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Thom polynomials of Morin singularities

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In memoriam Raoul Bott

Abstract

We prove a formula for Thom polynomials of A_d singularities in any codimension. We use a combination of the test-curve model of Porteous, and the localization methods in equivariant cohomology. Our formulas are independent of the codimension, and are computationally effective up to $d = 6$.

0. Introduction

We begin with a quick summary of the notions of global singularity theory and the theory of Thom polynomials. For a more detailed review we refer the reader to [1], [14].

Consider a holomorphic map $f : N \to K$ between two complex manifolds, of dimensions $n \leq k$. We say that $p \in N$ is a *singular* point of f if the rank of the differential $df_p : \mathrm{T}_pN \to \mathrm{T}_{f(p)}K$ is less than n.

Topology often forces f to be singular at some points of N, and we will be interested in studying such situations. Before we proceed, we introduce a finer classification of singular points. Choose local coordinates near $p \in N$ and $f(p) \in K$, and consider the resulting map-germ $\check{f}_p : (\mathbb{C}^n, 0) \to (\mathbb{C}^k, 0)$, which may be thought of as a sequence of k power series in n variables without constant terms. The group of infinitesimal local coordinate changes $\mathrm{Diff}(\mathbb{C}^k) \times \mathrm{Diff}(\mathbb{C}^n)$ acts on the space $\mathcal{J}(n, k)$ of all such map-germs. We will call $\mathrm{Diff}(\mathbb{C}^k) \times \mathrm{Diff}(\mathbb{C}^n)$-orbits or, more generally, $\mathrm{Diff}(\mathbb{C}^k) \times \mathrm{Diff}(\mathbb{C}^n)$-invariant subsets $O \subset \mathcal{J}(n, k)$ *singularities*. For a singularity O and holomorphic $f : N \to K$, we can define the set

$$Z_O[f] = \{p \in N;\ \check{f}_p \in O\},$$

which is independent of any coordinate choices. Then, under some additional technical assumptions (compact N, appropriately chosen closed O, and sufficiently generic f), $Z_O[f]$ is an analytic subvariety of N. The computation of the Poincaré dual class $\alpha_O[f] \in H^*(N, \mathbb{Z})$ of this set is one of the fundamental problems of global singularity theory. This is indeed useful: for example,

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if we can prove that $\alpha_O[f]$ does not vanish, then we can guarantee that the singularity O occurs at some point of the map f.

This problem was first studied by René Thom (cf. [28], [12]) in the category of smooth varieties and smooth maps; in this case cohomology with $\mathbb{Z}/2\mathbb{Z}$-coefficients is used. Thom discovered that to every singularity O one can associate a bivariant characteristic class τ_O, which, when evaluated on the pair (TN, f^*TK) produces the Poincaré dual class $\alpha_O[f]$. One of the consequences of this result is that the class $\alpha_O[f]$ depends only on the homotopy class of f.

A similar result, which we will call *Thom's principle*, has been used in the holomorphic category (cf. [14], [9] and §2 of the present paper). To formulate it in more concrete terms, denote by $\mathbb{C}[\boldsymbol{\lambda},\boldsymbol{\theta}]^{\mathcal{S}_n\times\mathcal{S}_k}$ the space of those polynomials in the variables $(\lambda_1,\dots,\lambda_n,\theta_1,\dots,\theta_k)$ which are invariant under the permutations of the λs and the permutations of the θs. According to the structure theorem of symmetric polynomials, $\mathbb{C}[\boldsymbol{\lambda},\boldsymbol{\theta}]^{\mathcal{S}_n\times\mathcal{S}_k}$ itself is a polynomial ring in the elementary symmetric polynomials

$$\mathbb{C}[\boldsymbol{\lambda},\boldsymbol{\theta}]^{\mathcal{S}_n\times\mathcal{S}_k}=\mathbb{C}[c_1(\boldsymbol{\lambda}),\dots,c_n(\boldsymbol{\lambda}),c_1(\boldsymbol{\theta}),\dots,c_k(\boldsymbol{\theta})].$$

Using the Chern-Weil map, a polynomial $Q\in\mathbb{C}[\boldsymbol{\lambda},\boldsymbol{\theta}]^{\mathcal{S}_n\times\mathcal{S}_k}$ and a pair of bundles (E,F) over N of ranks n and k, respectively, produces a characteristic class $Q(E,F)\in H^*(N,\mathbb{C})$. Then the complex variant of Thom's principle reads:

> *For appropriate* $\mathrm{Diff}(\mathbb{C}^k)\times\mathrm{Diff}(\mathbb{C}^n)$*-invariant subset* O *of codimension* m *in* $\mathcal{J}(n,k)$*, there exists a homogeneous polynomial* $\mathrm{Tp}_O\in\mathbb{C}[\boldsymbol{\lambda},\boldsymbol{\theta}]^{\mathcal{S}_n\times\mathcal{S}_k}$ *of degree* m*, such that for a sufficiently generic map* $f:N\to K$*, the cycle* $Z_O[f]\subset N$ *is Poincaré dual to the characteristic class* $\mathrm{Tp}_O(TN,f^*TK)$.

A precise version of this statement is described in Section 2. The polynomial Tp_O is called the *Thom polynomial* of O, and the computation of these polynomials is a central problem of singularity theory.

The structure of the $\mathrm{Diff}(\mathbb{C}^k)\times\mathrm{Diff}(\mathbb{C}^n)$-action on $\mathcal{J}(n,k)$ is rather complicated; even the parametrization of the orbits is difficult. There is, however, a simple invariant on the space of orbits: to each map-germ $\check{f}:(\mathbb{C}^n,0)\to(\mathbb{C}^k,0)$, we can associate the finite-dimensional nilpotent algebra $A_{\check{f}}$ defined as the quotient of the algebra of power series with no constant term $\mathbb{C}_0[[x_1,\dots,x_n]]$ by the ideal generated by the pull-back subalgebra $\check{f}^*(\mathbb{C}_0[[y_1,\dots,y_k]])$. This algebra $A_{\check{f}}$ is trivial if the map-germ $\check{f}$ is nonsingular, and it does not change along a $\mathrm{Diff}(\mathbb{C}^k)\times\mathrm{Diff}(\mathbb{C}^n)$-orbit (cf. §2 more details).

Combining Thom's principle with this observation, to each finite-dimensional nilpotent algebra A and pair of integers (n,k), one can associate a doubly symmetric polynomial $\mathrm{Tp}_A^{n\to k}\in\mathbb{C}[\boldsymbol{\lambda},\boldsymbol{\theta}]^{\mathcal{S}_n\times\mathcal{S}_k}$, which, in the sense described above, will serve as a universal Poincaré dual of those points in the source spaces of holomorphic maps whose local nilpotent algebra is A.

The computation of Thom polynomials associated to nilpotent algebras is difficult, but a few structural statements are known (cf. §2.6 for more details).

First, as discovered by Damon and Ronga ([8], [24]) in the 70's, the polynomial $\mathrm{Tp}_A^{n\to k}$ lies in the subring of $\mathbb{C}[\boldsymbol{\lambda},\boldsymbol{\theta}]^{\mathcal{S}_n\times\mathcal{S}_k}$ generated by the relative Chern classes defined by the generating series

$$1+c_1q+c_2q^2+\cdots=\frac{\prod_{j=1}^k(1+\theta_jq)}{\prod_{i=1}^n(1+\lambda_iq)}.$$

Next, the Thom polynomial, expressed in terms of these relative Chern classes, only depends on the codimension $j=k-n$. More precisely, there is a unique polynomial $\mathrm{TD}_A^j(c_1,c_2,\dots)$ such that

$$\mathrm{Tp}_A^{n\to k}(\boldsymbol{\lambda},\boldsymbol{\theta})=\mathrm{TD}_A^{k-n}(c_1(\boldsymbol{\lambda},\boldsymbol{\theta}),c_2(\boldsymbol{\lambda},\boldsymbol{\theta}),\dots).$$

Finally, in a recent paper, Fehér and Rimányi observed [9] that performing the substitution $c_i\mapsto c_{i-1}$ in TD_A^j produces TD_A^{j-1}. This implies that to each nilpotent algebra A one can associate a power series in infinitely many variables, which encodes all of the Thom polynomials associated to A. This observation served as the starting point for the present work.

In this paper, we will concentrate on the so-called Morin singularities [19], which correspond to the situation when the algebra A is generated by a single element. The list of these algebras is simple: $A_d=t\mathbb{C}[t]/t^{d+1}$, $d=1,2,\dots$.

The goal of our paper is to compute the Thom polynomial $\mathrm{Tp}_{A_d}^{n\to k}$ for arbitrary d,n and k. For simplicity of notation, we will denote this polynomial by $\mathrm{Tp}_d^{n\to k}$, or sometimes simply by Tp_d, omitting the dependence on the parameters n and k.

The problem of calculating $\mathrm{Tp}_d^{n\to k}$ goes back to Thom [28]. The case $d=1$ is the classical formula of Porteous: $\mathrm{Tp}_1=c_{k-n+1}$. The Thom polynomial in the $d=2$ case was computed by Ronga in [24]. More recently, an explicit formula for Tp_3 was proposed in [2], and P. Pragacz has given a sketch of a proof of this conjecture [22]. Finally, using his method of restriction equations, Rimányi [23] was able to treat the $n=k$ case, and he computed $\mathrm{Tp}_d^{n\to n}$ for $d\le 8$ (cf. [10] for the case $d=4$).

Our approach combines the test-curve model of Porteous [21] with localization techniques in equivariant cohomology [4], [26], [29]. We obtain a formula reducing the computation of $\mathrm{Tp}_d^{n\to k}$ to a certain problem of commutative algebra, which depends on d only. This problem is trivial for $d=1,2,3$ (cf. (0.1) below); hence we instantly recover all results known for arbitrary $n\le k$. An important feature of our formula is that it manifestly satisfies all three properties listed above. In particular, we obtain a tentative geometric interpretation for the Thom series introduced by Fehér and Rimányi.

The paper is structured as follows. We describe the basic setup and notions of singularity theory in Section 1, essentially repeating the above constructions using more formal notation. Next, in Section 2, using Vergne's integral formula, we introduce the notion of equivariant Poincaré dual, which provides us with a convenient language for describing Thom polynomials. In Section 3 we present the localization formulas of Berline-Vergne [4] and Rossmann [26] and develop a calculus localizing equivariant Poincaré duals. With these preparations, we proceed to describe the test curve model for Morin singularities in Section 4. This is the key part of our work, where we reinterpret and modify this model using a double fibration in a way which allows us to compactify our model space and apply the localization formulas. At the end of Section 5, we summarize our constructions and results in a diagram to orient the reader. Section 5 is a rather straightforward application of the localization techniques of Section 3 to the double fibration constructed in Section 4. The resulting formula (5.24), in principle, reduces the computation of our Thom polynomials to a finite problem, but this formula is difficult to use for concrete calculations. Remarkably, however, the formula undergoes several simplifications, which we explain in Section 6.

The simplifications bring us to our main result: Theorem 6.16 and formula (6.26). While this formula is rather simple, it still contains an unknown quantity: a certain homogeneous polynomial $\widehat{Q}_d$ in d variables, which does not depend on n and k. The list of these polynomials begins as follows:

$$\widehat{Q}_1 = \widehat{Q}_2 = \widehat{Q}_3 = 1,\ \widehat{Q}_4(z_1, z_2, z_3, z_4) = 2z_1 + z_2 - z_4, \dots . \tag{0.1}$$

In principle, $\widehat{Q}_d$ may be calculated for each concrete d using a computer algebra program, but, at the moment, we do not have an efficient algorithm for performing such calculations for large d. We discuss certain partial results in the final section of our paper. These, in particular, allow us to compute $\widehat{Q}_5$ by hand and $\widehat{Q}_6$ using the computer algebra program Macaulay.

We end the paper with an application of our theorem to positivity of Thom series. Rimányi conjectured in [23] that the Thom polynomials Tp_d expressed in terms of relative Chern classes have positive coefficients. Our formalism suggests a stronger positivity conjecture, which we formulate in Section 7.5 and check for the first few values of d.

In closing, we note that Morin singularities are special cases of the so-called Thom-Boardman singularities [28], [5], [17]. These are parametrized by finite nonincreasing sequences of integers, and Morin singularities correspond to sequences starting with 1. Our method extends to a wider class of Thom-Boardman singularities; we hope to report on new results in this direction in a later publication.

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1. Basic notions of singularity theory

1.1. *The setup.* We start with a brief introduction to singularity theory. We suggest [16], [1], [28] as references for the subject.

Let $(e_1, \dots, e_n)$ be the basis of $\mathbb{C}^n$, and denote the corresponding coordinates by $(x_1, \dots, x_n)$. Introduce the notation $\mathcal{J}(n) = \{h \in \mathbb{C}[[x_1, \dots, x_n]]; h(0)] = 0\}$ for the algebra of power series without a constant term, and let $\mathcal{J}_d(n)$ be the space of d-jets of holomorphic functions on $\mathbb{C}^n$ near the origin, i.e., the quotient of $\mathcal{J}(n)$ by the ideal of those power series whose lowest order term is of degree at least $d+1$. As a linear space, $\mathcal{J}_d(n)$ may be identified with polynomials on $\mathbb{C}^n$ of degree at most d without a constant term.

In this paper, we will call an algebra *nilpotent* if it is finite-dimensional and there exists a positive integer N such that the product of any N elements of the algebra vanishes. The algebra $\mathcal{J}_d(n)$, in particular, is nilpotent, since $\mathcal{J}_d(n)^{d+1} = 0$.

Our basic object is $\mathcal{J}_d(n,k)$, the space of d-jets of holomorphic maps $(\mathbb{C}^n, 0) \to (\mathbb{C}^k, 0)$. This is a finite-dimensional complex vector space, which one can identify $\mathcal{J}_d(n) \otimes \mathbb{C}^k$; hence $\dim \mathcal{J}_d(n,k) = k\binom{n+d}{d} - k$. We will call the elements of $\mathcal{J}_d(k,n)$ *map-jets of order* d, or simply map-jets. In this paper we will always assume $n \leq k$.

Eliminating the terms of degree d results in an algebra homomorphism $\mathcal{J}_d(n) \twoheadrightarrow \mathcal{J}_{d-1}(n)$, and the chain $\mathcal{J}_d(n) \twoheadrightarrow \mathcal{J}_{d-1}(n) \twoheadrightarrow \dots \twoheadrightarrow \mathcal{J}_1(n)$ induces an increasing filtration on $\mathcal{J}_d(n)^*$:

$$\mathcal{J}_1(n)^* \subset \mathcal{J}_2(n)^* \subset \dots \subset \mathcal{J}_d(n)^*. \tag{1.1}$$

The space $\mathcal{J}_i(n)^*$ may be interpreted as a set of differential operators with constant coefficients of degree at most i, and, in particular, by taking symbols, we have

$$\mathcal{J}_d(n)^* \cong \mathrm{Sym}_d^\bullet \mathbb{C}^n \stackrel{\text{def}}{=} \oplus_{l=1}^d \mathrm{Sym}^l \mathbb{C}^n, \tag{1.2}$$

where Sym^l stands for the symmetric tensor product and the isomorphism is that of filtered GL_n-modules.

One can compose map-jets via substitution and elimination of terms of degree greater than d. This leads to the composition maps

$$\mathcal{J}_d(n,k) \times \mathcal{J}_d(m,n) \to \mathcal{J}_d(m,k), \quad (\Psi_2, \Psi_1) \mapsto \Psi_2 \circ \Psi_1. \tag{1.3}$$

When $d = 1$, $\mathcal{J}_1(m,n)$ may be identified with n-by-m matrices, and (1.3) reduces to multiplication of matrices. By taking the linear parts of map-jets, we obtain a map

$$\mathrm{Lin} : \mathcal{J}_d(n,k) \to \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^k),$$

which is compatible with the compositions (1.3) and matrix multiplication. We define the set of jets with regular linear parts (for $n \le k$) as

$$\Sigma_0(n,k) = \{\Psi \in \mathcal{J}_d(n,k) : \mathrm{rank}(\mathrm{Lin}(\Psi)) = n\}. \tag{1.4}$$

Consider now

$$\mathrm{Diff}_d(n) = \{\Delta \in \mathcal{J}_d(n,n);\ \mathrm{Lin}(\Delta) \text{ invertible}\}.$$

The composition map (1.3) endows this set with the structure of an algebraic group, which has a faithful representation on $\mathcal{J}_d(n)$. Using the compositions (1.3) again, we obtain the so-called *left-right* action of the group $\mathrm{Diff}_d(k) \times \mathrm{Diff}_d(n)$ on $\mathcal{J}_d(k,n)$:

$$[(\Delta_L, \Delta_R), \Psi] \mapsto \Delta_L \circ \Psi \circ \Delta_R^{-1}.$$

Note that the action of $\mathrm{Diff}_d(n)$ is linear, while the action of $\mathrm{Diff}_d(k)$ is not. *Singularity theory*, in the sense that we are considering here, studies the left-right-invariant algebraic subsets of $\mathcal{J}_d(n,k)$.

A natural way to form such subsets is as follows. Observe that to each element $\Psi = (P_1, \dots, P_k) \in \mathcal{J}_d(n,k)$, where $P_i \in \mathcal{J}_d(n)$ for $i = 1, \dots, k$, we can associate the quotient algebra $A_\Psi = \mathcal{J}_d(n)/I\langle P_1, \dots, P_k\rangle$: the algebra $\mathcal{J}_d(n)$ modulo the ideal generated by the elements of the sequence. Since $\mathcal{J}_d(n)^{d+1} = 0$, we also have A_Ψ^{d+1}=0. We will call A_Ψ the *nilpotent algebra*[1] of the map-jet Ψ. For $\Psi = 0$, this nilpotent algebra is $\mathcal{J}_d(n)$, while for a generic Ψ (in fact, as soon as $\mathrm{rank}[\mathrm{Lin}(\Psi)] = n$), we have $A_\Psi = 0$.

Now let A be a nilpotent algebra, as defined above. Consider the subset

$$\Theta_A^{n\to k} = \{(P_1, \dots, P_k) \in \mathcal{J}_d(n,k);\ \mathcal{J}_d(n)/I\langle P_1, \dots, P_n\rangle \cong A\} \tag{1.5}$$

of the map-jets of order d.

It is easy to show that Θ_A^{k-n} is $\mathrm{Diff}_d(k) \times \mathrm{Diff}_d(n)$-invariant. A key observation is that although two map-jets with the same nilpotent algebra may be in different $\mathrm{Diff}_d(k) \times \mathrm{Diff}_d(n)$-orbits, there is a group acting on $\mathcal{J}_d(n,k)$ whose orbits are exactly the sets $\Theta_A^{n\to k}$ for various nilpotent algebras A. This group is defined as the semidirect product

$$\mathcal{K}_d(n,k) = \mathrm{GL}_k(\mathbb{C} \oplus \mathcal{J}_d(n)) \rtimes \mathrm{Diff}_d(n), \tag{1.6}$$

[1] Instead of this algebra, it is customary to use the so-called *local algebra* of Ψ, which is simply the augmentation of A_Ψ by the constants.

using the natural action of $\mathrm{Diff}_d(n)$ on $\mathcal{J}_d(n)$; the algebra $\mathbb{C} \oplus \mathcal{J}_d(n)$ is the augmentation of $\mathcal{J}_d(n)$ by constants. The vector space $\mathcal{J}_d(n)$ is naturally a module over $\mathbb{C} \oplus \mathcal{J}_d(n)$, and hence $\mathcal{K}_d(n,k)$ acts on $\mathcal{J}_d(n,k)$ linearly via

$$[(M,\Delta),\Psi] \mapsto (M \cdot \Psi) \circ \Delta^{-1}, \tag{1.7}$$

where "$\cdot$" stands for matrix multiplication.

PROPOSITION 1.1 ([17], [16], [1]). *Two map-jets in $\mathcal{J}_d(n,k)$ have the same nilpotent algebra if and only if they are in the same $\mathcal{K}_d$-orbit.*

Remark 1.2. Two jets in the same $\mathcal{K}_d$-orbit are called *contact equivalent*, or $\mathcal{K}$-equivalent (cf. [1]). The term V-equivalence is also used (e.g., [15]). The varieties Θ_A are called *contact singularity classes* or simply *contact singularities.*

Since $\mathcal{K}_d$ is connected, $\overline{\Theta}_A$ is an irreducible subvariety of $\mathcal{J}_d(n,k)$, and it is well known that its codimension depends only on $k-n$ and A; see [1]. In the present paper, we will study certain rough topological invariants of contact singularities; these invariants depend only on the closure of the singularity locus in $\mathcal{J}_d(n,k)$.

1.2. *Morin singularities.* In this paper, we will focus on nilpotent algebras A generated by a single element. Such algebras form a one-parameter family

$$A_d = t\mathbb{C}[t]/t^{d+1}, \quad d = 1, 2, \dots .$$

The corresponding singularity classes are called the *A_d-singularities* or *Morin singularities* [1], [19]. We introduce the simplified notation

$$\Theta_d^{n\to k} \text{ instead of } \Theta_{A_d}^{n\to k} \tag{1.8}$$

for these varieties, and we will omit the parameters n and k when this causes no confusion. The following statements about Morin singularities can be found in [1].

PROPOSITION 1.3.

- *The variety $\Theta_d^{n\to k}$ is nonempty for any $n \le k$, and its closure is an irreducible variety of codimension $d(k-n+1)$.*
- *For $n \ge d$, there exist map-jets (the so-called stable jets) in $\mathcal{J}_d(n,k)$ with nilpotent algebra A_d, whose left-right orbit is dense in $\Theta_d^{n\to k}$.*

The second statement will allow us to use left-right orbits and contact orbits interchangeably.

Finally, we recall that the A_d-singularities fit into the wider family of so-called *Thom-Boardman* singularity classes $\Theta_d = \Sigma^{1\dots,1,0}$ with d 1's (cf. [5], [1]).

2. Equivariant Poincaré duals and Thom polynomials

The goal of this paper is to compute certain topological invariants of the subvarieties $\Theta_d^{n\to k}$ introduced in the previous section. In this section, we define and describe these invariants in detail.

Let T be a complexified torus $T \cong (\mathbb{C}^*)^r$, and let W be a T-module. The *equivariant Poincaré dual* is an invariant $\Sigma \mapsto \mathrm{eP}[\Sigma, W]_T$ associated to algebraic or analytic T-invariant subvarieties of W. This invariant takes values in homogeneous polynomials on the Lie algebra $\mathrm{Lie}(T)$ of T. The central objects of the present work, Thom polynomials, are special cases of equivariant Poincaré duals (cf. [23], [14]).

The equivariant Poincaré dual has appeared in the literature in several guises: as Joseph polynomial, equivariant multiplicity, multidegree, etc. One of the first definitions was given by Joseph [13], who introduced it as the polynomial governing the asymptotic behavior of the character of the algebra of functions on the subvariety. Rossmann in [26] defined this invariant for analytic subvarieties via an integral-limit representation and then used it to write down a very general localization formula for equivariant integrals. This formula will play an important role in our computations.

In this paper, we follow the approach of Vergne [29], who, motivated by the work of Rossmann, defined the equivariant Poincaré dual as the integral of the Thom form in equivariant cohomology.

We begin thus with this definition in Section 2.1, where we also list the basic properties of this invariant. Then we describe an alternative, algebraic definition, due to Joseph, which is useful for computations. We continue by giving a simple example and then describing a universal property of the equivariant Poincaré dual. We then give the definition of the Thom polynomials as equivariant Poincaré duals in Section 2.5 and argue that this definition is consistent with Thom's principle from the introduction. We end the section with a list of properties of Thom polynomials of contact singularities.

2.1. *Equivariant cohomology and the equivariant Poincaré dual.* We begin with a brief introduction to equivariant cohomology. For more details, we refer the reader to [3].

Let $\breve{T} \cong U(1)^n$ be the maximal compact subgroup of the complex torus group $T \cong (\mathbb{C}^*)^n$, and denote by $\breve{\mathfrak{t}}$ the Lie algebra of $\breve{T}$. The weight lattice of T has a canonical basis: $\lambda_1, \dots, \lambda_n \in \breve{\mathfrak{t}}^*$.

For a manifold M endowed with the action of $\breve{T}$, one can define a differential $d_{\breve{T}}$ on the space $S^\bullet\breve{\mathfrak{t}}^* \otimes \Omega^\bullet(M)^{\breve{T}}$ of polynomial functions on $\breve{\mathfrak{t}}$ with values in $\breve{T}$-invariant differential forms by the formula

$$[d_{\breve{T}}\alpha](X) = d(\alpha(X)) - \iota_X[\alpha(X)],$$

where $X \in \breve{\mathfrak{t}}$, and ι_X is contraction by the corresponding vector field on M. A homogeneous polynomial of degree d with values in r-forms is placed in degree $2d+r$, and then $d_{\breve{T}}$ is an operator of degree 1. The cohomology of this complex, denoted by $H^\bullet_{\breve{T}}(M)$, is called the $\breve{T}$-equivariant cohomology of M. Restricting the complex to compactly supported (or quickly decreasing at infinity) differential forms, one obtains the compactly supported equivariant cohomology groups $H^\bullet_{\breve{T},\mathrm{cpt}}(M)$. Clearly $H^\bullet_{\breve{T},\mathrm{cpt}}(M)$ is a module over $H^\bullet_{\breve{T}}(M)$. For the case when $M = W$ is an N-dimensional complex vector space and the action is linear, one has $H^\bullet_{\breve{T}}(W) = S^\bullet\breve{\mathfrak{t}}^*$, and $H^\bullet_{\breve{T},\mathrm{cpt}}(W)$ is a free module over $H^\bullet_{\breve{T}}(W)$ generated by a single element of degree $2N$:

$$H^\bullet_{\breve{T},\mathrm{cpt}}(W) = H^\bullet_{\breve{T}}(W) \cdot \mathrm{Thom}_{\breve{T}}(W). \tag{2.1}$$

The equivariant class $\mathrm{Thom}_{\breve{T}}(W)$ may be normalized by the condition

$$\frac{1}{(2\pi)^N}\int_W \mathrm{Thom}_{\breve{T}}(W) = 1.$$

Fixing coordinates $y_1,\dots,y_N$ on W, in which the T-action is diagonal with weights $\eta_1,\dots,\eta_N$, one can write an explicit representative of $\mathrm{Thom}_{\breve{T}}(W)$ as follows:

$$\mathrm{Thom}_{\breve{T}}(W) = \exp\left(-\frac{1}{2}\sum_{i=1}^N |y_i|^2\right)\cdot \sum_{\sigma\subset\{1,\dots,N\}}\prod_{i\in\sigma}\eta_i \cdot \prod_{i\notin\sigma} dy_i\, d\bar{y}_i.$$

We will say that an algebraic variety has dimension d if its maximal-dimensional irreducible components are of dimension d. A T-invariant algebraic subvariety Σ of dimension d in W represents $\breve{T}$-equivariant $2d$-cycle in the sense that

- a compactly-supported equivariant form μ of degree $2d$ is absolutely integrable over the components of maximal dimension of Σ, and $\int_\Sigma \mu \in S^\bullet\breve{\mathfrak{t}}$;
- if $d_{\breve{T}}\mu = 0$, then $\int_\Sigma \mu$ depends only on the class of μ in $H^\bullet_{\breve{T},\mathrm{cpt}}(W)$;
- $\int_\Sigma \mu = 0$ if $\mu = d_{\breve{T}}\nu$ for a compactly-supported equivariant form ν.

Definition 2.1. Let Σ be an T-invariant algebraic subvariety of dimension d in the vector space W. Then the equivariant Poincaré dual of Σ is the polynomial on $\breve{\mathfrak{t}}$ defined by the integral

$$\mathrm{eP}[\Sigma, W]_T = \frac{1}{(2\pi)^d}\int_\Sigma \mathrm{Thom}_{\breve{T}}(W). \tag{2.2}$$

Remark 2.2. (1) The relation of this class to Poincaré duality will be explained in Section 2.4.

(2) This definition naturally extends to the case of an analytic subvariety of $\mathbb{C}^n$ defined in the neighborhood of the origin or, more generally, to any T-invariant cycle in $\mathbb{C}^n$.

We list some basic properties of the equivariant Poincaré dual (cf. Proposition 2.6). The proofs can be found in [26], [29], [18].

Proposition 2.3.

Positivity: *The equivariant Poincaré dual* $\mathrm{eP}[\Sigma, W]_T$ *of a* d*-dimensional subvariety* Σ *of* W *is a degree–*$(N-d)$ *homogeneous polynomial of* $\lambda_1, \dots, \lambda_n$*, which may be expressed as a polynomial of the weights* η_i*,* $i = 1, \dots, N$*, with nonnegative integer coefficients.*

Additivity: *If* $\Sigma_1, \Sigma_2 \subset W$ *are two* T*-invariant subvarieties of dimension* d *having no common components of top dimension, then* $\mathrm{eP}[\Sigma_1 \cup \Sigma_2, W]_T = \mathrm{eP}[\Sigma_1, W]_T + \mathrm{eP}[\Sigma_2, W]_T$.

Deformation invariance: *If* Σ_t *is a flat algebraic family of varieties with a* T *action, then, then* $\mathrm{eP}[\Sigma_t, W]_T$ *is independent of* t.

Symmetry: *Let* $T = (\mathbb{C}^*)^n$ *be the subgroup of diagonal matrices of the complex group* GL_n*, and denote by* $\lambda_1, \dots, \lambda_n$ *its basic weights. If* Σ *is a* GL_n*-invariant subvariety of the* GL_n*-module* W*, then the equivariant Poincaré dual* $\mathrm{eP}[\Sigma, W]_T$ *is a* symmetric *polynomial in* $\lambda_1, \dots, \lambda_n$.

Complete intersections: *Let the variety* $\Sigma \subset W$ *be a complete intersection defined by* r *relations* $f_1, \dots, f_r \in \mathbb{C}[y_1, \dots, y_N]$ *of degrees (weights)* $\alpha_1, \dots, \alpha_r \in \breve{\mathfrak{t}}^*$ *correspondingly. Then*

$$\mathrm{eP}[\Sigma, W]_T = \prod_{i=1}^{r} \alpha_i. \tag{2.3}$$

Elimination: *Let* $\Sigma \subset W$ *be a closed* T*-invariant subvariety, and denote by* $I(\Sigma)$ *the ideal of functions vanishing on* Σ*. Fix a polynomial* $f \in \mathbb{C}[y_1, \dots, y_N]$ *of weight* η_0*, and let* Σ_f *be the variety in* $W \oplus \mathbb{C}y_0$ *with ideal generated by* $I(\Sigma)$ *and* $y_0 - f$*. Then*

$$\mathrm{eP}[\Sigma_f, W \oplus \mathbb{C}y_0]_T = \eta_0 \cdot \mathrm{eP}[\Sigma, W]_T.$$

Remark 2.4. We can write down the formula for complete intersections in a different way as follows. Let E be a T-vector space with a list of weights $\alpha_1, \dots, \alpha_r$, and denote by $\mathrm{Euler}^T(E)$ the equivariant Euler class of E, i.e.,

$$\mathrm{Euler}^T(E) = \prod_{i=1}^{r} \alpha_i. \tag{2.4}$$

Suppose that $\gamma : W \to E$ is an equivariant polynomial map with the property that the differential $d\gamma : W \to E$ is surjective on a Zariski open part of $\gamma^{-1}(0)$.

Then

$$\mathrm{eP}[\gamma^{-1}(0), W]_T = \mathrm{Euler}^T(E). \tag{2.5}$$

Remark 2.5. An important special case of complete intersections are the linear subspaces. For these, the formula (2.3) takes the following form: For every subset $\mathbf{i} \subset \{1, \ldots, N\}$, we have

$$\mathrm{eP}[\{y_i = 0,\ i \in \mathbf{i}\}, W]_T = \prod_{i\in\mathbf{i}} \eta_i. \tag{2.6}$$

2.2. *Multidegrees and equivariant Poincaré duals.* Another incarnation of the equivariant Poincaré dual is the notion of *multidegree*, which is close in spirit to the original construction of Joseph [13].

Let Σ be a codimension-D, T-invariant subvariety of W. Introduce the notation $S = \mathbb{C}[y_1, \ldots, y_N]$ for the polynomial functions on W, and denote the ideal of the functions vanishing on Σ by $I(\Sigma)$; thus $I(\Sigma) = \{f \in \mathbb{C}[y_1, \ldots, y_N];\ f(p) = 0 \text{ if } p \in \Sigma\}$.

Consider a finite (length-M), T-graded resolution of $S/I(\Sigma)$ by free S-modules:

$$\oplus_{i=1}^{j[M]} Sw_i[M] \to \cdots \to \oplus_{i=1}^{j[m]} Sw_i[m] \to \cdots \to \oplus_{i=1}^{j[1]} Sw_i[1] \to S \to S/I(\Sigma) \to 0,$$

where $w_i[m]$ is a free generator of weight $\eta_i[m]$, $i = 1, \ldots j[m]$, $m = 1, \ldots, M$. Then the multidegree of the ideal $I(\Sigma)$ is defined by the formula

$$\mathrm{mdeg}[I, S]_T = \frac{1}{D!} \sum_{m=1}^{M} \sum_{i=1}^{j[m]} (-1)^{D-m} \eta_i[m]^D, \tag{2.7}$$

where D is the codimension of Σ.

Proposition 2.6 ([26]). *Let $\Sigma \subset W$ be a T-invariant subvariety. Then we have*

$$\mathrm{eP}[\Sigma, W]_T = \mathrm{mdeg}[I(\Sigma), \mathbb{C}[y_1, \ldots, y_N]].$$

2.3. *An example.* A simple way to construct T-invariant subvarieties of W is to take the orbit closures of points in W.

Consider the following example. Let $W = \mathbb{C}^4$ endowed with a $T = (\mathbb{C}^*)^3$-action, whose weights η_1, η_2, η_3, and η_4 span $\breve{\mathfrak{t}}^*$ and satisfy $\eta_1 + \eta_3 = \eta_2 + \eta_4$. The four weights, η_i, $i = 1, \ldots, 4$, for example, may form the vertices of a parallelogram in $\breve{\mathfrak{t}}^*$ lying in a hyperplane which does not pass through the origin. Choose $p = (1, 1, 1, 1) \in W$; then the closure of the T-orbit of p is given by a single equation:

$$\overline{T\cdot p} = \{(y_1, y_2, y_3, y_4) \in \mathbb{C}^4;\ y_1y_3 = y_2y_4\}. \tag{2.8}$$

This variety is defined by a single equation, hence it is a complete intersection. According to (2.3), $\mathrm{eP}[\Sigma, W]_T$ is the degree of this equation; thus

$$\mathrm{eP}[\Sigma, W]_T = \eta_1 + \eta_3 = \eta_2 + \eta_4. \tag{2.9}$$

This result may be obtained similarly from the 1-step resolution of the vanishing ideal of this variety.

We will calculate this equivariant Poincaré dual at the end of Section 3 in a completely different way.

2.4. *Universal Poincaré dual.* In this paragraph, we present the equivariant Poincaré dual as a universal obstruction class, which will explain its link with Thom's principle.

It will be more natural to consider the complex group GL_n and its maximal compact subgroup U_n instead of the torus groups we have used so far. We will denote the subgroup of diagonal elements of GL_n by T_n.

Let F be a principal U_n-bundle over a compact oriented manifold M. Then, using the Chern-Weil map, any symmetric polynomial $P \in \mathbb{C}[\lambda_1, \dots, \lambda_n]^{\mathcal{S}_n}$ defines a characteristic class $P(F) \in H^*(M, \mathbb{C})$. Now let Σ be GL_n-invariant subvariety of the GL_n-module W. Recall from Proposition 2.3 that in this case the polynomial $\mathrm{eP}[\Sigma, W]_{T_n}(F)$ is symmetric in the λs. Denote by W_F the associated vector bundle $F \times_{U_n} W$ over M and by Σ_F the subset of W_F corresponding to Σ:

$$\begin{array}{ccc} F \times_{U_n} W = W_F & \hookleftarrow & \Sigma_F = F \times_{U_n} \Sigma \\ s \uparrow\downarrow & \swarrow & \\ M. & & \end{array} \tag{2.10}$$

A key technical point is that the variety Σ_F defines a cycle in the manifold W_F, and as such it has a Poincaré dual class $\alpha_\Sigma \in H^{2\mathrm{codim}(\Sigma)}(W_F)$ satisfying

$$\int_{W_F} \alpha_\Sigma \cdot \beta = \int_{\Sigma_F} \beta$$

for any compactly supported cohomology class β on W_F. This class is linked to the equivariant Poincaré dual in the following remarkable way:

$$\iota_F^* \alpha_\Sigma = \mathrm{eP}[\Sigma, W]_{T_n}(F) \text{ in } H^*(M), \tag{2.11}$$

where ι_F is the embedding of M into W_F as the zero-section. In words, *the Chern-Weil image of the equivariant Poincaré dual is the restriction of the ordinary Poincaré dual of the induced variety.*

Now we reformulate this statement in a geometric language. In this setup, $\mathrm{eP}[\Sigma, W]_{T_n}(F)$ will appear as the Poincaré dual of $s^{-1}(\Sigma_F)$ in M for an appropriate section $s : M \to W_F$. To formulate this more precisely, we make the following

Definition 2.7. Consider diagram (2.10), and assume for simplicity that Σ is equidimensional. We say that a smooth section $s : M \to W_F$ is *transversal* to Σ_F at some point $p \in M$ if $s(p)$ is a smooth point of Σ_F and the subspaces $ds(\mathrm{T}_p M)$ and $\mathrm{T}_{s(p)}\Sigma_F$ span the vector space $\mathrm{T}_{s(p)}W_F$. We say that $s : M \to W_F$ is *generically transversal* to Σ_F if we have

$$\overline{\{p \in M; s \text{ is transversal to } \Sigma_F \text{ at } p\}} = s^{-1}(\Sigma_F).$$

Armed with this technical notion, we reformulate (2.11) as follows.

PROPOSITION 2.8. *For a smooth section $s : M \to W_F$, which is generically transversal to Σ_F, the cycle $s^{-1}(\Sigma_F) \subset M$ is Poincaré dual to the characteristic class* $\mathrm{eP}[\Sigma, W]_{T_n}(F)$ *of the bundle F corresponding to the symmetric polynomial* $\mathrm{eP}[\Sigma, W]_{T_n}$.

2.5. *Thom polynomials and equivariant Poincaré duals.* Let us apply our new-found invariant to the setup of global singularity theory described in Section 1. Recall that for integers d and $n \leq k$, we defined an algebraic subset $\Theta_d \subset \mathcal{J}_d(n, k)$, which is invariant under the natural action of the group $\mathrm{Diff}_d(k) \times \mathrm{Diff}_d(n)$.

Now observe that the quotient map $\mathrm{Lin} : \mathrm{Diff}_d(n) \to \mathrm{Diff}_1(n) = \mathrm{GL}_n$ has a canonical section, consisting of linear substitutions. In other words, we have a canonical group embedding

$$\mathrm{GL}_n \hookrightarrow \mathrm{Diff}_d(n),$$

and we can restrict the action of the diffeomorphism groups $\mathrm{Diff}_d(k) \times \mathrm{Diff}_d(n)$ on $\mathcal{J}_d(n, k)$ to the canonical subgroup $\mathrm{GL}_k \times \mathrm{GL}_n$. Denoting the subgroups of diagonal matrices of GL_k and GL_n by T_k and T_n, their basic weights by $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_k)$ and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$, respectively, we can introduce the central object of our paper.

Definition 2.9. Let A be a nilpotent algebra. The *Thom polynomial* of the A-singularity from n-to-k dimensions is defined to be the following equivariant Poincaré dual:

$$\mathrm{Tp}_A^{n \to k}(\boldsymbol{\lambda}, \boldsymbol{\theta}) \stackrel{\mathrm{def}}{=} \mathrm{eP}[\overline{\Theta}_A, \mathcal{J}_d(n, k)]_{T_k \times T_n}. \tag{2.12}$$

According to Proposition 2.3, the Thom polynomial is a homogeneous polynomial which is symmetric in the variables $\theta_1, \ldots, \theta_k$ and $\lambda_1, \ldots, \lambda_n$ separately.

Starting with the next section we will focus on the computation of the polynomial $\mathrm{Tp}_A^{n\to k}(\boldsymbol{\lambda}, \boldsymbol{\theta})$ for the case $A = t\mathbb{C}[t]/t^{d+1}$. In the remainder of this paragraph, however, we would like to argue that this polynomial is a reasonable candidate for the universal class satisfying Thom's principle quoted in Section 0. This is standard for the experts (cf. [23], [14], [9], [22]), but good references are hard to come by. In any case, we would like to stress that this material is not necessary for understanding the rest of the paper. The reader comfortable with Definition 2.9 may safely skip to Section 2.6.

Now let us consider the situation of singularity loci of holomorphic maps described in the introduction. For complex manifolds N and K of dimensions n and k, respectively, and a positive integer d, consider the principal $\mathrm{Diff}_d(k) \times \mathrm{Diff}_d(n)$-bundle $\mathrm{Diff}_d(K) \times \mathrm{Diff}_d(N)$ over the product space $N \times K$ consisting of local coordinate changes up to order d. Denote by $\mathcal{J}_d(N, K)$ the bundle over $N \times K$ associated to the representation $\mathcal{J}_d(n, k)$ of the group $\mathrm{Diff}_d(k) \times \mathrm{Diff}_d(n)$. Note that even though the space $\mathcal{J}_d(n, k)$ has a linear structure, the action of the group $\mathrm{Diff}_d(k) \times \mathrm{Diff}_d(n)$ on it is not linear, and hence this bundle is not a vector bundle. Then any holomorphic map $f : N \to K$ induces a section $s_f : N \to (1 \times f)^*\mathcal{J}_d(N, K)$ of the bundle pulled back from the graph. We need the following key fact.

LEMMA 2.10. *The structure group* $\mathrm{Diff}_d(k) \times \mathrm{Diff}_d(n)$ *of the bundle* $\mathcal{J}_d(n, k)$ *reduces to the subgroup* $\mathrm{GL}_k \times \mathrm{GL}_n$

This can be seen using that $\mathrm{Diff}_d(k) \times \mathrm{Diff}_d(n)$ is homotopy equivalent to $\mathrm{GL}_k \times \mathrm{GL}_n$ or, alternatively, by directly presenting the reduction, for example, by introducing Hermitian metrics on TN and TK (cf. [14, §2.2]).

Now, for a nilpotent algebra A satisfying $A^{d+1} = 0$, consider the subvariety

$$\mathcal{J}_d(\Theta_A^{N\to K}) \subset \mathcal{J}_d(N, K) \tag{2.13}$$

associated to the subvariety $\Theta_A^{n\to k} \subset \mathcal{J}_d(n, k)$.

Taking advantage of Lemma 2.10, we can present Thom's principle in the following formal manner.

PROPOSITION 2.11. *Let* N, K, A, *and* d *be as above. Let* $f : N \to K$ *be a* smooth *map and* $s : N \to (1 \times f)^*\mathcal{J}_d(N, K)$ *be an arbitrary smooth section, generically transversal to* $(1\times f)^*\mathcal{J}_d(\Theta_A^{N\to K})$. *Then the bivariant characteristic class* $\mathrm{Tp}_A^{n\to k}(TN, f^*TK) \in H^*(N)$, *where* $\mathrm{Tp}_A^{n\to k}$ *is the polynomial defined in* (2.12), *is Poincaré dual to the subvariety* $s_f^{-1}((1 \times f)^*\mathcal{J}_d(\Theta_A^{N\to K})) \subset N$.

2.6. *Thom polynomials of contact singularities.* One of the natural questions to ask is how the Thom polynomials for fixed A and different pairs (n, k) are related. We collect the known facts from [1], [8], [9] in Proposition 2.12 below. For simplicity, we will formulate the statements for the algebra

$A_d = t\mathbb{C}[t]/t^{d+1}$ we study, although essentially the same properties are satisfied by the Thom polynomials of any other contact singularity (see [9] for details).

Denote by $\mathbb{C}[\boldsymbol{\lambda}, \boldsymbol{\theta}]^{\mathcal{S}_n \times \mathcal{S}_k}$ the ring of bisymmetric polynomials in the λs and θs, and recall from §2.1 that for $1 \leq d$ and $1 \leq n \leq k$, $\Theta_d = \Theta_d^{n \to k}$ is a nonempty subvariety of $\mathcal{J}_d(n,k)$ of codimension $d(k-n+1)$. Consider the infinite sequence of homogeneous polynomials $c_i \in \mathbb{C}[\boldsymbol{\lambda}, \boldsymbol{\theta}]^{\mathcal{S}_n \times \mathcal{S}_k}$, $\deg c_i = i$, defined by the generating series

$$\text{RC}(q) = 1 + c_1 q + c_2 q^2 + \cdots = \frac{\prod_{m=1}^{k}(1+\theta_m q)}{\prod_{l=1}^{n}(1+\lambda_l q)}; \tag{2.14}$$

we will call c_i the ith *relative Chern class.*

Proposition 2.12 ([9]). *Let $1 \leq d$ and $1 \leq n \leq k$. Then for each nonnegative integer j, there is a polynomial $\text{TD}_d^j(b_0, b_1, b_2, \dots)$ in the indeterminates $b_0, b_1, b_2, \dots$ with the following properties:*

(1) *TD_d^j is homogeneous of degree d;*
(2) *if we set $\deg(b_i) = i$, then TD_d^j is homogeneous of degree $d(k-n+1)$;*
(3) *for all $1 \leq n \leq k$, we have*

$$\text{Tp}_d^{n \to k}(\boldsymbol{\lambda}, \boldsymbol{\theta}) = \text{TD}_d^{k-n}(1, c_1(\boldsymbol{\lambda}, \boldsymbol{\theta}), c_2(\boldsymbol{\lambda}, \boldsymbol{\theta}), \dots), \tag{2.15}$$

where the polynomials $c_i(\boldsymbol{\lambda}, \boldsymbol{\theta})$, $i = 1, \dots,$ are defined by (2.14);

(4) *the polynomial TD_d^{j-1} may be obtained from TD_d^j via the following substitution:*

$$\text{TD}_d^{j-1}(b_0, b_1, b_2, \dots) = \text{TD}_d^j(0, b_0, b_1, b_2, \dots).$$

The notation TD stands for Thom-Damon polynomial. The 3rd property (2.15) is an older result of Damon and Ronga ([8], [24]), while the 4th is a theorem of Fehér and Rimányi [9].

There is a somewhat confusing aspect of (2.15), which we would like to clarify now. For fixed j and sufficiently large n and k, the polynomials $c_i(\boldsymbol{\lambda}, \boldsymbol{\theta}) \in \mathbb{C}[\boldsymbol{\lambda}, \boldsymbol{\theta}]^{\mathcal{S}_n \times \mathcal{S}_k}$, $i = 1, \dots, d(j+1)$ are algebraically independent. This means that for fixed codimension j and large enough n, the Thom polynomial $\text{Tp}_d^{n \to n+j}(\boldsymbol{\lambda}, \boldsymbol{\theta})$ determines TD_d^j. However, for small values of n, the natural map

$$\mathbb{C}[c_1, c_2, \dots] \to \mathbb{C}[\boldsymbol{\lambda}, \boldsymbol{\theta}]^{\mathcal{S}_n \times \mathcal{S}_k}$$

is not surjective in degree $d(k-n+1)$, and in this case there are several expressions of the Thom polynomial in terms of relative Chern classes. Only one of these expressions remains valid for all n.

Example 2.13. For $d = 4, n = 1, k = 1$,

$$\text{RC}(q) = \frac{1+\theta q}{1+\lambda q} = 1 + (\theta - \lambda)q - \lambda(\theta - \lambda)q^2 + \dots .$$

Thus we have

$$c_0(\theta,\lambda)=1,\quad c_1(\theta,\lambda)=\theta-\lambda,\quad c_2(\theta,\lambda)=-\lambda(\theta-\lambda),$$

$$c_3(\theta,\lambda)=\lambda^2(\theta-\lambda),\quad c_4(\theta,\lambda)=-\lambda^3(\theta-\lambda)\dots.$$

We have (cf. [10, Th. 2.2], also §7.4)

$$\mathrm{TD}_4^0=c_1^4+6c_1^2c_2+2c_2^2+9c_1c_3+6c_4c_0,$$

and for $n>1$, this is the only possible expression for the Thom polynomial in terms of the relative Chern classes. However, since for $n=k=1$,

$$c_1(\theta,\lambda)c_3(\theta,\lambda)=c_2(\theta,\lambda)^2,$$

we can conclude that

$$\mathrm{Tp}_4^{1\to 1}(\theta,\lambda)=c_1^4+6c_1^2c_2+\alpha c_2^2+(11-\alpha)c_1c_3+6c_4c_0$$

holds for any $\alpha\in\mathbb{R}$.

Next, following [9], observe that property (4) allows us to define a universal object, the Thom series $\mathrm{Ts}(a_i,\, i\in\mathbb{Z})$, which is an infinite formal series in infinitely many variables with the following properties:

- $\mathrm{Ts}(a_i,\, i\in\mathbb{Z})$ is homogeneous of degree d;
- setting $\deg(a_i)=i$ for $i\in\mathbb{Z}$, the series $\mathrm{Ts}_d(a_i,\, i\in\mathbb{Z})$ is homogeneous of degree 0;
- the Thom-Damon polynomial maybe expressed via the following substitution:

$$\mathrm{TD}_d^j(b_0,b_1,b_2,\dots)=\mathrm{Ts}_d\left(\begin{cases}a_i=b_{i+k-n+1}, & \text{if } i\geq -(k-n+1),\\ a_i=0, & \text{otherwise.}\end{cases}\right).$$

For example, in this notation, Porteous's formula reads simply $\mathrm{Ts}_1=a_0$, while Ronga's formula takes the form $\mathrm{Ts}_2=a_0^2+\sum_{i=0}^{\infty}2^{i-1}a_ia_{-i}$. This suggestive way of expressing Thom polynomials, found by Fehér and Rimányi, served as a starting point for our work. We obtained a rather satisfactory answer, which manifestly has the structure described above. The final result (6.26) even gives some insight into the geometric meaning of the coefficients of the Thom series.

3. Localizing Poincaré duals

In this section we prove a localization formula for equivariant Poincaré duals. Roughly, we show that if the T-invariant subvariety $\Sigma\subset W$ is equivariantly fibered over a parameter space M, then the equivariant Poincaré dual $\mathrm{eP}[\Sigma,W]_T$ may be read off from local data near fixed points of the T action on M.

3.1. *Integration and equivariant multiplicities.* In [26], Rossmann made the important observation that the notion of equivariant Poincaré dual may be extended to the case of analytic T-invariant varieties defined in a neighborhood of the origin in T-modules, and further, to nonlinear actions, as we explain below.

Let Z be a complex manifold with a holomorphic T-action, and let $M \subset Z$ be a T-invariant analytic subvariety with an isolated fixed point $p \in M^T$. Then one can find local analytic coordinates near p, in which the action is linear and diagonal. Using these coordinates, one can identify a neighborhood of the origin in T_pZ with a neighborhood of p in Z. We denote by $\hat{\mathrm{T}}_pM$ the part of T_pZ which corresponds to M under this identification. Clearly, this is an analytic subvariety defined in a neighborhood of the origin; informally, we will call $\hat{\mathrm{T}}_pM$ the T-invariant *tangent cone* of M at p. This tangent cone is not quite canonical: it depends on the choice of coordinates. The equivariant Poincaré dual of $\Sigma = \hat{\mathrm{T}}_pM$ in $W = \mathrm{T}_pZ$, however, does not. Rossmann named this equivariant Poincaré dual the *equivariant multiplicity of M in Z at p*:

$$\mathrm{emult}_p[M, Z] \stackrel{\mathrm{def}}{=} \mathrm{eP}[\hat{\mathrm{T}}_pM, \mathrm{T}_pZ]_T. \tag{3.1}$$

An important application of the equivariant multiplicity is Rossmann's localization formula [26]. Assume that in addition to the setup above, the T-action on Z has a finite set of fixed points and that M is a compact T-invariant subvariety in Z. Let $\mu : \mathrm{Lie}(T) \to \Omega^\bullet(Z)^T$ be holomorphic map from the Lie algebra of T to the space of T-invariant differential forms on Z, which is equivariantly closed; i.e., $d_K\mu = 0$. Then *Rossmann's localization formula* states that at a regular element of $\mathrm{Lie}(T)$ one has

$$\int_M \mu = \sum_{p \in M^T} \frac{\mathrm{emult}_p[M, Z]}{\mathrm{Euler}^T(\mathrm{T}_pZ)} \cdot \mu^{[0]}(p), \tag{3.2}$$

where $\mu^{[0]}(p)$ is the differential-form-degree-zero component of μ evaluated at p. Recall that $\mathrm{Euler}^T(\mathrm{T}_pZ)$ stands for the product of the weights of the T-action on T_pZ.

This formula generalizes the equivariant integration formula of Berline and Vergne [4], which applies when M is smooth. In this case the tangent cone of M at p is a linear subspace $\mathrm{T}_pM \subset \mathrm{T}_pZ$ and $\mathrm{emult}_p[M]$ is the equivariant Poincaré dual of this subspace. Then the fraction in (3.2) simplifies: the ambient space Z is eliminated from the picture, and one arrives at (cf. [4])

$$\int_M \mu = \sum_{p \in M^T} \frac{\mu^{[0]}(p)}{\mathrm{Euler}^T(\mathrm{T}_pM)}. \tag{3.3}$$

Remark 3.1. Our presentation does not follow the history of the subject. Rossmann's original definition of equivariant multiplicity in (3.2) used a more complicated integral-limit formula, which he linked to the definition of Joseph following the method of Bott [6]. His work inspired Vergne to come up with the beautiful formula (2.2) (cf. Definition 2.1). In our paper, however, we have taken Vergne's integral formula as the definition of the equivariant Poincaré dual and equivariant multiplicity.

3.2. *The localization formula.* Recall (see, e.g., [7]) that if $f : X \to Y$ is a smooth proper map between connected oriented manifolds such that f restricted to some open subset of X is a diffeomorphism, then for a compactly supported form μ on Y, we have $\int_X f^*\mu = \int_Y \mu$.

We need a version of this statement for singular varieties and equivariant forms. This reads as follows.

Let T be a complex torus group and $f : M \to N$ be a smooth proper T-equivariant map between smooth quasiprojective varieties. Assume that $X \subset M$ and $Y \subset N$ are possibly singular T-invariant closed subvarieties such that f restricted to X is a birational map from X to Y. Next, let μ be an equivariantly closed differential form on N with values in polynomials on $\mathfrak{t}$. Then the integral of μ on the smooth part of Y is absolutely convergent; we denote this by $\int_Y \mu$. With this convention, we again have

$$\int_X f^*\mu = \int_Y \mu. \tag{3.4}$$

Let Σ be a T-invariant closed subvariety of the T-module W. Consider the following diagram describing the slicing of an affine subvariety Σ into a pieces parameterized by a variety M:

$$\begin{array}{ccccccc} & & S & \xrightarrow{\text{ev}} & W & \hookleftarrow & \Sigma \\ & & & \searrow^{\tau} & & & \\ M^T & \hookrightarrow & M & \hookrightarrow & Z. & & \end{array} \tag{3.5}$$

Here

- each arrow stands for a T-equivariant morphism;
- Z is a compact smooth variety, and S is a bundle of quasiprojective varieties on Z endowed with a proper map $\text{ev} : S \to W$;
- M is a compact, not necessarily smooth T-invariant subvariety of Z with a finite set M^T of fixed points.

PROPOSITION 3.2. *Let Σ be a closed T-invariant subvariety of the complex vector space W, and assume that the restriction of* ev *to $\tau^{-1}M$ is a birational* *map to* Σ*. Then*

$$\mathrm{eP}[\Sigma, W]_T = \sum_{p \in M^T} \frac{\mathrm{eP}[\mathrm{ev}(S_p), W]_T \cdot \mathrm{emult}_p[M, Z]}{\mathrm{Euler}^T(\mathrm{T}_p Z)}. \tag{3.6}$$

Remark 3.3. The equivariant Euler class in the denominator is a product of weights (cf. (2.4)), hence each term in the sum is a rational function whose denominator is a product of linear factors. After the summation, however, the denominators cancel, and one ends up with a polynomial result.

Proof. Combining Definition 2.1 with our assumption that ev establishes a birational map between $\tau^{-1}(M)$ and Σ, we obtain

$$\mathrm{eP}[\Sigma, W]_T = \int_{\tau^{-1}(M)} \mathrm{ev}^*\mathrm{Thom}(W).$$

The push-forward $\tau_*\mathrm{ev}^*\mathrm{Thom}(W)$ is a polynomial on $\check{\mathfrak{t}}$ tensored with integrals of a smooth form along the fibers of a locally trivial fibration, and as such, it is a polynomial on $\check{\mathfrak{t}}$ with values in smooth forms on Z. Thus we can represent the equivariant Poincaré dual as the integral of a smooth equivariant form on M:

$$\mathrm{eP}[\Sigma, W]_T = \int_M \tau_*\mathrm{ev}^*\mathrm{Thom}(W).$$

Applying Rossmann's localization formula (3.6) to this integral, we obtain

$$\mathrm{eP}[\Sigma, W]_T = \sum_{p \in M^T} \frac{(\tau_*\mathrm{ev}^*\mathrm{Thom}(W))^{[0]}(p) \cdot \mathrm{emult}_p[M, Z]}{\mathrm{Euler}^T(\mathrm{T}_p Z)}. \tag{3.7}$$

Clearly, $(\tau_*\mathrm{ev}^*\mathrm{Thom}(W))^{[0]}(p) = \int_{\mathrm{ev}(S_p)} \mathrm{Thom}(W)$, and this latter integral, by definition, is equal to $\mathrm{eP}[\mathrm{ev}(S_p), W]_T$. This completes the proof. □

Later in the paper, this formula will be the key tool in our calculations.

Again, just as in (3.3), formula (3.6) simplifies when M is a smooth subvariety of Z. In this case, one obtains

$$\mathrm{eP}[\Sigma, W]_T = \sum_{p \in M^T} \frac{\mathrm{eP}[\mathrm{ev}(S_p), W]_T}{\mathrm{Euler}^T(\mathrm{T}_p M)}. \tag{3.8}$$

3.3. *An interlude*: *the case of* $d = 1$. In this paragraph, we consider the case $d = 1$ of the A_d-singularities introduced in Section 1.2, and we recover the classical result of Porteous.

We have $\mathcal{J}_1(n, k) = \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^k)$, and $\Theta_1 \subset \mathcal{J}_1(n, k)$ consists of those linear maps $\mathbb{C}^n \to \mathbb{C}^k$ whose kernel is 1-dimensional. These maps may be identified with k-by-n matrices, and the weight of the action on the entry e_{ji} is equal to $\theta_j - \lambda_i$. Then the closure $\overline{\Theta}_1$ consist of those k-by-n matrices which have a nontrivial kernel:

$$\overline{\Theta}_1 = \{A \in \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n);\ \exists v \in \mathbb{C}^n,\ v \neq 0 :\ Av = 0\}. \tag{3.9}$$

This description immediately suggests an equivariant birational fibration of $\overline{\Theta}_1$ over $\mathbb{P}^{n-1}$, fitting the conditions of Proposition 3.2. In this case, $M = Z = \mathbb{P}^{n-1}$, and the fiber over a point $[v] \in \mathbb{P}^{n-1}$ is the linear subspace $\{A;\ Av = 0\} \subset \overline{\Theta}_1$, where $[v]$ stands for the point in $\mathbb{P}^{n-1}$ corresponding to the nonzero vector $v \in \mathbb{C}^n$.

We simply need to collect our fixed-point data and then apply (3.8). There are n fixed points, $p_1, \dots, p_n$ in $\mathbb{P}^{n-1}$, corresponding to the coordinate axes. The weights of $\mathrm{T}_{p_i}\mathbb{P}^{n-1}$ are $\{\lambda_s - \lambda_i;\ s \neq i\}$. The fiber at p_i is the set of matrices A with all entries in the ith column vanishing. Using (2.3), we deduce that the equivariant Poincaré dual of the fiber at p_i is $\prod_{j=1}^k(\theta_j - \lambda_i)$; hence our localization formula looks as follows:

$$\mathrm{eP}[\overline{\Theta}_1, \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^k)]_{T_n \times T_k} = \sum_{i=1}^n \frac{\prod_{j=1}^k(\theta_j - \lambda_i)}{\prod_{s \neq i}(\lambda_s - \lambda_i)}. \tag{3.10}$$

This is a finite sum for fixed n, but as n increases, the number of terms also increases. There is a way, however, to further "localize" this expression and obtain a formula that only depends on the local behavior of a certain function at a single point.

Indeed, consider the following rational differential form on $\mathbb{P}^1$:

$$-\frac{\prod_{j=1}^k(\theta_j - z)}{\prod_{i=1}^n(\lambda_i - z)}\, dz.$$

Observe that the residues of this form at finite poles: $\{z = \lambda_i;\ i = 1, \dots, n\}$ exactly recover the terms of the sum (3.10). Then, applying the Residue theorem, we obtain

$$\mathrm{eP}[\overline{\Theta}_1, \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^k)]_{T_n \times T_k} = \operatorname*{Res}_{z=\infty} \frac{\prod_{j=1}^k(\theta_j - z)}{\prod_{i=1}^n(\lambda_i - z)}\, dz.$$

Finally, after the change of variables $z \to -1/q$, we end up with

$$\mathrm{eP}[\overline{\Theta}_1, \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^k)]_{T_n \times T_k} = \operatorname*{Res}_{q=0} \frac{\prod_{j=1}^k(1 + q\theta_j)}{\prod_{i=1}^n(1 + q\lambda_i)} \frac{dq}{q^{k-n+2}},$$

which, according to (2.14), is exactly the relative Chern class c_{k-n+1}. Thus we recovered the well-known Giambelli-Thom -Porteous formula ([20]; [11, Ch. I.5]).

As a final remark, note that our basic example introduced in Section 2.3 is the special case of $\overline{\Theta}_1$ corresponding to the values $n = k = 2$. Applying our new method, we have

$$\mathrm{eP}[\Sigma, \mathbb{C}^4]_T = \frac{\eta_1\eta_2}{\eta_3 - \eta_2} + \frac{\eta_3\eta_4}{\eta_2 - \eta_3}. \tag{3.11}$$

Using $\eta_1 + \eta_3 = \eta_2 + \eta_4$, we recover formula (2.9).

4. The test curve model

In Section 1, we described the variety Θ_d as a contact singularity class (1.5). In this section, we recall another description of Θ_d — the so-called "test curve model" — which goes back to the works of Porteous, Ronga, and Gaffney [21], [25], [10]. Roughly, the idea of the construction is to generalize (3.9) to $d > 1$ by requiring that the map-jet $\Psi \in \mathcal{J}_d(n,k)$ carry a d-jet of a curve in $\mathbb{C}^n$ to zero. As we have not found a complete proof of the appropriate statement (Proposition 4.1) in the literature, we give one below.

4.1. *The model.* Recall the notation $\Sigma_0(n,k)$ in (1.4) for the set of map-jets whose linear part is of (the maximal possible) rank n. In particular, we will call an element $\gamma \in \Sigma_0(1,n)$ a *regular curve.* In turn, $\Sigma_1(n,k)$ denotes the set of map-jets whose linear part is a matrix with kernel of dimension 1. Note that we omit d from the notation here.

Proposition 4.1. *The variety $\Theta_d \subset \mathcal{J}_d(n,k)$ of map-jets with nilpotent algebra $A_d = t\mathbb{C}[t]/t^{d+1}$ allows for the following description*:

$$\Theta_d = \{\Psi \in \Sigma_1(n,k);\ \exists \gamma \in \Sigma_0(1,n) :\ \Psi \circ \gamma = 0\}. \tag{4.1}$$

Proof. First assume that the nilpotent algebra $\mathcal{J}_d(n)/I_\Psi$ of $\Psi = (P_1,\dots,P_k)$ is isomorphic to $A_d = \mathcal{J}_d(1)$, where $I_\Psi = \langle P_1,\dots,P_k\rangle_{\text{ideal}}$ is the ideal generated by the coordinate polynomials. Then there is a surjective algebra-morphism $\beta : \mathcal{J}_d(n) \to A_d$ with $\ker(\beta) = I_\Psi$, giving us the exact sequence of algebras

$$0 \longrightarrow I_\Psi \longrightarrow \mathcal{J}_d(n) \xrightarrow{\beta} \mathcal{J}_d(1) = A_d \longrightarrow 0. \tag{4.2}$$

Setting $\beta(x_i) = \gamma_i(t)$, $i = 1,\dots,n$, where $x_1,\dots,x_n$ are the generators of $\mathcal{J}_d(n)$, we obtain $\beta(P_j) = P_j(\gamma_1,\dots,\gamma_n) = 0$ for $j = 1,\dots,k$. This implies $\Psi \circ \gamma = 0$ for the curve $\gamma = (\gamma_1,\dots,\gamma_n) \in \mathcal{J}_d(1,n)$. Note that γ is in $\Sigma_0(1,n)$ since β is surjective.

To prove the converse statement, we reverse this argument. Assume that $\Psi \circ \gamma = 0$ for a $\Psi = (P_1,\dots,P_k) \in \Sigma_1(n,k)$, and $\gamma = (\gamma_1,\dots,\gamma_n) \in \Sigma_0(1,n)$. Then the pairing (1.3),

$$\mathcal{J}_d(n,1) \times \mathcal{J}_d(1,n) \to \mathcal{J}_d(1,1),$$

defines an algebra map

$$\beta_\gamma : \mathcal{J}_d(n) \to \mathcal{J}_d(1) = A_d \text{ for which } \beta_\gamma(x_i) = \gamma_i(t),\ i = 1,\dots,n. \tag{4.3}$$

Since $\gamma \in \Sigma_0(1,n)$ and A_d is generated by one element, the map β_γ is surjective. Then the obvious inclusion $I_\Psi \subset \ker(\beta_\gamma)$ induces the surjective algebra morphism $A_\Psi \to A_d$, where $A_\Psi = \mathcal{J}_d(n)/I_\Psi$ is also an algebra of depth d. Now the fact that $\Psi \in \Sigma_1(n,k)$ implies that A_Ψ is a rank-1 algebra and hence $A_\Psi \cong A_d$. □

Definition 4.2. For a vector space V, introduce the partial flag variety

$$\operatorname{Flag}_d(V) = \{(F_1 \subset \cdots \subset F_d \subset V);\ \dim F_i = i,\ i = 1, \dots, d)\}. \tag{4.4}$$

This space can also be described as a quotient

$$\operatorname{Flag}_d(V) = \operatorname{Hom}^{\mathrm{inj}}(\mathbb{C}^d, \mathbb{C}^n)/B_d, \tag{4.5}$$

where $\operatorname{Hom}^{\mathrm{inj}}(\mathbb{C}^d, \mathbb{C}^n)$ stands for the set of injective linear maps, or rank-d n-by-d matrices, and B_d is the group of upper-triangular d-by-d matrices acting on the right.

Now recall the filtrations (1.1) on the duals $\mathcal{J}_d(n)^*$ and $\mathcal{J}_d(1)^*$, and note that given $\gamma \in \Sigma_0(1, n)$, the dual map $\beta_\gamma^* : \mathcal{J}_d(1)^* \hookrightarrow \mathcal{J}_d(n)^*$ preserves this filtration. Since β_γ^* is injective, this allows us to associate to every regular γ a partial flag in $\mathcal{J}_d(n)^*$:

$$\gamma \mapsto \psi(\gamma) = (\beta_\gamma^*(\mathcal{J}_1(1)^*), \dots, \beta_\gamma^*(\mathcal{J}_d(1)^*) \in \operatorname{Flag}_d(\mathcal{J}_d(n)^*). \tag{4.6}$$

PROPOSITION 4.3. *Given $\Psi \in \Theta_d$, the partial flag $\psi(\gamma)$ is the same for every regular γ for which $\Psi \circ \gamma = 0$.*

Proof. Indeed, this follows from the fact that the ith element of the partial flag $\psi(\gamma)$ is the intersection of $\mathcal{J}_i(n)^* \subset \mathcal{J}_d(n)^*$ with the annihilator of I_Ψ (cf. (4.2)):

$$\beta_\gamma^*(\mathcal{J}_i(1)^*) = \{\delta \in \mathcal{J}_i(n)^*;\ \langle \delta, Q \rangle = 0 \text{ for all } Q \in I_\Psi\}.$$

Note that the right-hand side here depends only on Ψ, not on γ. □

We can summarize the situation in the following commutative diagram:

$$\begin{array}{ccc} \{(\Psi, \gamma) \in \Sigma_1(n,k) \times \Sigma_0(1,n);\ \Psi \circ \gamma = 0\} & \longrightarrow & \Theta_d \\ \big\downarrow & & \big\downarrow \alpha \\ \Sigma_0(1,n) & \xrightarrow{\ \psi\ } & \operatorname{Flag}_d(\mathcal{J}_d(n)^*). \end{array} \tag{4.7}$$

Here $\alpha(\Psi)$ is the flag associated to $\Psi \in \Theta_d$ by Proposition 4.3.

Our next goal is to write down the map ψ and the equation $\Psi \circ \gamma = 0$ explicitly. A curve $\gamma \in \Sigma_0(1, n)$ is parametrized by d vectors $v_1, \dots, v_d$ in $\mathbb{C}^n$:

$$\gamma(t) = tv_1 + t^2 v_2 + \cdots + t^d v_d,\ v_1 \neq 0. \tag{4.8}$$

Thus we have the identification

$$\Sigma_0(1, n) \cong \{\gamma = (v_1, \dots, v_d) \in \operatorname{Hom}(\mathbb{C}^d, \mathbb{C}^n);\ v_1 \neq 0\}. \tag{4.9}$$

According to (1.2), the space $\mathcal{J}_d(n)^*$ is isomorphic to the truncated symmetric algebra on $\mathbb{C}^n$: $\operatorname{Sym}_d^\bullet \mathbb{C}^n = \oplus_{m=1}^d \operatorname{Sym}^m \mathbb{C}^n$. To parametrize a basis of this space, we introduce the following notation.

Definition 4.4. We denote by $\Pi[m]$ the set of partitions of m into nonnegative integers. For a partition $\tau = (i_1, \dots, i_s)$, we write

- $\mathrm{sum}(\tau) = i_1 + \cdots, +i_s$ for the sum;
- $|\tau| = s$ for the length; and
- $\gamma_\tau = \mathrm{perm}(\tau) \cdot v_{i_1} \dots v_{i_s} \in \mathrm{Sym}^s\mathbb{C}^n$ for a curve of the form (4.8), where $\mathrm{perm}(\tau)$ denotes the cardinality of the set of all permutations of the sequence $(i_1, \dots, i_s)$.

Note that a map-jet $\Psi \in \mathcal{J}_d(n,k)$ may be interpreted as a map $\Psi : \mathcal{J}_d(n)^* \to \mathbb{C}^k$. Now we can write down our formulas.

LEMMA 4.5. *Let γ be a curve of the form (4.8), and consider the following sequence of symmetric tensors associated to γ:*

$$\phi(\gamma) = \left(v_1, v_2 + v_1^2, v_3 + 2v_1v_2 + v_1^3, \dots, \sum_{\mathrm{sum}(\tau)=m} \gamma_\tau, \dots \right). \tag{4.10}$$

Then the equation $\Psi \circ \gamma = 0$ is equivalent to the vanishing of the pairing between Ψ and $\phi(\gamma)$, which, in turn, may be written down explicitly as the following system of linear equations with values in $\mathbb{C}^k$:

$$\sum_{\tau \in \Pi[m]} \Psi(\gamma_\tau) = 0, \quad m = 1, 2, \dots, d. \tag{4.11}$$

The partial flag $\psi(\gamma)$ is simply the flag generated by the sequence $\phi(\gamma)$ (cf. (4.5)):

$$\psi(\gamma) = \phi(\gamma) \cdot B_d. \tag{4.12}$$

The proof is a straightforward substitution and will be omitted. For $d = 3$, the equations (4.11) have the following form:

$$\begin{aligned} \Psi^1(v_1) &= 0, \\ \Psi^1(v_2) + \Psi^2(v_1, v_1) &= 0, \\ \Psi^1(v_3) + 2\Psi^2(v_1, v_2) + \Psi^3(v_1, v_1, v_1) &= 0. \end{aligned} \tag{4.13}$$

Here, for clarity, we marked by Ψ^m the restriction of Ψ to $\mathrm{Sym}^m\mathbb{C}^n$.

LEMMA 4.6. *For a curve $\gamma \in \Sigma_0(1,n)$, we have*

$$S_\gamma \stackrel{\mathrm{def}}{=} \{\Psi \in \mathcal{J}_d(n,k);\ \Psi \circ \gamma = 0\} = \ker \beta_\gamma \otimes \mathbb{C}^k, \tag{4.14}$$

and this set is a codimension-dk linear subspace in $\mathcal{J}_d(n,k)$. Moreover, $S_\gamma \setminus \Sigma_1(n,k)$ is a codimension-$(k-n+2)$ algebraic subvariety of S_γ.

Proof. Equality (4.14) immediately follows from (4.3). Equations (4.11) form a system of d linear equations with values in $\mathbb{C}^k$. Their linear independence follows from the surjectivity of the map β_γ in (4.3), but also from the presence of the summand v_1^i in the ith equation ($v_1 \neq 0$). To prove the second part of the statement, we observe that fixing Ψ^1, the linear part of Ψ, the remainder of the system remains a nondegenerate (inhomogeneous) linear system. This means that S_γ is a vector bundle over the set $\{A \in \mathcal{J}_1(n,k);\ v_1 \in \ker(A)\}$, and in this base, the complement of the subset $\{A : \ker A = v_1\}$ is of codimension $k-n+2$. □

Now, using (4.14), we can summarize the "test-curve model" as follows:

$$\begin{array}{ccccc} \Theta_d & \hookrightarrow & \mathcal{J}_d(n,k) & \xleftarrow{\mathrm{ev}_S} & S^{\mathrm{Fl}} \\ & & & & \Big\downarrow \tau \\ & & \Sigma_0(1,n) & \xrightarrow{\psi} & \mathrm{Flag}_d(\mathrm{Sym}^\bullet_d \mathbb{C}^n). \end{array} \tag{4.15}$$

Proposition 4.7. *Let* $\tau : S^{\mathrm{Fl}} \to \mathrm{Flag}_d(\mathrm{Sym}^\bullet_d \mathbb{C}^n)$ *be the vector bundle* $S^{\mathrm{Fl}} = V^\perp \otimes \mathbb{C}^k$, *where* V *is the tautological rank-d vector bundle over* $\mathrm{Flag}_d(\mathrm{Sym}^\bullet_d \mathbb{C}^n)$. *Since* $V^\perp$ *is a linear subspace of* $(\mathrm{Sym}^\bullet_d \mathbb{C}^n)^* \cong \mathcal{J}_d(n)$, *we have a tautological evaluation* $\mathrm{ev}_S : S^{\mathrm{Fl}} \to \mathcal{J}_d(n,k)$. *Then*

$$\Theta_d = \mathrm{ev}_S[\tau^{-1}(\mathrm{im}\,\psi)] \quad \textit{and} \quad \overline{\Theta}_d = \mathrm{ev}_S[\tau^{-1}(\overline{\mathrm{im}\,\psi})]. \tag{4.16}$$

(*The map* ψ *was defined in* (4.6).)

The first equality of (4.16) immediately follows from Proposition 4.3 and Lemma 4.6, while the second follows from the fact that the map ev_S is proper.

Diagram (4.15) is somewhat reminiscent of the localization diagram (3.5), which we would like to use. We note that the map ψ is GL_n-invariant, but not generically injective, and the variety $\Sigma_0(1,n)$ is not compact. Indeed, given $\Psi \in \Theta_d$, $\gamma \in \Sigma_0(1,n)$ such that $\Psi \circ \gamma = 0$, and $\Delta \in \mathrm{Diff}_d(1) = \Sigma_0(1,1)$, clearly

$$\Psi \circ (\gamma \circ \Delta) = 0.$$

Thus the map ψ is constant on the $\mathrm{Diff}_d(1)$-orbits. In fact, we can make a more precise statement.

Proposition 4.8. *For* $\Psi \in \Theta_d$ *and* $\gamma, \delta \in \Sigma_0(1,n)$,

$$\Psi \circ \gamma_1 = \Psi \circ \gamma_2 = 0 \Leftrightarrow \exists \Delta \in \mathrm{Diff}_d(1) = \Sigma_0(1,1) \textit{ such that } \gamma = \delta \circ \Delta.$$

Proof. We prove this statement by induction. Let $\gamma = v_1 t + \cdots + v_d t^d$ and $\delta = w_1 t + \cdots + w_d t^d$. Since $\Psi \in \Sigma_1(n,k)$, we have $\dim \ker \Psi^1 = 1$, and hence $v_1 = \lambda w_1$, for some $\lambda \neq 0$. This proves the $d = 1$ case.

Suppose the statement is true for $d-1$. Then, using the appropriate order-$(d-1)$ diffeomorphism, we can assume that $v_m = w_m$, $m = 1, \ldots, d-1$. It is clear then from the explicit form (4.11) (cf. (4.13)) of the equation $\Psi \circ \gamma = 0$, that $\Psi^1(v_d) = \Psi^1(w_d)$; hence $w_d = v_d - \lambda v_1$ for some $\lambda \in \mathbb{C}$. Then $\gamma = \Delta \circ \delta$ for $\Delta = t + \lambda t^d$, and the proof is complete. □

It is clear from diagram (4.7) that Proposition 4.8 implies

COROLLARY 4.9. *For* $\gamma, \delta \in \Sigma_0(1,n)$, *we have*

$\psi(\gamma) = \psi(\delta) \in \mathrm{Flag}_d(\mathrm{Sym}_d^\bullet \mathbb{C}^n) \Leftrightarrow \exists \Delta \in \mathrm{Diff}_d(1) = \Sigma_0(1,1)$ *such that* $\gamma = \delta \circ \Delta$; *hence* ψ *induces an injection on the orbits*

$$\Sigma_0(1,n)/\mathrm{Diff}_d(1) \hookrightarrow \mathrm{Flag}_d(\mathrm{Sym}_d^\bullet \mathbb{C}^n).$$

Remark 4.10. The $\underline{\text{fibers of } S^{\mathrm{Fl}} \text{ are}}$ codimension-dk vector spaces in $\mathcal{J}_d(n,k)$, while the base $\psi(\mathrm{Hom}^{\mathrm{inj}}(\mathbb{C}^d, \mathbb{C}^n))$ has dimension $d(n-1)$ by Corollary 4.9. The dimension of $\Sigma_0(1,n)$ is dn, and the dimension of $\mathrm{Diff}_d(1)$ is d. Thus the codimension of $\overline{\Theta}_d$ in $\mathcal{J}_d(n,k)$ is $dk - dn + d = d(k-n+1)$. This is also well known from general arguments in global singularity theory; see [1].

The closure of the image $\psi(\Sigma_0(1,n))$ thus provides us with a compactification of the set $\Sigma_0(1,n)/\mathrm{Diff}_d(1)$. In the next paragraph, we will find a more efficient compactification.

4.2. *Localization over* $\mathrm{Flag}_d(\mathbb{C}^n)$. We begin with the observation that the formula $\overline{\Theta}_d = \mathrm{ev}_S[\tau^{-1}(\overline{\mathrm{im}\,\psi})]$ in (4.16) remains true if we replace the space of regular curves $\Sigma_0(1,n)$ by a dense open subset. A convenient such subset will be the set $\mathrm{Hom}^{\mathrm{inj}}(\mathbb{C}^d, \mathbb{C}^n)$ of injective linear maps, i.e., the set of linearly independent sequences $(v_1, \ldots, v_n)$ of vectors in $\mathbb{C}^n$ (cf. (4.9)). The key property of this dense open subset is that it forms a single orbit under the GL_n-action on $\Sigma_0(1,n)$, and it has a GL_n-equivariant fibration over the partial flag variety $\mathrm{Flag}_d(\mathbb{C}^n)$:

$$\mathbf{p} : \mathrm{Hom}^{\mathrm{inj}}(\mathbb{C}^d, \mathbb{C}^n) \to \mathrm{Flag}_d(\mathbb{C}^n),$$

associating to a sequence of d linearly independent vectors the corresponding partial flag. Observe that the map $\mathbf{p}$ is a principal B_d fibration, where B_d is the group of d-by-d upper triangular matrices, acting as filtration-preserving maps on $\mathbb{C}^d$ (see Definition 4.13 below). We will try to enhance our model (4.15) by incorporating this fibration into it.

We consider the map ϕ given in (4.10). Clearly, for $\gamma = \sum_{m=1} v_m t^m \in \mathrm{Hom}^{\mathrm{inj}}(\mathbb{C}^d, \mathbb{C}^n)$, the elements of the sequence of symmetric tensors $\phi(\gamma)$ belong to $\bigoplus_{\mathrm{sum}(\tau)\le d} \mathbb{C}\gamma_\tau$. Now we observe that this latter vector space does not change if we replace $\gamma \in \mathrm{Hom}^{\mathrm{inj}}(\mathbb{C}^d, \mathbb{C}^n)$ by $\gamma \cdot b$ where $b \in B_d$, since the action of $b \in B_d$ "lowers" indices; i.e., it takes v_i to a linear combination of $\{v_j,\, 1 \le j \le m\}$. We can formulate this observation as follows.

LEMMA 4.11. *For* $\gamma = (v_1, \ldots, v_d)$, *introduce the filtered linear subspace of* $\mathrm{Sym}_d^\bullet \mathbb{C}^n$:

$$\bigoplus_{\mathrm{sum}(\tau)\le d} \mathbb{C}\gamma_\tau \supset \bigoplus_{\mathrm{sum}(\tau)\le d-1} \mathbb{C}\gamma_\tau \supset \cdots \supset \mathbb{C}v_2 \oplus \mathbb{C}v_1^2 \oplus \mathbb{C}v_1. \supset \mathbb{C}v_1. \tag{4.17}$$

Then given $\gamma = (v_1, \dots, v_d), \delta = (w_1, \dots, w_d) \in \mathrm{Hom}^{\mathrm{inj}}(\mathbb{C}^d, \mathbb{C}^n)$ *satisfying* $\mathbf{p}(\gamma) = \mathbf{p}(\delta) \in \mathrm{Flag}_d(\mathbb{C}^n)$, *we have the equality of two filtered vector spaces*

$$\bigoplus_{\mathrm{sum}(\tau)\le d} \mathbb{C}\gamma_\tau = \bigoplus_{\mathrm{sum}(\tau)\le d} \mathbb{C}\delta_\tau. \tag{4.18}$$

Definition 4.12. For a flag $\mathbf{f} \in \mathrm{Flag}_d(\mathbb{C}^n)$, denote by $\mathrm{Sym}^\bullet_{\mathbf{f}}(\mathbb{C}^n)$ the filtered vector space (4.17) for some γ with $\mathbf{p}(\gamma) = \mathbf{f}$.

Definition 4.13. Consider the natural structure of a filtered vector space on $\mathbb{C}^d$:

$$\mathbb{C}^d = \bigoplus_{i=1}^d \mathbb{C}e_i \supset \ldots \supset \mathbb{C}e_1 \oplus \mathbb{C}e_2 \supset \mathbb{C}e_1,$$

where $(e_1, \dots, e_d)$ is the standard basis of $\mathbb{C}^d$, and for a filtered vector space $V^\bullet = V_d \supset V_{d-1} \supset \cdots \supset V_1$, introduce the linear space of filtration-preserving maps

$$\mathrm{Hom}^\triangle(\mathbb{C}^d, V^\bullet) = \{\varepsilon \in \mathrm{Hom}(\mathbb{C}^d, V^\bullet);\ \varepsilon(e_m) \in V_m\} \tag{4.19}$$

and the corresponding flag variety

$$\mathrm{Flag}_d^\triangle(V^\bullet) = \{(F_1 \subset \cdots \subset F_d) \in \mathrm{Flag}_d(V^\bullet); F_m \subset V_m, m = 1, \dots, d\}. \tag{4.20}$$

Lemma 4.14. *The partial flag variety* $\mathrm{Flag}_d^\triangle(V^\bullet)$ *is a smooth subvariety of* $\mathrm{Flag}_d(V^\bullet)$, *and it may be represented as a quotient as follows*:

$$\mathrm{Flag}_d^\triangle(V^\bullet) = \{\varepsilon \in \mathrm{Hom}^\triangle(\mathbb{C}^d, V^\bullet); \ker(\varepsilon) = 0\}/B_d. \tag{4.21}$$

Formula (4.21) is obvious, while the smoothness follows from the natural representation of $\mathrm{Flag}_d^\triangle(V^\bullet)$ as a tower of projective spaces.

We can apply this construction to $V^\bullet = \mathrm{Sym}^\bullet_{\mathbf{f}}(\mathbb{C}^n)$. For $\mathbf{f} \in \mathrm{Flag}_d(\mathbb{C}^n)$, we define

$$\mathcal{E}_{\mathbf{f}} = \{\varepsilon \in \mathrm{Hom}^\triangle(\mathbb{C}^d, \mathrm{Sym}^\bullet_{\mathbf{f}}(\mathbb{C}^n));\ \ker(\varepsilon) = 0\} \subset \mathrm{Hom}(\mathbb{C}^d, \mathrm{Sym}^\bullet_d \mathbb{C}^n) \tag{4.22}$$

and the corresponding flag space

$$\widetilde{\mathcal{E}}_{\mathbf{f}} \overset{\mathrm{def}}{=} \mathrm{Flag}_d^\triangle(\mathrm{Sym}^\bullet_{\mathbf{f}}(\mathbb{C}^n)) = \mathcal{E}_{\mathbf{f}}/B_d \subset \mathrm{Flag}_d(\mathrm{Sym}^\bullet_{\mathbf{f}}(\mathbb{C}^n)). \tag{4.23}$$

Globalizing this construction leads to our fibered model. The spaces $\mathcal{E}_{\mathbf{f}}$ and $\widetilde{\mathcal{E}}_{\mathbf{f}}$ form the fibers of two GL_n-equivariant bundles over $\mathrm{Flag}_d(\mathbb{C}^n)$:

$$\mathcal{E} = \{(\mathbf{f}, \varepsilon) \in \mathrm{Flag}_d(\mathbb{C}^n) \times \mathrm{Hom}^\triangle(\mathbb{C}^d_R, \mathrm{Sym}^\bullet_{\mathbf{f}}(\mathbb{C}^n));\ \varepsilon \in \mathcal{E}_{\mathbf{f}}\} \to \mathrm{Flag}_d(\mathbb{C}^n) \tag{4.24}$$

and

$$\widetilde{\mathcal{E}} = \mathcal{E}/B_d = \{(\mathbf{f}, \tilde{\varepsilon}) \in \mathrm{Flag}_d(\mathbb{C}^n) \times \mathrm{Flag}_d(\mathrm{Sym}^\bullet_d \mathbb{C}^n);\ \tilde{\varepsilon} \in \widetilde{\mathcal{E}}_{\mathbf{f}}\} \to \mathrm{Flag}_d(\mathbb{C}^n). \tag{4.25}$$

Note that there is a tautological map $\kappa : \widetilde{\mathcal{E}} \to \mathrm{Flag}_d(\mathrm{Sym}^\bullet_d \mathbb{C}^n)$.

LEMMA 4.15.

(1) *Given* $\gamma \in \mathrm{Hom}^{\mathrm{inj}}(\mathbb{C}^d, \mathbb{C}^n)$, *we have* $\psi(\gamma) \in \widetilde{\mathcal{E}}_{\mathbf{p}(\gamma)}$.
(2) *The map* ψ *factorizes as follows*:

$$\psi_{\widetilde{\mathcal{E}}} \circ \kappa = \psi_{\mathrm{Sym}},$$

where $\psi_{\widetilde{\mathcal{E}}} : \mathrm{Hom}^{\mathrm{inj}}(\mathbb{C}^d, \mathbb{C}^n) \to \widetilde{\mathcal{E}}$ *is given by* $\psi_{\widetilde{\mathcal{E}}}(\gamma) = (\pi(\gamma), \psi_{\mathrm{Sym}}(\gamma)$.

The first statement follows from the exact form of the map ϕ from (4.10), while the second is a corollary of the first.

Finally, we introduce some notation for the fibers over a fixed *reference* flag of $\mathrm{Flag}_d(\mathbb{C}^n)$:

- we denote by γ_{ref} the *reference* sequence

$$\gamma_{\mathrm{ref}} = (e_1, \ldots, e_d) \in \mathrm{Hom}^{\mathrm{inj}}(\mathbb{C}^d, \mathbb{C}^n)$$

  where e_i is the ith basis vector of $\mathbb{C}^n$;
- by $\mathbf{f}_{\mathrm{ref}} = \pi(\gamma_{\mathrm{ref}}) \in \mathrm{Flag}_d(\mathbb{C}^n)$ its flag;
- by $\mathrm{Sym}^{\bullet}_{\mathrm{ref}}(\mathbb{C}^n)$ the corresponding subspace $\mathrm{Sym}_{\gamma_{\mathrm{ref}}}(\mathbb{C}^n) \subset \mathrm{Sym}^{\bullet}_d \mathbb{C}^n$; and
- by $\mathcal{E}_{\mathrm{ref}}$ and $\widetilde{\mathcal{E}}_{\mathrm{ref}}$ the fibers of $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ over the reference flag $\mathbf{f}_{\mathrm{ref}}$.

Note that $\mathrm{Sym}^{\bullet}_{\mathrm{ref}}(\mathbb{C}^n) = \mathrm{Sym}^{\bullet}_{\mathrm{ref}}(\mathbb{C}^d)$; thus this space and $\widetilde{\mathcal{E}}_{\mathrm{ref}}$ does not depend on n. The space $\widetilde{\mathcal{E}}_{\mathrm{ref}}$ is endowed by a natural action of the Borel group B_d acting on $\mathrm{Sym}^{\bullet}_{\mathrm{ref}}(\mathbb{C}^n) \subset \mathrm{Sym}^{\bullet}_d(\mathbb{C}^d)$

Remark 4.16. There are two copies of the Borel group B_d acting on $\mathrm{Hom}^{\triangle}(\mathbb{C}^d, \mathrm{Sym}^{\bullet}_d(\mathbb{C}^d))$, and we will differentiate these two actions in our notation when confusion might arise. We will denote by B_L the copy of B_d acting on the left and by B_R the copy acting on the right. Thus we have $\widetilde{\mathcal{E}}_{\mathrm{ref}} = \mathcal{E}_{\mathrm{ref}}/B_R$, and B_L acts on $\widetilde{\mathcal{E}}_{\mathrm{ref}}$.

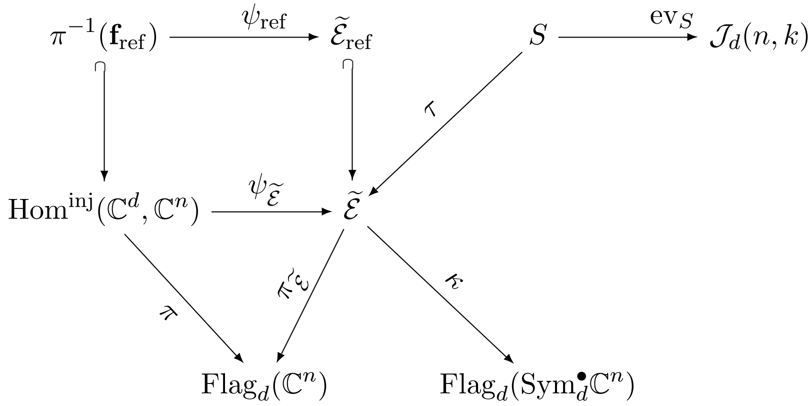
Using the B_L-action on $\widetilde{\mathcal{E}}_{\mathrm{ref}}$, we can represent $\widetilde{\mathcal{E}}$ as an induced space

$$\widetilde{\mathcal{E}} = \mathrm{Hom}^{\mathrm{inj}}(\mathbb{C}^d, \mathbb{C}^n) \times_{B_L} \widetilde{\mathcal{E}}_{\mathrm{ref}}.$$

The restriction $\psi_{\mathrm{ref}} : B_L\gamma_{\mathrm{ref}} \to \widetilde{\mathcal{E}}_{\mathrm{ref}}$ of the map ψ is B_L-equivariant, and hence we have

$$\psi(\pi^{-1}(\mathbf{f}_{\mathrm{ref}})) = B_L \cdot \tilde{\varepsilon}_{\mathrm{ref}}, \text{ where } \tilde{\varepsilon}_{\mathrm{ref}} = \psi_{\mathrm{ref}}(\gamma_{\mathrm{ref}}) \in \widetilde{\mathcal{E}}_{\mathrm{ref}}. \tag{4.26}$$

We arrive at the following picture:



(4.27)

Here

- The vector bundle $\tau : S \to \widetilde{\mathcal{E}}$ is the pull-back $\kappa^* S^{\text{Fl}}$ of the bundle S^{Fl} (cf. (4.15)).
- The map $\psi_{\widetilde{\mathcal{E}}}$ is GL_n-equivariant.
- The fibers of $\psi_{\widetilde{\mathcal{E}}}$ are d-dimensional varieties, copies of $\text{Diff}_d(1)$.
- ψ_{ref} is B_L-equivariant.
- π is a principal B_L-fibration, while $\pi_{\widetilde{\mathcal{E}}}$ is an $\widetilde{\mathcal{E}}_{\text{ref}}$-fibration.

This allows us to formulate our model as follows.

Proposition 4.17.

(1) *The map* ev_S *establishes a birational surjection*

$$\text{ev}_S : \tau^{-1}\left(\overline{\psi_{\widetilde{\mathcal{E}}}(\text{Hom}^{\text{inj}}(\mathbb{C}^d,\mathbb{C}^n))}\right) \twoheadrightarrow \overline{\Theta}_d.$$

(2) $\widetilde{\mathcal{E}}_{\text{ref}}$ *is a smooth projective variety endowed with a* B_d*-action.*

(3) *The subset* $\psi_{\text{ref}}(\pi^{-1}(\mathbf{f}_{\text{ref}})) \subset \widetilde{\mathcal{E}}_{\text{ref}}$*, the part of* $\psi_{\widetilde{\mathcal{E}}}(\text{Hom}^{\text{inj}}(\mathbb{C}^d,\mathbb{C}^n))$ *lying over* $\mathbf{f}_{\text{ref}}$*, is a* B_d*-orbit in* $\widetilde{\mathcal{E}}_{\text{ref}}$ *of dimension* $\binom{d}{2}$*.*

Proof. The first statement follows from Proposition 4.7 and Lemma 4.15, while the second follows from Lemma 4.14 (cf. (4.23)). For the last statement, observe that according to (4.26), the subvariety $\psi(\pi^{-1}(\mathbf{f}_{\text{ref}}))$ is a B_L-orbit, which, according to Corollary 4.9, has a d-dimensional stabilizer. □

5. Application of the localization formulas

Recall that our aim is the computation Thom polynomial $\text{Tp}_d^{n\to k}$, which we defined as the equivariant Poincaré dual of the subvariety $\Theta_d \subset \mathcal{J}_d(n,k)$, representing the A_d-singularity (cf. Definition 2.9). The symmetry group of the problem is the product of matrix groups $\text{GL}_n \times \text{GL}_k$. The respective subgroups of diagonal matrices are T_n with weights $(\lambda_1,\dots,\lambda_n)$ and T_k with weights $(\theta_1, \dots, \theta_k)$; hence $\mathrm{eP}[\overline{\Theta}_d, \mathcal{J}_d(n,k)]_{T_n \times T_k}$ is a bisymmetric polynomial in these two sets of variables.

In this section, we apply the localization techniques of Section 3 to the computation of $\mathrm{eP}[\overline{\Theta}_d, \mathcal{J}_d(n,k)]_{T_n \times T_k}$ using the model described in Section 4. As our model is a double fibration, the application of the localization formula is a 2-step process.

Before we proceed, we set the following convention. When describing the action of B_L on the B_R-quotient $\widetilde{\mathcal{E}}$, we will revert to the notation B_d, since here only one copy of the Borel group acts.

5.1. *Localization in* $\mathrm{Flag}_d(\mathbb{C}^n)$. The model of Proposition 4.17 is an equivariant fibration over the smooth homogeneous space $\mathrm{Flag}_d(\mathbb{C}^n)$; hence, in this case, we can use Proposition 3.2 with $M = Z$, which applies when the fibers of S are not necessarily linear and smooth. The result of our calculation is Proposition 5.3 below.

The data needed for formula (3.6) is

- the fixed point set of the T_n-action on $\mathrm{Flag}_d(\mathbb{C}^n)$,
- the weights of this action on the tangent spaces $\mathrm{T}_p\mathrm{Flag}_d(\mathbb{C}^n)$ at these fixed points,
- the equivariant Poincaré duals of the fibers at these fixed points.

The following general statement will be helpful in organizing our fixed point data. Its proof is straightforward and will be omitted.

Lemma 5.1. *Assume that the torus action in Proposition 3.2 is obtained by a restriction of a* GL_n*-action to its subgroup of diagonal matrices* T_n*. Then the Weyl group of permutation matrices* $\mathcal{S}_n$ *acts on* M^{T_n}*, and we have*

$$\mathrm{eP}[S_{\sigma\cdot p}, W]_{T_n} = \sigma \cdot \mathrm{eP}[S_p, W]_{T_n} \text{ and } \mathrm{Euler}^{T_n}(\mathrm{T}_{\sigma\cdot p}M) = \sigma \cdot \mathrm{Euler}^{T_n}(\mathrm{T}_p M)$$

for all $\sigma \in \mathcal{S}_n$ *and* $p \in M_{T_n}$.

Our situation is fortunate in the sense that the action of $\mathcal{S}_n$ on the fixed point set is transitive. Indeed, the fixed point set $\mathrm{Flag}_d(\mathbb{C}^n)^{T_n}$ is the set of partial flags obtained from sequences of d elements of the basis $(e_1, \dots, e_n)$ of $\mathbb{C}^n$; in particular, $|\mathrm{Flag}_d(\mathbb{C}^n)^{T_n}| = n(n-1)\dots(n-d+1)$.

Recall the notation $\mathbf{f}_{\mathrm{ref}}$ for the reference flag associated to the sequence $(e_1, \dots, e_d)$. The stabilizer subgroup of $\mathbf{f}_{\mathrm{ref}}$ in $\mathcal{S}_n$ is the subgroup $\mathcal{S}_{n-d}$ permuting the numbers starting with $d+1$, and the map $\sigma \mapsto \sigma \cdot \mathbf{f}_{\mathrm{ref}}$ induces a bijection between $\mathrm{Flag}_d(\mathbb{C}^n)^{T_n}$ and the quotient $\mathcal{S}_n/\mathcal{S}_{n-d}$.

According to Lemma 5.1, it is sufficient to compute the equivariant Poincaré dual of the fiber and the weights of the tangent space at the reference flag $\mathbf{f}_{\mathrm{ref}}$. The weights of $\mathrm{T}_{\mathbf{f}_{\mathrm{ref}}}\mathrm{Flag}_d(\mathbb{C}^n)$ are well known:

$$\{\lambda_i - \lambda_m;\ 1 \le m \le d,\ m < i \le n\};$$

the weights at the other fixed points are obtained by applying the corresponding permutation to this set.

The numerators of the summands of (3.6) in our case are much harder to compute, although, thanks to Lemma 5.1, it suffices to compute the numerator for the fixed point $\mathbf{f}_{\mathrm{ref}}$. The situation over $\mathbf{f}_{\mathrm{ref}}$ is presented in the following diagram:

$$\begin{array}{ccc} & S_{\mathrm{ref}} & \xrightarrow{\mathrm{ev}_S} \mathcal{J}_d(n,k) \\ & \Big\downarrow \tau & \\ \mathcal{O} = \overline{B_d \tilde{\varepsilon}_{\mathrm{ref}}} & \hookrightarrow & \widetilde{\mathcal{E}}_{\mathrm{ref}}. \end{array} \tag{5.1}$$

The fiber of our model over the fixed point $\mathbf{f}_{\mathrm{ref}}$ is the set $\tau^{-1}(\mathcal{O})$, where we introduced the notation $\mathcal{O}$ for the closure of the B_d-orbit of $\tilde{\varepsilon}_{\mathrm{ref}}$. Using this notation, we can write the numerator of the term corresponding to $\mathbf{f}_{\mathrm{ref}}$ in the sum (3.6) as follows:

$$\mathrm{eP}\left[\mathrm{ev}_S\left(\tau^{-1}(\mathcal{O})\right), \mathcal{J}_d(n,k)\right]_{T_n \times T_k}. \tag{5.2}$$

Recall that this is a polynomial in two sets of variables: $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$ and $\boldsymbol{\theta} = (\theta_1, \dots, \theta_k)$. Since $\mathcal{O}$ is invariant under B_d only, this polynomial is not necessarily symmetric in the λs. The following statement is straightforward.

LEMMA 5.2. *The equivariant Poincaré dual* (5.2) *does not depend on the last $n-d$ basic weights $\lambda_{d+1}, \dots, \lambda_n$.*

Proof. Indeed, recall that $\mathrm{ev}_S\tau^{-1}(B_d\tilde{\varepsilon}_{\mathrm{ref}})$ consists of all possible solutions of the systems of equations of the form $B_L\varepsilon_{\mathrm{ref}}$, and we saw in Section 4.2 that all these systems are in $\mathcal{E}_{\mathrm{ref}}$. The systems of equations in $\mathcal{E}_{\mathrm{ref}}$, however, impose conditions only on those components of Ψ which do not have indices higher than d, and this implies the statement of the lemma. □

As a consequence of Lemma 5.2, the equivariant Poincaré dual (5.2) may be considered as being taken with respect to the group $T_d \times T_k$, which has weights $\mathbf{z} = (z_1, \dots, z_d)$ and $\boldsymbol{\theta} = (\theta_1, \dots, \theta_k)$.

Putting together Lemmas 5.1 and 5.2 and the description of the fixed point set $\mathrm{Flag}_d(\mathbb{C}^n)^{T_d}$ given above, we arrive at the following form of (3.6) applied to our situation.

PROPOSITION 5.3. *We have*

$$\mathrm{eP}[\overline{\Theta}_d, \mathcal{J}_d(n,k)]_{T_n \times T_k} = \sum_{\sigma \in \mathcal{S}_n/\mathcal{S}_{n-d}} \frac{Q_{\mathrm{Flag}}(\lambda_{\sigma\cdot 1}, \dots, \lambda_{\sigma\cdot d}, \boldsymbol{\theta})}{\prod_{1 \le m \le d} \prod_{i=m+1}^{n} (\lambda_{\sigma\cdot i} - \lambda_{\sigma\cdot m})}, \tag{5.3}$$

where

$$Q_{\mathrm{Flag}}(\mathbf{z}, \boldsymbol{\theta}) = \mathrm{eP}\left[\mathrm{ev}_S\left(\tau^{-1}(\mathcal{O})\right), \mathcal{J}_d(n,k)\right]_{T_d \times T_k}. \tag{5.4}$$

5.2. *Residue formula for the cohomology pairings of* $\mathrm{Flag}_d(\mathbb{C}^n)$. Usually, formulas such as (5.3) are difficult to use. They have the form of a finite sum of rational functions, and only after adding up the terms of this sum and performing some cancellations do we obtain a polynomial. These computations often obscure the underlying structures, and they are rather unwieldy as the number of terms of the sum grows very quickly with n and d.

In this paragraph, we derive an efficient residue formula for the right-hand side of (5.3). While the geometric meaning of this formula is not entirely clear, our summation procedure yields an effective, "truly" localized formula. By this we mean that for its evaluation one only needs to know the behavior of a certain function at a single point, rather than at a large, albeit finite number of points.

To describe this formula, we will need the notion of an *iterated residue* (cf., e.g., [27]) at infinity. Let $\omega_1, \dots, \omega_N$ be affine linear forms on $\mathbb{C}^d$. Denoting the coordinates by $z_1, \dots, z_d$, this means that we can write $\omega_i = a_i^0 + a_i^1 z_1 + \dots + a_i^d z_d$. We will use the shorthand $h(\mathbf{z})$ for a function $h(z_1, \dots, z_d)$ and $d\mathbf{z}$ for the holomorphic d-form $dz_1 \wedge \dots \wedge dz_d$. Now, let $h(\mathbf{z})$ be an entire function, and define the *iterated residue at infinity* as follows:

$$\operatorname*{Res}_{z_1=\infty} \dots \operatorname*{Res}_{z_d=\infty} \frac{h(\mathbf{z})\, d\mathbf{z}}{\prod_{i=1}^N \omega_i} \stackrel{\text{def}}{=} \left(\frac{1}{2\pi i}\right)^d \int_{|z_1|=R_1} \dots \int_{|z_d|=R_d} \frac{h(\mathbf{z})\, d\mathbf{z}}{\prod_{i=1}^N \omega_i}, \tag{5.5}$$

where $1 \ll R_1 \ll \dots \ll R_d$. The torus $\{|z_m| = R_m;\ m = 1, \dots, d\}$ is oriented in such a way that $\operatorname{Res}_{z_1=\infty} \dots \operatorname{Res}_{z_d=\infty} d\mathbf{z}/(z_1 \dots z_d) = (-1)^d$.

We will also use the following simplified notation:

$$\operatorname*{Res}_{\mathbf{z}=\infty} \stackrel{\text{def}}{=} \operatorname*{Res}_{z_1=\infty} \operatorname*{Res}_{z_2=\infty} \dots \operatorname*{Res}_{z_d=\infty}.$$

In practice, the iterated residue (5.5) may be computed using the following algorithm. For each i, use the expansion

$$\frac{1}{\omega_i} = \sum_{j=0}^{\infty} (-1)^j \frac{(a_i^0 + a_i^1 z_1 + \dots + a_i^{q(i)-1} z_{q(i)-1})^j}{(a_i^{q(i)} z_{q(i)})^{j+1}}, \tag{5.6}$$

where $q(i)$ is the largest value of m for which $a_i^m \neq 0$, multiply the product of these expressions with $(-1)^d h(z_1, \dots, z_d)$, and then take the coefficient of $z_1^{-1} \dots z_d^{-1}$ in the resulting Laurent series.

We have the following *iterated residue theorem.*

PROPOSITION 5.4. *For a polynomial* $Q(\mathbf{z})$ *on* $\mathbb{C}^d$, *we have*

(5.7)

$$\sum_{\sigma \in \mathcal{S}_n/\mathcal{S}_{n-d}} \frac{Q(\lambda_{\sigma\cdot 1}, \dots, \lambda_{\sigma\cdot d})}{\prod_{1\le m\le d} \prod_{i=m+1}^n (\lambda_{\sigma\cdot i} - \lambda_{\sigma\cdot m})} = \operatorname*{Res}_{\mathbf{z}=\infty} \frac{\prod_{1\le m<l\le d} (z_m - z_l)\, Q(\mathbf{z})\, d\mathbf{z}}{\prod_{l=1}^d \prod_{i=1}^n (\lambda_i - z_l)}.$$

Proof. We compute the iterated residue (5.7) using the Residue Theorem on the projective line $\mathbb{C}\cup\{\infty\}$. The first residue, which is taken with respect to z_d, is a contour integral, whose value is minus the sum of the z_d-residues of the form in (5.7). These poles are at $z_d=\lambda_j$, $j=1,\dots,n$, and after canceling the signs that arise, we obtain the following expression for the right-hand side of (5.7):

$$\sum_{j=1}^{n}\frac{\prod_{1\le m<l\le d-1}(z_m-z_l)\prod_{l=1}^{d-1}(z_l-\lambda_j)\,Q(z_1,\dots,z_{d-1},\lambda_j)\,dz_1\dots dz_{d-1}}{\prod_{l=1}^{d-1}\prod_{i=1}^{n}(\lambda_i-z_l)\prod_{i\neq j}^{n}(\lambda_i-\lambda_j)}.$$

After cancellation and exchanging the sum and the residue operation, at the next step, we have

$$(-1)^{d-1}\sum_{j=1}^{n}\operatorname*{Res}_{z_{d-1}=\infty}\frac{\prod_{1\le m<l\le d-1}(z_m-z_l)\,Q(z_1,\dots,z_{d-1},\lambda_j)\,dz_1\dots dz_{d-1}}{\prod_{i\neq j}^{n}\left((\lambda_i-\lambda_j)\prod_{l=1}^{d-1}(\lambda_i-z_l)\right)}.$$

Now we again apply the Residue Theorem, with the only difference being that now the pole $z_{d-1}=\lambda_j$ has been eliminated. As a result, after converting the second residue to a sum, we obtain

$$(-1)^{2d-3}\sum_{j=1}^{n}\sum_{s=1,\,s\neq j}^{n}\frac{\prod_{1\le m<l\le d-2}(z_l-z_m)\,Q(z_1,\dots,z_{d-2},\lambda_s,\lambda_j)\,dz_1\dots dz_{d-2}}{(\lambda_s-\lambda_j)\prod_{i\neq j,s}^{n}\left((\lambda_i-\lambda_j)(\lambda_i-\lambda_s)\prod_{l=1}^{d-1}(\lambda_i-z_l)\right)}.$$

Iterating this process, we arrive at a sum very similar to (5.3). The differences between the two sums will be the sign $(-1)^{d(d-1)/2}$ and that the $d(d-1)/2$ factors of the form $(\lambda_{\sigma(i)}-\lambda_{\sigma(m)})$ with $1\le m<i\le d$ in the denominator will have opposite signs. These two differences cancel each other, and this completes the proof. □

Remark 5.5. Changing the order of the variables in iterated residues, usually, changes the result. In this case, however, because all the poles are normal crossing, formula (5.7) remains true no matter in what order we take the iterated residues.

5.3. *Localization in the fiber.* Combining Proposition 5.3 with Proposition 5.4, we arrive at the formula

$$\text{(5.8)}\qquad \mathrm{eP}[\overline{\Theta}_d,\mathcal{J}_d(n,k)]_{T_n\times T_k}=\operatorname*{Res}_{\mathbf{z}=\infty}\frac{\prod_{1\le m<l\le d}(z_m-z_l)\,Q_{\mathrm{Flag}}(\mathbf{z},\boldsymbol{\theta})\,d\mathbf{z}}{\prod_{l=1}^{d}\prod_{i=1}^{n}(\lambda_i-z_l)}.$$

The "only" unknown here is the polynomial $Q_{\mathrm{Flag}}(\mathbf{z},\boldsymbol{\theta})$ defined in (5.4), and, therefore, we now turn to its computation.

Let us briefly review the construction of $Q_{\mathrm{Flag}}(\mathbf{z},\boldsymbol{\theta})$ (cf. diagram (5.1) and Proposition 5.3). This polynomial is an equivariant Poincaré dual taken with respect to the group $T_d\times T_k$, which has weights $(z_1,\dots,z_d)$ and $(\theta_1,\dots,\theta_k)$. Consider the $B_L\times B_R$-module $\mathrm{Hom}^{\triangle}(\mathbb{C}^d_R,\mathrm{Sym}^{\bullet}_{\mathrm{ref}}(\mathbb{C}^n_L))$ and endow it with coordinates $u^l_\tau\in\mathrm{Hom}^{\triangle}(\mathbb{C}^d_R,\mathrm{Sym}^{\bullet}_{\mathrm{ref}}(\mathbb{C}^n_L))^*$, indexed by pairs $(\tau,l)\in\Pi\times\mathbb{Z}_{>0}$

satisfying $\mathrm{sum}(\tau) \le l \le d$. We will consider the dual space spanned by these coordinates as carrying a *right* action of $T_d \times T_k$; accordingly,

(5.9) the weight of $u^l_\tau = (z_{i_1} + z_{i_2} + \cdots + z_{i_m}, \theta_l)$, where $\tau = [i_1, i_2, \ldots, i_m]$.

For each nondegenerate system $\varepsilon \in \mathcal{E}_{\mathrm{ref}} \subset \mathrm{Hom}^{\triangle}(\mathbb{C}^d_R, \mathrm{Sym}^\bullet_{\mathrm{ref}}(\mathbb{C}^n_L))$, we denote the image $\mathrm{pr}_{\mathrm{ref}}(\varepsilon)$ in the quotient $\mathrm{pr}_{\mathrm{ref}} : \mathcal{E}_{\mathrm{ref}} \to \widetilde{\mathcal{E}}_{\mathrm{ref}} = \mathcal{E}/B_R$ by $\tilde{\varepsilon}$; in particular, we have a reference point $\tilde{\varepsilon}_{\mathrm{ref}} \in \widetilde{\mathcal{E}}_{\mathrm{ref}}$ corresponding to the system $\varepsilon_{\mathrm{ref}}$ given by

$$u^l_\pi(\varepsilon_{\mathrm{ref}}) = \begin{cases} 1, & \text{if } \mathrm{sum}(\pi) = l, \\ 0, & \text{otherwise.} \end{cases} \tag{5.10}$$

Next, consider the vector bundle

$$V = \mathcal{E}_{\mathrm{ref}} \times_{B_R} \mathbb{C}^d_R \longrightarrow \widetilde{\mathcal{E}}_{\mathrm{ref}} = \mathcal{E}_{\mathrm{ref}}/B_R$$

associated to the standard representation of B_R. We define a $T_d \times T_k$-equivariant linear bundle map from a trivial bundle

$$s : \widetilde{\mathcal{E}}_{\mathrm{ref}} \times \mathcal{J}_d(n,k) \longrightarrow V^* \otimes \mathbb{C}^k$$

as follows. Let

$$\varepsilon \in \widetilde{\mathcal{E}}_{\mathrm{ref}} = \mathrm{Flag}_d(\mathrm{Sym}^\bullet_d \mathbb{C}^n) = \mathrm{Flag}_d(\mathcal{J}_d(n)^*) = \mathrm{Hom}(\mathbb{C}^d, \mathcal{J}_d(n)^*)/B_d$$

be a point of the base, and $\alpha \in \mathcal{J}_d(n,k) = \mathcal{J}_d(n) \otimes \mathbb{C}^k$. Then the canonical pairing of $\mathcal{J}_d(n)^*$ and $\mathcal{J}_d(n)$ gives us an element $(\varepsilon, \alpha) \in V^* \otimes \mathbb{C}^k$. By Proposition 4.7, the bundle S fits into the short exact sequence

$$0 \longrightarrow S \xrightarrow{\mathrm{ev}_S} \mathcal{J}_d(n,k) \xrightarrow{s} V^* \times \mathbb{C}^k \longrightarrow 0,$$

and the polynomial $Q_{\mathrm{Flag}}(\mathbf{z}, \boldsymbol{\theta})$ is the *equivariant Poincaré dual in* $\mathcal{J}_d(n,k)$ *of the union of the vector spaces* $\ker(s)$ *lying over* $\mathcal{O} \subset \widetilde{\mathcal{E}}_{\mathrm{ref}}$ (cf. (5.4)).

While the variety $\mathcal{O}$ is highly singular, the set of T_d-fixed points of $\mathcal{O}$ is finite — as we will see shortly — and hence we can apply here Proposition 3.2 with $M = \mathcal{O}$ and $Z = \widetilde{\mathcal{E}}_{\mathrm{ref}}$. The result is

$$Q_{\mathrm{Flag}}(\mathbf{z}, \boldsymbol{\theta}) = \sum_{p \in \mathcal{O}^{T_d}} \frac{\mathrm{Euler}^{T_d \times T_k}(V^*_p \otimes \mathbb{C}^k)\,\mathrm{emult}_p[\mathcal{O}, \widetilde{\mathcal{E}}_{\mathrm{ref}}]}{\mathrm{Euler}^{T_d \times T_k}(\mathrm{T}_p \widetilde{\mathcal{E}}_{\mathrm{ref}})}. \tag{5.11}$$

Our task thus has reduced to the identification and computation of the objects in this formula. These are

- the set $\mathcal{O}^{T_d}$ of T_d-fixed points in $\mathcal{O} \subset \widetilde{\mathcal{E}}_{\mathrm{ref}}$,
- the weights of the T_d-action on the fibers V_p for $p \in \mathcal{O}^{T_d}$,
- the weights of the T_d-action on the tangent spaces $\mathrm{T}_p \widetilde{\mathcal{E}}_{\mathrm{ref}}$ for $p \in \mathcal{O}^{T_d}$,
- the equivariant multiplicities of $\mathcal{O}$ in $\widetilde{\mathcal{E}}$ at each fixed point $p \in \mathcal{O}^{T_d}$.

The most immediate problem we face is that we do not have an effective description of the set $\mathcal{O}^{T_d}$ of T_d-fixed points in $\mathcal{O}$. There is a formal way around this: We replace the fixed point set $\mathcal{O}^{T_d}$ with the larger set $\widetilde{\mathcal{E}}_{\mathrm{ref}}^{T_d}$ and define the equivariant multiplicity $\operatorname{emult}_p[\mathcal{O}, \widetilde{\mathcal{E}}_{\mathrm{ref}}]$ to be zero in the case when $p \in \widetilde{\mathcal{E}}_{\mathrm{ref}}^{T_d} \setminus \mathcal{O}^{T_d}$.

The fixed point set $\widetilde{\mathcal{E}}_{\mathrm{ref}}^{T_d}$ is fairly easy to determine: The fixed points correspond to those nondegenerate systems $\varepsilon \in \mathcal{E}_{\mathrm{ref}} \subset \mathrm{Hom}^{\triangle}(\mathbb{C}_R^d, \mathrm{Sym}_{\mathrm{ref}}^{\bullet}(\mathbb{C}_L^n))$ for which the tensors $\varepsilon(e_m) \in \mathrm{Sym}_{\mathrm{ref}}^{\bullet}(\mathbb{C}_L^n)$, $m = 1, \dots, d$ are of pure T_d-weight. These, in turn, may be enumerated as follows.

Definition 5.6. We will call a sequence of partitions $\boldsymbol{\pi} = (\pi_1, \dots, \pi_d) \in \Pi^{\times d}$ *admissible* if

(1) $\mathrm{sum}(\pi_l) \leq l$ for $l = 1, \dots, d$ and
(2) $\pi_l \neq \pi_m$ for $1 \leq l \neq m \leq d$.

We will denote the set of admissible sequences of length d by $\mathbf{\Pi}_d$; we also introduce the numerical characteristic

$$\mathrm{defect}(\boldsymbol{\pi}) = \sum_{l=1}^{d} (l - \mathrm{sum}(\pi_l)).$$

As an example, we list the admissible sequences in the case $d = 3$:

$$\begin{aligned}\mathbf{\Pi}_3 = \{&([1],[2],[3]),\ ([1],[2],[1,2]),\ ([1],[2],[1,1]),\ ([1],[2],[1,1,1])\\ &([1],[1,1],[3]),\ ([1],[1,1],[1,1,1]),\ ([1],[1,1],[2]),\ ([1],[1,1],[1,2])\}.\end{aligned}$$

For $\boldsymbol{\pi} = (\pi_1, \dots, \pi_d) \in \mathbf{\Pi}_d$, introduce the system $\varepsilon_{\boldsymbol{\pi}}$ given by

$$u_\tau^l(\varepsilon_{\boldsymbol{\pi}}) = \begin{cases} 1, & \text{if } \tau = \pi_l, \\ 0, & \text{otherwise.} \end{cases} \tag{5.12}$$

As usual, the point corresponding to $\varepsilon_{\boldsymbol{\pi}}$ in $\widetilde{\mathcal{E}}_{\mathrm{ref}}$ will be denoted by $\tilde{\varepsilon}_{\boldsymbol{\pi}} = \mathrm{pr}_{\mathrm{ref}}(\varepsilon_{\boldsymbol{\pi}})$.

The following statement follows from the definitions.

LEMMA 5.7.
- *The correspondence $\boldsymbol{\pi} \mapsto \tilde{\varepsilon}_{\boldsymbol{\pi}}$ establishes a bijection between the set $\mathbf{\Pi}_d$ of admissible sequences of partitions and the fixed point set $\widetilde{\mathcal{E}}_{\mathrm{ref}}^{T_d}$.*
- *For $\tau \in \Pi$ and an integer i, denote by $\mathrm{mult}(i, \tau)$ the number of times i occurs in τ and let $z_\tau = \sum_{i \in \tau} \mathrm{mult}(i, \tau)\, z_i$. Then, given an admissible sequence $\boldsymbol{\pi} \in \mathbf{\Pi}_d$, the weights of the T_d-action on the fiber of V at the fixed point $\tilde{\varepsilon}_{\boldsymbol{\pi}}$ are*

$$z_{\pi_1}, \dots, z_{\pi_d}.$$

Corollary 5.8. *The weights of the $T_d \times T_k$ action on fiber $V^*_{\tilde{\varepsilon}_{\boldsymbol{\pi}}} \otimes \mathbb{C}^k$ are*

$$\{\theta_j - z_{\pi_m};\ m = 1, \ldots, d,\ j = 1, \ldots, k\}.$$

Next we turn to the 3rd item on our list: the weights of the T_d-action on tangent space of $\widetilde{\mathcal{E}}_{\mathrm{ref}}$ at the fixed points $\tilde{\varepsilon}_{\boldsymbol{\pi}}$. We will use the simplified notation $\mathrm{T}_{\boldsymbol{\pi}}\widetilde{\mathcal{E}}_{\mathrm{ref}}$ for this tangent space. To compute the answer, it will be convenient to linearize the action near $\tilde{\varepsilon}_{\boldsymbol{\pi}}$.

Definition 5.9. For each $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_d) \in \boldsymbol{\Pi}_d$, introduce the affine-linear subspace $\mathcal{N}_{\boldsymbol{\pi}} \subset \mathrm{Hom}^{\triangle}(\mathbb{C}^d_R, \mathrm{Sym}^{\bullet}_d\mathbb{C}^n)$ given by

$$\mathcal{N}_{\boldsymbol{\pi}} = \left\{ \varepsilon \in \mathrm{Hom}^{\triangle}(\mathbb{C}^d_R, \mathrm{Sym}^{\bullet}_d\mathbb{C}^n);\ u^m_{\pi_l}(\varepsilon) = \begin{cases} 1, & \text{if } m = l \\ 0, & \text{if } m > l \end{cases} \text{ for } 1 \le l \le d \right\}.$$

Also, for $\boldsymbol{\pi} \in \boldsymbol{\Pi}_d$, introduce the map

$$\alpha_{\boldsymbol{\pi}} : \mathrm{Hom}^{\triangle}(\mathbb{C}^d_R, \mathrm{Sym}^{\bullet}_d\mathbb{C}^n) \to \mathrm{Mat}^{d\times d}$$

which associates to each system ε its $d \times d$ minor corresponding to the sequence of partitions $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_d)$.

A few comments are in order. First, we can rewrite the above definition of $\mathcal{N}_{\boldsymbol{\pi}}$ as follows:

$$\mathcal{N}_{\boldsymbol{\pi}} = \left\{ \varepsilon \in \mathrm{Hom}^{\triangle}(\mathbb{C}^d_R, \mathrm{Sym}^{\bullet}_d\mathbb{C}^n);\ \alpha_{\boldsymbol{\pi}}(\varepsilon) \in U_- \right\}, \tag{5.13}$$

where U_- is the subgroup of lower-triangular $d \times d$ matrices with 1s on the diagonal; this way it is apparent that $\mathcal{N}_{\boldsymbol{\pi}} \subset \mathcal{E}_{\mathrm{ref}}$.

Also, observe that $\varepsilon_{\boldsymbol{\pi}} \in \mathcal{N}_{\boldsymbol{\pi}}$. Considering this special point to be the origin, we may think of $\mathcal{N}_{\boldsymbol{\pi}}$ as a *linear* space. Then $\mathcal{N}_{\boldsymbol{\pi}}$ is endowed with a natural set of coordinates:

$$\hat{u}^l_{\tau|\boldsymbol{\pi}} = u^l_\tau | \mathcal{N}_{\boldsymbol{\pi}},\ \mathrm{sum}(\tau) \le l \le d,\ \tau \ne \pi_1, \ldots, \pi_l. \tag{5.14}$$

Proposition 5.10. *Let $\boldsymbol{\pi} \in \boldsymbol{\Pi}_d$ be an admissible sequence of partitions. Then*

1. *The restriction of the projection $\mathrm{pr}_{\mathrm{ref}} : \mathcal{E}_{\mathrm{ref}} \to \widetilde{\mathcal{E}}_{\mathrm{ref}}$ to $\mathcal{N}_{\boldsymbol{\pi}}$ is an embedding and the collection $\{\mathrm{pr}_{\mathrm{ref}}(\mathcal{N}_{\boldsymbol{\pi}});\ \boldsymbol{\pi} \in \boldsymbol{\Pi}_d\}$ forms an open cover of $\widetilde{\mathcal{E}}_{\mathrm{ref}}$.*
2. *For* any *$\boldsymbol{\pi} \in \boldsymbol{\Pi}_d$, the image $\mathrm{pr}_{\mathrm{ref}}(\mathcal{N}_{\boldsymbol{\pi}}) \subset \widetilde{\mathcal{E}}_{\mathrm{ref}}$ is T_d-invariant and the induced T_d-action on $\mathcal{N}_{\boldsymbol{\pi}}$ is linear and diagonal with respect to the coordinates* (5.14). *Considering T_d as acting on the* right *on these coordinates,*

$$\text{the weight of } \hat{u}^l_{\tau|\boldsymbol{\pi}} = z_\tau - z_{\pi_l}. \tag{5.15}$$

3. *If* defect$(\boldsymbol{\pi}) = 0$, *then $\mathrm{pr}_{\mathrm{ref}}(\mathcal{N}_{\boldsymbol{\pi}}) \subset \widetilde{\mathcal{E}}_{\mathrm{ref}}$ is B_d- invariant.*

Remark 5.11. We will denote by $T_{\boldsymbol{\pi}}$ and $B_{\boldsymbol{\pi}}$ the actions of T_d and B_d induced on $N_{\boldsymbol{\pi}}$ by the embedding $\mathrm{pr}_{\mathrm{ref}}$.

Proof. We first show that $\cup\{\mathrm{pr}_{\mathrm{ref}}(\mathcal{N}_{\boldsymbol{\pi}});\ \boldsymbol{\pi}\in\mathbf{\Pi}_d\}=\widetilde{\mathcal{E}}_{\mathrm{ref}}$. This means that for an arbitrary element $\varepsilon\in\mathcal{E}_{\mathrm{ref}}$, we have to find an admissible partition $\boldsymbol{\pi}\in\mathbf{\Pi}_d$ and an upper-triangular matrix $b_R=b_R(\varepsilon,\boldsymbol{\pi})\in B_R$ such that $\varepsilon\cdot b_R\in\mathcal{N}_{\boldsymbol{\pi}}$. This can be done by elementary column operations. Consider ε as a $\dim(\mathrm{Sym}^{\bullet}_{\mathrm{ref}}(\mathbb{C}^n_L))\times d$ matrix whose columns are linearly independent and whose rows are indexed by partitions. The only nonzero entry in the first column corresponds to the trivial partition [1]; hence, we can multiply the first column by a constant to rescale this entry to 1 and then annihilate all other entries in the same row by adding multiples of the first column to the others. Next, since ε is nonsingular, we can pick a nonzero entry in the second column of the resulting matrix — this entry will correspond to a partition π_2 — and, again, using column operations, we annihilate all entries in this row starting form column 3 and so on. Continuing this process, we obtain an admissible $\boldsymbol{\pi}=(\pi_1,\dots,\pi_d)$, and the described sequence of column operations produces an upper-triangular $b_R\in B_R$ such that $\varepsilon\cdot b_R\in\mathcal{N}_{\boldsymbol{\pi}}$.

The process described above finds an appropriate $\boldsymbol{\pi}\in\mathbf{\Pi}_d$ for each ε and brings $\alpha_{\boldsymbol{\pi}}(\varepsilon)$ to lower-triangular form. Moreover, if $\mathrm{pr}_{\mathrm{ref}}(\varepsilon_1)=\mathrm{pr}_{\mathrm{ref}}(\varepsilon_2)$ for $\varepsilon_1,\varepsilon_2\in\mathcal{N}_{\boldsymbol{\pi}}$, then $\varepsilon_1\cdot b_R=\varepsilon_2$ for some $b_R\in B_R$, and therefore $\alpha_{\boldsymbol{\pi}}(\varepsilon_1)\cdot b_R=\alpha_{\boldsymbol{\pi}}(\varepsilon_2)$. Since $\alpha_{\boldsymbol{\pi}}(\varepsilon_1),\alpha_{\boldsymbol{\pi}}(\varepsilon_2)$ are lower-triangular with 1s on the diagonal and B_R is upper-triangular, this can only happen when b_R is the unit matrix, so $\varepsilon_1=\varepsilon_2$. This proves that $\mathrm{pr}_{\mathrm{ref}}$ is injective on $\mathcal{N}_{\boldsymbol{\pi}}$; hence the restriction $\mathrm{pr}_{\mathrm{ref}}|\mathcal{N}_{\boldsymbol{\pi}}$ is an embedding.

To approach statements (2) and (3), we write down the action of B_d on $\widetilde{\mathcal{E}}$ in the chart $\mathcal{N}_{\boldsymbol{\pi}}$. Recall that the multiplication map $U_-\times B_d\to\mathrm{GL}_d$ is injective. This allows us to define the B_d-component a^B for an element $a\in U_-B_d$; in particular, for any such a, we have $a\cdot(a^B)^{-1}\in U_-$. Then, for $b\in B_d$ and $\varepsilon\in\mathcal{N}_{\boldsymbol{\pi}}$, we can define the partial action

$$(b,\varepsilon)\mapsto b_{\boldsymbol{\pi}}\varepsilon=b_L\cdot\varepsilon\cdot(\alpha_{\boldsymbol{\pi}}(b_L\cdot\varepsilon)^B)^{-1}, \tag{5.16}$$

which is valid if $\alpha_{\boldsymbol{\pi}}(b_L\cdot\varepsilon)\in U_-B_d$.

Now consider the case when $b=t\in T_d$ is a diagonal matrix. In this case, $\alpha_{\boldsymbol{\pi}}(b_L\cdot\varepsilon)$ remains lower-triangular, with the numbers $(t^{\pi_1},\dots,t^{\pi_d})$ on the diagonal, where t^τ is the character of T_d corresponding to the weight z_τ. This means that $\alpha_{\boldsymbol{\pi}}(b_L\cdot\varepsilon)\in U_-B_d$, and the Borel factor $\alpha_{\boldsymbol{\pi}}(b_L\cdot\varepsilon)^B$ is the diagonal matrix with these same entries:

$$\alpha_{\boldsymbol{\pi}}(b_L\cdot\varepsilon)^B=\mathrm{diag}[t^{\pi_1},\dots,t^{\pi_d}]. \tag{5.17}$$

Note that this matrix is independent of ε. Now statement (2) follows easily.

Finally, to prove (3), observe that if $\operatorname{defect}(\boldsymbol{\pi}) = 0$, then the filtration-preserving property implies that $\alpha_{\boldsymbol{\pi}}(\varepsilon)$ is upper-triangular for any $\varepsilon \in \operatorname{Hom}^{\triangle}(\mathbb{C}^d_R, \operatorname{Sym}^\bullet_d \mathbb{C}^n)$. Hence for $\varepsilon \in \mathcal{N}_{\boldsymbol{\pi}}$, the matrix $\alpha_{\boldsymbol{\pi}}(\varepsilon)$ is the identity matrix, and thus, using the condition $\operatorname{defect}(\boldsymbol{\pi}) = 0$ once again, we can conclude that $\alpha_{\boldsymbol{\pi}}(b_L \cdot \varepsilon)$ is upper-triangular with the numbers $(t^{\pi_1}, \dots, t^{\pi_d})$ on the diagonal, where t is the diagonal part of b. This means that $\alpha_{\boldsymbol{\pi}}(b_L \cdot \varepsilon)^B = \alpha_{\boldsymbol{\pi}}(b_L \cdot \varepsilon) \in B_d$, which implies statement (3). □

Remark 5.12. Clearly, $\alpha_{\boldsymbol{\pi}}(b_L \cdot \varepsilon)$ depends linearly on ε. In the case $\operatorname{defect}(\boldsymbol{\pi}) = 0$, we have $\alpha_{\boldsymbol{\pi}}(b_L \cdot \varepsilon)^B = \alpha_{\boldsymbol{\pi}}(b_L \cdot \varepsilon)$, and hence the action (5.16) of $B_{\boldsymbol{\pi}}$ on $\mathcal{N}_{\boldsymbol{\pi}}$ is quadratic, not linear as the $T_{\boldsymbol{\pi}}$-action. When $\operatorname{defect}(\boldsymbol{\pi}) > 0$, the action of $B_{\boldsymbol{\pi}}$ is not defined on the whole of $\mathcal{N}_{\boldsymbol{\pi}}$.

Proposition 5.10 provides us with a linearization of the T_d-action on $\widetilde{\mathcal{E}}_{\mathrm{ref}}$ near every fixed point. This allows us to compute equivariant multiplicities in (5.11) using (3.1). Indeed, if we introduce the notation

$$\mathcal{O}_{\boldsymbol{\pi}} \stackrel{\mathrm{def}}{=} (\mathrm{pr}_{\mathrm{ref}}|\mathcal{N}_{\boldsymbol{\pi}})^{-1}(\mathcal{O}) \tag{5.18}$$

or the part of $\mathcal{O}$ in the local chart $\mathcal{N}_{\boldsymbol{\pi}}$, then we can write

$$\operatorname{emult}_{\tilde{\varepsilon}_{\boldsymbol{\pi}}}[\mathcal{O}, \widetilde{\mathcal{E}}_{\mathrm{ref}}] = \operatorname{eP}[\mathcal{O}_{\boldsymbol{\pi}}, \mathcal{N}_{\boldsymbol{\pi}}]_{T_d}. \tag{5.19}$$

Next, we take a closer look at the set $\mathcal{O}_{\boldsymbol{\pi}}$.

Lemma 5.13. *For every $\boldsymbol{\pi} \in \boldsymbol{\Pi}_d$, we have*

$$\mathcal{O}_{\boldsymbol{\pi}} = \overline{B_L \varepsilon_{\mathrm{ref}} B_R} \cap \mathcal{N}_{\boldsymbol{\pi}}. \tag{5.20}$$

Moreover, $\varepsilon_{\mathrm{ref}} \in \mathcal{N}_{\boldsymbol{\pi}}$ if and only if $\operatorname{defect}(\boldsymbol{\pi}) = 0$, *and in this case $\mathcal{O}_{\boldsymbol{\pi}} = \overline{B_{\boldsymbol{\pi}} \varepsilon_{\mathrm{ref}}}$, where $B_{\boldsymbol{\pi}}$ stands for the action* (5.16).

Proof. By definition, $\mathcal{O}_{\boldsymbol{\pi}} = \overline{B_L \varepsilon_{\mathrm{ref}} B_R \cap N_{\boldsymbol{\pi}}}$, and hence (5.20) follows from the fact that B_d acts properly on the right on $U_- B_d \subset \mathrm{GL}_d$. The second statement then immediately follows from the comparison of (5.10) and Definition 5.9. □

Let us take stock of our results so far. Substituting the weights from Corollary 5.8 and (5.15) into (5.11), and taking into consideration (5.19), we obtain

$$Q_{\mathrm{Flag}}(\boldsymbol{\lambda}, \boldsymbol{\theta}) = \sum_{\boldsymbol{\pi} \in \boldsymbol{\Pi}_d} \frac{\prod_{m=1}^{d} \prod_{j=1}^{k} (\theta_j - z_{\pi_m})\, Q_{\boldsymbol{\pi}}(z_1, \dots, z_d)}{\displaystyle\prod_{l=1}^{d} \prod_{\substack{\tau \neq \pi_1, \dots, \pi_l \\ \mathrm{sum}(\tau) \leq l}} (z_\tau - z_{\pi_l})}, \tag{5.21}$$

where

$$Q_{\boldsymbol{\pi}} = \begin{cases} \operatorname{eP}[\mathcal{O}_{\boldsymbol{\pi}}, \mathcal{N}_{\boldsymbol{\pi}}]_{T_d}, & \text{if } \tilde{\varepsilon}_{\boldsymbol{\pi}} \in \mathcal{O}, \\ 0, & \text{if } \tilde{\varepsilon}_{\boldsymbol{\pi}} \notin \mathcal{O}. \end{cases} \tag{5.22}$$

Combining this formula with (5.3) and (5.7), we arrive at our first expression for $\mathrm{Tp}_d^{n\to k}$, which we defined as $\mathrm{eP}[\overline{\Theta}_d, \mathcal{J}_d(n,k)]_{T_n\times T_k}$:

$$\mathrm{Tp}_d^{n\to k}(\mathbf{z},\boldsymbol{\theta}) = \operatorname*{Res}_{\mathbf{z}=\infty} \frac{\prod_{m<l}(z_m - z_l)\, d\mathbf{z}}{\prod_{l=1}^{d}\prod_{i=1}^{n}(\lambda_i - z_l)} \cdot \sum_{\boldsymbol{\pi}\in\boldsymbol{\Pi}_d} \frac{\prod_{m=1}^{d}\prod_{j=1}^{k}(\theta_j - z_{\pi_m})\, Q_{\boldsymbol{\pi}}(\mathbf{z})}{\prod_{l=1}^{d}\prod\{(z_\tau - z_{\pi_l});\ \mathrm{sum}(\tau)\le l,\ \tau\neq\pi_1,\dots,\pi_l\}}. \tag{5.23}$$

This sum is finite; hence we are free to exchange the summation with the residue operation. Rearranging the formula accordingly, we arrive at the following statement.

Proposition 5.14. *We have*

$$\mathrm{Tp}_d^{n\to k}(\mathbf{z},\boldsymbol{\theta}) = \sum_{\boldsymbol{\pi}\in\boldsymbol{\Pi}_d} \operatorname*{Res}_{\mathbf{z}=\infty} \frac{Q_{\boldsymbol{\pi}}(\mathbf{z}) \prod_{m<l}(z_m - z_l)}{\prod_{l=1}^{d}\prod_{\substack{\tau\neq\pi_1,\dots,\pi_l\\ \mathrm{sum}(\tau)\le l}}} \frac{\prod_{m=1}^{d}\prod_{j=1}^{k}(\theta_j - z_{\pi_m})}{\prod_{l=1}^{d}\prod_{i=1}^{n}(\lambda_i - z_l)}\, d\mathbf{z}, \tag{5.24}$$

where the polynomial $Q_{\boldsymbol{\pi}}(\mathbf{z})$ is defined in (5.22) for each admissible sequence $\boldsymbol{\pi} = (\pi_1,\dots,\pi_d)$ of partitions of d.

This formula has the pleasant feature that the three parameters of our problem, n, k and d, enter in it in a separate manner. The first fraction only depends on d, the denominator of the second only depends on n, and the numerator of this latter fraction controls the k-dependence, with some interference from the sequence $\boldsymbol{\pi}$.

While this formula is a step forward, it is rather difficult to use in practice, since the number of terms and factors in it grows with d as the the number of elements in $\boldsymbol{\Pi}_d$. Also, the known properties of Thom polynomials listed in Proposition 2.12 are not manifest in (5.24).

In the next section, we will see that this formula goes through two dramatic simplifications, which will make it easy to evaluate it for small values of d.

Before proceeding, we present a schematic diagram of the main objects of our constructions (see Figure 1). We hope this will help the reader to navigate among the various spaces we have introduced.

Explanations.

- The lower circle is the flag variety $\mathrm{Flag}_d(\mathbb{C}^n)$; the fat dots inside represent the T_n-fixed flags in $\mathrm{Flag}_d(\mathbb{C}^n)$.
- The upper circle is $\widetilde{\mathcal{E}}_{\mathrm{ref}}$, the fiber of the bundle $\widetilde{\mathcal{E}}$ over the reference flag $\mathbf{f}_{\mathrm{ref}}$. The small circles inside represent the T_d-fixed points in $\widetilde{\mathcal{E}}_{\mathrm{ref}}$. One of these fixed points, $\tilde{\varepsilon}_{\mathrm{dst}} \in \widetilde{\mathcal{E}}_{\mathrm{ref}}$, will play an important role in what follows.

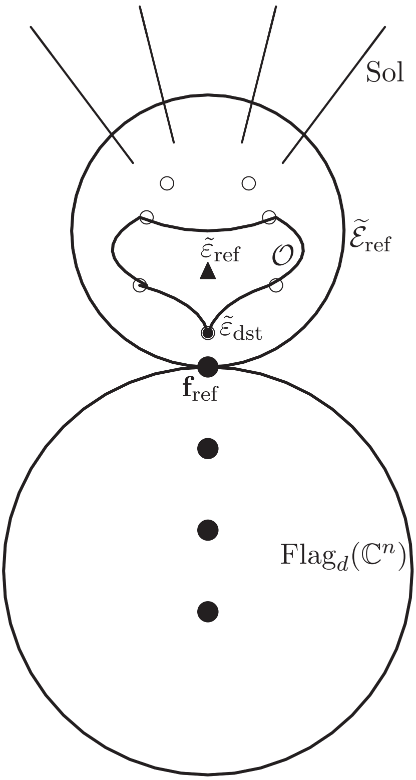


Figure 1.

- The region bounded by the curvy-linear pentagon represents the B_d-orbit of the reference point $\tilde{\varepsilon}_{\mathrm{ref}}$, which is marked by a triangle. The closure of the orbit is $\mathcal{O}$; this is a singular subvariety of $\widetilde{\mathcal{E}}_{\mathrm{ref}}$, which contains some of the fixed points of $\widetilde{\mathcal{E}}_{\mathrm{ref}}$, but not all of them.
- The straight lines on top are the linear solution spaces of the corresponding systems of equations in $\widetilde{\mathcal{E}}_{\mathrm{ref}}$. The union of these solution spaces lying over those points of the fiber bundle $\widetilde{\mathcal{E}}$ that correspond to $\mathcal{O}$ form the closure of our singularity locus Θ_d.

6. **Vanishing residues and the main result**

The terms on the right-hand side of formula (5.24) are enumerated by admissible sequences. There is a simplest one among these:

(6.1) $$\boldsymbol{\pi}_{\mathrm{dst}} = ([1], [2], \ldots, [d]),$$

which we will call *distinguished*. To avoid double indices, below, we will use the simplified notation Q_{dst} instead of $Q_{\boldsymbol{\pi}_{\mathrm{dst}}}$, and similarly $\tilde{\varepsilon}_{\mathrm{dst}}, \mathcal{N}_{\mathrm{dst}}, \mathcal{O}_{\mathrm{dst}}$, etc.

The following remarkable vanishing result holds.

Proposition 6.1. *Assume that $d \ll n \le k$. Then all terms of the sum in (5.24) vanish except for the term corresponding to the sequence of partitions $\boldsymbol{\pi}_{\mathrm{dst}} = ([1], [2], \dots, [d])$. Hence, formula (5.24) reduces to*

$$\text{(6.2)}\quad \mathrm{eP}[\overline{\Theta}_d, \mathcal{J}_d(n,k)]_{T_n \times T_k} = \operatorname*{Res}_{\mathbf{z}=\infty} \frac{Q_{\mathrm{dst}}(z_1,\dots,z_n)\prod_{m<l}(z_m - z_l)\; d\mathbf{z}}{\prod_{l=1}^{d}\prod\{(z_\tau - z_l);\, \mathrm{sum}(\tau) \le l,\, |\tau|>1\}} \frac{\prod_{l=1}^{d}\prod_{j=1}^{k}(\theta_j - z_l)}{\prod_{l=1}^{d}\prod_{i=1}^{n}(\lambda_i - z_l)},$$

where $Q_{\mathrm{dst}} = \mathrm{eP}[\mathcal{O}_{\mathrm{dst}}, \mathcal{N}_{\mathrm{dst}}]_{T_d}$.

Before turning to the proof, we make a few remarks. First, note that this simplification is dramatic: the number of terms in (5.24) grows exponentially with d, and of this sum now a single term survives. This is fortunate, because computing all the polynomials $Q_{\boldsymbol{\pi}}$, $\boldsymbol{\pi} \in \boldsymbol{\Pi}_d$ seems to be an insurmountable task; at the moment, we do not even have an algorithm to determine when $Q_{\boldsymbol{\pi}} = 0$, i.e., when $\tilde{\varepsilon}_{\boldsymbol{\pi}} \in \mathcal{O}$.

Our second observation is that after replacing z_l by $-z_l$, $l = 1, \dots, d$ in (6.2), we can rewrite the equation as

$$\text{(6.3)}\quad \mathrm{eP}[\overline{\Theta}_d, \mathcal{J}_d(n,k)]_{T_n \times T_k} = \operatorname*{Res}_{\mathbf{z}=\infty} \frac{(-1)^d \prod_{m<l}(z_m - z_l)\, Q_{\mathrm{dst}}(z_1,\dots,z_n)}{\prod_{l=1}^{d}\prod\{(z_\tau - z_l);\, \mathrm{sum}(\tau) \le l,\, |\tau|>1\}} \prod_{l=1}^{d} \mathrm{RC}\left(\frac{1}{z_l}\right) z_l^{k-n}\, dz_l,$$

where $\mathrm{RC}(z)$ is the generating series of the relative Chern classes introduced in (2.14). Indeed, the denominator and the numerator of the fraction in (6.3) are homogeneous polynomials of the same degree; hence this substitution will leave the fraction unchanged. We thus obtain an explicit formula for the Thom polynomial of the A_d-singularity in terms of the relative Chern classes. This is important, because the fact that (6.3) conforms to the result of Thom-Damon, Proposition 2.12 (3), suggests that we have the “right” formula.

Most of the present section will be taken up by the proof of Proposition 6.1. In Section 6.2, we derive a criterion for the vanishing of iterated residues of the form (5.5). Applying this criterion to the right-hand side of (5.24) reduces Proposition 6.1 to a statement about the factors of the polynomials $Q_{\boldsymbol{\pi}}$, $\boldsymbol{\pi} \in \boldsymbol{\Pi}_d$: Proposition 6.4. According to the elimination property in Proposition 2.3, such divisibility properties follow from the existence of relations of a certain form in the ideal of the subvariety $\mathcal{O}_{\boldsymbol{\pi}} \subset \mathcal{N}_{\boldsymbol{\pi}}$. We find a family of such relations in Section 6.3 (see (6.18)) and then convert the elimination conditions in Proposition 2.3 into a combinatorial condition on $\boldsymbol{\pi}$ (cf. Lemma 6.12). At the end of Section 6.3, we show that if a sequence $\boldsymbol{\pi}$ does not satisfy this combinatorial condition, then it is either $\boldsymbol{\pi}_{\mathrm{dst}}$ or $\tilde{\varepsilon}_{\boldsymbol{\pi}} \notin \mathcal{O}$, thus completing the proof of Proposition 6.1.

Introduce the subset $\mathbf{\Pi}_{\mathcal{O}} \subset \mathbf{\Pi}_d$ defined by

$$\mathbf{\Pi}_{\mathcal{O}} = \{\boldsymbol{\pi} \in \mathbf{\Pi}_d;\ \tilde{\varepsilon}_{\boldsymbol{\pi}} \in \mathcal{O}\}\,. \tag{6.4}$$

As we mentioned earlier, at the moment, we do not have an explicit description of this set. In the course of this proof, however, we obtain a rather efficient, albeit incomplete, criterion for a sequence $\boldsymbol{\pi} \in \mathbf{\Pi}_d$ *not* to belong to $\mathbf{\Pi}_{\mathcal{O}}$; we explain this criterion in Section 6.4. Finally, in Section 6.5, we further simplify (6.3) and formulate our main result, Theorem 6.16.

Before embarking on this rather tortuous route, we give a few examples in Section 6.1 that demonstrate the localization formulas and the vanishing property explicitly. Note that we devote the last chapter of the paper to the detailed study of (6.3) for small values of d, and hence the proofs in Section 6.1 will be omitted.

6.1. *The localization formulas for $d = 2, 3$.* The situation for $d = 2$ and 3 is simplified by the fact that in these cases the closure of the Borel-orbit $\mathcal{O} = \overline{B_d\tilde{\varepsilon}_{\mathrm{ref}}} \subset \widetilde{\mathcal{E}}_{\mathrm{ref}}$ is smooth. We will thus use the Berline-Vergne localization formula (3.3) instead of Rossmann's formula, and instead of (5.21), we can work with an explicit expression, not containing equivariant multiplicities which need to be computed. This allows us to write down the fixed point formula for $\mathrm{eP}[\overline{\Theta}_d, \mathcal{J}_d(n,k)]_{T_n\times T_k}$ obtained by substituting a simplified version of (5.21) into (5.8) and then compare it to the residue formula (6.2). In these cases we can easily describe the set $\mathbf{\Pi}_{\mathcal{O}}$ as well. The formulas below are justified in Section 7.

For $d = 2$, we have $\mathcal{O} = \widetilde{\mathcal{E}}_{\mathrm{ref}} \cong \mathbb{P}^1$. There are two fixed points in $\widetilde{\mathcal{E}}_{\mathrm{ref}}$:

$$\mathbf{\Pi}_{\mathcal{O}} = \mathbf{\Pi}_2 = \{([1],[2]),\quad ([1],[1,1])\}.$$

Then our fixed point formula reads as follows:

$$\mathrm{Tp}_2^{n\to k}(\mathbf{z},\boldsymbol{\theta}) = \sum_{s=1}^{n}\sum_{t\neq s}^{n} \frac{1}{\prod_{i\neq s}^{n}(\lambda_i-\lambda_s)\prod_{i\neq s,t}^{n}(\lambda_i-\lambda_t)}$$
$$\times\left(\frac{\prod_{j=1}^{k}(\theta_j-\lambda_s)\prod_{j=1}^{k}(\theta_j-\lambda_t)}{2\lambda_s-\lambda_t} + \frac{\prod_{j=1}^{k}(\theta_j-\lambda_s)\prod_{j=1}^{k}(\theta_j-2\lambda_s)}{\lambda_t-2\lambda_s}\right).$$

This is equal to the residue (5.24):

$$\operatorname*{Res}_{z_1=\infty}\operatorname*{Res}_{z_2=\infty} \frac{z_1-z_2}{\prod_{i=1}^{n}(\lambda_i-z_1)\prod_{i=1}^{n}(\lambda_i-z_2)}$$
$$\times\left(\frac{\prod_{j=1}^{k}(\theta_j-z_1)\prod_{j=1}^{k}(\theta_j-z_2)}{2z_1-z_2} + \frac{\prod_{j=1}^{k}(\theta_j-z_1)\prod_{j=1}^{k}(\theta_j-2z_1)}{z_2-2z_1}\right).$$

Proposition 6.1 states that the residue of the second term vanishes; this is easy to check by hand.

For $d = 3$, the orbit closure $\mathcal{O}$ is a smooth 3-dimensional hypersurface in $\widetilde{\mathcal{E}}_{\mathrm{ref}}$. There are six fixed points in $\mathcal{O}$, namely

$$\mathbf{\Pi}_{\mathcal{O}} = \{([1],[2],[3]),\ ([1],[2],[1,2]),\ ([1],[2],[1,1]),\\ ([1],[1,1],[3]),\ ([1],[1,1],[1,1,1]),\ ([1],[1,1],[2])\};$$

the remaining two fixed points in $\widetilde{\mathcal{E}}_{\mathrm{ref}}$ do not belong to $\mathcal{O}$ (see Proposition 6.14):

$$([1],[2],[1,1,1]),\ ([1],[1,1],[1,2]) \notin \mathbf{\Pi}_{\mathcal{O}}.$$

Hence the corresponding fixed point formula has six terms:

$$\begin{aligned}\mathrm{Tp}_3^{n\to k}(\mathbf{z},\boldsymbol{\theta}) = \sum_{s=1}^{n}\sum_{t\neq s}^{n}\sum_{u\neq s,t}^{n} &\frac{\prod_{j=1}^k(\theta_j-\lambda_s)}{\prod_{i\neq s}^n(\lambda_i-\lambda_s)\prod_{i\neq s,t}^n(\lambda_i-\lambda_t)\prod_{i\neq s,t,u}^n(\lambda_i-\lambda_u)}\\
\times\Bigg[\frac{\prod_{j=1}^k(\theta_j-\lambda_t)}{2\lambda_s-\lambda_t}\cdot\Bigg(&\frac{\prod_{j=1}^k(\theta_j-\lambda_u)}{(2\lambda_s-\lambda_u)(\lambda_s+\lambda_t-\lambda_u)}\\
+\frac{\prod_{j=1}^k(\theta_j-\lambda_s-\lambda_t)}{(\lambda_u-\lambda_s-\lambda_t)(2\lambda_s-\lambda_s-\lambda_t)}&+\frac{\prod_{j=1}^k(\theta_j-2\lambda_s)}{(\lambda_u-2\lambda_s)(\lambda_s+\lambda_t-2\lambda_s)}\Bigg)\\
+\frac{\prod_{j=1}^k(\theta_j-2\lambda_s)}{\lambda_t-2\lambda_s}\cdot\Bigg(&\frac{\prod_{j=1}^k(\theta_j-\lambda_u)}{(\lambda_t-\lambda_u)(3\lambda_s-\lambda_u)}+\frac{\prod_{j=1}^k(\theta_j-3\lambda_s)}{(\lambda_u-3\lambda_s)(\lambda_t-3\lambda_s)}\\
+\frac{\prod_{j=1}^k(\theta_j-\lambda_t)}{(\lambda_u-\lambda_t)(3\lambda_s-\lambda_t)}\Bigg)\Bigg].&\end{aligned}$$

The corresponding residue formula (5.24) also has six terms:

$$\begin{aligned}\mathrm{Tp}_3^{n\to k}(\mathbf{z},\boldsymbol{\theta}) = \operatorname*{Res}_{z_1=\infty}\operatorname*{Res}_{z_2=\infty}\operatorname*{Res}_{z_3=\infty} &\frac{(z_1-z_2)(z_1-z_3)(z_2-z_3)\prod_{j=1}^k(\theta_j-z_1)}{\prod_{i=1}^n(\lambda_i-z_1)\prod_{i=1}^n(\lambda_i-z_2)\prod_{i=1}^n(\lambda_i-z_3)}\\
\times\Bigg[\frac{\prod_{j=1}^k(\theta_j-z_2)}{2z_1-z_2}\cdot\Bigg(&\frac{\prod_{j=1}^k(\theta_j-z_3)}{(2z_1-z_3)(z_1+z_2-z_3)}\\
+\frac{\prod_{j=1}^k(\theta_j-z_1-z_2)}{(z_3-z_1-z_2)(2z_1-z_1-z_2)}&+\frac{\prod_{j=1}^k(\theta_j-2z_1)}{(z_3-2z_1)(z_1+z_2-2z_1)}\Bigg)\\
+\frac{\prod_{j=1}^k(\theta_j-2z_1)}{z_2-2z_1}\cdot\Bigg(&\frac{\prod_{j=1}^k(\theta_j-z_3)}{(z_2-z_3)(3z_1-z_3)}+\frac{\prod_{j=1}^k(\theta_j-3z_1)}{(z_3-3z_1)(z_2-3z_1)}\\
+\frac{\prod_{j=1}^k(\theta_j-z_2)}{(z_3-z_2)(3z_1-z_2)}\Bigg)\Bigg].&\end{aligned}$$

Here, again, the last five terms vanish and only the one corresponding to the distinguished fixed point $([1],[2],[3])$ remains, leaving us with (6.2).

For $d > 3$, the variety $\mathcal{O}_d \subset \widetilde{\mathcal{E}}_{\mathrm{ref}}$ is singular. This means that the analogs of these formulas involve calculation of equivariant multiplicities, which is a rather difficult problem. We present some of these computations in Section 7.

6.2. *The vanishing of residues.* In this paragraph, we describe the conditions under which iterated residues of the type appearing in the sum in (5.24) vanish.

We start with the 1-dimensional case, where the residue at infinity is defined by (5.5) with $d = 1$. By bounding the integral representation along a contour $|z| = R$ with R large, one can easily prove

LEMMA 6.2. *Let $p(z), q(z)$ be polynomials of one variable. Then*

$$\operatorname*{Res}_{z=\infty} \frac{p(z)\,dz}{q(z)} = 0 \quad \text{if } \deg(p(z)) + 1 < \deg(q).$$

Consider now the multidimensional situation. Let $p(\mathbf{z}), q(\mathbf{z})$ be polynomials in the d variables $z_1, \dots, z_d$, and assume that $q(\mathbf{z})$ is the product of linear factors $q = \prod_{i=1}^N L_i$, as in (6.2). We continue to use the notation $d\mathbf{z} = dz_1 \dots dz_d$. We would like to formulate conditions under which the iterated residue

$$\operatorname*{Res}_{z_1=\infty} \operatorname*{Res}_{z_2=\infty} \dots \operatorname*{Res}_{z_d=\infty} \frac{p(\mathbf{z})\; d\mathbf{z}}{q(\mathbf{z})} \tag{6.5}$$

vanishes. Introduce the following notation:

- For a set of indices $S \subset \{1, \dots, d\}$, denote by $\deg(p(\mathbf{z}); S)$ the degree of the one-variable polynomial $p_S(t)$ obtained from p via the substitution $z_m \to \begin{cases} t, & \text{if } m \in S, \\ 1, & \text{if } m \notin S. \end{cases}$
- For a nonzero linear function $L = a_0 + a_1 z_1 + \dots + a_d z_d$, denote by $\operatorname{coeff}(L, z_l)$ the coefficient a_l.
- Finally, for $1 \le m \le d$, set

  $$\operatorname{lead}(q(\mathbf{z}); m) = \#\{i;\ \max\{l;\ \operatorname{coeff}(L_i, z_l) \neq 0\} = m\},$$

  which is the number of those factors L_i in which the coefficient of z_m does not vanish, but the coefficients of $z_{m+1}, \dots, z_d$ are 0.

Thus we group the N linear factors of $q(\mathbf{z})$ according to the nonvanishing coefficient with the largest index. In particular, for $1 \le m \le d$, we have

$$\deg(q(\mathbf{z}); m) \ge \operatorname{lead}(q(\mathbf{z}); m), \text{ and } \sum_{m=1}^{d} \operatorname{lead}(q(\mathbf{z}); m) = N.$$

Now applying Lemma 6.2 to the first residue in (6.5), we see that

$$\operatorname*{Res}_{z_d=\infty} \frac{p(z_1, \dots, z_{d-1}, z_d)\, d\mathbf{z}}{q(z_1, \dots, z_{d-1}, z_d)} = 0$$

whenever $\deg(p(\mathbf{z}); d) + 1 < \deg(q(\mathbf{z}), d)$. In this case, of course, the entire iterated residue (6.5) vanishes.

Now we suppose the residue with respect to z_d does not vanish, and we look for conditions of vanishing of the next residue:

$$\operatorname*{Res}_{z_{d-1}=\infty} \operatorname*{Res}_{z_d=\infty} \frac{p(z_1,\dots,z_{d-2},z_{d-1},z_d)\,d\mathbf{z}}{q(z_1,\dots,z_{d-2},z_{d-1},z_d)}. \tag{6.6}$$

Now the condition $\deg(p(\mathbf{z}); d-1)+1 < \deg(q(\mathbf{z}), d-1)$ will be *insufficient*; for example,

(6.7)

$$\operatorname*{Res}_{z_{d-1}=\infty} \operatorname*{Res}_{z_d=\infty} \frac{dz_{d-1}dz_d}{z_{d-1}(z_{d-1}+z_d)} = \operatorname*{Res}_{z_{d-1}=\infty} \operatorname*{Res}_{z_d=\infty} \frac{dz_{d-1}dz_d}{z_{d-1}z_d}\left(1-\frac{z_{d-1}}{z_d}+\cdots\right) = 1.$$

After performing the expansions (5.6) to $1/q(\mathbf{z})$, we obtain a Laurent series with terms $z_1^{-i_1}\dots z_d^{-i_d}$ such that $i_{d-1}+i_d \geq \deg(q(z); d-1, d)$; hence, the condition

$$\deg(p(\mathbf{z}); d-1, d) + 2 < \deg(q(\mathbf{z}); d-1, d) \tag{6.8}$$

will suffice for the vanishing of (6.6).

There is another way to ensure the vanishing of (6.6). Suppose that for $i = 1,\dots,N$, every time we have $\operatorname{coeff}(L_i, z_{d-1}) \neq 0$, we also have $\operatorname{coeff}(L_i, z_d) = 0$, which is equivalent to the condition $\deg(q(\mathbf{z}), d-1) = \operatorname{lead}(q(\mathbf{z}); d-1)$. Now the Laurent series expansion of $1/q(\mathbf{z})$ will have terms $z_1^{-i_1}\dots z_d^{-i_d}$ satisfying $i_{d-1} \geq \deg(q(\mathbf{z}), d-1) = \operatorname{lead}(q(\mathbf{z}); d-1)$. Hence, in this case the vanishing of (6.6) is guaranteed by $\deg(p(\mathbf{z}), d-1)+1 < \deg(q(\mathbf{z}), d-1)$. This argument easily generalizes to the following statement.

PROPOSITION 6.3. *Let $p(\mathbf{z})$ and $q(\mathbf{z})$ be polynomials in $z_1,\dots,z_d$, and assume that $q(\mathbf{z})$ is a product of linear factors $q(\mathbf{z}) = \prod_{i=1}^N L_i$. Set $d\mathbf{z} = dz_1\dots dz_d$. Then*

$$\operatorname*{Res}_{z_1=\infty} \operatorname*{Res}_{z_2=\infty} \dots \operatorname*{Res}_{z_d=\infty} \frac{p(\mathbf{z})\,d\mathbf{z}}{q(\mathbf{z})} = 0$$

if for some $l \leq d$, either of the following two options hold:

- $\deg(p(\mathbf{z}); d, d-1,\dots,l) + d - l + 1 < \deg(q(\mathbf{z}); d, d-1,\dots,l)$,
  or
- $\deg(p(\mathbf{z}); l) + 1 < \deg(q(\mathbf{z}); l) = \operatorname{lead}(q(\mathbf{z}); l)$.

Note that in case of the second option, the equality $\deg(q(\mathbf{z}); l) = \operatorname{lead}(q(\mathbf{z}); l)$ means that

(6.9)

$$\text{for each } i = 1,\dots,N \text{ and } m > l,\ \operatorname{coeff}(L_i, z_l) \neq 0 \text{ implies } \operatorname{coeff}(L_i, z_m) = 0.$$

Recall that our goal is to show that all the terms of the sum in (5.24) vanish except for the one corresponding to $\boldsymbol{\pi}_{\mathrm{dst}} = ([1],\dots,[d])$. Let us apply our new-found tool, Proposition 6.3, to the terms of this sum.

Fix a sequence $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_d) \in \mathbf{\Pi}_d$ and consider the iterated residue corresponding to it on the right-hand side of (5.24). The expression under the residue is the product of two fractions:

$$\frac{p(\mathbf{z})}{q(\mathbf{z})} = \frac{p_1(\mathbf{z})}{q_1(\mathbf{z})} \cdot \frac{p_2(\mathbf{z})}{q_2(\mathbf{z})},$$

where

$$\frac{p_1(\mathbf{z})}{q_1(\mathbf{z})} = \frac{Q_{\boldsymbol{\pi}}(\mathbf{z}) \prod_{m<l} (z_m - z_l)}{\prod_{l=1}^{d} \prod_{\substack{\tau \neq \pi_1, \ldots, \pi_l \\ \mathrm{sum}(\tau) \leq l}} (z_\tau - z_{\pi_l})} \quad \text{and} \quad \frac{p_2(\mathbf{z})}{q_2(\mathbf{z})} = \frac{\prod_{m=1}^{d} \prod_{j=1}^{k} (\theta_j - z_{\pi_m})}{\prod_{l=1}^{d} \prod_{i=1}^{n} (\lambda_i - z_l)}. \tag{6.10}$$

Note that $p(\mathbf{z})$ is a polynomial, while $q(\mathbf{z})$ is a product of linear forms, and that $p_1(\mathbf{z})$ and $q_1(\mathbf{z})$ are independent of n and k and depend on d only.

As a warm-up, we show that if the last element of the sequence is not the trivial partition, i.e., if $\pi_d \neq [d]$, then already the first residue in the corresponding term on the right-hand side of (5.24) — the one with respect to z_d — vanishes. Indeed, if $\pi_d \neq [d]$, then $\deg(q_2(\mathbf{z}); d) \geq n$, while z_d does not appear in $p_2(\mathbf{z})$. Then, assuming that $d \ll n$, we have $\deg(p(\mathbf{z}); d) \ll \deg(q(\mathbf{z}); d)$, and this, in turn, implies the vanishing of the residue with respect to z_d (cf. Proposition 6.3).

We can thus assume that $\pi_d = [d]$ and proceed to the study of the next residue, the one taken with respect to z_{d-1}. Again, assume that $\pi_{d-1} \neq [d-1]$. As in the case of z_d above, $d \ll n$ implies $\deg(p(\mathbf{z}); d-1) \ll \deg(q(\mathbf{z}); d-1)$. However, now we cannot use the first option in Proposition 6.3, because $\deg(p_2(\mathbf{z}); d-1, d) = k \geq n$. In order to apply the second option, we have to exclude all linear factors from $q_1(\mathbf{z})$ that have nonzero coefficients in front of both z_{d-1} and z_d. The fact that $\pi_d = [d]$ and the restrictions $\mathrm{sum}(\pi_l) \leq l$ $l = 1, \ldots, d$ tell us that there are two such troublesome factors, $(z_d - z_{d-1})$ and $(z_d - z_{d-1} - z_1)$, which come from the two partitions $\tau = [d-1]$ and $\tau = [d-1, 1]$ in the $l = d$ part of $q_1(\mathbf{z})$. The first of the two fortunately cancels with a factor in the Vandermonde determinant in the numerator. As for the second factor, our only hope is to find it as a factor in the polynomial $Q_{\boldsymbol{\pi}}$.

Continuing this argument by induction, we can reduce Proposition 6.1 to the following statement about the equivariant multiplicities $Q_{\boldsymbol{\pi}}$, $\boldsymbol{\pi} \in \mathbf{\Pi}_d$.

Proposition 6.4. *Let $l \geq 1$, and let $\boldsymbol{\pi}$ be an admissible sequence of partitions of the form (6.12), where $\pi_l \neq [l]$. Then for $m > l$, and every partition τ such that $l \in \tau$, $\mathrm{sum}(\tau) \leq m$, and $|\tau| > 1$, we have*

$$(z_\tau - z_m) | Q_{\boldsymbol{\pi}}. \tag{6.11}$$

This statement will be proved in Section 6.3. For now, we will assume that it is true and give a quick proof of the main result of this section.

Proof of Proposition 6.1. Let $\boldsymbol{\pi} \neq \boldsymbol{\pi}_{\mathrm{dst}}$ be an admissible sequence of partitions. This means that there is $l > 1$ such that $\pi_l \neq [l]$, but $\pi_m = [m]$ for $m > l$:

$$\boldsymbol{\pi} = (\pi_1, \dots \pi_l, [l+1], [l+2], \dots, [d]). \tag{6.12}$$

Note that l does not appear anywhere in $\boldsymbol{\pi}$; thus we can conclude $\deg(p(\mathbf{z}); l) \ll \deg(q(\mathbf{z}); l)$ from $d \ll n$, as usual. This allows us to apply the second option of Proposition 6.3 to the residue taken with respect to z_l as long as we can cancel from $q_2(\mathbf{z})$ all factors which do not satisfy condition (6.9).

These factors are of the form $z_\tau - z_m$, where $m > l$ and $l \in \tau$. If $|\tau| = 1$, i.e., if $\tau = [l]$, then we can find this factor in the Vandermonde determinant in the numerator. We can use Proposition 6.4 to cancel the rest of the factors, as long as we make sure that such factors occur in $q_1(\mathbf{z})$ with multiplicity 1. This is straightforward in our case, since the variable z_m with $m \geq l$ may appear only in the mth factor of $q_1(\mathbf{z})$. □

6.3. *The homogeneous ring of $\widetilde{\mathcal{E}}_{\mathrm{ref}}$ and factorization of $Q_{\boldsymbol{\pi}}$.* Now we turn to the proof of Proposition 6.4. Let $\boldsymbol{\pi} \in \boldsymbol{\Pi}_d$ be an admissible sequence of partitions. Recall (cf. (5.22)) that $Q_{\boldsymbol{\pi}}$ is the T_d-equivariant Poincaré dual of the part $\mathcal{O}_{\boldsymbol{\pi}} = \mathrm{pr}_{\mathrm{ref}}^{-1}(\mathcal{O}) \cap \mathcal{N}_{\boldsymbol{\pi}}$ of the orbit closure $\mathcal{O}$ in the linear chart $\mathcal{N}_{\boldsymbol{\pi}}$ (cf. (5.19)); this latter linear space is endowed with coordinates $\hat{u}^l_{\tau|\boldsymbol{\pi}}$ defined in (5.14).

Our plan is to use the elimination property in Proposition 2.3, which, when applied to our situation, says that the divisibility relation (6.11) follows if we find a relation in the ideal of the subvariety $\mathcal{O}_{\boldsymbol{\pi}} \subset \mathcal{N}_{\boldsymbol{\pi}}$ expressing the appropriate variable $\hat{u}^m_{\tau|\boldsymbol{\pi}}$ as a polynomial of the rest of the variables.

We will lift the calculation from $\widetilde{\mathcal{E}}_{\mathrm{ref}}$ to the vector space

$$\mathrm{Hom}^{\triangle}(\mathbb{C}^d_R, \mathrm{Sym}^\bullet_{\mathrm{ref}}(\mathbb{C}^n_L)).$$

Denote by $\mathbb{C}[u^\bullet]$ the ring of polynomial functions on $\mathrm{Hom}^{\triangle}(\mathbb{C}^d_R, \mathrm{Sym}^\bullet_{\mathrm{ref}}(\mathbb{C}^n_L))$, i.e., the space of polynomials in the variables u^l_τ, $1 \leq l \leq d$, $\mathrm{sum}(\tau) \leq l$. As one can see from Definition 5.9 and (5.14), the relations on the two spaces are connected as follows.

LEMMA 6.5. *Let $Z \in \mathbb{C}[u^\bullet]$ be a polynomial on $\mathrm{Hom}^{\triangle}(\mathbb{C}^d_R, \mathrm{Sym}^\bullet_d\mathbb{C}^n)$, and let $M \subset \mathrm{Hom}^{\triangle}(\mathbb{C}^d_R, \mathrm{Sym}^\bullet_d\mathbb{C}^n)$ be a closed subvariety such that $Z|M$ vanishes. Then the restricted polynomial $\hat{Z} = Z|\mathcal{N}_{\boldsymbol{\pi}}$, written in terms of the coordinates $\hat{u}_{\cdot|\boldsymbol{\pi}}$, may be obtained from Z as follows:*

- *setting $u^l_{\pi_l}$ to 1 for $l = 1, \dots, d$,*
- *setting $u^m_{\pi_l}$ to 0 for $1 \leq l \leq m \leq d$,*
- *replacing the remaining variables u^l_τ by $\hat{u}^l_{\tau|\boldsymbol{\pi}}$.*

In addition, $\hat{Z}$ vanishes on $M \cap \mathcal{N}_{\boldsymbol{\pi}}$.

Eventually, using this lemma with $M = \overline{B_L \varepsilon_{\mathrm{ref}} B_R}$ and $M \cap \mathcal{N}_{\boldsymbol{\pi}} = \mathcal{O}_{\boldsymbol{\pi}}$, we will be able to produce the necessary relations in the defining ideal of $\mathcal{O}_{\boldsymbol{\pi}} \subset \mathcal{N}_{\boldsymbol{\pi}}$. As most of the action will take space in $\mathbb{C}[u^\bullet]$, our next task is to set up some convenient notation for this ring.

The ring $\mathbb{C}[u^\bullet]$ carries a right action of the group B_L and a left action of the group B_R. In particular, it has two multigradings induced from the T_L and T_R actions: the L-multigrading is the vector of multiplicities $(\mathrm{mult}(i,\pi),\ i = 1,\dots,d)$, while the R-multigrading is the lth basis vector in $\mathbb{Z}^d$. A combination of these gradings will be particularly important for us (cf. Definition 5.6):

$$\text{(6.13)} \qquad \mathrm{defect}(u^l_\pi) = l - \mathrm{sum}(\pi).$$

This induces a $\mathbb{Z}^{\geq 0}$-grading on $\mathbb{C}[u^\bullet]$.

Recall that the projection $B_d \to T_d$ is a group homomorphism whose kernel is the subgroup of unipotent matrices. We denote the corresponding nilpotent Lie algebras of strictly upper-triangular matrices by $\mathfrak{n}_R$ and $\mathfrak{n}_L$ for B_R and B_L, respectively..

The two Lie algebras, $\mathfrak{n}_L$ and $\mathfrak{n}_R$ are generated by the simple root vectors

$$\Delta_L = \{E^L_{l,l+1};\ l = 1,\dots,d-1\}, \text{ and } \Delta_R = \{E^R_{l,l+1};\ l = 1,\dots,d-1\},$$

respectively, where $E_{l,l+1}$ is the matrix whose only nonvanishing entry is a 1 in the lth row and $l+1$st column. Let us write down the action of these root vectors on $\mathbb{C}[u^\bullet]$ in the coordinates u^l_τ, $|\tau| \leq l \leq d$. We first define certain operations on partitions.

- Given a positive integer m and a partition $\tau \in \Pi$, denote by $\tau \cup m$ the partition with m added to τ, e.g., $[2,3,4] \cup 3 = [2,3,3,4]$.
- If $m \in \tau$, then denote by $\tau - m$ the partition τ with one of the ms deleted, e.g., $[2,4,4,5,5,5,6] - 5 = [2,4,4,5,5,6]$.
- More generally, we will write $[2,4,5,5] \cup [3,4] = [2,3,4,4,5,5]$ and $[2,4,5,5] - [4,5] = [2,5]$.

Returning to the Lie algebra actions, we have

(6.14)
$$\begin{cases} \mathfrak{n}_R u^l_\tau = u^l_\tau \mathfrak{n}_L = 0, \text{ if } \mathrm{sum}(\tau) = l, \\ E^R_{m,m+1} u^l_\tau = \delta_{l,m+1} u^{l-1}_\tau,\ u^l_\tau E^L_{m,m+1} = \mathrm{mult}(m,\tau)\, u^l_{\tau - m \cup m+1}, \text{ if } \mathrm{sum}(\tau) < l, \end{cases}$$

where $\delta_{a,b}$ is the Kronecker delta. Observe that both $\mathfrak{n}_R$ and $\mathfrak{n}_L$ act compatibly with the $T_R \times T_L$-multigrading and they both decrease the defect (6.13).

The following subspace will play a key role in our calculations:

$$\text{(6.15)} \quad I_{\mathcal{O}} = \left\{ Z \in \mathbb{C}[u^\bullet];\ \mathfrak{n}_R Z = 0 \text{ and } [Z \mathfrak{n}_L^N](\varepsilon_{\mathrm{ref}}) = 0 \text{ for } N = 0,1,2,\dots \right\},$$

where $\mathfrak{n}_L^N$ is the subset $\{X_1 \cdot \dots \cdot X_N;\ X_i \in \mathfrak{n}_L,\ i = 1,\dots,N\}$ of the universal enveloping algebra of $\mathfrak{n}_L$.

PROPOSITION 6.6. *If $Z \in I_{\mathcal{O}}$, then $Z(\varepsilon) = 0$ for every $\varepsilon \in B_L \varepsilon_{\mathrm{ref}} B_R$.*

Proof. First, observe that the actions of $\mathfrak{n}_R$ and $\mathfrak{n}_L$ described in (6.14) are compatible with the multigrading induced by the $T_R \times T_L$-action, and hence, if Z is in $I_{\mathcal{O}}$, then so are all of its $T_R \times T_L$-homogeneous components. This means that without loss of generality we may assume that Z is a homogeneous element of $I_{\mathcal{O}}$.

For such Z, clearly, $Z(\varepsilon) = 0 \Leftrightarrow t_R Z t_L(\varepsilon) = 0$ for any $t_L \in T_L$, $t_R \in T_R$. Combining this with the condition $\mathfrak{n}_R Z = 0$, we can conclude that the zero set of Z is B_R-invariant; hence it is sufficient to show $Z(\varepsilon) = 0$ for $B_L \varepsilon_{\mathrm{ref}}$. Now, since $\ker(B_L \to T_L) = \exp(\mathfrak{n}_L)$, the definition of $I_{\mathcal{O}}$ also implies $Z(b\varepsilon_{\mathrm{ref}}) = 0$ for all $b \in B_L$., and this completes the proof. □

Remark 6.7. Before we proceed, we make a comment on the geometric meaning of $I_{\mathcal{O}}$. The space $\{Z \in \mathbb{C}[u^\bullet];\ \mathfrak{n}_R Z = 0\}$ is the homogeneous coordinate ring of $\widetilde{\mathcal{E}}_{\mathrm{ref}}$ corresponding to the line bundles induced by the characters of T_R. Then Proposition 6.6 may be interpreted as saying that $I_{\mathcal{O}}$ is contained in the ideal of functions vanishing on $\mathcal{O}$. In fact, is not difficult to show that $I_{\mathcal{O}}$ is exactly this ideal.

We will be looking for polynomials $Z \in I_{\mathcal{O}}$ in a particular subspace of $\mathbb{C}[u^\bullet]$. To describe this space, for each $\boldsymbol{\pi} \in \mathbf{\Pi}_d$, introduce the monomial

$$\mathbf{u}^{\boldsymbol{\pi}} = \prod_{l=1}^{d} u^l_{\pi_l};\ \text{these satisfy } \mathbf{u}_{\boldsymbol{\pi}}(\varepsilon_{\boldsymbol{\pi}'}) = \begin{cases} 1, & \text{if } \boldsymbol{\pi} = \boldsymbol{\pi}', \\ 0, & \text{otherwise.} \end{cases} \tag{6.16}$$

Now consider the linear span of these monomials:

$$\Lambda = \left\{ \sum_{\boldsymbol{\pi} \in \mathbf{\Pi}_d} \alpha_{\boldsymbol{\pi}} \mathbf{u}^{\boldsymbol{\pi}} \in \mathbb{C}[u^\bullet];\ \alpha_{\boldsymbol{\pi}} \in \mathbb{C} \right\}. \tag{6.17}$$

In order to write down our formulas for certain elements of $\Lambda \cap I_{\mathcal{O}}$, we need to introduce two operations on $\mathbf{\Pi}_d$. For a sequence of partitions $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_d)$ and a permutation $\sigma \in \mathcal{S}_d$, define the the permuted sequence

$$\boldsymbol{\pi} \cdot \sigma = (\pi_{\sigma(1)}, \ldots, \pi_{\sigma(d)});$$

this defines a natural right action of $\mathcal{S}_d$ on $\Pi^{\times d}$. Note that permuting an admissible sequence $\boldsymbol{\pi} \in \mathbf{\Pi}_d$ does not necessarily result in an admissible sequence.

The second operation modifies just one entry of $\boldsymbol{\pi}$: For $\boldsymbol{\pi} \in \mathbf{\Pi}_d$ and $\tau \in \Pi$, define

$$\boldsymbol{\pi} \cup_m \tau = (\pi_1, \ldots, \pi_{m-1}, \pi_m \cup \tau, \pi_{m+1}, \ldots, \pi_d).$$

Now we are ready to write down our relations.

PROPOSITION 6.8. *Let $\boldsymbol{\pi} \in \mathbf{\Pi}_d$ be an admissible sequence of partitions and let $\tau \in \Pi$ be any partition. Then the following polynomial is an element of $I_{\mathcal{O}}$:*

$$\mathrm{Rel}(\boldsymbol{\pi}, \tau) = \sum \mathrm{sign}(\sigma)\, \mathbf{u}^{\boldsymbol{\pi}\cdot\sigma \cup_m \tau}, 1 \le m \le d,\ \sigma \in \mathcal{S}_d,\ \boldsymbol{\pi}\cdot\sigma \cup_m \tau \in \mathbf{\Pi}_d. \tag{6.18}$$

Remark 6.9. The sum in (6.18) may be empty. This happens when there are no pairs (σ, m) satisfying the conditions in (6.18). Note, however, that no two terms of this sum may cancel each other.

Proof. We begin by noting that $\mathrm{Rel}(\boldsymbol{\pi}, \tau)$ is of pure $T_R \times T_L$ weight. Indeed, the torus T_R acts on the whole space Λ with the same weight $(1, 1, \dots, 1)$, while the lth component of the T_L-weight of a term of $\mathrm{Rel}(\boldsymbol{\pi}, \tau)$ is equal to $\mathrm{mult}(l, \tau) + \sum_{m=1}^{d} \mathrm{mult}(l, \pi_m)$.

Next, we show that

$$E^R_{l,l+1}\mathrm{Rel}(\boldsymbol{\pi}, \tau) = 0,\ l = 1, \dots, d-1, \tag{6.19}$$

which implies that $\mathfrak{n}_R\mathrm{Rel}(\boldsymbol{\pi}, \tau) = 0$. Let us fix l. The terms of $\mathrm{Rel}(\boldsymbol{\pi}, \tau)$ in (6.18) are indexed by pairs (σ, m), and we can ignore those pairs for which $\mathrm{sum}(\pi_{l+1}) + \delta_{m,l+1}\mathrm{sum}(\tau) \ge l+1$, since in this case $E^R_{l,l+1}\mathbf{u}^{\boldsymbol{\pi}\cdot\sigma\cup_m\tau} = 0$. Then the vanishing (6.19) clearly follows if, on the set of the remaining pairs contributing to (6.18), we find an involution $(\sigma, m) \mapsto (\sigma', m')$ such that

$$E^R_{l,l+1}\mathbf{u}^{\boldsymbol{\pi}\cdot\sigma\cup_m\tau} = E^R_{l,l+1}\mathbf{u}^{\boldsymbol{\pi}\cdot\sigma'\cup_{m'}\tau} \text{ and } \mathrm{sign}(\sigma') = -\mathrm{sign}(\sigma).$$

Indeed, it is easy to check that this holds for the involution

$$(\sigma', m') = (\sigma \cdot \langle l \leftrightarrow l+1\rangle, \langle l \leftrightarrow l+1\rangle(m)),$$

where $\langle l \leftrightarrow l+1\rangle \in \mathcal{S}_d$ is the transposition of l and $l+1$. This proves (6.19).

Our second task is to show that $\mathrm{Rel}(\boldsymbol{\pi}, \tau)$ is in the linear space

$$I'_{\mathcal{O}} = \left\{ Z \in \mathbb{C}[u^\bullet];\ \left[Z\mathfrak{n}_L^N\right](\varepsilon_{\mathrm{ref}}) = 0 \text{ for } N = 0, 1, \dots \right\}.$$

Using the Leibniz rule, it is easy to see see that $I'_{\mathcal{O}} \subset \mathbb{C}[u^\bullet]$ is an ideal.

First we show that for partitions $\rho, \tau \in \Pi$ and $m \ge \mathrm{sum}(\rho) + \mathrm{sum}(\tau)$, the polynomial

$$Z^m_{\rho\tau} = u^m_{\rho\cup\tau} - \sum u^t_\rho u^r_\tau,\ t + r = m,\ t \ge \mathrm{sum}(\rho),\ r \ge \mathrm{sum}(\tau) \tag{6.20}$$

is in $I'_{\mathcal{O}}$. Indeed, a quick computation produces the equality

$$Z^m_{\rho\tau}E^L_{l,l+1} = \mathrm{mult}(l, \rho)Z^m_{\rho'\tau} + \mathrm{mult}(l, \tau)Z^m_{\rho\tau'},$$

where

$$\rho' = \rho - l \cup [l+1],\ \tau' = \tau - l \cup [l+1].$$

This equality implies that it is sufficient for us to prove $Z^m_{\rho\tau}(\varepsilon_{\mathrm{ref}}) = 0$ for the case $m = \mathrm{sum}(\rho) + \mathrm{sum}(\tau)$. In this case we have

$$Z^m_{\rho\tau} = u^m_{\rho\cup\tau} - u^{\mathrm{sum}(\rho)}_\rho u^{\mathrm{sum}(\tau)}_\tau, \tag{6.21}$$

and this polynomial clearly vanishes on $\varepsilon_{\mathrm{ref}}$, because all three coordinates appearing in this relation are equal to 1 according to (5.10).

Now we return to the proof of $\mathrm{Rel}(\boldsymbol{\pi}, \tau) \in I'_{\mathcal{O}}$. Using the fact that $Z^m_{\rho\tau}$ is in the ideal $I'_{\mathcal{O}}$, modulo the $I'_{\mathcal{O}}$, we can replace all the factors of the form $u^m_{\pi_{\sigma(m)}\cup\tau}$ in all the terms of $\mathrm{Rel}(\boldsymbol{\pi}, \tau)$ by the appropriate sum of quadratic terms in (6.20). Our claim is that the resulting polynomial is identically zero, which implies that $\mathrm{Rel}(\boldsymbol{\pi}, \tau) \in I'_{\mathcal{O}}$.

Indeed, let us perform this substitution. The terms of the resulting sum are parametrized by a triple (σ, m, r), which is obtained by applying (6.20) to the term of $\mathrm{Rel}(\boldsymbol{\pi}, \tau)$ indexed by (σ, m) and taking the term corresponding to r in (6.20). The correspondence is thus

$$(\sigma, m, r) \longrightarrow u^1_{\pi_{\sigma(1)}} \dots u^{m-1}_{\pi_{\sigma(m-1)}} u^{m-r}_{\pi_{\sigma(m)}} u^r_\tau u^{m+1}_{\pi_{\sigma(m+1)}} \dots u^d_{\pi_{\sigma(d)}}. \tag{6.22}$$

Just as above, we can see that the involution $(\sigma, m, r) \mapsto (\sigma \cdot \langle m \leftrightarrow m-r\rangle, m, r)$ provides us with a complete pairing of the terms of the sum described above; each pair consists of identical monomials with opposite signs. This implies that indeed, the result is zero. Hence $\mathrm{Rel}(\boldsymbol{\pi}, \tau)$ vanishes modulo $I'_{\mathcal{O}}$; i.e., $\mathrm{Rel}(\boldsymbol{\pi}, \tau) \in I'_{\mathcal{O}}$. □

Armed with these relations, we are ready to *prove Proposition* 6.4. Recall that according to the strategy described at the beginning of this paragraph, given $\boldsymbol{\pi} \in \mathbf{\Pi}_d$, m and τ as in Proposition 6.4, we need to find a relation of the form $\mathrm{Rel}(\cdot, \cdot)$ that, when restricted to $\mathcal{N}_{\boldsymbol{\pi}}$, expresses the variable $\hat{u}^m_{\tau|\boldsymbol{\pi}}$ in terms of the rest of the variables.

Thus the first thing is to study the conditions under which $\hat{u}^m_{\tau|\boldsymbol{\pi}}$ appears as the restriction of a monomial of the form $\mathbf{u}^{\boldsymbol{\pi}'}$. The following statement immediately follows form the prescription Lemma 6.5.

LEMMA 6.10. *Given* $\boldsymbol{\pi} = (\pi_1, \dots, \pi_d) \in \mathbf{\Pi}_d$, *a positive integer* $m \le d$, *and a partition* $\tau \in \Pi \setminus \{\pi_1, \dots, \pi_d\}$ *satisfying* $\mathrm{sum}(\tau) \le m$, *we have* $\mathbf{u}^{\boldsymbol{\pi}'}|\mathcal{N}_{\boldsymbol{\pi}} = \hat{u}^m_{\tau|\boldsymbol{\pi}}$ *for some* $\boldsymbol{\pi}' \in \mathbf{\Pi}_d$ *if and only if*

$$\boldsymbol{\pi}' = (\pi_1, \dots, \pi_{m-1}, \tau, \pi_{m+1}, \dots, \pi_d).$$

Now let us take a closer look at the conditions of Proposition 6.4. We are given $1 \le l < m \le d$ and $\tau \in \Pi$ satisfying

$$\mathrm{sum}(\tau) \le m,\ l \in \tau \text{ and } |\tau| > 1$$

and a sequence $\boldsymbol{\pi}$ of the form (6.12) with $\pi_l \neq [l]$. In view of Lemma 6.10, the variable $\hat{u}^m_{\tau|\boldsymbol{\pi}}$ will appear as the restriction to $\mathcal{N}_{\boldsymbol{\pi}}$ of the term $\mathbf{u}^{\boldsymbol{\rho}\cup_m\tau\setminus[l]}$ of a relation $\mathrm{Rel}(\boldsymbol{\rho}, \tau \setminus [l])$ as long as

$$\boldsymbol{\rho} = (\pi_1, \dots, \pi_l, [l+1], [l+2], \dots, [m-1], [l], [m+1], \dots, [d-1], [d])$$

is admissible, which is obvious. We leave it to the reader to check that the rest of the terms of $\mathrm{Rel}(\boldsymbol{\rho}, \tau \setminus [l])$ cannot contain $\hat{u}^m_{\tau|\boldsymbol{\pi}}$ as a factor. This completes the proof of Proposition 6.4 and thus also the proof of Proposition 6.1. □

This proof suggests a simple criterion for a monomial $\boldsymbol{\pi} \in \boldsymbol{\Pi}_d$ to appear in one of the relations (6.18).

Definition 6.11. We will call an admissible sequence of partitions $\boldsymbol{\pi} = (\pi_1, \dots, \pi_d)$ *complete* if for every $l \in \{1, \dots, d\}$ and every nontrivial subpartition $\tau \subset \pi_l$, there is $m \in \{1, \dots, d\}$ such that $\pi_m = \tau$.

Taking into account Remark 6.9, we have the following criterion.

LEMMA 6.12. *A monomial* $\mathbf{u}^{\boldsymbol{\pi}}$ *appears in a relation* $\mathrm{Rel}(\boldsymbol{\rho}, \tau)$ *for some* $\boldsymbol{\rho} \in \boldsymbol{\Pi}_d$ *and* $\tau \in \Pi$ *if and only if* $\boldsymbol{\pi}$ *is* not *complete.*

6.4. *The fixed points of the* T_L*-action on* $\mathcal{O}$. As a small detour, based on the results of the previous paragraph, we obtain a rather powerful criterion for $\boldsymbol{\pi} \in \boldsymbol{\Pi}_d$ *not* to belong to $\boldsymbol{\Pi}_{\mathcal{O}}$; i.e., we will construct a large number of T_L-fixed points that do not lie in $\mathcal{O}$. We note, however, that describing the set $\boldsymbol{\Pi}_{\mathcal{O}}$ remains an interesting open problem. Our starting point is (6.16).

LEMMA 6.13. *If the monomial* $\mathbf{u}^{\boldsymbol{\pi}}$ *appears with nonzero coefficient in a polynomial from* $\Lambda \cap I_{\mathcal{O}}$*, then the fixed point* $\tilde{\varepsilon}_{\boldsymbol{\pi}} \notin \mathcal{O}$*, i.e.,* $\boldsymbol{\pi} \notin \boldsymbol{\Pi}_{\mathcal{O}}$.

Proof. Indeed, let Z be such a polynomial. According to Proposition 6.6, a polynomial in $I_{\mathcal{O}}$ vanishes at all points of $\mathcal{O}$. On the other hand, it is clear from (6.16) that all but exactly one of the terms of Z vanishes at $\varepsilon_{\boldsymbol{\pi}}$, and hence $Z(\varepsilon_{\boldsymbol{\pi}}) \neq 0$. □

Combining this statement with Lemma 6.12, we have the following.

PROPOSITION 6.14. *If* $\boldsymbol{\pi} \in \boldsymbol{\Pi}_{\mathcal{O}}$*, i.e., if* $\tilde{\varepsilon}_{\boldsymbol{\pi}} \in \mathcal{O}$*, then the sequence* $\boldsymbol{\pi}$ *is complete.*

This Proposition provides us with a powerful necessary criterion for $\boldsymbol{\pi}$ to be in $\boldsymbol{\Pi}_{\mathcal{O}}$. As the an example below shows, this condition is not sufficient.

Example 6.15. (1) The sequence

$$([1], [2], \dots, [d-1], [l, m]), \quad \text{where } l + m \leq d,$$

is complete, and, in fact, it corresponds to a fixed point.

(2) For $d = 3, 4$, the reverse of Proposition 6.14 holds: if $\boldsymbol{\pi}$ is complete, then the fixed point $\tilde{\varepsilon}_{\boldsymbol{\pi}}$ lies in the orbit closure $\mathcal{O}_d$ (see §7).

(3) The completeness of $\boldsymbol{\pi}$ is a necessary but not sufficient condition for $\boldsymbol{\pi}$ to be in $\boldsymbol{\Pi}_{\mathcal{O}}$. An example is the following zero-defect sequence of

partitions: let $d = 60$, $\tau = [1, 12, 12, 15, 20]$ and set

$$\pi_l = \begin{cases} \rho, & \text{if } \rho \subset \tau \text{ and } \operatorname{sum}(\rho) = l, \\ [l], & \text{otherwise.} \end{cases}$$

By definition, this is a complete sequence of partitions, but it is not in $\mathcal{O}$, which is left as an exercise.

6.5. *The distinguished fixed point and the main result.* Now we turn our attention to our much simplified formula (6.2) for the Thom polynomial of the A_d-singularity.

The proof of the vanishing of the contributions to (5.24), naturally, fails at the fixed point $\tilde{\varepsilon}_{\text{dst}}$. Indeed, for the factors (6.10) in the case of the distinguished sequence $\boldsymbol{\pi}_{\text{dst}}$, we have $\deg(p_2(\mathbf{z}); l) > \deg(q_2(\mathbf{z}); l)$ for $l = 1, \dots, d$, and hence we cannot apply Proposition 6.3.

The factorization arguments of Section 6.3 may be partially saved, however. Indeed, for the case of the distinguished partition $\boldsymbol{\pi}_{\text{dst}}$, each T_L-weight $z_\tau - z_l$ of $\mathcal{N}_{\text{dst}}$ appears with multiplicity one (cf. end of Section 6.2). Hence, again, we can apply the elimination property on Proposition 2.3 and Lemmas 6.10 and 6.12 to conclude that for $|\tau| > 1$,

$$(z_\tau - z_l) \,|\, Q_{\text{dst}} \text{ if } ([1], [2], \dots, [l-1], \tau, [l+1], \dots, [d-1], [d]) \text{ is not complete.}$$

Clearly, such a sequence is complete if and only if $|\tau| = 2$, and this means that in the fraction on the right-hand side of (6.3), we can cancel all factors between the numerator and the denominator corresponding to partitions τ with $|\tau| > 2$. This reduces the denominator to the product of the factors with $|\tau| = 2$:

$$\prod (z_m + z_r - z_l),\ 1 \le m \le r,\ m + r \le l \le d,$$

while Q_{dst} is replaced by a polynomial $\widehat{Q}_d$, whose degree is much smaller than that of Q_{dst}. Note that in this case no factors of the Vandermonde in the numerator are canceled. The fraction in (6.3) thus simplifies to

$$\frac{(-1)^d \prod_{m<l}(z_m - z_l)\, \widehat{Q}_d(z_1, \dots, z_d)}{\prod_{l=1}^{d} \prod_{m=1}^{l-1} \prod_{r=1}^{\min(m,l-m)} (z_m + z_r - z_l)}.$$

The polynomial $\widehat{Q}_d$, just as Q_{dst}, only depends on d; we mark its d-dependence explicitly.

All that remains to do before we can formulate our final result is to describe the geometric meaning of this cancellation and that of the polynomial $\widehat{Q}_d$ itself.

First, note that $\boldsymbol{\pi}_{\text{dst}}$ is of the defect-0 type. Hence, according to Proposition 5.10 (3) and Lemma 5.13, we have an action of the upper-triangular group B_{dst} on $\mathcal{N}_{\text{dst}}$ given by (5.16); moreover, $\varepsilon_{\text{ref}} \in \mathcal{N}_{\text{dst}}$ and $\mathcal{O}_{\text{dst}} = \overline{B_{\text{dst}} \cdot \varepsilon_{\text{ref}}}$.

Remarkably, this action is also linear (cf. Remark 5.12), because the $B_L \times B_R$-action on $\mathrm{Hom}^{\triangle}(\mathbb{C}^d_R, \mathrm{Sym}^{\bullet}_d \mathbb{C}^n)$ preserves the length of the partitions, and $\boldsymbol{\pi}_{\mathrm{dst}}$ contains all the partitions of length 1.

Next, define the linear subspace $\widehat{\mathcal{N}}_d \subset \mathcal{N}_{\mathrm{dst}}$,

$$\widehat{\mathcal{N}}_d = \{\varepsilon \in \mathcal{N}_{\mathrm{dst}};\ \hat{u}^m_{\tau|\mathrm{dst}}(\varepsilon) = 0 \text{ for } |\tau| > 2\} \subset \mathrm{Hom}(\mathbb{C}^d, \mathrm{Sym}^2\mathbb{C}^d), \tag{6.23}$$

and let $\widehat{\mathrm{pr}} : \mathcal{N}_{\mathrm{dst}} \to \widehat{\mathcal{N}}_d$ be the natural projection. Then (cf. the elimination property in Proposition 2.3) we can conclude that

$$\widehat{Q}_d = \mathrm{eP}[\widehat{\mathcal{O}}_d, \widehat{\mathcal{N}}_d]_{T_d}, \quad \text{where } \widehat{\mathcal{O}}_d = \widehat{\mathrm{pr}}(\widehat{\mathcal{O}}_{\mathrm{dst}}). \tag{6.24}$$

In addition, it is easy to see that $\widehat{\mathrm{pr}}$ commutes with the B_{dst}-action; in particular, $\widehat{\mathcal{N}}_d$ in $\mathcal{N}_{\mathrm{dst}}$ is B_{dst}-invariant. The linear representation of B_{dst} on $\widehat{\mathcal{N}}_d$ is easily identified with an action of degree-3 tensors (see Theorem 6.16). In any case, we have

$$\widehat{\mathcal{O}}_d = \overline{B_d \hat{\varepsilon}_{\mathrm{ref}}}, \text{ where } \hat{\varepsilon}_{\mathrm{ref}} = \widehat{\mathrm{pr}}(\varepsilon_{\mathrm{ref}}).$$

Stripping our formulas of extraneous notation, we can formulate our main result in a self-contained manner as follows.

THEOREM 6.16. *Let $T_d \subset B_d \subset \mathrm{GL}_d$ be the subgroups of invertible diagonal and upper-triangular matrices, respectively. Denote the diagonal weights of T_d by $z_1, \dots, z_d$. Then the GL_d-module of* 3*-tensors* $\mathrm{Hom}(\mathbb{C}^d, \mathrm{Sym}^2\mathbb{C}^d)$ *has a diagonal decomposition*

$$\mathrm{Hom}(\mathbb{C}^d, \mathrm{Sym}^2\mathbb{C}^d) = \bigoplus \mathbb{C} q^{mr}_l,\ 1 \le m, r, l \le d,$$

where the tensors q^{mr}_l are of weight $(z_m + z_r - z_l)$, and one identifies q^{rm}_l with q^{mr}_l. Consider the reference element

$$\hat{\varepsilon}_{\mathrm{ref}} = \sum_{m=1}^{d} \sum_{r=1}^{d-m} q^{mr}_{m+r}$$

in the B_d-invariant subspace

$$\widehat{\mathcal{N}}_d = \bigoplus_{1 \le m+r \le l \le d} \mathbb{C} q^{mr}_l \subset \mathrm{Hom}(\mathbb{C}^d, \mathrm{Sym}^2\mathbb{C}^d). \tag{6.25}$$

Set the notation $\widehat{\mathcal{O}}_d$ for the orbit closure $\overline{B_d \hat{\varepsilon}_{\mathrm{ref}}} \subset \widehat{\mathcal{N}}_d$, and consider its T_d-equivariant Poincaré dual

$$\widehat{Q}_d(z_1, \dots, z_d) = \mathrm{eP}[\widehat{\mathcal{O}}_d, \widehat{\mathcal{N}}_d]_{T_d},$$

which is a homogeneous polynomial of degree $\dim(\widehat{\mathcal{N}}_d) - \dim(\widehat{\mathcal{O}}_d)$.

Then for arbitrary integers $n \le k$, the Thom polynomial for the A_d-singularity with n-dimensional source space and k-dimensional target space is given *by the following iterated residue formula:*

(6.26) $$\mathrm{Tp}_d^{n\to k}(\mathbf{z},\boldsymbol{\theta}) = \operatorname*{Res}_{\mathbf{z}=\infty} \frac{(-1)^d \prod_{m<l}(z_m - z_l)\, \widehat{Q}_d(z_1,\dots,z_d)}{\prod_{l=1}^d \prod_{m=1}^{l-1} \prod_{r=1}^{\min(m,l-m)} (z_m + z_r - z_l)} \prod_{l=1}^d \mathrm{RC}\left(\frac{1}{z_l}\right) z_l^{k-n}\, dz_l,$$

where $\mathrm{RC}(\cdot)$ *is the generating function of the relative Chern classes given in* (2.14), *and the residue operation is defined by* (5.5).

Proof. Recall the line of our argument so far. Using the model formulated in Proposition 4.17 and localizing over the flag variety, we obtained our initial formula (5.3). Next, we used localization along the fibers over the flag variety and some residue calculus to convert this formula into the form (5.24). Studying the relations of a certain Borel orbit in a single fiber of this fibration, we proved a cancellation phenomenon in Proposition 6.1. Finally, the argument at the beginning of the current paragraph leads to further simplifications of the formula, which is reflected in (6.26).

Note that Theorem 6.16 seems to claim more than to what we seem to be entitled: Proposition 6.1 is proved under the assumption $d \ll n$, while here we claim that our statement holds for any d and $n \leq k$. To finish the proof, we simply need to point out that according to Proposition 2.12, if an expression of a Thom polynomial in the relative Chern classes holds for large n, then the same expression works for any n. □

Let us make a few final comments. It is not difficult to see that formula (6.26) manifestly satisfies all properties listed in Proposition 2.12. In particular, it only depends on the codimension $k-n$, and reducing the codimension by 1 leads to shifting the indices of the relative Chern classes down by 1. Another benefit of the result is that it shows that the *Thom series* introduced in [9], which, in principle has infinitely many parameters, is governed by a finite object: $\widehat{Q}_d$.

Before we turn to examples in the final section of our paper, we point out an important aspect of our model of Θ_d. Consider the direct summand $\mathrm{Sym}_2^\bullet\mathbb{C}^n = \mathbb{C}^n \oplus \mathrm{Sym}^2\mathbb{C}^n$ of $\mathrm{Sym}_d^\bullet\mathbb{C}^n$, and introduce the rational map

$$\mathrm{pr} : \mathrm{Flag}_d(\mathrm{Sym}_d^\bullet\mathbb{C}^n) \to \mathrm{Flag}_d(\mathrm{Sym}_2^\bullet\mathbb{C}^n)$$

induced by the projection $\mathrm{Sym}_d^\bullet\mathbb{C}^n \to \mathrm{Sym}_2^\bullet\mathbb{C}^n$. The image $\psi(\mathrm{Hom}^{\mathrm{inj}}(\mathbb{C}^d,\mathbb{C}^n))$ is in the domain of definition of pr (cf. diagram (4.7)), inducing the map $\mathrm{pr}\circ\psi : \mathrm{Hom}^{\mathrm{inj}}(\mathbb{C}^d,\mathbb{C}^n) \to \mathrm{Flag}_d(\mathrm{Sym}_2^\bullet\mathbb{C}^n)$. Similarly, $\mathrm{pr}\circ\alpha : \Theta_d \to \mathrm{Flag}_d(\mathrm{Sym}_2^\bullet\mathbb{C}^n)$ is a well-defined map, and one can show that $\mathrm{Sym}_d^\bullet\mathbb{C}^n$ may be replaced by $\mathrm{Sym}_2^\bullet\mathbb{C}^n$ in diagram (4.7), preserving all its relevant properties. This, and the fact that the final formula (6.26) depends only on the linear and quadratic coordinates, suggests that it might have been more efficient to use $\mathrm{Flag}_d(\mathrm{Sym}_2^\bullet\mathbb{C}^n)$ instead of $\mathrm{Flag}_d(\mathrm{Sym}_d^\bullet\mathbb{C}^n)$ in our model from the very beginning.

However, a more delicate computation shows that the vanishing property does not hold in this smaller flag space. It turns out that there are fixed points in the closure of $\mathrm{im}(\mathrm{pr}\circ\psi)$ that do not come from the fixed points in the closure of $\mathrm{im}(\psi)$, and for which the residue in the analog of the the fixed point formula (5.24) does not vanish. Naturally, the sum of these contributions will be zero, but this cannot be shown with our methods.

7. How to calculate $\widehat{Q}_d$? Explicit formulas for Thom polynomials

Theorem 6.16 reduces the computation of the Thom polynomials of the algebra A_d to that of the polynomial $\widehat{Q}_d$, which is the equivariant Poincaré dual of a B_d-orbit in a certain B_d-invariant subspace of 3-tensors in d dimensions. Note that the parameters n and k do not enter this picture; in particular, $\widehat{Q}_d$ only depends on d.

Clearly, in principle, the computation of $\widehat{Q}_d$ is a finite problem in commutative algebra, which, for each value of d, can be handled by a computer algebra package such as Macaulay. However, the number of variables and the degree of $\widehat{Q}_d$ grow rather quickly: they are of order d^3. More importantly, computer algebra programs have difficulties dealing with parametrized subvarieties already in very small examples.

At this point, we do not have an efficient method of computation for $\widehat{Q}_d$ in general. The purpose of this section is to show how to compute $\widehat{Q}_d$ for small degrees: $d = 2, 3, 4, 5, 6$. At the end, we also present an application of our result to the conjectured positivity of the coefficients of the Thom polynomials in Section 7.5.

7.1. *The degree of* $\widehat{Q}_d$. The degree of the polynomial $\widehat{Q}_d$ is the codimension of the orbit $B_d\varepsilon_{\mathrm{ref}}$, or that of its closure $\widehat{\mathcal{O}}_d$, in $\widehat{\mathcal{N}}_d$.

Recall that $\widehat{\mathcal{N}}_d$ has a basis indexed by the set of indices $\{m+r\leq l\leq d\}$. An elementary computation shows that $\dim\widehat{\mathcal{N}}_d$ is given by a cubic quasi-polynomial in d with leading term $d^3/24$.

On the other hand, we have

$$\dim(B_d\hat{\varepsilon}_{\mathrm{ref}}) = \dim(B_d) - \dim(H_d) = \binom{d+1}{2} - d = \binom{d}{2}.$$

Next, denote by $\widehat{\mathcal{N}}_d^0$ the *minimal* (or defect-zero) part of $\widehat{\mathcal{N}}_d$ spanned by the vectors $\{q^l_{mr};\ m+r=l\leq d\}$, and let $\mathrm{pr}_0 : \widehat{\mathcal{N}}_d \to \widehat{\mathcal{N}}_d^0$ be the natural projection; note that $\hat{\varepsilon}_{\mathrm{ref}} \in \widehat{\mathcal{N}}_d^0$. Recall that $B_d = T_dU_d$, where $U_d \subset B_d$ is the subgroup of unipotent matrices. It is easy to check that U_d acts trivially on $\widehat{\mathcal{N}}_d^0$, and its action commutes with the projection pr_0. Now introduce the toric orbit $T_d\hat{\varepsilon}_{\text{ref}} \subset \widehat{\mathcal{N}}_d^0$ and its closure $\widehat{\mathcal{T}} \subset \widehat{\mathcal{N}}_d^0$. The following is a simple consequence of the preceding arguments.

LEMMA 7.1. *The projection* pr_0 *restricted to the orbit* $B_d\hat{\varepsilon}_{\text{ref}}$ *establishes a fibration over the toric orbit* $T_d\hat{\varepsilon}_{\text{ref}}$. *This map extends to a map between the closures* $\widehat{\mathcal{O}} \to \widehat{\mathcal{T}}$, *where* $\widehat{\mathcal{T}} = \overline{T_d\hat{\varepsilon}_{\text{ref}}}$.

Remark 7.2. We note that there are standard algorithms to compute the equivariant Poincaré dual of a toric orbit — we presented some of these in the example of the toric orbit in Section 2.3 — but no such simple algorithm is known for Borel orbits. The fibration in Lemma 7.1 suggests that, in our situation, one might be able to reduce this latter problem to the former.

Lemma 7.1 implies, in particular, that the codimension of $B_d\hat{\varepsilon}_{\text{ref}}$ is the sum of the codimensions of $\widehat{\mathcal{T}}$ in $\widehat{\mathcal{N}}_d^0$ and the codimension in the fiberwise directions. We collect the appropriate numeric values in the following table:

d	$\dim\widehat{\mathcal{O}}=\binom{d}{2}$	$\dim\widehat{\mathcal{N}}_d$	$\deg\widehat{Q}_d=\text{codim}(\widehat{\mathcal{O}})$	$\dim(\widehat{\mathcal{T}})=d-1$	$\dim\widehat{\mathcal{N}}_d^0$	$\text{codim}(\widehat{\mathcal{T}})$
1	0	0	0	0	0	0
2	1	1	0	1	1	0
3	3	3	0	2	2	0
4	6	7	1	3	4	1
5	10	13	3	4	6	2
6	15	22	7	5	9	4

The first 3 columns list the codimension of the closure of the Borel orbit $\widehat{\mathcal{O}}$ in $\widehat{\mathcal{N}}_d$, while the last three list the codimension of the closure of the toric orbit $\widehat{\mathcal{T}}$ in $\widehat{\mathcal{N}}_d^0$.

Now we are ready for the computations.

7.2. *The cases* $d = 1, 2, 3$. In these cases $\deg\widehat{Q}_d = 0$, and thus $\widehat{Q}_d = 1$; geometrically, this means that $\mathcal{O}_d = \widetilde{\mathcal{E}}_{\text{ref}}$, and thus $\widehat{\mathcal{O}}_d = \widehat{\mathcal{N}}_d$. The case of $d = 1$ was described in Section 3.3.

For $d = 2$, we obtain

$$\text{Tp}_2^{n\to k} = \operatorname*{Res}_{z_1=\infty}\operatorname*{Res}_{z_2=\infty} \frac{z_1 - z_2}{2z_1 - z_2}\text{RC}\left(\frac{1}{z_1}\right)\text{RC}\left(\frac{1}{z_2}\right) z_1^{k-n} z_2^{k-n}\, dz_1dz_2. \tag{7.1}$$

Expanding the iterated residue, one immediately recovers Ronga's formula [24]:

$$\text{Tp}_2^{n\to k} = c_{k-n+1}^2 + \sum_{i=1}^{k-n+1} 2^{i-1}c_{k-n+1-i}c_{k-n+1+i}. \tag{7.2}$$

For $d = 3$, the formula is

$$\text{Tp}_3^{n\to k} = (-1)\operatorname*{Res}_{z_1=\infty}\operatorname*{Res}_{z_2=\infty}\operatorname*{Res}_{z_3=\infty} \frac{(z_1-z_2)(z_1-z_3)(z_2-z_3)}{(2z_1-z_2)(z_1+z_2-z_3)(2z_1-z_3)} \text{RC}\left(\frac{1}{z_1}\right)\text{RC}\left(\frac{1}{z_2}\right)\text{RC}\left(\frac{1}{z_3}\right) z_1^{k-n} z_2^{k-n} z_3^{k-n}\, dz_1dz_2dz_3. \tag{7.3}$$

This is a more compact and conceptual formula for $\mathrm{Tp}_3^{n\to k}$ than the one given in [2].

7.3. *The basic equations in general.* As our table in Section 7.1 shows, the polynomial $\widehat{Q}_d$ is not trivial when $d > 3$. As a step towards its computation, we describe a set of equations satisfied by $\widehat{\mathcal{O}} \subset \widehat{\mathcal{N}}_d$ and $\widehat{\mathcal{T}} \subset \widehat{\mathcal{N}}_d^0$. We will call these equations *basic*.

The equations will be written in terms of the coordinates $\hat{u}^l_{\tau|\mathrm{dst}}$ on $\mathcal{N}_{\mathrm{dst}}$ introduced in (5.14), where now we assume that $|\tau| = 2$. Clearly, these variables form a dual basis to the basis $\{q^l_{mr}\}$ of $\widehat{\mathcal{N}}_d$. We will streamline our notation by writing $\hat{u}^l_{mr}$ instead of $\hat{u}^l_{[m,r]|\mathrm{dst}}$; naturally, we have $\hat{u}^l_{mr} = \hat{u}^l_{rm}$, and $r + m \le l$.

The construction is as follows. If $i + j + m \le l$, then the sequence

$$\boldsymbol{\pi}(i,j,m;l) = ([1],[2],\dots,[l-1],[i,j,m],[l+1],\dots,[d-1],[d])$$

is admissible but not complete; hence $\mathbf{u}^{\boldsymbol{\pi}(i,j,m;l)}$ will appear as a term of some of the relations $\mathrm{Rel}(\boldsymbol{\rho},\tau)$ introduced in Proposition 6.8. In fact, it appears in three different relations:

$$\text{for } \tau = [i],\ \rho_l = [j,m], \text{ for } \tau = [j],\ \rho_l = [i,m], \text{ and for } \tau = [m],\ \rho_l = [i,j];$$

in all cases $\rho_r = [r]$ for $r \neq l$. Next, we reduce the relation $\mathrm{Rel}(\boldsymbol{\rho},\tau)$ according to the prescription of Lemma 6.5. After the reduction, only the terms corresponding to the identity permutation and those corresponding to the transpositions of the form (s,l) survive; for example, in the case $\tau = [m]$, we obtain the "localized" relation

$$\hat{u}^l_{ijm} = \sum_{s=j+m}^{l-i} \hat{u}^s_{jm}\hat{u}^l_{is}. \tag{7.4}$$

Note that the number of terms on the right-hand side is $l - (i + j + m) + 1$, which is the defect of $\hat{u}^l_{ijm}$ plus 1.

We obtain two other expressions for $\hat{u}^l_{ijm}$ when we choose τ to be $[j]$ or $[k]$, and the resulting equalities provide us with quadratic relations among our variables $\hat{u}^l_{mr}$, $m + r \le l \le d$.

PROPOSITION 7.3. *Let $(i,j,m;l)$ be a quadruple of nonnegative integers satisfying $i + j + m \le l \le d$. Then the ideal of the variety $\widehat{\mathcal{O}} \subset \widehat{\mathcal{N}}_d$ contains the relations*

$$R(i,j,m;l):\quad \sum_{s=j+m}^{l-i} \hat{u}^s_{jm}\hat{u}^l_{is} = \sum_{s=i+m}^{l-j} \hat{u}^s_{im}\hat{u}^l_{js} = \sum_{s=i+j}^{l-m} \hat{u}^s_{ij}\hat{u}^l_{ms}. \tag{7.5}$$

Remark 7.4.
- In general, the quadruple $(i,j,m;l)$ gives us 2 relations. If $i = j \neq m$, then the number of relations reduces to 1, and if $i = j = m$, then (7.5) is vacuous.

- The equalities $R(i,j,m;l)$ with $i+j+m=l$ are relations of the toric orbit closure $\widehat{\mathcal{T}}\subset\widehat{\mathcal{N}}_d^0$. We will call these equations *toric*.

7.4. $d=4,5,6$. The first nontrivial case is $d=4$. Here $\deg\widehat{Q}_4=1$; i.e., $\widehat{\mathcal{O}}_4=\overline{B_4\hat{\varepsilon}_{\mathrm{ref}}}$ is a hypersurface in $\widehat{\mathcal{N}}_4$. Checking the table at the end of Section 7.1, we see that the codimension of the toric piece $\widehat{\mathcal{T}}_4$ in $\widehat{\mathcal{N}}_4^0$ is the same as the codimension of $\widehat{\mathcal{O}}_4$ in $\widehat{\mathcal{N}}_4$. This means that $\widehat{Q}_4=\mathrm{eP}[\widehat{\mathcal{T}}_4,\widehat{\mathcal{N}}_4^0]_{T_4}$.

It is not surprising then to find that the only basic equation is a toric one, corresponding to the quadruple $(1,1,2,4)$:

$$R(1,1,2;4):\quad \hat{u}_{11}^2\hat{u}_{22}^4=\hat{u}_{12}^3\hat{u}_{13}^4. \tag{7.6}$$

We note that this toric hypersurface is essentially our example from Section 2.3. The variety defined by (7.6) in $\widehat{\mathcal{N}}_4$ is irreducible and has the same dimension as $\widehat{\mathcal{O}}_4$; therefore it coincides with $\widehat{\mathcal{O}}_4$. We have already determined the equivariant Poincaré dual in this case in a number of ways: it is the sum of the weights of any of the monomials in the equation. This brings us to the formula

$$\widehat{Q}_4(z_1,z_2,z_3,z_4)=(2z_1-z_2)+(2z_2-z_4)=2z_1+z_2-z_4. \tag{7.7}$$

As a result, we obtain

$$\mathrm{Tp}_4^{n\to k}=\operatorname*{Res}_{z_1=\infty}\operatorname*{Res}_{z_2=\infty}\operatorname*{Res}_{z_3=\infty}\operatorname*{Res}_{z_4=\infty}\prod_{l=1}^{4}\mathrm{RC}\left(\frac{1}{z_l}\right)z_l^{k-n}\,dz_l$$
$$\frac{(z_1-z_2)(z_1-z_3)(z_1-z_4)(z_2-z_3)(z_2-z_4)(z_3-z_4)(2z_1+z_2-z_4)}{(2z_1-z_2)(z_1+z_2-z_3)(2z_1-z_3)(z_1+z_3-z_4)(2z_2-z_4)(z_1+z_2-z_4)(2z_1-z_4)}.$$

$d=5$. Again, we consult our table. We have $\dim\widehat{\mathcal{N}}_5=13$ and $\operatorname{codim}\widehat{\mathcal{O}}_5=3$, while $\dim\widehat{\mathcal{N}}_5^0=6$ and $\operatorname{codim}\widehat{\mathcal{T}}_5=2$.

Let us list our variables.

$$\begin{aligned}
6\text{ toric}:&\quad \hat{u}_{14}^5,\hat{u}_{23}^5,\hat{u}_{13}^4,\hat{u}_{22}^4,\hat{u}_{12}^3,\hat{u}_{11}^2,\\
4\text{ defect-1}:&\quad \hat{u}_{13}^5,\hat{u}_{22}^5,\hat{u}_{12}^4,\hat{u}_{11}^3,\\
2\text{ defect-2}:&\quad \hat{u}_{12}^5,\hat{u}_{11}^4,\ \text{and}\\
1\text{ defect-3}:&\quad \hat{u}_{11}^5.
\end{aligned}$$

There are 3 toric equations, which necessarily involve the toric variables only:

$$\begin{aligned}
R(1,1,2;4):&\quad \hat{u}_{12}^3\hat{u}_{13}^4=\hat{u}_{11}^2\hat{u}_{22}^4,\\
R(1,1,3;5):&\quad \hat{u}_{14}^5\hat{u}_{13}^4=\hat{u}_{23}^5\hat{u}_{11}^2,\\
R(1,2,2;5):&\quad \hat{u}_{14}^5\hat{u}_{22}^4=\hat{u}_{23}^5\hat{u}_{12}^3
\end{aligned} \tag{7.8}$$

and one defect-1 equation:

$$R(1,1,2;5):\quad \hat{u}_{13}^5\hat{u}_{12}^3+\hat{u}_{14}^5\hat{u}_{12}^4=\hat{u}_{11}^2\hat{u}_{22}^5+\hat{u}_{23}^5\hat{u}_{11}^3. \tag{7.9}$$

We observe that the toric equations (7.8) describe the vanishing of the three maximal minors of a 2×3 matrix. This is an irreducible toric variety; thus we can again argue that it coincides with $\widehat{\mathcal{T}_5}$. Fortunately, this variety is a special case of the A_1-singularity, this time with $n = 2$ and $k = 3$. Substituting the appropriate weights into (3.10), we obtain

(7.10)

$$\begin{aligned}
&\mathrm{eP}[\widehat{\mathcal{T}_5}, \widehat{\mathcal{N}_d^0}]_{T_5} \\
&= \frac{(z_1+z_2-z_3)(2z_1-z_2)(z_1+z_4-z_5)-(2z_2-z_4)(z_1+z_3-z_4)(z_2+z_3-z_5)}{z_1+z_4-z_2-z_3} \\
&= 2z_1^2+3z_1z_2-2z_1z_5+2z_2z_3-z_2z_4-z_2z_5-z_3z_4+z_4z_5.
\end{aligned}$$

Let M_5 denote the variety determined by the basic equations. Notice that for fixed $\hat{u}_{11}^2, \hat{u}_{12}^3, \hat{u}_{14}^5, \hat{u}_{23}^5$, (7.9) is linear in the remaining variables. This means that outside the codimension-2 subvariety $\widehat{\mathcal{T}_5'}$ in $\widehat{\mathcal{T}_5}$ where these four variables vanish, the natural projection $M_5 \to \widehat{\mathcal{T}_5}$ is the projection of a vector bundle onto its base, which implies that M_5 is irreducible, and thus $M_5 = \widehat{\mathcal{O}}_5$; the fibers of this vector bundle are hyperplanes in the 7-dimensional complement of $\widehat{\mathcal{N}_5^0}$ in $\widehat{\mathcal{N}_5}$. It is also clear from (7.9) that the variety determined by the relation $R(1,1,2,5)$ is transversal to $\mathrm{pr}_0^{-1}(\widehat{\mathcal{T}_5})$ outside the part lying over $\widehat{\mathcal{T}_5'}$, and hence we can conclude that $\mathrm{eP}[\widehat{\mathcal{O}}_5, \widehat{\mathcal{N}_5}]_{T_5}$ is the product of $\mathrm{eP}[\widehat{\mathcal{T}_5}, \widehat{\mathcal{N}_5^0}]_{T_5}$ and the weight of the relation $R(1,1,2;5)$. The latter equals $2z_1 + z_2 - z_5$, hence the final result is

$$\begin{aligned}
&\widehat{Q}_5(z_1, z_2, z_3, z_4, z_5) \\
&\quad = (2z_1 + z_2 - z_5)(2z_1^2 + 3z_1z_2 - 2z_1z_5 + 2z_2z_3 - z_2z_4 - z_2z_5 - z_3z_4 + z_4z_5).
\end{aligned}$$

$d = 6$. The polynomial $\widehat{Q}_6$ is of degree-7 in six variables, and one needs the help of a computer algebra program to do the calculations. Here we summarize our computations with Macaulay.

Let M_6 denote, again, the variety defined by the basic equations. It turns out that the codimension of M_6 in $\widehat{\mathcal{N}_6}$ is equal to the codimension of $\widehat{\mathcal{O}}_6$. However, M_6 contains two maximal dimensional components, namely,

$$M_6^1 = \langle \hat{u}_{11}^2, \hat{u}_{12}^3, \hat{u}_{11}^3, \hat{u}_{14}^5, \hat{u}_{14}^6, \hat{u}_{15}^6, \hat{u}_{24}^6 \rangle$$

and

$$M_6^2 = \langle \text{basic equations}, R \rangle,$$

where the extra relation is

$$\begin{aligned}
R = {}& \hat{u}_{12}^4\hat{u}_{12}^4\hat{u}_{23}^5\hat{u}_{33}^6 + \hat{u}_{22}^4\hat{u}_{13}^4\hat{u}_{12}^5\hat{u}_{33}^6 + \hat{u}_{13}^4\hat{u}_{13}^4\hat{u}_{22}^5\hat{u}_{23}^6 + \hat{u}_{22}^4\hat{u}_{13}^4\hat{u}_{23}^5\hat{u}_{13}^6 \\
&- \hat{u}_{22}^4\hat{u}_{11}^4\hat{u}_{23}^5\hat{u}_{33}^6 - \hat{u}_{13}^4\hat{u}_{12}^4\hat{u}_{22}^5\hat{u}_{33}^6 - \hat{u}_{22}^4\hat{u}_{13}^4\hat{u}_{13}^5\hat{u}_{23}^6 - \hat{u}_{13}^4\hat{u}_{13}^4\hat{u}_{23}^5\hat{u}_{22}^6
\end{aligned}$$

of weight $2z_1 + 3z_2 + 3z_3 - 2z_4 - z_5 - z_6$. Since $\widehat{\mathcal{O}}_6$ is irreducible, we have $\widehat{\mathcal{O}}_6 = M_6^2$. The other component, M_6^1, is a linear subspace, and we obtain $\widehat{Q}_6$ as

$$\widehat{Q}_6 = \mathrm{eP}[M_6, \widehat{\mathcal{N}}_6]_{T_6} - \mathrm{eP}[M_6^1, \widehat{\mathcal{N}}_6]_{T_6}.$$

Having described the vanishing ideal of $\widehat{\mathcal{O}}_6$ by explicit relations, using Macaulay, one then obtains $\widehat{Q}_6$; this formula is too long to present here.

7.5. *An application*: *the positivity of Thom polynomials*. It is conjectured in [23, Conj. 5.5] that all coefficients of the Thom polynomials $\mathrm{Tp}_d^{n\to k}$ expressed in terms of the relative Chern classes are nonnegative. Rimányi also proves that this property is special to the A_d-singularities. In this final paragraph, we would like to show that our formalism is well-suited to approach this problem. We will also formulate a more general positivity conjecture, which will imply this statement.

We start with a comment about the sign $(-1)^d$ in our main formula (6.26). Recall from (5.5) in Section 5.2 that, according to our convention, the iterated residue at infinity may be obtained by expanding the denominators in terms of z_i/z_j with $i < j$ and then *multiplying the result by* $(-1)^d$. This sign appears because of the change of orientation of the residue cycle when passing to the point at infinity. This means that if we compute (6.26) via expanding the denominators, then the sign in the formula cancels.

Now we are ready to formulate our positivity conjecture.

Conjecture. *Expanding the rational function*

$$\frac{\prod_{m<l}(z_m - z_l)\,\widehat{Q}_d(z_1,\ldots,z_d)}{\prod_{l=1}^d \prod_{m=1}^{l-1} \prod_{r=1}^{\min(m,l-m)} (z_m + z_r - z_l)}$$

in the domain $|z_1| \ll \cdots \ll |z_d|$, *one obtains a Laurent series with nonnegative coefficients.*

This statement clearly implies the nonnegativity of the coefficients of the Thom polynomial.

At the moment we do not know how to prove this conjecture in general. However, we observe that the expansion of a fraction of the form $(1 - f)/(1 - (f + g))$ with f and g small has positive coefficients. Indeed, this follows from the identity

$$\frac{1-f}{1-f-g} = 1 + \frac{g}{1-f-g}.$$

Now, introducing the variables $a = z_1/z_2$ and $b = z_2/z_3$, we can rewrite the above fraction in the $d = 3$ case as follows:

$$\frac{(z_1-z_2)(z_1-z_3)(z_2-z_3)}{(2z_1-z_2)(z_1+z_2-z_3)(2z_1-z_3)} = \frac{1-a}{1-2a}\cdot\frac{1-ab}{1-2ab}\cdot\frac{1-b}{1-b-ab}.$$

Applying the above identity to the right-hand side of this formula immediately implies our conjecture for $d = 3$. As a token reward for having followed our paper this far, we offer to the reader the rather amusing exercise of proving the same statement for $d = 4$.

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