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Multiplicity one theorems: the Archimedean case

By Binyong Sun and Chen-Bo Zhu

Abstract

Let G be one of the classical Lie groups $\mathrm{GL}_{n+1}(\mathbb{R})$, $\mathrm{GL}_{n+1}(\mathbb{C})$, $\mathrm{U}(p,q+1)$, $\mathrm{O}(p,q+1)$, $\mathrm{O}_{n+1}(\mathbb{C})$, $\mathrm{SO}(p,q+1)$, $\mathrm{SO}_{n+1}(\mathbb{C})$, and let G' be respectively the subgroup $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{GL}_n(\mathbb{C})$, $\mathrm{U}(p,q)$, $\mathrm{O}(p,q)$, $\mathrm{O}_n(\mathbb{C})$, $\mathrm{SO}(p,q)$, $\mathrm{SO}_n(\mathbb{C})$, embedded in G in the standard way. We show that every irreducible Casselman-Wallach representation of G' occurs with multiplicity at most one in every irreducible Casselman-Wallach representation of G. Similar results are proved for the Jacobi groups $\mathrm{GL}_n(\mathbb{R})\ltimes \mathrm{H}_{2n+1}(\mathbb{R})$, $\mathrm{GL}_n(\mathbb{C})\ltimes \mathrm{H}_{2n+1}(\mathbb{C})$, $\mathrm{U}(p,q)\ltimes \mathrm{H}_{2p+2q+1}(\mathbb{R})$, $\mathrm{Sp}_{2n}(\mathbb{R})\ltimes \mathrm{H}_{2n+1}(\mathbb{R})$, $\mathrm{Sp}_{2n}(\mathbb{C})\ltimes \mathrm{H}_{2n+1}(\mathbb{C})$, with their respective subgroups $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{GL}_n(\mathbb{C})$, $\mathrm{U}(p,q)$, $\mathrm{Sp}_{2n}(\mathbb{R})$, and $\mathrm{Sp}_{2n}(\mathbb{C})$.

Contents

1. Introduction and main results

Let G be one of the (five plus two) classical groups

$$\mathrm{GL}_{n+1}(\mathbb{R}),\ \mathrm{GL}_{n+1}(\mathbb{C}),\ \mathrm{U}(p,q+1),\ \mathrm{O}(p,q+1),\ \mathrm{O}_{n+1}(\mathbb{C}),\ \mathrm{SO}(p,q+1),\ \mathrm{SO}_{n+1}(\mathbb{C}), \tag{1}$$

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or one of the five Jacobi groups

(2) $\quad \mathrm{GL}_n(\mathbb{R}) \ltimes \mathrm{H}_{2n+1}(\mathbb{R})$, $\mathrm{GL}_n(\mathbb{C}) \ltimes \mathrm{H}_{2n+1}(\mathbb{C})$, $\mathrm{U}(p,q) \ltimes \mathrm{H}_{2p+2q+1}(\mathbb{R})$,

$$\mathrm{Sp}_{2n}(\mathbb{R}) \ltimes \mathrm{H}_{2n+1}(\mathbb{R}),\ \mathrm{Sp}_{2n}(\mathbb{C}) \ltimes \mathrm{H}_{2n+1}(\mathbb{C}), \quad p,q,n \geq 0.$$

Here "H_{2k+1}" indicates the appropriate Heisenberg group of dimension $2k+1$. A precise description of Jacobi groups is given in Section 8.

Let G' be respectively the subgroup

$$\mathrm{GL}_n(\mathbb{R}),\ \mathrm{GL}_n(\mathbb{C}),\ \mathrm{U}(p,q),\ \mathrm{O}(p,q),\ \mathrm{O}_n(\mathbb{C}),\ \mathrm{SO}(p,q),\ \mathrm{SO}_n(\mathbb{C}),$$

or

$$\mathrm{GL}_n(\mathbb{R}),\ \mathrm{GL}_n(\mathbb{C}),\ \mathrm{U}(p,q),\ \mathrm{Sp}_{2n}(\mathbb{R}),\ \mathrm{Sp}_{2n}(\mathbb{C})$$

embedded in G in the standard way. The main technical result of this paper is the following

THEOREM A. *There exists a real algebraic anti-automorphism σ on G preserving G' with the following property: every generalized function on G which is invariant under the adjoint action of G' is automatically σ-invariant.*

A set of anti-automorphisms which satisfy Theorem A is constructed in Section 8. For example, the matrix transpose is one such anti-automorphism when G is a general linear group.

By a representation of a Lie group, we mean a continuous linear action of the group on a (complete, Hausdorff, complex) locally convex topological vector space. When the Lie group is real reductive, a representation is said to be a Casselman-Wallach representation if it is Fréchet, smooth, of moderate growth, admissible and Z-finite. Here Z is the center of the universal enveloping algebra of its complexified Lie algebra. The reader may consult [Cas89], [Wal92, Chapter 11] or [BK] for more details about Casselman-Wallach representations.

By (a version of) the Gelfand-Kazhdan criterion ([SZ11, Cor. 2.5]), Theorem A for the classical groups implies the following result (which we call the multiplicity one theorem for Bessel models, as it implies uniqueness of the Bessel models ([GGP, JSZ10])).

THEOREM B. *Let G be one of the classical groups in* (1). *Let V* (*resp. V'*) *be an irreducible Casselman-Wallach representation of G* (*resp. G'*). *Then the space of G'-invariant continuous bilinear functionals on $V \times V'$ is at most one dimensional.*

Theorem B and its p-adic analog have been expected (by Bernstein, and Rallis) since the 1980's. When V' is the trivial representation, Theorem B is proved in [AGS08], [AGS09] and [vD09b], in the case of general linear, orthogonal, and unitary groups, respectively. The p-adic analog of Theorem B is proved in [AGRS10] (except for the case of special orthogonal groups, which is proved in [Wal]). When the initial manuscript of this paper was completed, the authors learned that A. Aizenbud and D. Gourevitch had proved the multiplicity one theorems for the pairs $(\mathrm{GL}_{n+1}(\mathbb{R}), \mathrm{GL}_n(\mathbb{R}))$ and $(\mathrm{GL}_{n+1}(\mathbb{C}), \mathrm{GL}_n(\mathbb{C}))$, independently and in a different approach. This has since appeared as [AG09b].

Now assume that G is one of the five Jacobi groups. Write $G = G' \ltimes H$, where H is an appropriate Heisenberg group. Fix a nontrivial unitary character ψ on the center of H. Let $\widetilde{G'}$ be a double cover of G' so that $\widetilde{G'} \ltimes H$ admits a smooth oscillator representation ω_ψ corresponding to ψ; that is, ω_ψ is a genuine smooth Fréchet moderate growth representation of $\widetilde{G'} \ltimes H$ which is irreducible with central character ψ when viewed as a representation of H. We say that a representation of G is a Casselman-Wallach ψ-representation if it is of the form $V = V_0 \widehat{\otimes} \omega_\psi$ (completed projective tensor product), where V_0 is a genuine Casselman-Wallach representation of $\widetilde{G'}$. This definition is independent of $\widetilde{G'}$ and ω_ψ. The representation V is irreducible if and only if V_0 is. See [Su10] for more details on Casselman-Wallach ψ-representations.

By the Gelfand-Kazhdan criterion for Jacobi groups ([Su10, Cor. D]), Theorem A for the Jacobi groups implies the following result (which we call the multiplicity one theorem for Fourier-Jacobi models, as it implies uniqueness of the Fourier-Jacobi models (cf. [GGP])).

Theorem C. *Let G be one of the Jacobi groups in (2). Let V be an irreducible Casselman-Wallach ψ-representation of G and let V' be an irreducible Casselman-Wallach representation of G'. Then the space of G'-invariant continuous bilinear functionals on $V \times V'$ is at most one dimensional.*

The p-adic analog of Theorem C was conjectured by D. Prasad ([Pra96, p. 20] in the case of symplectic groups) and is proved in [Su09].

2. **A uniform formulation**

We first introduce some general notation which will be used throughout the paper. For any (smooth) manifold M, denote by $C^{-\infty}(M)$ the space of generalized functions on M, which by definition consists of continuous linear functionals on $\mathrm{D}^\infty_c(M)$, the space of (complex) smooth densities on M with compact supports. The latter is equipped with the usual inductive smooth topology. For any locally closed subset Z of M, denote by

$$C^{-\infty}(M; Z) \subset C^{-\infty}(U) \tag{3}$$

the subspace consisting of all f which are supported in Z, where U is an open subset of M containing Z as a closed subset. This definition is independent of U.

If M is a Nash manifold, denote by $C^{-\xi}(M) \subset C^{-\infty}(M)$ the space of tempered generalized functions on M and by $C^{\varsigma}(M) \subset C^{-\xi}(M)$ the space of Schwartz functions. We refer the interested reader to [Shi87], [AG08] on generalities of Nash manifolds and their function spaces. Since the closure of every semialgebraic set is semialgebraic, given any locally closed semialgebraic subset Z of a Nash manifold M, we may find an open semialgebraic subset U of M containing Z as a closed subset. We define $C^{-\xi}(M;Z)$ as the subspace of $C^{-\xi}(U)$ consisting of all f which are supported in Z. Again this is independent of U.

If H is a Lie group acting smoothly on a manifold M, then for any character χ_H of H, denote by

$$C^{-\infty}_{\chi_H}(M) \subset C^{-\infty}(M) \tag{4}$$

the subspace consisting of all f which are χ_H-equivariant, i.e.,

$$f(h \cdot x) = \chi_H(h) f(x), \quad \text{for all } h \in H.$$

Similar notation (such as $C^{-\xi}_{\chi_H}(M;Z)$) will be used without further explanation.

We now proceed to describe a general set-up in order to work with all classical groups in a uniform manner.

Let A be a finite dimensional semi-simple commutative algebra over $\mathbb{R}$, which is thus a finite product of copies of $\mathbb{R}$ and $\mathbb{C}$. Let τ be a $\mathbb{R}$-algebra involution on A. We call (A, τ) (or A when τ is understood) a commutative involutive algebra (over $\mathbb{R}$). Let $\varepsilon = \pm 1$. Let E be an ε-Hermitian A-module, namely, it is a finitely generated A-module, equipped with a nondegenerate $\mathbb{R}$-bilinear map

$$\langle \, , \, \rangle_E : E \times E \to A$$

satisfying

$$\langle u, v \rangle_E = \varepsilon \langle v, u \rangle_E^{\tau}, \quad \langle au, v \rangle_E = a \langle u, v \rangle_E, \quad a \in A, \, u, v \in E.$$

Denote by $\mathrm{U}(E)$ the group of all A-module automorphisms of E which preserve the form $\langle \, , \, \rangle_E$, and by $\mathfrak{u}(E)$ its Lie algebra, which consists of all $x \in \mathrm{End}_A(E)$ such that

$$\langle xu, v \rangle_E + \langle u, xv \rangle_E = 0, \quad u, v \in E.$$

Write $E_{\mathbb{R}} := E$, viewed as a real vector space. Following Moeglin-Vigneras-Waldspurger ([MVW87]), we define a subgroup

$$\breve{\mathrm{U}}(E) \subset \mathrm{GL}(E_{\mathbb{R}}) \times \{\pm 1\} \tag{5}$$

consisting of pairs (g, δ) such that either

$$\delta = 1 \quad \text{and} \quad \langle gu, gv \rangle_E = \langle u, v \rangle_E, \;\; u, v \in E,$$

or

$$\delta = -1 \quad \text{and} \quad \langle gu, gv \rangle_E = \langle v, u \rangle_E, \;\; u, v \in E.$$

Note that for every element $(g, \delta) \in \breve{\mathrm{U}}(E)$, if $\delta = 1$, then g is automatically A-linear, and if $\delta = -1$, then g is τ-conjugate linear. Denote by

$$\chi_E : \breve{\mathrm{U}}(E) \to \{\pm 1\} \tag{6}$$

the quadratic character of $\breve{\mathrm{U}}(E)$ projecting to the second factor. It is a surjective homomorphism with kernel $\mathrm{U}(E)$.

Let $\breve{\mathrm{U}}(E)$ act on $\mathrm{U}(E)$ by

$$(g, \delta) \cdot x := g x^{\delta} g^{-1} \tag{7}$$

and act on $\mathfrak{u}(E)$ through its differential, i.e.,

$$(g, \delta) \cdot x := \delta g x g^{-1}.$$

It is known that every $\mathrm{U}(E)$-orbit in $\mathrm{U}(E)$ or $\mathfrak{u}(E)$ is $\breve{\mathrm{U}}(E)$-stable ([MVW87, Prop. 4.I.2]). Let $\breve{\mathrm{U}}(E)$ act on E by

$$(g, \delta) \cdot v := \delta g v \tag{8}$$

and act on $\mathrm{U}(E) \times E$ and $\mathfrak{u}(E) \times E$ diagonally.

The next five sections will be devoted to a proof of the following

THEOREM 2.1. *One has that*

$$C^{-\xi}_{\chi_E}(\mathfrak{u}(E) \times E) = 0.$$

3. Fourier transform and rigidity

Let F be a finite dimensional real vector space, which is canonically a Nash manifold. Denote by $\mathbb{C}[F]$ the algebra of (complex) polynomial functions and $D[F]$ the algebra of constant coefficient differential operators, on F. It is a classical result of L. Schwartz that

$$C^{-\xi}(F; \{0\}) = D[F]\delta_F.$$

Here δ_F is a Dirac function on F, which is characterized (up to a nonzero scalar) by the equation

$$\lambda \delta_F = 0 \quad \text{for all real linear functionals } \lambda \text{ on } F.$$

This is the simplest case of rigidity we have in mind.

From now on, we assume that F is equipped with a nondegenerate bilinear form $\langle \, , \, \rangle_F$ which is either symmetric or skew-symmetric.

We introduce one general notation which will be extensively used. If M is a Nash manifold, then we define the partial Fourier transform along F to be the topological linear automorphism

$$\mathcal{F}_F : C^{\varsigma}(M \times F) \to C^{\varsigma}(M \times F)$$

given by

$$(\mathcal{F}_F f)(m,y) = \int_F f(m,x) e^{2\pi\sqrt{-1}\,\langle x,y\rangle_F}\,dx, \quad m\in M,\ y\in F. \tag{9}$$

Here dx is any fixed Lebesgue measure on F. The partial Fourier transform uniquely extends to a topological linear isomorphism

$$\mathcal{F}_F : C^{-\xi}(M\times F) \to C^{-\xi}(M\times F).$$

When M reduces to a single point, we recover the usual Fourier transform.

When the factor F is understood, for any two closed semialgebraic subsets Z_1 and Z_2 of $M\times F$, put

(10)

$$C^{-\xi}(M\times F; Z_1, Z_2) := \{f\in C^{-\xi}(M\times F; Z_1) \mid \mathcal{F}_F(f)\in C^{-\xi}(M\times F; Z_2)\}.$$

For a subspace F' of F, let $F'^{\perp}$ denote its perpendicular space:

$$F'^{\perp} := \{v\in F \mid \langle v', v\rangle_F = 0,\ v'\in F'\}.$$

Lemma 3.1. *If $F = F'\oplus F''$ is a direct sum decomposition, then*

$$C^{-\xi}(F; F', F'^{\perp}) = \mathbb{C}[F']\otimes C^{-\xi}(F''; \{0\}).$$

Proof. Note that every tempered generalized function has a finite order. Hence by the well-known result of L. Schwartz about local representation of a generalized function with support, we have

$$C^{-\xi}(F; F') = C^{-\xi}(F')\otimes C^{-\xi}(F''; \{0\}).$$

The lemma then follows easily. □

For later use, we record the following

Proposition 3.2. *If F^0 is a nondegenerate subspace of F and*

$$(F^0)^{\perp} = F^+\oplus F^-$$

is a decomposition into totally isotropic subspaces F^+ and F^-, then

$$C^{-\xi}(F; F^+\oplus F^0, F^+\oplus F^0) = \mathbb{C}[F^+]\otimes C^{-\xi}(F^-; \{0\})\otimes C^{-\xi}(F^0).$$

Proof. The proof is similar to that of Lemma 3.1. □

We also need the following result, which is a special case of [SZ, Th. A].

Proposition 3.3. *Assume $\dim_{\mathbb{R}} F = 2k$. Let $F_1, F_2, \ldots, F_s$ be a set of (distinct) totally isotropic subspaces of F, each of dimension k. Then*

$$C^{-\xi}(F; F_1\cup F_2\cup\cdots\cup F_s, F_1\cup F_2\cup\cdots\cup F_s) = \bigoplus_{i=1}^{s} C^{-\xi}(F; F_i, F_i).$$

4. Nonnegativity of eigenvalues of an Euler vector field

We continue with the notation of Section 2. Set

$$\mathrm{U}(A) := \{a \in A^\times \mid a^\tau a = 1\}$$

and its Lie algebra

$$\mathfrak{u}(A) := \{a \in A \mid a^\tau + a = 0\}.$$

Scalar multiplication then yields a homomorphism $\mathrm{U}(A) \to \mathrm{U}(E)$ and its differential $\mathfrak{u}(A) \to \mathfrak{u}(E)$. Denote by $\mathrm{Z}(E)$ and $\mathfrak{z}(E)$ their respective images. Then $\mathrm{Z}(E)$ coincides with the center of $\mathrm{U}(E)$ (but $\mathfrak{z}(E)$ may not coincide with the center of $\mathfrak{u}(E)$).

Denote by

$$\mathrm{tr}_A : \mathrm{End}_A(E) \to A$$

the trace map. It is specified by requiring that the diagram

$$\begin{tikzcd} \mathrm{End}_A(E) \arrow[r, "\mathrm{tr}_A"] \arrow[d, "1_{A_0}\otimes"'] & A \arrow[d] \\ \mathrm{End}_{A_0}(A_0 \otimes_A E) \arrow[r, "\mathrm{tr}"] & A_0 \end{tikzcd}$$

commutes for every quotient field A_0 of A, where the bottom arrow is the usual trace map. Set

$$\mathfrak{sl}(E) := \{x \in \mathrm{End}_A(E) \mid \mathrm{tr}_A(x) = 0\}.$$

Then we have

$$\mathrm{End}_A(E) = \{\text{scalar multiplication by } a \in A\} \oplus \mathfrak{sl}(E) \tag{11}$$

and

$$\mathfrak{u}(E) = \mathfrak{z}(E) \oplus \mathfrak{su}(E),$$

where $\mathfrak{su}(E) := \mathfrak{u}(E) \cap \mathfrak{sl}(E)$.

We call the commutative involutive algebra A simple if it is nonzero and has no τ-stable ideal except $\{0\}$ and itself. Every simple commutative involutive algebra is isomorphic to one of the following:

$$(\mathbb{R}, 1),\ (\mathbb{C}, 1),\ (\mathbb{C}, ^{-}),\ (\mathbb{R} \times \mathbb{R}, \tau_{\mathbb{R}}),\ (\mathbb{C} \times \mathbb{C}, \tau_{\mathbb{C}}), \tag{12}$$

where $\tau_{\mathbb{R}}$ and $\tau_{\mathbb{C}}$ are the maps which interchange the coordinates.

Assume in the rest of the section that A is simple. We say that $(A, \tau; \varepsilon)$ is of orthogonal type if it is either $(\mathbb{R}, 1; 1)$ or $(\mathbb{C}, 1; 1)$. If $(A, \tau; \varepsilon)$ is not of orthogonal type, we fix a nonzero element $c_0 \in A$ so that

$$c_0 + \varepsilon c_0^\tau = 0.$$

For any $v \in E$, write

$$\phi_v(u) := \langle u, v\rangle_E\, v, \quad u \in E;$$

then $\phi_v \in \mathrm{End}_A(E)$. Denote by $\phi'_v \in \mathfrak{sl}(E)$ the projection of ϕ_v to the second factor according to the decomposition (11). For any $x \in \mathfrak{su}(E)$, set

$$\phi_{x,v} := \begin{cases} x\phi_v + \phi_v x, & \text{if } (A, \tau; \varepsilon) \text{ is of orthogonal type,} \\ c_0\, \phi'_v, & \text{otherwise.} \end{cases} \tag{13}$$

This is checked to be in $\mathfrak{su}(E)$.

Recall that an element of $\mathfrak{u}(E)$ is said to be nilpotent (semisimple) if it is nilpotent (semisimple) as a $\mathbb{R}$-linear operator on E. Recall also that a nilpotent element of $\mathfrak{u}(E)$ (which is automatically in $\mathfrak{su}(E)$) is said to be distinguished if it commutes with no nonzero semisimple element in $\mathfrak{su}(E)$ (cf. [CM93, §8.2]).

Fix a distinguished nilpotent element $\mathbf{e} \in \mathfrak{su}(E)$. Following [AGRS10], we define

$$E(\mathbf{e}) := \{\, v \in E \mid \phi_{\mathbf{e},v} \in [\mathfrak{su}(E), \mathbf{e}]\,\}. \tag{14}$$

Extend $\mathbf{e}$ (by Jacobson-Morozov Theorem) to a standard triple $\mathbf{h}, \mathbf{e}, \mathbf{f}$ in $\mathfrak{su}(E)$ so that

$$[\mathbf{h}, \mathbf{e}] = 2\mathbf{e}, \quad [\mathbf{h}, \mathbf{f}] = -2\mathbf{f}, \quad [\mathbf{e}, \mathbf{f}] = \mathbf{h}.$$

Denote by $E^i_{\mathbf{h}} \subset E$ the eigenspace of $\mathbf{h}$ with eigenvalue i, where $i \in \mathbb{Z}$. Write

$$E^+_{\mathbf{h}} := \bigoplus_{i>0} E^i_{\mathbf{h}}, \quad \text{and} \quad E^-_{\mathbf{h}} := \bigoplus_{i<0} E^i_{\mathbf{h}}.$$

Note that $\langle E^i_{\mathbf{h}}, E^j_{\mathbf{h}} \rangle_E = 0$ whenever $i + j \neq 0$.

Lemma 4.1. *If A is a field, then $E(\mathbf{e}) = E^+_{\mathbf{h}} \oplus E^0_{\mathbf{h}}$.*

Proof. We prove the lemma in the case of real orthogonal groups. The other cases are proved similarly. So assume that $(A, \tau; \varepsilon) = (\mathbb{R}, 1; 1)$. Then $\mathfrak{su}(E) = \mathfrak{o}(E)$ is a real orthogonal Lie algebra.

View E as a $\mathfrak{sl}_2(\mathbb{R})$-module via the standard triple. Let

$$E = E_1 \oplus E_2 \oplus \cdots \oplus E_s \tag{15}$$

be a decomposition of E into irreducible $\mathfrak{sl}_2(\mathbb{R})$-modules. By the classification of distinguished nilpotent orbits ([CM93, Th. 8.2.14]), we know that (15) is an orthogonal decomposition and $E_1, E_2, \ldots, E_s$ have pairwise distinct odd dimensions. Denote by $\mathbf{e}_i \in \mathfrak{o}(E_i) \subset \mathfrak{o}(E)$ the restriction of $\mathbf{e}$ to E_i.

View $\mathfrak{o}(E)$ as a real quadratic space under the trace form. For every $v \in E$, we have that $v \in E(\mathbf{e})$ if and only if

$$\begin{aligned} \phi_{\mathbf{e},v} \in [\mathfrak{o}(E), \mathbf{e}] &\Leftrightarrow \phi_{\mathbf{e},v} \perp [\mathfrak{o}(E), \mathbf{e}]^{\perp} \\ &\Leftrightarrow \phi_{\mathbf{e},v} \perp \mathfrak{o}(E)^{\mathbf{e}} \quad \text{(the centralizer of } \mathbf{e} \text{ in } \mathfrak{o}(E)) \\ &\Leftrightarrow \langle \mathbf{e}v, xv \rangle_E = 0 \quad \text{for all } x \in \mathfrak{o}(E)^{\mathbf{e}}. \end{aligned}$$

Thus if $v \in E(\mathbf{e})$, then we have $\langle \mathbf{e}v, \mathbf{e}_i^{2k+1} v\rangle_E = 0$ for all $1 \leq i \leq s$ and $k \geq 0$, and so v must be in $E^+_{\mathbf{h}} + E^0_{\mathbf{h}}$.

On the other hand, every element $x \in \mathfrak{o}(E)^{\mathbf{e}}$ stabilizes $E^+_{\mathbf{h}} + E^0_{\mathbf{h}}$. Therefore $v \in E^+_{\mathbf{h}} + E^0_{\mathbf{h}}$ implies that $\langle \mathbf{e}v, xv\rangle_E = 0$. This finishes the proof. $\square$

Denote by

$$\Gamma_E := \{v \in E \mid \langle v, v\rangle_E = 0\} \tag{16}$$

the null cone of E. Equip $E_{\mathbb{R}} = E$ with the (symmetric or skew-symmetric) bilinear form

$$\langle u, v\rangle_{E_{\mathbb{R}}} := \mathrm{tr}_{A/\mathbb{R}}(\langle u, v\rangle_E), \quad u, v \in E,$$

where $\mathrm{tr}_{A/\mathbb{R}} : A \to \mathbb{R}$ is the usual trace map.

Put

$$\mathcal{V}_{E,\mathbf{e}} := C^{-\xi}(E; E(\mathbf{e}) \cap \Gamma_E, E(\mathbf{e}) \cap \Gamma_E)^{\mathrm{Z}(E)}, \tag{17}$$

where, as usual, a superscript by a group indicates the group invariants. This space arises naturally when one carries out the reduction within the null cone. See Lemma 5.5.

For any finite dimensional real vector space F and any $x \in \mathrm{End}_{\mathbb{R}}(F)$, denote by $\varepsilon_{F,x}$ the vector field on F whose tangent vector at $v \in F$ is xv. When $x = 1$ is the identity operator, this is the usual Euler vector field $\varepsilon_F := \varepsilon_{F,1}$.

The main result of this section is the following

Proposition 4.2. *The vector field $\varepsilon_{E,\mathbf{h}}$ acts semisimply on $\mathcal{V}_{E,\mathbf{e}}$, and all its eigenvalues are nonnegative integers.*

If A is a field, then

$$\begin{aligned}\mathcal{V}_{E,\mathbf{e}} &\subset C^{-\xi}(E; E(\mathbf{e}), E(\mathbf{e}))\\ &= C^{-\xi}(E; E^+_{\mathbf{h}} \oplus E^0_{\mathbf{h}}, E^+_{\mathbf{h}} \oplus E^0_{\mathbf{h}}) \quad (\text{Lemma } 4.1)\\ &= \mathbb{C}[E^+_{\mathbf{h}}] \otimes C^{-\xi}(E^-_{\mathbf{h}}; \{0\}) \otimes C^{-\xi}(E^0_{\mathbf{h}}) \quad (\text{Proposition } 3.2),\end{aligned}$$

and Proposition 4.2 follows easily.

Otherwise assume that $(A, \tau) = (\mathbb{K} \times \mathbb{K}, \tau_{\mathbb{K}})$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Up to an isomorphism, every ε-Hermitian A-module is of the form

$$(E, \langle\,,\,\rangle_E) = (\mathbb{K}^n \oplus \mathbb{K}^n, \langle\,,\,\rangle_{\mathbb{K},n}), \quad n \geq 0,$$

where $\mathbb{K}^n$ is considered as a space of column vectors, and the ε-Hermitian form $\langle\,,\,\rangle_{\mathbb{K},n}$ is given by

$$\left(\begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} u' \\ v' \end{bmatrix} \right) \mapsto ({v'}^t u, \varepsilon {u'}^t v).$$

Then

$$\mathrm{U}(E) = \left\{ \begin{bmatrix} g & 0 \\ 0 & g^{-t} \end{bmatrix} \mid g \in \mathrm{GL}_n(\mathbb{K}) \right\} = \mathrm{GL}_n(\mathbb{K})$$

and

$$\mathfrak{u}(E) = \left\{ \begin{bmatrix} x & 0 \\ 0 & -x^t \end{bmatrix} \mid x \in \mathfrak{gl}_n(\mathbb{K}) \right\} = \mathfrak{gl}_n(\mathbb{K}).$$

In this case every distinguished nilpotent element of $\mathfrak{su}(E)$ is principal, and so we may assume that

$$\mathbf{e} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ & & \cdots & \cdots & & \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

and

$$\mathbf{h} = \mathrm{diag}(n-1, n-3, \dots, 3-n, 1-n).$$

As in the proof of Lemma 4.1, one easily calculates $E(\mathbf{e})$, and one has

$$E(\mathbf{e}) \cap \Gamma_E = \cup_{i=0}^n F_i, \quad \text{where } F_i = (\mathbb{K}^i \oplus \{0\}^{n-i}) \oplus (\{0\}^i \oplus \mathbb{K}^{n-i}). \tag{18}$$

Proposition 3.3 implies that

$$\mathcal{V}_{E,\mathbf{e}} = C^{-\xi}(E; E(\mathbf{e}) \cap \Gamma_E, E(\mathbf{e}) \cap \Gamma_E)^{\mathrm{Z}(E)} = \bigoplus_{i=0}^n C^{-\xi}(E; F_i, F_i)^{\mathrm{Z}(E)}.$$

To finish the proof of Proposition 4.2, it therefore suffices to prove the following

Lemma 4.3. *The vector field $\varepsilon_{E,\mathbf{h}}$ acts semisimply on $C^{-\xi}(E; F_i, F_i)^{\mathrm{Z}(E)}$, and all its eigenvalues are nonnegative integers* $(0 \le i \le n)$.

Proof. We prove the lemma for $\mathbb{K} = \mathbb{R}$. The complex case is proved in the same way.

Denote by $x_1, x_2, \dots, x_n$, $y_1, y_2, \dots, y_n$ the standard coordinates of $\mathbb{R}^n \oplus \mathbb{R}^n$, and write $\partial_j = \frac{\partial}{\partial x_j}$ and $d_j = \frac{\partial}{\partial y_j}$ for $j = 1, 2, \dots, n$.

By Lemma 3.1, the space $C^{-\xi}(E; F_i, F_i)$ has a basis consisting of generalized functions of the form

$$\begin{aligned} f = {} & x_1^{a_1} x_2^{a_2} \cdots x_i^{a_i}\, y_{i+1}^{b_{i+1}} y_{i+2}^{b_{i+2}} \cdots y_n^{b_n} \\ & \otimes \partial_{i+1}^{a_{i+1}-1} \partial_{i+2}^{a_{i+2}-1} \cdots \partial_n^{a_n-1}\, d_1^{b_1-1} d_2^{b_2-1} \cdots d_i^{b_i-1} \delta_{F_i'}, \end{aligned}$$

where $a_1, \dots, a_i, b_{i+1}, \dots, b_n$ are nonnegative integers, and the rest of a's and b's are positive integers. Here $\delta_{F_i'}$ is a fixed Dirac function on the space

$$F_i' := (\{0\}^i \oplus \mathbb{R}^{n-i}) \oplus (\mathbb{R}^i \oplus \{0\}^{n-i}), \quad \text{(a complement of } F_i\text{)}.$$

The generalized function f as above is an eigenvector for both $\mathrm{Z}(E)$ and $\varepsilon_{E,\mathbf{h}}$. For f to be $\mathrm{Z}(E)$-invariant, we must have

$$\sum_{j \le i} (a_j + b_j) = \sum_{j > i} (a_j + b_j).$$

Then the $\varepsilon_{E,\mathbf{h}}$-eigenvalue of f is

$$\begin{aligned}
&\sum_{j\leq i}(n-(2j-1))a_j - \sum_{j>i}(n-(2j-1))a_j \\
&\qquad + \sum_{j\leq i}(n-(2j-1))b_j - \sum_{j>i}(n-(2j-1))b_j \\
&\quad \geq (n-2i)\Big(\sum_{j\leq i} a_j\Big) - (n-2i)\Big(\sum_{j>i} a_j\Big) \\
&\qquad + (n-2i)\Big(\sum_{j\leq i} b_j\Big) - (n-2i)\Big(\sum_{j>i} b_j\Big) = 0.
\end{aligned}$$ □

5. Reduction within the null cone: distinguished orbits

We continue to assume that (A,τ) is simple. Since Theorem 2.1 is trivial when $E=0$, we will assume that E is nonzero. Denote by $\mathcal{N}_E \subset \mathfrak{su}(E)$ the null cone, which consists of all nilpotent elements in $\mathfrak{su}(E)$. Let

$$\mathcal{N}_E = \mathcal{N}_0 \supset \mathcal{N}_1 \supset \cdots \supset \mathcal{N}_r = \{0\} \supset \mathcal{N}_{r+1} = \emptyset \tag{19}$$

be a filtration of $\mathcal{N}_E$ by its closed subsets so that each difference

$$\mathcal{O}_i := \mathcal{N}_i \setminus \mathcal{N}_{i+1}, \quad 0 \leq i \leq r$$

is a $\breve{\mathrm{U}}(E)$-orbit (which is also a $\mathrm{U}(E)$-orbit). In this and the next section, we shall prove the following

Proposition 5.1. *If every element of $C^{-\xi}_{\chi_E}(\mathfrak{su}(E)\times E)$ is supported in $\mathcal{N}_i \times \Gamma_E$ for some fixed $0 \leq i \leq r$, then every element of $C^{-\xi}_{\chi_E}(\mathfrak{su}(E)\times E)$ is supported in $\mathcal{N}_{i+1}\times\Gamma_E$.*

For the ease of notation, denote $\mathfrak{s} := \mathfrak{su}(E)$. We shall view $\mathfrak{s}$ as a nondegenerate real quadratic space via the form

$$\langle x, y\rangle_{\mathfrak{s}} := \mathrm{tr}_{A/\mathbb{R}}(\mathrm{tr}_A(xy)).$$

It is easily verified (and important for us) that the partial Fourier transforms $\mathcal{F}_E$ and $\mathcal{F}_{\mathfrak{s}}$ both preserve the space $C^{-\xi}_{\chi_E}(\mathfrak{s}\times E)$.

Lemma 5.2. *Proposition 5.1 holds when $\mathfrak{s}=0$.*

Proof. For $\mathfrak{s}=0$, the assumption of Proposition 5.1 implies that $C^{-\xi}_{\chi_E}(\mathfrak{s}\times E) \subset C^{-\xi}(E;\Gamma_E,\Gamma_E)^{\mathrm{Z}(E)}$. The latter space is easily checked to be zero. □

For the remaining part of this section, assume that $\mathfrak{s}\neq 0$.

Before proceeding further, we introduce a version of pull back of generalized functions.

Definition 5.3. Let Z and Z' be locally closed subsets of manifolds M and M', respectively. A smooth map $\phi : M \to M'$ is said to be submersive from Z to Z' if

- ϕ is submersive at every point of Z, and
- for every $z \in Z$, there is an open neighborhood U of z in M such that

$$\phi^{-1}(Z') \cap U = Z \cap U.$$

The following lemma is elementary.

LEMMA 5.4. *If $\phi : M \to M'$ is submersive from Z to Z', as in Definition 5.3, then there is a unique linear map*

$$\phi^* : C^{-\infty}(M'; Z') \to C^{-\infty}(M; Z) \tag{20}$$

with the following property: for any open subset U of M and U' of M', if

- *ϕ restricts to a submersive map $\phi_U : U \to U'$,*
- *$Z' \cap U'$ is closed in U', and*
- *$\phi_U^{-1}(Z' \cap U') = Z \cap U$,*

then the diagram

$$\begin{array}{ccc} C^{-\infty}(M'; Z') & \xrightarrow{\phi^*} & C^{-\infty}(M; Z) \\ \downarrow & & \downarrow \\ C^{-\infty}(U'; Z' \cap U') & \xrightarrow{\phi_U^*} & C^{-\infty}(U; Z \cap U) \end{array}$$

commutes, where the two vertical arrows are restrictions, and the bottom arrow is the usual pull back map of generalized functions via a submersion.

The map ϕ^* in (20) is still called the pull back. It is injective if $\phi(Z) = Z'$. In this case, we say that ϕ is submersive from Z onto Z'. If M, M' are Nash manifolds, Z, Z' are locally closed semialgebraic subsets and ϕ is a Nash map which is submersive from Z to Z', then ϕ^* maps $C^{-\xi}(M'; Z')$ into $C^{-\xi}(M; Z)$ (cf. [AG09a, §B.2]).

We continue the proof of Proposition 5.1.

Fix $i \in \{0, 1, \ldots, r\}$ and assume that $\mathcal{O}_i$ is distinguished; namely, some (so all) elements of it are distinguished. We use the notation of last section. Put

$$\mathcal{Z}_i := (\mathcal{N}_{i+1} \times \Gamma_E) \cup \Big(\bigsqcup_{\mathbf{e} \in \mathcal{O}_i} \{\mathbf{e}\} \times (E(\mathbf{e}) \cap \Gamma_E) \Big). \tag{21}$$

One checks that $\mathcal{Z}_i$ is a closed semialgebraic subset of $\mathfrak{s} \times E$.

LEMMA 5.5. *Assume that every element of $C^{-\xi}_{\chi_E}(\mathfrak{s} \times E)$ is supported in $\mathcal{N}_i \times \Gamma_E$. Then every $f \in C^{-\xi}_{\chi_E}(\mathfrak{s} \times E)$ is supported in $\mathcal{Z}_i$.*

Proof. We follow the method of [AGRS10]. For every $t \in \mathbb{R}$, define a map

$$\begin{array}{rcl} \eta := \eta_t : \mathfrak{s} \times E & \to & \mathfrak{s} \times E, \\ (x, v) & \mapsto & (x - t\phi_{x,v}, v), \end{array}$$

which is checked to be submersive from $\mathfrak{s} \times \Gamma_E$ to $\mathfrak{s} \times \Gamma_E$. Therefore, by Lemma 5.4, it yields a pull back map

$$\eta^* : C^{-\infty}(\mathfrak{s} \times E; \mathfrak{s} \times \Gamma_E) \quad \to \quad C^{-\infty}(\mathfrak{s} \times E; \mathfrak{s} \times \Gamma_E).$$

Fix $f \in C^{-\xi}_{\chi_E}(\mathfrak{s} \times E)$. By our assumption,

$$f \in C^{-\xi}_{\chi_E}(\mathfrak{s} \times E; \mathcal{N}_i \times \Gamma_E) \subset C^{-\xi}_{\chi_E}(\mathfrak{s} \times E; \mathfrak{s} \times \Gamma_E).$$

Since the map η is algebraic and $\breve{\mathrm{U}}(E)$-equivariant,

$$\eta^*(f) \in C^{-\xi}_{\chi_E}(\mathfrak{s} \times E; \mathfrak{s} \times \Gamma_E).$$

Let $(\mathbf{e}, v) \in \mathcal{O}_i \times \Gamma_E$ be a point in the support of f. It is routine to check that η restricts to a bijection from $\mathfrak{s} \times \Gamma_E$ onto itself. Denote by

$$\mathbf{e}' := \mathbf{e}'(\mathbf{e}, v, t) \in \mathfrak{s}$$

the unique element so that

$$\eta(\mathbf{e}', v) = (\mathbf{e}, v).$$

Then $(\mathbf{e}', v)$ is in the support of $\eta^*(f)$, and therefore our assumption implies that

$$\mathbf{e}' \in \mathcal{N}_i. \tag{22}$$

An easy calculation shows that

$$\mathbf{e}' = \begin{cases} \mathbf{e} + t\phi_{\mathbf{e},v} + t^2\phi_v \mathbf{e} \phi_v, & \text{if } (A, \tau; \varepsilon) \text{ is of othogonal type,} \\ \mathbf{e} + t\phi_{\mathbf{e},v}, & \text{otherwise.} \end{cases}$$

Since $\mathcal{O}_i$ is open in $\mathcal{N}_i$, (22) implies that

$$\phi_{\mathbf{e},v} = \frac{d}{dt}|_{t=0}\, \mathbf{e}'(\mathbf{e}, v, t) \in \mathrm{T}_{\mathbf{e}}(\mathcal{O}_i) = [\mathfrak{u}(E), \mathbf{e}] = [\mathfrak{s}, \mathbf{e}],$$

i.e., $v \in E(\mathbf{e})$, and the proof is now complete. □

Denote by

$$\mathcal{V}_{\mathfrak{s} \times E, \mathcal{O}_i} \subset C^{-\xi}(\mathfrak{s} \times E; \mathcal{O}_i \times E)^{\mathrm{U}(E)} \tag{23}$$

the subspace consisting of those f such that both f and $\mathcal{F}_E(f)$ are supported in $\bigsqcup_{\mathbf{e} \in \mathcal{O}_i} \{\mathbf{e}\} \times (E(\mathbf{e}) \cap \Gamma_E)$.

Proposition 5.6. *The Euler vector field $\varepsilon_{\mathfrak{s}}$ acts semisimply on $\mathcal{V}_{\mathfrak{s} \times E, \mathcal{O}_i}$, and all its eigenvalues are real numbers $< -\frac{1}{2} \dim_{\mathbb{R}} \mathfrak{s}$.*

Let us first prove the following

LEMMA 5.7. *Proposition 5.6 implies Proposition 5.1 when $\mathcal{O}_i$ is distinguished.*

Proof. Denote by $q_{\mathfrak{s}}$ the quadratic form on $\mathfrak{s}$, i.e.,

$$q_{\mathfrak{s}}(x) = \langle x, x \rangle_{\mathfrak{s}} = \mathrm{tr}_{A/\mathbb{R}}(\mathrm{tr}_A(x^2)).$$

Denote by $\Delta_{\mathfrak{s}}$ the Laplacian operator associated to $q_{\mathfrak{s}}$. The operators

$$\varepsilon_{\mathfrak{s}} + \frac{1}{2}\dim_{\mathbb{R}} \mathfrak{s}, \quad -\frac{1}{2}q_{\mathfrak{s}}, \quad \frac{1}{2}\Delta_{\mathfrak{s}}$$

form a standard triple, and each of them leaves the space $\mathcal{V}_{\mathfrak{s}\times E,\mathcal{O}_i}$ stable. Proposition 5.6 says that $\varepsilon_{\mathfrak{s}} + \frac{1}{2}\dim_{\mathbb{R}} \mathfrak{s}$ is semisimple and has negative eigenvalues on $\mathcal{V}_{\mathfrak{s}\times E,\mathcal{O}_i}$, and so by [Wal88, Lemma 8.A.5.1], the map

$$\Delta_{\mathfrak{s}} : \mathcal{V}_{\mathfrak{s}\times E,\mathcal{O}_i} \to \mathcal{V}_{\mathfrak{s}\times E,\mathcal{O}_i}$$

is injective.

Let $f \in C^{-\xi}_{\chi_E}(\mathfrak{s} \times E)$. Applying Lemma 5.5 to f and $\mathcal{F}_E(f)$, we conclude that under the restriction map

$$r_{\mathfrak{s}\times E} : C^{-\xi}_{\chi_E}(\mathfrak{s} \times E) \subset C^{-\xi}(\mathfrak{s} \times E; \mathcal{N}_i \times E) \to C^{-\xi}(\mathfrak{s} \times E; \mathcal{O}_i \times E),$$

the image

$$r_{\mathfrak{s}\times E}(f) \in \mathcal{V}_{\mathfrak{s}\times E,\mathcal{O}_i}.$$

Since $\mathcal{F}_{\mathfrak{s}}(f) \in C^{-\xi}_{\chi_E}(\mathfrak{s} \times E)$ is supported in

$$\mathcal{N}_i \times \Gamma_E \subset (\text{the null cone of the real quadratic space } \mathfrak{s}) \times E,$$

we conclude that f and thus $r_{\mathfrak{s}\times E}(f)$ are annihilated by some positive power of $\Delta_{\mathfrak{s}}$. By the injectivity of $\Delta_{\mathfrak{s}}$ on $\mathcal{V}_{\mathfrak{s}\times E,\mathcal{O}_i}$, we conclude that $r_{\mathfrak{s}\times E}(f) = 0$, and we are done. □

The remaining part of this section is devoted to a proof of Proposition 5.6.

Pick any element $\mathbf{e} \in \mathcal{O}_i$ and extend it to a standard triple $\mathbf{h}, \mathbf{e}, \mathbf{f} \in \mathfrak{s}$. Then we have a vector space decomposition

$$\mathfrak{s} = [\mathfrak{s}, \mathbf{e}] \oplus \mathfrak{s}^{\mathbf{f}}.$$

Let $\mathrm{U}(E)$ act on $\mathrm{U}(E) \times \mathfrak{s}^{\mathbf{f}} \times E$ via the left translation on the first factor. Define a $\mathrm{U}(E)$-equivariant map

$$\begin{aligned} \theta : \mathrm{U}(E) \times \mathfrak{s}^{\mathbf{f}} \times E &\to \mathfrak{s} \times E, \\ (g, x, v) &\mapsto g \cdot (x + \mathbf{e}, v). \end{aligned} \tag{24}$$

LEMMA 5.8. *The vector field*

$$\iota_{\mathbf{h}/2} + \varepsilon_{\mathfrak{s}^{\mathbf{f}},1-\mathrm{ad}(\mathbf{h}/2)} - \varepsilon_{E,\mathbf{h}/2}$$

on $\mathrm{U}(E) \times \mathfrak{s}^{\mathbf{f}} \times E$ *is* θ*-related to the Euler vector field* $\varepsilon_{\mathfrak{s}}$ *on* $\mathfrak{s} \times E$*, where* $\iota_{\mathbf{h}/2}$ *is the left invariant vector field on* $\mathrm{U}(E)$ *whose tangent vector at the identity is* $\mathbf{h}/2$.

Proof. Since both vector fields under consideration are $\mathrm{U}(E)$-invariant, it suffices to prove the θ-relatedness at a point of the form

$$\mathbf{x} := (1, x, v) \in \mathrm{U}(E) \times \mathfrak{s}^{\mathbf{f}} \times E.$$

Under the differential of θ at $\mathbf{x}$, we have

$$\begin{array}{lcl} \iota_{\mathbf{h}/2}|_{\mathbf{x}} = (\mathbf{h}/2, 0, 0) & \mapsto & ([\mathbf{h}/2, x + \mathbf{e}], (\mathbf{h}/2)v), \\ \varepsilon_{\mathfrak{s}^{\mathbf{f}}, 1-\mathrm{ad}(\mathbf{h}/2)}|_{\mathbf{x}} = (0, x - [\mathbf{h}/2, x], 0) & \mapsto & (x - [\mathbf{h}/2, x], 0), \\ \varepsilon_{E, \mathbf{h}/2}|_{\mathbf{x}} = (0, 0, (\mathbf{h}/2)v) & \mapsto & (0, (\mathbf{h}/2)v). \end{array}$$

This implies the lemma since $\varepsilon_{\mathfrak{s}}|_{\theta(\mathbf{x})} = (x + \mathbf{e}, 0)$. □

Let $\mathrm{Z}(E)$ act on $\mathfrak{s}^{\mathbf{f}} \times E$ and $\mathrm{U}(E) \times \mathfrak{s}^{\mathbf{f}} \times E$ via its action on the factor E. Then the map θ is $\mathrm{Z}(E)$-equivariant as well. Note that θ is submersive from $\mathrm{U}(E) \times \{0\} \times E$ onto $\mathcal{O}_i \times E$ (cf. [Wal88, p. 299]). Therefore it yields an injective pull back map

$$\theta^* : C^{-\xi}(\mathfrak{s} \times E; \mathcal{O}_i \times E)^{\mathrm{U}(E)} \hookrightarrow C^{-\xi}(\mathrm{U}(E) \times \mathfrak{s}^{\mathbf{f}} \times E; \mathrm{U}(E) \times \{0\} \times E)^{\mathrm{U}(E) \times \mathrm{Z}(E)}.$$

Denote by

$$r_{\mathfrak{s}^{\mathbf{f}} \times E} : C^{-\xi}(\mathrm{U}(E) \times \mathfrak{s}^{\mathbf{f}} \times E; \mathrm{U}(E) \times \{0\} \times E)^{\mathrm{U}(E) \times \mathrm{Z}(E)} \to C^{-\xi}(\mathfrak{s}^{\mathbf{f}} \times E; \{0\} \times E)^{\mathrm{Z}(E)}$$

the linear isomorphism specified by the rule

$$f = 1 \otimes r_{\mathfrak{s}^{\mathbf{f}} \times E} f. \tag{25}$$

Recall the space $\mathcal{V}_{E, \mathbf{e}} \subset C^{-\xi}(E)$ from (17).

LEMMA 5.9. *The composition map* $r_{\mathfrak{s}^{\mathbf{f}} \times E} \circ \theta^*$ *sends* $\mathcal{V}_{\mathfrak{s} \times E, \mathcal{O}_i}$ *into* $C^{-\xi}(\mathfrak{s}^{\mathbf{f}}; \{0\}) \otimes \mathcal{V}_{E, \mathbf{e}}$*, and the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{V}_{\mathfrak{s} \times E, \mathcal{O}_i} & \xhookrightarrow{r_{\mathfrak{s}^{\mathbf{f}} \times E} \circ \theta^*} & C^{-\xi}(\mathfrak{s}^{\mathbf{f}}; \{0\}) \otimes \mathcal{V}_{E, \mathbf{e}} \\ \Big\downarrow {\scriptstyle \varepsilon_{\mathfrak{s}}} & & \Big\downarrow {\scriptstyle \varepsilon_{\mathfrak{s}^{\mathbf{f}}, 1-\mathrm{ad}(\mathbf{h}/2)} - \varepsilon_{E, \mathbf{h}/2}} \\ \mathcal{V}_{\mathfrak{s} \times E, \mathcal{O}_i} & \xhookrightarrow{r_{\mathfrak{s}^{\mathbf{f}} \times E} \circ \theta^*} & C^{-\xi}(\mathfrak{s}^{\mathbf{f}}; \{0\}) \otimes \mathcal{V}_{E, \mathbf{e}}. \end{array}$$

Proof. The first assertion follows by noting that both θ^* and $r_{\mathfrak{s}^{\mathbf{f}}, E}$ commute with the partial Fourier transform along E. The second assertion follows from Lemma 5.8. □

LEMMA 5.10. *The vector field* $\varepsilon_{\mathfrak{s}^{\mathbf{f}}, 1-\mathrm{ad}(\mathbf{h}/2)}$ *acts semisimply on* $C^{-\xi}(\mathfrak{s}^{\mathbf{f}}; \{0\})$*, and all its eigenvalues are real numbers* $< -\frac{1}{2} \dim_{\mathbb{R}} \mathfrak{s}$.

Proof. Recall that $\mathfrak{s}$ is assumed to be nonzero. We view $\mathfrak{s}$ as a $\mathfrak{sl}_2(\mathbb{R})$-module via the adjoint representation and the standard triple $\{\mathbf{h}, \mathbf{e}, \mathbf{f}\}$. We shall prove that the analog of Lemma 5.10 holds for any finite dimensional nonzero $\mathfrak{sl}_2(\mathbb{R})$-module F. Without loss of generality, we may assume that F is irreducible of real dimension $k+1$. Then

$$\varepsilon_{F^{\mathbf{f}},1-\mathbf{h}/2} = (1 + k/2)\varepsilon_{F^{\mathbf{f}}},$$

which clearly acts semisimply on $C^{-\xi}(F^{\mathbf{f}}; \{0\})$, with all its eigenvalues real numbers $\leq -(1 + k/2) = -\frac{1}{2}\dim_{\mathbb{R}} F - \frac{1}{2} < -\frac{1}{2}\dim_{\mathbb{R}} F$. □

In view of Lemma 5.9, Proposition 5.6 will follow from Proposition 4.2 and Lemma 5.10. We have thus proved Proposition 5.1 when $\mathcal{O}_i$ is distinguished.

6. Reduction within the null cone: metrically proper orbits

We are in the same setting as Section 5, so (A, τ) is simple and E is nonzero. Now assume that $\mathcal{O}_i$ is not distinguished. The purpose of this section is to prove Proposition 5.1 in this case.

If F is a nondegenerate finite dimensional real quadratic space, we say that a submanifold S of F is metrically proper ([JSZ11]) if for every $x \in S$, the tangent space $\mathrm{T}_x(S)$ is contained in a proper nondegenerate subspace of the real quadratic space F.

Lemma 6.1. *The orbit $\mathcal{O}_i$ is metrically proper in $\mathfrak{s}$.*

Proof. Let $x \in \mathcal{O}_i$. By definition, it commutes with a nonzero semisimple element $h \in \mathfrak{s}$. Denote by $\mathfrak{a}_h$ the center of $\mathfrak{s}^h$ (the centralizer of h in $\mathfrak{s}$), which is a nonzero nondegenerate subspace of $\mathfrak{s}$.

Using the fact that every element of $\mathfrak{a}_h$ commutes with x, we see that the tangent space

$$\mathrm{T}_x(\mathcal{O}_i) = [\mathfrak{u}(E), x] = [\mathfrak{s}, x]$$

is contained in the proper nondegenerate subspace $(\mathfrak{a}_h)^\perp \subset \mathfrak{s}$. □

The following lemma is a form of the uncertainty principle.

Lemma 6.2. *Let M be a Nash manifold and let F be a nondegenerate finite dimensional real quadratic space. Let $Z_1 \supset Z_2$ be closed semialgebraic subsets of F so that the difference $Z_1 \setminus Z_2$ is a metrically proper submanifold of F. Then*

$$C^{-\xi}(M \times F; M \times Z_1, M \times \Gamma_F) = C^{-\xi}(M \times F; M \times Z_2, M \times \Gamma_F),$$

where Γ_F is the null cone of F.

Proof. This is a direct consequence of [JSZ11, Lemma 2.2]. □

In view of Lemma 6.2, Proposition 5.1 in the metrically proper case follows by noting that the partial Fourier transform $\mathcal{F}_{\mathfrak{s}}$ preserves the space $C^{-\xi}_{\chi_E}(\mathfrak{s}\times E)$ and that $\mathcal{N}_i$ is contained in the null cone of the real quadratic space $\mathfrak{s}$.

7. Reduction to the null cone and proof of Theorem 2.1

Now let E be an ε-Hermitian A-module, with (A,τ) arbitrary. Define an involution on $\mathrm{End}_A(E)$, which is still denoted by τ, by requiring that

$$\langle xu, v\rangle_E = \langle u, x^\tau v\rangle_E, \quad x\in \mathrm{End}_A(E),\ u,v\in E. \tag{26}$$

For any x in $\mathrm{End}_A(E)$, let A_x be the subalgebra generated by x, x^τ and scalar multiplications by A. If x is a semisimple element in $\mathrm{U}(E)$ or $\mathfrak{u}(E)$, then (A_x,τ) is a commutative involutive algebra. Write $E_x := E$, to be viewed as an ε-Hermitian A_x-module with the form $\langle\,,\,\rangle_{E_x}$. The latter is specified by

$$\mathrm{tr}_{A_x/\mathbb{R}}(a\langle u,v\rangle_{E_x}) = \mathrm{tr}_{A/\mathbb{R}}(\langle au, v\rangle_E), \quad u,v\in E,\ a\in A_x.$$

Write

$$(A,\tau)=(A_1,\tau_1)\times(A_2,\tau_2)\times\cdots\times(A_l,\tau_l)$$

as a product of simple commutative involutive algebras. We then have

$$E = E_1\times E_2\times\cdots\times E_l, \tag{27}$$

where $E_j := A_j\otimes_A E$, which is naturally an ε-Hermitian A_j-module. Note that E_j is free as an A_j-module. Put

$$\mathrm{sdim}(E) := \sum_{j=1}^{l}\max\{\mathrm{rank}_{A_j}(E_j)-1,\,0\}+\dim_{\mathbb{R}}(E). \tag{28}$$

The following result may be considered as a case of Harish-Chandra descent.

Proposition 7.1. *Assume that for all commutative involutive algebra A' and all ε-Hermitian A'-module E',*

$$\mathrm{sdim}(E') < \mathrm{sdim}(E) \quad \textit{implies} \quad C^{-\xi}_{\chi_{E'}}(\mathfrak{u}(E')\times E') = 0. \tag{29}$$

Then every $f\in C^{-\xi}_{\chi_E}(\mathfrak{u}(E)\times E)$ is supported in $(\mathfrak{z}(E)+\mathcal{N}_E)\times E$.

Proof. Let x be a semisimple element in $\mathfrak{u}(E)\setminus\mathfrak{z}(E)$. Then $\mathrm{sdim}(E_x) < \mathrm{sdim}(E)$.

For any $y\in\mathfrak{u}(E_x)$, denote by $J(y)$ the determinant of the $\mathbb{R}$-linear map

$$[y,\cdot] : \mathfrak{u}(E)/\mathfrak{u}(E_x)\to\mathfrak{u}(E)/\mathfrak{u}(E_x).$$

Then J is a $\breve{\mathrm{U}}(E_x)$-invariant function on $\mathfrak{u}(E_x)$. Put

$$\mathfrak{u}(E_x)^\circ := \{y\in\mathfrak{u}(E_x)\mid J(y)\neq 0\},$$

which contains $x + \mathcal{N}_{E_x}$. The map

$$\begin{array}{rcl} \pi_x : \breve{\mathrm{U}}(E) \times (\mathfrak{u}(E_x)^\circ \times E_x) & \rightarrow & \mathfrak{u}(E) \times E, \\ (g, y, v) & \mapsto & g \cdot (y, v) \end{array}$$

is a submersion. Therefore we have a well-defined restriction map ([JSZ11, Lemma 4.4])

$$r_{E,E_x} : C^{-\xi}_{\chi_E}(\mathfrak{u}(E) \times E) \rightarrow C^{-\xi}_{\chi_{E_x}}(\mathfrak{u}(E_x)^\circ \times E_x), \tag{30}$$

which is specified by the rule

$$\pi_x^*(f) = \chi_E \otimes r_{E,E_x}(f).$$

The assumption (29) easily implies that the latter space in (30) is zero. Thus every $f \in C^{-\xi}_{\chi_E}(\mathfrak{u}(E) \times E)$ vanishes on the image of π_x. As x is arbitrary, the proposition follows. □

PROPOSITION 7.2. *Assume that A is simple, and for all commutative involutive algebra A' and all ε-Hermitian A'-module E',*

$$\mathrm{sdim}(E') < \mathrm{sdim}(E) \quad \textit{implies} \quad C^{-\xi}_{\chi_{E'}}(\mathfrak{u}(E') \times E') = 0.$$

Then every $f \in C^{-\xi}_{\chi_E}(\mathfrak{u}(E) \times E)$ is supported in $\mathfrak{u}(E) \times \Gamma_E$.

Proof. The proof is similar to that of [AGRS10, Prop. 5.2]. □

We are now ready to prove Theorem 2.1, which will be by induction on $\mathrm{sdim}(E)$. If $\mathrm{sdim}(E)$ is 0, then $E = 0$ and the theorem is trivial. Now assume that E is nonzero, and we have proved Theorem 2.1 when $\mathrm{sdim}(E)$ is smaller.

Without loss of generality, assume that E is faithful as an A-module. If A is not simple, then for $1 \leq j \leq l$,

$$\mathrm{sdim}(E_j) < \mathrm{sdim}(E) \quad \text{and thus} \quad C^{-\xi}_{\chi_{E_j}}(\mathfrak{u}(E_j) \times E_j) = 0.$$

This clearly implies that $C^{-\xi}_{\chi_E}(\mathfrak{u}(E) \times E) = 0$.

Otherwise assume that A is simple. Note that $\breve{\mathrm{U}}(E)$ acts trivially on $\mathfrak{z}(E)$. Propositions 7.1 and 7.2 imply that every element in $C^{-\xi}_{\chi_E}(\mathfrak{su}(E) \times E)$ is supported in $\mathcal{N}_E \times \Gamma_E$, and Proposition 5.1 further implies that $C^{-\xi}_{\chi_E}(\mathfrak{su}(E) \times E) = 0$. Therefore $C^{-\xi}_{\chi_E}(\mathfrak{u}(E) \times E) = 0$, and the proof of Theorem 2.1 is now complete.

8. Proof of Theorem A

The argument of this section is standard and we will thus be brief. As before, let (A, τ) be a commutative involutive algebra and let E be an ε-Hermitian A-module.

THEOREM 8.1. *One has that $C^{-\infty}_{\chi_E}(\mathfrak{u}(E) \times E) = 0$.*

Proof. In view of Theorem 2.1, this follows from a general principle of "distributions versus Schwartz distributions" ([AG09a, Th. 4.0.2]). $\square$

THEOREM 8.2. *One has that* $C^{-\infty}_{\chi_E}(\mathrm{U}(E) \times E) = 0$.

Proof. Again we prove by induction on $\mathrm{sdim}(E)$ and assume that the theorem holds when $\mathrm{sdim}(E)$ is smaller. As in the proof of Proposition 7.1, we show that

$$C^{-\infty}_{\chi_E}(\mathrm{U}(E) \times E) = C^{-\infty}_{\chi_E}(\mathrm{U}(E) \times E; (\mathrm{Z}(E)\mathcal{U}_E) \times E),$$

where $\mathcal{U}_E$ is the set of unipotent elements in $\mathrm{U}(E)$. Note that $\breve{\mathrm{U}}(E)$ acts on $\mathrm{Z}(E)$ trivially. The map

$$\begin{array}{rcl} \rho_E : \mathrm{Z}(E) \times \mathfrak{su}(E) \times E & \rightarrow & \mathrm{U}(E) \times E, \\ (z, x, v) & \mapsto & (z \exp(x), v) \end{array}$$

is $\breve{\mathrm{U}}(E)$-equivariant and yields an injective pull back map

$$C^{-\infty}_{\chi_E}(\mathrm{U}(E) \times E; (\mathrm{Z}(E)\mathcal{U}_E) \times E) \xhookrightarrow{\rho_E^*} C^{-\infty}_{\chi_E}(\mathrm{Z}(E) \times \mathfrak{su}(E) \times E; \mathrm{Z}(E) \times \mathcal{N}_E \times E).$$

The latter space vanishes by Theorem 8.1 and the result follows. $\square$

Assume for the moment that (A, τ) is simple and $\varepsilon = 1$. Let $E = E' \oplus Av_0$ be an orthogonal decomposition with $v_0 \notin \Gamma_E$ (the null cone of E). Then $\breve{\mathrm{U}}(E')$ is identified with the stabilizer of $v_0 \in E$ in $\breve{\mathrm{U}}(E)$ via the embedding

$$(g, \delta) \mapsto \left(\begin{bmatrix} \delta g & 0 \\ 0 & \tau_\delta \end{bmatrix}, \delta \right), \tag{31}$$

where $\tau_\delta : Av_0 \to Av_0$ is the $\mathbb{R}$-linear map given by

$$\tau_\delta(av_0) = \begin{cases} av_0, & \text{if } \delta = 1, \\ -a^\tau v_0, & \text{if } \delta = -1. \end{cases}$$

The following result is a consequence of Theorem 8.2 in the case of $\varepsilon = 1$. We refer the reader to [AGRS10, Prop. 5.1] for the necessary argument.

COROLLARY 8.3. *Let the notation be as in this section. Let* $\breve{\mathrm{U}}(E')$ *act on* $\mathrm{U}(E)$ *through the action of* $\breve{\mathrm{U}}(E)$. *Then* $C^{-\infty}_{\chi_{E'}}(\mathrm{U}(E)) = 0$.

Corollary 8.3 implies Theorem A for the first five classical groups of this paper.

Now assume that (A, τ) is simple and $\varepsilon = -1$. Write

$$\mathrm{H}(E) := E \times A^{\tau=1} \quad \text{(where } A^{\tau=1} \text{ is the set of } \tau\text{-fixed elements in } A)$$

for the Heisenberg group with group multiplication

$$(u, t)(u', t') = \left(u + u', t + t' + \frac{\langle u, u' \rangle_E}{2} - \frac{\langle u', u \rangle_E}{2} \right).$$

Let $\breve{\mathrm{U}}(E)$ act on $\mathrm{H}(E)$ as group automorphisms by

$$(g,\delta)\cdot(u,t):=(gu,\delta t).$$

We form the semidirect products (the Jacobi groups)

$$\breve{\mathrm{J}}(E):=\breve{\mathrm{U}}(E)\ltimes \mathrm{H}(E)\supset \mathrm{J}(E):=\mathrm{U}(E)\ltimes \mathrm{H}(E). \tag{32}$$

The following result is a consequence of Theorem 8.2 in the case of $\varepsilon=-1$. The necessary argument can be found in [vD09a, Th. 3.1] or [Su09, Th. D].

COROLLARY 8.4. *Let the notation be as in this section. Let $\breve{\mathrm{U}}(E)$ act on $\mathrm{J}(E)$ by*

$$g\cdot x:=gx^{\chi_E(g)}g^{-1}.$$

Then $C^{-\xi}_{\chi_E}(\mathrm{J}(E))=0$.

Corollary 8.4 implies Theorem A for all Jacobi groups.

Finally we come to the special orthogonal groups. The key idea to establish this variant is due to Waldspurger [Wal] and it is to introduce another extended group. Assume that $\varepsilon=1$. If E is a quadratic space (i.e., A is $\mathbb{R}$ or $\mathbb{C}$, and τ is trivial), then define

$$\breve{\mathrm{SO}}(E):=\left\{(g,\delta)\in \mathrm{O}(E)\times\{\pm1\}\mid \det(g)=\delta^{\left[\frac{\dim_A E+1}{2}\right]}\right\}\supset \mathrm{SO}(E).$$

Denote by $\chi_{\mathrm{s},E}$ the quadratic character on $\breve{\mathrm{SO}}(E)$ with kernel $\mathrm{SO}(E)$. Let $\breve{\mathrm{SO}}(E)$ act on $\mathrm{SO}(E)$ and E as in equations (7) and (8), respectively.

THEOREM 8.5. *One has that $C^{-\infty}_{\chi_{\mathrm{s},E}}(\mathrm{SO}(E)\times E)=0$.*

The descent process in the proof of Theorem 8.5 requires us to define a compatible family of extended groups. First assume that (A,τ) is simple. Define

$$(\breve{\mathrm{U}}_{\mathrm{s}}(E),\mathrm{U}_{\mathrm{s}}(E)):=\begin{cases}(\breve{\mathrm{SO}}(E),\mathrm{SO}(E)), & \text{if } \tau \text{ is trivial},\\ (\breve{\mathrm{U}}(E),\mathrm{U}(E)), & \text{if } \tau \text{ is nontrivial}.\end{cases}$$

In general, write $E=E_1\times E_2\times\cdots\times E_l$ as in (27). We put

$$\mathrm{U}_{\mathrm{s}}(E):=\mathrm{U}_{\mathrm{s}}(E_1)\times \mathrm{U}_{\mathrm{s}}(E_2)\times\cdots\times \mathrm{U}_{\mathrm{s}}(E_l)$$

and

$$\begin{aligned}\breve{\mathrm{U}}_{\mathrm{s}}(E)&:=\breve{\mathrm{U}}_{\mathrm{s}}(E_1)\times_{\{\pm1\}}\breve{\mathrm{U}}_{\mathrm{s}}(E_2)\times_{\{\pm1\}}\cdots\times_{\{\pm1\}}\breve{\mathrm{U}}_{\mathrm{s}}(E_l)\\ &:=\{(g_1,g_2,\dots,g_l,\delta)\mid (g_j,\delta)\in\breve{\mathrm{U}}_{\mathrm{s}}(E_j),\ j=1,2,\dots,l\}.\end{aligned}$$

The latter contains the former as a subgroup of index two. Let $\breve{\mathrm{U}}_{\mathrm{s}}(E)$ act on $\mathrm{U}_{\mathrm{s}}(E)$ and E, again as in (7) and (8).

In the notation of this paper, Waldspurger's observation may be stated as follows.

LEMMA 8.6. *Let x be a semisimple element of $\mathrm{U_s}(E)$ and let E_x be as in Section 7. Then $x \in \mathrm{U_s}(E_x)$, and $\breve{\mathrm{U}}_{\mathrm{s}}(E_x)$ is contained in the stabilizer of x in $\breve{\mathrm{U}}_{\mathrm{s}}(E)$.*

LEMMA 8.7. *Assume that A is simple. Let $v \in E \setminus \Gamma_E$. Then the stabilizer of v in $\breve{\mathrm{U}}_{\mathrm{s}}(E)$ is naturally isomorphic to $\breve{\mathrm{U}}_{\mathrm{s}}(E')$, where E' is the orthogonal complement of Av in E.*

The argument of this paper, together with the above two lemmas, will imply Theorem 8.5. We skip the details. Theorem 8.5 implies the analog of Corollary 8.3 for special orthogonal groups. Theorem A for special orthogonal groups then follows.

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AMSS, Chinese Academy of Sciences, Beijing, China
E-mail: sun@math.ac.cn

National University of Singapore, Singapore
E-mail: matzhucb@nus.edu.sg