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The link between the shape of the irrational Aubry-Mather sets and their Lyapunov exponents

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Abstract

We consider the irrational Aubry-Mather sets of an exact symplectic monotone C^1 twist map of the two-dimensional annulus, introduce for them a notion of "C^1-regularity" (related to the notion of Bouligand paratingent cone) and prove that

- a Mather measure has zero Lyapunov exponents if and only if its support is C^1-regular almost everywhere;
- a Mather measure has nonzero Lyapunov exponents if and only if its support is C^1-irregular almost everywhere;
- an Aubry-Mather set is uniformly hyperbolic if and only if it is irregular everywhere;
- the Aubry-Mather sets which are close to the KAM invariant curves, even if they may be C^1-irregular, are not "too irregular" (i.e., have small paratingent cones).

The main tools that we use in the proofs are the so-called Green bundles.

Contents

1. Introduction

The exact symplectic twist maps of the two-dimensional annulus were studied for a long time because they represent (via a symplectic change of coordinates) the dynamic of the generic symplectic diffeomorphisms of surfaces

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near their elliptic periodic points (see [8]). One motivating example of such a map was introduced by Poincaré for the study of the restricted 3-Body problem.

For these maps, the first invariant sets which were studied were the periodic orbits. The "last geometric Poincaré's theorem" was proved by G. D. Birkhoff in 1913 in [7]. Later, in the 50's, the K.A.M. theorems provided the existence of some invariant curves for sufficiently regular symplectic diffeomorphisms of surfaces near their elliptic fixed points (see [20], [3], [31] and [33]). Then, in the 80's, the Aubry-Mather sets were discovered simultaneously and independently by Aubry and Le Daeron (in [5]) and Mather (in [29]). These sets are the union of some quasi-periodic (in a weak sense) orbits, which are not necessarily on an invariant curve. We can define for each of these sets a *rotation number* and for every real number, there exists at least one Aubry-Mather set with this rotation number.

In 1988, Le Calvez proved in [22] that for every generic exact symplectic twist map f, there exists an open dense subset $U(f)$ of $\mathbb{R}$ such that every Aubry-Mather set for f whose rotation number belongs to $U(f)$ is hyperbolic. Of course this does not imply that *all* the Aubry-Mather sets are hyperbolic (in particular the K.A.M. curves are not hyperbolic).

Some results concerning these hyperbolic Aubry-Mather sets are known. It is proved in [26] that their projections have zero Lebesgue measure, and in [24] it is proved that they have zero Hausdorff dimension.

The main question in which we will be interested is then: given some Aubry-Mather set of a symplectic twist map, is there a link between the geometric shape of these set and the fact that it is hyperbolic? Or: Can we deduce the Lyapunov exponents of the measure supported on the Aubry-Mather set from the "shape" of this measure?

I did not hear of such results for any dynamical systems and I think that the ones contained in this article are the first in this direction.

Before explaining which kind of positive answers we can give to this question, let us introduce some notations and definitions. For classical results concerning exact symplectic twist map, the reader is referred to the books [13], [23], [35] and the article [30].

Notation.

- $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is the circle.
- $\mathbb{A} = \mathbb{T} \times \mathbb{R}$ is the annulus and an element of $\mathbb{A}$ is denoted by (θ, r).
- $\mathbb{A}$ is endowed with its usual symplectic form, $\omega = d\theta \wedge dr$ and its usual Riemannian metric.
- $\pi : \mathbb{T} \times \mathbb{R} \to \mathbb{T}$ is the first projection and $\tilde{\pi} : \mathbb{R}^2 \to \mathbb{R}$ its lift.
- $p : \mathbb{R}^2 \to \mathbb{A}$ is the usual covering map.

Definition. A C^1 diffeomorphism $f : \mathbb{A} \to \mathbb{A}$ of the annulus that is isotopic to identity is a *positive twist map* (resp. *negative twist map*) if, for any given lift $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}^2$ and for every $\tilde{\theta} \in \mathbb{R}$, the maps $r \mapsto \tilde{\pi} \circ \tilde{f}(\tilde{\theta}, r)$ are increasing (resp. decreasing) diffeomorphisms. A *twist map* may be positive or negative. Moreover, f is *exact symplectic* if the 1-form $f^*(rd\theta) - rd\theta$ is exact.

Notation. $\mathcal{ET}_\omega^+$ is the set of exact symplectic positive C^1 twist maps of $\mathbb{A}$ (C^1 ESPT), $\mathcal{ET}_\omega^-$ is the set of exact symplectic negative C^1 twist maps of $\mathbb{A}$ and $\mathcal{ET}_\omega = \mathcal{ET}_\omega^+ \cup \mathcal{ET}_\omega^-$ is the set of exact symplectic C^1 twist maps of $\mathbb{A}$.

Definition. Let M be a nonempty subset of $\mathbb{A}$, let $f : \mathbb{A} \to \mathbb{A}$ be an exact symplectic twist map and let $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}^2$ be one of its lifts. The set M is *f-ordered* if

- M is compact;
- M is f-invariant;
- for all $z, z' \in p^{-1}(M)$, $\tilde{\pi}(z) < \tilde{\pi}(z') \Leftrightarrow \tilde{\pi}(\tilde{f}(z)) < \tilde{\pi}(\tilde{f}(z'))$.

(Note that this definition does not depend on the choice of the lift $\tilde{f}$ of f.)

A classical result in the subject asserts that every f-ordered set is a Lipschitz graph above a compact subset of the circle. Moreover, if K is a compact subset or $\mathbb{A}$, then there exists a constant $k > 0$ depending only on K and f such that the Lipschitz constant of every f-ordered set meeting K is less than k.

Definition. An *Aubry-Mather set* for an exact symplectic twist map f is an f-ordered set M_0 that is minimal in the following sense:

$$\text{If } M \text{ is a } f\text{-ordered set such that } M \subset M_0, \text{ then } M = M_0.$$

Remark. The original definition of the Aubry-Mather sets was done by J. Mather in a variational setting and is a little different from this one. The orbits of such a set have to minimize a certain functional called the action. These sets are f-ordered, but not necessarily minimal in the previous sense (the dynamic restricted to such a set is not necessarily minimal). Moreover, the Aubry-Mather sets that we introduce here are not necessarily minimizing for the action.

Then it is well known that if M is an Aubry-Mather set of a $f \in \mathcal{ET}_\omega$, then there exists a bi-Lipschitz orientation-preserving homeomorphism $h : \mathbb{T} \to \mathbb{T}$ of the circle such that for all $(\theta, r) \in M$, $\pi \circ f(\theta, r) = h(\theta)$. The dynamic of f on M is conjugate via the first projection to the one of a bi-Lipschitz homeomorphism of the circle on a minimal invariant compact set. If we write the previous equality for a lift $\tilde{f}$ of f, then we can associate to every Aubry-Mather set M of f a rotation number (which is the rotation number of any $\tilde{h}$

such that for all $(\tilde\theta, r) \in \widetilde{M} = p^{-1}(M)$, $\tilde h(\tilde\theta) = \tilde\pi \circ \tilde f(\tilde\theta, r))$ denoted by $\rho(M, \tilde f)$. Then for every $\rho \in \mathbb{R}$, there exists at least one Aubry-Mather set M for f such that $\rho(M, \tilde f) = \rho$. With our definition of Aubry-Mather set (minimal), if $\rho(M, \tilde f)$ is rational, then M is a periodic orbit; in the other case, we will say that the Aubry-Mather set is *irrational* and two cases may happen:

- either M is a curve (and h is C^0-conjugate to a rotation);
- or M is a Cantor (and h is a Denjoy counter example).

Moreover, every Aubry-Mather set carries a unique f-invariant Borel probability measure denoted by $\mu(M, f)$. This measure is always ergodic (even uniquely ergodic on its support) and its support is M. Such a measure μ (associated to an Aubry-Mather set M for f) will be called a *Mather measure*. Let us now explain what we mean by "shape of a set" or of a measure. This notion is related to a notion of regularity.

Definition. Let $M \subset \mathbb{A}$ be a subset of $\mathbb{A}$ and $x \in M$ a point of M. The *paratingent cone* to M at x is the cone of $T_x\mathbb{A}$ denoted by $P_M(x)$ whose elements are the limits

$$v = \lim_{n\to\infty} \frac{x_n - y_n}{\tau_n},$$

where (x_n) and (y_n) are sequences of elements of M converging to x, (τ_n) is a sequence of elements of $\mathbb{R}_+^*$ converging to 0, and $x_n - y_n \in \mathbb{R}^2$ refers to the unique lift of this element of $\mathbb{A}$ that belongs to $[-\frac{1}{2}, \frac{1}{2}[^2$.

We will say that M is *C^1-regular* at x if there exists a line D of $T_x\mathbb{A}$ such that $P_M(x) \subset D$. If M is not C^1-regular at x, we say that M is *C^1-irregular* at x.

This notion of (Bouligand's) paratingent cone comes from nonsmooth analysis (see for example [4]). Of course, at an isolated point, the notion of regularity does not mean anything, and we will use it only for Aubry-Mather sets having no isolated point, i.e., irrational Aubry-Mather sets.

THEOREM 1. *Let f be an exact symplectic C^1 twist map and let μ be an irrational Mather measure of f. The following two assertions are equivalent*:

- *for μ-almost every x,* supp(μ) *is C^1-regular at x;*
- *the Lyapunov exponents of μ (for f) are zero.*

An alternative statement of this result is

THEOREM 2. *Let f be an exact symplectic C^1 twist map and let μ be an irrational Mather measure of f. The following two assertions are equivalent*:

- *for μ-almost every x,* supp(μ) *is C^1-irregular at x;*
- *the Lyapunov exponents of μ (for f) are nonzero.*

Hence we do not obtain exactly the kind of result we wanted. Knowing the measure μ (and not the diffeomorphism f!), we can say if the Lyapunov exponents are zero or not, but the *a priori* knowledge of the Aubry-Mather set is not sufficient to deduce if the Lyapunov exponents are zero or no.

We can easily construct two C^1 ESPT f and g for which the zero section is an irrational invariant curve, the dynamic restricted to the zero section is minimal, but the two invariant measures $\mu(M, f)$ and $\mu(M, g)$ are not mutually absolutely continuous. Indeed, there exists a C^∞ minimal diffeomorphism h of the circle that is transitive but whose invariant measure is not absolutely continuous with respect to the Lebesgue measure (see [16] for example). Then we choose f such that its restriction to the zero section is h and g such that its restriction to the zero section is an irrational rotation. For this example, "almost everywhere" for one measure is different from "almost everywhere" for the other one.

However, in the extreme cases, we obtain a result concerning the shape of the Aubry-Mather sets.

COROLLARY 1. *Let f be an exact symplectic C^1 twist map and let M be an irrational Aubry-Mather set of f. If for all $x \in M$, M is C^1-regular at x, then the Lyapunov exponents of $\mu(M, f)$ (for f) are zero.*

It is not hard to see that an Aubry-Mather set is C^1-regular everywhere if and only if there exists a C^1 map $\gamma : \mathbb{T} \to \mathbb{R}$ whose graph contains M. Therefore we can state Corollary 1 in the following nicer form:

> If an irrational Aubry-Mather set is contained in a (not necessarily invariant) C^1 curve, then the Lyapunov exponents of $\mu(M, f)$ are zero.

In [18], M. Herman gives some examples of irrational Aubry-Mather sets which are invariant by a twist map, contained in a C^1 graph but not contained in an *invariant* continuous curve. I do not know any example of an irrational Aubry-Mather set with zero Lyapunov exponents which is not contained in a C^1 curve.

Problem. Is it possible to build an irrational Aubry-Mather set with zero Lyapunov exponents which is not contained in a C^1 graph?

THEOREM 3. *Let f be an exact symplectic C^1 twist map and let M be an irrational Aubry-Mather set of f. The following two assertions are equivalent*:

- *for all $x \in M$, M is C^1-irregular at x;*
- *the set M is uniformly hyperbolic (for f).*

In the nonuniformly hyperbolic case, we can be more specific.

Theorem 4. *Let f be an exact symplectic C^1 twist map and let μ be an irrational Mather measure of f which is nonuniformly hyperbolic*; *i.e.*, *the Lyapunov exponents are nonzero but the corresponding Aubry-Mather set $M = \mathrm{supp}(\mu)$ is not* (*uniformly*) *hyperbolic. Then there exists a dense G_δ subset $\mathcal{G}$ of M such that M is C^1-regular at every point of $\mathcal{G}$.*

I must say that I do not know any example of an irrational Aubry-Mather set which is nonuniformly hyperbolic.

Questions.

- Is it possible to build an irrational nonuniformly hyperbolic Aubry-Mather set for an exact symplectic C^1 twist map?
- Is it possible to build an essential irrational nonsmooth invariant curve for an exact symplectic C^1 twist map? (This question is due to J. Mather.)
- Are there essential invariant irrational curves that support a measure with positive Lyapunov exponents?

Let us now consider what happens near a K.A.M. invariant curve $\mathcal{C}$ for a generic $f \in \mathcal{ET}_\omega$. If α is the rotation number of this K.A.M. curve, then for every neighbourhood $\mathcal{V}$ of $\mathcal{C}$ for the Hausdorff topology, there exists $\varepsilon > 0$ such that every Aubry-Mather set whose rotation number is in $]\alpha - \varepsilon, \alpha + \varepsilon[$ belongs to $\mathcal{V}$. (Indeed, a limit of f-ordered set is f-ordered and the rotation number is continuous on the set of f-ordered sets; moreover, a classical result asserts that if there is a KAM curve, it is the unique f-ordered set having this rotation number.) Hence, using Le Calvez's result mentioned before, we find in every neighbourhood $\mathcal{V}$ of $\mathcal{C}$ some irrational uniformly hyperbolic Aubry-Mather sets, and hence some C^1-irregular Cantor sets (see the beginning of the proof of Theorem 10 to see why it cannot be a curve). But even if these Cantor sets are C^1-irregular, the closest they are to $\mathcal{C}$, then the less irregular they are in the following sense.

Theorem 5. *Let $f \in \mathcal{ET}_\omega$ be an exact symplectic twist map and $\mathcal{C}$ be a C^1 invariant curve which is a graph such that $f_{|\mathcal{C}}$ is C^1 conjugate to a rotation. Let W be a neighbourhood of $T^1\mathcal{C}$, the unit tangent bundle to $\mathcal{C}$ in $T^1\mathbb{A}$, the unit tangent bundle to $\mathbb{A}$. Then there exists a neighbourhood V of $\mathcal{C}$ in $\mathbb{A}$ such that for every Aubry-Mather set M for f contained in V,*

$$\forall x \in M, P^1_M(x) \subset W,$$

where $P^1_M(x) = P_M(x) \cap T^1\mathbb{A}$ refers to the unit paratingent cone.

It implies that in this case, even if the paratingent cone at x to M is not a line, it is a thin cone close to a line.

Let us mention that we dealt only with *irrational* Aubry-Mather sets or invariant curves. Even in this case, it is interesting to understand the connection with the periodic orbits. Let us now explain this.

If we consider a sequence of periodic f-ordered orbits whose rotation numbers converge to an irrational one, then we can extract a subsequence of periodic orbits that converge to an irrational f-ordered set.

Hence, for example, if we want to "draw" (with a computer) our irregular (and hyperbolic) Aubry-Mather sets, then we can use some sequences of minimizing periodic orbits. But if we look at the pictures of Aubry-Mather sets that exist, we see Cantor sets or curves, but we never see angles of the tangent spaces. That is why the following question was raised by X. Buff.

Question (X. Buff). Is it possible (for example by using minimizing periodic orbits) to draw some Aubry-Mather sets with "corners"?

Another interesting connection is the so-called "Greene's criterion" (see [10], [15] or [25]). Greene introduced a quantity attached to any periodic orbit, called the residue, and stated that if a "good" sequence of minimizing periodic orbit converge to an irrational f-ordered set, then

(1) either this f-ordered set is contained in an invariant curve and the logarithms of the mean residues tend to 0;
(2) or there exist no invariant curve with this irrational number and the logarithms of the mean residues do not tend to 0.

The logarithm of the mean residue is closely related to the Lyapunov exponents. If Greene's criterion is true, then any sequence of periodic orbits tending to an invariant curves has its Lyapunov exponent that tends to 0. This seems to be related to the question "Are there essential invariant irrational curves that support a measure with positive Lyapunov exponents?" The problem is that Greene criterion is not completely proved, only certain particular cases have been considered in the mentioned articles. But we see that our problem has some connections with Greene's criterion.

To prove the results contained in this article, we will use a very useful mathematical object, the Green bundles. They were introduced by L. W. Green in [14] for Riemannian geodesic flows. Then P. Foulon extended this construction to Finsler metrics in [11], and G. Contreras and R. Iturriaga extended it in [9] to optical Hamiltonian flows. In [6], M. Bialy and R. S. Mackay give an analogous construction for the dynamics of sequence of symplectic twist maps of $T^*\mathbb{T}^d$ without conjugate point. Let us also cite a very short survey [19] of R. Iturriaga on the various applications of these bundles (problems of rigidity, measure of hyperbolicity...).

In [2] and [1], I constructed these bundles along invariant graphs and proved, under various dynamical assumptions, that they may be used to prove some results of C^1-regularity. In particular, the strongest result contained in [2] for twist maps is that the essential invariant curves are more regular than Lipschitz (more precisely C^1-regular on a dense G_δ subset that has full Lebesgue measure) or, equivalently that the C^1 solutions of the Hamilton-Jacobi equation $H(q, du(q)) = c$ for a Tonelli Hamiltonian $H \ : T^*\mathcal{S} \to \mathbb{R}$ defined on the cotangent bundle of a surface are C^2 on G_δ-dense subset of $\mathcal{S}$ that has full measure.

In the second section of this new article, I enlarge the construction of the Green bundles to a set that is called the Green set, which contains all the irrational Aubry-Mather sets and is denoted by $\mathrm{Green}(f)$. I then give some of their properties (semi-continuity…), introduce a notion of C^1-regularity (which is quite different from the one contained in [2]) and explain how the coincidence of the two Green bundles implies some regularity of the Aubry-Mather sets. The main statement in this section is the following theorem (the exact definition of the relation $\preceq$ will be given in next section, it is related to the order between the slopes of the lines), that explains the link between the Green bundles and the regularity.

THEOREM 6. *Let $f : \mathbb{A} \to \mathbb{A}$ be a C^1 ESPT. Then the Green bundles, defined at every point of $\mathrm{Green}(f)$, are invariant by Df.*

The map $(x \in \mathrm{Green}(f) \to G_+(x))$ is upper semi-continuous and the map $x \to G_-(x)$ is lower semi-continuous and we have that for all $x \in \mathrm{Green}(f), G_-(f) \preceq G_+(f)$. Therefore, the set

$$\mathcal{G}(f) = \{x \in \mathrm{Green}(f); G_-(x) = G_+(x)\}$$

is a G_δ subset of $\mathrm{Green}(f)$.

Moreover, for every irrational Aubry-Mather set M of f and every $x \in M$, we have $G_-(x) \preceq P_M(x) \preceq G_+(x)$, and for every $x_0 \in \mathcal{G}(f) \cap M$, M is C^1-regular at x_0 and $P_M(x_0) = G_+(x_0) = G_-(x_0)$. Moreover, G_- and G_+ are continuous at such an x_0.

In the third section, I explain how the transversality of the Green bundles implies some hyperbolicity. The same result with an extra dynamical assumption (the fact that the dynamic is nonwandering) is due to Contreras and Iturriaga.

THEOREM 7. *Let f be a C^1 ESPT and let $K \subset \mathrm{Green}(f)$ be an invariant compact subset of $\mathrm{Green}(f)$ such that, at every point of K, $G_-(x)$ and $G_+(x)$ are transverse. Then K is uniformly hyperbolic and at every $x \in K$, we have $G_-(x) = E^s(x)$ and $G_+(x) = E^u(x)$.*

To obtain Theorem 7, I prove a result concerning symplectic quasi-hyperbolic cocycles (a cocycle is quasi-hyperbolic if the orbit of every nonzero vector is unbounded).

THEOREM 8. *Let* (F_k) *be a continuous, symplectic quasi-hyperbolic cocycle on a linear and symplectic (finite dimensional) bundle* $P : E \to K$ *above a compact metric space* K. *Then* $(F_k)_{k\in\mathbb{Z}}$ *is hyperbolic.*

Then I prove a similar statement for nonuniformly hyperbolic measure.

THEOREM 9. *Let* $f \in \mathcal{ET}_\omega^+$ *be a* C^1 ESPT *and let* μ *be an irrational Mather measure for* f. *We assume that at* μ*-almost every point,* G_- *is transverse to* G_+. *Then the Lyapunov exponents of* μ *are nonzero.*

This result concerning Lyapunov exponents is completely new.

In the fourth section, I prove that hyperbolic Aubry-Mather sets are C^1-irregular. I deal with the uniformly and nonuniformly hyperbolic cases.

THEOREM 10. *Let* M *be a uniformly hyperbolic irrational Aubry-Mather set of a* C^1 ESPT f *of* $\mathbb{A}$. *Then at every* $x \in M$, M *is* C^1*-irregular.*

THEOREM 11. *Let* $f \in \mathcal{ET}_\omega$ *be a* C^1 ESPT *and let* μ *be an irrational Mather measure of* f *whose Lyapunov exponents are nonzero. Then, at* μ *almost every point,* $\text{supp}(\mu)$ *is* C^1*-irregular.*

Finally, in the last section, I prove the four first theorems contained in the introduction.

Acknowledgments. I am grateful to R. Perez-Marco who first suggested to me that the result for Aubry-Mather sets could be "hyperbolicity versus regularity", to J.-C. Yoccoz whose questions led me to the appropriate definition of regularity, to L. Rifford who pointed to me the notion of Bouligand's paratingent cone and to S. Crovisier who suggested me to send one Green bundle on the "horizontal" for the proof of the "dynamical criterion," which gives a significant improvement of the proof. I thank X. Buff for stimulating discussions and the referees for many improvements of the article.

2. Construction of the Green bundles along an irrational Aubry-Mather set, link with the C^1-regularity

Notation.

$\pi : \mathbb{T} \times \mathbb{R} \to \mathbb{T}$ is the projection.
If $x \in \mathbb{A}$, then $V(x) = \ker D\pi(x) \subset T_x\mathbb{A}$ is the *vertical* at x.
If $x \in \mathbb{A}$ and $k \in \mathbb{Z}$, then $G_k(x) = Df^k(f^{-k}(x))V(f^{-k}(x))$is a 1-dimensional linear subspace (or line) of $T_x\mathbb{A}$.

Definition. If we identify $T_x\mathbb{A}$ with $\mathbb{R}^2$ by using the standard coordinates $(\theta, r) \in \mathbb{R}^2$, then we may deal with the *slope* $s(L)$ of any line L of $T_x\mathbb{A}$ which is transverse to the vertical $V(x)$: this means that $L = \{(t, s(L)t); t \in \mathbb{R}\}$.

If $x \in \mathbb{A}$ and if L_1 and L_2 , are two lines of $T_x\mathbb{A}$ which are transverse to the vertical $V(x)$, then L_2 is *above* (resp. *strictly above*) L_1 if $s(L_2) \geq s(L_1)$ (resp.

$s(L_2) > s(L_1)$). In this case, we write: $L_1 \preceq L_2$ (resp. $L_1 \prec L_2$). Similarly, if $\mathcal{L}_1$ and $\mathcal{L}_2$ are two sets of lines of $T_x\mathbb{A}$ which are transverse to the vertical $V(x)$, $\mathcal{L}_2$ is *above* (resp. *strictly above*) $\mathcal{L}_1$ if $s(L_2) \geq s(L_1)$ (resp. $s(L_2) > s(L_1)$) for all $L_1 \in \mathcal{L}_1, L_2 \in \mathcal{L}_2$. In this case, we write: $\mathcal{L}_1 \preceq \mathcal{L}_2$ (resp. $\mathcal{L}_1 \prec \mathcal{L}_2$).

A sequence $(L_n)_{n\in\mathbb{N}}$ of lines of $T_x\mathbb{A}$ is *nondecreasing* (resp. *increasing*) if for every $n \in \mathbb{N}$, L_n is transverse to the vertical and L_{n+1} is above (resp. strictly above) L_n. We similarly, define the *nonincreasing* and *decreasing* sequences of lines of T_xM.

Remark. A decreasing sequence of lines corresponds to a decreasing sequence of slopes.

Definition. If K is a subset of $\mathbb{A}$ or of its universal covering $\mathbb{R}\times\mathbb{R}$ and if F is a not necessarily 1-dimensional sub-bundle of $T_K\mathbb{A}$ (resp. $T_K\mathbb{R}^2$) transverse to the vertical, we say that F is upper (resp. lower) semi-continuous if the map which maps $x \in K$ onto the slope $s(F(x))$ of $F(x)$ is upper (resp. lower) semi-continuous.

PROPOSITION 1. *Let $f : \mathbb{T}\times\mathbb{R} \to \mathbb{T}\times\mathbb{R}$ be an exact symplectic positive C^1 twist map (C^1 ESPT) and let M be a f-ordered set. Then, for every $x \in M$ which is not an isolated point of M, the lines $G_k(x)$ for $k \in \mathbb{Z}^*$ are transverse to the vertical $V(x)$ and we have*

$$\forall n \in \mathbb{N}^*, G_{-n}(x) \prec G_{-(n+1)}(x) \prec P_M(x) \prec G_{n+1}(x) \prec G_n(x).$$

In particular, if M is an irrational Aubry-Mather set, then it has no isolated point.

(In this statement we identify the cone $P_M(x)$ with the set of the lines which are contained in this cone.)

Proof. As M is an f-ordered set, it is the graph of a Lipschitz map γ above a nonempty and compact set K of $\mathbb{T}$. Now let $x = (t, \gamma(t))$ be a point of M. We will use the left and right paratingent cones to M at x, defined by

- the right paratingent cone of M at x, denoted by $P^r_M(x)$, is the set whose elements are the limit, $v = \lim_{n\to\infty} \frac{(u_n,\gamma(u_n))-(s_n,\gamma(s_n))}{\tau_n}$, where (u_n) and (s_n) are sequences of elements of K converging to t from above (i.e., $u_n, s_n \in [t, +\infty[$) and (τ_n) is a sequence of elements of $\mathbb{R}^*_+$ converging to 0;
- similarly, the left paratingent cone of M at x, denoted by $P^l_M(x)$, is the set whose elements are the limits $v = \lim_{n\to\infty} \frac{(u_n,\gamma(u_n))-(s_n,\gamma(s_n))}{\tau_n}$ where (u_n) and (s_n) are sequences of elements of K converging to t from below and (τ_n) is a sequence of elements of $\mathbb{R}^*_+$ converging to 0.

It is not hard to verify that every element of $P_M(x)$ is in the convex hull of $P_M^l(x) \cup P_M^r(x)$ (we identify the lines of $T_x\mathbb{A}$ transverse to the vertical with their slopes in order to deal with their convex hull). Hence, we only have to prove the inequalities of Proposition 1 for $P_M^r(x)$ and $P_M^l(x)$ (and even for those of these two cones that are not trivial) to deduce the inequalities of this proposition. Because the four proofs are similar, we will assume, for example, that $P_M^r(x) \neq \{0\}$ and we will prove that for all $n \in \mathbb{N}^*, P_M^r(x) \prec G_{n+1}(x) \prec G_n(x)$.

In fact we shall need to deal with half lines instead of lines. Hence we define $\mathcal{P}_M^r(x)$ as being the set of the half lines of $T_x\mathbb{A}$, which are contained in $P_M^r(x)$ such that their points have positive abscissa. Equivalently, $\mathcal{P}_M^r(x)$ is the set of the limits $v = \lim_{n\to\infty} \frac{(u_n,\gamma(u_n))-(s_n,\gamma(s_n))}{\tau_n}$, where (u_n) and (s_n) are sequences of elements of K converging to t such that for all $n, t \leq s_n < u_n$ and (τ_n) is a sequence of elements of $\mathbb{R}_+^*$ converging to 0. As M is f-ordered, for all $y \in M$, we have $Df(\mathcal{P}_M^r(y)) = \mathcal{P}_M^r(f(y))$ (in particular the image through Df of the right paratingent cone at y is the right paratingent cone at the image $f(y)$). Hence for all $k \in \mathbb{Z}$, $\mathcal{P}_M^r(f^k x) = Df^k(\mathcal{P}_M^r(x))$. Now let $V_+(x) = \{(0,R), R > 0\} \subset T_x\mathbb{A}$ be the *upper vertical* at x and let us denote by $g_k(x)$ the half line $g_k(x) = Df^k(f^{-k}(x))V_+(f^{-k}x)$.

Let us look at the action of Df on the half lines of the tangent linear spaces $T_{f^k(x)}\mathbb{A}$. As f is a positive twist map, we have (identifying as before $T_{f(x)}\mathbb{A}$ with $\mathbb{R}^2$) $Df(x)(0,1) = (a,b)$ with $a > 0$. As a result, $G_1(f(x))$ is transverse to $V(f(x))$. Now if $\mathbb{R}_+(\alpha,\beta) \in \mathcal{P}_M^r(x)$, then we know that $\alpha > 0$. Hence the base $((\alpha,\beta),(0,1))$ is a direct base (for ω) of $T_x\mathbb{A}$; as $Df(x)$ is symplectic, the image base $((\alpha',\beta'),(a,b))$ is also direct. It means exactly that the line $\mathbb{R}(a,b) = G_1(f(x))$ is strictly above the line $\mathbb{R}(\alpha',\beta')$ of $P_M^r(f(x))$. Repeating this argument for every half line of $\mathcal{P}_M^r(x)$ and every point of the orbit of x, we obtain that for all $k \in \mathbb{Z}$, $P_M^r(f^k(x)) \prec G_1(f^k(x))$.

Let us consider the action of Df on the circles bundle of the half lines along the orbit of x: as f is orientation-preserving, this action preserves the orientation of the circles. Moreover, if these circles are oriented in the direct sense, then any half line of $\mathcal{P}_M^r(f^k(x))$, $g_1(f^k(x))$ and $V_+(f^kx)$ are in the direct sense (let us recall that on the oriented circle, we can speak of the orientation of three points but not of a pair). Hence their images under Df, $Df^2, \dots$ are in the same order; i.e, any half line of $\mathcal{P}_M^r(f^k(x))$, $g_{n+1}(f^k(x))$ and $g_n(f^k(x))$ are in the direct sense, and then all the $G_n(f^k(x))$ are transverse to the vertical and $P_M^r(f^k(x)) \prec G_{n+1}(f^k(x)) \prec G_n(f^k(x)) \prec \cdots \prec G_1(f^k(x))$. □

Remark. Let us notice that in the proof of Proposition 1, we have seen that

$$\forall x \in M, \forall n \geq 1, \quad D\pi \circ Df^n(x)(0,1) > 0.$$

Similarly, we have

$$\forall x \in M, \forall n \geq 1, \quad D\pi \circ Df^{-n}(x)(0,1) < 0.$$

Hence $(G_n(x))$ is a strictly decreasing sequence of lines of $T_x\mathbb{A}$ which is bounded below. Then it tends to a limit $G_+(x)$. Similarly, the sequence $(G_{-n}(x))$ tends to a limit, $G_-(x)$.

Definition. If $x \in \mathbb{A}$ belongs to a f-ordered set M of $f \in \mathcal{ET}_\omega^+$ and if x is not an isolated point of M, then the bundles $G_-(x)$ and $G_+(x)$ are called the *Green bundles* at x associated to f.

Example. Let us assume that M is an f-ordered set and that $x \in M$ is a periodic hyperbolic periodic point of f that is not an isolated point of M. Then $G_+(x) = E^u(x)$ is the tangent space to the unstable manifold of x, and $G_-(x) = E^s(x)$ is the tangent space to the stable manifold.

In fact, in order to build the Green bundles for f at a point $x \in \mathbb{A}$, we do not need that x belongs to a f-ordered set. Let us introduce the exact set which will be useful for us (the one along which we can define the Green bundles).

Definition. Let $f \in \mathcal{ET}_\omega^+(f)$ be a C^1 ESPT. Then the *Green set* of f, denoted by $\mathrm{Green}(f)$, is the set of points $x \in \mathbb{A}$ such that

- for all $n \geq 1$ and $k \in \mathbb{Z}$,

$$D\pi \circ Df^n(f^k x)(0,1) > 0 \text{ and } D\pi \circ Df^{-n}(f^k x)(0,1) < 0;$$

- for all $n \geq 1$ and $k \in \mathbb{Z}$,

$$\begin{aligned} G_{-n}(f^k x) &= Df^{-n}(f^{n+k}x)V(f^{n+k}x) \prec Df^{-(n+1)}(f^{n+1+k}x)V(f^{n+1+k}x) \\ &= G_{-(n+1)}(f^k x) \prec G_{n+1}(f^k x) \\ &= Df^{n+1}(f^{-(n+1)+k}x)V(f^{-(n+1)+k}x) \\ &\prec Df^n(f^{-n+k}x)V(^{-n+k}x) = G_n(f^k x). \end{aligned}$$

Let us notice that the first point is not useful to define the Green bundles (we only need to ask that $G_n(f^k x)$ and $G_{-n}(f^k x)$ are transverse to the vertical), but will be used in the next section to prove the so-called "dynamical criterion." Then we have

Proposition 2. *Let $f \in \mathcal{ET}_\omega^+$ be a C^1* ESPT. *Then* $\mathrm{Green}(f)$ *is a nonempty, closed subset of $\mathbb{A}$ which contains every irrational Aubry-Mather set of f and is invariant by f. At every $x \in$* $\mathrm{Green}(f)$, *we can define $G_-(x)$ and $G_+(x)$.*

Remark. Let us notice that every essential invariant curve by $f \in \mathcal{ET}_\omega^+$ is a subset of Green(f) (see [2] or apply Proposition 1). Moreover, it can be proved by using the construction done by M. Bialy and R. MacKay in [6] that any Aubry-Mather set in the sense of Mather (i.e., minimizing for an action, for example, any minimizing periodic orbit) is in Green(f).

Proof of Proposition 2. The only thing we need to prove is that Green(f) is closed. Because f is a positive twist map, for every $x \in \mathbb{A}$, we have $D\pi \circ Df(x)(0,1) > 0$ and $D\pi \circ Df^{-1}(x)(0,1) < 0$. Hence for every $x \in \mathbb{A}$, $V(x)$ and $G_1(x)$ are transverse, and $V(x)$ and $G_{-1}(x)$ are also transverse. We deduce that for every $x \in \mathbb{A}$ and every $n \in \mathbb{N}^*$, $G_n(x) = Df^{-(n+1)}G_1(f^{n+1}x)$ and $G_{n+1}(x) = Df^{-(n+1)}V(f^{n+1}x)$ are transverse, and $G_{-(n+1)}(x)$ and $G_{-n}(x)$ are transverse. Let us now consider $\mathcal{C}(f)$ the set of $x \in \mathbb{A}$ such that

- for all $n \geq 1$, $D\pi \circ Df^n(x)(0,1) \geq 0$ and $D\pi \circ Df^{-n}(x)(0,1) \leq 0$;
- for all $n \in \mathbb{N}^*$, $G_{-1} \preceq \cdots \preceq G_{-n}(x) \preceq G_{-(n+1)}(x) \preceq G_{n+1}(x) \preceq G_n(x) \preceq \cdots \preceq G_1(x)$.

Then $\mathcal{C}(f)$ is closed. If we prove that $\mathcal{C}(f) = $ Green(f), we have finished the proof. We have Green$(f) \subset \mathcal{C}(f)$. Moreover, if $x \in \mathcal{C}(f)$, we know that for all $n \in \mathbb{N}^*$, $G_{n+1}(x) \preceq G_n(x)$. As $G_n(x)$ and $G_{n+1}(x)$ are transverse, we deduce that $G_{n+1}(x) \prec G_n(x)$. Similarly, we obtain that $G_{-n}(x) \prec G_{-(n+1)}(x)$. From $G_{-n}(x) \prec G_{-(n+1)}(x) \preceq G_{n+1}(x) \prec G_n(x)$, we deduce that $G_{-n}(x) \prec G_n(x)$. Thus if $x \in \mathcal{C}(f)$, then x satisfies the second point of the definition of Green(f). Hence every $G_k(x)$ for $k \in \mathbb{Z}^*$ is transverse to the vertical and for all $k \in \mathbb{Z}^*$ and $x \in \mathcal{C}(f)$, $D\pi \circ Df^k(x)(1,0) \neq 0$. Therefore, $x \in \mathcal{C}(f)$ satisfies the first point of the definition of Green(f) too. Finally, $\mathcal{C}(f) \subset$ Green(f) and then $\mathcal{C}(f) = $ Green(f). □

Having built the Green bundles on Green(f), we can give some of their properties, similar to the ones given in [2], which in particular give a link between these Green bundles and the notion of C^1-regularity. We recall the theorem that was given in the introduction.

THEOREM 6. *Let f be a C^1* ESPT *$f : \mathbb{A} \to \mathbb{A}$. Then the Green bundles, defined at every point of* Green(f), *are invariant by Df.*

The map $(x \in$ Green(f) *$\to G_+(x))$ is upper semi-continuous, the map $x \to G_-(x)$ is lower semi-continuous, and for all $x \in$* Green(f), *we have $G_-(f) \preceq G_+(f)$. Therefore, the set*

$$\mathcal{G}(f) = \{x \in \text{Green}(f); G_-(x) = G_+(x)\}$$

is a G_δ subset of Green(f).

Moreover, for every irrational Aubry-Mather set M of f and every $x \in M$, we have $G_-(x) \preceq P_M(x) \preceq G_+(x)$ and for every $x_0 \in \mathcal{G}(f) \cap M$, M is C^1-regular at x_0 and $P_M(x_0) = G_+(x_0) = G_-(x_0)$. Moreover, G_- and G_+ are continuous at such an x_0.

This theorem is a corollary of Proposition 1 and of usual properties of real functions (the fact that the (simple) limit of a decreasing sequence of continuous functions is upper semi-continuous).

Corollary 2. *Let M be an irrational Aubry-Mather set of a C^1 ESPT $f : \mathbb{A} \to \mathbb{A}$. We assume that*

$$\forall x \in M, G_-(x) = G_+(x).$$

Then M is C^1-regular at every $x \in M$, and therefore there exists a C^1 map $\gamma : \mathbb{T} \to \mathbb{R}$ whose graph contains M. Moreover, in this case, at every $x = (t, \gamma(t)) \in M$, the sequences $(G_n(x))_{n\in\mathbb{N}}$ and $(G_{-n}(x))_{n\in\mathbb{N}}$ converge uniformly to $\mathbb{R}(1, \gamma'(t))$.

Everything in this corollary is a consequence of Theorem 6; the fact that the convergence is uniform comes from Dini's theorem: if an increasing or decreasing sequence of real valued continuous functions defined on a compact set converges simply to a continuous function, then the convergence is uniform.

This corollary gives us some criterion using the Green bundles to prove that an Aubry-Mather set is C^1-regular. But of course we never said that the transversality of the Green bundles implies the irregularity of the corresponding Aubry-Mather set. This will be explained later.

3. Green bundles and Lyapunov exponents

3.1. *A dynamical criterion.* We begin by giving a criterion to determine if a given vector is in one of the two Green bundles.

Proposition 3. *Let f be a C^1 ESPT and let $x \in \mathrm{Green}(f)$ be a point of the Green set whose orbit $\{f^k(x), k \in \mathbb{Z}\}$ is relatively compact. Then*

$$\lim_{n\to+\infty} D\pi \circ Df^n(x)(1,0) = +\infty \quad \text{and} \quad \lim_{n\to+\infty} D\pi \circ Df^{-n}(x)(1,0) = -\infty.$$

Corollary 3 (dynamical criterion). *Let f be a C^1 ESPT and let $x \in \mathrm{Green}(f)$ be a point of the Green set whose orbit $\{f^k(x), k \in \mathbb{Z}\}$ is relatively compact. Let $v \in T_x\mathbb{A}$. Then*

- *if $v \notin G_-(x)$, then $\lim_{n\to+\infty} |D\pi \circ Df^n(x)v| = +\infty$;*
- *if $v \notin G_+(x)$, then $\lim_{n\to+\infty} |D\pi \circ Df^{-n}(x)v| = +\infty$.*

Proof of Proposition 3. We will only prove the part of the proposition corresponding to what happens in $+\infty$. We use the standard symplectic coordinates (θ, r) of $\mathbb{A}$, and we define $x_k = f^k(x)$ for every $k \in \mathbb{Z}$. In these coordinates, for $j \in \mathbb{Z}^*$, the line $G_j(x_k)$ is the graph of $(t \to s_j(x_k)t)$ ($s_j(x_k)$ is the slope of $G_j(x_k)$).

The matrix $M_n(x_k)$ of $Df^n(x_k)$ (for $n \geq 1$) is a symplectic matrix

$$M_n(x_k) = \begin{pmatrix} a_n(x_k) & b_n(x_k) \\ c_n(x_k) & d_n(x_k) \end{pmatrix}$$

with $\det M_n(x_k) = 1$. We know that the coordinate $D(\pi \circ f^n)(x_k)(0,1) = b_n(x_k)$ is strictly positive. Using the definition of $G_n(x_{k+n})$, we obtain $d_n(x_k) = s_n(x_{k+n})b_n(x_k)$.

The matrix $M_n(x_k)$ being symplectic, we have

$$M_n(x_k)^{-1} = \begin{pmatrix} d_n(x_k) & -b_n(x_k) \\ -c_n(x_k) & a_n(x_k) \end{pmatrix}.$$

From the definition of $G_{-n}(x_k)$, we deduce that $a_n(x_k) = -b_n(x_k)s_{-n}(x_k)$. Finally, if we use the fact that $\det M_n(x_k) = 1$, then we obtain

$$M_n(x_k) = \begin{pmatrix} -b_n(x_k)s_{-n}(x_k) & b_n(x_k) \\ -b_n(x_k)^{-1} - b_n(x_k)s_{-n}(x_k)s_n(x_{k+n}) & s_n(x_{k+n})b_n(x_k) \end{pmatrix}.$$

LEMMA 1. *Let K be a compact subset of* Green(f). *There exists a constant $A > 0$ such that*

$$\forall x \in K, \forall n \in \mathbb{N}^*, \max\{|s_n(x)|, |s_{-n}(x)|\} \leq A.$$

Proof. From the definition of Green(f), we deduce that

$$\forall x \in \mathrm{Green}(f), \forall n \in \mathbb{N}^*, s_{-1}(x) \leq s_{-n}(x) < s_n(x) \leq s_1(x).$$

Therefore, we only have to prove the inequalities of the lemma for $n = 1$.

The real number $s_{-1}(x)$, which is the slope of $Df^{-1}(f(x))V(f(x))$, depends continuously on x and is defined for every x belonging to the compact subset K. Hence it is uniformly bounded. The same argument proves that s_1 is uniformly bounded on K and concludes the proof of Lemma 1. □

LEMMA 2. *Let $x \in$* Green(f) *be such that its orbit is relatively compact. Then we have* $\lim\limits_{n\to\infty} b_n(x) = +\infty$.

Let us notice that this gives exactly the first part of Proposition 3.

Proof. We will use a change of basis along the orbit of x. Let us denote by $s_-(f^kx)$ the slope of $G_-(f^kx)$ and by $s_+(f^kx)$ the slope of $G_+(f^kx)$. We will choose $G_-(x)$ as new "horizontal line;" i.e., if the "old coordinates" in $T_y\mathbb{A}$ are (Θ, R), the new coordinates are given via the 2-by-2 matrix $P(y)$ as follows:

$$P(y).\begin{pmatrix} \Theta \\ R \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -s_-(y) & 1 \end{pmatrix}\begin{pmatrix} \Theta \\ R \end{pmatrix} = \begin{pmatrix} \Theta \\ -s_-(y)\Theta + R \end{pmatrix}.$$

In general, P does not depend continuously on the considered point, but by Lemma 1, $P(f^k x)$ and $P^{-1}(f^{-k}x)$ are bounded uniformly in $k \in \mathbb{Z}$ (because s_- is uniformly bounded). Moreover, P is symplectic. Let us compute in the new coordinates $N_n(x_k) = P(x_{n+k})M_n(x_k)P(x_k)^{-1}$.

$$N_n(x_k) = \begin{pmatrix} b_n(x_k)(s_-(x_k) - s_{-n}(x_k)) & b_n(x_k) \\ 0 & b_n(x_k)(s_n(x_{k+n}) - s_-(x_{k+n})) \end{pmatrix}.$$

We know that $\lim_{n\to\infty} \uparrow s_{-n}(x_k) = s_-(x_k)$. Hence, $\lim_{n\to+\infty}(s_-(x_k) - s_{-n}(x_k)) = 0^+$. By Lemma 1, we have $s_n(x_{k+n}) - s_-(x_{k+n}) \leq 2A$ for all $n \geq 1$. As N_n is symplectic, we have $1 = \det N_n(x_k) = b_n(x_k)^2(s_-(x_k) - s_{-n}(x_k))(s_n(x_{k+n}) - s_-(x_{k+n}))$. We deduce that

$$\forall n \in \mathbb{N}^*, 1 \leq 2Ab_n(x_k)^2(s_-(x_k) - s_{-n}(x_k)),$$

and then $\lim_{n\to\infty} b_n(x_k) = +\infty$. □

Proof of Corollary 3. We will only prove the part of the corollary that corresponds to what happens in $+\infty$. Let us assume that $v \in T_x\mathbb{A}\backslash G_-(x)$. We use the "old coordinates" (the usual ones) and write $v = (v_1, v_2)$. Because $v \notin G_-(x)$, we have $s_-(x)v_1 - v_2 \neq 0$. Let us compute $D\pi \circ Df^n(x)(v_1, v_2) = b_n(x)(v_2 - s_{-n}(x)v_1)$ with

$$\lim_{n\to+\infty}(v_2 - s_{-n}(x)v_1) = v_2 - s_-(x)v_1 \neq 0 \text{ and } \lim_{n\to+\infty} b_n(x) = +\infty.$$

We deduce that

$$\lim_{n\to+\infty} |D\pi \circ Df^n(x)v| = +\infty.$$ □

3.2. *Some easy consequences concerning* (*nonuniform*) *hyperbolicity*. All the results contained in this subsection are not new; see, for example, [9]. At first, an easy and well-known consequence of the dynamical criterion is the following:

PROPOSITION 4 (Contreras-Iturriaga). *Let M be an f-ordered and uniformly hyperbolic set where f is a C^1* ESPT. *Then at every $x \in M$, $G_-(x) = E^s(x)$ and $G_+(x) = E^u(x)$ are transverse.*

The proposition clearly follows from the characterization of the stable and unstable tangent spaces for a uniformly hyperbolic set and the dynamical criterion for G_- and G_+.

Let us now consider an irrational Mather measure μ for a positive twist map f. We have noticed that μ is ergodic. Hence we can associate to μ two Lyapunov exponents, $-\lambda$ and λ (because f is area preserving). If $\lambda \neq 0$, we say that the measure is (nonuniformly) hyperbolic and the Oseledet theorem asserts that at μ almost every point there exists a measurable splitting $T_x\mathbb{A} = E^s_x \oplus E^u_x$ in two transverse lines, invariant under Df such that

- $\forall v \in E^s_x, \lim_{n\to+\infty} \|Df^n(x)v\| = 0;$
- $\forall v \in E^u_x, \lim_{n\to+\infty} \|Df^{-n}(x)v\| = 0.$

Then we again deduce from the dynamical criterion that $G_-(x) = E^s(x)$ and $G_+(x) = E^u(x)$ are transverse μ almost everywhere:

PROPOSITION 5 (Contreras-Iturriaga). *Let μ be a Mather measure of a C^1 ESPT. If the Lyapunov exponents of μ are nonzero, then at μ almost all points, G_- and G_+ are transverse.*

We have explained why, for (nonuniformly) hyperbolic Mather measures, the Green bundles are transverse almost everywhere. We will now interest ourselves in the converse assertion: if the Green bundles are transverse (almost everywhere), is the dynamic (nonuniformly) hyperbolic?

We begin by the uniform case, and then consider the nonuniform one.

3.3. *What happens when the Green bundles are transverse everywhere.* It is known that, with some additional hypothesis, the transversality of the Green bundles implies hyperbolicity. For example, in [9], the authors prove that if $K \subset \mathrm{Green}(f)$ is an invariant compact subset such that on K, the Green bundles are transverse and such that $f_{|K}$ is nonwandering, then K is hyperbolic for f. As we know that the dynamic on Aubry-Mather sets is minimal and then nonwandering, we can deduce a result for the Aubry-Mather sets.

In fact, we prove that the hypothesis "$f_{|K}$ is nonwandering" is not necessary and that is why we give a new statement (it was also given in the introduction).

THEOREM 7. *Let f be a C^1 ESPT and let $K \subset \mathrm{Green}(f)$ be an invariant compact subset of $\mathrm{Green}(f)$ such that, at every point of K, $G_-(x)$ and $G_+(x)$ are transverse. Then K is uniformly hyperbolic and at every $x \in K$, we have $G_-(x) = E^s(x)$ and $G_+(x) = E^u(x)$.*

COROLLARY 4. *Let M be an irrational Aubry-Mather set for a C^1 ESPT f such that, at every point of M, $G_-(x)$ and $G_+(x)$ are transverse. Then M is uniformly hyperbolic and at every $x \in M$, we have $G_-(x) = E^s(x)$ and $G_+(x) = E^u(x)$.*

COROLLARY 5. *Let M be an irrational Aubry-Mather set for a C^1 ESPT f which is not uniformly hyperbolic. Then the set*

$$\mathcal{G}(M) = \{x \in M; G_-(x) = G_+(x)\}$$

is a dense G_δ-subset of M and at every $x \in \mathcal{G}(M)$, M is C^1-regular.

This corollary is a consequence of Theorems 7 and 6. In order to prove Theorem 7, let us give a definition.

Definition. Let $(F_k)_{k\in\mathbb{Z}}$ be a continuous cocycle on a linear normed bundle $P : E \to K$ above a compact metric space K. We say that the cocycle is *quasi-hyperbolic* if

$$\forall v \in E, v \neq 0 \Rightarrow \sup_{k\in\mathbb{Z}} \|F_k v\| = +\infty.$$

A consequence of the dynamical criterion (Corollary 3) is: If $K \subset \text{Green}(f)$ is a compact invariant subset of $\text{Green}(f)$ such that for every $x \in K$, $G_+(x)$ and $G_-(x)$ are transverse, then $(Df^k_{|K})_{k\in\mathbb{Z}}$ is a quasi-hyperbolic cocycle. Hence, we only have to prove the following statement to deduce the proof of Theorem 7.

THEOREM 8. *Let (F_k) be a continuous, symplectic and quasi-hyperbolic cocycle on a linear and symplectic (finite dimensional) bundle $P : E \to K$ above a compact metric space K. Then $(F_k)_{k\in\mathbb{Z}}$ is hyperbolic.*

We will deduce Theorem 8 from two lemmas we will now state and prove. The ideas of the two lemmas and their proofs are not really new. The reader can find similar statements in the setting of the so-called "quasi-Anosov diffeomorphisms," for example in [27].

LEMMA 3. *Let $(F_k)_{k\in\mathbb{Z}}$ be a continuous and quasi-hyperbolic cocycle on a linear normed bundle $P : E \to K$ above a compact metric space K. Let us define*

- $E^s = \{v \in E; \sup_{k\geq 0} \|F_k v\| < \infty\}$;
- $E^u = \{v \in E; \sup_{k\leq 0} \|F_k v\| < \infty\}$.

Then $(F_{n|E^s})_{n\geq 0}$ and $(F_{-n|E^u})_{n\geq 0}$ are uniformly contracting.

LEMMA 4. *Let $(F_k)_{k\in\mathbb{Z}}$ be a continuous and quasi-hyperbolic cocycle on a linear normed bundle $P : E \to K$ above a compact metric space K. If (x_n) is a sequence of points of K tending to x and (k_n) a sequence of integers tending to $+\infty$ such that $\lim_{n\to\infty} P \circ F_{k_n}(x_n) = y \in K$, then $\dim E^u(y) \geq \text{codim}E^s(x)$.*

Let us explain how to deduce Theorem 8 from these lemmas.

Proof of Theorem 8. If the dimension of E is $2d$, then we only have to prove that for all $x \in K$, $\dim E^u(x) = \dim E^s(x) = d$. Let us prove for example that $\dim E^u(x) = d$.

By Lemma 3, $(F_{n|E^s})_{n\geq 0}$ and $(F_{-n|E^u})_{n\geq 0}$ are uniformly contracting. As the cocycle is symplectic, we deduce that every $E^s(x)$ and $E^u(x)$ is isotropic for the symplectic form and then $\dim E^s(x) \leq d$ and $\dim E^u(x) \leq d$.
Let us now consider $x \in K$. As K is compact, we can find a sequence $(k_n)_{n\in\mathbb{N}}$ of integers tending to $+\infty$ such that the sequence $(P \circ F_{k_n}(x))_{n\in\mathbb{N}}$ converges to a point $y \in K$. Then, by Lemma 4, we have $\dim E^u(y) \geq \text{codim}E^s(x)$.

But we know that $\dim E^u(y) \le d$, hence $2d - \dim E^s(x) \le \dim E^u(y) \le d$ and $\dim E^s(x) = d$. □

Let us now prove the two lemmas.

Proof of Lemma 3. We will only prove the result for E^s. Let us assume that we know that

$$(*)\forall C > 1, \exists N_C \ge 1, \forall v \in E^s, \forall n \ge N_C, \|F_n v\| \le \frac{\sup\{\|F_k v\|; k \ge 0\}}{C}.$$

Then in this case,

$$\sup\{\|F_k v\|; k \ge 0\} = \sup\{\|F_k v\|; k \in |[0, N_C]|\}.$$

We define $M = \sup\{\|F_k(x)\|; x \in K, k \in |[0, N_C]|\}$. Then, if $j \in |[0, N_C - 1]|$ and $n \in \mathbb{N}$,

$$\begin{aligned}\|F_{nN_c+j} v\| &\le \frac{1}{C}\sup\{\|F_{(n-1)N_C+j+k} v\|; k \ge 0\}\\ &\le \frac{1}{C^2}\sup\{\|F_{(n-2)N_C+j+k} v\|; k \ge 0\}\\ \cdots &\le \frac{1}{C^n}\sup\{\|F_{j+k} v\|; k \ge 0\} \le \frac{1}{C^n}\sup\{\|F_k v\|; k \ge 0\} \le \frac{M}{C^n}\|v\|.\end{aligned}$$

This prove exponential contraction.

Let us now prove $(*)$. If $(*)$ is not true, then there exists $C > 1$, a sequence (k_n) in $\mathbb{N}$ tending to $+\infty$ and $v_n \in E^s$ with $\|v_n\| = 1$ such that

$$\forall n \in \mathbb{N}, \|F_{k_n} v_n\| \ge \frac{\sup\{\|F_k v_n\|; k \ge 0\}}{C}.$$

We define $w_n = \frac{F_{k_n}(v_n)}{\|F_{k_n}(v_n)\|}$. If we take a subsequence, we can assume that the sequence (w_n) converges to a limit $w \in E$. Then we have

$$\forall n \in \mathbb{N}, \forall k \in [-k_n, +\infty[, \|F_k w_n\| = \frac{\|F_{k+k_n}(v_n)\|}{\|F_{k_n} v_n\|} \le \frac{\sup\{\|F_j v_n\|; j \ge 0\}}{\|F_{k_n} v_n\|} \le C.$$

Hence, for all $k \in \mathbb{Z}$, $\|F_k w\| \le C$. This is impossible because $\|w\| = 1$, and the cocycle is quasi-hyperbolic. □

Proof of Lemma 4. With the notation of this lemma, we choose a linear subspace $V \subset E_x$ such that V is transverse to $E^s(x)$. We want to prove that $\dim E^u(y) \ge \dim V$. We choose $V_n \subset E_{x_n}$ such that $\lim\limits_{n\to\infty} V_n = V$. If we use a subsequence, we have $\lim\limits_{n\to\infty} F_{k_n}(V_n) = V' \subset E_y$. Then we will prove that $V' \subset E^u(y)$.

Let us assume that we have proved that there exists $C > 0$ such that

$$(*) \qquad \forall n, \forall 0 \le k \le k_n, \|F_{-k|F_{k_n}(V_n)}\| \le C.$$

Then, for all $w \in V'$ and $k \in \mathbb{Z}_-$, $\|F_k w\| \le C\|w\|$ and $w \in E^u(y)$.

Let us now assume that $(*)$ is not true. Replacing (k_n) by a subsequence, for all $n \in N$, we find an integer i_n between 0 and k_n such that $\|F_{-i_n|F_{k_n}(V_n)}\| \geq n$. We choose $w_n \in F_{k_n}(V_n)$ such that $\|w_n\| = 1$ and $\|F_{-i_n}(w_n)\| = \|F_{-i_n|F_{k_n}(V_n)}\|$. We may even assume that

$$\|F_{-i_n}(w_n)\| = \sup\{\|F_k(w_n)\|; -k_n \leq k \leq 0\} \geq n.$$

Then $\lim_{n\to+\infty} i_n = +\infty$. If $v_n = \frac{F_{-i_n}(w_n)}{\|F_{-i_n}(w_n)\|}$, we may extract a subsequence and assume that $\lim_{n\to\infty} v_n = v$, with $\|v\| = 1$. Then for all $k \in |[0, i_n]|$, we have $\|F_k v_n\| \leq \|v_n\|$ and therefore $\|F_k v\| \leq \|v\|$ for all $k \in \mathbb{N}$ and $v \in E^s$.

Now, we have two cases.

Case 1. $(k_n - i_n)$ *does not tend to* $+\infty$. We may extract a subsequence and assume that $\lim_{n\to+\infty}(k_n - i_n) = N \geq 0$. Then

$$F_{-N}v = \lim_{n\to\infty} F_{i_n-k_n}(v_n) = \lim_{n\to\infty} \frac{F_{-k_n}(w_n)}{\|F_{-i_n}(w_n)\|}.$$

We have $\frac{F_{-k_n}(w_n)}{\|F_{-i_n}(w_n)\|} \in V_n$, and then $F_{-N}v \in V$. Moreover, $F_{-N}v \in F_{-N}E^s = E^s$. As $\|v\| = 1$ and V is transverse to E^s_x, we obtain a contradiction.

Case 2. $\lim_{n\to\infty}(k_n - i_n) = +\infty$. In this case, for every $k = -k_n + i_n, \ldots, i_n$, we have $-k_n \leq k - i_n \leq 0$, and therefore $\|F_k v_n\| = \frac{\|F_{k-i_n} w_n\|}{\|F_{-i_n} w_n\|} \leq 1 = \|v_n\|$. Hence, since $v_n \to v$, $i_n \to +\infty$, and $-k_n + i_n \to -\infty$, when $n \to +\infty$, we obtain $\|F_k v\| \leq \|v\| = 1$ for all $k \in \mathbb{Z}$. This implies $v \in E^s \cap E^u$. This contradicts $\|v\| = 1$ and the fact that the cocycle is quasi-hyperbolic. □

3.4. *What happens for the Mather measures whose Green bundles are transverse almost everywhere.* Let us now consider a Mather measure of $f \in \mathcal{ET}^+_\omega$. The map $d : \mathrm{supp}(\mu) \to \{0, 1\}$ defined by $d(x) = \dim(G_-(x) \cap G_+(x))$ being measurable and constant along the orbits of f, we know that d is constant μ-almost everywhere. This constant is 0 or 1. In this subsection, we will study the case of a constant equal to zero and prove

Theorem 9. *Let $f \in \mathcal{ET}^+_\omega$ be a C^1* ESPT *and let μ be an irrational Mather measure for f. We assume that at μ-almost every point, G_- is transverse to G_+. Then the Lyapunov exponents of μ are nonzero.*

Corollary 6. *Let $f \in \mathcal{ET}^+_\omega$ be a C^1* ESPT *and let μ be an irrational Mather measure for f. We assume that the Lyapunov exponents of μ are zero. Then μ almost everywhere,* $\mathrm{supp}(\mu)$ *is C^1-regular.*

Indeed, in this case, $d = \dim(G_- \cap G_+)$ is μ-almost equal to 1, i.e., μ-almost everywhere, we have $G_- = G_+$. Hence we deduce from Theorem 6 that μ-almost everywhere, $\mathrm{supp}(\mu)$ is C^1-regular. We deduce

Corollary 7. *Let $f \in \mathcal{ET}_\omega^+$ be a C^1 ESPT and let μ be an irrational Mather measure for f. We assume that μ almost everywhere,* $\mathrm{supp}(\mu)$ *is C^1-irregular. Then the Lyapunov exponents of μ are nonzero.*

Proof of Theorem 9. We will use the same notations as in the proof of Proposition 3. At $x \in \mathrm{supp}(\mu)$, we have

$$M_n(x) = \begin{pmatrix} -b_n(x)s_{-n}(x) & b_n(x) \\ -b_n(x)^{-1} - b_n(x)s_{-n}(x)s_n(x_n) & s_n(x_n)b_n(x) \end{pmatrix}.$$

Instead of using a change of basis which sends G_- on the horizontal, we will use such a change which sends G_+ on the horizontal:

$$P(x) = \begin{pmatrix} 1 & 0 \\ -s_+(x) & 1 \end{pmatrix}.$$

In the new coordinates, the new matrix of $Df^n(x)$ is

$$N_n(x) = P(x_n)M_n(x)P(x)^{-1}$$

with

$$N_n(x) = \begin{pmatrix} b_n(x)(s_+(x) - s_{-n}(x)) & b_n(x) \\ 0 & b_n(x)(s_n(x_n) - s_+(x_n)) \end{pmatrix}.$$

In the proof we will use Lemma 1 and two other lemmas.

Lemma 5. *Let $\varepsilon > 0$. There exists a subset $K_\varepsilon \subset$* $\mathrm{supp}(\mu)$ *such that $\mu(K_\varepsilon) > 1 - \varepsilon$ and such that on K_ε, (s_{-n}) and (s_n) converge uniformly on K_ε to their limits s_- and s_+.*

This lemma is just a consequence of Egorov theorem (see, for example, [21]).

Lemma 6. *Let $\varepsilon > 0$. There exists a subset $F_\varepsilon \subset$* $\mathrm{supp}(\mu)$ *such that $\mu(F_\varepsilon) > 1 - \varepsilon$ and $\alpha > 0$ such that for all $x \in F_\varepsilon$, $s_+(x) - s_-(x) \geq \alpha$.*

Proof. We have assumed that at μ-almost every point $x \in \mathbb{A}$, $G_-(x)$ and $G_+(x)$ are transverse, i.e., $s_+(x) - s_-(x) > 0$. Hence

$$\mu\left(\bigcup_{n\geq 1}\left\{x; s_+(x) - s_-(x) \geq \frac{1}{n}\right\}\right) = 1.$$

As the previous union is monotone, we deduce that there exists $n \geq 1$ such that

$$\mu\left(\left\{x; s_+(x) - s_-(x) \geq \frac{1}{n}\right\}\right) \geq 1 - \varepsilon. \qquad \square$$

From these two lemmas we deduce that there exists J_ε and a constant $\alpha > 0$ such that $\mu(J_\varepsilon) \geq 1 - \varepsilon$, (s_n) and (s_{-n}) converge uniformly on J_ε and $s_+(x) - s_-(x) \geq \alpha$ for all $x \in J_\varepsilon$.

Lemma 7. *Let $A > 0$ and $\varepsilon > 0$. Then there exists $N = N(A, \varepsilon)$ such that*

$$\forall n \geq N, \forall x \in J_\varepsilon, f^n x \in J_\varepsilon \Rightarrow b_n(x) \geq A.$$

Proof. We use the matrix $N_n(x)$ and obtain

$$1 = \det N_n(x) = b_n(x)^2 (s_+(x) - s_{-n}(x))(s_n(x_n) - s_+(x_n))$$

with $x_n = f^n(x)$. By Lemma 1, there exists $B > 0$ such that for all $y \in \mathrm{supp}(\mu)$ and $k \in \mathbb{Z}$, $-B \leq s_k(x) \leq B$. Then for all $x \in \mathrm{supp}(\mu)$ and $n \in \mathbb{N}^*$, $0 < s_+(x) - s_{-n}(x) \leq 2B$. We deduce that for all $x \in \mathrm{supp}(\mu)$ and $n \in \mathbb{N}^*$, $1 \leq 2Bb_n(x)^2(s_n(x_n) - s_+(x_n))$.

By definition of J_ε, we know that s_n converge uniformly on J_ε to s_+. Hence there exists $N \geq 1$ such that for all $n \geq N$ and $y \in J_\varepsilon$, $0 < s_n(y) - s_+(y) < \frac{1}{2BA^2}$.

Let us now assume that $x, x_n = f^n(x) \in J_\varepsilon$. Then $1 \leq 2Bb_n(x)^2(s_n(x_n) - s_+(x_n)) \leq 2Bb_n(x)^2 \frac{1}{2BA^2} = \frac{b_n(x)^2}{A^2}$ and $b_n(x) \geq A$. □

To a given $\varepsilon > 0$, we have associated a set $J_\varepsilon \subset \mathrm{supp}(\mu)$ such that $\mu(J_\varepsilon) > 1 - \varepsilon$, (s_n) and (s_{-n}) converge uniformly on J_ε to their limits and $\forall x \in J_\varepsilon, s_+(x) - s_-(x) \geq \alpha > 0$. By Lemma 7, we find $N \geq 1$ such that

$$\forall x \in J_\varepsilon, \forall n \geq N, f^n(x) \in J_\varepsilon \Rightarrow b_n(x) \geq \frac{2}{\alpha}.$$

Let us notice that because μ is an irrational Mather measure, it is ergodic not only for f but also for f^N (we do not say in general that an ergodic measure for f is ergodic for f^N, but this is true for f homeomorphism of the circle with a irrational rotation number). If we denote by $\sharp Y$ the cardinal of a set Y, then we know by the ergodic theorem of Birkhoff (see, e.g., [28]) that for almost $x \in J_\varepsilon$,

$$\frac{1}{\ell} \sharp\{0 \leq k \leq \ell - 1; f^{kN}(x) \in J_\varepsilon\} \overset{\ell \to +\infty}{\longrightarrow} \mu(J_\varepsilon) \geq 1 - \varepsilon.$$

We denote by λ and $-\lambda$ the Lyapunov exponents of f (with $\lambda \geq 0$).

Then L_ε is the set of points of J_ε such that

- $\frac{1}{\ell} \sharp\{0 \leq k \leq \ell - 1; f^{kN}(x) \in J_\varepsilon\} \overset{\ell \to +\infty}{\longrightarrow} \mu(J_\varepsilon)$;
- x is a regular point for μ; i.e., at x there exists a splitting of the tangent space $T_x\mathbb{A}$ corresponding to the Lyapunov exponents (see, e.g., [28]).

Then $\mu(L_\varepsilon) = \mu(J_\varepsilon) \geq 1 - \varepsilon$ and if $x \in L_\varepsilon$, we have: $\lim\limits_{n \to +\infty} \frac{1}{n} \log \|Df^n(x)\| = \lambda$.

If $x \in L_\varepsilon$, we define

$$n(\ell) = \sharp\{0 \leq k \leq \ell - 1; f^{kN}(x) \in J_\varepsilon\},$$

and $0 = k(1) < k(2) < \cdots < k(n(\ell)) \leq \ell$ are such that $f^{k(i)N} x \in J_\varepsilon$.

The chain rule of derivatives implies that for all $x \in L_\varepsilon$,

$$Df^{k(n(\ell))N}(x) = Df^{(k(n(\ell))-k(n(\ell)-1))N}(f^{k(n(\ell)-1)N}x)$$
$$Df^{(k(n(\ell)-1)-k(n(\ell)-2))N}(f^{k(n(\ell)-2)N}x)\cdots Df^{k(1)N}(x).$$

We write this equality for the matrices N_k and especially for the terms a_k,

$$\begin{aligned}&b_{k(n(\ell))N}(x)(s_+(x) - s_{-k(n(\ell))N}(x))\\&\quad= b_{(k(n(\ell))-k(n(\ell)-1))N}(f^{k(n(\ell)-1)N}x)\\&\qquad\times \Delta s_{(k(n(\ell))-k(n(\ell)-1))N}(f^{k(n(\ell)-1)N}x)\cdots b_{k(1)N}(x)\Delta s_{k(1)N}(x),\end{aligned}$$

where $\Delta s_n(x) := s_+(x) - s_{-n}(x)$.

Let us notice that $\|Df^{k(n(\ell))N}(x)\| \geq b_{k(n(\ell))N}(x)(s_+(x) - s_{-k(n(\ell))N}) = a_{k(n(\ell))}$. Moreover, since we have $f^{k(j)N}(x) \in J_\varepsilon$ for every $0 \leq j \leq n(\ell)$, we know that $b_{(k(j+1)-k(j))N}(f^{k(j)N}(x)) \geq \frac{2}{\alpha}$ for every $0 \leq j \leq n(\ell) - 1$. Furthermore, $\Delta s_{(k(j+1)-k(j))N}(f^{k(j)N}(x)) > s_+((f^{k(j)N}(x))) - s_-(f^{k(j)N}(x)) \geq \alpha$. We deduce that

$$\|Df^{k(n(\ell))N}(x)\| \geq b_{k(n(\ell))N}(x)(s_+(x) - s_{-k(n(\ell))N}(x)) \geq (\frac{2}{\alpha}.\alpha)^{n(\ell)} = 2^{n(\ell)}.$$

We also deduce that

$$\frac{1}{k(n(\ell))N}\log\|Df^{k(n(\ell))N}(x)\| \geq \frac{n(\ell)}{k(n(\ell))N}\log 2.$$

But we have $k(n(\ell)) \leq \ell$; then $\frac{1}{k(n(\ell))N}\log\|Df^{k(n(\ell))N}(x)\| \geq \frac{n(\ell)}{\ell N}\log 2$.

As $\lim\limits_{\ell\to+\infty}\dfrac{n(\ell)}{\ell} = \mu(J_\varepsilon) \geq 1-\varepsilon$, we obtain

$$\lambda = \lim_{\ell\to+\infty}\frac{1}{k(n(\ell))N}\log\|Df^{k(n(\ell))N}(x)\| \geq \frac{1-\varepsilon}{N}\log 2 > 0;$$

hence the Lyapunov exponents are nonzero. □

4. **The hyperbolic case: proof of its irregularity**

4.1. *Case of uniform hyperbolicity.*

4.1.1. Theorem 10. *Let M be a uniformly hyperbolic irrational Aubry-Mather set of a C^1* ESPT *f of $\mathbb{A}$. Then at every $x \in M$, M is C^1-irregular.*

Proof of Theorem 10. At first, let us notice that such a M cannot be a curve. We proved in [2] that if the graph of a continuous map $\gamma : \mathbb{T} \to \mathbb{R}$ is invariant by f, then Lebesgue almost everywhere, we have $G_-(t, \gamma(t)) = G_+(t, \gamma(t))$, which contradicts Proposition 4 that asserts that $G_- = E^s$ and $G_+ = E^u$. Another argument is the fact, proved in [26], that $\pi(M)$ has zero Lebesgue measure.

Hence M is a Cantor and the dynamic on M is Lipschitz conjugate to the one of a Denjoy counterexample on its minimal invariant set. Then we consider two points $x \neq y$ of M such that there exists an open interval $I \subset \mathbb{T}$ whose ends are $\pi(x)$ and $\pi(y)$ and which does not meet $\pi(M)$: $I \cap \pi(M) = \emptyset$. From the dynamic of the Denjoy counter examples (see [17]) , we deduce that

- the positive and negative orbits of x and y under f are dense in M;
- $\lim_{n\to+\infty} d(f^n x, f^n y) = \lim_{n\to+\infty} d(f^{-n}x, f^{-n}y) = 0$.

As M is uniformly hyperbolic, we can define a local stable and unstable laminations on M (see, for example, [34]), W^s_{loc} and W^u_{loc}. Then for big enough n, $f^n x$ and $f^n y$ belong to the same local stable leaf, and $f^{-n}x$ and $f^{-n}y$ belong to the same local unstable leaf. Hence, because

$$\lim_{n\to+\infty} d(f^n x, f^n y) = \lim_{n\to+\infty} d(f^{-n}x, f^{-n}y) = 0,$$

for big enough n, the vector joining $f^n x$ to $f^n y$ (resp. $f^{-n}x$ to $f^{-n}y$) is close to the stable bundle E^s (resp. the unstable bundle E^u).

Now let $z \in M$ be any point. Then there exist two sequences (i_n) and (j_n) of integers which tend to $+\infty$ and are such that

$$\lim_{n\to+\infty} f^{i_n}x = \lim_{n\to+\infty} f^{i_n}y = \lim_{n\to+\infty} f^{-j_n}x = \lim_{n\to+\infty} f^{-j_n}y = z.$$

The direction of the "vector" joining $f^{i_n}x$ to $f^{i_n}y$ tends to $E^s(z)$, and the direction of the vector joining $f^{-j_n}x$ to $f^{-j_n}y$ tends to $E^u(z)$. Hence $E^u(z) \cup E^s(z) \subset P_M(z)$ and M is C^1-irregular at z. □

4.2. *Case of nonuniform hyperbolicity.*

THEOREM 11. *Let $f \in \mathcal{ET}_\omega$ be a C^1 ESPT and let μ be an irrational Mather measure of f whose Lyapunov exponents are nonzero. Then, at μ almost every point,* $\text{supp}(\mu)$ *is C^1-irregular.*

To prove this result, we will need some results concerning ergodic theory (see, for example, [32]). For us, every probability space (X, μ) will be such that X is a metric compact space endowed with its Borel σ-algebra.

Definition. Let (X, μ) be a probability space, T be a measure-preserving transformation of (X, μ) and $(f_n) \in L^1(X, \mu)$ be a sequence of μ-integrable functions from X to $\mathbb{R}$. Then (f_n) is T-*sub-additive* if for μ almost every $x \in X$ and $n, m \in \mathbb{N}$, we have $f_{n+m}(x) \leq f_n(x) + f_m(T^n x)$.

A useful result in ergodic theory is the following:

THEOREM (Sub-additive ergodic theorem, Klingman). *Let (X, μ) be a probability space, let T be a measure-preserving transformation of (X, μ) such that μ is ergodic for T and let $f = (f_n) \in L^1(X, \mu)$ be a T-sub-additive sequence. Then there exists a constant $\Lambda(f) \geq -\infty$ such that for μ-almost every*

$x \in X$, *we have*

$$\lim_{n\to+\infty} \frac{1}{n} f_n(x) = \Lambda(f).$$

Moreover, the constant $\Lambda(f)$ *satisfies*

$$\Lambda(f) = \lim_{n\to\infty} \frac{1}{n} \int f_n d\mu = \inf_n \frac{1}{n} \int f_n d\mu.$$

We will use the following refinement of this proposition, which concerns only the uniquely ergodic measures. A proof of it in the case of continuous functions is given in [12]; the proof for upper semi-continuous functions is exactly the same.

THEOREM (Furman). *Let* (X,μ) *be a probability space,* T *be a measure-preserving transformation of* (X,μ) *such that* μ *is uniquely ergodic for* T *and* $(f_n) \in L^1(X,\mu)$ *be a* T*-sub-additive sequence of upper semi-continuous functions. Let* $\Lambda(f)$ *be the constant associated to* f *via the sub-additive ergodic theorem. We assume that* $\Lambda(f) \in \mathbb{R}$*. Then*

$$\forall \varepsilon > 0, \exists N \geq 0, \forall n \geq N, \forall x \in X, \frac{1}{n} f_n(x) \leq \Lambda(f) + \varepsilon.$$

Proof of Theorem 11. At first, let us notice that the set R of points where $\mathrm{supp}(\mu)$ is C^1-regular is a G_δ subset of $\mathrm{supp}(\mu)$ and then is measurable. Let us assume that $\mu(R) = a > 0$. If $\mathrm{supp}(\mu)$ is the graph of γ above $\pi(\mathrm{supp}(\mu))$, then γ is differentiable at every $\theta \in \pi(R)$, and its derivative is continuous at such a θ. Moreover, R is invariant by f.

We know that there exists an orientation-preserving bi-Lipschitz homeomorphism $h : \mathbb{T} \to \mathbb{T}$ such that for all $(\theta, r) \in \mathrm{supp}(\mu)$, we have $\pi \circ f(\theta, r) = h(\theta)$. We denote by m the unique h-invariant probability measure on $\mathbb{T}$ (this measure is supported in $\pi(\mathrm{supp}(\mu))$).

We may choose h in a more precise way. If $I =]a, b[$ is an open interval which is a connected component of $\mathbb{T} \backslash \pi(\mathrm{supp}(\mu))$, then we may choose h affine on I. Let D be the (countable) set of the points of $\pi(\mathrm{supp}(\mu))$ which are ends of such intervals. Let us prove that every h^k is differentiable on $\pi(R)\backslash D$.

Let us consider $\theta \in \pi(R)\backslash D$ and $(\alpha_n) < (\beta_n)$ two sequences of elements of $\mathbb{T}$ converging to θ. Let $I_n = [\alpha_n^1, \alpha_n^2]$ (resp. $J_n = [\beta_n^1, \beta_n^2]$) be

- either the longest closed interval of $(\mathbb{T}\backslash\pi(\mathrm{supp}(\mu))) \cup D$ containing α_n (resp. β_n) if $\alpha_n \notin \pi(\mathrm{supp}(\mu))\backslash D$ (resp. $\beta_n \notin \pi(\mathrm{supp}(\mu))\backslash D$);
- or $\{\alpha_n\}$ (resp $\{\beta_n\}$) if $\alpha_n \in \pi(\mathrm{supp}(\mu))\backslash D$ (resp. $\beta_n \in \pi(\mathrm{supp}(\mu))\backslash D$).

As $\theta \notin D$, we have

$$\lim_{n\to\infty} \alpha_n^1 = \lim_{n\to\infty} \alpha_n^2 = \lim_{n\to\infty} \beta_n^1 = \lim_{n\to\infty} \beta_n^2 = \theta.$$

Moreover (we denote by CH the convex hull),

$$\frac{h^k(\alpha_n)-h^k(\beta_n)}{\alpha_n-\beta_n} \in \mathrm{CH}\left\{\frac{h^k(\alpha_n)-h^k(\alpha_n^2)}{\alpha_n-\alpha_n^2}, \frac{h^k(\alpha_n^2)-h^k(\beta_n^1)}{\alpha_n^2-\beta_n^1}, \frac{h^k(\beta_n^1)-h^k(\beta_n)}{\beta_n^1-\beta_n}\right\}.$$

(When the written slope is not defined, we do not write it.)

As h^k is affine on I_n and J_n, this last set is equal to

$$\mathrm{CH}\left\{\frac{h^k(\alpha_n^1)-h^k(\alpha_n^2)}{\alpha_n^1-\alpha_n^2}, \frac{h^k(\alpha_n^2)-h^k(\beta_n^1)}{\alpha_n^2-\beta_n^1}, \frac{h^k(\beta_n^1)-h^k(\beta_n^2)}{\beta_n^1-\beta_n^2}\right\}.$$

As $\alpha_n^1, \alpha_n^2, \beta_n^1, \beta_n^2 \in \pi(\mathrm{supp}(\mu))$ tend to $\theta \in \pi(R)$ when n goes to $+\infty$, we have (when the slope is defined, i.e., $\alpha_n^1 \neq \alpha_n^2$)

$$\begin{aligned}\lim_{n\to\infty} \frac{h^k(\alpha_n^1)-h^k(\alpha_n^2)}{\alpha_n^1-\alpha_n^2} &= \lim_{n\to\infty} \frac{\pi\circ f^k(\alpha_n^1,\gamma(\alpha_n^1)) - \pi\circ f^k(\alpha_n^2,\gamma(\alpha_n^2))}{\alpha_n^1-\alpha_n^2} \\ &= D\pi\circ Df^k(\theta,\gamma(\theta))(1,\gamma'(\theta))\end{aligned}$$

and similarly (if defined)

$$\lim_{n\to\infty} \frac{h^k(\alpha_n^2)-h^k(\beta_n^1)}{\alpha_n^2-\beta_n^1} = \lim_{n\to\infty} \frac{h^k(\beta_n^1)-h^k(\beta_n^2)}{\beta_n^1-\beta_n^2} = D\pi\circ Df^k(\theta,\gamma(\theta))(1,\gamma'(\theta)).$$

Hence $\lim_{n\to\infty} \frac{h^k(\alpha_n)-h^k(\beta_n)}{\alpha_n-\beta_n} = D\pi\circ Df^k(\theta,\gamma(\theta))(1,\gamma'(\theta))$.

Finally, every h^n is differentiable on $\pi(R)\backslash D$ and

$$\begin{aligned}\forall\theta\in\pi(R)\backslash D, \forall n\in\mathbb{N}, \lim_{\alpha,\beta\to\theta} \frac{h^n(\alpha)-h^n(\beta)}{\alpha-\beta} &= (h^n)'(\theta) \\ &= D\pi\circ Df^n(\theta,\gamma(\theta))(1,\gamma'(\theta)).\end{aligned}$$

For every $\theta \in \mathbb{T}$, we define $h_n'(\theta) = \liminf_{y\neq z\to\theta} \frac{h^n(z)-h^n(y)}{z-y} > 0$. Then every h_n' is lower semi-continuous and then measurable. As h is bi-Lipschitz, there exists $K_n > 1$ such that for every $x \in \mathbb{T}$, $\frac{1}{K_n} \leq h_n'(x) \leq K_n$. Hence every $g_n = -\log h_n'$ is bounded and measurable and thus belongs to $L^1(m)$, and the sequence $g = (g_n)_{n\geq 1}$ is a h-sub-additive sequence. Moreover, every g_n is upper semicontinuous. As m is uniquely ergodic for h, we may apply Furman's theorem:

$$\forall\varepsilon>0, \exists N\geq 0, \forall\theta\in\mathbb{T}, \forall n\geq N, \frac{1}{n}g_n(\theta) \leq \Lambda(g)+\varepsilon.$$

Let λ be the Lebesgue measure on $\mathbb{T}$. As $(-\log)$ is convex, we have by Jensen inequality

$$-\log\left(\int h_n' d\lambda\right) \leq -\int \log h_n' d\lambda = \int g_n d\lambda.$$

Moreover, h being Lipschitz is differentiable λ-almost everywhere and $\int h_n' d\lambda \leq \int (h^n)' d\lambda = \tilde{h}^n(1) - \tilde{h}^n(0) = 1$. Hence

$$0 = -\log 1 \leq -\log \left(\int h_n' d\lambda \right) \leq \int g_n d\lambda;$$

i.e., $\int g_n d\lambda \geq 0$. Let us now choose $\varepsilon > 0$. We know that there exists $N \geq 1$ such that for all $n \geq N$ and $x \in \mathbb{T}$, $\frac{1}{n} g_n(x) \leq \Lambda(g) + \varepsilon$ and thus for all $n \geq N$, $0 \leq \frac{1}{n} \int g_n d\lambda \leq \Lambda(g) + \varepsilon$. We deduce that $\Lambda(g) \geq 0$.

By Klingman theorem, we know that we have $\lim\limits_{n \to +\infty} \dfrac{1}{n} g_n(\theta) = \Lambda(g)$ for m-almost $\theta \in \mathbb{T}$. Hence for m-almost $\theta \in \pi(R) \backslash D$, we have $\lim\limits_{n \to +\infty} \dfrac{1}{n} g_n(\theta) \geq 0$; we denote by $A = \pi(R')$ the set of such θs. We have noticed that for such a θ, if $(\theta, r) \in \operatorname{supp}(\mu)$

- every h^n is differentiable at θ and even: $(h^n)'(\theta) = \lim\limits_{y \neq z \to \theta} \dfrac{h^n(z) - h^n(y)}{y - z}$ $= h_n(\theta)$ and then $g_n(\theta) = -\log((h^n)'(\theta))$;
- we have also seen that $(h^n)'(\theta) = D\pi \circ Df^n(\theta, r)(1, \gamma'(\theta))$.

Let us now denote by $\nu > 0, -\nu$ the Lyapunov exponents of μ for f. Then there exists a subset S of R' such that $\mu(S) = \mu(R') = a > 0$ and such that at every $(\theta, r) \in S$, we can define the Oseledet's splitting $E^s \oplus E^u$:

$$\begin{aligned} &\forall v \in E^u(\theta, r), \lim_{n \to \pm\infty} \frac{1}{n} \log \|Df^n v\| = \nu; \\ &\forall v \in E^s(\theta, r), \lim_{n \to \pm\infty} \frac{1}{n} \log \|Df^n v\| = -\nu. \end{aligned}$$

Then for $(\theta, r) \in S$, we have (we recall that γ' is bounded because $\operatorname{supp}(\mu)$ is Lipschitz)

$$\begin{aligned} \forall n \in \mathbb{N}^*, \frac{1}{n} \log \|Df^n(\theta, r)(1, \gamma'(\theta))\| &= \frac{1}{n} \log \|((h^n)'(\theta), \gamma'(h^n(\theta))(h^n)'(\theta))\| \\ &= \frac{1}{n} \log |(h^n)'(\theta)| + \frac{1}{n} \log \|(1, \gamma'(h^n(\theta)))\| \\ &= -\frac{1}{n} g_n(\theta) + \frac{1}{n} \log \|(1, \gamma'(h^n(\theta)))\| \\ &\overset{n \to \infty}{\longrightarrow} -\Lambda(g) \leq 0. \end{aligned}$$

We deduce that $(1, \gamma'(\theta)) \in E^s(\theta, r)$. A similar argument for n going to $-\infty$ (replacing f by f^{-1} and h by h^{-1}) proves that $(1, \gamma'(\theta)) \in E^u(\theta, r)$. As $E^u(\theta, r) \cap E^s(\theta, r) = \{0\}$, we obtain a contradiction. $\square$

5. Proof of the remaining results contained in the introduction

Notation. We denote the diffeomorphism $(\theta, r) \to (-\theta, r)$ of the 2-dimensional annulus $\mathbb{A}$ by I. The map $\mathcal{I} : f \in \mathcal{ET}_\omega \to I \circ f \circ I^{-1} \in \mathcal{ET}_\omega$ is then an involution $\mathcal{I}$ of $\mathcal{ET}_\omega$ such that $\mathcal{I}(\mathcal{ET}_\omega^+) = \mathcal{T}_\omega^-$.

Proof of Theorem 1. We assume that μ is an irrational Mather measure of $f \in \mathcal{ET}_\omega$; considering $\mathcal{I}(f)$ instead of f, we may assume that $f \in \mathcal{ET}_\omega^+$.

1) Let us assume that for μ-almost x, $\mathrm{supp}(\mu)$ is C^1-regular at x. Then by Theorem 11, the Lyapunov exponents of f are zero.

2) Let us assume that the Lyapunov exponents of μ are zero. Then we deduce from Corollary 6 that $\mathrm{supp}(\mu)$ is C^1-regular μ-almost everywhere. □

Proof of Theorem 2. We assume that μ is an irrational Mather measure of $f \in \mathcal{ET}_\omega$. Considering $\mathcal{I}(f)$ instead of f, we may assume that $f \in \mathcal{ET}_\omega^+$.

1) Let us assume that for μ-almost x, $\mathrm{supp}(\mu)$ is C^1-irregular at x. Then by Theorem 1, the Lyapunov exponents of μ are nonzero.

2) Let us assume that the Lyapunov exponents of μ are nonzero. Then by Theorem 11, $\mathrm{supp}(\mu)$ is C^1-irregular at μ-almost every point. □

Proof of Theorem 3. We assume that M is an irrational Aubry-Mather set of $f \in \mathcal{ET}_\omega$. Considering $\mathcal{I}(f)$ instead of f, we may assume that $f \in \mathcal{ET}_\omega^+$.

1) We assume that M is nowhere C^1-regular. By Theorem 6, at every $x \in M$, $G_+(x)$ and $G_-(x)$ are transverse. Hence, by Corollary 4, M is uniformly hyperbolic.

2) We assume that M is uniformly hyperbolic. Then by Theorem 10, M is nowhere C^1-regular. □

Proof of Theorem 4. Let $f \in \mathcal{ET}_\omega^+$ be a C^1 ESPT and let μ be an irrational Mather measure of f which is nonuniformly hyperbolic; i.e., the Lyapunov exponents are nonzero but the corresponding Aubry-Mather set $M = \mathrm{supp}(\mu)$ is not uniformly hyperbolic. The set $\mathcal{G}$ of the points x of M, where $G_-(x) = G_+(x)$, is a G_δ of M which is invariant by f. As $f_{|M}$ is minimal, either $\mathcal{G}$ is empty or it is a dense G_δ of M. Moreover, by Theorem 6, at every point of $\mathcal{G}$, M is C^1-regular.

Hence we only have to prove that $\mathcal{G} \neq \emptyset$. By Theorem 7, as M is not uniformly hyperbolic, $\mathcal{G} \neq \emptyset$. □

Proof of Theorem 5. Let $f \in \mathcal{ET}_\omega^+$ be a C^1 ESPT and let $\mathcal{C}$ be a C^1-invariant curve, which is a graph such that $f_{|\mathcal{C}}$ is C^1 conjugate to a rotation. Then we know (see [2], it is an easy consequence of the dynamical criterion) that at every $x \in \mathcal{C}$, $G_-(x) = G_+(x)$.

Then, by Theorem 6, the map $(x \in \mathrm{Green}(f) \to G_-(x))$ and $(x \in \mathrm{Green}(f) \to G_+(x))$ are continuous at every point of $\mathcal{C}$.

Let W be a neigbourhood of $T^1\mathcal{C}$, the unit tangent bundle to $\mathcal{C}$ in $T^1\mathbb{A}$ (the unit tangent bundle to $\mathbb{A}$). We may assume that W is "symmetrically fibered convex" (i.e., if $u, v \in W \cap T_x\mathbb{A}$, if $\mathbb{R}u \preceq \mathbb{R}w \preceq \mathbb{R}v$, then $w \in W$). We denote by G_-^1 and G_+^1 the unit Green bundles. Then there exists a neighbourhood V of $\mathcal{C}$ in $\mathbb{A}$ such that

$$\forall x \in \mathrm{Green}(f) \cap V, \;\; G_-^1(x) \cup G_+^1(x) \subset W.$$

Hence for every Aubry-Mather set M for f contained in V, $G_-^1(x), G_+^1(x) \subset W$ for all $x \in M$.

Moreover, by Theorem 6 we know that $G_-(x) \preceq P_M(x) \preceq G_+(x)$. For every Aubry-Mather set M for f contained in V, we deduce that

$$\forall x \in M, P_M^1(x) \subset W. \qquad \square$$

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