

The spherical Hecke algebra for affine Kac-Moody groups I

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Abstract

We define the spherical Hecke algebra for an (untwisted) affine Kac-Moody group over a local non-archimedean field. We prove a generalization of the Satake isomorphism for this algebra, relating it to integrable representations of the Langlands dual affine Kac-Moody group. In the next publication we shall use these results to define and study the notion of *Hecke eigenfunction* for the group G_{aff} .

1. Introduction

1.1. *Langlands duality and the Satake isomorphism.* Let F be a global field, and let \mathbb{A}_F denote its ring of adèles. Let G be a split reductive group over F . The classical *Langlands duality* predicts that irreducible automorphic representations of $G(\mathbb{A}_F)$ are closely related to the homomorphisms from the absolute Galois group Gal_F of F to the *Langlands dual group* G^\vee . Similarly, if G is a split reductive group over a local-nonarchimedean field \mathcal{K} , Langlands duality predicts a relation between irreducible representations of $G(\mathcal{K})$ and homomorphisms from $\text{Gal}_{\mathcal{K}}$ to G^\vee .

The starting point for Langlands duality, which allows one to relate the “simplest” irreducible representations of $G(\mathcal{K})$ (the so called *spherical representations*) to the “simplest” homomorphisms from $\text{Gal}_{\mathcal{K}}$ to G^\vee (the unramified homomorphisms), is the *Satake isomorphism*, whose formulation we now recall. Let $\mathcal{O} \subset \mathcal{K}$ denote the ring of integers of \mathcal{K} . Then the group $G(\mathcal{K})$ is a locally compact topological group and $G(\mathcal{O})$ is its maximal compact subgroup. One may study the *spherical Hecke algebra* \mathcal{H} of $G(\mathcal{O})$ -biinvariant compactly supported \mathbb{C} -valued measures on $G(\mathcal{K})$. The Satake isomorphism is a canonical isomorphism between \mathcal{H} and the complexified Grothendieck ring $K_0(\text{Rep}(G^\vee))$ of finite-dimensional representations of G^\vee . For future purposes it will be convenient to note that $K_0(\text{Rep}(G^\vee))$ is also naturally isomorphic to the algebra $\mathbb{C}(T^\vee)^W$ of polynomial functions on the maximal torus $T^\vee \subset G^\vee$ invariant under the Weyl group W of G (which is the same as the Weyl group of G^\vee).

1.2. *The group G_{aff} .* To a connected reductive group G as above one can associate the corresponding affine Kac-Moody group G_{aff} in the following way.

Let Λ denote the coweight lattice of G let Q be an integral, even, negative-definite symmetric bilinear form on Λ which is invariant under the Weyl group of G .

One can consider the polynomial loop group $G[t, t^{-1}]$ (this is an infinite-dimensional group ind-scheme). It is well known (cf. [10]) that a form Q as above gives rise to a central extension \tilde{G} of $G[t, t^{-1}]$:

$$1 \rightarrow \mathbb{G}_m \rightarrow \tilde{G} \rightarrow G[t, t^{-1}] \rightarrow 1.$$

(We will review the construction in §6.3.) Moreover, \tilde{G} has again a natural structure of a group ind-scheme.

The multiplicative group \mathbb{G}_m acts naturally on $G[t, t^{-1}]$, and this action lifts to \tilde{G} . We denote the corresponding semi-direct product by G_{aff} ; we also let $\mathfrak{g}_{\text{aff}}$ denote its Lie algebra. Thus if G is semi-simple, then $\mathfrak{g}_{\text{aff}}$ is an untwisted affine Kac-Moody Lie algebra in the sense of [7]; in particular, it can be described by the corresponding affine root system.

1.3. *The Hecke algebra for affine Kac-Moody groups.* Our dream is to develop some sort of Langlands theory in the case when G is replaced by an affine Kac-Moody group G_{aff} . As the very first step towards realizing this dream we are going to define the spherical Hecke algebra of G_{aff} and prove an analog of the Satake isomorphism for it.

Let \mathcal{K} and \mathcal{O} be as above. Then one may consider the group $G_{\text{aff}}(\mathcal{K})$ and its subgroup $G_{\text{aff}}(\mathcal{O})$. The group G_{aff} , by definition, maps to \mathbb{G}_m ; thus $G_{\text{aff}}(\mathcal{K})$ maps to \mathcal{K}^* . We denote this homomorphism by $\tilde{\zeta}$. In addition, the group \mathcal{K}^* is endowed with a natural (valuation) homomorphism to \mathbb{Z} . We denote its composition with $\tilde{\zeta}$ by π .

We now define the semigroup $G_{\text{aff}}^+(\mathcal{K})$ to be the subsemigroup of $G_{\text{aff}}(\mathcal{K})$ generated by

- the central $\mathcal{K}^* \subset G_{\text{aff}}(\mathcal{K})$;
- the subgroup $G_{\text{aff}}(\mathcal{O})$;
- all elements $g \in G_{\text{aff}}(\mathcal{K})$ such that $\pi(g) > 0$.

We show (cf. Theorem 4.6(1)) that the convolution of any two double cosets of $G_{\text{aff}}(\mathcal{O})$ inside $G_{\text{aff}}^+(\mathcal{K})$ is well defined in the appropriate sense and gives rise to an associative algebra structure on a suitable space of $G_{\text{aff}}(\mathcal{O})$ -biinvariant functions on $G_{\text{aff}}^+(\mathcal{K})$. This algebra turns out to be commutative and we call it *the spherical Hecke algebra of G_{aff}* and denote it by $\mathcal{H}(G_{\text{aff}})$. The algebra $\mathcal{H}(G_{\text{aff}})$ is graded by nonnegative integers (the grading comes from the map π which is well defined on double cosets with respect to $G_{\text{aff}}(\mathcal{O})$); it is also an algebra over the field $\mathbb{C}((v))$ of Laurent power series in a variable v , which comes from the central \mathcal{K}^* in $G_{\text{aff}}(\mathcal{K})$.

1.4. *The Satake isomorphism.* The statement of the Satake isomorphism for G_{aff} is very similar to that for G . First of all, in Section 4.2 we are going to define an analog of the algebra $\mathbb{C}(T^\vee)^W$ which we shall denote by $\mathbb{C}(\widehat{T}_{\text{aff}}^\vee)^{W_{\text{aff}}}$. (Here $T_{\text{aff}} = \mathbb{C}^* \times T^\vee \times \mathbb{C}^*$ is the dual of the maximal torus of G_{aff} , W_{aff} is the corresponding affine Weyl group and $\mathbb{C}(\widehat{T}^\vee)$ denotes certain completion of the algebra of regular functions on T_{aff}^\vee .) This is a finitely generated $\mathbb{Z}_{\geq 0}$ -graded commutative algebra over the field $\mathbb{C}((v))$ of Laurent formal power series in the variable v which should be thought of as a coordinate on the third factor in $T_{\text{aff}}^\vee = \mathbb{C}^* \times T^\vee \times \mathbb{C}^*$. (The grading has to do with the first factor.) Moreover, each component of the grading is finite-dimensional over $\mathbb{C}((v))$.

Assume that either G is simply connected or that G is a torus.¹ In this case we define (in §4) the *Langlands dual group* G_{aff}^\vee . This is a group ind-scheme over \mathbb{C} . If G is semi-simple, then G_{aff}^\vee is another Kac-Moody group whose Lie algebra $\mathfrak{g}_{\text{aff}}^\vee$ is an affine Kac-Moody algebra with root system dual to that of $\mathfrak{g}_{\text{aff}}$. (Thus, in particular, it might be a twisted affine Lie algebra.) The group G_{aff}^\vee contains the torus T_{aff}^\vee ; moreover the first \mathbb{C}^* -factor in T_{aff}^\vee is central in G_{aff}^\vee ; also the projection $T_{\text{aff}}^\vee \rightarrow \mathbb{C}^*$ to the last factor extends to a homomorphism $G_{\text{aff}}^\vee \rightarrow \mathbb{C}^*$. It makes sense to consider integrable highest weight representations of G_{aff}^\vee and one can define certain category $\text{Rep}(G_{\text{aff}}^\vee)$ of such representations which is stable under tensor product. (This category contains all highest weight integrable representations of finite length, but certain infinite direct sums must be included there as well.) The complexified Grothendieck ring $K_0(G_{\text{aff}}^\vee)$ of this category is naturally isomorphic to the algebra $\mathbb{C}(\widehat{T}_{\text{aff}}^\vee)^{W_{\text{aff}}}$ via the character map. The corresponding grading on $K_0(G_{\text{aff}}^\vee)$ comes from the central charge of G_{aff}^\vee -modules and the action of the variable v comes from tensoring G_{aff}^\vee -modules by the one-dimensional representation coming from the homomorphism $G_{\text{aff}}^\vee \rightarrow \mathbb{C}^*$, mentioned above.

The Satake isomorphism (cf. Theorem 4.6(2)) claims that the Hecke algebra $\mathcal{H}(G_{\text{aff}})$ is canonically isomorphic to $\mathbb{C}(\widehat{T}_{\text{aff}}^\vee)^{W_{\text{aff}}}$ (and thus, when it makes sense also to $K_0(G_{\text{aff}}^\vee)$).

As was mentioned, in the case when G is semi-simple and simply connected, the group G_{aff} is an affine Kac-Moody group. We expect that with slight modifications our Satake isomorphism should make sense for any symmetrizable Kac-Moody group. However, our proofs are really designed for the affine case and do not seem to generalize to more general Kac-Moody groups.

1.5. *Further questions.* It would definitely be interesting to extend the results of this paper to subgroups of $G_{\text{aff}}(\mathcal{K})$ other than $G_{\text{aff}}(\mathcal{O})$. Arguably

¹We believe that this assumption is not necessary, but at the moment we cannot remove it.

the most interesting of these subgroups is the analog of the Iwahori subgroup of $G_{\text{aff}}(\mathcal{O})$. In this case we expect that the Hecke algebra is well defined and will be closely related to Cherednik's double affine Hecke algebra. We plan to address this issue in [2].

Another natural question is this. By definition, the algebra $\mathcal{H}(G_{\text{aff}})$ is endowed with a natural basis given by the characteristic functions of $G_{\text{aff}}(\mathcal{O}) \times G_{\text{aff}}(\mathcal{O})$ orbits on $G_{\text{aff}}(\mathcal{K})$. It would be quite interesting to determine the image of this basis under the Satake isomorphism. When we deal with G instead of G_{aff} , the answer is given by the so-called *Hall polynomials* and it is essentially equivalent to the "Macdonald formula for the spherical function" (cf. [12]). A generalization of Macdonald formula to the affine case (closely related to the results of [4] on affine Hall polynomials) will also be discussed in [2].

Let us also point out that the idea to interpret the convolution in $\mathcal{H}(G_{\text{aff}})$ in geometric terms (which is used in order to prove the main results of this paper) came to us from the joint works of the first-named author with M. Finkelberg which give a partial generalization of the "geometric Satake correspondence" (cf. [13] and references therein) to the affine case.

1.6. *Organization of the paper.* This paper is organized as follows. In Section 2 we introduce some notions related to general Hecke algebras. In Section 3 we consider an example of (generalized) Hecke algebras (which later turns out to be closely related to the algebra \mathcal{H}_{aff} in the case when G is a torus). In Section 4 we formulate our main Theorem 4.6 describing the Hecke algebra \mathcal{H}_{aff} in the general case. The proof of Theorem 4.6 occupies the last four sections of the paper. Although the statement of Theorem 4.6 is quite elementary, the proof uses heavily the machinery of algebraic geometry related to moduli spaces of G -bundles on various algebraic surfaces. (In particular, we do not know how to generalize our proofs to the case when G_{aff} is replaced by a more general (nonaffine) Kac-Moody group.) In addition, in Section 8 we construct explicitly the Satake isomorphism for \mathcal{H}_{aff} by looking at its action on the principal series for G_{aff} (in the spirit of [9]).

1.7. *Acknowledgments.* This work grew out of an attempt to find a "classical counterpart" of the first author's joint work [1] with M. Finkelberg whom we would like to thank for very useful discussions and ideas. This paper has in fact grown out from an attempt to produce an elementary analog of [1]. Also we are deeply grateful to V. Drinfeld, who gave us many interesting ideas on the subject. In addition we would like to thank I. Cherednik, P. Etingof, H. Garland, V. Kac, M. Kapranov and C. Teleman for very illuminating discussions related to the contents of this paper.

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2. Generalities on Hecke algebras

2.1. *Good pairs.* Let Γ be a group, and let $\Gamma_0 \subset \Gamma$ be a subgroup. Consider the multiplication map

$$m : \Gamma \times_{\Gamma_0} \Gamma \rightarrow \Gamma.$$

(Here the subscript Γ_0 means taking quotient by the diagonal action of Γ_0 .) Let X (resp. Y) be a subset of Γ which is right (resp. left) invariant with respect to Γ_0 . Then we denote by $m_{X,Y}$ the restriction of m to $X \times_{\Gamma_0} Y$.

We say that the pair (Γ, Γ_0) is *good* if for any two double cosets X and Y in Γ with respect to Γ_0 , we have

- 1) The image of $m_{X,Y}$ consists of finite union of double cosets.
- 2) The map $m_{X,Y}$ has finite fibers.

In this case one can define the Hecke algebra $\mathcal{H}(\Gamma, \Gamma_0)$ as the convolution algebra of Γ_0 -bi-invariant functions on Γ supported on finitely many double cosets with respect to Γ_0 .

LEMMA 2.2. *The pair (Γ, Γ_0) is good if and only if, for any $x \in \Gamma$, the set $\Gamma_0 x \Gamma_0 / \Gamma_0$ is finite. Also, condition 2) above implies condition 1).*

Proof. Assume that the condition of Lemma 2.2 holds. Let us prove that (Γ, Γ_0) is a good pair. Take any $x, y \in \Gamma$ and set $X = \Gamma_0 x \Gamma_0, Y = \Gamma_0 y \Gamma_0$. Choose some $x_1, \dots, x_n \in X, y_1, \dots, y_k \in Y$ such that

$$X = \bigsqcup_{i=1}^n x_i \Gamma_0, \quad Y = \bigsqcup_{j=1}^k y_j \Gamma_0.$$

Then $X \times_{\Gamma_0} Y = \bigsqcup_{i=1}^n x_i Y$. Since the restriction of $m_{X,Y}$ to each $x_i Y$ is obviously injective, it follows that every fiber of $m_{X,Y}$ has at most n elements. This proves condition 2). On the other hand, $x_i Y$ is the union of all the $x_i y_j \Gamma_0$, and thus it follows that the image of $m_{X,Y}$ is a finite union of right Γ_0 -cosets. In particular, it is a finite union of double cosets with respect to Γ_0 .

Let us now show that condition 2) implies that $\Gamma_0 x \Gamma_0 / \Gamma_0$ is finite for all $x \in \Gamma$. (In view of the preceding paragraph this will also imply that 2) implies 1).) Indeed, let $X = \Gamma_0 x \Gamma_0$, and let $Y = \Gamma_0 x^{-1} \Gamma_0$. Consider the fiber of $m_{X,Y}$ over the unit element $e \in \Gamma$. It is clearly equal to X / Γ_0 , and thus condition 2) implies that it has to be finite. □

The main source of an example of good pairs is the following. Assume that Γ is actually a locally compact topological group, and let Γ_0 be any open compact subgroup. (In particular, Γ can be a discrete group and Γ_0 — a finite subgroup.)

2.3. *Generalization: almost good pairs.* We want to generalize the above notion in two directions. First, let Γ and Γ_0 be as above, and let $\Gamma^+ \subset \Gamma$ be a sub-semi-group containing Γ_0 . Then we may speak about the pair (Γ^+, Γ_0) being good, and the Hecke algebra $\mathcal{H}(\Gamma^+, \Gamma_0)$ makes sense.

Remark. In this case condition 2) no longer implies condition 1), since in the proof of this implication given above we used inversion in Γ .

Let us also consider the following situation. Assume that we are given a central extension

$$1 \rightarrow A \rightarrow \tilde{\Gamma} \rightarrow \Gamma \rightarrow 1$$

of Γ by means of an abelian group A . Let also $B \subset A$ be a subgroup of A such that $A/B \simeq \mathbb{Z}$. Assume that $\Gamma_0 \subset \Gamma$ can be lifted to a subgroup of $\tilde{\Gamma}_0 \subset \tilde{\Gamma}$ fitting into the short exact sequence

$$1 \rightarrow B \rightarrow \tilde{\Gamma}_0 \rightarrow \Gamma_0 \rightarrow 1.$$

Note that in this case the group $\mathbb{Z} = A/B$ acts naturally on the set of double cosets of $\tilde{\Gamma}$ with respect to $\tilde{\Gamma}_0$. We shall say that a subset $W \subset \tilde{\Gamma}$ is *almost a finite union of double cosets with respect to $\tilde{\Gamma}_0$* if W is contained in the set $\mathbb{Z}_+(Z)$, where $Z \subset \Gamma$ is a finite union of double cosets.

Let $\Gamma^+ \subset \Gamma$ be a semi-group as above, and let $\tilde{\Gamma}^+$ be its preimage in $\tilde{\Gamma}$. Then we say that the pair $(\tilde{\Gamma}^+, \tilde{\Gamma}_0)$ is *almost good* if for any two double cosets X and Y of $\tilde{\Gamma}^+$ with respect to $\tilde{\Gamma}_0$, condition 2) above is satisfied, and in addition, the following generalization of condition 1) is satisfied:

- 1') The image of the map $m_{X,Y}$ is almost a finite union of double cosets with respect to $\tilde{\Gamma}_0$.

Warning. It is by no means true that if the pair $(\tilde{\Gamma}^+, \tilde{\Gamma}_0)$ is almost good, then the pair (Γ^+, Γ_0) is good!

In the above situation one may define the algebra $\mathcal{H}(\tilde{\Gamma}^+, \tilde{\Gamma}_0)$ as the convolution algebra of $\tilde{\Gamma}_0$ -bi-invariant functions on $\tilde{\Gamma}^+$ whose support is almost a finite union of double cosets. Note that this algebra can naturally be regarded as an algebra over formal Laurent power series $\mathbb{C}((v))$; here multiplication by v corresponds to shifting the support of a function by $1 \in \mathbb{Z}$.

The algebra $\mathcal{H}(\tilde{\Gamma}^+, \tilde{\Gamma}_0)$ has a natural subspace $\mathcal{H}^{\text{fin}}(\tilde{\Gamma}^+, \tilde{\Gamma}_0)$ consisting of functions supported on finitely many double cosets; it is naturally a module over the ring $\mathbb{C}[v, v^{-1}]$ of Laurent polynomials. By definition, we have

$$\mathcal{H}(\tilde{\Gamma}^+, \tilde{\Gamma}_0) = \mathcal{H}^{\text{fin}}(\tilde{\Gamma}^+, \tilde{\Gamma}_0) \otimes_{\mathbb{C}[v, v^{-1}]} \mathbb{C}((v)).$$

In general $\mathcal{H}^{\text{fin}}(\tilde{\Gamma}^+, \tilde{\Gamma}_0)$ is not a subalgebra of $\mathcal{H}(\tilde{\Gamma}^+, \tilde{\Gamma}_0)$. Let R be any subring of $\mathbb{C}((v))$ containing $\mathbb{C}[v, v^{-1}]$. We say that $\mathcal{H}(\tilde{\Gamma}^+, \tilde{\Gamma}_0)$ is *defined over* R if $\mathcal{H}^{\text{fin}}(\tilde{\Gamma}^+, \tilde{\Gamma}_0) \otimes_{\mathbb{C}[v, v^{-1}]} R$ is a subalgebra of $\mathcal{H}(\tilde{\Gamma}^+, \tilde{\Gamma}_0)$. In this case we set $\mathcal{H}_R(\tilde{\Gamma}^+, \tilde{\Gamma}_0) = \mathcal{H}^{\text{fin}}(\tilde{\Gamma}^+, \tilde{\Gamma}_0) \otimes_{\mathbb{C}[v, v^{-1}]} R$.

2.4. *Good actions.* Let (Γ, Γ_0) be a (not necessarily good) pair, and let Ω be a Γ -set. Let $a : \Gamma \times_{\Gamma_0} \Omega \rightarrow \Omega$ be the natural map, coming from the action of Γ on Ω . Let now X , as before, be a subset of Γ which is right-invariant under Γ_0 , and let Z be a Γ_0 -invariant subset of Ω . Then we shall denote by $a_{X,Z}$ the restriction of a to $X \times_{\Gamma_0} Z$.

We say that the action of Γ on Ω is good if for any double coset X in Γ with respect to Γ_0 and for any Γ_0 -orbit Z in Ω , then the following conditions are satisfied:

- 1 $_{\Omega}$: The image of $a_{X,Z}$ is a union of finitely many Γ_0 -orbits.
- 2 $_{\Omega}$: The map $a_{X,Z}$ has finite fibers.

Let $\mathcal{F}(\Omega)$ denote the space of Γ_0 -invariant \mathbb{C} -valued functions on Ω , which are supported on finitely many Γ_0 -orbits. (In other words, $\mathcal{F}(\Omega)$ is the space of functions on $\Gamma_0 \backslash \Omega$ with finite support.) Then conditions 1 $_{\Omega}$ and 2 $_{\Omega}$ above guarantee that we can define an action of $\mathcal{H}(\Gamma, \Gamma_0)$ on $\mathcal{F}(\Omega)$ by putting

$$h(f)(z) = \sum_{(g,w) \in \Gamma \times_{\Gamma_0} \Omega, a(g,w)=z} h(g)f(w).$$

Here $h \in \mathcal{H}(\Gamma, \Gamma_0)$, $f \in \mathcal{F}(\Omega)$, $z \in \Omega$.

The following lemma is straightforward.

LEMMA 2.5. *Let (Γ, Γ_0) be any pair, and let Ω be any Γ -set. Assume that condition 2 $_{\Omega}$ is satisfied. Then the pair (Γ, Γ_0) satisfies condition 2 from Section 2.1.*

2.6. *Almost good actions.* Let $(\tilde{\Gamma}, \tilde{\Gamma}^+, \tilde{\Gamma}_0)$ be as in Section 2.3, and let Ω , as before, be a Γ -set. Let also $\tilde{\Omega}$ be an A -torsor over Ω on which $\tilde{\Gamma}$ acts in such a way that $A \subset \tilde{\Gamma}$ acts on it in the natural way and the action of $\tilde{\Gamma}$ on $\tilde{\Omega}$ is compatible with the action of Γ on Ω . Then we may define the notion of an almost good action of $\tilde{\Gamma}^+$ on $\tilde{\Omega}$ (similarly to the notion of an almost good pair). Namely, we say that a $\tilde{\Gamma}_0$ -invariant subset W of $\tilde{\Omega}$ is almost a finite union of $\tilde{\Gamma}_0$ -orbits (or just *almost finite* for brevity) if W is contained in $\mathbb{Z}_+(Z)$, where Z is a finite union of $\tilde{\Gamma}_0$ -orbits. (Note that $\mathbb{Z} = A/B$ acts naturally on the set of $\tilde{\Gamma}_0$ -orbits in $\tilde{\Omega}$.) We say that the action of $\tilde{\Gamma}$ on $\tilde{\Omega}$ is almost good if condition 2 $_{\tilde{\Omega}}$ is satisfied together with the following condition 1'_{\tilde{\Omega}}:

$1'_{\tilde{\Omega}}$: For any $\tilde{\Gamma}_0$ double coset X in $\tilde{\Gamma}^+$ and for any $\tilde{\Gamma}_0$ -orbit Z in $\tilde{\Omega}$, the image of $a_{X,Z}$ is almost a finite union of $\tilde{\Gamma}_0$ -orbits.

We now let $\mathcal{F}^{\text{fin}}(\tilde{\Omega})$ denote the space of $\tilde{\Gamma}_0$ -invariant \mathbb{C} -valued functions on $\tilde{\Omega}$ which are supported on a finite union of $\tilde{\Gamma}_0$ -orbits, and we also let $\mathcal{F}(\tilde{\Omega})$ denote the space of $\tilde{\Gamma}_0$ -invariant \mathbb{C} -valued functions on $\tilde{\Omega}$ which are supported on an almost finite union of $\tilde{\Gamma}_0$ -orbits. It is easy to see that if the action of $\tilde{\Gamma}^+$ on $\tilde{\Omega}$ is almost good, then the algebra $\mathcal{H}(\tilde{\Gamma}^+, \tilde{\Gamma}_0)$ acts on $\mathcal{F}(\tilde{\Omega})$.

2.7. *A generalization.* Let $(\Omega, \tilde{\Omega})$ be as above, and let $(\Omega', \tilde{\Omega}')$ be another such pair. Assume that we are given an $\tilde{\Gamma}$ -equivariant morphism $\varpi : \tilde{\Omega} \rightarrow \tilde{\Omega}'$. We shall say that a $\tilde{\Gamma}_0$ -invariant subset Z of $\tilde{\Omega}$ is *almost finite with respect to $\tilde{\Omega}'$* if

- a) $\varpi(Z)$ is almost finite.
- b) The natural map $\tilde{\Gamma}_0 \backslash Z \rightarrow \tilde{\Gamma}_0 \backslash \varpi(Z)$ has finite fibers.

We shall say that $\tilde{\Omega}$ is *almost good with respect to $\tilde{\Omega}'$* if the following two conditions hold:

- (i) For any $X \in \tilde{\Gamma}_0 \backslash \tilde{\Gamma}^+ / \tilde{\Gamma}_0$ and any $Z \in \tilde{\Gamma}_0 \backslash \tilde{\Omega}$, the image of the map $a_{X,Z}$ is almost finite with respect to $\tilde{\Omega}'$.
- (ii) $\tilde{\Omega}'$ is almost good.

We claim that condition (ii) implies condition $2'_{\tilde{\Omega}}$. Indeed, without loss of generality, we may assume that Ω and Ω' are homogenous spaces for Γ (resp. $\tilde{\Omega}$ and $\tilde{\Omega}'$ are homogenous spaces for $\tilde{\Gamma}$); i.e., $\Omega = \Gamma / \Delta, \tilde{\Omega} = \tilde{\Gamma} / \tilde{\Delta}, \Omega' = \Gamma / \Delta', \tilde{\Omega}' = \tilde{\Gamma} / \tilde{\Delta}'$, where $\Delta \subset \Delta' \subset \Gamma$ and $\tilde{\Delta} \subset \tilde{\Delta}' \subset \tilde{\Gamma}$. Let X be a double coset in $\tilde{\Gamma}^+$ with respect to $\tilde{\Gamma}_0$. Choose $z, \gamma \in \tilde{\Gamma}$. Set $X = \tilde{\Gamma}_0 x \tilde{\Gamma}_0, Z = \tilde{\Gamma}_0 z \tilde{\Delta} / \tilde{\Delta}$ and $Z' = \tilde{\Gamma}_0 z \tilde{\Delta}' / \tilde{\Delta}'$. Then the fiber of $a_{X,Z}$ over the image of γ in $\tilde{\Omega}$ consists of all triples $(x, \delta) \in X \times \tilde{\Delta}$ such that $xz\delta = \gamma$. Similarly, the fiber of $a_{X,Z'}$ over the image of γ in $\tilde{\Omega}'$ consists of all triples $(x, \delta') \in X \times \tilde{\Delta}'$ such that $xz\delta' = \gamma$. Since $\tilde{\Delta} \subset \tilde{\Delta}'$, it follows that the former fiber is embedded in the latter. Thus, the fact that the fibers of $a_{X,Z'}$ are finite implies that the fibers of $a_{X,Z}$ are finite.

In addition it is obvious that condition (ii) implies part a) of condition (i). Let now $\mathcal{F}_{\tilde{\Omega}'}(\tilde{\Omega})$ denote the space of all $\tilde{\Gamma}_0$ -invariant functions on $\tilde{\Omega}$ whose support is almost finite with respect to $\tilde{\Omega}'$. Then it is easy to see that conditions (i) and (ii) imply that the Hecke algebra $\mathcal{H}(\tilde{\Gamma}^+, \tilde{\Gamma}_0)$ acts on $\mathcal{F}_{\tilde{\Omega}'}(\tilde{\Omega})$.

3. An example

3.1. *The setup.* Let Λ be a lattice of finite rank (i.e., Λ is an abelian group isomorphic to \mathbb{Z}^l for some l). Let $\Gamma = (\Lambda \oplus \Lambda) \rtimes \mathbb{Z}$, where \mathbb{Z} acts on $\Lambda \oplus \Lambda$ by means of the automorphism σ sending (λ, μ) to $(\lambda, \lambda + \mu)$. We shall usually

write an element of Γ as (λ, μ, k) . Define the subgroup $\Gamma_0 \subset \Gamma$ as the subgroup consisting of all elements of the form $(\lambda, 0, 0)$.

Define also the semigroup Γ^+ by setting

$$\Gamma^+ = \{(\lambda, 0, 0) \mid \lambda \in \Lambda\} \cup \{(\lambda, \mu, k) \text{ where } \lambda \text{ and } \mu \text{ are arbitrary and } k > 0\}.$$

Let $b(\cdot, \cdot)$ be any \mathbb{Z} -valued bilinear form on Λ . We set

$$Q(\lambda, \mu) = b(\lambda, \mu) + b(\mu, \lambda).$$

Then Q is a symmetric \mathbb{Z} -valued form on Λ . Moreover, Q is even; i.e., $Q(\lambda, \lambda)$ is even for any $\lambda \in \Lambda$. We define the central extension $\widehat{\Lambda \oplus \Lambda}$ of $\Lambda \oplus \Lambda$ by means of \mathbb{Z} in the following way: $\widehat{\Lambda \oplus \Lambda} = \mathbb{Z} \times (\Lambda \oplus \Lambda)$ as a set and the multiplication is defined by

$$(a_1, \lambda_1, \mu_1)(a_2, \lambda_2, \mu_2) = (a_1 + a_2 + b(\lambda_1, \mu_2) - b(\mu_1, \lambda_2), \lambda_1 + \lambda_2, \mu_1 + \mu_2).$$

It is well known that $\widehat{\Lambda \oplus \Lambda}$ depends (canonically) only on Q and not on b .

The automorphism σ of $\Lambda \oplus \Lambda$ considered above extends naturally to $\widehat{\Lambda \oplus \Lambda}$. Abusing the notation, we shall denote the resulting automorphism of $\widehat{\Lambda \oplus \Lambda}$ also by σ . It is defined by

$$\sigma : (a, \lambda, \mu) \mapsto (a, \lambda, \lambda + \mu).$$

We define $\widetilde{\Gamma} = \widehat{\Lambda \oplus \Lambda} \rtimes \mathbb{Z}$, where the generator of \mathbb{Z} acts on $\widehat{\Lambda \oplus \Lambda}$ by means of σ . Explicitly, we have $\widetilde{\Gamma} = \mathbb{Z} \times (\Lambda \oplus \Lambda) \times \mathbb{Z}$ as a set and the multiplication is given by

$$(3.1) \quad (a_1, \lambda_1, \mu_1, k_1)(a_2, \lambda_2, \mu_2, k_2) = (a_1 + a_2 + b(\lambda_1, \mu_2) - b(\mu_1, \lambda_2) + k_1 b(\lambda_1, \lambda_2), \lambda_1 + \lambda_2, \mu_1 + \mu_2 + k_1 \lambda_2, k_1 + k_2).$$

We set $\widetilde{\Gamma}_0$ to consist of all elements of $\widetilde{\Gamma}$ of the form $(0, \lambda, 0, 0)$. Note that in this case $\widetilde{\Gamma}_0$ is naturally isomorphic to Γ_0 .

As before we define $\widetilde{\Gamma}^+$ to be the preimage of Γ^+ in $\widetilde{\Gamma}$.

3.2. *The variety $X_{Q,v}$.* From now on we assume that Q is nondegenerate. Let Λ^\vee denote the dual lattice to Λ . Then Q defines a homomorphism $e : \Lambda \rightarrow \Lambda^\vee$, which is injective since Q is nondegenerate. Let $T = \Lambda \otimes \mathbb{C}^*$, $T^\vee = \Lambda^\vee \otimes \mathbb{C}^*$. Then Λ^\vee is a lattice of characters of T and Λ is the lattice of characters of T^\vee .

For every $v \in \mathbb{C}^*$ and any $\lambda^\vee \in \Lambda^\vee$, we denote by v^{λ^\vee} the corresponding element of T^\vee . Assume that $|v| < 1$. In this case we set

$$X_{Q,v} = T^\vee / v^{e(\Lambda)}.$$

It is well known that $X_{Q,v}$ is a complex abelian variety. The form Q also defines a holomorphic line bundle $\mathcal{L}_{Q,v}$ on $X_{Q,v}$ in the following way. Let $\pi : T^\vee \rightarrow X_{Q,v}$ be the natural projection. Then for any open subset U of

$X_{Q,v}$ we define the space of holomorphic sections of $\mathcal{L}_{Q,v}$ to be the space of holomorphic functions F on $\pi^{-1}(U)$ satisfying

$$F(xv^{e(\nu)}) = v^{-\frac{Q(\nu,\nu)}{2}} x^{-\nu} F(x).$$

Here $x \in T^V$ and $x^\nu = \nu(x)$. The line bundle $\mathcal{L}_{Q,v}$ is ample if and only if Q is negative definite. In this case we set

$$\mathbb{C}[X_{Q,v}] = \bigoplus_{k=0}^{\infty} \Gamma(X_{Q,v}, \mathcal{L}_{Q,v}^{\otimes k}).$$

Then $\mathbb{C}[X_{Q,v}]$ is a graded holomorphic vector bundle of algebras on the punctured unit disc $\mathcal{D}^* = \{v \in \mathbb{C}^*, |v| < 1\}$. More precisely, let R denote the subring of $\mathbb{C}((v))$ consisting of those formal series which are convergent for $0 < |v| < 1$. Then there exists a pair (X_Q, \mathcal{L}_Q) , where X_Q is an abelian variety over R and \mathcal{L}_Q is a line bundle over it, such that the specialization of (X_Q, \mathcal{L}_Q) to any $v \in \mathcal{D}^*$ is isomorphic to $(X_{Q,v}, \mathcal{L}_{Q,v})$. We set $\mathbb{C}[X_Q]$ to be the direct image of the sheaf $\bigoplus_{k \geq 0} \mathcal{L}_Q^{\otimes k}$ to $\text{Spec}(R)$; it can be naturally regarded as a graded R -algebra. We also set $\widehat{\mathbb{C}[X_Q]} = \mathbb{C}[X_Q] \otimes_R \mathbb{C}((v))$.

Below is the main result of this section.

THEOREM 3.3. *Assume that Q is negative-definite. Then*

- (1) *The pair $(\tilde{\Gamma}^+, \tilde{\Gamma}_0)$ defined above is almost good.*
- (2) *There is a natural isomorphism*

$$\mathcal{H}(\tilde{\Gamma}^+, \tilde{\Gamma}_0) \simeq \widehat{\mathbb{C}[X_Q]}$$

of graded $\mathbb{C}((v))$ -algebras.

- (3) *The algebra $\mathcal{H}(\tilde{\Gamma}^+, \tilde{\Gamma}_0)$ is defined over the ring R , and we have the natural isomorphism*

$$\mathcal{H}_R(\tilde{\Gamma}^+, \tilde{\Gamma}_0) \simeq \mathbb{C}[X_Q]$$

of graded R -algebras.

Proof. The proof is a straightforward calculation. Let us first describe $\mathcal{H}(\tilde{\Gamma}^+, \tilde{\Gamma}_0)$ as a vector space. It follows from (3.1) that we have the following formula:

$$(3.2) \quad (0, \nu, 0, 0)(a, \lambda, \mu, k) = (a + b(\nu, \mu), \lambda + \nu, \mu, k).$$

Hence it follows that every left $\tilde{\Gamma}_0$ -coset contains unique element of the form $(a, 0, \mu, k)$. Let us investigate when two such elements lie in the same double coset with respect to $\tilde{\Gamma}_0$. For any $\nu \in \Lambda$, we have

$$(a, 0, \mu, k)(0, \nu, 0, 0) = (a - b(\mu, \nu), \nu, \mu + k\nu, k).$$

Multiplying the result on the left by $(0, -\nu, 0, 0)$ and applying (3.2) again we get

$$\begin{aligned} (a - b(\nu, \mu) - b(\mu, \nu) - kb(\nu, \nu), 0, \mu + k\nu, k) \\ = \left(a - Q(\mu, \nu) - \frac{kQ(\nu, \nu)}{2}, 0, \mu + k\nu, k \right). \end{aligned}$$

In other words, we see that an element of $\mathcal{H}(\tilde{\Gamma}^+, \tilde{\Gamma}_0)_k$ is a function f on $\mathbb{Z} \times \Lambda$ (“corresponding” to a and μ) satisfying the relation

$$(3.3) \quad f(a, \mu) = f\left(a - Q(\mu, \nu) - \frac{kQ(\nu, \nu)}{2}, \mu + k\nu\right)$$

for any $\nu \in \Lambda$ and such that

- 1) For fixed a , the number $f(a, \mu)$ is nonzero only for finitely many values of μ .
- 2) There exists some $a_0 \in \mathbb{Z}$ such that $f(a, \mu) = 0$ for any μ and any $a < a_0$.²

Now set

$$F(v, x) = \sum_{(a, \mu) \in \mathbb{Z} \times \Lambda} f(a, \mu) v^a x^\mu.$$

Then (3.3) together with 1) and 2) above imply that F can naturally be regarded as an element of $\widehat{\mathbb{C}[X_Q]}_k$. In other words, we get an isomorphism of graded vector spaces $\mathcal{H}(\tilde{\Gamma}^+, \tilde{\Gamma}_0) \simeq \widehat{\mathbb{C}[X_Q]}$. Let us prove that this is actually an isomorphism of algebras.

Let L be the collection of all the elements of $\tilde{\Gamma}$ of the form $(a, 0, \mu, k)$. It is easy to see that L is actually a normal subgroup of $\tilde{\Gamma}$ isomorphic to $\mathbb{Z} \times \Lambda \times \mathbb{Z}$, which can also be identified with the kernel of the homomorphism $\tilde{\Gamma} \rightarrow \Lambda$ sending (a, λ, μ, k) to λ . It is also easy to see that L and Γ_0 satisfy the following properties:

- 1) $L \cap \Gamma_0 = \{e\}$.
- 2) $\tilde{\Gamma} = L \cdot \Gamma_0 = \Gamma_0 \cdot L$.

In other words L is a normal subgroup of $\tilde{\Gamma}$ whose elements provide unique representatives for both left and right cosets of $\tilde{\Gamma}$ with respect to Γ_0 . In particular, since L is normal, we have a natural (conjugation) action of H on L . Let $\mathbb{C}[L]$ denote the space \mathbb{C} -valued functions on L with almost finite support. This is an algebra with respect to convolution (this algebra is a completion of the group algebra of L). Let also $\mathbb{C}[L]^{\Gamma_0}$ be the space Γ_0 -invariants in $\mathbb{C}[L]$. (Note that it follows from (3.3) that every Γ_0 -orbit in L is almost finite.) Then it is easy to see that conditions 1) and 2) above imply that the restriction map

²Note that here we use the assumption that Q is negative-definite.

from $\tilde{\Gamma}$ to L defines an isomorphism of algebras $\mathcal{H}(\tilde{\Gamma}^+, \tilde{\Gamma}_0) \simeq \mathbb{C}[L]^{\Gamma_0}$. However, L is isomorphic to $\mathbb{Z} \times \Lambda \times \mathbb{Z}$, which implies that $\mathbb{C}[L]^{\Gamma_0} \simeq \widehat{\mathbb{C}[X_Q]}$. \square

3.4. *A reformulation.* The above result can be reformulated as follows. Define an action of Λ on the torus $T^\vee \times \mathbb{C}^*$ by the following formula:

$$(3.4) \quad \nu(x, t) = \left(v^{e(\nu)} x, tx^\nu v^{\frac{Q(\nu, \nu)}{2}} \right).$$

Then the quotient $(T^\vee \times \mathbb{C}^*)/\Lambda$ can be naturally identified with the total space of the complement to the zero section in the line $\mathcal{L}_{Q, v}^{-1}$ on $X_{Q, v}$. In particular, the algebra $\mathbb{C}[X_{Q, v}]$ is equal to the algebra of regular functions on $(T^\vee \times \mathbb{C}^*)/\Lambda$. In other words, let us consider the torus $T_{\text{aff}}^\vee = \mathbb{C}^* \times T^\vee \times \mathbb{C}^*$ with ‘‘coordinates’’ (t, x, v) . There is a natural action of Λ on T_{aff}^\vee , defined by the obvious analog of (3.4):

$$(3.5) \quad \nu(t, x, v) = \left(tx^\nu v^{\frac{Q(\nu, \nu)}{2}}, v^{e(\nu)} x, v \right).$$

Also, for every $k \in \mathbb{Z}$, let $\mathbb{C}(T_{\text{aff}}^\vee)_k$ denote the space of regular functions on T_{aff}^\vee which are homogeneous of degree k with respect to the first \mathbb{C}^* . Let $\widehat{\mathbb{C}(T_{\text{aff}}^\vee)_k}$ denote the completion of this space in the v -adic topology. We also set

$$(3.6) \quad \widehat{\mathbb{C}(T_{\text{aff}}^\vee)} = \bigoplus_{k \in \mathbb{Z}} \widehat{\mathbb{C}(T_{\text{aff}}^\vee)_k}.$$

Then there is natural identification $\widehat{\mathbb{C}[X_Q]} \simeq \widehat{\mathbb{C}(T_{\text{aff}}^\vee)}^\Lambda$. (It is easy to see that $\mathbb{C}(T_{\text{aff}}^\vee)_k^\Lambda \neq 0$ if and only if $k \geq 0$.)

4. The main result

4.1. *Loop groups and their cousins.* In this paper, for convenience, we adopt the polynomial version of loop groups (as opposed to formal loops version; the case of formal loops can also be treated with some modifications).

Let G be a split connected reductive algebraic group over a field k with Lie algebra \mathfrak{g} . Let T be a maximal (split) torus in G , and let \mathfrak{t} denote the corresponding Cartan subalgebra of \mathfrak{g} . We also denote by W the Weyl group of G .

Let Λ denote the coweight lattice of G . Note that we can regard Λ as a subset of \mathfrak{h} . We shall assume that there exists an integral, negative-definite symmetric bilinear form Q on Λ which is W -invariant. In this case $Q \otimes k$ is a restriction to Λ of a G -invariant form on \mathfrak{g} , which we shall denote by $Q_{\mathfrak{g}}$.

Consider the corresponding polynomial Lie algebra $\mathfrak{g}[t, t^{-1}]$ and the group $G[t, t^{-1}]$. As was mentioned in the introduction, the form Q gives rise to a

central extension \tilde{G} of $G[t, t^{-1}]$:

$$1 \rightarrow \mathbb{G}_m^* \rightarrow \tilde{G} \rightarrow G[t, t^{-1}] \rightarrow 1.$$

Moreover, both $G[t, t^{-1}]$ and \tilde{G} have a natural structure of a group ind-scheme over k .³ We denote by $\hat{\mathfrak{g}}$ the corresponding Lie algebra; it fits into the exact sequence

$$0 \rightarrow k \rightarrow \tilde{\mathfrak{g}} \rightarrow \mathfrak{g}[t, t^{-1}] \rightarrow 0.$$

The group \mathbb{G}_m acts naturally on $G[t, t^{-1}]$, and this action lifts to \tilde{G} . We denote the corresponding semi-direct product by G_{aff} ; we also let $\mathfrak{g}_{\text{aff}}$ denote its Lie algebra. Thus if G is semi-simple, then $\mathfrak{g}_{\text{aff}}$ is an untwisted affine Kac-Moody Lie algebra in the sense of [7]; in particular, it can be described by the corresponding affine root system.

We also let G'_{aff} denote the semi-direct product $\mathbb{G}_m \times G[t, t^{-1}]$.

4.2. *The algebra $\mathbb{C}(\widehat{T}_{\text{aff}}^{\vee})^{W_{\text{aff}}}$.* Let us recall that for G as above one can consider the Langlands dual group G^{\vee} . One of its crucial properties says that the complexified Grothendieck group $K_0(\text{Rep } G^{\vee})$ of the category $\text{Rep}(G^{\vee})$ of finite-dimensional representations is naturally isomorphic to the algebra $\mathbb{C}(T^{\vee})^W$ of W -invariant polynomials on the torus T^{\vee} .

We would like to define analogous notions when G is replaced by G_{aff} . First of all, let us describe the analog of the algebra $\mathbb{C}(T^{\vee})$. Similarly to Section 3, let us set $T_{\text{aff}} = \mathbb{G}_m \times T \times \mathbb{G}_m$ and $T_{\text{aff}}^{\vee} = \mathbb{C}^* \times T^{\vee} \times \mathbb{C}^*$. (We shall only work with the dual torus T_{aff}^{\vee} over \mathbb{C} .) It will be convenient for us to think about the first \mathbb{G}_m -factor in T_{aff} as dual to the *second* \mathbb{C}^* -factor in T_{aff}^{\vee} (and vice versa). We also define $\mathbb{C}(\widehat{T}_{\text{aff}}^{\vee})$ as in (3.6).

The weight lattice Λ_{aff} of T_{aff}^{\vee} is naturally identified with $\mathbb{Z} \times \Lambda \times \mathbb{Z}$. It is also the coweight lattice of T_{aff} . (However, let us stress that according to our conventions, the first multiple in $\Lambda_{\text{aff}} = \mathbb{Z} \times \Lambda \times \mathbb{Z}$ corresponds to the loop rotation in G_{aff} and the last multiple corresponds to the center of G_{aff} .) Let $\Lambda'_{\text{aff}} = \mathbb{Z} \times \Lambda$. For any $k \in \mathbb{Z}$ we denote by $\Lambda_{\text{aff},k}$ (resp. by $\Lambda'_{\text{aff},k}$) the set of all elements of Λ_{aff} (resp. of Λ'_{aff}) whose first coordinate is equal to k .

Let W_{aff} denote affine Weyl group of G which is the semi-direct product of W and Λ . It acts on the lattice Λ_{aff} (resp. Λ'_{aff}) preserving each $\Lambda_{\text{aff},k}$ (resp. each $\Lambda'_{\text{aff},k}$). In order to describe this action explicitly, it is convenient to set $W_{\text{aff},k} = W \ltimes k\Lambda$ which naturally acts on Λ . Then the restriction of the W_{aff} -action to $\Lambda_{\text{aff},k} \simeq \Lambda \times \mathbb{Z}$ comes from the natural $W_{\text{aff},k}$ -action on the first multiple.

In the case when G is semi-simple and simply connected, the set $\Lambda_{\text{aff},k}/W_{\text{aff}}$ admits the following description. Let us denote by Λ_{aff}^+ the set of dominant

³This group-scheme is nonreduced if G is not semi-simple.

coweights of G_{aff} (i.e., those coweights whose product with every positive root of G_{aff} is nonnegative). We put $\Lambda_{\text{aff},k}^+ = \Lambda_{\text{aff}}^+ \cap \Lambda_{\text{aff},k}$.

Let $\Lambda_k^+ \subset \Lambda$ denote the set of dominant coweights of G such that $\langle \lambda, \alpha \rangle \leq k$ when α is the highest root of \mathfrak{g} . It is well known that a weight (k, λ, n) of G_{aff}^\vee lies in $\Lambda_{\text{aff},k}^+$ if and only if $\lambda \in \Lambda_k^+$ (thus $\Lambda_{\text{aff},k}^+ = \Lambda_k^+ \times \mathbb{Z}$).

Then (under the assumption that G is simply connected) every W_{aff} -orbit on $\Lambda_{\text{aff},k}$ contains unique element of $\Lambda_{\text{aff},k}^+$. This is equivalent to saying that $\Lambda_k^+ \simeq \Lambda/W_{\text{aff},k}$.

4.3. *The Langlands dual group.* We would like to interpret the algebra $\mathbb{C}(\widehat{T_{\text{aff}}^\vee})^{W_{\text{aff}}}$ as the Grothendieck ring of a certain category of representations of the appropriately defined affine Langlands dual group G_{aff}^\vee . Unfortunately, we do not know a good definition of G_{aff}^\vee for general G . However, we can define G_{aff}^\vee (and the appropriate category of representations) in the following two cases:

- 1) G is torus
- 2) G is simply connected.

Let us explain these definitions. (We are not going to use the group G_{aff}^\vee in the remaining part of the paper; however, we think that interpreting the algebra $\mathbb{C}(\widehat{T_{\text{aff}}^\vee})^{W_{\text{aff}}}$ (that will one of the most important players in our main Theorem 4.6) at least in some cases is very instructive.)

Case 1: G is a torus. Let us use the notations of the previous section. In particular, we let $e : \Lambda \rightarrow \Lambda^\vee$ denote the map given by the form Q . We let G_Q^\vee denote the torus over \mathbb{C} whose lattice of cocharacters is $e(\Lambda)$. Thus G_Q^\vee fits into the short exact sequence

$$0 \rightarrow Z \rightarrow G_Q^\vee \rightarrow G^\vee \rightarrow 0,$$

where $Z = \Lambda^\vee/e(\Lambda)$. Note that the form Q can be used to define the dual form Q^\vee on $e(\Lambda)$ which is also integral and even. Thus we can form the group $(G_Q^\vee)_{\text{aff}}$. It contains the group G_Q^\vee as a subgroup; it is easy to see that $Z \subset G_Q^\vee$ is central in $(G_Q^\vee)_{\text{aff}}$. Thus we may define $G_{\text{aff}}^\vee = (G_Q^\vee)_{\text{aff}}/Z$.

The quotient of G_{aff}^\vee by the central \mathbb{C}^* can be described as follows. Consider the group ind-scheme $(G^\vee)'_{\text{aff}}$. It is easy to see that its connected components are numbered by Λ^\vee . Then $G_{\text{aff}}^\vee/\mathbb{C}^*$ is equal to the union of the above connected components corresponding to the elements of $e(\Lambda) \subset \Lambda^\vee$.

Case 2: G is semi-simple and simply connected. In this case let $\mathfrak{g}_{\text{aff}}^\vee$ denote the *Langlands dual* affine Lie algebra of $\mathfrak{g}_{\text{aff}}$ (considered as a Lie algebra over \mathbb{C}). By definition, this is an affine Kac-Moody Lie algebra whose root system is dual to that of $\mathfrak{g}_{\text{aff}}$. Moreover, it comes also with a fixed central subalgebra $\mathbb{C} \subset \mathfrak{g}_{\text{aff}}^\vee$ which is determined uniquely by the choice of $Q_{\mathfrak{g}}$. Note that in general

(when \mathfrak{g} is not simply laced) the algebra $\mathfrak{g}_{\text{aff}}^\vee$ is not isomorphic to $(\mathfrak{g}^\vee)_{\text{aff}}$. (Here \mathfrak{g}^\vee as before denotes the Langlands dual Lie algebra of \mathfrak{g} .) Moreover, if \mathfrak{g} is not simply laced, then $\mathfrak{g}_{\text{aff}}^\vee$ is a twisted affine Lie algebra. However, the algebra $\mathfrak{g}_{\text{aff}}^\vee$ always contains $\mathfrak{g}^\vee \times \mathbb{C}^2$ as a Levi subalgebra.

We now let G_{aff}^\vee denote any connected group ind-scheme over \mathbb{C} such that

- a) The Lie algebra of G_{aff}^\vee is $\mathfrak{g}_{\text{aff}}^\vee$
- b) The above embedding $\mathfrak{g}^\vee \hookrightarrow \mathfrak{g}_{\text{aff}}^\vee$ extends to an embedding $G^\vee \hookrightarrow G_{\text{aff}}^\vee$.

The existence (and uniqueness) of G_{aff}^\vee follows immediately from [14].

In both cases 1) and 2), the group G_{aff}^\vee contains the torus $T_{\text{aff}}^\vee = \mathbb{C}^* \times T^\vee \times \mathbb{C}^*$. Here the first \mathbb{Z} -factor is responsible for the center of G_{aff}^\vee ; it can also be thought of as coming from the loop rotation in G_{aff} . The second \mathbb{Z} -factor is responsible for the loop rotation in G_{aff}^\vee . Similarly, we shall denote by $\Lambda'_{\text{aff}} = \Lambda \times \mathbb{Z}$ the weight lattice of G'_{aff} .

It follows easily from the above definitions that (in both cases 1) and 2)) the group G_{aff}^\vee maps naturally to \mathbb{C}^* . (This homomorphism is dual to the central embedding $\mathbb{C}^* \rightarrow G_{\text{aff}}$.) We denote the kernel of this homomorphism by \widetilde{G}^\vee and we let $\widetilde{\mathfrak{g}}^\vee$ denote its Lie algebra. It is clear that in fact $G_{\text{aff}}^\vee = \widetilde{G}^\vee \rtimes \mathbb{C}^*$. We shall fix such an isomorphism, which, in particular, endows G_{aff}^\vee with a subgroup \mathbb{C}^* , which we shall call the loop rotation subgroup in G_{aff}^\vee .

Also G_{aff}^\vee has a natural central subgroup isomorphic to \mathbb{C}^* . In addition, it is easy to see that G_{aff}^\vee is generated by $(G_{\text{aff}}^\vee)^0$ and by T_{aff}^\vee .

4.4. *Representations of G_{aff}^\vee .* The Lie algebra $\mathfrak{g}_{\text{aff}}$ is endowed with a canonical “parabolic” subalgebra \mathfrak{p}_0 which is equal to $\mathfrak{k} \oplus \mathfrak{g}[\mathfrak{t}] \oplus \mathfrak{k}$. We let \mathfrak{p}_0^\vee denote the corresponding dual subalgebra of $\mathfrak{g}_{\text{aff}}^\vee$ and we set $\mathfrak{n}_0^\vee = [\mathfrak{p}_0^\vee, \mathfrak{p}_0^\vee]$.

By an integrable highest weight representation of G_{aff}^\vee , we shall mean an algebraic representation of G_{aff}^\vee whose restriction to the Lie algebra \mathfrak{n}_0^\vee is locally nilpotent. It is easy to see that any such representation is semi-simple. We say that such a representation L is of level k if the central \mathbb{C}^* acts on L by means of the character $z \mapsto z^k$. Then it is well known that

- a) If L is a nonzero representation of level k , then $k \geq 0$.
- b) Any representation of level 0 is a pull-back of a representation of \mathbb{C}^* under the above-mentioned map $G_{\text{aff}}^\vee \rightarrow \mathbb{C}^*$.
- c) The irreducible representations of level $k > 0$ are in one-to-one correspondence with elements of $\Lambda_{\text{aff},k}/W_{\text{aff}}$.

Let $\text{Rep}(G_{\text{aff}}^\vee)$ denote the category of integrable highest weight representations L of G_{aff}^\vee satisfying the following conditions:

- (i) L is a direct sum of irreducible representations of G_{aff}^\vee with finite multiplicities.

- (ii) For every $n \in \mathbb{Z}$, let us denote by L_n the subspace of L on which the loop rotation subgroup \mathbb{C}^* acts by means of the character $z \mapsto z^n$. Then $L_n = 0$ for $0 \ll n$.

The category $\text{Rep}(G_{\text{aff}}^\vee)$ is stable under a tensor product (cf. [7]). Also the group \mathbb{Z} acts naturally on this category by multiplying every L by the corresponding character of G_{aff}^\vee coming from the homomorphism $G_{\text{aff}}^\vee \rightarrow \mathbb{G}_m$. We denote by $K_0(G_{\text{aff}}^\vee)$ the complexified Grothendieck ring of $\text{Rep}(G_{\text{aff}}^\vee)$ tensored with \mathbb{C} . It is clear that the above \mathbb{Z} -action gives rise to a $\mathbb{C}((v))$ -module structure on $K_0(G_{\text{aff}}^\vee)$. In addition, the algebra $K_0(G_{\text{aff}}^\vee)$ is \mathbb{Z}_+ -graded. By definition, elements of degree k in $K_0(G_{\text{aff}}^\vee)$ correspond to representations of G_{aff}^\vee of level k . In this way, $K_0(G_{\text{aff}}^\vee)$ becomes a graded $\mathbb{C}((v))$ -algebra.

To every $L \in \text{Rep}(G_{\text{aff}}^\vee)$, we can consider its character $\text{ch}(L)$ which is an element of $\mathbb{C}(\widehat{T}_{\text{aff}}^\vee)$. The assignment $L \mapsto \text{ch}(L)$ extends to an isomorphism

$$(4.1) \quad K_0(G_{\text{aff}}^\vee) \simeq \mathbb{C}(\widehat{T}_{\text{aff}}^\vee)^{W_{\text{aff}}}.$$

In the future we are going to work with the algebra $\mathbb{C}(\widehat{T}_{\text{aff}}^\vee)^{W_{\text{aff}}}$, which is defined for any reductive G . However, we think that intuitively it is important keep in mind the isomorphism (4.1).

4.5. *The spherical Hecke algebra of G_{aff} .* Let now \mathfrak{k} be a finite field. We shall choose a ring \mathcal{K} and its subring \mathcal{O} , which will come from one of the following two situations:

- a) \mathcal{K} is a local non-archimedean field with residue field \mathfrak{k} and \mathcal{O} is its ring of integers
- b) $\mathcal{K} = \mathfrak{k}[s, s^{-1}]$, $\mathcal{O} = \mathfrak{k}[s]$.

Case a) is somewhat more interesting and natural. However, we still include case b), since our proofs are a little more transparent in this case (although the difference between the two cases is not essential). In both cases we are given the natural valuation homomorphism $\text{val} : \mathcal{K}^* \rightarrow \mathbb{Z}$, whose kernel is \mathcal{O}^* .

It now makes sense to consider the group $\tilde{\Gamma} = G_{\text{aff}}(\mathcal{K})$ and its subgroup $\tilde{\Gamma}_0 = G_{\text{aff}}(\mathcal{O})$. (We also have $\Gamma = G'_{\text{aff}}(\mathcal{K})$, $\Gamma_0 = G'_{\text{aff}}(\mathcal{O})$.) Note that this pair satisfies the conditions of Section 2.3 with $A = \mathcal{K}^*$ which, in both cases a) and b), has a natural map to \mathbb{Z} with kernel being $B = \mathcal{O}^*$.

We have the natural homomorphisms

$$\zeta : \Gamma = G'_{\text{aff}}(\mathcal{K}) \rightarrow \mathcal{K}^* \quad \text{and} \quad \text{val} : \mathcal{K}^* \rightarrow \mathbb{Z}.$$

We set $\pi = \text{val} \circ \zeta$. Similarly we have the homomorphisms $\tilde{\zeta} : G_{\text{aff}}(\mathcal{K}) \rightarrow \mathcal{K}^*$ and $\tilde{\pi} : G_{\text{aff}}(\mathcal{K}) \rightarrow \mathbb{Z}$.

We set

$$\Gamma^+ = G'_{\text{aff}}(\mathcal{O}) \cup \pi^{-1}(\mathbb{Z}_{>0}).$$

As in Section 2.3 we let $\tilde{\Gamma}^+$ denote the preimage of Γ^+ in $\tilde{\Gamma}$.

Below is the main result of this paper.

- THEOREM 4.6. (1) *The pair $(\tilde{\Gamma}^+, \tilde{\Gamma}_0)$ defined above is almost good in the terminology of Section 2.3.*
 (2) *The Hecke algebra $\mathcal{H}(\tilde{\Gamma}^+, \Gamma_0)$ is naturally isomorphic to $\mathbb{C}(\widehat{T}_{\text{aff}}^\vee)^{W_{\text{aff}}}$ as a graded $\mathbb{C}((v))$ -algebra.*

In fact, we are going to construct the isomorphism (2) explicitly; we shall call it the Satake isomorphism for the group G_{aff} . We expect that it should be possible to describe this isomorphism explicitly in terms of representation theory of G_{aff}^\vee in the spirit of [11] (cf. also [1] for some closely related conjectures). However, we shall postpone the detailed discussion of this question for another publication.

The proof of the first assertion occupies Sections 5, 6 and 7. Although the first assertion seems to be quite elementary, the only proof of Theorem 4.6(1) that we know requires interpreting the fibers of the corresponding maps $m_{X,Y}$ in terms of algebraic geometry. The proof of the second assertion occupies Section 8.

5. The algebra $\mathcal{H}(\tilde{\Gamma}^+, \tilde{\Gamma}_0)$ via bundles on surfaces

5.1. *Double cosets and Kleinian singularities.* For any $k > 0$, let S_k be the scheme $\text{Spec}(\mathbb{k}[x, y, z]/xy - z^k)$; let also S_k^0 be the complement to the point $(0, 0, 0)$ in S_k . This is a smooth surface over \mathbb{k} .

Let \mathbb{A}^2 denote the 2-dimensional affine plane over \mathbb{k} ; the multiplicative group \mathbb{G}_m acts on \mathbb{A}^2 by $t : (x, y) \mapsto (tx, t^{-1}y)$. Let μ_k denote the group-scheme of roots of unity of order k ; this is a group sub-scheme of \mathbb{G}_m . Then the natural map $\mathfrak{p}_k : \mathbb{A}^2 \rightarrow S_k$ given by

$$(u, v) \mapsto (u^k, v^k, uv)$$

identifies S_k with $(\mathbb{A}^2 \setminus \{0\})/\mu_k$. (Note that the action of \mathbb{G}_m (and thus also of μ_k) on $(\mathbb{A}^2 \setminus \{0\})$ is free.)

Let Σ be some other surface over \mathbb{k} , and let a be a point of Σ . We shall say that a is of type A_k if there is an étale neighborhood of a in Σ which is isomorphic to an étale neighborhood of $(0, 0, 0)$ in S_k . In particular, if $k = 1$, then this will just mean that a is a smooth point of Σ .

5.2. *A variant: the case of an arbitrary local field.* The surface S_k introduced above will play a crucial role in the proof of Theorem 4.6 in case b) from Section 4.5. Let us explain how to define it in case a), i.e., when we work over a local nonarchimedean field \mathcal{K} with ring of integers \mathcal{O} . Let z be a uniformizer in \mathcal{O} . (It is convenient to choose it, though nothing will actually depend on this choice.) In this case we shall set $S_k = \text{Spec}(\mathcal{O}[x, y]/xy - z^k)$. This is a scheme over \mathcal{O} . It is easy to see that if $k = 1$, then this scheme is regular.

(Warning: it is, however, not regular as a scheme over \mathcal{O} .) We set S_k^0 to be the complement in S_k of the (closed) point given by the equations $x = y = z = 0$. The scheme S_k^0 is regular for all $k > 0$. We define a map $\mathfrak{p}_k : S_1 \rightarrow S_k$ of \mathcal{O} -schemes which sends (u, v) to $(x = u^k, y = v^k)$. It is clear that under this identification we get $S_k^0 = S_1^0/\mu_k$ where μ_k is the group-scheme of k -th roots of unity (over \mathcal{O}).

We shall say that a two-dimensional scheme Σ has a singularity of type A_k at a closed point p if near p it is étale-locally isomorphic to S_k , with p corresponding to the point $(0, 0, 0)$. (In particular, if $k = 1$, then this implies that p is a smooth point of Σ .)

In all the proofs that follow, we shall always assume that we are in case b) (in the terminology of §4.5). The extension to case a) will be straightforward using the notation introduced in this subsection.

For any k -variety S , we shall denote by $\text{Bun}_G(S)$ the set of isomorphism classes of principal G -bundles on S .

PROPOSITION 5.3. *The following sets are in natural bijection:*

- (1) *the set $\text{Bun}_G(S_k^0)$;*
- (2) *the set $G'_{\text{aff}}(\mathcal{O}) \backslash \pi^{-1}(k) / G'_{\text{aff}}(\mathcal{O})$;*
- (3) *the set $\Lambda / W_{\text{aff},k}$;*
- (4) *the set of G -conjugacy classes of homomorphisms $\mu_k \rightarrow G$.*

Proof. Let us first establish the bijection between (1) and (4). Let \mathcal{F} denote a G -bundle on S_k . Consider the G -bundle $\mathfrak{p}_k^*(\mathcal{F})$ on $\mathbb{A}^2 \setminus \{0\}$. It extends uniquely to the whole of \mathbb{A}^2 and thus it is trivial. On the other hand, the bundle $\mathfrak{p}_k^*(\mathcal{F})$ is μ_k -equivariant. Since this bundle is trivial, such an equivariant structure gives rise to a homomorphism $\mu_k \rightarrow G$ defined uniquely up to conjugacy. This defines a map $\text{Bun}_G(S_k^0) \rightarrow \text{Hom}(\mu_k, G)/G$. It is clear that this is actually a bijection, since a μ_k -equivariant G -bundle on $\mathbb{A}^2 \setminus \{0\}$ descends uniquely to S_k .

Let us now establish the bijection between (1) and (2). To do that let us denote by σ the automorphism of $G(\mathcal{K})$ sending $g(t, s)$ to $g(ts, s)$. Let us now identify $\pi^{-1}(k)$ with $G(\mathcal{K})$, with right $G(\mathcal{K})$ action being the standard one (by right shifts), and with left $G(\mathcal{K})$ -action given by $g(h) = \sigma^k(g)h$. Thus we have the bijection

$$(5.1) \quad G'_{\text{aff}}(\mathcal{O}) \backslash \pi^{-1}(k) / G'_{\text{aff}}(\mathcal{O}) = G[ts^k, t^{-1}s^{-k}, s] \backslash G[t, t^{-1}, s, s^{-1}] / G[t, t^{-1}, s].$$

Let $U_k = \text{Spec } k[ts^k, t^{-1}s^{-k}, s]$, $V_k = \text{Spec } k[t, t^{-1}, s]$. Both U_k and V_k are isomorphic to $\mathbb{G}_m \times \mathbb{A}^1$, and thus every G -bundle on either of these surfaces is trivial. Both U_k and V_k contain $W = \text{Spec } k[t, t^{-1}, s, s^{-1}] \simeq \mathbb{G}_m \times \mathbb{G}_m$ as a Zariski open subset. Thus the right-hand side of (5.1) can be identified with

$\text{Bun}_G(S')$, where S' is obtained by gluing U_k and V_k along W . Hence it remains to construct an isomorphism $S' \xrightarrow{\sim} S_k$. Such an isomorphism can be obtained by setting $x = ts^k, y = t^{-1}, z = s$.

It remains to construct a bijection between (3) and (4). Recall that $\Lambda = \text{Hom}(\mathbb{G}_m, T)$. Thus, since μ_k is a closed subscheme of \mathbb{G}_m , given every $\lambda \in \Lambda$, we may restrict it to μ_k and get a homomorphism $\mu_k \rightarrow T$. By composing it with the embedding $T \hookrightarrow G$, we get a homomorphism $\mu_k \rightarrow G$ which clearly depends only on the image of λ in $\Lambda/W_{\text{aff},k}$. Thus we get a well-defined map $\Lambda/W_{\text{aff},k} \rightarrow (\text{Hom}(\mu_k, G)/G)$. The surjectivity of this map follows from the fact that μ_k is a diagonalizable group-scheme. For the injectivity, note that any two elements in T which are conjugate in G lie in the same W -orbit in T . Thus it is enough to show that for any two homomorphisms $\lambda, \mu : \mathbb{G}_m \rightarrow T$ whose restrictions to μ_k coincide the difference, $\lambda - \mu$ is divisible by k . This is enough to check for $T = \mathbb{G}_m$ where it is obvious. \square

5.4. *Groupoids.* Recall that a groupoid is a category in which all morphisms are isomorphisms. All groupoids that will appear in this paper will be small. A typical example of a groupoid for us will be as follows. If a group H acts on a set X , then we can consider the groupoid X/H , where the objects are points of X and a morphism from x to y is an element $h \in H$ such that $h(x) = y$. Using the terminology common in the theory of algebraic stacks, we shall say that a functor $f : \mathcal{Y} \rightarrow \mathcal{Y}'$ between two groupoids is representable if any $y \in \mathcal{Y}$ has no nontrivial automorphisms which act trivially on $f(y)$. In this case, for any $y' \in \mathcal{Y}'$, we define the fiber $f^{-1}(y')$ to be the set of isomorphism classes of pairs (y, α) , where $y \in \mathcal{S}$ and α is an isomorphism between $f(y)$ and y' .

By a groupoid structure on a set Y we shall mean a groupoid \mathcal{Y} whose set of isomorphism classes is identified with Y .

The double quotient $G'_{\text{aff}}(\mathcal{O}) \backslash \pi^{-1}(k) / G'_{\text{aff}}(\mathcal{O})$ and the set $\text{Bun}_G(S_k^0)$ have natural groupoid structures and it is easy to see that the above identification between the them is actually an equivalence of groupoids.

From now on we shall treat these sets as groupoids. (Abusing the notation, we shall denote them in the same way as before.)

5.5. *The convolution diagram.* Choose any two positive integers k and l . Consider the following “convolution diagram:”

$$(5.2) \quad G'_{\text{aff}}(\mathcal{O}) \backslash \pi^{-1}(k) \times_{G'_{\text{aff}}(\mathcal{O})} \pi^{-1}(l) / G'_{\text{aff}}(\mathcal{O}) \xrightarrow{m} G'_{\text{aff}}(\mathcal{O}) \backslash \pi^{-1}(s^{k+l}) / G'_{\text{aff}}(\mathcal{O}),$$

where m denotes the multiplication map. As was mentioned above we shall think about this diagram as a map of groupoids. We want to give an interpretation of this diagram in geometric terms, i.e., in terms of G -bundles on some

surfaces. The right-hand side of (5.2) is clearly equal to $\text{Bun}_G(S_{k+l}^0)$. Let us interpret the left-hand side.

The surface S_{k+l} has canonical crepant resolution \tilde{S}_{k+l} . Its fiber over $0 \in S_{k+l}$ consists of a tree of $k+l-1$ rational curves E_1, \dots, E_{k+l-1} , where the self-intersection of each E_j is -2 . Thus it is possible to blow down all the E_j except E_k . Let us call the resulting surface $S_{k,l}$. It has two potentially singular points corresponding to 0 and ∞ in E_k . They are of type A_k and A_l respectively (i.e., these points are really singular if the corresponding integer (k or l) is greater than 1). More precisely, we claim that $S_{k,l}$ can be covered by two open subsets which are isomorphic to S_k, S_l , respectively. Let us describe how this is done in more detail.

The surface S_k can be covered by two open subsets U_k and V_k , each one isomorphic to $\mathbb{A}^1 \times \mathbb{G}_m$. Namely, if $S_k = \text{Spec } \mathbb{k}[x, y, z]/xy - z^k$, then $U_k = \text{Spec } \mathbb{k}[x, x^{-1}, z]$ and $V_k = \text{Spec } \mathbb{k}[y, y^{-1}, z]$. (Note that U_k and V_k can be considered as open subsets both S_k and S_k^0 .) Similarly, let us consider $S_l = \text{Spec } \mathbb{k}[u, v, w]/uv - w^l$. Let us identify V_k with U_l by setting $z = w$ and $y = u^{-1}$. Let us now glue S_k and S_l along V_k and U_l respectively. The resulting surface Σ maps to $S_{k+l} = \text{Spec } \mathbb{k}[p, q, r]/pq - r^{k+l}$ in the following way. We map S_k to S_{k+l} by sending (x, y, z) to (x, yz^l, z) , and we map S_l to S_{k+l} by sending (u, v, w) to (uw^k, v, w) . These formulas are compatible, since by multiplying the equality $y = u^{-1}$ by xu on both sides, we get $uxy = uz^k$ in the left-hand side and x in the right-hand side, which implies that $x = uz^k = uw^k$. Similarly, multiplying $u = y^{-1}$ by vy on both sides, we get the equality $yz^l = v$. These equalities make sense as long as y and u are invertible. It is now easy to see that the resulting map $\Sigma \rightarrow S_{k+l}$ is an isomorphism away from the point $(0, 0, 0)$ and the fiber over $(0, 0, 0)$ is naturally isomorphic to \mathbb{P}^1 . From this it is easy to deduce that $\Sigma \simeq S_{k,l}$. We let $\mathfrak{p}_{k,l}$ denote the natural map $S_{k,l} \rightarrow S_{k+l}$. Note that $S_{k,l} \setminus \mathfrak{p}_{k,l}^{-1}(0, 0, 0) \simeq S_{k+l}$.

We let $S_{k,l}^0$ denote the complement to the two singular points in $S_{k,l}$.

PROPOSITION 5.6. a) *The left-hand side of (5.2) can be naturally identified with $\text{Bun}_G(S_{k,l}^0)$ (as a groupoid).*

b) *The corresponding map $\text{Bun}_G(S_{k,l}^0) \rightarrow \text{Bun}_G(S_{k+l}^0)$ is just the restriction to the complement of $\mathfrak{p}_{k,l}^{-1}(0, 0, 0)$.*

Proof. According to the proof of Proposition 5.3, the convolution diagram (5.2) can be identified with

$$(5.3) \quad G[ts^{k+l}, t^{-1}s^{-k-l}, s] \backslash G[t, s, t^{-1}, s^{-1}] \times_{G[ts^l, t^{-1}s^{-l}, s]} G[t, t^{-1}, s, s^{-1}] / G[t, t^{-1}, s] \\ \xrightarrow{m} G[ts^{k+l}, t^{-1}s^{-k-l}, s] \backslash G[t, s, t^{-1}, s^{-1}] / G[t, t^{-1}, s].$$

We claim now that the left-hand side of (5.3) can be naturally identified with $\text{Bun}_G(S_{k,l}^0)$. Indeed, let us set

$$x = ts^{k+l}, y = t^{-1}s^{-l}, u = ts^l, v = t^{-1}, z = w = s.$$

Then we have $xy = z^k$ and $uv = w^l$ and $\mathbb{k}[ts^{k+l}, t^{-1}s^{-k-l}, s] = \mathbb{k}[x, x^{-1}, z]$, $\mathbb{k}[ts^l, t^{-1}s^{-l}, s] = \mathbb{k}[y, y^{-1}, z]$. Thus the product

$$G[ts^{k+l}, t^{-1}s^{-k-l}, s] \backslash G[t, t^{-1}, s, s^{-1}] \times G[t, t^{-1}, s, s^{-1}] / G[t, t^{-1}, s]$$

classifies $(\mathcal{F}_k, \mathcal{F}_l, \beta_k, \alpha_l)$, where

- a) \mathcal{F}_k is a G -bundle on S_k and β_k is a trivialization of \mathcal{F}_k on V_k ;
- b) \mathcal{F}_l is a G -bundle on S_l and β_l is a trivialization of \mathcal{F}_l on U_l .

It follows from this that the product

$$G[ts^{k+l}, t^{-1}s^{-k-l}, s] \backslash G[t, s, t^{-1}, s^{-1}] \times_{G[ts^l, t^{-1}s^{-l}, s]} G[t, t^{-1}, s, s^{-1}] / G[t, t^{-1}, s]$$

classifies the triples $(\mathcal{F}_k, \mathcal{F}_l, \gamma)$, where \mathcal{F}_k and \mathcal{F}_l are as above and γ is an isomorphism between $\mathcal{F}_k|_{V_k}$ and $\mathcal{F}_l|_{U_l}$ (with respect to the identification of V_k and U_l discussed in the previous subsection). But such a triple is the same as a G -bundle $\mathcal{F}_{k,l}$ on $S_{k,l}$ since the latter surface is obtained by gluing S_k and S_l by identifying V_k with U_l . □

5.7. *A generalization.* Fix now some positive integers k_1, \dots, k_n , and let $k = k_1 + \dots + k_n$. In this case we can define the surface S_{k_1, \dots, k_n} which is obtained by gluing the surfaces S_{k_1}, \dots, S_{k_n} by identifying V_{k_1} with U_{k_2} , V_{k_2} with U_{k_3} , etc. We have the map $\mathfrak{p}_{k_1, \dots, k_n} : S_{k_1, \dots, k_n} \rightarrow S_k$. The surface S_{k_1, \dots, k_n} is smooth away from n singular points which all lie in the preimage of zero under p_{k_1, \dots, k_n} and are of types A_{k_1}, \dots, A_{k_n} respectively. We shall denote by S_{k_1, \dots, k_n}^0 the complement to these singular points.

It is easy to see that away from $\mathfrak{p}_{k_1, \dots, k_n}^{-1}(0, 0, 0)$ the map $\mathfrak{p}_{k_1, \dots, k_n}$ is an isomorphism. In particular, S_k^0 embeds into S_{k_1, \dots, k_n}^0 as an open subset.

On the other hand, we can consider the map (of groupoids)

$$\begin{aligned} G'_{\text{aff}}(\mathcal{O}) \backslash \pi^{-1}(k_1) \times_{G'_{\text{aff}}(\mathcal{O})} \pi^{-1}(k_2) \times_{G'_{\text{aff}}(\mathcal{O})} \dots \times_{G'_{\text{aff}}(\mathcal{O})} \pi^{-1}(k_n) / G'_{\text{aff}}(\mathcal{O}) \\ \rightarrow G'_{\text{aff}}(\mathcal{O}) \backslash \pi^{-1}(k) / G'_{\text{aff}}(\mathcal{O}). \end{aligned}$$

Then, using the same argument as above, we can identify this map with the restriction map

$$\text{Bun}_G(S_{k_1, \dots, k_n}^0) \rightarrow \text{Bun}_G(S_k^0).$$

5.8. *Fibers of the convolution morphism.* Consider again the convolution morphism (5.2). It is easy to see that this morphism is representable and we would like to understand its fibers. According to Section 5.5, we can instead look at the fibers of the restriction map $\text{Bun}_G(S_{k,l}^0) \rightarrow \text{Bun}_G(S_{k+l}^0)$. By definition, the fiber of this map over a bundle \mathcal{F}_{k+l} consists of all extensions $\mathcal{F}_{k,l}$ of \mathcal{F}_{k+l} to $S_{k,l}$. If the pair (Γ^+, Γ_0) were good, this would imply that this set is finite. However, it is easy to see that this set is actually infinite. In Section 7 we shall see that this problem can be remedied by changing the group G'_{aff} by G_{aff} . First we need to recall some (mostly well-known) constructions from algebraic geometry.

6. Determinant torsors

6.1. *Relative determinants.* Let X be a smooth variety (over an arbitrary field), and let \mathcal{Q} be a vector bundle on X . Then we shall denote by $\det(\mathcal{Q}) \in \text{Pic}(X)$ the top exterior power of \mathcal{Q} . More generally, let \mathcal{Q} be an arbitrary coherent sheaf on X . Then locally \mathcal{Q} has a resolution

$$0 \rightarrow \mathcal{E}_n \rightarrow \cdots \rightarrow \mathcal{E}_1 \rightarrow \mathcal{E}_0 \rightarrow \mathcal{Q} \rightarrow 0,$$

where all \mathcal{E}_i are locally free. In this case we set

$$\det(\mathcal{Q}) = \bigotimes_{i=0}^n \det(\mathcal{E}_i)^{(-1)^i}.$$

It is well known that the result is canonically independent of the choice of the resolution. (Thus, in particular, it makes sense globally.)

Let now S be a smooth variety, and let X be a smooth divisor in S . (In the future, S will usually be a surface and X will usually be a closed curve inside S .) Set $S^0 = S \setminus X$.

Let $\mathcal{F}_1, \mathcal{F}_2$, be two locally free sheaves on S together with an isomorphism $\alpha : \mathcal{F}_1|_{S^0} \xrightarrow{\sim} \mathcal{F}_2|_{S^0}$. In this case we can form the *relative determinant* $\det_X(\mathcal{F}_1, \mathcal{F}_2, \alpha) \in \text{Pic}(X)$ of \mathcal{F}_1 and \mathcal{F}_2 in the following way.⁴

Assume first that α defines an embedding of \mathcal{F}_1 into \mathcal{F}_2 as coherent sheaves. Then we may consider the quotient $\mathcal{Q} = \mathcal{F}_2/\mathcal{F}_1$. This is a coherent sheaf on S which is set-theoretically concentrated on X . Thus we can find a filtration $0 \subset \mathcal{Q}_1 \subset \mathcal{Q}_2 \subset \cdots \subset \mathcal{Q}_r = \mathcal{Q}$ of \mathcal{Q} by coherent subsheaves such that each successive quotient $\mathcal{Q}_i/\mathcal{Q}_{i-1}$ is *scheme-theoretically* concentrated on X . In particular, we may regard each quotient $\mathcal{Q}_i/\mathcal{Q}_{i-1}$ just as a coherent sheaf

⁴In the sequel we shall just write $\det_X(\mathcal{F}_1, \mathcal{F}_2)$ or $\det_X(\alpha)$ when it does not lead to a confusion.

on X . We set

$$\det_X(\mathcal{F}_1, \mathcal{F}_2) = \bigotimes_{i=1}^r \det(\mathcal{Q}_i/\mathcal{Q}_{i-1}).$$

It is easy to see that the result does not depend on the choice of the above filtration.

In the general case, there exists another locally free sheaf \mathcal{F}_3 and embeddings $\beta_1, \beta_2 : \mathcal{F}_1, \mathcal{F}_2 \hookrightarrow \mathcal{F}_3$ such that

- a) Both β_1 and β_2 are isomorphisms away from X .
- b) $\alpha = \beta_1|_{S^0} \circ \beta_2|_{S^0}^{-1}$.

In this case we set $\det_X(\mathcal{F}_1, \mathcal{F}_2, \alpha) = \det_X(\mathcal{F}_1, \mathcal{F}_3, \beta_1) \otimes \det_X(\mathcal{F}_2, \mathcal{F}_3, \beta_2)^{-1}$. It is easy to see that the result is canonically independent of the choice of \mathcal{F}_3 and β_1, β_2 . (This definition is certainly well known, but we were unable to find a good reference for it.)

Let now G be an algebraic group, and let V be a finite-dimensional representation of G . In this case for any G -bundle \mathcal{F} (on any variety) we can consider the associated vector bundle \mathcal{F}_V . Given two G -bundles \mathcal{F}_1 and \mathcal{F}_2 on S with an isomorphism between their restrictions to S^0 , we set

$$\det_{X,V}(\mathcal{F}_1, \mathcal{F}_2) = \det_X((\mathcal{F}_1)_V, (\mathcal{F}_2)_V).$$

6.2. Representations and bilinear forms. We now go back to the case when G is reductive and keep all the notations from the previous sections. Then any finite-dimensional representation V of G as above defines a symmetric bilinear W -invariant form on Λ in the following way. By restricting V to T we get a collection of $n = \dim V$ weights $\lambda_1^\vee, \dots, \lambda_n^\vee$ of T , which gives an action of Λ on \mathbb{Z}^n . We set

$$Q_V(\lambda, \mu) = -\text{Tr}_{\mathbb{Z}^n}(\lambda \cdot \mu).$$

It is clear that this form is nonpositive definite, integral and W -invariant. Also, if we assume that the determinant of V is an even character of G (i.e., it is a square of another character), then this form is even. (In particular, that is always the case when G is semi-simple.) This form is negative-definite if V is almost faithful (i.e., if V is a faithful representation of a quotient of G by a finite central subgroup).

Moreover, it is well known that for any negative-definite W -invariant form Q , there exists a positive integer c and a representation V as above such that $cQ = Q_V$. It is easy to see that Theorem 4.6 holds for the form Q if and only if it holds for cQ . Thus we may assume that $Q = Q_V$.

6.3. Description of the central extension. We now want to define the central extension G_{aff} of the group $G[t, t^{-1}] \rtimes \mathbb{G}_m$ (which we shall only do in the case $Q = Q_V$). For any test k -scheme \mathcal{W} , we need to define $G_{\text{aff}}(\mathcal{W})$. Let us first do it just for $G[t, t^{-1}]$; i.e., let us define $\tilde{G}(\mathcal{W})$.

Consider some element $g \in G[t, t^{-1}](\mathcal{W})$. By definition, this is the same as a map $\mathcal{W} \times \mathbb{G}_m \rightarrow G$. Let $S = \mathbb{A}^1 \times \mathcal{W}$. Then to g there corresponds a triple $(\mathcal{F}_1, \mathcal{F}_2, \alpha_g)$ as above, where \mathcal{F}_1 and \mathcal{F}_2 are both trivial bundles on $\mathcal{W} \times \mathbb{A}^1$ and α_g is an isomorphism between their restrictions to $\mathcal{W} \times \mathbb{G}_m$ which is “equal to g ” in the obvious sense. Then we can consider $\mathcal{L}_g := \det_{\mathcal{W}}(\mathcal{F}_1, \mathcal{F}_2)$ which is a line bundle on \mathcal{W} . It is clear that \mathcal{L}_e is canonically trivial (here e is the identity element). Moreover, for any $g_1, g_2 \in G[t, t^{-1}](\mathcal{W})$, we have the natural isomorphism

$$(6.1) \quad \mathcal{L}_{g_1} \otimes \mathcal{L}_{g_2} \simeq \mathcal{L}_{g_1 g_2}.$$

We set $\widetilde{G}(\mathcal{W})$ to be the set of pairs (g, κ) where κ is a trivialization of \mathcal{L}_g . The group structure on $\widetilde{G}(\mathcal{W})$ comes from (6.1).

It is easy to see that with the above definition of \widetilde{G} the action of \mathbb{G}_m on $G[t, t^{-1}]$ by loop rotations extends to \widetilde{G} (this follows from the fact that the line bundle \mathcal{L}_g considered above does not change when we change g by a loop rotation), and thus we may consider the semi-direct product $G_{\text{aff}} = \widetilde{G} \ltimes \mathbb{G}_m$.

Remark. In [5] Faltings explains how to construct geometrically the central extension (and the determinant torsor) associated with any even symmetric invariant form Q . In principle, in what follows we could work with arbitrary Q using the construction of [5]. However, this would considerably complicate the exposition. As mentioned above, in order to prove Theorem 4.6(1), it is enough to assume that $Q = Q_V$ for some V .

6.4. *Torsors of extension.* Here is another source of examples of \mathbb{Z} -torsors that will be important in the future. Let X be a curve, and let x be a smooth point of X . Set $X^0 = X \setminus \{x\}$. Assume that we are given a line bundle \mathcal{L}^0 on X^0 . Then the set of all possible extensions \mathcal{L} of \mathcal{L}^0 to the whole of X (as a line bundle) is naturally a \mathbb{Z} -torsor, which we shall call the torsor of extensions of \mathcal{L}^0 to X .

The following remark will become very important later. Assume that X is projective. Then the above torsor of extensions is canonically trivial. Indeed to any extension \mathcal{L} we can associate the integer $\deg \mathcal{L}$ (the degree of \mathcal{L}).

6.5. *More \mathbb{Z} -torsors.* Let now S be a smooth surface, and let X_1, \dots, X_n be a collection of smooth curves S , intersecting at a point $y \in S$. Let us denote their union by X . Let now \mathcal{F}_1 and \mathcal{F}_2 be two vector bundles on S with an isomorphism away from X . Let $X_i^0 = X \setminus \{y\}$. Then we can consider the relative determinants $\det_{X_i^0}(\mathcal{F}_1, \mathcal{F}_2) \in \text{Pic}(X_i^0)$. Let \mathcal{T}_i be the \mathbb{Z} -torsor of extensions of $\det_{X_i^0}(\mathcal{F}_1, \mathcal{F}_2)$ to the whole of X_i . The following result is probably well known, but we were not able to find a reference. The proof given below is due to V. Drinfeld.

LEMMA 6.6. *The \mathbb{Z} -torsor*

$$\bigotimes_{i=1}^n \mathcal{T}_i$$

is canonically trivial.

Proof. Clearly, we can replace S by the formal neighbourhood of y . Let also \mathcal{D} denote the one-dimensional formal disc (formal neighbourhood of 0 in \mathbb{A}^1). Let us choose formal coordinates u, v , around y . Then for any $a, b \in k$, we can consider the map $f_{a,b} : S \rightarrow \mathcal{D}$ sending (u, v) to $au + bv$. Then for generic pair (a, b) , this map induces an isomorphism $X_i \simeq \mathcal{D}$. Let us choose such a pair.

As before we may assume that the identification $\mathcal{F}_1|_{S \setminus X} \simeq \mathcal{F}_2|_{S \setminus X}$ comes from an embedding $\mathcal{F}_1 \hookrightarrow \mathcal{F}_2$. Let $\mathcal{Q} = \mathcal{F}_2/\mathcal{F}_1$. This is a coherent sheaf on S , which is set-theoretically supported on X . Consider the direct image $\mathcal{R} = (f_{a,b})_* \mathcal{Q}$. Then we may consider $\det \mathcal{R}$. This is a line bundle on \mathcal{D} ; it is clear that its restriction to \mathcal{D}^0 is canonically isomorphic to the tensor product of all the $\det_{X_i^0}(\mathcal{F}_1, \mathcal{F}_2)$ (where we identify X_i^0 with \mathcal{D}^0 by means of $f_{a,b}$). Thus the \mathbb{Z} -torsor

$$\bigotimes_{i=1}^n \mathcal{T}_i$$

is canonically isomorphic to the torsor of extensions of $\det \mathcal{R}|_{\mathcal{D}^0}$ to \mathcal{D} . But since $\det \mathcal{R}$ is a line bundle defined on all of \mathcal{D} , it defines such an extension canonically and thus our \mathbb{Z} -torsor is canonically trivial.

The collection of all pairs (a, b) for which the above arguments works is a Zariski open subset \mathcal{X} of \mathbb{A}^2 ; in particular, \mathcal{X} is an irreducible algebraic variety. It is clear that the above trivialization depends regularly on $(a, b) \in \mathcal{X}$; thus the irreducibility of \mathcal{X} implies that the trivialization is independent of the choice of the pair (a, b) . Similar argument shows that the above trivialization is independent of the choice of coordinates (u, v) . \square

In the sequel, we shall say that a curve $X \subset S$ is *good* if it is a union of smooth irreducible components.

6.7. *Relative c_2 .* Let again S be a smooth surface, and let X be a smooth connected projective curve inside S . Let also $\mathcal{F}_1, \mathcal{F}_2$, be two G -bundles on S . Let also α be an identification between \mathcal{F}_1 and \mathcal{F}_2 on $S^0 = S \setminus X$ (compatible with the trivialization of the determinants). Then we set

$$c_2(\mathcal{F}_1, \mathcal{F}_2) = \deg(\det_X(\mathcal{F}_1, \mathcal{F}_2)).$$

More generally, assume that X is a good connected curve. In this case $\det_X(\mathcal{F}_1, \mathcal{F}_2)$ is not well defined as a line bundle on X . (It is only well defined on the smooth part X^0 of X .) It follows, however, from Lemma 6.6 that the degree of $\det_X(\mathcal{F}_1, \mathcal{F}_2)$ does make sense and we again denote it by $c_2(\mathcal{F}_1, \mathcal{F}_2)$.

Assume now that $G = \text{GL}(n)$. In this case we can think about \mathcal{F}_1 and \mathcal{F}_2 as vector bundles of rank n , and we can define the *relative Euler characteristic* $\chi(\mathcal{F}_1, \mathcal{F}_2)$ in the following way. First, as before, we choose some vector bundle \mathcal{F}_3 on S of rank n containing both \mathcal{F}_1 and \mathcal{F}_2 as locally free subsheaves. We set $\mathcal{Q}_i = \mathcal{F}_3/\mathcal{F}_i$ for $i = 1, 2$. Then both \mathcal{Q}_1 and \mathcal{Q}_2 are set-theoretically supported on X and, therefore, their cohomology is finite-dimensional. We set

$$\chi(\mathcal{F}_1, \mathcal{F}_2) = \chi(\mathcal{Q}_1) - \chi(\mathcal{Q}_2),$$

where $\chi(\mathcal{Q}_i)$ denotes the Euler characteristic of \mathcal{Q}_i . It is clear that the result does not depend on the choice of \mathcal{F}_3 .

PROPOSITION 6.8. *Assume that the triple $(\mathcal{F}_1, \mathcal{F}_2, \alpha)$ is such that $\det(\alpha)$ (which is an isomorphism between $\det \mathcal{F}_1|_{S^0}$ and $\det \mathcal{F}_2|_{S^0}$) extends (as an isomorphism) to the whole of S . Then*

$$\chi(\mathcal{F}_1, \mathcal{F}_2) = c_2(\mathcal{F}_1, \mathcal{F}_2).$$

Proof. Assume first that X is irreducible. Let us choose \mathcal{F}_3 as above. Since both \mathcal{Q}_1 and \mathcal{Q}_2 have a filtration with quotients supported on X , it follows that it makes sense to speak about the rank and the degree of \mathcal{Q}_i on X , which we shall denote by $\text{rk } \mathcal{Q}_i$ and $\text{deg } \mathcal{Q}_i$. Then the fact that $\det(\alpha)$ extends to the whole of X implies that $\text{rk } \mathcal{Q}_1 = \text{rk } \mathcal{Q}_2$. On the other hand, by definition, we have

$$c_2(\mathcal{F}_1, \mathcal{F}_2) = \text{deg } \mathcal{Q}_1 - \text{deg } \mathcal{Q}_2.$$

Thus Proposition 6.8 follows from the Riemann-Roch theorem for the curve X .

In the general case (i.e., when X is not necessarily irreducible) we can find a sequence $\mathcal{F}_0 = \mathcal{F}, \mathcal{F}_1, \dots, \mathcal{F}_m = \mathcal{F}'$ of vector bundles of rank n on S together with the isomorphisms $\alpha_i : \mathcal{F}_i|_{S^0} \xrightarrow{\sim} \mathcal{F}_{i+1}|_{S^0}$ for all $i = 0, \dots, m - 1$ such that

- a) The composition of all the α_i is equal to α .
- b) Each α_i is an isomorphism away from one irreducible component of X .

In this case the above proof shows that for all $i = 0, \dots, m - 1$, we have $c_2(\mathcal{F}_i, \mathcal{F}_{i+1}) = \chi(\mathcal{F}_i, \mathcal{F}_{i+1})$. Since we have

$$c_2(\mathcal{F}, \mathcal{F}') = \sum_{i=0}^{m-1} c_2(\mathcal{F}_i, \mathcal{F}_{i+1}) \quad \text{and} \quad \chi(\mathcal{F}, \mathcal{F}') = \sum_{i=0}^{m-1} \chi(\mathcal{F}_i, \mathcal{F}_{i+1}),$$

the assertion of Proposition 6.8 follows. □

Proposition 6.8 implies the following result, which in some sense explains the notation $c_2(\mathcal{F}, \mathcal{F}_2)$. (This result will not be used in this paper and we leave the proof to the reader.)

LEMMA 6.9. *Assume that S is projective. Assume also that $G = \text{SL}(n)$. (In other words, \mathcal{F}_1 and \mathcal{F}_2 are vector bundles, $\det \mathcal{F}_1$ and $\det \mathcal{F}_2$ are trivialized and α is compatible with these trivializations.) Then*

$$c_2(\mathcal{F}_1, \mathcal{F}_2) = c_2(\mathcal{F}_2) - c_2(\mathcal{F}_1).$$

6.10. *The torsor $\mathcal{T}_{\mathcal{F}}$.* Let now S and X be as above, and let \mathcal{F} be a G -bundle on S^0 . Then to this data we can associate a \mathbb{Z} -torsor $\mathcal{T}_{\mathcal{F}}$, which is uniquely characterized by the following properties:

- 1) Any extension \mathcal{F}_1 of \mathcal{F} to S defines a trivialization of $\kappa_{\mathcal{F}_1}$ of $\mathcal{T}_{\mathcal{F}}$.
- 2) Let \mathcal{F}_1 and \mathcal{F}_2 be two extensions of \mathcal{F} to S . Then the difference between $\kappa_{\mathcal{F}_1}$ and $\kappa_{\mathcal{F}_2}$ is equal to $c_2(\mathcal{F}_1, \mathcal{F}_2)$.

6.11. *The case of disconnected X .* We would like to refine slightly the above definitions. Namely, assume X has r connected components. Then the relative Chern class $c_2(\mathcal{F}_1, \mathcal{F}_2)$ naturally takes values in \mathbb{Z}^r (since it makes sense to talk about $c_2(\mathcal{F}_1, \mathcal{F}_2)$ in the formal neighbourhood of each connected component of X). In particular, in this case we shall denote by $\mathcal{T}_{\mathcal{F}}$ the corresponding \mathbb{Z}^r -torsor.

6.12. Consider the groupoid $\text{Bun}_G(S^0)$. We let $\widetilde{\text{Bun}}_G(S^0)$ denote the “total space of all the $\mathcal{T}_{\mathcal{F}}$ ’s.” (In other words, a point of $\widetilde{\text{Bun}}_G(S^0)$ is given by a G -bundle \mathcal{F} on S^0 together with the trivialization of the torsor $\mathcal{T}_{\mathcal{F}}$.) By definition, we have a natural map $q : \text{Bun}_G(S) \rightarrow \widetilde{\text{Bun}}_G(S^0)$ which is representable.

THEOREM 6.13. *Assume that X can be blown down, i.e., that there exists a (not necessarily smooth) surface Σ with a point $y \in \Sigma$ and a proper birational map $f : S \rightarrow \Sigma$ such that*

- a) X is equal to the set-theoretic preimage of y .
- b) The restriction of f to S^0 is an isomorphism between S^0 and $\Sigma^0 = \Sigma \setminus \{y\}$.

Assume also that G is semi-simple. Then

- (1) Let $\tilde{\mathcal{F}} \in \widetilde{\text{Bun}}_G(S^0)$ and assume that X has r connected components. Then there exists $A \in \mathbb{Z}$ such that for any $\vec{a} = (a_1, \dots, a_r) \in \mathbb{Z}^r$ such that $a_i > A$ for some $i = 1, \dots, r$, we have

$$\vec{a} \cdot \tilde{\mathcal{F}} \notin \text{im}(q).$$

(Here, by $\vec{a} \cdot \tilde{\mathcal{F}}$, we mean the action of \vec{a} on $\tilde{\mathcal{F}}$ as an element of the \mathbb{Z}^r -torsor $\widetilde{\text{Bun}}_G(S^0)$.)

- (2) q has finite fibers.

Remark. When X is smooth this is exactly Theorem 2.2.1 of [8], since in this case our condition is equivalent to the fact that X has negative self-intersection.

Proof. First of all, without loss of generality, we may assume that X is connected. We shall denote by $y \in \Sigma$ the image of X in Σ .

The first assertion of Theorem 6.13 is equivalent to the following statement. Fix some $\mathcal{F} \in \text{Bun}_G(S)$. Then we must show that for all triples

$(\mathcal{F}, \mathcal{F}', \alpha)$ (where $\mathcal{F}' \in \text{Bun}_G(S)$ and α is an isomorphism between the restrictions of \mathcal{F} and \mathcal{F}' to S^0), the set of all possible values of $c_2(\mathcal{F}, \mathcal{F}')$ is bounded above.

For any $\mathcal{F} \in \text{Bun}_G(S)$ and $l \in \mathbb{Z}$, let us consider the set of isomorphism classes of triples $(\mathcal{F}, \mathcal{F}', \alpha)$ as above such that

$$(6.2) \quad c_2(\mathcal{F}, \mathcal{F}') = l.$$

Then the second assertion of Theorem 6.13 says that this set is finite.

Since G is semi-simple, we can choose an embedding $G \hookrightarrow \text{SL}(n)$. Without loss of generality, we may assume that the quadratic form Q comes from this embedding. Then the set of all possible (\mathcal{F}', α) for G embeds into the similar set for $\text{SL}(n)$. Hence we may assume that $G = \text{SL}(n)$. In this case \mathcal{F} and \mathcal{F}' above should be thought of as vector bundles of rank n with trivial determinant.

Let us first concentrate on Theorem 6.13(2). Then

$$\chi(\mathcal{F}, \mathcal{F}') = c_2(\mathcal{F}, \mathcal{F}') = l.$$

The proof of the required finiteness is based on the following result.

LEMMA 6.14. *There exists a locally free subsheaf \mathcal{E} of \mathcal{F} , which is equal to \mathcal{F} on S^0 and such that for any \mathcal{F}' as above, the resulting identification $\mathcal{E}|_{S^0} \simeq \mathcal{F}'|_{S^0}$ extends to an embedding $\mathcal{E} \hookrightarrow \mathcal{F}'$.*

Proof. Let j denote the (open) embedding of S^0 into Σ . Then we can consider the sheaf $\mathcal{G} = j_*(\mathcal{F}|_{S^0})$ on Σ . Since the complement of S^0 in Σ consists of one point, it follows that this sheaf is coherent. Also, for any \mathcal{F}' as above, we have the natural embedding (of coherent sheaves) $f_*\mathcal{F}' \hookrightarrow \mathcal{G}$ which is an isomorphism away from y . Hence there exists some $N > 0$ such that $f_*\mathcal{F}'$ contains $\mathfrak{m}_{\Sigma,y}^N \mathcal{G}$, where $\mathfrak{m}_{\Sigma,y}$ denotes the maximal ideal of the point y in Σ .

We claim now that the number N as above can be chosen uniformly for all \mathcal{F}' satisfying (6.2). To prove this let us define

$$\eta(\mathcal{F}') = \text{length}(\text{coker}(f_*\mathcal{F}' \rightarrow \mathcal{G})) + \text{length}(R^1 f_*\mathcal{F}').$$

Then it is clear that $l = \chi(\mathcal{F}, \mathcal{F}') = \eta(\mathcal{F}) - \eta(\mathcal{F}')$. Thus

$$\text{length}(\text{coker}(f_*\mathcal{F}' \rightarrow \mathcal{G})) = \eta(\mathcal{F}) - \text{length}(R^1 f_*(\mathcal{F}')) - l.$$

Hence $\text{length}(\text{coker}(f_*\mathcal{F}' \rightarrow \mathcal{G})) \leq \eta(\mathcal{F}) - l$. Hence (by Nakayama lemma) if we choose a number N such that $\text{length}(\mathcal{G}/\mathfrak{m}_{\Sigma,y}^N \mathcal{G}) \geq \eta(\mathcal{F}) - l$, then the sheaf $f_*\mathcal{F}'$ must contain $\mathfrak{m}_{\Sigma,y}^N \mathcal{G}$.

Let now $\mathcal{H} = \mathfrak{m}_{\Sigma,y}^N \mathcal{G}$. This a coherent sheaf on Σ . The embedding $\mathcal{H} \hookrightarrow f_*\mathcal{F}'$ gives rise to a morphism $f^*\mathcal{H} \rightarrow \mathcal{F}'$, which is an isomorphism on S^0 . Let \mathcal{E}' denote the quotient of $f^*\mathcal{H}$ by torsion. This is a torsion-free coherent sheaf on S . Let \mathcal{E} denote its saturation (i.e., minimal locally free sheaf, containing \mathcal{E}'). Then the morphism $f^*\mathcal{H} \rightarrow \mathcal{F}'$ gives rise to an embedding $\mathcal{E} \hookrightarrow \mathcal{F}'$ of locally free sheaves, which is an isomorphism on S^0 . \square

Let us now explain why Lemma 6.14 implies Theorem 6.13(1). Without loss of generality, we may assume that $\mathcal{E} = \mathcal{F}(N \cdot X)$. Applying the same argument to the dual of \mathcal{F} we see that we may also assume that any \mathcal{F}' as above is contained in $\mathcal{F}(-N \cdot X)$. Let $\mathcal{Q} = \mathcal{F}(-N \cdot X)/\mathcal{F}(N \cdot X)$. This is a coherent sheaf on S , which is set-theoretically supported on X . Any \mathcal{F}' as above is uniquely determined by its image $\mathcal{F}'_{\mathcal{Q}} = \mathcal{F}'/\mathcal{F}(-N \cdot X)$ in \mathcal{Q} . Moreover, the Euler characteristic $\chi(\mathcal{F}'_{\mathcal{Q}})$ is independent of \mathcal{F}' . (It is equal to $\chi(\mathcal{F}, \mathcal{F}') + \chi(\mathcal{F}(N \cdot X), \mathcal{F}) = l + \chi(\mathcal{F}(N \cdot X), \mathcal{F})$.)

Hence it is enough for us to show that there exists a scheme of finite type over k whose k points are subsheaves of \mathcal{Q} with given Euler characteristic. (Indeed, since k is finite, the set of k -points of a scheme of finite type is finite.) Without loss of generality, we can assume that S is projective. We shall choose an embedding of S into some projective space. In other words, we are going to choose some very ample line bundle \mathcal{L} on S . Thus \mathcal{Q} is a coherent sheaf on a projective scheme S ; hence, according to [6], there exists a scheme $\text{Quot}(\mathcal{Q})$ whose k -points are subsheaves \mathcal{G} of \mathcal{Q} . Moreover, the subscheme of $\text{Quot}(\mathcal{Q})$ corresponding to looking at \mathcal{G} as above with fixed Hilbert polynomial is of finite type over k . For every irreducible component X_i of X we can define the generic rank $\text{rk}_i(\mathcal{G})$ of \mathcal{G} as follows. First, we can find a filtration $0 \subset \mathcal{G}_1 \subset \dots \subset \mathcal{G}_s = \mathcal{G}$ of \mathcal{G} such that every quotient $\mathcal{G}_j/\mathcal{G}_{j-1}$ is scheme-theoretically concentrated on X . Then we define $\text{rk}_i(\mathcal{G}) = \sum_j \text{rk}_i(\mathcal{G}_j/\mathcal{G}_{j-1})$, where $\text{rk}_i(\mathcal{G}_j/\mathcal{G}_{j-1})$ is the rank of the restriction of $\mathcal{G}_j/\mathcal{G}_{j-1}$ to the generic point of X_i . It is clear that $0 \leq \text{rk}_i(\mathcal{G}) \leq \text{rk}_i(\mathcal{Q})$; hence there are finitely many possibilities for each of the numbers $\text{rk}_i(\mathcal{G})$. Thus to finish the proof, it is enough to show the following.

LEMMA 6.15. *The Hilbert polynomial of \mathcal{F} is uniquely determined by $\chi(\mathcal{G})$ and by the numbers $\text{rk}_i(\mathcal{G})$.*

Proof. For each X_i as above, let us set $\mathcal{L}_i = \mathcal{L}|_{X_i}$. Let $d_i = \deg_{X_i} \mathcal{L}_i$. Also, let us choose a filtration \mathcal{G}_j of \mathcal{G} as above such that every quotient $\mathcal{G}_j/\mathcal{G}_{j-1}$ is concentrated scheme-theoretically on some irreducible component X_{i_j} of X . We shall denote by \mathcal{S}_j the corresponding coherent sheaf on X_{i_j} . By definition, the Hilbert polynomial of \mathcal{G} is determined by the numbers $\dim H^0(\mathcal{G} \otimes \mathcal{L}^{\otimes n})$ for n sufficiently large. However, if n is large, then $H^p(\mathcal{G} \otimes \mathcal{L}^{\otimes n}) = 0$ for $p > 0$, and thus we have

$$\begin{aligned} \dim H^0(\mathcal{G} \otimes \mathcal{L}^{\otimes n}) &= \chi(\mathcal{G} \otimes \mathcal{L}^{\otimes n}) = \sum_j \chi(\mathcal{S}_j \otimes \mathcal{L}_{i_j}^{\otimes n}) \\ &= \sum_j \chi(\mathcal{S}_j) + \text{rk}(\mathcal{S}_j) \cdot n \cdot d_{i_j} = \chi(\mathcal{G}) + \sum_i \text{rk}_i(\mathcal{G}) \cdot n \cdot d_i, \end{aligned}$$

which finishes the proof. □

Remark. The same proof works for arbitrary reductive G if we fix the $G/[G, G]$ -bundle obtained from \mathcal{F}' by push-forward with respect to the natural map $G \rightarrow G/[G, G]$.

Let us now prove Theorem 6.13(1). We are going to use the notation introduced in the proof of Lemma 6.14. We need to show that $\chi(\mathcal{F}, \mathcal{F}')$ is bounded above. However, we have

$$\chi(\mathcal{F}, \mathcal{F}') = \eta(\mathcal{F}) - \eta(\mathcal{F}').$$

By definition, $\eta(\mathcal{F}') \geq 0$. Hence $\chi(\mathcal{F}, \mathcal{F}') \leq \eta(\mathcal{F})$. □

7. Turning on the central extension

7.1. We now want to generalize Proposition 5.3 to the case where the group G'_{aff} is replaced by its central extension using the results of the previous section. For this, we have to first interpret the \mathbb{Z} -torsor $G_{\text{aff}}(\mathcal{O}) \backslash \tilde{\pi}^{-1}(k) / G_{\text{aff}}(\mathcal{O})$ over $G'_{\text{aff}}(\mathcal{O}) \backslash \pi^{-1}(k) / G'_{\text{aff}}(\mathcal{O})$ in geometric terms. To do this we are going to apply the constructions of the previous Section to $S = \tilde{S}_k$, where the role of X will be played by the exceptional fiber of the morphism $\tilde{S}_k \rightarrow S_k$ which we shall denote by E . (We want to reserve the notation X for something else.) Note that in this case $S^0 = S_k^0$. We shall write $\widetilde{\text{Bun}}_G(S_k^0)$ instead of writing $\widetilde{\text{Bun}}_G(S^0)$.

PROPOSITION 7.2. *The groupoid $G_{\text{aff}}(\mathcal{O}) \backslash \tilde{\pi}^{-1}(k) / G_{\text{aff}}(\mathcal{O})$ is canonically equivalent to the groupoid $\widetilde{\text{Bun}}_G(S_k^0)$.*

Proof. Consider the quotient $\tilde{G}(\mathcal{K}) / \mathcal{O}^*$, where we take the quotient by the central $\mathcal{O}^* \subset \mathcal{K}^*$. This is a central extension of $G(\mathcal{K}[t, t^{-1}])$ by \mathbb{Z} . In particular, it defines a \mathbb{Z} -torsor over every $g(t, s) \in G(\mathcal{K})$, which we shall denote by \mathcal{T}_g . By definition, this torsor can be described as follows.

Consider the plane \mathbb{A}^2 with coordinates t and s . Let $C \subset \mathbb{A}^2$ be given by the equation $s = 0$, and let $D \subset \mathbb{A}^2$ be given by the equation $t = 0$. Let $y \in \mathbb{A}^2$ be the point $t = 0, s = 0$. Set C^0 (resp. D^0) to be the complement of y in C (resp. D). Let now $\mathcal{M}_1, \mathcal{M}_2$, be two trivial bundles on \mathbb{A}^2 . Let ξ be the isomorphism between their restrictions on $\mathbb{G}_m \times \mathbb{G}_m$ given by g . (ξ is the natural isomorphism between two trivial bundles on $\mathbb{G}_m \times \mathbb{G}_m$ multiplied by $g(t, s)$.) Consider $\det_{D^0}(\mathcal{M}_1, \mathcal{M}_2)$; this is a line bundle on D^0 (which is isomorphic to \mathbb{G}_m with coordinate s). Then by definition, the \mathbb{Z} -torsor \mathcal{T}_g is the torsor of extensions of $\det_{D^0}(\mathcal{M}_1, \mathcal{M}_2)$ from D^0 to D . On the other hand, by Lemma 6.6 this torsor is canonically isomorphic to the inverse of the torsor of extensions of $\det_{C^0}(\mathcal{M}_1, \mathcal{M}_2)$ to C .

On the other hand, given $g(t, s)$ as above, we can construct a G -bundle \mathcal{F} on S_k^0 together with trivializations on U_k and V_k as in the proof of Proposition 5.3. What we need to do is construct a trivialization of \mathcal{T}_g starting from

a trivialization of $\mathcal{T}_{\mathcal{F}}$. (Recall that we consider S_k^0 as an open subset of \tilde{S}_k and the torsor $\mathcal{T}_{\mathcal{F}}$ is constructed via this embedding.) In other words, we need to start with an extension \mathcal{F}' of \mathcal{F} to \tilde{S}_k and construct a trivialization of \mathcal{T}_g . (The fact that this construction gives rise to an isomorphism $\mathcal{T}_{\mathcal{F}} \simeq \mathcal{T}_g$ will eventually be obvious.)

Let $X \subset S_k$ be given by the equation $x = 0$ in S_k . Let $X^0 = X \setminus \{0, 0, 0\} = S_k^0 \cap X$. Let α denote the trivialization of \mathcal{F} on $U_k = \text{Spec } k[x, x^{-1}, y]$. Let us also identify X^0 with C^0 by setting $y = t^{-1}$. Then it is easy to see that $\det_{C^0}(\mathcal{M}_1, \mathcal{M}_2) = \det_{X^0}(\alpha)$. Thus the torsor of extensions of $\det_{C^0}(\mathcal{M}_1, \mathcal{M}_2)$ to 0 is canonically isomorphic to the torsor of extensions of $\det_{X^0}(\alpha)$ to ∞ . The latter torsor is inverse to the torsor of extensions of $\det_{X^0}(\alpha)$ to 0. In other words, we get the canonical isomorphism between \mathcal{T}_g and the torsor of extensions of $\det_{X^0}(\alpha)$ to 0.

We claim now that an extension \mathcal{F}' of \mathcal{F} to \tilde{S}_k as above provides a canonical extension of $\det_{X^0}(\alpha)$ to the whole of X . Indeed, let us identify X with its proper preimage in \tilde{S}_k , and let also E denote the exceptional divisor in S_k . Then E and X intersect at a unique point p in \tilde{S}_k , and $Y = E \cup X$ is a good curve in \tilde{S}_k . We set $E^0 = E \setminus \{p\}$. Let \mathcal{F}^0 denote the trivial bundle on \tilde{S}_k . We may regard α as an isomorphism between $\mathcal{F}'|_{S_k^0}$ and $\mathcal{F}^0|_{S_k^0}$. Then by Lemma 6.6 the torsor of extensions of $\det_{X^0}(\alpha)$ to X is inverse to the torsor of extensions of $\det_{E^0}(\mathcal{F}', \mathcal{F}^0, \alpha)$ to E . However, since E is proper, by Section 6.4 the latter torsor acquires a canonical trivialization and hence the same is true for the former. \square

7.3. Geometric interpretation of the convolution diagram. We now want to generalize the results of Section 5.5 in order to give a geometric interpretation of the convolution diagram in the presence of a central extension. Namely, let us choose as before two positive integers k and l . Recall that in this case we have the partial resolution $\mathfrak{p}_{k,l} : S_{k,l} \rightarrow S_{k+l}$. Also we have the full resolution $\mathfrak{p}_{1,\dots,1} : \tilde{S}_{k+l} = S_{1,\dots,1} \rightarrow S_{k+l}$ which factorizes through a map $\mathfrak{r} : \tilde{S}_{k+l} \rightarrow S_{k,l}$ composed with $\mathfrak{p}_{k,l}$.

The map \mathfrak{r} is an isomorphism over $S_{k,l}^0$. Note that the complement of $S_{k,l}^0$ in \tilde{S}_{k+l} is disconnected. (It has two connected components, which correspond to the two singular points of $S_{k,l}$.) Hence the groupoid $\widetilde{\text{Bun}}_G(S_{k,l}^0)$ (here we view $S_{k,l}^0$ as a complement to a divisor in the smooth surface \tilde{S}_{k+l}) is $\mathbb{Z} \times \mathbb{Z}$ -torsor over $\text{Bun}_G(S_{k,l}^0)$. Since $S_{k,l}$ contains S_{k+l} as an open subset, it follows that the restriction map $\text{Bun}_G(S_{k,l}^0) \rightarrow \text{Bun}_G(S_{k+l}^0)$ extends to a map $\widetilde{\text{Bun}}_G(S_{k,l}^0) \rightarrow \widetilde{\text{Bun}}_G(S_{k+l}^0)$. This map is $\mathbb{Z} \times \mathbb{Z}$ -equivariant, where $\mathbb{Z} \times \mathbb{Z}$ acts naturally on the left-hand side and on the right-hand side it acts by means of the homomorphism $\mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$ sending (a, b) to $a + b$.

PROPOSITION 7.4. *The groupoid*

$$(7.1) \quad G_{\text{aff}}(\mathcal{O}) \backslash \tilde{\pi}^{-1}(k) \times_{G_{\text{aff}}(\mathcal{O})} \tilde{\pi}^{-1}(l) / G_{\text{aff}}(\mathcal{O})$$

can be canonically identified with $\widetilde{\text{Bun}}_G(S_{k,l}^0)$. Under this identification the natural $\mathbb{Z} \times \mathbb{Z}$ -action on (7.1) corresponds to the $\mathbb{Z} \times \mathbb{Z}$ -action on $\widetilde{\text{Bun}}_G(S_{k,l}^0)$.

Moreover, under this identification the multiplication map

$$G_{\text{aff}}(\mathcal{O}) \backslash \tilde{\pi}^{-1}(k) \times_{G_{\text{aff}}(\mathcal{O})} \tilde{\pi}^{-1}(l) / G_{\text{aff}}(\mathcal{O}) \rightarrow G_{\text{aff}}(\mathcal{O}) \backslash \tilde{\pi}^{-1}(s^{k+l}) / G_{\text{aff}}(\mathcal{O})$$

corresponds to the map $\widetilde{\text{Bun}}_G(S_{k,l}^0) \rightarrow \widetilde{\text{Bun}}_G(S_{k+l}^0)$ discussed above.

The proof of Proposition 7.4 is essentially the same as the proof of Proposition 7.2 and we leave it to the reader.

7.5. *Proof of Theorem 4.6(1).* We now claim that Proposition 7.4 together with Theorem 6.13 imply Theorem 4.6(1). First of all, this is clear if G is semi-simple. In this case condition 1') of Section 2 follows immediately from Theorem 6.13(1) and condition 2) follows from Theorem 6.13(2). Also, when G is a torus, Theorem 4.6(1) follows from Section 3. Hence it is easy to see that Theorem 4.6(1) holds for any reductive group G' which is isomorphic to a product of a semi-simple group and a torus. Also assume that we are given an isogeny $G \rightarrow G'$ of split reductive groups. Then it is easy to see that Theorem 4.6(1) for G' implies Theorem 4.6(1) for G . By taking G' to be the product of the adjoint group of G and of $G/[G, G]$, we see that Theorem 4.6(1) is true for any G .

8. Action on the principal series and proof of Theorem 4.6(2)

8.1. *Principal series.* Consider first the subgroup $\tilde{\Delta}'$ of $\tilde{\Gamma}$ which is equal to $\mathcal{O}^* \times G(\mathcal{K}[t]) \rtimes \mathcal{O}^*$. We shall denote by Δ' its image in Γ .

Now we define subgroups $\Delta \subset \Delta'$ and $\tilde{\Delta} \subset \tilde{\Delta}'$. First of all note that $\tilde{\Delta}'$ maps naturally to $\mathcal{O}^* \times G(\mathcal{K}) \times \mathcal{O}^*$ (by setting $t = 0$). Let B be a Borel subgroup of G . It has the decomposition $B = TU$, where T is a maximal torus in G and U is the unipotent radical of B . We set $\tilde{\Delta}$ to be the preimage of $\mathcal{O}^* \times T(\mathcal{O})U(\mathcal{K}) \times \mathcal{O}^*$ in $\tilde{\Delta}'$. We denote by Δ the image of $\tilde{\Delta}$ in Γ .

We now set

$$\Omega = \Gamma / \Delta, \quad \tilde{\Omega} = \tilde{\Gamma} / \tilde{\Delta}, \quad \Omega' = \Gamma / \Delta', \quad \tilde{\Omega}' = \tilde{\Gamma} / \tilde{\Delta}'.$$

Clearly, Γ acts on Ω and Ω' , and $\tilde{\Gamma}$ acts on $\tilde{\Omega}$ and $\tilde{\Omega}'$. In addition, we have the natural map $\varpi : \tilde{\Omega} \rightarrow \tilde{\Omega}'$, which is $\tilde{\Gamma}$ -equivariant.

THEOREM 8.2. *The action of $\tilde{\Gamma}^+$ on $\tilde{\Omega}$ is good with respect to $\tilde{\Omega}'$.*

As before, it is enough to prove Theorem 8.2 in the case $Q = Q_V$ for some representation V ; so we are going to assume this until the end of Section 8.9.

8.3. *The structure of Ω .* Before we go to the proof of Theorem 8.2, we would like to discuss how $\tilde{\Omega}$ and $\tilde{\Omega}'$ look. First of all, let us introduce another pair of groups $(\Delta'', \tilde{\Delta}'')$ such that $\tilde{\Delta} \subset \Delta'' \subset \Delta'$ (resp. $\tilde{\Delta} \subset \tilde{\Delta}'' \subset \tilde{\Delta}'$). These are defined exactly as Δ and $\tilde{\Delta}$ except that the group $T(\mathcal{O})U(\mathcal{K})$, used in the definition, has to be replaced by $B(\mathcal{K})$. It is easy to see that Δ is normal in Δ'' (resp. $\tilde{\Delta}$ is normal in $\tilde{\Delta}''$) and we have the natural isomorphism $\Delta''/\Delta = \tilde{\Delta}''/\tilde{\Delta} = T(\mathcal{K})/T(\mathcal{O}) \simeq \Lambda$. Thus Λ acts on Ω on the right. In addition, the group \mathbb{Z} acts on Ω . (This action comes from the center of Γ .) Altogether, we get an action of $\mathbb{Z} \times \Lambda = \Lambda'_{\text{aff}}$ on Ω , and this action commutes with the action of Γ . In particular, Λ'_{aff} acts on $\Gamma_0 \backslash \Omega$. Similarly, the lattice $\Lambda_{\text{aff}} = \mathbb{Z} \times \Lambda \times \mathbb{Z}$ acts on $\tilde{\Omega}$ and this action commutes with the action of $\tilde{\Gamma}$. In particular, Λ_{aff} acts on $\tilde{\Gamma}_0 \backslash \tilde{\Omega}$. The following lemma follows easily from [9, Prop. 1.4.5].

- LEMMA 8.4. (1) *The action of Λ'_{aff} on $\Gamma_0 \backslash \Omega$ is simply transitive. Similarly, the action of Λ_{aff} on $\tilde{\Gamma}_0 \backslash \tilde{\Omega}$ is simply transitive.*
 (2) *The action of $\mathbb{Z} \times \mathbb{Z}$ on $\tilde{\Gamma}_0 \backslash \tilde{\Omega}'$ is simply transitive.*
 (3) *Let us identify $\Gamma_0 \backslash \tilde{\Omega}$ with $\Lambda_{\text{aff}} = \mathbb{Z} \times \Lambda \times \mathbb{Z}$ by acting on the unit element. Similarly, let us identify $\tilde{\Gamma}_0 \backslash \tilde{\Omega}'$ with $\mathbb{Z} \times \mathbb{Z}$. Then the natural map $\tilde{\Gamma}_0 \backslash \tilde{\Omega} \rightarrow \tilde{\Gamma}_0 \backslash \tilde{\Omega}'$ induced by $\varpi : \tilde{\Omega} \rightarrow \tilde{\Omega}'$ corresponds to the projection onto the first and the third factor.*

Lemma 8.4 implies that $\mathcal{F}^{\text{fin}}(\tilde{\Omega}) \simeq \mathbb{C}[\Lambda_{\text{aff}}]$. Similarly, $\mathcal{F}_{\tilde{\Omega}'}(\tilde{\Omega})$ is the v -adic completion of $\mathbb{C}[\Lambda_{\text{aff}}]$ (here as before the action of v on the right-hand side comes from the “central” copy of \mathbb{Z} in Λ_{aff}), which consists of all functions $f(k, \lambda, n)$ such that

- 1) $f(k, \lambda, n) = 0$ for $k \ll 0$.
- 2) For almost all n , we have $f(k, \lambda, n) = 0$ for all λ, k .
- 3) For given k , the function $f(k, \lambda, n)$ is not equal to 0 only for finitely many pairs (λ, n) .

8.5. *$\tilde{\Omega}'$ is almost good.* Let us now move to the proof of Theorem 8.2. By definition we first have to prove that the action of $\tilde{\Gamma}$ on $\tilde{\Omega}'$ is almost good. The proof is similar to that of Theorem 4.6. Namely, we are going to give a geometric interpretation of the convolution diagram for the action of $\tilde{\Gamma}$ on $\tilde{\Omega}'$. More precisely, for any $l \in \mathbb{Z}$, let $\tilde{\Omega}'_l = \tilde{\pi}^{-1}(l)/\tilde{\Delta}'$ (resp. $\Omega'_l = \pi^{-1}(l)/\Delta'$). Then for any $k > 0$, we want to give a a geometric interpretation of the morphism (of groupoids)

$$\tilde{\Gamma}_0 \backslash \tilde{\Omega}'_l \leftarrow \tilde{\Gamma}_0 \backslash \tilde{\pi}^{-1}(k) \times_{\tilde{\Gamma}_0} \tilde{\Omega}'_l \rightarrow \Gamma_0 \backslash \tilde{\Omega}'_{k+l}.$$

First let us do this for Γ and Ω' . In other words, we want to give a geometric interpretation of the morphism (of groupoids)

$$\Gamma_0 \backslash \Omega'_l \leftarrow \Gamma_0 \backslash \pi^{-1}(k) \times_{\Gamma_0} \Omega'_l \rightarrow \Gamma_0 \backslash \Omega'_{k+l}.$$

First, we need some preparatory material.

8.6. *Blow ups.* Let Σ be a smooth surface, and let Y be a smooth curve in Σ . Let x be some point in Y . Set $\Sigma^0 = \Sigma \setminus \{x\}$. Let \mathcal{O}_Σ (resp. \mathcal{O}_Y) denote the structure sheaf of S (resp. of Y), and let $\mathfrak{m}_{Y,x}$ be the ideal sheaf of the point x in \mathcal{O}_Y . Abusing the notation, we shall regard \mathcal{O}_Y as a sheaf on S with support on Y . For each $k > 0$, let \mathcal{J}_k be the preimage of $\mathfrak{m}_{Y,y}^k$ under the natural map $\mathcal{O}_\Sigma \rightarrow \mathcal{O}_Y$. We shall denote by $\text{Bl}_{x,k}(\Sigma)$ the blow up of Σ at the ideal \mathcal{J}_k . This is a new surface which has unique singular point y of type A_k .⁵ It is endowed with a proper birational map $f : \text{Bl}_{x,k}(\Sigma) \rightarrow \Sigma$, which is an isomorphism away from x and whose fiber over x is isomorphic to \mathbb{P}^1 . We shall denote by $\text{Bl}_{x,k}(\Sigma)^0$ the complement to the point y in $\text{Bl}_{x,k}(\Sigma)$. Also we shall identify the proper preimage of Y in $\text{Bl}_{x,k}(\Sigma)$ with Y . Let $\Sigma' = \text{Bl}_{x,k}(\Sigma) \setminus Y$. Thus we have the natural morphism of groupoids $\mathfrak{q} : \text{Bun}_G(\text{Bl}_{x,k}(\Sigma)^0) \rightarrow \text{Bun}_G(\Sigma)$ and $\mathfrak{p} : \text{Bun}_G(\text{Bl}_{x,k}(\Sigma)^0) \rightarrow \text{Bun}_G(\Sigma')$. (The first morphism is obtained by restricting a G -bundle to Σ^0 and then taking the unique extension to Σ ; the second morphism is obtained just by restriction to Σ' .)

The surface $\text{Bl}_{x,k}(\Sigma)$ admits canonical resolution of singularities $\widetilde{\text{Bl}}_{x,k}(\Sigma)$. This is a smooth surface, containing $\text{Bl}_{x,k}(\Sigma)^0$ as an open subset. Moreover, the complement to $\text{Bl}_{x,k}(\Sigma)^0$ in $\widetilde{\text{Bl}}_{x,k}(\Sigma)$ is a good connected projective curve. Thus it makes sense to consider $\widetilde{\text{Bun}}_G(\text{Bl}_{x,k}(\Sigma)^0)$. We have canonical morphism $\widetilde{\mathfrak{q}} : \widetilde{\text{Bun}}_G(\text{Bl}_{x,k}(\Sigma)^0) \rightarrow \text{Bun}_G(\Sigma) \times \mathbb{Z}$ (which modulo \mathbb{Z} give rise to \mathfrak{q}) constructed as follows. Let us pick up an element in $\widetilde{\text{Bun}}_G(\text{Bl}_{x,k}(\Sigma))$. We may assume that it comes from a G -bundle $\widetilde{\mathcal{F}}$ on $\widetilde{\text{Bl}}_{x,k}(\Sigma)$. To describe $\widetilde{\mathfrak{q}}(\widetilde{\mathcal{F}})$ we only need to describe its projection to \mathbb{Z} . Consider $\mathfrak{q}(\widetilde{\mathcal{F}}|_{\text{Bl}_{x,k}(\Sigma)^0})$. Let us denote by \mathcal{F}' its lift to $\widetilde{\text{Bl}}_{x,k}(\Sigma)$. Then the desired integer is $c_2(\mathcal{F}, \mathcal{F}')$.

We shall also denote by $\widetilde{\mathfrak{p}} : \widetilde{\text{Bun}}_G(\text{Bl}_{x,k}(\Sigma)^0) \rightarrow \text{Bun}_G(\Sigma')$ the composition of the projection $\widetilde{\text{Bun}}_G(\text{Bl}_{x,k}(\Sigma)^0) \rightarrow \text{Bun}_G(\text{Bl}_{x,k}(\Sigma)^0)$ with \mathfrak{p} .

Here is an example of such a situation. Let X be another smooth curve, and let \mathcal{L} be a line bundle over X . Let also x be some point of X . We want to consider the above construction in the case when $S = \mathcal{L}$ and $Y = \mathcal{L}_x$ — the fiber of \mathcal{L} over x . In this case it is easy to see that $\text{Bl}_{x,k}(\Sigma) \setminus Y$ is naturally isomorphic to $\mathcal{L}(-kx)$.

⁵In fact, if $k = 1$, then this point is smooth; in this case $\text{Bl}_{x,k}(\Sigma)$ is the usual blow up of S at the point x and the point y corresponds to the tangent space to Y at x .

Let us now consider the case $X = \mathbb{A}^1$. In this case the bundle \mathcal{L} is automatically trivial and thus we can identify Σ with \mathbb{A}^2 with coordinates (p, s) , where the point x corresponds to $(0, 0)$ and the curve Y is given by the equation $s = 0$. In this case the surface $\text{Bl}_{x,k}(\Sigma)$ can be described as follows. Consider two surfaces Σ_1 and Σ_2 , where $\Sigma_1 = \text{Spec } k[p, q, s]/pq - s^k$ and $\Sigma_2 = \text{Spec } k[u, s]$. (Here p, q and u are some additional variables.) Let us identify the open subset of Σ_1 , given by the equation $q \neq 0$, with the open subset of Σ_2 , given by the equation $u \neq 0$ by setting $u = q^{-1}$. Then $\text{Bl}_{x,k}(\Sigma)$ is obtained by gluing Σ_1 and Σ_2 along this common open subset. The corresponding map $\text{Bl}_{x,k}(\Sigma) \rightarrow \Sigma$ is obvious. On Σ_1 it is given by $(p, q, s) \mapsto (p, s)$, and on Σ_2 it is given by $(u, s) \mapsto (us^k, s)$. The proper preimage of Y lies inside Σ_1 and is given there by the equation $q = 0$. Thus in this case $\Sigma' = \Sigma_2$ and it can be naturally identified with \mathbb{A}^2 with coordinates (u, s) .

In what follows we shall only work with Σ, Σ' , in this case. We shall also set $\Sigma_1^0 = \Sigma_1 \cap \text{Bl}_{x,k}(\Sigma)^0$. We have natural isomorphisms $\Sigma_1 \simeq S_k$ and $\Sigma_1^0 \simeq S_k^0$.

Now we claim the following proposition.

PROPOSITION 8.7. *The diagram of groupoids*

$$\Gamma_0 \backslash \Omega'_l \leftarrow \Gamma_0 \backslash \pi^{-1}(k) \times_{\Gamma_0} \Omega'_l \rightarrow \Gamma_0 \backslash \Omega'_{k+l}$$

is equivalent to the diagram

$$\text{Bun}_G(\Sigma') \xleftarrow{p} \text{Bun}_G(\text{Bl}_{x,k}(\Sigma)^0) \xrightarrow{q} \text{Bun}_G(\Sigma).$$

Similarly, the diagram

$$\tilde{\Gamma}_0 \backslash \tilde{\Omega}'_l \leftarrow \tilde{\pi}^{-1}(k) \times_{\tilde{\Gamma}_0} \tilde{\Omega}'_l \rightarrow \tilde{\Gamma}_0 \backslash \tilde{\Omega}'_{k+l}$$

is equivalent to

$$\text{Bun}_G(\Sigma') \times \mathbb{Z} \xleftarrow{p \times \text{id}} \widetilde{\text{Bun}}_G(\text{Bl}_{x,k}(\Sigma)^0) \times \mathbb{Z} \xrightarrow{\tilde{q} + \text{id}} \text{Bun}_G(\Sigma) \times \mathbb{Z}.$$

(Here, by $\tilde{q} + \text{id}$, we mean the map which sends a pair $(\tilde{\mathcal{F}}, a) \in \widetilde{\text{Bun}}_G(\text{Bl}_{x,k}(\Sigma)^0) \times \mathbb{Z}$, such that $\tilde{q}(\tilde{\mathcal{F}}) = (\mathcal{G}, b)$, to $(\mathcal{G}, a + b)$.)

Proof. The proof is very similar to the proof of Propositions 5.6 and 7.4. First, for any $n \in \mathbb{Z}$, let us consider the quotient

$$\Gamma_0 \backslash \Omega'_n = G[ts^n, t^{-1}s^{-n}, s] \backslash G[t, t^{-1}, s, s^{-1}] / G[t, s, s^{-1}].$$

We claim that (as a groupoid) it can naturally be identified with $\text{Bun}_G(\mathbb{A}^2)$ with coordinates (u, s) , where $u = ts^n$. The proof is exactly the same as the proof of the bijection (1) \Leftrightarrow (2) from Proposition 5.3.

Next, we claim that the \mathbb{Z} -torsor $\tilde{\Gamma}_0 \backslash \tilde{\Omega}'_n \rightarrow \Gamma \backslash \Omega'_n$ is canonically trivialized. In other words, we get an equivalence

$$\tilde{\Gamma}_0 \backslash \tilde{\Omega}'_n \simeq \text{Bun}_G(\Sigma) \times \mathbb{Z}.$$

This would have been obvious if we were dealing with sets, rather than with groupoids, since the set of isomorphism classes of points of $\Gamma \backslash \Omega'_n$ consists of one element. In order to construct the required trivialization on the level of groupoids we must prove the following result.

LEMMA 8.8. *The group of automorphisms of the unique point of $\Gamma \backslash \Omega'_n$ acts trivially on \mathbb{Z} .*

Proof. Note that by definition, this group is equal to $\Delta' = G(\mathcal{K}[t]) \times \mathcal{O}^*$. It is easy to see that it is enough to prove this separately in the case when G is a torus and when G is semi-simple. If G is a torus, then our statement follows from the calculations of Section 3. When G is semi-simple, then Δ' is actually the group of k -points of a connected group ind-scheme $\mathcal{G}_{\Delta'}$ over k and it is clear that the above homomorphism from $\Delta' \rightarrow \mathbb{Z}$ comes from an algebraic homomorphism $\mathcal{G}_{\Delta'} \rightarrow \mathbb{Z}$, which has to be trivial since $\mathcal{G}_{\Delta'}$ is connected. \square

Let us now recall that $\text{Bl}_{x,k}(\Sigma)^0$ is glued from $\Sigma_1^0 \simeq S_k^0$ and $\Sigma_2 \simeq \mathbb{A}^2$. Thus setting $p = ts^{k+n}, q = ts^{-n}$, and arguing as in the proof of Proposition 5.6, we get the the equivalence

$$\Gamma_0 \backslash \pi^{-1}(k) \times_{\Gamma_0} \Omega'_l \simeq \text{Bun}_G(\text{Bl}_{x,k}(\Sigma)^0).$$

The rest of the proof is essentially a word-by-word repetition of the proof of Propositions 5.6 and 7.4. It is left to the reader. \square

8.9. *Proof of Theorem 8.2.* We can now prove Theorem 8.2. Indeed, the fact that the action of $\tilde{\Gamma}$ on $\tilde{\Omega}'$ is good follows from Proposition 8.7 together with Theorem 6.13 (as in the proof of Theorem 4.6(1)). What is left to show is property (i) from Section 2.7. For this we need some additional geometric construction.

Recall that Σ is a (trivial) line bundle over $X \simeq \mathbb{A}^1$. In particular, X is embedded into Σ . In coordinates (p, s) it is given by the equation $s = 0$.

Let now $\text{Bun}_{G,\Delta}(\Sigma)$ denote the groupoid of G -bundles \mathcal{F} on Σ endowed with the following additional data:

- 1) A reduction of $\mathcal{F}|_{X^0=X \setminus \{x\}}$ to B . We shall denote the corresponding B -bundle on X^0 by \mathcal{F}_B^0 . We shall also denote by \mathcal{F}_T^0 the corresponding T -bundle.
- 2) An extension \mathcal{F}'_T of \mathcal{F}_T^0 to X .

Then arguing as in the previous subsection, for any $n \in \mathbb{Z}$, we can construct an equivalence of groupoids

$$\Gamma_0 \backslash \Omega_n \simeq \text{Bun}_{G,\Delta}(\Sigma), \quad \tilde{\Gamma}_0 \backslash \tilde{\Omega}_n \simeq \text{Bun}_{G,\Delta}(\Sigma) \times \mathbb{Z}.$$

Recall that the set of isomorphism classes of the groupoid $\Gamma_0 \backslash \Omega_n$ is Λ . Geometrically the corresponding map $\text{Bun}_{G,\Delta}(\Sigma) \rightarrow \Lambda$ is constructed as follows.

Since the quotient G/B is proper, any B -structure on X^0 extends uniquely to X . Let us denote by \mathcal{F}_B the corresponding B -bundle on X and by \mathcal{F}_T the induced T -bundle. In particular, this shows that $\text{Bun}_{G,\Delta}(\Sigma)$ classifies triples $(\mathcal{F}, \mathcal{F}_B, \mathcal{F}'_T)$, where \mathcal{F}_B is a reduction of \mathcal{F} to B on X and \mathcal{F}'_T is a modification of \mathcal{F}_T at x ; i.e., \mathcal{F}'_T is a T -bundle on X endowed with an isomorphism $\mathcal{F}'_T|_{X^0} \simeq \mathcal{F}_T|_{X^0}$. It is well known that the set of isomorphism classed of pairs of T -bundles $(\mathcal{F}_T, \mathcal{F}'_T)$ on (any smooth curve) X together with an isomorphism on X^0 is in one-to-one correspondence with elements of Λ . Given some $(\mathcal{F}, \mathcal{F}_B, \mathcal{F}'_T) \in \text{Bun}_{G,\Delta}(\Sigma)$ we shall denote by $d(\mathcal{F}, \mathcal{F}_B, \mathcal{F}'_T)$ the corresponding element of Λ , and we shall call it *the defect* of $(\mathcal{F}, \mathcal{F}_B, \mathcal{F}'_T)$.

Similarly, we can define $\text{Bun}_{G,\Delta}(\text{Bl}_{x,k}^0)$ (in that case we should work with the proper preimage of X in $\text{Bl}_{x,k}(\Sigma)$ which we shall identify with X) and

$$\widetilde{\text{Bun}}_{G,\Delta}(\text{Bl}_{x,k}^0) = \text{Bun}_{G,\Delta}(\text{Bl}_{x,k}^0) \times_{\text{Bun}_G(\text{Bl}_{x,k}^0)} \widetilde{\text{Bun}}_G(\text{Bl}_{x,k}^0)$$

together with natural equivalences

$$\Gamma_0 \backslash \pi^{-1}(k) \times_{\Gamma_0} \Omega_l \simeq \text{Bun}_{G,\Delta}(\text{Bl}_{x,k}(\Sigma)^0), \quad \widetilde{\Gamma}_0 \backslash \widetilde{\pi}^{-1}(k) \times_{\widetilde{\Gamma}_0} \widetilde{\Omega}_l \simeq \widetilde{\text{Bun}}_{G,\Delta}(\text{Bl}_{x,k}(\Sigma)^0).$$

The corresponding maps

$$\text{Bun}_{G,\Delta}(\Sigma') \xrightarrow{\text{q}\Delta} \text{Bun}_{G,\Delta}(\text{Bl}_{x,k}(\Sigma)^0) \xrightarrow{\text{q}\Delta} \text{Bun}_{G,\Delta}(\Sigma)$$

and

$$\widetilde{\text{Bun}}_{G,\Delta}(\text{Bl}_{x,k}(\Sigma)^0) \xrightarrow{\text{q}\Delta} \text{Bun}_{G,\Delta}(\Sigma) \times \mathbb{Z}$$

are constructed in the obvious way.

With this notation, in order to prove property (i), we must show the following. Fix some $a \in \mathbb{Z}$ and $\lambda \in \Lambda$. Consider the set Ξ of isomorphism classes of all points $\xi \in \widetilde{\text{Bun}}_{G,\Delta}(\text{Bl}_{x,k}(\Sigma)^0)$ such that

- a) $d(\xi) = \lambda$.
- b) The projection of $\widetilde{\text{q}}_\Delta(\xi)$ to \mathbb{Z} is equal to a .

Consider now the set $d(\widetilde{\text{q}}_\Delta(\Xi))$ of all possible defects of elements of $\widetilde{\text{q}}_\Delta(\Xi)$. We must show that this set is finite. To see this, assume that we have a point ξ as above of the form $(\widetilde{\mathcal{F}}, \mathcal{F}_B, \mathcal{F}'_T)$, where $\widetilde{\mathcal{F}}$ is an element of $\widetilde{\text{Bun}}_{G,\Delta}(\text{Bl}_{x,k}(\Sigma)^0)$ lying over some $\mathcal{F} \in \text{Bun}_{G,\Delta}(\text{Bl}_{x,k}(\Sigma)^0)$ and \mathcal{F}_B and \mathcal{F}'_T are as before. Let \mathcal{F} denote the underlying G -bundle on $\text{Bl}_{x,k}(\Sigma)^0$. Let \mathcal{G} denote the unique extension of $\mathcal{F}|_{\Sigma^0}$ to Σ , and let \mathcal{F}' denote the pull-back of \mathcal{G} to $\text{Bl}_{x,k}(\Sigma)^0$. Note that $\mathcal{F}|_{X^0} = \mathcal{F}'|_{X^0}$; thus \mathcal{F}' comes with canonical Δ -structure. Moreover, it is clear that $d(\mathcal{F}', \mathcal{F}_B, \mathcal{F}'_T)$ is equal to $d(\mathfrak{q}(\mathcal{F}, \mathcal{F}_B, \mathcal{F}'_T))$. Thus what we have to show is that the defect of $(\mathcal{F}', \mathcal{F}_B, \mathcal{F}'_T)$ lies in some finite subset of Λ . This follows from condition b) together with Theorem 6.13.

8.10. *Construction of the Satake isomorphism.* We are now in the position to construct the isomorphism claimed in Theorem 4.6(2). As mentioned above, the quotient $\tilde{\Gamma}_0 \backslash \tilde{\Omega}$ can be naturally identified with Λ_{aff} . Thus the space of $\tilde{\Gamma}_0$ -invariant functions on $\tilde{\Omega}$, supported on finitely many $\tilde{\Gamma}_0$ -orbits, is naturally isomorphic to $\mathbb{C}[\Lambda_{\text{aff}}] = \mathbb{C}(T_{\text{aff}}^{\vee})$. This together with Theorem 8.2 implies that the action of $\mathcal{H}(\tilde{\Gamma}, \Gamma_0)$ on $\mathcal{F}(\tilde{\Gamma}_0 \backslash \tilde{\Omega})$ gives rise to a homomorphism $\mathcal{H}(\tilde{\Gamma}, \tilde{\Gamma}_0) \rightarrow \mathbb{C}(\widehat{T}_{\text{aff}}^{\vee})$. Let ρ_{aff}^{\vee} denote any element of $\Lambda_{\text{aff}}^{\vee}$ whose scalar product with every simple root of G_{aff}^{\vee} is equal to 1. Then Theorem 3.3.5 of [9] implies that the composition of the above homomorphism with the shift by $q^{-\rho_{\text{aff}}^{\vee}}$ lands in $\mathbb{C}(\widehat{T}_{\text{aff}}^{\vee})^{W_{\text{aff}}}$. We claim that the resulting homomorphism

$$(8.1) \quad \iota : \mathcal{H}(\tilde{\Gamma}^+, \tilde{\Gamma}_0) \rightarrow \mathbb{C}(\widehat{T}_{\text{aff}}^{\vee})^{W_{\text{aff}}}$$

is an isomorphism.

PROPOSITION 8.11. *ι is injective.*

Proof. Recall that

$$\mathcal{H}(\tilde{\Gamma}^+, \tilde{\Gamma}_0) = \mathcal{H}(\tilde{\Gamma}^+, \tilde{\Gamma}_0)^{\text{fin}} \otimes_{\mathbb{C}[v, v^{-1}]} \mathbb{C}((v)),$$

where $\mathcal{H}(\tilde{\Gamma}, \tilde{\Gamma}_0)^{\text{fin}}$ is the space of finitely supported functions on $\tilde{\Gamma}_0 \backslash \tilde{\Gamma}^+ / \tilde{\Gamma}_0$. Moreover, for any $k > 0$, we have $\tilde{\Gamma}_0 \backslash \tilde{\pi}^{-1}(k) / \tilde{\Gamma}_0 = \Lambda_{\text{aff}, k} / W_{\text{aff}}$. In particular, $\mathcal{H}(\tilde{\Gamma}, \tilde{\Gamma}_0)^{\text{fin}} \simeq \mathbb{C}[\Lambda_{\text{aff}}^+]$ as a vector space. It is enough to show that the restriction of ι to $\mathcal{H}^{\text{fin}}(\tilde{\Gamma}^+, \tilde{\Gamma}_0)$ is injective, since ι is a morphism of graded $\mathbb{C}((v))$ -algebras and every graded component of $\mathcal{H}(\tilde{\Gamma}, \tilde{\Gamma}_0)^{\text{fin}}$ is of finite rank over $\mathbb{C}[v, v^{-1}]$. For any $\lambda_{\text{aff}} \in \Lambda_{\text{aff}}$, let $\delta_{\lambda_{\text{aff}}}$ denote the characteristic function of $\tilde{\Gamma}_0 s^{\lambda_{\text{aff}}} \tilde{\Gamma}_0$. Then we have to show that the functions $\iota(\delta_{\lambda})$ are linearly independent for all $\lambda \in \Lambda_{\text{aff}} / W_{\text{aff}}$.

To simplify the discussion, let us prove this in the case when G is simply connected. (The general case is similar, but notationally a bit more cumbersome; we shall leave it to the reader.) Recall that when G is simply connected, the quotient $\Lambda_{\text{aff}} / W_{\text{aff}}$ can be identified with the set Λ_{aff}^+ of dominant coweights of G_{aff} . Let us define a partial ordering on $\Lambda_{\text{aff}} / W_{\text{aff}}$ by saying that $\lambda_{\text{aff}} \geq \mu_{\text{aff}}$ if $\lambda_{\text{aff}} - \mu_{\text{aff}}$ is a sum of positive roots of $\mathfrak{g}_{\text{aff}}^{\vee}$. Then we claim that for any $\lambda_{\text{aff}} \in \Lambda_{\text{aff}}^+$, we have

$$(8.2) \quad \iota(\delta_{\lambda_{\text{aff}}}) = s_{\lambda_{\text{aff}}} + \sum_{\mu_{\text{aff}} < \lambda_{\text{aff}}} a_{\mu_{\text{aff}}} s_{\mu_{\text{aff}}}.$$

It is clear that (8.2) implies that all the $\iota(\delta_{\lambda_{\text{aff}}})$ are linearly independent. The proof of (8.2) is a word-by-word repetition of the corresponding statement for finite-dimensional semi-simple groups (cf., e.g., [3, p. 148]). \square

To prove the surjectivity, let us note that ι is a morphism of graded $\mathbb{C}((v))$ -algebras. On the other hand, we have

$$\dim_{\mathbb{C}((v))}(\mathcal{H}(\tilde{\Gamma}^+, \tilde{\Gamma}_0)_k) = \#(\Lambda_{\text{aff},k}/W_{\text{aff}}) = \dim_{\mathbb{C}((v))}(\mathbb{C}(\widehat{T}_{\text{aff}}^V)_k^{W_{\text{aff}}}).$$

Thus the injectivity of ι implies its surjectivity.

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