74, 5.2], [Tsu98, (6.2)]).

If  $\mathcal{T}$  is a topos, we denote by  $\mathcal{T}^\Delta$  (resp.  $\mathcal{T}^{\Delta[m]}$ ) the topos of cosimplicial objects (resp.  $m$ -truncated cosimplicial objects) in  $\mathcal{T}$ . Let  $\mathcal{A}$  be a ring in  $\mathcal{T}$  and  $\mathcal{A}^\bullet$  the constant cosimplicial ring defined by  $\mathcal{A}$ . If  $\mathcal{E}^\bullet$  is an  $\mathcal{A}^\bullet$ -module of  $\mathcal{T}^\Delta$  (resp.  $\mathcal{T}^{\Delta[m]}$ ), one associates to  $\mathcal{E}^\bullet$  the complex

$$\begin{aligned} \varepsilon_* \mathcal{E}^\bullet &= \mathcal{E}^0 \rightarrow \mathcal{E}^1 \rightarrow \dots \rightarrow \mathcal{E}^r \xrightarrow{\sum_j (-1)^j \partial^j} \mathcal{E}^{r+1} \rightarrow \dots \\ (\text{resp. } \varepsilon_*^m \mathcal{E}^\bullet &= \mathcal{E}^0 \rightarrow \mathcal{E}^1 \rightarrow \dots \rightarrow \mathcal{E}^m \rightarrow 0 \rightarrow \dots). \end{aligned}$$

One views  $\varepsilon_* \mathcal{E}^\bullet$  (resp.  $\varepsilon_*^m \mathcal{E}^\bullet$ ) as a filtered complex of  $\mathcal{A}$ -modules using the naive filtration. The functors  $\varepsilon_*$  and  $\varepsilon_*^m$  are exact functors from the category of  $\mathcal{A}^\bullet$ -modules to the category of filtered complexes of  $\mathcal{A}$ -modules (which means that they transform a short exact sequence of  $\mathcal{A}^\bullet$ -modules into a short exact sequence of filtered complexes, i.e., such that the sequence of  $\text{Fil}^i$ 's is exact for all  $i$ ). Hence, they factorize so as to define exact functors  $R\varepsilon_*$  and  $R\varepsilon_*^m$  from  $D^+(\mathcal{T}^\Delta, \mathcal{A}^\bullet)$  (resp.  $D^+(\mathcal{T}^{\Delta[m]}, \mathcal{A}^\bullet)$ ) to  $D^+F(\mathcal{T}, \mathcal{A})$ . For any complex  $\mathcal{E}^{\bullet, \bullet} \in D^+(\mathcal{A}^\bullet)$ , they provide functorial spectral sequences

$$(2.3.1) \quad E_1^{r,q} = \mathcal{H}^q(\mathcal{E}^{r, \bullet}) \Rightarrow \mathcal{H}^{r+q}(R\varepsilon_*(\mathcal{E}^{\bullet, \bullet}))$$

and similarly for  $R\varepsilon_*^m$  with  $E_1^{r,q} = 0$  for  $r > m$ . (We use here the first index to denote the simplicial degree.) Note that the truncation functor induces a functorial morphism

$$(2.3.2) \quad R\varepsilon_*(\mathcal{E}^{\bullet, \bullet}) \longrightarrow R\varepsilon_*^m(\text{sk}_m(\mathcal{E}^{\bullet, \bullet}))$$

and thus a morphism between the corresponding spectral sequences (2.3.1). It follows that if  $\mathcal{H}^q(\mathcal{E}^{r, \bullet}) = 0$  for  $q < 0$  and all  $r$ , then the morphism (2.3.2) is a quasi-isomorphism in degrees  $< m$ .

Let  $Y_\bullet$  be a simplicial scheme (resp.  $m$ -truncated simplicial scheme) and  $\underline{\text{Sets}}$  the topos of sets. If  $R$  is a commutative ring and  $\mathcal{E}^\bullet$  a (Zariski, étale, ...) sheaf of  $R$ -modules on  $Y_\bullet$ , one can associate to  $\mathcal{E}^\bullet$  a cosimplicial  $R^\bullet$ -module  $\Gamma^\bullet(Y_\bullet, \mathcal{E}^\bullet) \in \underline{\text{Sets}}^\Delta$  (resp.  $\underline{\text{Sets}}^{\Delta[m]}$ ) by setting for all  $r \geq 0$ ,

$$\Gamma^r(Y_\bullet, \mathcal{E}^\bullet) = \Gamma(Y_r, \mathcal{E}^r).$$

The functor  $\Gamma^\bullet$  can be derived, and its right derived functor  $R\Gamma^\bullet$  can be computed using resolutions by complexes  $\mathcal{I}^{\bullet, \bullet}$  such that for each  $r, q$ , the sheaf  $\mathcal{I}^{r,q}$  is acyclic on  $Y_r$ . The cohomology of  $Y_\bullet$  with coefficients in a complex  $\mathcal{E}^{\bullet, \bullet}$  is then, by definition,

$$\begin{aligned} R\Gamma(Y_\bullet, \mathcal{E}^{\bullet, \bullet}) &= R\varepsilon_* R\Gamma^\bullet(Y_\bullet, \mathcal{E}^{\bullet, \bullet}) \quad (\text{resp. } R\varepsilon_*^m), \\ H^q(Y_\bullet, \mathcal{E}^{\bullet, \bullet}) &= H^q(R\Gamma(Y_\bullet, \mathcal{E}^{\bullet, \bullet})). \end{aligned}$$

If  $Y_\bullet$  is a smooth simplicial (resp.  $m$ -truncated simplicial)  $R$ -scheme, this can be applied to the complex  $\Omega_{Y_\bullet/R}^\bullet$  and to its sub-complexes  $\sigma_{\geq i} \Omega_{Y_\bullet/R}^\bullet$ , defining

the naive filtration. This provides the definition of the de Rham cohomology of  $Y_\bullet$  and of its Hodge filtration.

PROPOSITION 2.4. *Let  $K$  be a field of characteristic 0,  $X$  a proper and smooth  $K$ -scheme, and  $Y_\bullet \rightarrow X$  an  $m$ -truncated proper hypercovering of  $X$  over  $K$  such that  $Y_r$  is proper and smooth for all  $r$ . Then, for all  $q < m$ , the canonical homomorphism*

$$(2.4.1) \quad H^q(X, \Omega_{X/K}^\bullet) \longrightarrow H^q(Y_\bullet, \Omega_{Y_\bullet/K}^\bullet)$$

is an isomorphism of filtered  $K$ -vector spaces for the Hodge filtrations.

*Proof.* Since algebraic de Rham cohomology (endowed with the Hodge filtration) commutes with base field extensions, standard limit arguments allow us to assume that  $K$  is of finite type over  $\mathbb{Q}$ . Choosing an embedding  $\iota : K \hookrightarrow \mathbb{C}$ , we are reduced to the case where  $K = \mathbb{C}$ . Using resolution of singularities, we can find a proper and smooth hypercovering  $Z_\bullet$  of  $X$  such that  $\text{sk}_m(Z_\bullet) = Y_\bullet$ . As the morphism (2.3.2) for  $\sigma_{\geq i} \Omega_{Z_\bullet/\mathbb{C}}^\bullet$  is a quasi-isomorphism in degrees  $< m$  for all  $i$ , it suffices to prove the proposition with  $Y_\bullet$  replaced by  $Z_\bullet$ . This now follows from [Del74, Prop. (8.2.2)].  $\square$

COROLLARY 2.5. *Under the assumptions (a) and (b) of Lemma 2.2 assume, in addition, that  $X_K$  is proper and smooth and that  $H^q(X_K, \mathcal{O}_{X_K}) = 0$  for some  $q < m$ . Then the smallest Hodge slope of  $H^q(Y_{\bullet, K'}, \Omega_{Y_{\bullet, K'}}^\bullet)$  is at least 1.*

*Proof.* Assumptions (a) and (b) imply that the hypotheses of the proposition are satisfied by  $Y_{\bullet, K'} \rightarrow X_{K'}$ , and the corollary is then clear.  $\square$

2.6. Let  $\underline{\Sigma}_n, \underline{\Sigma}$  be as in the proof of Theorem 2.1. We now denote by  $\underline{Y}_\bullet = (Y_\bullet, M_{Y_\bullet})$  an  $m$ -truncated simplicial log scheme over  $\underline{\Sigma}_1$ . We assume that each  $\underline{Y}_r$  is smooth of Cartier type over  $\underline{\Sigma}_1$ , so that, for all  $n \geq 1$ , its de Rham-Witt complex  $W_n \Omega_{\underline{Y}_r}^\bullet$  is defined [HK94, (4.1)]. When  $r$  varies, the functoriality of the de Rham-Witt complex turns the family of complexes  $(W_n \Omega_{\underline{Y}_r}^\bullet)_{0 \leq r \leq m}$  into a complex  $W_n \Omega_{\underline{Y}_\bullet}^\bullet$  on  $Y_\bullet$ . One defines its cohomology as in 2.3, and one has similar definitions for the de Rham-Witt complex  $W \Omega_{\underline{Y}_\bullet}^\bullet = \varprojlim_n W_n \Omega_{\underline{Y}_\bullet}^\bullet$ .

For a morphism  $\alpha : [r] \rightarrow [s]$  in  $\Delta[m]$ , let  $\alpha_{\text{crys}} : (\underline{Y}_s/\underline{\Sigma}_n)_{\text{crys}} \rightarrow (\underline{Y}_r/\underline{\Sigma}_n)_{\text{crys}}$  be the morphism between the log crystalline topos induced by the corresponding morphism  $\underline{Y}_s \rightarrow \underline{Y}_r$ . One defines the log crystalline topos  $(\underline{Y}_\bullet/\underline{\Sigma}_n)_{\text{crys}}$  as being the topos of families of sheaves  $(E^r)_{0 \leq r \leq m}$ , where  $E^r$  is a sheaf on the log crystalline site  $\text{Crys}(\underline{Y}_r/\underline{\Sigma}_n)$ , endowed with a transitive family of morphisms  $\alpha_{\text{crys}}^{-1} E^r \rightarrow E^s$  for morphisms  $\alpha$  in  $\Delta[m]$ . In particular, the family of sheaves  $\mathcal{O}_{\underline{Y}_r/\underline{\Sigma}_n}$  defines the structural sheaf of  $(\underline{Y}_\bullet/\underline{\Sigma}_n)_{\text{crys}}$ , denoted by  $\mathcal{O}_{\underline{Y}_\bullet/\underline{\Sigma}_n}$ . There is a canonical morphism  $u_{\underline{Y}_\bullet/\underline{\Sigma}_n} : (\underline{Y}_\bullet/\underline{\Sigma}_n)_{\text{crys}} \rightarrow Y_{\bullet, \text{Zar}}$  such that  $u_{\underline{Y}_\bullet/\underline{\Sigma}_n}^*(E^\bullet)^r = u_{\underline{Y}_r/\underline{\Sigma}_n}^*(E^r)$  for all  $r$ . If  $E^{\bullet, \bullet}$  is a complex of

abelian sheaves in  $(\underline{Y}_\bullet/\underline{\Sigma}_n)_{\text{crys}}$ , one proceeds as in 2.3 to define its log crystalline cohomology  $R\Gamma_{\text{crys}}(\underline{Y}_\bullet/\underline{\Sigma}_n, E^{\bullet,\bullet})$  and its projection on the Zariski topos  $Ru_{\underline{Y}_\bullet/\underline{\Sigma}_n*}(E^{\bullet,\bullet})$ . One gives similar definitions for the log crystalline topos  $(\underline{Y}_\bullet/\underline{\Sigma})_{\text{crys}}$  relative to  $\underline{\Sigma}$ . By construction, there are canonical isomorphisms

$$(2.6.1) \quad R\Gamma(Y_\bullet, Ru_{\underline{Y}_\bullet/\underline{\Sigma}_n*}(E^{\bullet,\bullet})) \xrightarrow{\sim} R\Gamma_{\text{crys}}(\underline{Y}_\bullet/\underline{\Sigma}_n, E^{\bullet,\bullet}),$$

$$(2.6.2) \quad R\Gamma(Y_\bullet, Ru_{\underline{Y}_\bullet/\underline{\Sigma}*}(E^{\bullet,\bullet})) \xrightarrow{\sim} R\Gamma_{\text{crys}}(\underline{Y}_\bullet/\underline{\Sigma}, E^{\bullet,\bullet}).$$

If  $\underline{Y}_\bullet \hookrightarrow \underline{P}_\bullet$  is a closed immersion of the  $m$ -truncated simplicial log scheme  $\underline{Y}_\bullet$  into a smooth  $m$ -truncated simplicial  $\Sigma_n$ -log scheme  $\underline{P}_\bullet$  (resp.  $\underline{\Sigma}$ -formal log scheme), the family of PD-envelopes  $\mathcal{P}_{\underline{Y}_r}^{\text{log}}(\underline{P}_r)$  (resp. completed PD-envelopes) [Kat89, (5.4)] defines a sheaf  $\mathcal{P}_{\underline{Y}_\bullet}^{\text{log}}(\underline{P}_\bullet)$  on  $\underline{Y}_\bullet$ , and one can form the de Rham complex  $\mathcal{P}_{\underline{Y}_\bullet}^{\text{log}}(\underline{P}_\bullet) \otimes_{\mathcal{O}_{\underline{P}_\bullet}} \Omega_{\underline{P}_\bullet/\underline{\Sigma}_n}^\bullet$  (resp.  $\mathcal{P}_{\underline{Y}_\bullet}^{\text{log}}(\underline{P}_\bullet) \otimes_{\mathcal{O}_{\underline{P}_\bullet}} \Omega_{\underline{P}_\bullet/\underline{\Sigma}}^\bullet$ ), which is supported in  $\underline{Y}_\bullet$ . Because the linearization functor  $L$  used in the proof of the comparison theorem between crystalline and de Rham cohomologies [Kat89, (6.9)] makes sense simplicially, this theorem extends to the simplicial case and there is a canonical isomorphism in  $D^+(Y_\bullet, W_n)$  (resp.  $D^+(Y_\bullet, W)$ )

$$(2.6.3) \quad Ru_{\underline{Y}_\bullet/\underline{\Sigma}_n*}(\mathcal{O}_{\underline{Y}_\bullet/\underline{\Sigma}_n}) \xrightarrow{\sim} \mathcal{P}_{\underline{Y}_\bullet}^{\text{log}}(\underline{P}_\bullet) \otimes_{\mathcal{O}_{\underline{P}_\bullet}} \Omega_{\underline{P}_\bullet/\underline{\Sigma}_n}^\bullet$$

$$(2.6.4) \quad (\text{resp. } Ru_{\underline{Y}_\bullet/\underline{\Sigma}*}(\mathcal{O}_{\underline{Y}_\bullet/\underline{\Sigma}}) \xrightarrow{\sim} \mathcal{P}_{\underline{Y}_\bullet}^{\text{log}}(\underline{P}_\bullet) \otimes_{\mathcal{O}_{\underline{P}_\bullet}} \Omega_{\underline{P}_\bullet/\underline{\Sigma}}^\bullet).$$

**PROPOSITION 2.7.** *With the hypotheses of 2.6, assume that  $\underline{Y}_\bullet$  is split. Then there exists in  $D^+(Y_\bullet, W_n)$  (resp.  $D^+(Y_\bullet, W)$ ) canonical isomorphisms compatible with the transition morphisms and the Frobenius actions*

$$(2.7.1) \quad Ru_{\underline{Y}_\bullet/\underline{\Sigma}_n*}(\mathcal{O}_{\underline{Y}_\bullet/\underline{\Sigma}_n}) \xrightarrow{\sim} W_n \Omega_{\underline{Y}_\bullet}^\bullet$$

$$(2.7.2) \quad (\text{resp. } Ru_{\underline{Y}_\bullet/\underline{\Sigma}*}(\mathcal{O}_{\underline{Y}_\bullet/\underline{\Sigma}}) \xrightarrow{\sim} W \Omega_{\underline{Y}_\bullet}^\bullet).$$

The proof will use the next lemma, due to Nakajima [Nak09, Lemma 6.1].

**LEMMA 2.8.** *Under the assumptions of 2.7, there exists an  $m$ -truncated simplicial log scheme  $\underline{Z}_\bullet$  and a morphism of  $m$ -truncated simplicial log schemes  $\underline{Z}_\bullet \rightarrow \underline{Y}_\bullet$  such that, for  $0 \leq r \leq m$ ,  $Z_r$  is a disjoint union of affine open subsets of  $Y_r$  covering  $Y_r$ , and the morphism  $Z_r \rightarrow Y_r$  induces the natural inclusion on each of these subsets.*

**Definition 2.9.** Let  $X$  be a scheme on which  $p$  is locally nilpotent, and  $n \geq 1$  an integer. We denote by  $|X|$  the topological space underlying  $X$ , and by  $W_n(X)$  the ringed space  $(|X|, W_n(\mathcal{O}_X))$ , which is a scheme ([Ill79, 0, 1.5] and [LZ04, 1.10]). The ideal  $VW_{n-1}(\mathcal{O}_X)$  carries a canonical PD-structure ([Ill79, 0, 1.4] and [LZ04, 1.1]), which turns the nilpotent immersion  $u : X \hookrightarrow W_n(X)$  into a PD-thickening of  $X$ .

If  $\underline{X} = (X, M_X)$  is a log scheme, we denote by  $W_n(\underline{X}) = (W_n(X), M_{W_n(X)})$  the log scheme obtained by sending  $M_X$  to  $W_n(\mathcal{O}_X)$  by the Teichmüller representative map and taking the associated log structure [HK94, Def. (3.1)]. The immersion  $u$  is then, in a natural way, an exact closed immersion  $u : \underline{X} \hookrightarrow W_n(\underline{X})$ , functorial with respect to  $\underline{X}$ .

LEMMA 2.10. *Under the assumptions of 2.7, there exists a bisimplicial log scheme  $\underline{Z}_{\bullet, \bullet}$ ,  $m$ -truncated with respect to the first index and augmented towards  $\underline{Y}_{\bullet}$  with respect to the second index, a bisimplicial formal log scheme  $\underline{T}_{\bullet, \bullet}$  over  $\underline{\Sigma}$ ,  $m$ -truncated with respect to the first index, and a closed immersion of bisimplicial formal log schemes  $i_{\bullet, \bullet} : \underline{Z}_{\bullet, \bullet} \hookrightarrow \underline{T}_{\bullet, \bullet}$  such that the following conditions are satisfied:*

(a) *For  $0 \leq r \leq m$ ,  $Z_{r,0}$  is a disjoint union of affine open subsets of  $Y_r$  covering  $Y_r$ , the augmentation morphism  $Z_{r,0} \rightarrow Y_r$  induces the natural inclusion on each of these subsets, and the canonical morphism  $\underline{Z}_{r, \bullet} \rightarrow \text{cosk}_0^{Y_r}(\text{sk}_0^{Y_r}(\underline{Z}_{r, \bullet}))$  is an isomorphism.*

(b) *For  $0 \leq r \leq m$  and  $t \geq 0$ , the formal log scheme  $\underline{T}_{r,t}$  is smooth over  $\underline{\Sigma}$  (i.e., its reduction mod  $p^n$  is smooth over  $\underline{\Sigma}_n$  for all  $n$ ), and the canonical morphism  $\underline{T}_{r, \bullet} \rightarrow \text{cosk}_0^{\underline{\Sigma}}(\text{sk}_0^{\underline{\Sigma}}(\underline{T}_{r, \bullet}))$  is an isomorphism.*

(c) *Let  $i_{\bullet, \bullet; n} : \underline{Z}_{\bullet, \bullet} \hookrightarrow \underline{T}_{\bullet, \bullet; n}$  be the reduction mod  $p^n$  of  $i_{\bullet, \bullet}$ , and let  $u_{\bullet, \bullet; n} : \underline{Z}_{\bullet, \bullet} \hookrightarrow W_n(\underline{Z}_{\bullet, \bullet})$  denote the morphism of bisimplicial log schemes defined by the canonical immersions. For variable  $n$ , there exists a compatible family of  $\underline{\Sigma}_n$ -morphisms of bisimplicial schemes  $h_{\bullet, \bullet; n} : W_n(\underline{Z}_{\bullet, \bullet}) \rightarrow \underline{T}_{\bullet, \bullet; n}$  such that  $h_{\bullet, \bullet; n} \circ u_{\bullet, \bullet; n} = i_{\bullet, \bullet; n}$ .*

*Proof.* Let  $j_{\bullet} : \underline{Z}_{\bullet} \rightarrow \underline{Y}_{\bullet}$  be a morphism of  $m$ -truncated simplicial log schemes satisfying the conclusions of Lemma 2.8. One chooses a decomposition  $\underline{Z}_r = \coprod_{\alpha} \underline{Z}_r^{\alpha}$ , with  $Z_r^{\alpha} \subset Y_r$  open affine such that  $j_r|_{Z_r^{\alpha}}$  is the natural inclusion.

Let  $\underline{Z}_{r;1}^{\alpha} = \underline{Z}_r^{\alpha}$ . Since  $\underline{Z}_r^{\alpha}$  is affine and smooth over  $\underline{\Sigma}_1$  and  $\underline{\Sigma}_{n-1} \hookrightarrow \underline{\Sigma}_n$  is a nilpotent exact closed immersion, for each  $r, \alpha$  and each  $n \geq 2$  there exists a smooth log scheme  $\underline{Z}_{r;n}^{\alpha}$  over  $\underline{\Sigma}_n$  endowed with an isomorphism  $\underline{Z}_{r;n-1}^{\alpha} \xrightarrow{\sim} \underline{\Sigma}_{n-1} \times_{\underline{\Sigma}_n} \underline{Z}_{r;n}^{\alpha}$  [Kat89, Prop. (3.14) (1)]. Taking limits when  $n \rightarrow \infty$ , we obtain a smooth formal log scheme  $\underline{Z}_r^{\alpha}$  over  $\underline{\Sigma}$  and an isomorphism  $\underline{Z}_r^{\alpha} \xrightarrow{\sim} \underline{\Sigma}_1 \times_{\underline{\Sigma}} \underline{Z}_r^{\alpha}$ . Moreover, the smoothness of  $\underline{Z}_{r;n}^{\alpha}$  over  $\underline{\Sigma}_n$  for all  $n$  implies that we can find inductively a compatible family of  $\underline{\Sigma}_n$ -morphisms  $g_{r;n}^{\alpha} : W_n(\underline{Z}_r^{\alpha}) \rightarrow \underline{Z}_{r;n}^{\alpha}$  such that the composition  $\underline{Z}_r^{\alpha} \hookrightarrow W_n(\underline{Z}_r^{\alpha}) \rightarrow \underline{Z}_{r;n}^{\alpha}$  is the chosen immersion  $\underline{Z}_r^{\alpha} \hookrightarrow \underline{Z}_{r;n}^{\alpha}$ .

Let  $\underline{Z}_{r;n} = \coprod_{\alpha} \underline{Z}_{r;n}^{\alpha}$ ,  $\underline{Z}_r = \coprod_{\alpha} \underline{Z}_r^{\alpha}$ , let  $v_{r;n} : \underline{Z}_r \hookrightarrow \underline{Z}_{r;n}$ ,  $v_r : \underline{Z}_r \hookrightarrow \underline{Z}_r$  be defined by the immersions  $\underline{Z}_r^{\alpha} \hookrightarrow \underline{Z}_{r;n}^{\alpha}$  and  $\underline{Z}_r^{\alpha} \hookrightarrow \underline{Z}_r^{\alpha}$ , and let  $g_{r;n} : W_n(\underline{Z}_r) \rightarrow \underline{Z}_{r;n}$  be defined by the morphisms  $g_{r;n}^{\alpha}$ . We now use the method of Chiarellotto and Tsuzuki ([CT03, 11.2], [Tsu04, 7.3]) to deduce from these data a closed immersion  $i_{\bullet}$  of  $\underline{Z}_{\bullet}$  into an  $m$ -truncated simplicial formal log scheme



$\mathcal{T}_\bullet$ , smooth over  $\underline{\Sigma}$ , with reduction  $\mathcal{T}_{\bullet;n}$  over  $\underline{\Sigma}_n$ , and a compatible family of  $\underline{\Sigma}_n$ -morphisms of  $m$ -truncated simplicial log schemes  $h_{\bullet;n} : W_n(\underline{\mathcal{Z}}_\bullet) \rightarrow \mathcal{T}_{\bullet;n}$  such that  $h_{\bullet;n} \circ u_{\bullet;n} = i_{\bullet;n}$ , where  $u_{\bullet;n} : \underline{\mathcal{Z}}_{\bullet;n} \hookrightarrow W_n(\underline{\mathcal{Z}}_{\bullet;n})$  is the canonical morphism, and  $i_{\bullet;n}$  is the reduction mod  $p^n$  of  $i_\bullet$ . First, for  $0 \leq s \leq m$ , we set

$$\Gamma_s(\underline{\mathcal{Z}}_r) = \prod_{\gamma: [r] \rightarrow [s]} \underline{\mathcal{Z}}_{r,\gamma},$$

where the product is taken over  $\underline{\Sigma}$  and indexed by the set of morphisms  $\gamma : [r] \rightarrow [s]$  in  $\Delta[m]$ , and where  $\underline{\mathcal{Z}}_{r,\gamma} = \underline{\mathcal{Z}}_r$  for all  $\gamma$ . Then any morphism  $\eta : [s'] \rightarrow [s]$  in  $\Delta[m]$  defines a morphism  $\Gamma_s(\underline{\mathcal{Z}}_r) \rightarrow \Gamma_{s'}(\underline{\mathcal{Z}}_r)$  having as component of index  $\gamma'$  the projection of  $\Gamma_s(\underline{\mathcal{Z}}_r)$  to the factor of index  $\eta \circ \gamma'$ . In this way, one obtains an  $m$ -truncated simplicial formal log scheme  $\Gamma_\bullet(\underline{\mathcal{Z}}_r)$  over  $\underline{\Sigma}$ , the terms of which are smooth over  $\underline{\Sigma}$ .

For each  $\gamma : [r] \rightarrow [s]$ , there is a commutative diagram

$$\begin{array}{ccccccc} W_n(\underline{\mathcal{Z}}_s) & \xrightarrow{W_n(\gamma)} & W_n(\underline{\mathcal{Z}}_r) & & & & \\ \uparrow u_{s;n} & & \uparrow u_{r;n} & \searrow g_{r;n} & & & \\ \underline{\mathcal{Z}}_s & \xrightarrow{\gamma} & \underline{\mathcal{Z}}_r & \xhookrightarrow{v_{r;n}} & \underline{\mathcal{Z}}_{r;n} & \hookrightarrow & \underline{\mathcal{Z}}_r. \end{array}$$

For fixed  $r$  and variable  $s$ , the family of morphisms  $\underline{\mathcal{Z}}_s \rightarrow \Gamma_s(\underline{\mathcal{Z}}_r)$  having the composition  $\underline{\mathcal{Z}}_s \xrightarrow{\gamma} \underline{\mathcal{Z}}_r \hookrightarrow \underline{\mathcal{Z}}_r$  as component of index  $\gamma$  defines a morphism of  $m$ -truncated simplicial formal log schemes  $\underline{\mathcal{Z}}_\bullet \rightarrow \Gamma_\bullet(\underline{\mathcal{Z}}_r)$ . We set

$$\mathcal{T}_\bullet = \prod_{0 \leq r \leq m} \Gamma_\bullet(\underline{\mathcal{Z}}_r),$$

and we define  $i_\bullet : \underline{\mathcal{Z}}_\bullet \rightarrow \mathcal{T}_\bullet$  as having the previous morphism as component of index  $r$  for  $0 \leq r \leq m$ . For each  $r$ , the morphism  $\underline{\mathcal{Z}}_r \rightarrow \Gamma_r(\underline{\mathcal{Z}}_r)$  has the closed immersion  $v_r : \underline{\mathcal{Z}}_r \hookrightarrow \underline{\mathcal{Z}}_r$  as component of index  $\text{Id}_{[r]}$ . It follows that  $\underline{\mathcal{Z}}_r \rightarrow \mathcal{T}_r$  is a closed immersion for all  $r$ .

Similarly, the family of morphisms  $W_n(\underline{\mathcal{Z}}_s) \rightarrow \Gamma_s(\underline{\mathcal{Z}}_r)$  having the composition  $W_n(\underline{\mathcal{Z}}_s) \xrightarrow{W_n(\gamma)} W_n(\underline{\mathcal{Z}}_r) \xrightarrow{g_{r;n}} \underline{\mathcal{Z}}_{r;n} \hookrightarrow \underline{\mathcal{Z}}_r$  as component of index  $\gamma$  defines a morphism of  $m$ -truncated simplicial log schemes  $W_n(\underline{\mathcal{Z}}_\bullet) \rightarrow \Gamma_\bullet(\underline{\mathcal{Z}}_r)$ . We define  $h_\bullet : W_n(\underline{\mathcal{Z}}_\bullet) \rightarrow \mathcal{T}_\bullet$  as having the previous morphism as component of index  $r$  for  $0 \leq r \leq m$  and  $h_{\bullet;n} : W_n(\underline{\mathcal{Z}}_\bullet) \rightarrow \mathcal{T}_{\bullet;n}$  as being the reduction of  $h_\bullet$  mod  $p^n$ . It is clear that  $h_{\bullet;n} \circ u_{\bullet;n} = i_{\bullet;n}$  and that the morphisms  $h_{\bullet;n}$  form a compatible family when  $n$  varies.

We now set  $\underline{\mathcal{Z}}_{\bullet,0} = \underline{\mathcal{Z}}_\bullet$ ,  $\mathcal{T}_{\bullet,0} = \mathcal{T}_\bullet$ , and we define

$$\underline{\mathcal{Z}}_{\bullet,\bullet} = \text{cosk}_0^Y(\underline{\mathcal{Z}}_{\bullet,0}), \quad \mathcal{T}_{\bullet,\bullet} = \text{cosk}_0^\Sigma(\mathcal{T}_{\bullet,0}),$$

the coskeletons being taken respectively in the category of simplicial  $m$ -truncated simplicial log schemes over  $\underline{Y}_\bullet$  and of simplicial  $m$ -truncated simplicial

formal log schemes over  $\underline{\Sigma}$ . The augmentation morphism  $\underline{Z}_{\bullet,0} \rightarrow \underline{Y}_{\bullet}$  is given by  $j_{\bullet}$ , and the morphism  $i_{\bullet,\bullet}$  is defined by setting  $i_{\bullet,0} = i_{\bullet} : \underline{Z}_{\bullet,0} \hookrightarrow \underline{T}_{\bullet,0}$  and extending  $i_{\bullet,0}$  by functoriality to the coskeletons. As seen above,  $i_{\bullet,0}$  is a closed immersion, and it follows from the construction of coskeletons that  $i_{\bullet,t}$  is a closed immersion for all  $t$ . Since  $\text{cosk}_0^{\underline{\Sigma}}(\underline{T}_{r,0})_t = \underline{T}_r \times_{\underline{\Sigma}} \times \cdots \times_{\underline{\Sigma}} \underline{T}_r$  ( $t+1$  times),  $\underline{T}_{r,t}$  is smooth over  $\underline{\Sigma}$  for all  $r, t$ . Finally, we define  $h_{\bullet,\bullet;n} : W_n(\underline{Z}_{\bullet,\bullet}) \rightarrow \underline{T}_{\bullet,\bullet;n}$  as being the composition

$$\begin{aligned} W_n(\text{cosk}_0^{\underline{Y}_{\bullet}}(\underline{Z}_{\bullet,0})) &\rightarrow \text{cosk}_0^{W_n(\underline{Y}_{\bullet})}(W_n(\underline{Z}_{\bullet,0})) \\ &\rightarrow \text{cosk}_0^{\underline{\Sigma}^n}(\underline{T}_{\bullet,0;n}) \simeq \underline{\Sigma}_n \times_{\underline{\Sigma}} \text{cosk}_0^{\underline{\Sigma}}(\underline{T}_{\bullet,0}), \end{aligned}$$

where the first map is defined by the universal property of the coskeleton (and is actually an isomorphism), the second one is defined by functoriality by the morphism  $h_{\bullet;n} : W_n(\underline{Z}_{\bullet,0}) \rightarrow \underline{T}_{\bullet;n} = \underline{T}_{\bullet,0;n}$ , and the last one is the base change isomorphism for coskeletons. The relations  $h_{\bullet,\bullet;n} \circ u_{\bullet,\bullet;n} = i_{\bullet,\bullet;n}$  and the compatibility for variable  $n$  follow from the similar properties for the morphisms  $h_{\bullet,n}$ . Properties (a)–(c) of the lemma are then satisfied.  $\square$

2.11. *Proof of Proposition 2.7.* Let

$$\begin{array}{ccc} \underline{Z}_{\bullet,\bullet} & \xhookrightarrow{i_{\bullet,\bullet}} & \underline{T}_{\bullet,\bullet} \\ j_{\bullet,\bullet} \downarrow & & \downarrow \\ \underline{Y}_{\bullet} & \longrightarrow & \underline{\Sigma} \end{array}$$

be a commutative diagram satisfying the properties of Lemma 2.10. Since, for all  $r \leq m$ , the morphism  $j_{r,0}$  is locally an open immersion, the scheme underlying  $\underline{Z}_{r,t}$  is the usual fibred product  $Z_{r,0} \times_{Y_r} \cdots \times_{Y_r} Z_{r,0}$  ( $t+1$  times). Keeping the notation of the proof of Lemma 2.10, let  $\mathfrak{U}_r = (Z_r^\alpha)_\alpha$  be an affine covering of  $Y_r$  such that  $Z_{r,0} = \coprod_\alpha Z_r^\alpha$  and  $j_{r,0}|_{Z_r^\alpha}$  is the natural inclusion. Then, for any abelian sheaf  $\mathcal{E}$  on  $Y_r$ , the complex

$$\varepsilon_{r*}(j_{r,\bullet}*j_{r,\bullet}^{-1}\mathcal{E}) = \left[ j_{r,0}*j_{r,0}^{-1}\mathcal{E} \rightarrow \cdots \rightarrow j_{r,t}*j_{r,t}^{-1}\mathcal{E} \xrightarrow{\sum_k (-1)^k \partial_k} j_{r,t+1}*j_{r,t+1}^{-1}\mathcal{E} \rightarrow \cdots \right]$$

is the Čech resolution of  $\mathcal{E}$  defined by the covering  $\mathfrak{U}_r$ . If  $\mathcal{E}^\bullet$  is an abelian sheaf on  $Y_\bullet$ , the fact that  $j_{\bullet,0}$  is an augmentation morphism in the category of  $m$ -truncated simplicial schemes implies that the complex  $\varepsilon_{r*}(j_{r,\bullet}*j_{r,\bullet}^{-1}\mathcal{E}^r)$  is functorial with respect to  $[r] \in \Delta[m]$ , and we obtain a resolution  $\varepsilon_{\bullet*}(j_{\bullet,\bullet}*j_{\bullet,\bullet}^{-1}\mathcal{E}^\bullet)$  of  $\mathcal{E}^\bullet$  in the category of abelian sheaves on  $Y_\bullet$ . In particular, taking into account that each  $j_{q,q'}$  is locally an open immersion, we obtain for all  $n$  a resolution of the de Rham-Witt complex of  $\underline{Y}_\bullet$  given by

$$(2.11.1) \quad W_n \Omega_{\underline{Y}_\bullet}^\bullet \xrightarrow{\text{qis}} \varepsilon_{\bullet*}(j_{\bullet,\bullet}*W_n \Omega_{\underline{Z}_{\bullet,\bullet}}^\bullet).$$

On the other hand, one can also define for all  $r$  a complex on  $\text{Crys}(\underline{Y}_r/\underline{\Sigma}_n)$  by setting

$$\begin{aligned} & \varepsilon_{r*}(j_{r,\bullet} \text{crys}^*(\mathcal{O}_{\underline{Z}_{r,\bullet}/\underline{\Sigma}_n})) \\ &= \left[ j_{r,0} \text{crys}^*(\mathcal{O}_{\underline{Z}_{r,0}/\underline{\Sigma}_n}) \rightarrow \cdots \rightarrow j_{r,t} \text{crys}^*(\mathcal{O}_{\underline{Z}_{r,t}/\underline{\Sigma}_n}) \xrightarrow{\sum_k (-1)^k \partial_k} \cdots \right]. \end{aligned}$$

Since  $Z_{r,\bullet} \rightarrow Y_r$  is the Čech simplicial scheme defined by an affine open covering of  $Y_r$ , this complex is a resolution of  $\mathcal{O}_{\underline{Y}_r/\underline{\Sigma}_n}$  [Ber74, III, Prop. 3.1.2 and V, Prop. 3.1.2]. Since  $\underline{Z}_{\bullet,\bullet}$  is a bisimplicial scheme, these resolutions are functorial with respect to  $[r]$  and yield a resolution  $\varepsilon_{\bullet,*}(j_{\bullet,\bullet} \text{crys}^*(\mathcal{O}_{\underline{Z}_{\bullet,\bullet}/\underline{\Sigma}_n}))$  of  $\mathcal{O}_{\underline{Y}_\bullet/\underline{\Sigma}_n}$ . Let  $\underline{T}_{\bullet,\bullet;n}$  be the reduction mod  $p^n$  of  $\underline{T}_{\bullet,\bullet}$ . The linearization functor  $L$  [Kat89, (6.9)] is functorial with respect to embeddings, hence it provides a complex  $L(\Omega_{\underline{T}_{\bullet,\bullet;n}/\underline{\Sigma}_n}^\bullet)$  on  $\text{Crys}(\underline{Z}_{\bullet,\bullet}/\underline{\Sigma}_n)$ . This complex is a resolution of  $\mathcal{O}_{\underline{Z}_{\bullet,\bullet}/\underline{\Sigma}_n}$  thanks to the log Poincaré lemma, which follows from [Kat89, Prop. (6.5)]. For each  $(r,t)$  and each  $i$ , one checks easily that the term  $j_{r,t} \text{crys}^*(L(\Omega_{\underline{T}_{r,t;n}/\underline{\Sigma}_n}^i))$  is acyclic with respect to  $u_{\underline{Y}_r/\underline{\Sigma}_n}$ . (Use [Ber74, V, (2.2.3)] and the equality  $u_{\underline{Y}_r/\underline{\Sigma}_n} \circ j_{r,t} \text{crys}^* = j_{r,t} \circ u_{\underline{Z}_{r,t}/\underline{\Sigma}_n}$ .) Hence, the complex  $\varepsilon_{\bullet,*}(j_{\bullet,\bullet} \text{crys}^*(L(\Omega_{\underline{T}_{\bullet,\bullet;n}/\underline{\Sigma}_n}^\bullet)))$  is an  $u_{\underline{Y}_\bullet/\underline{\Sigma}_n}$ -acyclic resolution of  $\mathcal{O}_{\underline{Y}_\bullet/\underline{\Sigma}_n}$ . Moreover, the closed immersion of bisimplicial schemes  $i_{\bullet,\bullet}$  defines a family of PD-envelopes  $\mathcal{P}_{\underline{Z}_{\bullet,\bullet}}^{\log}(\underline{T}_{\bullet,\bullet;n})$ , supported in  $Z_{\bullet,\bullet}$ . They provide a de Rham complex  $\mathcal{P}_{\underline{Z}_{\bullet,\bullet}}^{\log}(\underline{T}_{\bullet,\bullet;n}) \otimes \Omega_{\underline{T}_{\bullet,\bullet;n}/\underline{\Sigma}_n}^\bullet$ , which can be viewed as a complex of abelian sheaves on  $Z_{\bullet,\bullet}$ , and it follows from [Ber74, V, (2.2.3)] that

$$u_{\underline{Y}_\bullet/\underline{\Sigma}_n}(j_{\bullet,\bullet} \text{crys}^*(L(\Omega_{\underline{T}_{\bullet,\bullet;n}/\underline{\Sigma}_n}^\bullet))) = j_{\bullet,\bullet}(\mathcal{P}_{\underline{Z}_{\bullet,\bullet}}^{\log}(\underline{T}_{\bullet,\bullet;n}) \otimes \Omega_{\underline{T}_{\bullet,\bullet;n}/\underline{\Sigma}_n}^\bullet).$$

As the  $j_{r,t}$ 's are affine morphisms (as a consequence of 1) in our general conventions), we finally get in  $D^+(Z_\bullet, W_n)$  an isomorphism

$$(2.11.2) \quad Ru_{\underline{Y}_\bullet/\underline{\Sigma}_n}(\mathcal{O}_{\underline{Y}_\bullet/\underline{\Sigma}_n}) \xrightarrow{\sim} \varepsilon_{\bullet,*}(j_{\bullet,\bullet}(\mathcal{P}_{\underline{Z}_{\bullet,\bullet}}^{\log}(\underline{T}_{\bullet,\bullet;n}) \otimes \Omega_{\underline{T}_{\bullet,\bullet;n}/\underline{\Sigma}_n}^\bullet)).$$

To prove Proposition 2.7, it suffices to define a quasi-isomorphism between the right-hand sides of (2.11.1) and (2.11.2). Note that, for each  $r,t,i$ , the sheaves  $W_n \Omega_{\underline{Z}_{r,t}}^i$  and  $\mathcal{P}_{\underline{Z}_{r,t}}^{\log}(\underline{T}_{r,t;n}) \otimes \Omega_{\underline{T}_{r,t;n}/\underline{\Sigma}_n}^i$  are  $j_{r,t}$ -acyclic. Indeed,  $Z_{r,t}$  is a disjoint union of affine open subsets of  $Y_r$ , and on the one hand  $W_n \Omega_{\underline{Z}_{r,t}}^i$  has a finite filtration with subquotients that are coherent over suitable Frobenius pullbacks of  $Z_{r,t}$  [HK94, Th. (4.4)], on the other hand  $\mathcal{P}_{\underline{Z}_{r,t}}^{\log}(\underline{T}_{r,t;n}) \otimes \Omega_{\underline{T}_{r,t;n}/\underline{\Sigma}_n}^i$  is a quasi-coherent  $\mathcal{O}_{\underline{T}_{r,t;n}}$ -module with support in  $Z_{r,t}$ , hence is a direct limit of submodules that have a finite filtration with subquotients that are coherent over  $Z_{r,t}$ . Therefore, it suffices to construct a quasi-isomorphism

$$(2.11.3) \quad \mathcal{P}_{\underline{Z}_{\bullet,\bullet}}^{\log}(\underline{T}_{\bullet,\bullet;n}) \otimes \Omega_{\underline{T}_{\bullet,\bullet;n}/\underline{\Sigma}_n}^\bullet \longrightarrow W_n \Omega_{\underline{Z}_{\bullet,\bullet}}^\bullet$$

in the category of complexes of  $W_n$ -modules over  $Z_{\bullet,\bullet}$ .

We can now argue as in the proof of [HK94, Th. (4.19)]. Since the PD-immersion  $u_{r,t;n} : \underline{Z}_{r,t} \hookrightarrow W_n(\underline{Z}_{r,t})$  is an exact closed immersion for all  $r, t$ , the morphism  $h_{\bullet,\bullet;n} : W_n(\underline{Z}_{\bullet,\bullet}) \rightarrow \underline{T}_{\bullet,\bullet;n}$  defines uniquely a PD-morphism  $\mathcal{P}_{\underline{Z}_{\bullet,\bullet}}^{\log}(\underline{T}_{\bullet,\bullet;n}) \rightarrow W_n(\mathcal{O}_{\underline{Z}_{\bullet,\bullet}})$  in the category of sheaves of  $W$ -modules on the bisimplicial scheme  $T_{\bullet,\bullet;n}$ . As  $h_{\bullet,\bullet;n}$  is a morphism of bisimplicial log schemes, it defines by functoriality a morphism of complexes  $\Omega_{\underline{T}_{\bullet,\bullet;n}/\Sigma_n}^{\bullet} \rightarrow \Omega_{W_n(\underline{Z}_{\bullet,\bullet})/\Sigma_n}^{\bullet}$  on  $T_{\bullet,\bullet;n}$ . This morphism extends as a morphism of complexes with support in  $\underline{Z}_{\bullet,\bullet}$

$$\mathcal{P}_{\underline{Z}_{\bullet,\bullet}}^{\log}(\underline{T}_{\bullet,\bullet;n}) \otimes \Omega_{\underline{T}_{\bullet,\bullet;n}/\Sigma_n}^{\bullet} \longrightarrow \Omega_{W_n(\underline{Z}_{\bullet,\bullet})/\Sigma_n}^{\bullet} / \mathcal{N}_{\bullet,\bullet}^{\bullet},$$

where  $\mathcal{N}_{\bullet,\bullet}^{\bullet} \subset \Omega_{W_n(\underline{Z}_{\bullet,\bullet})/\Sigma_n}^{\bullet}$  denotes the graded ideal generated by the sections  $d(a^{[i]}) - a^{[i-1]}da$  for all sections  $a$  of  $VW_{n-1}(\mathcal{O}_{\underline{Z}_{\bullet,\bullet}})$  and all  $i \geq 1$ . The differential graded algebra  $W_n\Omega_{\underline{Z}_{\bullet,\bullet}}^{\bullet}$  is a quotient of  $\Omega_{W_n(\underline{Z}_{\bullet,\bullet})/\Sigma_n}^{\bullet}$  [HK94, Prop. (4.7)], and the generators of  $\mathcal{N}_{\bullet,\bullet}^{\bullet}$  vanish in  $W_n\Omega_{\underline{Z}_{\bullet,\bullet}}^{\bullet}$  (because  $W\Omega_{\underline{Z}_{\bullet,\bullet}}^{\bullet}$  is  $p$ -torsion free), so we finally get the morphism (2.11.3). To check that it is a quasi-isomorphism, it suffice to do so on each  $Z_{r,t}$ , and this follows from [HK94, Th. (4.19)]. In this way we obtain the isomorphism (2.7.1).

To construct the isomorphism (2.7.2), it suffices to observe that the compatibility of the previous constructions when  $n$  varies implies that they make sense in the category of inverse systems indexed by  $n \in \mathbb{N}$ . Then one can apply the functor  $R\varprojlim_n$  to the isomorphism (2.7.1) viewed as an isomorphism in the derived category of inverse systems of sheaves of  $W$ -modules on  $Y_{\bullet}$ , and this provides the isomorphism (2.7.2), since the local structure of the  $W_n\Omega^i$  recalled above implies that they form a  $\varprojlim$ -acyclic inverse system.

The isomorphisms (2.7.1) and (2.7.2) do not depend upon the choices made in their construction. If

$$(\underline{Z}_{\bullet,\bullet}, \underline{T}_{\bullet,\bullet}, j_{\bullet,\bullet}, i_{\bullet,\bullet}, h_{\bullet,\bullet;n}) \quad \text{and} \quad (\underline{Z}'_{\bullet,\bullet}, \underline{T}'_{\bullet,\bullet}, j'_{\bullet,\bullet}, i'_{\bullet,\bullet}, h'_{\bullet,\bullet;n})$$

are two sets of data provided by Lemma 2.10, one can construct a third set of data  $(\underline{Z}''_{\bullet,\bullet}, \underline{T}''_{\bullet,\bullet}, j''_{\bullet,\bullet}, i''_{\bullet,\bullet}, h''_{\bullet,\bullet;n})$  mapping to the two previous ones by setting

$$\underline{Z}''_{\bullet,\bullet} = \underline{Z}_{\bullet,\bullet} \times_{\underline{Y}_{\bullet}} \underline{Z}'_{\bullet,\bullet}, \quad \underline{T}''_{\bullet,\bullet} = \underline{T}_{\bullet,\bullet} \times_{\underline{\Sigma}} \underline{T}'_{\bullet,\bullet},$$

and defining  $j''_{\bullet,\bullet}$ ,  $i''_{\bullet,\bullet}$  and  $h''_{\bullet,\bullet;n}$  by functoriality. Then the independence property of (2.7.1) and (2.7.2) follows from the functoriality of the canonical isomorphisms used in their construction with respect to the projections from  $(\underline{Z}''_{\bullet,\bullet}, \underline{T}''_{\bullet,\bullet})$  to  $(\underline{Z}_{\bullet,\bullet}, \underline{T}_{\bullet,\bullet})$  and  $(\underline{Z}'_{\bullet,\bullet}, \underline{T}'_{\bullet,\bullet})$ . Moreover, one can also prove the functoriality of (2.7.1) and (2.7.2) with respect to  $\underline{Y}_{\bullet}$  by similar arguments using the graph construction: for a morphism  $\varphi_{\bullet} : \underline{Y}'_{\bullet} \rightarrow \underline{Y}_{\bullet}$  between two  $m$ -truncated simplicial log schemes satisfying the assumptions of Lemma 2.7, one can find sets of data  $(\underline{Z}_{\bullet,\bullet}, \underline{T}_{\bullet,\bullet}, j_{\bullet,\bullet}, i_{\bullet,\bullet}, h_{\bullet,\bullet;n})$  and  $(\underline{Z}'_{\bullet,\bullet}, \underline{T}'_{\bullet,\bullet}, j'_{\bullet,\bullet}, i'_{\bullet,\bullet}, h'_{\bullet,\bullet;n})$  satisfying the conditions of Lemma 2.10 relatively to  $\underline{Y}_{\bullet}$  and  $\underline{Y}'_{\bullet}$ , and such that there exists morphisms of bisimplicial log schemes  $\psi_{\bullet,\bullet} : \underline{Z}'_{\bullet,\bullet} \rightarrow \underline{Z}_{\bullet,\bullet}$ ,

$\theta_{\bullet,\bullet} : \mathcal{T}'_{\bullet,\bullet} \rightarrow \mathcal{T}_{\bullet,\bullet}$  satisfying the obvious compatibilities. Then the functoriality of (2.7.1) and (2.7.2) with respect to  $\varphi_{\bullet}$  follows from the functoriality of the canonical isomorphisms used in their construction with respect to  $\varphi_{\bullet}, \psi_{\bullet,\bullet}$  and  $\theta_{\bullet,\bullet}$ . In particular, in this way one obtains that the isomorphisms (2.7.1) and (2.7.2) are compatible with the Frobenius actions.  $\square$

2.12. *Proof of Theorem 1.3, assuming Theorem 1.5.* To conclude this section, we prove that Theorem 1.5 implies Theorem 1.3. We keep the notation of 1.1, and we first observe that if Theorem 1.3 holds when  $R$  is complete, then it holds in general. Indeed, let  $\widehat{R}$  be the completion of  $R$ , and let  $\widetilde{X} = X_{\widehat{R}}$ . Then  $\widetilde{X}$  is a regular scheme: on the one hand, its generic fibre is smooth over  $\widehat{K} = \text{Frac}(\widehat{R})$ ; on the other hand, its special fibre is isomorphic to  $X_k$ , and the completions of the local rings of  $X$  and  $\widetilde{X}$  are isomorphic at any corresponding points of their special fibres. It follows that  $\widetilde{X}$  satisfies the assumptions of Theorem 1.3 relatively to  $\widehat{R}$ , and the theorem for  $\widetilde{X}$  implies the theorem for  $X$ .

Therefore, we assume in the rest of the proof that  $R$  is complete. We fix an integer  $m > q$ . Let  $K'$  be a finite extension of  $K$ , with ring of integers  $R'$  and residue field  $k'$ , such that there exists an  $m$ -truncated simplicial log scheme  $\underline{Y}_{\bullet}$  over  $R'$ , with an augmentation morphism  $u : Y_0 \rightarrow X_{R'}$ , such that properties (a)–(c) of Lemma 2.2 are satisfied. Let  $W'_n = W_n(k')$ ,  $W' = W(k')$ ,  $K'_0 = \text{Frac}(W')$ , and let  $\underline{\Sigma}'_n, \underline{\Sigma}'$  be the log schemes defined by  $W'_n, W'$  as in 2.1.

Thanks to property (a) of Lemma 2.2, the log schemes  $(\underline{Y}_r)_{k'}$  are smooth of Cartier type over  $\underline{\Sigma}'_1$ . Therefore, we can consider the log crystalline cohomology of  $\underline{Y}_{\bullet,k'}$

$$R\Gamma_{\text{crys}}(\underline{Y}_{\bullet,k'}/\underline{\Sigma}', \mathcal{O}_{\underline{Y}_{\bullet,k'}/\underline{\Sigma}'}) := R\varepsilon_*^m R\Gamma_{\text{crys}}^{\bullet}(\underline{Y}_{\bullet,k'}/\underline{\Sigma}', \mathcal{O}_{\underline{Y}_{\bullet,k'}/\underline{\Sigma}'}),$$

as defined in 2.6. Using the naive filtration on the functor  $R\varepsilon_*^m$  (see 2.3), its basic properties follow from those of the log crystalline cohomology of the proper and smooth log schemes  $(Y_r)_{k'}$ . In particular, since  $Y_r$  is proper over  $\underline{\Sigma}'_1$  for all  $r$ , the complex  $R\Gamma_{\text{crys}}(\underline{Y}_{\bullet,k'}/\underline{\Sigma}', \mathcal{O}_{\underline{Y}_{\bullet,k'}/\underline{\Sigma}'})$  is a perfect complex of  $W'$ -modules, and the cohomology space  $H^q_{\text{crys}}(\underline{Y}_{\bullet,k'}/\underline{\Sigma}', \mathcal{O}_{\underline{Y}_{\bullet,k'}/\underline{\Sigma}'}) \otimes K'_0$  is a finite dimensional  $K'_0$ -vector space. By functoriality, it is endowed with the semi-linear Frobenius action defined by the absolute Frobenius endomorphism of  $\underline{Y}_{\bullet,k'}$ .

From (2.6.2) and (2.7.2), we deduce an isomorphism

$$H^q_{\text{crys}}(\underline{Y}_{\bullet,k'}/\underline{\Sigma}', \mathcal{O}_{\underline{Y}_{\bullet,k'}/\underline{\Sigma}'}) \otimes K'_0 \xrightarrow{\sim} H^q(\underline{Y}_{\bullet,k'}, W\Omega_{\underline{Y}_{\bullet,k'}}^{\bullet}) \otimes K'_0,$$

which is compatible with the Frobenius actions thanks to Proposition 2.7. The filtration of the complex  $W\Omega_{\underline{Y}_{\bullet,k'}}^{\bullet}$  by the subcomplexes  $\sigma_{\geq i} W\Omega_{\underline{Y}_{\bullet,k'}}^{\bullet}$  provides a spectral sequence

$$E_1^{i,j} = H^j(\underline{Y}_{\bullet,k'}, W\Omega_{\underline{Y}_{\bullet,k'}}^i) \otimes K'_0 \implies H^{i+j}_{\text{crys}}(\underline{Y}_{\bullet,k'}/\underline{\Sigma}', \mathcal{O}_{\underline{Y}_{\bullet,k'}/\underline{\Sigma}'}) \otimes K'_0,$$

which is endowed with a Frobenius action. Using the naive filtration on  $R\varepsilon_*^m$ , from the case of a single log scheme we deduce that each term  $E_1^{i,j}$  is a finite dimensional  $K_0$ -vector space on which the Frobenius action is bijective with slopes in  $[i, i + 1[$ . Therefore the spectral sequence degenerates at  $E_1$  and, taking (2.1.1) into account, we get in particular an isomorphism

$$(2.12.1) \quad (H_{\text{crys}}^q(\underline{Y}_{\bullet, k'}/\underline{\Sigma}', \mathcal{O}_{\underline{Y}_{\bullet, k'}/\underline{\Sigma}'}) \otimes K'_0)^{<1} \xrightarrow{\sim} H^q(Y_{\bullet, k'}, W\mathcal{O}_{Y_{\bullet, k'}, \mathbb{Q}}).$$

Since  $\underline{Y}_{\bullet}$  satisfies property (a) of 2.2, the construction of the monodromy operator  $N$  on log crystalline cohomology can be extended to the case of  $\underline{Y}_{\bullet, k'}$  [Tsu98, (6.3)]. Moreover, the Hyodo-Kato isomorphism  $\rho$  can also be extended to the case of  $\underline{Y}_{\bullet, k'}$  [Tsu98, (6.3.2)], providing an isomorphism

$$(2.12.2) \quad \rho : H_{\text{crys}}^q(\underline{Y}_{\bullet, k'}/\underline{\Sigma}', \mathcal{O}_{\underline{Y}_{\bullet, k'}/\underline{\Sigma}'}) \otimes K' \xrightarrow{\sim} H^q(Y_{\bullet, K'}, \Omega_{Y_{\bullet, K'}}^{\bullet}).$$

Thus,  $H_{\text{crys}}^q(\underline{Y}_{\bullet, k'}/\underline{\Sigma}', \mathcal{O}_{\underline{Y}_{\bullet, k'}/\underline{\Sigma}'}) \otimes K'$  inherits a filtered  $(\varphi, N)$ -module structure.

It follows from [Tsu98, Th. 7.1.1] (generalizing [Tsu99, Th. 0.2]) that, endowed with this structure,  $H_{\text{crys}}^q(\underline{Y}_{\bullet, k'}/\underline{\Sigma}', \mathcal{O}_{\underline{Y}_{\bullet, k'}/\underline{\Sigma}'}) \otimes K'_0$  is an admissible filtered  $(\varphi, N)$ -module, corresponding to the Galois representation  $H_{\text{ét}}^q(Y_{\bullet, \bar{K}}, \mathbb{Q}_p)$ . Thus, it is weakly admissible. In particular, either  $H_{\text{crys}}^q(\underline{Y}_{\bullet, k'}/\underline{\Sigma}', \mathcal{O}_{\underline{Y}_{\bullet, k'}/\underline{\Sigma}'}) \otimes K'_0 = 0$ , or its smallest Newton slope is greater or equal to its smallest Hodge slope. Since  $H^q(X_K, \mathcal{O}_{X_K}) = 0$ , Corollary 2.5 implies that the smallest Hodge slope is at least 1. Therefore, the part of Newton slope  $< 1$  vanishes. By (2.12.1), we obtain

$$(2.12.3) \quad H^q(Y_{\bullet, k'}, W\mathcal{O}_{Y_{\bullet, k'}, \mathbb{Q}}) = 0.$$

As  $Y_{\bullet} \rightarrow X_{R'}$  satisfies property 2.2(c), there exists a sub- $K$ -extension  $K_1 \subset K'$ , with ring of integers  $R_1$  and residue field  $k_1$ , a semi-stable scheme  $Y$  over  $R_1$ , a projective  $R$ -alteration  $f : Y \rightarrow X$ , and finitely many  $R$ -embeddings  $\sigma_i : R_1 \hookrightarrow K'$  such that if  $u_1 : Y \rightarrow X_{R_1}$  denotes the  $R$ -morphism defined by  $f$  and if  $Y_{\sigma_i}$  (resp.  $u_{\sigma_i} : Y_{\sigma_i} \rightarrow X_{R'}$ ) denotes the  $R'$ -scheme (resp.  $R'$ -morphism) deduced by base change via  $\sigma_i$  from  $Y$  (resp.  $u_1$ ), then  $Y_0 = \coprod_i Y_{\sigma_i}$ , and the augmentation morphism  $u : Y_0 \rightarrow X_{R'}$  is defined by  $u|_{Y_{\sigma_i}} = u_{\sigma_i}$ . This provides a commutative diagram

$$(2.12.4) \quad \begin{array}{ccccc} & & Y_{\bullet, k'} & \xrightarrow{u_{\bullet, k'}} & X_{k'} \\ & s_0 \nearrow & & & \searrow \\ Y_{0, k'} = \coprod_i Y_{\sigma_i, k'} & & & & X_k \\ & \searrow & Y_{k_1} & \hookrightarrow & Y_k \\ & & & \xrightarrow{f_k} & \end{array}$$

in which we identify schemes with their Zariski topoi,  $Y_k := \text{Spec } k \times_{\text{Spec } R} Y$ , and

(i) the morphism  $u_{\bullet, k'}$  is such that, for any sheaf  $E$  on  $X_{k'}$ ,  $u_{\bullet, k'}^{-1}E$  is the family of sheaves  $(u_r)_{k'}^{-1}E$ , with  $u_r : Y_r \rightarrow X_{R'}$  defined by the augmentation morphism;

(ii) the morphism  $s_0$  is such that, for any sheaf  $F^\bullet$  on  $Y_{\bullet, k'}$ ,  $s_0^{-1}F^\bullet = F^0$ ;

(iii) the morphism  $Y_{\sigma_i, k'} \rightarrow Y_{k_1}$  is the projection corresponding to  $\sigma_i$ .

By functoriality, we obtain a commutative diagram for the corresponding Witt cohomology spaces

(2.12.5)

$$\begin{array}{ccc}
 & H^q(X_{k'}, W\mathcal{O}_{X_{k'}, \mathbb{Q}}) \rightarrow H^q(Y_{\bullet, k'}, W\mathcal{O}_{Y_{\bullet, k'}, \mathbb{Q}}) & \\
 & \nearrow & \searrow \\
 H^q(X_k, W\mathcal{O}_{X_k, \mathbb{Q}}) & & \bigoplus_i H^q(Y_{\sigma_i, k'}, W\mathcal{O}_{Y_{\sigma_i, k'}, \mathbb{Q}}). \\
 & \searrow^{f_k^*} & \nearrow \\
 & H^q(Y_k, W\mathcal{O}_{Y_k, \mathbb{Q}}) \xrightarrow{\sim} H^q(Y_{k_1}, W\mathcal{O}_{Y_{k_1}, \mathbb{Q}}) & 
 \end{array}$$

In this diagram, the lower horizontal arrow is an isomorphism because  $Y_{k_1} \hookrightarrow Y_k$  is a nilpotent immersion [BBE07, Prop. 2.1 (i)]. The lower right arrow is injective on each summand, because each  $\sigma_i$  turns  $k'$  into a finite separable extension of  $k_1$ . Hence it follows from [Ill79, 0, Prop. 1.5.8] that

$$W(k') \otimes_{W(k_1)} \Gamma(U, W\mathcal{O}_{Y_{k_1}}) \xrightarrow{\sim} \Gamma(U_{\sigma_i}, W\mathcal{O}_{Y_{\sigma_i, k'}})$$

for any affine open subset  $U \subset Y_{k_1}$  with inverse image  $U_{\sigma_i} \subset Y_{\sigma_i, k'}$ ; as one can compute Witt cohomology using Čech cohomology, this implies that

$$W(k') \otimes_{W(k_1)} H^q(Y_{k_1}, W\mathcal{O}_{Y_{k_1}}) \xrightarrow{\sim} H^q(Y_{\sigma_i, k'}, W\mathcal{O}_{Y_{\sigma_i, k'}}).$$

Finally,  $f : Y \rightarrow X$  is a projective alteration between two flat regular schemes of finite type over  $R$ , so Theorem 1.5 implies that  $f_k^*$  is injective. Therefore, the functoriality map  $H^q(X_k, W\mathcal{O}_{X_k, \mathbb{Q}}) \rightarrow \bigoplus_i H^q(Y_{\sigma_i, k'}, W\mathcal{O}_{Y_{\sigma_i, k'}, \mathbb{Q}})$  is injective. But (2.12.3) implies that the composition of the upper path in the diagram is 0. It follows that  $H^q(X_k, W\mathcal{O}_{X_k, \mathbb{Q}}) = 0$ .  $\square$

### 3. An injectivity theorem for coherent cohomology

We now begin our preliminary work in view of the proof of Theorem 1.5. One of the key ingredients in this proof is a theorem that bounds the order of elements in the kernel of the functoriality map induced on coherent cohomology by a proper surjective complete intersection morphism  $f : Y \rightarrow X$  of virtual relative dimension 0. Such a result is a consequence of the existence of a “trace morphism”  $\tau_f : Rf_*\mathcal{O}_Y \rightarrow \mathcal{O}_X$  that satisfies the properties stated in the following theorem.

**THEOREM 3.1.** *Let  $X$  be a noetherian scheme with a dualizing complex, and let  $f : Y \rightarrow X$  be a proper complete intersection morphism of virtual*

relative dimension 0. There exists a morphism  $\tau_f : Rf_*\mathcal{O}_Y \rightarrow \mathcal{O}_X$  that satisfies the following properties:

(i) If  $g : Z \rightarrow Y$  is a second proper complete intersection morphism of virtual relative dimension 0, then the composed morphism

$$(3.1.1) \quad R(f \circ g)_*\mathcal{O}_Z \cong Rf_*Rg_*\mathcal{O}_Z \xrightarrow{Rf_*(\tau_g)} Rf_*\mathcal{O}_Y \xrightarrow{\tau_f} \mathcal{O}_X$$

is equal to  $\tau_{fg}$ .

(ii) Let  $X'$  be another noetherian scheme with a dualizing complex,  $u : X' \rightarrow X$  a morphism such that  $X'$  and  $Y$  are Tor-independent over  $X$ , and  $f' : Y' \rightarrow X'$  the pull-back of  $f$  by  $u$ . If  $f$  is projective, or if either  $f$  is flat, or  $u$  is residually stable [Con00, p. 132], then the morphism

$$(3.1.2) \quad Rf'_*\mathcal{O}_{Y'} \cong Lu^*Rf_*\mathcal{O}_Y \xrightarrow{Lu^*(\tau_f)} \mathcal{O}_{X'},$$

defined by the base change isomorphism (A.1.2), is equal to  $\tau_{f'}$ .

(iii) If  $f$  is finite and flat then, for any section  $b \in f_*\mathcal{O}_Y$ ,

$$(3.1.3) \quad \tau_f(b) = \text{trace}_{f_*\mathcal{O}_Y/\mathcal{O}_X}(b).$$

As explained in the introduction, we refer to B.7 for the definition of  $\tau_f$  and to B.9 for the proof of the theorem.

It may be worth recalling a few examples of complete intersection morphisms of virtual relative dimension 0 (in short: ci0):

1) If  $X$  and  $Y$  are two regular schemes with the same Krull dimension, any morphism  $f : Y \rightarrow X$  that is locally of finite type is ci0. This is the situation where we will use Theorem 3.1 in this article.

2) If  $X$  and  $Y$  are smooth over a third scheme  $S$ , with the same relative dimension, any  $S$ -morphism  $Y \rightarrow X$  is ci0.

3) If  $X$  is a scheme,  $Z \hookrightarrow X$  a regularly embedded closed subscheme, and  $f : Y \rightarrow X$  the blowing up of  $X$  along  $Z$ , then  $f$  is ci0 [SGA6, VII, Prop. 1.8].

The existence of  $\tau_f$  has a remarkable consequence for the functoriality maps induced on coherent cohomology.

**THEOREM 3.2.** *Let  $X$  be a noetherian scheme with a dualizing complex, and  $f : Y \rightarrow X$  a proper complete intersection morphism of virtual relative dimension 0. Assume that there exists a scheme-theoretically dense open subset  $U \subset X$  such that  $f^{-1}(U) \rightarrow U$  is finite locally free of constant rank  $r \geq 1$ . Then, for any complex  $\mathcal{E}^\bullet \in D_{\text{qc}}^b(\mathcal{O}_X)$  and any  $q \geq 0$ , the kernel of the functoriality map*

$$(3.2.1) \quad H^q(X, \mathcal{E}^\bullet) \rightarrow H^q(Y, Lf^*\mathcal{E}^\bullet)$$

is annihilated by  $r$ . In particular, when  $r$  is invertible on  $X$ , the functoriality maps are injective.



*Proof.* By 3.1(iii), the composition  $\mathcal{O}_X \rightarrow Rf_*\mathcal{O}_Y \xrightarrow{\tau_f} \mathcal{O}_X$  is multiplication by  $r$  over  $U$ . Since  $U$  is scheme-theoretically dense in  $X$ , it is multiplication by  $r$  over  $X$ .

The complete intersection hypothesis implies that  $f$  has finite Tor-dimension; hence,  $Lf^*\mathcal{E}^\bullet$  belongs to  $D_{\text{qc}}^b(\mathcal{O}_Y)$ . Moreover, we can apply the projection formula [SGA6, III, 3.7] to obtain a commutative diagram

$$\begin{array}{ccccc} \mathcal{E}^\bullet & \longrightarrow & Rf_*\mathcal{O}_Y \overset{L}{\otimes}_{\mathcal{O}_X} \mathcal{E}^\bullet & \xrightarrow{\sim} & Rf_*Lf^*\mathcal{E}^\bullet \\ & \searrow \times r & \downarrow \tau_f \otimes \text{Id} & & \swarrow \\ & & \mathcal{E}^\bullet & & \end{array}$$

in which the upper composed morphism is the adjunction morphism. Applying the functors  $H^q(X, -)$  to the diagram, the theorem follows.  $\square$

**4. Koszul resolutions and local description of the trace morphism  $\tau_f$**

We recall here some well-known explicit constructions based on the Koszul complex that enter in the definition of the trace morphism  $\tau_f$ . Later on, this will allow us to define generalizations of  $\tau_f$  for sheaves of Witt vectors. As in the whole article, we follow Conrad’s constructions and conventions [Con00].

4.1. Let  $P$  be a scheme, and let  $\mathbf{t} = (t_1, \dots, t_d)$  be a regular sequence of sections of  $\mathcal{O}_P$ , defining an ideal  $\mathcal{I} \subset \mathcal{O}_P$ . We denote by  $Y \subset P$  the closed subscheme defined by  $\mathcal{I}$  and by  $i : Y \hookrightarrow P$  the corresponding closed immersion. Classically, the Koszul complex  $K_\bullet(\mathbf{t})$  defined by the sequence  $(t_1, \dots, t_d)$  is the chain complex concentrated in homological degrees  $[0, d]$  such that  $\mathcal{E} := K_1(\mathbf{t})$  is a free  $\mathcal{O}_P$ -module of rank  $d$  with basis  $e_1, \dots, e_d$ ,  $K_k(\mathbf{t}) = \wedge^k \mathcal{E}$  for all  $k$  and such that the differential is given in degree  $k$  by

$$d_k(e_{i_1} \wedge \dots \wedge e_{i_k}) = \sum_{j=1}^k (-1)^{j-1} t_{i_j} e_{i_1} \wedge \dots \wedge \widehat{e_{i_j}} \wedge \dots \wedge e_{i_k}.$$

It is often more convenient to consider  $K_\bullet(\mathbf{t})$  as a cochain complex concentrated in cohomological degrees  $[-d, 0]$ , by setting  $(K_\bullet(\mathbf{t}))^k = K_{-k}(\mathbf{t})$  and leaving the differential unchanged [Con00, p. 17].

Since  $\mathbf{t}$  is a regular sequence,  $K_\bullet(\mathbf{t})$  is a free resolution of  $\mathcal{O}_Y$  over  $\mathcal{O}_P$ . For any  $\mathcal{O}_P$ -module  $\mathcal{M}$ , this resolution provides an isomorphism

(4.1.1)

$$\text{Ext}_{\mathcal{O}_P}^d(\mathcal{O}_Y, \mathcal{M}) := H^d(\text{Hom}_{\mathcal{O}_P}^\bullet(K_\bullet(\mathbf{t}), \mathcal{M})) \xrightarrow[\sim]{\psi_{\mathbf{t}, \mathcal{M}}} \frac{\text{Hom}_{\mathcal{O}_P}(\wedge^d \mathcal{E}, \mathcal{M})}{\mathcal{I}\text{Hom}_{\mathcal{O}_P}(\wedge^d \mathcal{E}, \mathcal{M})},$$

where  $\psi_{\mathbf{t}, \mathcal{M}}$  is the tautological isomorphism multiplied by  $(-1)^{d(d+1)/2}$  (see [Con00, definition of (1.3.28) and (2.5.2)]). For any section  $m$  of  $\mathcal{M}$ , we will

denote by

$$(4.1.2) \quad \begin{bmatrix} m \\ t_1, \dots, t_d \end{bmatrix} \in \mathcal{E}xt_{\mathcal{O}_P}^d(\mathcal{O}_Y, \mathcal{M})$$

the section corresponding by (4.1.1) to the class of the homomorphism  $u_{\mathbf{t},m}$  that sends  $e_1 \wedge \dots \wedge e_d$  to  $(-1)^d m$  (the  $(-1)^d$  sign being needed to obtain relation (4.5.1) later). Note that this section is linear with respect to  $m$ , only depends on the class of  $m \bmod \mathcal{I}\mathcal{M}$ , and is functorial with respect to  $\mathcal{M}$ . Its dependence on the regular sequence  $\mathbf{t}$  is given by the following lemma.

LEMMA 4.2. *Let  $\mathbf{t}' = (t'_1, \dots, t'_d)$  be another regular sequence of sections of  $\mathcal{O}_P$ , generating an ideal  $\mathcal{I}'$  such that  $\mathcal{I}' \subset \mathcal{I}$ . Let  $C = (c_{i,j})_{1 \leq i,j \leq d}$  be a matrix with entries in  $\mathcal{O}_P$  such that  $t'_i = \sum_{j=1}^d c_{i,j} t_j$  for all  $i$ . If  $\alpha : \mathcal{E}xt_{\mathcal{O}_P}^d(\mathcal{O}_P/\mathcal{I}, \mathcal{M}) \rightarrow \mathcal{E}xt_{\mathcal{O}_P}^d(\mathcal{O}_P/\mathcal{I}', \mathcal{M})$  is the functoriality homomorphism, then*

$$(4.2.1) \quad \alpha \left( \begin{bmatrix} m \\ t_1, \dots, t_d \end{bmatrix} \right) = \begin{bmatrix} \det(C)m \\ t'_1, \dots, t'_d \end{bmatrix}.$$

*Proof.* Let  $K_\bullet(\mathbf{t}')$  be the Koszul resolution of  $\mathcal{O}_P/\mathcal{I}'$ , and  $\mathcal{E}' = K_1(\mathbf{t}')$ , with basis  $e'_1, \dots, e'_d$ . One defines a morphism of resolutions  $\phi : K_\bullet(\mathbf{t}') \rightarrow K_\bullet(\mathbf{t})$  by setting  $\phi_1(e'_i) = \sum_j c_{i,j} e_j$ , and  $\phi_k = \wedge^k \phi_1$  for  $0 \leq k \leq d$ . Then  $\phi$  provides a commutative diagram

$$\begin{array}{ccc} \mathcal{H}om_{\mathcal{O}_P}(\wedge^d \mathcal{E}, \mathcal{M}) & \longrightarrow & \mathcal{E}xt_{\mathcal{O}_P}^d(\mathcal{O}_P/\mathcal{I}, \mathcal{M}) \\ \phi_d = \det(C) \downarrow & & \downarrow \alpha \\ \mathcal{H}om_{\mathcal{O}_P}(\wedge^d \mathcal{E}', \mathcal{M}) & \longrightarrow & \mathcal{E}xt_{\mathcal{O}_P}^d(\mathcal{O}_P/\mathcal{I}', \mathcal{M}). \end{array}$$

The lemma follows. □

4.3. Under the assumptions of 4.1, the morphism  $d_1 : \mathcal{E} \twoheadrightarrow \mathcal{I}$  defines an isomorphism  $\mathcal{E}/\mathcal{I}\mathcal{E} \xrightarrow{\sim} \mathcal{I}/\mathcal{I}^2$ . Using the canonical isomorphisms, this provides

$$(4.3.1) \quad \begin{aligned} \frac{\mathcal{H}om_{\mathcal{O}_P}(\wedge^d \mathcal{E}, \mathcal{M})}{\mathcal{I}\mathcal{H}om_{\mathcal{O}_P}(\wedge^d \mathcal{E}, \mathcal{M})} &\xrightarrow{\sim} (\wedge^d \mathcal{E})^\vee / \mathcal{I}(\wedge^d \mathcal{E})^\vee \otimes_{\mathcal{O}_Y} \mathcal{M}/\mathcal{I}\mathcal{M} \\ &\xrightarrow{\sim} \wedge^d((\mathcal{E}/\mathcal{I}\mathcal{E})^\vee) \otimes_{\mathcal{O}_Y} \mathcal{M}/\mathcal{I}\mathcal{M} \\ &\xrightarrow{\sim} \omega_{Y/P} \otimes_{\mathcal{O}_Y} i^* \mathcal{M}. \end{aligned}$$

Note that, due to the commutation between dual and exterior power, the composition (4.3.1) maps the class of the homomorphism  $u_{\mathbf{t},m}$  used in the definition of (4.1.2) to  $(-1)^d (\bar{t}_d^\vee \wedge \dots \wedge \bar{t}_1^\vee) \otimes i^*(m)$ , where  $\bar{t}_k$  denotes the class of  $t_k \bmod \mathcal{I}^2$ .

Composing (4.1.1) and (4.3.1), one obtains the *fundamental local isomorphism* [Har66, III, 7.2] as defined by Conrad [Con00, (2.5.2)] in the local case:

$$(4.3.2) \quad \eta_{Y/P} : \mathcal{E}xt_{\mathcal{O}_P}^d(\mathcal{O}_Y, \mathcal{M}) \xrightarrow{\sim} \omega_{Y/P} \otimes_{\mathcal{O}_Y} i^* \mathcal{M}.$$

Applying Lemma 4.2 to the case of two regular sequences of generators of the ideal  $\mathcal{I}$ , one sees that the isomorphism  $\eta_{Y/P}$  does not depend on the sequence  $\mathbf{t}$ , so that local constructions can be glued to define  $\eta_{Y/P}$  for any regular immersion  $i : Y \hookrightarrow P$ , without assuming that  $\mathcal{I}$  is defined globally by a regular sequence. One obtains in this way the fundamental local isomorphism in the general case [Con00, (2.5.1)].

Let us now recall from [Con00, 2.5] how the isomorphism (4.3.2) allows us to define functorially for any  $\mathcal{M}^\bullet \in D(\mathcal{O}_P)$  the isomorphism [Con00, (2.5.3)]

$$(4.3.3) \quad \eta_i : R\mathcal{H}om_{\mathcal{O}_P}(\mathcal{O}_Y, \mathcal{M}^\bullet) \xrightarrow{\sim} \omega_{Y/P}[-d] \otimes_{\mathcal{O}_P}^L Li^*(\mathcal{M}^\bullet).$$

Applying [Con00, Lemma 2.1.1] and using the isomorphism of functors defined by  $\eta_{Y/P}$ , one gets the isomorphism

$$R\mathcal{H}om_{\mathcal{O}_P}(\mathcal{O}_Y, \mathcal{M}^\bullet) \xrightarrow{\sim} (\omega_{Y/P} \otimes_{\mathcal{O}_P}^L Li^*(\mathcal{M}^\bullet))[-d].$$

The isomorphism  $\eta_i$  is then obtained by composition with the canonical isomorphism

$$(\omega_{Y/P} \otimes_{\mathcal{O}_P}^L Li^*(\mathcal{M}^\bullet))[-d] \xrightarrow{\sim} \omega_{Y/P}[-d] \otimes_{\mathcal{O}_P}^L Li^*(\mathcal{M}^\bullet)$$

defined by the general convention [Con00, p. 11]. From the discussion on p. 53 of [Con00], it follows that  $\eta_i$  satisfies the following properties:

- (a) If  $\mathcal{M}^\bullet = \mathcal{M}[0]$  for an  $\mathcal{O}_P$ -module  $\mathcal{M}$ , then the homomorphism induced by  $\eta_i$  between the cohomology sheaves in degree  $d$  is the isomorphism  $\eta_{Y/P}$ .
- (b) The isomorphism  $\eta_i$  commutes with translations in  $D(\mathcal{O}_P)$  (using the general convention [Con00, (1.3.6)] for the right-hand side of (4.3.3)).

4.4. Let  $\pi : P \rightarrow X$  be a smooth morphism of relative dimension  $d$ ,  $i : Y \hookrightarrow P$  a regular immersion of codimension  $d$ , and  $f = \pi \circ i$ . Let

$$\zeta'_{i,\pi} : \omega_{Y/X} \xrightarrow{\sim} \omega_{Y/P}[-d] \otimes_{\mathcal{O}_Y}^L Li^*(\omega_{P/X}[d])$$

be the canonical isomorphism (A.2.6), which induces in degree 0 the tautological isomorphism  $\zeta'_{i,\pi}$  provided in (A.2.5) by the construction of  $\omega_{Y/X}$  in A.2. Let  $\delta_f$  be the canonical section of  $\omega_{Y/X}$  (defined by (A.7.2)) and  $\varphi_f : \mathcal{O}_Y \rightarrow \omega_{Y/X}$  the morphism sending 1 to  $\delta_f$ . We define the morphism

$$(4.4.1) \quad \gamma_f : \mathcal{O}_Y \rightarrow \omega_{P/X}[d]$$

as being the composition

$$\begin{array}{ccc}
\mathcal{O}_Y \xrightarrow{\varphi_f} \omega_{Y/X} \xrightarrow{\zeta'_{i,\pi}} \omega_{Y/P}[-d] \overset{L}{\otimes}_{\mathcal{O}_Y} Li^*(\omega_{P/X}[d]) \xrightarrow{\eta_i^{-1}} R\mathcal{H}om_{\mathcal{O}_P}(\mathcal{O}_Y, \omega_{P/X}[d]) \\
\searrow \gamma_f \qquad \downarrow \text{can} \\
\qquad \omega_{P/X}[d].
\end{array}$$

PROPOSITION 4.5. *Under the assumptions of 4.4, the isomorphism  $\eta_i^{-1} \circ \zeta'_{i,\pi}$  entering in the definition of  $\gamma_f$  induces in degree 0 an isomorphism*

$$\eta_i^{-1} \circ \zeta'_{i,\pi} : \omega_{Y/X} \xrightarrow{\sim} \mathcal{E}xt_{\mathcal{O}_P}^d(\mathcal{O}_Y, \omega_{P/X}),$$

which is such that

$$(4.5.1) \quad \eta_i^{-1} \circ \zeta'_{i,\pi}(\delta_f) = \begin{bmatrix} dt_1 \wedge \cdots \wedge dt_d \\ t_1, \dots, t_d \end{bmatrix}.$$

*Proof.* Let us consider first the isomorphism

$$\eta_{Y/P}^{-1} \circ \zeta'_{i,\pi} : \omega_{Y/X} \xrightarrow{\sim} \mathcal{E}xt_{\mathcal{O}_P}^d(\mathcal{O}_Y, \omega_{P/X}),$$

where  $\eta_{Y/P}$  is defined by (4.3.2). By definition,  $\begin{bmatrix} dt_1 \wedge \cdots \wedge dt_d \\ t_1, \dots, t_d \end{bmatrix}$  is mapped to  $u_{\mathbf{t}, dt_1 \wedge \cdots \wedge dt_d}$  by (4.1.1), and we observed in 4.3 that  $u_{\mathbf{t}, dt_1 \wedge \cdots \wedge dt_d}$  is mapped to  $(-1)^d (\bar{t}_d^\vee \wedge \cdots \wedge \bar{t}_1^\vee) \otimes i^*(dt_1 \wedge \cdots \wedge dt_d)$  by (4.3.1). Since  $\zeta'_{i,\pi}(\delta_f) = (\bar{t}_1^\vee \wedge \cdots \wedge \bar{t}_d^\vee) \otimes i^*(dt_d \wedge \cdots \wedge dt_1)$  by construction, we get the relation

$$(4.5.2) \quad \eta_{Y/P}^{-1} \circ \zeta'_{i,\pi}(\delta_f) = (-1)^d \begin{bmatrix} dt_1 \wedge \cdots \wedge dt_d \\ t_1, \dots, t_d \end{bmatrix}.$$

Let  $\eta_{i, \omega_{P/X}}$  and  $\eta_{i, \omega_{P/X}[d]}$  be the isomorphisms (4.3.3) relative to the complexes  $\omega_{P/X}[0]$  and  $\omega_{P/X}[d]$ . By 4.3(a),  $\eta_{i, \omega_{P/X}}$  induces in degree  $d$  the isomorphism  $\eta_{Y/P}$ . On the other hand, 4.3(b) shows that  $\eta_{i, \omega_{P/X}[d]}$  is identified with  $\eta_{i, \omega_{P/X}}[d]$  when using the canonical isomorphisms

$$R\mathcal{H}om(\mathcal{O}_Y, \omega_{P/X}[d]) \xrightarrow{\sim} R\mathcal{H}om(\mathcal{O}_Y, \omega_{P/X})[d]$$

and

$$\omega_{Y/P}[-d] \overset{L}{\otimes} Li^*(\omega_{P/X}[d]) \xrightarrow{\sim} (\omega_{Y/P}[-d] \overset{L}{\otimes} Li^*(\omega_{P/X}))[d].$$

The first one involves no sign and, as  $\omega_{Y/P}[-d]$  is concentrated in degree  $d$ , the second one is given by multiplication by  $(-1)^{d^2} = (-1)^d$  on  $\omega_{Y/P} \otimes i^*(\omega_{P/X})$ . Thus relation (4.5.2) implies relation (4.5.1).  $\square$

PROPOSITION 4.6. *Let  $X$  be a separated noetherian scheme with a dualizing complex,  $P = \mathbb{P}_X^d$  a projective space over  $X$ ,  $\pi : P \rightarrow X$  the structural morphism,  $i : Y \hookrightarrow P$  a regular immersion of codimension  $d$ , and  $f = \pi \circ i$ . Then the trace morphism  $\tau_f : Rf_* \mathcal{O}_Y \rightarrow \mathcal{O}_X$  of Theorem 3.1 is equal to the composition*

$$(4.6.1) \quad Rf_*(\mathcal{O}_Y) \xrightarrow{R\pi_*(\gamma_f)} R\pi_*(\omega_{P/X}[d]) \xrightarrow{\text{Trp}_\pi} \mathcal{O}_X,$$

where  $\text{Trp}_\pi$  is the trace morphism for the projective space defined in [Con00, (2.3.1)–(2.3.5)].

*Proof.* By construction (see B.7),  $\tau_f$  is the composition  $\text{Tr}_f \circ Rf_*(\lambda_f) \circ Rf_*(\varphi_f)$  in the commutative diagram

$$\begin{array}{ccccc}
 Rf_*(\mathcal{O}_Y) & & & & \\
 \downarrow Rf_*(\varphi_f) & & & & \\
 Rf_*(\omega_{Y/X}) & \xrightarrow[\sim]{Rf_*(\lambda_f)} & Rf_*(f^!(\mathcal{O}_X)) & \xrightarrow{\text{Tr}_f} & \mathcal{O}_X \\
 \downarrow Rf_*(\zeta'_{i,\pi}) \wr & & \uparrow Rf_*(c_{i,\pi}^{-1}) \wr & & \uparrow \text{Tr}_\pi \\
 Rf_*(\omega_{Y/P}[-d] \otimes^L Li^*(\omega_{P/X}[d])) & & Rf_*(i^!\pi^!(\mathcal{O}_X)) & \xrightarrow{R\pi_*(\text{Tr}_i)} & R\pi_*(\pi^!(\mathcal{O}_X)) \\
 \downarrow Rf_*(\eta_i^{-1}) \wr & & \uparrow Rf_*(i^!(e_\pi)) \wr & & \uparrow R\pi_*(e_\pi) \wr \\
 Rf_*(R\mathcal{H}om_{\mathcal{O}_P}(\mathcal{O}_Y, \omega_{P/X}[d])) & \xrightarrow[\sim]{Rf_*(d_i)} & Rf_*(i^!(\omega_{P/X}[d])) & \xrightarrow{R\pi_*(\text{Tr}_i)} & R\pi_*(\omega_{P/X}[d]),
 \end{array}$$

in which the isomorphism  $\lambda_f$  is defined by the commutativity of the left rectangle before applying  $Rf_*$  (cf. B.1) and  $d_i$ ,  $e_\pi$ ,  $c_{i,\pi}$  are defined as follows:

(a)  $d_i$  is the canonical isomorphism of functors  $i^b := R\mathcal{H}om_{\mathcal{O}_P}(\mathcal{O}_Y, -) \xrightarrow{\sim} i^!$ , defined by [Con00, (3.3.19)];

(b)  $e_\pi$  is the canonical isomorphism of functors  $\pi^\sharp := \omega_{P/X}[d] \otimes^L \pi^*(-) \xrightarrow{\sim} \pi^!$ , defined by [Con00, (3.3.21)].

(c)  $c_{i,\pi}$  is the transitivity isomorphism  $f^! \xrightarrow{\sim} i^!\pi^!$ , defined by [Con00, (3.3.14)].

Moreover, the upper right square commutes because of the transitivity of the trace morphism [Con00, 3.4.3, (TRA1)], and the lower right square commutes by functoriality of the trace morphism  $\text{Tr}_i$  with respect to  $e_\pi$ .

In this diagram, the composition of the right vertical arrows is the projective trace morphism  $\text{Tr}_\pi$  [Con00, 3.4.3, (TRA3)] and the isomorphism  $d_i$  on the bottom row identifies  $\text{Tr}_i$  with the trace morphism  $\text{Tr}_i$  for finite morphisms [Con00, 3.4.3, (TRA2)]. As the latter is the canonical morphism  $i_*R\mathcal{H}om_{\mathcal{O}_P}(\mathcal{O}_Y, -) \rightarrow \text{Id}$  defined by  $\mathcal{O}_P \rightarrow \mathcal{O}_Y$ , it follows that the composition of the left column and the bottom row of the diagram is equal to  $R\pi_*(\gamma_f)$ , which proves the proposition.  $\square$

## 5. Preliminaries on the relative de Rham-Witt complex

We extend here to the relative de Rham-Witt complex constructed by Langer and Zink [LZ04] structure theorems that are classical when the base is a perfect scheme of characteristic  $p$  ([Ill79], [IR83]). We begin by recalling some basic facts from their construction.

From now on, we fix a prime number  $p$ . We denote by  $\mathbb{Z}_{(p)}$  the localization of  $\mathbb{Z}$  at the prime ideal  $(p)$ . Although many results of [LZ04] are valid for  $\mathbb{Z}_{(p)}$ -schemes, we limit our exposition to the case of schemes on which  $p$  is locally nilpotent, which will suffice for our applications.

5.1. Let  $S$  be a scheme on which  $p$  is locally nilpotent, and let  $f : X \rightarrow S$  be a morphism of schemes. An  $F$ - $V$ -pro-complex of  $X/S$ , as defined in [LZ04], is a pro-complex  $\{R : E_{n+1}^\bullet \rightarrow E_n^\bullet\}_{n \geq 1}$  of sheaves on  $X$ , where  $E_n^\bullet$  is a differential graded  $W_n(\mathcal{O}_X)/f^{-1}W_n(\mathcal{O}_S)$ -algebra (i.e.,  $E_n^\bullet$  is a graded  $W_n(\mathcal{O}_X)$ -algebra together with an  $f^{-1}W_n(\mathcal{O}_S)$ -linear map  $d : E_n^\bullet \rightarrow E_n^\bullet(1)$  such that  $d^2 = 0$ , satisfying  $\eta\omega = (-1)^{\deg \omega \deg \eta} \omega\eta$  and  $d(\omega\eta) = (d\omega)\eta + (-1)^{\deg \omega} \omega d\eta$  for any homogeneous sections  $\omega, \eta \in E_n^\bullet$ ), which is equipped with a map of graded pro-rings

$$F : E_{\bullet+1}^\bullet \rightarrow E_\bullet^\bullet,$$

called the Frobenius morphism, and with a map of graded abelian groups

$$V : E_\bullet^\bullet \rightarrow E_{\bullet+1}^\bullet,$$

called the Verschiebung morphism, such that the following properties hold:

- (i) The structure map  $W_\bullet(\mathcal{O}_X) \rightarrow E_\bullet^0$  is compatible with  $F$  and  $V$ .
- (ii) The following relations hold:

$$(5.1.1) \quad FV = p, \quad FdV = d,$$

$$(5.1.2) \quad V(\omega F(\eta)) = V(\omega)\eta \quad \text{for all } \omega \in E_n^\bullet, \eta \in E_{n+1}^\bullet, n \geq 1,$$

$$(5.1.3) \quad F(d[a]) = [a]^{p-1}d[a] \quad \text{for all } a \in \mathcal{O}_X,$$

where  $[a]$  denotes the Teichmüller lift of  $a$  to  $W_n(\mathcal{O}_X)$  for any  $n$ .

A morphism between two  $F$ - $V$ -pro-complexes of  $X/S$  is a map of pro-differential graded  $W_\bullet(\mathcal{O}_X)/f^{-1}W_\bullet(\mathcal{O}_S)$ -algebras compatible with  $F$  and  $V$ . By [LZ04, Prop. 1.6, Rem. 1.10], there exists an initial object in the category of  $F$ - $V$ -pro-complexes of  $X/S$ , which is called the *relative de Rham-Witt complex of  $X/S$*  and is denoted by  $\{R : W_{n+1}\Omega_{X/S}^\bullet \rightarrow W_n\Omega_{X/S}^\bullet\}_{n \geq 1}$ . Each sheaf  $W_n\Omega_{X/S}^q$  is a quasi-coherent sheaf on the scheme  $W_n(X) := (|X|, W_n(\mathcal{O}_X))$  defined in 2.9, and the transition morphisms  $R$  are epimorphisms. When  $S$  is a perfect scheme of characteristic  $p$ , the relative de Rham-Witt complex coincides with the one defined in [Ill79]. Notice that we have the following properties:

$$W_n\Omega_{X/S}^0 = W_n(\mathcal{O}_X), \quad W_1\Omega_{X/S}^\bullet = \Omega_{X/S}^\bullet,$$

and note that, by [LZ04, (1.16), (1.17) and (1.19)], relations (5.1.1) and (5.1.2) imply that

$$(5.1.4) \quad V(\omega d\eta) = V(\omega)dV(\eta) \quad \text{for all } \omega, \eta \in W_n\Omega_{X/S}^\bullet, n \geq 1,$$

$$(5.1.5) \quad Vd = p dV, \quad dF = p Fd.$$

In addition, when  $S$  is an  $\mathbb{F}_p$ -scheme, the operators  $F$  and  $V$  satisfy the relation  $VF = p$ .

We also recall the behaviour of the de Rham-Witt complex with respect to étale pull-backs. Let

$$\begin{array}{ccc} X' & \xrightarrow{h} & X \\ \downarrow & & \downarrow \\ S' & \xrightarrow{g} & S \end{array}$$

be a commutative diagram in which  $h$  is étale and  $g$  unramified. Then, for all  $q \geq 0$  and  $r \geq n \geq 1$ ,  $W_n(X')$  is étale over  $W_n(X)$  and we have the  $W_r(\mathcal{O}_{X'})$ -linear isomorphisms

$$(5.1.6) \quad W_r(\mathcal{O}_{X'}) \otimes_{h^{-1}W_r(\mathcal{O}_X)} h^{-1}W_n\Omega_{X/S}^q \xrightarrow{\sim} W_n\Omega_{X'/S'}^q,$$

$$(5.1.7) \quad W_r(\mathcal{O}_{X'}) \otimes_{h^{-1}W_r(\mathcal{O}_X)} h^{-1}(F_*^{r-n}W_n\Omega_{X/S}^q) \xrightarrow{\sim} F_*^{r-n}W_n\Omega_{X'/S'}^q, \quad a \otimes \omega \mapsto F^{r-n}(a)\omega,$$

where, for any  $W_n$ -module  $M$ ,  $F_*^{r-n}M$  denotes  $M$  viewed as a  $W_r$ -module via  $F^{r-n} : W_r \rightarrow W_n$  [LZ04, Props. 1.11, A.8 and Cor. A.11].

Finally, the completed relative de Rham-Witt complex is defined by  $W\Omega_{X/S}^\bullet := \varprojlim_n W_n\Omega_{X/S}^\bullet$ ; the canonical morphisms  $W\Omega_{X/S}^\bullet \rightarrow W_n\Omega_{X/S}^\bullet$  are still epimorphisms.

5.2. Let  $S = \text{Spec } A$  be affine. We want to recall the calculation of  $W\Omega_{A[x_1, \dots, x_d]/A}^q := \Gamma(\mathbb{A}_S^d, W\Omega_{\mathbb{A}_S^d/S}^q)$ . We need some notation for this.

A *weight* is a function  $k : [1, d] = \{1, 2, \dots, d\} \rightarrow \mathbb{Z}[\frac{1}{p}]_{\geq 0}$ . We write  $k_i := k(i)$  for  $i \in [1, d]$ . The support of  $k$ ,  $\text{supp } k$ , consists of those  $i \in [1, d]$  with  $k_i \neq 0$ . For any weight  $k$ , we choose once and for all a total ordering on the elements of the support of  $k$ ,

$$(5.2.1) \quad \text{supp } k = \{i_1, \dots, i_r\},$$

such that

- (i)  $\text{ord}_p k_{i_1} \leq \text{ord}_p k_{i_2} \leq \dots \leq \text{ord}_p k_{i_r}$ .
- (ii) The ordering on  $\text{supp } k$  and on  $\text{supp } p^a k$  agree for any  $a \in \mathbb{Z}$ .

We say  $k$  is *integral* if  $k_i \in \mathbb{Z}$  for all  $i \in [1, d]$ . We say  $k$  is *primitive* if it is integral and not all  $k_i$  are divisible by  $p$ . We set

$$(5.2.2) \quad t(k_i) := -\text{ord}_p k_i \quad \text{and} \quad t(k) := \begin{cases} \max \{ t(k_i) \mid i \in \text{supp } k \} & \text{if } \text{supp } k \neq \emptyset, \\ 0 & \text{if } k = 0. \end{cases}$$

If  $k \neq 0$ , then  $t(k)$  is the smallest integer such that  $p^{t(k)}k$  is primitive, and we have

$$t(k) = t(k_{i_1}) \geq t(k_{i_2}) \geq \dots \geq t(k_{i_r}).$$

We denote by  $u(k)$  the smallest nonnegative integer such that  $p^{u(k)}k$  is integral, i.e.,  $u(k) = \max\{0, t(k)\}$ . Notice that  $k$  is integral if and only if  $u(k) = 0$  if and only if  $t(k) \leq 0$ , and  $k$  is primitive if and only if  $t(k) = 0$ . An *interval of the support of  $k$*  is by definition a subset  $I \subset \text{supp } k$  of the form

$$I = \{i_s, i_{s+1}, \dots, i_{s+m}\}.$$

We denote by  $k_I$  the weight that equals  $k$  on  $I$  and is zero on  $[1, d] \setminus I$ . If  $k$  is fixed and  $I$  is an interval of the support of  $k$ , we write  $u(I) := u(k_I)$  and  $t(I) := t(k_I)$ . An *admissible partition  $\mathcal{P}$  of length  $q$  of  $\text{supp } k$*  (or just of  $k$ ) is a tuple of intervals of  $\text{supp } k$ ,  $\mathcal{P} = (I_0, I_1, \dots, I_q)$  such that

- (i)  $\text{supp } k = I_0 \sqcup I_1 \sqcup \dots \sqcup I_q$ .
- (ii) The elements in  $I_j$  are smaller than the elements in  $I_{j+1}$  (with respect to the ordering (5.2.1)) for all  $j = 0, \dots, q - 1$ .
- (iii) The intervals  $I_1, \dots, I_q$  are nonempty (but  $I_0$  may be).

Notice that  $u(k) = u(I_0)$  if  $I_0 \neq \emptyset$  and  $u(k) = u(I_1)$  if  $I_0 = \emptyset$ .

For any  $n \leq \infty$ , we write  $X_i := [x_i] \in W_n(A[x_1, \dots, x_d])$ . If  $k$  is an integral weight as above, we write  $X^k = X_{i_1}^{k_{i_1}} \dots X_{i_r}^{k_{i_r}} \in W_n(A[x_1, \dots, x_d])$ .

Let  $k$  be any weight and  $\eta \in W(A)$ . We define

$$(5.2.3) \quad e^0(\eta, k) := V^{u(k)}(\eta X^{p^{u(k)}k}) \in W(A[x_1, \dots, x_d])$$

and

$$(5.2.4) \quad e^1(\eta, k) := \begin{cases} dV^{u(k)}(\eta X^{p^{u(k)}k}) & \text{if } k \text{ is not integral} \\ \eta F^{-t(k)} dX^{p^{t(k)}k} & \text{if } k \text{ is integral} \end{cases} \in W\Omega_{A[x_1, \dots, x_d]/A}^1.$$

*Definition 5.3* (Basic Witt differentials [LZ04, 2.2]). Let  $k$  be a weight,  $\mathcal{P} = (I_0, I_1, \dots, I_q)$  an admissible partition of  $k$ , and  $\xi = V^{u(k)}(\eta) \in W(A)$ . The basic Witt differential  $e(\xi, k, \mathcal{P}) \in W\Omega_{A[x_1, \dots, x_d]/A}^q$  is defined as follows:

$$e(\xi, k, \mathcal{P}) := \begin{cases} e^0(\eta, k_{I_0})e^1(1, k_{I_1}) \cdots e^1(1, k_{I_q}) & \text{if } I_0 \neq \emptyset, \\ e^1(\eta, k_{I_1})e^1(1, k_{I_2}) \cdots e^1(1, k_{I_q}) & \text{if } I_0 = \emptyset. \end{cases}$$

*Rules 5.4* ([LZ04, Props. 2.5, 2.6]). Let  $k$  be a weight,  $\mathcal{P} = (I_0, I_1, \dots, I_q)$  a partition of  $k$  and  $\xi = V^{u(k)}(\eta) \in W(A)$ . Note that  $u(k) \geq 1$  when  $k$  is not integral, so that one can then define  $V^{-1}\xi := V^{u(k)-1}\eta$ . Then

- (i)  $\rho e(\xi, k, \mathcal{P}) = e(\rho\xi, k, \mathcal{P})$  for all  $\rho \in W(A)$ .
- (ii)  $F e(\xi, k, \mathcal{P}) = \begin{cases} e(F\xi, pk, \mathcal{P}) & \text{if } I_0 \neq \emptyset \text{ or } k \text{ integral,} \\ e(V^{-1}\xi, pk, \mathcal{P}) & \text{if } I_0 = \emptyset \text{ and } k \text{ not integral.} \end{cases}$



$$\begin{aligned}
 \text{(iii)} \quad Ve(\xi, k, \mathcal{P}) &= \begin{cases} e(V\xi, \frac{1}{p}k, \mathcal{P}) & \text{if } I_0 \neq \emptyset \text{ or } \frac{1}{p}k \text{ integral,} \\ e(pV\xi, \frac{1}{p}k, \mathcal{P}) & \text{if } I_0 = \emptyset \text{ and } \frac{1}{p}k \text{ not integral.} \end{cases} \\
 \text{(iv)} \quad de(\xi, k, \mathcal{P}) &= \begin{cases} 0 & \text{if } I_0 = \emptyset, \\ e(\xi, k, (\emptyset, \mathcal{P})) & \text{if } I_0 \neq \emptyset \text{ and } k \text{ not integral,} \\ p^{-t(k)}e(\xi, k, (\emptyset, \mathcal{P})) & \text{if } I_0 \neq \emptyset \text{ and } k \text{ integral.} \end{cases}
 \end{aligned}$$

THEOREM 5.5 ([LZ04, Th. 2.8]). *Every  $\omega \in W\Omega_{A[x_1, \dots, x_d]/A}^q$  can uniquely be written as*

$$\omega = \sum_{k, \mathcal{P}} e(\xi_{k, \mathcal{P}}, k, \mathcal{P}),$$

where the sum is over all weights  $k$  with  $|\text{supp } k| \geq q$  and over all admissible partitions of length  $q$  of  $k$ , and the sum converges in the sense that, for any  $m \geq 0$ , we have  $\xi_{k, \mathcal{P}} \in V^m W(A)$  for all but finitely many  $\xi_{k, \mathcal{P}}$ .

For a weight  $k$ ,  $n \geq 1$  and  $\eta \in W_{n-u(k)}(A)$ , we define

$$e_n^0(\eta, k) \in W_n(A[x_1, \dots, x_d]), \quad e_n^1(\eta, k) \in W_n\Omega_{A[x_1, \dots, x_d]/A}^1$$

by the same formulas as in (5.2.3) and (5.2.4). For  $\mathcal{P}$  an admissible partition of length  $q$  of  $k$  and  $\xi = V^{u(k)}(\eta) \in W_n(A)$ , we then define  $e_n(\xi, k, \mathcal{P}) \in W_n\Omega_{A[x_1, \dots, x_d]/A}^q$  by the same formula as in Definition 5.3 but with  $e^i$  replaced by  $e_n^i$ ,  $i = 0, 1$ .

COROLLARY 5.6 ([LZ04, Prop. 2.17]). *Every  $\omega \in W_n\Omega_{A[x_1, \dots, x_d]/A}^q$  may uniquely be written as a finite sum*

$$\omega = \sum_{k, \mathcal{P}} e_n(\xi_{k, \mathcal{P}}, k, \mathcal{P}), \quad \xi_{k, \mathcal{P}} \in V^{u(k)}W_{n-u(k)}(A),$$

where the sum is over all weights  $k$  with  $|\text{supp } k| \geq q$  and such that  $p^{n-1}k$  is integral and over all admissible partitions  $\mathcal{P}$  of  $k$  of length  $q$ .

We now assume that  $S$  is an  $\mathbb{F}_p$ -scheme; the absolute Frobenius endomorphism of  $S$  will then be denoted  $F_S$ , or  $F$  when no confusion can arise. The following proposition is known if  $S$  is perfect (see [IR83, II, (1.2.2)]); a similar result has been proved by M. Olsson when, étale locally,  $S$  has a flat lifting over  $\mathbb{Z}_p$  to which  $F_S$  can be lifted [Ols07, Th. 4.2.15].

PROPOSITION 5.7. *Let  $S$  be a locally noetherian  $\mathbb{F}_p$ -scheme and  $X$  a smooth  $S$ -scheme. Then the sequence*

$$\begin{aligned}
 F_*^n \mathcal{O}_S \otimes_{W_{n+1}(\mathcal{O}_S)} W_{n+1}\Omega_{X/S}^{q-1} &\xrightarrow{(1 \otimes F^n, -1 \otimes F^n d)} F_*^n \Omega_{X/S}^{q-1} \oplus F_*^n \Omega_{X/S}^q \\
 &\xrightarrow{dV^n + V^n} W_{n+1}\Omega_{X/S}^q \longrightarrow R_* W_n \Omega_{X/S}^q \longrightarrow 0
 \end{aligned}$$

is an exact sequence of  $W_{n+1}(\mathcal{O}_S)$ -modules.

*Proof.* The question is local; we thus assume  $S = \text{Spec } A$ ,  $X = \text{Spec } B$  and  $B$  is étale over  $B_1 = A[x_1, \dots, x_d]$ . As  $W\Omega_{X/S}^\bullet \rightarrow W_{n+1}\Omega_{X/S}^\bullet$  is an epimorphism, [LZ04, Prop. 2.19] provides the exactness of the second line, and we only have to show that

$$(*_{B/A}) : F_*^n A \otimes W_{n+1}\Omega_{B/A}^{q-1} \xrightarrow{(1 \otimes F^n, -1 \otimes F^n d)} F_*^n \Omega_{B/A}^{q-1} \oplus F_*^n \Omega_{B/A}^q \xrightarrow{dV^n + V^n} W_{n+1}\Omega_{B/A}^q$$

is exact. Notice that it is a complex, as for  $a \in A$  and  $\omega \in W_{n+1}\Omega_{B/A}^{q-1}$ , we have

$$dV^n(aF^n\omega) - V^n(aF^n d\omega) = 0.$$

Notice also that if we let  $W_{2n+2}(B)$  act through  $F^{n+1}: W_{2n+2}(B) \rightarrow W_{n+1}(B)$ , the differentials of this complex are  $W_{2n+2}(B)$ -linear, since  $dF^{n+1} = p^{n+1}F^{n+1}d = 0$  in  $W_{n+1}$ . We claim that

$$(5.7.1) \quad (*_{B/A}) = F_*^{n+1}(*_{B_1/A}) \otimes_{W_{2n+2}(B_1)} W_{2n+2}(B).$$

We have the following diagrams (where the tensor products with  $W_{2n+2}(B)$  are taken over  $W_{2n+2}(B_1)$ ):

$$\begin{array}{ccc} F_*^{n+1}(F_*^n A \otimes W_{n+1}\Omega_{B/A}^{q-1}) & \xrightarrow{1 \otimes F^n} & F_*^{2n+1}\Omega_{B/A}^{q-1} \\ \uparrow & & \uparrow \\ F_*^{n+1}(F_*^n A \otimes W_{n+1}\Omega_{B_1/A}^{q-1}) \otimes W_{2n+2}(B) & \xrightarrow{(1 \otimes F^n) \otimes 1} & F_*^{n+1}(F_*^n \Omega_{B_1/A}^{q-1}) \otimes W_{2n+2}(B), \\ F_*^{n+1}(F_*^n A \otimes W_{n+1}\Omega_{B/A}^{q-1}) & \xrightarrow{-1 \otimes F^n d} & F_*^{2n+1}\Omega_{B/A}^q \\ \uparrow & & \uparrow \\ F_*^{n+1}(F_*^n A \otimes W_{n+1}\Omega_{B_1/A}^{q-1}) \otimes W_{2n+2}(B) & \xrightarrow{(-1 \otimes F^n d) \otimes 1} & F_*^{n+1}(F_*^n \Omega_{B_1/A}^q) \otimes W_{2n+2}(B), \end{array}$$

both with vertical maps

$$(a \otimes \omega) \otimes b \mapsto a \otimes F^{n+1}(b)\omega, \quad \eta \otimes b \mapsto F^{2n+1}(b)\eta,$$

and

$$\begin{array}{ccc} F_*^{2n+1}\Omega_{B/A}^{q-1} \oplus F_*^{2n+1}\Omega_{B/A}^q & \xrightarrow{dV^n + V^n} & F_*^{n+1}W_{n+1}\Omega_{B/A}^q \\ \uparrow & & \uparrow \\ F_*^{n+1}(F_*^n \Omega_{B_1/A}^{q-1} \oplus F_*^n \Omega_{B_1/A}^q) \otimes W_{2n+2}(B) & \xrightarrow{(dV^n + V^n) \otimes 1} & F_*^{n+1}W_{n+1}\Omega_{B_1/A}^q \otimes W_{2n+2}(B), \end{array}$$

with vertical maps

$$(\eta, \omega) \otimes b \mapsto (F^{2n+1}(b)\eta, F^{2n+1}(b)\omega), \quad \omega \otimes b \mapsto F^{n+1}(b)\omega.$$

Using again the relation  $dF^{n+1} = p^{n+1}F^{n+1}d = 0$  in  $W_{n+1}$ , one checks immediately that all three diagrams commute. Now the claim (5.7.1) follows, since the

vertical maps are isomorphisms by (5.1.7). As  $W_{2n+2}(B_1) \rightarrow W_{2n+2}(B)$  is étale [LZ04, Prop. A.8], we are thus reduced to the case  $B = B_1 = A[x_1, \dots, x_d]$ .

Now take  $\alpha \in \Omega_{B/A}^q$  and  $\beta \in \Omega_{B/A}^{q-1}$  with  $V^n(\alpha) = -dV^n(\beta)$ . We have to show that there exists an element  $\gamma \in F_*^n A \otimes W_{n+1}\Omega_{B/A}^{q-1}$  with

$$(5.7.2) \quad -(1 \otimes F^n d)(\gamma) = \alpha \quad \text{and} \quad (1 \otimes F^n)(\gamma) = \beta.$$

By Corollary 5.6 (and keeping the notation used there), we can write  $\alpha$  and  $\beta$  uniquely as finite sums

$$(5.7.3) \quad \alpha = \sum_{k, \mathcal{P}} e_1(\xi_{k, \mathcal{P}}, k, \mathcal{P}), \quad \beta = \sum_{k, \mathcal{Q}} e_1(\eta_{k, \mathcal{Q}}, k, \mathcal{Q}), \quad \text{with } \xi_{k, \mathcal{P}}, \eta_{k, \mathcal{Q}} \in A,$$

where the sums are over all integral weights  $k$  and all admissible partitions  $\mathcal{P} = (I_0, \dots, I_q)$  of length  $q$  (resp. over all admissible partitions  $\mathcal{Q} = (J_0, \dots, J_{q-1})$  of length  $q - 1$ ). Using the rules 5.4(iii) and (iv), we obtain

$$(5.7.4) \quad \begin{aligned} V^n(\alpha) &= \sum_{i=0}^{n-1} \sum_{\substack{\frac{k}{p^i} \text{ primitive} \\ \text{and } I_0 = \emptyset}} e_{n+1}(p^{n-i} V^n(\xi_{k, \mathcal{P}}), \frac{k}{p^n}, \mathcal{P}) \\ &+ \sum_{\substack{\frac{k}{p^n} \text{ integral} \\ \text{or } I_0 \neq \emptyset}} e_{n+1}(V^n(\xi_{k, \mathcal{P}}), \frac{k}{p^n}, \mathcal{P}) \end{aligned}$$

and

$$(5.7.5) \quad \begin{aligned} -dV^n(\beta) &= \sum_{\substack{\frac{k}{p^n} \text{ integral} \\ \text{and } J_0 \neq \emptyset}} -p^{t(\frac{k}{p^n})} e_{n+1}(V^n(\eta_{k, \mathcal{Q}}), \frac{k}{p^n}, (\emptyset, \mathcal{Q})) \\ &+ \sum_{\substack{\frac{k}{p^n} \text{ not integral} \\ \text{and } J_0 \neq \emptyset}} -e_{n+1}(V^n(\eta_{k, \mathcal{Q}}), \frac{k}{p^n}, (\emptyset, \mathcal{Q})), \end{aligned}$$

where  $t(k/p^n)$  is defined as in (5.2.2). By the uniqueness of this presentation, and since  $V^n : A \rightarrow W_{n+1}(A)$  is injective, the equality  $V^n(\alpha) = -dV^n(\beta)$  thus gives the following set of equations:

$$(5.7.6) \quad \begin{aligned} \xi_{k, \mathcal{P}} &= -p^{-t(\frac{k}{p^n})} \eta_{k, \mathcal{Q}} && \text{if } \frac{k}{p^n} \text{ is integral, } \mathcal{P} = (\emptyset, \mathcal{Q}) \text{ and } J_0 \neq \emptyset, \\ \eta_{k, \mathcal{Q}} &= -p^{n-i} \xi_{k, \mathcal{P}} && \text{if } \frac{k}{p^i} \text{ is primitive, } \mathcal{P} = (\emptyset, \mathcal{Q}), J_0 \neq \emptyset \text{ and } 0 \leq i \leq n-1, \\ \xi_{k, \mathcal{P}} &= 0 && \text{if } I_0 \neq \emptyset. \end{aligned}$$

We claim that (5.7.2) holds for the following choice of  $\gamma \in F_*^n A \otimes_{W_{n+1}(A)} W_{n+1} \Omega_{B/A}^{q-1}$ :

$$\gamma := \sum_{i=0}^{n-1} \left( \sum_{\substack{\frac{k}{p^i} \text{ primitive} \\ \text{and } J_0 \neq \emptyset}} (-\xi_{k,(\emptyset, \mathcal{Q})} \otimes e_{n+1}(V^{n-i}(1), \frac{k}{p^n}, \mathcal{Q})) \right. \\ \left. + \sum_{\substack{\frac{k}{p^i} \text{ primitive} \\ \text{and } J_0 = \emptyset}} (\eta_{k, \mathcal{Q}} \otimes e_{n+1}(V^{n-i}(1), \frac{k}{p^n}, \mathcal{Q})) \right) \\ + \sum_{\substack{\frac{k}{p^n} \text{ integral, } \mathcal{Q}}} \eta_{k, \mathcal{Q}} \otimes e_{n+1}(1, \frac{k}{p^n}, \mathcal{Q}).$$

Indeed the rules 5.4(ii) and (iv) yield the following formulas for  $k$  an integral weight,  $\xi \in V^{u(\frac{k}{p^n})} W_{n+1-u(\frac{k}{p^n})}(A)$  and  $\mathcal{Q} = (J_0, \dots, J_{q-1})$  a partition of length  $q-1$  of  $\text{supp } k$ :

$$F^n e_{n+1}(\xi, \frac{k}{p^n}, \mathcal{Q}) = \begin{cases} e_1(F^n(\xi), k, \mathcal{Q}) & \text{if } J_0 \neq \emptyset \text{ or } \frac{k}{p^n} \text{ integral,} \\ e_1(F^i V^{-(n-i)}(\xi), k, \mathcal{Q}) & \text{if } J_0 = \emptyset \text{ and } \frac{k}{p^i} \text{ is primitive} \\ & \text{for } 0 \leq i \leq n-1 \end{cases}$$

and

$$F^n d e_{n+1}(\xi, \frac{k}{p^n}, \mathcal{Q}) = \begin{cases} 0 & \text{if } J_0 = \emptyset, \\ p^{-t(\frac{k}{p^n})} e_1(F^n(\xi), k, (\emptyset, \mathcal{Q})) & \text{if } J_0 \neq \emptyset \text{ and } \frac{k}{p^n} \text{ integral,} \\ e_1(F^i V^{-(n-i)}(\xi), k, (\emptyset, \mathcal{Q})) & \text{if } J_0 \neq \emptyset \text{ and } \frac{k}{p^i} \text{ is primitive} \\ & \text{for } 0 \leq i \leq n-1. \end{cases}$$

Using this, rule 5.4(i), and relations (5.7.6), we obtain

$$\begin{aligned} (-1 \otimes F^n d)(\gamma) &= \sum_{i=0}^{n-1} \sum_{\substack{\frac{k}{p^i} \text{ primitive} \\ \text{and } J_0 \neq \emptyset}} e_1(\xi_{k,(\emptyset, \mathcal{Q})}, k, (\emptyset, \mathcal{Q})) \\ &\quad + \sum_{\substack{\frac{k}{p^n} \text{ integral} \\ \text{and } J_0 \neq \emptyset}} e_1(-p^{-t(\frac{k}{p^n})} \eta_{k, \mathcal{Q}}, k, (\emptyset, \mathcal{Q})) \\ &= \alpha \end{aligned}$$

and

$$\begin{aligned}
 (1 \otimes F^n)(\gamma) &= \sum_{i=0}^{n-1} \left( \sum_{\substack{\frac{k}{p^i} \text{ primitive} \\ \text{and } J_0 \neq \emptyset}} e_1(-p^{n-i}\xi_{k,(\emptyset, \mathcal{Q})}, k, \mathcal{Q}) + \sum_{\substack{\frac{k}{p^i} \text{ primitive} \\ \text{and } J_0 = \emptyset}} e_1(\eta_{k, \mathcal{Q}}, k, \mathcal{Q}) \right) \\
 &+ \sum_{\substack{\frac{k}{p^n} \text{ integral, } \mathcal{Q}}} e_1(\eta_{k, \mathcal{Q}}, k, \mathcal{Q}) \\
 &= \beta.
 \end{aligned}$$

This proves the proposition.  $\square$

5.8. We now recall some facts from [Ill79, 0, 2] about the Cartier operator and its iterates. Let  $S$  be an  $\mathbb{F}_p$ -scheme,  $X \rightarrow S$  a smooth morphism, and set  $X^{(p^n)} := S \times_{S, F_S^n} X$ . We have the usual diagram that defines the iterates  $F_{X/S}^n$  of the relative Frobenius morphism (we write  $F_{X/S} = F_{X/S}^1$ ,  $W = W^1$ ):

$$\begin{array}{ccccc}
 & & F_X^n & & \\
 & & \curvearrowright & & \\
 & & X & \xrightarrow{F_{X/S}^n} & X^{(p^n)} & \xrightarrow{W^n} & X \\
 & & \searrow & & \downarrow & & \downarrow \\
 & & & & S & \xrightarrow{F_S^n} & S.
 \end{array}$$

Notice that

$$F_{X/S}^n = F_{X^{(p^{n-1})}/S} \circ \cdots \circ F_{X/S}.$$

For an  $S$ -morphism  $f : X' \rightarrow X$ , we denote by  $f^{(p^n)}$  the base-change morphism  $f^{(p^n)} = \text{Id}_S \times f : X'^{(p^n)} \rightarrow X^{(p^n)}$ .

The inverse Cartier operator is an isomorphism of graded  $\mathcal{O}_{X^{(p)}}$ -algebras

$$C_{X/S}^{-1} : \Omega_{X^{(p)}/S}^\bullet \rightarrow \mathcal{H}^\bullet(\Omega_{X/S}^\bullet)$$

that is uniquely determined by

$$(5.8.1) \quad C_{X/S}^{-1}|_{\mathcal{O}_{X^{(p)}}} = F_{X/S}^* \quad \text{and} \quad C_{X/S}^{-1}(W^* dx) = x^{p-1} dx \quad \text{for all } x \in \mathcal{O}_X.$$

For  $n \geq 0$ , one defines abelian subsheaves of  $\Omega_{X/S}^q$ :

$$(5.8.2) \quad B_n \Omega_{X/S}^q \subset Z_n \Omega_{X/S}^q \subset \Omega_{X/S}^q$$

via

$$\begin{aligned}
 B_0 \Omega_{X/S}^q &= 0, & Z_0 \Omega_{X/S}^q &= \Omega_{X/S}^q, \\
 B_1 \Omega_{X/S}^q &= B \Omega_{X/S}^q = d \Omega_{X/S}^{q-1}, & Z_1 \Omega_{X/S}^q &= Z \Omega_{X/S}^q = \text{Ker}(d : \Omega_{X/S}^q \rightarrow \Omega_{X/S}^{q+1})
 \end{aligned}$$

and, for  $n \geq 1$ ,

$$(5.8.3) \quad C_{X/S}^{-1} : B_n \Omega_{X^{(p)}/S}^q \xrightarrow{\cong} B_{n+1} \Omega_{X/S}^q / B_1 \Omega_{X/S}^q,$$

$$(5.8.4) \quad C_{X/S}^{-1} : Z_n \Omega_{X^{(p)}/S}^q \xrightarrow{\cong} Z_{n+1} \Omega_{X/S}^q / B_1 \Omega_{X/S}^q.$$

We obtain a chain of inclusions

$$(5.8.5) \quad 0 \subset B_1 \Omega_{X/S}^q \subset \cdots \subset B_n \Omega_{X/S}^q \subset B_{n+1} \Omega_{X/S}^q \\ \subset \cdots \subset Z_{n+1} \Omega_{X/S}^q \subset Z_n \Omega_{X/S}^q \subset \cdots \subset Z_1 \Omega_{X/S}^q \subset \Omega_{X/S}^q.$$

PROPOSITION 5.9 ([Ill79, 0, (2.2.7), Prop. 2.2.8]). *Let  $S$  be an  $\mathbb{F}_p$ -scheme and  $X$  a smooth  $S$ -scheme. Then, for all  $q \geq 0$  and  $n \geq 1$ , the sheaves  $Z_n \Omega_{X/S}^q$  and  $B_n \Omega_{X/S}^q$  satisfy the following properties:*

(i)  $Z_n \Omega_{X/S}^q$  and  $B_n \Omega_{X/S}^q$  are locally free  $\mathcal{O}_{X^{(p^n)}}$ -modules of finite type and, for any  $h : S' \rightarrow S$ , we have

$$h_X^{(p^n)*} Z_n \Omega_{X/S}^q \xrightarrow{\cong} Z_n \Omega_{X'/S'}^q, \quad h_X^{(p^n)*} B_n \Omega_{X/S}^q \xrightarrow{\cong} B_n \Omega_{X'/S'}^q,$$

where  $h_X : X' := S' \times_S X \rightarrow X$  is the base-change map.

(ii) If  $f : X' \rightarrow X$  is an étale  $S$ -morphism, then there are natural isomorphisms

$$f^{(p^n)*} Z_n \Omega_{X/S}^q \xrightarrow{\cong} Z_n \Omega_{X'/S}^q, \quad f^{(p^n)*} B_n \Omega_{X/S}^q \xrightarrow{\cong} B_n \Omega_{X'/S}^q.$$

(iii)  $B_n \Omega_{X/S}^q$  is the sub- $\mathcal{O}_S$ -module of  $\Omega_{X/S}^q$  locally generated by sections of the form  $a_1^{p^r-1} \cdots a_q^{p^r-1} da_1 \cdots da_q$ , with  $a_i \in \mathcal{O}_X$  and  $0 \leq r \leq n-1$ .

(iv)  $Z_n \Omega_{X/S}^q$  is the sub- $\mathcal{O}_S$ -module of  $\Omega_{X/S}^q$  locally generated by  $B_n \Omega_{X/S}^q$  and sections of the form  $ba_1^{p^n-1} \cdots a_q^{p^n-1} da_1 \cdots da_q$ , with  $a_i \in \mathcal{O}_X$  and  $b \in \mathcal{O}_{X^{(p^n)}}$ .

PROPOSITION 5.10 (cf. [Ill79, I, Prop. 3.3]). *For  $X/S$  smooth as above, there is a unique map of  $W_n(\mathcal{O}_S)$ -modules*

$$C_n^{-1} : F_{S*} W_n(\mathcal{O}_S) \otimes_{W_n(\mathcal{O}_S)} W_n \Omega_{X/S}^q \longrightarrow \frac{W_n \Omega_{X/S}^q}{dV^{n-1} \Omega_{X/S}^q}$$

that makes the following diagram commutative:

$$\begin{array}{ccc} F_* W_n(\mathcal{O}_S) \otimes_{W_{n+1}(\mathcal{O}_S)} W_{n+1} \Omega_{X/S}^q & \xrightarrow{1 \otimes F} & W_n \Omega_{X/S}^q \\ \downarrow 1 \otimes R & & \downarrow \\ F_* W_n(\mathcal{O}_S) \otimes_{W_n(\mathcal{O}_S)} W_n \Omega_{X/S}^q & \xrightarrow{C_n^{-1}} & \frac{W_n \Omega_{X/S}^q}{dV^{n-1} \Omega_{X/S}^q}. \end{array}$$

For  $n = 1$ , we have  $F_{S*}\mathcal{O}_S \otimes_{\mathcal{O}_S} \Omega_{X/S}^q = \Omega_{X^{(p)}/S}^q$ , and  $C_n^{-1} = C^{-1} : \Omega_{X^{(p)}/S}^q \rightarrow \frac{\Omega_{X/S}^q}{d\Omega_{X/S}^{q-1}}$  is the inverse Cartier operator.

*Proof.* Since  $1 \otimes R$  is surjective, it is enough to see that the kernel of  $1 \otimes R$  is mapped to  $dV^{n-1}\Omega_{X/S}^q$  under  $1 \otimes F$ . But an element in the kernel of  $1 \otimes R$  is a sum of elements of the form  $a \otimes V^n\omega$  and  $a \otimes dV^n\eta$ , with  $a \in W_n(\mathcal{O}_S)$ ,  $\omega \in \Omega_{X/S}^q$ , and  $\eta \in \Omega_{X/S}^{q-1}$ . In  $W_n\Omega_{X/S}^q$ , we have

$$(1 \otimes F)(a \otimes V^n\omega) = aV^{n-1}(p\omega) = 0, \quad (1 \otimes F)(a \otimes dV^n\eta) = dV^{n-1}(F^{n-1}(a)\eta).$$

This gives the existence and the uniqueness of  $C_n^{-1}$ . The second statement follows from the fact that  $1 \otimes F$  is compatible with products and from the formula  $1 \otimes F(a \otimes d[x]) = ax^{p-1}dx$  for  $a \in \mathcal{O}_S$ ,  $x \in \mathcal{O}_X$ .  $\square$

COROLLARY 5.11 (cf. [Ill79, I, Prop. 3.11]). *Let  $X/S$  be as above. Then*

- (i)  $\text{Im}(1 \otimes F^n : F_*^n \mathcal{O}_S \otimes_{W_{n+1}(\mathcal{O}_S)} W_{n+1}\Omega_{X/S}^q \rightarrow \Omega_{X/S}^q) = Z_n\Omega_{X/S}^q$ .
- (ii)  $\text{Im}(1 \otimes F^{n-1}d : F_*^n \mathcal{O}_S \otimes_{W_{n+1}(\mathcal{O}_S)} F_*W_n\Omega_{X/S}^{q-1} \rightarrow \Omega_{X/S}^q) = B_n\Omega_{X/S}^q$ .

*Proof.* We do induction on  $n$ . For  $n = 1$ , (i) follows from Proposition 5.10 and the relation  $d = FdV$ , and (ii) holds by definition. Now assume the statements are proven for  $n$ . To prove (i) for  $n + 1$ , we consider the following commutative diagram of abelian sheaves on  $X$ :

$$\begin{array}{ccccc} F_*^{n+1}\mathcal{O}_S \otimes_{W_{n+2}(\mathcal{O}_S)} W_{n+2}\Omega_{X/S}^q & \xrightarrow{1 \otimes F^n} & F_*\mathcal{O}_S \otimes_{W_2(\mathcal{O}_S)} W_2\Omega_{X/S}^q & \xrightarrow{1 \otimes F} & \Omega_{X/S}^q \\ \downarrow 1 \otimes R & & \downarrow 1 \otimes R & & \downarrow \\ F_*^{n+1}\mathcal{O}_S \otimes_{W_{n+1}(\mathcal{O}_S)} W_{n+1}\Omega_{X/S}^q & \xrightarrow{1 \otimes F^n} & F_*\mathcal{O}_S \otimes_{\mathcal{O}_S} \Omega_{X/S}^q = \Omega_{X^{(p)}/S}^q & \xrightarrow{C^{-1}} & \frac{\Omega_{X/S}^q}{d\Omega_{X/S}^{q-1}}. \end{array}$$

By induction hypothesis, we have

$$\text{Im}\left((1 \otimes R) \circ (1 \otimes F^n)\right) = \text{Im}\left((1 \otimes F^n) \circ (1 \otimes R)\right) = F_{S*}\mathcal{O}_S \otimes_{\mathcal{O}_S} Z_n\Omega_{X/S}^q = Z_n\Omega_{X^{(p)}/S}^q,$$

where the last equality follows from the compatibility with base-change. Now, thanks to the relation  $d = F^{n+1}dV^{n+1}$ , (i) follows from the definition of  $Z_{n+1}\Omega_{X/S}^q$ . The proof of (ii) is similar.  $\square$

LEMMA 5.12. *Let  $X/S$  be as above. The sheaf  $B_n\Omega_{X/S}^q$  is given by*

$$\begin{aligned} & \text{Im}(1 \otimes F^{n-1}d : F_*^n \mathcal{O}_S \otimes_{W_{n+1}(\mathcal{O}_S)} F_*W_n\Omega_{X/S}^{q-1} \rightarrow \Omega_{X/S}^q) \\ & = \{(1 \otimes F^n d)(\alpha) \mid \alpha \in F_*^n \mathcal{O}_S \otimes_{W_{n+1}(\mathcal{O}_S)} W_{n+1}\Omega_{X/S}^{q-1} \text{ with } (1 \otimes F^n)(\alpha) = 0\}. \end{aligned}$$

*Proof.* We call the left-hand side  $\mathcal{A}$  and the right-hand side  $\mathcal{B}$ . We know from the previous corollary that  $B_n\Omega_{X/S}^q = \mathcal{A}$ , and now we want to show that

$\mathcal{A} = \mathcal{B}$ . In the following, all nonspecified tensor products are over  $W_{n+1}(\mathcal{O}_S)$ . We have the commutative diagram

$$\begin{array}{ccc} F_*^n \mathcal{O}_S \otimes F_* W_n \Omega_{X/S}^{q-1} & \xrightarrow{1 \otimes V} & F_*^n \mathcal{O}_S \otimes W_{n+1} \Omega_{X/S}^{q-1} \\ & \searrow 1 \otimes F^{n-1} d & \swarrow 1 \otimes F^n d \\ & \Omega_{X/S}^q & \end{array}$$

Since we also have  $(1 \otimes F^n) \circ (1 \otimes V) = 0$ , it follows that  $\mathcal{A} \subset \mathcal{B}$ . It remains to show

$$(5.12.1) \quad \begin{aligned} & \text{Ker}\left(1 \otimes F^n : F_*^n \mathcal{O}_S \otimes W_{n+1} \Omega_{X/S}^{q-1} \rightarrow F_*^n \Omega_{X/S}^{q-1}\right) \\ & \subset \text{Im}\left(F_*^n \mathcal{O}_S \otimes (F_* W_n \Omega_{X/S}^{q-1} \oplus F_* W_n \Omega_{X/S}^{q-2}) \xrightarrow{1 \otimes (V+dV)} F_*^n \mathcal{O}_S \otimes W_{n+1} \Omega_{X/S}^{q-1}\right). \end{aligned}$$

Indeed, if we take an element  $\alpha$  in the kernel on the left-hand side and we write it as an element in the right-hand side,  $\alpha = (1 \otimes V)(\beta) + (1 \otimes dV)(\gamma)$ , then  $(1 \otimes F^n d)(\alpha) = (1 \otimes F^{n-1} d)(\beta)$ , i.e.,  $\mathcal{B} \subset \mathcal{A}$ . The question is local in  $X$ ; we may thus assume  $X$  is étale over  $\mathbb{A}_S^d$ . For a  $W_n(\mathcal{O}_X)$ -module  $\mathcal{M}$ , we write  $F_*^r \mathcal{M}_* F^s$  for  $\mathcal{M}$  viewed as a left  $W_{n+r}(\mathcal{O}_S)$ -module via  $F^r$  and as a right  $W_{n+s}(\mathcal{O}_X)$ -module via  $F^s$ . Then we have the following commutative diagram, in which the most right tensor product in the upper line is over  $W_{2n+2}(\mathcal{O}_{\mathbb{A}_S^d})$ :

$$\begin{array}{ccc} \left(F_*^n \mathcal{O}_S \otimes F_*(W_n \Omega_{\mathbb{A}_S^d/S}^{q-2})_* F^{n+2} \xrightarrow{1 \otimes dV} F_*^n \mathcal{O}_S \otimes (W_{n+1} \Omega_{\mathbb{A}_S^d/S}^{q-1})_* F^{n+1}\right) \otimes W_{2n+2}(\mathcal{O}_X) & & \\ \downarrow (1 \otimes \text{can}) \otimes 1 & & \downarrow (1 \otimes \text{can}) \otimes 1 \\ F_*^n \mathcal{O}_S \otimes F_*(W_n \Omega_{X/S}^{q-2})_* F^{n+2} \xrightarrow{1 \otimes dV} F_*^n \mathcal{O}_S \otimes (W_{n+1} \Omega_{X/S}^{q-1})_* F^{n+1}. & & \end{array}$$

If we write  $V$  instead of  $dV$  and  $q-1$  on the left-hand side instead of  $q-2$ , we obtain again a commutative diagram. Since  $X/\mathbb{A}_S^d$  is étale, the vertical maps are isomorphisms (in both diagrams). Thus, if we denote the image in (5.12.1) by  $\text{Im}(X/S)$ , we obtain

$$\text{Im}(X/S) \cong \text{Im}(\mathbb{A}_S^d/S)_* F^{n+1} \otimes_{W_{2n+2}(\mathcal{O}_{\mathbb{A}_S^d})} W_{2n+2}(\mathcal{O}_X).$$

Similarly, denoting the kernel in (5.12.1) by  $\text{Ker}(X/S)$ , one finds

$$\text{Ker}(X/S) \cong \text{Ker}(\mathbb{A}_S^d/S)_* F^{n+1} \otimes_{W_{2n+2}(\mathcal{O}_{\mathbb{A}_S^d})} W_{2n+2}(\mathcal{O}_X).$$

And, since  $W_{2n+2}(\mathcal{O}_X)$  is étale over  $W_{2n+2}(\mathcal{O}_{\mathbb{A}_S^d})$  [LZ04, Prop. A.8], it is thus enough to prove (5.12.1) in the case  $S = \text{Spec } A$ , with  $A$  an  $\mathbb{F}_p$ -algebra, and  $X = \text{Spec } B$ , with  $B = A[x_1, \dots, x_d]$ .



Now, using the notation of Corollary 5.6, any element  $\alpha \in F_*^n A \otimes W_{n+1} \Omega_{B/A}^{q-1}$  can be written as a finite sum

$$(5.12.2) \quad \alpha = \sum_i \sum_{\substack{p^n k \text{ integral} \\ \mathcal{P}=(I_0, \dots, I_{q-1})}} a_i \otimes e_{n+1}(V^{u(k)}(\eta_{k, \mathcal{P}, i}), k, \mathcal{P}), \quad \eta_{k, \mathcal{P}, i} \in W_{n+1-u(k)}(A).$$

By rule 5.4(ii), we have

$$F^n e_{n+1}(V^{u(k)}(\eta), k, \mathcal{P}) = \begin{cases} e_1(F^{n-u(k)}(\eta), p^n k, \mathcal{P}) & \text{if } I_0 = \emptyset \text{ or } (I_0 \neq \emptyset, k \text{ integral}), \\ 0 & \text{if } I_0 \neq \emptyset \text{ and } k \text{ not integral.} \end{cases}$$

It follows that an element  $\alpha$  as in (5.12.2) lies in  $\text{Ker}(1 \otimes F^n) = \text{Ker}(B/A)$  if and only if it satisfies

$$(5.12.3) \quad \sum_i a_i F^{n-u(k)}(\eta_{k, \mathcal{P}, i}) = 0, \quad \text{for } I_0 = \emptyset \text{ or } (I_0 \neq \emptyset, k \text{ integral}).$$

We consider the following three cases:

1)  $k$  is integral, i.e.,  $u(k) = 0$ . Then, by Definition 5.3,  $e_{n+1}(\eta, k, \mathcal{P}) = \eta e_{n+1}(1, k, \mathcal{P})$ . By (5.12.3), we get

$$\sum_i a_i \otimes e_{n+1}(\eta_{i, k, \mathcal{P}}, k, \mathcal{P}) = \left( \sum_i a_i F^n(\eta_{i, k, \mathcal{P}}) \right) \otimes e_{n+1}(1, k, \mathcal{P}) = 0.$$

2)  $k$  is not integral and  $I_0 = \emptyset$ . In this case  $e_{n+1}(\eta, k, \mathcal{P}) \in \text{Im}(dV)$  by Definition 5.3. Thus

$$\sum_i a_i \otimes e_{n+1}(\eta_{i, k, \mathcal{P}}, k, \mathcal{P}) \in \text{Im}(1 \otimes dV).$$

3)  $k$  is not integral and  $I_0 \neq \emptyset$ . Now  $e_{n+1}(\eta, k, \mathcal{P}) \in \text{Im}(V)$  by Definition 5.3. Hence

$$\sum_i a_i \otimes e_{n+1}(\eta_{i, k, \mathcal{P}}, k, \mathcal{P}) \in \text{Im}(1 \otimes V).$$

Putting the three cases together, we see that  $\alpha \in \text{Ker}(1 \otimes F^n)$  implies  $\alpha \in \text{Im}(1 \otimes V + 1 \otimes dV) = \text{Im}(B/A)$ . This gives the statement.  $\square$

**THEOREM 5.13** (cf. [Ill79, I, Cor. 3.9], [Ols07, Th. 4.2.15]). *Let  $S$  be an  $\mathbb{F}_p$ -scheme, and let  $X$  be a smooth  $S$ -scheme. For  $n, q \geq 0$ , denote by  $\text{gr}^n W\Omega_{X/S}^q$  the  $n$ -th graded piece of the canonical filtration*

$$\text{Fil}^n W\Omega_{X/S}^q = V^n W\Omega_{X/S}^q + dV^n W\Omega_{X/S}^q = \text{Ker}(W\Omega_{X/S}^q \rightarrow W_n \Omega_{X/S}^q).$$

*Then we have an exact sequence of  $\mathcal{O}_X$ -modules*

$$(5.13.1) \quad 0 \longrightarrow F_{X^*}^{n+1} \frac{\Omega_{X/S}^q}{B_n \Omega_{X/S}^q} \xrightarrow{V^n} \text{gr}^n W\Omega_{X/S}^q \xrightarrow{U_n} F_{X^*}^{n+1} \frac{\Omega_{X/S}^{q-1}}{Z_n \Omega_{X/S}^{q-1}} \longrightarrow 0,$$

where the map  $U_n$  is given by  $V^n(\alpha) + dV^n(\beta) \mapsto \beta$  and the  $\mathcal{O}_X$ -module structure on  $\mathrm{gr}^n W\Omega_{X/S}^q$  is given via

$$\mathcal{O}_X = \frac{W_n \mathcal{O}_X}{VW_{n-1} \mathcal{O}_X} \xrightarrow{F} \frac{W_{n+1} \mathcal{O}_X}{pW_n \mathcal{O}_X}.$$

Furthermore,  $F_{X/S*}^n \frac{\Omega_{X/S}^q}{B_n \Omega_{X/S}^q}$  and  $F_{X/S*}^n \frac{\Omega_{X/S}^{q-1}}{Z_n \Omega_{X/S}^{q-1}}$  are locally free  $\mathcal{O}_{X^{(p^n)}}$ -modules.

*Proof.* The exactness of the sequence follows from Proposition 5.7, Corollary 5.11, and Lemma 5.12. The second statement is proven as in [Ill79, I, Cor. 3.9]. By étale base change (Proposition 5.9(ii)), we reduce the question of the local freeness of the two extreme  $\mathcal{O}_{X^{(p^n)}}$ -modules in the exact sequence to the case  $X = \mathbb{A}_S^d$ . Since everything is compatible with arbitrary base change in the base  $S$  (by Proposition 5.9(i)), we may also assume  $S = \mathrm{Spec} \mathbb{F}_p$ , and even  $S = \mathrm{Spec} k$  with  $k$  algebraically closed. But now the sheaves in question are coherent on  $(\mathbb{A}_k^d)^{(p^n)} \cong \mathbb{A}_k^d$ , hence locally free in some nonempty open subset, whose translates under certain closed points cover the whole of  $(\mathbb{A}_k^d)^{(p^n)}$ . As they are invariant under translation, this gives the statement.  $\square$

*Remark.* There is also an analog to the vertical exact sequence in [Ill79, I, Cor. 3.9]. This follows from Proposition 5.7, Corollary 5.11, and the following statement (which is an analog of Lemma 5.12): for an  $\mathbb{F}_p$ -scheme  $S$  and a smooth  $S$ -scheme  $X$ , the sheaf  $Z_{n+1} \Omega_{X/S}^{q-1}$  is given by

$$\begin{aligned} & \mathrm{Im}(1 \otimes F^{n+1} : F_*^{n+1} \mathcal{O}_S \otimes_{W_{n+2}(\mathcal{O}_S)} W_{n+2} \Omega_{X/S}^{q-1} \rightarrow \Omega_{X/S}^{q-1}) \\ & = \{1 \otimes F^n(\alpha) \mid \alpha \in F_*^n \mathcal{O}_S \otimes_{W_{n+1}(\mathcal{O}_S)} W_{n+1} \Omega_{X/S}^{q-1} \text{ with } 1 \otimes F^n d(\alpha) = 0\}. \end{aligned}$$

This statement is proven by direct inspection of the basic Witt differentials after reducing to the case  $A[x_1, \dots, x_d]/A$ . As the proof is rather long and technical, and the result is not needed in this article, we do not include it here.

### 6. The Hodge-Witt trace morphism for projective spaces

Let  $X$  be a noetherian  $\mathbb{F}_p$ -scheme with a dualizing complex, and let  $f : Y \rightarrow X$  be a projective complete intersection morphism of virtual relative dimension 0. Our goal in the next two sections is to prove that, given a factorization  $f = \pi \circ i$ , where  $\pi : P = \mathbb{P}_X^d \rightarrow X$  is the structural morphism of some projective space over  $X$  and  $i : Y \hookrightarrow P$  is a closed immersion, one can define for all  $n \geq 1$  a morphism

$$\tau_{i,\pi,n} : Rf_* W_n \mathcal{O}_Y \longrightarrow W_n \mathcal{O}_X$$

so as to satisfy the following properties:

- (i) For  $n = 1$ ,  $\tau_{i,\pi,n}$  is the morphism  $\tau_f$  of Theorem 3.1.
- (ii) For variable  $n$ ,  $\tau_{i,\pi,n}$  commutes with  $R, F$  and  $V$ .

Our construction of  $\tau_{i,\pi,n}$  will be based on a generalization for arbitrary  $n$  of the description of  $\tau_f$  given in Proposition 4.6: we will construct on the one hand a trace morphism  $R\pi_*W_n\Omega_{P/X}^d[d] \rightarrow W_n\mathcal{O}_X$  that will be a generalization of the trace morphism  $\text{Trp}_\pi$  for the projective space, and on the other hand a morphism  $i_*W_n\mathcal{O}_Y \rightarrow W_n\Omega_{P/X}^d[d]$  that will be a generalization of the morphism  $\gamma_f : \mathcal{O}_Y \rightarrow \omega_{P/X}[d]$  defined in (4.4.1).

We begin with the trace morphism for projective spaces.

6.1. We recall first from [Ill90, D ef. 1.1] that a smooth proper  $\mathbb{F}_p$ -morphism  $f : X \rightarrow S$  is called *ordinary* if it satisfies

$$R^i f_* B\Omega_{X/S}^q = 0 \quad \text{for all } i, q \geq 0.$$

This notion is compatible with arbitrary base-change in the base  $S$ , and  $\mathbb{P}_{\mathbb{F}_p}^d$  is ordinary over  $\text{Spec } \mathbb{F}_p$  [Ill90, Props. 1.2, 1.4]. Hence if  $\mathcal{E}$  is a locally free  $\mathcal{O}_X$ -module of finite rank on some  $\mathbb{F}_p$ -scheme  $X$ , then  $\mathbb{P}(\mathcal{E}) = \text{Proj}(\text{Sym}_{\mathcal{O}_X}\mathcal{E})$  is ordinary over  $X$ .

LEMMA 6.2. *Let  $f : X \rightarrow S$  be ordinary. Then, for all  $n \geq 1$  and  $q \geq 0$ ,*

$$V^n : F_{S^*}^{n+1} Rf_*\Omega_{X/S}^q \xrightarrow{\sim} Rf_*\text{gr}^n W\Omega_{X/S}^q$$

*is an isomorphism in the derived category of quasi-coherent  $\mathcal{O}_S$ -modules (where the  $\mathcal{O}_S$ -module structure on the right-hand side comes from the  $\mathcal{O}_X$ -module structure defined in Theorem 5.13).*

*Proof.* This follows immediately from Theorem 5.13 and the following claim:

$$(6.2.1) \quad R^i f_* Z_n \Omega_{X/S}^q \xrightarrow{\sim} R^i f_* \Omega_{X/S}^q, \quad R^i f_* B_n \Omega_{X/S}^q = 0 \quad \text{for all } i, q \geq 0, n \geq 1.$$

We prove this by induction on  $n$ . The statement for  $B_1$  holds by definition of ordinarity and for  $Z_1$  follows from the exact sequence

$$0 \longrightarrow Z\Omega_{X/S}^q \longrightarrow \Omega_{X/S}^q \xrightarrow{d} B\Omega_{X/S}^{q+1} \longrightarrow 0.$$

Now, for the general case, consider the following commutative diagram (in which  $f_*$  is viewed as a functor on the category of abelian sheaves for the Zariski topology on  $|X| = |X^{(p)}|$ ):

$$\begin{array}{ccccc} R^i f_* Z_n \Omega_{X^{(p)}/S}^q & \xrightarrow{C_{X/S}^{-1}} & R^i f_* \frac{Z_{n+1} \Omega_{X/S}^q}{B_1 \Omega_{X/S}^q} & \longleftarrow & R^i f_* Z_{n+1} \Omega_{X/S}^q \\ \downarrow & & \downarrow & & \downarrow \\ R^i f_* \Omega_{X^{(p)}/S}^q & \xrightarrow{C_{X/S}^{-1}} & R^i f_* \frac{Z_1 \Omega_{X/S}^q}{B_1 \Omega_{X/S}^q} & \longleftarrow & R^i f_* Z_1 \Omega_{X/S}^q \end{array}$$

The horizontal maps are isomorphisms, as is the vertical map on the left by induction (notice that  $X^{(p)}/S$  is also ordinary). Hence all maps in the diagram are isomorphisms, which yields the claim for  $Z_{n+1}$ . To prove the statement for  $B_{n+1}$ , it is enough to consider the upper line in the diagram, with  $Z$  replaced by  $B$ , and one immediately obtains the statement.  $\square$

6.3. Let  $S$  be a scheme on which  $p$  is locally nilpotent, and  $X$  an  $S$ -scheme. As in the classical case [Ill79, I, 3.23], for any  $n \geq 1$ , we define the log derivation  $\mathrm{dlog}_n$  to be the morphism of abelian sheaves

$$\mathrm{dlog}_n : \mathcal{O}_X^\times \longrightarrow W_n \Omega_{X/S}^1, \quad a \mapsto \mathrm{dlog}_n(a) := \frac{d[a]}{[a]}.$$

We may write simply  $\mathrm{dlog}$  if  $n$  is fixed.

For variable  $n$ , the maps  $\mathrm{dlog}_n$  satisfy the following relations:

$$(6.3.1) \quad R(\mathrm{dlog}_n(a)) = \mathrm{dlog}_{n-1}(a), \quad F(\mathrm{dlog}_n(a)) = \mathrm{dlog}_{n-1}(a).$$

The maps  $\mathrm{dlog}_n$  allow us to define Chern classes for line bundles and to prove for relative Hodge-Witt cohomology the analog of the classical theorem on the cohomology of projective bundles (cf. [SGA7II, XI, Th. 1.1]).

**THEOREM 6.4.** *Let  $X$  be an  $\mathbb{F}_p$ -scheme,  $\mathcal{E}$  a locally free  $\mathcal{O}_X$ -module of rank  $d + 1$ ,  $P = \mathbb{P}(\mathcal{E})$ , and let  $\pi : P \rightarrow X$  be the canonical projection. Denote by  $\eta_n \in H^0(X, R^1 \pi_* W_n \Omega_{P/X}^1)$  the image under  $\mathrm{dlog}_n$  of the class of  $\mathcal{O}_P(1)$  in  $R^1 \pi_* \mathcal{O}_P^\times$  and by  $\eta_n^q \in H^0(X, R^q \pi_* W_n \Omega_{P/X}^q)$  its  $q$ -fold cup product. Then, for all  $n \geq 1$  and all  $q$  such that  $0 \leq q \leq d$ , we have*

$$(6.4.1) \quad R^j \pi_* W_n \Omega_{P/X}^q = 0 \quad \text{for } j \neq q,$$

and multiplication with  $\eta_n^q$  induces an isomorphism in the derived category of  $W_n(\mathcal{O}_X)$ -modules

$$(6.4.2) \quad W_n(\mathcal{O}_X)[-q] \xrightarrow{\sim} R\pi_* W_n \Omega_{P/X}^q.$$

Furthermore, these isomorphisms are compatible with restriction, Frobenius, and Verschiebung on both sides.

*Proof.* To prove (6.4.1), we can argue by induction using the exact sequences

$$0 \longrightarrow \mathrm{gr}^n W_{n+1} \Omega_{P/X}^q \longrightarrow W_{n+1} \Omega_{P/X}^q \longrightarrow W_n \Omega_{P/X}^q \longrightarrow 0.$$

For  $n = 1$ , the claim follows from [SGA7II, XI, Th. 1.1] and, since  $\mathbb{P}(\mathcal{E})$  is ordinary over  $X$ , Lemma 6.2 implies similarly the claim for all  $n$ .

Therefore, we obtain a canonical isomorphism

$$(6.4.3) \quad R\pi_* W_n \Omega_{P/X}^q \xrightarrow{\sim} R^q \pi_* W_n \Omega_{P/X}^q[-q],$$

and we can define the morphism (6.4.2) as corresponding via (6.4.3) and translation to the morphism

$$(6.4.4) \quad W_n(\mathcal{O}_X) \longrightarrow R^q \pi_* W_n \Omega_{P/X}^q, \quad w \mapsto w \eta_n^q.$$

This reduces the proof of the theorem to proving that (6.4.4) is an isomorphism, compatible with  $R$ ,  $F$  and  $V$ .

From (6.3.1), for all  $w \in W_{n+1}(\mathcal{O}_X)$ , we get the relations

$$(6.4.5) \quad R(w \eta_{n+1}^q) = R(w) \eta_n^q, \quad F(w \eta_{n+1}^q) = F(w) \eta_n^q$$

in  $R^q \pi_* W_n \Omega_{P/X}^q$ . From the second relation, we also get

$$(6.4.6) \quad V(w \eta_{n-1}^q) = V(w F(\eta_n^q)) = V(w) \eta_n^q$$

for all  $w \in W_{n-1}(\mathcal{O}_X)$ . So the homomorphisms (6.4.4) satisfy the required compatibilities.

To prove that the homomorphisms (6.4.4) are isomorphisms, we may now again argue by induction on  $n$ , using the compatibility with  $R$  and  $V$ . Then Lemma 6.2 reduces the proof to the case  $n = 1$ , which is known by [SGA7II, Exp. XI, Th. 1.1].  $\square$

*Definition 6.5.* Under the assumptions of Theorem 6.4, we define the Hodge-Witt trace morphism for the projective space  $\mathbb{P}(\mathcal{E})$  to be the  $W_n \mathcal{O}_X$ -linear map

$$(6.5.1) \quad \text{Trp}_{\pi,n} : R\pi_* W_n \Omega_{\mathbb{P}(\mathcal{E})/X}^d[d] \xrightarrow{\sim} W_n \mathcal{O}_X$$

obtained by inverting the isomorphism (6.4.2), shifting by  $d$  and multiplying by  $(-1)^{d(d-1)/2}$ . Theorem 6.4 implies that  $\text{Trp}_{\pi,n}$  is compatible with restriction, Frobenius, and Verschiebung.

**PROPOSITION 6.6.** *With the hypotheses of Theorem 6.4, assume in addition that  $X$  is locally noetherian. Then the morphism*

$$(6.6.1) \quad \text{Trp}_{\pi,1} : R\pi_* \Omega_{\mathbb{P}(\mathcal{E})/X}^d[d] \xrightarrow{\sim} \mathcal{O}_X$$

*defined by (6.5.1) for  $n = 1$  is equal to the morphism  $\text{Trp}_{\pi}$  defined by [Con00, (2.3.5)] for  $\mathcal{O}_X$ .*

*Proof.* By (6.4.2), it suffices to prove the proposition locally on  $X$ . So we may assume that  $\mathbb{P}(\mathcal{E}) = \mathbb{P}_X^d$ . Let  $X_0, \dots, X_d$  be the standard homogeneous coordinates on  $\mathbb{P}_X^d$ ,  $x_i = X_i/X_0$ ,  $U_i = D_+(X_i)$ , and let  $\mathfrak{U} = (U_i)_{i=0, \dots, d}$  be the corresponding covering of  $\mathbb{P}_X^d$ . Using Čech cohomology relative to  $\mathfrak{U}$ ,  $\eta_1$  is defined by the 1-cocycle  $(d \log(X_j/X_i))_{i < j} = (d(X_j/X_i)/(X_j/X_i))_{i < j}$  and  $\eta_1^d$

by the  $d$ -cocycle given by

$$\begin{aligned} & \mathrm{dlog}(X_1/X_0) \wedge \cdots \wedge \mathrm{dlog}(X_d/X_{d-1}) \\ &= dx_1/x_1 \wedge (dx_2/x_2 - dx_1/x_1) \wedge \cdots \wedge (dx_d/x_d - dx_{d-1}/x_{d-1}) \\ &= dx_1 \wedge \cdots \wedge dx_d/x_1 \cdots x_d \end{aligned}$$

on  $U_0 \cap \cdots \cap U_d$ . Thus  $\mathrm{Trp}_{\pi,1}$  is the only morphism that induces on degree-0 cohomology the isomorphism mapping the class  $dx_1 \wedge \cdots \wedge dx_d/x_1 \cdots x_d$  to  $(-1)^{d(d-1)/2}$ .

To prove the proposition, it suffices to check that, with Conrad's definitions, the map induced by  $\mathrm{Trp}_{\pi} : Rf_*(f^{\sharp}(\mathcal{O}_X)) = Rf_*(\omega_{P/X}[d]) \rightarrow \mathcal{O}_X$  on degree-0 cohomology is such that

$$(6.6.2) \quad \mathrm{Trp}_{\pi}(dx_1 \wedge \cdots \wedge dx_d/x_1 \cdots x_d) = (-1)^{d(d-1)/2}.$$

As  $(-1)^d(-1)^{d(d-1)/2} = (-1)^{d(d+1)/2}$ , this follows from the definition of the isomorphism [Con00, (2.3.1)]

$$(6.6.3) \quad \gamma : R^d\pi_*(\omega_{P/X}) \xrightarrow{\sim} \mathcal{O}_X,$$

which sends  $dx_1 \wedge \cdots \wedge dx_d/x_1 \cdots x_d$  to  $(-1)^{d(d+1)/2}$  [Con00, (2.3.3)], and from the discussion on pages 35–36 of [Con00], which explains that an additional  $(-1)^d$  sign is required to recover (6.6.3) from the map induced in degree 0 by  $\mathrm{Trp}_{\pi}$ . (Note that by “induced” we mean that we use here as we always do the standard identifications [Con00, (1.3.1), (1.3.4)] to compute the cohomology objects of a translated complex.)

This ends the proof of the proposition, but, as formula (6.6.2) is only implicit in the discussion [Co00, pp. 35–36], it may be worth adding a few lines to give a proof explaining where this extra  $(-1)^d$  sign comes from. Conrad's construction of the projective trace  $\mathrm{Trp}_{\pi}$  is the same as Hartshorne's in [Har66, III, 4.3], but using [Con00, Lemma 2.1.1] instead of [Har66, I, Prop. 7.4]. Because  $\pi_*$  has cohomological dimension  $d$  on the category of quasi-coherent  $\mathcal{O}_P$ -modules, and any quasi-coherent  $\mathcal{O}_P$ -module can be written as a quotient of modules for which the functors  $R^i\pi_*$  vanish for  $i \neq d$  [Har66, III, Lemmas 4.1 and 4.2], Lemma 2.1.1 of [Con00] provides an isomorphism of functors on  $D(\mathrm{Qcoh}(\mathcal{O}_P))$ :

$$\psi : R\pi_* \xrightarrow{\sim} L(R^d\pi_*)[-d].$$

For complexes of the form  $\mathcal{F}^{\bullet} = \mathcal{F}[0]$ , where  $\mathcal{F}$  is a quasi-coherent  $\mathcal{O}_P$ -module,  $\psi_{\mathcal{F}^{\bullet}}$  induces in degree  $d$  the identity of  $R^d\pi_*(\mathcal{F})$  [Con00, Cor. 2.1.2]. Moreover, the compatibility of  $\psi$  with translations, given by [Con00, (2.1.1)], implies that, for any  $m \in \mathbb{Z}$ , we have

$$\psi_{\mathcal{F}^{\bullet}[m]} = (-1)^{md}\psi_{\mathcal{F}^{\bullet}}[m].$$

In particular,  $\psi_{\omega_{P/X}[d]}$  induces in degree 0 multiplication by  $(-1)^{d^2} = (-1)^d$  on  $R^d\pi_*(\omega_{P/X})$ . Now, for  $\mathcal{G}^\bullet \in D_{\text{qc}}^+(\mathcal{O}_X)$ , the trace morphism for  $\mathcal{G}^\bullet$  is the composition

$$\begin{array}{ccc} R\pi_*(\pi^\sharp\mathcal{G}^\bullet) & \xlongequal{\quad} & R\pi_*(\omega_{P/X}[d] \otimes_{\mathcal{O}_P} \pi^*\mathcal{G}^\bullet) \xrightarrow[\sim]{\psi_{\pi^\sharp\mathcal{G}^\bullet}} L(R^d\pi_*)(\omega_{P/X}[d] \otimes_{\mathcal{O}_P} \pi^*\mathcal{G}^\bullet)[-d] \\ \downarrow \text{Tr}_{P,\pi,\mathcal{G}^\bullet} & & \sim \downarrow \\ \mathcal{G}^\bullet & \xleftarrow[\sim]{\gamma \otimes \text{Id}} (R^d\pi_*)(\omega_{P/X}) \otimes_{\mathcal{O}_X} \mathcal{G}^\bullet & \xleftarrow[\sim]{} (R^d\pi_*)(\omega_{P/X} \otimes_{\mathcal{O}_P} \pi^*\mathcal{G}^\bullet) \end{array}$$

(see [Har66, III, 4.3] for details). Taking  $\mathcal{G}^\bullet = \mathcal{O}_X[0]$  and applying the previous remark to  $\pi^\sharp\mathcal{O}_X = \omega_{P/X}[d]$ , we obtain that  $\text{Tr}_{P,\pi,\mathcal{O}_X}$  induces  $(-1)^d\gamma$  in degree 0, which gives (6.6.2).  $\square$

*Remark.* Due to differences in sign conventions between Hartshorne [Har66, III, Th 3.4] and Conrad [Con00, 2.3], our trace morphism  $\text{Tr}_{\pi,n}$  differs by  $(-1)^{d(d-1)/2}$  from the trace morphism defined by Ekedahl [Eke84, I, Lemma 3.2] when  $X = \text{Spec } k$ ,  $k$  being a perfect field.

### 7. A Hodge-Witt local class for regularly embedded subschemes

In this section, we assume that  $X$  is a locally noetherian scheme of characteristic  $p$ , and we consider a regular immersion  $i : Y \hookrightarrow P$  of codimension  $d$ , where  $P$  is a smooth  $X$ -scheme. Under these assumptions, we want to associate to  $Y$  a canonical class  $\gamma_Y \in \Gamma(P, \mathcal{H}_Y^d(W_n\Omega_{P/X}^d))$  for each  $n \geq 1$ .

PROPOSITION 7.1. *Under the previous assumptions,*

- (i) *If  $t_1, \dots, t_d$  is a regular sequence of sections of  $\mathcal{O}_P$ , then, for all  $n \geq 1$  and all  $r \geq 1$ ,  $[t_1]^r, \dots, [t_d]^r$  is a regular sequence of sections of  $W_n(\mathcal{O}_P)$ .*
- (ii) *For all  $n \geq 1$  and all  $q$ ,  $\mathcal{H}_Y^j(W_n\Omega_{P/X}^q) = 0$  for  $j \neq d$ .*

*Proof.* We proceed by induction on  $n$ . In the exact sequence of  $W_{n+1}(\mathcal{O}_P)$ -modules

$$0 \longrightarrow F_*^n\mathcal{O}_P \xrightarrow{V^n} W_{n+1}(\mathcal{O}_P) \xrightarrow{R} W_n(\mathcal{O}_P) \longrightarrow 0,$$

the action of  $[t_i]^r$  on  $F_*^n\mathcal{O}_P$  is given by multiplication by  $t_i^{rp^n}$  on  $\mathcal{O}_P$ . As  $P$  is a locally noetherian scheme, the sequence  $t_1^{rp^n}, \dots, t_d^{rp^n}$  is regular in  $\mathcal{O}_P$ , and the first claim follows easily.

For  $n = 1$ , the second one is a well-known consequence of the regularity of the sequence  $t_1, \dots, t_d$ . As  $\mathcal{O}_P$  is locally free of finite rank over  $\mathcal{O}_{P^{(p^n)}}$ , we also have  $\mathcal{H}_Y^j(\mathcal{O}_{P^{(p^n)}}) = 0$  for  $j \neq d$ . In the exact sequence

$$0 \longrightarrow \text{gr}^n W_{n+1}\Omega_{P/X}^q \longrightarrow W_{n+1}\Omega_{P/X}^q \xrightarrow{R} W_n\Omega_{P/X}^q \longrightarrow 0,$$

Theorem 5.13 allows us to endow the kernel  $\text{gr}^n W_{n+1}\Omega_{P/X}^q$  with an  $\mathcal{O}_P$ -module structure for which it is an extension of two  $\mathcal{O}_P$ -modules that are locally free over  $\mathcal{O}_{P^{(p^n)}}$ . Therefore,  $\mathcal{H}_Y^j(\text{gr}^n W_{n+1}\Omega_{P/X}^q) = 0$  for  $j \neq d$ . The second claim follows by induction.  $\square$

**THEOREM 7.2.** *Under the assumptions of this section, let  $\mathbf{t} = (t_1, \dots, t_d)$  and  $\mathbf{t}' = (t'_1, \dots, t'_d)$  be two regular sequences of sections of  $\mathcal{O}_P$  generating the ideal  $\mathcal{I}$  of  $Y$  in  $P$ . Let  $n \geq 1$  be an integer, and let  $\mathcal{J} = ([t_1], \dots, [t_d])$ ,  $\mathcal{J}' = ([t'_1], \dots, [t'_d])$  be the ideals of  $W_n(\mathcal{O}_P)$  generated by the Teichmüller representatives of these generators. If*

$$\beta_{\mathcal{J}} : \mathcal{E}xt_{W_n(\mathcal{O}_P)}^d(W_n(\mathcal{O}_P)/\mathcal{J}, W_n\Omega_{P/X}^d) \longrightarrow \mathcal{H}_Y^d(W_n\Omega_{P/X}^d)$$

*is the canonical homomorphism (and similarly for  $\beta_{\mathcal{J}'}$ ) then, with the notation of 4.1,*

$$(7.2.1) \quad \beta_{\mathcal{J}} \left( \begin{bmatrix} d[t_1] \cdots d[t_d] \\ [t_1], \dots, [t_d] \end{bmatrix} \right) = \beta_{\mathcal{J}'} \left( \begin{bmatrix} d[t'_1] \cdots d[t'_d] \\ [t'_1], \dots, [t'_d] \end{bmatrix} \right).$$

*Proof.* It suffices to prove (7.2.1) in a neighbourhood of each point  $y \in Y$ . Localizing, one can reduce the proof of Theorem 7.2 to the case of a very simple change of generators in  $\mathcal{I}$ , thanks to the following remarks (see also [SGA4 $\frac{1}{2}$ , Cycle, Lemme 2.2.3]):

(a) If the sequence  $(t'_1, \dots, t'_d)$  is deduced from  $(t_1, \dots, t_d)$  by permutation, then  $\mathcal{J} = \mathcal{J}'$ , and formula (4.2.1) implies the theorem.

(b) If there exists invertible sections  $a_1, \dots, a_d \in \mathcal{O}_P^\times$  such that  $t'_i = a_i t_i$  for all  $i$ , then  $[t'_i] = [a_i][t_i]$  for all  $i$ . So  $\mathcal{J} = \mathcal{J}'$ , we can apply Lemma 4.2, and we can choose the matrix  $C$  to be the diagonal matrix with entries  $[a_i]$ . Then the theorem follows from formula (4.2.1), because an element such as (4.1.2) only depends upon the class of  $m \bmod (t_1, \dots, t_d)M$ , and here we have the congruence

$$d[t'_1] \cdots d[t'_d] \equiv \left( \prod_{i=1}^d [a_i] \right) d[t_1] \cdots d[t_d] \pmod{\mathcal{J}W_n\Omega_{P/X}^d}.$$

(c) Given  $y \in Y$ , there exists a permutation  $\sigma \in \mathfrak{S}_d$  such that, for any  $i$ ,  $1 \leq i \leq d$ , the sequence  $\mathbf{t}^{(i)} = (t'_{\sigma(1)}, \dots, t'_{\sigma(i)}, t_{i+1}, \dots, t_d)$  is a regular sequence of generators of  $\mathcal{I}$  around  $y$ . Indeed, a sequence of elements of  $\mathcal{I}_y$  is a regular sequence of generators if and only if it gives a basis of  $\mathcal{I}_y/\mathfrak{m}_y\mathcal{I}_y$ , and this reduces the claim to an elementary result in linear algebra over a field. If we set  $\mathbf{t}^{(0)} = (t_1, \dots, t_d)$ , then  $\mathbf{t}^{(0)} = \mathbf{t}$ , and  $\mathbf{t}^{(d)}$  is deduced from  $\mathbf{t}'$  by permutation. So, using remark (a), it suffices to prove the theorem for the couple of sequences  $\mathbf{t}^{(i-1)}$  and  $\mathbf{t}^{(i)}$ , for all  $i$ ,  $1 \leq i \leq d$ .

This reduces the proof to the case where there exists an integer  $i_0 \in \{1, \dots, d\}$  such that

$$t'_i = t_i \quad \text{for } i \neq i_0, \quad t'_{i_0} = \sum_{j=1}^d c_{i_0,j} t_j.$$

Using remark (a), we may assume that  $i_0 = 1$ . Moreover, the fact that  $\mathbf{t}$  and  $\mathbf{t}'$  induce bases of the vector space  $\mathcal{I}_y/\mathfrak{m}_y\mathcal{I}_y$  implies that the coefficient  $c_{1,1}$  is invertible around  $y$ .



(d) In this last case, we define inductively elements  $t_1^{(j)}$  for  $0 \leq j \leq d$  by setting

$$t_1^{(0)} = t_1, \quad t_1^{(1)} = c_{1,1}t_1^{(0)}, \quad t_1^{(j)} = t_1^{(j-1)} + c_{1,j}t_j \quad \text{for } 1 < j.$$

If, for  $0 \leq j \leq d$ , we define  $\mathbf{t}^{(j)} = (t_1^{(j)}, t_2, \dots, t_d)$ , then  $\mathbf{t}^{(0)} = \mathbf{t}$ ,  $\mathbf{t}^{(d)} = \mathbf{t}'$ , and it suffices to prove the theorem for each of the couples  $\mathbf{t}^{(j-1)}, \mathbf{t}^{(j)}$  for  $1 \leq j \leq d$ . The theorem is true for  $\mathbf{t}^{(0)}, \mathbf{t}^{(1)}$ , thanks to remark (b) and, applying again remark (a), we can write all the remaining couples as changes of generators of the form

$$(7.2.2) \quad t'_1 = t_1 + ct_2 \quad \text{for some } c \in \mathcal{O}_P, \quad t'_i = t_i \quad \text{for } i \geq 2.$$

Thus it suffices to prove the theorem for the change of generators of  $I$  given by (7.2.2). Let  $h \in VW_{n-1}(\mathcal{O}_P)$  be defined by setting

$$(7.2.3) \quad [t_1] + [c][t_2] = [t_1 + ct_2] + h = [t'_1] + h$$

in  $W_n(\mathcal{O}_P)$ . Since  $[t'_2] = [t_2]$ , this can be rewritten as

$$(7.2.4) \quad [t_1] = [t'_1] - [c][t'_2] + h.$$

The binomial formula gives

$$(7.2.5) \quad [t_1]^{p^{n-1}} = ([t'_1] - [c][t'_2])^{p^{n-1}} + \sum_{i=1}^{p^{n-1}} \frac{p^{n-1}!}{(p^{n-1} - i)!i!} h^i ([t'_1] - [c][t'_2])^{p^{n-1}-i}.$$

Because the ideal  $VW_{n-1}(\mathcal{O}_P) \subset W_n(\mathcal{O}_P)$  is a PD-ideal, we can write  $h^i = i!h^{[i]}$ , with  $h^{[i]} \in VW_{n-1}(\mathcal{O}_P)$  when  $i \geq 1$ . Therefore the numerical coefficient of  $h^{[i]}$  in the  $i$ -th term of the sum is divisible by  $p^{n-1}$  for all  $i \geq 1$ . Since  $p^{n-1}$  kills  $VW_{n-1}(\mathcal{O}_P)$ , equation (7.2.5) reduces to

$$(7.2.6) \quad [t_1]^{p^{n-1}} = ([t'_1] - [c][t'_2])^{p^{n-1}}.$$

If, for all  $k \geq 1$ , we denote by  $\mathcal{J}^{(k)}$  the ideal  $([t_1]^k, \dots, [t_d]^k)$ , this shows that  $\mathcal{J}^{(p^{n-1})} \subset \mathcal{J}'$ . So we can apply Lemma 4.2 to the sequences  $([t'_1], \dots, [t'_d])$  and  $([t_1]^{p^{n-1}}, \dots, [t_d]^{p^{n-1}})$ , which are regular by Lemma 7.1. Moreover, we can write equation (7.2.6) as

$$[t_1]^{p^{n-1}} = [t'_1]^{p^{n-1}-1} \cdot [t'_1] + c_{1,2} \cdot [t'_2]$$

so that we can use as matrix  $C$  in Lemma 4.2 an upper triangular matrix with diagonal entries  $[t'_1]^{p^{n-1}-1}, \dots, [t'_d]^{p^{n-1}-1}$  (since  $[t_i]^{p^{n-1}} = [t'_i]^{p^{n-1}-1} \cdot [t'_i]$  for  $i \geq 2$ ). In particular,  $\det(C) = [t'_1]^{p^{n-1}-1} \dots [t'_d]^{p^{n-1}-1}$ . Thus, formula (4.2.1) provides the equality

$$(7.2.7) \quad \alpha' \left( \begin{bmatrix} d[t'_1] \cdots d[t'_d] \\ [t'_1], \dots, [t'_d] \end{bmatrix} \right) = \begin{bmatrix} [t'_1]^{p^{n-1}-1} \cdots [t'_d]^{p^{n-1}-1} d[t'_1] \cdots d[t'_d] \\ [t_1]^{p^{n-1}}, \dots, [t_d]^{p^{n-1}} \end{bmatrix},$$

where  $\alpha'$  is the canonical homomorphism

$$\mathcal{E}xt_{W_n(\mathcal{O}_P)}^d(W_n(\mathcal{O}_P)/\mathcal{J}', W_n\Omega_{P/X}^d) \longrightarrow \mathcal{E}xt_{W_n(\mathcal{O}_P)}^d(W_n(\mathcal{O}_P)/\mathcal{J}^{(p^{n-1})}, W_n\Omega_{P/X}^d).$$

On the other hand, we also have  $\mathcal{J}^{(p^{n-1})} \subset \mathcal{J}$ . So we can also apply Lemma 4.2 to the regular sequences  $([t_1], \dots, [t_d])$  and  $([t_1]^{p^{n-1}}, \dots, [t_d]^{p^{n-1}})$ , using now for  $C$  the diagonal matrix with entries  $[t_1]^{p^{n-1}-1}, \dots, [t_d]^{p^{n-1}-1}$ . If we denote by

$$\begin{aligned} \alpha : \mathcal{E}xt_{W_n(\mathcal{O}_P)}^d(W_n(\mathcal{O}_P)/\mathcal{J}, W_n\Omega_{P/X}^d) \\ \longrightarrow \mathcal{E}xt_{W_n(\mathcal{O}_P)}^d(W_n(\mathcal{O}_P)/\mathcal{J}^{(p^{n-1})}, W_n\Omega_{P/X}^d) \end{aligned}$$

the canonical homomorphism, formula (4.2.1) provides the second equality

$$(7.2.8) \quad \alpha \left( \begin{bmatrix} d[t_1] \cdots d[t_d] \\ [t_1], \dots, [t_d] \end{bmatrix} \right) = \begin{bmatrix} [t_1]^{p^{n-1}-1} \cdots [t_d]^{p^{n-1}-1} d[t_1] \cdots d[t_d] \\ [t_1]^{p^{n-1}}, \dots, [t_d]^{p^{n-1}} \end{bmatrix}.$$

As  $\beta_{\mathcal{J}} = \beta_{\mathcal{J}^{(p^{n-1})}} \circ \alpha$  and  $\beta_{\mathcal{J}'} = \beta_{\mathcal{J}^{(p^{n-1})}} \circ \alpha'$ , relation (7.2.1) will follow if we prove the equality

$$(7.2.9) \quad \begin{aligned} & \begin{bmatrix} [t'_1]^{p^{n-1}-1} \cdots [t'_d]^{p^{n-1}-1} d[t'_1] \cdots d[t'_d] \\ [t_1]^{p^{n-1}}, \dots, [t_d]^{p^{n-1}} \end{bmatrix} \\ &= \begin{bmatrix} [t_1]^{p^{n-1}-1} \cdots [t_d]^{p^{n-1}-1} d[t_1] \cdots d[t_d] \\ [t_1]^{p^{n-1}}, \dots, [t_d]^{p^{n-1}} \end{bmatrix} \end{aligned}$$

in  $\mathcal{E}xt_{W_n(\mathcal{O}_P)}^d(W_n(\mathcal{O}_P)/\mathcal{J}^{(p^{n-1})}, W_n\Omega_{P/X}^d)$ . To prove this it suffices to prove in  $W_n\Omega_{P/X}^d$  the congruence

$$(7.2.10) \quad \begin{aligned} & [t'_1]^{p^{n-1}-1} [t'_2]^{p^{n-1}-1} \cdots [t'_d]^{p^{n-1}-1} d[t'_1] d[t'_2] \cdots d[t'_d] \\ & \equiv [t_1]^{p^{n-1}-1} [t_2]^{p^{n-1}-1} \cdots [t_d]^{p^{n-1}-1} d[t_1] d[t_2] \cdots d[t_d] \end{aligned}$$

mod  $([t_1]^{p^{n-1}}, [t_2]^{p^{n-1}}, \dots, [t_d]^{p^{n-1}})W_n\Omega_{P/X}^d$ . As  $t_i = t'_i$  for  $i > 2$ , it suffices by multiplicativity to prove in  $W_n\Omega_{P/X}^2$  the congruence

$$[t'_1]^{p^{n-1}-1} [t'_2]^{p^{n-1}-1} d[t'_1] d[t'_2] \equiv [t_1]^{p^{n-1}-1} [t_2]^{p^{n-1}-1} d[t_1] d[t_2]$$

mod  $([t_1]^{p^{n-1}}, [t_2]^{p^{n-1}})W_n\Omega_{P/X}^2$  and, thanks to (5.1.3), the latter will follow by applying  $F^{n-1}$  if we prove the congruence

$$(7.2.11) \quad d[t'_1] d[t'_2] \equiv d[t_1] d[t_2] \pmod{([t_1], [t_2])W_{2n-1}\Omega_{P/X}^2}.$$

So let us prove (7.2.11). We still denote by  $h \in VW_{2n-2}\mathcal{O}_P$  the difference  $h = [t_1] + [c][t_2] - [t'_1] = [t_1] + [ct_2] - [t_1 + ct_2]$  computed in  $W_{2n-1}\mathcal{O}_P$ . Since  $t'_2 = t_2$ , it suffices to prove the congruence

$$(7.2.12) \quad dh d[t_2] \equiv 0 \pmod{([t_1], [t_2])W_{2n-1}\Omega_{P/X}^2}.$$

For all  $i$ , let

$$S_i(X_0, \dots, X_i, Y_0, \dots, Y_i) \in \mathbb{Z}[X_0, \dots, X_i, Y_0, \dots, Y_i]$$

be the universal polynomial defining the  $i$ -th component of the sum of two Witt vectors, and let

$$(7.2.13) \quad s_i(X_0, Y_0) = S_i(X_0, 0, \dots, 0, Y_0, 0, \dots, 0) \in \mathbb{Z}[X_0, Y_0].$$

Note that, for  $i \geq 1$ , the polynomial  $s_i(X_0, Y_0)$  is divisible by  $X_0 Y_0$ , since  $(0, \dots, 0)$  is the zero element in a Witt vector ring. By definition, we have

$$[t_1] + [ct_2] = (t_1 + ct_2, s_1(t_1, ct_2), \dots, s_{2n-2}(t_1, ct_2))$$

and

$$h = (0, s_1(t_1, ct_2), \dots, s_{2n-2}(t_1, ct_2)).$$

Since  $s_i(X_0, Y_0)$  is divisible by  $Y_0$ , we can write  $s_i(t_1, ct_2) = z_i t_2$  for some section  $z_i \in \mathcal{O}_P$ . We obtain

$$h = (0, z_1 t_2, \dots, z_{2n-2} t_2),$$

which we can write as

$$h = \sum_{i=1}^{2n-2} V^i([z_i][t_2]).$$

For each  $i$ ,  $1 \leq i \leq 2n - 2$ , we now obtain the relations

$$\begin{aligned} dV^i([z_i][t_2]) d[t_2] &= dV^i([z_i][t_2] F^i(d[t_2])) = dV^i([z_i][t_2]^{p^i} d[t_2]) \\ &= dV^i([z_i] F^i([t_2]) d[t_2]) = d([t_2] V^i([z_i] d[t_2])), \\ d([t_2] V^i([z_i] d[t_2])) &\equiv d[t_2] V^i([z_i] d[t_2]) \pmod{[t_2] W_{2n-1} \Omega_{P/X}^2}, \\ d[t_2] V^i([z_i] d[t_2]) &= V^i(F^i(d[t_2])[z_i] d[t_2]) = V^i([t_2]^{p^i-1} d[t_2][z_i] d[t_2]) = 0, \end{aligned}$$

which imply (7.2.12). □

*Definition 7.3.* Under the assumptions of this section, we define the local class  $\gamma_{Y,n} \in \Gamma(P, \mathcal{H}_Y^d(W_n \Omega_{P/X}^d))$  as being the section obtained by glueing the sections  $\beta_{\mathcal{J}} \left( \begin{bmatrix} d[t_1] \cdots d[t_d] \\ [t_1], \dots, [t_d] \end{bmatrix} \right)$  defined locally by regular sequences of generators of the ideal  $\mathcal{I}$  of  $Y$  in  $P$ .

**PROPOSITION 7.4.** For  $n \geq 1$ , let

$$\begin{aligned} R &: \mathcal{H}_Y^d(W_{n+1} \Omega_{P/X}^d) \longrightarrow \mathcal{H}_Y^d(W_n \Omega_{P/X}^d), \\ F &: \mathcal{H}_Y^d(W_{n+1} \Omega_{P/X}^d) \longrightarrow \mathcal{H}_Y^d(W_n \Omega_{P/X}^d), \\ V &: \mathcal{H}_Y^d(W_n \Omega_{P/X}^d) \longrightarrow \mathcal{H}_Y^d(W_{n+1} \Omega_{P/X}^d) \end{aligned}$$

be the homomorphisms defined by functoriality. Then

$$(7.4.1) \quad R(\gamma_{Y,n+1}) = \gamma_{Y,n}, \quad F(\gamma_{Y,n+1}) = \gamma_{Y,n}, \quad V(\gamma_{Y,n}) = p\gamma_{Y,n+1}.$$

*Proof.* We may assume that there exists a regular sequence  $t_1, \dots, t_d$  such that  $\mathcal{I} = (t_1, \dots, t_d)$ . For each  $n \geq 1$ , let  $\mathcal{J}_n$  be the ideal of  $W_n(\mathcal{O}_P)$  generated by the Teichmüller representatives  $[t_i]$  of the  $t_i$ 's, and let  $K_\bullet([t]_n)$  be the Koszul complex defined by the  $[t_i]$ 's over  $W_n(\mathcal{O}_P)$ . Since  $R([t_i]) = [t_i]$ , scalar extension through  $R$  yields an isomorphism

$$W_n(\mathcal{O}_P) \otimes_{W_{n+1}(\mathcal{O}_P)} K_\bullet([t]_{n+1}) \xrightarrow{\sim} K_\bullet([t]_n).$$

Using the fact that the  $[t_i]$ 's form a regular sequence both in  $W_{n+1}(\mathcal{O}_P)$  and in  $W_n(\mathcal{O}_P)$ , it can be seen in the derived category of  $W_n(\mathcal{O}_P)$ -modules as an isomorphism

$$(7.4.2) \quad W_n(\mathcal{O}_P) \overset{L}{\otimes}_{W_{n+1}(\mathcal{O}_P)} W_{n+1}(\mathcal{O}_P)/\mathcal{J}_{n+1} \xrightarrow{\sim} W_n(\mathcal{O}_P)/\mathcal{J}_n.$$

By adjunction, for any  $W_n(\mathcal{O}_P)$ -module  $\mathcal{M}$  and any  $q \geq 0$ , (7.4.2) defines an isomorphism

$$(7.4.3) \quad \begin{aligned} \mathcal{E}xt_{W_n(\mathcal{O}_P)}^q(W_n(\mathcal{O}_P)/\mathcal{J}_n, \mathcal{M}) \\ \xrightarrow{\sim} \mathcal{E}xt_{W_{n+1}(\mathcal{O}_P)}^q(W_{n+1}(\mathcal{O}_P)/\mathcal{J}_{n+1}, \mathcal{M}), \end{aligned}$$

and we obtain the diagram

$$(7.4.4) \quad \begin{array}{ccccc} \mathcal{H}^d(\mathcal{H}om_{W_{n+1}(\mathcal{O}_P)}^\bullet(K_\bullet([t]_{n+1}), W_{n+1}\Omega_{P/X}^d)) & & & & \\ \downarrow R & \searrow \sim & & & \\ \mathcal{E}xt_{W_{n+1}(\mathcal{O}_P)}^d(W_{n+1}(\mathcal{O}_P)/\mathcal{J}_{n+1}, W_{n+1}\Omega_{P/X}^d) & \xrightarrow{\beta_{\mathcal{J}_{n+1}}} & \mathcal{H}_Y^d(W_{n+1}\Omega_{P/X}^d) & & \\ \downarrow & & \downarrow R & & \\ \mathcal{H}^d(\mathcal{H}om_{W_{n+1}(\mathcal{O}_P)}^\bullet(K_\bullet([t]_{n+1}), W_n\Omega_{P/X}^d)) & \searrow \sim & & & \\ \downarrow \wr & & \mathcal{E}xt_{W_{n+1}(\mathcal{O}_P)}^d(W_{n+1}(\mathcal{O}_P)/\mathcal{J}_{n+1}, W_n\Omega_{P/X}^d) & \xrightarrow{\beta_{\mathcal{J}_{n+1}}} & \mathcal{H}_Y^d(W_n\Omega_{P/X}^d) \\ \uparrow & & \downarrow & & \parallel \\ \mathcal{H}^d(\mathcal{H}om_{W_n(\mathcal{O}_P)}^\bullet(K_\bullet([t]_n), W_n\Omega_{P/X}^d)) & \searrow \sim & \wr (7.4.3) & & \\ \downarrow & & \uparrow & & \\ \mathcal{E}xt_{W_n(\mathcal{O}_P)}^d(W_n(\mathcal{O}_P)/\mathcal{J}_n, W_n\Omega_{P/X}^d) & \xrightarrow{\beta_{\mathcal{J}_n}} & \mathcal{H}_Y^d(W_n\Omega_{P/X}^d), & & \end{array}$$

in which the lower left-hand square commutes by construction. On the other hand, (7.4.3) implies that injective  $W_n(\mathcal{O}_P)$ -modules are acyclic for the functor  $\mathcal{H}om_{W_{n+1}(\mathcal{O}_P)}(W_{n+1}(\mathcal{O}_P)/\mathcal{J}_{n+1}, -)$ . Replacing  $W_n\Omega_{P/X}^d$  by an injective resolution over  $W_n(\mathcal{O}_P)$ , it is then easy to check that the lower right square commutes. As the upper part of the diagram commutes by functoriality, and  $R(d[t_1] \cdots d[t_d]) = d[t_1] \cdots d[t_d]$ , the first relation of (7.4.1) follows.

Viewing now  $W_n(\mathcal{O}_P)$  as a  $W_{n+1}(\mathcal{O}_P)$ -algebra via  $F$ , one proceeds similarly to prove the second one. Since  $F([t_i]) = [t_i^p] = [t_i]^p$ , and the sequence

$[t_i]^p, \dots, [t_d]^p$  is a regular sequence in  $W_n(\mathcal{O}_P)$ , we obtain isomorphisms

$$(7.4.5) \quad \begin{aligned} &W_n(\mathcal{O}_P) \otimes_{W_{n+1}(\mathcal{O}_P)} K_\bullet([t]_{n+1}) \xrightarrow{\sim} K_\bullet([t]_n^p), \\ &W_n(\mathcal{O}_P) \otimes_{W_{n+1}(\mathcal{O}_P)}^L W_{n+1}(\mathcal{O}_P)/\mathcal{J}_{n+1} \xrightarrow{\sim} W_n(\mathcal{O}_P)/\mathcal{J}_n^{(p)}, \end{aligned}$$

and

$$(7.4.6) \quad \text{Ext}_{W_n(\mathcal{O}_P)}^q(W_n(\mathcal{O}_P)/\mathcal{J}_n^{(p)}, \mathcal{M}) \xrightarrow{\sim} \text{Ext}_{W_{n+1}(\mathcal{O}_P)}^q(W_{n+1}(\mathcal{O}_P)/\mathcal{J}_{n+1}, \mathcal{M})$$

for any  $W_n(\mathcal{O}_P)$ -module  $\mathcal{M}$  and any  $q \geq 0$ . They provide a commutative diagram similar to (7.4.4):

$$(7.4.7) \quad \begin{array}{ccccc} \mathcal{H}^d(\text{Hom}_{W_{n+1}(\mathcal{O}_P)}^\bullet(K_\bullet([t]_{n+1}), W_{n+1}\Omega_{P/X}^d)) & & & & \\ \downarrow F & \searrow \sim & & & \\ \text{Ext}_{W_{n+1}(\mathcal{O}_P)}^d(W_{n+1}(\mathcal{O}_P)/\mathcal{J}_{n+1}, W_{n+1}\Omega_{P/X}^d) & \xrightarrow{\beta_{\mathcal{J}_{n+1}}} & \mathcal{H}_Y^d(W_{n+1}\Omega_{P/X}^d) & & \\ \downarrow F & & \downarrow F & & \\ \mathcal{H}^d(\text{Hom}_{W_{n+1}(\mathcal{O}_P)}^\bullet(K_\bullet([t]_{n+1}), W_n\Omega_{P/X}^d)) & & \mathcal{H}_Y^d(W_n\Omega_{P/X}^d) & & \\ \uparrow \wr & \searrow \sim & \xrightarrow{\beta_{\mathcal{J}_{n+1}}} & & \\ \text{Ext}_{W_{n+1}(\mathcal{O}_P)}^d(W_{n+1}(\mathcal{O}_P)/\mathcal{J}_{n+1}, W_n\Omega_{P/X}^d) & & & & \\ \uparrow (7.4.6) & & & & \\ \mathcal{H}^d(\text{Hom}_{W_n(\mathcal{O}_P)}^\bullet(K_\bullet([t]_n^p), W_n\Omega_{P/X}^d)) & & & & \\ \downarrow \wr & \searrow \sim & & & \\ \text{Ext}_{W_n(\mathcal{O}_P)}^d(W_n(\mathcal{O}_P)/\mathcal{J}_n^{(p)}, W_n\Omega_{P/X}^d) & \xrightarrow{\beta_{\mathcal{J}_n^{(p)}}} & \mathcal{H}_Y^d(W_n\Omega_{P/X}^d) & & \end{array}$$

Since  $F(d[t_1] \cdots d[t_d]) = [t_1]^{p-1} \cdots [t_d]^{p-1} d[t_1] \cdots d[t_d]$ , it follows that

$$F\left(\beta_{\mathcal{J}_{n+1}}\left(\begin{bmatrix} d[t_1] \cdots d[t_d] \\ [t_1], \dots, [t_d] \end{bmatrix}\right)\right) = \beta_{\mathcal{J}_n^{(p)}}\left(\begin{bmatrix} [t_1]^{p-1} \cdots [t_d]^{p-1} d[t_1] \cdots d[t_d] \\ [t_1]^p, \dots, [t_d]^p \end{bmatrix}\right).$$

On the other hand, if  $\alpha$  denotes the canonical homomorphism

$$\begin{aligned} \alpha : \text{Ext}_{W_n(\mathcal{O}_P)}^d(W_n(\mathcal{O}_P)/\mathcal{J}_n, W_n\Omega_{P/X}^d) & \\ \longrightarrow \text{Ext}_{W_n(\mathcal{O}_P)}^d(W_n(\mathcal{O}_P)/\mathcal{J}_n^{(p)}, W_n\Omega_{P/X}^d), & \end{aligned}$$

then by (4.2.1), we have

$$\alpha\left(\begin{bmatrix} d[t_1] \cdots d[t_d] \\ [t_1], \dots, [t_d] \end{bmatrix}\right) = \begin{bmatrix} [t_1]^{p-1} \cdots [t_d]^{p-1} d[t_1] \cdots d[t_d] \\ [t_1]^p, \dots, [t_d]^p \end{bmatrix}.$$

As  $\beta_{\mathcal{J}_n^{(p)}} \circ \alpha = \beta_{\mathcal{J}_n}$ , it follows that  $F(\gamma_{Y,n+1}) = \gamma_{Y,n}$ .

The last relation of (7.4.1) follows formally because  $V(\gamma_{Y,n}) = V(F(\gamma_{Y,n+1})) = p\gamma_{Y,n+1}$ .  $\square$

PROPOSITION 7.5. *Let  $n \geq 1$  be an integer, and let  $\gamma_{Y,n} \in \mathcal{H}_Y^d(W_n\Omega_{P/X}^d)$  be the local class defined in 7.3.*

(i) *The linear homomorphism  $W_n(\mathcal{O}_P) \rightarrow \mathcal{H}_Y^d(W_n\Omega_{P/X}^d)$  sending 1 to  $\gamma_{Y,n}$  vanishes on  $W_n(\mathcal{I}) := \text{Ker}(W_n(\mathcal{O}_P) \rightarrow i_*W_n(\mathcal{O}_Y))$ .*

(ii) *Let  $\gamma_{i,\pi,n}$  be the composition*

$$(7.5.1) \quad \gamma_{i,\pi,n} : i_*W_n(\mathcal{O}_Y) \longrightarrow \mathcal{H}_Y^d(W_n\Omega_{P/X}^d) \xrightarrow{\sim} R\Gamma_Y(W_n\Omega_{P/X}^d[d]) \longrightarrow W_n\Omega_{P/X}^d[d],$$

*where the first morphism is defined thanks to the previous assertion. Then  $\gamma_{i,\pi,n}$  commutes with  $R$ ,  $F$ , and  $V$ .*

(iii) *For  $n = 1$ , we have  $\gamma_{i,\pi,1} = \gamma_f$ , where  $\gamma_f$  is the morphism defined by (4.4.1).*

*Proof.* To prove assertion (i), we may again assume that  $\mathcal{I}$  is generated by a regular sequence  $t_1, \dots, t_d$ . Any section  $w$  of  $W_n(\mathcal{I})$  can then be written as a sum

$$w = \sum_{i=0}^{n-1} V^i([a_{i,1}][t_1] + \dots + [a_{i,d}][t_d]),$$

with  $a_{i,j} \in \mathcal{I}$  and  $[a_{i,j}], [t_j] \in W_{n-i}(\mathcal{O}_P)$ . By functoriality, we have  $V(a)\omega = V(aF(\omega))$  for any  $a \in W_i(\mathcal{O}_P)$ ,  $\omega \in \mathcal{H}_Y^d(W_{i+1}\Omega_{P/X}^d)$ ,  $i \geq 1$ . Using (7.4.1), we obtain

$$V^i([a_{i,j}][t_j])\gamma_{Y,n} = V^i([a_{i,j}][t_j]F^i(\gamma_{Y,n})) = V^i([a_{i,j}][t_j]\gamma_{Y,n-i}).$$

The symbol (4.1.2) is linear with respect to  $m$ ; therefore, we have

$$[a_{i,j}][t_j]\gamma_{Y,n-i} = \beta_{\mathcal{J}} \left( \begin{bmatrix} [a_{i,j}][t_j] d[t_1] \cdots d[t_d] \\ [t_1], \dots, [t_d] \end{bmatrix} \right) = 0,$$

since the upper entry in the symbol belongs to  $([t_1], \dots, [t_d])W_{n-i}\Omega_{P/X}^d$ .

In the definition of  $\gamma_{i,\pi,n}$ , the last two arrows commute with  $R$ ,  $F$ , and  $V$  by functoriality. Relations (7.4.1) imply that the first one also commutes with  $R$ ,  $F$ , and  $V$ , since  $R(1) = F(1) = 1$  and  $V(1) = p$ .

Let us assume that  $n = 1$  and check assertion (iii). By construction,  $\gamma_{i,\pi,1}$  is the composition of the morphism  $i_*\mathcal{O}_Y \rightarrow \mathcal{H}_Y^d(\Omega_{P/X}^d)$  sending 1 to  $\gamma_{Y,1}$  with the canonical morphism

$$\mathcal{H}_Y^d(\Omega_{P/X}^d) \xrightarrow{\sim} R\Gamma_Y(\Omega_{P/X}^d[d]) \longrightarrow \Omega_{P/X}^d[d].$$

Comparing with the definition of  $\gamma_f$  in 4.4, and using the same notation, it suffices to show that the composed morphism

$$\mathcal{O}_Y \xrightarrow{\varphi_f} \omega_{Y/X} \xrightarrow{\eta_i^{-1} \circ \zeta'_{i,\pi}} \mathcal{E}xt_{\mathcal{O}_P}^d(\mathcal{O}_Y, \Omega_{P/X}^d) \xrightarrow{\beta_{\mathcal{I}}} \mathcal{H}_Y^d(\Omega_{P/X}^d)$$

sends 1 to  $\gamma_{Y,1}$ . Since this is a morphism of sheaves (rather than complexes in the derived category), it is a local verification, which is provided by Proposition 4.5. □

*Definition 7.6.* Let  $X$  be a noetherian  $\mathbb{F}_p$ -scheme with a dualizing complex,  $\mathcal{E}$  a locally free  $\mathcal{O}_X$ -module of rank  $d + 1$ ,  $P = \mathbb{P}(\mathcal{E})$ ,  $\pi : P \rightarrow X$  the canonical projection, and  $i : Y \hookrightarrow P$  a regular closed immersion of codimension  $d$ . For each integer  $n \geq 1$ , we define a trace morphism  $\tau_{i,\pi,n}$  by

$$(7.6.1) \quad \tau_{i,\pi,n} : Rf_*(W_n(\mathcal{O}_Y)) \xrightarrow{R\pi_*(\gamma_{i,\pi,n})} R\pi_*(W_n\Omega_{P/X}^d[d]) \xrightarrow{\mathrm{Trp}_{\pi,n}} W_n(\mathcal{O}_X),$$

where  $\gamma_{i,\pi,n}$  is the morphism (7.5.1) and  $\mathrm{Trp}_{\pi,n}$  is the Hodge-Witt trace morphism defined in (6.5.1).

*Remark.* As mentioned in the introduction, we expect that  $\tau_{i,\pi,n}$  depends only on  $f$  and not on the factorization  $f = \pi \circ i$ . We also expect that the analog of Theorem 3.1 holds for the trace morphisms  $\tau_{f,n}$  that would be thus defined. More generally, one can hope that these constructions are part of a theory of canonical classes for relative de Rham-Witt cohomology (see [EZ78], [Eke84], [Grs85] for such results over a field). In order to develop this program, generalizations and nontrivial properties of our constructions are needed (even for the independence statement), which would lead us to expand this article too much. As most of them are not needed for the proof of our main results, we do not include them here, and we hope to return to these questions elsewhere. However, in the next section we will give a partial generalization of Theorem 3.1(iii) that is the key to the injectivity property of Theorem 1.5.

**PROPOSITION 7.7.** *Under the assumptions of 7.6, the morphisms  $\tau_{i,\pi,n}$  satisfy the following properties:*

- (i) For variable  $n$ ,  $\tau_{i,\pi,n}$  commutes with  $R$ ,  $F$  and  $V$ .
- (ii) For  $n = 1$ ,  $\tau_{i,\pi,1} = \tau_f$ .

*Proof.* Taking into account Proposition 4.6, both assertions follow from the similar properties of  $\gamma_{i,\pi,n}$  and  $\mathrm{Trp}_{\pi,n}$  proved in 7.5 and 6.6.  $\square$

*Definition 7.8.* Under the assumptions of 7.6, we can use the previous constructions to define a morphism  $\tau_{i,\pi} : Rf_*(W(\mathcal{O}_Y)) \rightarrow W(\mathcal{O}_X)$  that commutes with  $F$  and  $V$  and is such that  $R_n \circ \tau_{i,\pi} = \tau_{i,\pi,n} \circ R_n$  for all  $n$ ,  $R_n$  denoting both restriction maps  $W(\mathcal{O}_X) \rightarrow W_n(\mathcal{O}_X)$  and  $W(\mathcal{O}_Y) \rightarrow W_n(\mathcal{O}_Y)$ .

To construct  $\tau_{i,\pi}$ , we first recall that, for any scheme  $X$ , the inverse system  $(W_n(\mathcal{O}_X))_{n \geq 0}$  is  $\varprojlim$ -acyclic, as the cohomology of each term vanishes on affine open subsets, and the inverse system of sections on such a subset has surjective transition maps. So, if  $f_{\bullet,*}$  denotes the obvious extension of the direct image functor to the category of inverse systems, it suffices to define a morphism

$$(7.8.1) \quad \tau_{i,\pi,\bullet} : Rf_{\bullet,*}(W_{\bullet}(\mathcal{O}_Y)) \rightarrow W_{\bullet}(\mathcal{O}_X)$$

in the derived category of inverse systems on  $X$  and to apply the functor  $R\varprojlim$  and the canonical isomorphism  $Rf_* \circ R\varprojlim \simeq R\varprojlim \circ Rf_{\bullet,*}$ . On the one hand, the relations  $R(\gamma_{Y,n+1}) = \gamma_{Y,n}$  imply that, for variable  $n$ , these classes define a

morphism of inverse systems  $i_{\bullet*}(W_{\bullet}(\mathcal{O}_Y)) \rightarrow \mathcal{H}_Y^d(W_{\bullet}\Omega_{P/X}^d)$ . As the canonical morphisms

$$\mathcal{H}_Y^d(W_{\bullet}\Omega_{P/X}^d) \xrightarrow{\sim} R\Gamma_Y(W_{\bullet}\Omega_{P/X}^d[d]) \longrightarrow W_{\bullet}\Omega_{P/X}^d[d]$$

make sense in the derived category of inverse systems, in this derived category we can define a morphism  $\gamma_{i,\pi,\bullet} : i_{\bullet*}(W_{\bullet}(\mathcal{O}_Y)) \rightarrow W_{\bullet}\Omega_{P/X}^d[d]$  that has the morphisms  $\gamma_{i,\pi,n}$  defined in (7.5.1) as components. On the other hand, the homomorphisms  $\text{dlog}_n$  used to define Chern classes for invertible bundles form an inverse system of homomorphisms. Hence, for variable  $n$ , the powers of the Chern classes of  $\mathcal{O}_P(1)$  define a morphism  $W_{\bullet}(\mathcal{O}_P)[-d] \rightarrow R\pi_{\bullet*}(W_{\bullet}\Omega_{P/X}^d)$  that is an isomorphism of the derived category of inverse systems. Composing its inverse with the projection by  $R\pi_{\bullet*}$  of  $\gamma_{i,\pi,\bullet}$  provides  $\tau_{i,\pi,\bullet}$ . It is clear that  $\tau_{i,\pi,\bullet}$  has the morphisms  $\tau_{i,\pi,n}$  as components and commutes with  $F$  and  $V$ . Then the morphism

$$(7.8.2) \quad \tau_{i,\pi} : Rf_*(W(\mathcal{O}_Y)) \xrightarrow{\sim} R\varprojlim Rf_{\bullet*}(W_{\bullet}(\mathcal{O}_Y)) \xrightarrow{R\varprojlim(\tau_{i,\pi,\bullet})} W(\mathcal{O}_X)$$

has the required properties.

Finally, as  $f$  is a morphism of noetherian schemes,  $f_*$  and  $Rf_*$  commute with tensorization with  $\mathbb{Q}$ . So we can define a morphism, again denoted  $\tau_{i,\pi} : Rf_*(W\mathcal{O}_{Y,\mathbb{Q}}) \rightarrow W\mathcal{O}_{X,\mathbb{Q}}$ , by

$$(7.8.3) \quad \tau_{i,\pi} : Rf_*(W\mathcal{O}_{Y,\mathbb{Q}}) \xrightarrow{\sim} Rf_*(W\mathcal{O}_Y) \otimes \mathbb{Q} \xrightarrow{\tau_{i,\pi} \otimes \mathbb{Q}} W\mathcal{O}_{X,\mathbb{Q}}.$$

This morphism also commutes with  $F$  and  $V$ .

### 8. Proof of the injectivity theorem for Witt vector cohomology

The main result of this section is Theorem 8.1 below, which gives an injectivity property for the functoriality morphisms induced on Witt vector cohomology by some complete intersection morphisms of virtual relative dimension 0. As explained in Remark 8.2, Theorem 1.5 is a particular case of this result.

**THEOREM 8.1.** *Let  $f : Y \rightarrow X$  be a projective morphism between two flat noetherian  $\mathbb{Z}_{(p)}$ -schemes with dualizing complexes, that is complete intersection of virtual relative dimension 0. We assume that there exists a scheme-theoretically dense open subscheme  $U \subset X$  such that  $f^{-1}(U) \rightarrow U$  is finite locally free of constant rank  $r \geq 1$ . Let  $f_n : Y_n \rightarrow X_n$  be the reduction of  $f$  mod  $p^{n+1}$ .*

(i) *For all  $q \geq 0$ , the kernels of the functoriality homomorphisms*

$$(8.1.1) \quad f^* : H^q(X, \mathcal{O}_X) \longrightarrow H^q(Y, \mathcal{O}_Y),$$

$$(8.1.2) \quad f_n^* : H^q(X_n, \mathcal{O}_{X_n}) \longrightarrow H^q(Y_n, \mathcal{O}_{Y_n}),$$



$$(8.1.3) \quad f_0^* : H^q(X_0, W_n(\mathcal{O}_{X_0})) \longrightarrow H^q(Y_0, W_n(\mathcal{O}_{Y_0})),$$

$$(8.1.4) \quad f_0^* : H^q(X_0, W(\mathcal{O}_{X_0})) \longrightarrow H^q(Y_0, W(\mathcal{O}_{Y_0}))$$

are annihilated by  $r$ .

(ii) For all  $q \geq 0$ , the functoriality homomorphism

$$(8.1.5) \quad f_0^* : H^q(X_0, W\mathcal{O}_{X_0, \mathbb{Q}}) \longrightarrow H^q(Y_0, W\mathcal{O}_{Y_0, \mathbb{Q}})$$

is injective.

*Remark 8.2.* Theorem 8.1 implies Theorem 1.5. Indeed, let  $f : Y \rightarrow X$  be as in 1.5. The morphisms  $X_k \hookrightarrow X_0$  and  $Y_k \hookrightarrow Y_0$  are nilpotent immersions. Hence the canonical homomorphisms

$$\begin{aligned} H^q(X_0, W\mathcal{O}_{X_0, \mathbb{Q}}) &\longrightarrow H^q(X_k, W\mathcal{O}_{X_k, \mathbb{Q}}), \\ H^q(Y_0, W\mathcal{O}_{Y_0, \mathbb{Q}}) &\longrightarrow H^q(Y_k, W\mathcal{O}_{Y_k, \mathbb{Q}}) \end{aligned}$$

are isomorphisms [BBE07, Prop. 2.1]. Therefore it suffices to check that  $f$  satisfies the hypotheses of Theorem 8.1. We may assume that  $X$  is connected, and we replace  $Y$  by one of its connected components mapping surjectively to  $X$  so that  $X$  and  $Y$  are integral schemes. At any closed point  $y \in Y$ , with image  $x = f(y)$ , we may choose a closed immersion  $Y \hookrightarrow P$  around  $y$ , with  $P$  smooth over  $X$ . If  $\dim \mathcal{O}_{X,x} = n$ , then  $\mathcal{O}_{P,y}$  is a regular local ring of dimension  $n + d$  for  $d = \dim(P/X)$ , and  $\mathcal{O}_{Y,y}$  is a regular quotient of  $\mathcal{O}_{P,y}$  of dimension  $n$ . Therefore, the ideal  $\mathcal{I}$  of  $Y$  in  $P$  is regular of codimension  $d$  around  $y$ , and it follows that  $f$  is complete intersection of virtual relative dimension 0. Moreover, the function field extension  $K(X) \hookrightarrow K(Y)$  is finite. Hence,  $f$  is finite and locally free of constant rank  $\geq 1$  above a nonempty open subset  $U$ . As  $X$  is integral,  $U$  is scheme-theoretically dense and the hypotheses of Theorem 8.1 are satisfied.

In order to prove Theorem 8.1, we will choose a factorization  $f = \pi \circ i$ , where  $i : Y \hookrightarrow P = \mathbb{P}_X^d$  is a closed immersion and  $\pi : P \rightarrow X$  the structural morphism. Let  $i_0, \pi_0$  be the reductions mod  $p$  of  $i, \pi$ . The key point will be to relate the trace morphisms  $\tau_{i_0, \pi_0, n}$  constructed in 7.6 to the trace morphism  $\tau_f$  given by Theorem 3.1, and this is made possible by the following constructions.

**LEMMA 8.3.** *Let  $X$  be a scheme on which  $p$  is locally nilpotent,  $P$  a smooth  $X$ -scheme,  $\mathfrak{a} \subset \mathcal{O}_X$  a quasi-coherent ideal,  $X' \hookrightarrow X$  the closed subscheme defined by  $\mathfrak{a}$ ,  $P' = X' \times_X P$ . For each  $n \geq 1$ , let  $\mathcal{N}_n^\bullet \subset W_n \Omega_{P/X}^\bullet$  be the additive subgroup generated by sections of the form*

$$(8.3.1) \quad V^r([a]\omega), \quad dV^r([a]\omega), \quad \text{with } a \in \mathfrak{a}, \quad \omega \in W_{n-r} \Omega_{P/X}^\bullet, \quad 0 \leq r \leq n-1.$$

*Then, for variable  $n$ , the canonical homomorphisms  $W_n \Omega_{P/X}^\bullet \rightarrow W_n \Omega_{P'/X'}^\bullet$  induce a transitive family of isomorphisms*

$$(8.3.2) \quad W_n \Omega_{P/X}^\bullet / \mathcal{N}_n^\bullet \xrightarrow{\sim} W_n \Omega_{P'/X'}^\bullet.$$

*Proof.* Thanks to (5.1.2), one first notices that  $\mathcal{N}_n^\bullet$  is a differential graded ideal of  $W_n\Omega_{P/X}^\bullet$ . Using (5.1.5), one sees that, for all  $n \geq 1$ ,  $V(\mathcal{N}_n^\bullet) \subset \mathcal{N}_{n+1}^\bullet$ . Using (5.1.1) (and a direct computation for  $r = 0$ ), one sees that  $F(\mathcal{N}_{n+1}^\bullet) \subset \mathcal{N}_n^\bullet$ . Therefore, the projective system  $\{W_n\Omega_{P/X}^\bullet/\mathcal{N}_n^\bullet\}$  is an  $F$ - $V$ -procomplex over  $P/X$ . In degree 0, it is easy to see by induction on  $n$  that the ideal  $\mathcal{N}_n^0 \subset W_n(\mathcal{O}_P)$  is the kernel of  $W_n(\mathcal{O}_P) \rightarrow W_n(\mathcal{O}_{P'})$ . It follows that  $\{W_n\Omega_{P/X}^\bullet/\mathcal{N}_n^\bullet\}$  is actually an  $F$ - $V$ -procomplex over  $P'/X'$ . It is then clear that it satisfies the universal property that defines  $\{W_n\Omega_{P'/X'}^\bullet\}$ , which implies that (8.3.2) is an isomorphism of  $F$ - $V$ -procomplexes.  $\square$

PROPOSITION 8.4 (see also [Ols07, Th. 4.2.3]). *Let  $X$  be a  $\mathbb{Z}_{(p)}$ -scheme, and denote  $X_n = X \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_{(p)}/p^{n+1}$ .*

(i) *For all  $n \geq 1$ , there exists a unique homomorphism of sheaves of rings*

$$\tilde{F}^n : W_n(\mathcal{O}_{X_0}) \longrightarrow \mathcal{O}_{X_{n-1}}$$

*making the following diagram commute*

$$\begin{array}{ccc} W_{n+1}(\mathcal{O}_{X_{n-1}}) & \xrightarrow{F^n} & \mathcal{O}_{X_{n-1}} \\ \downarrow & \nearrow \tilde{F}^n & \\ W_n(\mathcal{O}_{X_0}) & & \end{array}$$

*where the vertical map is the natural reduction map. Furthermore, if we assume  $X$  to be flat over  $\mathbb{Z}_{(p)}$  and denote by  $R_n : W(\mathcal{O}_{X_0}) \rightarrow W_n(\mathcal{O}_{X_0})$  the natural reduction map, then*

$$(8.4.1) \quad \text{Ker}(F - \text{Id} : W(\mathcal{O}_{X_0}) \rightarrow W(\mathcal{O}_{X_0})) \cap \left( \bigcap_{n \geq 1} \text{Ker}(\tilde{F}^n \circ R_n) = 0 \right).$$

(ii) *Let  $P$  be a smooth  $X$ -scheme, and denote  $P_n = P \times_X X_n$ . For all  $n \geq 1$ , there exists a unique homomorphism of sheaves of graded algebras*

$$\tilde{F}^n : W_n\Omega_{P_0/X_0}^\bullet \longrightarrow \mathcal{H}^\bullet(\Omega_{P_{n-1}/X_{n-1}}^\bullet),$$

*making the following diagram commute:*

$$\begin{array}{ccc} W_{n+1}\Omega_{P_{n-1}/X_{n-1}}^\bullet & \xrightarrow{F^n} & Z\Omega_{P_{n-1}/X_{n-1}}^\bullet \\ \downarrow & & \downarrow \\ W_n\Omega_{P_0/X_0}^\bullet & \xrightarrow{\tilde{F}^n} & \mathcal{H}^\bullet(\Omega_{P_{n-1}/X_{n-1}}^\bullet). \end{array}$$

*Furthermore, for all  $a \in \mathcal{O}_{P_0}^\times$  and all  $\tilde{a} \in \mathcal{O}_{P_{n-1}}^\times$  lifting  $a$ , we have*

$$(8.4.2) \quad \tilde{F}^n(\text{dlog}([a])) = \text{cl}(d\tilde{a}/\tilde{a}).$$

When  $X_0$  is a perfect scheme and  $X_{n-1} = W_n(X_0)$ ,  $\widetilde{F}^n$  is the isomorphism

$$(8.4.3) \quad \theta_n : W_n \Omega_{P_0/X_0}^\bullet \xrightarrow{\sim} \mathcal{H}^\bullet(\Omega_{P_{n-1}/X_{n-1}}^\bullet),$$

defined by Illusie-Raynaud [IR83, III, (1.5)].

Note that, in formula (8.4.2), the class of  $d\tilde{a}/\tilde{a}$  does not depend upon the choice of the lifting  $\tilde{a}$ : if  $\tilde{b} = \tilde{a} + pw$ , then

$$d\tilde{b}/\tilde{b} = d\tilde{a}/\tilde{a} + d\left(\log\left(1 + p\frac{w}{\tilde{a}}\right)\right),$$

where  $\log(1 + pw/\tilde{a})$  is defined thanks to the canonical divided powers of  $p$ .

*Proof.* (i) We may assume  $X$  is affine. The kernel of the vertical map in the diagram is locally generated (as an abelian group) by elements of the form  $V^n([a])$  and  $V^r([pb])$  for some  $a, b \in \mathcal{O}_{P_{n-1}}$  and  $0 \leq r \leq n$ . As these elements are clearly mapped to 0 under  $F^n$ , this gives the unique existence of  $\widetilde{F}^n$ .

To prove (8.4.1), let  $w \in \text{Ker}(F - \text{Id}) \cap \left(\bigcap_n \text{Ker}(\widetilde{F}^n \circ R_n)\right)$ . If  $w \neq 0$ , we can write

$$w = \sum_{i \geq s}^\infty V^i([a_i]), \quad \text{with } a_i \in \mathcal{O}_{X_0} \text{ and } a_s \neq 0.$$

Then  $R_{s+1}(w) = V^s([a_s]) \in W_{s+1}(\mathcal{O}_{X_0})$ . If  $\tilde{a}_s \in \mathcal{O}_{X_s}$  is any lifting of  $a_s$ , and if  $[\tilde{a}_s]$  is the Teichmüller representative of  $\tilde{a}_s$  in  $W_2(\mathcal{O}_{X_s})$ , so that  $V^s([\tilde{a}_s])$  is a lifting of  $V^s([a_s])$  in  $W_{s+2}(\mathcal{O}_{X_s})$ , we have

$$\widetilde{F}^{s+1}(V^s([a_s])) = F^{s+1}(V^s([\tilde{a}_s])) = p^s F([\tilde{a}_s]) = p^s \tilde{a}_s^p$$

in  $\mathcal{O}_{X_s}$ . Thus  $\widetilde{F}^{s+1}(R_{s+1}(w)) = 0$  if and only if  $p^s \tilde{a}_s^p = 0$  in  $\mathcal{O}_{X_s}$ . Since  $X_s$  is flat over  $\mathbb{Z}/p^{s+1}\mathbb{Z}$ , we obtain  $\tilde{a}_s^p \in p\mathcal{O}_{X_s}$ ; in particular,  $a_s^p = 0 \in \mathcal{O}_{X_0}$ . But by assumption, we have

$$F(w) = \sum_{i \geq s} V^i([a_i^p]) = \sum_{i \geq s} V^i([a_i]) = w.$$

Hence  $a_s = a_s^p = 0$ , a contradiction.

(ii) First of all, since  $dF^n = p^n F^n d$ , the image of  $F^n : W_{n+1} \Omega_{P_{n-1}/X_{n-1}}^\bullet \rightarrow \Omega_{P_{n-1}/X_{n-1}}^\bullet$  is clearly contained in  $Z \Omega_{P_{n-1}/X_{n-1}}^\bullet$ . Thus, the diagram makes sense. Now, Lemma 8.3 and [LZ04, Prop. 2.19] imply that, in degree  $q$ , the kernel of the vertical map on the left-hand side is locally generated (as an abelian group) by sections of the following form:

$$(8.4.4) \quad V^n(\alpha), \quad dV^n(\beta), \quad V^r([p]\omega), \quad dV^r([p]\eta),$$

with  $\alpha \in \Omega_{P_{n-1}/X_{n-1}}^q$ ,  $\beta \in \Omega_{P_{n-1}/X_{n-1}}^{q-1}$ ,  $0 \leq r \leq n$ ,  $\omega \in W_{n+1-r} \Omega_{P_{n-1}/X_{n-1}}^q$ , and  $\eta \in W_{n+1-r} \Omega_{P_{n-1}/X_{n-1}}^{q-1}$ . One immediately sees that, via  $F^n$ , the first two sections are mapped to 0 in  $\mathcal{H}^q(\Omega_{P_{n-1}/X_{n-1}}^\bullet)$ . Since, for any  $m \geq 1$  and any

Teichmüller representative  $[a]_m$  in  $W_m(\mathcal{O}_{P_{n-1}})$ , we have  $F([a]_m) = [a]_{m-1}^p \in W_{m-1}(\mathcal{O}_{P_{n-1}})$ , we obtain for the last two sections

$$F^n V^r([p]\omega) = p^r F^{n-r}([p]\omega) = p^{p^{n-r}+r} F^{n-r}(\omega) = 0,$$

$$F^n dV^r([p]\eta) = F^{n-r} d([p]\eta) = F^{n-r}([p]d\eta) = p^{p^{n-r}-(n-r)} d(F^{n-r}(\eta)) = 0$$

in  $\mathcal{H}^q(\Omega_{P_{n-1}/X_{n-1}}^\bullet)$ . Thus  $F^n$  maps all elements in the kernel of the vertical map to 0 in  $\mathcal{H}^q(\Omega_{P_{n-1}/X_{n-1}}^\bullet)$ . Since the vertical map is surjective, this yields the statement.

If  $\tilde{a} \in \mathcal{O}_{P_{n-1}}^\times$  lifts  $a$ , we get

$$\tilde{F}^n(d[a]/[a]) = \text{cl}(F^n(d[\tilde{a}]/[\tilde{a}])) = \text{cl}([\tilde{a}]^{p^n-1} d[\tilde{a}]/[\tilde{a}]^{p^n}),$$

which gives (8.4.2).

Finally, let us assume that  $X_0$  is perfect and  $X_{n-1} = W_n(X_0)$ . By [IR83, III, (1.5)],  $\mathcal{H}^\bullet(\Omega_{P_{n-1}/X_{n-1}}^\bullet)$  has the structure of a differential graded algebra (dga) with the differential  $d : \mathcal{H}^i(\Omega_{P_{n-1}/X_{n-1}}^\bullet) \rightarrow \mathcal{H}^{i+1}(\Omega_{P_{n-1}/X_{n-1}}^\bullet)$  given by the boundary of the long exact cohomology sequence coming from the short exact sequence

$$0 \longrightarrow \Omega_{P_{n-1}/X_{n-1}}^\bullet \xrightarrow{p^n} \Omega_{P_{2n-1}/X_{2n-1}}^\bullet \longrightarrow \Omega_{P_{n-1}/X_{n-1}}^\bullet \longrightarrow 0.$$

The isomorphism  $\theta_n$  is compatible with the differential and the product, and it thus induces an isomorphism of dga's  $\theta_n : W_n \Omega_{P_0/X_0}^\bullet \xrightarrow{\sim} \mathcal{H}^\bullet(\Omega_{P_{n-1}/X_{n-1}}^\bullet)$ . On the other hand, it follows from the relation  $dF^n = p^n F^n d$  that the morphism  $\tilde{F}^n$  is compatible with the differentials. Therefore,  $\tilde{F}^n$  also induces a morphism of dga's,  $\tilde{F}^n : W_n \Omega_{P_0/X_0}^\bullet \xrightarrow{\sim} \mathcal{H}^\bullet(\Omega_{P_{n-1}/X_{n-1}}^\bullet)$ . In degree 0,  $\theta_n$  is defined by

$$\theta_n(a_0, \dots, a_{n-1}) = \tilde{a}_0^{p^n} + p\tilde{a}_1^{p^{n-1}} + \dots + p^{n-1}\tilde{a}_{n-1}^p,$$

where  $\tilde{a}_0, \dots, \tilde{a}_{n-1}$  are liftings to  $\mathcal{O}_{P_{n-1}}$  of  $a_0, \dots, a_{n-1}$  [IR83, p. 142, l. 8]. This definition shows that, in degree 0,  $\theta_n$  is the factorization of the  $n$ -th ghost component  $w_n : W_{n+1}(\mathcal{O}_{P_{n-1}}) \rightarrow \mathcal{O}_{P_{n-1}}$ , given by  $w_n(a_0, \dots, a_n) = \sum_{i=0}^n p^i a_i^{p^{n-i}}$ , with  $p^n a_n = 0$  in  $\mathcal{O}_{P_{n-1}}$ . From the definition of the morphism of functors  $F^n : W_{n+1} \rightarrow W_1$ , we also get that, in degree 0,  $\tilde{F}^n$  is the factorization of the  $n$ -th ghost component. Since  $\tilde{F}^n = \theta_n$  in degree 0 and  $W_n \Omega_{P_{n-1}/X_{n-1}}^\bullet$  is generated as dga by its sections in degree 0,  $\tilde{F}^n$  and  $\theta_n$  have to be equal.  $\square$

LEMMA 8.5. *Let  $S$  be  $\text{Spec } \mathbb{Z}_{(p)}$ ,  $X$  an  $S$ -scheme, and  $\pi : P := \mathbb{P}_X^d \rightarrow X$  the structural morphism of a projective space over  $X$ . For  $n \geq 0$ , denote by  $S_n, X_n, P_n, \pi_n$  the reductions modulo  $p^{n+1}$ , and let  $B\Omega_{P_n/X_n}^d \subset \Omega_{P_n/X_n}^d$  be the subsheaf of exact differential forms.*

(i) *For all  $n \geq 0$ , the canonical homomorphism*

$$(8.5.1) \quad b_n^d : R^d \pi_{n*}(\Omega_{P_n/X_n}^d) \longrightarrow R^d \pi_{n*}(\Omega_{P_n/X_n}^d / B\Omega_{P_n/X_n}^d)$$

*is an isomorphism.*

(ii) Assume that  $X$  is flat over  $S$ , and let  $Y_0 \hookrightarrow P_0$  be a regular closed immersion of codimension  $m$ . Then,

$$(8.5.2) \quad \forall j \neq m, \forall n \geq 0, \quad \mathcal{H}_{Y_0}^j(\Omega_{P_n/X_n}^d/B\Omega_{P_n/X_n}^d) = 0.$$

*Proof.* Let  $Q = \mathbb{P}_S^d$ , and let  $T_0, \dots, T_d$  be homogeneous coordinates on  $Q$ . We define an  $S$ -endomorphism  $\phi : Q \rightarrow Q$  by sending  $T_i$  to  $T_i^p$ ,  $0 \leq i \leq d$ . By base change by  $u : X \rightarrow S$ , we obtain an  $X$ -endomorphism of  $P$ , for which we will keep the notation  $\phi$ , as well as for its reduction mod  $p^{n+1}$ .

Let us fix  $n \geq 0$ . We can use the morphism  $\phi^{n+1}$  and view  $\phi_*^{n+1}\Omega_{P_n/X_n}^\bullet$  as a complex of quasi-coherent  $\mathcal{O}_{P_n}$ -modules, the differential of which is then  $\mathcal{O}_{P_n}$ -linear. But  $P_n$  has an open covering by  $d + 1$  open subsets that are relatively affine with respect to  $X_n$ , and therefore  $R^d\pi_{n*}$  is a right exact functor on the category of quasi-coherent  $\mathcal{O}_{P_n}$ -modules. As  $R^d\pi_{n*}(\Omega_{P_n/X_n}^{d-1}) = 0$ , assertion (i) follows.

To prove assertion (ii), we use  $\phi^{n+2}$  to view  $\phi_*^{n+2}\Omega_{P_n/X_n}^\bullet$  as a complex of quasi-coherent  $\mathcal{O}_{P_n}$ -modules with an  $\mathcal{O}_{P_n}$ -linear differential, and we claim that the sheaf of  $\mathcal{O}_{P_n}$ -modules

$$\mathcal{H}^d(\phi_*^{n+2}\Omega_{P_n/X_n}^\bullet) = \phi_*(\phi_*^{n+1}\Omega_{P_n/X_n}^d/B\phi_*^{n+1}\Omega_{P_n/X_n}^d)$$

has a filtration by sub- $\mathcal{O}_{P_n}$ -modules, the graded of which is locally free over  $\mathcal{O}_{P_0}$ . As  $Y_0$  is locally defined in  $P_0$  by a regular sequence of  $m$  sections, the claim clearly implies assertion (ii).

To prove the existence of this filtration, we may replace  $X, P$  by  $S, Q$ , because the projection  $v : P \rightarrow Q$  is flat, and

$$v^*(\phi_*^{n+2}\Omega_{Q_n/S_n}^\bullet) \xrightarrow{\sim} \phi_*^{n+2}\Omega_{P_n/X_n}^\bullet.$$

Now  $S_0$  is a perfect scheme, and  $S_n = W_{n+1}(S_0)$ . Thanks to the last assertion of Proposition 8.4(ii),  $F^{n+1}$  defines an isomorphism of graded algebras

$$\tilde{F}^{n+1} : W_{n+1}\Omega_{Q_0/S_0}^\bullet \xrightarrow{\sim} \mathcal{H}^\bullet(\Omega_{Q_n/S_n}^\bullet).$$

We may view  $\tilde{F}^{n+1}$  as an  $\mathcal{O}_{Q_n}$ -linear isomorphism by endowing  $\mathcal{H}^\bullet(\Omega_{Q_n/S_n}^\bullet)$  with the  $\mathcal{O}_{Q_n}$ -module structure provided by the homomorphism

$$\mathcal{O}_{Q_n} \rightarrow \mathcal{H}^0(\Omega_{Q_n/S_n}^\bullet)$$

defined by  $\phi^{n+2}$ , and  $W_{n+1}\Omega_{Q_0/S_0}^\bullet$  with the structure corresponding to the previous one via  $(\tilde{F}^{n+1})^{-1} : \mathcal{H}^0(\Omega_{Q_n/S_n}^\bullet) \xrightarrow{\sim} W_{n+1}(\mathcal{O}_{Q_0})$ . The canonical filtration of  $W_{n+1}\Omega_{Q_0/S_0}^d$  is then a filtration by sub- $\mathcal{O}_{Q_n}$ -modules, which can be transported to  $\mathcal{H}^d(\Omega_{Q_n/S_n}^\bullet)$  via  $\tilde{F}^{n+1}$ . As we know by [Ill79, I, Cor. 3.9] that the corresponding graded pieces are locally free  $\mathcal{O}_{Q_0}$ -modules for the structure defined by the homomorphism

$$(8.5.3) \quad \bar{F} : \mathcal{O}_{Q_0} \longrightarrow W_{n+1}(\mathcal{O}_{Q_0})/pW_{n+1}(\mathcal{O}_{Q_0})$$

factorizing  $F : W_{n+1}(\mathcal{O}_{Q_0}) \rightarrow W_{n+1}(\mathcal{O}_{Q_0})$ , the proof will be complete if we check the commutativity of the diagram

$$(8.5.4) \quad \begin{array}{ccc} \mathcal{O}_{Q_n} & \xrightarrow{\phi^{n+2*}} & \mathcal{H}^0(\Omega_{Q_n/S_n}^\bullet) \xrightarrow[\sim]{(\widetilde{F}^{n+1})^{-1}} & W_{n+1}(\mathcal{O}_{Q_0}) \\ \downarrow & & & \downarrow \\ \mathcal{O}_{Q_0} & \xrightarrow{\quad \quad \quad \bar{F} \quad \quad \quad} & & W_{n+1}(\mathcal{O}_{Q_0})/pW_{n+1}(\mathcal{O}_{Q_0}). \end{array}$$

It is enough to check that the diagram induced on sections over  $D_+(T_i) \subset Q_n$  commutes, for  $0 \leq i \leq d$ . So we may replace  $\mathcal{O}_{Q_n}$  by  $A = (\mathbb{Z}/p^{n+1}\mathbb{Z})[\underline{x}]$ , with  $\underline{x} = (x_1, \dots, x_d)$  and  $\phi^*(x_j) = x_j^p$ ,  $1 \leq j \leq d$ . Take  $f = \sum_I a_I \underline{x}^I \in A$ , with  $a_I \in \mathbb{Z}/p^{n+1}\mathbb{Z}$ . Then,

$$(\widetilde{F}^{n+1})^{-1} \circ \phi^{n+2*}(f) = \sum_I a_I (\widetilde{F}^{n+1})^{-1}(\underline{x}^{p^{n+2}I}).$$

As  $\widetilde{F}^{n+1}$  is the factorization of the  $(n+2)$ -th ghost component  $w_{n+1} : W_{n+2}(A) \rightarrow A$ , we see that  $(\widetilde{F}^{n+1})^{-1}(x_j^{p^{n+1}}) = [x_j]$ ,  $1 \leq j \leq d$ . Therefore, we obtain

$$(\widetilde{F}^{n+1})^{-1} \circ \phi^{n+2*}(f) = \sum_I a_I [\underline{x}]^{pI}.$$

Since  $\bar{F}$  is given by lifting an element of  $A_0$  to  $W_{n+1}(A_0)$ , applying Frobenius and reducing modulo  $p$ , this gives the commutativity of (8.5.4).  $\square$

**PROPOSITION 8.6.** *Under the assumptions of Theorem 8.1, let  $f = \pi \circ i$  be a factorization of  $f$  as the composition of a regular closed immersion  $i : Y \hookrightarrow P = \mathbb{P}_X^d$  of  $Y$  into a projective space on  $X$ , followed by the canonical projection  $\pi : P \rightarrow X$ . For all  $n \geq 1$ , let  $f_n, i_n, \pi_n$  be the reductions of  $f, i, \pi$  modulo  $p^{n+1}$ . Then, the compositions*

$$(8.6.1) \quad \mathcal{O}_X \xrightarrow{f^*} Rf_*(\mathcal{O}_Y) \xrightarrow{\tau_f} \mathcal{O}_X,$$

$$(8.6.2) \quad \mathcal{O}_{X_n} \xrightarrow{f_n^*} Rf_{n*}(\mathcal{O}_{Y_n}) \xrightarrow{\tau_{f_n}} \mathcal{O}_{X_n},$$

$$(8.6.3) \quad W_n(\mathcal{O}_{X_0}) \xrightarrow{f_0^*} Rf_{0*}(W_n(\mathcal{O}_{Y_0})) \xrightarrow{\tau_{i_0, \pi_0, n}} W_n(\mathcal{O}_{X_0}),$$

$$(8.6.4) \quad W(\mathcal{O}_{X_0}) \xrightarrow{f_0^*} Rf_{0*}(W(\mathcal{O}_{Y_0})) \xrightarrow{\tau_{i_0, \pi_0}} W(\mathcal{O}_{X_0}),$$

$$(8.6.5) \quad W\mathcal{O}_{X_0, \mathbb{Q}} \xrightarrow{f_0^*} Rf_{0*}(W\mathcal{O}_{Y_0, \mathbb{Q}}) \xrightarrow{\tau_{i_0, \pi_0}} W\mathcal{O}_{X_0, \mathbb{Q}}$$

are given by multiplication by  $r$ .

*Proof.* Since the restriction of  $f$  above  $U$  is finite locally free of rank  $r$ , it follows from (3.1.3) that the endomorphism of  $\mathcal{O}_U$  induced by  $\tau_f \circ f^*$  is multiplication by  $r$ . But  $U$  is scheme-theoretically dense in  $X$ , therefore the same relation holds on  $X$  itself. So (8.6.1) is multiplication by  $r$ .

Thanks to the flatness of  $X$  and  $Y$  over  $\mathbb{Z}_{(p)}$ , the spectral sequence for the composition of Tor's implies that, for all  $n \geq 1$ ,  $X_n$  and  $Y$  are Tor-independent over  $X$ . Therefore, by Theorem 3.1(ii), the morphism  $\tau_{f_n} \circ f_n^*$  is deduced from  $\tau_f \circ f^*$  by base change from  $X$  to  $X_n$ , and (8.6.2) is also multiplication by  $r$ .

We want to deduce from this result that (8.6.3) is also multiplication by  $r$ . We observe first that the homomorphisms  $\widetilde{F}^n$  defined by Lemma 8.4 provide morphisms

$$\begin{aligned} \widetilde{F}_X^n &: W_n(\mathcal{O}_{X_0}) \longrightarrow \mathcal{O}_{X_{n-1}}, \\ f_*(\widetilde{F}_Y^n) &: f_{0*}(W_n(\mathcal{O}_{Y_0})) \longrightarrow f_{n-1*}(\mathcal{O}_{Y_{n-1}}), \\ R^d\pi_*(\widetilde{F}_P^n) &: R^d\pi_{0*}(W_n\Omega_{P_0/X_0}^d) \longrightarrow R^d\pi_{n-1*}(\Omega_{P_{n-1}/X_{n-1}}^d/B\Omega_{P_{n-1}/X_{n-1}}^d). \end{aligned}$$

Moreover, we can use the isomorphism (8.5.1) and define

$$\widetilde{G}_P^n := (b_n^d)^{-1} \circ R^d\pi_*(\widetilde{F}_P^n) : R^d\pi_{0*}(W_n\Omega_{P_0/X_0}^d) \longrightarrow R^d\pi_{n-1*}(\Omega_{P_{n-1}/X_{n-1}}^d).$$

We consider the diagram

(8.6.6)

$$\begin{array}{ccccccc} W_n(\mathcal{O}_{X_0}) & \xrightarrow{f_0^*} & f_{0*}(W_n(\mathcal{O}_{Y_0})) & \xrightarrow{\pi_{0*}(\gamma_{i_0, \pi_{0,n}})} & R^d\pi_{0*}(W_n\Omega_{P_0/X_0}^d) & \xrightarrow{\sim \text{Trp}_{\pi_{0,n}}} & W_n(\mathcal{O}_{X_0}) \\ \widetilde{F}_X^n \downarrow & & f_*(\widetilde{F}_Y^n) \downarrow & & \downarrow \widetilde{G}_P^n & & \downarrow \widetilde{F}_X^n \\ \mathcal{O}_{X_{n-1}} & \xrightarrow{f_{n-1}^*} & f_{n-1*}(\mathcal{O}_{Y_{n-1}}) & \xrightarrow{\pi_{n-1*}(\gamma_{f_{n-1}})} & R^d\pi_{n-1*}(\Omega_{P_{n-1}/X_{n-1}}^d) & \xrightarrow{\sim \text{Trp}_{\pi_{n-1}}} & \mathcal{O}_{X_{n-1}}, \end{array}$$

where the compositions of the upper and lower rows are respectively the maps induced by (8.6.3) and (8.6.2) on degree-0 cohomology. Let us prove that this diagram is commutative. The left square commutes because the morphism  $\widetilde{F}_X^n$  is functorial with respect to  $X$ . To prove that the right square commutes, it suffices to show that if  $\xi_{\text{dRW}}$  and  $\xi_{\text{dR}}$  are the de Rham-Witt and de Rham Chern classes of  $\mathcal{O}_P(1)$ , then  $\xi_{\text{dRW}}^d$  and  $\xi_{\text{dR}}^d$  have same image in  $R^d\pi_{n-1*}(\Omega_{P_{n-1}/X_{n-1}}^d/B\Omega_{P_{n-1}/X_{n-1}}^d)$ . As  $R^\bullet\pi_{n-1*}(\widetilde{F}_P^n)$  and  $b_n^\bullet$  are compatible with cup-products, it suffices to show that the diagram

$$\begin{array}{ccc} R^1\pi_{0*}(\mathcal{O}_{P_0}^\times) & \xrightarrow{\text{dlog}} & R^1\pi_{0*}(W_n\Omega_{P_0/X_0}^1) \\ \uparrow & & \downarrow R^1\pi_*(\widetilde{F}_P^n) \\ R^1\pi_{n-1*}(\mathcal{O}_{P_{n-1}}^\times) & \xrightarrow{\text{dlog}} & R^1\pi_{n-1*}(\mathcal{H}^1(\Omega_{P_{n-1}/X_{n-1}}^\bullet)) \end{array}$$

is commutative, which follows from (8.4.2).

To simplify notation, we drop the base scheme from the indices and denote  $C_{P_{n-1}}^d = \Omega_{P_{n-1}}^d/B\Omega_{P_{n-1}}^d$ . To prove the commutativity of the central square of

(8.6.6), it suffices to prove the commutativity of the diagram

$$\begin{array}{ccccccc}
 i_{0*}(W_n(\mathcal{O}_{Y_0})) & \longrightarrow & \mathcal{H}_{Y_0}^d(W_n\Omega_{P_0}^d) & \xleftarrow{\sim} & R\underline{\Gamma}_{Y_0}(W_n\Omega_{P_0}^d)[d] & \longrightarrow & W_n\Omega_{P_0}^d[d] \\
 \downarrow i_*(\tilde{F}_Y^n) & & \mathcal{H}_Y^d(\tilde{F}_P^n) \downarrow & & \downarrow R\underline{\Gamma}_Y(\tilde{F}_P^n)[d] & & \downarrow \tilde{F}_P^n[d] \\
 & & \mathcal{H}_{Y_{n-1}}^d(C_{P_{n-1}}^d) & \xleftarrow{\sim} & R\underline{\Gamma}_{Y_{n-1}}(C_{P_{n-1}}^d)[d] & \longrightarrow & C_{P_{n-1}}^d[d] \\
 & & \uparrow & & \uparrow & & \uparrow \\
 i_{n-1*}(\mathcal{O}_{Y_{n-1}}) & \longrightarrow & \mathcal{H}_{Y_{n-1}}^d(\Omega_{P_{n-1}}^d) & \xleftarrow{\sim} & R\underline{\Gamma}_{Y_{n-1}}(\Omega_{Y_{n-1}}^d)[d] & \longrightarrow & \Omega_{Y_{n-1}}^d[d],
 \end{array}$$

to apply the functor  $R\pi_{n-1*}$ , and to pass to cohomology sheaves in degree 0. In this diagram, the upper left (resp. lower left) horizontal arrow maps 1 to  $\gamma_{Y_0,n}$  (resp.  $\gamma_{Y_{n-1},1}$ ), and the middle horizontal arrow is an isomorphism thanks to Lemma 8.5(ii). The middle and right squares commute by functoriality, and it suffices to prove that the left rectangle commutes. This part of the diagram comes from a diagram of morphisms of sheaves; therefore, the verification is local on  $P$ . Thus we may assume that  $Y$  is defined by a regular sequence  $t_1, \dots, t_d$  in  $P$ . Then, since  $Y$  and  $P$  are flat over  $\mathbb{Z}_{(p)}$ , the images of this sequence in  $\mathcal{O}_{P_{n-1}}$  and  $\mathcal{O}_{P_0}$  (still denoted  $t_1, \dots, t_d$ ) are regular sequences defining  $Y_{n-1}$  and  $Y_0$ . It is enough to show that the symbols

$$\begin{bmatrix} d[t_1] \cdots d[t_d] \\ [t_1], \dots, [t_d] \end{bmatrix} \in \mathcal{E}xt_{W_n(\mathcal{O}_{P_0})}^d(W_n(\mathcal{O}_{Y_0}), W_n\Omega_{P_0}^d)$$

and

$$\begin{bmatrix} dt_1 \cdots dt_d \\ t_1, \dots, t_d \end{bmatrix} \in \mathcal{E}xt_{\mathcal{O}_{P_{n-1}}}^d(\mathcal{O}_{Y_{n-1}}, \Omega_{P_{n-1}}^d)$$

have same image in  $\mathcal{H}_Y^d(C_{P_{n-1}}^d)$ . By functoriality, the image of  $\begin{bmatrix} dt_1 \cdots dt_d \\ t_1, \dots, t_d \end{bmatrix}$  in  $\mathcal{E}xt_{\mathcal{O}_{P_{n-1}}}^d(\mathcal{O}_{Y_{n-1}}, C_{P_{n-1}}^d)$  is  $\begin{bmatrix} \text{cl}(dt_1 \cdots dt_d) \\ t_1, \dots, t_d \end{bmatrix}$ . On the other hand, it follows from the construction of  $\tilde{F}^n$  in Proposition 8.4 that  $\tilde{F}_P^n([t_i]) = t_i^{p^n} \in \mathcal{O}_{P_{n-1}}$  and  $\tilde{F}_P^n(d[t_i]) = \text{cl}(t_i^{p^n-1} dt_i) \in \mathcal{H}^1(\Omega_{P_{n-1}}^\bullet)$ . Since the  $t_i^{p^n}$ 's form a regular sequence in  $\mathcal{O}_{P_{n-1}}$ , we may argue as in the proof of Proposition 7.4 to show that the symbols  $\begin{bmatrix} d[t_1] \cdots d[t_d] \\ [t_1], \dots, [t_d] \end{bmatrix}$  and  $\begin{bmatrix} \text{cl}(t_1^{p^n-1} \cdots t_d^{p^n-1} dt_1 \cdots dt_d) \\ t_1^{p^n}, \dots, t_d^{p^n} \end{bmatrix}$  have same image in  $\mathcal{H}_{Y_{n-1}}^d(C_{P_{n-1}}^d)$ . The wanted equality is then a consequence of Lemma 4.2, and the commutativity of (8.6.6) follows.

Returning to the homomorphism (8.6.3), we observe that it is defined by multiplication by a section  $\kappa_n$  of  $W_n(\mathcal{O}_{X_0})$ . Proposition 7.7(i) implies that, for variable  $n$ , the sections  $\kappa_n$  form a compatible family under restriction and satisfy  $F(\kappa_n) = \kappa_{n-1}$ . If  $\kappa = \varprojlim_n \kappa_n \in \Gamma(X_0, W(\mathcal{O}_{X_0}))$ , then  $F(\kappa - r) = \kappa - r$ . On the other hand, the commutativity of (8.6.6) implies that  $\tilde{F}_X^n(\kappa_n - r) = 0$ . So, if  $R_n : W(\mathcal{O}_{X_0}) \rightarrow W_n(\mathcal{O}_{X_0})$  is the restriction homomorphism, we obtain



that

$$\kappa - r \in \text{Ker}(F - \text{Id}) \cap \left( \bigcap_{n \geq 1} \text{Ker}(\widetilde{F}_X^n \circ R_n), \right)$$

which is zero by (8.4.1). Thus,  $\kappa = r$ ; hence,  $\kappa_n = r$  for all  $n$ .

If we now consider in the derived category of inverse systems the composition

$$W_\bullet(\mathcal{O}_{X_0}) \xrightarrow{f_0^*} Rf_{0\bullet*}(W_\bullet(\mathcal{O}_{Y_0})) \xrightarrow{\tau_{i,\pi,\bullet}} W_\bullet(\mathcal{O}_{X_0}),$$

we obtain a morphism that has (8.6.3) as component of degree  $n$ . Therefore, this composition is multiplication by  $r$  on the inverse system  $W_\bullet(\mathcal{O}_{X_0})$ . It follows that the composition

$$W(\mathcal{O}_{X_0}) \xrightarrow{R\varprojlim f_0^*} R\varprojlim Rf_{0\bullet*}(W_\bullet(\mathcal{O}_{Y_0})) \xrightarrow{R\varprojlim \tau_{i,\pi,\bullet}} W(\mathcal{O}_{X_0})$$

is multiplication by  $r$ . Using the isomorphism  $R\varprojlim \circ Rf_{0\bullet*} \simeq Rf_{0*} \circ R\varprojlim$ , we obtain that (8.6.4) is multiplication by  $r$ . Tensoring by  $\mathbb{Q}$  and using the commutation of  $Rf_{0*}$  with tensorization by  $\mathbb{Q}$ , we obtain that (8.6.5) is multiplication by  $r$ . □

8.7. *Proof of Theorem 8.1.* The first assertion is a particular case of Theorem 3.2. To prove the other ones, we choose a factorization  $f = \pi \circ i$ , where  $i$  is a closed immersion of  $Y$  into a projective space  $P = \mathbb{P}_X^d$  over  $X$  and  $\pi$  is the structural morphism, and we keep the notation of the previous subsections. Applying the functor  $H^q(X_n, -)$  (resp.  $H^i(X_0, -)$ ), the morphisms  $\tau_{f_n}$ ,  $\tau_{i,\pi,n}$ , and  $\tau_{i,\pi}$  define homomorphisms

$$\begin{aligned} H^q(Y_n, \mathcal{O}_{Y_n}) &\xrightarrow{\tau_{f_n}} H^q(X_n, \mathcal{O}_{X_n}), \\ H^q(Y_0, W_n(\mathcal{O}_{Y_0})) &\xrightarrow{\tau_{i,\pi,n}} H^q(X_0, W_n(\mathcal{O}_{X_0})), \\ H^q(Y_0, W(\mathcal{O}_{Y_0})) &\xrightarrow{\tau_{i,\pi}} H^q(X_0, W(\mathcal{O}_{X_0})), \\ H^q(Y_0, W\mathcal{O}_{Y_0,\mathbb{Q}}) &\xrightarrow{\tau_{i,\pi}} H^q(X_0, W\mathcal{O}_{X_0,\mathbb{Q}}). \end{aligned}$$

Proposition 8.6 implies that the composition of these homomorphisms with the functoriality homomorphisms defined by  $f_n$  (resp.  $f_0$ ) is multiplication by  $r$ , and this implies Theorem 8.1. □

This also completes the proof of Theorems 1.5, 1.3, and 1.1.

### 9. An example

Because Theorem 1.1 was previously known in some cases, and can be proved in some other cases without using the most difficult results of this paper, it may be worth giving an example for which we would not know how to prove congruence (1.1.1) without using them. Here, we give such an example for each  $p \geq 7$ , except perhaps when  $p$  is a Fermat number.

9.1. We begin with a list of conditions that we want our example to satisfy. In these conditions,  $R$ ,  $K$ , and  $k$  are as in Theorem 1.1, and  $X$  is an  $R$ -scheme.

- (1)  $X$  is a regular scheme, projective and flat over  $R$ .
- (2)  $H^0(X_K, \mathcal{O}_{X_K}) = K$ , and  $H^q(X_K, \mathcal{O}_{X_K}) = 0$  for all  $q \geq 1$ .
- (3) There exists  $q \geq 1$  such that  $H^q(X_k, \mathcal{O}_{X_k}) \neq 0$ .
- (4)  $X$  is not a semi-stable  $R$ -scheme (in particular, not smooth).
- (5)  $\dim X_K \geq 3$ .
- (6)  $X_K$  is a variety of general type.

Conditions (1) and (2) will ensure that  $X$  satisfies the hypotheses of Theorem 1.1. Condition (3) will ensure that we are not in the trivial situation described in the first paragraph of Section 1.4. Condition (4) will ensure that Theorem 2.1 does not suffice to conclude. Condition (5) will rule out the case of surfaces, for which Theorem 1.1 is already known by [Esn06, Th. 1.3]. Condition (6) rules out rationally connected varieties, for which Theorem 1.1 is also known because they satisfy the coniveau condition of [Esn06, Th. 1.1]. It also grants that if  $X$  can be embedded as a global complete intersection in some projective space over  $R$ , then congruence (1.1.1) cannot be proved by applying Katz's theorem [Ktz71, Th. 1.0] to  $X_k$ , since a smooth complete intersection in a  $K$ -projective space for which Katz's  $\mu$  invariant is  $\geq 1$  is a Fano variety.

*Remarks 9.2.* We begin with a few remarks that make it easier to find an example satisfying the previous conditions.

(i) Examples such that  $\dim_k H^1(X_k, \mathcal{O}_{X_k}) > \dim_K H^1(X_K, \mathcal{O}_{X_K}) = 0$  have been known since Serre's construction of a counter-example to Hodge symmetry in characteristic  $p$  [Ser58, Prop. 16]. The general principle behind such examples, which goes back to Grothendieck (see [SGA1, XI, 6.11, (\*)] over an algebraically closed field and [Ray70, Prop. 6.2.1] for a general statement), is that the datum of a torsor  $Y$  on  $X$  under a finite group  $G$  defines a morphism  $G' \rightarrow \underline{\text{Pic}}_{X/R}$ , where  $G'$  is the Cartier dual of  $G$ . Then, under certain conditions, the Lie algebra of  $G'_k$  can have a nonzero image in the tangent space  $H^1(X_k, \mathcal{O}_{X_k})$  to  $\underline{\text{Pic}}_{X_k/k}$ . The simplest case (which was the one considered by Serre) is when  $G$  is the étale group  $\mathbb{Z}/p\mathbb{Z}$ . Then the Artin-Schreier exact sequence shows that when the torsor  $Y_k$  remains nontrivial after extension to an algebraic closure  $\bar{k}$  of  $k$ , its class gives a nonzero element in  $H^1(X_{\bar{k}}, \mathcal{O}_{X_{\bar{k}}})$ , and therefore  $H^1(X_k, \mathcal{O}_{X_k}) \neq 0$ . This happens in particular when  $Y_k$  is a complete intersection in some projective space, since we then have  $\dim_{\bar{k}} H^0(Y_{\bar{k}}, \mathcal{O}_{Y_{\bar{k}}}) = 1$ .

To simplify our quest, we will therefore replace condition (3) (and condition (5)) by the more restrictive condition:

(3')  $X$  is the quotient of an hypersurface  $Y$  in a projective space  $\mathbb{P}_R^n$  of relative dimension  $n \geq 4$  over  $R$  by a free action of the group  $\mathbb{Z}/p\mathbb{Z}$ .

(ii) Assume that  $X$  satisfies condition (3'). Then  $H^0(Y_K, \mathcal{O}_{Y_K}) = K$ , and  $H^q(Y_K, \mathcal{O}_{Y_K}) = 0$  for  $q \neq 0, n - 1$ . Because  $\text{char}(K) = 0$ , we have  $H^q(X_K, \mathcal{O}_{X_K}) = H^q(Y_K, \mathcal{O}_{Y_K})^G$ . Hence,  $H^0(X_K, \mathcal{O}_{X_K}) = K$ , and condition (2) is satisfied if and only if  $\chi(\mathcal{O}_{X_K}) = 1$ . As  $Y_K$  is an étale cover of  $X_K$  of degree  $p$ , the Riemann-Roch-Hirzebruch formula implies that

$$(9.2.1) \quad \chi(\mathcal{O}_{Y_K}) = p\chi(\mathcal{O}_{X_K}).$$

Then condition (2) is satisfied if and only if  $\chi(\mathcal{O}_{Y_K}) = p$ . If  $d$  is the degree of the hypersurface  $Y$ , we obtain

$$\begin{aligned} (-1)^{n-1}(p - 1) &= \dim_K H^{n-1}(Y_K, \mathcal{O}_{Y_K}) \\ &= \dim_K H^n(\mathbb{P}_K^n, \mathcal{O}_{\mathbb{P}_K^n}(-d)) \\ &= \dim_K H^0(\mathbb{P}_K^n, \mathcal{O}_{\mathbb{P}_K^n}(d - n - 1)). \end{aligned}$$

The simplest choice for checking this equation is  $d - n - 1 = 1$ , so that we get  $\dim_K H^0(\mathbb{P}_K^n, \mathcal{O}_{\mathbb{P}_K^n}(d - n - 1)) = n + 1$ . Then we have to satisfy the conditions

$$(9.2.2) \quad p > 2, \quad n = p - 2, \quad d = p.$$

Therefore, we will simplify our quest even further by replacing condition (3') by the following more precise condition, which implies (2), (3) and (5):

(3'')  $X$  is the quotient of an hypersurface  $Y$  of degree  $p$  in the projective space  $\mathbb{P}_R^n$  of relative dimension  $n = p - 2$  over  $R$  by a free action of the group  $\mathbb{Z}/p\mathbb{Z}$ , with  $p \geq 7$ .

(iii) Assuming that  $X$  satisfies conditions (1) and (3''), then condition (6) follows automatically. Indeed,  $Y_K$  is smooth over  $K$  since  $\text{char}(K) = 0$ , and its canonical sheaf is then  $\mathcal{O}_{Y_K}(-n - 1 + d) = \mathcal{O}_{Y_K}(1)$ . Since  $Y_K$  is an étale covering of  $X_K$ , it is the inverse image of the canonical sheaf on  $X$ , which therefore is ample too.

So it suffices for our purpose to construct an example satisfying conditions (1), (3''), and (4).

9.3. We now begin the construction of our example. Assume that  $p \geq 5$ , and let  $E$  be the free  $\mathbb{Z}_{(p)}$ -module  $(\mathbb{Z}_{(p)})^p$ . We denote by  $e_0, \dots, e_{p-1}$  its canonical basis. Let  $\sigma$  be a generator of  $G := \mathbb{Z}/p\mathbb{Z}$ . We let  $\sigma$  act on  $E$  by cyclic permutation of the basis

$$(9.3.1) \quad \sigma : e_0 \mapsto e_1 \mapsto \dots \mapsto e_{p-1} (\mapsto e_0).$$

Let  $H \subset E$  be the hyperplane consisting of elements for which the sum of coordinates is 0. It is stable under the action of  $G$ , and we endow it with the basis  $v_1, \dots, v_{p-1}$  defined by  $v_i = e_i - e_{i-1}$ . We take as projective space the space  $\mathbb{P}(H) \simeq \mathbb{P}_{\mathbb{Z}_{(p)}}^{p-2}$ , with the induced  $G$ -action, and we denote by  $X_1, \dots, X_{p-1}$

the homogeneous coordinates on  $\mathbb{P}(H)$  defined by the dual basis to the basis  $v_1, \dots, v_{p-1}$  of  $H$ . Letting  $G$  act by composition on functions on  $H$ , one checks easily that the orbit of  $X_1$  is described by

$$(9.3.2) \quad X_1 \mapsto -X_{p-1} \mapsto X_{p-1} - X_{p-2} \mapsto X_{p-2} - X_{p-3} \mapsto \dots \mapsto X_2 - X_1 \ (\mapsto X_1).$$

Let  $g_0(X_1, \dots, X_{p-1})$  be the sum of the elements of the orbit of  $X_1^p$ , i.e.,

$$(9.3.3) \quad g_0(X_1, \dots, X_{p-1}) = X_1^p + (-X_{p-1})^p + \sum_{i=2}^{p-1} (X_i - X_{i-1})^p.$$

Then  $g_0 \in p\mathbb{Z}[X_1, \dots, X_{p-1}]$ , and we can define a polynomial  $g(X_1, \dots, X_{p-1}) \in \mathbb{Z}[X_1, \dots, X_{p-1}]$  by

$$(9.3.4) \quad g(X_1, \dots, X_{p-1}) = \frac{1}{p}g_0(X_1, \dots, X_{p-1}).$$

Let  $Z \subset \mathbb{P}(H)$  be the hypersurface defined by  $g$ . Since  $g$  is  $G$ -invariant, the action of  $G$  on  $\mathbb{P}(H)$  induces an action on  $Z$ . We denote by  $\bar{g}$  the reduction of  $g$  in  $\mathbb{F}_p[X_1, \dots, X_{p-1}]$ . We first study the singular points of  $Z_{\mathbb{F}_p}$ . They are solutions of the system of homogeneous equations  $\partial\bar{g}/\partial X_i = 0$ ,  $1 \leq i \leq p-1$ , which can be written as

$$(9.3.5) \quad \begin{cases} X_1^{p-1} = (X_2 - X_1)^{p-1} \\ (X_2 - X_1)^{p-1} = (X_3 - X_2)^{p-1} \\ \vdots = \vdots \\ (X_{p-1} - X_{p-2})^{p-1} = (-X_{p-1})^{p-1} \end{cases}.$$

LEMMA 9.4. *Let  $\bar{\mathbb{F}}_p$  be an algebraic closure of  $\mathbb{F}_p$ .*

(i) *The solutions of (9.3.5) in  $\mathbb{P}^n(\bar{\mathbb{F}}_p)$  belong to  $\mathbb{P}^n(\mathbb{F}_p)$ , and they correspond bijectively to the families  $(u_1, \dots, u_{p-1}) \in (\mathbb{F}_p^\times)^{p-1}$  such that*

$$(9.4.1) \quad 1 + u_1 + \dots + u_{p-1} = 0.$$

(ii) *For  $u \in \mathbb{F}_p^\times$ , let  $\tilde{u} = [u] \in \mu_{p-1}(\mathbb{Z}_p)$  be its Teichmüller representative. Then a point  $x \in \mathbb{P}^n(\mathbb{F}_p)$  that is a solution of (9.3.5) belongs to  $Z_{\mathbb{F}_p}$  if and only if*

$$(9.4.2) \quad 1 + \tilde{u}_1 + \dots + \tilde{u}_{p-1} \in p^2\mathbb{Z}_p,$$

where  $(u_1, \dots, u_{p-1}) \in (\mathbb{F}_p^\times)^{p-1}$  corresponds to  $x$  by (i).

*Proof.* Given  $(u_1, \dots, u_{p-1}) \in (\mathbb{F}_p^\times)^{p-1}$  satisfying (9.4.1), the corresponding solution  $x = (\xi_1 : \dots : \xi_{p-1}) \in \mathbb{P}^n(\mathbb{F}_p)$  of the system (9.3.5) is obtained by choosing  $\xi_1 \in \mathbb{F}_p^\times$ , setting

$$(9.4.3) \quad \xi_i - \xi_{i-1} = u_{i-1}\xi_1 \quad \text{for } 2 \leq i \leq p-1,$$

and observing that (9.4.1) implies that  $-\xi_{p-1} = u_{p-1}\xi_1$ . Assertion (i) is then straightforward.

Let  $\eta_1 \in \mathbb{Z}_p$  be a lifting of  $\xi_1$ , and let  $\eta_i$  be defined inductively for  $2 \leq i \leq p-1$  by

$$(9.4.4) \quad \eta_i - \eta_{i-1} = \tilde{u}_{i-1}\eta_1.$$

Define  $\alpha \in \mathbb{Z}_p$  by

$$(9.4.5) \quad 1 + \tilde{u}_1 + \cdots + \tilde{u}_{p-1} = p\alpha.$$

Then by adding the equations in (9.4.4), we get

$$(9.4.6) \quad \eta_{p-1} = (1 + \cdots + \tilde{u}_{p-2})\eta_1 = (p\alpha - \tilde{u}_{p-1})\eta_1.$$

We can now substitute (9.4.4) and (9.4.6) in  $g_0$ , and we obtain the relation

$$(9.4.7) \quad \begin{aligned} g_0(\eta_1, \dots, \eta_{p-1}) &= \eta_1^p(1 + \tilde{u}_1^p + \cdots + \tilde{u}_{p-2}^p + (\tilde{u}_{p-1} - p\alpha)^p) \\ &= \eta_1^p(1 + \tilde{u}_1 + \cdots + \tilde{u}_{p-2} + \tilde{u}_{p-1} + \sum_{j=1}^p \binom{p}{j} \tilde{u}_{p-1}^{p-j} (-p\alpha)^j) \\ &\equiv p\alpha\eta_1^p \pmod{p^2\mathbb{Z}_p}. \end{aligned}$$

Hence we get

$$(9.4.8) \quad g(\eta_1, \dots, \eta_{p-1}) \equiv \alpha\eta_1^p \pmod{p\mathbb{Z}_p},$$

and assertion (ii) follows. □

LEMMA 9.5. (i) *The action of  $G$  on  $Z_{\mathbb{F}_p}$  is free.*

(ii) *If  $p$  is not a Fermat number, then  $Z_{\mathbb{F}_p}$  is singular and is not the special fibre of a semi-stable scheme.*

Let us recall that the Fermat numbers are the integers of the form  $2^{2^n} + 1$  with  $n \geq 0$ , that any prime number of the form  $2^n + 1$  with  $n > 0$  is a Fermat number, and that the only known prime Fermat numbers are 3, 5, 17, 257, and 65537.

*Proof.* Over  $\overline{\mathbb{F}_p}$ , the matrix of the action of  $\sigma$  on  $(\overline{\mathbb{F}_p})^p$  has 1 as unique eigenvalue, with a corresponding eigenspace of dimension 1, generated by the eigenvector  $(1, \dots, 1)$ . This eigenvector belongs to  $H/pH$ , where it has coordinates  $(p-1, p-2, \dots, 1) = -(1, \dots, p-1)$  in the basis  $v_1, \dots, v_{p-1}$ . Therefore, the only fixed point of  $\sigma$  in  $\mathbb{P}^n(\overline{\mathbb{F}_p})$  is the point  $x_0 = (1 : 2 : \cdots : p-1)$ . This point is the solution of (9.3.5) corresponding to  $u_1 = \cdots = u_{p-1} = 1$ . Lemma 9.4(ii) implies that it does not belong to  $Z_{\mathbb{F}_p}$ , which proves assertion (i).

As the system (9.3.5) has only a finite number of solutions, the singular points of  $Z_{\mathbb{F}_p}$  are isolated. In particular, since  $\dim Z_{\mathbb{F}_p} \geq 4$ ,  $Z_{\mathbb{F}_p}$  cannot be the special fibre of a semi-stable scheme if it has a singular point. To find a singular point on  $Z_{\mathbb{F}_p}$ , Lemma 9.4 shows that it suffices to construct a family

$(\tilde{u}_i)_{1 \leq i \leq p-1}$  of  $(p-1)$ -th roots of unity in  $\mathbb{Z}_p$  such that  $1 + \sum_i \tilde{u}_i \in p^2\mathbb{Z}_p$ . Since  $p$  is not a Fermat number,  $p-1$  has an odd prime factor  $q$ . We can choose a primitive  $q$ -th root of unity  $\zeta$  and set  $\tilde{u}_i = \zeta^i$  for  $1 \leq i \leq q-1$ ,  $\tilde{u}_i = 1$  for  $q \leq i \leq q + (p-q)/2 - 1$ ,  $\tilde{u}_i = -1$  for  $q + (p-q)/2 \leq i \leq p-1$ . So  $Z_{\mathbb{F}_p}$  is singular.  $\square$

9.6. We now address the regularity condition in 9.1(1). We replace  $Z$  by another equivariant lifting of  $Z_{\mathbb{F}_p}$  defined as follows. Let  $R$  be the ring of integers of a finite extension  $K$  of  $\mathbb{Q}_p$ , of degree  $> 1$ , with residue field  $k$ . If  $K/\mathbb{Q}_p$  is unramified, we set  $\pi = p$ ; otherwise, we choose a uniformizer  $\pi$  of  $R$ . Let  $\lambda \in R$  be an element satisfying the following condition:

- (a) If  $K/\mathbb{Q}_p$  is unramified, then the reduction of  $\lambda \pmod p$  does not belong to  $\mathbb{F}_p$ .
- (b) If  $K/\mathbb{Q}_p$  is ramified, then  $\lambda \in R^\times$ .

Let  $h \in \mathbb{Z}[X_1, \dots, X_{p-1}]$  be the product of the elements of the orbit of  $X_1$ , i.e.,

$$(9.6.1) \quad h(X_1, \dots, X_{p-1}) = X_1(-X_{p-1}) \prod_{i=2}^{p-1} (X_i - X_{i-1}),$$

and let  $f \in R[X_1, \dots, X_{p-1}]$  be defined by

$$(9.6.2) \quad f = g + \pi\lambda h.$$

We define  $Y \subset \mathbb{P}_R^n$  to be the hypersurface with equation  $f$ . Since  $f$  is invariant under  $G$ , the action of  $G$  on  $\mathbb{P}_R^n$  induces an action on  $Y$ . Its special fibre  $Y_k$  is equal to  $Z_k$ , on which  $G$  acts freely by Lemma 9.5. Then the fixed locus of  $\sigma$  is a closed subscheme of  $Y$ , and its projection on  $\text{Spec } R$  is a closed subset that does not contain the closed point. Therefore it is empty, and the action of  $G$  on  $Y$  is free. We define  $X$  to be the quotient scheme  $X = Y/G$ .

**PROPOSITION 9.7.** *Assume that  $p$  is an odd prime that is not a Fermat number. Then the scheme  $X$  defined above satisfies conditions (1)–(6) of 9.1.*

*Proof.* As observed in 9.2(iii), it suffices to check that  $X$  satisfies conditions (1), (3''), and (4). Condition (3'') holds by construction.

The hypersurface  $Y$  is projective and flat over  $R$ , since  $g$  is not divisible by  $\pi$ . So  $X$  is also projective and flat. As  $Y_k = Z_k$ , Lemma 9.5(ii) implies that  $Y$  is not semi-stable. Since  $Y \rightarrow X$  is étale and semi-stability is a local property for the étale topology,  $X$  is not semi-stable either. So we only have to prove that  $X$  is regular. This is again a local property for the étale topology. Hence, it suffices to prove that  $Y$  is regular. Because  $Y$  is excellent, its singular locus is closed, and the same holds for its projection to  $\text{Spec } R$ . So it is enough to check the regularity of  $Y$  at the points of its special fibre. The regularity is clear at the smooth points of  $Y_k$ , and we need to prove it at the singular points.

Let  $x = (\xi_1 : \cdots : \xi_{p-1}) \in \mathbb{P}^n(k)$  be a singular point of  $Y_k$ . As  $Y_k = Z_k$ ,  $x$  corresponds by Lemma 9.4 to a family  $(u_1, \dots, u_{p-1}) \in (\mathbb{F}_p^\times)^{p-1}$  such that

$$(9.7.1) \quad 1 + \tilde{u}_1 + \cdots + \tilde{u}_{p-1} = p^2\beta$$

for some  $\beta \in \mathbb{Z}_p$ . We have seen in the proof of Lemma 9.4 that  $\xi_1 \in \mathbb{F}_p^\times$ , so we may assume that  $\xi_1 = 1$ . We set  $\eta_1 = 1$ , and we define inductively  $\eta_i$  for  $2 \leq i \leq p - 1$  by (9.4.4). This allows us to work on the affine space  $\mathbb{A}_R^n = D_+(X_1) \subset \mathbb{P}_R^n$ , and we will denote

$$a_*(X_2, \dots, X_{p-1}) := a(1, X_2, \dots, X_{p-1})$$

for any homogeneous polynomial  $a(X_1, \dots, X_{p-1}) \in R[X_1, \dots, X_{p-1}]$ . For  $2 \leq i \leq p - 1$ , we set

$$X_i = \eta_i + Y_i,$$

so that  $(\pi, Y_2, \dots, Y_{p-1})$  is a regular sequence of generators of the maximal ideal  $\mathfrak{m}_x$  of the regular local ring  $\mathcal{O}_{\mathbb{A}_R^n, x}$ .

We want to prove that  $\mathcal{O}_{\mathbb{A}_R^n, x}/(f_*)$  is regular, i.e., that  $f_* \notin \mathfrak{m}_x^2$ . We first claim that

$$(9.7.2) \quad g_* \equiv p\beta \pmod{\mathfrak{m}_x^2}.$$

Indeed, applying (9.4.7) with  $\alpha = p\beta$ , we obtain the congruence

$$g_{0*}(\eta_2, \dots, \eta_{p-1}) \equiv p^2\beta \pmod{p^3\mathbb{Z}_p};$$

hence

$$(9.7.3) \quad g_*(\eta_2, \dots, \eta_{p-1}) \equiv p\beta \pmod{p^2\mathbb{Z}_p \subset \mathfrak{m}_x^2}.$$

On the other hand, equations (9.4.4) show that, for  $2 \leq i \leq p - 2$ ,

$$(9.7.4) \quad \frac{\partial g_*}{\partial X_i}(\eta_2, \dots, \eta_{p-1}) = 0.$$

Finally, equations (9.4.4) and (9.4.6) show that

$$(9.7.5) \quad \begin{aligned} \frac{\partial g_*}{\partial X_{p-1}}(\eta_2, \dots, \eta_{p-1}) &= (\eta_{p-1} - \eta_{p-2})^{p-1} - \eta_{p-1}^{p-1} \\ &= 1 - (p^2\beta - \tilde{u}_{p-1})^{p-1} \\ &\equiv 0 \pmod{p^2\mathbb{Z}_p \subset \mathfrak{m}_x^2}. \end{aligned}$$

Applying (9.7.3), (9.7.4), and (9.7.5) to the Taylor development of  $g_*$  proves (9.7.2).

From the definition of  $h$ , we obtain

$$(9.7.6) \quad h_*(\eta_2, \dots, \eta_{p-1}) = -(p^2\beta - \tilde{u}_{p-1}) \prod_{i=1}^{p-2} \tilde{u}_i \equiv \prod_{i=1}^{p-1} \tilde{u}_i \pmod{\mathfrak{m}_x}.$$

As  $h_* \equiv h_*(\eta_2, \dots, \eta_{p-1}) \pmod{\mathfrak{m}_x}$ ,  $f_*$  satisfies the congruence

$$(9.7.7) \quad f_* = g_* + \pi \lambda h_* \equiv \pi \left( \frac{p}{\pi} \beta + \lambda \prod_{i=1}^{p-1} \tilde{u}_i \right) \pmod{\mathfrak{m}_x^2}.$$

Let  $w = \frac{p}{\pi} \beta + \lambda \prod_i \tilde{u}_i$ . If  $K/\mathbb{Q}_p$  is ramified, then condition 9.6(b) implies that  $w$  is a unit. If  $K/\mathbb{Q}_p$  is unramified, then  $\pi = p$ , and condition 9.6(a) implies that the reduction mod  $p$  of  $w$  is nonzero; hence,  $w$  is again a unit. In each case,  $f_* \notin \mathfrak{m}_x^2$ , and  $\mathcal{O}_{Y,x}$  is regular.  $\square$

### Appendix: Complete intersection morphisms of virtual relative dimension 0

As mentioned in the introduction, we explain here the construction of the morphism  $\tau_f : Rf_* \mathcal{O}_Y \rightarrow \mathcal{O}_X$  for a proper complete intersection morphism  $f : Y \rightarrow X$  of virtual dimension 0, and we give a proof of Theorem 3.1.

The appendix consists of two sections. In Section A, we recall the construction of the invertible sheaf  $\omega_{Y/X}$  associated to a complete intersection morphism  $f : Y \rightarrow X$ , and we prove some of its properties. We do not use duality theory here, even if we keep for convenience the terminology “relative dualizing sheaf.” Instead, we use the complete intersection assumption to deduce our constructions from the elementary properties of smooth morphisms and regular immersions, thanks to the canonical isomorphisms defined by Conrad [Con00, 2.2]. It is then easy to define the canonical section  $\delta_f$  of  $\omega_{Y/X}$  when  $f$  has virtual relative dimension 0 and to prove its basic properties.

In Section B, we assume that  $X$  is noetherian and has a dualizing complex. We then use duality theory and the identification  $\omega_{Y/X} \xrightarrow{\sim} f^! \mathcal{O}_X$  to deduce  $\tau_f$  from the canonical section  $\delta_f$ . To translate the properties of  $\delta_f$  into the properties of  $\tau_f$  listed in Theorem 3.1, we need to use the fundamental identifications of duality theory, as well as the various compatibilities between these identifications. Our proofs rely in an essential way on Conrad’s exposition [Con00].

It may be worth pointing out that in this article we need the compatibility of  $\tau_f$  with base change in a context that is not covered by the base change results of [Con00]. Indeed, we consider morphisms  $f$  that are not flat in general (such as in Theorem 1.5) and base change morphisms that are not flat either (such as reduction mod  $p^n$  in the proof of Proposition 8.6). The key property we use here, which is familiar to the experts, but not so well documented in the literature, is the Tor-independence of  $f$  and the base change morphism.

#### A. The canonical section of the relative dualizing sheaf

We recall now the construction of the invertible sheaf  $\omega_{Y/X}$  for a complete intersection morphism, and we explain some of its properties. As often, the



main work is to prove that the constructions are well defined and, in particular, to check the sign conventions. As the details are easy but tedious, we leave most of them as exercises and only sketch the main steps of the verifications.

We first recall a standard base change result for complete intersection morphisms (see [SGA6, 3.7.1] for the finite Tor-dimension of  $Rf_*\mathcal{E}^\bullet$ ).

PROPOSITION A.1. *Let  $f : Y \rightarrow X$  be a complete intersection morphism of virtual relative dimension  $m$ , and let*

$$(A.1.1) \quad \begin{array}{ccc} Y' & \xrightarrow{v} & Y \\ f' \downarrow & & \downarrow f \\ X' & \xrightarrow{u} & X \end{array}$$

*be a cartesian square such that  $X'$  and  $Y$  are Tor-independent over  $X$ .*

(i) *The morphism  $f'$  is a complete intersection morphism of virtual relative dimension  $m$ .*

(ii) *Assume that  $X$  is quasi-compact, and that  $f$  is separated of finite type. If  $\mathcal{E}^\bullet \in D_{qc}^b(\mathcal{O}_Y)$  is of finite Tor-dimension over  $\mathcal{O}_Y$ , then  $Rf_*\mathcal{E}^\bullet$  is of finite Tor-dimension over  $\mathcal{O}_X$ , and the base change morphism*

$$(A.1.2) \quad Lu^*Rf_*\mathcal{E}^\bullet \longrightarrow Rf'_*Lv^*\mathcal{E}^\bullet$$

*is an isomorphism.*

*Proof.* The first claim is local on  $Y'$ , so we may assume that there exists a factorization  $f = \pi \circ i$  such that  $\pi : P \rightarrow X$  is a smooth morphism of relative dimension  $n$  and  $i : Y \hookrightarrow P$  is a closed immersion of codimension  $d = n - m$ . Then  $i$  is a regular immersion, defined by an ideal  $\mathcal{I} \subset \mathcal{O}_P$  and, since the claim is local, we may assume that  $\mathcal{I}$  is generated by a regular sequence  $t_1, \dots, t_d$  of sections of  $\mathcal{O}_P$ . Then the Koszul complex  $K_\bullet(t_1, \dots, t_d)$  is a resolution of  $\mathcal{O}_Y$  by  $\mathcal{O}_P$ -modules that are flat relatively to  $X$ . Let  $P' = X' \times_X P$ , and let  $t'_1, \dots, t'_d$  be the images of  $t_1, \dots, t_d$  in  $\mathcal{O}_{P'}$ . Since  $X'$  and  $Y$  are Tor-independent over  $X$ , the Koszul complex  $K(t'_1, \dots, t'_d)$  is a resolution of  $\mathcal{O}_{Y'}$  over  $\mathcal{O}_{P'}$ , which shows that  $f'$  is a complete intersection morphism of virtual relative dimension  $m$ .

Assume now that the hypotheses of (ii) are satisfied. Since  $X$  is quasi-compact, it suffices to check that  $Rf_*\mathcal{E}^\bullet$  is of finite Tor-dimension when  $X$  is affine. We can then choose a finite covering  $\mathfrak{U}$  of  $Y$  by affine open subsets  $U_\alpha$ , and we may assume that the  $U_\alpha$  are small enough so that the restriction  $f_\alpha$  of  $f$  to  $U_\alpha$  can be factorized as  $f_\alpha = \pi_\alpha \circ i_\alpha$ , where  $\pi_\alpha : P_\alpha \rightarrow X$  is smooth and  $i_\alpha : U_\alpha \hookrightarrow P_\alpha$  is a closed immersion defined by a regular sequence of sections of  $\mathcal{O}_{P_\alpha}$ . For each sequence  $\alpha_0 < \dots < \alpha_r$ , denote  $U_{\underline{\alpha}} = U_{\alpha_0} \cap \dots \cap U_{\alpha_r}$ ,  $j_{\underline{\alpha}} : U_{\underline{\alpha}} \hookrightarrow Y$ , and let  $f_{\underline{\alpha}}$  be the restriction of  $f$  to  $U_{\underline{\alpha}}$ . If  $\mathcal{I}^\bullet$  is an injective resolution of  $\mathcal{E}^\bullet$ , then the alternating Čech complex  $\check{C}^\bullet(\mathfrak{U}, \mathcal{I}^\bullet)$  is a resolution of  $\mathcal{E}^\bullet$ . Since  $j_{\underline{\alpha}}$  is an affine open immersion, the complex  $j_{\underline{\alpha}*}j_{\underline{\alpha}}^*\mathcal{I}^\bullet = Rj_{\underline{\alpha}*}j_{\underline{\alpha}}^*\mathcal{E}^\bullet$

belongs to  $D_{\text{qc}, \text{fTd}}^{\text{b}}(\mathcal{O}_Y)$  for each  $\underline{\alpha}$ . Therefore, it suffices to prove that  $Rf_*\mathcal{E}^\bullet \in D_{\text{qc}, \text{fTd}}^{\text{b}}(\mathcal{O}_X)$  for complexes  $\mathcal{E}^\bullet$  of the form  $Rj_*\mathcal{F}^\bullet$ , where  $j$  is the inclusion of an affine open subscheme  $U$  and  $\mathcal{F}^\bullet \in D_{\text{qc}, \text{fTd}}^{\text{b}}(\mathcal{O}_U)$ . This reduces the proof to the case where  $Y$  is affine. Then there exists a bounded complex of  $\mathcal{O}_Y$ -modules  $\mathcal{P}^\bullet$  with flat quasi-coherent terms, and a quasi-isomorphism  $\mathcal{P}^\bullet \rightarrow \mathcal{E}^\bullet$ . Since  $\mathcal{O}_Y$  has finite Tor-dimension over  $\mathcal{O}_X$ , so does any flat  $\mathcal{O}_Y$ -module, and the first assertion of (ii) follows.

The complex  $Lv^*\mathcal{E}^\bullet$  belongs to  $D_{\text{qc}, \text{fTd}}^{\text{b}}(\mathcal{O}_{Y'})$ , and the base change morphism (A.1.2) can be defined by adjunction as usual. Arguing as before, it suffices to prove that it is an isomorphism when  $X$  is affine and  $\mathcal{E}^\bullet$  is of the form  $Rj_*\mathcal{F}^\bullet$ , where  $j$  is the inclusion of an affine open subscheme  $U \subset Y$ , and  $\mathcal{F}^\bullet \in D_{\text{qc}, \text{fTd}}^{\text{b}}(\mathcal{O}_U)$ . Let  $U' = X' \times_X U$ , and let  $w : U' \rightarrow U$  be the projection,  $j' : U' \hookrightarrow Y'$  the pull-back of  $j$ . Since  $j$  is an affine morphism and  $\mathcal{F}^\bullet \in D_{\text{qc}, \text{fTd}}^{\text{b}}(\mathcal{O}_U)$ , the base change morphism  $Lv^*Rj_*\mathcal{F}^\bullet \rightarrow Rj'_*Lw^*\mathcal{F}^\bullet$  is an isomorphism. This implies that the base change morphism for  $f$  and  $\mathcal{E}^\bullet$  is an isomorphism if and only if the base change morphism for  $f \circ j$  and  $\mathcal{F}^\bullet$  is an isomorphism. If one chooses a bounded, flat, quasi-coherent resolution  $\mathcal{P}^\bullet$  of  $\mathcal{F}^\bullet$ , the Tor-independence assumption implies that, for each  $n$ ,  $(f \circ j)_*\mathcal{P}^n$  is  $u^*$ -acyclic. It follows easily that the base change morphism for  $\mathcal{P}^\bullet$  is an isomorphism, which ends the proof.  $\square$

*Remark.* Assertion (ii) holds more generally if one replaces the complete intersection hypothesis on  $f$  by the assumption that  $\mathcal{E}^\bullet$  has finite Tor-dimension over  $\mathcal{O}_X$ . It is also standard to extend the assertion to the case where  $f$  is only assumed to be coherent, i.e., quasi-compact and quasi-separated.

A.2. Let  $f : Y \rightarrow X$  be a complete intersection morphism of relative dimension  $m$ . We now recall how one can associate to  $f$  an invertible  $\mathcal{O}_Y$ -module  $\omega_{Y/X}$ , called the *relative dualizing sheaf* of  $Y/X$  (or  $f$ ). Here we will use the direct construction based on elementary algebra,<sup>1</sup> which makes explicit the existence of the canonical section when  $m = 0$ , and is a natural extension of Conrad's constructions for the canonical isomorphisms  $\zeta'_{f,g}$  [Con00, 2.2].

If  $f = \pi \circ i$  is a factorization of  $f$  where  $\pi : P \rightarrow X$  is a smooth morphism of relative dimension  $n$  and  $i : Y \hookrightarrow P$  a closed immersion of codimension  $d = n - m$ , defined by a regular ideal  $\mathcal{I} \subset \mathcal{O}_P$ , then one defines  $\omega_{Y/X}$  by

<sup>1</sup> For a more intrinsic construction, one can use the general properties of the cotangent complex  $L_{Y/X}$  [Ill71]. Here,  $L_{Y/X}$  is a perfect complex, of perfect amplitude in  $[-1, 0]$ , and of rank  $m$  [Ill71, 3.2.6]. Taking its (graded) determinant in the sense of Knudsen-Mumford [KM76], one obtains the complex  $\omega_{Y/X}[m]$ . In this construction, special attention should be paid to sign compatibilities, as, for historical reasons, the sign conventions used in [Ill71] and [KM76] conflict with those of [Con00].

setting

$$(A.2.1) \quad \begin{aligned} \omega_{Y/X} &= \omega_{Y/P} \otimes_{\mathcal{O}_Y} i^* \omega_{P/X} \\ &= \wedge^d((\mathcal{I}/\mathcal{I}^2)^\vee) \otimes_{\mathcal{O}_Y} i^* \Omega_{P/X}^n. \end{aligned}$$

Up to canonical isomorphism, this construction is made independent of the choice of the factorization as follows. Let  $f = \pi' \circ i'$  be another factorization of  $f$  through a smooth morphism  $\pi' : P' \rightarrow X$ , and let  $\omega_{Y/X}^P$  and  $\omega_{Y/X}^{P'}$  be the invertible  $\mathcal{O}_Y$ -modules defined by (A.2.1) using the two factorizations. Assume first that there exists an  $X$ -morphism  $u : P' \rightarrow P$  that is either a smooth morphism or a regular immersion and is such that  $u \circ i' = i$ . Then, one defines an isomorphism  $\varepsilon^{P',P}(u) : \omega_{Y/X}^P \xrightarrow{\sim} \omega_{Y/X}^{P'}$  by the commutative diagram

$$(A.2.2) \quad \begin{array}{ccc} \omega_{Y/X}^P = \omega_{Y/P} \otimes i^* \omega_{P/X} & \xrightarrow[\sim]{\zeta'_{i',u} \otimes \text{Id}} & \omega_{Y/P'} \otimes i'^* \omega_{P'/P} \otimes i'^* u^* \omega_{P/X} \\ & \searrow[\sim]{\varepsilon^{P',P}(u)} & \uparrow[\sim]{\text{Id} \otimes i'^*(\zeta'_{u,\pi})} \\ & & \omega_{Y/P'} \otimes i'^* \omega_{P'/X} = \omega_{Y/X}^{P'}. \end{array}$$

The definitions of  $\zeta'_{i',u}$  and  $\zeta'_{u,\pi}$  depend upon whether  $u$  is a smooth morphism or a regular immersion (the two definitions agree when  $u$  is an open and closed immersion).

(a) If  $u$  is smooth, then  $\zeta'_{i',u}$  is defined by [Con00, p. 29, (d)] and  $\zeta'_{u,\pi}$  is defined by [Con00, p. 29, (a)].

(b) If  $u$  is a regular immersion, then  $\zeta'_{i',u}$  is defined by [Con00, p. 29, (b)] and  $\zeta'_{u,\pi}$  is defined by [Con00, p. 29, (c)].

Let  $f = \pi'' \circ i''$  be a third factorization of  $f$  through a smooth morphism  $\pi'' : P'' \rightarrow X$ , let  $\omega_{Y/X}^{P''}$  be defined by (A.2.1) using this factorization, and assume that there exists an  $X$ -morphism  $v : P'' \rightarrow P'$  such that  $v \circ i'' = i'$  and such that each of the morphisms  $v$  and  $u \circ v$  is either a smooth morphism or a regular immersion. Then it follows readily from Conrad's general transitivity relation for compositions of smooth morphisms and regular immersions [Con00, (2.2.4)] that

$$(A.2.3) \quad \varepsilon^{P'',P'}(v) \circ \varepsilon^{P',P}(u) = \varepsilon^{P'',P}(u \circ v).$$

If  $f = \pi \circ i = \pi' \circ i'$  are any factorizations as above, let now  $P'' = P' \times_X P$ , and let  $i'' : Y \hookrightarrow P''$  be the diagonal immersion and  $q : P'' \rightarrow P, q' : P'' \rightarrow P'$  the two projections. One defines the canonical isomorphism  $\varepsilon^{P',P} : \omega_{Y/X}^P \xrightarrow{\sim} \omega_{Y/X}^{P'}$  by setting

$$(A.2.4) \quad \varepsilon^{P',P} := \varepsilon^{P'',P'}(q')^{-1} \circ \varepsilon^{P'',P}(q).$$

Whenever there exists a smooth morphism or a regular immersion  $u : P' \rightarrow P$  as above, it follows from (A.2.3) that  $\varepsilon^{P',P}(u) = \varepsilon^{P',P}$ . One checks similarly that the isomorphisms  $\varepsilon_{P',P}$  satisfy the usual cocycle condition for three factorizations.

Thanks to this cocycle condition, one can then define the invertible sheaf  $\omega_{Y/X}$  even when there does not exist a global factorization  $f = \pi \circ i$  as above, by choosing local factorizations and glueing the invertible sheaves obtained locally by the previous construction. By construction, the sheaf  $\omega_{Y/X}$  commutes with Zariski localization and is equipped with a canonical isomorphism for which we keep the notation  $\zeta'$ :

$$(A.2.5) \quad \zeta'_{\pi,i} : \omega_{Y/X} \xrightarrow{\sim} \omega_{Y/P} \otimes_{\mathcal{O}_Y} i^* \omega_{P/X}$$

for any factorization  $f = \pi \circ i$  where  $\pi$  is a smooth morphism and  $i$  is a regular immersion.

If  $m$  is the virtual relative dimension of  $Y$  over  $X$ , we will need to work with the complex  $\omega_{Y/X}[m]$ , which is the single  $\mathcal{O}_Y$ -module  $\omega_{Y/X}$  sitting in degree  $-m$ . If  $f = \pi \circ i$  as above, we define in  $D^b(\mathcal{O}_Y)$  the isomorphism of complexes

$$(A.2.6) \quad \zeta'_{i,\pi} : \omega_{Y/X}[m] \xrightarrow{\sim} \omega_{Y/P}[-d] \overset{L}{\otimes}_{\mathcal{O}_Y} Li^*(\omega_{P/X}[n])$$

by (A.2.5) in degree  $-m$ , without any sign modification. If  $f$  is a smooth morphism or a regular immersion, this definition is consistent with [Con00, (2.2.6)]. By [Con00, (1.3.6)], the isomorphism (A.2.6) is equal to the composed isomorphism

$$\begin{aligned} \omega_{Y/X}[m] &\xrightarrow[\text{(A.2.5)[}m\text{]}{\sim} (\omega_{Y/P} \overset{L}{\otimes}_{\mathcal{O}_Y} Li^*(\omega_{P/X})) [m] \xrightarrow{\sim} (\omega_{Y/P} \overset{L}{\otimes}_{\mathcal{O}_Y} Li^*(\omega_{P/X}[n])) [-d] \\ &\xrightarrow{\sim} \omega_{Y/P}[-d] \overset{L}{\otimes}_{\mathcal{O}_Y} Li^*(\omega_{P/X}[n]) \end{aligned}$$

and differs from the composed isomorphism

$$\begin{aligned} \omega_{Y/X}[m] &\xrightarrow[\text{(A.2.5)[}m\text{]}{\sim} (\omega_{Y/P} \overset{L}{\otimes}_{\mathcal{O}_Y} Li^*(\omega_{P/X})) [m] \xrightarrow{\sim} (\omega_{Y/P}[-d] \overset{L}{\otimes}_{\mathcal{O}_Y} Li^*(\omega_{P/X})) [n] \\ &\xrightarrow{\sim} \omega_{Y/P}[-d] \overset{L}{\otimes}_{\mathcal{O}_Y} Li^*(\omega_{P/X}[n]) \end{aligned}$$

by multiplication by  $(-1)^{dn}$ .

LEMMA A.3. *Under the assumptions of Proposition A.1, there exists a canonical isomorphism*

$$(A.3.1) \quad Lv^*(\omega_{Y/X}) \cong v^*(\omega_{Y/X}) \xrightarrow{\sim} \omega_{Y'/X'}$$

*Moreover, if the assumptions of Proposition A.1(ii) are satisfied, the canonical base change morphism*

$$(A.3.2) \quad Lu^* Rf_*(\omega_{Y/X}) \rightarrow Rf'_*(\omega_{Y'/X'})$$

*is an isomorphism.*

*Proof.* Since  $\omega_{Y/X}$  is invertible,  $Lv^*(\omega_{Y/X}) \xrightarrow{\sim} v^*(\omega_{Y/X})$ . To prove the isomorphism (A.3.1), assume first that there exists a factorization  $f = \pi \circ i$ , where  $\pi$  is smooth and  $i$  is a regular immersion. Let  $f' = \pi' \circ i'$  be the factorization deduced from  $f = \pi \circ i$  by base change. Then, if  $\mathcal{I}$  and  $\mathcal{I}'$  are the ideals defining  $i$  and  $i'$ , the Tor-independence assumption implies that the canonical homomorphism  $u^*(\mathcal{I}/\mathcal{I}^2) \rightarrow \mathcal{I}'/\mathcal{I}'^2$  is an isomorphism, which defines (A.3.1). It is easy to check that, for two factorizations of  $f$ , the corresponding isomorphisms are compatible with the identifications (A.2.4). This provides the isomorphism (A.3.1) in the general case.

When the assumptions of A.1(ii) are satisfied, the isomorphism (A.3.2) follows from (A.3.1) and (A.1.2).  $\square$

A.4. Let

$$\begin{array}{ccc} Y' & \xrightarrow{v} & Y \\ f' \downarrow & & \downarrow f \\ X' & \xrightarrow{u} & X \end{array}$$

be a cartesian square, and assume that

- (a)  $f$  and  $u$  are complete intersection morphisms of relative dimensions  $m$  and  $n$ ;
- (b)  $X'$  and  $Y$  are Tor-independent over  $X$ .

Then Lemma A.3 provides canonical isomorphisms

$$v^*(\omega_{Y/X}) \xrightarrow{\sim} \omega_{Y'/X'}, \quad f'^*(\omega_{X'/X}) \xrightarrow{\sim} \omega_{Y'/Y}.$$

One defines the canonical isomorphism

$$(A.4.1) \quad \chi_{f,u} : \omega_{Y'/Y} \otimes_{\mathcal{O}_{Y'}} v^*(\omega_{Y/X}) \xrightarrow{\sim} \omega_{Y'/X'} \otimes_{\mathcal{O}_{Y'}} f'^*(\omega_{X'/X})$$

as being the product by  $(-1)^{mn}$  of the composite

$$\omega_{Y'/Y} \otimes_{\mathcal{O}_{Y'}} v^*(\omega_{Y/X}) \xrightarrow{\sim} f'^*(\omega_{X'/X}) \otimes_{\mathcal{O}_{Y'}} \omega_{Y'/X'} \xrightarrow{\sim} \omega_{Y'/X'} \otimes_{\mathcal{O}_{Y'}} f'^*(\omega_{X'/X}),$$

where the first isomorphism is the product of the previous base change isomorphisms and the second one is the usual commutativity isomorphism of the tensor product (see [Del84, Appendix, (a)] and [Con00, p. 215-216]).

The following relations follow easily from the local description of the isomorphisms  $\zeta'_{f,g}$  given in [Con00, p. 30, (a)–(d)]:

- (i) In the above cartesian square, assume that each of the three morphisms  $u$ ,  $f$ , and  $u \circ f' = f \circ v$  is either a smooth morphism or a regular immersion. Then, the following isomorphisms  $\omega_{Y'/X} \xrightarrow{\sim} \omega_{Y'/X'} \otimes_{\mathcal{O}_{Y'}} f'^*(\omega_{X'/X})$  are equal:

$$(A.4.2) \quad \zeta'_{f',u} = \chi_{f,u} \circ \zeta'_{v,f}.$$

(ii) Let

$$\begin{array}{ccccc}
 & & Y' & \xrightarrow{v} & Y \\
 & \nearrow j & \downarrow f' & & \downarrow f \\
 X'' & \xrightarrow{i} & X' & \xrightarrow{u} & X
 \end{array}$$

be a commutative diagram in which the square is cartesian,  $f$  is smooth, and  $i$  and  $u$  are regular immersions. Then the following isomorphisms

$$\omega_{X''/X} \xrightarrow{\sim} \omega_{X''/Y'} \otimes_{\mathcal{O}_{X''}} j^*(\omega_{Y'/X'}) \otimes_{\mathcal{O}_{X''}} i^*(\omega_{X'/X})$$

are equal:

$$(A.4.3) \quad (\zeta'_{j,f'} \otimes \text{Id}) \circ \zeta'_{i,u} = (\text{Id} \otimes j^*(\chi_{f,u})) \circ (\zeta'_{j,v} \otimes \text{Id}) \circ \zeta'_{v,j,f}.$$

(iii) Let

$$\begin{array}{ccccc}
 Y'' & \xrightarrow{v'} & Y' & \xrightarrow{v} & Y \\
 f'' \downarrow & & f' \downarrow & & \downarrow f \\
 X'' & \xrightarrow{u'} & X' & \xrightarrow{u} & X
 \end{array}$$

be a commutative diagram in which both squares are cartesian, each of the morphisms  $f$ ,  $u$ ,  $u'$  and  $u \circ u'$  is either a smooth morphism or a regular immersion,  $X'$  and  $Y$  are Tor-independent over  $X$ , and  $X''$  and  $Y'$  are Tor-independent over  $X'$  (so that  $X''$  and  $Y$  are Tor-independent over  $X$  and all immersions are regular). Then the following isomorphisms

$$\omega_{Y''/Y} \otimes_{\mathcal{O}_{Y''}} (vv')^*(\omega_{Y/X}) \xrightarrow{\sim} \omega_{Y''/X''} \otimes_{\mathcal{O}_{Y''}} f''^*(\omega_{X''/X'} \otimes_{\mathcal{O}_{X''}} u'^*(\omega_{X'/X}))$$

are equal:

$$(A.4.4) \quad (\text{Id} \otimes f''^*(\zeta'_{u',u})) \circ \chi_{f,uu'} = (\chi_{f',u'} \otimes \text{Id}) \circ (\text{Id} \otimes v'^*(\chi_{f,u})) \circ (\zeta'_{v',v} \otimes \text{Id}).$$

We will also need to extend the isomorphism  $\chi_{f,u}$  to the derived category. We define

$$(A.4.5) \quad \chi_{f,u} : \omega_{Y'/Y}[n] \overset{L}{\otimes}_{\mathcal{O}_{Y'}} Lv^*(\omega_{Y/X}[m]) \xrightarrow{\sim} \omega_{Y'/X'}[m] \overset{L}{\otimes}_{\mathcal{O}_{Y'}} Lf'^*(\omega_{X'/X}[n])$$

by (A.4.1) in degree  $-(m+n)$ , without any further sign modification. Because of the sign convention in the commutativity isomorphism for the derived tensor product [Con00, p. 11],  $\chi_{f,u}$  can also be described as the composite

$$\begin{aligned}
 \omega_{Y'/Y}[n] \overset{L}{\otimes}_{\mathcal{O}_{Y'}} Lv^*(\omega_{Y/X}[m]) &\xrightarrow{\sim} Lf'^*(\omega_{X'/X}[n]) \overset{L}{\otimes}_{\mathcal{O}_{Y'}} \omega_{Y'/X'}[m] \\
 &\xrightarrow{\sim} \omega_{Y'/X'}[m] \overset{L}{\otimes}_{\mathcal{O}_{Y'}} Lf'^*(\omega_{X'/X}[n]),
 \end{aligned}$$

where the first isomorphism is the tensor product of the base change isomorphisms and the second one is the commutativity isomorphism. With this definition, the previous relations (A.4.2)–(A.4.4) remain valid in  $D^b(\mathcal{O}_{Y'})$ .

A.5. We now consider the composition of two complete intersection morphisms  $f : Y \rightarrow X$ ,  $g : Z \rightarrow Y$ , and define a canonical isomorphism

$$(A.5.1) \quad \zeta'_{g,f} : \omega_{Z/X} \xrightarrow{\sim} \omega_{Z/Y} \otimes_{\mathcal{O}_Z} g^*(\omega_{Y/X})$$

extending the isomorphism (A.2.5).

Assume first that there exists a factorization  $f = \pi \circ i$ , where  $\pi : P \rightarrow X$  is a smooth morphism, and a factorization  $i \circ g = \pi'' \circ j$ , where  $\pi'' : P'' \rightarrow P$  is a smooth morphism. (Such factorizations always exist when  $X$ ,  $Y$  and  $Z$  are affine.) Let  $\pi' : P' \rightarrow Y$  be the pull-back of  $\pi''$  so that we get a commutative diagram

$$(A.5.2) \quad \begin{array}{ccccc} & & & P'' & \\ & & & \nearrow & \searrow \\ & & & i'' & \pi'' \\ & & & \nearrow & \searrow \\ & & & P' & P \\ & & \square & \nearrow & \searrow \\ & & & i' & i \\ & & & \nearrow & \searrow \\ Z & \xrightarrow{g} & Y & \xrightarrow{f} & X, \end{array}$$

where the middle square is cartesian. Using (A.2.5) for  $(j, \psi)$ , and the isomorphisms  $\zeta'_{i',i''} \otimes j^*(\zeta'_{\pi'',\pi})$ , we obtain an isomorphism

$$\begin{aligned} \omega_{Z/X} &\cong \omega_{Z/P''} \otimes j^*(\omega_{P''/X}) \\ &\xrightarrow{\sim} (\omega_{Z/P'} \otimes i'^*(\omega_{P'/P''})) \otimes j^*(\omega_{P''/P} \otimes \pi''^*(\omega_{P/X})) \\ &\xrightarrow{\sim} \omega_{Z/P'} \otimes i'^*(\omega_{P'/P''} \otimes i''^*(\omega_{P''/P})) \otimes g^*i^*(\omega_{P/X}). \end{aligned}$$

Using the isomorphism

$$\chi_{\pi'',i} : \omega_{P'/P''} \otimes i''^*(\omega_{P''/P}) \xrightarrow{\sim} \omega_{P'/Y} \otimes \pi'^*(\omega_{Y/P})$$

defined in A.4 and  $(\zeta'_{i',\pi'} \otimes g^*(\zeta'_{i,\pi}))^{-1}$ , we then obtain the composed isomorphism

$$\begin{aligned} \omega_{Z/X} &\xrightarrow{\sim} \omega_{Z/P'} \otimes i'^*(\omega_{P'/Y} \otimes \pi'^*(\omega_{Y/P})) \otimes g^*i^*(\omega_{P/X}) \\ &\xrightarrow{\sim} (\omega_{Z/P'} \otimes i'^*(\omega_{P'/Y})) \otimes g^*(\omega_{Y/P} \otimes i^*(\omega_{P/X})) \\ &\xrightarrow{\sim} \omega_{Z/Y} \otimes g^*(\omega_{Y/X}), \end{aligned}$$

which defines (A.5.1).

To prove that this isomorphism is well defined and to glue the local constructions to obtain a global one when a diagram (A.5.2) does not exist globally, we must check that it does not depend on the chosen factorizations. If we have two diagrams (A.5.2), with factorizations  $f = \pi_k \circ i_k$ ,  $i_k \circ g = \pi''_k \circ j_k$  for  $k = 1, 2$ ,

then we can embed  $Y$  diagonally in  $P_1 \times_X P_2$  and  $Z$  in  $P_1'' \times_X P_2''$ . This allows us to reduce the verification to the case where there exists a smooth  $X$ -morphism  $u : P_2 \rightarrow P_1$  such that  $u \circ i_2 = i_1$ , and a smooth morphism  $u'' : P_2'' \rightarrow P_1''$  such that  $\pi_1'' \circ u'' = u \circ \pi_2''$ , and  $j_1 = u'' \circ j_2$ . Moreover, the same argument shows that we may assume that the morphism  $P_2'' \rightarrow P_1'' \times_{P_1} P_2$  is smooth. The verification can then be reduced to the following two cases:

- (i) The morphism  $P_2'' \rightarrow P_1'' \times_{P_1} P_2$  is an isomorphism.
- (ii) The morphism  $P_2 \rightarrow P_1$  is an isomorphism.

In each of these cases, the equality of the two definitions of (A.5.1) breaks down to a succession of elementary commutative diagrams involving isomorphisms of the form  $\zeta'_{f,g}$  and  $\chi_{f,u}$ . We omit details here and only point out that, in addition to [Con00, (2.2.4)], the first case uses relation (A.4.2) and the second one uses relation (A.4.3). In particular, the sign convention introduced in the definition of  $\chi_{f,u}$  in A.4 is necessary for this independence result.

If  $m$  and  $m'$  are the virtual relative dimensions of  $f$  and  $g$ , we define as in A.2 the derived category variant of (A.5.1) as being the morphism

$$(A.5.3) \quad \zeta'_{g,f} : \omega_{Z/X}[m + m'] \xrightarrow{\sim} \omega_{Z/Y}[m'] \otimes_{\mathcal{O}_Z}^L Lf^*(\omega_{Y/X}[m]),$$

defined by applying (A.5.1) to the underlying modules (sitting in degree  $-m - m'$ ), without any sign modification.

With the definition of  $\zeta'_{g,f}$  provided by (A.5.1) (resp. (A.5.3)), we now extend to complete intersection morphisms Conrad's transitivity relation [Con00, (2.2.4)].

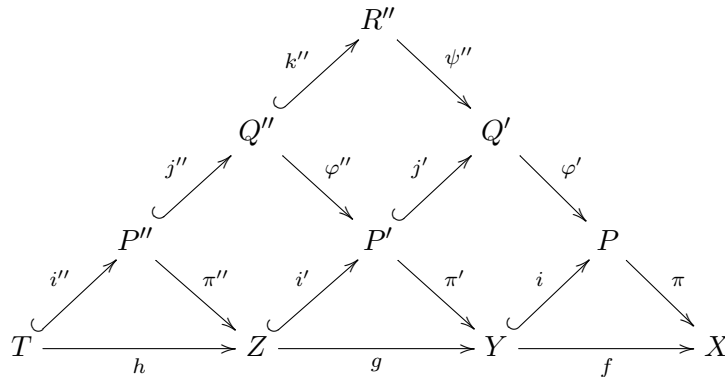
PROPOSITION A.6. *Let*

$$T \xrightarrow{h} Z \xrightarrow{g} Y \xrightarrow{f} X$$

*be three complete intersection morphisms. Then*

$$(A.6.1) \quad (\text{Id} \otimes h^*(\zeta'_{g,f})) \circ \zeta'_{h,fg} = (\zeta'_{h,g} \otimes \text{Id}) \circ \zeta'_{gh,f}.$$

*Proof.* As the verification is local on  $T$ , we may assume that there exists a commutative diagram





in which the three squares are cartesian, the morphisms  $\pi, \varphi', \psi''$  are smooth, and the immersions  $i, i', i''$  are regular. Using [Con00, (2.2.4)] and the relation (A.4.4), the proof of (A.6.1) again breaks into a succession of elementary commutative diagrams that we do not detail here.  $\square$

A.7. We now assume that  $f : Y \rightarrow X$  is a complete intersection morphism of (virtual) relative dimension 0 and, under this hypothesis, we define a section  $\delta_f \in \Gamma(Y, \omega_{Y/X})$  that we call the *canonical section*.

We first assume that there is a factorization  $f = \pi \circ i$  such that  $\pi : P \rightarrow X$  is a smooth morphism of relative dimension  $n$  and  $i : Y \hookrightarrow P$  is a regular closed immersion, necessarily of codimension  $n$  since  $f$  has relative dimension 0. Let  $\mathcal{I} \subset \mathcal{O}_P$  be the ideal defining  $i$ . The canonical derivation  $d : \mathcal{O}_P \rightarrow \Omega_{P/X}^1$  induces an  $\mathcal{O}_Y$ -linear homomorphism  $\bar{d} : \mathcal{I}/\mathcal{I}^2 \rightarrow i^*\Omega_{P/X}^1$ . Taking its  $n$ -th exterior power, we obtain a linear homomorphism

$$(A.7.1) \quad \wedge^n \bar{d} : \wedge^n(\mathcal{I}/\mathcal{I}^2) \longrightarrow i^*\Omega_{P/X}^n.$$

Through the canonical isomorphisms

$$\begin{aligned} \text{Hom}_{\mathcal{O}_Y}(\wedge^n(\mathcal{I}/\mathcal{I}^2), i^*\Omega_{P/X}^n) &\cong (\wedge^n(\mathcal{I}/\mathcal{I}^2))^\vee \otimes_{\mathcal{O}_Y} i^*\Omega_{P/X}^n \\ &\cong \wedge^n((\mathcal{I}/\mathcal{I}^2)^\vee) \otimes_{\mathcal{O}_Y} i^*\Omega_{P/X}^n \\ &= \omega_{Y/X}, \end{aligned}$$

it can be seen as a section of  $\omega_{Y/X}$ , which is the section  $\delta_f$ . If  $(t_1, \dots, t_n)$  is a regular sequence of generators of  $\mathcal{I}$  on a neighbourhood  $U$  of some point  $y \in Y$ , then

$$(A.7.2) \quad \delta_f = (\bar{t}_1^\vee \wedge \dots \wedge \bar{t}_n^\vee) \otimes i^*(dt_n \wedge \dots \wedge dt_1) \in \Gamma(U, \omega_{Y/X}),$$

since the canonical isomorphism  $(\wedge^n(\mathcal{I}/\mathcal{I}^2))^\vee \cong \wedge^n((\mathcal{I}/\mathcal{I}^2)^\vee)$  maps  $(\bar{t}_n \wedge \dots \wedge \bar{t}_1)^\vee$  to  $\bar{t}_1^\vee \wedge \dots \wedge \bar{t}_n^\vee$ .

To end the construction of  $\delta_f$ , it suffices to check that the section obtained in this way does not depend on the chosen factorization. Using the diagonal embedding, it suffices as usual to compare the sections  $\delta_f$  and  $\delta'_f$  defined by two factorizations  $f = \pi \circ i = \pi' \circ i'$  when there exists a smooth  $X$ -morphism  $u : P' \rightarrow P$  such that  $u \circ i' = i$ . Let  $\mathcal{I}'$  be the ideal of  $Y$  in  $P'$ , and let

$$\omega'_{Y/X} = \wedge^{n'}((\mathcal{I}'/\mathcal{I}'^2)^\vee) \otimes_{\mathcal{O}_Y} i'^*\Omega_{P'/X}^{n'},$$

where  $n'$  is the codimension of  $Y$  in  $P'$ . Then the canonical identification  $\omega_{Y/X} \cong \omega'_{Y/X}$  is given by (A.2.2) case (a) and, thanks to (A.7.2), the equality  $\delta_f = \delta'_f$  follows from [Con00, p. 30, (a) and (d)].

PROPOSITION A.8. *Let  $f : Y \rightarrow X$  be a complete intersection morphism of virtual relative dimension 0.*

(i) Let  $g : Z \rightarrow Y$  be a second complete intersection morphism of virtual relative dimension 0. The image of  $\delta_{fg}$  under the isomorphism  $\zeta'_{g,f}$  defined in (A.5.1) is given by

$$(A.8.1) \quad \zeta'_{g,f}(\delta_{fg}) = \delta_g \otimes g^*(\delta_f).$$

(ii) For any cartesian square (A.1.1), the isomorphism (A.3.1)

$$v^*(\omega_{Y/X}) \xrightarrow{\sim} \omega_{Y'/X'}$$

maps  $v^*(\delta_f)$  to  $\delta_{f'}$ .

*Proof.* As the first claim is local on  $Z$ , we may assume that there exists a diagram (A.5.2) in which the immersion  $i$  is defined by a regular sequence  $(t_1, \dots, t_n)$  and the immersion  $j = i'' \circ i'$  is defined by a regular sequence  $(t'_1, \dots, t'_{n'}, t''_1, \dots, t''_n)$ , with  $t''_i = \pi''^*(t_i)$ . If we set  $\bar{t}'_i = i'^*(t'_i)$ , then  $i'$  is defined by the regular sequence  $(\bar{t}'_1, \dots, \bar{t}'_{n'})$ . By construction,  $\delta_{fg}$  corresponds by  $\zeta'_{j,\psi}$  to the section

$$(t''_n{}^\vee \wedge \dots \wedge t''_1{}^\vee \wedge t'_{n'}{}^\vee \wedge \dots \wedge t'_1{}^\vee) \otimes j^*(dt'_1 \wedge \dots \wedge dt'_{n'} \wedge dt''_1 \wedge \dots \wedge dt''_n)$$

of  $\omega_{Z/P''} \otimes j^*(\omega_{P''/X})$ , which is mapped by  $\zeta'_{i',i''} \otimes j^*(\zeta'_{\pi'',\pi})$  to the section

$$((-1)^{nn'} (\bar{t}'_{n'}{}^\vee \wedge \dots \wedge \bar{t}'_1{}^\vee) \otimes i'^*(t''_n{}^\vee \wedge \dots \wedge t''_1{}^\vee))$$

$$\otimes j^*((dt'_1 \wedge \dots \wedge dt'_{n'}) \otimes \pi''^*(dt_1 \wedge \dots \wedge dt_n))$$

of  $(\omega_{Z/P'} \otimes i'^*(\omega_{P'/P''})) \otimes j^*(\omega_{P''/P} \otimes \pi''^*(\omega_{P/X}))$ . Then, via  $\chi_{\pi''^*,i}$ , we get the section

$$(\bar{t}'_{n'}{}^\vee \wedge \dots \wedge \bar{t}'_1{}^\vee) \otimes i'^*(d\bar{t}'_1 \wedge \dots \wedge d\bar{t}'_{n'}) \otimes i'^*\pi'^*(t_n{}^\vee \wedge \dots \wedge t_1{}^\vee) \otimes j^*\pi''^*(dt_1 \wedge \dots \wedge dt_n)$$

of  $\omega_{Z/P'} \otimes i'^*(\omega_{P'/Y}) \otimes i'^*\pi'^*(\omega_{Y/P}) \otimes j^*\pi''^*(\omega_{P/X})$ , which, by construction, corresponds by  $(\zeta'_{i',\pi'} \otimes g^*(\zeta'_{i,\pi}))^{-1}$  to the section  $\delta_g \otimes g^*(\delta_f)$  of  $\omega_{Z/Y} \otimes g^*(\omega_{Y/X})$ .

The second claim follows from (A.7.2).  $\square$

### B. The trace morphism $\tau_f$ on $Rf_*(\mathcal{O}_Y)$

Let  $f : Y \rightarrow X$  be a proper complete intersection morphism of virtual relative dimension 0. This section is devoted to the construction of the “trace morphism”  $\tau_f : Rf_*\mathcal{O}_Y \rightarrow \mathcal{O}_X$ , derived from the canonical section of  $\omega_{Y/X}$  defined in A.7. The key step is to define an identification  $\lambda_f$  between  $\omega_{Y/X}$ , as defined in A.2, and  $f^!\mathcal{O}_X$ . The construction is then a straightforward application of the relative duality theorem, and the properties of  $\tau_f$  listed in Theorem 3.1 follow from corresponding properties of  $\delta_f$  and  $\lambda_f$ .

B.1. For the whole section, we assume that  $X$  is a noetherian scheme with a dualizing complex. If  $f : Y \rightarrow X$  is a morphism of finite type and  $K^\bullet$  a residual complex on  $X$ , let  $f^\Delta K^\bullet$  be its inverse image on  $Y$  in the sense of residual complexes, which is a residual complex on  $Y$ . Then  $K^\bullet$  and  $f^\Delta K^\bullet$  define respectively duality  $\delta$ -functors  $D_X$  on  $D_{\text{coh}}(\mathcal{O}_X)$  and  $D_Y$  on  $D_{\text{coh}}(\mathcal{O}_Y)$ .

We recall that, following [Con00, 3.3], the functor  $f^! : D_{\text{coh}}^+(\mathcal{O}_X) \rightarrow D_{\text{coh}}^+(\mathcal{O}_Y)$  is defined by  $f^! = D_Y \circ Lf^* \circ D_X$ . We also recall that, when  $f$  is smooth of relative dimension  $m$ ,  $f^\sharp : D_{\text{coh}}^+(\mathcal{O}_X) \rightarrow D_{\text{coh}}^+(\mathcal{O}_Y)$  denotes the functor defined by

$$(B.1.1) \quad f^\sharp(\mathcal{E}^\bullet) := \omega_{Y/X}[m] \overset{L}{\otimes}_{\mathcal{O}_Y} Lf^*(\mathcal{E}^\bullet)$$

while, when  $f$  is finite,  $f^\flat : D_{\text{coh}}^+(\mathcal{O}_X) \rightarrow D_{\text{coh}}^+(\mathcal{O}_Y)$  denotes the functor defined by

$$(B.1.2) \quad f^\flat(\mathcal{E}^\bullet) := \bar{f}^* R\mathcal{H}om_{\mathcal{O}_X}(f_*\mathcal{O}_Y, \mathcal{E}^\bullet),$$

where  $\bar{f}$  is the (flat) morphism of ringed spaces  $(Y, \mathcal{O}_Y) \rightarrow (X, f_*\mathcal{O}_Y)$ .

Assume now that  $f : Y \rightarrow X$  is a complete intersection morphism of virtual relative dimension  $m$ . We first explain the relation between the relative dualizing module  $\omega_{Y/X}$  defined in the previous section and the extraordinary inverse image functor  $f^!$ . We will consider complexes of the form  $\mathcal{E}^\bullet = \mathcal{L}[r] \in D_{\text{coh}}^b(\mathcal{O}_X)$ , where  $r \in \mathbb{Z}$  is some integer and  $\mathcal{L}$  is an invertible  $\mathcal{O}_X$ -module. For such a complex, we generalize the above notation  $f^\sharp$ , and we again define

$$(B.1.3) \quad f^\sharp(\mathcal{E}^\bullet) := \omega_{Y/X}[m] \overset{L}{\otimes}_{\mathcal{O}_Y} Lf^*(\mathcal{E}^\bullet).$$

We observe that  $f^\sharp(\mathcal{E}^\bullet)$  is another complex concentrated in a single degree, with an invertible cohomology sheaf. We can then construct a canonical isomorphism

$$(B.1.4) \quad \lambda_{f, \mathcal{E}^\bullet} : f^\sharp(\mathcal{E}^\bullet) \xrightarrow{\sim} f^!(\mathcal{E}^\bullet)$$

as follows.

(a) If  $f$  is smooth, then definitions (B.1.1) and (B.1.3) coincide, and we set

$$(B.1.5) \quad \lambda_{f, \mathcal{E}^\bullet} = e_f : f^\sharp(\mathcal{E}^\bullet) \xrightarrow{\sim} f^!(\mathcal{E}^\bullet),$$

where  $e_f$  is the isomorphism defined by [Con00, (3.3.21)].

(b) If  $f$  is a regular immersion, then we define  $\lambda_{f, \mathcal{E}^\bullet}$  to be the composition

$$(B.1.6) \quad \lambda_{f, \mathcal{E}^\bullet} : f^\sharp(\mathcal{E}^\bullet) \xrightarrow[\sim]{\eta_f^{-1}} f^\flat(\mathcal{E}^\bullet) \xrightarrow[\sim]{d_f} f^!(\mathcal{E}^\bullet),$$

where  $\eta_f$  is defined by [Con00, (2.5.3)] and  $d_f$  by [Con00, (3.3.19)].

(c) In the general case, let us assume first that there exists a factorization  $f = \pi \circ i$ , where  $\pi : P \rightarrow X$  is a smooth morphism of relative dimension  $n$  and  $i$  is a regular immersion of codimension  $d = n - m$ . Then we define  $\lambda_{f, \mathcal{E}^\bullet}$  by the commutative diagram

$$(B.1.7) \quad \begin{array}{ccc} \omega_{Y/X}[m] \overset{L}{\otimes}_{\mathcal{O}_Y} Lf^*\mathcal{E}^\bullet & \xrightarrow[\sim]{\lambda_{f, \mathcal{E}^\bullet}} & f^!\mathcal{E}^\bullet \\ \zeta'_{i, \pi} \otimes \text{Id} \downarrow \sim & & \sim \downarrow c_{i, \pi} \\ \omega_{Y/P}[-d] \overset{L}{\otimes}_{\mathcal{O}_Y} Li^*\pi^\sharp\mathcal{E}^\bullet & \xrightarrow[\sim]{\lambda_{i, \pi^\sharp\mathcal{E}^\bullet}} i^!\pi^\sharp\mathcal{E}^\bullet \xrightarrow[\sim]{i^!(\lambda_{\pi, \mathcal{E}^\bullet})} & i^!\pi^!\mathcal{E}^\bullet, \end{array}$$

where  $c_{i, \pi}$  is the transitivity isomorphism [Con00, (3.3.14)].

This isomorphism is actually independent of the chosen factorization. To check it, one can argue as in A.2 to reduce the comparison between the isomorphisms (B.1.4) defined by two factorizations  $f = \pi \circ i = \pi' \circ i'$  to the case where there is a smooth  $X$ -morphism  $u : P' \rightarrow P$  such that  $u \circ i' = i$ . It is then a long but straightforward verification, using various functorialities, the compatibility between  $\zeta'_{i',u}$  and the isomorphism  $i^b \simeq i'^b u^\sharp$  [Con00, (2.7.4)], the compatibility between  $\zeta'_{u,\pi}$  and the isomorphism  $\pi'^\sharp \simeq u^\sharp \pi^\sharp$  [Con00, (2.2.7)], and the properties (VAR1), (VAR3), and (VAR5) of the functor  $f^!$  (see [Har66, III, Th. 8.7] and [Con00, p. 139]).

Since  $f^! \mathcal{E}^\bullet$  is acyclic outside degree  $-m - r$ , a morphism  $\omega_{Y/X}[m] \otimes^L Lf^* \mathcal{E}^\bullet \rightarrow f^! \mathcal{E}^\bullet$  in  $D(\mathcal{O}_Y)$  is simply a module homomorphism  $\omega_{Y/X} \otimes f^* \mathcal{L} \rightarrow \mathcal{H}^{-m-r}(f^! \mathcal{E}^\bullet)$ . Therefore, in the general case, the previous construction provides local isomorphisms that can be glued to define a global isomorphism even if there does not exist a global factorization  $f = \pi \circ i$  as above.

When  $\mathcal{E}^\bullet = \mathcal{O}_X[0]$ , the isomorphism (B.1.4) will simply be denoted

$$(B.1.8) \quad \lambda_f : \omega_{Y/X}[m] \xrightarrow{\sim} f^! \mathcal{O}_X.$$

If  $f$  is flat, hence is a CM map, it provides the identification between the construction of  $\omega_{Y/X}$  used in this article and the construction of Conrad for CM maps [Con00, 3.5, p. 157].

We now give a transitivity property of the isomorphisms  $\lambda_{f,\mathcal{E}^\bullet}$  which generalizes (B.1.7).

**PROPOSITION B.2.** *Let  $f : Y \rightarrow X$ ,  $g : Z \rightarrow Y$  be two complete intersection morphisms, with virtual relative dimensions  $m, m'$ , and let  $\mathcal{E}^\bullet = \mathcal{L}[r]$  for some invertible  $\mathcal{O}_X$ -module  $\mathcal{L}$  and some integer  $r$ . Then the diagram*

(B.2.1)

$$\begin{array}{ccc}
 \omega_{Z/X}[m + m'] \otimes_{\mathcal{O}_Z}^L L(fg)^* \mathcal{E}^\bullet & \xrightarrow[\sim]{\lambda_{fg,\mathcal{E}^\bullet}} & (fg)^! \mathcal{E}^\bullet \\
 \zeta'_{g,f} \otimes \text{Id} \downarrow \sim & & \sim \downarrow c_{g,f} \\
 \omega_{Z/Y}[m'] \otimes_{\mathcal{O}_Z}^L Lg^* f^\sharp \mathcal{E}^\bullet & \xrightarrow[\sim]{\lambda_{g,f^\sharp \mathcal{E}^\bullet}} g^! f^\sharp \mathcal{E}^\bullet \xrightarrow[\sim]{g^!(\lambda_{f,\mathcal{E}^\bullet})} & g^! f^! \mathcal{E}^\bullet \\
 \parallel & & \parallel \\
 \omega_{Z/Y}[m'] \otimes_{\mathcal{O}_Z}^L Lg^* f^\sharp \mathcal{E}^\bullet & \xrightarrow[\sim]{g^\sharp(\lambda_{f,\mathcal{E}^\bullet})} \omega_{Z/Y}[m'] \otimes_{\mathcal{O}_Z}^L Lg^* f^! \mathcal{E}^\bullet \xrightarrow[\sim]{\lambda_{g,f^! \mathcal{E}^\bullet}} & g^! f^! \mathcal{E}^\bullet
 \end{array}$$

commutes.

*Proof.* The commutativity of the lower part of the diagram is due to the functoriality of the isomorphism  $\lambda_g$  with respect to morphisms between two complexes concentrated in the same degree. We first observe that the commutativity of (B.2.1) is clear in the following cases:

(a) If  $f$  is smooth and  $g$  is a closed immersion, the diagram is (B.1.7), which commutes by construction.

(b) If  $f$  and  $g$  are smooth, the isomorphism  $(fg)^\# \cong g^\# f^\#$  is defined by  $\zeta'_{g,f}$ . Hence, the commutativity of (B.2.1) is the compatibility of the isomorphisms  $e_f$  with composition, i.e., property (VAR3) of the functor  $f^!$  [Con00, p. 139].

(c) If  $f$  and  $g$  are regular immersions, then isomorphisms such as  $\eta_{fg}$  commute with  $\zeta'_{g,f}$  and  $c_{g,f}$  [Con00, Th. 2.5.1], and the commutativity of (B.2.1) follows from the compatibility of the isomorphisms  $d_f$  with composition, i.e., property (VAR2) of the functor  $f^!$  [Con00, p. 139].

We will also use the following remark. Let  $h : T \rightarrow Z$  be a third complete intersection morphism, yielding the four couples of composable complete intersection morphisms  $(h, g)$ ,  $(g, f)$ ,  $(gh, f)$ , and  $(h, fg)$ . Then, if the diagrams (B.2.1) for the couples  $(h, g)$  and  $(g, f)$  are commutative, the commutativity of (B.2.1) for  $(gh, f)$  is equivalent to the commutativity of (B.2.1) for  $(h, fg)$ . This is a consequence of (A.6.1) and of the compatibility of the isomorphisms  $c_{g,f}$  with triple composites (i.e., property (VAR1) of the functor  $f^!$  [Con00, p. 139]).

In the general case, the complexes entering in (B.2.1) are concentrated in the same degree; hence, its commutativity can be checked locally. So we may assume that there exists a diagram (A.5.2). Thanks to the three particular cases listed above, one can then deduce the commutativity of (B.2.1) for  $(f, g)$  from the commutativity of (B.2.1) for  $(\pi', i)$ , by applying the previous remark successively to the triples  $(i', i'', \pi'')$ ,  $(i''i', \pi'', \pi)$ ,  $(i', \pi', i)$ , and  $(g, i, \pi)$ .

To prove the commutativity of (B.2.1) for  $(\pi', i)$ , we use the factorization  $i \circ \pi' = \pi'' \circ i''$  to define  $\lambda_{i\pi', \mathcal{E}}$ . Let  $d$  be the codimension of  $Y$  in  $P$  and  $n'$  the relative dimension of  $P''$  over  $P$ . Then, if  $\mathcal{E}$  is a complex on  $P$  as in B.1, (B.2.1) for  $(\pi', i)$  is made of the exterior composites in the following diagram: (B.2.2)

$$\begin{array}{ccc}
 & \omega_{P'/P}[n'-d] \otimes^L L(i\pi')^* \mathcal{E}^\bullet & \\
 & \swarrow \zeta'_{\pi', i} \otimes \text{Id} \sim & \searrow \zeta'_{i'', \pi''} \otimes \text{Id} \sim \\
 \omega_{P'/Y}[n'] \otimes^L \pi'^*(\omega_{Y/P}[-d] \otimes^L Li^* \mathcal{E}^\bullet) & \xrightarrow{\sim} \chi_{\pi'', i} \otimes \text{Id} & \omega_{P'/P''}[-d] \otimes^L Li''^*(\omega_{P''/P}[n'] \otimes^L \pi''^* \mathcal{E}^\bullet) \\
 \downarrow \pi'^\#(\lambda_{i, \mathcal{E}}) & \sim & \downarrow \eta_{i''}^{-1} \sim \\
 \pi'^\# i^\# \mathcal{E}^\bullet & \xrightarrow{\sim} & i''^\# \pi''^\# \mathcal{E}^\bullet \\
 \downarrow \pi'^\#(d_i) & & \downarrow d_{i''} \sim \\
 \pi'^\# i^! \mathcal{E}^\bullet & & i''^\# \pi''^\# \mathcal{E}^\bullet \\
 \downarrow e_{\pi'} = \lambda_{\pi', i^! \mathcal{E}} & & \downarrow i''^\#(\lambda_{\pi'', \mathcal{E}}) = i''^\#(e_{\pi''}) \sim \\
 \pi'^! i^! \mathcal{E}^\bullet & \xleftarrow{\sim} (i\pi')^! \mathcal{E}^\bullet = (\pi''i'')^! \mathcal{E}^\bullet \xrightarrow{\sim} & i''^\# \pi''^\# \mathcal{E}^\bullet
 \end{array}$$

Here, the middle horizontal arrow is the standard isomorphism [Con00, Lemma 2.7.3], and the lower rectangle commutes thanks to property (VAR4) of the functor  $f^!$  [Con00, Theorem 3.3.1]. The upper triangle commutes thanks to (A.4.2). To check the commutativity of the middle rectangle, one observes on the one hand that  $\eta_i$  commutes with the flat base change  $\pi''$  and that  $\eta_{i''}$  commutes with tensorization by the invertible sheaf  $\omega_{P''/P}$  (see [Con00, last paragraph of p. 54]). On the other hand,  $\eta_{i''}$  commutes also with the translation by  $n'$ , provided that the convention [Con00, (1.3.6)] is used for the commutation of the tensor product with translations applied to the second argument (see the discussion on [Con00, p. 53]). Here, this requires multiplication by  $(-1)^{dn'}$  on  $\omega_{P'/P''} \otimes i''^* \omega_{P''/P}$ , since  $\omega_{P'/P''}$  sits in degree  $d$ . As this is the sign entering in the definition of  $\chi_{\pi'',i}$ , this ends the proof.  $\square$

B.3. Assume now that  $f$  is proper. As in B.1, let  $\mathcal{E}^\bullet = \mathcal{L}[r] \in D_{\text{coh}}^b(\mathcal{O}_X)$ ,  $\mathcal{L}$  being an invertible  $\mathcal{O}_X$ -module and  $r$  an integer. Using (B.1.4), we can define the trace morphism  $\text{Tr}_{f,\mathcal{E}^\bullet}^\sharp$  on  $Rf_* f^\sharp \mathcal{E}^\bullet = Rf_*(\omega_{Y/X}[m] \otimes_{\mathcal{O}_Y}^L Lf^* \mathcal{E}^\bullet)$  as the composite

$$(B.3.1) \quad \text{Tr}_{f,\mathcal{E}^\bullet}^\sharp : Rf_* f^\sharp \mathcal{E}^\bullet \xrightarrow[\sim]{Rf_*(\lambda_{f,\mathcal{E}^\bullet})} Rf_* f^! \mathcal{E}^\bullet \xrightarrow{\text{Tr}_f} \mathcal{E}^\bullet,$$

where  $\text{Tr}_f$  denotes the classical trace morphism defined in [Har66, VII, Cor. 3.4] and [Con00, 3.4]. When  $\mathcal{E}^\bullet = \mathcal{O}_X[0]$ , we will use the shorter notation

$$(B.3.2) \quad \text{Tr}_f^\sharp : Rf_*(\omega_{Y/X}[m]) \rightarrow \mathcal{O}_X.$$

We first give some basic properties of the morphism  $\text{Tr}_f^\sharp$ .

LEMMA B.4. *With the previous hypotheses, let*

$$(B.4.1) \quad \mu_{f,\mathcal{E}^\bullet} : Rf_*(\omega_{Y/X}[m]) \otimes_{\mathcal{O}_X}^L \mathcal{E}^\bullet \xrightarrow{\sim} Rf_*(\omega_{Y/X}[m] \otimes_{\mathcal{O}_Y}^L Lf^* \mathcal{E}^\bullet) = Rf_* f^\sharp \mathcal{E}^\bullet$$

be the isomorphism given by the projection formula [Har66, II, Prop. 5.6]. Then the diagram

$$(B.4.2) \quad \begin{array}{ccc} Rf_*(\omega_{Y/X}[m]) \otimes_{\mathcal{O}_X}^L \mathcal{E}^\bullet & \xrightarrow[\sim]{\mu_{f,\mathcal{E}^\bullet}} & Rf_* f^\sharp \mathcal{E}^\bullet \\ & \searrow \sim \text{Tr}_f^\sharp \otimes \text{Id} & \swarrow \sim \text{Tr}_{f,\mathcal{E}^\bullet}^\sharp \\ & & \mathcal{E}^\bullet \end{array}$$

commutes.

*Proof.* When  $f$  is flat, it suffices to invoke [Con00, Th. 4.4.1]. Since we make no such assumption on  $f$ , we give a direct argument that is made a lot simpler by the very special nature of the complex  $\mathcal{E}^\bullet$ .

Using the fact that  $\mathcal{E}^\bullet = \mathcal{L}[r]$ , with  $\mathcal{L}$  invertible, one easily sees that there is a canonical isomorphism that commutes with translations acting on  $\mathcal{E}^\bullet$

$$(B.4.3) \quad f^! \mathcal{O}_X \overset{L}{\otimes}_{\mathcal{O}_Y} Lf^* \mathcal{E}^\bullet \xrightarrow{\sim} f^! \mathcal{E}^\bullet.$$

On the other hand, we have by definition a canonical isomorphism

$$(B.4.4) \quad f^\# \mathcal{O}_X \overset{L}{\otimes}_{\mathcal{O}_Y} Lf^* \mathcal{E}^\bullet \xrightarrow{\sim} f^\# \mathcal{E}^\bullet$$

that also commutes with translations. A first observation is that the diagram

$$(B.4.5) \quad \begin{array}{ccc} f^\# \mathcal{O}_X \overset{L}{\otimes}_{\mathcal{O}_Y} Lf^* \mathcal{E}^\bullet & \xrightarrow[\sim]{(B.4.4)} & f^\# \mathcal{E}^\bullet \\ \lambda_f \otimes \text{Id} \downarrow \sim & & \sim \downarrow \lambda_{f, \mathcal{E}^\bullet} \\ f^! \mathcal{O}_X \overset{L}{\otimes}_{\mathcal{O}_Y} Lf^* \mathcal{E}^\bullet & \xrightarrow[\sim]{(B.4.3)} & f^! \mathcal{E}^\bullet \end{array}$$

commutes. Indeed, all complexes are concentrated in the same degree  $m - r$ . Hence, the verification can be done locally. This allows us to assume that  $\mathcal{L} = \mathcal{O}_X$ , which reduces the verification to the commutation of the vertical arrows with translations acting on  $\mathcal{E}^\bullet$ . This now follows from the fact that the isomorphisms  $e_f$ ,  $\eta_f$  and  $d_f$  used in the construction of  $\lambda_f$  commute with translations.

Applying  $Rf_*$  to this diagram and using the functoriality of the projection formula isomorphism, the proof is reduced to proving the commutativity of the diagram

$$(B.4.6) \quad \begin{array}{ccccc} Rf_* f^! \mathcal{O}_X \overset{L}{\otimes}_{\mathcal{O}_X} \mathcal{E}^\bullet & \xrightarrow[\sim]{\nu_f} & Rf_*(f^! \mathcal{O}_X \overset{L}{\otimes}_{\mathcal{O}_Y} Lf^* \mathcal{E}^\bullet) & \xrightarrow[\sim]{(B.4.3)} & Rf_* f^! \mathcal{E}^\bullet \\ & \searrow \sim & & \swarrow \sim & \\ & & \mathcal{E}^\bullet & & \end{array}$$

$\text{Tr}_f \otimes \text{Id}$  (left arrow),  $\text{Tr}_f$  (right arrow)

where  $\nu_f$  is the projection formula isomorphism. As all morphisms of the diagram commute with translations, we may assume that  $r = 0$ . We recall that  $\text{Tr}_f$  is defined as the morphism of functors defined by the composite

$$Rf_* f^! (\cdot) \xrightarrow{\sim} R\mathcal{H}om_{\mathcal{O}_X}(D_X(\cdot), f_* f^\Delta K) \xrightarrow{\text{Tr}_{f, K}} R\mathcal{H}om_{\mathcal{O}_X}(D_X(\cdot), K) \xleftarrow{\sim} \text{Id},$$

where the first isomorphism follows from the definition of  $f^!$  and the adjunction between  $Lf^*$  and  $Rf_*$ , the second morphism is defined by the trace morphism for residual complexes  $\text{Tr}_{f, K}$ , and the last isomorphism is the local biduality isomorphism (see [Con00, p. 146]). Each of these morphisms has a natural compatibility with respect to the tensor product of the argument by an invertible sheaf. Putting together these compatibilities yields the commutativity of (B.4.6).  $\square$

PROPOSITION B.5. *Let  $g : Z \rightarrow Y$  be a second proper complete intersection morphism, with virtual relative dimension  $m'$ . Then the diagram (B.5.1)*

$$\begin{array}{ccc}
 Rf_*Rg_*(\omega_{Z/X}[m' + m]) & \xrightarrow[\sim]{Rf_*Rg_*(\zeta'_{g,f})} & Rf_*Rg_*(\omega_{Z/Y}[m'] \overset{L}{\otimes}_{\mathcal{O}_Z} Lg^*(\omega_{Y/X}[m])) \\
 \downarrow \text{Tr}_{fg}^\# & & \downarrow \sim \\
 & & Rf_*(Rg_*(\omega_{Z/Y}[m']) \overset{L}{\otimes}_{\mathcal{O}_Y} \omega_{Y/X}[m]) \\
 & & \downarrow Rf_*(\text{Tr}_g^\# \otimes \text{Id}) \\
 \mathcal{O}_X & \xleftarrow{\text{Tr}_f^\#} & Rf_*(\omega_{Y/X}[m])
 \end{array}$$

(where the second isomorphism is given by the projection formula) is commutative.

*Proof.* It follows from Lemma B.4 that the right vertical arrow is equal to the morphism

$$Rf_*Rg_*(\omega_{Z/Y}[m'] \otimes g^*(\omega_{Y/X}[m])) \xrightarrow{\text{Tr}_{g,\omega_{Y/X}[m]}^\#} Rf_*(\omega_{Y/X}[m]).$$

Then, using adjunction between  $Rf_*$  and  $f^!$ , and adjunction between  $Rg_*$  and  $g^!$ , one sees that the commutativity of (B.5.1) is equivalent to the commutativity of (B.2.1).  $\square$

PROPOSITION B.6. *With the hypotheses of Proposition A.1 assume, in addition, that  $X$  and  $X'$  are noetherian schemes with dualizing complexes, and that one of the following conditions is satisfied:*

- (a)  $f$  is projective;
- (b)  $f$  is proper and  $u$  is residually stable [Con00, p. 132];
- (c)  $f$  is proper and flat.

*Then the triangle*

$$\begin{array}{ccc}
 Lu^*Rf_*(\omega_{Y/X}[m]) & & \\
 \downarrow \text{(A.3.2)} \sim & \searrow^{Lu^*(\text{Tr}_f^\#)} & \mathcal{O}_{X'} \\
 Rf'_*(\omega_{Y'/X'}[m]) & \nearrow_{\text{Tr}_{f'}^\#} & 
 \end{array}$$

*is commutative.*

*Proof of Case (a).* We can choose a factorization  $f = \pi \circ i$ , where  $\pi : P \rightarrow X$  is the structural morphism of some projective space  $P = \mathbb{P}_X^n$  over  $X$ , and  $i$



is a regular immersion of codimension  $d = n - m$ . Let  $f' = \pi' \circ i'$  be the factorization of  $f'$  defined by base change, with  $\pi' : P' = \mathbb{P}_{X'}^n \rightarrow X'$ , and let  $w : P' \rightarrow P$  be the projection.

The isomorphisms  $\zeta'_{i,\pi}$  and  $\zeta'_{i',\pi'}$  are clearly compatible with the base change isomorphisms (A.3.1) relative to  $f$  and  $u$ , and the same holds for the projection formula isomorphisms  $\mu_{i,\omega_{P/X}[n]}$  and  $\mu_{i',\omega_{P'/X'}[n]}$ , and the base change isomorphisms (A.3.1) relative to  $i$  and  $w$ . Then, using Proposition B.5, one sees that it suffices to prove the proposition for  $f = i$  and for  $f = \pi$ .

When  $f = \pi : \mathbb{P}_X^n \rightarrow X$ , let  $X_0, \dots, X_n$  be the canonical coordinates on  $\mathbb{P}_X^n$ , and let  $x_i = X_i/X_0$ ,  $1 \leq i \leq n$ . If  $\mathfrak{U}$  is the relatively affine covering of  $\mathbb{P}_X^n$  defined by  $X_0, \dots, X_n$ , the corresponding alternating Čech resolution provides a canonical isomorphism

$$(B.6.2) \quad f_*(\check{C}^\bullet(\mathfrak{U}, \omega_{P/X})[n]) \xrightarrow{\sim} Rf_*(\omega_{P/X}[n]).$$

Recall that  $e_\pi : \pi^\sharp \cong \pi^!$  identifies the trace morphism for projective spaces  $\text{Trp}_\pi$  with the general trace morphism  $\text{Tr}_\pi$  [Con00, Lemma 3.4.3, (TRA3)]. Then the commutativity of (B.6.1) for  $\pi$  follows from the fact that, thanks to (6.6.2),  $\text{Trp}_\pi$  can be characterized as the only morphism that, via (B.6.2), induces on  $\mathcal{H}^0$  the map sending the cohomology class  $dx_1 \wedge \dots \wedge dx_n/x_1 \cdots x_n$  to  $(-1)^{n(n-1)/2}$ .

When  $f = i : Y \hookrightarrow P$ , recall that  $d_i : i^\flat \cong i^!$  identifies the trace morphism for finite morphisms  $\text{Trf}_i$  with the general trace morphism  $\text{Tr}_i$  [Con00, Lemma 3.4.3, (TRA2)], and recall that  $\text{Trf}_i : R\mathcal{H}om_{\mathcal{O}_P}(\mathcal{O}_Y, \mathcal{O}_P) \rightarrow \mathcal{O}_P$  is the canonical morphism induced by  $\mathcal{O}_P \rightarrow \mathcal{O}_Y$ . Using local cohomology with supports in  $Y$ , it can be factorized as

$$(B.6.3) \quad \text{Trf}_i : R\mathcal{H}om_{\mathcal{O}_P}(\mathcal{O}_Y, \mathcal{O}_P) \rightarrow R\underline{\Gamma}_Y(\mathcal{O}_P) \rightarrow \mathcal{O}_P.$$

On the other hand, there exists a canonical morphism

$$(B.6.4) \quad Lw^*R\underline{\Gamma}_Y(\mathcal{O}_P) \longrightarrow R\underline{\Gamma}_{Y'}(\mathcal{O}_{P'}),$$

which is an isomorphism. To check this, it suffices to choose a finite affine covering  $\mathfrak{V}$  of  $V = P \setminus Y$  and to identify  $R\underline{\Gamma}_Y(\mathcal{O}_P)$  with its flat resolution provided by the total complex

$$\mathcal{O}_P \rightarrow j_*\check{C}(\mathfrak{V}, \mathcal{O}_V),$$

where  $j$  denotes the inclusion of  $V$  in  $P$  and  $\mathcal{O}_P$  sits in degree 0. Moreover, this shows that the diagram

$$\begin{array}{ccc} Lw^*R\underline{\Gamma}_Y(\mathcal{O}_P) & \longrightarrow & Lw^*(\mathcal{O}_P) \\ \sim \downarrow & & \downarrow \sim \\ R\underline{\Gamma}_{Y'}(\mathcal{O}_{P'}) & \longrightarrow & \mathcal{O}_{P'} \end{array}$$

commutes. Therefore, it suffices to prove the commutativity of the diagram

$$(B.6.5) \quad \begin{array}{ccc} Lw^*i_*(\omega_{Y/P}[-d]) & \longrightarrow & Lw^*R\Gamma_Y(\mathcal{O}_P) \\ \sim \downarrow & & \downarrow \sim \\ \omega_{Y'/P'}[-d] & \longrightarrow & R\Gamma_{Y'}(\mathcal{O}_{P'}). \end{array}$$

Since  $Y' \hookrightarrow P'$  is a regular immersion of codimension  $d$ , all complexes in this diagram are acyclic except in degree  $d$ , so that, up to translation by  $-d$ , the diagram is actually a diagram of morphisms of  $\mathcal{O}_{P'}$ -modules. It follows that its commutativity can be checked locally on  $P'$ . Thus, we may assume that  $P$  is affine and that the ideal  $\mathcal{I}$  of  $Y$  in  $P$  is generated by a regular sequence  $t_1, \dots, t_d$ . Then, the ideal  $\mathcal{I}'$  of  $Y'$  in  $P'$  is generated by the images  $t'_1, \dots, t'_d$  of  $t_1, \dots, t_d$  in  $\mathcal{O}_{P'}$ , which form a regular sequence. Let  $\mathfrak{V} = (V_1, \dots, V_d)$  be the open covering of  $P \setminus Y$  defined by the sequence  $(t_1, \dots, t_d)$ . For any section  $a \in \Gamma(P, \mathcal{O}_P)$ , let us still denote by  $a/t_1 \cdots t_d$  the image of  $a/t_1 \cdots t_d \in \Gamma(V_1 \cap \cdots \cap V_d, \mathcal{O}_P)$  under the canonical homomorphisms

$$\Gamma(V_1 \cap \cdots \cap V_d, \mathcal{O}_P) \rightarrow H^{d-1}(P \setminus Y, \mathcal{O}_P) \rightarrow H_Y^d(P, \mathcal{O}_P) = \Gamma(P, \mathcal{H}_Y^d(\mathcal{O}_P)).$$

Then the canonical morphism

$$\omega_{Y/P} \xrightarrow{\sim} \mathcal{E}xt_{\mathcal{O}_P}^d(\mathcal{O}_Y, \mathcal{O}_P) \rightarrow \mathcal{H}_Y^d(\mathcal{O}_P)$$

maps  $(\bar{t}_1^Y \wedge \cdots \wedge \bar{t}_d^Y) \otimes a$  to  $\varepsilon(d)a/t_1 \cdots t_d$ , where  $\varepsilon(d) \in \{\pm 1\}$  only depends upon  $d$  (see [Con00, p. 252-254]). The commutativity of (B.6.5) follows.  $\square$

*Proof of Case (b).* When  $u$  is residually stable, the diagram analogous to (B.6.1) commutes, thanks to [Con00, 3.4.3, (TRA4)]. Moreover, the isomorphisms  $e_\pi$  and  $d_i$  entering in the local definition of  $\lambda_f$  in B.1.4(c) also commute with base change by  $u$ , thanks to [Con00, p. 139, (VAR6)]. Then it suffices to observe that  $\eta_i$  commutes with flat base change, which is clear.  $\square$

*Proof of Case (c).* When  $f$  is flat,  $f$  is a CM map, and the results of [Con00, 3.5 –3.6] can be applied. Then the commutativity of (B.6.1) follows from [Con00, Th. 3.6.5], provided one checks that  $\lambda_f$  identifies the base change isomorphism (A.3.1) for  $\omega_{Y/X}$  with the more subtle base change isomorphism  $\beta_{f,u}$  for  $\omega_f$  defined in [Con00, Th. 3.6.1]. As we will not use Case (c) in this article, we leave the details to the reader.  $\square$

B.7. Let  $X$  be a noetherian scheme with a dualizing complex, and  $f : Y \rightarrow X$  a proper complete intersection morphism of virtual relative dimension 0. In  $D_{\text{coh}}^b(X)$  one can define a “trace morphism”

$$(B.7.1) \quad \tau_f : Rf_*(\mathcal{O}_Y) \longrightarrow \mathcal{O}_X$$

as follows. Thanks to the relative duality theorem (see [Har66, VII, 3.4] or [Con00, Th. 3.4.4]), defining  $\tau_f$  is equivalent to defining a morphism  $\mathcal{O}_Y \rightarrow f^! \mathcal{O}_X$ . Using the isomorphism  $\lambda_f$ , this is also equivalent to defining a morphism

$$(B.7.2) \quad \varphi_f : \mathcal{O}_Y \longrightarrow \omega_{Y/X},$$

i.e., a section of the invertible sheaf  $\omega_{Y/X}$ . We define  $\varphi_f$  as being the morphism that maps 1 to the canonical section  $\delta_f$  of  $\omega_{Y/X}$ , defined in A.7.

From this construction, it follows that the morphism  $\tau_f$  can be described equivalently either as the composition

$$(B.7.3) \quad \tau_f : Rf_*(\mathcal{O}_Y) \xrightarrow{Rf_*(\lambda_f \circ \varphi_f)} Rf_*(f^! \mathcal{O}_X) \xrightarrow{\text{Tr}_f} \mathcal{O}_X,$$

or as the composition

$$(B.7.4) \quad \tau_f : Rf_*(\mathcal{O}_Y) \xrightarrow{Rf_*(\varphi_f)} Rf_*(\omega_{Y/X}) \xrightarrow{\text{Tr}_f^\#} \mathcal{O}_X,$$

where  $\text{Tr}_f^\#$  is the trace map defined in (B.3.1).

Before proving Theorem 3.1, we relate  $\tau_f$  to the residue symbol defined in [Con00, (A.1.4)] (which differs by a sign from Hartshorne’s definition in [Har66]).

**PROPOSITION B.8.** *With the hypotheses of B.7 assume, in addition, that  $f$  is finite and flat and that  $f = \pi \circ i$ , where  $\pi$  is smooth of relative dimension  $d$  and  $i$  is a closed immersion, globally defined by a regular sequence  $(t_1, \dots, t_d)$  of sections of  $\mathcal{O}_P$ . Then, for any section  $a$  of  $\mathcal{O}_P$ , with reduction  $\bar{a}$  on  $Y$ , we have*

$$(B.8.1) \quad \tau_f(\bar{a}) = \text{Res}_{P/X} \left[ \begin{array}{c} a dt_1 \wedge \cdots \wedge dt_d \\ t_1, \dots, t_d \end{array} \right].$$

*Proof.* Let  $\omega = a dt_1 \wedge \cdots \wedge dt_d$ . By construction, the residue symbol is given by

$$(B.8.2) \quad \text{Res}_{P/X} \left[ \begin{array}{c} \omega \\ t_1, \dots, t_d \end{array} \right] = (-1)^{d(d-1)/2} \varphi_\omega(1),$$

where  $\varphi_\omega : f_* \mathcal{O}_Y \rightarrow \mathcal{O}_X$  is the image of  $(t_1^\vee \wedge \cdots \wedge t_d^\vee) \otimes i^*(\omega)$  by the isomorphism of complexes concentrated in degree 0 [Con00, (A.1.3)]:

$$(B.8.3) \quad \omega_{Y/P}[-d] \otimes_{\mathcal{O}_Y}^L Li^*(\omega_{P/X}[d]) \xrightarrow[\sim]{\eta_i^{-1}} i^b \pi^\# \mathcal{O}_X \xrightarrow[\sim]{\psi_{i,\pi}^{-1}} f^b \mathcal{O}_X.$$

Here,  $f^b \mathcal{O}_X = \mathcal{H}om_{\mathcal{O}_X}(f_* \mathcal{O}_Y, \mathcal{O}_X)$ , viewed as a  $\mathcal{O}_Y$ -module, and  $\psi_{i,\pi}$  is the canonical isomorphism of functors  $f^b \xrightarrow{\sim} i^b \pi^\#$ . Since  $\text{Tr}_f$  is the morphism  $f_* \mathcal{H}om_{\mathcal{O}_X}(f_* \mathcal{O}_Y, \mathcal{O}_X) \rightarrow \mathcal{O}_X$  given by evaluation at 1, we can use the isomorphism  $d_f : f^b \xrightarrow{\sim} f^!$  and the equality  $\text{Tr}_f \circ f_*(d_f) = \text{Tr}_f$  [Con00, 3.4.3, (TRA2)]

to write

(B.8.4)

$$\mathrm{Res}_{P/X} \left[ \begin{array}{c} \omega \\ t_1, \dots, t_d \end{array} \right] = (-1)^{d(d-1)/2} \mathrm{Tr}_f(f_*(d_f \circ \psi_{i,\pi}^{-1} \circ \eta_i^{-1})(t_1^\vee \wedge \dots \wedge t_d^\vee \otimes i^*(\omega))).$$

On the other hand, by definition, we have

$$\zeta'_{i,\pi}(\delta_f) = (-1)^{d(d-1)/2} t_1^\vee \wedge \dots \wedge t_d^\vee \otimes i^*(dt_1 \wedge \dots \wedge dt_d),$$

so from (B.7.3), we deduce the equality

$$\begin{aligned} \tau_f(\bar{a}) &= \mathrm{Tr}_f(f_*(\lambda_f \circ \varphi_f)(\bar{a})) \\ &= (-1)^{d(d-1)/2} \mathrm{Tr}_f(f_*(c_{i,\pi}^{-1} \circ i^!(e_\pi) \circ d_i \circ \eta_i^{-1})(t_1^\vee \wedge \dots \wedge t_d^\vee \otimes i^*(\omega))). \end{aligned}$$

Therefore, it suffices to check that

$$d_f \circ \psi_{i,\pi}^{-1} = c_{i,\pi}^{-1} \circ i^!(e_\pi) \circ d_i,$$

and this results from (VAR5) [Con00, (3.3.26)]. □

B.9. *Proof of Theorem 3.1.*

(i) The transitivity formula (3.1.1) is the equality of the exterior composites in the diagram

$$\begin{array}{ccccc} Rf_*Rg_*\mathcal{O}_Z & \xrightarrow{Rf_*Rg_*(\varphi_g)} & Rf_*Rg_*\omega_{Z/Y} & & \\ \downarrow Rf_*Rg_*(\varphi_{fg}) & & \downarrow Rf_*Rg_*(\mathrm{Id} \otimes Lg^*(\varphi_f)) & & \\ Rf_*Rg_*\omega_{Z/X} & \xrightarrow{Rf_*Rg_*(\zeta'_{g,f})} & Rf_*Rg_*\left(\omega_{Z/Y} \otimes_{\mathcal{O}_Z}^L Lg^*\omega_{Y/X}\right) & \xrightarrow{Rf_*(\mathrm{Tr}_g^\sharp)} & Rf_*\mathcal{O}_Y \\ \downarrow \mathrm{Tr}_{fg}^\sharp & & \downarrow \sim & & \\ & & Rf_*\left((Rg_*\omega_{Z/Y}) \otimes_{\mathcal{O}_Y}^L \omega_{Y/X}\right) & & \\ & & \downarrow Rf_*(\mathrm{Tr}_g^\sharp \otimes \mathrm{Id}) & & \\ \mathcal{O}_X & \xleftarrow{\mathrm{Tr}_f^\sharp} & Rf_*\omega_{Y/X} & \xleftarrow{Rf_*(\varphi_f)} & Rf_*\mathcal{O}_Y \end{array}$$

where the upper left square commutes thanks to (A.8.1), the lower left square is the commutative square (B.5.1), and the right triangle commutes by functoriality.

(ii) Thanks to Proposition B.6 and to the description (B.7.4) of  $\tau_f$ , the assertion follows from the compatibility of the canonical section  $\delta_f$  with Tor-independent pull-backs (proved in Proposition A.8(ii)) and the functoriality of the base change morphism.

(iii) To prove (3.1.3), it suffices to prove that the equality holds in the henselization  $\mathcal{O}_{X,x}^h$  of the local ring of  $X$  at each point  $x$ . As the morphism  $\mathrm{Spec} \mathcal{O}_{X,x}^h \rightarrow X$  is residually stable [Con00, p. 132], Proposition B.6 and the commutation with base change of the classical trace map for the finite locally free algebra  $f_*\mathcal{O}_Y$  allow to assume that  $X = \mathrm{Spec} A$ , where  $A$  is a henselian,

noetherian local ring. Then  $Y$  is a disjoint union of open subschemes  $Y_i = \text{Spec } B_i$ , where  $B_i$  is a finite local algebra over  $A$ . Each of the morphisms  $Y_i \rightarrow X$  is a complete intersection morphism of virtual relative dimension 0 (since this is a local condition on  $Y$ ), and the additivity of the trace (valid both for  $\text{Tr}_f$ , hence for  $\tau_f$ , and for  $\text{trace}_{f_*\mathcal{O}_Y/\mathcal{O}_X}$ ) shows that it suffices to prove (3.1.3) for each morphism  $Y_i \rightarrow X$ . So we may assume that  $B$  is local. We can choose a presentation  $B \cong C/I$ , where  $C$  is a smooth  $A$ -algebra and  $I$  is an ideal in  $C$ . Let  $P = \text{Spec } C$ ,  $\mathcal{I} = I\mathcal{O}_P$ , and let  $y \in Y \subset P$  be the closed point. Then  $\mathcal{I}_y$  is generated by a regular sequence  $(t_1, \dots, t_d)$ . Shrinking  $P$  if necessary, we may assume that  $t_1, \dots, t_d$  generate  $\mathcal{I}$  globally on  $P$ , so that the hypotheses of B.8 are satisfied. Then (3.1.3) follows from (B.8.1) and from property (R6) of the residue symbol [Con00, p. 240].  $\square$

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