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On a problem in simultaneous Diophantine approximation: Schmidt's conjecture

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Dedicated to the memory of Graham Everest and Antonia J. Jones

Abstract

For any $i,j \geqslant 0$ with $i+j=1$, let $\mathbf{Bad}(i,j)$ denote the set of points $(x,y) \in \mathbb{R}^2$ for which $\max\{\|qx\|^{1/i}, \|qy\|^{1/j}\} > c/q$ for all $q \in \mathbb{N}$. Here $c = c(x,y)$ is a positive constant. Our main result implies that any finite intersection of such sets has full dimension. This settles a conjecture of Wolfgang M. Schmidt in the theory of simultaneous Diophantine approximation.

1. Introduction

A real number x is said to be *badly approximable* if there exists a positive constant $c(x)$ such that

$$\|qx\| \ > \ c(x)\ q^{-1} \quad \forall\ q \in \mathbb{N}\,.$$

Here and throughout $\|\cdot\|$ denotes the distance of a real number to the nearest integer. It is well known that the set $\mathbf{Bad}$ of badly approximable numbers is of Lebesgue measure zero. However, a result of Jarník (1928) states that

$$\dim \mathbf{Bad} = 1\ , \tag{1}$$

where $\dim X$ denotes the Hausdorff dimension of the set X. Thus, in terms of dimension the set of badly approximable numbers is maximal; it has the same dimension as the real line. For details regarding Hausdorff dimension the reader is referred to [3].

In higher dimensions there are various natural generalizations of $\mathbf{Bad}$. Restricting our attention to the plane $\mathbb{R}^2$, given a pair of real numbers i and j such that

$$0 \leqslant i,j \leqslant 1 \quad \text{and} \quad i+j=1\,, \tag{2}$$

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a point $(x, y) \in \mathbb{R}^2$ is said to be (i, j)-*badly approximable* if there exists a positive constant $c(x, y)$ such that

$$\max\{\|qx\|^{1/i},\ \|qy\|^{1/j}\} \ > \ c(x,y)\ q^{-1} \quad \forall\, q \in \mathbb{N}\,.$$

Denote by $\mathbf{Bad}(i, j)$ the set of (i, j)-badly approximable points in $\mathbb{R}^2$. If $i = 0$, then we use the convention that $x^{1/i} := 0$, and so $\mathbf{Bad}(0, 1)$ is identified with $\mathbb{R} \times \mathbf{Bad}$. That is, $\mathbf{Bad}(0, 1)$ consists of points (x, y) with $x \in \mathbb{R}$ and $y \in \mathbf{Bad}$. The roles of x and y are reversed if $j = 0$. It easily follows from classical results in the theory of metric Diophantine approximation that $\mathbf{Bad}(i, j)$ is of (two-dimensional) Lebesgue measure zero. Building upon the work of Davenport [2] from 1964, it has recently been shown in [7] that $\dim \mathbf{Bad}(i, j) = 2$. For further background and various strengthenings of this full dimension statement the reader is referred to [4], [5], [7]. A consequence of the main result obtained in this paper is the following statement.

THEOREM 1. *Let* $(i_1, j_1), \ldots, (i_d, j_d)$ *be a finite number of pairs of real numbers satisfying* (2). *Then*

$$\dim\Big(\bigcap_{t=1}^{d} \mathbf{Bad}(i_t, j_t)\Big) = 2\,.$$

Therefore, the intersection of any finitely many badly approximable sets $\mathbf{Bad}(i, j)$ is trivially nonempty and thereby establishes the following conjecture of Wolfgang M. Schmidt [8] from the eighties.

SCHMIDT'S CONJECTURE *For any* (i_1, j_1) *and* (i_2, j_2) *satisfying* (2), *we have that*

$$\mathbf{Bad}(i_1, j_1) \cap \mathbf{Bad}(i_2, j_2) \neq \emptyset\,.$$

To be precise, Schmidt stated the specific problem with $i_1 = 1/3$ and $j_1 = 2/3$, and even this has previously resisted attack. Indeed, the statement

$$\dim(\mathbf{Bad}(1, 0) \cap \mathbf{Bad}(0, 1) \cap \mathbf{Bad}(i, j)) = 2$$

first obtained in [7] sums up all previously known results.

As noted by Schmidt, a counterexample to his conjecture would imply the famous Littlewood conjecture: for any $(x, y) \in \mathbb{R}^2$,

$$\liminf_{q \to \infty} q\,\|qx\|\,\|qy\| = 0\,.$$

Indeed, the same conclusion is valid if there exists any finite (or indeed countable) collection of pairs (i_t, j_t) satisfying (2) for which the intersection of the sets $\mathbf{Bad}(i_t, j_t)$ is empty. However, Theorem 1 implies that no such finite collection exists and Littlewood's conjecture remains very much alive and kicking.

For background and recent developments regarding Littlewood's conjecture, see [6], [9].

1.1. *The main theorem.* The key to establishing Theorem 1 is to investigate the intersection of the sets $\mathbf{Bad}(i_t, j_t)$ along fixed vertical lines in the (x, y)-plane. With this in mind, let L_x denote the line parallel to the y-axis passing through the point $(x, 0)$. Next, for any real number $0 \leqslant i \leqslant 1$, define the set

$$\mathbf{Bad}(i) := \{x \in \mathbb{R} : \exists\, c(x) > 0 \text{ so that } \|qx\| > c(x)\, q^{-1/i} \quad \forall\, q \in \mathbb{N}\}\,.$$

Clearly,

$$\mathbf{Bad} = \mathbf{Bad}(1) \subseteq \mathbf{Bad}(i)\,, \tag{3}$$

which together with (1) implies that

$$\dim \mathbf{Bad}(i) = 1 \quad \forall\, i \in [0, 1]\ . \tag{4}$$

In fact, a straightforward argument involving the Borel-Cantelli lemma from probability theory enables us to conclude that, for $i < 1$, the complement of $\mathbf{Bad}(i)$ is of Lebesgue measure zero. In other words, for $i < 1$, the set $\mathbf{Bad}(i)$ is not only of full dimension but of full measure.

We are now in the position to state our main theorem.

THEOREM 2. *Let (i_t, j_t) be a countable number of pairs of real numbers satisfying* (2) *and let $i := \sup\{i_t : t \in \mathbb{N}\}$. Suppose that*

$$\liminf_{t \to \infty} \min\{i_t, j_t\} > 0\,. \tag{5}$$

Then, for any $\theta \in \mathbf{Bad}(i)$, we have that

$$\dim \Big(\bigcap_{t=1}^{\infty} \mathbf{Bad}(i_t, j_t) \cap \mathrm{L}_\theta \Big) = 1\,.$$

The hypothesis imposed on θ is absolutely necessary. Indeed, for $\theta \notin \mathbf{Bad}(i)$, it is readily verified that the intersection of the sets $\mathbf{Bad}(i_t, j_t)$ along the line L_θ is empty; see Section 1.3 for the details. However, in view of (3), the dependence of θ on i and therefore the pairs (i_t, j_t) can be entirely removed by insisting that $\theta \in \mathbf{Bad}$. Obviously, the resulting statement is cleaner but nevertheless weaker than Theorem 2.

On the other hand, the statement of Theorem 2 is almost certainly valid without imposing the 'lim inf' condition. Indeed, this is trivially true if the number of (i_t, j_t) pairs is finite. In the course of establishing the theorem, it will become evident that in the countable 'infinite' case we require (5) for an important but nevertheless technical reason. It would be desirable to remove (5) from the statement of the theorem.

The following corollary is technically far easier to establish than the theorem and is more than adequate for establishing Schmidt's conjecture.

Corollary 1. *Let* $(i_1, j_1), \ldots, (i_d, j_d)$ *be a finite number of pairs of real numbers satisfying* (2). *Then, for any* $\theta \in \mathbf{Bad}$, *we have that*

$$\bigcap_{t=1}^{d} \mathbf{Bad}(i_t, j_t) \cap \mathrm{L}_\theta \neq \emptyset \,.$$

We give a self-contained proof of the corollary during the course of establishing Theorem 2.

Remark. The corollary is of independent interest even when $d = 1$. Since the work of Davenport [2], it has been known that there exist badly approximable numbers x and y such that (x, y) is also a badly approximable pair; i.e., $\mathbf{Bad}(1, 0) \cap \mathbf{Bad}(0, 1) \cap \mathbf{Bad}(1/2, 1/2) \neq \emptyset$. However it was not possible, using previous methods, to specify which x one might take. Corollary 1 implies that we can take x to be any badly approximable number. So, for example, there exist $y \in \mathbf{Bad}$ such that $(\sqrt{2}, y) \in \mathbf{Bad}(1/2, 1/2)$. Moreover, Theorem 2 implies that

$$\dim \Big(\{ y \in \mathbf{Bad} : (\sqrt{2}, y) \in \mathbf{Bad}(1/2, 1/2) \} \Big) = 1 \,.$$

1.2. *Theorem* 2 $\implies$ *Theorem* 1. We show that Theorem 2 implies the following countable version of Theorem 1.

Theorem 1′ . *Let* (i_t, j_t) *be a countable number of pairs of real numbers satisfying* (2). *Suppose that* (5) *is also satisfied. Then*

$$\dim \Big(\bigcap_{t=1}^{\infty} \mathbf{Bad}(i_t, j_t) \Big) = 2 \,.$$

Note that if the number of (i_t, j_t) pairs is finite, the 'lim inf' condition is trivially satisfied and Theorem 1′ reduces to Theorem 1.

We proceed to establish Theorem 1′ modulo Theorem 2. Since any set $\mathbf{Bad}(i, j)$ is a subset of $\mathbb{R}^2$, we immediately obtain the upper bound result that

$$\dim \Big(\bigcap_{t=1}^{\infty} \mathbf{Bad}(i_t, j_t) \Big) \leqslant 2 \,.$$

The following general result that relates the dimension of a set to the dimensions of parallel sections, enables us to establish the complementary lower bound estimate; see [3, p. 99].

Proposition . *Let* F *be a subset of* $\mathbb{R}^2$ *and let* E *be a subset of the* x*-axis. If* $\dim(F \cap \mathrm{L}_x) \geqslant t$ *for all* $x \in E$, *then* $\dim F \geqslant t + \dim E$.

With reference to the proposition, let F be a countable intersection of $\mathbf{Bad}(i,j)$ sets and let E be the set $\mathbf{Bad}$. In view of (1) and Theorem 2, the lower bound result immediately follows. Since (1) is classical and the upper bound statement for the dimension is trivial, the main ingredient in establishing Theorem 1′ (and therefore Theorem 1) is Theorem 2.

Remark. It is self-evident that removing (5) from the statement of Theorem 2 would enable us to remove (5) from the statement of Theorem 1′. In other words, it would enable us to established in full the countable version of Schmidt's conjecture.

1.3. *The dual form.* At the heart of the proof of Theorem 2 is an intervals construction that enables us to conclude that

$$\mathbf{Bad}(i,j) \cap \mathrm{L}_\theta \neq \emptyset \qquad \forall\, \theta \in \mathbf{Bad}(i)\,.$$

Note that this is essentially the statement of Corollary 1 with $d = 1$. The case when either $i = 0$ or $j = 0$ is relatively straightforward, so let us assume that

$$0 < i, j < 1 \quad \text{and} \quad i + j = 1\,. \tag{6}$$

In order to carry out the construction alluded to above, we shall work with the equivalent dual form representation of the set $\mathbf{Bad}(i,j)$. In other words, a point $(x,y) \in \mathbf{Bad}(i,j)$ if there exists a positive constant $c(x,y)$ such that

$$\max\{|A|^{1/i}, |B|^{1/j}\}\ \|Ax - By\| > c(x,y) \qquad \forall\ (A,B) \in \mathbb{Z}^2 \backslash \{(0,0)\}\ . \tag{7}$$

Consider for the moment the case $B = 0$. Then, (7) simplifies to the statement that

$$|A|^{1/i}\ \|Ax\| > c(x) \qquad \forall\ A \in \mathbb{Z}\backslash\{0\}\ .$$

It now becomes obvious that for a point (x,y) in the plane to have any chance of being in $\mathbf{Bad}(i,j)$, we must have that $x \in \mathbf{Bad}(i)$. Otherwise, (7) is violated and $\mathbf{Bad}(i,j) \cap \mathrm{L}_x = \emptyset$. This justifies the hypothesis imposed on θ in Theorem 2.

For i and j satisfying (6), the equivalence of the 'simultaneous' and 'dual' forms of $\mathbf{Bad}(i,j)$ is a consequence of the transference principle described in [1, Chap. 5]. To be absolutely precise, without obvious modification, the principle as stated in [1] only implies the equivalence in the case $i = j = 1/2$. In view of this and for the sake of completeness, we have included the modified statement and its proof as an appendix.

Notation. For a real number r we denote by $[r]$ its integer part and by $\lceil r \rceil$ the smallest integer not less than r. For a subset X of $\mathbb{R}^n$ we denote by $|X|$ its Lebesgue measure.

2. The overall strategy

Fix i and j satisfying (6) and $\theta \in \mathbf{Bad}(i)$ satisfying $0 < \theta < 1$. Let Θ denote the segment of the vertical line L_θ lying within the unit square; i.e.,

$$\Theta := \{(x, y) : x = \theta, y \in [0, 1]\}\,.$$

In this section we describe the basic intervals construction that enables us to conclude that

$$\mathbf{Bad}(i, j) \cap \Theta \;\neq\; \emptyset\,.$$

As mentioned in Section 1.3, the basic construction lies at the heart of establishing Theorem 2.

2.1. *The sets* $\mathbf{Bad}_c(i, j)$. For any constant $c > 0$, let $\mathbf{Bad}_c(i, j)$ denote the set of points $(x, y) \in \mathbb{R}^2$ such that

$$\max\{|A|^{1/i}, |B|^{1/j}\}\, \|Ax - By\| > c \qquad \forall\ (A, B) \in \mathbb{Z}^2 \backslash \{(0, 0)\}\ . \tag{8}$$

It is easily seen that $\mathbf{Bad}_c(i, j) \subset \mathbf{Bad}(i, j)$ and

$$\mathbf{Bad}(i, j) \;=\; \bigcup_{c>0} \mathbf{Bad}_c(i, j)\,.$$

Geometrically, given integers A, B, C with $(A, B) \neq (0, 0)$, consider the line $L = L(A, B, C)$ defined by the equation

$$Ax - By + C = 0\,.$$

The set $\mathbf{Bad}_c(i, j)$ simply consists of points in the plane that avoid the

$$\frac{c}{\max\{|A|^{1/i}, |B|^{1/j}\}\, \sqrt{A^2 + B^2}}$$

thickening of each line L; alternatively, points in the plane that lie within any such neighbourhood are removed. With reference to our fixed $\theta \in \mathbf{Bad}(i)$, let us assume that

$$c(\theta) \geqslant c > 0\,. \tag{9}$$

Then, by definition

$$|A|^{1/i}\, \|A\theta\| > c \qquad \forall\ A \in \mathbb{Z} \backslash \{0\} \tag{10}$$

and the line L_θ (and therefore the segment Θ) avoids the thickening of any vertical line $L = L(A, 0, C)$. *Thus, without loss of generality, we can assume that* $B \neq 0$. With this in mind, it is easily verified that the thickening of a line $L = L(A, B, C)$ will remove from Θ an interval $\Delta(L)$ centered at (θ, y) with

$$y = \frac{A\theta + C}{B}$$

and length

$$(11)\quad |\Delta(L)| = \frac{2c}{H(A,B)} \qquad \text{where} \qquad H(A,B) := |B| \max\{|A|^{1/i}, |B|^{1/j}\}\,.$$

For reasons that will soon become apparent, the quantity $H(A,B)$ will be referred to as the *height of the line* $L(A,B,C)$. In short, the height determines the amount of material a line removes from the fixed vertical line L_θ and therefore from Θ.

The upshot of the above analysis is that the set

$$\mathbf{Bad}_c(i,j) \cap \Theta$$

consists of points (θ, y) in the unit square which avoid all intervals $\Delta(L)$ arising from lines $L = L(A,B,C)$ with $B \neq 0$. Since

$$\mathbf{Bad}_c(i,j) \cap \Theta \subset \mathbf{Bad}(i,j) \cap \Theta\ ,$$

the name of the game is to show that we have something left after removing these intervals.

Remark 1. The fact that we have restricted our attention to Θ rather than working on the whole line L_θ is mainly for convenience. It also means that for any fixed A and B, there are only a finite number of lines $L = L(A,B,C)$ of interest; i.e., lines for which $\Delta(L) \cap \Theta \neq \emptyset$. Indeed, with $c \leqslant 1/2$ the number of such lines is bounded above by $|B| + 2$.

Remark 2. Without loss of generality, when considering lines $L = L(A,B,C)$ we will assume that

$$(12)\qquad (A,B,C) = 1 \quad \text{and} \quad B > 0\,.$$

Otherwise we can divide the coefficients of L by their common divisor or by -1. Then the resulting line L' will satisfy the required conditions and moreover $\Delta(L') \supseteq \Delta(L)$. Therefore, removing the interval $\Delta(L')$ from Θ takes care of removing $\Delta(L)$.

Note that in view of (12), for any line $L = L(A,B,C)$, we always have that $H(A,B) \geqslant 1$.

2.2. *Description of basic construction.* Let $R \geqslant 2$ be an integer. Choose $c_1 = c_1(R)$ sufficiently small so that

$$(13)\qquad c_1 \leqslant \tfrac{1}{4} R^{-\frac{3i}{j}}$$

and

$$(14)\qquad c := \frac{c_1}{R^{1+\alpha}}$$

satisfies (9) with

$$\alpha := \tfrac{1}{4}\, ij \,. \tag{15}$$

We now describe the basic construction that enables us to conclude that

$$\mathbf{Bad}_c(i,j) \cap \Theta \neq \emptyset \,. \tag{16}$$

We start by subdividing the segment Θ from the $(\theta, 0)$ end into closed intervals J_0 of equal length c_1. Denote by $\mathcal{J}_0$ the collection of intervals J_0. Thus,

$$\#\mathcal{J}_0 = [c_1^{-1}] \,.$$

The idea is to establish, by induction on n, the existence of a collection $\mathcal{J}_n$ of closed intervals J_n such that $\mathcal{J}_n$ is nested in $\mathcal{J}_{n-1}$; that is, each interval J_n in $\mathcal{J}_n$ is contained in some interval J_{n-1} in $\mathcal{J}_{n-1}$. The length of an interval J_n will be given by

$$|J_n| := c_1\, R^{-n} \ ,$$

and each interval J_n in $\mathcal{J}_n$ will satisfy the condition that

$$J_n \cap \Delta(L) = \emptyset \qquad \forall \ \ L = L(A,B,C) \ \text{ with } \ H(A,B) < R^{n-1} \,. \tag{17}$$

In particular, we put

$$\mathbf{K}_c = \mathbf{K}_{c(R)} := \bigcap_{n=1}^{\infty} \bigcup_{J \in \mathcal{J}_n} J \,.$$

By construction, we have that

$$\mathbf{K}_c \subset \mathbf{Bad}_c(i,j) \cap \Theta \,.$$

Moreover, since the intervals J_n are nested, in order to establish (16) it suffices to show that each $\mathcal{J}_n$ is nonempty; i.e.,

$$\#\mathcal{J}_n \geqslant 1 \qquad \forall \ n = 0, 1, \dots \ .$$

The induction. For $n = 0$, we trivially have that (17) is satisfied for any interval $J_0 \in \mathcal{J}_0$. The point is that in view of (12), there are no lines satisfying the height condition $H(A,B) < 1$. For the same reason, (17) with $n = 1$ is trivially satisfied for any interval J_1 obtained by subdividing each J_0 in $\mathcal{J}_0$ into R closed intervals of equal length $c_1 R^{-1}$. Denote by $\mathcal{J}_1$ the resulting collection of intervals J_1 and note that

$$\#\mathcal{J}_1 = [c_1^{-1}]\, R \,.$$

In general, given $\mathcal{J}_n$ satisfying (17) we wish to construct a nested collection $\mathcal{J}_{n+1}$ of intervals J_{n+1} for which (17) is satisfied with n replaced by $n+1$. By definition, any interval J_n in $\mathcal{J}_n$ avoids intervals $\Delta(L)$ arising from lines with height bounded above by R^{n-1}. Since any 'new' interval J_{n+1} is to be nested in some J_n, it is enough to show that J_{n+1} avoids intervals $\Delta(L)$ arising from lines $L = L(A, B, C)$ with height satisfying

$$R^{n-1} \leqslant H(A, B) < R^n \ . \tag{18}$$

Denote by $\mathcal{C}(n)$ the collection of all lines satisfying this height condition. Throughout, we are already assuming that lines satisfy (12). Thus, formally

$$\mathcal{C}(n) := \{L = L(A, B, C) : L \ \text{ satsifies (12) and (18)}\} \ ,$$

and it is precisely this collection of lines that comes into play when constructing $\mathcal{J}_{n+1}$ from $\mathcal{J}_n$. We now proceed with the construction.

Stage 1: *The collection* $\mathcal{I}_{n+1}$. We subdivide each J_n in $\mathcal{J}_n$ into R closed intervals I_{n+1} of equal length and denote by $\mathcal{I}_{n+1}$ the collection of such intervals. Thus,

$$|I_{n+1}| = c_1 R^{-n-1} \qquad \text{and} \qquad \#\mathcal{I}_{n+1} = R \times \#\mathcal{J}_n \ .$$

In view of the nested requirement, the collection $\mathcal{J}_{n+1}$ which we are attempting to construct will be a subcollection of $\mathcal{I}_{n+1}$. In other words, the intervals I_{n+1} represent possible candidates for J_{n+1}. The goal now is simple — it is to remove those 'bad' intervals I_{n+1} from $\mathcal{I}_{n+1}$ for which

$$I_{n+1} \cap \Delta(L) \neq \emptyset \quad \text{for some} \ \ L \in \mathcal{C}(n) \, . \tag{19}$$

Note that the number of bad intervals that can be removed by any single line $L = L(A, B, C)$ is bounded by

$$\frac{|\Delta(L)|}{|I_{n+1}|} + 2 = 2\frac{cR^{n+1}}{c_1 H(A, B)} + 2 = \frac{2R^{n-\alpha}}{H(A, B)} + 2 \, . \tag{20}$$

Thus any single line L in $\mathcal{C}(n)$ can remove up to $[2R^{1-\alpha}] + 2$ intervals from $\mathcal{I}_{n+1}$. Suppose that we crudely remove this maximum number for each L in $\mathcal{C}(n)$. Then, for n large enough, a straightforward calculation shows that all the intervals from $\mathcal{I}_{n+1}$ are eventually removed and the construction comes to a halt. In other words, we need to be much more sophisticated in our approach.

Stage 2: *Trimming.* Even before considering the effect that lines from $\mathcal{C}(n)$ have on intervals in $\mathcal{I}_{n+1}$, we trim the collection $\mathcal{I}_{n+1}$ by removing from each J_n the first $\lceil R^{1-\alpha} \rceil$ subintervals I_{n+1} from each end. Let us denote by J_n^- the resulting 'trimmed' interval and by $\mathcal{I}_{n+1}^-$ the resulting 'trimmed' collection. This process removes $\#\mathcal{J}_n \times 2 \lceil R^{1-\alpha} \rceil$ intervals I_{n+1} from $\mathcal{I}_{n+1}$ regardless of whether an interval is bad or not. However, it ensures that for any remaining interval I_{n+1} in $\mathcal{I}_{n+1}^-$ which satisfies (19), the line L itself must intersect the associated interval J_n within which I_{n+1} is nested. The upshot of 'trimming' is that when considering (19), we only need to consider those lines L from $\mathcal{C}(n)$ for which

$$J_n \cap L \neq \emptyset \quad \text{for some} \ \ J_n \in \mathcal{J}_n \, .$$

The intervals $\Delta(L)$ arising from the 'other' lines are either removed by the trimming process or they do not even intersect intervals in $\mathcal{J}_n$ and therefore they cannot possibly remove any intervals from $\mathcal{I}_{n+1}$.

The sought after collection $\mathcal{J}_{n+1}$ is precisely that obtained by removing those 'bad' intervals I_{n+1} from $\mathcal{I}_{n+1}^-$ which satisfy (19). Formally, for $n \geqslant 1$, we let

$$\mathcal{J}_{n+1} := \{I_{n+1} \in \mathcal{I}_{n+1}^- : \Delta(L) \cap I_{n+1} = \emptyset \ \ \forall \ \ L \in \mathcal{C}(n)\}\,. \tag{21}$$

For any strictly positive $\varepsilon < \frac{1}{2}\alpha^2$ and $R > R_0(\varepsilon)$ sufficiently large, we claim that

$$\#\mathcal{J}_{n+1} \geqslant (R - 5R^{1-\varepsilon}) \times \#\mathcal{J}_n \qquad \forall \ n = 0, 1, \dots\,. \tag{22}$$

Clearly, this implies that

$$\#\mathcal{J}_{n+1} \geqslant (R - 5R^{1-\varepsilon})^{n+1} > 1$$

which in turn completes the proof of the induction step and therefore establishes (16). Thus, our goal now is to justify (22).

Stage 3: *The subcollection* $\mathcal{C}(n,l)$. In the first instance, we subdivide the collection $\mathcal{C}(n)$ of lines into various subcollections that reflect a common geometric configuration. For any integer $l \geqslant 0$, let $\mathcal{C}(n,l) \subset \mathcal{C}(n)$ denote the collection of lines $L = L(A,B,C)$ satisfying the additional condition that

$$R^{-\lambda(l+1)} R^{\frac{nj}{j+1}} \leqslant B < R^{-\lambda l} R^{\frac{nj}{j+1}}\,, \tag{23}$$

where

$$\lambda := 3/j > 1\,.$$

Thus the B variable associated with any line in $\mathcal{C}(n,l)$ is within a tight range governed by (23). In view of (18), it follows that $B^{1+1/j} < R^n$, and so $1 \leqslant B < R^{\frac{nj}{j+1}}$. Therefore,

$$0 \leqslant l < \frac{nj}{\lambda(j+1)} < n\,.$$

A useful 'algebraic' consequence of imposing (23) is that

$$H(A,B) = |B|\,|A|^{1/i} \qquad \forall \ \ L(A,B,C) \in \mathcal{C}(n, l > 0)\,. \tag{24}$$

To see this, suppose that the $B^{1/j}$ term is the maximum term associated with $H(A,B)$. Then, by (18) we have that

$$B \cdot B^{1/j} \geqslant R^{n-1} \implies B \geqslant R^{\frac{(n-1)j}{j+1}}\,.$$

Thus, by definition, $L(A,B,C) \in \mathcal{C}(n,0)$. Moreover, in view of (24) and the definition of $\mathcal{C}(n,l)$, it follows that

$$R^{(\lambda l-1)i} \cdot R^{\frac{ni}{j+1}} < |A| < R^{\lambda(l+1)i} R^{\frac{ni}{j+1}} \qquad \forall \ \ L(A,B,C) \in \mathcal{C}(n, l > 0)\,. \tag{25}$$

The upshot is that for $l > 0$, both the A and B variables associated with lines in $\mathcal{C}(n,l)$ are tightly controlled. The above consequences of imposing (23) are important but are out weighed by the significance of the following 'geometric' consequence.

THEOREM 3. *All lines from $\mathcal{C}(n,l)$ that intersect a fixed interval $J_{n-l} \in \mathcal{J}_{n-l}$ pass through a single rational point P.*

The theorem is proved in Section 4. It implies that if we have three or more lines from $\mathcal{C}(n,l)$ passing through any fixed interval J_{n-l}, then the lines cannot possibly enclose a triangular region. In short, triangles are not allowed. The theorem represents a crucial ingredient towards establishing the following counting statement. *Let $l \geqslant 0$ and $J_{n-l} \in \mathcal{J}_{n-l}$. Then, for any strictly positive $\varepsilon < \frac{1}{2}\alpha^2$ and $R > R_0(\varepsilon)$ sufficiently large, we have that*

$$\#\{I_{n+1} \in \mathcal{I}_{n+1}^- : J_{n-l} \cap \Delta(L) \cap I_{n+1} \neq \emptyset \text{ for some } L \in \mathcal{C}(n,l)\} \leqslant R^{1-\varepsilon}\,. \tag{26}$$

Armed with this estimate it is reasonably straightforward to establish (22). We use induction. For $n = 0$, we have that

$$\#\mathcal{J}_1 = R \times \#\mathcal{J}_0\ ,$$

and so (22) is obviously true. For $n \geqslant 1$, we suppose that

$$\#\mathcal{J}_{k+1} \geqslant (R - 5R^{1-\varepsilon}) \times \#\mathcal{J}_k \qquad \forall\ k = 0, 1, \ldots, n-1$$

and proceed to establish the statement for $k = n$. In view of (26), we have that the total number of intervals I_{n+1} removed from $\mathcal{I}_{n+1}^-$ by lines from $\mathcal{C}(n,l)$ is bounded above by

$$R^{1-\varepsilon} \times \#\mathcal{J}_{n-l}\,.$$

It now follows that

$$\begin{aligned} &\#\{I_{n+1} \in \mathcal{I}_{n+1}^- : \Delta(L) \cap I_{n+1} \neq \emptyset \text{ for some } L \in \mathcal{C}(n)\} \\ &\qquad \leqslant \sum_{l=0}^{n} R^{1-\varepsilon}\, \#\mathcal{J}_{n-l} \leqslant R^{1-\varepsilon}\, \#\mathcal{J}_n + R^{1-\varepsilon} \sum_{l=1}^{n} \#\mathcal{J}_{n-l}\,. \end{aligned} \tag{27}$$

In view of the induction hypothesis, for R sufficiently large, we have that

$$\sum_{l=1}^{n} \#\mathcal{J}_{n-l} \leqslant \#\mathcal{J}_n \sum_{l=1}^{\infty} (R - 5R^{1-\varepsilon})^{-l} \leqslant 2\, \#\mathcal{J}_n\,,$$

and so

$$\text{l.h.s. of (27)} \leqslant 3\, R^{1-\varepsilon}\, \#\mathcal{J}_n. \tag{28}$$

Therefore, for R sufficiently large,

$$\begin{aligned}\#\mathcal{J}_{n+1} &= \#\mathcal{I}_{n+1}^{-} - \text{l.h.s. of (27)} \\ &\geqslant (R - 2\lceil R^{1-\alpha}\rceil)\,\#\mathcal{J}_n - 3\,R^{1-\varepsilon}\,\#\mathcal{J}_n \\ &= (R - 5R^{1-\varepsilon})\,\#\mathcal{J}_n\,.\end{aligned}$$

This completes the induction step and therefore establishes (22). Thus, our goal now is to justify (26).

Stage 4: *The subcollection* $\mathcal{C}(n,l,k)$. Clearly, when attempting to establish (26), we are only interested in lines $L = L(A,B,C)$ in $\mathcal{C}(n,l)$ which remove intervals. In other words, $\Delta(L) \cap I_{n+1} \neq \emptyset$ for some $I_{n+1} \in \mathcal{I}_{n+1}^{-}$. Now the total number of intervals that a line L can remove depends on the actual value of its height. In the situation under consideration, the height satisfies (18). Therefore, in view of (20), the total number of intervals I_{n+1} removed by L can vary anywhere between 1 and $[2R^{1-\alpha}] + 2$. In a nutshell, this variation is too large to handle and we need to introduce a tighter control on the height. For any integer $k \geqslant 0$, let $\mathcal{C}(n,l,k) \subset \mathcal{C}(n,l)$ denote the collection of lines $L = L(A,B,C)$ satisfying the additional condition that

$$2^k R^{n-1} \leqslant H(A,B) < 2^{k+1} R^{n-1}\,. \tag{29}$$

In view of (18), it follows that

$$0 \leqslant k < \frac{\log R}{\log 2}\,. \tag{30}$$

The following counting result implies (26) and indeed represents the technical key to unlocking Schmidt's conjecture.

THEOREM 4. *Let $l,k \geqslant 0$ and $J_{n-l} \in \mathcal{J}_{n-l}$. Then, for any strictly positive $\varepsilon < \alpha^2$ and $R > R_0(\varepsilon)$ sufficiently large, we have that*

$$\#\{I_{n+1} \in \mathcal{I}_{n+1}^{-} : J_{n-l} \cap \Delta(L) \cap I_{n+1} \neq \emptyset \ \text{ for some } \ L \in \mathcal{C}(n,l,k)\} \leqslant R^{1-\varepsilon}\,. \tag{31}$$

Theorem 4 is proved in Section 6. It is in this proof that we make use of Theorem 3. Note that the latter is applicable since $\mathcal{C}(n,l,k) \subset \mathcal{C}(n,l)$. Also note that in view of the 'trimming' process, when considering (31) we can assume that $J_{n-l} \cap L \neq \emptyset$. With Theorem 4 at our disposal, it follows that for R sufficiently large,

$$\text{l.h.s. of (26)} \leqslant \frac{\log R}{\log 2} \times R^{1-\varepsilon} \leqslant R^{1-\frac{1}{2}\varepsilon}\,.$$

This establishes (26) and completes the description of the basic construction.

Remark. We emphasize that from the onset of this section we have fixed i and j satisfying (6). Thus this condition on i and j is implicit within the statements of Theorems 3 and 4.

3. **Proof of Corollary 1: Modulo Theorems 3 and 4**

Modulo Theorems 3 and 4, the basic construction of Section 2.2 yields the statement of Corollary 1 for any single (i, j) pair satisfying (6). We now show that with very little extra effort, we can modify the basic construction to simultaneously incorporate any finite number of (i, j) pairs satisfying (2). In turn, this will prove Corollary 1 in full and thereby establish Schmidt's conjecture.

3.1. *Modifying the basic construction for finite pairs*. To start with we suppose that the d given pairs $(i_1, j_1), \ldots, (i_d, j_d)$ in Corollary 1 satisfy (6). Note that for each $t = 1, \ldots, d$, the height $H(A, B)$ of a given line $L = L(A, B, C)$ is dependent on the pair (i_t, j_t). In view of this and with reference to Section 2, let us write $H_t(A, B)$ for $H(A, B)$, $\Delta_t(L)$ for $\Delta(L)$ and $\mathcal{C}_t(n)$ for $\mathcal{C}(n)$. With this in mind, let $R \geqslant 2$ be an integer. Choose $c_1 = c_1(R)$ sufficiently small so that

$$c_1 \leqslant \tfrac{1}{4} R^{-3i_t/j_t} \qquad \forall \quad 1 \leqslant t \leqslant d\,,$$

and for each $t = 1, \ldots, d$,

$$c(t) := \frac{c_1}{R^{1+\alpha_t}}$$

satisfies (9) with

$$\alpha_t := \tfrac{1}{4}\, i_t\, j_t\,.$$

Note that with this choice of c_1 we are able to separately carry out the basic construction of Section 2.2 for each (i_t, j_t) pair and therefore conclude that

$$\mathbf{Bad}_{c(t)}(i_t, j_t) \cap \Theta \neq \emptyset \qquad \forall \quad 1 \leqslant t \leqslant d\,.$$

We now describe the minor modifications to the basic construction that enable us to simultaneously deal with the d given (i_t, j_t) pairs and therefore conclude that

$$\bigcap_{t=1}^{d} \mathbf{Bad}_{c(t)}(i_t, j_t) \cap \Theta \neq \emptyset\,.$$

The modifications are essentially at the 'trimming' stage and in the manner in which the collections $\mathcal{J}_n$ for $n \geqslant 2$ are defined.

Let c_1 be as above. Define the collections $\mathcal{J}_0$ and $\mathcal{J}_1$ as in the basic construction. Also Stage 1 of the 'induction' in which the collection $\mathcal{I}_{n+1}$ is introduced remains unchanged. However, the goal now is to remove those 'bad' intervals I_{n+1} from $\mathcal{I}_{n+1}$ for which

$$I_{n+1} \cap \Delta_t(L) \neq \emptyset \quad \text{for some } t = 1, \ldots, d \text{ and } L \in \mathcal{C}_t(n)\,. \tag{32}$$

Regarding Stage 2, we trim the collection $\mathcal{I}_{n+1}$ by removing from each J_n the first $\lceil R^{1-\alpha_{\min}} \rceil$ subintervals I_{n+1} from each end. Here

$$\alpha_{\min} := \min\{\alpha_1, \dots, \alpha_d\}\,.$$

This gives rise to the trimmed collection $\mathcal{I}_{n+1}^-$, and we define $\mathcal{J}_{n+1}$ to be the collection obtained by removing those 'bad' intervals I_{n+1} from $\mathcal{I}_{n+1}^-$ which satisfy (32). In other words, for $n \geqslant 1$, we let

$$\mathcal{J}_{n+1} := \{I_{n+1} \in \mathcal{I}_{n+1}^- : \Delta_t(L) \cap I_{n+1} = \emptyset \quad \forall\ 1 \leqslant t \leqslant d \ \text{ and } \ L \in \mathcal{C}_t(n)\}\,.$$

Apart from obvious notational modifications, Stages 3 and 4 remain pretty much unchanged and enable us to establish (28) for each $t = 1, \dots, d$. That is, for any strictly positive $\varepsilon < \frac{1}{2}\alpha_t^2$ and $R > R_0(\varepsilon)$ sufficiently large,

$$\#\{I_{n+1} \in \mathcal{I}_{n+1}^- : \Delta_t(L) \cap I_{n+1} \neq \emptyset \quad \text{for some } \ L \in \mathcal{C}_t(n)\} \leqslant 3R^{1-\varepsilon}\, \#\mathcal{J}_n\,. \tag{33}$$

It follows that for any strictly positive $\varepsilon < \frac{1}{2}\,\alpha_{\min}^2$ and $R > R_0(\varepsilon)$ sufficiently large,

$$\begin{aligned} \#\mathcal{J}_{n+1} &= \#\mathcal{I}_{n+1}^- - \sum_{t=1}^{d} \text{l.h.s. of (33)} \\ &\geqslant (R - 2\lceil R^{1-\alpha_{\min}} \rceil)\, \#\mathcal{J}_n - 3d\, R^{1-\varepsilon}\, \#\mathcal{J}_n \\ &= (R - 5dR^{1-\varepsilon})\ \#\mathcal{J}_n \qquad \forall\ n = 0, 1, \dots\ . \end{aligned}$$

The upshot is that

$$\#\mathcal{J}_n \geqslant (R - 5dR^{1-\varepsilon})^n \geqslant 1 \qquad \forall\ n = 0, 1, \dots\ ,$$

and therefore

$$\bigcap_{t=1}^{d} \mathbf{Bad}_{c(t)}(i_t, j_t) \cap \Theta \supset \bigcap_{n=1}^{\infty} \bigcup_{J \in \mathcal{J}_n} J \neq \emptyset\,.$$

This establishes Corollary 1 in the case the pairs (i_t, j_t) satisfy (6). In order to complete the proof in full, we need to deal with the pairs $(1, 0)$ and $(0, 1)$.

3.2. *Dealing with* $(1, 0)$ *and* $(0, 1)$. By definition, $\mathbf{Bad}(1, 0) = \{(x, y) \in \mathbb{R}^2 : x \in \mathbf{Bad}\}$. Thus, the condition that $\theta \in \mathbf{Bad}$ imposed in Corollary 1 implies that

$$\mathbf{Bad}(1, 0) \cap \mathrm{L}_\theta = \mathrm{L}_\theta\,.$$

In other words, the pair $(1, 0)$ has absolutely no effect when considering the intersection of any number of different $\mathbf{Bad}(i, j)$ sets with L_θ nor does it in anyway effect the modified construction of Section 3.1.

In order to deal with intersecting $\mathbf{Bad}(0, 1)$ with L_θ, we show that the pair $(0, 1)$ can be easily integrated within the modified construction. To start with, note that

$$\mathbf{Bad}(0, 1) \cap \Theta = \{(\theta, y) \in [0, 1)^2 : y \in \mathbf{Bad}\}.$$

With c_1 as in Section 3.1, let

$$c := \frac{c_1}{2R^2}\,. \tag{34}$$

For the sake of consistency with the previous section, for $n \geqslant 0$, let

$$\mathcal{C}(n) := \left\{p/q \in \mathbb{Q} : R^{n-1} \leqslant H(p/q) < R^n\right\} \qquad \text{where} \qquad H(p/q) := q^2\,.$$

Furthermore, let $\Delta(p/q)$ be the interval centered at $(\theta, p/q)$ with length

$$|\Delta(p/q)| := \frac{2c}{H(p/q)}\,.$$

With reference to Section 3.1, suppose that (i_t, j_t) is $(0,1)$ for some $t = 1, \ldots, d$. Since $\mathcal{C}(n) = \emptyset$ for $n = 0$, the following analogue of (33) allows us to deal with the pair $(0,1)$ within the modified construction. *For $R \geqslant 4$, we have that*

$$\#\{I_{n+1} \in \mathcal{I}_{n+1}^{-} : \Delta(p/q) \cap I_{n+1} \neq \emptyset \ \text{ for some } \ p/q \in \mathcal{C}(n)\} \ \leqslant \ 3\,\#\mathcal{J}_n\,. \tag{35}$$

To establish this estimate we proceed as follows. First note that in view of (34), we have that

$$\frac{|\Delta(p/q)|}{|I_{n+1}|} \leqslant 1\,.$$

Thus, any single interval $\Delta(p/q)$ removes at most three intervals I_{n+1} from $\mathcal{I}_{n+1}$. Next, for any two rationals $p_1/q_1, p_2/q_2 \in \mathcal{C}(n)$, we have that

$$\left|\frac{p_1}{q_1} - \frac{p_2}{q_2}\right| \geqslant \frac{1}{q_1 q_2} \geqslant R^{-n} > c_1 R^{-n}\,.$$

Thus, there is at most one interval $\Delta(p/q)$ that can possibly intersect any given interval J_n from $\mathcal{J}_n$. This together with the previous fact establishes (35).

4. **Proof of Theorem 3**

Let $R \geqslant 2$ be an integer. We start by showing that two parallel lines from $\mathcal{C}(n,l)$ cannot intersect J_{n-l}. For any line $L(A,B,C) \in \mathcal{C}(n,l)$, we have that

$$R^{\lambda l}\, R^{-\frac{nj}{j+1}} \overset{(23)}{<} B^{-1}\,.$$

Thus, if two parallel lines $L_1(A_1,B_1,C_1)$ and $L_2(A_2,B_2,C_2)$ from $\mathcal{C}(n,l)$ intersect J_{n-l}, we must have that

$$R^{2\lambda l}\, R^{-\frac{2nj}{j+1}} \leq \frac{1}{B_1 B_2} \leq \left|\frac{C_1}{B_1} - \frac{C_2}{B_2}\right| \leqslant |J_{n-l}| = c_1 R^{-n+l}\,.$$

However, this is clearly false since $c_1 < 1 < \lambda$ and $2j < j+1$.

Now suppose we have three lines L_1, L_2 and L_3 from $\mathcal{C}(n,l)$ that intersect J_{n-l} but do not intersect one another at a single point. In view of the above discussion, the three lines $L_m = L(A_m, B_m, C_m)$ corresponding to $m = 1, 2$ or 3

cannot be parallel to one another, and therefore we must have three distinct intersection points:

$$P_{12} = L_1 \cap L_2\,, \quad P_{13} = L_1 \cap L_3 \quad \text{and} \quad P_{23} = L_2 \cap L_3\,.$$

Since P_{12}, P_{13} and P_{23} are rational points in the plane, they can be represented in the form

$$P_{st} = \left(\frac{p_{st}}{q_{st}}, \frac{r_{st}}{q_{st}}\right) \qquad (1 \leqslant s < t \leqslant 3)\,,$$

where

$$\frac{p_{st}}{q_{st}} = \frac{B_sC_t - B_tC_s}{A_sB_t - A_tB_s} \qquad \text{and} \qquad \frac{r_{st}}{q_{st}} = \frac{A_sC_t - A_tC_s}{A_sB_t - A_tB_s}\,.$$

In particular, there exists an integer $k_{st} \neq 0$ such that

$$k_{st}q_{st} = A_sB_t - A_tB_s \qquad \text{and} \qquad k_{st}p_{st} = B_sC_t - B_tC_s$$

and, without loss of generality, we can assume that $q_{st} > 0$. On a slightly different note, the three intersection points $Y_m := L_m \cap J_{n-l}$ are obviously distinct and it is easily verified that

$$Y_m = \left(\theta, \frac{A_m\theta + C_m}{B_m}\right) \qquad (1 \leqslant m \leqslant 3)\,.$$

Let $\mathrm{T}(P_{12}P_{13}P_{23})$ denote the triangle subtended by the points P_{12}, P_{23} and P_{13}. Then twice the area of the triangle is equal to the absolute value of the determinant

$$\det := \begin{vmatrix} 1 & p_{12}/q_{12} & r_{12}/q_{12} \\ 1 & p_{13}/q_{13} & r_{13}/q_{13} \\ 1 & p_{23}/q_{23} & r_{23}/q_{23} \end{vmatrix}\,.$$

It follows that

$$\textbf{area}\,\mathrm{T}(P_{12}P_{13}P_{23}) \geqslant \frac{1}{2q_{12}q_{13}q_{23}}\,. \tag{36}$$

On the other hand, notice that the triangle $\mathrm{T}(P_{12}P_{13}P_{23})$ is covered by the union of triangles $\mathrm{T}(Y_1Y_2P_{12}) \cup \mathrm{T}(Y_1Y_3P_{13}) \cup \mathrm{T}(Y_2Y_3P_{23})$. Thus

$$\textbf{area}\,\mathrm{T}(P_{12}P_{13}P_{23}) \leqslant \textbf{area}\,\mathrm{T}(Y_1Y_2P_{12}) + \textbf{area}\,\mathrm{T}(Y_1Y_3P_{13}) + \textbf{area}\,\mathrm{T}(Y_2Y_3P_{23}).$$

Without loss of generality, assume that $\mathrm{T}(Y_1Y_2P_{12})$ has the maximum area. Then

$$\textbf{area}\,\mathrm{T}(P_{12}P_{13}P_{23}) \leqslant 3 \cdot \textbf{area}\,\mathrm{T}(Y_1Y_2P_{12}) = \frac{3}{2}|Y_1 - Y_2| \cdot \left|\theta - \frac{p_{12}}{q_{12}}\right|.$$

Now observe that

$$c_1 R^{-n+l} = |J_{n-l}| \geqslant |Y_1 - Y_2| = \frac{|(A_1B_2 - A_2B_1)\theta - (B_1C_2 - B_2C_1)|}{|B_1B_2|}$$
$$= \frac{|k_{12}\, q_{12}\theta - k_{12}\, p_{12}|}{B_1B_2}$$
$$\geqslant \frac{|q_{12}\theta - p_{12}|}{B_1B_2} \, .$$

Hence

$$\mathbf{area}\, \mathrm{T}(P_{12}P_{13}P_{23}) \leqslant \frac{3}{2}\, c_1^2 R^{-2(n-l)} \frac{1}{q_{12}} B_1B_2 \, .$$

Therefore, on combining with (36), we have that

$$R^{2n} \;\leqslant\; 3c_1^2 R^{2l} B_1B_2 q_{13} q_{23} \; . \tag{37}$$

On making use of the fact that c_1 satisfies (13) and so

$$4c_1 R^{\lambda i} \leqslant 1 \, , \tag{38}$$

we now show that the previous inequality (37) is in fact false. As a consequence, the triangle $\mathrm{T}(P_{12}P_{13}P_{23})$ has zero area and therefore cannot exist. Thus, if there are two or more lines from $\mathcal{C}(n,l)$ that intersect J_{n-l}, then they are forced to intersect one another at a single point.

On using the fact that $q_{st} \leqslant |A_s|B_t + |A_t|B_s$, it follows that

$$\text{r.h.s. of (37)} \leqslant 3\, c_1^2 R^{2l} B_1B_2 \left(|A_1|B_3 + |A_3|B_1\right) \left(|A_2|B_3 + |A_3|B_2\right) \tag{39}$$
$$= 3\, c_1^2 R^{2l} B_1B_2 \Big(|A_1|B_3|A_2|B_3 c|A_1|B_3|A_3|B_2$$
$$+ |A_3|B_1|A_2|B_3 + |A_3|B_1|A_3|B_2 \Big) \, .$$

By making use of (18) and (23), it is easily verified that

$$|A_t|B_t = |A_t|B_t^i \; B_t^j < R^{ni} \; R^{-\lambda jl} \; R^{n\frac{j^2}{j+1}} = R^{-\lambda jl} R^{\frac{n}{j+1}} \, .$$

In turn it follows that each of the first three terms associated with (39) is bounded above by

$$3c_1^2 R^{2l(1-(1+j)\lambda)} R^{2n} \;\overset{\lambda>1}{\leqslant}\; 3c_1^2 R^{2n} \, .$$

Turning our attention to the fourth term, since $L_1, L_3 \in \mathcal{C}(n,l)$, we have via (23) that $B_1 \leqslant R^{\lambda} B_3$. Therefore,

$$3c_1^2 R^{2l} |A_3|^2 B_1^2 B_2^2 \;\leqslant\; 3c_1^2 R^{2l+2\lambda i} \left(|A_3|B_3^i\right)^2 B_1^{2j} B_2^2$$
$$\leqslant\; 3c_1^2 R^{2l(1-\lambda j-\lambda)+2\lambda i} R^{2n}$$
$$\overset{\lambda>1}{\leqslant}\; 3c_1^2 R^{2\lambda i} R^{2n} \, .$$

On combining this with the estimate for the first three terms, we have that

$$\text{r.h.s. of } (37) \leqslant R^{2n}(9c_1^2 + 3c_1^2 R^{2\lambda i}) \ < \ R^{2n}\, 12c_1^2 R^{2\lambda i} \ \overset{(38)}{<} \ R^{2n}\,.$$

Clearly this is not compatible with the left-hand side of (37), and therefore we must have that (37) is false.

Remark. It is evident from the proof that the statement of Theorem 3 is true for any fixed interval of length $|J_{n-l}| := c_1\, R^{-(n-l)}$.

5. **Preliminaries for Theorem 4**

In this section, we make various observations and establish results that are geared towards proving Theorem 4. Throughout, $R \geqslant 2$ is an integer, and for $n \in \mathbb{N}$ and $\tau \in \mathbb{R}^{>0}$, we let

$$\mathrm{J} = \mathrm{J}(n,\tau)$$

denote a generic interval contained within Θ of length τR^{-n}. Note that the position of J within Θ is not specified. Also, for an integer $k \geqslant 0$, we let $\mathcal{C}(n,k)$ denote the collection of lines from $\mathcal{C}(n)$ with height satisfying the additional condition given by (29); that is,

$$\mathcal{C}(n,k) := \left\{ L = L(A,B,C) \in \mathcal{C}(n) : 2^k R^{n-1} \leqslant H(A,B) < 2^{k+1} R^{n-1} \right\}.$$

Trivially, for any $l \geq 0$, we have that

$$\mathcal{C}(n,l,k) \subset \mathcal{C}(n,k)\,.$$

No confusion with the collection $\mathcal{C}(n,l)$ introduced earlier in Section 2.2 should arise. The point is that beyond Theorem 3, the collection $\mathcal{C}(n,l)$ plays no further role in establishing Theorem 4 and therefore will not be explicitly mentioned.

5.1. *A general property.* The following is a general property concerning points in the set $\mathbf{Bad}(i)$ and lines passing through a given rational point in the plane.

Lemma 1. *Let $\theta \in \mathbf{Bad}(i)$ and $P := (\frac{p}{q}, \frac{r}{q})$ be a rational point such that*

$$|q\theta - p| < c(\theta)\, q^{-i}\,.$$

Then there exists a line $L = L(A,B,C)$ passing through P with $|A| \leqslant q^i$ and $0 < B \leqslant q^j$.

Proof. Consider the set

$$ap - br \pmod q \quad \text{where} \quad 0 \leqslant a \leqslant [q^i] \quad \text{and} \quad 0 \leqslant b \leqslant [q^j]\,.$$

The number of such pairs (a,b) is

$$(q^i + 1 - \{q^i\})(q^j + 1 - \{q^j\}) > q.$$

Therefore, by the 'pigeon hole' principle, there exist pairs (a_1, b_1) and (a_2, b_2) such that

$$a_1 p - b_1 r \equiv a_2 p - b_2 r \pmod q.$$

Thus, there is clearly a choice of integers A, B, C with

$$Ap - Br + Cq = 0 \quad \text{where} \quad |A| \leqslant q^i \quad \text{and} \quad 0 \leqslant B \leqslant q^j.$$

It remains to show that we may choose $B > 0$. This is where the Diophantine condition on θ comes into play. Suppose $B = 0$. Then $Ap + Cq = 0$ and without loss of generality, we may assume that $(A, C) = 1$. Put $d := (p, q)$ and define $q_* := q/d$ and $p_* := p/d$. Then

$$Ap_* = -Cq_* \quad \text{and} \quad |A| = q_* \,.$$

Hence $q_* \leqslant q^i$ and $d \geqslant q^j \geqslant {q_*}^{j/i}$. However

$$d|q_*\theta - p_*| = |q\theta - p| < c(\theta)\, q^{-i}\,.$$

Thus, it follows that

$$|q_*\theta - p_*| \ < \ c(\theta){q_*}^{-i} d^{-1-i} \leqslant c(\theta) q_*^{-1/i}\,.$$

But this contradicts the hypothesis that $\theta \in \mathbf{Bad}(i)$, and so we must have that $B > 0$. □

5.2. *Two nonparallel lines intersecting* $\mathrm{J}(n, \tau)$. Let $P := (\frac{p}{q}, \frac{r}{q})$ be a rational point in the plane and consider two nonparallel lines

$$\begin{aligned} L_1 : \quad & A_1 x - B_1 y + C_1 = 0, \\ L_2 : \quad & A_2 x - B_2 y + C_2 = 0 \end{aligned}$$

that intersect one another at P. It follows that

$$\frac{p}{q} = \frac{B_1 C_2 - B_2 C_1}{A_1 B_2 - A_2 B_1} \qquad \text{and} \qquad \frac{r}{q} = \frac{A_1 C_2 - A_2 C_1}{A_1 B_2 - A_2 B_1}\,.$$

Thus, there exists an integer $t \neq 0$ such that

$$A_1 B_2 - A_2 B_1 = tq \qquad \text{and} \qquad B_1 C_2 - B_2 C_1 = tp\,. \tag{40}$$

Without loss of generality, we will assume that $q > 0$. In this section, we investigate the situation in which both lines pass through a generic interval $\mathrm{J} = \mathrm{J}(n, \tau)$. Trivially, for this to happen we must have that

$$|\mathrm{J}| \geqslant |Y_1 - Y_2| = \frac{|(A_1 B_2 - A_2 B_1)\theta - (B_1 C_2 - B_2 C_1)|}{|B_1 B_2|},$$

where

$$Y_m := L_m \cap \mathrm{J} = \Big(\theta, \frac{A_m \theta + C_m}{B_m}\Big) \qquad m = 1, 2\,.$$

This together with (40) implies that

$$\frac{|q\theta - p|}{B_1 B_2} \leqslant \frac{|tq\theta - tp|}{B_1 B_2} \leqslant \tau\, R^{-n}\,. \tag{41}$$

In the case that the lines L_1 and L_2 are from the collection $\mathcal{C}(n,k)$, this general estimate leads to the following statement.

LEMMA 2. *Let $L_1, L_2 \in \mathcal{C}(n,k)$ be two lines that intersect at $P := (\frac{p}{q}, \frac{r}{q})$ and let $\mathrm{J} = \mathrm{J}(n,\tau)$ be a generic interval. Suppose*

$$L_1 \cap \mathrm{J} \neq \emptyset \qquad \textit{and} \qquad L_2 \cap \mathrm{J} \neq \emptyset\,.$$

Then

$$|q\theta - p| < 2^i \tau \frac{2^{k+1}}{R} q^{-i}\,. \tag{42}$$

Proof. With reference to the lines

$$L_1 = L(A_1, B_1, C_1) \quad \text{ and } \quad L_2 = L(A_2, B_2, C_2)\,,$$

there is no loss of generality in assuming that $B_1 \leqslant B_2$. With this mind, by (41) we have that

$$\begin{aligned} |q\theta - p| \;&<\; \tau R^{-n} B_1 B_2 \\ &\overset{(29)}{\leqslant} \tau\, R^{-n}\, B_1\, \left(2^{k+1} R^{n-1}\right)^{\frac{j}{j+1}} \\ &=\; \tau\, 2^{k+1}\, R^{-1}\, B_1 \left(2^{k+1} R^{n-1}\right)^{-\frac{1}{1+j}}\,. \end{aligned} \tag{43}$$

On the other hand, by (40) we have that

$$\begin{aligned} q \;&\leqslant\; |tq| = \; |A_1 B_2 - A_2 B_1| \leqslant |A_1 B_2| + |A_2 B_1| \\ &\overset{(29)}{\leqslant} \left(2^{k+1} R^{n-1}\right)^{\frac{j}{j+1}} \left(\frac{2^{k+1} R^{n-1}}{B_1}\right)^i \;+\; \left(2^{k+1} R^{n-1}\right)^{\frac{j}{j+1}} \left(\frac{2^{k+1} R^{n-1}}{B_2}\right)^i \\ &=\; \left(B_1^{-i} + B_2^{-i}\right)\, \left(2^{k+1} R^{n-1}\right)^{i + \frac{j}{j+1}} \\ &\leqslant\; 2\, B_1^{-i} \left(2^{k+1} R^{n-1}\right)^{\frac{1+ij}{1+j}}\,. \end{aligned}$$

Therefore

$$\begin{aligned} q^{-i} \;&\geqslant\; 2^{-i}\, B_1^{i^2} \left(2^{k+1} R^{n-1}\right)^{-\frac{i+i^2 j}{1+j}} \\ &=\; 2^{-i} B_1\, B_1^{-j(i+1)} \left(2^{k+1} R^{n-1}\right)^{-\frac{i+i^2 j}{1+j}} \\ &\overset{(29)}{\geqslant} 2^{-i}\, B_1 \left(2^{k+1} R^{n-1}\right)^{-\frac{j^2(i+1)}{1+j} - \frac{i+i^2 j}{1+j}} \\ &=\; 2^{-i}\, B_1 \left(2^{k+1} R^{n-1}\right)^{-\frac{1}{1+j}}\,. \end{aligned}$$

This estimate together with (43) yields the desired statement. □

Remark. It is evident from the proof that the statement of Lemma 2 is actually true for lines L_1, L_2 with height bounded above by $2^{k+1} R^{n-1}$.

5.3. *The figure F*. In this section, we give a geometric characterization of lines from $\mathcal{C}(n,l,k)$ that pass through a given rational point and intersect a generic interval. Let $L_1 = L(A_1,B_1,C_1)$ and $L_2 = L(A_2,B_2,C_2)$ be two lines from $\mathcal{C}(n,l,k)$ that pass through $P := (\frac{p}{q},\frac{r}{q})$ and intersect $\mathrm{J} = \mathrm{J}(n,\tau)$. Without loss of generality, assume that $B_1 \leqslant B_2$. Then, in view of (41), we have that

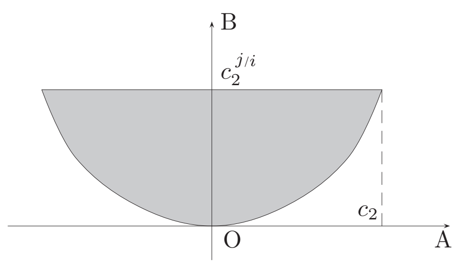
$$\frac{|q\theta - p|}{B_1B_2} \leqslant \tau R^{-n} \overset{(29)}{<} \tau\frac{2^{k+1}}{R}\frac{1}{H(A_2,B_2)}\,.$$

Thus

$$\frac{2^{k+1}\tau}{R|q\theta-p|} > \frac{H(A_2,B_2)}{B_1B_2} = \frac{\max\{|A_2|^{1/i}, B_2^{1/j}\}}{B_1} \geqslant \max\left\{\frac{|A_2|^{1/i}}{B_2}, B_2^{i/j}\right\}\,. \tag{44}$$

Given a rational point P, the upshot is that if two lines from $\mathcal{C}(n,l,k)$ pass through P and intersect J, then the point $(A,B) \in \mathbb{Z}^2$ associated with the coordinates A and B of at least one of the lines lies inside the figure F defined by

$$|A| < c_2^iB^i, \qquad 0 < B < c_2^{j/i} \qquad \text{with} \qquad c_2 := \frac{2^{k+1}\tau}{R|q\theta-p|}\,. \tag{45}$$



The figure F

Notice that the figure F is independent of l, and therefore the above discussion is actually true for lines coming from the larger collection $\mathcal{C}(n,k)$. As a consequence, *apart from one possible exception, all lines $L(A,B,C) \in \mathcal{C}(n,k)$ passing through P and intersecting a generic interval* J *will have A and B coordinates corresponding to points (A,B) lying inside the figure F.* Additionally, notice that the triple (A,B,C) associated with any line L passing through P belongs to the lattice

$$\mathcal{L} = \mathcal{L}(P) := \{(A,B,C) \in \mathbb{Z}^3 : Ap - Br + Cq = 0\}\,.$$

We will actually be interested in the projection of $\mathcal{L}$ onto the (A,B) plane within which the figure F is embedded. By an abuse of notation we will also refer to this projection as $\mathcal{L}$.

Remark. Note that the figure F is independent of the actual position of the generic interval J. However, it is clearly dependent on the position of the rational point P.

Now assume that $L_1, L_2 \in \mathcal{C}(n,l,k)$ with $l>0$. In this case we have that

$$B^{1/j} \overset{(23)}{<} R^{-\frac{\lambda l}{j}} R^{\frac{n}{j+1}} \overset{(25)}{<} R^{1-\frac{\lambda l(j+1)}{j}} |A|^{1/i} \overset{(45)}{<} R^{1-\frac{\lambda l(j+1)}{j}} c_2 B\,.$$

Therefore

$$0 < B < c_3\, c_2^{j/i} \qquad \text{with} \qquad c_3 := R^{\frac{j}{i}-\frac{\lambda l(j+1)}{i}}. \tag{46}$$

Note that $c_3 < 1$ and that

$$|A| \overset{(45)}{<} c_2^i B^i < c_3^i \cdot c_2\,. \tag{47}$$

The upshot is that if two lines from $\mathcal{C}(n, l>0, k)$ pass through P and intersect J, then the point $(A,B) \in \mathbb{Z}^2$ associated with the coordinates A and B of at least one of the lines lies inside the figure $F_l \subset F$ defined by (46) and (47).

5.4. *Lines intersecting* $\Delta(L_0)$. Let $L_0 = L(A_0, B_0, C_0)$ be an arbitrary line passing through the rational point $P := (\frac{p}{q}, \frac{r}{q})$ and intersecting Θ. It is easily verified that the point $Y_0 := L_0 \cap \Theta$ has y-coordinate

$$\frac{A_0\theta + C_0}{B_0} = \frac{A_0\frac{p}{q} + C_0}{B_0} + \frac{A_0}{B_0}\left(\theta - \frac{p}{q}\right) = \frac{r}{q} + \frac{A_0}{B_0}\left(\theta - \frac{p}{q}\right).$$

Now, assume there is another line $L = L(A,B,C)$ with

$$H(A,B) \geqslant H(A_0,B_0)$$

passing through P and intersecting Θ. Let

$$Y = Y(A,B,C) := L \cap \Theta$$

and notice that

$$Y \in \Delta(L_0) \Longleftrightarrow |Y - Y_0| = \left|\frac{A}{B} - \frac{A_0}{B_0}\right| \left|\theta - \frac{p}{q}\right| \leqslant \frac{c}{H(A_0,B_0)}\,.$$

In other words,

$$Y \in \Delta(L_0) \Longleftrightarrow \frac{A}{B} \in \left[\frac{A_0}{B_0} - \frac{c}{H(A_0,B_0)\left|\theta - \frac{p}{q}\right|}, \frac{A_0}{B_0} + \frac{c}{H(A_0,B_0)\left|\theta - \frac{p}{q}\right|}\right]. \tag{48}$$

Geometrically, points $(A,B) \in \mathbb{Z}^2$ satisfying the right-hand side of (48) form a cone $C(A_0,B_0)$ with apex at origin. The upshot is that all lines $L = L(A,B,C)$ with A and B coordinates satisfying $H(A,B) \geqslant H(A_0,B_0)$ and $A/B \in C(A_0,B_0)$ will have $Y(A,B,C) \in \Delta(L_0)$.

In addition, let F be the figure associated with P, a generic interval $\mathrm{J} = \mathrm{J}(n,\tau)$ and the collection $\mathcal{C}(n,k)$. Suppose that

$$F \cap \mathcal{L} \subset C(A_0,B_0) \text{ and } H(A,B) \geqslant H(A_0,B_0) \quad \forall\,(A,B) \in F \cap \mathcal{L}. \tag{49}$$

Then, in view of the discussion above, any line $L = L(A,B,C)$ passing through P such that $(A,B) \in F \cap \mathcal{L}$ will have $Y(A,B,C) \in \Delta(L_0)$. In particular, it follows via Section 5.3 that if we have two lines $L_1, L_2 \in \mathcal{C}(n,k)$ passing through P and intersecting J, then one of them has coordinates corresponding to a point in $F \cap \mathcal{L}$ and therefore it intersects J inside $\Delta(L_0)$. Thus, apart from one possible exceptional line L', all lines $L = L(A,B,C) \in \mathcal{C}(n,k)$ passing through P and intersecting J will have the property that $(A,B) \in F \cap \mathcal{L}$ and $Y(A,B,C) \in \Delta(L_0)$. Note that for $L' = L(A',B',C')$, we have that $(A',B') \notin F \cap \mathcal{L}$, and therefore we cannot guarantee that $H(A',B') \geqslant H(A_0,B_0)$. Also, L' may or may not intersect J inside $\Delta(L_0)$.

5.5. *The key proposition.* Under the hypothesis of Lemma 2, we know that there exists some $\delta \in (0,1)$ such that

$$|q\theta - p| = \delta 2^i \tau \frac{2^{k+1}}{R} q^{-i}\,.$$

Hence

$$c_2 \overset{(45)}{=} \frac{2^{k+1}\tau}{R|q\theta - p|} = \delta^{-1} 2^{-i} q^i\,. \tag{50}$$

The following statement is at the heart of the proof of Theorem 4.

PROPOSITION 1. *Let $P = (\frac{p}{q}, \frac{r}{q})$ be a rational point and $\mathrm{J} = \mathrm{J}(n,\tau)$ be a generic interval. Let $\mathcal{C}$ be the collection of lines $L = L(A,B,C)$ passing through P with height $H(A,B) < R^n$. Let $\mathcal{C}_k \subset \mathcal{C}(n,k)$ denote the collection of lines passing through P and intersecting* J. *Suppose that $\#\mathcal{C}_k \geqslant 2$, $\tau \geqslant cR2^{-k}$ and*

$$\delta \leqslant c_4 \left(\frac{cR}{2^k\tau}\right)^{2/j} \qquad \textit{where} \qquad c_4 := 4^{-2/j}\, 2^{-i}\,. \tag{51}$$

Then there exists a line $L_0 \in \mathcal{C}$ satisfying (49). *Furthermore, apart from one possible exceptional line, for all other $L \in \mathcal{C}_k$, we have that $(A,B) \in F \cap \mathcal{L}$ and $L \cap \mathrm{J} \in \Delta(L_0)$.*

Remark. We stress that the line L_0 of the proposition is completely independent of the actual position of the generic interval J, and therefore the furthermore part of the proposition is also valid irrespective of the position of J.

Proof. Notice that since $\#\mathcal{C}_k \geqslant 2$, there exists at least one line $L(A,B,C) \in \mathcal{C}_k$ with A and B coordinates corresponding to (A,B) lying within F; see

Section 5.3. Thus, there is at least one point in $F \cap \mathcal{L}$ corresponding to a line with height bounded above by R^n.

A consequence of Section 5.4 is that if there exists a line L_0 satisfying (49), then the furthermore part of the statement of the proposition is automatically satisfied. In order to establish (49), we consider the following two cases.

Case A. Suppose there exists a point $(A,B) \in F \cap \mathcal{L}$ such that

$$B \leqslant \sigma \cdot \delta \cdot q^j \qquad \text{where} \qquad \sigma := \left(2^{k+2+ij}\frac{\tau}{Rc}\right)^{1/j}.$$

Now let (A_0', B_0') denote such a point in $F \cap \mathcal{L}$ with B_0' minimal. It follows that for all points $(A,B) \in F \cap \mathcal{L}$,

$$\begin{aligned}\left|\frac{A}{B}\right| &\overset{(45)}{<} \frac{c_2^i}{B^{1-i}} \leqslant \frac{(c_2\, B_0')^i}{B_0'} \\ &\leqslant \frac{(\delta^{-1}2^{-i}q^i\ \sigma\delta q^j)^i}{B_0'} = \frac{2^{-i^2}\sigma^i q^i}{B_0'},\end{aligned}$$

and therefore

$$\left|\frac{A}{B} - \frac{A_0'}{B_0'}\right| < 2\frac{2^{-i^2}\sigma^i q^i}{B_0'}.$$

This together with (48) implies that if

$$\frac{c}{H(A_0', B_0')\left|\theta - \frac{p}{q}\right|} \geqslant 2\frac{2^{-i^2}\sigma^i q^i}{B_0'}, \tag{52}$$

then $F \cap \mathcal{L} \subset C(A_0', B_0')$. In other words, the first condition of (49) is satisfied. Therefore, modulo (52), if the point (A_0', B_0') has minimal height among all $(A,B) \in F \cap \mathcal{L}$, then the second condition of (49) is also valid and we are done. Suppose this is not the case and let (A_0, B_0) denote the minimal height point within $F \cap \mathcal{L}$. Then $H(A_0, B_0) \leqslant H(A_0', B_0')$ and so

$$\begin{aligned}\frac{c}{H(A_0, B_0)\left|\theta - \frac{p}{q}\right|} &\geqslant \frac{c}{H(A_0', B_0')\left|\theta - \frac{p}{q}\right|} \\ &\overset{(52)}{\geqslant} 2\frac{2^{-i^2}\sigma^i q^i}{B_0'} \geqslant \left|\frac{A}{B} - \frac{A_0}{B_0}\right| \qquad \forall \quad (A,B) \in F \cap \mathcal{L}.\end{aligned}$$

Thus, by (48) we have that $F \cap \mathcal{L} \subset C(A_0, B_0)$. The upshot is that if (52) holds, then there exists a line from the collection $\mathcal{C}$ satisfying (49). We now establish (52). Note that

$$\begin{aligned}(52) &\Longleftrightarrow \frac{c \cdot q^{1+i}}{B_0' \max\{|A_0'|^{1/i}, {B_0'}^{1/j}\} \cdot 2^i \tau\delta\left(\frac{2^{k+1}}{R}\right)} \geqslant \frac{2^{1-i^2}\sigma^i q^i}{B_0'} \\ &\Longleftrightarrow \left(\frac{cR}{2^{k+2+i-i^2}\tau\delta\sigma^i}\right) q \geqslant \max\{|A_0'|^{1/i}, {B_0'}^{1/j}\}.\end{aligned} \tag{53}$$

Note that

$$|A_0'|^{1/i} \overset{(45)}{<} c_2 B_0' \leqslant 2^{-i}\sigma q \quad \text{ and } \quad {B_0'}^{1/j} \leqslant \sigma^{1/j}\delta^{1/j} q\,.$$

- Suppose that $|A_0'|^{1/i} > {B_0'}^{1/j}$. Then

$$\text{r.h.s. of (53)} \Longleftarrow \frac{cR}{2^{k+2+i-i^2}\tau\delta\sigma^i} \geqslant 2^{-i}\sigma$$
$$\Longleftrightarrow \delta \leqslant \frac{cR}{2^{k+2-i^2}\tau\sigma^{1+i}} = c_4\left(\frac{cR}{\tau 2^k}\right)^{\frac{2}{j}}\,.$$

  This is precisely (51) and therefore verifies (52) when $|A_0'|^{1/i} > {B_0'}^{1/j}$.
- Suppose that $|A_0'|^{1/i} \leqslant {B_0'}^{1/j}$. Then

$$\text{r.h.s. of (53)} \Longleftarrow \frac{cR}{2^{k+2+i-i^2}\tau\delta\sigma^i} \geqslant \delta^{1/j}\sigma^{1/j} \tag{54}$$
$$\Longleftrightarrow \delta^{1+1/j} \leqslant \left(\frac{1}{2^{2+ij}}\right)^{1+\frac{ij+1}{j^2}}\left(\frac{cR}{\tau 2^k}\right)^{1+\frac{ij+1}{j^2}}$$
$$\Longleftrightarrow \delta \leqslant c_4\left(\frac{cR}{\tau 2^k}\right)^{1/j}\,.$$

  By the hypothesis imposed on τ, it follows that

$$\frac{cR}{\tau 2^k} \leqslant 1\,. \tag{55}$$

  Therefore, in view of (51), the lower bound for δ given by (54) is valid. In turn, this verifies (52) when $|A_0'|^{1/i} \leqslant {B_0'}^{1/j}$.

Case B. Suppose that for all points (A,B) within $F\cap\mathcal{L}$, we have that

$$B > \sigma\delta q^j\,.$$

Then, in view of (45), it follows that

$$\left|\frac{A}{B}\right| < \frac{2^{-i^2}\sigma^i q^i}{\sigma\delta q^j} = \frac{q^{i-j}}{2^{i^2}\sigma^j\delta} \qquad \forall \quad (A,B)\in F\cap\mathcal{L}\,. \tag{56}$$

By making use of (9), (51) and (55), it is readily verified that

$$|q\theta - p| < c(\theta)\,q^{-i}\,.$$

Thus, Lemma 1 is applicable and there exists a point $(A_0',B_0')\in\mathcal{L}$ satisfying

$$H(A_0',B_0') \leqslant q^{1+j}\,.$$

As a consequence,

$$\frac{c}{H(A_0',B_0')\left|\theta-\frac{p}{q}\right|} \geqslant 2\frac{q^{i-j}}{2^{i^2}\sigma^j\delta}. \tag{57}$$

Indeed,

$$(57) \Longleftarrow \frac{cq^{1+i}}{2^i\tau\delta\left(\frac{2^{k+1}}{R}\right)q^{1+j}} \geqslant \frac{q^{i-j}}{2^{i2-1}\sigma^j\delta}$$
$$\Longleftrightarrow \sigma^j \geqslant 2^{k+2+ij}\frac{\tau}{Rc}.$$

By the definition, the last inequality concerning σ is valid, and therefore so is (57). We now show that $F \cap \mathcal{L} \subset C(A_0', B_0')$. In view of (48), this will be the case if

$$(58) \qquad \left|\frac{A}{B} - \frac{A_0'}{B_0'}\right| \leqslant \frac{c}{H(A_0', B_0')\left|\theta - \frac{p}{q}\right|} \qquad \forall \quad (A, B) \in F \cap \mathcal{L}.$$

- Suppose that $(A_0', B_0') \in F \cap \mathcal{L}$. Then, clearly

$$(58) \Longleftarrow (56) \text{ and } (57).$$

- Suppose that $(A_0', B_0') \notin F \cap \mathcal{L}$. Then

$$(58) \Longleftarrow \frac{c}{H(A_0', B_0')\left|\theta - \frac{p}{q}\right|} \geqslant \left|\frac{A}{B}\right| + \left|\frac{A_0'}{B_0'}\right|$$
$$\Longleftarrow \frac{c}{H(A_0', B_0')\left|\theta - \frac{p}{q}\right|} \geqslant \left|\frac{A_0'}{B_0'}\right| + \frac{q^{i-j}}{2^{i^2}\sigma^j\delta}$$
$$\overset{(57)}{\Longleftarrow} \frac{c}{2H(A_0', B_0')\left|\theta - \frac{p}{q}\right|} \geqslant \left|\frac{A_0'}{B_0'}\right|$$
$$\Longleftarrow \frac{cq^{1+i}}{2^i\tau\delta\left(\frac{2^{k+1}}{R}\right)B_0'\,q} \geqslant \frac{2q^i}{B_0'}$$
$$\Longleftrightarrow \delta \leqslant \frac{1}{4\cdot 2^i}\,\frac{cR}{2^k\tau}.$$

In view of (51) and (55), this lower bound for δ is valid, and therefore so is (58).

The upshot of the above is that $F \cap \mathcal{L} \subset C(A_0', B_0')$. In other words, the first condition of (49) is satisfied. Therefore if the pair (A_0', B_0') has the minimal height among all $(A, B) \in F \cap \mathcal{L}$, the second condition of (49) is also valid and we are done. Suppose this is not the case and let $(A_0, B_0) \in F \cap \mathcal{L}$ denote the minimal height point within $F \cap \mathcal{L}$. By assumption,

$$H(A_0, B_0) < H(A_0', B_0'),$$

and so

$$\frac{c}{H(A_0,B_0)\left|\theta-\frac{p}{q}\right|} \geqslant \frac{c}{H(A_0',B_0')\left|\theta-\frac{p}{q}\right|}$$
$$\overset{(57)}{\geqslant} 2\frac{q^{i-j}}{2^{i^2}\sigma^j\delta} \overset{(56)}{\geqslant} \left|\frac{A}{B}-\frac{A_0}{B_0}\right| \qquad \forall \quad (A,B)\in F\cap\mathcal{L}\,.$$

Thus, by (48) we have that $F\cap\mathcal{L}\subset C(A_0,B_0)$. The upshot is that (58) holds; thus there exists a line from the collection $\mathcal{C}$ satisfying (49). □

6. **Proof of Theorem 4**

Let $l,k\geqslant 0$ and $J_{n-l}\in\mathcal{J}_{n-l}$. Let $\varepsilon>0$ be sufficiently small and $R=R(\varepsilon)$ be sufficiently large. In view of the trimming process, Theorem 4 will follow on showing that no more than $R^{1-\varepsilon}$ intervals I_{n+1} from $\mathcal{I}_{n+1}$ can be removed by the intervals $\Delta(L)$ arising from lines $L\in\mathcal{C}(n,l,k)$ that intersect J_{n-l}. Let $L_1,\dots,L_M$ denote these lines of interest and let

$$Y_m := L_m\cap J_{n-l} \qquad (1\leqslant m\leqslant M)\,.$$

Indeed, then

$$\text{l.h.s. of (31)} \leqslant \#\{I_{n+1}\in\mathcal{I}_{n+1}^{-} : I_{n+1}\cap\Delta(L_m)\neq\emptyset \ \text{ for some } \ 1\leqslant m\leqslant M\}\,.$$

A consequence of Theorem 3 is that the lines $L_1,\dots,L_M$ pass through a single rational point $P=(\frac{p}{q},\frac{r}{q})$. This is an absolutely crucial ingredient within the proof of Theorem 4.

In view of (20), the number of intervals $I_{n+1}\in\mathcal{I}_{n+1}$ that can be removed by any single line L_m is bounded above by

$$\frac{2R^{n-\alpha}}{H(A,B)}+2 \overset{(29)}{\leqslant} K := \frac{2R^{1-\alpha}}{2^k}+2\,.$$

Notice that $K\geqslant 2$ is independent of l. Motivated by the quantity K, we consider the following two cases.

Case A. Suppose that $2^k<R^{1-\alpha}$.

Case B. Suppose that $2^k\geqslant R^{1-\alpha}$.

Then

$$K\leqslant\begin{cases}\dfrac{4R^{1-\alpha}}{2^k} & \text{in Case A}\\ 4 & \text{in Case B}\,.\end{cases}$$

Also, let

$$\tilde{c}_1 := \begin{cases}\dfrac{4c_1R^{l+\varepsilon-\alpha}}{2^k} & \text{in Case A}\\ 4c_1R^{l+\varepsilon-1} & \text{in Case B}\,.\end{cases}$$

We now subdivide the given interval J_{n-l} into d intervals $\tilde{I}_{nl}$ of equal length $c_1 R^{l-n} \lceil R^{1-\varepsilon}/K \rceil^{-1}$. It follows that

$$d := \frac{|J_{n-l}|}{|\tilde{I}_{nl}|} = \left\lceil \frac{R^{1-\varepsilon}}{K} \right\rceil$$

and that

$$|\tilde{I}_{nl}| := c_1 R^{l-n} \lceil R^{1-\varepsilon}/K \rceil^{-1} \leqslant \tilde{c}_1 R^{-n} .$$

By choosing R sufficiently large and $\varepsilon < \alpha$ so that

$$R^{\alpha-\varepsilon} \geqslant 8 , \tag{59}$$

we can guarantee that

$$2 \leqslant d \leqslant \frac{2\, R^{1-\varepsilon}}{K} . \tag{60}$$

To proceed, we divide the d intervals $\tilde{I}_{nl}$ into the following two classes.

Type 1. Intervals $\tilde{I}_{nl}$ that intersect no more than one line among $L_1, \ldots, L_M$.

Type 2. Intervals $\tilde{I}_{nl}$ that intersect two or more lines among $L_1, \ldots, L_M$.

6.1. *Dealing with Type* 1 *intervals.* Trivially, the number of Type 1 intervals is bounded above by d. By definition, each Type 1 interval has no more than one line L_m intersecting it. The total number of intervals $I_{n+1} \in \mathcal{I}_{n+1}$ removed by a single line L_m is bounded above by K. Thus, for any strictly positive $\varepsilon < \alpha$ and R sufficiently large so that (59) is valid, the total number of intervals $I_{n+1} \in \mathcal{I}_{n+1}$ removed by the lines $L_1, \ldots, L_M$ associated with Type 1 intervals is bounded above by

$$d\, K \overset{(60)}{\leqslant} 2\, R^{1-\varepsilon} . \tag{61}$$

6.2. *Dealing with Type* 2 *intervals.* Consider an interval $\tilde{I}_{nl}$ of Type 2. By definition, there are at least two lines $L_s, L_t \in \mathcal{C}(n,l,k)$ passing through P which intersect $\tilde{I}_{nl}$. With reference to Section 5, let J be a generic interval of length $\tilde{c}_1 R^{-n}$. Clearly $|\mathrm{J}|$ is the same for k and l fixed and $|\tilde{I}_{nl}| \leqslant |\mathrm{J}|$. Also, in view of (59), we have that $|\mathrm{J}| < |J_{n-l}|$. Thus, given an interval $\tilde{I}_{nl}$ there exists a generic interval $\mathrm{J} = \mathrm{J}(n,\tau)$ with $\tau := \tilde{c}_1$ such that $\tilde{I}_{nl} \subset \mathrm{J} \subset J_{n-l}$. By Lemma 2, there exists some $\delta \in (0,1)$ such that

$$|q\theta - p| = \delta 2^i \tilde{c}_1 q^{-i} \left(\frac{2^{k+1}}{R} \right) .$$

As a consequence of Section 5.3, apart from one possible exception, all lines $L \in \mathcal{C}(n,l,k)$ passing through P and intersecting J will have A and B coordinates corresponding to points (A,B) lying inside the figure F defined by (45) with $c_2 := \delta^{-1} 2^{-i} q^i$. The upshot is that among the lines $L_1, \ldots, L_M$ passing through any $\tilde{I}_{nl}$ of Type 2, all but possibly one line L' will have coordinates corresponding to points in $F \cap \mathcal{L}$. Moreover, if $l > 0$, then F can be replaced by the smaller figure F_l defined by (46) and (47).

6.2.1. *Type* 2 *intervals with* δ *small.* Suppose that

$$\delta \leqslant c_4 \left(\frac{cR}{2^k \tilde{c_1}}\right)^{2/j} \qquad \text{where} \qquad c_4 := 4^{-2/j}\, 2^{-i}\,. \tag{62}$$

With reference to the hypotheses of Proposition 1, the above guarantees (51) and it is easily verified that $\tilde{c}_1 > cR2^{-k}$ and that $\mathcal{C}_k \geqslant 2$ since $\tilde{I}_{nl} \subset \mathrm{J}$ is of Type 2. Hence, Proposition 1 implies the existence of a line $L_0 \in \mathcal{C}(n')$ with $n' \leqslant n$ passing through P and satisfying (49). Furthermore, among the lines L_m from $L_1, \dots, L_M$ that intersect J, all apart from possibly one exceptional line L' will satisfy $L_m \cap \mathrm{J} = Y_m \in \Delta(L_0)$ and have coordinates corresponding to points $(A, B) \in F \cap \mathcal{L}$. Note that L_0 is independent of the position of J, and therefore it is the same for each generic interval associated with a Type 2 interval. The point is that P is fixed and all the lines of interest pass though P. However, in principle, the possible exceptional line L' may be different for each Type 2 interval. Fortunately, it is easy to deal with such lines. There are at most d exceptional lines L' — one for each of the d intervals $\tilde{I}_{nl}$. The number of intervals $I_{n+1} \in \mathcal{I}_{n+1}$ that can be removed by any single line L' is bounded above by K. Thus, no more than $d\,K \leq 2\,R^{1-\varepsilon}$ intervals I_{n+1} are removed in total by the exceptional lines L'. Now consider those lines $L_m = L(A_m, B_m, C_m)$ among $L_1, \dots, L_M$ that intersect some Type 2 interval and are not exceptional. It follows that

$$Y_m \in \Delta(L_0) \qquad \text{and} \qquad H(A_m, B_m) \geqslant H(A_0, B_0)\,.$$

- Suppose that $L_0 \in \mathcal{C}(n')$ for some $n' < n$. Denote by $\Delta^+(L_0)$ the interval with the same center as $\Delta(L_0)$ and length $|\Delta(L_0)| + 2\lceil R^{1+\alpha}\rceil |I_{n'+2}|$. It is readily verified that $\Delta(L_m) \subset \Delta^+(L_0)$ for any nonexceptional line L_m. Now observe that the interval $\Delta(L_0)$ is removed (from the segment Θ) at level n' of the basic construction; i.e., during the process of removing those 'bad' intervals $I_{n'+1}$ from $\mathcal{I}^-_{n'+1}$ that intersect some $\Delta(L)$ with $L \in \mathcal{C}(n')$. The set $\Delta^+(L_0) \setminus \Delta(L_0)$ is removed (from the segment Θ) by the 'trimming' process at level $n'+1$ of the basic construction. In other words, the interval $\Delta^+(L_0)$ has been totally removed from Θ even before we consider the effect of lines from $\mathcal{C}(n)$ on the remaining part of Θ; i.e., on intervals $I_{n+1} \in \mathcal{I}^-_{n+1}$. In a nutshell, there are no intervals $I_{n+1} \in \mathcal{I}^-_{n+1}$ that lie in $\Delta^+(L_0)$, and therefore any nonexceptional line L_m will have absolutely no 'removal' effect.
- Suppose that $L_0 \in \mathcal{C}(n)$. Denote by $\Delta^+(L_0)$ the interval with the same center as $\Delta(L_0)$ and length $2|\Delta(L_0)|$. It is readily verified that $\Delta(L_m) \subset \Delta^+(L_0)$ for any nonexceptional line L_m. In view of (20), the interval $\Delta^+(L_0)$ can remove no more than $4R^{1-\alpha} + 2$ intervals $I_{n+1} \in \mathcal{I}_{n+1}$.

The upshot when δ satisfies (62) is as follows. For any strictly positive $\varepsilon < \alpha$ and R sufficiently large so that (59) is valid, the total number of intervals $I_{n+1} \in \mathcal{I}_{n+1}$ removed by the lines $L_1, \dots, L_M$ associated with Type 2 intervals is bounded above by

$$4R^{1-\alpha} + 2 + K \cdot d \,=\, 4R^{1-\alpha} + 2 + 2R^{1-\varepsilon} \leqslant 6R^{1-\varepsilon} + 2 \leqslant 8R^{1-\varepsilon}\,. \tag{63}$$

Naturally, we now proceed by dealing with the situation when (62) is not satisfied.

6.2.2. *Type* 2 *intervals with* δ *large.* Suppose that

$$\delta > c_4 \left(\frac{cR}{2^k \tilde{c_1}}\right)^{2/j}. \tag{64}$$

In Case A it follows that

$$\delta > c_4 4^{-2/j}\, R^{-2(l+\varepsilon)/j}, \tag{65}$$

and in Case B, using the fact that $2^k < R$ (see (30)), it follows that

$$\delta \;>\; c_4 \left(\frac{R^{1-l-\alpha-\varepsilon}}{4 \cdot 2^k}\right)^{2/j} > c_4 4^{-2/j} R^{-\frac{2(l+\alpha+\varepsilon)}{j}}\,. \tag{66}$$

Recall that for the generic interval J associated with a Type 2 interval $\tilde{I}_{nl}$, there exists at most one exceptional line L' among $L_1, \dots, L_M$ that intersects J and has coordinates corresponding to a point not in $F \cap \mathcal{L}$. We have already observed that no more than $d\,K \leq 2\,R^{1-\varepsilon}$ intervals $I_{n+1} \in \mathcal{I}_{n+1}$ are removed in total by the d possible exceptional lines L'. Indeed, the latter are exactly the same as in the δ small case, and therefore the corresponding removed intervals I_{n+1} coincide.

We now consider those lines $L_m = L(A_m, B_m, C_m)$ among $L_1, \dots, L_M$ with $(A_m, B_m) \in F \cap \mathcal{L}$. Suppose we have two such lines L_m and $L_{m'}$ so that the points (A_m, B_m) and $(A_{m'}, B_{m'})$ lie on a line passing through the lattice point $(0,0)$. Clearly all points $(A, B) \in F \cap \mathcal{L}$ on this line have the same ratio A/B. Thus the lines L_m and $L_{m'}$ are parallel. However, this is impossible since L_m and $L_{m'}$ intersect at the rational point P. The upshot of this is that the points (A_m, B_m), $(A_{m'}, B_{m'})$ and $(0,0)$ do not lie on the same line. Recall that the lines L_m of interest are from within the collection $\mathcal{C}(n, l, k)$. To proceed we need to consider the $l = 0$ and $l > 0$ situations separately.

- Suppose that $l = 0$. Let

$$M^* := \#\{L_m \in \{L_1, \dots, L_M\} : (A_m, B_m) \in F \cap \mathcal{L}\}.$$

Observe that the figure F is convex. In view of the discussion above, it then follows that the lattice points in $F \cap \mathcal{L}$ together with the lattice point $(0,0)$ form the vertices of $(M^* - 1)$ disjoint triangles lying within F. Since the area of the fundamental domain of $\mathcal{L}$ is equal to q, the area of each of these disjoint triangles is at least $q/2$ and therefore the area of F is at least $q/2 \cdot (M^* - 1)$. Thus

$$\frac{q}{2}(M^* - 1) \leqslant \mathbf{area}(F) < 2c_2^{1+j/i} \overset{(50)}{=} \frac{q}{\delta^{1/i}}\,,$$

and therefore

$$M^* < 2\,\delta^{-1/i} + 1\,.$$

◦ In Case A it follows via (65) that

$$M^* < 4^{\frac{4}{ij}+1}\, R^{\frac{2\varepsilon}{ij}} + 1\,.$$

Hence

$$M^*K < 20 \cdot 4^{\frac{4}{ij}}\, R^{1-\alpha+\frac{2\varepsilon}{ij}}.$$

Moreover, if

$$\varepsilon \leqslant \frac{\alpha\, ij}{ij+2}\,,$$

then we have that

$$M^*K < 20 \cdot 4^{\frac{4}{ij}}\, R^{1-\varepsilon}\,. \tag{67}$$

◦ In Case B it follows via (66) that

$$M^* < 4^{\frac{4}{ij}+1}\, R^{\frac{2(\alpha+\varepsilon)}{ij}} + 1\,,$$

and thus the number of removed intervals is bounded by

$$M^*K < 20 \cdot 4^{\frac{4}{ij}}\, R^{\frac{2(\alpha+\varepsilon)}{ij}}\,.$$

It is readily verified that if

$$\varepsilon \leqslant \frac{ij-2\alpha}{ij+2}\,,$$

then the upper bound for M^*K given by (67) is valid in Case B.

• Suppose that $l > 0$. Instead of working with the figure F as in the $l = 0$ situation, we work with the 'smaller' convex figure $F_l \subset F$. Let

$$M^* := \#\{L_m \in \{L_1, \ldots, L_M\} : (A_m, B_m) \in F_l \cap \mathcal{L}\}.$$

The same argument as in the $l = 0$ situation yields that

$$\frac{q}{2}(M^* - 1) \leqslant \mathbf{area}(F_l) < 2c_3^{1+i}c_2^{1+j/i} = R^{-\left(\frac{\lambda l(j+1)}{j}-1\right)\cdot\frac{j(i+1)}{i}}\,\frac{q}{\delta^{1/i}}\,.$$

◦ In Case A we have

$$M^* < 4^{\frac{4}{ij}+1}\, R^{\frac{2(l+\varepsilon)}{ij}}\, R^{\frac{j(i+1)}{i}-\frac{\lambda l(i+1)(j+1)}{i}} + 1\,.$$

Since $l > 0$ and by definition $\lambda = 3/j$, it follows that

$$\frac{\lambda l(i+1)(j+1)}{i} - \frac{j(i+1)}{i} - \frac{2l}{ij} \;>\; 0\,. \tag{68}$$

Thus

$$M^* < 4^{\frac{4}{ij}+1} R^{\frac{2\varepsilon}{ij}} + 1$$

as in the $l = 0$ situation. In turn, the upper bound for M^*K given by (67) is valid for $l > 0$.

- In Case B we have

$$\begin{aligned} M^* &< 4^{\frac{4}{ij}+1} R^{\frac{2(l+\alpha+\varepsilon)}{ij}} \cdot R^{\frac{j(i+1)}{i} - \frac{\lambda l(i+1)(j+1)}{i}} + 1 \\ &\overset{(68)}{<} 4^{\frac{4}{ij}+1} R^{\frac{2(\alpha+\varepsilon)}{ij}} + 1 \end{aligned}$$

as in the $l = 0$ situation. In turn, the upper bound for M^*K given by (67) is valid in Case B for $l > 0$.

The upshot when δ satisfies (64) is as follows. For any strictly positive

$$\varepsilon \leqslant \frac{\alpha\, ij}{ij+2} \overset{(15)}{=} \min\left\{\frac{\alpha\, ij}{ij+2}\,,\ \frac{ij-2\alpha}{ij+2}\right\}$$

and R sufficiently large so that (59) is valid, the total number of intervals $I_{n+1} \in \mathcal{I}_{n+1}$ removed by the lines $L_1, \ldots, L_M$ associated with Type 2 intervals is bounded above by

$$K\, M^* \ + \ K \cdot d < 20 \cdot 4^{\frac{4}{ij}}\ R^{1-\varepsilon} \ + \ 2\, R^{1-\varepsilon} < 21 \cdot 4^{\frac{4}{ij}}\ R^{1-\varepsilon} . \tag{69}$$

6.3. *The finale.* On combining the upper bound estimates given by (61), (63) and (69), for any strictly positive $\varepsilon \leqslant \alpha ij/(ij+2)$ and $R > R_0(\varepsilon)$ sufficiently large, we have that

$$\begin{aligned} \text{l.h.s. of } (31) &\leqslant \#\{I_{n+1} \in \mathcal{I}^-_{n+1} : I_{n+1} \cap \Delta(L_m) \neq \emptyset \ \text{ for some } \ 1 \leqslant m \leqslant M\} \\ &< 2R^{1-\varepsilon} + 8R^{1-\varepsilon} + 21 \cdot 4^{\frac{4}{ij}}\ R^{1-\varepsilon} . \end{aligned}$$

This together with the fact that

$$\alpha^2 \ < \ \frac{\alpha\, ij}{ij+2}$$

completes the proof of Theorem 4.

7. Proof of Theorem 2

With reference to the statement of Theorem 2, since the set under consideration is a subset of a line, we immediately obtain the upper bound result that

$$\dim \Big(\bigcap_{t=1}^{\infty} \mathbf{Bad}(i_t, j_t) \cap \mathrm{L}_\theta \Big) \leqslant 1 . \tag{70}$$

Thus, the proof of Theorem 2 follows on establishing the following complementary lower bound estimate.

THEOREM 5. *Let* (i_t, j_t) *be a countable number of pairs of real numbers satisfying* (6) *and let* $i := \sup\{i_t : t \in \mathbb{N}\}$. *Suppose that* (5) *is also satisfied. Then, for any* $\theta \in \mathbf{Bad}(i)$, *we have that*

$$\dim\Big(\bigcap_{t=1}^{\infty} \mathbf{Bad}(i_t, j_t) \cap \mathrm{L}_\theta\Big) \geqslant 1\,.$$

Remark. Strictly speaking, in order to deduce Theorem 2 we should replace (6) by (2) in the above statement of Theorem 5. However, given the arguments set out in Section 3.2, the proof of Theorem 5 as stated can easily be adapted to deal with the 'missing' pairs $(1, 0)$ and $(0, 1)$.

A general and classical method for obtaining a lower bound for the Hausdorff dimension of an arbitrary set is the following mass distribution principle; see [3, p. 55].

LEMMA 3 (Mass Distribution Principle). *Let* μ *be a probability measure supported on a subset* X *of* $\mathbb{R}$. *Suppose there are positive constants* a, s *and* l_0 *such that*

$$\mu(I) \leqslant a\,|I|^s \tag{71}$$

for any interval I *with length* $|I| \leqslant l_0$. *Then*, $\dim X \geqslant s$.

The overall strategy for establishing Theorem 5 is simple enough. For each $t \in \mathbb{N}$, let

$$\alpha_t := \tfrac{1}{4}\, i_t\, j_t \qquad \text{and} \qquad \varepsilon_0 := \inf_{t\in\mathbb{N}} \tfrac{1}{2}\alpha_t^2\,. \tag{72}$$

In view of condition (5) imposed in the statement of the theorem, we have that ε_0 is strictly positive. Then for any strictly positive $\varepsilon < \varepsilon_0$, we construct a 'Cantor-type' subset $\mathbf{K}(\varepsilon)$ of $\bigcap_{t=1}^{\infty} \mathbf{Bad}(i_t, j_t) \cap \mathrm{L}_\theta$ and a probability measure μ supported on $\mathbf{K}(\varepsilon)$ satisfying the condition that

$$\mu(I) \leqslant a\,|I|^{1-\varepsilon/2}\,, \tag{73}$$

where the constant a is absolute and $I \subset \Theta$ is an arbitrary small interval. Hence by construction and the mass distribution principle we have that

$$\dim\Big(\bigcap_{t=1}^{\infty} \mathbf{Bad}(i_t, j_t) \cap \mathrm{L}_\theta\Big) \geqslant \dim\big(\mathbf{K}(\varepsilon)\big) \geqslant 1 - \varepsilon/2\,.$$

Now suppose that $\dim\big(\bigcap_{t=1}^{\infty} \mathbf{Bad}(i_t, j_t) \cap \mathrm{L}_\theta\big) < 1$. Then

$$\dim\Big(\bigcap_{t=1}^{\infty} \mathbf{Bad}(i_t, j_t) \cap \mathrm{L}_\theta\Big) = 1 - \eta$$

for some $\eta > 0$. However, by choosing $\varepsilon < 2\eta$ we obtain a contradiction and thereby establish Theorem 5.

In view of the above outline, the whole strategy of our proof is centred around the construction of a 'right type' of Cantor set $\mathbf{K}(\varepsilon)$ which supports a measure μ with the desired property. It should come as no surprise, that the first step involves modifying the basic construction to simultaneously incorporate any countable number of (i,j) pairs satisfying (5) and (6).

7.1. *Modifying the basic construction for countable pairs.* With reference to Section 2, for each $t \in \mathbb{N}$, let us write $H_t(A,B)$ for $H(A,B)$, $\Delta_t(L)$ for $\Delta(L)$ and $\mathcal{C}_t(n)$ for $\mathcal{C}(n)$. Furthermore, write $\mathcal{J}_n(t)$ for $\mathcal{J}_n$ and $\mathcal{I}_n^-(t)$ for $\mathcal{I}_n^-$. With this in mind, let $R \geqslant 2$ be an integer. Choose $c_1(t) = c_1(R,t)$ sufficiently small so that

$$c_1(t) \leqslant \tfrac{1}{4} R^{-3i_t/j_t}, \tag{74}$$

and

$$c(t) := \frac{c_1(t)}{R^{1+\alpha_t}}$$

satisfies (9) with α_t given by (72). With this choice of $c_1(t)$, the basic construction of Section 2.2 enables us to conclude that $\mathbf{Bad}(i_t,j_t) \cap \mathrm{L}_\theta \neq \emptyset$, and in the process we establish the all important 'counting' estimate given by (26). Namely, let $l \geqslant 0$ and $J_{n-l} \in \mathcal{J}_{n-l}(t)$. Then, for any strictly positive $\varepsilon < \frac{1}{2}\alpha_t^2$ and $R > R_0(\varepsilon,t)$ sufficiently large, we have that

(75)

$$\#\{I_{n+1} \in \mathcal{I}_{n+1}^-(t) : J_{n-l} \cap \Delta_t(L) \cap I_{n+1} \neq \emptyset \quad \text{for some} \quad L \in \mathcal{C}_t(n,l)\} \leqslant R^{1-\varepsilon} .$$

With ε_0 given by (72), this estimate is clearly valid for any strictly positive $\varepsilon < \varepsilon_0$. The first step towards simultaneously dealing with the countable number of (i_t,j_t) pairs is to modify the basic construction in such a manner so that corresponding version of (75) remains intact. The key is to start the construction with the (i_1,j_1) pair and then introduce at different levels within it the other pairs. Beyond this, the modifications are essentially at the 'trimming' stage and in the manner in which the collections $\mathcal{J}_n$ are defined.

Fix some strictly positive $\varepsilon < \varepsilon_0$ and let R be an arbitrary integer satisfying

$$R > R_0(\varepsilon,1) \ . \tag{76}$$

Then, with

$$c_1 := c_1(1)$$

we are able to carry out the basic construction for the (i_1,j_1) pair. For each $t \geqslant 2$, the associated basic construction for the (i_t,j_t) pair is carried out with respect to a sufficiently large integer $R_t > R_0(\varepsilon,t)$, where R_t is some power of R. This enables us to embed the construction for each $t \geqslant 2$ within the construction for $t = 1$. More precisely, for $t \geqslant 1$, we let

$$R_t := R^{m_t} \ ,$$

where the integer m_t satisfies

$$m_1 = 1\,,$$

and for $t \geq 2$,

$$R^{m_t} \geqslant \max\{R_0(\varepsilon, t), R^{1+m_{t-1}}\}\,.$$

Notice that

$$m_t \geqslant t \quad \text{for} \quad t \geqslant 2. \tag{77}$$

Now for each $t \geqslant 2$, we fix an integer k_t sufficiently large such that

$$c_1(t) := c_1 R^{-k_t}$$

satisfies (74); for consistency we let $k_1 = 0$. Then for each $t \geqslant 1$, with this choice of $c_1(t)$ we are able to carry out the basic construction for the pair (i_t, j_t). Moreover, for each integer $s \geqslant 0$, let

$$n_s(t) := k_t + s m_t\,.$$

Then intervals at level s of the construction for (i_t, j_t) can be described in terms of intervals at level $n_s(t)$ of the construction for (i_1, j_1). In particular, an interval of length $c_1(t) R_t^{-s}$ at level s for (i_t, j_t) corresponds to an interval of length $c_1 R^{-n_s(t)}$ at level $n_s(t)$ for (i_1, j_1).

We are now in the position to modify the basic construction for the pair (i_1, j_1) so as to simultaneously incorporate each (i_t, j_t) pair. Let c_1 be as above. Define the collections $\mathcal{J}_0 := \mathcal{J}_0(1)$ and $\mathcal{J}_1 := \mathcal{J}_1(1)$. Also Stage 1 of 'the induction' in which the collection $\mathcal{I}_{n+1}$ is introduced remains unchanged. However, the goal now is to remove those 'bad' intervals $I_{n+1} \in \mathcal{I}_{n+1}$ for which

$$I_{n+1} \cap \Delta_t(L) \neq \emptyset \quad \text{for some } t \in \mathbb{N} \quad \text{and } L \in \mathcal{C}_t\left(\left\lceil \tfrac{n+1-k_t}{m_t} \right\rceil - 1\right). \tag{78}$$

Regarding Stage 2, we trim the collection $\mathcal{I}_{n+1}$ in the following manner. To begin with we remove from each $J_n \in \mathcal{J}_n$ the first $\lceil R^{1-\alpha_1} \rceil$ subintervals I_{n+1} from each end. In other words, we implement the basic trimming process associated with the pair (i_1, j_1). Then for any integer $t \geqslant 2$, if $n+1 = n_{s+1}(t)$ for some s, we incorporate the basic trimming process associated with the pair (i_t, j_t). This involves removing any interval I_{n+1} that coincides with one of the $\lceil R_t^{1-\alpha_t} \rceil$ subintervals of length $|I_{n+1}|$ at either end of some $J_{n_s(t)} \in \mathcal{J}_{n_s(t)}$. It follows that for each such t, the number of intervals I_{n+1} from $\mathcal{I}_{n+1}$ that are removed by this 'modified' trimming process is bounded above by

$$\#\mathcal{J}_{n_s(t)} \times 2\,\lceil R_t^{1-\alpha_t} \rceil := \#\mathcal{J}_{n+1-m_t} \times 2\,\lceil R^{m_t(1-\alpha_t)} \rceil\;;$$

i.e., the number removed by the basic trimming process associated with the pair (i_t, j_t). Note that this bound is valid for $t = 1$. The intervals I_{n+1} from $\mathcal{I}_{n+1}$ that survive the above trimming process give rise to the trimmed collection

$\mathcal{I}_{n+1}^-$. We define $\mathcal{J}_{n+1}$ to be the collection obtained by removing those 'bad' intervals I_{n+1} from $\mathcal{I}_{n+1}^-$ which satisfy (78). In other words, for $n \geqslant 1$ we let

(79)
$$\mathcal{J}_{n+1} := \Big\{ I_{n+1} \in \mathcal{I}_{n+1}^- : \Delta_t(L) \cap I_{n+1} = \emptyset \quad \forall\, t \in \mathbb{N} \text{ and } L \in C_t\left(\left[\tfrac{n+1-k_t}{m_t}\right] - 1\right) \Big\}.$$

Here, it is understood that the collection of lines $C_t(n)$ is the empty set whenever n is negative. Note that by construction, the collection $\mathcal{J}_n$ is a subcollection of $\mathcal{J}_s(t)$ whenever $n = n_s(t)$ for some $t \in \mathbb{N}$.

Apart from obvious notational modifications, Stages 3 and 4 remain pretty much unchanged and give rise to (75) for each $t \in \mathbb{N}$ with R replaced by R_t. As consequence, for any $l \geqslant 0$ and $J_{n+1-(l+1)m_t} \in \mathcal{J}_{n+1-(l+1)m_t}$, we have that

$$\text{(80)} \quad \#\Big\{ I_{n+1} \in \mathcal{I}_{n+1}^- : J_{n+1-(l+1)m_t} \cap \Delta_t(L) \cap I_{n+1} \neq \emptyset \\ \text{for some } \ L \in C_t\left(\left[\tfrac{n+1-k_t}{m_t}\right] - 1, l\right) \Big\} \leqslant R^{m_t(1-\varepsilon)}.$$

To see this, let $s+1 := [(n+1-k_t)/m_t]$. Now if $s+1 = (n+1-k_t)/m_t$, then the statement is a direct consequence of (75) with $n = s$ and R replaced by R_t. Here we use the fact that $\mathcal{I}_{n+1}^- \subseteq \mathcal{I}_{n+1}^-(t)$. Now suppose that $s+1 < (n+1-k_t)/m_t$. Then I_{n+1} is contained in some interval $J_{k_t+(s+1)m_t}$. By construction the latter does not intersect any interval $\Delta_t(L)$ with $L \in \mathcal{C}_t(s,l)$. Thus the set on the left-hand side of (80) is empty and the inequality is trivially satisfied.

For fixed $\varepsilon < \varepsilon_0$ and any R satisfying (76), the upshot of the modified basic construction is the existence of nested collections $\mathcal{J}_n$ of intervals J_n given by (79) such that

$$\text{(81)} \qquad \mathbf{K}^*(\varepsilon, R) := \bigcap_{n=0}^{\infty} \bigcup_{J \in \mathcal{J}_n} J \subset \bigcap_{t=1}^{\infty} \mathbf{Bad}(i_t, j_t) \cap \mathrm{L}_\theta\,.$$

Moreover, for R sufficiently large, the counting estimate (80) can be used to deduce that

$$\text{(82)} \qquad \#\mathcal{J}_n \geqslant (R - R^{1-\varepsilon/2})^n\,;$$

see the remark following the proof of Lemma 4 below. Clearly, (82) is more than sufficient to conclude that $\mathbf{K}^*(\varepsilon, R)$ is nonempty which together with (81) implies that

$$\bigcap_{t=1}^{\infty} \mathbf{Bad}(i_t, j_t) \cap \mathrm{L}_\theta \neq \emptyset\,.$$

Recall that, as long as (5) is valid, this enables us to establish the countable version of Schmidt's conjecture. However, counting alone is not enough to obtain the desired dimension result. For this we need to adapt the collections $\mathcal{J}_n$ arising from the modified construction. The necessary 'adaptation' will be the subject of the next section.

We end this section by investigation the distribution of intervals within a given collection $\mathcal{J}_n$. Let J_0 be an arbitrary interval from $\mathcal{J}_0$ and define $\mathcal{T}_0 := \{J_0\}$. For $n \geqslant 1$, we construct the nested collections $\mathcal{T}_n, \mathcal{T}_{n-1}, \ldots, \mathcal{T}_1, \mathcal{T}_0$ as follows. Take an arbitrary interval in $\mathcal{T}_{n-1}$ and subdivide it into R closed intervals of equal length. Choose any $[2R^{1-\varepsilon/2}]$ of the R subintervals and disregard the others. Repeat this procedure for each interval in $\mathcal{T}_{n-1}$ and let $\mathcal{T}_n$ denote the collection of all chosen subintervals. Clearly,

$$\#\mathcal{T}_n = \#\mathcal{T}_{n-1} \times [2R^{1-\varepsilon/2}] .$$

Loosely speaking, the following result shows that the intervals J_n from $\mathcal{J}_n$ are ubiquitous within each of the intervals $J_0 \subset \Theta$ and thus within the whole of Θ. It is worth emphasizing that both the collections $\mathcal{J}_n$ and $\mathcal{T}_n$ are implicitly dependent on R.

LEMMA 4. *For R sufficiently large*,

$$\mathcal{T}_n \cap \mathcal{J}_n \neq \emptyset \qquad \forall \quad n = 0, 1, \ldots . \tag{83}$$

Proof. For an integer $m \geqslant 0$, let $f(m)$ denote the cardinality of the set $\mathcal{T}_m \cap \mathcal{J}_m$. Trivially, $f(0) = 1$ and the lemma would follow on showing that

$$f(m) \geqslant R^{1-\varepsilon/2} f(m-1) \qquad \forall \quad m \in \mathbb{N} . \tag{84}$$

This we now do via induction. To begin with, note that $\#\mathcal{J}_1 = \#\mathcal{J}_0 \times R$, and so

$$f(1) = [2R^{1-\varepsilon/2}] > R^{1-\varepsilon/2} .$$

In other words, (84) is satisfied for $m = 1$. Now assume that (84) is valid for all $1 \leqslant m \leqslant n$. In order to establish the statement for $m = n+1$, observe that each of the $f(n)$ intervals in $\mathcal{T}_n \cap \mathcal{J}_n$ gives rise to $[2R^{1-\varepsilon/2}]$ intervals in $\mathcal{T}_{n+1} \cap \mathcal{I}_{n+1}$. Now consider some $t \in \mathbb{N}$ and an integer $l \geqslant 0$ such that $n+1-(l+1)m_t \geqslant k_t$. Then in view of (80), for any interval

$$J_{n+1-(l+1)m_t} \in \mathcal{J}_{n+1-(l+1)m_t} \cap \mathcal{T}_{n+1-(l+1)m_t} ,$$

the number of intervals from $\mathcal{I}^-_{n+1}$ removed by lines $L \in C_t\Big([\frac{n+1-k_t}{m_t}] - 1, l\Big)$ is bounded above by $R^{m_t(1-\varepsilon)}$. By the induction hypothesis,

$$\#(\mathcal{J}_{n+1-(l+1)m_t} \cap \mathcal{T}_{n+1-(l+1)m_t}) = f(n+1-(l+1)m_t) .$$

Thus the total number of intervals from $\mathcal{T}_{n+1} \cap \mathcal{I}^-_{n+1}$ removed by lines from $C_t\Big([\frac{n+1-k_t}{m_t}] - 1, l\Big)$ is bounded above by

$$R^{m_t(1-\varepsilon)} \; f\Big(n+1-(l+1)m_t\Big) .$$

Furthermore, the number of intervals from $\mathcal{T}_{n+1} \cap \mathcal{I}_{n+1}$ removed by the modified trimming process associated with the pair (i_t, j_t) is bounded above by

$$2\,\lceil R^{m_t(1-\alpha_t)} \rceil \; f(n+1-m_t) \leqslant 2\,R^{m_t(1-\varepsilon)} \; f(n+1-m_t).$$

Here we have made use of the fact that $R^{m_t} > R_0(\varepsilon, t)$, and so $\lceil R^{m_t(1-\alpha_t)} \rceil \leq R^{m_t(1-\varepsilon)}$.

On combining the above estimates for intervals removed by 'lines' and those removed by 'trimming', it follows that

$$\begin{aligned} f(n+1) \geqslant [2R^{1-\varepsilon/2}] \;\; f(n) \\ - \sum_{t=1}^{\infty} R^{m_t(1-\varepsilon)} \sum_{l=1}^{\infty} f(n+1-lm_t) - 2\sum_{t=1}^{\infty} R^{m_t(1-\varepsilon)} f(n+1-m_t) \ . \end{aligned}$$

Here, it is understood that $f(k) = 0$ whenever k is negative. Then, in view of our induction hypothesis, we have that

$$\begin{aligned} f(n+1) &\geqslant [2R^{1-\varepsilon/2}] \;\; f(n) \\ &\quad - \sum_{t=1}^{\infty} R^{m_t(1-\varepsilon)} \; f(n) \; (R^{-1+\varepsilon/2})^{m_t-1} \left(2 + \sum_{l=0}^{\infty} (R^{-1+\varepsilon/2})^{lm_t}\right) \\ &\geqslant f(n) \left([2R^{1-\varepsilon/2}] - R^{1-\varepsilon} \; C(R) \; - \; \sum_{t=2}^{\infty} R^{1-\frac{m_t}{2}\varepsilon - \frac{\varepsilon}{2}} \; C(R) \right), \end{aligned}$$

where

$$C(R) := 2 + \sum_{k=0}^{\infty} (R^{-1+\varepsilon/2})^k.$$

In addition to R satisfying (76) we assume that R is sufficiently large so that

$$(85) \qquad C(R) < 4\,, \qquad [2R^{1-\varepsilon/2}] \geqslant \tfrac{5}{3} R^{1-\varepsilon/2} \qquad \text{and} \qquad \sum_{k=1}^{\infty} R^{-\frac{k}{2}\varepsilon} < \tfrac{1}{6}\,.$$

Then, by making use of (77), it follows that

$$f(n+1) \geqslant \; R^{1-\varepsilon/2} f(n) \left(\tfrac{5}{3} - 4 \sum_{t=1}^{\infty} R^{-\frac{t}{2}\varepsilon} \right) \geqslant R^{1-\varepsilon/2} f(n)\,.$$

This completes the proof of the lemma. □

Remark. For any R satisfying (76) and (85), a straightforward consequence of (84) is that

$$\#\mathcal{J}_n \geqslant f(n) \times \#\mathcal{J}_0 \; \geqslant R^{1-\varepsilon/2} \, f(n-1) \times \#\mathcal{J}_0 \geqslant (R^{1-\varepsilon/2})^n > 1\,.$$

This is sufficient to show that $\mathbf{K}^*(\varepsilon, R)$ is nonempty and in turn enables us to establish the countable version of Schmidt's conjecture. However, the proof of the lemma can be naturally modified and adapted to deduce the stronger counting estimate given by (82); essentially replace $f(m)$ by $\#\mathcal{J}_m$ and $[2R^{1-\varepsilon/2}]$ by R.

7.2. *The set* $\mathbf{K}(\varepsilon)$ *and the measure* μ. Fix some strictly positive $\varepsilon < \varepsilon_0$ and an integer R satisfying (76) and (85). The modified construction of the previous section enables us to conclude that the set $\mathbf{K}^*(\varepsilon) := \mathbf{K}^*(\varepsilon, R)$ defined by (81) is nonempty and in turn implies the weaker nonempty analogue of Theorem 5. To obtain the desired dimension statement we construct a regular 'Cantor-type' subset $\mathbf{K}(\varepsilon)$ of $\mathbf{K}^*(\varepsilon)$ and a measure μ satisfying (73). The key is to refine the collections $\mathcal{J}_n$ arising from the modified construction in such a manner that the refined nested collections $\mathcal{M}_n \subseteq \mathcal{J}_n$ are nonempty and satisfy the following property. For any integer $n \geqslant 0$ and $J_n \in \mathcal{M}_n$,

$$\#\{J_{n+1} \in \mathcal{M}_{n+1} : J_{n+1} \subset J_n\} \geqslant R - 2R^{1-\varepsilon/2}\,.$$

Suppose for the remaining part of this section the desired collections $\mathcal{M}_n$ exist and let

$$\mathbf{K}(\varepsilon) := \bigcap_{n=0}^{\infty} \bigcup_{J \in \mathcal{M}_n} J\,.$$

We now construct a probability measure μ supported on $\mathbf{K}(\varepsilon)$ in the standard manner. For any $J_n \in \mathcal{M}_n$, we attach a weight $\mu(J_n)$ defined recursively as follows.

For $n = 0$,

$$\mu(J_0) := \frac{1}{\#\mathcal{M}_0}$$

and for $n \geqslant 1$,

$$\mu(J_n) := \frac{\mu(J_{n-1})}{\#\{J \in \mathcal{M}_n : J \subset J_{n-1}\}}\,, \tag{86}$$

where $J_{n-1} \in \mathcal{M}_{n-1}$ is the unique interval such that $J_n \subset J_{n-1}$. This procedure thus defines inductively a mass on any interval appearing in the construction of $\mathbf{K}(\varepsilon)$. In fact a lot more is true; μ can be further extended to all Borel subsets F of $\mathbb{R}$ to determine $\mu(F)$ so that μ constructed as above actually defines a measure supported on $\mathbf{K}(\varepsilon)$. We now state this formally.

> *Fact.* The probability measure μ constructed above is supported on $\mathbf{K}(\varepsilon)$ and, for any Borel set F,
>
> $$\mu(F) := \mu(F \cap \mathbf{K}(\varepsilon)) = \inf \sum_{J \in \mathcal{J}} \mu(J)\,.$$
>
> The infimum is over all coverings $\mathcal{J}$ of $F \cap \mathbf{K}(\varepsilon)$ by intervals $J \in \{\mathcal{M}_n : n = 0, 1, \ldots\}$.

For further details, see [3, Prop. 1.7]. It remains to show that μ satisfies (73). Firstly, notice that for any interval $J_n \in \mathcal{M}_n$, we have that

$$\begin{aligned}\mu(J_n) &\leqslant \left(R\left(1 - 2R^{-\varepsilon/2}\right)\right)^{-1} \mu(J_{n-1}) \\ &\leqslant \left(R\left(1 - 2R^{-\varepsilon/2}\right)\right)^{-n}\,.\end{aligned}$$

Next, let d_n denote the length of a generic interval $J_n \in \mathcal{M}_n$ and consider an arbitrary interval $I \subset \Theta$ with length $|I| < d_0$. Then there exists a nonnegative integer n such that

$$d_{n+1} \leqslant |I| < d_n \,. \tag{87}$$

It follows that

$$\begin{aligned}
\mu(I) &\leqslant \sum_{\substack{J_{n+1}\in\mathcal{M}_{n+1}\\ J_{n+1}\cap I\neq\emptyset}} \mu(J_{n+1}) \\
&\leqslant \left\lceil \frac{|I|}{d_{n+1}} \right\rceil \Big(R\,(1-R^{-\varepsilon/2})\Big)^{-n-1} \\
&\leqslant 2\frac{|I|}{c_1 R^{-n-1}}\; R^{-n-1}\, \Big(1-2R^{-\varepsilon/2}\Big)^{-n-1} \\
&\overset{(87)}{<} 2\,c_1^{\varepsilon/2-1} R^{\varepsilon/2}\,\Big(R^{\varepsilon/2}\Big(1-2R^{-\varepsilon/2}\Big)\Big)^{-n-1}\,|I|^{1-\varepsilon/2} \\
&\overset{(85)}{\leqslant} 2\,c_1^{\varepsilon/2-1} R^{\varepsilon/2}|I|^{1-\varepsilon/2} \,.
\end{aligned}$$

Thus (73) follows with $a = 2c_1^{\varepsilon/2-1}R^{\varepsilon/2}$, and this completes the proof of Theorem 5 modulo the existence of the collection $\mathcal{M}_n$.

7.3. *Constructing the collection* $\mathcal{M}_n$. For any integer $n \geq 0$, the goal of this section is to construct the desired nested collection $\mathcal{M}_n \subseteq \mathcal{J}_n$ alluded to in the previous section. This will involve constructing auxiliary collections $\mathcal{M}_{n,m}$ and $\mathcal{R}_{n,m}$ for integers n, m satisfying $0 \leqslant n \leqslant m$. For a fixed m, let

$$\mathcal{J}_0,\ \mathcal{J}_1, \ldots, \mathcal{J}_m$$

be the collections arising from the modified construction of Section 7.1. We will require $\mathcal{M}_{n,m}$ to satisfy the following conditions.

C1. For any $0 \leqslant n \leqslant m$, we have that $\mathcal{M}_{n,m} \subseteq \mathcal{J}_n$.

C2. For any $0 \leqslant n < m$, the collections $\mathcal{M}_{n,m}$ are nested; that is,

$$\bigcup_{J\in\mathcal{M}_{n+1,m}} J \quad \subset \quad \bigcup_{J\in\mathcal{M}_{n,m}} J.$$

C3. For any $0 \leqslant n < m$ and $J_n \in \mathcal{M}_{n,m}$, we have that there are at least $R - [2R^{1-\varepsilon/2}]$ intervals $J_{n+1} \in \mathcal{M}_{n+1,m}$ contained within J_n; that is,

$$\#\{J_{n+1} \in \mathcal{M}_{n+1,m} : J_{n+1} \subset J_n\} \geqslant R - [2R^{1-\varepsilon/2}]\,.$$

In addition, define $\mathcal{R}_{0,0} := \emptyset$ and, for $m \geqslant 1$,

$$\mathcal{R}_{m,m} := \{I_m \in \mathcal{I}_m \backslash \mathcal{J}_m : I_m \subset J_{m-1} \text{ for some } J_{m-1} \in \mathcal{M}_{m-1,m-1}\}\ . \tag{88}$$

Furthermore, for $0 \leqslant n < m$, define

(89)

$$\mathcal{R}_{n,m} := \mathcal{R}_{n,m-1} \cup \{J_n \in \mathcal{M}_{n,m-1} : \#\{J_{n+1} \in \mathcal{R}_{n+1,m} : J_{n+1} \subset J_n\} \geqslant [2R^{1-\varepsilon/2}]\}\,.$$

Loosely speaking and with reference to condition (C3), the collections $\mathcal{R}_{n,m}$ are the 'dumping ground' for those intervals $J_n \in \mathcal{M}_{n,m-1}$ which do not contain enough subintervals J_{n+1}. Note that for m fixed, the collections $\mathcal{R}_{n,m}$ are defined in descending order with respect to n. In other words, we start with $\mathcal{R}_{m,m}$ and finish with $\mathcal{R}_{0,m}$.

The construction is as follows.

Stage 1. Let $\mathcal{M}_{0,0} := \mathcal{J}_0$ and $\mathcal{R}_{0,0} := \emptyset$.

Stage 2. Let $0 \leqslant t \leqslant n$. Suppose we have constructed the desired collections

$$\mathcal{M}_{0,t} \subseteq \mathcal{J}_0,\ \mathcal{M}_{1,t} \subseteq \mathcal{J}_1, \dots, \mathcal{M}_{t,t} \subseteq \mathcal{J}_t$$

and

$$\mathcal{R}_{0,t}, \dots, \mathcal{R}_{t,t}\,.$$

We now construct the corresponding collections for $t = n+1$.

Stage 3. Define

$$\mathcal{M}'_{n+1,n+1} := \{J_{n+1} \in \mathcal{J}_{n+1} : J_{n+1} \subset J_n \text{ for some } J_n \in \mathcal{M}_{n,n}\}$$

and let $\mathcal{R}_{n+1,n+1}$ be given by (88) with $m = n+1$. Thus the collection $\mathcal{M}'_{n+1,n+1}$ consists of 'good' intervals from $\mathcal{J}_{n+1}$ that are contained within some interval from $\mathcal{M}_{n,n}$. Our immediate task is to construct the corresponding collections $\mathcal{M}'_{u,n+1}$ for each $0 \leqslant u \leqslant n$. These will be constructed together with the 'complementary' collections $\mathcal{R}_{u,n+1}$ in descending order with respect to u.

Stage 4. With reference to Stage 3, suppose we have constructed the collections $\mathcal{M}'_{u+1,n+1}$ and $\mathcal{R}_{u+1,n+1}$ for some $0 \leqslant u \leqslant n$. We now construct $\mathcal{M}'_{u,n+1}$ and $\mathcal{R}_{u,n+1}$. Consider the collections $\mathcal{M}_{u,n}$ and $\mathcal{R}_{u,n}$. Observe that some of the intervals J_u from $\mathcal{M}_{u,n}$ may contain less than $R - [2R^{1-\varepsilon/2}]$ subintervals from $\mathcal{M}'_{u+1,n+1}$ (or in other words, at least $[2R^{1-\varepsilon/2}]$ intervals from $\mathcal{R}_{u+1,n+1}$). Such intervals J_u fail the counting condition (C3) for $\mathcal{M}_{u,n+1}$ and informally speaking are moved out of $\mathcal{M}_{u,n}$ and into $\mathcal{R}_{u,n}$. The resulting subcollections are $\mathcal{M}'_{u,n+1}$ and $\mathcal{R}_{u,n+1}$ respectively. Formally,

$$\mathcal{M}'_{u,n+1} := \{J_u \in \mathcal{M}_{u,n} : \#\{J_{u+1} \in \mathcal{R}_{u+1,n+1} : J_{u+1} \subset J_u\} < [2R^{1-\varepsilon/2}]\}$$

and $\mathcal{R}_{u,n+1}$ is given by (89) with $n = u$ and $m = n+1$.

Stage 5. By construction the collections $\mathcal{M}'_{u,n+1}$ satisfy conditions (C1) and (C3). However, for some $J_{u+1} \in \mathcal{M}'_{u+1,n+1}$, it may be the case that J_{u+1}

is not contained in any interval $J_u \in \mathcal{M}'_{u,n+1}$, and thus the collections $\mathcal{M}'_{u,n+1}$ are not necessarily nested. The point is that during Stage 4 above the interval $J_u \in \mathcal{J}_u$ containing J_{u+1} may be 'moved' into $\mathcal{R}_{u,n+1}$. In order to guarantee the nested condition (C2), such intervals J_{u+1} are removed from $\mathcal{M}'_{u+1,n+1}$. The resulting subcollection is the required auxiliary collection $\mathcal{M}_{u+1,n+1}$. Note that $\mathcal{M}_{u+1,n+1}$ is constructed via $\mathcal{M}'_{u+1,n+1}$ in ascending order with respect to u. Formally,

$$\mathcal{M}_{0,n+1} := \mathcal{M}'_{0,n+1},$$

and for $1 \leqslant u \leqslant n+1$,

$$\mathcal{M}_{u,n+1} := \{J_u \in \mathcal{M}'_{u,n+1} : J_u \subset J_{u-1} \text{ for some } J_{u-1} \in \mathcal{M}_{u-1,n+1}\}.$$

With reference to Stage 2, this completes the induction step and thereby the construction of the auxiliary collections.

For any integer $n \geq 0$, it remains to construct the sought after collection $\mathcal{M}_n$ via the auxiliary collections $\mathcal{M}_{n,m}$. Observe that since

$$\mathcal{M}_{n,n} \supset \mathcal{M}_{n,n+1} \supset \mathcal{M}_{n,n+2} \supset \cdots$$

and the cardinality of each collection $\mathcal{M}_{nm}$ with $n \leqslant m$ is finite, there exists some integer $N(n)$ such that

$$\mathcal{M}_{n,m} = \mathcal{M}_{n,m'} \qquad \forall \quad m, m' \geqslant N(n).$$

Now simply define

$$\mathcal{M}_n := \mathcal{M}_{n,N(n)}.$$

Unfortunately, there remains one slight issue. The collection $\mathcal{M}_n$ defined in this manner could be empty.

The goal now is to show that $\mathcal{M}_{n,m} \neq \emptyset$ for any $n \leqslant m$. This clearly implies that $\mathcal{M}_n \neq \emptyset$ and thereby completes the construction.

PROPOSITION 2. *For all integers satisfying* $0 \leqslant n \leqslant m$, *the collection* $\mathcal{M}_{n,m}$ *is nonempty.*

Proof. Suppose on the contrary that $\mathcal{M}_{n,m} = \emptyset$ for some integers satisfying $0 \leqslant n \leqslant m$. In view of the construction of $\mathcal{M}_{n,m}$, every interval from $\mathcal{M}_{n-1,m}$ contains at least $R - 2R^{1-\varepsilon/2}$ subintervals from $\mathcal{M}_{n,m}$. Therefore $\mathcal{M}_{0,m}$ is empty and it follows that $\mathcal{R}_{0,m} = \mathcal{J}_0$.

Now consider the set $\mathcal{R}_{n,m}$. Note that

$$\mathcal{R}_{n,m} \supseteq \mathcal{R}_{n,m-1} \supseteq \cdots \supseteq \mathcal{R}_{n,n}$$

and that in view of (88), elements of $\mathcal{R}_{n,n}$ are intervals from $\mathcal{I}_n \backslash \mathcal{J}_n$. Consider any interval $J_n \in \mathcal{R}_{n,m} \backslash \mathcal{R}_{n,n}$. Then there exists an integer m_0 with $n < m_0 \leqslant m$ such that $J_n \in \mathcal{R}_{n,m_0}$ but $J_n \notin \mathcal{R}_{n,m_0-1}$. In view of (89), any interval from $\mathcal{R}_{n,m_0}$ contains at least $[2R^{1-\varepsilon/2}]$ subintervals from $\mathcal{R}_{n+1,m_0}$ and therefore from

$\mathcal{R}_{n+1,m}$. The upshot is that for any interval $I_n \in \mathcal{R}_{n,m}$, we either have that $I_n \in \mathcal{I}_n \backslash \mathcal{J}_n$ or that I_n contains at least $[2R^{1-\varepsilon/2}]$ intervals $I_{n+1} \in \mathcal{R}_{n+1,m}$.

Next we exploit Lemma 4. Choose an arbitrary interval J_0 from $\mathcal{R}_{0,m} = \mathcal{J}_0$ and define $\mathcal{T}_0 := \{J_0\}$. For $0 \leqslant n < m$, we define inductively the nested collections

$$\mathcal{T}_{n+1} := \{I_{n+1} \in \mathcal{T}(I_n) : I_n \in \mathcal{T}_n\}$$

with $\mathcal{T}(I_n)$ given by one of the following three scenarios.

- $I_n \in \mathcal{R}_{n,m}$ and I_n contains at least $[2R^{1-\varepsilon/2}]$ subintervals I_{n+1} from $\mathcal{R}_{n+1,m}$. Let $\mathcal{T}(I_n)$ be any collection consisting of $[2R^{1-\varepsilon/2}]$ such subintervals. Note that when $n = m-1$, we have $\mathcal{T}(I_n) \subset \mathcal{R}_{m,m} \subset \mathcal{I}_m \backslash \mathcal{J}_m$. Therefore $\mathcal{T}(I_{m-1}) \cap \mathcal{J}_m = \emptyset$.
- $I_n \in \mathcal{R}_{n,m}$ and I_n contains strictly less than $[2R^{1-\varepsilon/2}]$ subintervals I_{n+1} from $\mathcal{R}_{n+1,m}$. Then the interval $I_n \in \mathcal{I}_n \backslash \mathcal{J}_n$, and we subdivide I_n into R closed intervals I_{n+1} of equal length. Let $\mathcal{T}(I_n)$ be any collection consisting of $[2R^{1-\varepsilon/2}]$ such subintervals. Note that $\mathcal{T}(I_n) \cap \mathcal{J}_{n+1} = \emptyset$.
- $I_n \notin \mathcal{R}_{n,m}$. Then the interval I_n does not intersect any interval from $\mathcal{J}_n$ and we subdivide I_n into R closed intervals I_{n+1} of equal length. Let $\mathcal{T}(I_n)$ be any collection consisting of $[2R^{1-\varepsilon/2}]$ such subintervals. Note that $\mathcal{T}(I_n) \cap \mathcal{J}_{n+1} = \emptyset$.

The upshot is that

$$\#\mathcal{T}_n = \#\mathcal{T}_{n-1} \times [2R^{1-\varepsilon/2}] \qquad \forall \quad 0 < n \leqslant m$$

and that

$$\mathcal{T}_m \cap \mathcal{J}_m = \emptyset \, .$$

However, in view of Lemma 4, the latter is impossible and therefore the starting premise that $\mathcal{M}_{n,m} = \emptyset$ is false. This completes the proof of the proposition. □

Appendix: The dual and simultaneous forms of Bad(i,j)

Given a pair of real numbers i and j satisfying (6), the following statement allows us to deduce that the dual and simultaneous forms of **Bad**(i,j) are equivalent.

THEOREM 6. *Let*

$$L_t(\mathbf{q}) := \sum_s \theta_{ts} \, q_s \qquad (1 \leqslant s \leqslant m, \, 1 \leqslant t \leqslant n)$$

be n linear forms in m variables and let

$$M_s(\mathbf{u}) := \sum_t \theta_{ts} \, u_t$$

be the transposed set of m linear forms in n variables. Suppose that there are integers $\mathbf{q} \neq \mathbf{0}$ such that

$$||L_t(\mathbf{q})|| \leqslant C_t\,, \qquad |q_s| \leqslant X_s\,,$$

for some constants C_t and X_s satisfying

$$\max_s \{\, D_s := (l-1)\, X_s^{-1}\, d^{1/(l-1)} \} < 1\,,$$

where

$$d := \prod_t C_t \prod_s X_s \quad \text{and} \quad l := m+n\,.$$

Then there are integers $\mathbf{u} \neq \mathbf{0}$ such that

$$||M_s(\mathbf{u})|| \leqslant D_s\,, \qquad |u_t| \leqslant U_t\,, \tag{90}$$

where

$$U_t := (l-1)\, C_t^{-1}\, d^{1/(l-1)}\,.$$

This theorem is essentially a generalization of Theorem II in [1, Chap. V]. In short, compared to the latter, the above theorem allows the upper bounds for $||L_t(\mathbf{q})||$ and $|q_s|$ to vary with t and s respectively. The proof of Theorem 6 makes use of the following result which appears as Theorem I in [1, Chap. V].

PROPOSITION 3. *Let $f_k(\mathbf{z})$ $(1 \leqslant k \leqslant l)$ be l linearly independent homogeneous linear forms in the l variables $\mathbf{z} = (z_1, \ldots, z_l)$ and let $g_k(\mathbf{w})$ be l linearly independent homogeneous linear forms in the l variables $\mathbf{w} = (w_1, \ldots, w_l)$ of determinant d. Suppose that all the products $z_i w_j$ $(1 \leqslant i, j \leqslant l)$ have integer coefficients in*

$$\Phi(\mathbf{z}, \mathbf{w}) := \sum_k f_k(\mathbf{z})\, g_k(\mathbf{w})\,.$$

If the inequalities

$$|f_k(\mathbf{z})| \leqslant \lambda \qquad (1 \leqslant k \leqslant l)$$

are soluble with integral $\mathbf{z} \neq \mathbf{0}$, then the inequalities

$$|g_k(\mathbf{w})| \leqslant (l-1)\, |\lambda\, d|^{1/(l-1)}$$

are soluble with integral $\mathbf{w} \neq \mathbf{0}$.

Armed with this proposition, the proof of Theorem 6 is relatively straightforward. Indeed, apart from obvious modifications the proof is essentially as in [1].

Proof of Theorem 6. We start by introducing the new variables

$$\mathbf{p} = (p_1, \ldots, p_n) \qquad \text{and} \qquad \mathbf{v} = (v_1, \ldots, v_m)\,.$$

Now let

$$f_k(\mathbf{q}, \mathbf{p}) := \begin{cases} C_k^{-1}\,(L_k(\mathbf{q}) + p_k) & \text{if} \quad 1 \leqslant k \leqslant n \\ X_{k-n}^{-1}\, q_{k-n} & \text{if} \quad n < k \leqslant l \end{cases}$$

and

$$g_k(\mathbf{u},\mathbf{v}) := \begin{cases} C_k u_k & \text{if} \quad 1 \leqslant k \leqslant n \\ X_{k-n}(-M_{k-n}(\mathbf{u}) + v_{k-n}) & \text{if} \quad n < k \leqslant l\,. \end{cases}$$

Then the f_k are linearly independent forms in the $l := m+n$ variables $\mathbf{z} = (\mathbf{q},\mathbf{p})$ and the g_k are linearly independent forms in the l variables $\mathbf{w} = (\mathbf{u},\mathbf{v})$ with determinant

$$d := \prod_{t=1}^{n} C_t \prod_{s=1}^{m} X_s\,.$$

Furthermore,

$$\sum_{k \leqslant l} f_k g_k \;=\; \sum_{t \leqslant n} u_t p_t + \sum_{s \leqslant m} v_s q_s$$

since the terms in $u_t q_s$ all cancel out. By hypothesis there are integers $\mathbf{q} \neq \mathbf{0}$ and $\mathbf{p}$ such that

$$|f_k(\mathbf{q},\mathbf{p})| \leqslant 1,$$

so we may apply Proposition 3 with $\lambda = 1$. It follows that there are integers $(\mathbf{u},\mathbf{v}) \neq (\mathbf{0},\mathbf{0})$ such that

$$\left.\begin{array}{l} C_t|u_t| \\ X_s| - M_s(\mathbf{u}) + v_s| \end{array}\right\} \leqslant (l-1)\, d^{1/(l-1)}\,,$$

and so the inequalities given by (90) hold. It remains to show that $\mathbf{u} \neq \mathbf{0}$. By hypothesis $D_s < 1$ for all s and so if $\mathbf{u} = \mathbf{0}$, we must have that $v_s = 0$ for all s. However $(\mathbf{u},\mathbf{v}) = (\mathbf{0},\mathbf{0})$ is excluded. □

Given Theorem 6, it is relatively straightforward to show that the dual and simultaneous forms of $\mathbf{Bad}(i,j)$ are equivalent.

Suppose that the point $(x,y) \in \mathbb{R}^2$ does not belong to the simultaneous $\mathbf{Bad}(i,j)$ set. It follows from the definition of the latter that for any constant $c > 0$, there exists an integer $q_0 \geq 1$ such that

$$\begin{aligned} ||q_0 x|| &\leqslant c\, q_0^{-i}, \\ ||q_0 y|| &\leqslant c\, q_0^{-j}\,. \end{aligned}$$

Without loss of generality, assume that $c < 1/2$. With reference to Theorem 6, let $m = 1$, $n = 2$, $L_1(\mathbf{q}) = qx$, $L_2(\mathbf{q}) = qy$, $C_1 = cq_0^{-i}$, $C_2 = cq_0^{-j}$ and $X_1 = q_0$. Hence there exists an integer pair $(u_1,u_2) \neq (0,0)$ such that

$$\begin{aligned} &||xu_1 + yu_2|| \leqslant 2cq_0^{-1}, \\ &|u_1| \leqslant 2q_0^{i}, \\ &|u_2| \leqslant 2q_0^{j}\,. \end{aligned}$$

This in turn implies that

$$\max\{|u_1|^{1/i}, |u_2|^{1/j}\}\ ||xu_1 + yu_2|| \leqslant 2^{1/i+1/j+1}\, c\,. \tag{91}$$

In other words, for any arbitrary small constant $c > 0$, there exists $(u_1, u_2) \in \mathbb{Z}^2 \backslash \{(0,0)\}$ for which (91) is satisfied. It follows that the point (x, y) does not belong to the dual $\mathbf{Bad}(i,j)$ set. The upshot is that the dual $\mathbf{Bad}(i,j)$ set is a subset of the simultaneous $\mathbf{Bad}(i,j)$ set.

Suppose the point $(x, y) \in \mathbb{R}^2$ does not belong to the dual $\mathbf{Bad}(i,j)$ set. It follows from the definition of the latter that for any constant $c > 0$, there exists $(a, b) \in \mathbb{Z}^2 \backslash \{(0,0)\}$ such that

$$\max\{|a|^{1/i},\, |b|^{1/j}\}\ ||ax + by|| \leqslant c.$$

Without loss of generality, assume that $c < 1/4$ and let $q_0 := \max\{|a|^{1/i},\, |b|^{1/j}\}$. With reference to Theorem 6, let $m = 2$, $n = 1$, $L_1(\mathbf{q}) = q_1 x + q_2 y$, $C_1 = c q_0^{-1}$, $X_1 = q_0^i$ and $X_2 = q_0^j$. Hence there exists an integer $u \neq 0$ such that

$$\begin{aligned} ||ux|| &\leqslant 2\, c^{1/2}\, q_0^{-i} \\ ||uy|| &\leqslant 2\, c^{1/2}\, q_0^{-j} \\ |u| &\leqslant 2\, c^{-1/2}\, q_0\, . \end{aligned}$$

This in turn implies that there exists and integer $q = |u| \geqslant 1$ such that

$$\max\{||qx||^{1/i}, ||qy||^{1/j}\} \leqslant \max\left\{2^{\frac{1+i}{i}} c^{\frac{j}{2i}},\, 2^{\frac{1+j}{j}} c^{\frac{i}{2j}}\right\}\ q^{-1}\, . \tag{92}$$

In other words, for any arbitrary small constant $c > 0$, there exists $q \in \mathbb{N}$ for which (92) is satisfied. It follows that the point (x, y) does not belong to the simultaneous $\mathbf{Bad}(i,j)$ set. The upshot is that the simultaneous $\mathbf{Bad}(i,j)$ set is a subset of the dual $\mathbf{Bad}(i,j)$ set.

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